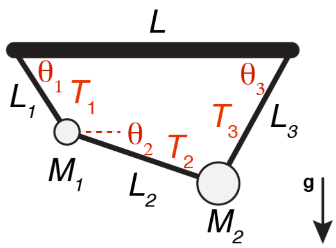
13_String_Problem

March 18, 2022

1 13 Linear Algebra: String Problem

1.1 Motivating problem: Two masses on three strings

Two masses M_1 and M_2 are hung from a horizontal rod with length L in such a way that a rope of length L_1 connects the left end of the rod to M_1, a rope of length L_2 connects M_1 and M_2, and a rope of length L_3 connects M_2 to the right end of the rod. The system is at rest (in equilibrium under gravity).



Find the angles that the ropes make with the rod and the tension forces in the ropes.

1.2 Theoretical background

Treat $\sin\theta_i$ and $\cos\theta_i$ together with T_i, $1 \le i \le 3$, as unknowns that have to simultaneously fulfill the nine equations

$$-T_1\cos\theta_1 + T_2\cos\theta_2 = 0 \tag{1}$$
$$T_1\sin\theta_1 - T_2\sin\theta_2 - W_1 = 0 \tag{2}$$
$$-T_2\cos\theta_2 + T_3\cos\theta_3 = 0 \tag{3}$$
$$T_2\sin\theta_2 + T_3\sin\theta_3 - W_2 = 0 \tag{4}$$
$$L_1\cos\theta_1 + L_2\cos\theta_2 + L_3\cos\theta_3 - L = 0 \tag{5}$$
$$-L_1\sin\theta_1 - L_2\sin\theta_2 + L_3\sin\theta_3 = 0 \tag{6}$$
$$\sin^2\theta_1 + \cos^2\theta_1 - 1 = 0 \tag{7}$$
$$\sin^2\theta_2 + \cos^2\theta_2 - 1 = 0 \tag{8}$$
$$\sin^2\theta_3 + \cos^2\theta_3 - 1 = 0 \tag{9}$$

Consider the nine equations a vector function $\mathbf{f}$ that takes a 9-vector $\mathbf{x}$ of the unknowns as argument:

$$\mathbf{f}(\mathbf{x}) = 0 \tag{10}$$

$$\mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} = \begin{pmatrix} \sin\theta_1 \\ \sin\theta_2 \\ \sin\theta_3 \\ \cos\theta_1 \\ \cos\theta_2 \\ \cos\theta_3 \\ T_1 \\ T_2 \\ T_3 \end{pmatrix} \tag{11}$$

$$\mathbf{L} = \begin{pmatrix} L \\ L_1 \\ L_2 \\ L_3 \end{pmatrix}, \quad \mathbf{W} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \tag{12}$$

Using the unknowns from above, our system of 9 coupled equations is:

$$-x_6 x_3 + x_7 x_4 = 0 \tag{13}$$
$$x_6 x_0 - x_7 x_1 - W_1 = 0 \tag{14}$$
$$-x_7 x_4 + x_8 x_5 = 0 \tag{15}$$
$$x_7 x_1 + x_8 x_2 - W_2 = 0 \tag{16}$$
$$L_1 x_3 + L_2 x_4 + L_3 x_5 - L = 0 \tag{17}$$
$$-L_1 x_0 - L_2 x_1 + L_3 x_2 = 0 \tag{18}$$
$$x_0^2 + x_3^2 - 1 = 0 \tag{19}$$
$$x_1^2 + x_4^2 - 1 = 0 \tag{20}$$
$$x_2^2 + x_5^2 - 1 = 0 \tag{21}$$

Solve the root-finding problem $\mathbf{f}(\mathbf{x}) = 0$ with the **generalized Newton-Raphson** algorithm:

$$\mathsf{J}(\mathbf{x})\Delta\mathbf{x} = -\mathbf{f}(\mathbf{x})$$

and

$$\mathbf{x} \leftarrow \mathbf{x} + \Delta\mathbf{x}.$$

1.3 Problem setup

Set the problem parameters and the objective function $\mathbf{f}(\mathbf{x})$

[1]:
```python
import numpy as np

# problem parameters
```

```python
W = np.array([10, 20])
L = np.array([8, 3, 4, 4])

def f_2masses(x, L, W):
    return np.array([
            -x[6]*x[3] + x[7]*x[4],
             x[6]*x[0] - x[7]*x[1] - W[0],
            -x[7]*x[4] + x[8]*x[5],
             x[7]*x[1] + x[8]*x[2] - W[1],
             L[1]*x[3] + L[2]*x[4] + L[3]*x[5] - L[0],
            -L[1]*x[0] - L[2]*x[1] + L[3]*x[2],
            x[0]**2 + x[3]**2 - 1,
            x[1]**2 + x[4]**2 - 1,
            x[2]**2 + x[5]**2 - 1,
        ])

def fLW(x, L=L, W=W):
    return f_2masses(x, L, W)
```

1.3.1 Initial values

Guess some initial values (they don't have to fullfil the equations!):

[2]:
```python
# initial parameters
#theta0 = np.deg2rad([45, 45, 90])
#T0 = np.array([1, 1, 2])
#x0 = np.concatenate([np.sin(theta0), np.cos(theta0), T0])

x0 = np.array([1.5, 0.5, 0.5, 0.5, 0.5, 0.5, 1, 1, 1])
```

[3]:
```python
x0
```

[3]:
```
array([1.5, 0.5, 0.5, 0.5, 0.5, 0.5, 1. , 1. , 1. ])
```

[4]:
```python
f_2masses(x0, L, W)
```

[4]:
```
array([  0. ,  -9. ,   0. , -19. ,  -2.5,  -4.5,   1.5,  -0.5,  -0.5])
```

1.3.2 Visualization

Plot the positions of the 2 masses and the 3 strings for any solution vector **x**:

[5]:
```python
import matplotlib
import matplotlib.pyplot as plt
%matplotlib inline
```

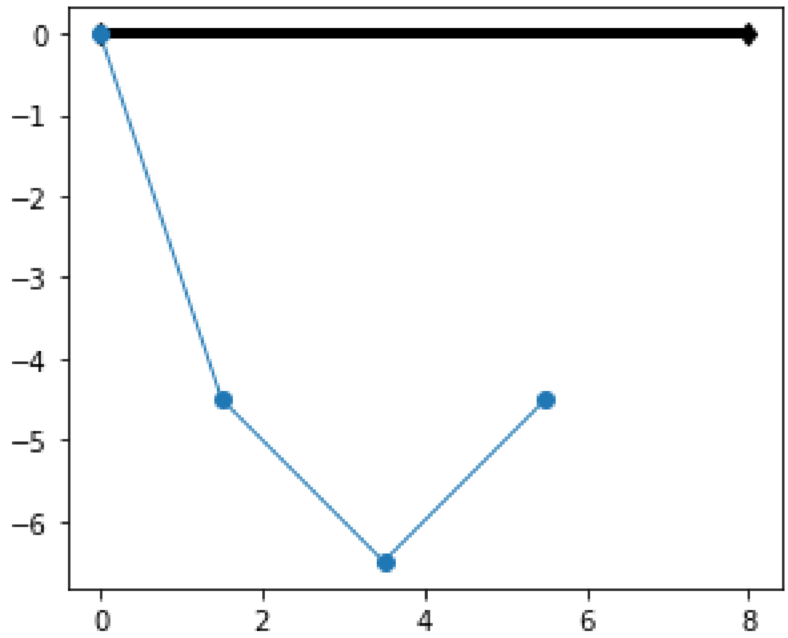
[6]:

```python
def plot_2masses(x, L, W, **kwargs):
    """Plot 2 mass/3 string problem for parameter vector x and parameters L and␣
↪W"""

    kwargs.setdefault('linestyle', '-')
    kwargs.setdefault('marker', 'o')
    kwargs.setdefault('linewidth', 1)

    ax = kwargs.pop('ax', None)
    if ax is None:
        ax = plt.subplot(111)

    r0 = np.array([0, 0])
    r1 = r0 + np.array([L[0], 0])
    rod = np.transpose([r0, r1])

    L1 = r0 + np.array([L[1]*x[3], -L[1]*x[0]])
    L2 = L1 + np.array([L[2]*x[4], -L[2]*x[1]])
    L3 = L2 + np.array([L[3]*x[5], L[3]*x[2]])
    strings = np.transpose([r0, L1, L2, L3])

    ax.plot(rod[0], rod[1], color="black", marker="d", linewidth=4)
    ax.plot(strings[0], strings[1], **kwargs)
    ax.set_aspect(1)
    return ax
```

What does the initial guess look like?

[7]:
```python
plot_2masses(x0, L, W)
```

[7]: <AxesSubplot:>



1.4 Jacobian

Write a function `Jacobian(f, x, h=1e-5)` that computes the Jacobian matrix numerically (use the central difference algorithm).

$$\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \tag{22}$$

$$J_{ij} = \frac{\partial f_i(x_1, \dots, x_j, \dots)}{\partial x_j} \tag{23}$$

$$\approx \frac{f_i(x_1, \dots, x_j + \frac{h}{2}, \dots) - f_i(x_1, \dots, x_j - \frac{h}{2}, \dots)}{h} \tag{24}$$

[8]:
```
def Jacobian(f, x, h=1e-5):
    """df_i/dx_j with central difference (fi(xj+h/2)-fi(xj-h/2))/h"""
    J = np.zeros((len(f(x)), len(x)), dtype=np.float64)
    hvec = np.zeros_like(x, dtype=np.float64)
    for j in range(len(x)):
        hvec *= 0
        hvec[j] = 0.5*h
        J[:, j] = (f(x + hvec) - f(x - hvec))/h
    return J
```

Test `Jacobian()` on

$$\mathbf{g}(\mathbf{x}) = \begin{pmatrix} x_0^2 - x_1 \\ x_0 \end{pmatrix}$$

with analytical result

$$\mathsf{J} = \left[\frac{\partial g_i}{\partial x_j}\right] = \begin{pmatrix} \frac{\partial g_0}{\partial x_0} & \frac{\partial g_0}{\partial x_1} \\ \frac{\partial g_1}{\partial x_0} & \frac{\partial g_1}{\partial x_1} \end{pmatrix} = \begin{pmatrix} 2x_0 & -1 \\ 1 & 0 \end{pmatrix}$$

Given a test vector $\mathbf{x}_{\text{test}} = (1, 0)$, what is the numerical answer for $\mathsf{J}(\mathbf{x}_{\text{test}})$?

$$\mathsf{J}(\mathbf{x}_{\text{test}}) = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

Test your `Jacobian()` function with $\mathbf{x}_{\text{test}}$ and check that you get the same answer:

[9]:
```python
def g(x):
    return np.array([
            x[0]**2 - x[1],
            x[0]
        ])
x_test = np.array([1, 0])
J = Jacobian(g, x_test)
print(J)
```

```
[[ 2. -1.]
 [ 1.  0.]]
```

Test that it also works for our starting vector:

[10]:
```python
J0 = Jacobian(fLW, x0)
J0
```

[10]:
```
array([[ 0. ,  0. ,  0. , -1. ,  1. ,  0. , -0.5,  0.5,  0. ],
       [ 1. , -1. ,  0. ,  0. ,  0. ,  0. ,  1.5, -0.5,  0. ],
       [ 0. ,  0. ,  0. ,  0. , -1. ,  1. ,  0. , -0.5,  0.5],
       [ 0. ,  1. ,  1. ,  0. ,  0. ,  0. ,  0. ,  0.5,  0.5],
       [ 0. ,  0. ,  0. ,  3. ,  4. ,  4. ,  0. ,  0. ,  0. ],
       [-3. , -4. ,  4. ,  0. ,  0. ,  0. ,  0. ,  0. ,  0. ],
       [ 3. ,  0. ,  0. ,  1. ,  0. ,  0. ,  0. ,  0. ,  0. ],
       [ 0. ,  1. ,  0. ,  0. ,  1. ,  0. ,  0. ,  0. ,  0. ],
       [ 0. ,  0. ,  1. ,  0. ,  0. ,  1. ,  0. ,  0. ,  0. ]])
```

[11]:
```python
J0.shape
```

[11]:
```
(9, 9)
```

1.5 n-D Newton-Raphson Root Finding

Write a function `newton_raphson(f, x, Nmax=100, tol=1e-8, h=1e-5)` to find a root for a vector function `f(x)=0`. (See also 12 Root-finding by trial-and-error and the *1D Newton-Raphson algorithm* in 12-Root-finding.ipynb.) As a convergence criterion we demand that the length of the vector `f(x)` (the norm — see `np.linalg.norm`) be less than the tolerance.

[12]:
```python
def newton_raphson(f, x, Nmax=100, tol=1e-8, h=1e-5):
    """n-D Newton-Raphson: solves f(x) = 0.

    Iterate until |f(x)| < tol or nmax steps.
    """
    x = x.copy()
    for istep in range(Nmax):
        fx = f(x)
        if np.linalg.norm(fx) < tol:
            break
        J = Jacobian(f, x, h=h)
        Delta_x = np.linalg.solve(J, -fx)
        x += Delta_x
    else:
        print("Newton-Raphson: no root found after {0} iterations (eps={1}); "
            "best guess is {2} with error {3}".format(Nmax, tol, x, fx))
    return x
```

1.5.1 Solve 2 masses/3 strings string problem

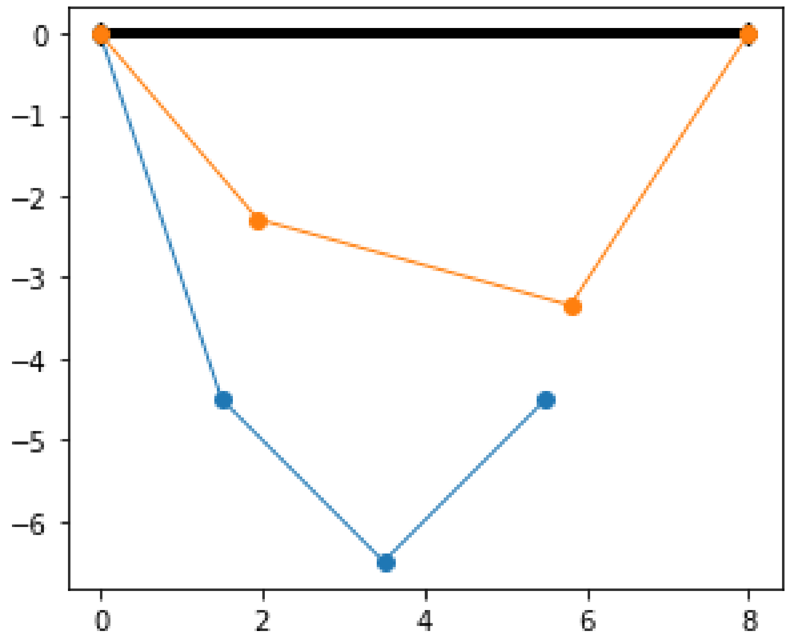
Solution

[13]:
```python
x = newton_raphson(fLW, x0)
print(x0)
print(x)
```

```
[1.5 0.5 0.5 0.5 0.5 0.5 1.  1.  1. ]
[ 0.76100269  0.26495381  0.83570583  0.64874872  0.9642611   0.54917735
 17.16020978 11.54527968 20.27152804]
```

Plot the starting configuration and the solution:

[14]:
```python
ax = plot_2masses(x0, L, W)
ax = plot_2masses(x, L, W, ax=ax)
```



Pretty-print the solution (angles in degrees):

[15]:
```
def pretty_print(x):
    theta = np.rad2deg(np.arcsin(x[0:3]))
    tensions = x[6:]
    print("theta1 = {0[0]:.1f} \t theta2 = {0[1]:.1f} \t theta3 = {0[2]:.1f}".
 ↪format(theta))
    print("T1     = {0[0]:.1f} \t T2     = {0[1]:.1f} \t T3     = {0[2]:.1f}".
 ↪format(tensions))
```

[16]:
```
print("Starting values")
pretty_print(x0)
print()
print("Solution")
pretty_print(x)
```

```
Starting values
theta1 = nan     theta2 = 30.0   theta3 = 30.0
T1     = 1.0     T2     = 1.0    T3     = 1.0

Solution
theta1 = 49.6    theta2 = 15.4   theta3 = 56.7
T1     = 17.2    T2     = 11.5   T3     = 20.3
```

```
/var/folders/sm/37rm_wm16tq9qsf4n5md98ph0000gp/T/ipykernel_97127/3554523550.py:2
: RuntimeWarning: invalid value encountered in arcsin
  theta = np.rad2deg(np.arcsin(x[0:3]))
```

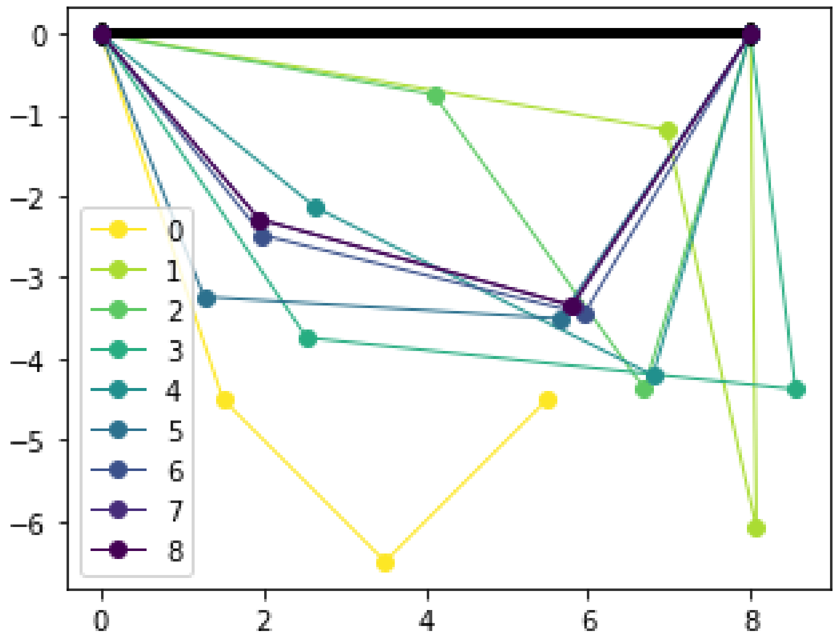
Show intermediate steps Create a new function `newton_raphson_intermediates()` based on `newton_raphson()` that returns *all* trial `x` values including the last one.

[17]:
```
def newton_raphson_intermediates(f, x, Nmax=100, tol=1e-8, h=1e-5):
    """n-D Newton-Raphson: solves f(x) = 0.

    Iterate until |f(x)| < tol or nmax steps.

    Returns all intermediates.
    """
    intermediates = []
    x = x.copy()
    for istep in range(Nmax):
        fx = f(x)
        if np.linalg.norm(fx) < tol:
            break
        J = Jacobian(f, x, h=h)
        Delta_x = np.linalg.solve(J, -fx)
        intermediates.append(x.copy())
        x += Delta_x
    else:
        print("Newton-Raphson: no root found after {0} iterations (eps={1}); "
            "best guess is {2} with error {3}".format(Nmax, tol, x, fx))
    return np.array(intermediates)
```

Visualize the intermediate configurations:

[18]:
```
x_series = newton_raphson_intermediates(fLW, x0)
```

[19]:
```
ax = plt.subplot(111)
ax.set_prop_cycle("color", [plt.cm.viridis_r(i) for i in np.linspace(0, 1,␣
 ↪len(x_series))])
for iteration, x in enumerate(x_series):
    plot_2masses(x, L, W, label=str(iteration), ax=ax)
ax.legend(loc="best");
```



It's convenient to turn the above plotting code into a function that we can reuse:

[20]:
```
def plot_series(x_series, L, W):
    """Plot all N masses/strings solution vectors in x_series (N, 9) array"""
    ax = plt.subplot(111)
    ax.set_prop_cycle("color", [plt.cm.viridis_r(i) for i in np.linspace(0, 1,␣
 ↪len(x_series))])
    for iteration, x in enumerate(x_series):
        plot_2masses(x, L, W, label=str(iteration), ax=ax)
    ax.legend(loc="best")
    return ax
```

1.6 Additional work

Try different masses, e.g. M1 = M2 = 10, or M1= 0 , M2 = 10.

Use nicer starting parameters that already fulfill the angle equations (7) - (9) (but it works with pretty much any guess):

[21]:
```
# initial parameters
theta0 = np.deg2rad([45, 45, 90])
T0 = np.array([1, 1, 2])
x0 = np.concatenate([np.sin(theta0), np.cos(theta0), T0])
```

1.6.1 M1 = M2 = 10

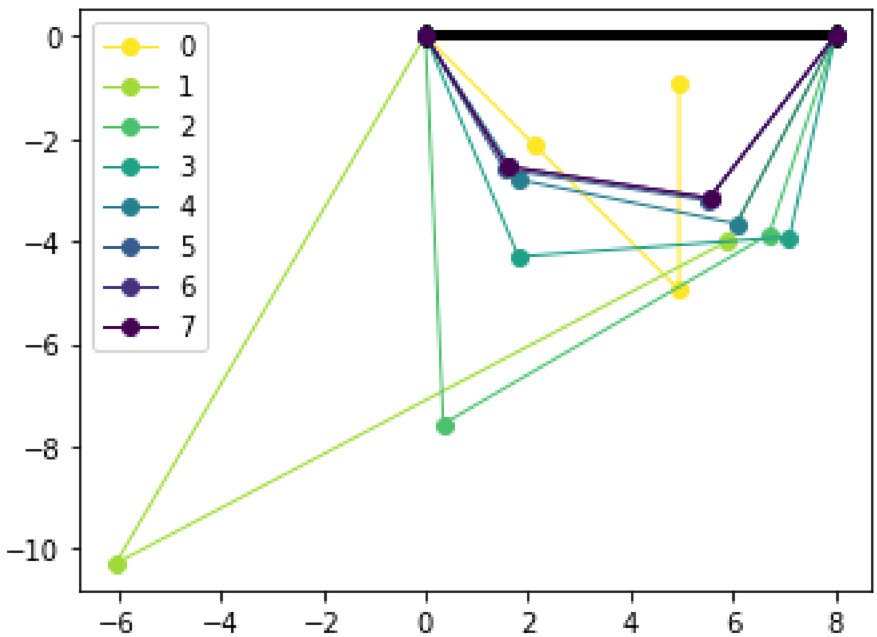
[22]:
```
W_2 = np.array([10, 10])
def fLW_2(x, L=L, W=W_2):
    return f_2masses(x, L, W)
```

[23]:
```
x_series_2 = newton_raphson_intermediates(fLW_2, x0)
pretty_print(x_series_2[-1])
```

```
theta1 = 57.9    theta2 = 8.8    theta3 = 52.1
T1     = 13.1    T2     = 7.0    T3     = 11.3
```

[24]:
```
plot_series(x_series_2, L, W_2)
```

[24]: <AxesSubplot:>



1.6.2 M1 = 0, M2 = 10

[25]:
```
W_3 = np.array([0, 10])
def fLW_3(x):
    return f_2masses(x, L=L, W=W_3)
```

[26]:
```
x_series_3 = newton_raphson_intermediates(fLW_3, x0)
pretty_print(x_series_3[-1])
```

```
theta1 = 30.0    theta2 = 30.0    theta3 = 61.0
T1     = 4.8     T2     = 4.8     T3     = 8.7
```

[27]:
```
plot_series(x_series_3, L, W_3)
```

[27]: <AxesSubplot:>