A Fast and Scalable Joint Estimator for Learning Multiple Related Sparse Gaussian Graphical Models

Beilun Wang, Ji Gao, Yanjun Qi

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R package: fasjem

http://jointggm.org

```
install.packages("fasjem")
library(fasjem)
demo(fasjem)
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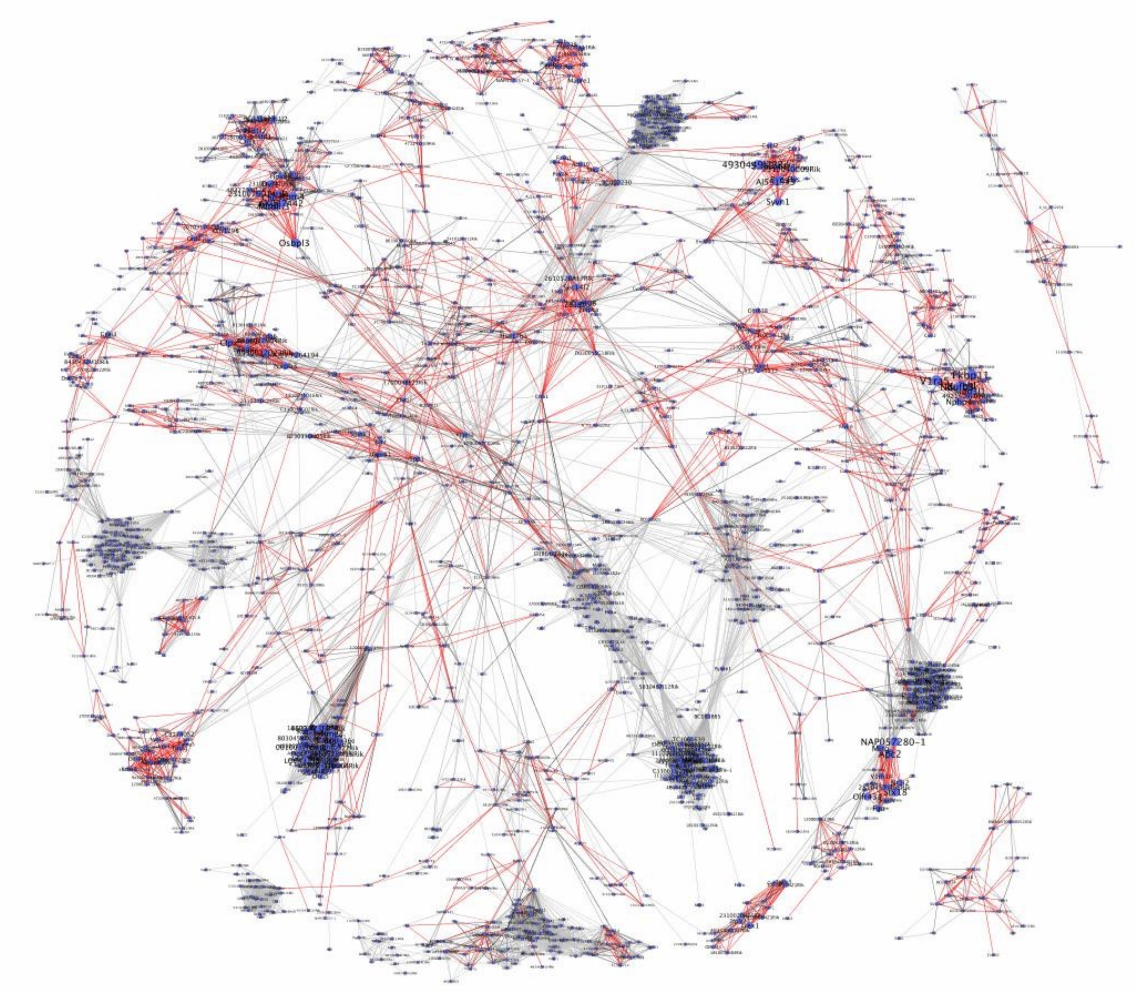
Motivation: Entity Graph

Interaction among genes

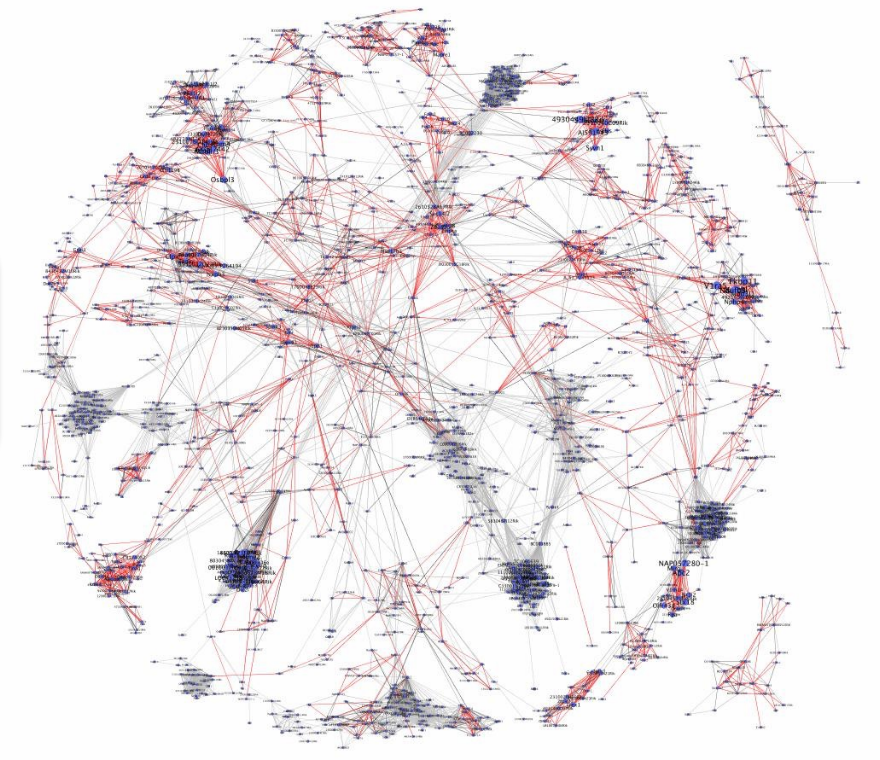


Social Network

Motivation: Data to Graph

Samples-->	T1	T2	T3	T4	T5	T6	T7	N1	N2	N3	N4	*P* value
Genes				Expression level relative to non-tumor pool								
Gene 1	2.4		2	2.5	1.5	2	2.3	1.7	1	1	0.81	9.5E-06
Gene 2	2.9	3.2	1.4	1.7	2.6	3.7	2.5	1	0.9	0.7	0.7	8.0E-05
Gene 3	2	3.5	1.4	1.8	2	2.5	2.2	0.7	1	0.7	0.6	8.6E-05
Gene 4	2.2	2.2	2.7	1.3	2.3	3.7	1.7	0.7	0.9	0.9	0.9	0.00012
Gene 5	5.2	2	3.7	2	5.8	3.2	1.6	0.7	0.7	0.7	0.6	0.00014
Gene 6	6.9	15	18	5.8	12	21	2.3	0.9	1.6	0.7	1.2	0.00015
Gene 7	5.4	2.1	3	2.2	3.5	2.8	1.5	0.8	1.2	0.9	0.8	0.00022
Gene 8	3.1	2.3	1.8	1.7	2.9	1.5	1.2	0.7	0.7	1	0.7	0.00023
Gene 9	9.7	25	23	6.1	9.5	23	2.4	1.2	1.6	0.8	1.1	0.00024
Gene 10	7.6	14	13	4.7	8.2	24	2.1	0.9	1	0.8	1.1	0.00025
Gene 11	4.8	7.7	2.1	2.3	6.6	3.7	7.4	1.1	1.2	0.6	1.2	0.00028
Gene 12	3.6	5.7	3.8	3.3	4.7	6.6	1.8	1.7	0.9	1.1	1	0.00029
Gene 13	5.7	9.8	12	4.5	6	17	1.7	1	1.3	0.8	0.8	0.00031
Gene 14	1.5	2.1	1	1.1	1.2	1.2	1.4	0.6	0.8	0.6	0.8	0.00031
Gene 15	2.5	2.9	1.9	1.8	5.5	2	1.3	1	0.8	0.7	0.7	0.0004
Gene 16	2.2	1.5	1.3	1.2	1.4	1.8	1.1	0.8	0.8	0.8	0.8	0.00042
Gene 17	5.9	2	3.4	2.5	4.3	3.1	2.1	1.2	1.3	1.5	1	0.00048
Gene 18	4	1.6	2.8	1.4	2.9	2.2	1.6	0.9	0.9	1	0.8	0.00052
Gene 19	1.6	1.5	2.3	1.4	1.4	1.8	1.6	1.1	1.2	1	1.1	0.00059
Gene 20	3.9	6.7	6.6	2.3	4	11	1.5	0.8	1.1	0.8	0.8	0.00059
Gene 21	5.3	1.8	2.6	1.4	3.4	2.2	1.6	0.7	0.8	0.8	0.8	0.0006
Gene 22	4.2	1.9	1	2	4.2	4.3	7.9	16	1.1	1.3	10.9	0.00061
Gene 23	2.3	1.3	2.5	1.8	5.7	2.2	1.8	1.1	0.7	0.6	0.7	0.00066
Gene 24	2.9	1	2.9	2.2	3.8	1.9	2.3	0.9	0.8	0.9	0.8	0.00071
Gene 25	2.6	1.4	1.7	1.4	2.4	1.7	1.3	0.9	0.9	0.9	0.9	0.00079
Gene 26	5.8	2.3	3.4	2.1	5	4.7	1.7	1.2	0.9	1.2	0.6	0.0009
Gene 27	5.7	2	4	2.4	5	3.5	1.5	0.8	1.2	1.3	1.1	0.00093
Gene 28	1.6	2.8	1.7	1.7	1.5	2.8	1.9	1.2	1.1	0.8	1.1	0.00094

trends in Biotechnology



Many entities' data

Inference Important

Few entities graph

Motivation: Data Heterogeneity across context

Samples of the same set of genes(human genes)
Vary across Normal vs Leukemia vs Stem

Normal	Leukemia	Stem
rows 1, 2, ..., p: $X^{1}_{p*n_1}$	rows 1, 2, ..., p: $X^{2}_{p*n_2}$	rows 1, 2, ..., p: $X^{3}_{p*n_3}$

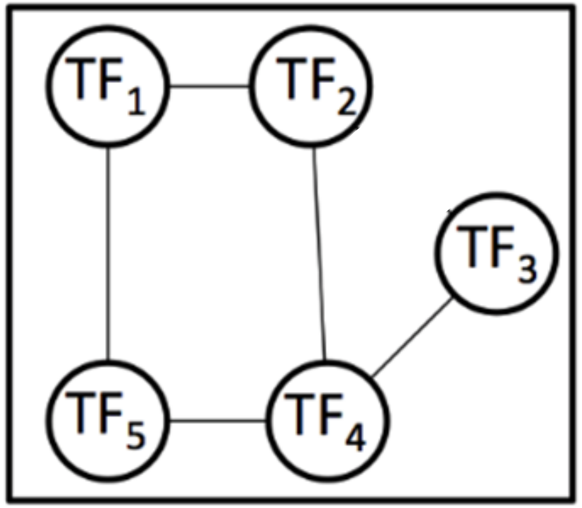
Notation

p represents the number of nodes or features

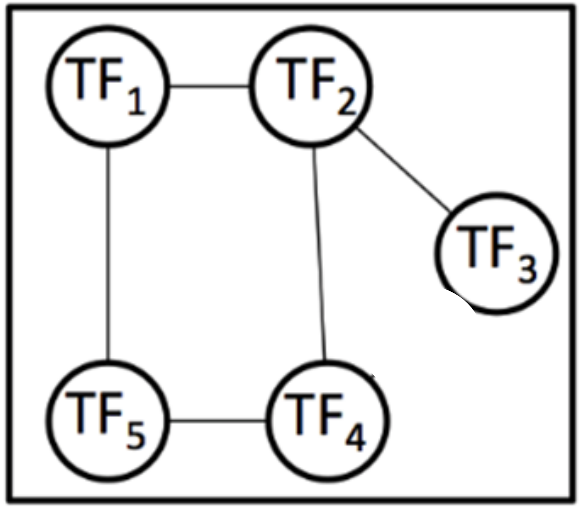
K represents the number of tasks or contexts

Motivation: Entity Graphs vary across contexts

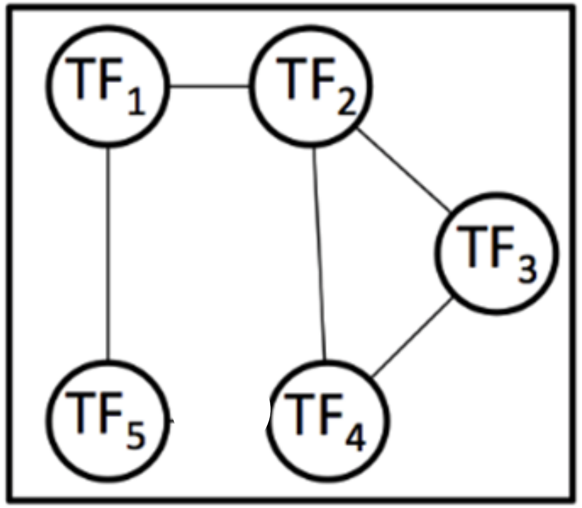
Different but related entity graphs



Normal



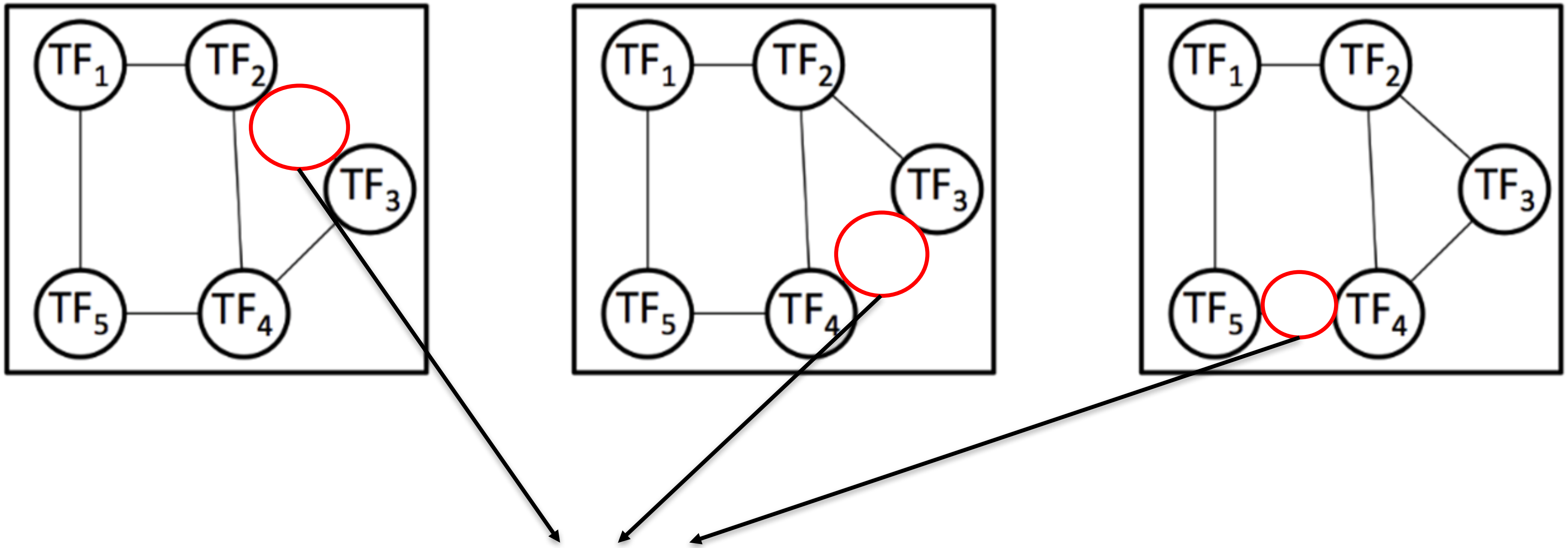
Leukemia



Stem

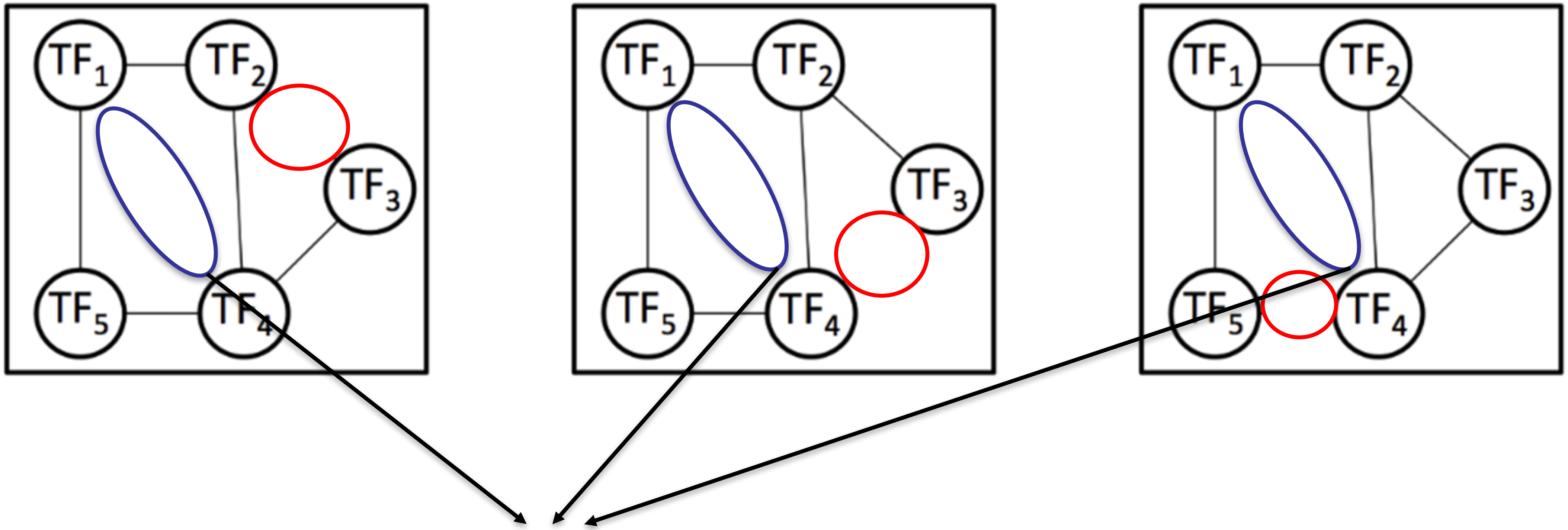
Graphs are sparse

Difference among related graphs: Sparsity



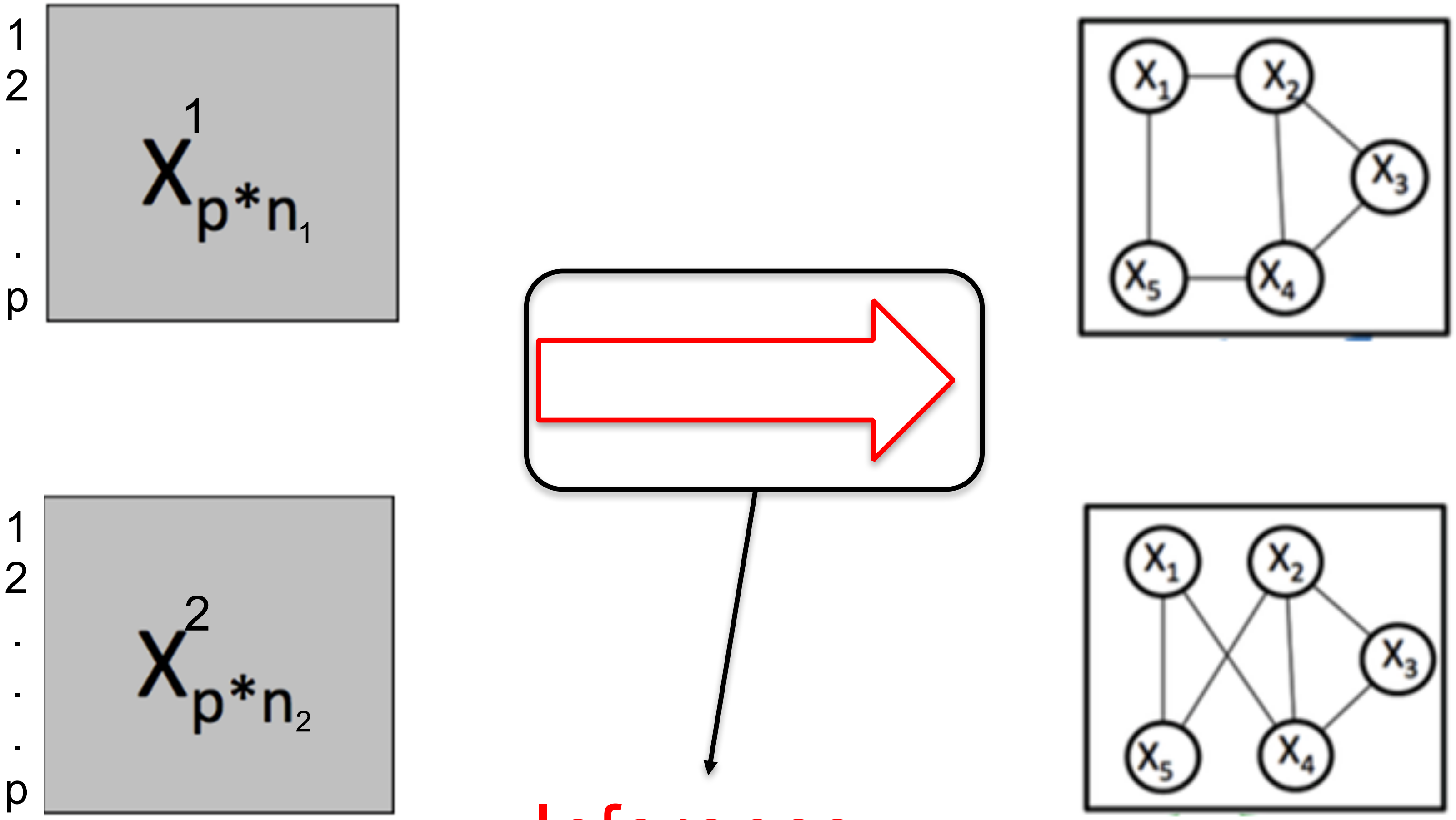
Sparsity Patterns are different
e.g., (TF_2, TF_3)

Similarity among related graphs: Group Sparsity



Group Sparsity means
e.g., (TF_1, TF_4) no edge pattern across three

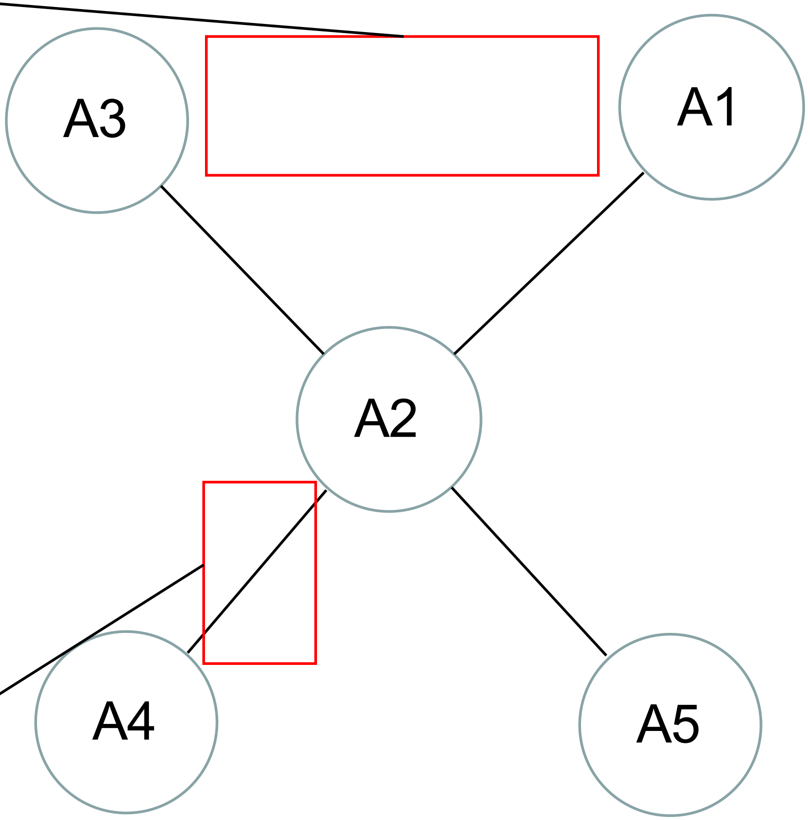
Motivation: Data to Graphs across context



Inference Important

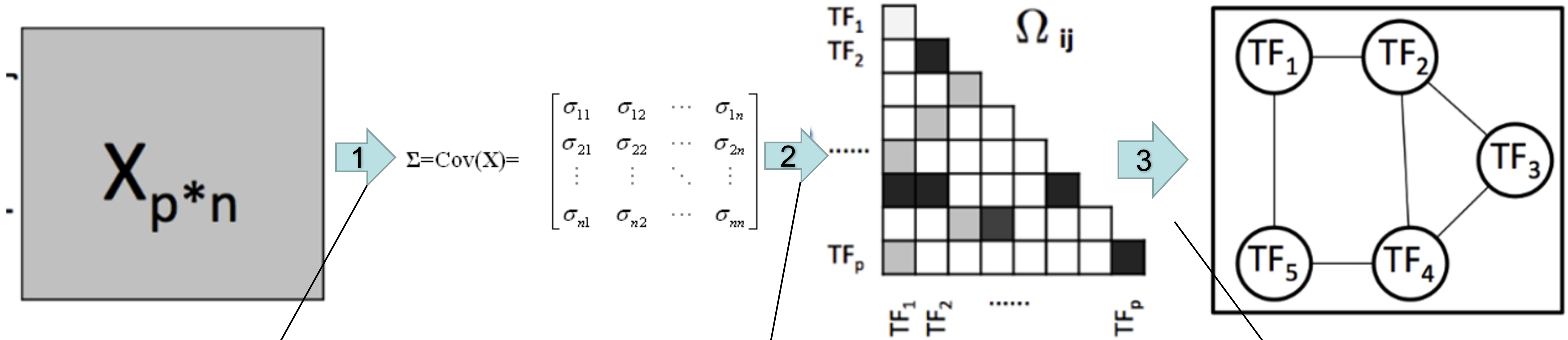
Motivation: Entity Graph – Conditional Independence Graph

Conditional Independent



Conditional Dependent

Background: sparse Gaussian Graphical Model(sGGM) to derive Conditional Independence Graph from data



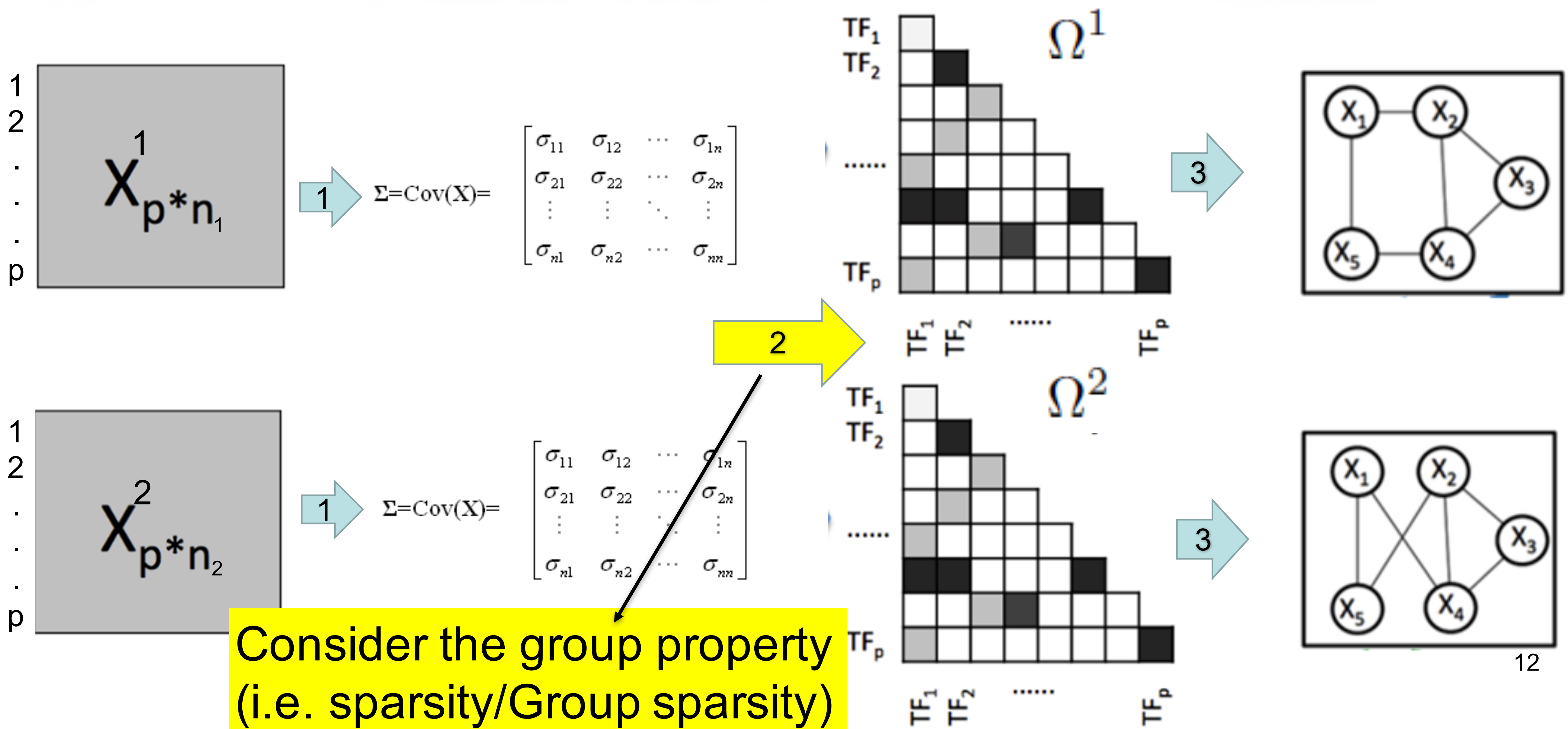
Step1:
Calculate the Covariance matrix
X → Σ

Step2:
Estimate Sparse Inverse of Covariance matrix

$$\boxed{\Sigma^{-1} = \Omega}$$

Step3:
Transfer sparse matrix to Conditional Independence graph

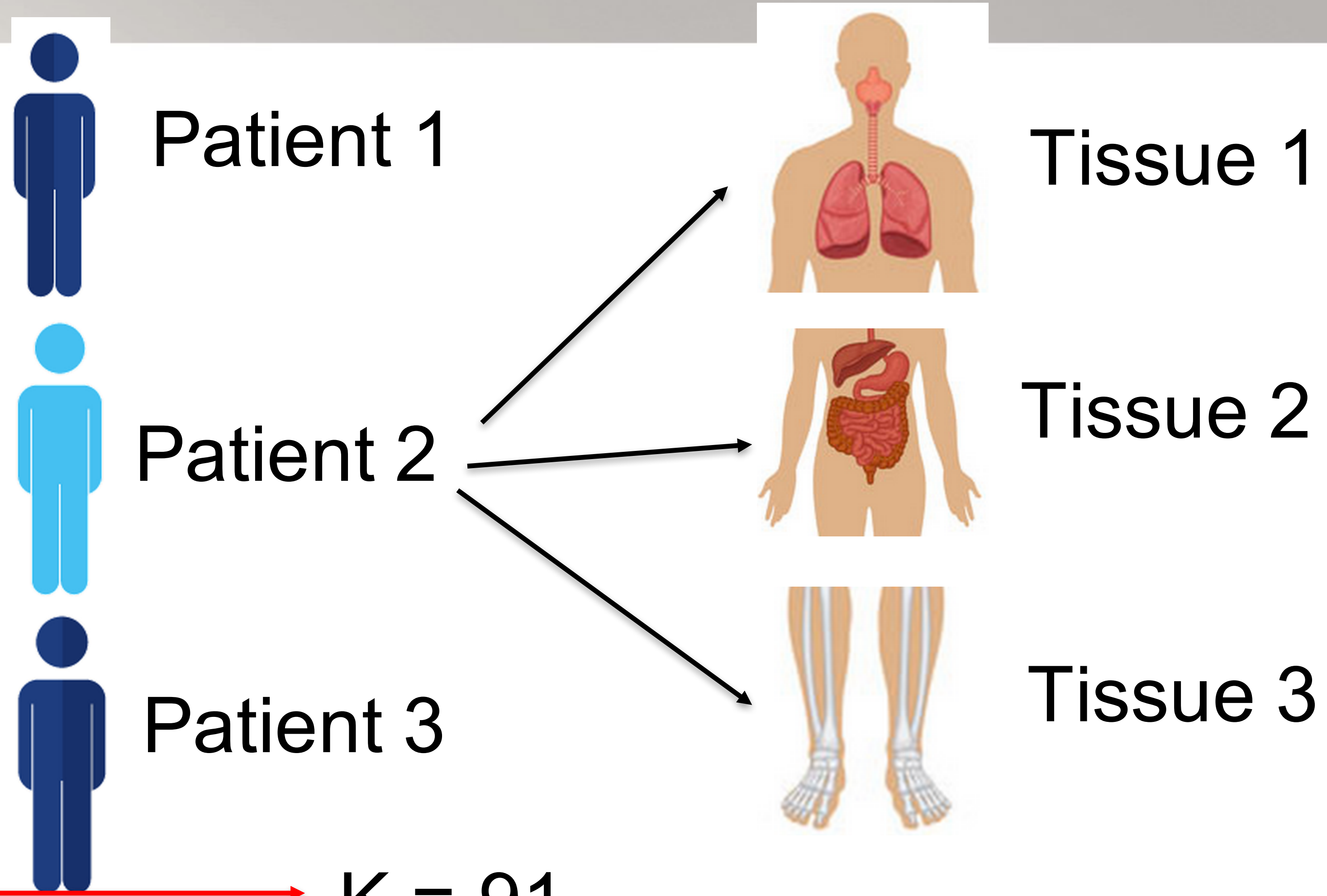
Background Model: Multi-task sGGM



Consider the group property
(i.e. sparsity/Group sparsity)

Motivation: More tasks(K) to be considered

Normal vs Cancer



K = 2 → K = 91

ENCODE Project Consortium et al. An integrated encyclopedia of dna elements in the human genome. *Nature*, 489(7414):57–74, 2012.

Motivation: More Num of features(p) to consider

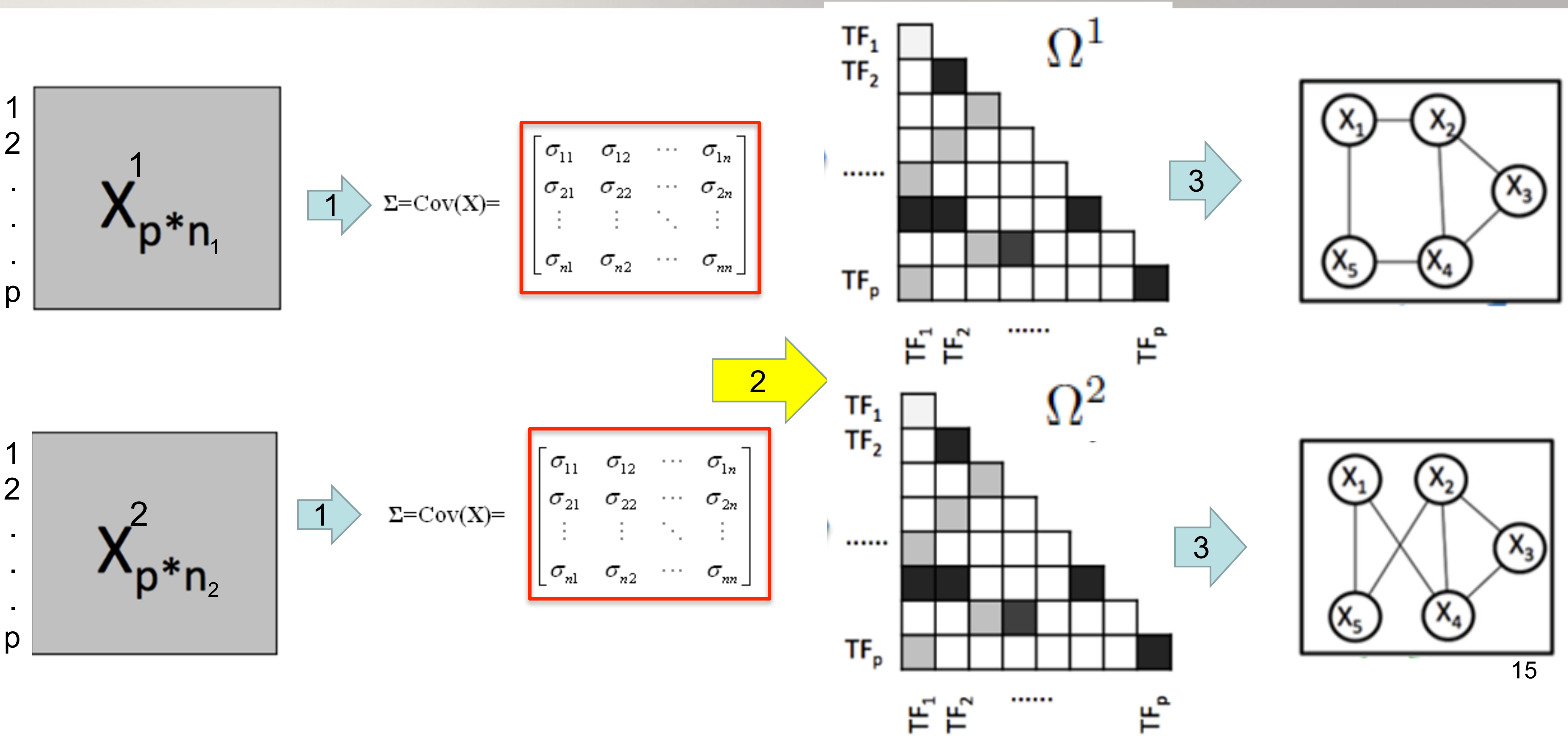
e.g. Yeast gene: 6K

↓

Human gene: 30K

ENCODE Project Consortium et al. An integrated encyclopedia of dna elements in the human genome. *Nature*, 489(7414):57–74, 2012.

Limitation of Previous Methods: Storage



Limitation of Previous Methods : Storage

e.g., calculate the gradient

$$\Sigma=\mathrm{Cov}(X)=\boxed{\begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix}}$$

$$\Sigma=\mathrm{Cov}(X)=\boxed{\begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix}}$$

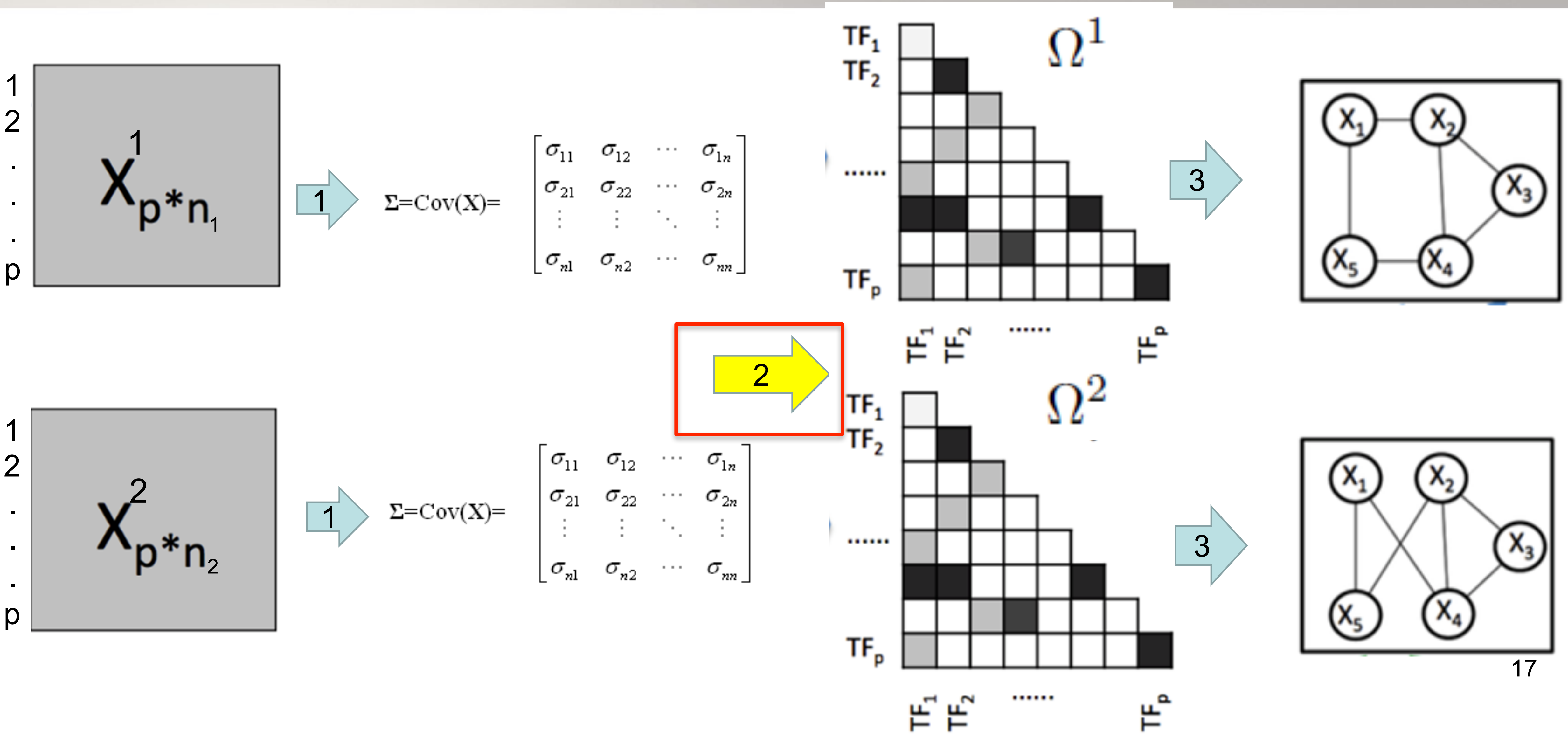
$$\Sigma=\mathrm{Cov}(X)=\boxed{\begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix}}$$

K = 91, p= 30K

$O(Kp^2)$ in memory

Double type: 65 TB

Limitation of Previous Methods: Speed



Limitation of Previous Methods: Speed

Suppose they have same iteration number T

Traditional Optimization Method

K = 91, p= 30K

---- Block Coordinate Descent : $O(K^3p^4)$ / Itera

more than 2 billion years

Improved Optimization:

---- Still needs SVD for each covariance matrix

SVD for the matrices needs $O(Kp^3)$ / Itera $\longrightarrow$ 3.5 days

Roadmap

1. Goal & Background
2. Proposed
3. Evaluation
4. Conclusion

Goal

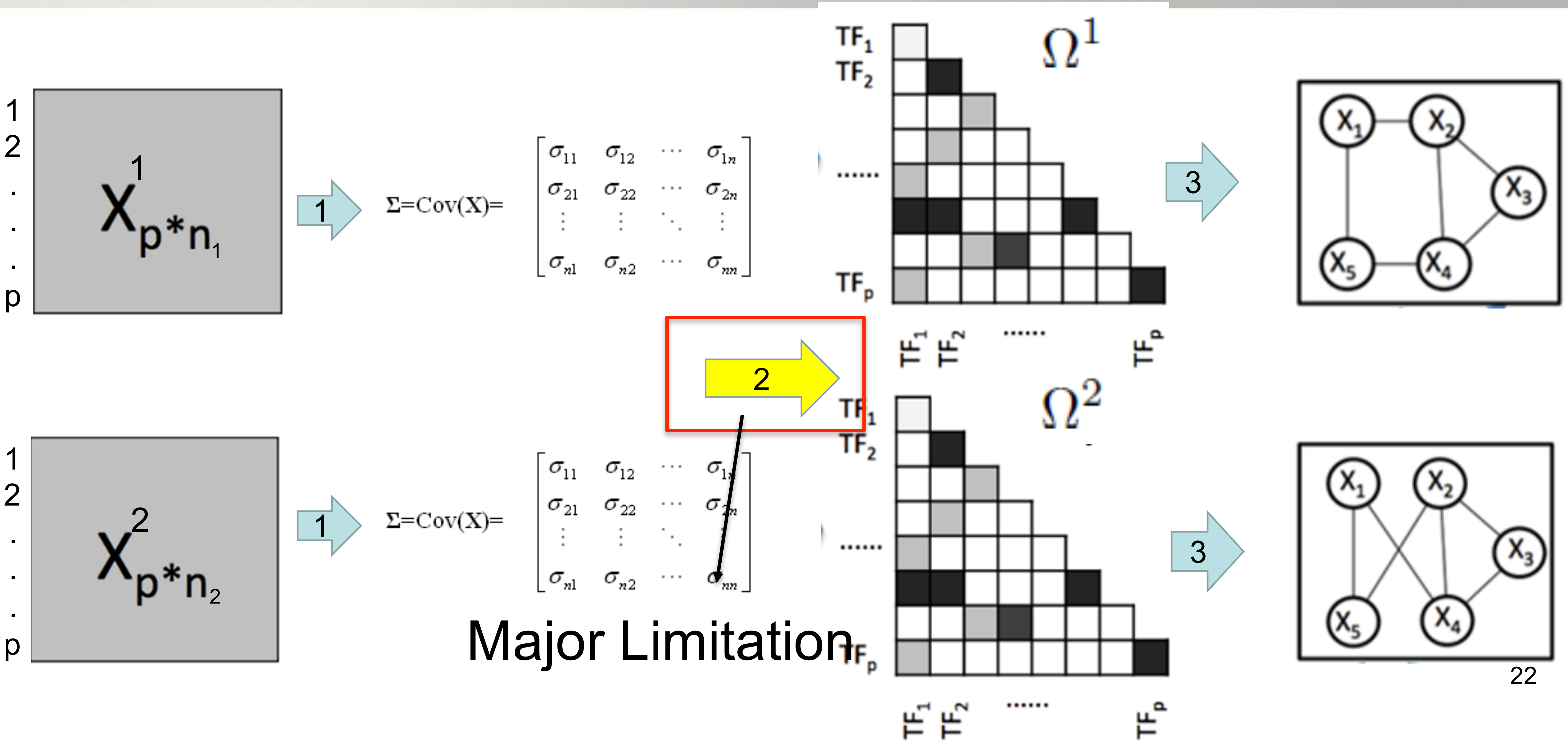
1. Design a fast and scalable joint estimator for multi-task sparse Gaussian Graphical Model

2. Prove the theoretical Bound for our estimator

Roadmap

1. Goal & Background
2. <u>Proposed</u>
3. Evaluation
4. Conclusion

Major Limitation of Previous : Optimization

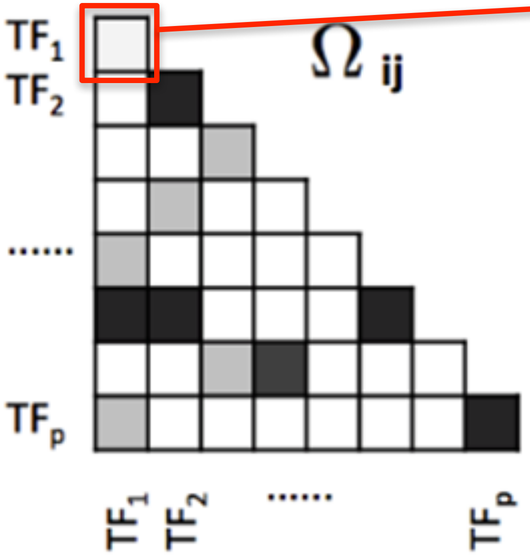


Notation: Entry

$$\Sigma = \mathrm{Cov}(X) = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix}$$

Ω^1

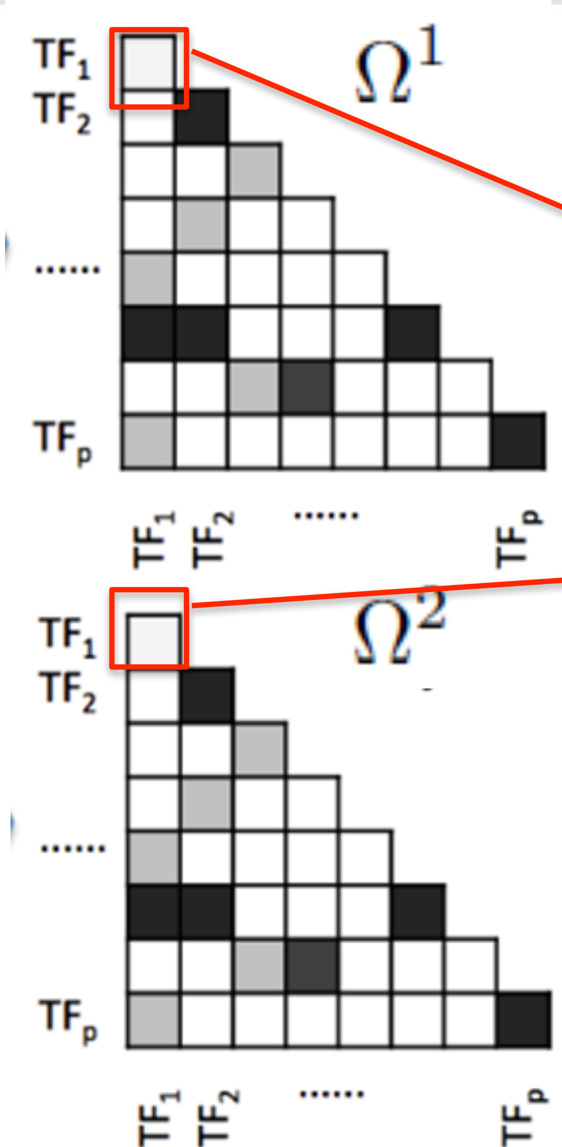
Ω_{ij}



Entry

(Ω_{ij})

Notation: Group Entries among all tasks



Group Entries

$$(\Omega^1_{ij}, \Omega^2_{ij}, \ldots, \Omega^K_{ij})$$

Our Model

- ✓ Traditional Models:

Penalized log-likelihood model $\longrightarrow$ Better optimization method

Some expensive computation is because of model itself!

- ✓ Proposed Model:

New model $\longrightarrow$ Entry-wise(group entry-wise) optimization method

Equivalent Forms of Constrained Optimization

$$\boxed{\begin{aligned}&\min_x f(x)\\&\text{subject to: } g(x) \le c_1\end{aligned}} \Leftrightarrow \boxed{\min_x f(x) + \lambda g(x)} \Leftrightarrow \min_x \tfrac{1}{\lambda} f(x) + g(x) \Leftrightarrow \begin{aligned}&\min_x g(x)\\&\text{subject to: } f(x) \le c_2\end{aligned}$$

$$\underset{\Omega^{(i)} \succ 0}{\operatorname{argmin}} \sum_i \boxed{\left(-L(\Omega^{(i)})\right)}$$

Subject to: 1. sparse
2. group sparse

Initial problem

$$\underset{\Omega^{(i)} \succ 0}{\operatorname{argmin}} \sum_i \boxed{\left(-L(\Omega^{(i)})\right)} + \lambda_1 \sum_i ||\Omega^{(i)}||_1 + \lambda_2 P(\Omega^{(1)}, \Omega^{(2)}, \dots, \Omega^{(K)})$$

Traditional Models

Equivalent Forms of Constrained Optimization

$$\min_x f(x) \quad \text{subject to: } g(x) \le c_1 \iff \boxed{\min_x f(x) + \lambda g(x)} \iff \min_x \tfrac{1}{\lambda} f(x) + g(x) \iff \boxed{\min_x g(x) \quad \text{subject to: } f(x) \le c_2}$$

$$\operatorname*{argmin}_{\Omega^{(i)}>0} \sum_i (-L(\Omega^{(i)}) + \lambda_1 \sum_i ||\Omega^{(i)}||_1 + \lambda_2 P(\Omega^{(1)}, \Omega^{(2)}, \ldots, \Omega^{(K)})$$

Traditional Models

$$\operatorname*{argmin}_{\Omega_{tot}} |\Omega_{tot}|_1 + \epsilon R'(\Omega_{tot})$$

$$s.t. |\Omega_{tot} - inv(T_v(\Sigma_{tot}))|_\infty \le \lambda_n$$

$$\mathcal{R}'^*(\Omega_{tot} - inv(T_v(\Sigma_{tot}))) \le \epsilon\lambda_n$$

Proposed Models

Our Model: FASJEM

Fast and Scalable Joint Estimator for Multiple related sparse Gaussian Graphical Model

$$\underset{\Omega_{tot}}{\operatorname{argmin}} |\Omega_{tot}|_1 + \epsilon R'(\Omega_{tot})$$

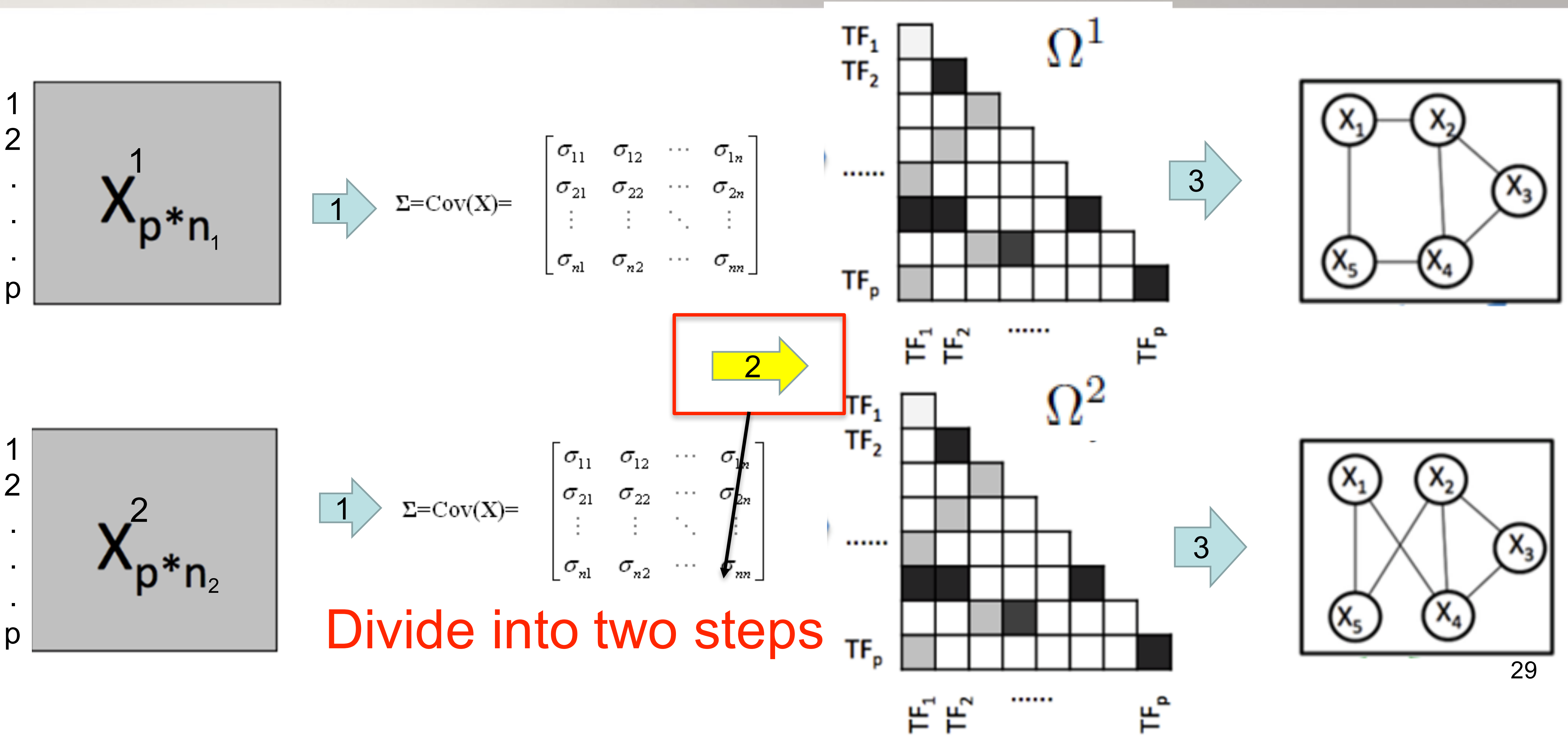
$$s.t. |\Omega_{tot} - inv(T_v(\Sigma_{tot}))|_\infty \leq \lambda_n$$

$$\mathcal{R}'^*(\Omega_{tot} - inv(T_v(\Sigma_{tot}))) \leq \epsilon\lambda_n$$

Here R' is another penalty norm and R'* is the dual norm of R'

$$\Omega_{tot} = (\Omega^{(1)}, \Omega^{(2)}, \ldots, \Omega^{(K)}) \qquad \Sigma_{tot} = (\Sigma^{(1)}, \Sigma^{(2)}, \ldots, \Sigma^{(K)})$$

Optimization: Structure



Divide into two steps

Optimization: Structure

- Step I: Pre-compute and pre-store(not in the memory) approximated backward mapping matrix $\mathcal{B}^*(\Sigma_{tot})$

$$\Sigma_{tot} \longrightarrow \mathcal{B}^*(\Sigma_{tot})$$

- Step II: Use proximity algorithm(entry-wise and group entry-wise) to solve the optimization problem.

$$\mathcal{B}^*(\Sigma_{tot}) \longrightarrow \Omega_{tot}$$

Optimization: Structure

$$\Sigma = \mathrm{Cov}(X) = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix}$$

$$\mathcal{B}^{*(1)} := \begin{pmatrix} b_{11}^{(1)} & b_{12}^{(1)} & \cdots & b_{1p}^{(1)} \\ b_{21}^{(1)} & b_{22}^{(1)} & \cdots & b_{2p}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1}^{(1)} & b_{p2}^{(1)} & \cdots & b_{pp}^{(1)} \end{pmatrix}$$

Pre-compute

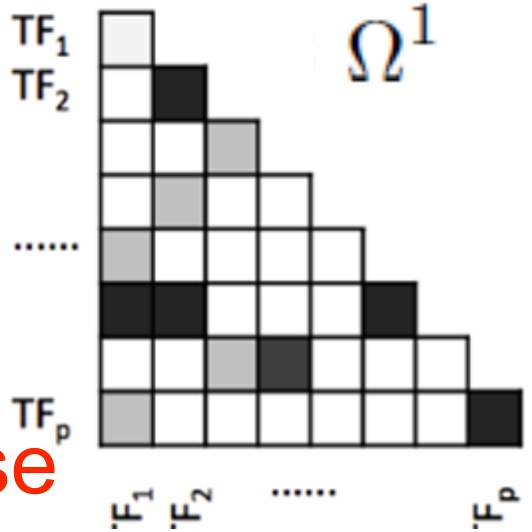
1

Pre-store

entry-wise

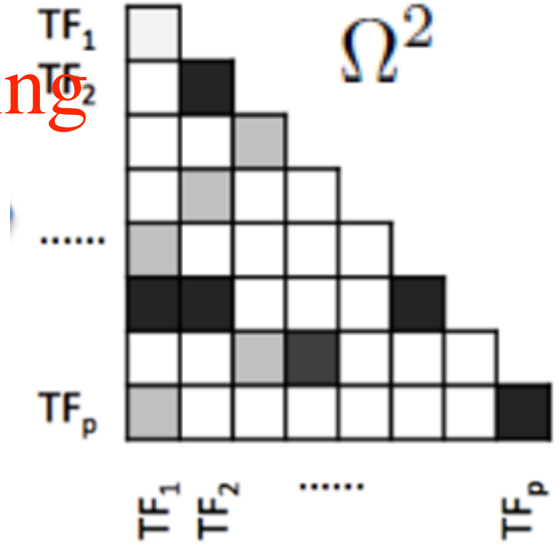
2

soft-thresholding



$$\Sigma = \mathrm{Cov}(X) = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix}$$

$$\mathcal{B}^{*(2)} := \begin{pmatrix} b_{11}^{(2)} & b_{12}^{(2)} & \cdots & b_{1p}^{(2)} \\ b_{21}^{(2)} & b_{22}^{(2)} & \cdots & b_{2p}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1}^{(2)} & b_{p2}^{(2)} & \cdots & b_{pp}^{(2)} \end{pmatrix}$$



Optimization: Step I

Precompute $\mathcal{B}^*(\Sigma_{tot}) = \boxed{inv}(\boxed{T_v(\Sigma_{tot})})$

A matrix inversion + A soft-thresholding operator

Note: this only compute and pre-store(no need to store them in the memory) once. No need to use the whole matrix Σ_{tot} repeatedly.

Here $inv(\Sigma_{tot}) := ({\Sigma^{(1)}}^{-1}, {\Sigma^{(2)}}^{-1}, \ldots, {\Sigma^{(K)}}^{-1})$

Optimization: Step2

$$\underset{\Omega_{tot}}{\operatorname{argmin}} |\Omega_{tot}|_1 + \epsilon \boxed{R'}(\Omega_{tot})$$

$$s.t. |\Omega_{tot} - \mathcal{B}^*(\Sigma_{tot})|_\infty \leq \lambda_n$$

$$\mathcal{R}'^*(\Omega_{tot} - \mathcal{B}^*(\Sigma_{tot})) \leq \epsilon\lambda_n$$

Multiple choices for R'( )

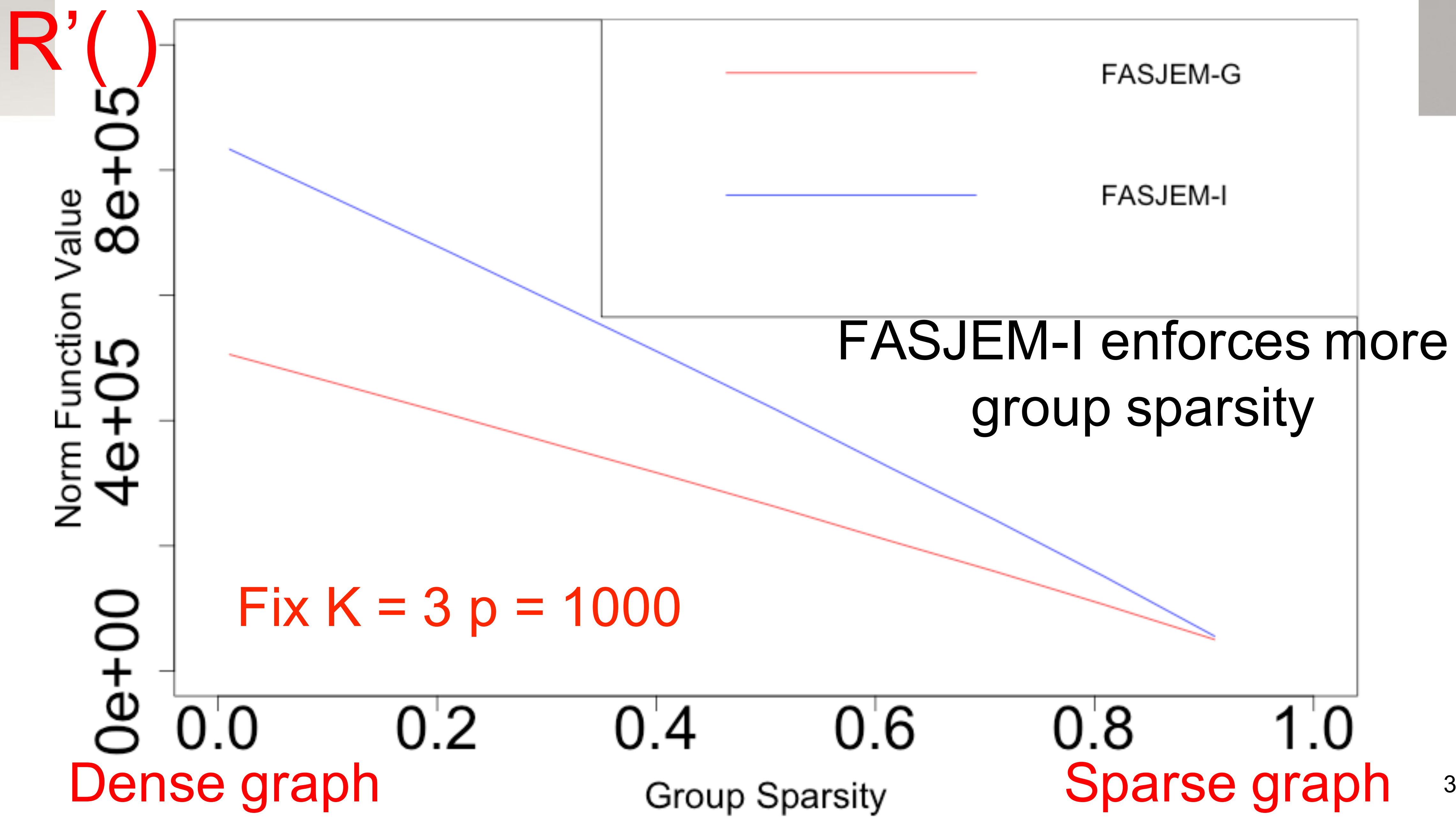
Two Variations

- Case I -- FASJEM-G:

$$\mathcal{R}'(\cdot) = |\cdot|_{\mathcal{G},2}$$

- Case II – FASJEM-I:

$$R'(\cdot) = |\cdot|_{\mathcal{G},\infty}$$


R'( )
FASJEM-G
FASJEM-I
FASJEM-I enforces more group sparsity
Fix K = 3 p = 1000
Norm Function Value
0e+00
4e+05
8e+05
0.0
0.2
0.4
0.6
0.8
1.0
Group Sparsity
Dense graph
Sparse graph

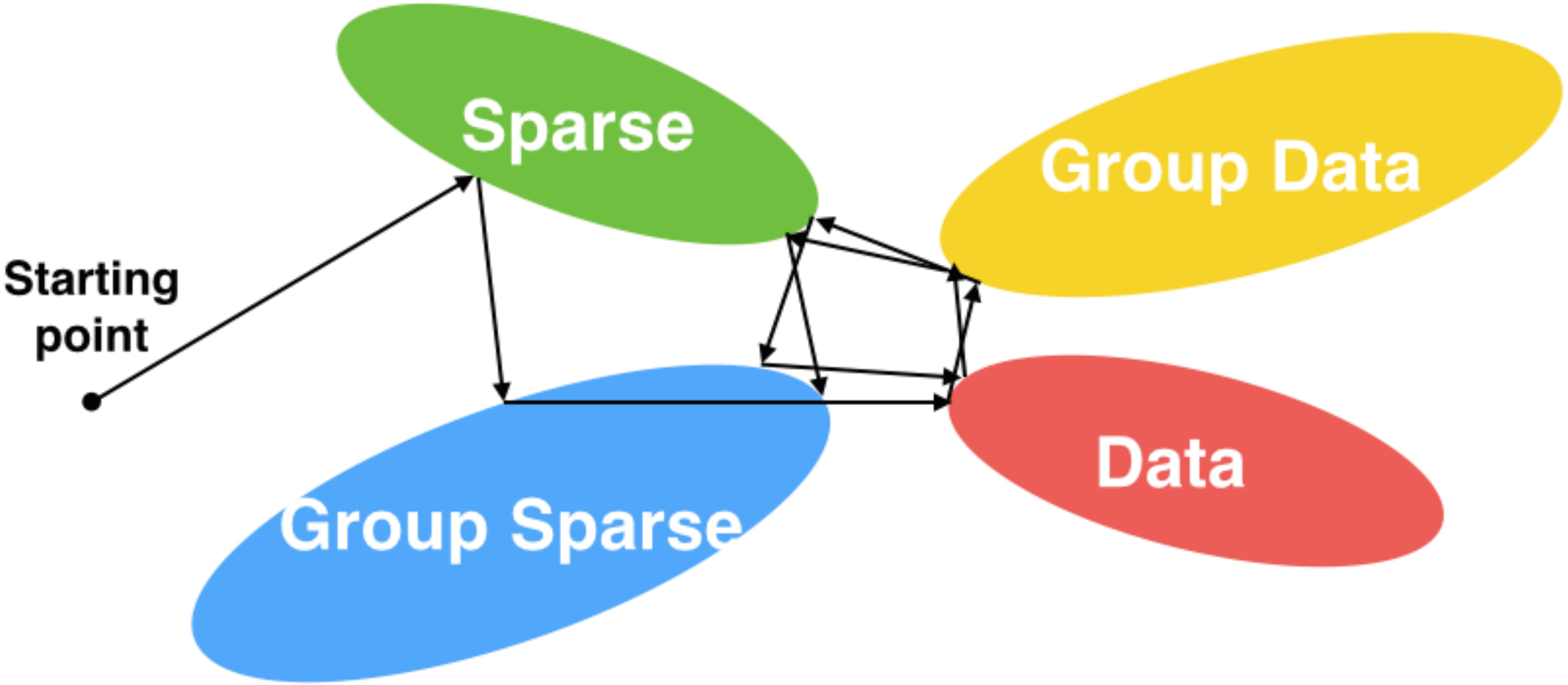
Optimization: Step II for FASJEM-G

We only need to compute the following four soft-thresholding operators for each entry(element) or group entries(element)

We choose FASJEM-G as an example.

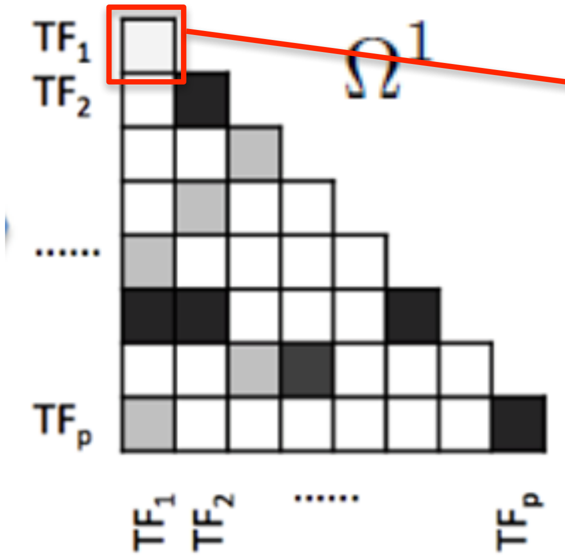
Other second norm is similar to this one.

Optimization Step2: Overall



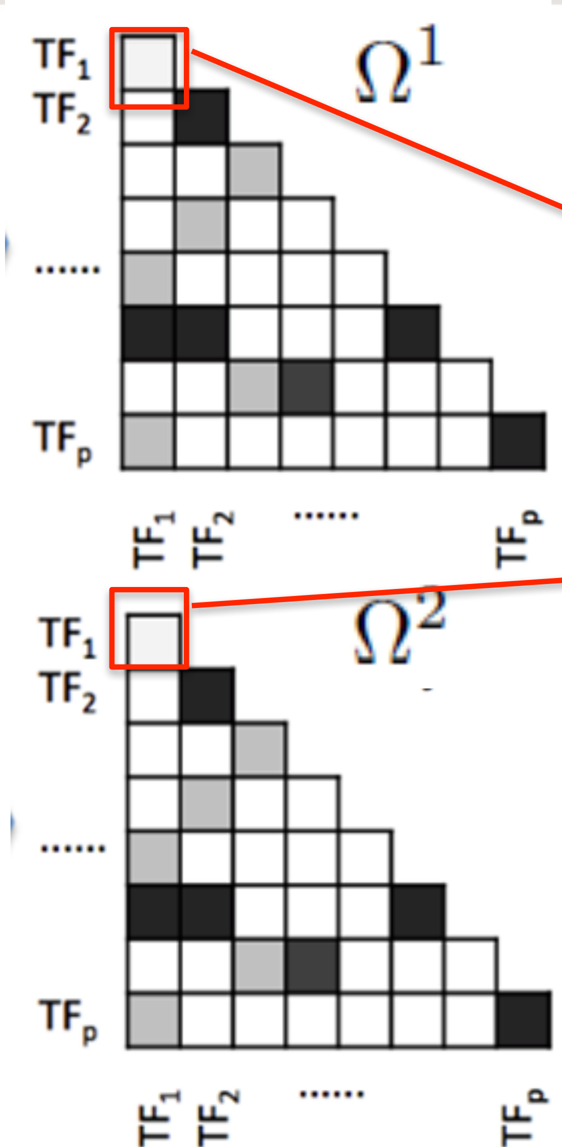
Optimization: Step II(1) - Sparse

In each iteration,



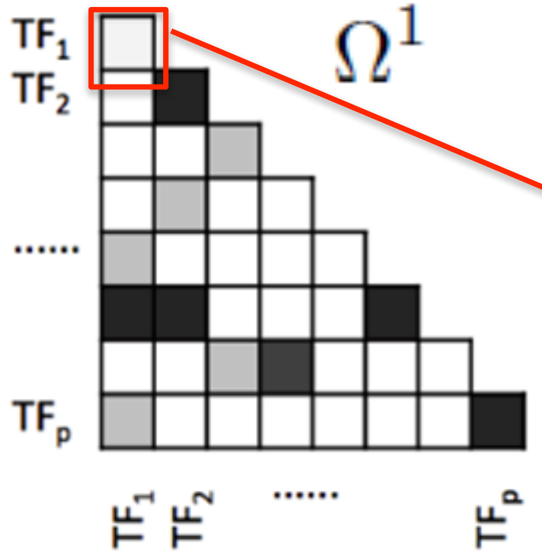
$$(\text{prox}_{\gamma||\cdot||_1}(x))_i = \begin{cases} x_i - \gamma,\, x_i > \gamma \\ 0,\, |x_i| \le \gamma \\ x_i + \gamma,\, x_i < -\gamma \end{cases}$$

Optimization: Step II(2) – Group sparse



$$\text{prox}_{\gamma||\cdot||_{\mathcal{G},2}}(x_g) = \begin{cases} x_g - \gamma\frac{x_g}{||x_g||_2}, ||x_g||_2 > \gamma \\ 0, ||x_g||_2 \leq \gamma \end{cases}$$

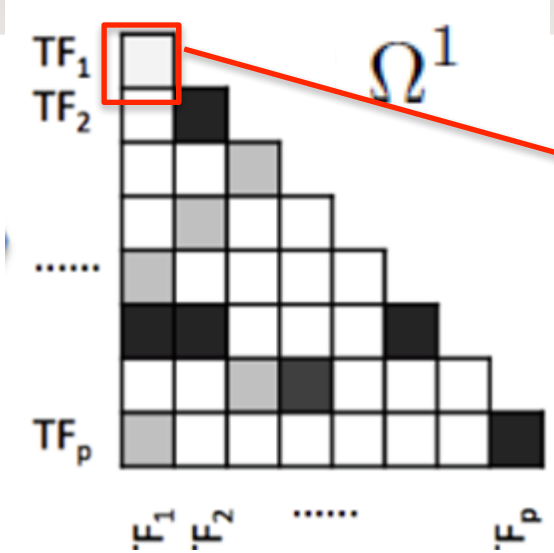
Optimization: Step II (3) – Data



$$\mathrm{proj}_{||x-a||_\infty \leq \lambda} = \begin{cases} x_i, |x_i - a_i| \leq \lambda \\ a_i + \lambda, x_i > a_i + \lambda \\ a_i - \lambda, x_i < a_i - \lambda \end{cases}$$

$$\mathcal{B}^{*(1)} := \begin{pmatrix} \boxed{b_{11}^{(1)}} & b_{12}^{(1)} & \cdots & b_{1p}^{(1)} \\ b_{21}^{(1)} & b_{22}^{(1)} & \cdots & b_{2p}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1}^{(1)} & b_{p2}^{(1)} & \cdots & b_{pp}^{(1)} \end{pmatrix}$$

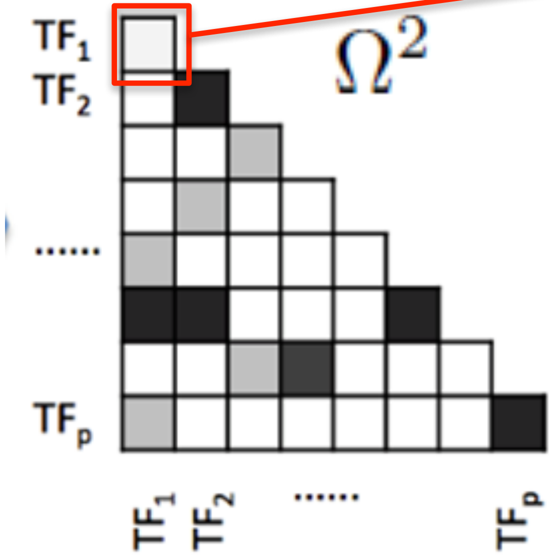
Optimization: Step II (4) – Group Data



$$\mathcal{B}^{*(1)} := \begin{pmatrix} \boxed{b_{11}^{(1)}} & b_{12}^{(1)} & \cdots & b_{1p}^{(1)} \\ b_{21}^{(1)} & b_{22}^{(1)} & \cdots & b_{2p}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1}^{(1)} & b_{p2}^{(1)} & \cdots & b_{pp}^{(1)} \end{pmatrix}$$

$$\mathrm{prox}_{\gamma g_2}(x) = \mathrm{proj}_{||x-a||^*_{\mathcal{G},2} \leq \lambda}$$

$$= \begin{cases} x_g, & ||x_g - a_g||_2 \leq \lambda \\ \lambda \frac{x_g - a_g}{||x_g - a_g||_2} + a_g, & ||x_g - a_g||_2 > \lambda \end{cases}$$



$$\mathcal{B}^{*(2)} := \begin{pmatrix} \boxed{b_{11}^{(2)}} & b_{12}^{(2)} & \cdots & b_{1p}^{(2)} \\ b_{21}^{(2)} & b_{22}^{(2)} & \cdots & b_{2p}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1}^{(2)} & b_{p2}^{(2)} & \cdots & b_{pp}^{(2)} \end{pmatrix}$$

Advantage of Optimization: Space

Suppose they have same iteration number T

K = 91, p= 30K

$O(Kp^2)$ space to store / Itera ⇨ Pre-store, only $O(K)$ / Itera

Double type: 65 TB ⇨ Double type: 728B < 1KB

Advantage of Optimization: Time

Suppose they have same iteration number T

K = 91, p= 30K

Previous Multi-sGGM (SVD) needs $O(Kp^3)$/ Itera

Totally entry-wise, $O(Kp^2)$/ Itera also can be paralled

300000 times faster

3.5 days ⟷ 1 second

Roadmap

1. Goal & Background
2. Proposed
3. <u>Evaluation</u>
4. Conclusion

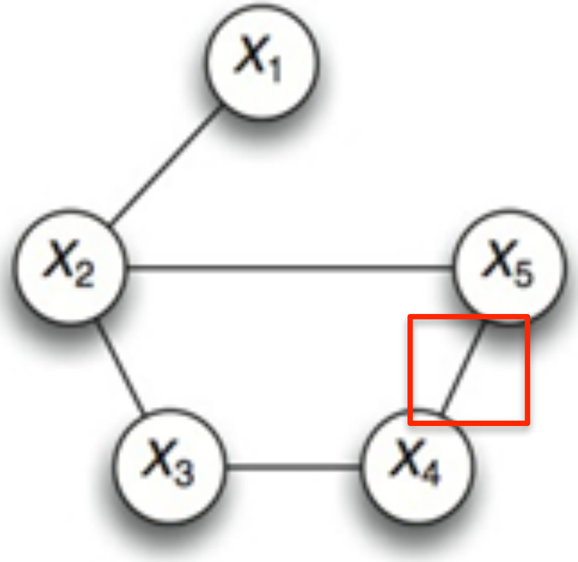
My Work: Evaluation

1. Simulation test
    - random graph models

2. Real world datasets

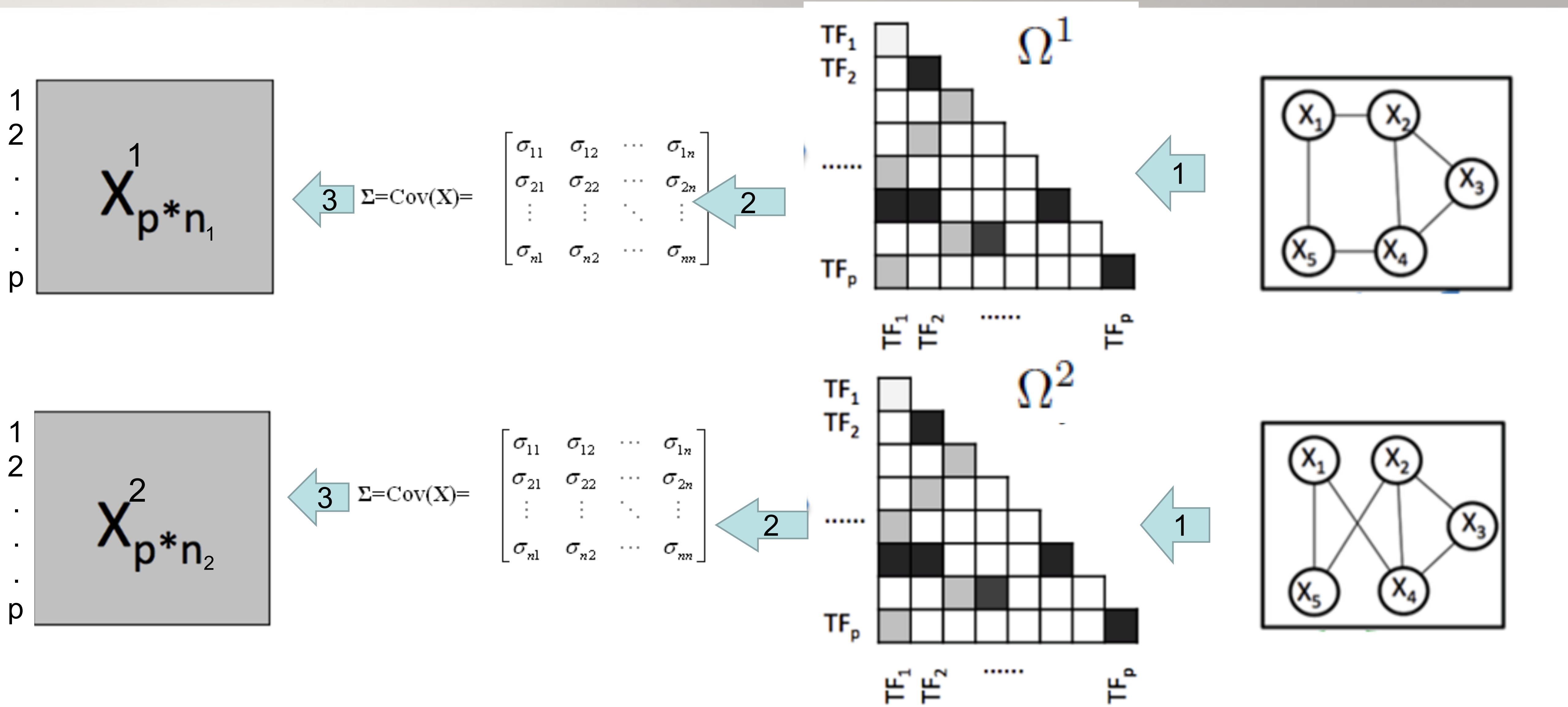
3. Theoretical Performance
    - e.g., Convergence rate

EXP I – Generating Simulation Data



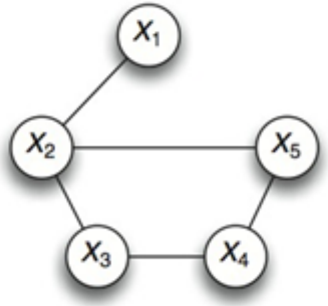
Generate edges randomly by Bernoulli distribution with probability q

EXP I – Generating Simulation Data



EXP I – Generating Simulation Data

Step1:



Non-edge → Zero entry

Edge → Nonzero entry

(b) Inverse Covariance Matrix

$$\begin{pmatrix} 1 & 0.2 & 0 & 0 & 0 \\ 0.2 & 1 & 0.2 & 0 & 0.2 \\ 0 & 0.2 & 1 & 0.2 & 0 \\ 0 & 0 & 0.2 & 1 & 0.2 \\ 0 & 0.2 & 0 & 0.2 & 1 \end{pmatrix}$$

EXP I – Generating Simulation Data

Step2:

$$\begin{pmatrix} 1 & 0.2 & 0 & 0 & 0 \\ 0.2 & 1 & 0.2 & 0 & 0.2 \\ 0 & 0.2 & 1 & 0.2 & 0 \\ 0 & 0 & 0.2 & 1 & 0.2 \\ 0 & 0.2 & 0 & 0.2 & 1 \end{pmatrix} \xrightarrow{\text{Inverse}} \Sigma = \begin{pmatrix} 1.05 & -0.23 & 0.05 & -0.02 & 0.05 \\ -0.23 & 1.45 & -0.25 & 0.10 & -0.25 \\ 0.05 & -0.25 & 1.10 & -0.24 & 0.10 \\ -0.02 & 0.10 & -0.24 & 1.10 & -0.24 \\ 0.05 & -0.25 & 0.10 & -0.24 & 1.10 \end{pmatrix}$$

EXP I – Generating Simulation Data

Step3:

Suppose $X \sim N(\mu, \Sigma)$ and X is a p-dimensional vector

Use MCMC to simulate data set based on Covariance matrix

Evaluation: metric

1. ROC curve varying different tuning parameter, compare AUC(area under curve)

2. computation time
   - fix K varying p
   - fix p varying K

3. memory
   - how large p and K will cause the programme to terminate

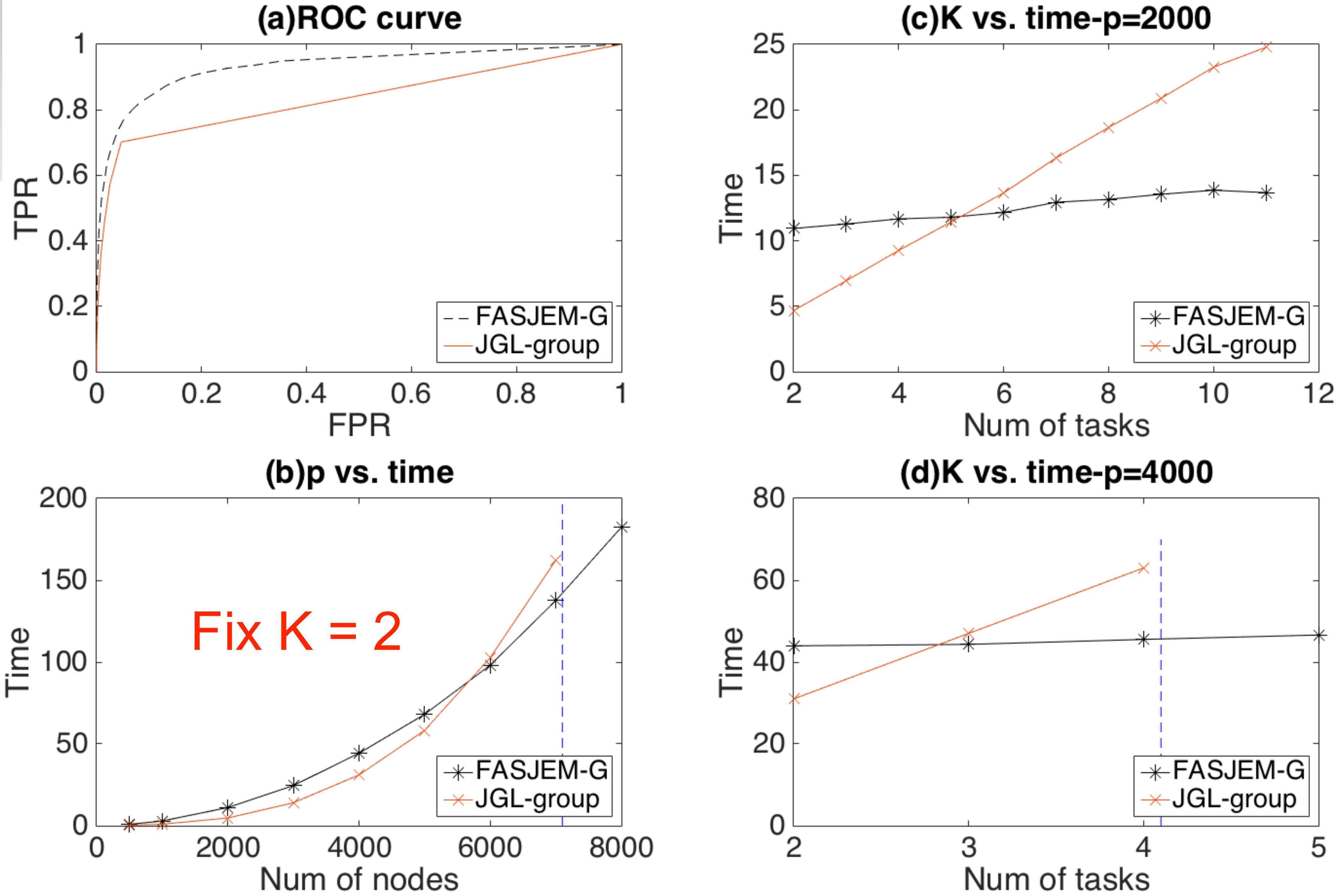
Evaluation : Experimental Setting

- simulation test datasets:
  - Model : random sparse graph model
  - Case I -- FASJEM-G:

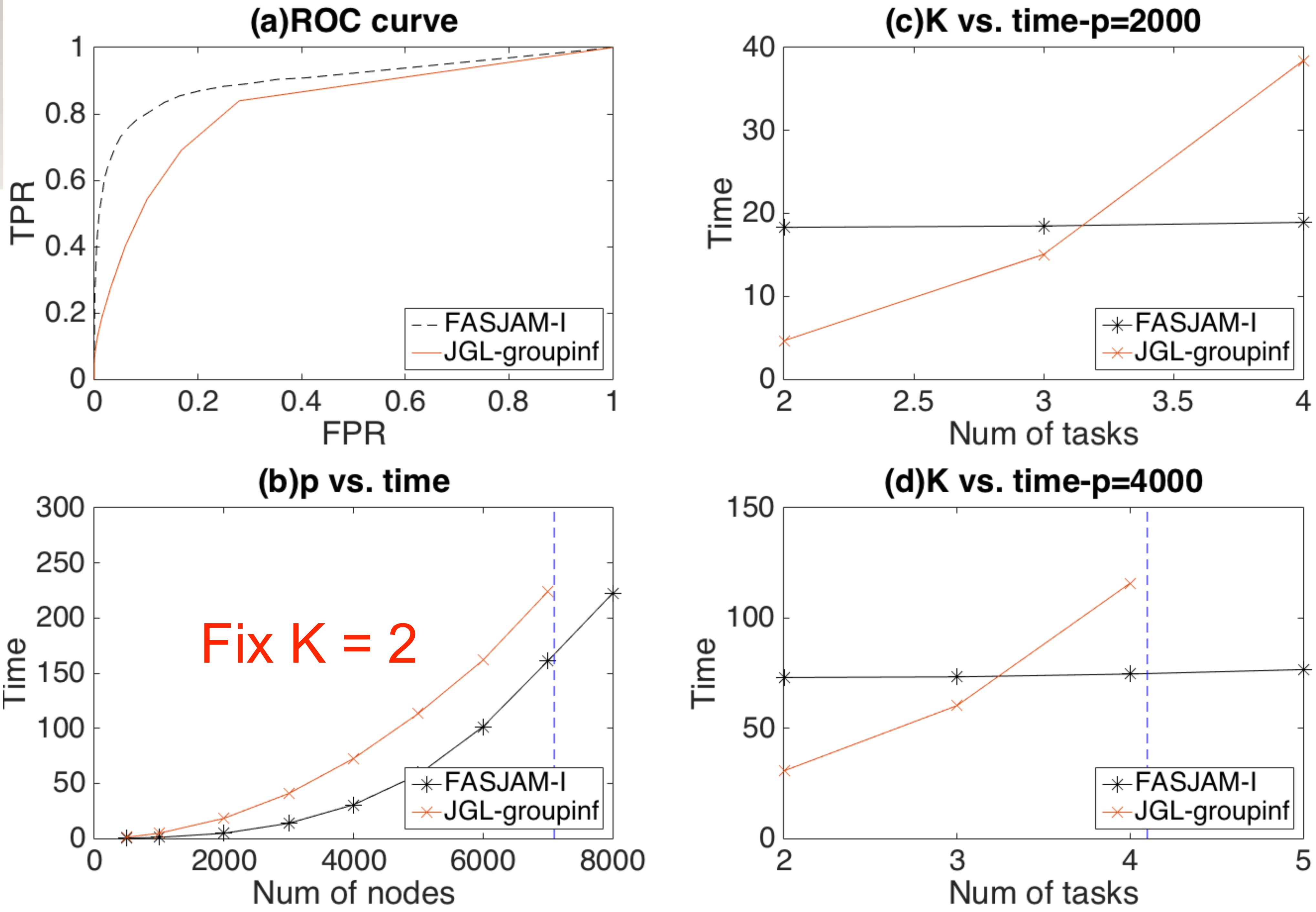
$$\mathcal{R}'(\cdot) = |\cdot|_{\mathcal{G},2}$$

  - Case II – FASJEM-I:

$$R'(\cdot) = |\cdot|_{\mathcal{G},\infty}$$



(a)ROC curve
TPR
FPR
FASJEM-G
JGL-group
(c)K vs. time-p=2000
Time
Num of tasks
FASJEM-G
JGL-group
(b)p vs. time
Time
Num of nodes
Fix K = 2
FASJEM-G
JGL-group
(d)K vs. time-p=4000
Time
Num of tasks
FASJEM-G
JGL-group



(a)ROC curve
TPR
FPR
FASJAM-I
JGL-groupinf
(c)K vs. time-p=2000
Time
Num of tasks
FASJAM-I
JGL-groupinf
(b)p vs. time
Time
Num of nodes
Fix K = 2
FASJAM-I
JGL-groupinf
(d)K vs. time-p=4000
Time
Num of tasks
FASJAM-I
JGL-groupinf

Evaluation-Experiment Result I

- Our models obtain the best accuracy result.
- Our models are faster than the baseline methods when $p>6000$&$K=2$ or $p = 2000$&$K > 6$
- Our models still work when $p>8000$&$K=2$ or $p=4000$&$K>4$
  - On 8GB memory desktop

Evaluation-Experiment Result II

Two existing
gene interactions databases

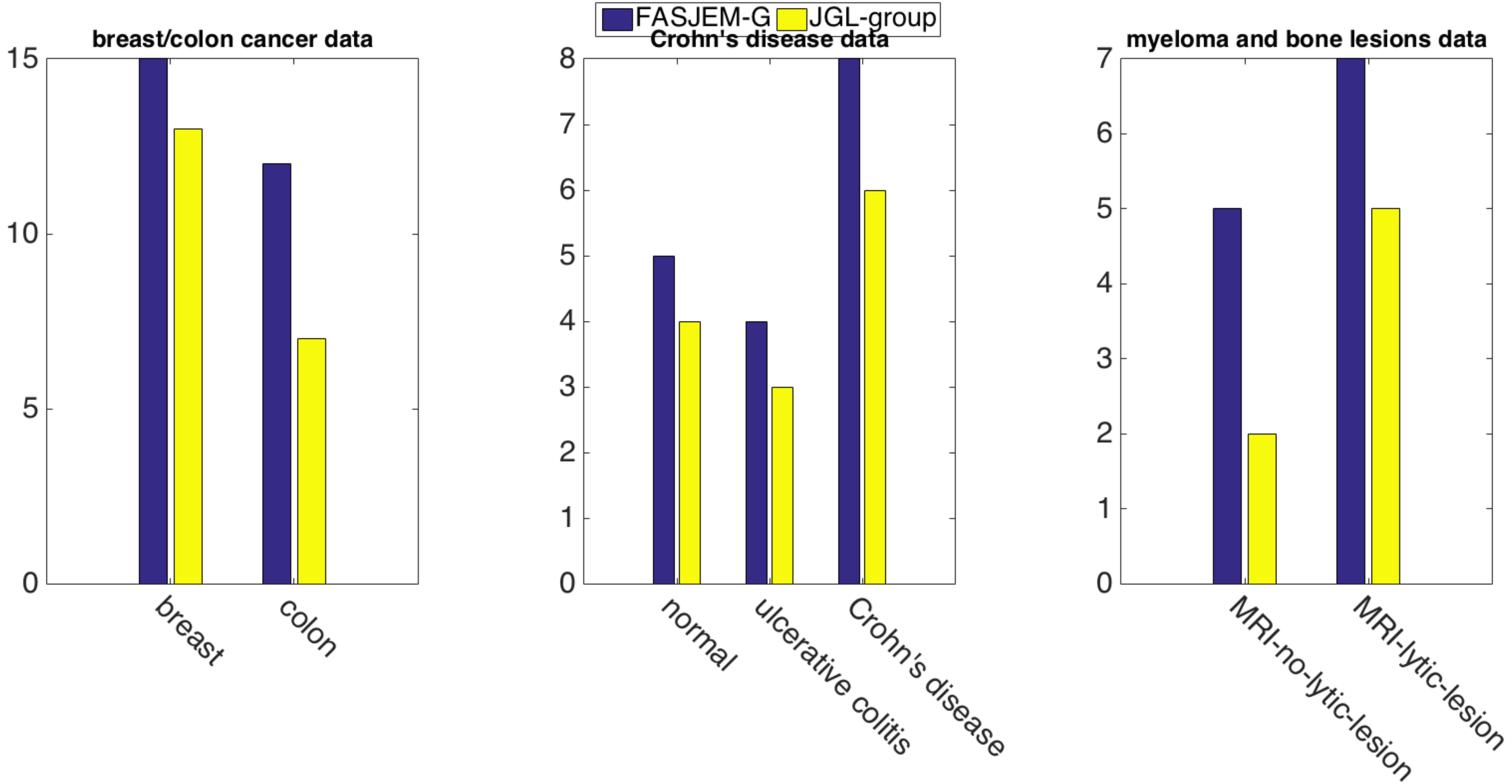
Three real world datasets

implement
different
methods

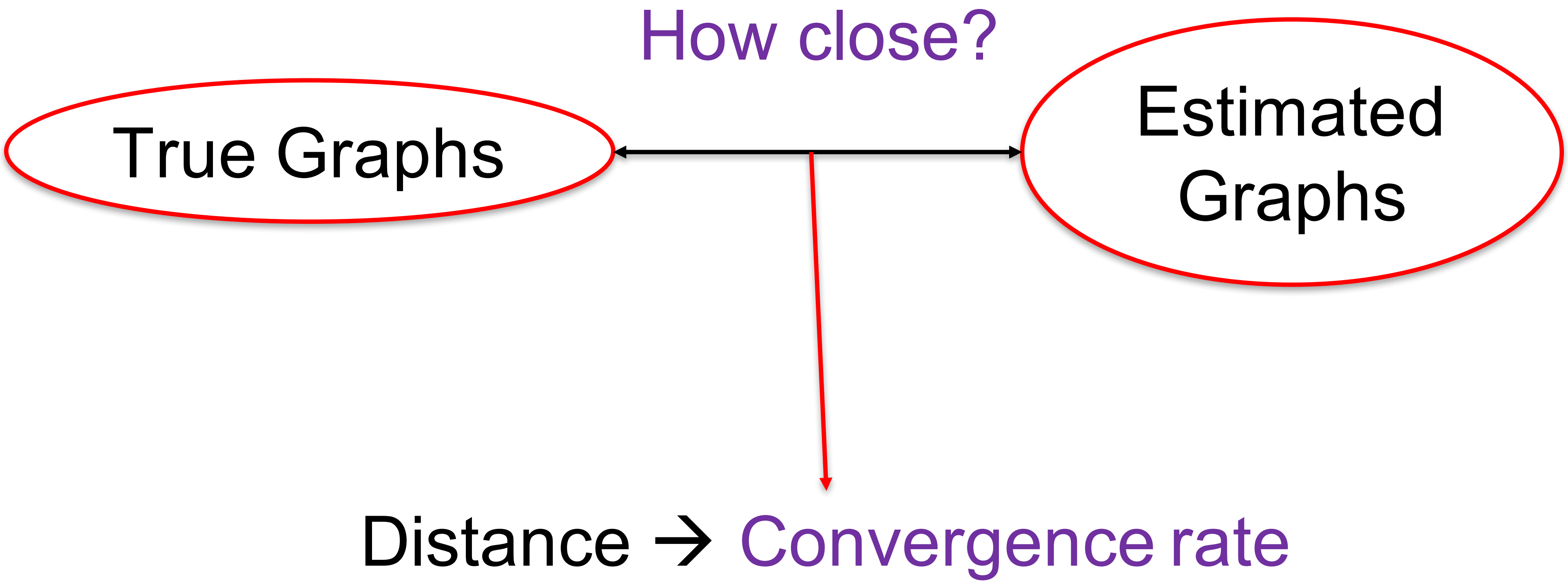
Results ⟺ Interactions

matching

Evaluation-Experiment Result II



Evaluation-Theoretical Analysis



Theoretical Analysis – Best convergence rate

- General case > some experiments

- The best convergence rate:

$$|\widehat{\mu} - \mu^*|_F \leq 8 \boxed{\max\{M_1\sqrt{\frac{k_1 \log Kp}{n_{tot}}}, M_2\sqrt{\frac{k_2 p \log Kp}{n_{tot}}}\}}$$

Sahand Negahban, Bin Yu, Martin J Wainwright, and Pradeep K Ravikumar. A unified framework for high-dimensional analysis of m-estimators with decom- posable regularizers. In *Advances in Neural Information Processing Systems*, pages 1348–1356, 2009

Theoretical Analysis – Best convergence rate

- General cases' conclusion

- The best convergence rate:

$$\|\widehat{\mu} - \mu^*\|_F \leq 8 \max\{M_1 \sqrt{\frac{k_1 \log Kp}{n_{tot}}}, M_2 \sqrt{\frac{k_2 p \log Kp}{n_{tot}}}\}$$

- We prove it!

Theoretical Analysis – When compared to single task case

- Single task:

$$|\widehat{\mu} - \mu^*|_F \le O(\sqrt{\frac{\log p}{n}})$$

- Our case:

$$|\widehat{\mu} - \mu^*|_F \le 8 \max\{M_1\sqrt{\frac{k_1 \log Kp}{n_{tot}}}, M_2\sqrt{\frac{k_2 p \log Kp}{n_{tot}}}\}$$

Theoretical Analysis – Multi-task helps !

Suppose $n_i = n_1$

$$\frac{\log Kp}{Kn_1} \leq \frac{\log p}{n_1}$$

$K = 91$, $p = 30K$, $n_1 = 1K$

Our case 0.0001 << 0.01 Single case

Multi-task —> n increase —> closer distance

Conclusion

- We design a novel algorithm to solve the Multi-task sGGM
- We have the best simulation test result in
  - Accuracy
  - Time
  - Memory
- Our method achieve the best convergence rate

Thank You!

http://jointggm.org

R package: fasjem

```
install.packages("fasjem")
library(fasjem)
demo(fasjem)
```

Background: Multi-task sGGM to derive Conditional Independence Graph from data

Step1：

Suppose $X \sim N(\mu, \Sigma)$ and X is a p-dimensional vector

$$\widehat{\Sigma} = (X - \bar{X})^T (X - \bar{X})$$

Background: Multi-task sGGM to derive Conditional Independence Graph from data

Step2: We solve the following optimization problem:

$$\min_{\Omega^k > 0} \sum_k \left(-\boxed{L(\Omega^k)} + \rho_1 \boxed{|\Omega^k|_1}\right) + \rho_2 \boxed{P(\Omega^1, \Omega^2, \ldots, \Omega^K)}$$

where $k = 1, 2, \ldots, K$.

Log likelihood

L1 penalty

Group penalty

Background: Multi-task sGGM to derive Conditional Independence Graph from data

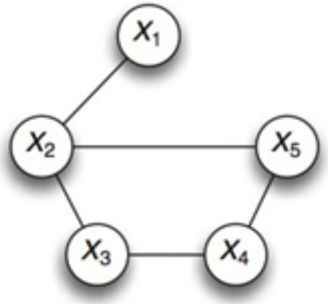
Step3:

(b) Inverse Covariance Matrix

$$\begin{pmatrix} 1 & 0.2 & 0 & 0 & 0 \\ 0.2 & 1 & 0.2 & 0 & 0.2 \\ 0 & 0.2 & 1 & 0.2 & 0 \\ 0 & 0 & 0.2 & 1 & 0.2 \\ 0 & 0.2 & 0 & 0.2 & 1 \end{pmatrix}$$

Zero entry → Non-edge

Nonzero entry → Edge



Background: Dual Norm

$$\mathcal{R}^*(v) := \sup_{u \in \mathbb{R}^p \setminus \{0\}} \frac{< u, v >}{\mathcal{R}(u)} = \sup_{\mathcal{R}(u) \leq 1} < u, v >$$

Experiment: Real world datasets

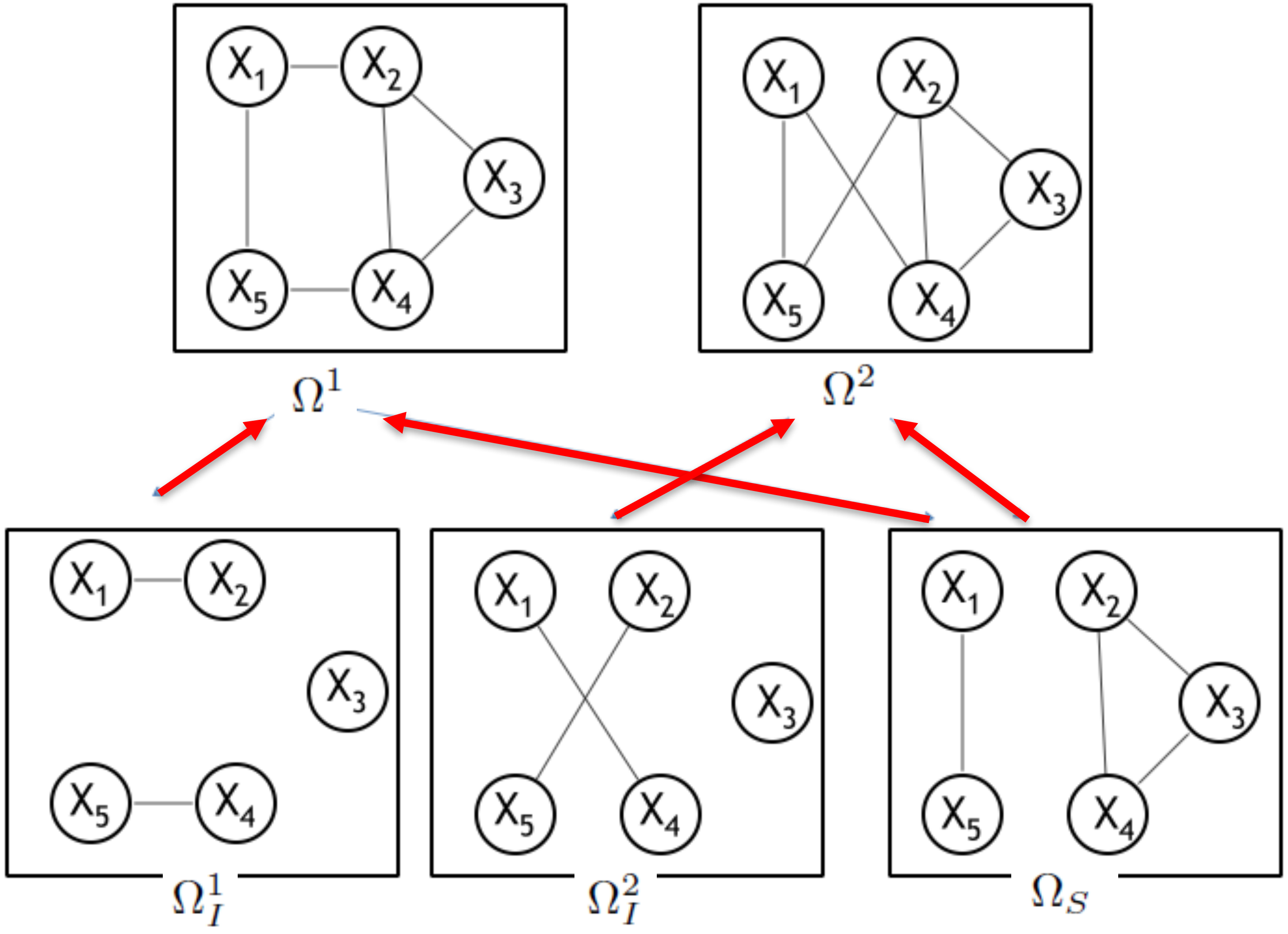
(1) The breast/colon cancer data (with 2 cell type and 104 samples, each of which has 22283 features);

(2) Chrohn's disease data (with 3 cell type and 127 samples, each of which has 22283 features)

(3) The myeloma and bone lesions data set (with 2 cell type and 173 samples, each of which has 12625 features)

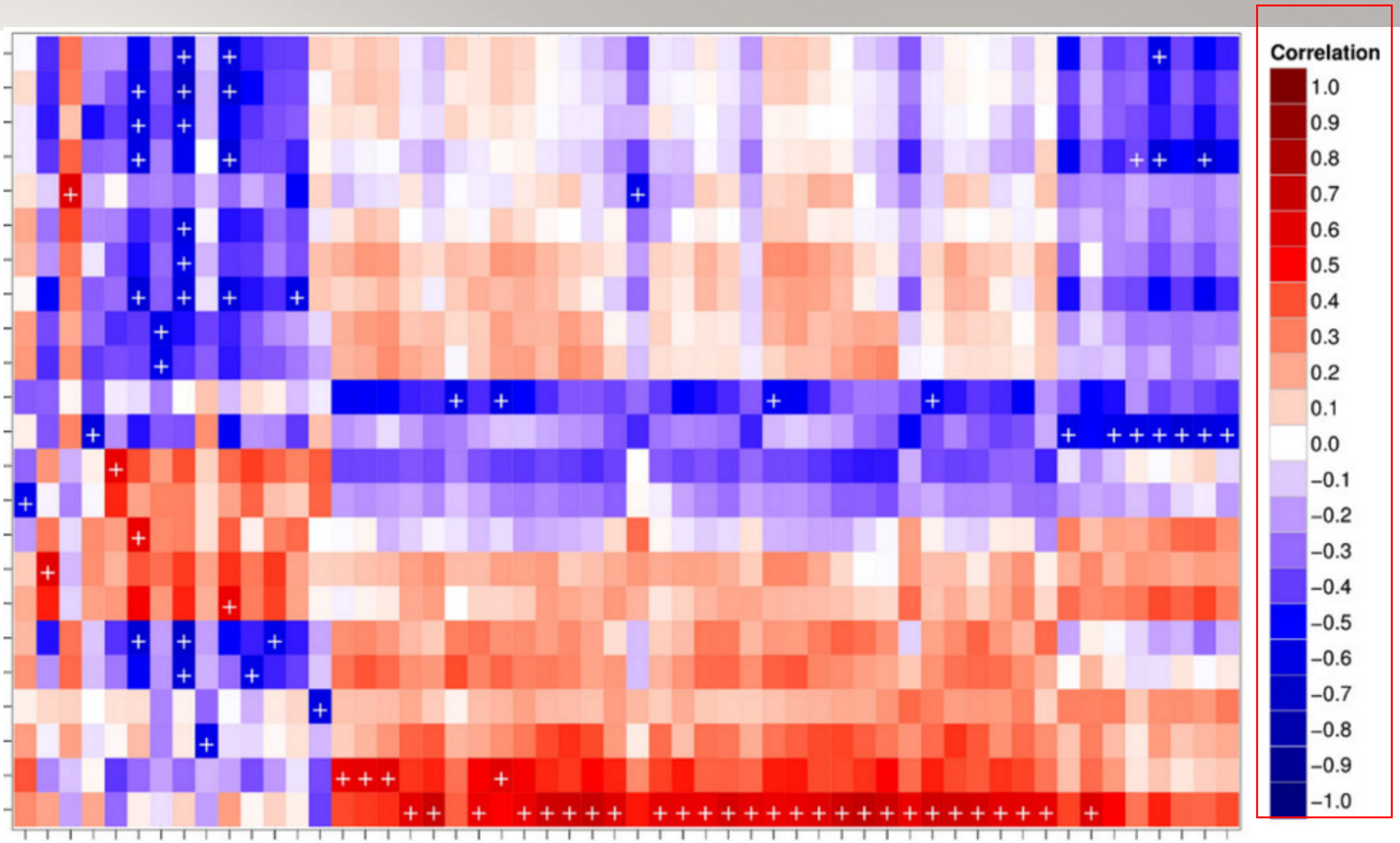
We select top 500 features based variable variance for all three datasets.

Experiment: Synthesizing Data With Random Graph Model



Motivation: Correlation is not enough

Correlation Heatmap:



Motivation: Correlation is not enough

Example:

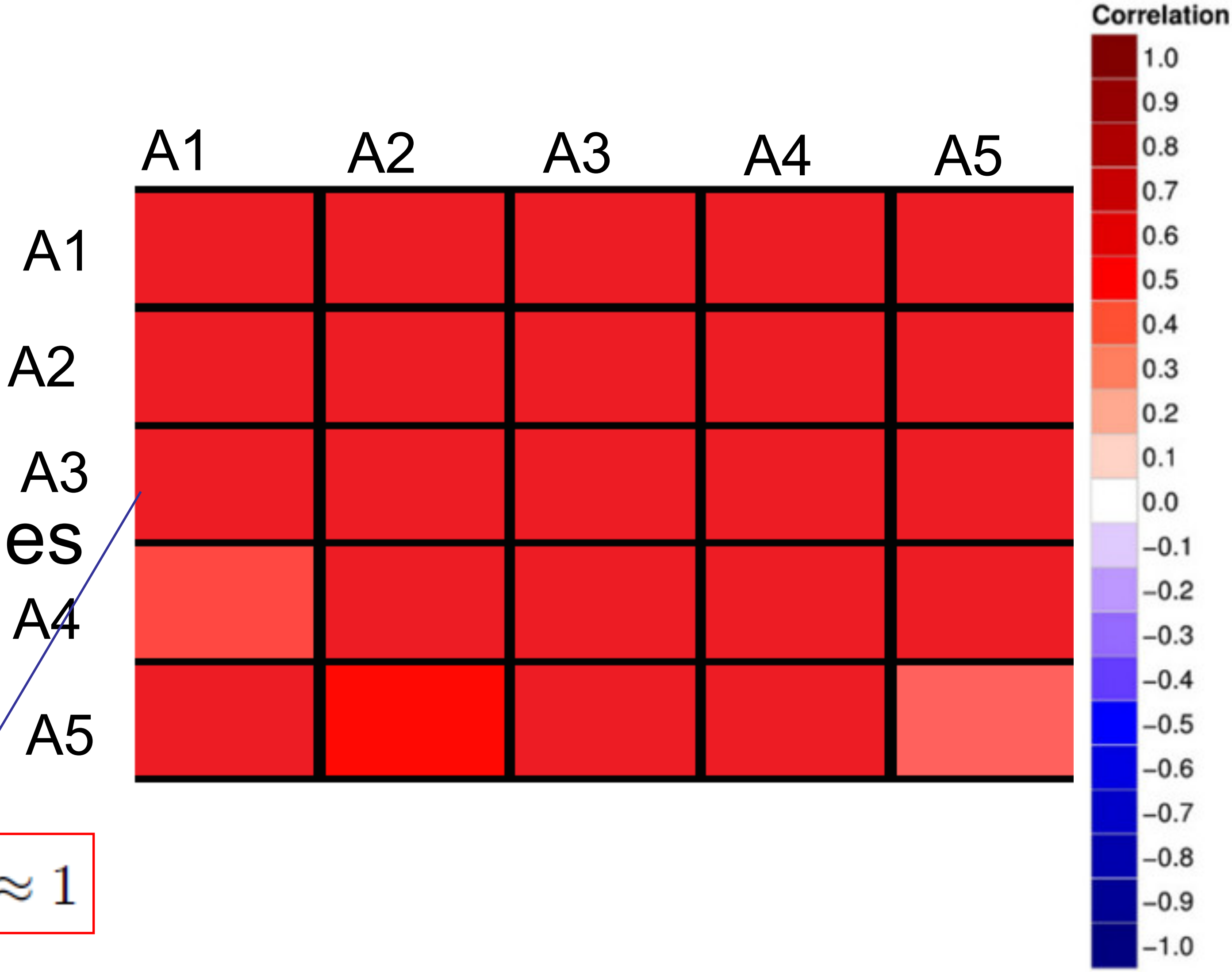
If X is a random variable follows $N(0,1)$, let $Y = X^2$.

Then $Cor(X,Y) = 0$, but X and Y have dependent relationship.

Motivation: Correlation is not enough

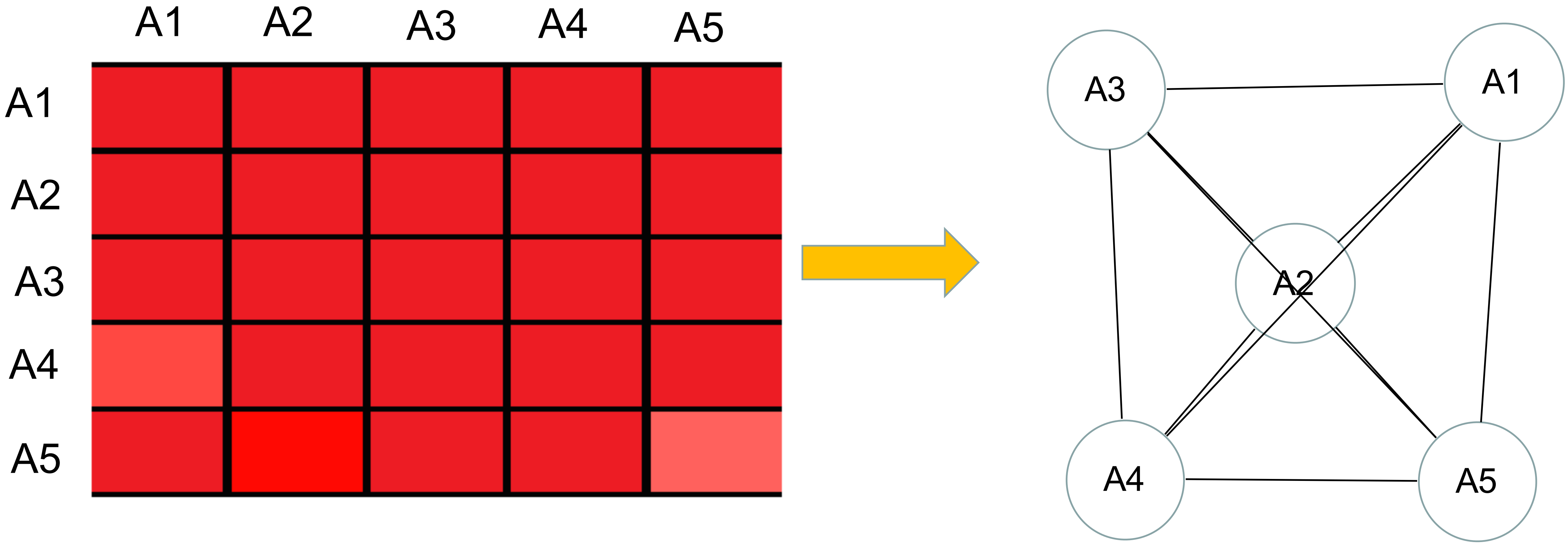
Another Example:

A1: Children try swim
A2: Weather is hot
A3: High sale of ice cream
A4: Wear less amount of clothes
A5: High Electricity Consumption

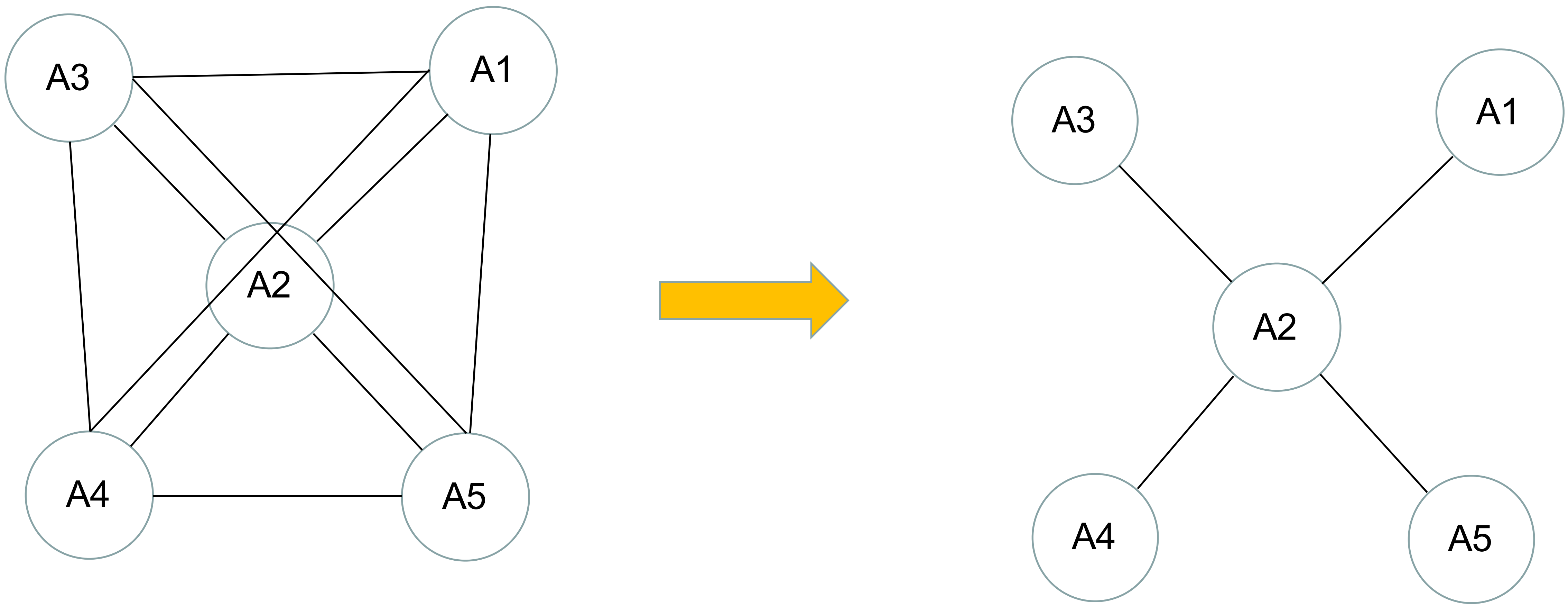


$$\boxed{Cor(A_1, A_3) \approx 1}$$

Motivation: Correlation is not enough



Motivation: Conditional independence is better



Motivation: Conditional independence is better

A1: Children are drown

A2: Weather is hot

A3: High sale of ice cream

A4: Wear less amount of clothes

A5: High Electricity Consumption