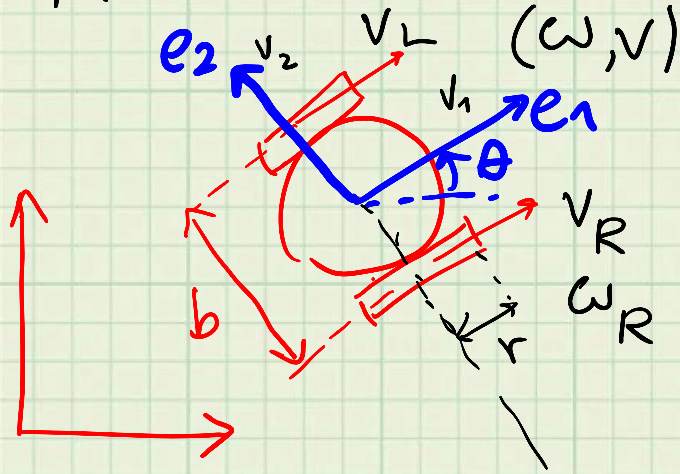
Robot motion in exp coordinates



$$v_R = r\,\omega_R$$

$$v_L = r\,\omega_L$$

$$v = \frac{v_L + v_R}{2}$$

$$\omega = \frac{v_R - v_L}{b}$$

twist $\xi = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \omega \\ v_1 \\ v_2 \end{bmatrix}_{3\times 1}$

Remark: Non-holonomic constaint

$$v_2 = 0$$

2D Rigid Body

$$so(2) = \{ A \in \mathbb{R}^{2\times 2} \mid A^T = -A \}$$

$$\omega \in \mathbb{R}, \quad \hat{\omega} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \in so(2)$$

$$se(2) = \left\{ \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \;\middle|\; \hat{\omega} \in so(2), v \in \mathbb{R}^2 \right\}$$

$$\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in se(2)$$

$$H = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix} \in SE(2)$$

Deterministic process

$$X_{k+1} = X_k U_k =: f_u(X_k)$$

$$\hat{u}_k \in se(2), \quad U_k = exp(\hat{u}_k \Delta t)$$

$$\Delta t = t_{k+1} - t_k$$

Noisy process:

$$X_{k+1} = X_k U_k \exp(w_k^\wedge)$$

$$w_k \in \mathbb{R}^3, \quad w_k \sim N(0, \Sigma_{w_k})$$

$$w_k^\wedge \in se(2)$$

Define the left-invariant error

$$\eta = \exp(\xi^\wedge) = X^{-1} \bar{X}$$

for any $L \in SE(2)$

$$(LX)^{-1}(L\bar{X}) = X^{-1}L^{-1}L\bar{X} = X^{-1}\bar{X} = \eta$$

X : true , $\bar{X}$: estimate

Why this is the correct way to define the error?

$$X = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}, \quad X^{-1} = \begin{bmatrix} R^T & -R^T P \\ 0 & 1 \end{bmatrix}$$

$$1)\ \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R & Rp \\ 0 & 1 \end{bmatrix}$$

then rotate first translate

$$2)\ \begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

then translate first rotate

e.g. $$\begin{bmatrix} R & Rp \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Rx + Rp \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Rx + p \\ 1 \end{bmatrix}$$

$$1)\ \eta = \begin{bmatrix} R & Rp \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \bar{R} & \bar{p} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R^T\bar{R} & \underline{R^T\bar{p} - p} \\ 0 & 1 \end{bmatrix}$$

$$\left(\begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} I & -p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R^T & 0 \\ 0 & 1 \end{bmatrix}$$

$$2)\ \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{R} & \bar{p} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R^T\bar{R} & R^T\bar{p} - R^T p \\ 0 & 1 \end{bmatrix}$$

$$\eta = X^{-1}\bar{X} \qquad X_k^{-1} X_{k+1} = U_k$$

$$\dot{R} = R\omega^\wedge \Rightarrow \exp(\omega^{b\wedge}\Delta t) = R_k^{-1} R_{k+1}$$

$$\dot{R} = \omega^\wedge R \Rightarrow \exp(\omega^{s\wedge}\Delta t) = R_{k+1} R_k^{-1}$$

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history $\begin{cases} e_p = \bar{p} - p \\ e_R = R^T \bar{R} \end{cases}$

Next time!

$$\eta = \exp(\xi^\wedge) = X^{-1}\bar{X}$$

$\exp(A)^{-1} = \exp(-A)$

$$X = \bar{X}\exp(-\xi^\wedge)$$

$$\bar{X}_{k+1}\exp(-\xi^\wedge_{k+1}) = \bar{X}_k \exp(-\xi^\wedge_k) U_k \exp(w^\wedge_k)$$

$AB \neq BA$

we need to shift the error terms to one side.

$f(x)$

$$Ad_X(\xi) = X\xi^\wedge X^{-1}$$

Ax

$$*\ (Ad_X\,\xi)^\wedge = X\xi^\wedge X^{-1}$$

$$\exp\left((Ad_X\,\xi)^\wedge\right) = X\exp(\xi^\wedge)X^{-1}$$

$$\text{I}\quad \exp\left((Ad_X\,\xi)^\wedge\right)X = X\exp(\xi^\wedge)$$

$X \mapsto X^{-1}$

$$\exp\left((Ad_{X^{-1}}\,\xi)^\wedge\right) = X^{-1}\exp(\xi^\wedge)X$$

$$\text{II}\quad X\exp\left((Ad_{X^{-1}}\,\xi)^\wedge\right) = \exp(\xi^\wedge)X$$

$$\bar{X}_{k+1}\exp(-\xi^\wedge_{k+1}) = \bar{X}_k U_k \exp\left((-Ad_{U_k^{-1}}\,\xi_k)^\wedge\right)\exp(w_k^\wedge)$$

II

$$\exp(-\xi^\wedge_{k+1}) = \exp\left((Ad_{U_k^{-1}}\,\xi_k)^\wedge\right)\cdot\exp(w_k^\wedge)$$

Q. How to combine two exp terms so we can work in the Lie algebra?

BCH series:

$$\exp(X)\exp(Y) = \exp(Z)$$

$$X, Y, Z \in \mathfrak{g}$$

$$Z = ?$$

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \dots$$

if both $\hat{\xi}_1$ and $\hat{\xi}_2$ are small

then $$\exp(\hat{\xi}_1)\exp(\hat{\xi}_2) \approx \exp(\hat{\xi}_1 + \hat{\xi}_2) + \text{HOT.}$$

$$\exp(-\hat{\xi}_{k+1}) = \exp((-Ad_{U_k^{-1}}\xi_k)^{\wedge}) \cdot \exp(\hat{w}_k)$$

$$\xi_{k+1} = Ad_{U_k^{-1}}\xi_k - w_k$$

$$\Sigma_{k+1} = Ad_{U_k^{-1}}\Sigma_k Ad_{U_k^{-1}}^T + \Sigma_{w_k}$$

$$\Sigma_k = \mathbb{E}\left([\xi - \xi_k][\xi - \xi_k]^T\right)$$

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what is Ad_X ?

$$\ast\ R\omega^\wedge R^{-1} = R\omega^\wedge R^T = (R\omega)^\wedge$$

$$\Rightarrow Ad_R = R$$

for rigid body

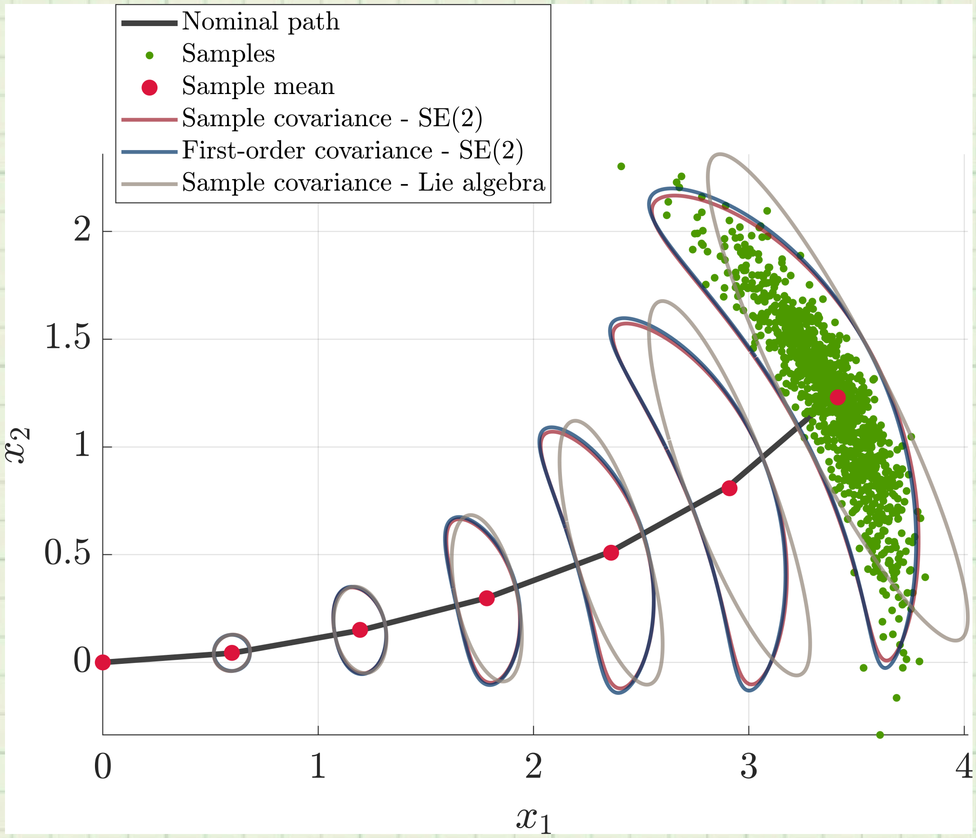
$$\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \omega^\wedge & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R\omega^\wedge R^T & -R\omega^\wedge R^T p + Rv \\ 0 & 0 \end{bmatrix}$$

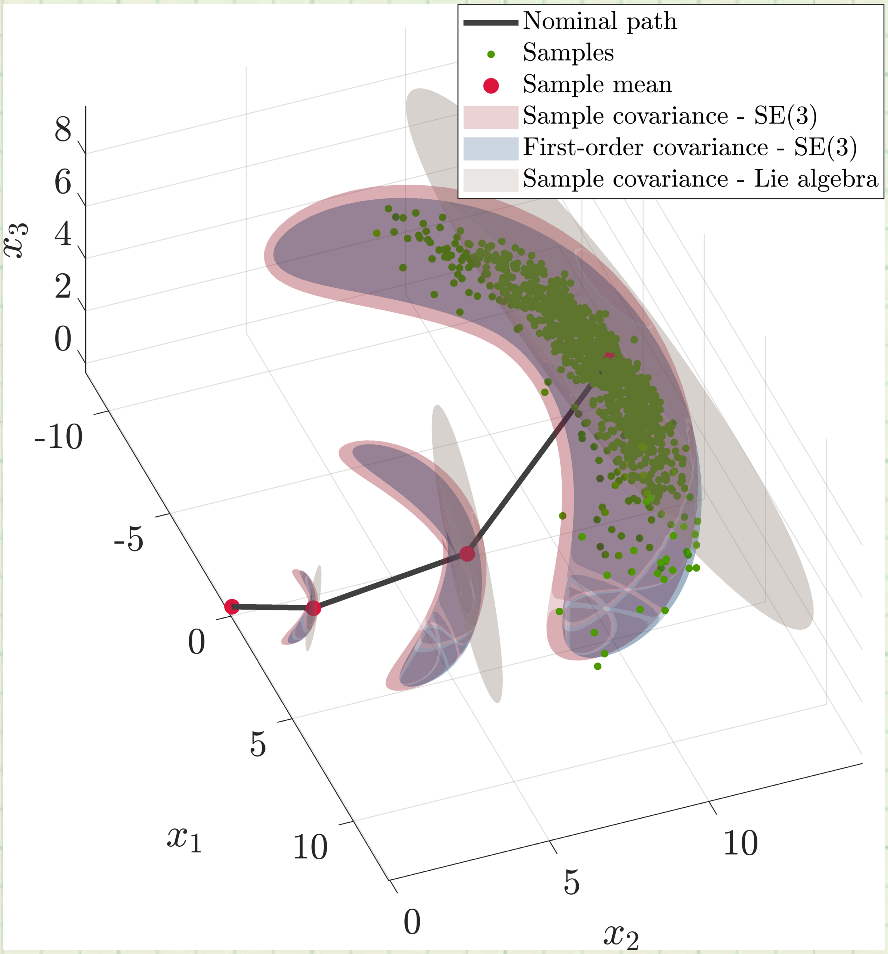
$$= \begin{bmatrix} (R\omega)^\wedge & \hat{p}(R\omega) + Rv \\ 0 & 0 \end{bmatrix}$$

$$= \left(\begin{bmatrix} R & 0 \\ p^\wedge R & \underline{R} \end{bmatrix} \begin{bmatrix} \omega \\ v \end{bmatrix} \right)^\wedge \Rightarrow Ad_X = \begin{bmatrix} R & 0 \\ \underline{\hat{p}R} & R \end{bmatrix}$$

$$r \times \omega^s + v^s = r \times R\omega^b + Rv^b$$



Nominal path
Samples
Sample mean
Sample covariance - SE(2)
First-order covariance - SE(2)
Sample covariance - Lie algebra
x_2
x_1



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