Willoughby Seago

Theoretical Physics

Particle Physics

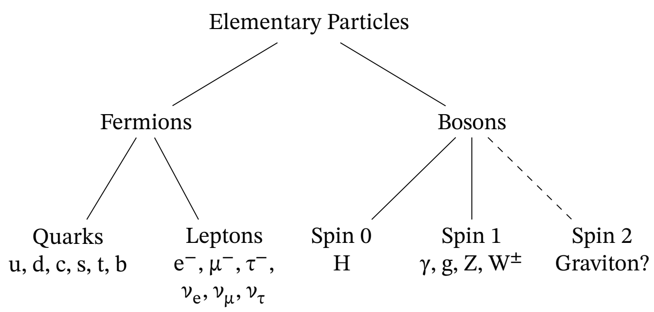
COURSE NOTES

Particle Physics

Willoughby Seago

These are my notes from the course particle physics. I took this course as a part of the theoretical physics degree at the University of Edinburgh.

These notes were last updated at 16:35 on October 29, 2022. For notes on other topics see https://github.com/WilloughbySeago/Uni-Notes.



Chapters

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One

Preliminaries

This course is not interested in the finer mathematical details, but the broader principles. For details see the *Quantum Field Theory* course. Knowledge is assumed of quantum mechanics (see the *Principles of Quantum Mechanics* and *Quantum Theory* courses), and special relativity (see the *Relativity, Nuclear, and Particle Physics* course, or *Classical Electrodynamics*).

We use natural units where $c = \hbar = \varepsilon_0 = \mu_0 = 1$. This means that masses, momentums, and energies are all measured in electron volts, force in electron volts squared, lengths and times in inverse electron volts, and speeds and angular momentums are dimensionless. Electric charge is also dimensionless, but the charge of an electron is *not* set to unity, although particle charges are given in terms of the (absolute value of the) electron charges. Instead we have the fine structure constant,

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \quad \text{(SI)} \qquad \alpha = \frac{e^2}{4\pi} \quad (c = \hbar = \varepsilon_0 = 1). \tag{1.0.1}$$

The charge of an electron then has magnitude

$$e = \sqrt{4\pi\alpha}. \tag{1.0.2}$$

Since α is dimensionless its value is the same in any system of units, and is approximately $1/137$, so

$$e \approx 0.303. \tag{1.0.3}$$

We use the Minkowski metric with the $(+---)$ convention:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.0.4}$$

Open Question 1.0.5 Particle physics is full of many unanswered questions. Relevant questions will be flagged in a box like this one. These are typically beyond the scope of the course, and are mentioned simply for interest.

Two

Introduction

2.1 The Standard Model

The **standard model of particle physics**, or the standard model for short, is our best model of the fundamental constituents of matter and fundamental forces. The standard model actually refers to a collection of quantum field theories (QFT) describing these forces as due to particle exchanges. The standard model is theoretically self consistent, all of the maths checks out. It can be used to make predictions, which can then be checked against measurements. So far all such tests have validated the standard model. As a model the standard model does not predict everything, instead there are about 20 parameters that have to be measured separately, including the masses of various particles, the coupling strengths of various fields, and mixing parameters.

The standard model cannot explain everything, a nonexhaustive list of unexplained phenomena by the standard model is as follows:

- general relativity/accelerating expansion of the universe;
- dark energy/dark matter;
- neutrino masses;
- matter/antimatter imbalance;
- why the parameters have the values they do.

2.2 Fundamental Particles

A particle with no charge. What's the point in that?

Victoria Marting

Spin is a property that every particle has, it's a quantum number, just a label, like charge or mass, but without such a simple interpretation. There are two spin operators, the total spin operator, $\hat{S}^2$, and the component of spin in the z-direction, $\hat{S}_z$. If $|\psi\rangle$ is a spin eigenstate then the action of these operators on $|\psi\rangle$ is

$$\hat{S}^2|\psi\rangle = s(s+1)|\psi\rangle, \quad \text{and} \quad \hat{S}_z|\psi\rangle = m_s\hbar|\psi\rangle. \tag{2.2.1}$$

Here s takes on a nonnegative half integer value, $s = 0, \pm 1/2, \pm 1, \pm 3/2, \ldots$. The value of m_s is then constrained to lie between $-s$ and s increasing in integer steps, so $m_s = -s, 1-s, \ldots, 0, \ldots, s-1, s$.

The fundamental particles in the standard model have spin $s = 0, 1/2, 1$. Specifically, the Higgs boson has spin 0, the quarks and leptons have spin 1/2, and the force carriers have spin 1. When a particle has spin 1/2 there are two possible values of m_s, $\pm 1/2$, which we call spin up $(+1/2)$ and spin down $(-1/2)$. When a particle has spin 1 there are three possible values of m_s, 0 and ± 1, which we refer to as polarisations. A photon can only have $m_s = \pm 1$, which is where this terminology comes from.

We broadly split all particles into two types, **fermions**, with half integer spin, and **bosons**, with integer spin. The fermions in the standard model further split into **quarks** and **leptons**.

There are 6 quarks, which split into two types up-type, **up**, **charm**, and **top quarks**, or u, c, and t, and down-type quarks, the **down**, **strange**, and **bottom quarks**, or d, s, and b. Up-type quarks have charge $+2/3$ in units of electron charge, and bottom-type quarks have charge $-1/3$. All types of quarks also have "colour charge", relating to the strong force, and "weak isospin", relating to the weak force, we'll see this in more detail later in the course. Each type (up/down) consists of three generations of quarks, and as we go down the generations, from u, to c, to t, they get more massive.

Similarly the leptons are split into two, first we have **electrons**, e^-, **muons**, μ^-, and **tau** particles, τ^-, these all have a charge of -1. Then, there are the three **neutrinos**, the **electron**, **muon**, and **tau neutrinos**, ν_e, ν_μ, and ν_τ, these are all electrically neutral. All leptons have zero colour charge, but they do have weak isospin.

We split the bosons into two parts, the spin zero bosons, or **scalar bosons**, which is just the **Higgs boson**, H. Then there are the force carrying bosons, also known as **gauge bosons** or **vector bosons**, which have spin 1. These consist of the **photon**, γ, gluons, g, Z **boson**, and $W^\pm$ **bosons**. The photon is the force carrier in electromagnetism and quantum electrodynamics (QED). The gluons are the force carriers in quantum chromodynamics (QCD). The Z-boson gives neutral currents in weak interactions and $W^\pm$-bosons give charged currents. Finally, there is the hypothetical graviton, which if it exists will be a spin 2 boson, or a **tensor boson**. Note that these names, scalar, vector, and gauge, come from what type of object the fields describing the particle are, so the photon is described by the electromagnetic field, A^μ, whereas the graviton is described by the energy-momentum tensor, $T^{\mu\nu}$.

2.2.1 Antiparticles

Every particle has a corresponding **antiparticle**, although some particles are there own antiparticles, such as the photon, Higgs boson, and Z boson. The W^+ and W^- are mutually each others antiparticles, and the antiparticle of a gluon is another gluon. The antiparticles are defined by being identical, but with opposite charges. By charges here we mean electric charge, colour charge, and weak isospin, which are the charges telling us how strongly the particle couples to the electromagnetic field, strong force, and weak force respectively.

The naming convention is to just stick the prefix "anti" in front of the particle's name. The one exception to the naming convention is the electron, whose antiparticle is the **positron**. Most antiparticles are denoted as the same symbol

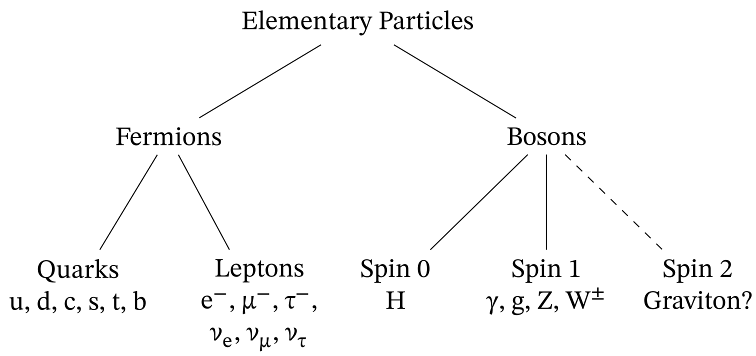


Figure 2.1: All of the particles in the standard model. The graviton is hypothesised but has not been observed.

with a bar, so $\bar{u}$ is an antiup quark. If the symbol usually has a sign, such as e^-, or W^+, then the antiparticle is denoted with the opposite sign, so e^+, or W^-.

Open Question 2.2.2 It is not known if the neutrinos are their own antiparticles or not. Fermions which are their own antiparticles are called Majorana fermions, whereas fermions which which aren't are called Dirac fermions.

2.3 Composite Particles

As well as the fundamental particles, of which there are 18 particles, there are *a lot* of **composite particles**, particles formed by combining quarks and/or antiquarks into a bound state. Note that there aren't composite leptons because the strong force is required to form bound states.

We call a composite particle formed from quarks a **hadron**. The hadrons split two types, first **baryons**, which are formed of three quarks[1], qqq, and **antibaryons**, which are formed of three antiquarks, $\bar{q}\bar{q}\bar{q}$. The other class is **mesons**, which are quark-antiquark pairs, $q\bar{q}$.

[1]here q denotes an arbitrary quark, in particular there is no requirement that all three quarks in a baryon are the same, even if we call all three q.

The antiparticle of a composite particle has its quarks replaced with the equivalent antiquarks, and antiquarks replaced with the equivalent quarks. For example, the **negative pion**, π^-, is a meson with quark content $d\bar{u}$, and its antiparticle is the **positive pion**, π^+, another meson with quark content $u\bar{d}$.

As well as the hadrons and mesons there have been other composite particles observed recently, although these aren't on the course as they aren't well understood yet. We've seen **tetraquarks**, formed from two quarks and two antiquarks, $qq\bar{q}\bar{q}$, and pentaquarks, formed from four quarks and an antiquark, $qqqq\bar{q}$. It is possible that these states aren't really new particles, but instead are "molecules" of either pairs of mesons in the tetraquark case or a hadron and a meson in the pentaquark case. The difference being that in the "molecules" not all quarks would be equally bound to each other, whereas if they truly are composite particles there will be no difference in how bound any pair is, apart from, for example, differences due to differing electric charges.

Three

Feynman Diagrams

Feynman diagrams are a way of depicting interactions between particles. They also correspond to the amplitude for that interaction occurring in that way. Each piece of the diagram can be converted into a term in some expression for the amplitude. More abstractly we can view Feynman diagrams as terms in a series expansion of some amplitude.

When reading Feynman diagrams in this course we follow the convention that time increases to the right.

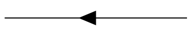
3.1 Currents

The most basic part of a Feynman diagram is a "current". This is a line representing the movement of a particle through spacetime. The phrase current comes from electrons, where an electron moving through space is interpreted as a current.

The way we depict a current depends on the type of particle. Fermions are depicted as a straight line with an arrow, so a fermion current looks like

(3.1.1)

Antifermions are then depicted as fermions travelling back in time, so an antifermion current looks like

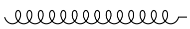


(3.1.2)

Spin 1 bosons, apart from gluons, so photons, Z bosons, and $W^{\pm}$ bosons, are depicted with a wavy line:

(3.1.3)

Gluons are depicted with a curly line,



(3.1.4)

The Higgs boson is depicted with a dashed line,

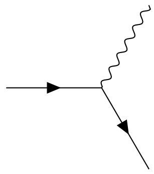
(3.1.5)

It's fine not to draw an arrow for bosons as either the particle is its own antiparticle, in which case the direction does not matter, or in the case of the $W^{\pm}$ bosons or gluons the antiparticle is of the same type, so changing direction corresponds to changing the charge (electric charge for $W^{\pm}$, and colour charge for gluons).

3.2 Vertex

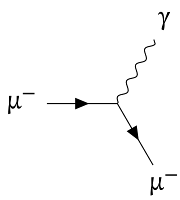
The building block of any Feynman diagram is the **vertex**, this is where currents come together. The exact rules for which currents can form a vertex depends on the theory in question. A vertex is not, by itself, a valid process. Instead, a diagram is a combination of vertices, connected in such a way that all conservation laws are obeyed.

For now we focus on vertices involving a fermion. These will always have exactly two fermion currents and a boson current, so for example,



(3.2.1)

If we want to specify particular particles we can do so by labelling the lines:



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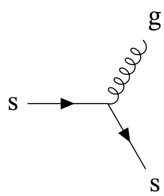


(3.2.2)

These represent the processes $\mu^- \to \mu^-\gamma$ and $\mathrm{s} \to \mathrm{sg}$.

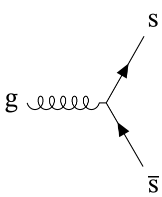
At every vertex there are conservation laws we have to apply. Again, the details depend on exactly what the fermions and bosons are. We must always conserve energy, momentum, and charge. There are also other quantities that must be conserved, such as muon lepton number, which is the number of muons and muon neutrinos minus the number of antimuons and antimuon neutrinos. We must also conserve the quark number, which is the number of quarks minus the number of antiquarks.

Energy and momentum conservation can be combined into conservation of four-momentum, where

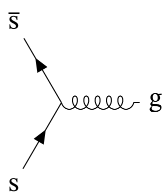
$$p^\mu = (E/c, \boldsymbol{p}) \tag{3.2.3}$$

is the four-momentum of a particle with energy E and momentum $\boldsymbol{p}$.

Given a valid vertex if we rotate it we will get another valid vertex. For example, rotating the $\mathrm{s} \to \mathrm{sg}$ vertex above we get the vertices



and



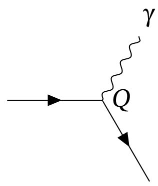
(3.2.4)

The only thing that changes here is the interpretation, these processes now represent $\mathrm{g} \to \mathrm{s\bar{s}}$ and $\mathrm{s\bar{s}} \to \mathrm{g}$.

3.2.1 Forces

3.2.1.1 Electromagnetic Vertex

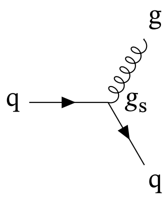
An electromagnetic vertex has a photon and any charged fermion. If the fermion has charge Q then the coupling constant, which we'll see later relates to how strong the interaction is, is Q. There will be no fermion flavour change.



(3.2.5)

3.2.1.2 Strong Vertex

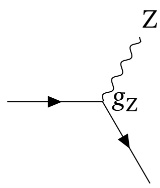
A strong vertex has a gluon and a quark. The coupling constant is g_s. There will be no quark flavour change.



(3.2.6)

3.2.1.3 Neutral Current Weak Vertex

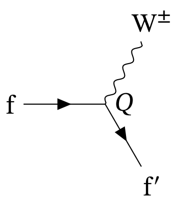
A neutral current weak vertex has a Z boson and any fermion. The coupling constant is g_Z. There will be no fermion flavour change.



(3.2.7)

3.2.1.4 Charged Current Weak Vertex

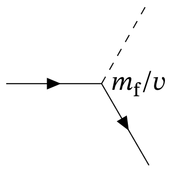
A charged current weak vertex has a $W^\pm$ boson and any fermion. The coupling constant is g_W. There will always be a fermion flavour change since it is required for charge conservation.



(3.2.8)

3.2.1.5 Higgs

A Higgs vertex has a Higgs boson and any fermion, except, at least for the purposes of this course, neutrinos. If the fermion has mass m_f then the coupling constant is m_f/v, where v is a constant of value $v \approx 246\,\text{GeV}$. There will be no fermion flavour change.



(3.2.9)

> **Open Question 3.2.10** It is not known whether neutrinos couple to the Higgs field, or get their mass through another mechanism. Since neutrino masses are incredibly small no coupling has ever been observed, but cannot be ruled out.

3.2.2 Coupling Constants

The coupling constants come with related quantities, which are also referred to as the coupling constants, given by

$$\alpha = \frac{g^2}{4\pi}, \tag{3.2.11}$$

where g is the relevant coupling constant:

- $g_{EM} = e$ for an electromagnetic vertex, where e is the charge of the fermion involved in the interaction, not the charge of an electron;
- g_s for a strong vertex;
- g_Z for a neutral current weak vertex; and
- g_W for a charged current weak vertex.

The value of α is a measure of how likely a particular vertex is to occur. The strong force is, unsurprisingly, the strongest by which we mean that, assuming it's an allowed interaction, a strong vertex is more likely to occur than any other sort. This is expressed through the (relatively) high value of α for the strong force:

$$\alpha_S \sim 1, \tag{3.2.12}$$

as an order of magnitude estimate. This causes problems when we try to do perturbation theory with the strong force, as it means very often things don't converge.

Next up in the list is the weak force, which has

$$\alpha_W \approx \alpha_Z \sim \frac{1}{40}. \tag{3.2.13}$$

Last, is electromagnetism, with

$$\alpha_{EM} = \alpha \approx \frac{1}{137}. \tag{3.2.14}$$

The actual likelihood of any particular interaction depends on much more than the value of the coupling constant. In reality we have to consider all possible ways the interaction can occur, compute the sum of the amplitudes for these processes, and square it to get a probability. This means that things like the small range of the strong and weak interactions means that they are actually far less relevant on a human scale than electromagnetism and gravity. However, on the scale of fundamental particles, and small composite particles, the strong and weak interactions cannot be ignored.

Notice that we aren't giving precise values for the coupling constants. This is because, despite the name, these aren't really constants. Instead their values depend on the energy scale at which the interaction occurs. This is called running of the coupling constant, and we'll see more detail later.

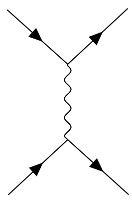
3.2.3 Boson Vertices

As well as vertices involving a fermion we can vertices with bosons. These are more varied, and we'll meet examples throughout the course. For now, a few examples of valid boson vertices are shown in Figure 3.1.

3.3 Feynman Diagrams

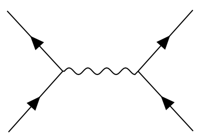
A complete Feynman diagram is then a combination of two or more vertices such that all conservation laws are obeyed. Exactly what the conservation laws are we'll get to later, but the basic conserved properties are charge, lepton number, quark number, and of course momentum and energy.

We can consider three broad classes of interactions, first, we have **scattering**, where two particles come in, interact, then go their separate ways. A simple scattering diagram is



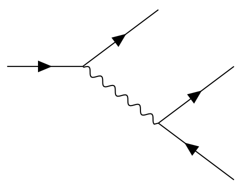
(3.3.1)

Another possibility is **annihilation**, where two particles come in, annihilate into a boson, and then the boson produced decays into another two particles. A simple annihilation diagram is



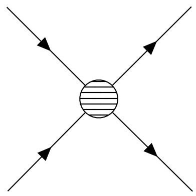
(3.3.2)

The third possibility is **decay**, where a single particle comes in, decays into a fermion and a boson, and the boson then decays into another two fermions. A simple decay diagram is



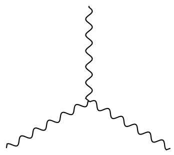
(3.3.3)

It is important to note that Feynman diagrams are *not* physical. We set up an experiment with some ingoing particles, and measure the outgoing particles. What happens between is a black box process. The actual process should be viewed more like the diagram

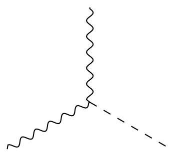


(3.3.4)

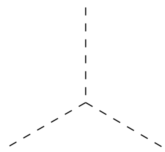
where the blob in the middle is all possible intermediate processes, all occurring at once. After all, from the perspective of quantum field theory a Feynman diagram is just a single term in an infinite expansion summing over all possible processes.



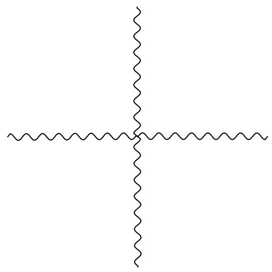
(a) Two W bosons and a Z boson or a photon.



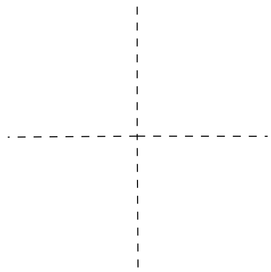
(b) Either two W bosons or two Z bosons and a Higgs boson.



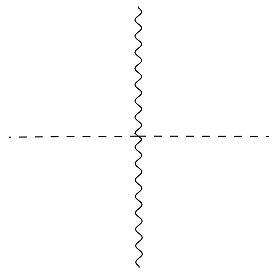
(c) Three Higgs bosons.



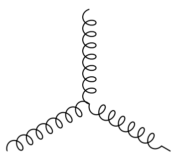
(d) Two W bosons and either two W bosons, two Z bosons, a Z boson and a photon, or two photons.



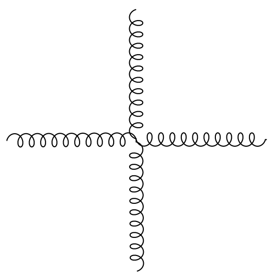
(e) Four Higgs bosons.



(f) Two W bosons or two Z bosons and two Higgs bosons.



(g) Three gluons.



(h) Four gluons.

Figure 3.1: Some allowed boson vertices.

Four

What do we Measure

4.1 Types of Interactions

There are three different things that can occur and we can measure to get information about the particles and forces involved. The first is particles can decay. That is, we start with some particle, A, and end up with particles $1, 2, \dots, n$,

$$A \to 1 + 2 + \cdots + n. \tag{4.1.1}$$

We measure the decay rate, that is the rate at which a given decay from one particle into some other set of other particles, occurs, and compare this to theoretical predictions. Often we compare ratios of decay rates to get rid of constant factors and make comparison easier.

The second thing that can occur is scattering. Here two particles, say A and B, interact and produce particles $1, 2, \dots, n$,

$$A + B \to 1 + 2 + \cdots + n. \tag{4.1.2}$$

Note that its possible that particles $1, 2, \dots, n$ contain particles that are the same as the incoming particles, A and B. However, we shouldn't think of these as the same particles, since from a quantum field theory perspective all scattering occurs by annihilating all particles, and then creating new particles. That said, one of the most common scattering occurrences is $A + B \to A + B$, which is essentially the two particles bouncing off each other.

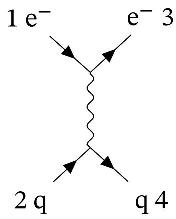
The third thing that can occur is bound states, such as **positronium** ($\mathrm{e^-e^+}$), **charmonium** ($\mathrm{c\bar{c}}$), or **bottomonium** ($\mathrm{b\bar{b}}$). Measuring the properties of these, such as their lifetimes, can also tell us a lot about the particles and the forces binding them.

More generally we measure the transition rate for the system to go from an initial quantum state to some other final quantum state.

4.2 Feynman Diagrams vs. the Lab Picture

Suppose we are doing a scattering experiment scattering electrons off quarks, the quarks in term being wrapped up in a proton, since we can't get unbound quarks. Consider a particular scattering event in which no new particles are produced, so

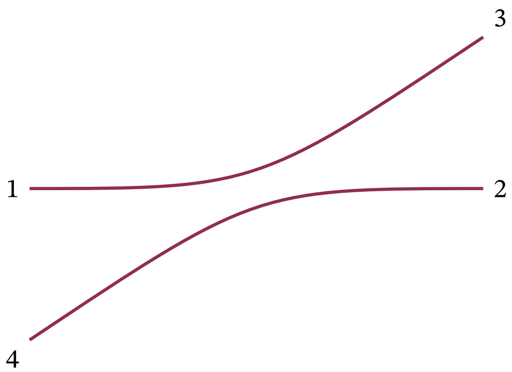
$e^- q \to e^- q$. A Feynman diagram for this is



(4.2.1)

Here we label the particles, with 1 and 2 being the incoming particles, and 3 and 4 being the outgoing particles.

The scattering experiment could be of two types, either two beams, or a fixed target. However these are really the same but related by a change of frame. So, for simplicity, we assume two beams. We also assume that, like most real scattering experiments, the two beams are head on. Then the interaction looks something like this:



(4.2.2)

Here the labels 1 and 2 refer to the incoming electron and quark, and the labels 3 and 4 are the outgoing electron and quark.

Momentum is conserved in this interaction. This is a relativistic statement, so we mean that the four-momentum is conserved. Let p_1 be the four-momentum of the incoming electron, p_2 the four-momentum of the incoming quark, p_3 the four-momentum of the outgoing electron, and p_4 the four momentum of the outgoing quark. Then we have

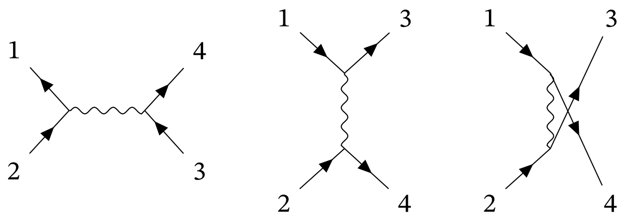
$$p_1 + p_2 = p_3 + p_4. \tag{4.2.3}$$

We will generally be interested in measuring/predicting the rate of the process, say $1 + 2 \to 3 + 4$, that is, of all scatterings with particles 1 and 2 incoming with four-momenta p_1 and p_2 how many result in particles 3 and 4 with four-momenta p_3 and p_4.

In this experiment the initial state is something that we control, we choose the particles to scatter and set their momenta based on how we create the beams. The final state is then what we measure.

4.3 Mandelstam Variables

Consider the following three processes:



(4.3.1)

These are called the ***s*-channel**, ***t*-channel**, and ***u*-channel** processes. For each one we can define a Lorentz invariant from the four momenta of the particles:

$$s := (p_1 + p_2)^2, \qquad t := (p_1 - p_3)^2, \qquad \text{and} \qquad u := (p_1 - p_4)^2. \tag{4.3.2}$$

These are called the **Mandelstam variables**

The most useful of these quantities is s. Expanding it out we get

$$s = p_1^2 + p_2^2 + 2p_1 \cdot p_2. \tag{4.3.3}$$

Now, if we work in the rest frame of a particle with mass m its four momentum is simply $p = (E, \mathbf{0})$, and $E^2 = m^2 + \mathbf{0}^2$, so $E = m$. Hence $p = (m, \mathbf{0})$ and so $p^2 = m^2$. This is true in the rest frame, but p^2 is Lorentz invariant so it must be true in any frame. Thus, we have

$$s = m_1^2 + m_2^2 + 2p_1 \cdot p_2, \tag{4.3.4}$$

where m_1 and m_2 are the masses of particles 1 and 2.

In the lab frame the four-momentum of each particle is $p = (E, \boldsymbol{p})$, where E is the energy and $\boldsymbol{p}$ the three-momentum. In our set up the two beams are head on, so the angle between the incoming particles is 180°. This is one of the reasons s is more useful than t or v, these involve p_3 or p_4, and we can't control the direction of the outgoing particles. Using this we have

$$p_1 \cdot p_2 = E_1 E_2 - \boldsymbol{p_1} \cdot \boldsymbol{p_2} = E_1 E_2 - |\boldsymbol{p_1}||\boldsymbol{p_2}| \sin(180°) = E_1 E_2 + |\boldsymbol{p_1}||\boldsymbol{p_2}|. \tag{4.3.5}$$

Now, we know that $E^2 = m^2 + |\boldsymbol{p}|^2$. Suppose we have very energetic particles, so that $m \ll |\boldsymbol{p}|$, then $E \approx |\boldsymbol{p}|$. Hence, we have

$$p_1 \cdot p_2 = E_1 E_2 + |\boldsymbol{p_1}||\boldsymbol{p_2}| \approx E_1 E_2 + E_1 E_2 = 2E_1 E_2. \tag{4.3.6}$$

Further, assuming that $m_1 \ll E_1$ and $m_2 \ll E_2$ we have

$$s = m_1^2 + m_2^2 + 2p_1 \cdot p_2 \approx m_1^2 + m_2^2 + 4E_1 E_2 \approx 4E_1 E_2. \tag{4.3.7}$$

If both beams have the same energy, E, then $s = 4E^2$, and so $\sqrt{s} = 2E$, which is the total energy of the two particles scattering. This means that $\sqrt{s}$ has a nice interpretation as the energy of the collision.

4.4 Fermi's Golden Rule

We are interested in the transition rate for going from some state $|i\rangle$ to some state $|f\rangle$. We suppose that the system evolves under the Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}'(x), \tag{4.4.1}$$

where $\hat{H}_0$ is a time independent Hamiltonian for which we can solve the Schrödinger equation and $\hat{H}'(x)$ is the interaction Hamiltonian, which induces the transitions between states. The interaction is generally position dependent. The state of the system then evolves according to the Schrödinger equation:

$$i\frac{\partial}{\partial t}|\psi, t\rangle = \hat{H}|\psi, t\rangle = [\hat{H}_0 + \hat{H}'(x)]|\psi, t\rangle. \tag{4.4.2}$$

Fermi's golden rule says that the transition rate from state $|i\rangle$ to state $|f\rangle$ is given by

$$\Gamma_{fi} = 2\pi|T_{fi}|^2\rho(E_i) \tag{4.4.3}$$

where Γ_{fi} is the transition rate, T_{fi} is the transition matrix, and $\rho(E_i)$ is the density of accessible states. The full derivation is in Chapter B.

To first order the transition matrix is simply the matrix elements of the perturbation Hamiltonian,

$$T_{fi} = \langle f|\hat{H}'|i\rangle. \tag{4.4.4}$$

This represents a process where the state evolves directly from $|i\rangle$ to $|f\rangle$. To second order the transition matrix is

$$T_{fi} = \langle f|\hat{H}'|i\rangle + \sum_{k\neq i}\frac{\langle f|\hat{H}'|k\rangle\langle k|\hat{H}'|i\rangle}{E_i - E_k}. \tag{4.4.5}$$

The second term accounts for all transitions via an intermediate state, $|k\rangle$. Continuing on the next term would have transitions via two intermediate states and so on.

4.4.1 Density of States

We want our processes to be physical. This means they have to obey several conservation laws, such as energy and momentum conservation. We enforce this by having a Dirac delta in our terms, such as

$$\delta(E_i - E_f), \qquad \delta^3(\boldsymbol{p_i} - \boldsymbol{p_f}), \qquad \text{and} \qquad \delta^4(p_i - p_f), \tag{4.4.6}$$

where E_i and E_f are the total energy of the initial and final states, $\boldsymbol{p_i}$ and $\boldsymbol{p_f}$ are the total three-momenta of the initial and final states, and p_i and p_f are the total four-momenta of the initial and final states.

The density of final accessible states depends on the kinematics of the process being considered. In particular, the initial energy, E_i, is the total amount of energy available to the final states. The density of states is given by integrating over all states with energy conservation enforced by a Dirac delta:

$$\rho(E_i) = \int \delta(E_i - E)\,\mathrm{d}n = \int \delta(E_i - E)\frac{\mathrm{d}n}{\mathrm{d}E}\,\mathrm{d}E = \left.\frac{\mathrm{d}n}{\mathrm{d}E}\right|_{E=E_i}. \tag{4.4.7}$$

Here $\mathrm{d}n$ gives the number of states with energies between E and $E + \mathrm{d}E$.

Fermi's golden rule is then that

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E)\, \mathrm{d}n. \tag{4.4.8}$$

The complication with this is that states are quantised, so this integral is non-trivial. A spinless, massless particle can be described with a plane wave

$$\psi = A\mathrm{e}^{-ip\cdot x}. \tag{4.4.9}$$

We consider the particle to be in a box of side length a and impose vanishing boundary conditions. Normalising the wave function gives us $A = 1/V$ where $V = a^3$ is the volume of the box. The momentum of the particle is then quantised as

$$p_i = \frac{2\pi n_i}{a}. \tag{4.4.10}$$

The volume occupied by a state in momentum space is then

$$\mathrm{d}^3\boldsymbol{p} = \mathrm{d}p_x\, \mathrm{d}p_y\, \mathrm{d}p_z = \left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}. \tag{4.4.11}$$

The number of states, $\mathrm{d}n$, with a momentum in the range p to $\mathrm{d}p$ is the volume in momentum space of the spherical shell of inner radius p and thickness $\mathrm{d}p$ divided by the volume of a single state:

$$\mathrm{d}n = 4\pi p^2\, \mathrm{d}p \frac{V}{(2\pi)^3}. \tag{4.4.12}$$

We can choose our box such that we have one particle per unit volume, and call this volume 1, we then have

$$\mathrm{d}n = \frac{4p^2\, \mathrm{d}p}{(2\pi)^3} = \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}. \tag{4.4.13}$$

For example, consider a decay $A \to 1 + 2 + \cdots + N$. Then

$$\mathrm{d}n = (2\pi)^3 \prod_{i=1}^{N} \frac{\mathrm{d}^3\boldsymbol{p_i}}{(2\pi)^3} \delta^3\left(\boldsymbol{p_A} - \sum_{i=1}^{N} \boldsymbol{p_i}\right). \tag{4.4.14}$$

So far we haven't considered the effects of relativity. Our box of unit volume will be Lorentz contracted in the direction of motion. The contraction is proportional to γ, which for a single particle of mass m and energy E is given by $\gamma = E/m$. The fix is to redefine the wave function to ψ', so that

$$\int \psi'^* \psi'\, \mathrm{d}V \propto E. \tag{4.4.15}$$

The convention is to choose the constant of proportionality to be 2, and then set $\psi' = \sqrt{2E}\psi$. We can then define a Lorentz invariant matrix element:

$$\mathcal{M}_{fi} = \sqrt{2E_1 \cdot 2E_2 \cdots 2E_N}\, T_{fi} \tag{4.4.16}$$

where the product is over the energies of all particles, in both the initial and final states.

The transition rate is then

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E)\,\mathrm{d}n. \tag{4.4.17}$$

For simplicity consider the decay $A \to 1 + 2$, the transition rate is

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_A - E_1 - E_2)\,\mathrm{d}n. \tag{4.4.18}$$

Moving to momentum space we have

$$\mathrm{d}n = (2\pi)^3 \frac{\mathrm{d}^3\boldsymbol{p_1}}{(2\pi)^3}\frac{\mathrm{d}^3\boldsymbol{p_2}}{(2\pi)^3}\delta^3(\boldsymbol{p_A} - \boldsymbol{p_1} - \boldsymbol{p_2}). \tag{4.4.19}$$

Making this Lorentz invariant we have

$$\mathcal{M}_{fi} = \sqrt{2E_1 2E_2 2E_A}\,T_{fi} \implies T_{fi} = \frac{\mathcal{M}_{fi}}{\sqrt{2E_1 2E_2 2E_A}} \tag{4.4.20}$$

and so

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_A}\int |\mathcal{M}_{fi}|^2 \delta^4(p_A - p_1 - p_2)\frac{\mathrm{d}^3\boldsymbol{p_1}}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3\boldsymbol{p_2}}{(2\pi)^3 2E_2}. \tag{4.4.21}$$

This is just a rewriting of Fermi's golden rule, but now in an explicitly Lorentz invariant form. There are several key features:

- The Lorentz invariant phase space measure, $\mathrm{d}^3\boldsymbol{p}/[(2\pi)^3 2E]$.
- Energy and Momentum conservation from the Dirac delta.
- The decay rate is inversely proportional to the energy fo the decaying particle.
- The Lorentz invariant $\mathcal{M}_{fi}$ is the part that encodes the physics of the process, and working out what value this should have for a given interaction is the goal of a large part of quantum field theory.

Consider the $A \to 1 + 2$ decay again, we can partially evaluate the integral and we find that

$$\Gamma_{fi} = \frac{|\boldsymbol{p}^*|}{32\pi^2 m_A^2}\int |\mathcal{M}_{fi}|^2\,\mathrm{d}\Omega \tag{4.4.22}$$

where $|\boldsymbol{p}^*|$ is the magnitude of the three-momenta of particles 1 and 2 in the rest frame of A, note that they must have the same magnitude of three-momenta as the initial momentum is zero in this frame. The integral is over all angles, with $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\vartheta$. This is valid for all two body decays.

If instead we consider scattering, $A + B \to 1 + 2$, again in the centre of mass frame then we find that

$$\sigma = \frac{1}{64\pi^2 s}\frac{|\boldsymbol{p}_f^*|}{|\boldsymbol{p}_i^*|}\int |\mathcal{M}_{fi}|^2\,\mathrm{d}\Omega \tag{4.4.23}$$

where $|\boldsymbol{p}_i^*|$ and $|\boldsymbol{p}_f^*|$ are the magnitude of the three-momenta of the initial and final states in the centre of mass frame, and $s = (p_A + p_B)^2$ is a Mandelstam variable. Note that this is Lorentz invariant, despite the appearance of three-vectors.

Five

Deriving the Dirac Equation

5.1 Schrödinger

Start with the nonrelativistic energy-momentum relation

$$E = \frac{p^2}{2m} + V, \tag{5.1.1}$$

where p is the magnitude of the particles momentum, m is its mass, and V is some potential. To get the Schrödinger equation we substitute in operators:

$$\hat{p}_j = -i\frac{\partial}{\partial x^j} = -i\partial_j, \qquad \hat{\boldsymbol{p}} = -i\nabla, \qquad \text{and} \qquad E = i\frac{\partial}{\partial t} = i\partial_t. \tag{5.1.2}$$

Acting with the resulting operator on a wave function we get the **Schrödinger equation**:

$$i\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2m}\nabla^2 + V\right)\psi. \tag{5.1.3}$$

Note that this is first order in time, and second order in space, which makes it not relativistic.

5.2 Klein–Gordon Equation

If we want a relativistic equation we should start with the relativistic energy-momentum relation,

$$E^2 = \boldsymbol{p}^2 + m^2. \tag{5.2.1}$$

Substituting in the same operators we get

$$\frac{\partial^2\varphi}{\partial t^2} = (\nabla^2 - m^2)\varphi \implies (\partial^2 + m^2)\varphi = 0. \tag{5.2.2}$$

Here $\partial^2 = \partial_\mu\partial^\mu$ is the d'Alembert operator. This is the **Klein–Gordon equation**, it describes free spin 0 bosons. The problem is that we have negative energy solutions, and worse still, if we try to interpret φ as a wave function then we can get negative probabilities.

5.3 Dirac Equation

The negative energies appear due to the presence of E^2, giving a positive and negative branch when we take the square root. To avoid this Dirac looked for a first order relationship between energy and momentum. He started with

$$\hat{E}\psi = (\boldsymbol{\alpha} \cdot \hat{\boldsymbol{p}} + \beta m)\psi \tag{5.3.1}$$

where $\boldsymbol{\alpha}$ and β are some constants. Inserting the operators we have

$$i\frac{\partial\psi}{\partial t} = \left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi. \tag{5.3.2}$$

Defining $\gamma^0 = \beta$, $\gamma^i = \beta\alpha_i$ some rearranging gives this equation in the form

$$\left(i\gamma^0\frac{\partial}{\partial t} + i\gamma^1\frac{\partial}{\partial x} + i\gamma^2\frac{\partial}{\partial y} + i\gamma^3\frac{\partial}{\partial z} - m\right)\psi = 0. \tag{5.3.3}$$

This can be written more compactly as

$$(i\gamma^\mu\partial_\mu - m)\psi = 0. \tag{5.3.4}$$

Defining $\not{a} \coloneqq \gamma^\mu a_\mu$ we can write this even more compactly as

$$(i\not{\partial} - m)\psi = 0. \tag{5.3.5}$$

This is the **Dirac equation**, it describes spin 1/2 fermions.

5.4 The γ Coefficients

Suppose that ψ satisfies both the Dirac equation, we can then write

$$(-i\not{\partial} - m)(i\not{\partial} - m)\psi = 0. \tag{5.4.1}$$

Expanding this we get

$$(\not{\partial}^2 + m^2)\psi = 0. \tag{5.4.2}$$

This looks slightly like the Klein–Gordon equation, so suppose that ψ also satisfies this, that is

$$(\partial^2 + m^2)\psi = 0. \tag{5.4.3}$$

Expanding the the $\not{\partial} = \gamma^\mu\partial_\mu$ we get

$$\begin{aligned}
\not{\partial}^2 &= \gamma^\mu\partial_\mu\gamma^\nu\partial_\nu && (5.4.4)\\
&= (\gamma^0)^2\partial_t^2 + (\gamma^1)^2\partial_x^2 + (\gamma^2)^2\partial_y^2 + (\gamma^3)^2\partial_z^2 && (5.4.5)\\
&+ \gamma^0\gamma^1\partial_t\partial_x + \gamma^0\gamma^2\partial_t\partial_y + \gamma^0\gamma^3\partial_t\partial_z && (5.4.6)\\
&+ \gamma^1\gamma^2\partial_x\partial_y + \gamma^1\gamma^3\partial_x\partial_z + \gamma^2\gamma^3\partial_y\partial_z. && (5.4.7)
\end{aligned}$$

Compare this to the expanded d'Alembertian:

$$\partial^2 = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2. \tag{5.4.8}$$

In order for ψ to satisfy both the Klein–Gordon equation and the Dirac equation we must have

$$(\gamma^0)^2 = 1, \qquad (\gamma^i)^2 = -1, \qquad \text{and} \qquad \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu. \tag{5.4.9}$$

More compactly we can write these requirements as

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{5.4.10}$$

where $\{A, B\} := AB + BA$ is the **anticommutator** and η is the metric tensor.

While this isn't a proof that γ^μ must satisfy this we take it to be so. The relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\varepsilon^{\mu\nu} \tag{5.4.11}$$

defines a Clifford algebra, $\mathrm{Cl}_{1,3}(\mathbb{R})$*, which can be thought of as the space of all linear operators from spacetime, V, to itself, that is,* $\mathrm{End}(V)$*. We can then think of γ^μ as the generators of this algebra, or a representation of this algebra.*

The lowest dimensional set of matrices satisfying the required properties[1] is

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{5.4.12}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{5.4.13}$$

This can be written more compactly as

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \sigma^3 \otimes I, \qquad \text{and} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & \end{pmatrix} = i\sigma^2 \otimes \sigma^i \tag{5.4.14}$$

where I is the 2-dimensional identity matrix and σ^i are the **Pauli spin matrices**:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.4.15}$$

[1] That is, the lowest dimensional nontrivial representation of $\mathrm{Cl}_{1,3}(\mathbb{R})$.

Six

Solving the Dirac Equation

6.1 Spinors

The Dirac equation,

$$(i\partial\!\!\!/ - m)\psi = 0, \tag{6.1.1}$$

can be written in terms of 4×4 matrices, $\partial\!\!\!/ = \gamma^\mu \partial_\mu$ and m (with an implicit factor of the 4×4 identity, I_4), acting on ψ. This means that ψ must have four components. We can think of it as a 4×1 matrix, but it is *not* a four-vector[1]. We call objects like this **spinors**.

[1] That is, it doesn't transform as a four-vector under $SO^+(1,3)$, they transform under a different representation which cannot be constructed from tensor products of the fundamental representation in the usual way.

We make the ansatz that ψ has the form

$$\psi(x) = u(p)e^{-ip\cdot x}, \tag{6.1.2}$$

where $u(p)$ is some spinor and $\exp[-ip \cdot x]$ is a phase factor. This means that

$$\psi(x) = \begin{pmatrix} \psi^1(x) \\ \psi^2(x) \\ \psi^3(x) \\ \psi^4(x) \end{pmatrix} = \begin{pmatrix} \varphi^1(p) \\ \varphi^2(p) \\ \varphi^3(p) \\ \varphi^4(p) \end{pmatrix} e^{-ip\cdot x} = u(p)e^{-ip\cdot x}. \tag{6.1.3}$$

Now notice that $i\partial_\mu$ acting on $e^{-ip\cdot x}$ gives $p_\mu e^{-ip\cdot x}$, and in particular ∂_μ has no effect on $u(p)$, since it has no position dependence. We can then write the Dirac equation as

$$(p\!\!\!/ - m)u(p) = 0, \tag{6.1.4}$$

having divided through by the phase factor.

We can write this in the significantly less compact form

$$\begin{pmatrix} E-m & 0 & -p_z & -p_x + ip_y \\ 0 & E-m & -p_x - ip_y & p_z \\ p_z & p_x - ip_y & -E-m & 0 \\ p_x + ip_y & -p_z & 0 & -E-m \end{pmatrix} \begin{pmatrix} \varphi^1(p) \\ \varphi^2(p) \\ \varphi^3(p) \\ \varphi^4(p) \end{pmatrix} = 0. \tag{6.1.5}$$

To solve this move to the rest frame of the particle, so $p_i = 0$, this then becomes

$$\begin{pmatrix} E-m & 0 & 0 & 0 \\ 0 & E-m & 0 & 0 \\ 0 & 0 & -E-m & 0 \\ 0 & 0 & 0 & -E-m \end{pmatrix} \begin{pmatrix} \varphi^1(p) \\ \varphi^2(p) \\ \varphi^3(p) \\ \varphi^4(p) \end{pmatrix} = 0. \tag{6.1.6}$$

This is just an eigenvalue equation, for the matrix $\mathrm{diag}(E, E, -E, -E)$, and there are four eigenspinors

$$u^1 = N\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad u^2 = N\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad u^3 = N\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \text{and} \quad u^4 = N\begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{6.1.7}$$

Here N is a normalisation factor, which is usually $\sqrt{E+m}$ so $u^\dagger u = 2E$ for all four spinors. The first two solutions here correspond to eigenvalues of $E = +m$, whereas the second two correspond to eigenvalues of $E = -m$. This means that despite Dirac's best attempts we *still* have negative energy solutions. The full solution to the Dirac equation for a particle at rest is then

$$\psi^1 = N\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\mathrm{e}^{-imt}, \quad u^2 = N\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\mathrm{e}^{-imt}, \quad u^3 = N\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\mathrm{e}^{imt}, \quad \text{and} \quad u^4 = N\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\mathrm{e}^{imt}. \tag{6.1.8}$$

We get four solutions, which is quite different to the Klein–Gordon equation. The reason for this is the Klein–Gordon equation represents spin 0 particles, whereas the Dirac equation represents spin 1/2 particles, and the four solutions correspond to four types of spin 1/2 particles:

- u^1 corresponds to a positive energy spin up solution.
- u^2 corresponds to a positive energy spin down solution.
- u^3 corresponds to a negative energy spin down solution.
- u^4 corresponds to a negative energy spin up solution.

6.2 Interpreting Negative Energies

6.2.1 Dirac Sea

The Dirac sea interpretation says that the vacuum state corresponds to all negative energy states being filled. The exclusion principle then prevents the entire system from decaying to arbitrarily large negative energies. What we view as a particle is then actually a particle being excited to a positive energy state from a negative energy state. At the same time this leaves a hole in the negative energy states, which is what we view as an antiparticle.

This interpretation means that the process $\gamma \to \mathrm{e}^-\mathrm{e}^+$ is just an electron in a negative energy state absorbing a photon, and being excited to a positive energy state, leaving behind a hole. Similarly the process $\mathrm{e}^-\mathrm{e}^+ \to \gamma$ is the electron falling back to the negative energy state, emitting a photon and filling the hole.

This interpretation lead to Dirac predicting the existence of the positron before it was experimentally discovered.

6.2.2 Feynman–Stückelberg

The more modern interpretation is that the negative energy solutions correspond to negative energy particles propagating backwards in time, which correspond to positive energy antiparticles propagating forward in time. This is (part of) the reason we use arrows going backwards in time to represent antiparticles in a Feynman diagram.

In this interpretation the process $e^- e^+ \to \gamma$ corresponds either to an electron and positron annihilating, or to an electron with positive energy coming in, emitting a photon to have negative energy, and then leaving going backwards in time.

6.3 General Solution

It is possible to solve the Dirac equation in a general frame[2], not just the rest frame of the particle. The result is again four different spinors, first

[2] see *Quantum Theory* or *Quantum Field Theory* for details

$$u^1(p) = N\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}, \qquad \text{and} \qquad u^2(p) = N\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \tag{6.3.1}$$

where $E = +\sqrt{m^2 + \boldsymbol{p}^2}$. Then we also have the negative energy solutions

$$u^3(p) = N\begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad u^4(p) = N\begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix} \tag{6.3.2}$$

where $E = -\sqrt{m^2 + \boldsymbol{p}^2}$.

To get rid of negative energy solutions we instead define explicit antiparticle spinors,

$$v^1(p) = u^4(-p), \qquad \text{and} \qquad v^2(p) = u^3(-p). \tag{6.3.3}$$

Note that 4 and 1 go together and 2 and 3 go together, which is probably not what one would expect, and corresponds to the weirdness with u^1, u^2, u^3, and u^4 representing spin up, down, down, and up solutions respectively. The antiparticle spinors are solutions to

$$(\not{p} + m)v = 0. \tag{6.3.4}$$

6.4 Helicity

Recall that we have two operators measuring spin:

$$\hat{\boldsymbol{S}}^2|\psi\rangle = s(s+1)|\psi\rangle, \qquad \text{and} \qquad \hat{S}_z|\psi\rangle = m_s|\psi\rangle. \tag{6.4.1}$$

Here $\hat{S}_z$ measures the spin along the z-axis. Rather than measure along an arbitrarily chosen axis it is usually preferable to measure spin relative to the direction of travel of the particle. This can be done by defining the **helicity**

$$\hat{h} \coloneqq \frac{\hat{\boldsymbol{S}} \cdot \hat{\boldsymbol{p}}}{|\boldsymbol{p}|}. \tag{6.4.2}$$

For a spin 1/2 particle m_s can be $\pm 1/2$. Similarly h, defined by $\hat{h}|\psi\rangle = h|\psi\rangle$, can take the values[3] $\pm 1/2$. If it takes the value 1/2 then the spin is aligned in the direction of travel and we say the particle is **right handed**. If h takes the value $-1/2$ then the spin is antialigned in the direction of travel and we say the particle is **left handed**.

[3]some texts define helicity with a factor of 2, so that h takes values of ± 1

Note that the helicity is not a Lorentz invariant, except for in the case where the particle is massless and helicity coincides with chirality.

If we define the direction of the momentum by a unit vector,

$$\hat{\boldsymbol{n}} = (\sin\vartheta\cos\varphi, \sin\vartheta\cos\varphi, \cos\vartheta), \tag{6.4.3}$$

then we can work with helicity eigenstates given by

$$u_\uparrow = N\begin{pmatrix} \cos(\vartheta/2) \\ e^{i\varphi}\sin(\vartheta/2) \\ \xi\cos(\vartheta/2) \\ \xi e^{i\varphi}\sin(\vartheta/2) \end{pmatrix}, \qquad u_\downarrow = N\begin{pmatrix} -\sin(\vartheta/2) \\ e^{i\varphi}\cos(\vartheta/2) \\ \xi\sin(\vartheta/2) \\ -\xi e^{i\varphi}\cos(\vartheta/2) \end{pmatrix}, \tag{6.4.4}$$

$$v_\uparrow = N\begin{pmatrix} \xi\sin(\vartheta/2) \\ -\xi e^{i\varphi}\cos(\vartheta/2) \\ -\sin(\vartheta/2) \\ e^{i\varphi}\cos(\vartheta/2) \end{pmatrix}, \qquad v_\downarrow = N\begin{pmatrix} \xi\cos(\vartheta/2) \\ \xi e^{i\varphi}\sin(\vartheta/2) \\ \cos(\vartheta/2) \\ e^{i\varphi}\sin(\vartheta/2) \end{pmatrix}, \tag{6.4.5}$$

where $\xi = |\boldsymbol{p}|/(E+m)$.

Notice that the transformation

$$\hat{\boldsymbol{n}} \mapsto (\sin(\vartheta+2\pi)\cos\varphi, \sin(\vartheta+2\pi), \cos(\vartheta+2\pi)) \tag{6.4.6}$$

doesn't result in the same spinors, instead it gives us back their negatives. It requires a 4π "rotation" to take the spinors back to their original value.

One nice thing about these eigenstates is that for low momentum particles they become significantly simpler if we neglect the $|\boldsymbol{p}|/(E+m)$ terms. Many of our calculations will use these helicity eigenstates as a basis.

Seven

Getting Probabilities

7.1 Interpreting the Spinor Wave Function

In nonrelativistic quantum mechanics we have a wave function, ψ, which is a complex number valued function encoding all of the information we have about a quantum system. This wave function is the solution to the Schrödinger equation. We have the further restriction that ψ must be square integrable, to allow us to normalise probabilities. To extract this information we consider $\rho(\boldsymbol{x}, t) \coloneqq \psi^*(t, \boldsymbol{x})\psi(t, \boldsymbol{x}) = |\psi(t, \boldsymbol{x})|^2$, which we interpret as a probability density. We can also define a three-vector probability current, $\boldsymbol{j}(t, \boldsymbol{x})$ To do this we consider the rate of change of the total probability in some region, V, of spacetime:

$$\frac{\partial}{\partial t} \int_V \rho \, \mathrm{d}V. \tag{7.1.1}$$

The rate of change of the total probability is equal to the flux out through the boundary of V, denoted ∂V. This is given by

$$\frac{\partial}{\partial t} \int_V \rho \, \mathrm{d}V = -\int_{\partial V} \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{S} = -\int_V \nabla \cdot \boldsymbol{j} \, \mathrm{d}V \tag{7.1.2}$$

where the last step is the divergence theorem. We then have the continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \boldsymbol{j} \implies \frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0. \tag{7.1.3}$$

It is also possible to express $\boldsymbol{j}$ in terms of the wave function:

$$\boldsymbol{j} = -\frac{i}{2m} \left(\psi^* \nabla \psi - \psi \nabla \psi^*\right). \tag{7.1.4}$$

We can write this relativistically by defining $j^\mu = (\rho, \boldsymbol{j})$ and then the continuity equation is

$$\partial_\mu j^\mu = 0. \tag{7.1.5}$$

Things don't change that much when we consider spinors. We still have a wave function, ψ, but now it's spinor valued[1].

So, to complete our probability finding step we just need to find a way to "square" a spinor to get a probability density and a probability current. Appealing to the

[1] Formally if $\rho : \mathrm{Spin}(p, q) \to \mathrm{GL}(V)$ is a representation of the spin group, which is the double cover of $\mathrm{SO}^+(p, q, \mathbb{R})$, which is the (proper orthochronous) Lorentz group when $p = 1$ and $q = 3$, then spinors are elements of V.

normal wave function case the obvious generalisation from the one (complex) dimensional complex-number-valued wave function to the four (in the Dirac representation) dimensional spinor-valued wave function is to replace $\psi^*\psi$ with $\psi^\dagger\psi$. If we have a four-component spinor

$$\psi = \begin{pmatrix} \varphi^1 \\ \varphi^2 \\ \varphi^3 \\ \varphi^4 \end{pmatrix} \mathrm{e}^{-ip\cdot x}, \tag{7.1.6}$$

then we can define $\psi^\dagger$ to be

$$\psi^\dagger \coloneqq \begin{pmatrix} \varphi^{1*} & \varphi^{2*} & \varphi^{3*} & \varphi^{4*} \end{pmatrix} \mathrm{e}^{ip\cdot x}. \tag{7.1.7}$$

Since we want to work relativistically we look directly for j^μ, rather than finding ρ and $\boldsymbol{j}$ separately. To do so we need to combine our spinor with something with a Lorentz index, in order to get a four vector. The obvious choice when it comes to spinors is a gamma matrix, γ^μ. After some consideration we see that if we consider $\psi^\dagger\gamma^\mu\psi$ then, for a fixed value of μ, this just gives a number, so allowing μ to take values $\mu = 0, 1, 2, 3$ we get a four-vector.

However, there is a problem. This quantity is not Lorentz invariant[2]. It turns out that a small modification fixes this, we should instead consider $j^\mu \coloneqq \psi^\dagger\gamma^0\gamma^\mu\psi$. This is a four-vector with components

[2] see *Quantum Field Theory* for details

$$j^\mu = \psi^\dagger\gamma^0\gamma^\mu\psi = (\psi^\dagger\psi, \psi^\dagger\gamma^0\gamma^1\psi, \psi^\dagger\gamma^0\gamma^2\psi, \psi^\dagger\gamma^0\gamma^3\psi), \tag{7.1.8}$$

where we've used $(\gamma^0)^2 = 1$. Quantities of the form $a^\dagger\gamma^0$ are common enough that we define $\bar{a} \coloneqq a^\dagger\gamma^0$. Hence,

$$j^\mu = \bar{\psi}\gamma^\mu\psi = (\bar{\psi}\gamma^0\psi, \bar{\psi}\gamma^1\psi, \bar{\psi}\gamma^2\psi, \bar{\psi}\gamma^3\psi), \tag{7.1.9}$$

which looks more symmetric in time and space. We call $\bar{\psi}$ the **Dirac adjoint** of ψ (as opposed to the Hermitian adjoint, $\psi^\dagger$). It is then possible to show that $\partial_\mu j^\mu = 0$ as expected.

7.2 Calculating Probabilities

Our goal now is to use Fermi's golden rule,

$$\Gamma_{fi} = 2\pi|T_{fi}|^2\rho(E_i), \tag{7.2.1}$$

to compute transition rates. We'll do so using the Lorentz invariant form

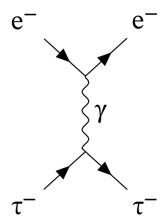
$$\Gamma_{fi} = \frac{|\boldsymbol{p}^*|}{32\pi^2 m_i^2}\int |\mathcal{M}_{fi}|^2 \,\mathrm{d}\Omega \tag{7.2.2}$$

for a decay rate for a decay $a \to 1 + 2$, and

$$\sigma = \frac{1}{64\pi^2 s}\frac{|\boldsymbol{p}_f^*|}{|\boldsymbol{p}_i^*|}\int |\mathcal{M}_{fi}|^2 \,\mathrm{d}\Omega \tag{7.2.3}$$

for the cross section for the scattering $a + b \to 1 + 2$.

Our goal for the next couple of chapters will be to compute the amplitude for the process $e^-\tau^- \to e^-\tau^-$ with a photon as the intermediate state. We'll consider the t-channel process



(7.2.4)

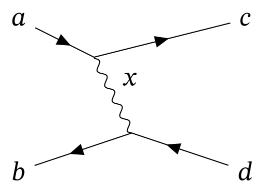
For now we'll focus on kinematic details of an arbitrary t-channel process, and in the next chapter we'll look at the specifics of this interaction.

To second order in perturbation theory the transition matrix element, T_{fi}, is given by

$$T_{fi} = \langle f|V|i\rangle + \sum_{j\neq i} \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j}. \tag{7.2.5}$$

The first term represents no interaction occurring, but we're considering the t-channel process, where there is an interaction, so its the second term here that we're interested in.

Now consider the explicitly time-ordered interaction $a + b \to c + d$ where a emits some intermediate particle, x, and turns into c, then b absorbs it and turns into d. We can draw this as



(7.2.6)

We then have three states, the initial state, $|a, b\rangle$, the intermediate state, $|c, x, b\rangle$, and the final state $|c, d\rangle$. The energy of the initial state is $E_a + E_b$, and the energy of the intermediate state is $E_c + E_x + E_b$. Putting this together we get the contribution to the second term from this time ordering:

$$T_{fi}^{ab} = \frac{\langle c, d|V|c, x, b\rangle\langle c, x, b|V|a, b\rangle}{(E_a + E_b) - (E_c + E_x + E_b)} = \frac{\langle c, d|V|c, x, b\rangle\langle c, x, b|V|a, b\rangle}{E_a - E_c - E_x}. \tag{7.2.7}$$

Here we write T_{fi}^{ab} for the process $a + b \to c + d$ specifically time ordered so that a emits a particle which is then absorbed by x.

Recall that T_{fi} is given by the amplitude, $\mathcal{M}_{fi}$, divided by the square root of the product of twice the energies of all particles in the initial and final state. We can use this to work out the amplitudes $\langle c, d|V|c, x, b\rangle$ and $\langle c, x, b|V|a, b\rangle$, which are just transition matrix elements for the processes $x + b \to d$ and $a \to c + x$ respectively. We then have

$$\langle c, d|V|c, x, b\rangle = \frac{\mathcal{M}_{fi}^{a\to c+x}}{\sqrt{2E_a 2E_c 2E_x}} = \frac{g_a}{\sqrt{2E_a 2E_c 2E_x}}, \tag{7.2.8}$$

where g_a measures the strength of the interaction $a \to c + x$. Similarly,

$$\langle c, x, b|V|a, b\rangle = \frac{\mathcal{M}_{fi}^{x+b\to d}}{\sqrt{2E_d 2E_x 2E_b}} = \frac{g_b}{\sqrt{2E_d 2E_x 2E_b}}, \tag{7.2.9}$$

where g_b measures the strength of the interaction $x + b \to d$.

We can then compute the total amplitude for this particular time ordering of this process. First we compute the total transition matrix element:

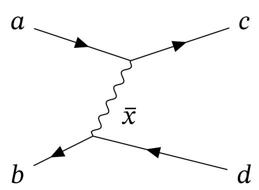
$$T_{fi}^{ab} = \frac{g_a g_b}{2E_x\sqrt{2E_a 2E_b 2E_c 2E_d}} \frac{1}{\sqrt{E_a - E_x - E_c}}, \tag{7.2.10}$$

and then the amplitude is

$$\mathcal{M}_{fi}^{ab} = g_a g_b \sqrt{2E_a 2E_b 2E_c 2E_d} T_{fi}^{ab} = \frac{g_a g_b}{2E_x(E_a - E_c - E_x)}. \tag{7.2.11}$$

Note that $\mathcal{M}_{fi}^{ab}$ is *not* Lorentz invariant, since we are explicitly defining a time ordering for the processes occurring, which a Lorentz transformation may change.

We can similarly consider the process where b emits the antiparticle $\bar{x}$, which is then absorbed by a:



(7.2.12)

An identical calculation gives

$$\mathcal{M}_{fi}^{ba} = \frac{g_a g_b}{2E_x(E_b - E_d - E_x)}. \tag{7.2.13}$$

The total amplitude for this t-channel process is

$$\begin{aligned}\mathcal{M}_{fi}^{t} &= \mathcal{M}_{fi}^{ab} + \mathcal{M}_{fi}^{ba} && (7.2.14)\\ &= \frac{g_a g_b}{2E_x}\left(\frac{1}{E_a - E_c - E_x} + \frac{1}{E_b - E_d - E_x}\right). && (7.2.15)\end{aligned}$$

Conservation of energy tells us that $E_a + E_b = E_c + E_d$, and so we can write this as

$$\begin{aligned}\mathcal{M}_{fi}^{t} &= \frac{g_a g_b}{2E_x}\left(\frac{1}{E_a - E_c - E_x} - \frac{1}{E_a - E_c + E_x}\right) && (7.2.16)\\ &= \frac{g_a g_b}{2E_x}\frac{2E_x}{(E_a - E_c)^2 - E_x^2} && (7.2.17)\\ &= \frac{g_a g_b}{(E_a - E_c)^2 - E_x^2}. && (7.2.18)\end{aligned}$$

The relativistic energy-momentum relation tells us that $E_x^2 = \boldsymbol{p_x}^2 + m_x^2$ and if we consider the first time ordering then we have $\boldsymbol{p_x} = \boldsymbol{p_a} - \boldsymbol{p_c}$, so

$$\mathcal{M}^t_{fi} = \frac{g_a g_b}{(E_a - E_c)^2 - (\boldsymbol{p_a} - \boldsymbol{p_c})^2 - m_x^2} \tag{7.2.19}$$

$$= \frac{g_a g_b}{(p_a - p_b)^2 - m_x} \tag{7.2.20}$$

$$= \frac{g_a g_b}{q^2 - m_x^2} \tag{7.2.21}$$

$$= \frac{g_a g_b}{t - m_x^2}. \tag{7.2.22}$$

Here $q = p_a - p_c$ is the four-momentum of the exchange particle and t is the Mandelstam invariant. This is a fairly common form for the amplitude to take.

Notice that we do *not* require $q^2 = m_x^2$, in fact, if this were the case then the amplitude would be singular. Suppose we have elastic scattering, then $p_a = (E, \boldsymbol{p_a})$ and $p_c = (E, \boldsymbol{p_c})$, and so we have

$$q^2 = (p_a - p_c)^2 = (E - E)^2 - (\boldsymbol{p_a} - \boldsymbol{p_c})^2 = -(\boldsymbol{p_a} - \boldsymbol{p_c})^2 \leq 0. \tag{7.2.23}$$

So we don't even need to have q^2 be positive. This is all because x is a **virtual particle**, which is a particle which is **off-shell**, meaning $q^2 \neq m_x^2$. If $q^2 = m_x^2$ then we say that the particle is **on-shell**. We cannot have virtual particles in the initial and final states, but there perfectly valid intermediate states. This is mostly just an artefact of us reading too literally into what's happening in a process defined by a Feynman diagram.

Eight

Quantum Electrodynamics

8.1 Modifying the Dirac Equation

We can modify the Dirac equation to allow for electromagnetic interactions by the **minimal coupling prescription**[1], where we modify the momentum and energy according to

$$\boldsymbol{p} \mapsto \boldsymbol{p} - q\boldsymbol{A}, \qquad \text{and} \qquad E \mapsto E - q\varphi \tag{8.1.1}$$

where q is the charge of the particle, $\boldsymbol{A}$ is the vector potential and φ the electrostatic potential. We can write this modification as

$$p^\mu \mapsto p^\mu - qA^\mu \tag{8.1.2}$$

where $A^\mu = (\varphi, \boldsymbol{A})$ is the four-potential[2]. We then make the corresponding modification to the quantum mechanical operators, $\boldsymbol{p} = -i\nabla$ and $E = i\partial_t$, or $p^\mu = i\partial^\mu$ to get

$$p^\mu = i\partial^\mu \mapsto p^\mu - qA^\mu = i\partial^\mu - qA^\mu. \tag{8.1.3}$$

Hence, the Dirac equation in the presence of an electric field is modified according to

$$(i\not{\partial} - m)\psi = 0 \mapsto (i\not{\partial} - q\not{A} - m)\psi = 0. \tag{8.1.4}$$

As motivation consider what we get if we rearrange this slightly. First multiply through by γ^0 to get

$$i\gamma^0\not{\partial}\psi - q\gamma^0\not{A}\psi - m\gamma^0\psi = 0. \tag{8.1.5}$$

Then using $(\gamma^0)^2 = 1$ we have

$$0 = i\gamma^0\not{\partial}\psi - q\gamma^0\not{A}\psi - m\gamma^0\psi \tag{8.1.6}$$
$$= i\gamma^0\gamma^\mu\partial_\mu\psi - q\gamma^0\not{A}\psi - m\gamma^0\psi \tag{8.1.7}$$
$$= i(\gamma^0)^2\partial_t\psi - i\gamma^0\gamma^i\partial_i\psi - q\gamma^0\not{A} - m\gamma^0\psi \tag{8.1.8}$$
$$= i\frac{\partial\psi}{\partial t} - i\gamma^0\boldsymbol{\gamma}\cdot\nabla\psi - q\gamma^0\not{A} - m\gamma^0\psi. \tag{8.1.9}$$

Moving the first term to the other side we have

$$i\frac{\partial\psi}{\partial t} = i\gamma^0\boldsymbol{\gamma}\cdot\nabla\psi + m\gamma^0\psi + q\gamma^0\not{A}. \tag{8.1.10}$$

[1] See *Quantum Theory*

[2] See *Classical Electrodynamics*

We then interpret the left hand side, $i\partial_t\psi$, as the Hamiltonian acting on the spinor, $\hat{H}\psi$. The terms on the right then correspond to the kinetic energy, mass-energy, and potential energy respectively.

We can then write the potential energy operator for a spinor satisfying the Dirac equation in an electromagnetic field as

$$V_\mathrm{D} = q\gamma^0 \not{A}. \tag{8.1.11}$$

Note that the time-like contribution, $\mu = 0$, to $q\gamma^0\gamma^\mu A_\mu$ is $q(\gamma^0)^2 A_0 = q\varphi$, which is just the usual electrostatic potential.

8.2 Photons

Before we get to the calculation we should consider what's happening in the interaction. We start with two fermions, these are described by the Dirac equation. We end with two fermions, also described by the Dirac equation. In between we'll have a photon, which is a spin 1 boson. Noninteracting photons are described by Maxwell's equation[3], $\partial^2 A^\nu - \partial^\nu \partial_\mu A^\mu = 0$. This has plane wave solutions,

[3] for relativistic use of Maxwell's equations see *Classical Electrodynamics*, for use in relativistic quantum theory see *Quantum Field Theory*

$$A_\mu = \varepsilon_\mu^{(\lambda)} \mathrm{e}^{-ip\cdot x}. \tag{8.2.1}$$

Here $\varepsilon_\mu^{(\lambda)}$ is the polarisation vector, with λ labelling the polarisation state. This is just a normal four vector, since A_μ is just a four-vector. Since polarisation is a purely spatial phenomenon the time component of $\varepsilon^{(\lambda)}$ is zero (at least we can always choose for it to be such). For massless particles, like the photon, it's not possible for them to be polarised in the direction of travel, since this would, intuitively, require the field to oscillate in the direction of travel, so when oscillating forward it would propagate a signal faster than the speed of light. There are therefore two polarisation states,

$$\varepsilon^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \varepsilon^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{8.2.2}$$

A virtual photon has two more polarisation states, it can be polarised in the time direction or in the direction of travel, corresponding to the states

$$\varepsilon^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \varepsilon^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{8.2.3}$$

respectively. A general photon state will be a superposition of polarisations, meaning the solution will be of the form

$$A^\mu = \sum_\lambda \varepsilon_\mu^{(\lambda)} \mathrm{e}^{-ip\cdot x}. \tag{8.2.4}$$

8.3 Calculating the Amplitude

Recall that the amplitude for t-channel scattering $a + b \to c + d$ is

$$\mathcal{M}^t_{fi} = \frac{g_a g_b}{q^2 - m_x^2}. \tag{8.3.1}$$

The values g_a and g_b are supposed to stand for the strength of the interactions $a \to x + c$ and $x + b \to d$ respectively. They are simply the matrix elements for these processes:

$$g_a = \langle\psi_c|V|\psi_a\rangle, \qquad \text{and} \qquad g_b = \langle\psi_d|V|\psi_b\rangle. \tag{8.3.2}$$

Here $|\psi\rangle$ is the spinor state corresponding to the relevant particles.

We now consider the process $\mathrm{e}^-\tau^- \to \mathrm{e}^-\tau^-$ via a single photon as an intermediate state in a t-channel diagram. Since the exchange particle is a photon it is massless, and so we can replace the $1/(q^2 - m_x^2)$ term with $1/q^2$. The potential is replaced with the potential $V_{\mathrm{D}} = q\gamma^0 \not{A}$. The initial state corresponds to an electron and a tau particle, and so does the final state, although the momenta may have changed (although the total momentum is of course conserved). Hence,

$$g_{\mathrm{e}^-} = \langle\psi_{\mathrm{e}^-,c}|V|\psi_{\mathrm{e}^-,a}\rangle. \tag{8.3.3}$$

The state for the electron before and after then interaction is

$$|\psi_{\mathrm{e}^-,a}\rangle = u(p_a), \qquad \text{and} \qquad |\psi_{\mathrm{e}^-,c}\rangle = u(p_c). \tag{8.3.4}$$

Hence, we have

$$g_{\mathrm{e}^-} = u^\dagger(p_c) e\gamma^0 \not{A} u(p_a) = e\bar{u}(p_c)\not{A}u(p_a). \tag{8.3.5}$$

Similarly for the tau particle we have

$$g_{\tau^-} = e\bar{u}(p_d)\not{A}^* u(p_b). \tag{8.3.6}$$

The only difference is the complex conjugate here. This arises because one particle emits the photon and the other absorbs it, however these processes are related by time reversal, and when we reverse time phase factors change as

$$\mathrm{e}^{-iEt} \mapsto \mathrm{e}^{-iE(-t)} = \mathrm{e}^{iEt} = (\mathrm{e}^{-iEt})^*, \tag{8.3.7}$$

so we have to take complex conjugates[4]. We don't know which of the particles absorbs the photon and which emits it, but it doesn't matter, the important thing is the relative complex conjugate. Putting this together we have

[4] See *Quantum Field Theory* for details

$$\mathcal{M}^t_{fi} = e^2\bar{u}(p_c)\gamma^0\not{A}u(p_b)\frac{1}{q^2}\bar{u}(p_d)\gamma^0\not{A}^*u(p_d). \tag{8.3.8}$$

Now we can deal with the $\not{A}$ terms. First notice that A_μ is just a number, its the component of some four-vector, and so commutes with everything. Since it is the same photon that leaves one particle and is absorbed by the other, with no other interactions, its polarisation cannot change. This means that our sum over polarisations should be just a single sum, rather than summing each factor of A_μ over polarisations, so we have

$$A_\mu A_\nu^* = \sum_\lambda \varepsilon_\mu^{(\lambda)}\mathrm{e}^{-ip\cdot x}(\varepsilon_\nu^{(\lambda)}\mathrm{e}^{-ip\cdot x})^* = \sum_\lambda \varepsilon_\mu^{(\lambda)}(\varepsilon_\nu^{(\lambda)})^*. \tag{8.3.9}$$

We then have

$$\mathcal{M}^t_{fi} = e^2\bar{u}(p_c)\gamma^0\gamma^\mu A_\mu u(p_b)\frac{1}{q^2}\bar{u}(p_d)\gamma^0\gamma^\nu A^*_\nu u(p_d) \tag{8.3.10}$$

$$= \sum_\lambda e^2\bar{u}(p_c)\gamma^0\gamma^\mu u(p_b)\frac{\varepsilon^{(\lambda)}_\mu \varepsilon^{(\lambda)}_\nu}{q^2}\bar{u}(p_d)\gamma^0\gamma^\nu u(p_d) \tag{8.3.11}$$

$$= -e^2\bar{u}(p_c)\gamma^0\gamma^\mu u(p_b)\frac{\eta_{\mu\nu}}{q^2}\bar{u}(p_d)\gamma^0\gamma^\nu u(p_d). \tag{8.3.12}$$

In the last step we used the result

$$\sum_\lambda \varepsilon^{(\lambda)}_\mu (\varepsilon_\nu)^{(\lambda)} = -g_{\mu\nu} \tag{8.3.13}$$

derived in Equation (C.4.26).

8.4 Feynman Rules

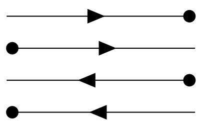
Good news! We don't have to do this calculation every time. Instead, since the process is relatively formulaic, we can use **Feynman rules** which allow us to look at a Feynman diagram and convert it into the matrix element. The exact Feynman rules depend on what theory we're considering, so here we'll do the Feynman rules for QED.

There are three parts to the Feynman rules[5]. We have a rule for each external line, that is a line representing a real particle, a rule for each internal particle, that is a virtual particle, and a rule for each vertex. The rules for the lines depend on the momentum of the particle, and the type of particle. We also give each vertex a Lorentz index, and the rules may depend on the indices at either end of the line, and the vertex term will also typically depend on the index of the vertex.

[5]We do not concern ourselves with symmetry factors here, which form a fourth part to the Feynman rules, see *Quantum Field Theory* for details.

So, without further ado here are the Feynman rules for QED. First, we have the rules for spin 1/2 external (anti)particles with momentum p, note that • corresponds to a vertex, whereas a lack of a • corresponds to an external particle:

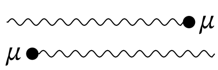
Incoming particle $u(p)$
Outgoing particle $\bar{u}(p)$
Incoming antiparticle $\bar{v}(p)$
Outgoing antiparticle $v(p)$



(8.4.1)

Next, we have the rules for spin 1 external particles, which in QED is just photons. For a photon with momentum p the rules are as follows:

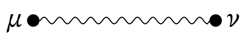
Incoming photon $\varepsilon^\mu(p)$
Outgoing photon $\varepsilon^\mu(p)$



(8.4.2)

Then there are internal lines, we've already seen the photon propagator, for an internal photon of momentum q:

Photon propagator $-\dfrac{i\eta_{\mu\nu}}{q^2}$



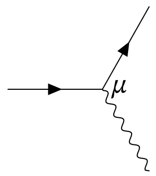
(8.4.3)

For a spin 1/2 fermion propagator of mass m and momentum q we have

Photon propagator $-\dfrac{i(\not{q} + m)}{q^2 - m^2}$

(8.4.4)

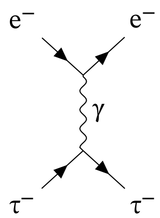
Finally we have a vertex factor. We've only considered vertices with two fermions and a photon so far, so for a fermion of charge $-e$, where e is the magnitude of the charge of the electron, we have the following rule:

Vertex $ie\gamma^\mu$  (8.4.5)

We find the amplitude from the Feynman rules by multiplying together all of the terms appearing in the diagram, which gives $-i\mathcal{M}$.

8.4.1 Using the Feynman Rules

We can consider the following diagram again:



(8.4.6)

Call the top vertex μ and the bottom one ν. At the top vertex we have an incoming electron, corresponding to a factor of $u(p_a)$, a vertex factor, $ie\gamma^\mu$, and an outgoing electron, corresponding to a factor of $\bar{u}(p_c)$. Similarly at the bottom vertex we get $\bar{u}(p_d)ie\gamma^\nu u(p_b)$. We also have a photon propagator giving a factor of $-i\eta/q^2$. So we recover the result

$$-i\mathcal{M} = ie^2\bar{u}(p_c)\gamma^\nu u(p_a)\frac{\eta_{\mu\nu}}{q^2}\bar{u}(p_d)\gamma^\nu u(p_b). \tag{8.4.7}$$

Nine

$e^+e^- \to \mu^+\mu^-$

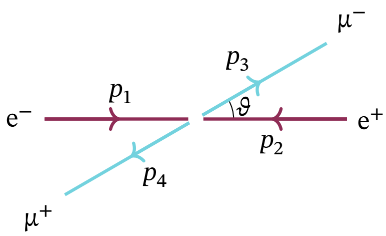
9.1 Setup

In this chapter we'll calculate the cross section for the interaction

$$e^+e^- \to \mu^+\mu^- \tag{9.1.1}$$

to lowest order in QED. Since the particles change the lowest order process has a single internal particle, and since this is QED it will be a photon. Note that this calculation is repeated in Section D.4 using the *Mathematica* packages *FeynCalc* and *FeynArt* to do most of the work.

In the lab this process may look like



(9.1.2)

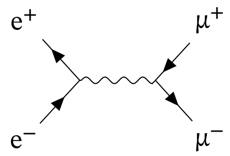
We assume that the electron and positron beams have the same energy, E, and then the muon and antimuon must also have energy E as total energy must be conserved and symmetries demands that the energy is shared evenly between the two particles. In the centre of mass frame the momenta of the particles are

$$p_1 = (E, 0, 0, p), \quad p_2 = (E, 0, 0, -p), \quad p_3 = (E, \boldsymbol{p_f}), \quad \text{and} \quad p_4 = (E, -\boldsymbol{p_f})$$

where $\boldsymbol{p_f}$ has magnitude p and is at an angle ϑ to the z-axis, which we take to be the axis along which the beams travel. We'll assume that the interaction occurs at high energies, so $m \ll E$ and $|\boldsymbol{p}| \approx E$ so we can neglect the masses in most equations.

9.2 Feynman Diagram and Amplitude

We consider only the lowest order diagram for this process. We also restrict ourselves to QED, if this wasn't the case then the internal propagator could also be a Z boson, or a Higgs boson. Since the particles change this is necessarily a second order process. There are two second order diagrams, s, t, and u-channel. Since this is an annihilation interaction producing different products to the incoming particles it can only proceed via an s-channel process. So we consider the diagram



(9.2.1)

We can convert this into an amplitude using the Feynman rules for QED. First focus on the incoming particles, we have an electron, giving a factor of $u(p_1)$, then a vertex giving a factor of $ie\gamma^\mu$, and then an antielectron, giving a factor of $\bar{v}(p_2)$. The photon propagator gives a factor of $-i\eta_{\mu\nu}/q^2$. The outgoing muon gives a factor of $\bar{u}(p_3)$, the vertex gives a factor of $ie\gamma^\nu$ and the outgoing antimuon gives a factor of $v(p_4)$.

Combining this we have the amplitude

$$-i\mathcal{M} = [\bar{v}(p_2)i\varepsilon\gamma^\mu u(p_1)]\left(-\frac{i\eta_{\mu\nu}}{q^2}\right)[\bar{u}(p_3)ie\gamma^\nu v(p_4)] \tag{9.2.2}$$

$$\mathcal{M} = -\frac{e^2}{q^2}\eta_{\mu\nu}\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(p_3)\gamma^\nu v(p_4) \tag{9.2.3}$$

$$= -\frac{e^2}{q^2} j_{\mathrm{e}} \cdot j_{\mu} \tag{9.2.4}$$

$$= -\frac{e^2}{s} j_{\mathrm{e}} \cdot j_{\mu} \tag{9.2.5}$$

where

$$j_{\mathrm{e}} = \bar{v}(p_2)\gamma^\mu u(p_1), \qquad \text{and} \qquad j_{\mu} = \bar{u}(p_3)\gamma^\nu v(p_4). \tag{9.2.6}$$

9.3 Spin

In general the electrons and positrons won't be polarised, instead we expect an even mix of left and right helicity states. This means that there are four options for the incoming helicity, RL, RR, LL, and LR, where the first letter is the helicity of the electron and the second the helicity of the positron. There are then 16 possible helicity combinations, since the helicity of the muon and antimuon can also take these four values.

The helicity of the final particles is something to measure. The helicity of the initial states must be averaged over, so we have

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{\text{helicities}} |M_i|^2 \tag{9.3.1}$$

$$= (|\mathcal{M}_{LL\to LL}|^2 + |\mathcal{M}_{LL\to LR}|^2 + |\mathcal{M}_{LL\to RL}|^2 + |\mathcal{M}_{LL\to RR}|^2 \tag{9.3.2}$$

$$+ |\mathcal{M}_{LR\to LL}|^2 + |\mathcal{M}_{LR\to LR}| + \cdots). \tag{9.3.3}$$

Now consider the helicity spinors

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ s\mathrm{e}^{i\varphi} \\ c \\ s\mathrm{e}^{i\varphi} \end{pmatrix}, \qquad u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ \mathrm{e}^{i\varphi} \\ s \\ -c\mathrm{e}^{i\varphi} \end{pmatrix}, \tag{9.3.4}$$

$$v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -c\mathrm{e}^{i\varphi} \\ -s \\ c\mathrm{e}^{i\varphi} \end{pmatrix}, \qquad v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ s\mathrm{e}^{i\varphi} \\ c \\ s\mathrm{e}^{i\varphi} \end{pmatrix} \tag{9.3.5}$$

where $c = \cos(\Theta/2)$ and $s = \sin(\Theta/2)$, where Θ is the polar angle defining the direction of momentum, and we've assumed $E \gg m$.

We choose to define our coordinates so that the electron moves along the z-axis, then the polar angle for the electron is 0, the polar angle for the positron is π, the polar angle for the muon is ϑ, and the polar angle for the antimuon is $\pi - \vartheta$. The particles move in a plane, which we may as well take to be the (x, y)-plane, so that $\varphi = 0, \pi$, depending on the direction.

The initial electron can be in either a left or right-handed helicity state and with $\Theta = 0$ and $\varphi = 0$ this gives the two possible states

$$u_\uparrow(p_1) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \text{or} \qquad u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}. \tag{9.3.6}$$

Similarly the positron can be left or right-handed, and with $\Theta = \pi$ and $\varphi = 0$ this gives

$$v_\uparrow(p_2) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \qquad \text{or} \qquad v_\downarrow(p_2) = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}. \tag{9.3.7}$$

The muon has polar angle $\Theta = \vartheta$ and $\varphi = 0$ giving possible states

$$u_\uparrow(p_3) = \sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix}, \qquad \text{or} \qquad u_\downarrow(p_3) = \sqrt{E}\begin{pmatrix} -s \\ c \\ s \\ -c \end{pmatrix}, \tag{9.3.8}$$

where $c = \cos(\vartheta/2)$ and $s = \sin(\vartheta/2)$. Finally, for the antimuon we have $\Theta = \pi - \vartheta$ and $\varphi = \pi$, and we can use the identities $\sin([\pi - \vartheta]/2) = \cos(\vartheta/2)$ and $\cos([\pi - \vartheta]/2) = \sin(\vartheta/2)$. Hence, the possible states of the antimuon are

$$v_\uparrow(p_4) = \sqrt{E}\begin{pmatrix} c \\ s \\ -c \\ -s \end{pmatrix}, \qquad \text{or} \qquad v_\downarrow(p_4) = \sqrt{E}\begin{pmatrix} -s \\ c \\ s \\ -c \end{pmatrix}. \tag{9.3.9}$$

9.4 Muon Current

We want to evaluate the muon current,

$$(j_\mu)^\nu = \bar{u}(p_3)\gamma^\nu v(p_4), \tag{9.4.1}$$

for each helicity combination. To start with we need a few identities. For arbitrary four-component spinors, Ψ and Φ, in the Dirac representation we have

$$\bar{\Psi}\gamma^0\Phi = \Psi^\dagger\gamma^0\gamma^0\Phi = \Psi_1^*\Phi_1 + \Psi_2^*\Phi_2 + \Psi_3^*\Phi_3 + \Psi_4^*\Phi_4, \tag{9.4.2}$$
$$\bar{\Psi}\gamma^1\Phi = \Psi^\dagger\gamma^0\gamma^1\Phi = \Psi_1^*\Phi_4 + \Psi_2^*\Phi_3 + \Psi_3^*\Phi_2 + \Psi_4^*\Phi_1, \tag{9.4.3}$$
$$\bar{\Psi}\gamma^2\Phi = \Psi^\dagger\gamma^0\gamma^2\Phi = -i(\Psi_1^*\Phi_4 - \Psi_2^*\Phi_3 + \Psi_3^*\Phi_2 - \Psi_4^*\Phi_1, \tag{9.4.4}$$
$$\bar{\Psi}\gamma^4\Phi = \Psi^\dagger\gamma^0\gamma^4\Phi = \Psi_1^*\Phi_3 - \Psi_2^*\Phi_4 + \Psi_3^*\Phi_1 - \Psi_4^*\Phi_2. \tag{9.4.5}$$
$$\tag{9.4.6}$$

These follow by writing

$$\Psi^\dagger = \begin{pmatrix}\Psi_1^* & \Psi_2^* & \Psi_3^* & \Psi_4^*\end{pmatrix}, \qquad \text{and} \qquad \Phi = \begin{pmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\\ \Phi_4\end{pmatrix}, \tag{9.4.7}$$

and then computing the matrix products.

Consider the case where the muon is right-handed and the antimuon is left-handed. Then we have $\Psi = u_\uparrow$ and $\Phi = v_\downarrow$ giving

$$\bar{u}_\uparrow\gamma^0\bar{v}_\downarrow = E(cs - sc + cs - sc) = 0, \tag{9.4.8}$$
$$\bar{u}_\uparrow\gamma^1\bar{v}_\downarrow = E(-c^2 + s^2 - c^2 + s^2) = 2E(s^2 - c^2) = -2E\cos\vartheta, \tag{9.4.9}$$
$$\bar{u}_\uparrow\gamma^2\bar{v}_\downarrow = -iE(-c^2 - s^2 - c^2 - s^2) = 2iE, \tag{9.4.10}$$
$$\bar{u}_\uparrow\gamma^3\bar{v}_\downarrow = E(cs + sc + cs + sc) = 4Ecs = 2E\sin\vartheta. \tag{9.4.11}$$

The four-vector current for the RL helicity combination is

$$\bar{u}_\uparrow\gamma^\nu\bar{v}_\downarrow = 2E(0, -\cos\vartheta, i, \sin\vartheta). \tag{9.4.12}$$

Doing the same with the other helicity combinations we get the results

$$RL: \quad \bar{u}_\uparrow(p_3)\gamma^\nu\bar{v}_\downarrow(p_4) = 2E(0, -\cos\vartheta, i, \sin\vartheta), \tag{9.4.13}$$
$$RR: \quad \bar{u}_\uparrow(p_3)\gamma^\nu\bar{v}_\uparrow(p_4) = (0,0,0,0), \tag{9.4.14}$$
$$LL: \quad \bar{u}_\downarrow(p_3)\gamma^\nu\bar{v}_\uparrow(p_4) = (0,0,0,0), \tag{9.4.15}$$
$$LR: \quad \bar{u}_\downarrow(p_3)\gamma^\nu\bar{v}_\downarrow(p_4) = 2E(0, -\cos\vartheta, -i, \sin\vartheta). \tag{9.4.16}$$

We see that, in the limit where $E \gg m$, only two helicity states are nonzero.

9.5 Electron Current

The electron current can be computed using the muon current by noticing that

$$[\bar{u}(p_3)\gamma^\nu v(p_4)]^\dagger = [u^\dagger(p_3)\gamma^0\gamma^\nu v(p_4)]^\dagger \tag{9.5.1}$$
$$= v^\dagger(p_4)(\gamma^\nu)^\dagger(\gamma^0)^\dagger u(p_3) \tag{9.5.2}$$
$$= v^\dagger(p_4)(\gamma^\nu)^\dagger\gamma^0 u(p_3) \tag{9.5.3}$$
$$= v^\dagger(p_4)\gamma^0\gamma^\nu u(p_3) \tag{9.5.4}$$
$$= \bar{v}(p_4)\gamma^0\gamma^\nu u(p_3). \tag{9.5.5}$$

This follows since $(\gamma^0)^\dagger = \gamma^0$, and $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$, which can be written as $(\gamma^\mu)^\dagger\gamma^0 = \gamma^0\gamma^\mu$ using $(\gamma^0)^2 = 1$.

This means that if we take the complex conjugate of the muon current we get the same form as the electron current. We just need to substitute in the correct momenta and the correct polar angle, $\Theta = 0$. Hence, the nonvanishing electron currents are

$$RL: \quad \bar{v}_\downarrow(p_2)\gamma^\nu u_\uparrow(p+1) = 2E(0,-1,-i,0), \tag{9.5.6}$$
$$LR: \quad \bar{v}_\uparrow(p_2)\gamma^\nu u_\downarrow(p+1) = 2E(0,-1,i,0). \tag{9.5.7}$$

Consider the process $LR \to LR$, for this choice of helicities we have

$$j_e \cdot j_\mu = 4E\begin{pmatrix}0 & -1 & i & 0\end{pmatrix}\begin{pmatrix}0 \\ -\cos\vartheta \\ -i \\ \sin\vartheta\end{pmatrix} = 4E(\cos\vartheta - 1). \tag{9.5.8}$$

Similarly,

$$RL \to RL: \quad j_e \cdot j_\mu = 4E^2(1-\cos\vartheta), \tag{9.5.9}$$
$$RL \to LR: \quad j_e \cdot j_\mu = 4E^2(1+\cos\vartheta), \tag{9.5.10}$$
$$LR \to LR: \quad j_e \cdot j_\mu = 4E^2(1+\cos\vartheta). \tag{9.5.11}$$

The $1-\cos\vartheta$ and $1+\cos\vartheta$ terms can be interpreted as conserving the angular momentum along the z-axis in the cases where $\vartheta = 0, \pi$, since the amplitudes where both helicities of the ingoing particles point in the z direction and both helicities of the outgoing particles point in the negative z-direction, or vice versa, vanish when $\vartheta = \pi$, which is exactly when this wouldn't conserve angular momentum, and similar for the other term.

9.6 Amplitude

The amplitudes for each process are then

$$\mathcal{M}_{RL\to RL} = \mathcal{M}_{RR} = e^2(1+\cos\vartheta)^2, \tag{9.6.1}$$
$$\mathcal{M}_{LR\to RL} = \mathcal{M}_{LR} = e^2(1-\cos\vartheta)^2, \tag{9.6.2}$$
$$\mathcal{M}_{RL\to LR} = \mathcal{M}_{RL} = e^2(1-\cos\vartheta)^2, \tag{9.6.3}$$
$$\mathcal{M}_{LR\to LR} = \mathcal{M}_{LL} = e^2(1+\cos\vartheta)^2. \tag{9.6.4}$$

The factor of $4E^2$ cancels with the factor of $1/s$ in the high energy approximation. We've introduce the shorthand $\mathcal{M}_{RL}$, for example, where the first letter tells us the helicity of the electron and the second the helicity of the muon. The antiparticles must then have opposite helicity for a nonvanishing contribution.

The total amplitude for this interaction, to second order in QED, is then given by

$$\begin{aligned}
|\mathcal{M}|^2 &= \frac{1}{4}e^4(|\mathcal{M}_{RR}|^2 + |\mathcal{M}_{LR}|^2 + |\mathcal{M}_{RL}|^2 + |\mathcal{M}_{LL}|^2) && (9.6.5)\\
&= \frac{1}{4}e^4[(1+\cos\vartheta)^2 + (1-\cos\vartheta)^2 + (1-\cos\vartheta)^2 + (1-\cos\vartheta)^2] \\
&= e^4(1+\cos^2\vartheta) && (9.6.6)
\end{aligned}$$

9.7 Cross Section

The cross section is given by the formula

$$\sigma = \frac{1}{64\pi^2 s}\frac{|\boldsymbol{p}_f^*|}{|\boldsymbol{p}_i^*|}\int |\mathcal{M}_{fi}|^2 \, \mathrm{d}\Omega. \tag{9.7.1}$$

In our case $|\boldsymbol{p}_f^*| = |\boldsymbol{p}_i^*| = p$, and at high energies we can approximate $s = 4E^2$.

The differential cross section is then

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}|\mathcal{M}|^2 = \frac{1}{64\pi^2 s}e^4(1 + \cos^2\vartheta). \tag{9.7.2}$$

Now we can use $\alpha = e^2/(4\pi)$, and so $e^4 = 16\pi^2\alpha^2$, to write this as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}(1 + \cos^2\vartheta). \tag{9.7.3}$$

This result is to first order in QED. We expect the next order term to on the order of α^3, which is approximately one hundredth the size of this term, the next term is then one ten thousandth the size of this term and so on. We therefore expect that our result will be within about 1 % of the experimental result, and indeed this is what we see.

Another improvement that can be made is to include the electroweak interaction, where this process happens via a Z boson. This introduces a small change, and also a slight asymmetry in the angular distribution about $\vartheta = 0$, since the electroweak interaction doesn't treat left and right-handed particles equally. This is indeed what we see in the results.

9.7.1 Total Cross Section

We compute the total cross section by integrating the differential cross section. To do so consider the integral

$$\int_{S^2} (1 + \cos^2\vartheta)\,\mathrm{d}\Omega = \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\vartheta \, \sin\vartheta(1 + \cos^2\vartheta) \tag{9.7.4}$$

$$= 2\pi \int_{-1}^{1} (1 + \cos^2\vartheta)\,\mathrm{d}(\cos\vartheta) \tag{9.7.5}$$

$$= \frac{16\pi}{3}. \tag{9.7.6}$$

We therefore have the total cross section to leading order in QED as

$$\sigma = \frac{4\pi\alpha^2}{3s}. \tag{9.7.7}$$

This is very close to what we actually measure.

9.8 Invariant Form of the Amplitude

In the centre of mass frame the amplitude squared is

$$|\mathcal{M}|^2 = e^4(1 + \cos^2\vartheta). \tag{9.8.1}$$

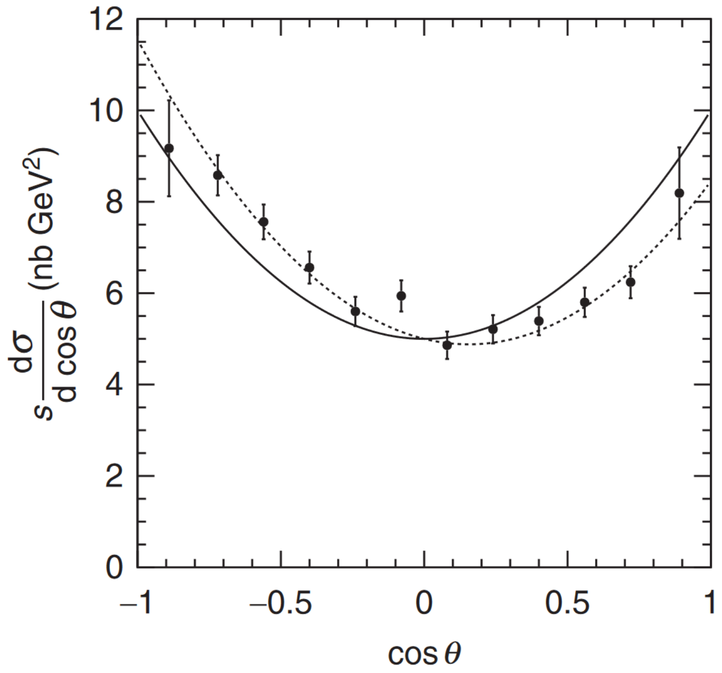


Figure 9.1: The differential cross section for the $e^-e^+ \to \mu^-\mu^+$ process. The line is the lowest order QED calculation, the dotted line includes electroweak interactions, and the data points are experimental measurements taken at $\sqrt{s} = 34.4$ GeV. Taken from [1], which adapted it from [2].

In the centre of mass frame the momenta of the particles are

$$p_1 = (E, 0, 0, E), \quad p_2 = (E, 0, 0, -E), \quad p_3 = (E, E\sin\vartheta, 0, E\cos\vartheta), \tag{9.8.2}$$
$$\text{and} \quad p_4 = (E, -E\sin\vartheta, 0, -E\cos\vartheta). \tag{9.8.3}$$

In the high energy limit, so $E \gg m_e, m_\mu$, we have $s = 2E^2$, $t = E^2(1 - \cos\vartheta)$, and $t = E^2(1 + \cos\vartheta)$. We then have

$$t^2 + u^2 = E^4[(1 - \cos\vartheta)^2 + (1 + \cos\vartheta)^2] \tag{9.8.4}$$
$$= E^4[1 + \cos^2\vartheta - 2\cos\vartheta + 1 + \cos^2\vartheta + 2\cos\vartheta]$$
$$= 2E^4(1 + \cos^2\vartheta). \tag{9.8.5}$$

Hence,

$$1 + \cos^2\vartheta = \frac{t^2 + u^2}{2E^4} = 2\frac{t^2 + u^2}{s^2}. \tag{9.8.6}$$

So we can rewrite the amplitude squared as

$$\langle|\mathcal{M}|^2\rangle = 2e^4\frac{t^2 + u^2}{s^2}. \tag{9.8.7}$$

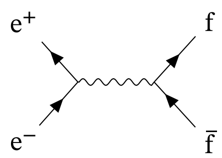
This is written in terms of Lorentz invariants, and so is valid in any frame in the high energy limit.

Ten

Chirality

10.1 $e^-e^+ \to f\bar{f}$

Consider the slightly more general process of an electron and positron annihilating and forming a photon, which decays into a charged fermion, f, and its antiparticle, $\bar{f}$. To first order this process occurs via the s-channel diagram



(10.1.1)

Suppose that f has charge $-iQ_f e$. Then the amplitude is

$$-i\mathcal{M} = \bar{v}_e(p_2) ie\gamma^\mu u_e(p_1) \frac{-ig_{\mu\nu}}{q^2} \bar{u}_f(p_3)(-iQ_f e\gamma^\nu) v_f(p_4) = \frac{Q_f e^2}{s} j_e \cdot j_f. \quad (10.1.2)$$

The only difference from the muon case is the charge.

For this process to be allowed we need $\sqrt{s} \geq 2m_f$. This is seen in experimental results where up to $\sqrt{s} = 2m_f$ the cross section for this process is zero. We can use this to measure the mass of the fermion, as shown in Figure 10.1.

A full calculation, not just in the high energy level, gives the cross section for this process as

$$\sigma(e^-e^+ \to f\bar{f}) = \frac{4\pi\alpha^2 Q_f^2}{3s} \beta \left(\frac{3 - \beta^2}{2} \right). \quad (10.1.3)$$

Here $\beta = p_3/E_f = 1 - 4m_f^2/s$ is the velocity of the fermion divided by the speed of light.

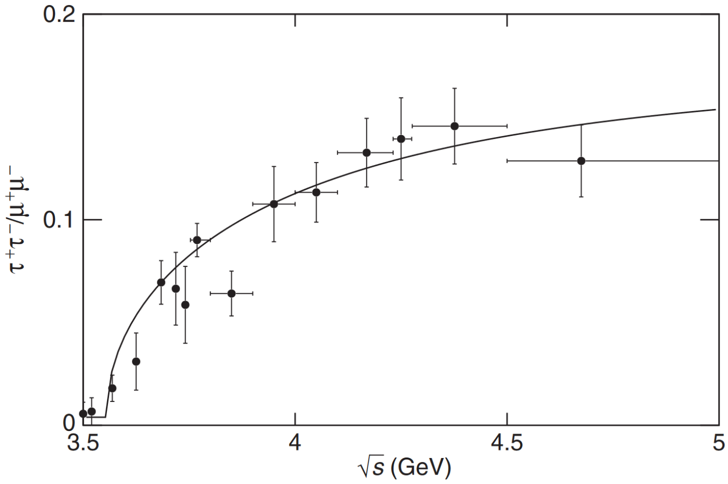
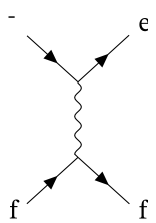


Figure 10.1: Plot showing the experimental measurements and theoretical predictions of the ratio $\sigma(\mathrm{e}^-\mathrm{e}^+ \to \tau^-\tau^+)/\sigma(\mathrm{e}^-\mathrm{e}^+ \to \mu^-\mu^+)$. Notice that below about 3.6 GeV this quantity vanishes. This corresponds to the tau particle having a mass of about 1.8 GeV, so below this energy it is not possible to produce two tau particles. This figure was taken from [1], which modified it from [3].

10.2 e$^-$f → e$^-$f

Consider the related process of an electron and a charged fermion scattering. This can happen as a t-channel process,



(10.2.1)

This is related to the $\mathrm{e}^-\mathrm{e}^+ \to \mathrm{f}\bar{\mathrm{f}}$ process by **crossing symmetry**. To see this consider the amplitude for this process:

$$-i\mathcal{M} = \bar{u}_\mathrm{e}(p_3) i e \gamma^\mu u_\mathrm{e}(p_1) \frac{-i\gamma_{\mu\nu}}{q^2} \bar{u}_\mathrm{f}(p_4)(-iQ_\mathrm{f} e\gamma^\nu) u_\mathrm{f}(p_1) = \frac{Q_\mathrm{f} e^2}{t} j_\mathrm{e} \cdot j_\mathrm{f}. \tag{10.2.2}$$

Here $q^2 = (p_1 - p_3)^2 = t$. The only difference is the momenta of the particles. If we make the substitution

$$p_1 \to p_1, \quad p_2 \to -p_3, \quad p_3 \to p_r, \quad \text{and} \quad p_4 \to -p_2 \tag{10.2.3}$$

in the amplitude for $\mathrm{e}^-\mathrm{e}^+ \to \mathrm{f}\bar{\mathrm{f}}$ and swap antiparticles to be particles then we get the amplitude for e$^-$f → e$^-$f. This corresponds to swapping the Mandelstam invariants according to

$$s \to t, \qquad t \to u, \quad \text{and} \quad u \to s. \tag{10.2.4}$$

So, the spin averaged amplitude squared is

$$\langle|\mathcal{M}|^2\rangle = 2Q_f^2 e^4 \frac{u^2+s^2}{t^2}. \tag{10.2.5}$$

10.3 The Fifth Gamma Matrix

We now define the matrix

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}. \tag{10.3.1}$$

This is a unitary matrix, $(\gamma^5)^\dagger\gamma^5 = 1$, and it anticommutes with all of the gamma matrices, $\{\gamma^5, \gamma^\mu\} = 0$. It is also Hermitian, $(\gamma^5)^\dagger = \gamma^5$, and so $(\gamma^5)^2 = 1$.

10.4 Chirality

Recall that helicity is defined as the spin along the direction of motion of the particle. This is determined in terms of the three-vector $\boldsymbol{p}$, and so is *not* Lorentz invariant. Instead we define a new quantity which is Lorentz invariant, called the **chirality**, which is the eigenvalue of the fifth gamma matrix, γ^5.

Chirality turns out to be very important when it comes to which interactions can occur. Recall that, in the high energy limit, only four of the possible sixteen helicity combinations contributed a nonzero amount to the $e^-e^+ \to \mu^-\mu^+$ scattering amplitude. This turns out to be due to the fact that in the high energy limit, or equivalently, for massless particles, chirality and helicity are the same. This is because when $m = 0$ the helicity operator commutes with the Hamiltonian, and is Lorentz invariant. The chiral structure of the electromagnetic interaction is then what restricts which helicity combinations contribute.

Suppose that u_R, u_L, v_R, and v_L are spinors which are eigenspinors of γ^5, such that

$$\gamma^5 u_R = +u_R, \quad \gamma^5 u_L = -u_L, \quad \gamma^5 v_R = -v_R, \quad \text{and} \quad \gamma^5 v_L = +v_L. \tag{10.4.1}$$

We call these right and left-handed **chiral spinors**. In the high energy/zero mass limit we find that $u_R = u_\uparrow$, $u_L = u_\downarrow$, $v_R = v_\uparrow$, and $v_L = v_\downarrow$.

We define the **chiral projection operators**

$$P_R := \frac{1}{2}(1+\gamma^5), \qquad \text{and} \qquad P_L := \frac{1}{2}(1-\gamma^5). \tag{10.4.2}$$

These project spinors as follows:

- P_R projects onto the right-handed chiral state of a particle,
- P_R projects onto the left-handed chiral state of an antiparticle,
- P_L projects onto the left-handed chiral state of a particle,
- P_L projects onto the right-handed chiral state of an antiparticle.

This can be shown by considering an arbitrary particle spinor, $\alpha u_{\mathrm{R}} + \beta u_{\mathrm{L}}$, or an arbitrary antiparticle spinor, $\alpha v_{\mathrm{R}} + \beta v_{\mathrm{L}}$, for $\alpha, \beta \in \mathbb{C}$. This works since the chiral spinors form a basis, so we can decompose an arbitrary spinor as

$$\psi = \psi_{\mathrm{L}} + \psi_{\mathrm{R}} = P_{\mathrm{L}}\psi + P_{\mathrm{R}}\psi. \tag{10.4.3}$$

Another way of saying this is that $\{P_{\mathrm{L}}, P_{\mathrm{R}}\}$ is a complete set of operators, so $P_{\mathrm{L}} + P_{\mathrm{R}} = 1$. The chiral projectors are also orthogonal, so $P_{\mathrm{R}}P_{\mathrm{L}} = P_{\mathrm{L}}P_{\mathrm{R}} = 0$. As projectors they are also idempotent, $P_{\mathrm{R}}^2 = P_{\mathrm{R}}$ and $P_{\mathrm{L}}^2 = P_{\mathrm{L}}$.

Consider an arbitrary fermion current, $j^\mu = \bar{\varphi}\gamma^\mu\psi$. We can write this as

$$j^\mu = (\bar{\varphi}_{\mathrm{L}} + \bar{\varphi}_{\mathrm{R}})\gamma^\mu(\psi_{\mathrm{L}} + \psi_{\mathrm{R}}) \tag{10.4.4}$$
$$= \bar{\varphi}_{\mathrm{L}}\gamma^\mu\psi_{\mathrm{L}} + \bar{\varphi}_{\mathrm{R}}\gamma^\mu\psi_{\mathrm{R}}. \tag{10.4.5}$$

Here we've used the fact that $\bar{\varphi}_{\mathrm{L}}\gamma^\mu\psi_{\mathrm{R}} = \bar{\varphi}_{\mathrm{R}}\gamma^\mu\psi_{\mathrm{L}} = 0$. This shows that only certain combinations of chiral spinors contribute to the fermion current. This is why only certain helicity combinations don't vanish in our earlier calculation.

QED treats left and right handed chiral spinors equally. The only important aspect of chirality when it comes to QED is this cancelling of mixed chirality components. If we were to swap all left chiral spinors for right chiral spinors, and vice versa, nothing would change as far as QED is concerned. In terms of the $\mathrm{e}^-\mathrm{e}^+ \to \mu^-\mu^+$ process the chiral structure means that we don't produce muons with physically unaligned spins. Since the muons go in opposite directions this means we produce one left-handed muon and one right-handed antimuon, or vice versa. Conservation of angular momentum then requires that we have a left-handed electron and right-handed positron, or vice versa.

Eleven

Symmetries

11.1 Parity

Parity is the operation

$$x^\mu = (t, \boldsymbol{x}) \mapsto x'^\mu = \mathcal{P}x^\mu = (t, -\boldsymbol{x}), \qquad \text{or} \tag{11.1.1}$$

$$t \mapsto t' = \mathcal{P}t = t, \qquad \text{and} \qquad \boldsymbol{x} \mapsto \boldsymbol{x}' = \mathcal{P}\boldsymbol{x} = -\boldsymbol{x}. \tag{11.1.2}$$

That is, parity inverts space, and leaves time unaffected. Parity acts on the wave function according to

$$\psi \mapsto \psi'(t', \boldsymbol{x}') = \mathcal{P}\psi(\mathcal{P}t, \mathcal{P}\boldsymbol{x}) = \psi(t, -\boldsymbol{x}). \tag{11.1.3}$$

Inverting space and then inverting it again gets us back to where we started, so we have $\mathcal{P}^2 = 1$.

Consider the Dirac equation, and some spinor, ψ, satisfying it. Then, by definition

$$(i\partial\!\!\!/ - m)\psi = 0. \tag{11.1.4}$$

Let $\psi' = \mathcal{P}\psi$. Under parity $\psi_i \mapsto -\partial_i$, and $\partial_0 \mapsto \partial_0$. So, the Dirac equation becomes

$$i\gamma^0\partial_0\psi - i\boldsymbol{\gamma} \cdot \nabla\psi - m\psi = 0. \tag{11.1.5}$$

Multiply through by γ^0, and we have

$$i(\gamma^0)^2\partial_0\psi - i\gamma^0\boldsymbol{\gamma} \cdot \nabla\psi - m\gamma^0\psi = 0. \tag{11.1.6}$$

Now using $(\gamma^0)^2 = 0$ and $\gamma^0\gamma^i = -\gamma^i\gamma^0$ we have

$$i\partial_0\psi + i\gamma^i\gamma^0\partial_i\psi - m\gamma^0\psi = 0. \tag{11.1.7}$$

We can now replace ψ with $\mathcal{P}\psi'$, and so we have

$$i\mathcal{P}\partial_0\psi' + i\gamma^i\gamma^0\mathcal{P}\partial_i\psi' - m\gamma^0\mathcal{P}\psi = 0. \tag{11.1.8}$$

We expect that ψ' satisfies the Dirac equation in it's own frame, so

$$i\gamma^0\partial_0\psi' + i\gamma^i\partial_i\psi' - m\psi' = 0. \tag{11.1.9}$$

Comparing these two equations we see that we must have $\gamma^0\mathcal{P} \propto 1$. Since $\mathcal{P}^2 = 1$ we can choose $\mathcal{P} = \gamma^0$. So, for spinors, we have

$$\psi' = \mathcal{P}\psi = \gamma^0\psi. \tag{11.1.10}$$

There is actually some ambiguity in the sign, we could also have chosen $\mathcal{P} = -\gamma^0$, or in fact $\mathcal{P} = \mathrm{e}^{i\vartheta}\gamma^0$ for some $\vartheta \in \mathbb{R}$.

If we apply $\mathcal{P} = \pm\gamma^0$ to the spinors in the rest frame then we find that

$$\mathcal{P}u_1 = \pm u_1, \quad \mathcal{P}u_2 = \pm u_2, \quad \mathcal{P}v_1 = \mp\mathcal{P}v_1, \quad \text{and} \quad \mathcal{P}v_2 = \mp\mathcal{P}v_2. \tag{11.1.11}$$

We are free to choose the sign, and the conventional choice is that we choose $\mathcal{P} = +\gamma^0$, so

- the intrinsic parity quantum number of fundamental spin 1/2 particles is positive,
- the intrinsic parity quantum number of fundamental spin 1/2 antiparticles is negative.

Under parity $\boldsymbol{x} \mapsto -\boldsymbol{x}$, and $\boldsymbol{p} \mapsto -\boldsymbol{p}$. So,

$$\boldsymbol{S} = \boldsymbol{x} \times \boldsymbol{p} \mapsto (-\boldsymbol{x}) \times (-\boldsymbol{p}) = \boldsymbol{x} \times \boldsymbol{p} = \boldsymbol{S}. \tag{11.1.12}$$

This means the direction of spin doesn't change under parity. We say that spin is an axial or pseudovector. The result is that helicity, which is spin in the $\boldsymbol{p}$ direction, flips under parity transformations. So parity transforms right-handed states into left-handed states and vice versa. This corresponds to the gamma matrices identity $\gamma^0\gamma^5 = -\gamma^5\gamma^0$.

11.2 Charge Conjugation

Consider an electron in an electromagnetic field. This is described by the modified Dirac equation

$$(i\not{\partial} - q\not{A} - m)\psi = 0. \tag{11.2.1}$$

For an electron with charge $q = -e$ we then have

$$(i\not{\partial} + e\not{A} - m)\psi = 0. \tag{11.2.2}$$

For a positron, with charge $q = e$ we instead have

$$(i\not{\partial} - e\not{A} - m)\psi = 0. \tag{11.2.3}$$

Take the complex conjugate of this, and we get

$$(-i\not{\partial}^* - e\not{A}^* - m)\psi^* = 0. \tag{11.2.4}$$

Multiply through by γ^2 and we get

$$(-i\gamma^2\not{\partial}^* - e\gamma^2\not{A}^* - m\gamma^2)\psi^* = 0. \tag{11.2.5}$$

Now use $(\gamma^2)^* = -\gamma^2$ we have $\gamma^2(\gamma^2)^* = -(\gamma^2)^2$ and $\gamma^2(\gamma^\mu)^* = \gamma^2\gamma^\mu = -\gamma^\mu\gamma^2$ for $\mu \neq 2$, since $(\gamma^\mu)^* = \gamma^\mu$ for $\mu \neq 2$ and γ^μ anticommutes with γ^2 for $\mu \neq 2$. We then have

$$(-i\not\partial\gamma^2 - e\not A\gamma^2 - m\gamma^2)\psi^* = 0, \tag{11.2.6}$$
$$(-i\not\partial + e\not A - m)\gamma^2\psi^* = 0, \tag{11.2.7}$$
$$(\not\partial + ie\not A + im)\psi' = 0, \tag{11.2.8}$$

where $\psi' = i\gamma^2\psi'$. We interpret this as ψ' describing another spin 1/2 particle with the same mass but opposite charge.

We call the operation

$$\psi \mapsto \psi' = \mathcal{C}\psi = i\gamma^2\psi^* \tag{11.2.9}$$

charge conjugation. By considering specific cases we can show that this maps particles to antiparticles and vice versa, and hence maps positive charge to negative charge and vice versa. For example, consider $\psi = u_1 \exp[-ip \cdot x]$. Under charge conjugation u_1 mapsto

$$i\gamma^1 u_1^* = i\sqrt{E+m}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}^* = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} = v_1, \tag{11.2.10}$$

and so

$$\psi = u_1 \mathrm{e}^{-ip\cdot x} \mapsto v_1 \mathrm{e}^{+ip\cdot x}. \tag{11.2.11}$$

Charge conjugation is such that $\mathcal{C}^2 = 1$, swapping matter and antimatter twice gets us back to where we started.

11.3 Time Reversal

Time reversal changes the direction of time, so

$$x^\mu = (t, \boldsymbol{x}) \mapsto x'^\mu = \mathcal{T}x^\mu = (-t, \boldsymbol{x}). \tag{11.3.1}$$

Since the direction of time reverses we have $\partial_0 \mapsto -\partial_0$, and so $\boldsymbol{p} \mapsto -\boldsymbol{p}$. However, $\boldsymbol{x}$ is unchanged, so $\boldsymbol{S} \mapsto -\boldsymbol{S}$. Hence, helicity is flipped by time reversal.

Experimentally it is much harder to reverse time than it is to either use antiparticles or do the experiment in the opposite direction. It is also theorised that the combined symmetry $\mathcal{CPT}$ is a symmetry of all systems, and if this is the case then $\mathcal{T}$ is necessarily equivalent to $\mathcal{CP}$. So, we often talk about charge conjugation and parity in place of time reversal.

Time reversal is such that $\mathcal{T}^2 = 1$. It can be shown that we can represent time reversal on spinors as $\mathcal{T} = i\gamma^1\gamma^3$.

11.4 Gauge Symmetry

11.4.1 Classical Electrodynamics

R See *Classical Electrodynamics* for details in the covariant formulation and *Electromagnetism* for details in the noncovariant formalism.

Consider the electric and magnetic fields defined in terms of the potentials:

$$\boldsymbol{E} = -\nabla\varphi - \frac{\partial \boldsymbol{A}}{\partial t}, \qquad \text{and} \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{11.4.1}$$

Alternatively, in a covariant notation

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{11.4.2}$$

where $A^\mu = (\varphi, \boldsymbol{A})$.

Now consider the transformation $\varphi \mapsto \varphi - \partial_t\chi$ and $\boldsymbol{A} \mapsto \boldsymbol{A} + \nabla\chi$, where χ is some scalar field with everywhere continuous second derivatives. Then

$$\boldsymbol{E} \mapsto -\nabla\varphi - \nabla(\partial_t\chi) - \partial_t\boldsymbol{A} - \partial_t(\nabla\chi) = -\nabla\varphi - \partial_t\boldsymbol{A} = \boldsymbol{E} \tag{11.4.3}$$

and

$$\boldsymbol{B} \mapsto \nabla \times \boldsymbol{A} + \nabla \times (\nabla\chi) = \nabla \times \boldsymbol{A} = \boldsymbol{B}. \tag{11.4.4}$$

So the physics doesn't change under this transformation of the potentials. We call a transformation like this a **gauge transformation**.

In covariant notation this gauge transformation is $A_\mu \mapsto A_\mu - \partial_\mu\chi$. We then have

$$\begin{aligned} F_{\mu\nu} &\mapsto \partial_\mu(A_\nu - \partial_\nu\chi) - \partial_\nu(A_\mu - \partial_\mu\chi) && (11.4.5)\\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - \partial_\mu\partial_\nu\chi + \partial_\nu\partial_\mu\chi && (11.4.6)\\ &= \partial_\mu A_\nu - \partial_\nu A_\mu && (11.4.7)\\ &= F_{\mu\nu}. && (11.4.8) \end{aligned}$$

So, we see exactly the same invariance under a gauge transformation.

11.4.2 Local U(1) Gauge Transformation

Suppose that we transform the state by some phase depending on the charge and some scalar field, χ, with everywhere continuous second derivatives:

$$\psi \mapsto \psi' = \mathrm{e}^{iq\chi}\psi. \tag{11.4.9}$$

This is called a U(1) symmetry, where

$$\mathrm{U}(1) := \{\mathrm{e}^{i\varphi} \in \mathbb{C} \mid \varphi \in [0, 2\pi)\} \cong S^1 \tag{11.4.10}$$

is the unitary group in one dimension, and is the group of symmetries of the circle[1].

[1] for details see *Symmetries of Quantum Mechanics* or *Symmetries of Particles and Fields*

Consider the Dirac equation for this transformed field:

$$(i\partial\!\!\!/ - m)\psi = 0 \mapsto i\partial\!\!\!/(\psi\mathrm{e}^{iq\chi}) - m\psi\mathrm{e}^{iq\chi} = 0. \tag{11.4.11}$$

Taking the derivative we get

$$i(\partial\!\!\!/\psi)\mathrm{e}^{iq\chi} + i\psi(iq\partial\!\!\!/\chi)\mathrm{e}^{iq\chi} - m\mathrm{e}^{iq\chi} = 0. \tag{11.4.12}$$

Hence, we have

$$i(\partial\!\!\!/ + iq\partial\!\!\!/\chi)\psi - m\psi = 0. \tag{11.4.13}$$

Suppose that at the same time as we transform the spinor we also modify the derivative $\partial_\mu \mapsto \partial_\mu + iqA_\mu$ where A_μ transforms under this transformation as $A_\mu \mapsto A_\mu - \partial_\mu\chi$. Then we instead have the modified Dirac equation

$$i(\partial\!\!\!/ + iqA\!\!\!/)\psi - m\psi = 0. \tag{11.4.14}$$

The modified derivative then transforms as

$$\partial\!\!\!/ + iqA\!\!\!/ \mapsto \partial\!\!\!/ + iqA\!\!\!/ - iq\partial\!\!\!/\chi. \tag{11.4.15}$$

So, we see that under these transformations the Dirac equation transforms as

$$0 = i(\partial\!\!\!/ + iqA\!\!\!/)\psi - m\psi \mapsto i(\partial\!\!\!/ + iqA\!\!\!/ - iq\partial\!\!\!/\chi + iq\partial\!\!\!/\chi) - m\psi \tag{11.4.16}$$
$$= i(\partial\!\!\!/ + iqA\!\!\!/)\psi - m\psi = 0. \tag{11.4.17}$$

So, the modified Dirac equation is invariant under this $\mathrm{U}(1)$ symmetry.

Note that we can write the modified Dirac equation as

$$(iD\!\!\!\!/ - m)\psi = 0 \tag{11.4.18}$$

where

$$D_\mu = \partial_\mu + iqA_\mu \tag{11.4.19}$$

is the **gauge covariant derivative**[2]

If we interpret A_μ as the photon field then we see that demanding $\mathrm{U}(1)$ symmetry invariance automatically implies the existence of photons and much of electrodynamics. Formally, we say that QED is an Abelian gauge theory with an $\mathrm{U}(1)$ symmetry group.

[2] see *General Relativity* for another use of covariant derivatives, where the gauge symmetry is invariance under diffeomorphisms.

Twelve

Quantum Chromodynamics

12.1 Comparison to QED

Quantum chromodynamics, or QCD, is the quantum field theory of the strong force. Formally, QCD is a non-Abelian gauge theory with SU(3) symmetry group. We'll see what this means shortly. First, we make some comparisons between QCD and QED.

- QED
- Charged particles carry electric charge.
- Electric charge has a single degree of freedom.
- Antiparticles have the opposite electric charge.
- Force is mediated by massless spin 1 photons.
- The strength is dictated by the value of $e = \sqrt{\alpha/(4\pi)}$.
- $\alpha \approx 1/137$
- U(1) gauge theory.

- QCD
- Quarks carry colour charge.
- Colour charge has three degrees of freedom. We call them red, green, and blue, or r, g, b.
- Antiquarks have anticolour charge, $\bar{\text{r}}$, $\bar{\text{g}}$, or $\bar{\text{b}}$.
- Force is mediated by massless spin 1 gluons.
- The strength is dictated by the value of $g_S = \sqrt{\alpha_S/(4\pi)}$.
- $\alpha_S \approx 1$.
- SU(3) gauge theory.

12.2 QCD Gauge Theory

R The group theory gets a bit complex here. I've included the simple definitions given in lectures, and more complicated, nonexaminable points in *italics*. See *Symmetries of Quantum Mechanics* and *Symmetries of Particles and Fields* for more details.

QCD treats all three colours the same. This means we can mix the colours and the state only changes up to an unobservable phase. The symmetry group which does this mixing is SU(3), where

$$\mathrm{SU}(3) \coloneqq \{U \in \mathcal{M}_3(\mathbb{C}) \mid U^\dagger U = 1 \text{ and } \det U = 1\} \tag{12.2.1}$$

is the special unitary group of 3×3 matrices. Note that the special part corresponds to the $\det U = 1$ requirement, and the unitary part to the $U^\dagger U = 1$ requirement. SU(3) *is the symmetry group preserving the metric* $x \cdot y = x_i^* y_i$ *on* $\mathbb{C}^3$*, this is one particular representation of it.*

In QED the symmetry group U(1) has one possible transformation, a phase factor, $e^{iq\chi}$. *In QED the symmetry group* U(1) *has the Lie algebra* $\mathfrak{u}(1)$*, which has a single generator,* $q\chi$. This corresponds to a single field, A_μ, in QED, and so a single exchange particle. In QCD the symmetry group SU(3) has eight possible transformations, $T_a = \lambda_a/2$, where $a = 1, \dots, 8$ and

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{12.2.2}$$

These are called the **Gell–Mann matrices**. *In QCD the symmetry group* SU(3) *has the Lie algebra* $\mathfrak{su}(3)$*, which has eight generators,* T_a*. The fundamental representation of this Lie algebra consists of* 3×3 *complex traceless Hermitian matrices,* M*, such that* $\exp[iM] \in \mathrm{SU}(3)$*. In this representation one choice is* $T_a = \lambda_a/2$*. We can then express an arbitrary element of* $\mathfrak{su}(3)$ *as* $\alpha^a T_a$ *and so an arbitrary element of* SU(3) *can be expressed as* $\exp[ig_s\alpha^a T_a]$. A general gauge transformation of the spinor field is then

$$\psi \mapsto \psi' = \mathrm{e}^{ig_s\alpha^a T_a}\psi \tag{12.2.3}$$

where g_s is a constant corresponding to the strength of the strong force, α^a is a set of parameters and $T_a = \lambda_a/2$ are the matrices defined above. *Since* ψ *above is a four-component spinor and* $\exp[ig_s\alpha^a T_a]$ *is a* 3×3 *matrix there is no way to actually take this product, its just a formal statement that we have two components now to our wave function, a spinor,* ψ*, and a colour phase,* $\exp[ig_s\alpha^a T_a]$*. Note that* $\mathfrak{su}(3)$ *is a semisimple Lie algebra, and so it's Killing form is proportional to the identity. This means we don't need to distinguish raised and lowered indices, and sometimes we won't, but we do here as it makes the use of the summation convention more obvious.*

The eight generators correspond to there being eight gluons. For each gluon, indexed by $a = 1, \dots, 8$, we introduce a gluon field, G_μ^a, which corresponds to A_μ in QED. Like photons these G_μ^a fields correspond to massless exchange particles, and we have to modify the Dirac equation as we did for QED, changing the derivative to be

$$\partial_\mu \mapsto D_\mu = \partial_\mu + ig_s G_\mu^a T_a, \tag{12.2.4}$$

and so the Dirac equation becomes

$$i(\partial\!\!\!/ + ig_s G\!\!\!/^a T_a)\psi - m\psi = 0. \tag{12.2.5}$$

To go from QED to QCD we replace $q\chi$ with $g_s T_a$, and A_μ with G^a_μ. Thus, in a QCD interaction we replace the vertex term $-iQ_f e\gamma^\mu$ with

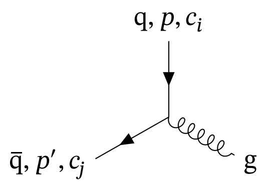
$$-g_s\gamma^\mu T_a. \tag{12.2.6}$$

In order to do QCD we need to include a colour factor in our spinors. We do so by replacing $u(p)$ with $u_j(p) = c_i u(p)$, where

$$c_1 = \mathrm{r} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad c_2 = \mathrm{g} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad \text{and} \qquad c_3 = \mathrm{b} = \begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{12.2.7}$$

Again, this is just a formal product, there's no way to actually compute it. We assume that in this formal product all colour terms commute with all spinor terms.

The QCD quark current corresponding to the vertex



(12.2.8)

is given by making the relevant substitutions to the QED fermion current:

$$\begin{aligned} j^\mu &= \bar{u}(p')c_j^\dagger\left(-\frac{1}{2}ig_s\lambda_a\gamma^\mu\right)c_i u(p) && (12.2.9)\\ &= -\frac{1}{2}ig_s c_j^\dagger\lambda_a c_i\bar{u}(p')\gamma^\mu u(p) && (12.2.10)\\ &= -\frac{1}{2}ig_s(\lambda_a)_{ji}\bar{u}(p')\gamma^\mu u(p). && (12.2.11)\end{aligned}$$

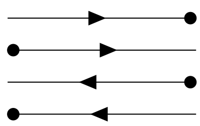
Here $(\lambda_a)_{ji}$ is the entry of the jth row and ith column of λ_a.

Colour is conserved. This means if we have an internal gluon then the colour at one end must be the same as the colour at the other end. To enforce this we include a factor of δ^{ab} for each internal gluon.

12.3 Feynman Rules

The Feynman rules for QCD are very similar to the Feynman rules for QED. For each external quark of momentum p we have

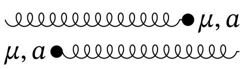
Incoming quark	$u(p)$
Outgoing quark	$\bar{u}(p)$
Incoming antiquark	$\bar{v}(p)$
Outgoing antiquark	$v(p)$



(12.3.1)

Next, we have the rules for spin 1 external particles, which in QCD are just gluons. For a gluon with momentum p we have

Incoming gluon	$\varepsilon^\mu(p)$
Outgoing gluon	$\varepsilon^\mu(p)$



(12.3.2)

Then there are internal lines, for an internal gluon of momentum q:

Gluon propagator $-\frac{i\eta_{\mu\nu}}{q^2}\delta^{ab}$ μ, a ν, b (12.3.3)

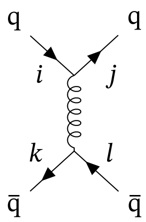
Then there is a vertex factor of

$$-i\frac{1}{2}g_s(\lambda_a)_{ji}\gamma^\mu \tag{12.3.4}$$

where c_j is the colour of the incoming quark and c_i the colour of the outgoing quark.

12.4 Quark-Antiquark Scattering

Consider the process of a quark and antiquark scattering. The diagram for this is



(12.4.1)

The amplitude for this is given by

$$-i\mathcal{M} = \left[\bar{u}_j\left(-i\frac{1}{2}g_s(\lambda_a)_{ji}\gamma^\mu u_i\right)\right]\frac{-i\eta_{\mu\nu}}{q^2}\delta^{ab}\left[\bar{v}_k\left(-i\frac{1}{2}g_s(\lambda_b)_{kl}\gamma^\nu\right)v_l\right], \tag{12.4.2}$$

which simplifies to

$$\mathcal{M} = -\frac{1}{4}g_s^2(\lambda_a)_{ji}(\lambda_a)_{kl}\frac{\eta_{\mu\nu}}{q^2}\bar{u}_j\gamma^\mu u_i\bar{v}_k\gamma^\nu v_l. \tag{12.4.3}$$

Here we use the δ^{ab} to change b into a.

Notice how similar this is to electron/positron scattering, but with e swapped for g_s and an extra factor of $(\lambda_a)_{ji}(\lambda^a)_{kl}/4$. This suggests that, to lowest order, we can approximate the dynamics of the $q\bar{q} \to q\bar{q}$ scattering process as occurring in a Coulomb-like potential:

$$V_{q\bar{q}} = -\frac{f\alpha_S}{r} \qquad \text{where} \qquad f = \frac{1}{4}(\lambda_a)_{ji}(\lambda_a)_{kl}. \tag{12.4.4}$$

Appendices

A

List of Particles

A.1 Fundamental Particles

A.1.1 Fermions

A.1.1.1 Leptons

Particle	Symbol	Charge	Mass	
Electron	e^-	-1	511.0	keV
Muon	μ^-	-1	105.6	MeV
Muon	τ^-	-1	1776.9	MeV
Electron Neutrino	ν_e	0	<0.120	eV
Muon Neutrino	ν_μ	0	<0.120	eV
Tau Neutrino	ν_τ	0	<0.120	eV

A.1.1.2 Quarks

Particle	Symbol	Charge	Mass	
Up Quark	u	$+2/3$	2.3	MeV
Down Quark	d	$-1/3$	4.8	MeV
Charm Quark	c	$+2/3$	1.275	GeV
Strange Quark	s	$-1/3$	95	MeV
Top Quark	t	$+2/3$	173.2	GeV
Bottom Quark	b	$-1/3$	4.18	GeV

A.1.1.3 Vector Bosons

Particle	Symbol	Charge	Mass	
Photon	γ	0	0	
Gluon	g	0	4.8	
Z boson	Z	0	91.19	GeV
$W^\pm$ boson	$W^\pm$	±1	80.38	GeV

Note that recent measurements place the mass of the $W^\pm$ boson at the slightly higher value of 80.43 GeV.

A.1.1.4 Scalar Bosons

Particle	Symbol	Charge	Mass	
Higgs Boson	H	0	125.25	GeV

B

Fermi's Golden Rule

R This derivation is repeated, in varying levels of detail with differing formalisms, in *Principles of Quantum Mechanics* and *Quantum Theory*.

B.1 Derivation to First Order

Consider the Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}'(t, \boldsymbol{x}) \tag{B.1.1}$$

where $\hat{H}_0$ is a time independent Hamiltonian for which we can solve the Schrödinger equation and $\hat{H}'(t, \boldsymbol{x})$ is the interaction Hamiltonian. Let $\varphi_k(t, \boldsymbol{x})$ be a normalised solution to the Schrödinger equation for the unperturbed Hamiltonian, that is

$$\hat{H}_0 \varphi_k = E_k \varphi_k, \qquad \text{and} \qquad \langle j|k\rangle = \delta_{jk} \tag{B.1.2}$$

where $\varphi_k(\boldsymbol{x}) = \langle \boldsymbol{x}|k\rangle$.

The Schrödinger equation in the presence of the interaction Hamiltonian is

$$i\frac{\partial \psi}{\partial t} = [\hat{H}_0 + \hat{H}'(t, \boldsymbol{x})]\psi. \tag{B.1.3}$$

The wave function, ψ, can be expressed in terms of the complete set of states of the eigenstates of the unperturbed Hamiltonian:

$$\psi(t, \boldsymbol{x}) = \sum_k c_k(t)\varphi_k(\boldsymbol{x})\mathrm{e}^{-iE_k t}. \tag{B.1.4}$$

Substituting this into the Schrödinger equation we get

$$i\sum_k \left[\frac{\mathrm{d}c_k}{\mathrm{d}t}\varphi_k \mathrm{e}^{-iE_k t} - iE_k c_k \varphi_k \mathrm{e}^{-iE_k t}\right] = \sum_k \left[c_k \hat{H}_0 \varphi_k \mathrm{e}^{-iE_k t} + \hat{H}' c_k \varphi_k \mathrm{e}^{-iE_k t}\right] \tag{B.1.5}$$

Using $\hat{H}_0\varphi_k = E_k\varphi_k$ the second term on the left cancels with the first on the right and we're left with

$$i\sum_k \frac{\mathrm{d}c_k}{\mathrm{d}t}\varphi_k \mathrm{e}^{-iE_k t} = \sum_k \hat{H}' c_k(t)\varphi_k \mathrm{e}^{-iE_k t}. \tag{B.1.6}$$

Suppose that at time $t = 0$ the initial state is φ_i, and the coefficients are $c_k(0) = \delta_{ik}$, that is $c_k(0) = 0$ for all $k \neq i$. Suppose also that the perturbing Hamiltonian is constant for $t > 0$, we can imagine switching it on at time $t = 0$ and then leaving it on, and that its small enough that at all times $c_i(t) \approx 1$ and $c_k(t) \approx 0$ for $k \neq i$. Then, we have

$$i \sum_k \frac{\mathrm{d}c_k}{\mathrm{d}t} \varphi_k \mathrm{e}^{-iE_k t} \approx \hat{H}' \varphi_i \mathrm{e}^{-iE_i t}. \tag{B.1.7}$$

Rewriting this in terms of kets we have

$$i \sum_k \frac{\mathrm{d}c_k}{\mathrm{d}t} \mathrm{e}^{-iE_k t} |k\rangle \approx \mathrm{e}^{-iE_i t} \hat{H}' |i\rangle. \tag{B.1.8}$$

Taking the product with some particular final state, $\langle f|$, we have

$$i \sum_k \frac{\mathrm{d}c_k}{\mathrm{d}t} \mathrm{e}^{-iE_k t} \langle \varphi_f | \varphi_i \rangle = i \frac{\mathrm{d}c_f}{\mathrm{d}t} \mathrm{e}^{-iE_f t} \approx \mathrm{e}^{-iE_i t} \langle f | \hat{H}' | i \rangle. \tag{B.1.9}$$

Rearranging this we get

$$\frac{\mathrm{d}c_f}{\mathrm{d}t} = -i \langle f | \hat{H}' | i \rangle \mathrm{e}^{i(E_f - E_i)t}. \tag{B.1.10}$$

Here we use the usual inner product

$$\langle f | \hat{H}' | i \rangle = \int \varphi_f^*(\boldsymbol{x}) \hat{H}' \varphi_i(\boldsymbol{x}) \, \mathrm{d}^3 \boldsymbol{x}. \tag{B.1.11}$$

Define the transition matrix element, $T_{fi} = \langle f | \hat{H}' i$. At time t the amplitude for transitions to the state $|f\rangle$ is given by

$$c_f(t) = -i \int_0^t T_{fi} \mathrm{e}^{i(E_f - E_i)t'} \, \mathrm{d}t'. \tag{B.1.12}$$

If the perturbing Hamiltonian is time independent then so is $\langle f | \hat{H}' | i \rangle$, and so

$$c_f(t) = -i T_{fi} \int_0^t \mathrm{e}^{i(E_f - E_i)t'} \, \mathrm{d}t'. \tag{B.1.13}$$

The probability to transition to the state $|f\rangle$ if we start in the state $|i\rangle$ is then

$$\mathbb{P}(f \to i) = c_f^*(t) c_f(t) = |T_{fi}|^2 \int_0^t \int_0^t \mathrm{e}^{i(E_f - E_i)t'} \mathrm{e}^{-i(E_f - E_i)t''} \, \mathrm{d}t' \, \mathrm{d}t''. \tag{B.1.14}$$

The transition rate, $\mathrm{d}\varGamma_{fi}$, to transition from the given initial state to the particular final state, $|f\rangle$, is then

$$\mathrm{d}\varGamma_{fi} = \frac{1}{t} \mathbb{P}(f \to i) = \frac{1}{t} |T_{fi}|^2 \int_0^t \int_0^t \mathrm{e}^{i(E_f - E_i)t'} \mathrm{e}^{-i(E_f - E_i)t''} \, \mathrm{d}t' \, \mathrm{d}t''. \tag{B.1.15}$$

We can make the substitution $t' \to t' + t/2$ and $t'' \to t'' + t/2$. This shifts the integration limits, without changing the integrand, since

$$\mathrm{e}^{kt'} \mathrm{e}^{-kt''} \to \mathrm{e}^{kt' + kt/2} \mathrm{e}^{-kt'' - kt/2} = \mathrm{e}^{kt'} \mathrm{e}^{kt''}, \tag{B.1.16}$$

with $k = i(E_f - E_i)$.

Performing this integral we get

$$4 \operatorname{sinc}^2 \left(\frac{t}{2} [E_f - E_i] \right). \tag{B.1.17}$$

Code B.1.18

```
Integrate[
Integrate[
Exp[I k t1] Exp[-I k t2],
{t1, -t/2, t/2}],
{t2, -t/2, t/2},
Assumptions → {k ∈ Reals, t ∈ Reals}]
4 Sin[kt/2]^2/k^2
```

It is well known[1] that $\operatorname{sinc}^2 x$ approximates $\delta(x)$. This means that the transition rate is only significant to states with $E_f \approx E_i$. This is energy conservation within the limits of the uncertainty relation, $\varDelta E \varDelta t \approx 1/2$. We use this to symmetrically extend the limits to $\pm\infty$:

$$\mathrm{d}\Gamma_{fi} = |T_{fi}|^2 \lim_{t\to\infty}\left[\frac{1}{t}\int_{-t/2}^{t/2}\int_{-t/2}^{t/2} \mathrm{e}^{i(E_f-E_i)t'}\mathrm{e}^{-i(E_f-E_i)t''}\,\mathrm{d}t'\,\mathrm{d}t''\right] \tag{B.1.19}$$

We can then recognise the integral representation of the Dirac delta:

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{ipx}\,\mathrm{d}p. \tag{B.1.20}$$

This comes from realising that the Fourier transform of $\delta(x)$ is

$$\mathcal{F}\{\delta(x)\} = \int_{-\infty}^{\infty}\delta(x)\mathrm{e}^{-ipx}\,\mathrm{d}x = \mathrm{e}^0 = 1, \tag{B.1.21}$$

and then the above expression for $\delta(x)$ is simply $\mathcal{F}^{-1}\{\mathcal{F}\{\delta(x)\}\}$.

We can use this to write

$$\mathrm{d}\Gamma_{fi} = 2\pi|T_{fi}|^2 \lim_{t\to\infty}\left[\frac{1}{t}\int_{-t/2}^{t/2}\mathrm{e}^{i(E_f-E_i)t'}\delta(E_f-E_i)\,\mathrm{d}t'\right]. \tag{B.1.22}$$

Suppose there are $\mathrm{d}n$ accessible final states in the range $[E_f, E_f + \mathrm{d}E_f]$. Then the total transition rate is given by integrating over the final states with accessible energy, which we convert to an integral over their energies using the density of states[2]:

$$\Gamma_{fi} = 2\pi\int|T_{fi}|^2\frac{\mathrm{d}n}{\mathrm{d}E_f}\delta(E_f-E_i)\lim_{t\to\infty}\left[\frac{1}{t}\int_{-t/2}^{t/2}\mathrm{d}t\right]\mathrm{d}E_f \tag{B.1.23}$$

$$= 2\pi\int|T_{fi}|\frac{\mathrm{d}n}{\mathrm{d}E_f}\delta(E_f-E_i)\,\mathrm{d}E_f \tag{B.1.24}$$

$$= 2\pi|T_{fi}|\left.\frac{\mathrm{d}n}{\mathrm{d}E_f}\right|_{E_i}. \tag{B.1.25}$$

The last term here is the density of states:

$$\rho(E_i) = \left.\frac{\mathrm{d}n}{\mathrm{d}E_f}\right|_{E_i}. \tag{B.1.26}$$

Thus, we have derived Fermi's golden rule

$$\Gamma_{fi} = 2\pi|T_{fi}|^2\rho(E_i). \tag{B.1.27}$$

[1] see *Methods of Mathematical Physics*

[2] see the *Statistical Mechanics* part of the *Thermal Physics* course.

B.2 Improvement to Second Order

To first order we have

$$T_{fi} = \langle f|\hat{H}'|i\rangle. \tag{B.2.1}$$

We assumed that $c_k(t) \approx 0$ for $k \neq i$. An improved derivation would again take $c_i(t) \approx 1$ and substitute the expression for $c_k(t)$ from Equation (B.1.13) and substitute it into Equation (B.1.6). Then again taking the inner product with $|f\rangle$ we get

$$\frac{\mathrm{d}c_f}{\mathrm{d}t} \approx -i\langle f|\hat{H}|i\rangle \mathrm{e}^{i(E_f-E_i)t} + (-i)^2 \sum_{k\neq i} \langle f|\hat{H}'|k\rangle \mathrm{e}^{i(E_f-E_k)t} \int_0^t \langle k|\hat{H}'|i\rangle \mathrm{e}^{i(E_k-E_i)t'}\,\mathrm{d}t'. \tag{B.2.2}$$

The perturbation is not present at time $t = 0$ and for $t > 0$ the perturbation is constant, which lets us write

$$\int_0^t \langle k|\hat{H}'|i\rangle \mathrm{e}^{i(E_k-E_i)t'}\,\mathrm{d}t' = \langle k|\hat{H}'|i\rangle \frac{\mathrm{e}^{i(E_k-E_i)t}}{i(E_k-E_i)}. \tag{B.2.3}$$

This gives us an improved approximate differential equation for the coefficients $c_f(t)$:

$$\frac{\mathrm{d}c_f}{\mathrm{d}t} \approx -i\left[\langle f|\hat{H}'|i\rangle + \sum_{k\neq i} \frac{\langle f|\hat{H}'|k\rangle\langle k|\hat{H}'|i\rangle}{E_i - E_k}\right] \mathrm{e}^{i(E_f-E_i)t}. \tag{B.2.4}$$

Then, to second order, the transition matrix elements are

$$T_{fi} = \langle f|\hat{H}'|i\rangle + \sum_{k\neq i} \frac{\langle f|\hat{H}'|k\rangle\langle k|\hat{H}'|i\rangle}{E_i - E_k}. \tag{B.2.5}$$

We interpret the first term as a direct transition, $|i\rangle \to |f\rangle$. The second term then corresponds to some indirect transition, $|i\rangle \to |k\rangle \to |f\rangle$, where we sum over all possible intermediate states, $|k\rangle$. The next higher order term would then allow two intermediate states, and so on.

C

Gauge Boson Polarisations

C.1 Classical Electrodynamic Gauge Invariance

Maxwell's equations are given by

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad \text{and} \qquad \partial_\mu F^{*\mu\nu} = 0 \tag{C.1.1}$$

where

$$F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu \tag{C.1.2}$$

is the electric field strength tensor, $A^\mu = (\varphi, \boldsymbol{A})$ is the four-potential, $j^\nu = (\rho, \boldsymbol{j})$ is the four-current, and

$$F^{*\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{C.1.3}$$

is the dual field strength tensor.

The first equation can be expanded in terms of A to get

$$\partial_\mu\partial^\mu A^\nu - \partial_\mu\partial^\nu A^\mu = \partial^2 A^\nu - \partial^\nu\partial_\mu A^\mu = j^\nu. \tag{C.1.4}$$

We know that in electromagnetism we have a gauge freedom where the transformation

$$\varphi \mapsto \varphi' - \frac{\partial\chi}{\partial t}, \qquad \text{and} \qquad \boldsymbol{A} \mapsto \boldsymbol{A} + \nabla\chi \tag{C.1.5}$$

for some function χ with continuous second derivatives doesn't change Maxwell's equations. This can easily be shown by substituting these transformed potentials into Maxwell's equations, through $\boldsymbol{B} = \nabla\times(\boldsymbol{A}+\nabla\chi)$ and $\boldsymbol{E} = -\nabla\varphi - \partial_t(\boldsymbol{A}+\nabla\chi)$ and then using the identities $\nabla\cdot(\nabla\times\boldsymbol{X}) = 0$ and $\nabla\times(\nabla f) = 0$.

Relativistically this gives the gauge freedom

$$A_\mu \mapsto A_\mu - \partial_\mu\chi. \tag{C.1.6}$$

Under this we have

$$\partial_\mu A^\mu \mapsto \partial_\mu A'^\mu = \partial_\mu(A^\mu - \partial^\mu\chi) = \partial_\mu A^\mu - \partial^2\chi. \tag{C.1.7}$$

We can choose $\partial^2\chi = \partial_\mu A^\mu$, then our transformed potential has $\partial_\mu A'^\mu = 0$. This is called the **Lorenz gauge** condition. In this gauge Maxwell's equation becomes

$$\partial^2 A^\mu = j^\mu. \tag{C.1.8}$$

C.2 Photon Polarisations

In a vacuum, $j^\nu = 0$, the photon field satisfies $\partial^2 A^\mu = 0$. This has plane wave solutions:

$$A^\mu = \varepsilon^\mu(q)\mathrm{e}^{-iq\cdot x} \tag{C.2.1}$$

where q is the four-momentum of the photon and $\varepsilon^\mu(p)$ describes the polarisation of the electromagnetic field. Substituting this into the equation of motion we get

$$\partial^2 A^\mu = -q^2\varepsilon^\mu(q)\mathrm{e}^{-iq\cdot x} = 0. \tag{C.2.2}$$

Hence the plane wave solutions must have $q^2 = 0$ for a free photon field.

A spin 1 boson, such as the photon, has three degrees of freedom, spins -1, 0, and $+1$. It's not immediately obvious how these correspond to the four degrees of freedom in defining ε^μ. The solution is that gauge freedom fixes one of the degrees of freedom. In the Lorenz gauge we must have $\partial_\mu A^\mu = 0$, and so

$$0 = \partial_\mu(\varepsilon^\mu \mathrm{e}^{-iq\cdot x}) = -iq_\mu\varepsilon^\mu\mathrm{e}^{-iq\cdot x}. \tag{C.2.3}$$

We must therefore have

$$q_\mu\varepsilon^\mu = 0, \tag{C.2.4}$$

fixing one degree of freedom in ε^μ.

There is still further freedom to make the gauge transformation

$$A_\mu \mapsto A_\mu - \partial_\mu\Lambda(x) \tag{C.2.5}$$

where Λ is some function satisfying $\partial^2\Lambda = 0$. Consider the gauge transformation defined by $\Lambda = -ia\mathrm{e}^{-iq\cdot x}$, which satisfies $\partial^2\Lambda = -q^2\Lambda = 0$, since $q^2 = 0$. Under this transformation A_μ becomes

$$\begin{aligned} A_\mu \mapsto A'_\mu &= A_\mu - \partial_\mu\Lambda && \text{(C.2.6)}\\ &= \varepsilon_\mu\mathrm{e}^{-iq\cdot x} + ia\partial_\mu\mathrm{e}^{-iq\cdot x} && \text{(C.2.7)}\\ &= \varepsilon_\mu\mathrm{e}^{-iq\cdot x} + ia(-iq_\mu)\mathrm{e}^{-iq\cdot x} && \text{(C.2.8)}\\ &= (\varepsilon_\mu + aq_\mu)\mathrm{e}^{-iq\cdot x}. && \text{(C.2.9)} \end{aligned}$$

We can then interpret this gauge freedom as the ability to make the transformation

$$\varepsilon_\mu \mapsto \varepsilon_\mu + aq_\mu \tag{C.2.10}$$

without changing the physics. The **Coulomb gauge** corresponds to the choice of a making the time component of the polarisation, ε_0, vanish. The Lorenz gauge condition, $\varepsilon_\mu q^\mu$, then becomes $\boldsymbol{\varepsilon}\cdot\boldsymbol{q} = 0$.

These requirements are satisfied by

$$\varepsilon^{(1)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \qquad \text{and} \qquad \varepsilon^{(2)} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}. \tag{C.2.11}$$

C.3 Polarisation of Massive Spin-1 Particles

A massless noninteracting spin 1 field has the Lagrangian[1]

$$\mathcal{L}_0 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{C.3.1}$$

Here $F^{\mu\nu}$ is the field-strength tensor, given by $F^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$, where we now use B instead of A because we no longer want to focus only on photons. For massive particles we add in a mass term[2]:

$$\mathcal{L}_m = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 B^\mu B_\mu. \tag{C.3.2}$$

The Euler–Lagrange equations then give[3]

$$(\partial^2 + m^2)B^\mu - \partial^\mu\partial_\nu B^\nu = 0. \tag{C.3.3}$$

[1] see *Classical Electrodynamics* for a derivation

[2] see *Quantum Field Theory* for more details

[3] see *Classical Electrodynamics* for a detailed calculation of $\partial F^{\mu\nu}F_{\mu\nu}/\partial(\partial_\mu B^\nu)$.

An alternative way to obtain this equation is to note that for a massless scalar particle the Klein–Gordon equation is $\partial^2\varphi = 0$, whereas for a massive scalar field it's $(\partial^2 + m^2)\varphi = 0$, suggesting that applying the prescription $\partial^2 \mapsto \partial^2 + m^2$ to a massless field equation gives the equivalent equation for a massive particle. Which we can then apply to the massless photon free equation

$$\partial^2 A^\mu - \partial^\mu\partial_\nu A^\nu = 0 \tag{C.3.4}$$

to get the result.

Taking the derivative of the resulting equation of motion we get

$$0 = (\partial^2+m^2)\partial_\mu B^\mu - \partial_\mu\partial^\mu\partial_\nu B^\nu = (\partial^2+m^2)\partial_\mu B^\mu - \partial^2\partial_\mu B^\mu = m^2\partial_\mu B^\mu = 0. \tag{C.3.5}$$

This implies that massive spin 1 particles automatically satisfy the Lorenz condition,

$$\partial_\mu B^\mu = 0. \tag{C.3.6}$$

Using this the equation of motion becomes

$$(\partial^2 + m^2)B^\mu = 0. \tag{C.3.7}$$

So we just get a Klein–Gordon equation for each component.

A massive particle with four-momentum q has $q^2 = m^3$ and so

$$\partial^2 e^{-iq\cdot x} = -q^2 e^{-q\cdot x} = -m^2 e^{-iq\cdot x} \tag{C.3.8}$$

implying plane wave solutions of the form

$$B^\mu = \varepsilon^\mu e^{-iq\cdot x}. \tag{C.3.9}$$

The Lorenz condition implies that

$$q_\mu \varepsilon^\mu = 0, \tag{C.3.10}$$

exactly the same as in the massless case.

There is no further gauge freedom, there isn't a massive equivalent of the Coulomb gauge, and so we have three different polarisations, we can take the two used for the photon and as the third

$$\varepsilon^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{C.3.11}$$

C.4 Polarisation Sums

A polarisation sum is a sum of the form

$$\sum_\lambda \varepsilon_\mu^{(\lambda)}(\varepsilon_\nu^{(\lambda)})^*. \tag{C.4.1}$$

C.4.1 Massive Gauge Bosons

We can interpret $\varepsilon_\mu^{(\lambda)}(\varepsilon_\nu^{(\lambda)})^*$ as the components of some two-index tensor, or a matrix. Rather than work with the polarisation vectors we've discussed, instead we use the left and right circular polarisations

$$\varepsilon^{(-)} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\\0\end{pmatrix}, \quad \text{and} \quad \varepsilon^{(+)} = -\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\\0\end{pmatrix}. \tag{C.4.2}$$

As the third polarisation state we then choose something orthogonal to both of these, so it must be of the form $(\alpha, 0, 0, \beta)^{\mathsf{T}}$. The relationship between α and β is fixed by the requirement that $q_\mu \varepsilon^\mu = 0$, so we must have $\alpha E - \beta p_z = 0$. Hence we choose the longitudinal polarisation to be

$$\varepsilon^{(\mathrm{L})} = \frac{1}{m}\begin{pmatrix}p_z\\0\\0\\E\end{pmatrix}, \tag{C.4.3}$$

where the normalisation is chosen such that in the rest frame we just have $(0,0,0,1)^{\mathsf{T}}$.

Using this our polarisation sum becomes

$$\sum_\lambda \varepsilon_\mu^{(\lambda)}(\varepsilon_\nu^{(\lambda)})^* = \varepsilon_\mu^{(+)}(\varepsilon_\nu^{(+)})^* + \varepsilon_\mu^{(-)}(\varepsilon_\nu^{(-)})^* + \varepsilon_\mu^{(\mathrm{L})}(\varepsilon_\nu^{(\mathrm{L})})^* \tag{C.4.4}$$

$$= \frac{1}{2}\begin{pmatrix}0&0&0&0\\0&1&-i&0\\0&-i&1&0\\0&0&0&0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0&0&0&0\\0&1&-i&0\\0&i&1&0\\0&0&0&0\end{pmatrix} \tag{C.4.5}$$

$$+ \frac{1}{m^2}\begin{pmatrix}p_z^2&0&0&Ep_z\\0&0&0&0\\0&0&0&0\\Ep_z&0&0&E^2\end{pmatrix} \tag{C.4.6}$$

$$= \begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix} + \frac{1}{m^2}\begin{pmatrix}(q^0)^2&0&0&q^0q^3\\0&0&0&0\\0&0&0&0\\q^0q^3&0&0&(q^3)^2\end{pmatrix}. \tag{C.4.7}$$

In the last step we consider that the particle is moving in the z-direction, by definition, and wrote $E^2 = m^2 + p_z^2$, so $p_z^2/m^2 = (E^2 - m^2)/m^2 = E^2 - 1$, and similarly $E^2/m^2 = (m^2 + p_z^2)/m^2 = 1 + p_z^2$, and then we wrote p_z and E in terms of components of the four-momentum, q. This generalises to a particle moving along an arbitrary direction:

$$\sum_\lambda \varepsilon_\mu^{(\lambda)}(\varepsilon_\nu^{(\lambda)})^* = -\eta_{\mu\nu} + \frac{q_\mu q_\nu}{m^2}. \tag{C.4.8}$$

C.4.2 Real Photons

For photons the gauge freedom makes matters slightly more complicated. For a real photon travelling in the z-direction we can use the polarisations $\varepsilon^{(1)} = (0,1,0,0)^{\mathsf{T}}$ and $\varepsilon^{(2)} = (0,0,1,0)^{\mathsf{T}}$, and these are the only polarisation states. Hence,

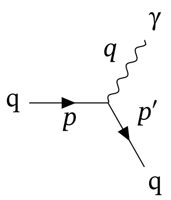
$$\sum_{\lambda} \varepsilon_{\mu}^{(\lambda)} (\varepsilon_{\nu}^{(\lambda)})^{*} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{C.4.9}$$

As before this generalises to a particle travelling in an arbitrary direction:

$$\sum_{T} \varepsilon_{i}^{(T)} (\varepsilon_{j}^{(T)})^{*} = \delta_{ij} - \frac{q_i q_j}{|\boldsymbol{q}|^2}. \tag{C.4.10}$$

Here the sum is over the transverse polarisation states, indexed by T.

If we want to sum over all possible components of the polarisation vector then we need to account for extra gauge freedom. To do so consider the process $\mathrm{q} \to \mathrm{q}\gamma$, such as occurs as part of the decay $\rho^0 \to \pi^0\gamma$. This is shown in the diagram



(C.4.11)

The amplitude for this vertex is

$$\mathcal{M} = Q_{\mathrm{q}} \bar{u}(p') \gamma^{\mu} u(p) \varepsilon_{\mu}^{(\lambda)}(q), \tag{C.4.12}$$

which follows in a similar way to Equation (8.3.5). We can write this as

$$\mathcal{M} = j^{\mu} \varepsilon_{\mu}^{(\lambda)}(q), \qquad \text{where} \qquad j^{\mu} = Q_{\mathrm{q}} \bar{u}(p') \gamma^{\mu} u(p). \tag{C.4.13}$$

Then, summing over all polarisations, we have

$$\sum_{\lambda} |\mathcal{M}|^2 = \sum_{\lambda} j^{\mu} (j^{\nu}) * \varepsilon_{\mu}^{(\lambda)} (\varepsilon_{\nu}^{(\lambda)})^{*}. \tag{C.4.14}$$

In the Coulomb gauge this takes the form

$$\sum_{T=1}^{2} |\mathcal{M}|^2 = j^{\mu} (j^{\nu})^{*} \sum_{T=1}^{2} \varepsilon_{\mu}^{(T)} (\varepsilon_{\nu}^{(T)})^{*}. \tag{C.4.15}$$

In the frame where the photon travels along the z-direction this becomes

$$\sum_{T=1}^{2} |\mathcal{M}|^2 = j_1 j_1^{*} + j_2 j_2^{*}. \tag{C.4.16}$$

We can write this as a sum over all four components by subtracting off the unwanted components:

$$\sum_{T=1}^{2} |\mathcal{M}|^2 = j_1 j_1^{*} + j_2 j_2^{*} = -\eta^{\mu\nu} j_{\mu} j_{\nu}^{*} + j_0 j_0^{*} - j_3 j_3^{*}. \tag{C.4.17}$$

We saw that the polarisation vectors ε_μ and $\varepsilon_\mu + aq^\mu$ describe the same physics, so the amplitude must be invariant under this, meaning we must have

$$\mathcal{M} = j^\mu \varepsilon_\mu^{(\lambda)}(q) = j^\mu \varepsilon_\mu^{(\lambda)}(q) + j^\mu q_\mu. \tag{C.4.18}$$

Hence, we must have

$$j_\mu q^\mu = 0. \tag{C.4.19}$$

For a photon with $q^\mu = (q, 0, 0, q)$ (so $q^2 = 0$) we have $qj_0 - qj_3 = 0$, and so the invariance condition gives $j_0 = j_3$. Hence,

$$\sum_{T=1}^{2} |\mathcal{M}|^2 = -\eta^{\mu\nu} j_\mu j_\nu. \tag{C.4.20}$$

Thus, the sum over polarisation states for a real photon gives

$$\sum_{T=1}^{2} \varepsilon_\mu^{(\lambda)} (\varepsilon_\nu^{(\lambda)})^* = -\eta_{\mu\nu}. \tag{C.4.21}$$

C.4.3 Virtual Photons

For off-shell photons $q^2 \neq 0$ and we cannot ignore the other two polarisation states. The amplitude for $e^-\tau^- \to e^-\tau^-$ scattering via a photon in a t-channel process is, as given in Equation (8.3.12),

$$\mathcal{M} \propto \sum_{\lambda=1}^{4} j_\mu^{(e^-)} j_\nu^{(\tau^-)} \frac{\varepsilon_\mu^{(\lambda)} (\varepsilon_\nu^{(\lambda)})^*}{q^2}. \tag{C.4.22}$$

We can treat the virtual photon as having effective mass $m^2 = q^2$, and so we can use the massive spin 1 result:

$$\sum_{\lambda=1}^{4} \varepsilon_\mu^{(\lambda)} (\varepsilon_\nu^{(\lambda)})^* = -\eta_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}. \tag{C.4.23}$$

Now, the t-channel process is the only one at this order in in QED. There is no s channel, as we'd need a boson with charge -2 to propagate between the two vertices. There is no u channel, since the two particles are distinct. This means that this term must be gauge invariant on its own. So, consider the gauge transformation $\varepsilon^\mu \mapsto \varepsilon^\mu + aq^\mu$ and $\varepsilon^\nu \mapsto \varepsilon^\nu + bq^\nu$, we can take different gauge transformations at different points in space time as long as the variation between a and b occurs linearly in spacetime, so vanishes when we take the second derivative. Hence,

$$\sum_{\lambda=1}^{4} j_\mu^{(e^-)} j_\nu^{(\tau^-)} \varepsilon_\mu^{(\lambda)} (\varepsilon_\nu^{(\lambda)})^* = \sum_{\lambda=1}^{4} j_\mu^{(e^-)} j_\nu^{(\tau^-)} (\varepsilon_\mu^{(\lambda)} + aq^\mu)((\varepsilon_\nu^{(\lambda)})^* + bq^\nu) \tag{C.4.24}$$

for all a and b. This means we have

$$j_\mu^{(e^-)} j_\nu^{(\tau^-)} q^\mu q^\nu = 0. \tag{C.4.25}$$

We can then conclude that the $q_\mu q_\nu / q^2$ term does not contribute to the amplitude. Hence, we have

$$\sum_{\lambda=1}^{4} \varepsilon_\mu^{(\lambda)} (\varepsilon_\nu^{(\lambda)})^* = -\eta^{\mu\nu}. \tag{C.4.26}$$

The Feynman rule for a photon propagator in a first order diagram is then

$$-i\frac{\eta_{\mu\nu}}{q^2}. \tag{C.4.27}$$

For higher order diagrams we cannot guarantee the gauge invariance of a single diagram's contribution to the amplitude and instead the photon propagator is

$$-\frac{i}{q^2}\left[\eta_{\mu\nu}+(1-\xi)\frac{q^\mu q^\nu}{q^2}\right] \tag{C.4.28}$$

where ξ is a gauge dependent parameter. Most calculations are performed in the Feynman gauge in which $\xi = 1$.

D

FeynCalc and FeynArt

FeynCalc and *FeynArt* are *Mathematica* packages for doing QFT computations. I don't know these packages very well but this appendix compiles some of the parts that I've found useful in this course.

To start the file use

```
$LoadAddOns = {"FeynArts"};
Get["FeynCalc`"]
```

D.1 Basics

A four-vector, such as p^μ, is represented in *FeynCalc* as one of the following

```
FourVector[p, μ]
FV[p, μ]
```

This gives p^μ, all indices are up in *FeynCalc*, which is fine as long as we keep it in mind when converting from *FeynCalc* results to normal notation. The product $p^\mu q_\mu$ is represented by one of the following

```
FV[p, μ] FV[q, μ] // Contract
ScalarProduct[p, q]
SP[p, q]
```

We can also use the shorthand SP[p] to get p^2. We can assign values to scalar products, for example, if we have p as the momentum of an on-shell particle of mass m we may wish to define

```
ScalarProduct[p, p] = m^2
```

Then evaluating SP[p] will give m^2.

The derivative is given using

```
FourDivergence[expr, FV[x, μ]]
```

to compute the derivative of expr with respect to x^μ. Similarly the d'Alembert operator is implemented as

```
FourLaplacian[expr, p, p]
```

to compute the d'Alembert acting on expr in momentum space. For example,

```
FourDivergence[Exp[−iSP[p,x]], FV[p,μ]]
FourLaplacian[Exp[−iSP[p,x]], x, x]
```

gives

$$-ip^{\mu}e^{-ip\cdot x}, \qquad \text{and} \qquad -p^2e^{-ip\cdot x} \tag{D.1.1}$$

respectively.

D.2 Gamma Matrices

The gamma matrices are given by

```
GA[μ]
```

Slash notation can be done using

```
GS[p]
```

to produce $\not{p}$, although this is then written as $\gamma \cdot p$. Note that since the gamma matrices don't commute when doing products with them we need to use . (Dot) as the product, instead of the default product (Times) which is assumed to commute.

The command DiracOrder places a product of gamma matrices in a canonical order, for example the following

```
GA[μ].GA[ν] // DiracOrder
GA[ν].GA[μ] // DiracOrder
```

gives

$$\gamma^{\mu}\gamma^{\nu}, \qquad \text{and} \qquad 2g^{\mu\nu} - \gamma^{\mu}\gamma^{\nu} \tag{D.2.1}$$

respectively, which are equivalent to the initial expressions and use the anticommutation relations to reorder products of gamma matrices.

Terms involving gamma matrices can be simplified using DiracSimplify, for example,

```
GA[μ].GA[μ] // DiracSimplify
GA[μ].GS[a].GA[μ] // DiracSimplify
GA[μ].GS[a].GS[b].GS[c].GS[d].GA[μ] // DiracSimplify
```

give

$$4, \qquad -2\not{a}, \qquad \text{and} \qquad 2(\not{d}\not{a}\not{b}\not{c} + \not{c}\not{b}\not{a}\not{d}) \tag{D.2.2}$$

respectively.

D.3 FeynArt

FeynArt is a package for creating Feynman diagrams. The first command we'll need is

```
InsertFields[CreateTopologies[...],
    {incoming particles} → {outgoing paritlces}
]
```

Here

```
CreateTopologies[l, i → o]
```

tells *FeynArt* to generate all diagrams with i incoming particles, o outgoing particles, and l loops. We then list the incoming and outgoing particles for `InsertFields`. A fermion is represented by `F` with some argument. The first argument of `F` is an integer from 1 to 4, these represent generic neutrinos, massive leptons, up-type quarks, and down-type quarks, in that order. The second argument is a list of integers. The simplest case being a list of one integer from 1 to 3, which tells us the generation. So, for example, an electron is `F[2, {1}]` and a charm quark is `F[3, {2}]`. The antiparticle is then given by `−F[...]`, so `−F[2, {1}]` is a positron and `F[1, {3}]` is an antitau neutrino.

There are two arguments to `InsertFields` that we'll use, one is `InsertionLevel → {Classes}`. This is related to how different levels of generality are implemented in *FeynArt*. The second is `Restrictions → QEDOnly`, which tells *FeynArt* to ignore any processes that aren't QED.

Finally, we need

```
CreateFeynAmp[...]
```

which takes the result of `InsertFields` and creates a list of corresponding Feynman amplitudes.

D.4 $e^-e^+ \to \mu^-\mu^+$

In Chapter 9 we computed the leading order QED contribution to the $e^-e^+ \to \mu^-\mu^+$ process. Here we do the same calculation using *FeynCalc* and *FeynArt*. First, define the interaction:

```
feynmanDiagram = InsertFields[
    CreateTopologies[0, 2 → 2],
    {F[2, {1}], −F[2, {1}]} → {F[2, {2}], −F[2, {2}]},
    InsertionLevel → {Classes},
    Restrictions → QEDOnly
]
```

We can print the diagram with the following command:

```
Paint[feynmanDiagram,
    ColumnsXRows → {1, 1}, Numbering → Simple,
    SheetHeader → None, ImageSize → {256, 256}
]
```

This produces the diagram shown in Figure D.1.

We can turn this into the amplitude as follows:

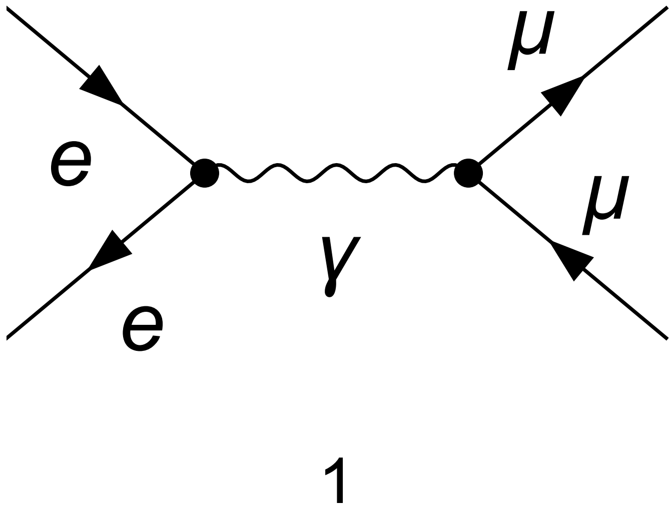


Figure D.1: The Feynman diagram generated by *FeynArt*.

```
amp[0] = FCFAConvert[CreateFeynAmp[diags],
    IncomingMomenta → {p1, p2},
    OutgoingMomenta → {p3, p4},
    UndoChiralSplittings → True,
    ChangeDimension → 4, List → False,
    SMP → True, Contract → True
]
```

here we use the *FeynArt* function CreateFeynAmp to create the Feynman amplitude and then the FeynCalc function FCFAConvert to turn the amplitude into a *FeynCalc* compatible result. This sets the momenta of the particles and does some simplifications.

Next we run FCClearScalarProducts[] which just clears any defined values of scalar products, and is good practice to run at the start of each new calculation. We can run the following commands:

```
MakeBoxes[p1, TraditionalForm] :=
    "\!\(\*SubscriptBox[\(p\),\(1\)]\)";
MakeBoxes[p2, TraditionalForm] :=
    "\!\(\*SubscriptBox[\(p\),\(2\)]\)";
MakeBoxes[p3, TraditionalForm] :=
    "\!\(\*SubscriptBox[\(p\),\(3\)]\)";
MakeBoxes[p4, TraditionalForm] :=
    "\!\(\*SubscriptBox[\(p\),\(4\)]\)";
```

which just tell *Mathematica* to pretty print p1 as p_1 and so on.

Then we can run

```
SetMandelstam[s, t, u, p1, p2, -p3, -p4,
    SMP["m_e"], SMP["m_e"], SMP["m_mu"], SMP["m_mu"]]
```

The command

```
SetMandelstam[s, t, u, p1, p2, p3, p4,
    m1, m2, m3, m4]
```

sets the Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, and $u = (p_1 + p_4)^2$. It also sets $p_i^2 = m_i^2$. Note that SMP["par"] displays a parameter to the standard model, such as the mass of an electron, in a standard way.

Next we compute the amplitude squared:

```
ampSquared[0] = Simplify[DiracSimplify[
    (FermionSpinSum[#1, ExtraFactor → 1/4] &)
    [FeynAmpDenominatorExplicit[amp[0]
    *ComplexConjugate[amp[0]]]]
]]
```

This squares the amplitude, computing $\mathcal{M}\mathcal{M}^*$, then sums over the helicities with FermionSpinSum, manually including the factor of 1/4 from averaging over initial helicity states. We then use *FeynCalc*'s DiracSimplify to simplify the spinor part and then *Mathematica*'s Simplify to simplify the normal algebra. The result is

$$\frac{8e^4[m_e^2(p_3\cdot p_4)+m_\mu^2(p_1\cdot p_2)+(p_1\cdot p_4)(p_2\cdot p3)+(p_1\cdot p_3)(p_2\cdot p_4)+2m_e^2m_\mu^2]}{[2(p_3\cdot p_4)+p_3^2+p_4^2]^2}.$$

Now we make the approximation that the particle masses are negligible, which we do through

```
ampSquaredMassless[0] = Simplify[
    (#1/.{SMP["m_e"] → 0, SMP["m_mu"] → 0} &)
    [ampSquared[0]]
]
```

giving the result

$$\frac{2e^4(t^2+u^2)}{s^2}. \tag{D.4.1}$$

Next we use the approximations

$$t \approx -\frac{s}{2}(1-\cos\vartheta), \qquad \text{and} \qquad u \approx -\frac{s}{2}(1+\cos\vartheta) \tag{D.4.2}$$

where ϑ is the scattering angle. This holds in the high energy regime. We also replace the electron charge, e, with the fine structure constant. To calculate the differential cross section we need two components, the prefactor and the bit that needs to be integrated to get the total cross section, we define these as

```
prefactor = 1 / (64 π^2 s);
integral = Factor[ampSquaredMassless[0] /.
    {t → (−s/2)(1 − Cos[ϑ]),
        u → (−s/2)(1 + Cos[ϑ]),
        SMP["e"]^4 → (4π SMP["alpha_fs"])^2}
]
```

This gives the result

$$16\pi^2\alpha^2(1+\cos^2\vartheta). \tag{D.4.3}$$

The differential cross section is then

```
differentialCrossSection = prefactor * integral
```

giving

$$\frac{\partial\sigma}{\partial\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\vartheta). \tag{D.4.4}$$

To calculate the total cross section we can do the φ integral by hand, giving a factor of 2π, and then we have

```
2π Integrate[
    differentialCrossSection * Sin[ϑ], {ϑ, 0, π}
]
```

which gives the result

$$\sigma = \frac{4\pi\alpha^2}{3s}. \tag{D.4.5}$$

This is exactly the result found in Chapter 9.

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Acronyms

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