Impedance controller and design of desired parameters

Senthur Raj

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Introduction

This report describes the way of approach, the progress and the results obtained. The research objective is to find a novel way to design the impedance parameters (M_d, B_d and K_d) with the help of passivity analysis.

Passivity

Let us consider a system H with input $u(t)$ and output $y(t)$, where H is regarded as a mapping from input space to output space. Then, passivity of the map can be defined as [1],

The system $H : u \implies y$ is said to be passive if there exists a positive constant β such that,

$$\int_0^\tau y^T(t)u(t)dt \geq -\beta$$

for all input signals $u(t)$ and for all $\tau \in R^+$. In addition, H is said to be

- **Input strictly passive** if there exists a positive scalar δ_u such that

$$\int_0^\tau y^T(t)u(t)dt \geq -\beta + \delta_u \int_0^\tau ||u(t)||^2 dt,$$

- **Output strictly passive** if there exists a positive scalar δ_y such that

$$\int_0^\tau y^T(t)u(t)dt \geq -\beta + \delta_y \int_0^\tau ||y(t)||^2 dt,$$

For example, let us consider a n-link manipulator, the robot dynamics can be written as

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = T \tag{1}$$

where q, $\dot{q}$ and $\ddot{q}$ represent joint angles, velocities, and accelerations respectively, and T represents the input torque vector. The matrices $M(q)$ and $C(q, \dot{q})$ represent the manipulator inertia matrix and Coriolis matrix respectively and the gravity vector $G(q)$ can be written as

$$G(q) = \left(\frac{\partial P(q)}{\partial q} \right)^T$$

where $P(q)$ is potential energy of the system.

It is well known that $M(q)$ is positive definite and the matrix

$$\dot{M(q)} - 2C(q, \dot{q})$$

is skew-symmetric by defining $C(q, \dot{q})$ using the Christoffel symbols.
Let us take the summation of the kinetic energy and potential energy as the storage function as

$$S(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$$

Then, the time derivative of S along the trajectories of $Eq : (1)$ satisfies

$$\dot{S} = \dot{q}^T M(q)\ddot{q} + \tfrac{1}{2}\dot{q}^T \dot{M(q)}\dot{q} + \left(\tfrac{\partial P(q)}{\partial q} \right) \dot{q}$$

$$= \dot{q}^T (T - C(q, \dot{q} - G(q)) + \tfrac{1}{2}\dot{q}^T \dot{M(q)}\dot{q} + G(q)^T \dot{q})$$

$$= \dot{q}^T T - \tfrac{1}{2}\dot{q}^T (\dot{M(q)} - 2C(q, \dot{q}))\dot{q}$$

$$\implies \dot{S} = \dot{q}^T T$$

The last equation holds because of the skew symmetry of $\dot{M(q)} - 2C(q, \dot{q})$. This means passivity of the manipulator dynamics from the input torque T to the joint velocity $\dot{q}$.

Impedance control

The designed controller is capable of imposing the desired behavior defined by impedance on the original complicated behavior of the end-effector [2]. The actual dynamic model of a 5-bar manipulator can be expressed as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = T_{inp} - T_{ext} \tag{2}$$

The desired impedance model according to the specific work requirements which is usually specified as a second-order dynamic equation:

$$M_d \ddot{e} + B_d \dot{e} + K_d e = -T_{ext} \tag{3}$$

where $e = (q - q_d)$.

After substituting the target impedance $[Eq:3]$ into the actual manipulator dynamics $[Eq:2]$ by eliminating $\ddot{q}$, a specific control law is obtained as

$$T_{inp} = u + (1 + MM_d^{-1})T_{ext} + C(q,\dot{q})\dot{q} + G(q) + M(q)[\ddot{q}_d - M_d^{-1}[B_d\dot{e} + K_d e + T_{ext}]] \tag{4}$$

since

$$\ddot{q} = \ddot{q}_d - M_d^{-1}[B_d\dot{e} + k_d e + T_{ext}]$$

The control law $[Eq:4]$ is the desired one to make the actuators produce torque T_{inp}. Then the manipulator driven by T_{inp} will perform the behavior defined by impedance in $[Eq:3]$.

Passivity analysis

The governing equation becomes,

$$M\ddot{e} + MM_d^{-1}B_d\dot{e} + MM_d^{-1}K_d e = u \tag{5}$$

Now let us take the summation of the kinetic energy and potential energy as the storage function as

$$S(e,\dot{e}) = \frac{1}{2}\dot{e}^T M(q)\dot{e} + P(e)$$

where $P(e)$ can be written as

$$P(e) = \frac{1}{2}e^T MM_d^{-1}K_d e$$

Therefore $S > 0$.

Then, the time derivative of S along the trajectories of $[Eq:(5)]$ satisfies

$$\dot{S} = \dot{e}^T M\ddot{e} + \frac{1}{2}\dot{e}^T\dot{M}\dot{e} + e^T MM_d^{-1}K_d\dot{e}$$

$$= \dot{e}^T(u - MM_d^{-1}B_d\dot{e} - MM_d^{-1}K_d e) + \frac{1}{2}\dot{e}^T\dot{M}\dot{e} + e^T MM_d^{-1}K_d\dot{e}$$

$$= \dot{e}^T u - \dot{e}^T MM_d^{-1}B_d\dot{e} + \frac{1}{2}\dot{e}^T\dot{M}\dot{e}$$

$$= \dot{e}^T u + \frac{1}{2}\dot{e}^T(\dot{M} - 2MM_d^{-1}B_d)\dot{e}$$

$$\implies \dot{S} = \dot{e}^T u \tag{6}$$

As $(\dot{M} - 2MM_d^{-1}B_d)$ is considered to be a **skew symmetric matrix**.
$[Eq:6]$ implies that the system $[Eq:5]$ is **passive** from input u to output $\dot{e}$. Also notice that the storage function is shaped so that it takes the minimal value at $e = 0$ and $\dot{e} = 0$.

Substituting $\dot{e} = 0$ and $u = 0$ in $[Eq : (5)]$, we get

$$e = 0$$

and hence the system is **zero-state observable** and **lyapunov stable**. Thus, the damping injection

$$u = -k_q \dot{e} \tag{7}$$

If k_q is a **diagonal positive gain matrix**, then it guarantees **asymptotic stability** of the origin $\dot{e} = 0$ and $e = 0$.

Modified dynamics

The dynamic model of the system as represented in $[Eq : 5]$:

$$\begin{aligned} M\ddot{e} + MM_d^{-1}B_d\dot{e} + MM_d^{-1}K_d e &= u \\ \ddot{e} + \frac{B_d}{M_d}\dot{e} + \frac{K_d}{M_d}e &= \frac{u}{M} \end{aligned}$$

Lyapunov stability

For lyapunov stability, $u = 0$. Therefore the system becomes

$$\ddot{e} + \frac{B_d}{M_d}\dot{e} + \frac{K_d}{M_d}e = 0$$

Natural frequency, $\omega_n = \sqrt{\frac{K_d}{M_d}}$ and Damping ratio, $\gamma = \frac{B_d}{2M_d\omega_n}$

Asymptotic stability

For asymptotic stability, $u = -k_q\dot{e}$ where k_q is a positive diagonal matrix,

$$\ddot{e} + \left(\frac{B_d}{M_d} + \frac{k_q}{M}\right)\dot{e} + \frac{K_d}{M_d}e = 0$$

The natural frequency ω_n is same as the system with lyapunov stability but the Damping ratio is

$$\gamma = \frac{B_d M + k_q M_d}{2MM_d\omega_n}$$

References

[1] Takeshi Hatanaka, Nikhil Chopra, Masayuki Fujita, and M.W. Spong. *Passivity-Based Control and Estimation in Networked Robotics*. 01 2015.

[2] P. Song, Y. Yu, and X. Zhang. Impedance control of robots: An overview. In *2017 2nd International Conference on Cybernetics, Robotics and Control (CRC)*, pages 51–55, July 2017.