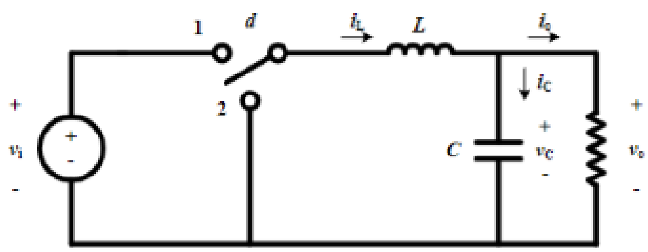
Robert Bara Assignment 3:

Thursday, March 3, 2022 10:57 AM

Assignment #3: Converter Small-Signal Modeling

- **Ideal Buck Converter**
  - *See the separate document for detailed descriptions*



For the ideal buck converter in Fig. 1,

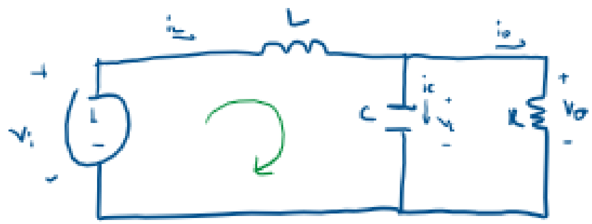
1) Derive the state space model with state variables selected as (i_L, v_C), input variables selected as (v_i, i_o), and output variable vector selected as v_o. Derive the coefficient matrices in the model below.

$$\begin{pmatrix} \frac{di_L}{dt} \\ \frac{dv_C}{dt} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \cdot \begin{pmatrix} v_i \\ i_o \end{pmatrix}$$

$$v_o = [C_{11} \quad C_{12}] \cdot \begin{pmatrix} i_L \\ v_C \end{pmatrix} + [D_{11} \quad D_{12}] \cdot \begin{pmatrix} v_i \\ i_o \end{pmatrix}$$

2) In the derived results in Question 1), select the line of $\frac{di_L}{dt}$ (i.e., $\frac{di_L}{dt} = A_{11} \cdot i_L + A_{12} \cdot v_C + B_{11} \cdot v_i + B_{12} \cdot i_o$), and use perturbation-and-linearization approach to derive its small-signal representation. Note that the derived matrix coefficients in 1) should be 'spelled out' and used here.

1) Circuit 1



$v_o \approx v_c$

State variable: $i_L\ v_c$ input: v_i, i_o output: v_o

KVL $\Rightarrow$ $v_L + v_c - v_i = 0$ 1) Perform KVL/KCL

KCL $\Rightarrow$ $i_L - i_c - i_o = 0$

$\therefore$ $v_L = L\frac{di_L}{dt} = v_i - v_c \Rightarrow \boxed{\frac{di_L}{dt} = \frac{v_i}{L} - \frac{v_c}{L}}$ 2) Solve for state variables i_L and v_c, get derivatives

$i_L - C\frac{dv_c}{dt} - i_o = 0$

$\Rightarrow C\frac{dv_c}{dt} = i_o - i_L$

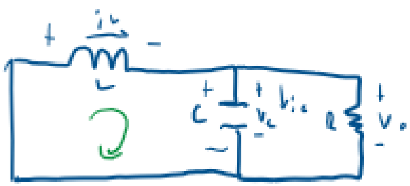
$\Rightarrow \boxed{\frac{dv_c}{dt} = \frac{i_o}{C} - \frac{i_L}{C}}$

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv_c}{dt} \end{bmatrix} = \overset{A}{\begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}} \begin{bmatrix} i_L \\ v_c \end{bmatrix} + \overset{B}{\begin{bmatrix} \frac{1}{L} & 0 \\ 0 & -\frac{1}{C} \end{bmatrix}} \begin{bmatrix} v_i \\ i_o \end{bmatrix}$$ 3) Derive Matrices

$$v_o = v_c = \overset{C}{\begin{bmatrix} 0 & 1 \end{bmatrix}} \begin{bmatrix} i_L \\ v_c \end{bmatrix} + \overset{D}{\begin{bmatrix} 0 & 0 \end{bmatrix}} \begin{bmatrix} v_i \\ i_o \end{bmatrix}$$

$$\Rightarrow \begin{cases} \dot{x} = A_1 x + B_1 U \\ Y = C_1 x + D_1 U \end{cases}$$

Circuit 2:



KVL => $v_L + v_c = 0$ $\qquad$ $V = IR$

KCL => $i_L - i_c - i_o = 0$

$i_c = i_L - i_o$

$C\frac{dv_c}{dt} = i_L - i_o$

$\therefore \quad v_L = L\frac{di_L}{dt} = -v_c$

$$\Rightarrow \boxed{\frac{di_L}{dt} = \frac{-v_c}{L}}$$

$$\boxed{\frac{dv_c}{dt} = \frac{i_L}{C} - \frac{i_o}{C}}$$

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv_c}{dt} \end{bmatrix} = \overset{A}{\begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}} \begin{bmatrix} i_L \\ v_c \end{bmatrix} + \overset{B}{\begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{C} \end{bmatrix}} \begin{bmatrix} v_i \\ i_o \end{bmatrix}$$

$$v_o = v_c = \overset{C}{\begin{bmatrix} 0 & 1 \end{bmatrix}} \begin{bmatrix} i_L \\ v_c \end{bmatrix} + \overset{D}{\begin{bmatrix} 0 & 0 \end{bmatrix}} \begin{bmatrix} v_i \\ i_o \end{bmatrix}$$

$$= \begin{cases} \dot{X} = A_2 X + B_2 U \\ Y = C_2 X + D_2 U \end{cases}$$

Find Large Signal model

$$\therefore \bar{A} = dA_1 + (1-d)A_2 = \begin{bmatrix} 0 & -\frac{d}{L} \\ \frac{d}{C} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-(1-d)}{L} \\ \frac{(1-d)}{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L}[-1+\cancel{d}-\cancel{d}] \\ \frac{1-\cancel{d}+\cancel{d}}{C} & 0 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}$$

$$\bar{B} = dB_1 + (1-d)B_2 = \begin{bmatrix} \frac{d}{L} & 0 \\ 0 & -\frac{d}{C} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{-(1-d)}{C} \end{bmatrix} = \begin{bmatrix} \frac{d}{L} & 0 \\ 0 & -\frac{1}{C} \end{bmatrix}$$

$$\bar{C} = dC_1 + (1-d)C_2 = [0 \;\; d] + [0 \;\; (1-d)] = [0 \;\; 1]$$

$$\bar{D} = dD_1 + (1-d)D_2 = [0 \;\; 0] + [0 \;\; 0] = [0 \;\; 0]$$

$$\therefore \begin{bmatrix} \frac{di_L}{dt} \\ \frac{dV_c}{dt} \end{bmatrix} = \overset{A}{\begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}} \begin{bmatrix} i_L \\ v_c \end{bmatrix} + \overset{B}{\begin{bmatrix} \frac{d}{L} & 0 \\ 0 & -\frac{1}{C} \end{bmatrix}} \begin{bmatrix} v_i \\ i_o \end{bmatrix}$$

$$V_o = \overset{C}{[0 \;\; 1]} \begin{bmatrix} i_L \\ V_c \end{bmatrix} + \overset{D}{[0 \;\; 0]} \begin{bmatrix} v_i \\ i_o \end{bmatrix}$$

Large Signal model

$$\begin{cases} x' = Ax + Bu \\ y = Cx + Du \end{cases}$$

Deriving Small signal model

2) $$\frac{di_L}{dt} = 0 i_L - \frac{1}{L} v_c + \frac{d}{L} V_i + 0 i_o \Rightarrow \boxed{\frac{di_L}{dt} = -\frac{1}{L} v_c + \frac{d}{L} v_i}$$

$$\frac{dV_c}{dt} = \frac{1}{C} i_L + 0 v_c + 0 v_i - \frac{1}{C} i_o \Rightarrow \boxed{\frac{dv_c}{dt} = \frac{1}{C} i_L - \frac{1}{C} i_o}$$

$$\Rightarrow \begin{cases} \frac{d(I_L + \hat{i}_L)}{dt} = -\frac{1}{L}(V_c + \hat{v}_c) + \frac{1}{L}(D + \hat{d})(V_i + \hat{v}_i) \\ \frac{d(V_c + \hat{v}_c)}{dt} = \frac{1}{C}(I_L + \hat{i}_L) - \frac{1}{C}(I_o + \hat{i}_o) \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{dI_L}{dt} + \dfrac{d\hat{i}_L}{dt} = -\dfrac{V_c}{L} - \dfrac{\hat{v}_c}{L} + \dfrac{D V_i}{L} + \dfrac{D \hat{v}_i}{L} + \dfrac{\hat{d} V_i}{L} + \dfrac{\hat{d}\hat{v}_i}{L} \\[2ex] \dfrac{dV_c}{dt} + \dfrac{d\hat{v}_c}{dt} = \dfrac{I_L}{C} + \dfrac{\hat{i}_L}{C} - \dfrac{I_o}{C} - \dfrac{\hat{i}_o}{C} \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{d\hat{i}_L}{dt} = -\dfrac{\hat{v}_c}{L} + \dfrac{D\hat{v}_i}{L} + \dfrac{\hat{d} V_i}{L} \\[2ex] \dfrac{d\hat{v}_c}{dt} = \dfrac{\hat{i}_L}{C} - \dfrac{\hat{i}_o}{C} \end{cases}$$

Derive as Matricies

$$\begin{bmatrix} \dfrac{d\hat{i}_L}{dt} \\[2ex] \dfrac{d\hat{v}_c}{dt} \end{bmatrix} = \overset{A}{\begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}} \begin{bmatrix} \hat{i}_L \\ \hat{v}_c \end{bmatrix} + \overset{B}{\begin{bmatrix} \frac{V_i}{L} & \frac{D}{L} & 0 \\ 0 & 0 & -\frac{1}{C} \end{bmatrix}} \begin{bmatrix} \hat{d} \\ \hat{v}_i \\ \hat{i}_o \end{bmatrix}$$

$$\hat{v}_o = \overset{C}{\begin{bmatrix} 0 & 1 \end{bmatrix}} \begin{bmatrix} \hat{i}_L \\ \hat{v}_c \end{bmatrix} + \overset{D}{\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}} \begin{bmatrix} \hat{d} \\ \hat{v}_i \\ \hat{i}_o \end{bmatrix}$$

$$\begin{cases} \hat{\dot{X}} = A' \cdot \hat{X} + B' \cdot \hat{U} \\ \hat{Y} = C' \cdot \hat{X} + D' \cdot \hat{U} \end{cases}$$

Small signal representation