$\LaTeX 2_\varepsilon$ Physics Exercises

Eiaki Morooka (Aki)
Department of IT Engineering
Metropolia University of Applied Sciences
Karaportti 2, 02610 Espoo, Finland

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SOAP Spectrum

$$
\begin{align}
c^i_{nlm} &= \int dV g_{nl}(r_i)\rho(\vec{r}_i)Y_{lm}(\theta,\phi) \tag{1}\\
&= \int dV \sum_k \beta_{nk} r^l e^{-\alpha_{kl} r_i^2} e^{(\vec{r}-\vec{r_i})^2} Y_{lm}(\theta,\phi) \tag{2}\\
&= \int dV \sum_k \beta_{nk} e^{-\alpha_{kl} r_i^2} e^{(\vec{r}-\vec{r_i})^2} \Phi_{lm}(r,\theta,\phi) \tag{3}\\
&= \int dV \sum_k \beta_{nk} e^{-(1+\alpha_{kl})(\vec{r}-\frac{\vec{r_i}}{1+\alpha_{kl}})^2} e^{-\frac{\alpha_{kl}}{1+\alpha_{kl}} r_i^2} \Phi_{lm}(x,y,z) \tag{4}\\
&= \sum_k \beta_{nk} e^{-\frac{\alpha_{kl}}{1+\alpha_{kl}} r_i^2} \int dV e^{-(1+\alpha_{kl})(\vec{r}-\frac{\vec{r_i}}{1+\alpha_{kl}})^2} \Phi_{lm}(x,y,z) \tag{5}\\
&= \sum_k \beta_{nk} e^{-\frac{\alpha_{kl}}{1+\alpha_{kl}} r_i^2} \int dV e^{-(1+\alpha_{kl})\vec{r}^2} \Phi_{lm}(x+\frac{x_i}{1+\alpha_{kl}}, y+\frac{y_i}{1+\alpha_{kl}}, z+\frac{z_i}{1+\alpha_{kl}}) \tag{6}\\
&= \sum_k \beta_{nk} e^{-\frac{\alpha_{kl}}{1+\alpha_{kl}} r_i^2} \Phi_{lm}(\frac{x_i}{1+\alpha_{kl}}, \frac{y_i}{1+\alpha_{kl}}, \frac{z_i}{1+\alpha_{kl}}) \int dV e^{-(1+\alpha_{kl})\vec{r}^2} \tag{7}\\
&= \frac{\pi^{3/2}}{(1+\alpha_{lm})^{3/2}} \sum_k \beta_{nk} e^{-\frac{\alpha_{kl}}{1+\alpha_{kl}} r_i^2} \Phi_{lm}(\frac{x_i}{1+\alpha_{kl}}, \frac{y_i}{1+\alpha_{kl}}, \frac{z_i}{1+\alpha_{kl}}) \tag{8}\\
&= \frac{\xi_l \pi^{3/2}}{(1+\alpha_{lm})^{3/2+l}} \sum_k \beta_{nk} e^{-\frac{\alpha_{kl}}{1+\alpha_{kl}} r_i^2} P_{lm}(x_i,y_i,z_i) \tag{9}
\end{align}
$$

eq.6 is possible because x,y and z go to plus and minus infinity, and because it is in the form of a polynomial. And eq.9 comes from the characteristics of the spherical harmonics,

$$
\Phi_{lm}(\frac{x}{a},\frac{y}{a},\frac{z}{a}) = r^l Y_{lm}(\frac{x}{a},\frac{y}{a},\frac{z}{a}) = \frac{\xi_l}{a^l} P_{lm}(x,y,z) \tag{10}
$$

where ξ_l is a constant and P_{lm} takes a polynomial form. Hence the derivatives are

$$
\begin{align}
\frac{\partial c^i_{nlm}}{\partial x_i} &= \frac{\xi_l \pi^{3/2}}{(1+\alpha_{lm})^{3/2+l}} \sum_k \beta_{nk} \left(\frac{\partial P_{lm}(x_i,y_i,z_i)}{\partial x_i} e^{-\frac{\alpha_{kl}}{1+\alpha_{kl}} r_i^2} - \frac{2\alpha_{kl} x_i}{1+\alpha_{kl}} e^{-\frac{\alpha_{kl}}{1+\alpha_{kl}} r_i^2} P_{lm}(x_i,y_i,z_i) \right) \tag{11}\\
\frac{\partial c^i_{nlm}}{\partial y_i} &= \frac{\xi_l \pi^{3/2}}{(1+\alpha_{lm})^{3/2+l}} \sum_k \beta_{nk} \left(\frac{\partial P_{lm}(x_i,y_i,z_i)}{\partial y_i} e^{-\frac{\alpha_{kl}}{1+\alpha_{kl}} r_i^2} - \frac{2\alpha_{kl} y_i}{1+\alpha_{kl}} e^{-\frac{\alpha_{kl}}{1+\alpha_{kl}} r_i^2} P_{lm}(x_i,y_i,z_i) \right) \tag{12}\\
\frac{\partial c^i_{nlm}}{\partial z_i} &= \frac{\xi_l \pi^{3/2}}{(1+\alpha_{lm})^{3/2+l}} \sum_k \beta_{nk} \left(\frac{\partial P_{lm}(x_i,y_i,z_i)}{\partial z_i} e^{-\frac{\alpha_{kl}}{1+\alpha_{kl}} r_i^2} - \frac{2\alpha_{kl} z_i}{1+\alpha_{kl}} e^{-\frac{\alpha_{kl}}{1+\alpha_{kl}} r_i^2} P_{lm}(x_i,y_i,z_i) \right) \tag{13}
\end{align}
$$

$$
\begin{align*}
\Phi_{0\,0} &= \frac{1}{2\sqrt{\pi}}\\
\Phi_{1\,-1} &= \frac{\sqrt{3}y}{2\sqrt{\pi}}
\end{align*}
$$

$$\Phi_{1\,0} = \frac{\sqrt{3}z}{2\sqrt{\pi}}$$

$$\Phi_{1\,+1} = \frac{\sqrt{3}x}{2\sqrt{\pi}}$$

$$\Phi_{2\,-2} = \frac{3\sqrt{5}xy}{\sqrt{2}\sqrt{6}\sqrt{\pi}}$$

$$\Phi_{2\,-1} = \frac{3\sqrt{5}yz}{\sqrt{2}\sqrt{6}\sqrt{\pi}}$$

$$\Phi_{2\,0} = \frac{\sqrt{5}\left(\frac{3z^2}{2} - \frac{r^2}{2}\right)}{2\sqrt{\pi}}$$

$$\Phi_{2\,+1} = \frac{3\sqrt{5}xz}{\sqrt{2}\sqrt{6}\sqrt{\pi}}$$

$$\Phi_{2\,+2} = \frac{3\sqrt{5}\left(x^2 - y^2\right)}{2^{\frac{3}{2}}\sqrt{6}\sqrt{\pi}}$$

$$\Phi_{3\,-3} = \frac{\sqrt{5}\sqrt{7}\left(3x^2y - y^3\right)}{2^{\frac{5}{2}}\sqrt{\pi}}$$

$$\Phi_{3\,-2} = \frac{15\sqrt{7}xyz}{\sqrt{2}\sqrt{30}\sqrt{\pi}}$$

$$\Phi_{3\,-1} = \frac{\sqrt{7}y\left(\frac{15z^2}{2} - \frac{3\left(r^2\right)}{2}\right)}{2^{\frac{3}{2}}\sqrt{3}\sqrt{\pi}}$$

$$\Phi_{3\,0} = \frac{\sqrt{7}\left(\frac{5z^3}{2} - \frac{3z\left(r^2\right)}{2}\right)}{2\sqrt{\pi}}$$

$$\Phi_{3\,+1} = \frac{\sqrt{7}x\left(\frac{15z^2}{2} - \frac{3\left(r^2\right)}{2}\right)}{2^{\frac{3}{2}}\sqrt{3}\sqrt{\pi}}$$

$$\Phi_{3\,+2} = \frac{15\sqrt{7}\left(x^2 - y^2\right)z}{2^{\frac{3}{2}}\sqrt{30}\sqrt{\pi}}$$

$$\Phi_{3\,+3} = \frac{\sqrt{5}\sqrt{7}\left(x^3 - 3xy^2\right)}{2^{\frac{5}{2}}\sqrt{\pi}}$$

$$\Phi_{4\,-4} = \frac{105\left(4x^3y - 4xy^3\right)}{2^{\frac{7}{2}}\sqrt{70}\sqrt{\pi}}$$

$$\Phi_{4\,-3} = \frac{3\sqrt{35}\left(3x^2y - y^3\right)z}{2^{\frac{5}{2}}\sqrt{\pi}}$$

$$\Phi_{4\,-2} = \frac{xy\left(\frac{105z^2}{2} - \frac{15\left(r^2\right)}{2}\right)}{\sqrt{2}\sqrt{10}\sqrt{\pi}}$$

$$\Phi_{4\,-1} = \frac{3y\left(\frac{35z^3}{2} - \frac{15z\left(r^2\right)}{2}\right)}{2^{\frac{3}{2}}\sqrt{5}\sqrt{\pi}}$$

$$\Phi_{4\,0} = \frac{3\left(\frac{3\left(r^2\right)^2}{8} + \frac{35z^4}{8} - \frac{15z^2\left(r^2\right)}{4}\right)}{2\sqrt{\pi}}$$

$$\Phi_{4\,+1} = \frac{3x\left(\frac{35z^3}{2} - \frac{15z\left(r^2\right)}{2}\right)}{2^{\frac{3}{2}}\sqrt{5}\sqrt{\pi}}$$

$$\Phi_{4\,+2} = \frac{\left(x^2 - y^2\right)\,\left(\frac{105z^2}{2} - \frac{15\left(r^2\right)}{2}\right)}{2^{\frac{3}{2}}\sqrt{10}\sqrt{\pi}}$$

$$\Phi_{4\,+3} = \frac{3\sqrt{35}\left(x^3 - 3xy^2\right)z}{2^{\frac{5}{2}}\sqrt{\pi}}$$

$$\Phi_{4\,+4} = \frac{105\left(y^4 - 6x^2y^2 + x^4\right)}{2^{\frac{7}{2}}\sqrt{70}\sqrt{\pi}}$$

$$\Phi_{5\,-5} = \frac{3\sqrt{7}\sqrt{11}\left(y^5 - 10x^2y^3 + 5x^4y\right)}{2^{\frac{9}{2}}\sqrt{\pi}}$$

$$\Phi_{5\,-4} = \frac{105\sqrt{11}\left(4x^3y - 4xy^3\right)z}{2^{\frac{7}{2}}\sqrt{70}\sqrt{\pi}}$$

$$\Phi_{5\,-3} = \frac{\sqrt{11}\left(3x^2y - y^3\right)\,\left(\frac{945z^2}{2} - \frac{105\left(r^2\right)}{2}\right)}{32^{\frac{7}{2}}\sqrt{35}\sqrt{\pi}}$$

$$\Phi_{5\,-2} = \frac{\sqrt{11}xy\,\left(\frac{315z^3}{2} - \frac{105z\left(r^2\right)}{2}\right)}{\sqrt{2}\sqrt{210}\sqrt{\pi}}$$

$$\Phi_{5\,-1} = \frac{\sqrt{11}y\,\left(\frac{15\left(r^2\right)^2}{8} + \frac{315z^4}{8} - \frac{105z^2\left(r^2\right)}{4}\right)}{\sqrt{2}\sqrt{30}\sqrt{\pi}}$$

$$\Phi_{5\,0} = \frac{\sqrt{11}\left(\frac{15z\left(r^2\right)^2}{8} + \frac{63z^5}{8} - \frac{35z^3\left(r^2\right)}{4}\right)}{2\sqrt{\pi}}$$

$$\Phi_{5\,+1} = \frac{\sqrt{11}x\,\left(\frac{15\left(r^2\right)^2}{8} + \frac{315z^4}{8} - \frac{105z^2\left(r^2\right)}{4}\right)}{\sqrt{2}\sqrt{30}\sqrt{\pi}}$$

$$\Phi_{5\,+2} = \frac{\sqrt{11}\left(x^2 - y^2\right)\,\left(\frac{315z^3}{2} - \frac{105z\left(r^2\right)}{2}\right)}{2^{\frac{3}{2}}\sqrt{210}\sqrt{\pi}}$$

$$\Phi_{5\,+3} = \frac{\sqrt{11}\left(x^3 - 3xy^2\right)\,\left(\frac{945z^2}{2} - \frac{105\left(r^2\right)}{2}\right)}{32^{\frac{7}{2}}\sqrt{35}\sqrt{\pi}}$$

$$\Phi_{5\,+4} = \frac{105\sqrt{11}\left(y^4 - 6x^2y^2 + x^4\right)z}{2^{\frac{7}{2}}\sqrt{70}\sqrt{\pi}}$$

$$\Phi_{5\,+5} = \frac{3\sqrt{7}\sqrt{11}\left(5xy^4 - 10x^3y^2 + x^5\right)}{2^{\frac{9}{2}}\sqrt{\pi}}$$

$$\Phi_{6\,-6} = \frac{\sqrt{13}\sqrt{231}\left(6xy^5 - 20x^3y^3 + 6x^5y\right)}{2^{\frac{11}{2}}\sqrt{\pi}}$$

$$\Phi_{6\,-5} = \frac{3\sqrt{13}\sqrt{77}\left(y^5 - 10x^2y^3 + 5x^4y\right)z}{2^{\frac{9}{2}}\sqrt{\pi}}$$

$$\Phi_{6\,-4} = \frac{\sqrt{13}\left(4x^3y - 4xy^3\right)\,\left(\frac{10395z^2}{2} - \frac{945\left(r^2\right)}{2}\right)}{452^{\frac{7}{2}}\sqrt{14}\sqrt{\pi}}$$

$$\Phi_{6\,-3} = \frac{\sqrt{13}\left(3x^2y - y^3\right)\,\left(\frac{3465z^3}{2} - \frac{945z\left(r^2\right)}{2}\right)}{32^{\frac{7}{2}}\sqrt{105}\sqrt{\pi}}$$

$$\Phi_{6\,-2} = \frac{\sqrt{13}xy\,\left(\frac{105\left(r^2\right)^2}{8} + \frac{3465z^4}{8} - \frac{945z^2\left(r^2\right)}{4}\right)}{2^{\frac{3}{2}}\sqrt{105}\sqrt{\pi}}$$

$$\Phi_{6\,-1} = \frac{\sqrt{13}y\,\left(\frac{105z\left(r^2\right)^2}{8} + \frac{693z^5}{8} - \frac{315z^3\left(r^2\right)}{4}\right)}{\sqrt{2}\sqrt{42}\sqrt{\pi}}$$

$$\Phi_{6\,0} = \frac{\sqrt{13}\left(-\frac{5\left(r^2\right)^3}{16} + \frac{105z^2\left(r^2\right)^2}{16} + \frac{231z^6}{16} - \frac{315z^4\left(r^2\right)}{16}\right)}{2\sqrt{\pi}}$$

$$\Phi_{6\,+1} = \frac{\sqrt{13}x\,\left(\frac{105z\left(r^2\right)^2}{8} + \frac{693z^5}{8} - \frac{315z^3\left(r^2\right)}{4}\right)}{\sqrt{2}\sqrt{42}\sqrt{\pi}}$$

$$\Phi_{6\,+2} = \frac{\sqrt{13}\left(x^2 - y^2\right)\left(\frac{105\left(r^2\right)^2}{8} + \frac{3465z^4}{8} - \frac{945z^2\left(r^2\right)}{4}\right)}{2^{\frac{5}{2}}\sqrt{105}\sqrt{\pi}}$$

$$\Phi_{6\,+3} = \frac{\sqrt{13}\left(x^3 - 3xy^2\right)\left(\frac{3465z^3}{2} - \frac{945z\left(r^2\right)}{2}\right)}{32^{\frac{7}{2}}\sqrt{105}\sqrt{\pi}}$$

$$\Phi_{6\,+4} = \frac{\sqrt{13}\left(y^4 - 6x^2y^2 + x^4\right)\left(\frac{10395z^2}{2} - \frac{945\left(r^2\right)}{2}\right)}{452^{\frac{7}{2}}\sqrt{14}\sqrt{\pi}}$$

$$\Phi_{6\,+5} = \frac{3\sqrt{13}\sqrt{77}\left(5xy^4 - 10x^3y^2 + x^5\right)z}{2^{\frac{9}{2}}\sqrt{\pi}}$$

$$\Phi_{6\,+6} = \frac{\sqrt{13}\sqrt{231}\left(-y^6 + 15x^2y^4 - 15x^4y^2 + x^6\right)}{2^{\frac{11}{2}}\sqrt{\pi}}$$

$$\Phi_{7\,-7} = \frac{429\sqrt{15}\left(-y^7 + 21x^2y^5 - 35x^4y^3 + 7x^6y\right)}{2^{\frac{11}{2}}\sqrt{858}\sqrt{\pi}}$$

$$\Phi_{7\,-6} = \frac{\sqrt{15}\sqrt{3003}\left(6xy^5 - 20x^3y^3 + 6x^5y\right)z}{2^{\frac{11}{2}}\sqrt{\pi}}$$

$$\Phi_{7\,-5} = \frac{\left(y^5 - 10x^2y^3 + 5x^4y\right)\left(\frac{135135z^2}{2} - \frac{10395\left(r^2\right)}{2}\right)}{32^{\frac{9}{2}}\sqrt{15}\sqrt{462}\sqrt{\pi}}$$

$$\Phi_{7\,-4} = \frac{\left(4x^3y - 4xy^3\right)\left(\frac{45045z^3}{2} - \frac{10395z\left(r^2\right)}{2}\right)}{2^{\frac{7}{2}}\sqrt{15}\sqrt{462}\sqrt{\pi}}$$

$$\Phi_{7\,-3} = \frac{\left(3x^2y - y^3\right)\left(\frac{945\left(r^2\right)^2}{8} + \frac{45045z^4}{8} - \frac{10395z^2\left(r^2\right)}{4}\right)}{2^{\frac{5}{2}}\sqrt{15}\sqrt{42}\sqrt{\pi}}$$

$$\Phi_{7\,-2} = \frac{\sqrt{15}xy\,\left(\frac{945z\left(r^2\right)^2}{8} + \frac{9009z^5}{8} - \frac{3465z^3\left(r^2\right)}{4}\right)}{32^{\frac{3}{2}}\sqrt{21}\sqrt{\pi}}$$

$$\Phi_{7\,-1} = \frac{\sqrt{15}y\,\left(-\frac{35\left(r^2\right)^3}{16} + \frac{945z^2\left(r^2\right)^2}{16} + \frac{3003z^6}{16} - \frac{3465z^4\left(r^2\right)}{16}\right)}{2^{\frac{3}{2}}\sqrt{14}\sqrt{\pi}}$$

$$\Phi_{7\,0} = \frac{\sqrt{15}\left(-\frac{35z\left(r^2\right)^3}{16} + \frac{315z^3\left(r^2\right)^2}{16} + \frac{429z^7}{16} - \frac{693z^5\left(r^2\right)}{16}\right)}{2\sqrt{\pi}}$$

$$\Phi_{7\,+1} = \frac{\sqrt{15}x\,\left(-\frac{35\left(r^2\right)^3}{16} + \frac{945z^2\left(r^2\right)^2}{16} + \frac{3003z^6}{16} - \frac{3465z^4\left(r^2\right)}{16}\right)}{2^{\frac{3}{2}}\sqrt{14}\sqrt{\pi}}$$

$$\Phi_{7\,+2} = \frac{\sqrt{15}\left(x^2 - y^2\right)\left(\frac{945z\left(r^2\right)^2}{8} + \frac{9009z^5}{8} - \frac{3465z^3\left(r^2\right)}{4}\right)}{32^{\frac{5}{2}}\sqrt{21}\sqrt{\pi}}$$

$$\Phi_{7\,+3} = \frac{\left(x^3 - 3xy^2\right)\left(\frac{945\left(r^2\right)^2}{8} + \frac{45045z^4}{8} - \frac{10395z^2\left(r^2\right)}{4}\right)}{2^{\frac{5}{2}}\sqrt{15}\sqrt{42}\sqrt{\pi}}$$

$$\Phi_{7\,+4} = \frac{\left(y^4 - 6x^2y^2 + x^4\right)\left(\frac{45045z^3}{2} - \frac{10395z\left(r^2\right)}{2}\right)}{2^{\frac{7}{2}}\sqrt{15}\sqrt{462}\sqrt{\pi}}$$

$$\Phi_{7\,+5} = \frac{\left(5xy^4 - 10x^3y^2 + x^5\right)\left(\frac{135135z^2}{2} - \frac{10395\left(r^2\right)}{2}\right)}{32^{\frac{9}{2}}\sqrt{15}\sqrt{462}\sqrt{\pi}}$$

$$\Phi_{7\,+6} = \frac{\sqrt{15}\sqrt{3003}\left(-y^6 + 15x^2y^4 - 15x^4y^2 + x^6\right)z}{2^{\frac{11}{2}}\sqrt{\pi}}$$

$$\Phi_{7\,+7} = \frac{429\sqrt{15}\left(-7xy^6 + 35x^3y^4 - 21x^5y^2 + x^7\right)}{2^{\frac{11}{2}}\sqrt{858}\sqrt{\pi}}$$

$$\Phi_{8\,-8} = \frac{2145\sqrt{17}\left(-8xy^7 + 56x^3y^5 - 56x^5y^3 + 8x^7y\right)}{2^{\frac{15}{2}}\sqrt{1430}\sqrt{\pi}}$$

$$\Phi_{8\,-7} = \frac{2145\sqrt{17}\left(-y^7 + 21x^2y^5 - 35x^4y^3 + 7x^6y\right)z}{2^{\frac{11}{2}}\sqrt{1430}\sqrt{\pi}}$$

$$\Phi_{8\,-6} = \frac{\sqrt{17}\left(6xy^5 - 20x^3y^3 + 6x^5y\right)\left(\frac{2027025z^2}{2} - \frac{135135\left(r^2\right)}{2}\right)}{3152^{\frac{11}{2}}\sqrt{429}\sqrt{\pi}}$$

$$\Phi_{8\,-5} = \frac{\sqrt{17}\left(y^5 - 10x^2y^3 + 5x^4y\right)\left(\frac{675675z^3}{2} - \frac{135135z\left(r^2\right)}{2}\right)}{452^{\frac{9}{2}}\sqrt{2002}\sqrt{\pi}}$$

$$\Phi_{8\,-4} = \frac{\sqrt{17}\left(4x^3y - 4xy^3\right)\left(\frac{10395\left(r^2\right)^2}{8} + \frac{675675z^4}{8} - \frac{135135z^2\left(r^2\right)}{4}\right)}{452^{\frac{7}{2}}\sqrt{154}\sqrt{\pi}}$$

$$\Phi_{8\,-3} = \frac{\sqrt{17}\left(3x^2y - y^3\right)\left(\frac{10395z\left(r^2\right)^2}{8} + \frac{135135z^5}{8} - \frac{45045z^3\left(r^2\right)}{4}\right)}{32^{\frac{5}{2}}\sqrt{2310}\sqrt{\pi}}$$

$$\Phi_{8\,-2} = \frac{\sqrt{17}xy\left(-\frac{315\left(r^2\right)^3}{16} + \frac{10395z^2\left(r^2\right)^2}{16} + \frac{45045z^6}{16} - \frac{45045z^4\left(r^2\right)}{16}\right)}{32^{\frac{3}{2}}\sqrt{35}\sqrt{\pi}}$$

$$\Phi_{8\,-1} = \frac{\sqrt{17}y\left(-\frac{315z\left(r^2\right)^3}{16} + \frac{3465z^3\left(r^2\right)^2}{16} + \frac{6435z^7}{16} - \frac{9009z^5\left(r^2\right)}{16}\right)}{12\sqrt{\pi}}$$

$$\Phi_{8\,0} = \frac{\sqrt{17}\left(\frac{35\left(r^2\right)^4}{128} - \frac{315z^2\left(r^2\right)^3}{32} + \frac{3465z^4\left(r^2\right)^2}{64} + \frac{6435z^8}{128} - \frac{3003z^6\left(r^2\right)}{32}\right)}{2\sqrt{\pi}}$$

$$\Phi_{8\,+1} = \frac{\sqrt{17}x\left(-\frac{315z\left(r^2\right)^3}{16} + \frac{3465z^3\left(r^2\right)^2}{16} + \frac{6435z^7}{16} - \frac{9009z^5\left(r^2\right)}{16}\right)}{12\sqrt{\pi}}$$

$$\Phi_{8\,+2} = \frac{\sqrt{17}\left(x^2 - y^2\right)\left(-\frac{315\left(r^2\right)^3}{16} + \frac{10395z^2\left(r^2\right)^2}{16} + \frac{45045z^6}{16} - \frac{45045z^4\left(r^2\right)}{16}\right)}{32^{\frac{5}{2}}\sqrt{35}\sqrt{\pi}}$$

$$\Phi_{8\,+3} = \frac{\sqrt{17}\left(x^3 - 3xy^2\right)\left(\frac{10395z\left(r^2\right)^2}{8} + \frac{135135z^5}{8} - \frac{45045z^3\left(r^2\right)}{4}\right)}{32^{\frac{5}{2}}\sqrt{2310}\sqrt{\pi}}$$

$$\Phi_{8\,+4} = \frac{\sqrt{17}\left(y^4 - 6x^2y^2 + x^4\right)\left(\frac{10395\left(r^2\right)^2}{8} + \frac{675675z^4}{8} - \frac{135135z^2\left(r^2\right)}{4}\right)}{452^{\frac{7}{2}}\sqrt{154}\sqrt{\pi}}$$

$$\Phi_{8\,+5} = \frac{\sqrt{17}\left(5xy^4 - 10x^3y^2 + x^5\right)\left(\frac{675675z^3}{2} - \frac{135135z\left(r^2\right)}{2}\right)}{452^{\frac{9}{2}}\sqrt{2002}\sqrt{\pi}}$$

$$\Phi_{8\,+6} = \frac{\sqrt{17}\left(-y^6 + 15x^2y^4 - 15x^4y^2 + x^6\right)\left(\frac{2027025z^2}{2} - \frac{135135\left(r^2\right)}{2}\right)}{3152^{\frac{11}{2}}\sqrt{429}\sqrt{\pi}}$$

$$\Phi_{8\,+7} = \frac{2145\sqrt{17}\left(-7xy^6 + 35x^3y^4 - 21x^5y^2 + x^7\right)z}{2^{\frac{11}{2}}\sqrt{1430}\sqrt{\pi}}$$

$$\Phi_{8\,+8} = \frac{2145\sqrt{17}\left(y^8 - 28x^2y^6 + 70x^4y^4 - 28x^6y^2 + x^8\right)}{2^{\frac{15}{2}}\sqrt{1430}\sqrt{\pi}}$$

$$\Phi_{9\,-9} = \frac{\sqrt{19}\sqrt{12155}\left(y^9 - 36x^2y^7 + 126x^4y^5 - 84x^6y^3 + 9x^8y\right)}{2^{\frac{17}{2}}\sqrt{\pi}}$$

$$\Phi_{9\,-8} = \frac{36465\sqrt{19}\left(-8xy^7 + 56x^3y^5 - 56x^5y^3 + 8x^7y\right)z}{2^{\frac{15}{2}}\sqrt{24310}\sqrt{\pi}}$$

$$\Phi_{9\,-7} = \frac{\sqrt{19}\left(-y^7 + 21x^2y^5 - 35x^4y^3 + 7x^6y\right)\left(\frac{34459425z^2}{2} - \frac{2027025\left(r^2\right)}{2}\right)}{9452^{\frac{15}{2}}\sqrt{715}\sqrt{\pi}}$$

$$\Phi_{9\,-6} = \frac{\sqrt{19}\left(6xy^5 - 20x^3y^3 + 6x^5y\right)\left(\frac{11486475z^3}{2} - \frac{2027025z\left(r^2\right)}{2}\right)}{3152^{\frac{11}{2}}\sqrt{2145}\sqrt{\pi}}$$

$$\Phi_{9\,-5} = \frac{\sqrt{19}\left(y^5 - 10x^2y^3 + 5x^4y\right)\left(\frac{135135\left(r^2\right)^2}{8} + \frac{11486475z^4}{8} - \frac{2027025z^2\left(r^2\right)}{4}\right)}{3152^{\frac{9}{2}}\sqrt{143}\sqrt{\pi}}$$

$$\Phi_{9\,-4} = \frac{\sqrt{19}\left(4x^3y - 4xy^3\right)\left(\frac{135135z\left(r^2\right)^2}{8} + \frac{2297295z^5}{8} - \frac{675675z^3\left(r^2\right)}{4}\right)}{92^{\frac{7}{2}}\sqrt{10010}\sqrt{\pi}}$$

$$\Phi_{9\,-3} = \frac{\sqrt{19}\left(3x^2y - y^3\right)\left(-\frac{3465\left(r^2\right)^3}{16} + \frac{135135z^2\left(r^2\right)^2}{16} + \frac{765765z^6}{16} - \frac{675675z^4\left(r^2\right)}{16}\right)}{32^{\frac{7}{2}}\sqrt{1155}\sqrt{\pi}}$$

$$\Phi_{9\,-2} = \frac{\sqrt{19}xy\left(-\frac{3465z\left(r^2\right)^3}{16} + \frac{45045z^3\left(r^2\right)^2}{16} + \frac{109395z^7}{16} - \frac{135135z^5\left(r^2\right)}{16}\right)}{32^{\frac{3}{2}}\sqrt{55}\sqrt{\pi}}$$

$$\Phi_{9\,-1} = \frac{\sqrt{19}y\left(\frac{315\left(r^2\right)^4}{128} - \frac{3465z^2\left(r^2\right)^3}{32} + \frac{45045z^4\left(r^2\right)^2}{64} + \frac{109395z^8}{128} - \frac{45045z^6\left(r^2\right)}{32}\right)}{3\sqrt{2}\sqrt{10}\sqrt{\pi}}$$

$$\Phi_{9\,0} = \frac{\sqrt{19}\left(\frac{315z\left(r^2\right)^4}{128} - \frac{1155z^3\left(r^2\right)^3}{32} + \frac{9009z^5\left(r^2\right)^2}{64} + \frac{12155z^9}{128} - \frac{6435z^7\left(r^2\right)}{32}\right)}{2\sqrt{\pi}}$$

$$\Phi_{9\,+1} = \frac{\sqrt{19}x\left(\frac{315\left(r^2\right)^4}{128} - \frac{3465z^2\left(r^2\right)^3}{32} + \frac{45045z^4\left(r^2\right)^2}{64} + \frac{109395z^8}{128} - \frac{45045z^6\left(r^2\right)}{32}\right)}{3\sqrt{2}\sqrt{10}\sqrt{\pi}}$$

$$\Phi_{9\,+2} = \frac{\sqrt{19}\left(x^2 - y^2\right)\left(-\frac{3465z\left(r^2\right)^3}{16} + \frac{45045z^3\left(r^2\right)^2}{16} + \frac{109395z^7}{16} - \frac{135135z^5\left(r^2\right)}{16}\right)}{32^{\frac{5}{2}}\sqrt{55}\sqrt{\pi}}$$

$$\Phi_{9\,+3} = \frac{\sqrt{19}\left(x^3 - 3xy^2\right)\left(-\frac{3465\left(r^2\right)^3}{16} + \frac{135135z^2\left(r^2\right)^2}{16} + \frac{765765z^6}{16} - \frac{675675z^4\left(r^2\right)}{16}\right)}{32^{\frac{7}{2}}\sqrt{1155}\sqrt{\pi}}$$

$$\Phi_{9\,+4} = \frac{\sqrt{19}\left(y^4 - 6x^2y^2 + x^4\right)\left(\frac{135135z\left(r^2\right)^2}{8} + \frac{2297295z^5}{8} - \frac{675675z^3\left(r^2\right)}{4}\right)}{92^{\frac{7}{2}}\sqrt{10010}\sqrt{\pi}}$$

$$\Phi_{9\,+5} = \frac{\sqrt{19}\left(5xy^4 - 10x^3y^2 + x^5\right)\left(\frac{135135\left(r^2\right)^2}{8} + \frac{11486475z^4}{8} - \frac{2027025z^2\left(r^2\right)}{4}\right)}{3152^{\frac{9}{2}}\sqrt{143}\sqrt{\pi}}$$

$$\Phi_{9\,+6} = \frac{\sqrt{19}\left(-y^6 + 15x^2y^4 - 15x^4y^2 + x^6\right)\left(\frac{11486475z^3}{2} - \frac{2027025z\left(r^2\right)}{2}\right)}{3152^{\frac{11}{2}}\sqrt{2145}\sqrt{\pi}}$$

$$\Phi_{9\,+7} = \frac{\sqrt{19}\left(-7xy^6 + 35x^3y^4 - 21x^5y^2 + x^7\right)\left(\frac{34459425z^2}{2} - \frac{2027025\left(r^2\right)}{2}\right)}{9452^{\frac{15}{2}}\sqrt{715}\sqrt{\pi}}$$

$$\Phi_{9\,+8} = \frac{36465\sqrt{19}\left(y^8 - 28x^2y^6 + 70x^4y^4 - 28x^6y^2 + x^8\right)z}{2^{\frac{15}{2}}\sqrt{24310}\sqrt{\pi}}$$

$$\Phi_{9\,+9} = \frac{\sqrt{19}\sqrt{12155}\left(9xy^8 - 84x^3y^6 + 126x^5y^4 - 36x^7y^2 + x^9\right)}{2^{\frac{17}{2}}\sqrt{\pi}}$$