Inertial Measurement Unit (IMU)

State variables:

- Rotation, $R \in SO(3)$
- velocity, $v \in \mathbb{R}^3$
- position, $p \in \mathbb{R}^3$

$X = (R, v, p)$ State tuple

Inputs / Measurements:

Angular velocity : $\omega \in \mathbb{R}^3$, $\hat{\omega} \in so(3)$

Linear acceleration: $a \in \mathbb{R}^3$

Define $u = \begin{bmatrix} \omega \\ a \end{bmatrix} \in \mathbb{R}^6$

u is directly measured by an IMU in the body frame.

Matrix Lie Group $SE_2(3)$

This is a group of double-direct isometries of $\mathbb{R}^3$. Dim $SE_2(3) = 9$.

$$X = \begin{bmatrix} R & V & P \\ 0_{1\times3} & 1 & 0 \\ 0_{1\times3} & 0 & 1 \end{bmatrix}_{5\times5} \in SE_2(3) \quad \mathbb{R}^3$$

$$\xi = \begin{bmatrix} \xi^R \\ \xi^V \\ \xi^P \end{bmatrix} \in \mathbb{R}^9$$

rotation

velocity

position

$$\xi^\wedge = \begin{bmatrix} \xi^{R\wedge} & \xi^V & \xi^P \\ 0_{1\times3} & 0 & 0 \\ 0_{1\times3} & 0 & 0 \end{bmatrix} \in se_2(3)$$

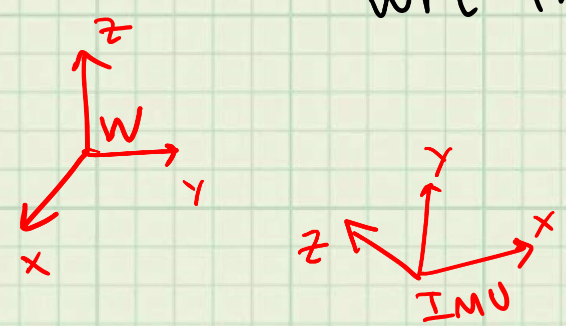
$$X = \exp(\xi^\wedge), \quad Ad_X = \begin{bmatrix} R & 0_{3\times3} & 0_{3\times3} \\ \hat{V}R & R & 0_{3\times3} \\ \hat{P}R & 0_{3\times3} & R \end{bmatrix}_{9\times9} \in \mathbb{R}^{9\times9}$$

Continuous-Time IMU process model:

$$
\begin{cases}
\dfrac{d}{dt} R_t = \dot{R}_t = R_t \omega_t^\wedge \\[2ex]
\dfrac{d}{dt} v_t = \dot{v}_t = R_t a_t + g \\[2ex]
\dfrac{d}{dt} p_t = v_t
\end{cases}
\qquad
g = \begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix}
$$

↳ gravity vector (constant)

R_t, v_t, p_t are rotation, velocity, and position of body/IMU frame wrt the world frame.



Let $X_t = \begin{bmatrix} R_t & v_t & p_t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in SE_2(3)$

$$
\frac{d}{dt} X_t = \begin{bmatrix} R_t \omega_t^\wedge & \underline{R a_t + g} & v_t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =: f_{u_t}(X_t)
$$

$$
\boxed{\dot{X} = X \xi^\wedge_{9\times 1} \quad \text{(this doesn't describe an IMU)}}
$$

$SO(3)$: $\dot{R} = R\omega^\wedge$, $SE(3)$: $\dot{X} = X\xi^\wedge$, $SE_2(3)$: $\underline{\dot{X} = X\xi^\wedge}$

Remark: The deterministic process model $f_{u_t}(X_t)$ satisfies the group-affine property.

Log-Linear Righ-Invariant Error Dynamics:

$$\frac{d}{dt}\xi_t = A_t^r \xi_t \quad , \quad A_t^r = \begin{bmatrix} 0 & 0 & 0 \\ \hat{g} & 0 & 0 \\ 0 & I & 0 \end{bmatrix}_{9\times 9}$$

Log-Linear Left-Invariant Error Dynamics:

$$\frac{d}{dt}\xi_t = A_t^l \xi_t \quad , \quad A_t^l = \begin{bmatrix} -\hat{\omega}_t & 0 & 0 \\ -\hat{a}_t & -\hat{\omega}_t & 0 \\ 0 & I & -\hat{\omega}_t \end{bmatrix}_{9\times 9}$$

Noisy process:

gyroscope: $\tilde{\omega}_t = \omega_t + w_t^g \Rightarrow \omega_t = \tilde{\omega}_t - w_t^g$

w_t^g is an additive white Gaussian noise (process)

accelerometer: $\tilde{a}_t = a_t + w_t^a \Rightarrow a_t = \tilde{a}_t - w_g^a$

w_t^a 〃 〃 〃 〃 〃

$$\frac{d}{dt} R_t = \underline{\dot{R}} = R_t \left(\tilde{\omega}_t - w_t^g \right)^\wedge$$

$R a \times R b = R(a \times b)$

$$\frac{d}{dt} v_t = \underline{\dot{v}_t} = R_t \left(\tilde{a}_t - w_t^a \right) + g$$

$$\dot{X} = \begin{bmatrix} \dot{R} & \dot{v} & \dot{p} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{d}{dt} p_t = \underline{\dot{p}_t} = v_t$$

$$\frac{d}{dt} X_t = \dot{X} = \begin{bmatrix} R\tilde{\omega}_t^\wedge & R\tilde{a}_t + g & v_t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} R_t & v_t & p_t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_t^{g\wedge} & w_t^a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$=: f_{u_t}(X) - X_t w_t^\wedge$$

We defined $w_t = [w_t^g, w_t^a, 0]^T \in \mathbb{R}^9$

Left-Invanriant EKF

propagation:

$$\frac{d}{dt}\bar{X} = f_{u_t}(\bar{X}_t) \quad , \quad t_{k-1} \leq t < t_k$$

$$\frac{d}{dt}P_t^l = A_t^l P_t^l + P_t^l A_t + Q_t$$

Continuos-time version of

$$P_{k+1} = \phi_k P_k \phi_k^T + Q_k$$

Let $\dot{x} = A_t x + w_t$

$$\begin{cases} x_{k+1} = \phi_k x_k + w_k \\ P_{k+1} = \phi_k P_k \phi_k^T + Q_k \end{cases}$$

$$\phi_k = \exp(A_t \Delta t) \approx I + A_t \Delta t$$

$$P_{k+1} = (I + A_t \Delta t) P_k (I + A_t \Delta t)^T + Q_t \Delta t$$

$$P_{k+1} = P_k + A_t P_k \Delta t + P_k A_t^T \Delta t + Q_t \Delta t$$

$$\frac{P_{k+1} - P_k}{\Delta t} = A_t P_k + P_k A_t^T + Q_t$$

$$\Delta t \to 0, \quad \lim_{\Delta t \to 0} \frac{P_{k+1} - P_k}{\Delta t} = \dot{P}_t$$

$$\dot{P}_t = A_t P_t + P_t A_t^T + Q_t$$