Maths For Physics 1 Overview

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This is an overview of important topics I created for the *maths for physics 1* course from the University of Edinburgh as part of the first year of the theoretical physics degree. When I took this course in the 2018/19 academic year it was taught by Dr Kristel Torokoff[1]. These notes are based on the lectures delivered as part of this course, and the notes provided as part of this course. The content within is correct to the best of my knowledge but if you find a mistake or just disagree with something or think it could be improved please let me know.

These notes were produced using LaTeX[2].

This is version 1.0 of these notes, which is up to date as of 04/01/2021.

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[1]https://www.ph.ed.ac.uk/people/kristel-torokoff
[2]https://www.latex-project.org/

Algebra

- Power rules:
$$a^b a^c = a^{b+c} \text{ and } (a^b)^c = a^{bc}$$

- Log rules:
$$\ln(ab) = \ln a + \ln b \text{ and } \ln a^b = b \ln a$$

- Quadratics:
$$x^2 + ax + c = 0 \implies \left(x + \frac{a}{2}\right)^2 - \frac{a^2}{4} + c = 0$$
$$ax^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Factor theorem (FT) - If $P(x)$ has $(x - a)$ as a factor then $P(a) = 0$
- Fundamental theorem of algebra (FTA) - If $P(x)$ is a polynomial of n^{th} degree then $P(x)$ has n roots
- Partial fractions
$$\frac{A}{(x+a)(x+b)} = \frac{B}{x+a} + \frac{C}{x+b} \implies A = B(x+b) + C(x+a)$$
Equate coefficients or set $x = -a, -b$ and solve for B and C
- Sketching graphs:
  1. Lable axis
  2. Where does it cross the x axis? ($f(x) = 0$)
  3. Where does it cross the y axis? ($f(0)$)
  4. Where are the minima/maxima? ($f'(x) = 0$)
  5. What is the nature of these points? ($f''(x) < 0 \implies$ maxima, $f''(x) = 0 \implies$ point of inflection and $f''(x) > 0 \implies$ minima)
  6. Are there any vertical asymptotes? Is anything divided by a value that can be 0?
  7. Are there any horizontal asymptotes? What happens as $x \to \pm\infty$?
- Conic sections:
  1. Ellipse
  $$\left(\frac{x - x_0}{a}\right)^2 + \left(\frac{y - y_0}{b}\right)^2 = 1$$
  This is an ellipse centre (x_0, y_0) height $2b$ and width $2a$. If $a = b$ then it is a circle radius a
  2. Parabola
  $$ax^2 + bx + c = y \text{ or } ay^2 + by + c = x$$
  The former will give a north or south facing parabola and the latter will give a east or west facing parabola.
  3. Hyperbola
  $$\left(\frac{x - x_0}{a}\right)^2 - \left(\frac{y - y_0}{b}\right)^2 = 1 \text{ or } \left(\frac{y - y_0}{a}\right)^2 - \left(\frac{x - x_0}{b}\right)^2 = 1$$
  This will give a hyperbola centre (x_0, y_0) The former will give a east/west facing hyperbola and the latter will give a north/south facing hyperbola.

Trigonometry

- Must know sine and cosine of common angles:

Angle, ϑ (rad)	$\sin\vartheta$	$\cos\vartheta$	$\tan\vartheta$
0	0	1	0
π	0	-1	0
$\frac{\pi}{2}$	1	0	Undefined
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$

- Must know common trig identities:
  1. $\sin(a+b) = \sin a \cos b + \sin b \cos a$
  2. $\cos(a+b) = \cos a \cos b - \sin a \sin b$
  3. $\sin^2 x + \cos^2 x = 1$
  4. $\tan x = \frac{\sin x}{\cos x}$
- If $\sin(ax+b) = c$ where a, b and c are constants then $ax+b = \arcsin(c) + 2n\pi$ where $n \in \mathbb{Z}$. Remember that $\arcsin c$ has two principal values.
- If $a\cos\varphi + b\sin\varphi = A\sin(\varphi+\vartheta)$ or $A\cos(\varphi+\vartheta)$ then expand the RHS using addition formulae and solve for A and φ

Complex numbers

- $i^2 \triangleq -1$
- If z is a complex number then $z = x + iy = re^{i\vartheta} = r(\cos\vartheta + i\sin\vartheta)$ Where $r = |z|$ and $\vartheta = \arg(z)$
- $x = r\cos\vartheta,\ y = r\sin\vartheta,\ r = \sqrt{x^2+y^2},$ and $\vartheta = \arccos\frac{x}{r} = \arcsin\frac{y}{r}$
- The complex conjugate of z is defined as $\bar{z} \triangleq x - iy = re^{-i\vartheta}$
- De Moivre's theorem:
$$z^n = r^n e^{in\vartheta} = r^n(\cos n\vartheta + i\sin n\vartheta)$$
- To write the n^{th} power of a trig function as multiple angles expand $(\cos\vartheta + i\sin\vartheta)^n$ and take the real part for cosine, the imaginary part for sine and the real part over the imaginary part for tangent.
- To write multiple angles as powers you need to know:
$$\left(z+z^{-1}\right)^n = (2\cos\vartheta)^n,\ \left(z^n+z^{-n}\right) = 2\cos n\vartheta,\ \left(z-z^{-1}\right)^n = (2\sin\vartheta)^n \text{ and } \left(z^n-z^{-n}\right) = 2\sin n\vartheta$$
Then expand and simplify $(z+z^{-1})^n$ collect powers of equal size together and replace with trig functions

Power series expansions

- Must know standard series:
  1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \quad n \in \{0, \mathbb{N}\} \wedge x \in \mathbb{R}$
  2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \quad n \in \{0, \mathbb{N}\} \wedge x \in \mathbb{R}$

3. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \quad n \in \{0, \mathbb{N}\} \wedge x \in \mathbb{R}$
4. $(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots + \frac{\prod_{r=0}^{n}(p-r)}{n!}x^n + \cdots \quad n \in \{0, \mathbb{N}\} \wedge |x| < 1 \wedge x \in \mathbb{R}$
5. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots \quad n \in \mathbb{N} \wedge x \in \mathbb{R}$

- To find expansions of other functions there are two methods:
  1. Combine the above standard series taking care that x must be small.
  2. Use the general formula for $f(x)$ about x_0:
     $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots$
     $n \in \{0, \mathbb{N}\} \wedge x,\, x_0 \in \mathbb{R}$
- If x isn't small and takes some value $\approx x_0$ it is possible to use the standard series expansions by replacing x with $x_0 + x - x_0$ and then rearranging to get $c + f(x - x_0)$ where c is a constant and f is a function such that it is possible to use the standard series to find the expansion of it. Then find the series expansion of $f(x - x_0)$

Limits

- Iff $\lim\limits_{x \to a_+} f(x) = \lim\limits_{x \to a_-} f(x) \neq \pm\infty$ then the function f is continuous at point a
- If $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ is of an indeterminate form then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)} = \lim\limits_{x \to a} \dfrac{f''(x)}{g''(x)} = \lim\limits_{x \to a} \dfrac{f^{(n)}(x)}{g^{(n)}(x)}$

Calculus

- The limit definition for the derivative of $y = f(x)$ is:

$$\frac{\text{Change in y}}{\text{change in x}} = \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{df}{dx} = \frac{dy}{dx} = f'(x) = y'$$

- For the limit to exist it must be the same when approached from either side, ie:

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x) - f(x - \Delta x)}{\Delta x}$$

- Genral remarks on derivatives:
  - If $f'(x)$ exists in $[a, b] \implies f(x)$ is continuous in $[a, b]$
  - If $f(x)$ is continuous in $[a, b] \;\not\Longrightarrow\; f'(x)$ exists in $[a, b]$
  - $f'(x)$ represents rate of change
  - Higher order derivatives are represented as:

$$\frac{d^n y}{dx^n} = f^{(n)}(x) = \lim_{\Delta x \to 0} \frac{f^{(n-1)}(x + \Delta x) - f^{(n-1)}(x)}{\Delta x}$$

- Must know basic derivatives:

Function $f(x)$	Derivative $f'(x)$
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
e^x	e^x
$\ln x$	$\frac{1}{x}$

- Must know general properties of derivatives:
  - Linearity: $f(x) = g(x) + h(x) \implies f'(x) = g'(x) + h'(x)$
  - Constants: $f(x) = cg(x) \implies f'(x) = cg'(x)$
  - Product rule: $f(x) = g(x)h(x) \implies f'(x) = g'(x)h(x) + g(x)h'(x)$
  - Chain rule: $f(x) = f(g(x)) \implies f'(x) = \frac{df}{dx} = \frac{df(g(x))}{dx} = \frac{df(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx}$
  - Quotient rule: $f(x) = \frac{g(x)}{h(x)} \implies f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$
  - Reciprocal rule: $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$
- To differentiate an inverse function set it equal to something and then rearrange to get rid of inverses. Next differentiate and rearrange to have it in terms of the variable that you started with.
- To differentiate something of the form a^x rearrange it to $e^{x \ln a}$ and differentiate
- Curve sketching:
  1. Check for symmetry. Is it odd, even or asymmetric?
  2. Where does it cross the x axis? Where is $f(x) = 0$?
  3. Where does it cross the y axis? Where is $f(0)$?
  4. Check for vertical asymptotes. What happens as $x \to \pm\infty$?
  5. Check for horizontal asymptotes. What happens as the divisor $\to 0$?
  6. Find stationary points. Where does $f'(x) = 0$?
  7. What is the nature of stationary points? What is the value of $f''(x)$ at the stationary points?
- Fundamental theorem of calculus: If $f(x)$ is integrable over $[a.b]$ then:

  $$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

  This is a deffinite or Riemann integral.
- An indefinate integral has no limits and only the only constraint is that $\frac{dF}{dx} = f$ This means that $\int f(x)\,dx = F(x) + c$ where c is a constant of integration.
- General properties of integrals:
  - Linearity: $\int f(x) + g(x)\,dx \equiv \int f(x)\,dx + \int g(x)\,dx$
  - If c is a constant then $\int cf(x)\,dx \equiv c\int f(x)\,dx$
  - $\int_a^b f(x)\,dx \equiv -\int_b^a f(x)\,dx$
  - If $a < b < c$ then $\int_a^c f(x)\,dx \equiv \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$
  - If $f(x)$ is even about a and the interval is even about a then:

    $$\int_{a-b}^{a+b} f(x)\,dx \equiv 2\int_0^{a+b} f(x)\,dx \equiv 2\int_{a-b}^0 f(x)\,dx$$

  - If $f(x)$ is odd about a and the interval is even about a then:

    $$\int_{a-b}^{a+b} f(x)\,dx \equiv 0$$

- Must know basic integrals:

Function $f(x)$	Integral $\int f(x)\,dx$
x^n	$\frac{x^{n+1}}{n+1} + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
e^x	$e^x + c$
$\frac{1}{x}$	$\ln x + c$

- Methods of integration:
  1. Algebra - simplify to a form that can be integrated
  2. Substitution - Replace a function with a different variable. Remember to change limits and replace dx
  3. Integration by parts:
$$\int u(x)v'(x)\,dx \equiv u(x)v(x) - \int u'(x)v(x)\,dx$$