Methods of Theoretical Physics

Complex Analysis

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These are my notes for the *complex analysis* part of the *methods of theoretical physics* course from the University of Edinburgh as part of the third year of the theoretical physics degree. When I took this course in the 2020/21 academic year it was taught by Dr Miguel Martìnez-Canales[1]. These notes are based on the lectures delivered as part of this course and the notes provided as part of this course. The content within is correct to the best of my knowledge but if you find a mistake or just disagree with something or think it could be improved please let me know.

These notes were produced using LaTeX[2]. Graphs where plotted using Matplotlib[3], NumPy[4], and SciPy[5]. As well as Mathematica[6]. Diagrams were drawn with tikz[7].

This is version 1.2 of these notes, which is up to date as of 04/07/2021.

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[1] https://www.ph.ed.ac.uk/people/miguel-martinez-canales
[2] https://www.latex-project.org/
[3] https://matplotlib.org/
[4] https://numpy.org/
[5] https://scipy.org/scipylib/
[6] https://www.wolfram.com/mathematica/?source=nav
[7] https://www.ctan.org/pkg/pgf

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1 Complex Numbers

1.1 Complex Numbers Definition

The complex numbers, $\mathbb{C}$, are an algebraic extension to the real numbers, $\mathbb{R}$. That is the complex numbers are such that all complex numbers are the root of some complex polynomial with complex coefficients. Note that this is not true for real numbers, for example, -1 is not the root of some real polynomial with real coefficients.

Definition 1: Complex numbers

A **complex number**, z, is defined as an ordered pair of real numbers, $(x, y) \in \mathbb{R}^2$.

The set of all complex numbers is denoted $\mathbb{C}$. For all $x, y \in \mathbb{R}$ $z = (x, y) \in \mathbb{C}$. Two complex numbers, $z_1, z_2 \in \mathbb{C}$, such that $z_1 = (x_1, y_1)$ and $z_2 = (x_1, y_2)$, are equal if and only if $x_1 = x_2$ and $y_1 = y_2$. That is

$$z_1 = z_2 \iff [x_1 = x_2 \wedge y_1 = y_2].$$

If $z = (x, y) \in \mathbb{C}$ then we say that x is the **real part** of z and y is the **imaginary part** of z. We denote this

$$\operatorname{Re} z = \Re z = x, \qquad \text{and} \qquad \operatorname{Im} z = \Im z = y$$

respectively.

Definition 2: Addition and multiplication

Let $z_1, z_2 \in \mathbb{C}$ such that $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. Then we define **addition** as

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$$

and **multiplication** as

$$z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1).$$

Lemma 1: Addition and multiplication properties

Addition and multiplication in $\mathbb{C}$ are **associative**, that is

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3, \qquad \text{and} \qquad z_1(z_2 z_3) = (z_1 z_2) z_3$$

for all $z_1, z_2, z_3 \in \mathbb{C}$. Addition and multiplication in $\mathbb{C}$ are **commutative**, that is

$$z_1 + z_2 = z_2 + z_1, \qquad \text{and} \qquad z_1 z_2 = z_2 z_1$$

for all $z_1, z_2 \in \mathbb{C}$. Multiplication is **distributive** over addition, that is

$$z_1(z_1 + z_2) = z_1 z_2 + z_1 z_3.$$

Proof. Let $z_i \in \mathbb{C}$ be given by $z_i = (x_i, y_i)$ for arbitrary $x_i, y_i \in \mathbb{R}$. First we will show that addition is associative:

$$\begin{aligned}
z_1 + (z_2 + z_3) &= (x_1, y_1) + (x_2 + x_3, y_1 + y_3) \\
&= (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3)) \\
&= ((x_1 + x_2) + x_3, (y_1 + y_2) + y_3) \\
&= (x_1 + x_2, y_1 + y_2) + (x_3, y_3)
\end{aligned}$$

$$= (z_1 + z_2) + z_3$$

assuming associativity of addition in $\mathbb{R}$. Next we will show that addition is commutative.

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = z_2 + z_1$$

assuming commutativity of addition in $\mathbb{R}$.

Next we will show that multiplication is associative:

$$\begin{aligned}
z_1(z_2z_3) &= (x_1, y_1)(x_2x_3 - y_2y_3, x_2y_3 + x_3y_2)\\
&= (x_1(x_2x_3 - y_2y_3) - y_1(x_2y_3 + x_3y_2), x_1(x_2y_3 + x_3y_2) + y_1(x_2x_3 - y_2y_3))\\
&= ((x_1x_2 - y_1y_2)x_3 - (x_1y_2 + x_2y_1)y_3, (x_1x_2 - y_1y_2)y_3 + (x_1y_2 + x_2y_1)x_3)\\
&= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)(x_3, y_3)\\
&= (z_1z_2)z_3
\end{aligned}$$

assuming associativity and commutativity of multiplication in $\mathbb{R}$ and distributivity of multiplication over addition in $\mathbb{R}$. Next we will show that multiplication is commutative:

$$z_1z_2 = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) = (x_2x_1 - y_2y_1, x_2y_1 + x_1y_2) = z_2z_1$$

assuming commutativity of addition and multiplication in $\mathbb{R}$.

Finally we will show that multiplication distributes over addition:

$$\begin{aligned}
z_1(z_2 + z_3) &= (x_1, y_1)(x_2 + x_3, y_2 + y_3)\\
&= (x_1(x_2 + x_3) - y_1(y_2 + y_3), x_1(y_2 + y_3) + (x_2 + x_3)y_1)\\
&= (x_1x_2 - y_1y_2 + x_1x_3 - y_1y_3, x_1y_2 + x_2y_1 + x_1y_3 + x_3y_1)\\
&= z_1z_2 + z_1z_3
\end{aligned}$$

assuming distributivity of multiplication over addition in $\mathbb{R}$. $\square$

The element $0 = (0, 0) \in \mathbb{C}$ acts as an **additive identity** in that

$$0 + z = (x + 0, y + 0) = (x, y) = z = (0 + x, 0 + y) = (0, 0) + (x, y) = z + 0$$

for all $z \in \mathbb{C}$. Similarly $1 = (1, 0) \in \mathbb{C}$ acts as a **multiplicative identity** in that

$$1z = (1, 0)(x, y) = (1x - 0y, 1y + 0x) = (x, y) = z = (x1 - y0, y0 + x1) = z1$$

for all $z \in \mathbb{C}$.

For $z = (x, y) \in \mathbb{C}$ the element $-z = (-x, -y)$ acts as an **additive inverse** in that

$$z + (-z) = (x + (-x), y + (-y)) = (0, 0).$$

Similarly for $z \in \mathbb{C} \setminus \{0\}$ the element

$$z^{-1} = \frac{1}{z} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right)$$

acts as a **multiplicative inverse** in that

$$\begin{aligned}
zz^{-1} &= (x, y)\left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right)\\
&= \left(\frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}, \frac{xy}{x^2 + y^2} - \frac{xy}{x^2 + y^2}\right)\\
&= (1, 0).
\end{aligned}$$

This allows us to define subtraction and division as follows

Definition 3: Subtraction and division

Let $z_1, z_2 \in \mathbb{C}$ such that $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. Then **subtraction** is defined as

$$z_1 - z_2 = z_1 + (-z_2) = (x_1, y_1) - (x_2, y_2) = (x_1, y_1) + (-x_2, y_2) = (x_1 - x_2, y_1 - y_2)$$

and division is defined for $z_2 \neq 0$ as

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = (x_1, y_1)\left(\frac{x_2}{x_2^2 + y_2^2}, \frac{-y_2}{x_2^2 + y_2^2}\right) = \left(\frac{x_1 x_2}{x_2^2 + y_2^2} + \frac{y_1 y_2}{x_2^2 + y_2^2}, \frac{-x_1 y_2}{x_2^2 + y_2^2} + \frac{x_2 y_1}{x_2^2 + y_2^2}\right).$$

With addition, multiplication, subtraction, and division defined like this the properties listed above make $\mathbb{C}$ a field. What this means informally is that although in some ways $\mathbb{C}$ is like $\mathbb{R}^2$ when it comes to operations they behave very similarly to operations in $\mathbb{R}$.

Note that with these definitions we have

$$(0, 1)^2 = (-1, 0).$$

We give the element $(0, 1)$ the special name i which we define to be such that $i^2 = -1$. We can then factor any generic complex number, $z = (x, y)$ as

$$z = (x, y) = (x, 0) + (0, 1)(y, 0) = x + iy.$$

From now on we will use the notation $z = x + iy$ to mean $z = (x, y)$.

1.2 Argand Diagrams

Since $\mathbb{C}$ is in some ways $\mathbb{R}^2$ it is natural to represent $\mathbb{C}$ geometrically with a plane. Diagrams that do this are called **Argand diagrams**. For example, see figure 1.1. In this Argand diagram we see the point

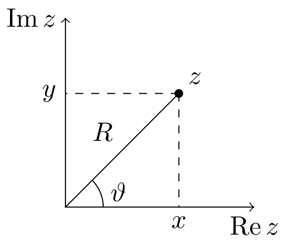


Figure 1.1: An Argand diagram showing the complex number $z = x + iy = Re^{i\vartheta}$.

$z \in \mathbb{C}$. This also gives us another natural way to represent z. A simple bit of geometry shows us that

$$z = x + iy = R(\cos\vartheta + i\sin\vartheta)$$

and that

$$R = \sqrt{x^2 + y^2}, \qquad \text{and} \qquad \vartheta = \arctan\left(\frac{y}{x}\right).$$

Definition 4: Modulus and argument

We call $R = \sqrt{x^2 + yr}$ the **modulus** of z, denoted $|z|$. We call ϑ *an* **argument** of z, denoted $\arg z$. For each $z \in \mathbb{C}$ we have an infinite number of different values that ϑ can take. In general if ϑ is an argument of z then so is $\vartheta + 2n\pi$ for any $n \in \mathbb{Z}$. The set of all ϑ which satisfy $z = |z|(\cos\vartheta + i\sin\vartheta)$ for some specific point z is called *the* **argument** of z, denoted $\operatorname{Arg} z$:

$$\operatorname{Arg} z = \{\vartheta \mid z = |z|(\cos\vartheta + i\sin\vartheta)\}.$$

We often choose to restrict ϑ to be in $(-\pi, \pi]$ in which case we call ϑ the **principle argument** of z.

One fact that will become important later is that if we traverse around a closed curve in $\mathbb{C}$ and zero is inside that curve then after we go around the entire curve we will come back to the same spot but the argument will be 2π greater than it was before (if we traverse in an anticlockwise direction).

1.3 Euler's Formula

Definition 5: Complex exponential

The **complex exponential** of a purely imaginary number, $i\vartheta$, for some $\vartheta \in \mathbb{R}$ is defined as

$$z = e^{i\vartheta} = \cos\vartheta + i\sin\vartheta.$$

This is called **Euler's formula**. In addition to this we also require that

$$e^{x+iy} = e^x e^{iy}$$

for all $x, y \in \mathbb{R}$.

The second requirement means that many of the expected properties of real exponentials carry over to the complex exponential. This is an extension of the real exponential in the sense that the real and complex exponential both give the same result when we exponentiate a real number. For this reason we don't usually make a distinction between the real and complex exponential.

We can use this to write z as

$$z = Re^{i\vartheta} = R(\cos\vartheta + i\sin\vartheta)$$

where $R = |z|$ and $\vartheta = \arg z$. While the complex exponential is defined in this way we can motivate the definition. We first assume that differentiation in $\mathbb{C}$ is the same as it is in $\mathbb{R}$ and we see that

$$\frac{\partial}{\partial\vartheta}[\cos\vartheta + i\sin\vartheta] = -\sin\vartheta + i\cos\vartheta$$

and also

$$\frac{\partial}{\partial\vartheta}e^{i\vartheta} = ie^{i\vartheta} = i(\cos\vartheta + i\sin\vartheta) = i\cos\vartheta - \sin\vartheta.$$

Both $e^{i\vartheta}$ and $\cos\vartheta + i\sin\vartheta$ have the same derivative and therefore it is reasonable to conclude that they are the same.

Another way to motivate this definition is by extending the Taylor series to $\mathbb{C}$ and noting that

$$\begin{aligned}\cos\vartheta + i\sin\vartheta &= \left[1 - \frac{\vartheta^2}{2!} + \frac{\vartheta^4}{4!} + \mathcal{O}(\vartheta^6)\right] + i\left[\vartheta - \frac{\vartheta^3}{3!} + \frac{\vartheta^5}{5!} + \mathcal{O}\big((\vartheta^7)\big)\right] \\ &= 1 + i\vartheta + \frac{(i\vartheta)^2}{2!} + \frac{(i\vartheta)^3}{3!} + \frac{(i\vartheta)^4}{4!} + \frac{(i\vartheta)^5}{5!} + \mathcal{O}(\vartheta^6) \\ &= e^{i\vartheta}.\end{aligned}$$

Some useful values to remember are

$$e^0 = e^{2\pi i} = 1, \qquad e^{-\pi} = -1, \qquad e^{\pm\pi/2} = \pm i, \qquad e^{\pm i\pi/4} = \frac{\sqrt{2}}{2}(1 \pm i).$$

Also note that

$$e^{i\vartheta} = e^{i\vartheta + 2n\pi i} \qquad \forall\vartheta \in \mathbb{R}, \forall n \in \mathbb{Z}.$$

Theorem 1: De Moivre's Formula

Let $z \in \mathbb{C}$ such that $|z| = 1$, then

$$z^n = (\cos\vartheta + i\sin\vartheta)^n = \cos(n\vartheta) + i\sin(n\vartheta) \qquad \forall n \in \mathbb{Z}$$

where $\vartheta = \arg z$.

Proof. We will prove this by induction for all natural numbers and later extend the result to negative numbers. The base case is $n = 0$ in which case

$$z^0 = 1 = \cos(0\vartheta) + i\sin(0\vartheta).$$

Suppose that this formula holds for some $k \in \mathbb{N}$. That is

$$z^k = (\cos\vartheta + i\sin\vartheta)^k = \cos(k\vartheta) + i\sin(k\vartheta).$$

Then

$$\begin{aligned}
z^{k+1} &= (\cos\vartheta + i\sin\vartheta)^{k+1} \\
&= (\cos\vartheta + i\sin\vartheta)^k(\cos\vartheta + i\sin\vartheta) \\
&= (\cos(k\vartheta) + i\sin(k(\vartheta))(\cos\vartheta + i\sin\vartheta) \\
&= \cos(k\vartheta)\cos(\vartheta) - \sin(k\vartheta)\sin(\vartheta) + i(\cos(k\vartheta)\sin(\vartheta) + \sin(k\vartheta)\cos(\vartheta)) \\
&= \cos((k+1)\vartheta) + i\sin((k+1)\vartheta)
\end{aligned}$$

using

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B), \qquad \text{and} \qquad \sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A).$$

Thus by mathematical induction the formula holds for all $n \in \mathbb{N}$. Now consider some specific $k \in \mathbb{N}$.

$$\begin{aligned}
z^{-k} &= (\cos\vartheta + i\sin\vartheta)^{-k} \\
&= \left[(\cos\vartheta + i\sin\vartheta)^k\right]^{-1} \\
&= (\cos(k\vartheta) + i\sin(k\vartheta))^{-1} \\
&= \frac{\cos(k\vartheta) - i\sin(k\vartheta)}{\cos^2(k\vartheta) + \sin^2(k\vartheta)} \\
&= \cos(k\vartheta) - i\sin(k\vartheta) \\
&= \cos(-k\vartheta) + i\sin(-k\vartheta)
\end{aligned}$$

using the trig identity

$$\cos^2(A) + \sin^2(A) = 1$$

and the fact that cos and sin are even and odd functions respectively so

$$\cos(-A) = \cos(A), \qquad \text{and} \qquad \sin(-A) = -\sin(A).$$

Hence the formula holds for all $n \in \mathbb{Z}$. □

1.4 Complex Conjugate

Definition 6: Complex conjugate

The complex conjugate of $z = x + iy = Re^{i\vartheta} \in \mathbb{C}$ is

$$z^* = \bar{z} = x - iy = Re^{-i\vartheta}.$$

Lemma 2: Conjugate properties

The following hold for all $z, z_1, z_2 \in \mathbb{C}$:

1. $(z_1 + z_2)^* = z_1^* + z_2^*$
2. $(z_1 z_2)^* = z_1^* z_2^*$
3. $(z^*)^* = z$
4. $\operatorname{Re} z = \frac{1}{2}(z + z^*)$
5. $\operatorname{Im} z = \frac{1}{2i}(z - z^*)$
6. $|z| = |z^*|$
7. $\arg z = -\arg(z^*)$
8. $|z|^2 = zz^* = x^2 + y^2$

Proof. We will prove these in the order provided:

1. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then
$$(z_1 + z_2)^* = (x_1 + iy_1 + x_2 + iy_2)^* = ((x_1 + x_2) + i(y_1 + y_2))^*$$
$$= (x_1 + x_2) - i(y_1 + y_2) = x_1 - iy_1 + x_2 - iy_2 = z_1^* + z_2^*.$$

2. Let $z_1 = R_1 e^{i\vartheta_1}$ and $z_2 = R_2 e^{i\vartheta_2}$. Then
$$(z_1 z_2)^* = (R_1 e^{i\vartheta_1} R_2 e^{i\vartheta_2})^* = (R_1 R_2 e^{i(\vartheta_1+\vartheta_2)}) = R_1 R_2 e^{-i(\vartheta_1+\vartheta_2)} = R_1 e^{-i\vartheta_1} R_2 e^{-i\vartheta_2} = z_1^* z_2^*.$$

3. Let $z = x + iy$. Then
$$(z^*)^* = [(x + iy)^*]^* = [x - iy]^* = x + iy = z.$$

4. Let $z = x + iy$. Then
$$\frac{1}{2}(z + z^*) = \frac{1}{2}[x + iy + (x + iy)^*] = \frac{1}{2}(x + iy + x - iy) = \frac{1}{2}2x = x = \operatorname{Re} z.$$

5. Let $z = x + iy$. Then
$$\frac{1}{2i}(z - z^*) = \frac{1}{2i}[x + iy - (x + iy)^*] = \frac{1}{2i}(x + iy - (x - iy)) = \frac{1}{2i}(x + iy - x + iy) = \frac{1}{2i}2iy = y = \operatorname{Im} z.$$

6. Let $z = Re^{i\vartheta}$. Then
$$|z^*| = \left|(Re^{-\vartheta})^*\right| = \left|Re^{-i\vartheta}\right| = R = |z|.$$

7. Let $z = Re^{i\vartheta}$. Then
$$-\arg(z^*) = -\arg([Re^{i\vartheta}]^*) = -\arg(Re^{-i\vartheta}) = -(-\vartheta) = \vartheta = \arg z.$$

8. Let $z = Re^{i\vartheta} = x + iy$. Then
$$zz^* = Re^{i\vartheta}(Re^{i\vartheta})^* = Re^{i\vartheta}Re^{-i\vartheta} = R^2 e^{i(\vartheta-\vartheta)} = R^2 e^0 = R^2 = |z|^2.$$

   Also
$$zz^* = (x + iy)(x + iy)^* = (x + iy)(x - iy) = x^2 - ixy + ixy - i^2 y = x^2 + y^2.$$

$\square$

Using the complex conjugate we can see how the multiplicative inverse was found:
$$\frac{1}{z} = \frac{z^*}{zz^*} = \frac{x - iy}{x^2 + y^2}.$$

1.5 Inequalities

When we move from $\mathbb{R}$ to $\mathbb{C}$ we lose the natural ordering that we have for real numbers. By this we mean that phrases like $z < w$ are not defined for $z, w \in \mathbb{C}$. To compare two complex numbers we instead compare their moduli and arguments.

Lemma 3: Modulus inequalities

The following inequalities hold for all $z, w \in \mathbb{C}$:
1. $\operatorname{Re} z \le |z|$ and $\operatorname{Im} z \le |z|$.
2. $|z + w| \le |z| + |w|$, this is known as the triangle inequality.
3. $|z - w| \ge |z| - |w|$.

Proof. First consider

$$|zw| = \sqrt{|zw|^2} = \sqrt{(zw)(zw)^*} = \sqrt{zwz^*w^*} = \sqrt{zz^*ww^*} = \sqrt{|z|^2|w|^2} = |z||w|.$$

Let $z = x + iy$, then

$$x^2 + y^2 = |z|^2 \implies x^2 = |z|^2 - y^2 \le |z|^2 \implies x \le |z|.$$

Similarly

$$x^2 + y^2 = |z|^2 \implies y^2 = |z|^2 - x^2 \le |z|^2 \implies y \le |z|.$$

This proves the first statement.

For the triangle inequality consider

$$\begin{aligned} |z+w|^2 &= (z+w)(z+w)^* \\ &= (z+w)(z^* + w^*) \\ &= zz^* + zw^* + z^*w + ww^* \\ &= |z|^2 + |w|^2 + zw^* + z^*w. \end{aligned}$$

Notice that

$$\mathrm{Re}(zw^*) = \frac{1}{2}(zw^* + (zw^*)^*) = \frac{1}{2}(zw^* + z^*w) \implies zw^* + z^*w = 2\,\mathrm{Re}(zw^*).$$

Hence

$$|z+w|^2 = |z|^2 + |w|^2 + 2\,\mathrm{Re}(zw^*).$$

We then use the first point to conclude that $2\,\mathrm{Re}(zw^*) \le 2|zw^*| = 2|z||w|$ and so

$$|z+w|^2 \le |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2 \implies |z+w| \le |z| + |w|.$$

Finally note that

$$|z| = |z - w + w| \le |z - w| + |w|$$

using the triangle inequality on the two complex number $z - w$ and w. Rearranging this we get

$$|z| - |w| \le |z - w|.$$

□

The triangle inequality is called the triangle inequality for a geometrical reason. If we consider a triangle in the complex plane then we can think of each side as a complex number. The triangle inequality then states that the longest side is shorter than, or the same length as, the sum of the two shorter sides. Specifically equality holds when the two sides are co-linear which corresponds to both complex numbers having the same argument.

The triangle inequality generalises to the addition of n complex numbers as

$$\left|\sum_{i=1}^{n} z_i\right| \le \sum_{i=1}^{n} |z_i|.$$

Note that these properties make $d\colon \mathbb{C}^2 \to [0, \infty)$ defined by $d(z, w) = |z - w|$ a metric which makes $(\mathbb{C}, d)$ a metric space.

1.6 Topology

We will regularly deal with subsets of $\mathbb{C}$. It will be useful to give some of them special names based on various properties. We start with some discs which are some combination of circles in the complex plane and the points inside them.

Definition 7: Open disc

The **open disc** centred on $a \in \mathbb{C}$ with radius $r \in \mathbb{R}$, $r > 0$, is

$$D(a; r) = \{z \in \mathbb{C} \mid |z - a| < r\}.$$

That is all the points contained in the circle in the complex plane that is centred on a and has a radius of r but *not* the circle itself. This is the generalisation of an open interval, (a, b), from the real line.

Definition 8: Closed disc

The **closed disc** centred on $a \in \mathbb{C}$ with radius $r \in \mathbb{R}$, $r > 0$, is

$$\bar{D}(a; r) = \{z \in \mathbb{C} \mid |z - a| \le r\}.$$

That is all the points contained in the circle in the complex plane that is centred on a and has a radius of r *and* the circle itself. This is the generalisation of a closed interval, $[a, b]$, from the real line.

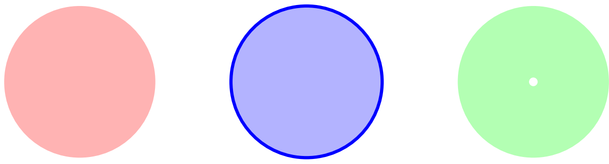
Definition 9: Punctured disc

The **punctured disc** centred on $a \in \mathbb{C}$ with radius $r \in \mathbb{R}$, $r > 0$, is

$$D'(a; r) = \{z \in \mathbb{C} \mid 0 < |z - a| < r\} = D(a; r) \setminus \{a\}.$$

That is the open disc centred on a with radius r with the centre of the disc removed.

See figure 1.2.



Open Disc Closed Disc Punctured Disc

Figure 1.2: An open, closed, and punctured disc.

We now define some properties that a set can have:

Definition 10: Open set

A set, $S \subseteq \mathbb{C}$, is **open** if for all $z \in S$ there exists $r > 0$ such that $D(z; r) \subseteq S$.

We call $D(z; r)$ for an arbitrarily small (but non-zero) r the **neighbourhood** of z. What the definition above says is that S is open if for all points we can find sufficiently small r such that the neighbourhood of any point, $z \in S$, is entirely inside of S. This is a useful property as it allows us wiggle room as whatever point we pick in the set we can always get closer to the edge. Some examples of open sets are the open disc, $D(a; r)$, the complex numbers, $\mathbb{C}$, and (vacuously) the empty set, $\emptyset$.

Definition 11: Closed set

A set, $S \subseteq \mathbb{C}$, is **closed** if its **complement**, $\mathbb{C} \setminus S = S^{\complement}$, is open.

Intuitively a closed set has an edge that we can reach. Examples of closed sets includes the closed disc, $\bar{D}(a; r)$, the empty set as $\emptyset^{\complement} = \mathbb{C} \setminus \emptyset = \mathbb{C}$ is open, and the complex numbers as $\mathbb{C}^{\complement} = \mathbb{C} \setminus \mathbb{C} = \emptyset$ is open. Notice how $\mathbb{C}$ and $\emptyset$ are both open and closed. In fact being open and closed are not mutually exclusive properties. Sometimes sets that are both open and closed are called **clopen** sets.

Definition 12: Isolated point

$z \in S \subseteq \mathbb{C}$ is an **isolated point** if the neighbourhood of z contains no points in S apart from z. That is there exists $r > 0$ such that $S \cap D'(z; r) = \emptyset$.

Intuitively an isolated point is not 'touching' any other points in S. For example consider the set $S = D(0; 1) \cup \{2\}$ has one isolated point, 2, as $S \cap D'(2; 0.5) = \emptyset$, see figure 1.3.

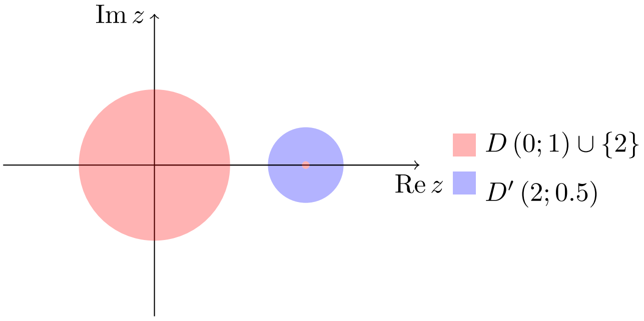


Figure 1.3: The set $S = D(0; 1) \cup \{2\}$ has 2 as an isolated point.

Definition 13: Limit point

$z \in \mathbb{C}$ is a **limit point** if for all $r > 0$ the punctured disc, $D'(a; r)$ has points both in S and in $\mathbb{C} \setminus S$, i.e. $D'(a; r) \cap S \neq \emptyset \neq D'(a; r) \cap S^{\complement}$.

Intuitively we can think of the limit points of a set as being the 'boundary' of the set as a small step in one direction takes us into the set and a small set in another direction takes us out of the set. Notice that the limit points of a set needn't be in the set. For example all z with $|z| = 1$ are limit points of $D(0; 1)$ despite the fact that $z \notin D(0; 1)$. The fact that the definition specifies a punctured disc ensures that isolated points are not limit points. See figure 1.4.

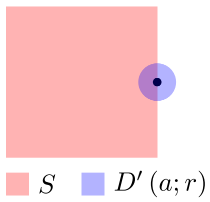


Figure 1.4: A limit point, a, of a set, S. Notice that no matter how small we make r the punctured disc $D'(a; r)$ will always overlap both S and $S^{\complement}$.

Definition 14: Connected set

A set S is **connected** if for all $z, w \in S$ there exists a continuous path from z to w.

Intuitively a set is connected if we can join any two points without a break. See figure 1.5. For example all discs form connected sets, as does the union of overlapping discs, e.g. $D(0; 1) \cup D(1 + i; 2)$ is a connected set. On the other hand if the discs don't overlap then they aren't a connected set, for example $D(0; 1) \cup D(1 + i; 0.1)$ is not a connected set.

Definition 15: Region

A **region**, R, is an open, connected set.

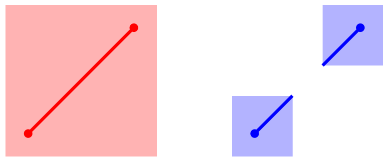


Figure 1.5: A connected set and an disconnected set. Both show an example of two points in the set which can be joined continuously in the connected case and not in the disconnected case.

Part II

Complex Functions

2 Complex Functions and Derivatives

2.1 Complex Functions

Definition 16: Complex function

A **complex function**, defined on a domain $S \subseteq \mathbb{C}$, is a mapping, $w\colon S \to \mathbb{C}$, which assigns to each $z \in S$ a value $w(z) \in \mathbb{C}$.

Importantly a function, w, assigns to each z in its domain precisely one value. Sometimes it is necessary to restrict the codomain of w for this to be the case. As we did with complex numbers we can split complex functions into real and imaginary parts. Let

$$S' = \{(x, y) \in \mathbb{R}^2 \mid x + iy \in S\} \subseteq \mathbb{R}^2$$

then we can define two functions, $u, v\colon S' \to \mathbb{R}$ such that

$$w(z) = w(x + iy) = u(x, y) + iv(x, y) = \operatorname{Re} w(z) + i \operatorname{Im} w(z).$$

Consider for example the function

$$\begin{aligned} w\colon \mathbb{C} &\to \mathbb{C} \\ z &\mapsto z^2. \end{aligned}$$

We can expand this in terms of x and y as

$$w(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2xyi.$$

We see that if we define $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$ then w, u, and v will have the desired relationship.

Another example that we may consider is

$$\begin{aligned} w\colon \mathbb{C} &\to \mathbb{C} \\ z &\mapsto e^z. \end{aligned}$$

Expanding this in terms of x and y we have

$$\begin{aligned} w(z) &= e^z \\ &= e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x(\cos y + i \sin y). \end{aligned}$$

If we define $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$ then again we will have the desired relationship between w, u, and v. Notice that u and v are periodic in y with period 2π, which means w is periodic in z with period $2\pi i$. This corresponds to the argument moving once around the unit circle. This is fine for this definition but becomes an issue if we want to invert w.

Notation

In this course we will denote the natural log, that is the inverse of the exponential, by log.

We define

$$\log z = \log|z| + i\vartheta = \frac{1}{2}\log\left(x^2 + y^2\right) + i \arctan\left(\frac{y}{x}\right).$$

Notice that since $\arctan\colon \mathbb{R} \to [-\pi/2, \pi/2]$ has a restricted codomain the value of ϑ above is also restricted to be in $[-\pi/2, \pi/2]$ which guarantees that if $w(z) = \log z$ then w is single valued for all values in its domain.

2.2 Limits and Continuity

Definition 17: Limit

Let $w \colon S \subseteq \mathbb{C} \to \mathbb{C}$ be a complex function. Then for $z_0 \in S$ we define

$$\lim_{z \to z_0} w(z) = w_0$$

for $z \in S$, to be the **limit** if for $\varepsilon > 0$

$$0 < |z - z_0| < \delta \implies |w(z) - w_0| < \varepsilon.$$

Intuitively what this definition means is that by making z and z_0 sufficiently close we can make $w(z)$ and w_0 arbitrarily close. Note that we use a punctured disc around z_0 because the value of $w(z_0)$ is not important to the definition of the limit, i.e. it needn't be the case that $w(z_0) = w_0$.

Definition 18: Continuity

Let $w \colon S \subseteq \mathbb{C} \to \mathbb{C}$ be a complex function. Then for $z_0 \in S$ we say that w is **continuous** at z_0 if

1. w is defined at z_0 (which we already know it is as $z_0 \in S$)
2. The limit,
$$\lim_{z \to z_0} w(z)$$
exists.
3. The limit and the function evaluated at z_0 match:
$$\lim_{z \to z_0} w(z) = w(z_0).$$

We say that w is continuous if it is continuous at all $z_0 \in S$.

Viewing the limit as approaching a point and the evaluation, $w(z_0)$ as the evaluation at that point intuitively a function is continuous at z_0 if there isn't a sudden jump when we reach the point.

2.2.1 Infinity

We often see limits of the form

$$\lim_{x \to \infty} f(x)$$

in real analysis. The natural question is is there an equivalent in complex analysis. In real analysis there are two disjoint points, $\pm\infty$, that get the title 'infinity'. These are not points in $\mathbb{R}$ rather they are things that we consider values to tend to. These two points can be thought of as the 'ends' of the real numbers in the sense that $\mathbb{R} = (-\infty, \infty)$. When we expand to the complex numbers we don't have a line with ends any more we have a plane.

Definition 19: Extended complex plane

We define the **extended complex plane**, $\mathbb{C}^*$, to be the complex numbers and a symbol representing infinity, that is $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$.

Infinity in complex analysis is best viewed using the Riemann sphere.

Definition 20: Riemann sphere

Let S^2 be the unit sphere,

$$S^2 = \{(\xi, \eta, \zeta) \in \mathbb{R}^3 \mid \xi^2 + \eta^2 + \zeta^2 = 1\}.$$

The **Riemann sphere**, $P \colon \mathbb{C}^* \to S^2$, then defines a one to one mapping of the complex plane. This mapping is defined as follows. First embed $\mathbb{C}$ in the (ξ, η)-plane by associating $z = x + iy$ with $(x, y, 0)$. Next define $P(\infty) = (0, 0, 1)$. Then define $P(z) = (\xi, \eta, \zeta)$ as the point on the sphere other than $(0, 0, 1)$ which intersects the line from $(0, 0, 1)$ to $(x, y, 0)$. It can be shown then that

$$(x, y) = \left(\frac{\xi}{1-\zeta}, \frac{\eta}{1-\zeta}\right)$$

and

$$(\xi, \eta, \zeta) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right).$$

Important points to consider are that $P(z = 0) = (0, 0, -1)$ and $\lim_{|x|\to\infty} P(x + iy) = \lim_{|y|\to\infty} P(x + iy) = (0, 0, 1)$. This justifies identifying $P(\infty) = (0, 0, 1)$. Intuitively we can view the Riemann sphere as wrapping the complex plane around a unit sphere so that all of the points we would consider to be 'at infinity' meet at the top of the sphere.

2.3 Derivatives

Definition 21: Derivative

Let $w\colon S \subseteq \mathbb{C} \to \mathbb{C}$ be a complex function. Then for $z_0 \in \mathbb{C}$ the **derivative** of the w at z_0, denoted $w'(z_0)$, is defined as

$$w'(z_0) = \lim_{z\to z_0} \frac{w(z) - w(z_0)}{z - z_0} = \lim_{\Delta z\to 0} \frac{w(z_0 + \Delta z) - w(z_0)}{\Delta z},$$

assuming that this limit exists and is independent of the direction of approach to z_0. When this is the case we say that w is **differentiable** at z_0.

This doesn't, at first, appear to be that different to the derivative in real analysis. This means that derivatives of functions of z will appear similar to how they appear for the equivalent real function.

Example 2.1. *Find the derivative of the complex function w defined by $w(z) = z^2$.*

$$\begin{aligned} w'(z) &= \lim_{\Delta z\to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} \\ &= \lim_{\Delta z\to 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z} \\ &= \lim_{\Delta z\to 0} \frac{2z\Delta z + (\Delta z)^2}{\Delta z} \\ &= \lim_{\Delta z\to 0} (2z + \Delta z) \\ &= 2z. \end{aligned}$$

Which is exactly what we would expect if $w(z) = z^2$ was a real function.

However the extra 'room' that we have in the complex numbers which allows us to approach a point not just from above and below but anywhere else in the plane means that many seemingly simple functions don't have derivatives.

Example 2.2. *Show that the complex function w, defined by $w(z) = z^*$, is nowhere differentiable.*

First we consider the derivative at some $z \in \mathbb{C}$ approached along a line parallel to the real axis:

$$L_1 = \lim_{\Delta x\to 0} \frac{(z + \Delta x)^* - z^*}{\Delta x} = \lim_{\Delta x\to 0} \frac{z^* + \Delta x - z^*}{\Delta z} = \lim_{\Delta x\to 0} \frac{\Delta x}{\Delta x} = 1.$$

Now consider the derivative at the same point but approached along a line perpendicular to the real axis:

$$L_2 = \lim_{\Delta y\to 0} \frac{(z + i\Delta y)^* - z}{i\Delta y} = \lim_{\Delta y\to 0} \frac{z^* - i\Delta y - z^*}{i\Delta y} = \lim_{\Delta y\to 0} \frac{-i\Delta y}{i\Delta y} = -1.$$

More generally we can approach along a line at an angle ϑ to the real axis:

$$L_\vartheta = \lim_{\Delta R\to 0} \frac{(z + \Delta Re^{i\vartheta})^* - z^*}{\Delta Re^{i\vartheta}} = \lim_{\Delta R\to 0} \frac{z^* + \Delta Re^{-i\vartheta} - z^*}{\Delta Re^{i\vartheta}} = \lim_{\Delta R\to 0} \frac{\Delta Re^{-i\vartheta}}{\Delta Re^{i\vartheta}} = e^{-2i\vartheta}.$$

Since this is not independent of ϑ the direction of approach matters so w is not differentiable at any point in $\mathbb{C}$.

2.4 Cauchy–Riemann Relations

It was quite a pain showing that a function as simple as $w(z) = z^*$ is nowhere differentiable. Fortunately there are other tests that we can use which are faster and easier than looking at the limit definition.

Notation

Suppose u is a function of the real variables x and y. Then we will use the notation

$$\frac{\partial u}{\partial x} = u_x, \qquad \text{and} \qquad \frac{\partial u}{\partial y} = u_y.$$

Theorem 2: Cauchy–Riemann relations

If $w = u + iv \colon S \subseteq \mathbb{C} \to \mathbb{C}$ is differentiable at a point $z = x + iy \in S$ then the **Cauchy–Riemann relations**,

$$u_x(x, y) = v_y(x, y), \qquad \text{and} \qquad u_y(x, y) = -v_x(x, y)$$

hold.

Proof. Let $w \colon S \subseteq \to \mathbb{C}$ be a complex function which is differentiable at $z = x + iy \in S$. We can write $w(z) = u(x, y) + iv(x, y)$. Since w is differentiable the following limit exists and is independent of the direction of approach:

$$\begin{aligned} w'(z) &= \lim_{\Delta z \to 0} \frac{w(z + \Delta z) - w(z)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - v(x, y)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{u(x + \Delta x, y + \Delta y) - u(x, y) + i[v(x + \Delta x, y + \Delta y) - v(x, y)]}{\Delta z}. \end{aligned} \tag{2.1}$$

We can approach this limit from any direction. If we approach it along a line parallel to the real axis, in particular choosing $\Delta z = \Delta x$, then we get

$$\begin{aligned} w'(z) &= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y) + i[v(x + \Delta x, y) - v(x, y)]}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left[\frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \right] \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \\ &= u_x(x, y) + iv_x(x, y). \end{aligned}$$

If instead we approach along a line perpendicular to the real axis, in particular choosing $\Delta z = i\Delta y$, then we get

$$\begin{aligned} w'(z) &= \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y) + i[v(x, y + \Delta y) - v(x, y)]}{i\Delta y} \\ &= \lim_{\Delta y \to 0} \left[\frac{1}{i} \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y} + \frac{v(x, y + \Delta y) - v(x, y)}{\Delta y} \right] \\ &= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\ &= -iu_y(x, y) + v_y(x, y). \end{aligned}$$

Since w is differentiable the result of these two approaches must be equal. Equating real and imaginary parts we have

$$u_x = v_y, \qquad \text{and} \qquad u_y = -v_x.$$

These are known as the **Cauchy–Riemann relations**. $\square$

We can use these to calculate the derivative of w as

$$w' = u_x + iv_x = v_y - iu_y.$$

We may be attempted to view w as a function $w\colon \mathbb{R}^2 \to \mathbb{R}^2$ and then apply the chain rule

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z}$$

but this is *wrong*. We can apply the chain rule separately to u and v as these are functions of real values or we can apply the chain rule to functions of the form $w(f(z))$ but to the variable z, not to x and y.

Note that satisfying the Cauchy–Riemann relations is a necessary condition for differentiability, not a sufficient one. There are functions that satisfy the Cauchy–Riemann relations but *aren't* differentiable. However we can add a condition to get sufficient conditions.

Theorem 3: Sufficient conditions for differentiability

Let $w\colon S \to \mathbb{C}$ be a complex function. Then w is differentiable at $z_0 = x_0 + iy_0$ if w satisfies the Cauchy–Riemann relations and u and v are differentiable at (x_0, y_0).

Proof. If u and v are differentiable at (x_0, y_0) then we can write

$$\begin{aligned}u(x,y) &= u(x_0,y_0) + u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y + \varepsilon_1\sqrt{(\Delta x)^2 + \Delta y^2}\\ v(x,y) &= v(x_0,y_0) + v_x(x_0,y_0)\Delta x + v_y(x_0,y_0)\Delta y + \varepsilon_2\sqrt{(\Delta x)^2 + \Delta y^2}\end{aligned}$$

where ε_1 and ε_2 tend to 0 as we move toward (x_0, y_0). We can substitute these into equation 2.1 and we find that

$$\begin{aligned}w'(x_0+iy_0) = \lim_{\Delta z\to 0}\Bigg[&\frac{u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y + i[v_x(x_0,y_0)\Delta x + v_y(x_0,y_0)\Delta y]}{\Delta x + i\Delta y}\\ &+(\varepsilon_1 + i\varepsilon_2)\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta z}\Bigg]\end{aligned}$$

where $\Delta z = \Delta x + i\Delta y$. Assuming that u and v satisfy the Cauchy–Riemann relations this becomes

$$\begin{aligned}w'(x_0+iy_0) &= \lim_{\Delta z\to 0}\Bigg[\frac{u_x(x_0,y_0)\Delta x - v_x(x_0,y_0)\Delta y + i[u_x(x_0,y_0)\Delta x + u_x(x_0,y_0)\Delta y]}{\Delta x + i\Delta y}\\ &\qquad+(\varepsilon_1 + i\varepsilon_2)\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta z}\Bigg]\\ &= \lim_{\Delta z\to 0}\Bigg[\frac{u_x(x_0,y_0)(\Delta x + i\Delta y) + iv_x(x_0,y_0)(\Delta x + i\Delta y)}{\Delta x + i\Delta y}\\ &\qquad+(\varepsilon_1 + i\varepsilon_2)\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta z}\Bigg]\\ &= u_x(x_0,y_0) + iv_x(x_0,y_0) + \lim_{\Delta z\to 0}(\varepsilon_1 + \varepsilon_2)\frac{|\Delta z|}{\Delta z}.\end{aligned}$$

Notice now that $||\Delta z|/\Delta z| = 1$ and therefore $|\Delta z|/\Delta z$ is bounded so the limit term vanishes as $\Delta z \to 0$ since $\varepsilon_1, \varepsilon_2 \to 0$. Thus

$$w'(x_0 + iy_0) = u_x(x_0,y_0) + iv_x(x_0,y_0)$$

so w is differentiable at $x_0 + iy_0$. □

2.5 Properties of the Derivative

Lemma 4: Linearity of the derivative

The derivative is linear. By this we mean that if w and f are complex functions which are continuous and differentiable at z and $\alpha \in \mathbb{C}$ then

$$(\alpha w)'(z) = \alpha w(z), \qquad \text{and} \qquad (w+f)'(z) = w'(z) + f'(z).$$

Proof. Let w, f, and α be as in the hypothesis. Then

$$\begin{aligned}
(\alpha w)'(z) &= \lim_{\Delta z \to 0} \frac{(\alpha w)(z+\Delta z) - (\alpha w)(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{\alpha w(z+\Delta z) - \alpha w(z)}{\Delta z} \\
&= \alpha \lim_{\Delta z \to 0} \frac{w(z+\Delta z) - w(z)}{\Delta z} \\
&= \alpha w'(z).
\end{aligned}$$

Also

$$\begin{aligned}
(w+f)'(w) &= \lim_{\Delta z \to 0} \frac{(w+f)(z+\Delta z) - (w+f)(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{w(z+\Delta z) + f(z+\Delta z) - w(z) - f(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{w(z+\Delta z) - w(z)}{\Delta z} + \lim_{\Delta z \to -} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\
&= w'(z) + f'(z).
\end{aligned}$$

□

Lemma 5: Product rule

Let w and f be complex functions which are continuous and differentiable at z. Then

$$(wf)'(z) = w'(z)f(z) + w(z)f'(z).$$

Proof. Let w and f be as in the hypothesis. Then

$$\begin{aligned}
(wf)' &= \lim_{\Delta z \to 0} \frac{(wf)(z+\Delta z) - (wf)(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{w(z+\Delta z)f(z+\Delta z) - w(z)f(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{w(z+\Delta z)f(z+\Delta z) - w(z+\Delta z)f(z) + w(z+\Delta z)f(z) - w(z)f(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} w(z+\Delta z)\frac{f(z+\Delta z) - f(z)}{\Delta z} + \lim_{\Delta z \to 0} f(z)\frac{w(z+\Delta z) - w(z)}{\Delta z} \\
&= w(z)f'(z) + f(z)w'(z)
\end{aligned}$$

□

Lemma 6: Chain rule

Let w and f be complex functions such that f is continuous and differentiable at z and w is continuous and differentiable at $f(z)$. Then

$$(w \circ f)'(z) = w'[f(z)]f'(z).$$

Proof. Let w and f be as in the hypothesis. Then

$$\begin{aligned}(w \circ f)'(z) &= \lim_{\Delta z \to 0} \frac{(w \circ f)(z + \Delta z) + (w \circ f)(z)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{w[f(z + \Delta z)] - w[f(z)]}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{w[f(z + \Delta z)] - w[f(z)]}{f(z + \Delta z) - f(z)} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\ &= w'[f(z)]f'(z).\end{aligned}$$

This assumes that $f(z+\Delta z) \neq f(z)$. If this isn't the case then clearly $f'(z) = 0$ and also $w[f(z+\Delta z)] - w[f(z)] = w[f(z)] - w[f(z)] = 0$ so $(w \circ f)'(z) = 0$ so the chain rule still holds. □

3 Analytic Functions

3.1 Examples

Example 3.1. *Where is $w(z) = z^2$ differentiable?*

$$w(z) = w(x + iy) = x^2 - y^2 + 2ixy \implies u(x, y) = x^2 - y^2, \qquad \text{and} \qquad v(x, y) = 2xy.$$

The first derivatives of u and v are then

$$\begin{aligned} u_x &= 2x, & v_y &= 2x, \\ u_y &= -2y, & v_x &= 2y. \end{aligned}$$

Recall that the Cauchy–Riemann relations are $u_x = v_y$ and $u_y = -v_x$ we see that w satisfies these at all $z \in \mathbb{C}$. Further each u and v are differentiable on $\mathbb{R}^2$ so w is differentiable on $\mathbb{C}$.

Example 3.2. *Where is $w(z) = |z|^2$ differentiable?*

$$w(z) = w(x + iy) = x^2 + y^2 \implies u(x, y) = x^2 + y^2, \qquad \text{and} \qquad v(x, y) = 0.$$

The first derivatives of u and v are then

$$\begin{aligned} u_x &= 2x, & v_y &= 0, \\ u_y &= 2y, & v_x &= 0. \end{aligned}$$

The Cauchy–Riemann relations are thus only satisfied at $z = 0$. Both u and v are differentiable at $(0, 0)$ so w is differentiable at $z = 0$ only.

Example 3.3. *Where is $w(z) = e^z$ differentiable?*

$$w(z) = w(x + iy) = e^x(\cos y + i \sin y) \implies u(x, y) = e^x \cos y, \qquad \text{and} \qquad v(x, y) = e^x \sin y.$$

The first derivatives of u and v are then

$$\begin{aligned} u_x &= e^x \cos y, & v_y &= e^x \cos y, \\ u_y &= -e^x \sin y, & v_x &= e^x \sin y. \end{aligned}$$

The Cauchy–Riemann relations are satisfied at all $z \in \mathbb{C}$ and u and v are differentiable on $\mathbb{R}^2$ so w is differentiable on $\mathbb{C}$.

3.2 Analytic Functions

We have seen that it is easy to accidentally define innocent looking functions, such as $w(z) = z^*$, which are continuous everywhere but differentiable almost nowhere. The nest definition defines a type of function which will behave very similarly to real functions when it comes to differentiability.

Definition 22: Analytic function

A function, $w\colon S \subseteq \mathbb{C} \to \mathbb{C}$, is said to be **analytic** or **holomorphic** at $z_0 \in S$ if w is differentiable at z_0 and at every point in the neighbourhood of z_0. We say that w is analytic on an open subset $S' \subseteq S$ if w is differentiable at all $z_0 \in S'$.

Notice that the neighbourhood of z_0 is a region of $\mathbb{C}$, that is an infinite, connected, open set. The *open* part of this requirement is important. If the set isn't open then we can take z_0 to be a limit point and then it is possible that $w(z_0 + \Delta z)$ won't be defined even as $\Delta z \to 0$.

Definition 23: Entire function

A function, $w\colon \mathbb{C} \to \mathbb{C}$ is **entire** or **regular** if it is analytic on $\mathbb{C}$, that is it is analytic at all $z_0 \in \mathbb{C}$.

For example if $w(z) = z^2$ we have already seen that this is differentiable for all $z \in \mathbb{C}$. Therefore w is entire. Similarly if $w(z) = e^z$ then w is entire. It can also be shown that $w(z) = e^{\alpha z}$ for $\alpha \in \mathbb{C}$ is entire.

Lemma 7: z^n is entire

The function

$$\begin{aligned} w\colon \mathbb{C} &\to \mathbb{C} \\ z &\mapsto z^n \end{aligned}$$

for $n \in \mathbb{N}$ is entire.

Proof. First consider the case $n = 1$. That is

$$w(z) = w(x + iy) = z = x + iy \implies u(x, y) = x, \qquad \text{and} \qquad v(x, y) = y.$$

The derivatives of u and v are

$$\begin{aligned} u_x &= 1, & v_y &= 1 \\ u_y &= 0, & v_y &= 0. \end{aligned}$$

Clearly u and v are differentiable and satisfy the Cauchy–Riemann relations for all $(x, y) \in \mathbb{R}^2$. Therefore w is differentiable for all $z \in \mathbb{C}$ so w is entire.

Now assume that the hypothesis is true for some specific $n = k$. That is that $w(z) = z^k$ is entire. Therefore w' exists everywhere. Using this we consider $f(z) = z^{k+1} = zz^k = zw(z)$. Note that

$$f(z) = zw(z) = (x + iy)(u + iv) = xu - yv + i(xv + yu).$$

Hence

$$f(z) = f(x + iy) = g(x, y) + ih(x, y) = xu(x, y) - yv(x, y)i(xv(x, y) + yu(x, y)).$$

Where

$$g(x, y) = xu(x, y) - yv(x, y), \qquad \text{and} \qquad h(x, y) = xv(x, y) + yu(x, y).$$

Thus we can see if the Cauchy–Riemann relations hold for g and h:

$$\begin{aligned} g_x &= u + xu_x - yv_x, & h_y &= xv_y + u + yu_y, \\ g_y &= xu_y - v - yv_y, & h_x &= v + xv_x + yu_x. \end{aligned}$$

The derivatives of u and v are guaranteed to exist as w is differentiable. Using the Cauchy–Riemann relations for u and v we have

$$\begin{aligned} g_x &= u + xu_x + yu_y, & h_y &= xu_x + u + yu_y, \\ g_y &= xu_y - v - yu_x, & h_x &= v - xu_y + yu_x \end{aligned}$$

So we see that the Cauchy–Riemann relations are satisfied everywhere and that g and h are differentiable everywhere. Thus f is differentiable everywhere so is entire.

Hence by mathematical induction $w(z) = z^n$ is entire for all $n \in \mathbb{N}$. □

Lemma 8: Sum of entire functions is entire

The sum of two entire functions is entire.

Proof. Let $w, f\colon \mathbb{C} \to \mathbb{C}$ be entire functions and $u, v, g, h\colon \mathbb{R}^2 \to \mathbb{R}$ be such that $w = u+iv$ and $f = g+ih$. Then

$$w + f = (u + g) + i(v + h).$$

Checking the Cauchy–Riemann relations for these we see that

$$\begin{aligned}(u+g)_x &= u_x + g_x, & (v+h)_y &= v_y + h_y,\\ (u+g)_y &= u_y + g_y, & (v+h)_x &= v_x + h_x.\end{aligned}$$

Now using the Cauchy–Riemann relations for u, v, g, and h we have

$$\begin{aligned}(u+g)_x &= u_x + g_x, & (v+h)_y &= u_x + g_x,\\ (u+g)_y &= u_y + g_y, & (v+h)_x &= -u_y - g_y.\end{aligned}$$

So we see that $u + g$ and $v + h$ satisfy the Cauchy–Riemann relations and are differentiable everywhere meaning that $f + w$ is differentiable everywhere and therefore $f + w$ is entire. $\square$

Corollary 1: Polynomials are entire

All polynomials, $P_n \in \mathbb{C}[z]$,

$$P_n(z) = \sum_{k=0}^{n} a_k z^k$$

for $a_k \in \mathbb{C}$ are entire.

Proof. First if w is an entire function then by the linearity of the derivative aw for $a \in \mathbb{C}$ is also an entire function. By lemma 7 z^k is entire and therefore $a_k z^k$ is also entire. Considering the sum

$$P_n(z) = \sum_{k=0}^{n} a_k z^k = (a_0 + (a_1 z + (a_2 z^2 + (\cdots (a_{n-2} z^{n-2} + (a_{n-1} z^{n-1} + a_n z^n))))))$$

we see that we can take any finite sum and reduce it to repeatedly summing pairs of functions. Each function is entire and each of these sums of pairs of functions returns an entire function by lemma 8. Hence $P_n(z)$ is entire. $\square$

Definition 24: Complex trigonometric functions

We can define $\sin, \cos\colon \mathbb{C} \to \mathbb{C}$ as

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \qquad \text{and} \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{iz}).$$

Importantly this means that these complex trig functions evaluate to the same value as their real counterparts with a real input. Notice the resemblance to the hyperbolic trig functions:

$$\cosh z = \frac{1}{2}(e^{z} + e^{-z}), \qquad \text{and} \qquad \sinh z = \frac{1}{2}(e^{z} - e^{-z}).$$

Corollary 2: Trigonometric functions are entire

Trig functions and hyperbolic trig functions are entire.

Proof. The trig functions and hyperbolic trig functions defined on $\mathbb{C}$ are defined as a sum of two exponentials of the form $e^{\alpha z}$ with $\alpha \in \mathbb{C}$. These exponentials are entire and therefore the trig and hyperbolic trig functions are entire. $\square$

Example 3.4. *Where is $w(z) = |z|^2$ analytic?*

Recall that $w(z) = |z|^2$ is differentiable only at $z = 0$. Therefore it is not analytic anywhere as it is differentiable at a point and not in its neighbourhood.

Similarly to above a function that is differentiable only along a line will not be analytic anywhere as the neighbourhood of any point on the line will contain points not on the line. This was the purpose of our definition, to exclude functions that are differentiable at a point or a few points but not in a region. This turns out to be a very strict requirement which allows us to prove some very strong results. As part of this we will consider functions analytic on a region apart from a few points inside. These points tend to require special consideration so we give them a name:

Definition 25: Isolated singular point

Let $w\colon S \subseteq \mathbb{C} \to \mathbb{C}$. We say that z_0 is an **isolated singular point** of w if there exists an open domain $S \ni z_0$ such that w is analytic for all $z \in S$, $z \neq z_0$.

Definition 26: Stationary point

If $w\colon S \subseteq \mathbb{C} \to \mathbb{C}$ is analytic in an open domain, $D \subseteq S$, and $z_0 \in D$ then we say that z_0 is a **stationary point** if $w'(z_0) = 0$.

Theorem 4: Properties of an analytic function

Let $w\colon S \subseteq \mathbb{C} \to \mathbb{C}$ be analytic in $S' \subseteq S$ and let $w(x + iy) = u(x, y) + iv(x, y)$. Then u and v have the following properties in S':

(i) Both u and v are harmonic, that is they satisfy Laplace's equation, $\nabla^2 u = \nabla^2 v = 0$ where $\nabla^2 = \partial_x^2 + \partial_y^2$ is the two dimensional Laplacian.
(ii) The level curves of u and v are orthogonal.
(iii) A stationary point is always a saddle point.

Proof. We will prove these in the order stated.

(i) Since w is analytic in the region of interest we know that they are continuous and differentiable. This means that we can change the order of partial derivatives. Thus

$$u_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial v_y}{\partial y} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial v_x}{\partial y} = -\frac{\partial u_y}{\partial y} = -u_{yy}.$$

Hence

$$\nabla^2 u = u_x x + u_y y = u_x x - u_x x = 0.$$

Similarly

$$v_{xx} = \frac{\partial v_x}{\partial x} = -\frac{\partial u_y}{\partial x} = -\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial u_x}{\partial y} = -\frac{\partial v_y}{\partial y} = -v_y y$$

so

$$\nabla^2 v = v_{xx} + v_{yy} = v_{xx} - v_{xx} = 0.$$

(ii) The level curves of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ are perpendicular to $\nabla f = f_x \boldsymbol{e_x} + f_y \boldsymbol{e_y}$. Therefore if the curves of constant u and v are orthogonal we expect their gradients to be as well. This is easy to show:

$$(\nabla u) \cdot (\nabla v) = (u_x \boldsymbol{e_x} + u_y \boldsymbol{e_y}) \cdot (v_x \boldsymbol{e_x} + v_y \boldsymbol{e_y}) = u_x v_x + u_y v_y = u_x v_x - u_x v_x = 0.$$

So ∇u and ∇v are perpendicular so the level curves of u and v are perpendicular.

(iii) A function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a local maximum (or minimum) at (x, y) if $f'(x, y) = 0$ and the Hessian matrix, $H_{ij} = \partial_i \partial_j f$, is positive definite (or negative definite). If the Hessian is indefinite then (x, y) is a saddle point. Recall that a matrix, M, is positive (or negative) definite if all of its eigenvalues are positive (or negative) and indefinite if some of its eigenvalues are positive and some are negative. So the third part of this theorem equates to stating that the Hessian matrices for u and v are indefinite, i.e. of their two eigenvalues one is positive and one is negative. The Hessian matrices for u and v are

$$H(u) = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{pmatrix}, \qquad \text{and} \qquad H(v) = \begin{pmatrix} v_{xx} & v_{xy} \\ v_{yx} & v_{yy} \end{pmatrix}.$$

We know from the first part of this theorem that $u_{xx} = -u_{yy}$ and $v_{xx} = -v_{yy}$. Also applying the Cauchy–Riemann relations we have $u_{xy} = v_{yy} = -v_{xx}$ and $v_{xy} = -u_{yy} = u_{xx}$.

$$H(u) = \begin{pmatrix} u_{xx} & -v_{xx} \\ -v_{xx} & -u_{xx} \end{pmatrix}, \qquad \text{and} \qquad H(v) = \begin{pmatrix} v_{xx} & u_{xx} \\ u_{xx} & -v_{xx} \end{pmatrix}.$$

Now notice that $|H(u)| = |H(v)| = -u_x x^2 - v_x x^2 \le 0$. If the determinants are zero then H are indefinite. If the determinants are non-zero then it is possible to diagonalise the matrix. Then the determinant will be the product of the leading diagonal, which is all of the eigenvalues. For the determinant to be negative since there are two eigenvalues we need one eigenvalue to be negative and the other to be positive. This makes H an indefinite matrix.

□

4 Consequences of Analyticity

Theorem 5: Constant analytic functions

Let $w\colon S \subseteq \mathbb{C} \to \mathbb{C}$ be analytic on S. Then

(i) If $w'(z) = 0$ for all $z \in S$ then $w(z)$ is constant for all $z \in S$.

(ii) If $|w(z)|$ is constant for all $z \in S$ then $w(z)$ is also constant for all $z \in S$.

Proof. We will prove these in the order stated.

(i) The mean value theorem (MVT) for several variables states that for $f\colon \mathbb{R}^n \to \mathbb{R}$ and $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ there exists $\boldsymbol{c} \in \mathbb{R}^n$ such that

$$f(\boldsymbol{b}) - f(\boldsymbol{a}) = [\nabla f(\boldsymbol{c})] \cdot (\boldsymbol{b} - \boldsymbol{a}).$$

Specifically in the one dimensional case this takes the form

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This theorem doesn't hold for complex functions, however we can apply it to the real and imaginary parts of w, u and v respectively.

Let $w\colon S \to \mathbb{C}$ be such that $w'(z) = 0$ for all $z \in S$ and $w(z) = w(x+iy) = u(x,y) + iv(x,y)$ where $u, v\colon \mathbb{R}^2 \to \mathbb{R}$. Then for $\boldsymbol{a}, \boldsymbol{b} \in S' = \{(x,y) \in \mathbb{R}^2 \mid x + iy \in S\}$ we can say that there exists $\boldsymbol{c} \in \mathbb{R}^2$ such that

$$u(\boldsymbol{b}) - u(\boldsymbol{a}) = [\nabla u(\boldsymbol{c})] \cdot (\boldsymbol{b} - \boldsymbol{a}).$$

However in this case we are assuming $w'(z) = 0$ which necessitates that all partial derivatives are also zero. Thus $\nabla u(\boldsymbol{c}) = 0$ for all $\boldsymbol{c} \in S'$. This means that $u(\boldsymbol{b}) - u(\boldsymbol{a}) = 0$ hence $u(\boldsymbol{b}) = u(\boldsymbol{a})$. Since this holds for any pair of points, $\boldsymbol{a}, \boldsymbol{b} \in S'$ we must have that $u(x,y)$ is constant for all $(x,y) \in S'$. By the same logic we must also have that $v(x,y)$ is constant for all $(x,y) \in S'$ and hence $w(z)$ is constant for all $z \in S$.

(ii) Let $w\colon S \to \mathbb{C}$ be such that $|w(z)| = C$ is constant for all $z \in S$. Then $u^2 + v^2 = C^2$. Differentiating with respect to x and separately with respect to y we get

$$2uu_x + 2vv_x = 0, \qquad \text{and} \qquad 2uu_y + 2vv_y = 0.$$

Applying the Cauchy–Riemann relations we get

$$uu_x - vu_y = 0, \qquad \text{and} \qquad uu_y + vu_x = 0.$$

Since both equations hold simultaneously we can solve them and find that $(u^2+v^2)u_y = C^2 u_y = 0$. If u and v are both zero then $C = 0$. If this is not the case then u_x and u_y must be zero. Similarly we can show that v_x and v_y must be zero and hence $w(z)$ is constant for all $z \in S$.

□

4.1 Reconstruction

Definition 27: Harmonic conjugate

Let $u, v\colon S \subseteq \mathbb{R}^2 \to \mathbb{R}$. We say v is the **harmonic conjugate** of u if u and v are both harmonic and they satisfy the Cauchy–Riemann relations in an open subset, $S \subseteq \mathbb{R}^2$.

Note that if v is the harmonic conjugate of u then $w(x+iy) = u(x,y) + iv(x,y)$ is analytic on S. For a given harmonic function $u\colon \mathbb{R}^2 \to \mathbb{R}$ its harmonic conjugate, if it exists, is unique up to a constant. The process of finding the harmonic conjugate is best demonstrated with an example.

Example 4.1. *Given the function $u(x,y) = y^3 - 3x^2y$ find its harmonic conjugate.*

We start by computing the partial derivatives of u:

$$u_x = -6xy, \qquad \text{and} \qquad u_y = 3y^2 - 3x^2.$$

From the Cauchy–Riemann relations and the fundamental theorem of calculus we have that

$$v = \int v_y \,\mathrm{d}y + f(x) = \int u_x \,\mathrm{d}y + f(x) = -6x \int y \,\mathrm{d}y + f(x) = -3xy^2 + h(x)$$

for some 'constant' of integration h. Hence

$$v_x = -3y^2 + h'(x).$$

We can also compute this value using the Cauchy–Riemann relations which gives

$$v_x = -u_y = 3x^2 - 3y^2.$$

For both of these to be equal we require

$$h'(x) = 3x^2 \implies h(x) = \int 3x^2 \,\mathrm{d}x = x^3 + C$$

Finally

$$v(x,y) = -3xy^2 + x^3 + C.$$

Thus

$$w(z) = w(x+iy) = y^3 - 3x^2y + i(x^3 - 3xy^2 + C) = i(z^3 + C)$$

is entire, which we already know as w is a polynomial.

We made a decision at the start to integrate v_y, we could have instead integrated v_x and we will arrive at the same result:

$$v = \in v_x \,\mathrm{d}x + f(y) = -\int u_y \,\mathrm{d}x + f(y) = \int (3x^2 - 3y^2) \,\mathrm{d}x + f(y) = x^3 - 3xy^2 + h(y).$$

Thus

$$v_y = -6xy + h'(y)$$

or by the Cauchy–Riemann relations

$$v_y = u_x = -6xy$$

so we see that this time $h'(y) = 0$ meaning that $h(y) = C$ is constant. Thus

$$v(x,y) = x^3 - 3xy^2 + C$$

which is the same result as we got the first time.

---

5 Branch Points

5.1 Elementary Functions

5.1.1 Polynomials

A polynomial of order n,

$$P_n(z) = \sum_{k=0}^{n} a_k z^k,$$

is a linear combination of powers of z which are entire so P_n is entire. The fundamental theorem of algebra, which we will prove later, states that P_n has exactly n roots. For example $w(z) = z^4$ has roots at $z = 1, -1, i, -i$. z^4 will map any wedge covering $\pi/2$ rad of $\mathbb{C}$ to the whole of $\mathbb{C}$. For example consider figure 5.1 which shows the complex plane divided into four wedges and the image[8] of one of these wedges after the mapping $z \mapsto z^4$. The image of one of these wedges is $\mathbb{C}$. While this isn't particularly

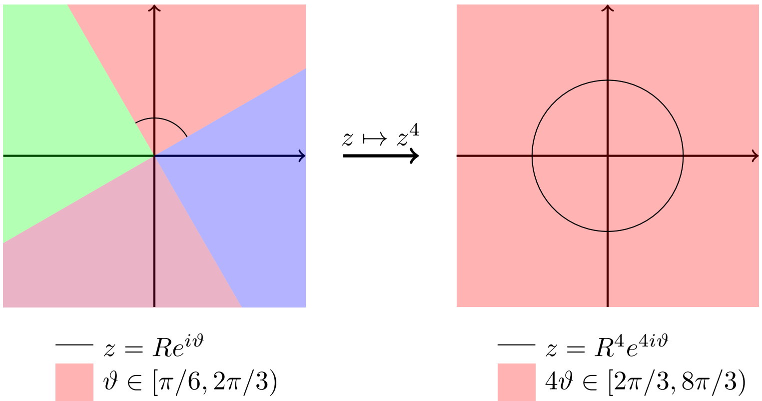


Figure 5.1: Four possible wedges that all map to $\mathbb{C}$ under $z \mapsto z^4$. The red wedge is $\vartheta \in [\pi/6, 2\pi/3)$. The curve $Re^{i\vartheta}$ within this wedge maps to a circle of radius R^4 as $2\vartheta \in [4\pi/3, 8\pi/3)$ which is a 2π range.

interesting for the mapping $z \mapsto z^4$ it causes problems for the inverse mapping, $z \mapsto z^{1/4}$. We have to ask the question which of the four wedges that map to $\mathbb{C}$ in the forward mapping should we map to in the reverse mapping. Or we could even map to a mixture of different wedges. In this and the next few sections we will discover the consequences of such choices.

One immediate choice is if we want $1^{1/3} = 1$ then we could choose the $\pi/3$ wedge given by $\vartheta \in [0, \pi/3]$. However this means that $(-1)^{1/3} \neq -1$ even though this is the value we would normally consider since $\arg(-1) = \pi \notin [0, \pi/3]$. Instead we have

$$(-1)^{1/3} = (e^{i\pi})^{1/3} = e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

We have even more of a problem if we consider irrational roots. For example the rth roots of z for some $r \in \mathbb{R}$ is

$$\sqrt[r]{z} = \sqrt[r]{|z|e^{i\vartheta + 2\pi k i}} = \sqrt[r]{z} \exp\left[i\frac{\vartheta}{r} + \frac{2\pi k i}{r}\right]$$

for all $k \in \mathbb{Z}$. If $r \in \mathbb{Z}$ then eventually we will get to a point where the argument starts overlapping with previous values as we increase k however if, for example, $r = e$ then this doesn't happen and we will end up with an infinite number of different values.

5.1.2 Exponential and Logarithm

The exponential function has many of the expected properties, for example

$$e^{z_1 + z_2} = e^{z_1} e^{z_2}, \qquad e^z \neq 0 \forall z \in \mathbb{C}, \qquad \text{and} \qquad e^{x+iy} = e^x(\cos y + i \sin y).$$

[8]Recall that the image of a set, $A \subseteq S$, under the function $w \colon S \subseteq \mathbb{C} \to \mathbb{C}$ is the set $w(A) = \{w(z) \mid z \in A\}$.

Likewise the logarithm also has many of the expected properties, for example

$$\log(e^z) = z, \qquad \log(z_1 z_2) = \log z_1 + \log z_2, \qquad \log\left(\frac{1}{z}\right) = -\log z,$$
$$\text{and} \qquad \log z = \log\big(|z|e^{i\vartheta}\big) = \log|z| \log e^{-\vartheta} = \log|z| + i\vartheta.$$

However e^z is periodic with period $2\pi i$. This means that $z \mapsto e^z$ maps a strip of width 2π to $\mathbb{C}\setminus\{0\}$. See figure 5.2. In order for $\log z$ to be single valued we have to make a choice to restrict ϑ to some 2π

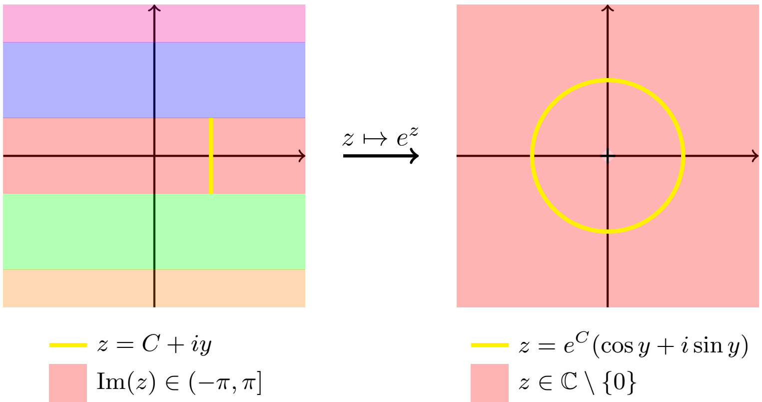


Figure 5.2: Stripes of width 2π which all map to $\mathbb{C}\setminus\{0\}$ under $z \mapsto e^z$. The red stripe is $\mathrm{Im}(z) \in (-\pi, \pi]$. The curve $z = C + iy$ for some constant $C \in \mathbb{R}$ maps to a circle of radius e^C.

interval. Common choices are $(-\pi, \pi]$ and $[0, 2\pi)$.

There is another problem however. Suppose we choose $\vartheta \in (-\pi, pi]$. Now move smoothly around a circle of arbitrary radius, R. This corresponds to increasing ϑ smoothly and moving along the curve given by $Re^{i\vartheta}$. Most of the time $\mathrm{Re}(\log z)$ and $\mathrm{Im}(\log z)$ change smoothly as we move around the circle up until the point $\vartheta = \pi$. At this point if we move from above the real axis to just below, even if we do so be an arbitrarily small amount, there will be a jump in ϑ of 2π which means there will be a jump in $\mathrm{Im}(\log z)$ of 2π.

5.1.3 Hyperbolic and Trigonometric Functions

Hyperbolic and trigonometric functions on $\mathbb{C}$ are defined using complex exponentials as

$$\cosh z = \frac{1}{2}(e^{-z} + e^{-z}), \qquad \sinh z = \frac{1}{2}(e^{-z} - e^{-z}), \qquad \tanh = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{1 - e^{-2z}}{1 + e^{-2z}},$$
$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}),$$
$$\text{and} \qquad \tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} = \frac{1 - e^{-2iz}}{i(1 - e^{-2iz})}.$$

We can see from these formulas that the hyperbolic and trigonometric functions are inextricably linked by

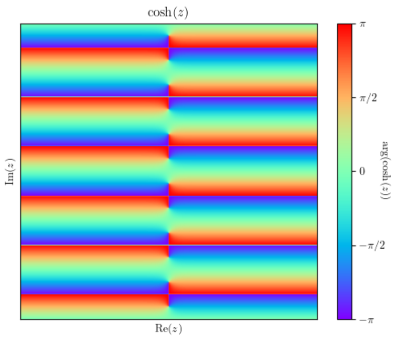
$$\cos z = \cosh(iz), \qquad \text{and} \qquad \sin z = -i\sinh(iz).$$

Note that multiplication by $\pm i$ is a rotation by $\pm\pi/2$ in the complex plane. Therefore we expect sin to be a rotated version of sinh with a phase shift. Similarly cos is a phase shifted version of cosh. The following trig identities still hold for the complex versions:

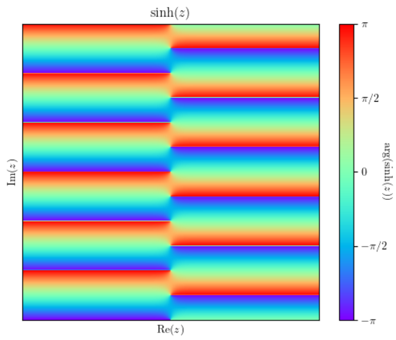
$$\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \sin(z_2)\cos(z_1)$$
$$\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) + \sin(z_1)\sin(z_2)$$
$$1 = \sin^2(z) + \cos^2(z)$$

The inverse hyperbolic and trigonometric functions are given by

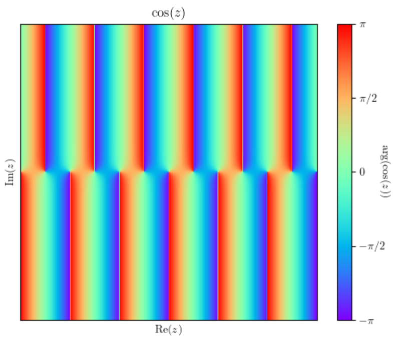
$$\operatorname{arcosh}(z) = \log\left(z + \sqrt{z^2 - 1}\right), \qquad \operatorname{arsinh}(z) = \log\left(z + \sqrt{z^2 + 1}\right),$$
$$\arccos(z) = \frac{1}{i}\log\left(z + \sqrt{1 - z^2}\right), \qquad \text{and} \qquad \arcsin(z) = \frac{1}{i}\log\left(iz + \sqrt{1 - z^2}\right).$$



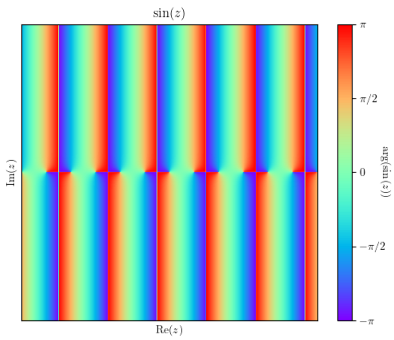
(a) The phase of $\cosh(z)$



(b) The phase of $\sinh(z)$



(c) The phase of $\cos(z)$



(d) The phase of $\sin(z)$

Figure 5.3: The phases of $\cosh(z)$, $\sinh(z)$, $\cos(z)$, and $\sin(z)$.

5.2 Branch Points

Recall that we mentioned earlier a discontinuity in $\log(z)$ when traversing a circle about the origin. This is because $z = 0$ is a branch point of $\log z$.

Definition 28: Branch point

Let $w\colon S \subseteq \mathbb{C} \to \mathbb{C}$. We say that $z_0 \in \mathbb{C}^*$ is a **branch point** of w if the neighbourhood, $D'(z_0; \varepsilon)$, contains points where w is discontinuous for arbitrarily small $\varepsilon > 0$.

To show that a point is a branch point it suffices to show that a circle about that point with arbitrarily small radius contains a discontinuity. A branch point is *not* an isolated singularity as w is, by definition, continuous in the neighbourhood of an isolated singularity. Again notice that the value of w at z_0 is unimportant in the question of whether z_0 is a branch point of w. Notice that $z = \infty$ can be a branch point. We can check for this by considering the transformation $z \to 1/t$ as $t \to 0$.

Example 5.1. *We have already seen that $w(z) = \log z$ has a branch point at $z = 0$. Does it have a branch point at $z = \infty$?*

Make the transformation $z \to 1/t$:

$$\log z \to \log \frac{1}{t} = -\log t.$$

Since this has a factor of $\log t$ it will have a branch point at $t = 0$ and therefore $\log z$ has a branch point at $z = \infty$.

Example 5.2. *Where are the branch points of $w(z) = \sqrt{z}$?*

We can write an arbitrary real power of z as

$$z^\alpha = e^{\alpha \log z} = \exp[\alpha \log |z| + i\alpha(\vartheta + 2\pi n)].$$

We see that this contains a logarithm and therefore has a branch point at $z = 0, \infty$.

From these two examples if we have a function of square roots and logarithms we can find the branch points simply by looking for when the arguments of said logarithms and square roots are zero.

6 Branch Cuts

Example 6.1. *Where are the branch points of* $w(z) = \sqrt{z^2 - 1}$*?*

We can write

$$w(z) = \sqrt{z^2 - 1} = \sqrt{z - 1}\sqrt{z + 1}.$$

First we consider when the argument of the square roots may be 0, we see this happens at $z = \pm 1$. What about $z = \infty$? Again we make the transformation $z \to 1/t$:

$$\sqrt{z^2 - 1} \to \sqrt{\frac{1}{t^2} - 1} = \sqrt{\frac{1 - t^2}{t^2}} = \frac{\sqrt{1 - t^2}}{t}.$$

Now as $t \to 0$ this is a singularity, however that doesn't matter as the value of w is unimportant at a branch point. Since the argument of the square root is non-zero at $t = 0$ this is not a branch point. Therefore the branch points of w are $z = \pm 1$.

We will use this example to develop an important concept. A point in the plane can be expressed in polar coordinates centred at $z = \pm 1$, see figure 6.1. In these coordinates

$$z + 1 = R^- e^{i\vartheta^-}, \qquad \text{and} \qquad z - 1 = R^{i\vartheta^+}.$$

This allows us to express $w(z) = \sqrt{z^2 - 1}$ as

$$w(z) = \sqrt{z^2 - 1} = \sqrt{z + 1}\sqrt{z - 1} = \sqrt{R^+ R^-} \exp\left[i\frac{1}{2}(\vartheta^+ + \vartheta^-)\right].$$

We will use this form to investigate the discontinuities of w. What we will see is that there are discon-

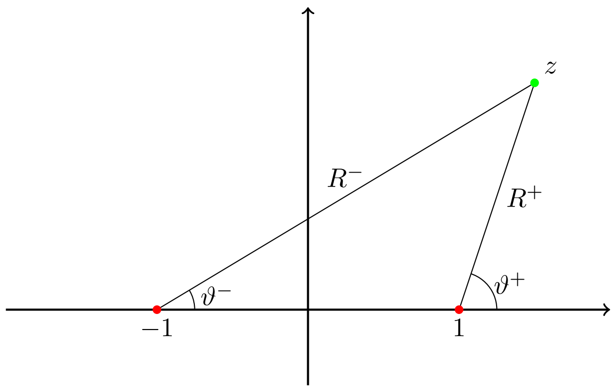


Figure 6.1: A point, z, can be thought of in two sets of polar coordinates centred at ± 1.

tinuities regardless of how we restrict $\vartheta^\pm$ to have a single valued function. We consider three different paths across the real axis, each starting at a point $x + i\delta$ above the axis for some $\delta > 0$ and ending at $x + i\delta$ below the axis. If there is a discontinuity above the axis then even as $\delta \to 0$ there will be a non-zero jump in the value of w.

- The first choice we make is $\vartheta^\pm \in (-\pi, \pi]$.
  - The first path we consider has $x \in (1, \infty)$.

    Above the axis in the limit $\delta \to 0$ we have $\vartheta^\pm = 0$. Hence

$$\lim_{\delta \to 0} w(x + i\delta) = \sqrt{R^+ R^-} \exp\left[i\frac{1}{2}(0 + 0)\right] = \sqrt{R^+ R^-}.$$

Below the axis in the limit $\delta \to 0$ we have $\vartheta^{\pm} \to 0$. Hence

$$\lim_{\delta \to 0} w(x - i\delta) = \sqrt{R^+ R^-} \exp\left[i\frac{1}{2}(0 + 0)\right] = \sqrt{R^+ R^-}.$$

Both of these are equal so there is no discontinuity in the $(1, \infty)$ interval of the real axis.

– The second path we consider has $x \in (-1, 1)$.

Above the axis in the limit $\delta \to 0$ we have $\vartheta^+ = \pi$ and $\vartheta^- = 0$. Hence

$$\lim_{\delta \to 0} w(x + i\delta) = \sqrt{R^+ R^-} \exp\left[i\frac{1}{2}(\pi + 0)\right] = e^{i\pi/2}\sqrt{R^+ R^-} = i\sqrt{R^+ R^-}.$$

Below the axis in the limit $\delta \to 0$ we have $\vartheta^+ = -\pi$ and $\vartheta^- = 0$. Hence

$$\lim_{\delta \to 0} w(x - i\delta) = \sqrt{R^+ R^-} \exp\left[i\frac{1}{2}(-\pi + 0)\right] = e^{-i\pi/2}\sqrt{R^+ R^-} = -i\sqrt{R^+ R^-}.$$

These are not equal so there is a discontinuity in the $(-1, 1)$ interval of the real axis.

– The third path we consider has $x \in (-\infty, -1)$.

Above the axis in the limit $\delta \to 0$ we have $\vartheta^{\pm} = \pi$. Hence

$$\lim_{\delta \to 0} w(x + i\delta) = \sqrt{R^+ R^-} \exp\left[i\frac{1}{2}(\pi + \pi)\right] = e^{i\pi}\sqrt{R^+ R^-} = -\sqrt{R^+ R^-}.$$

Below the axis in the limit $\delta \to 0$ we have $\vartheta^{\pm} = -\pi$. Hence

$$\lim_{\delta \to 0} w(x - i\delta) = \sqrt{R^+ R^-} \exp\left[i\frac{1}{2}(-\pi + -\pi)\right] = e^{-i\pi}\sqrt{R^+ R^-} = \sqrt{R^+ R^-}.$$

Both of these are equal so there is no discontinuity in the $(-\infty, 1)$ interval of the real axis.

We conclude that with the choice $\vartheta^{\pm} \in (-\pi, \pi]$ there is a discontinuity only crossing the interval $(-1, 1)$ of the real axis.

- The second choice we make is $\vartheta^+ \in (-\pi, \pi]$ and $\vartheta^- \in [0, 2\pi)$.

– The first path we consider has $x \in (1, \infty)$.

Above the axis in the limit $\delta \to 0$ we have $\vartheta^{\pm} = 0$. Hence

$$\lim_{\delta \to 0} w(x + i\delta) = \sqrt{R^+ R^-} \exp\left[i\frac{1}{2}(0 + 0)\right] = \sqrt{R^+ R^-}.$$

Below the axis in the limit $\delta \to 0$ we have $\vartheta^+ = 0$ and $\vartheta^- = 2\pi$. Hence

$$\lim_{\delta \to 0} w(x - i\delta) = \sqrt{R^+ R^-} \exp\left[i\frac{1}{2}(0 + 2\pi)\right] = e^{i\pi}\sqrt{R^+ R^-} = -\sqrt{R^+ R^-}.$$

These are not equal so there is a discontinuity in the $(1, \infty)$ interval of the real axis.

– The second path we consider has $x \in (-1, 1)$.

Above the axis in the limit $\delta \to 0$ we have $\vartheta^+ = -\pi$ and $\vartheta^- = 2\pi$. Hence

$$\lim_{\delta \to 0} w(x + i\delta) = \sqrt{R^+ R^-} \exp\left[i\frac{1}{2}(2\pi - \pi)\right] = e^{i\pi/2}\sqrt{R^+ R^-} = i\sqrt{R^+ R^-}.$$

Below the axis in the limit $\delta \to 0$ we have $\vartheta^+ = -\pi$ and $\vartheta^- = 2\pi$. Hence

$$\lim_{\delta \to 0} w(x - i\delta) = \sqrt{R^+ R^-} \exp\left[i\frac{1}{2}(-\pi + 2\pi)\right] = e^{i\pi/2}\sqrt{R^+ R^-} = i\sqrt{R^+ R^-}.$$

Both of these are equal so there is no discontinuity in the $(-1, 1)$ interval of the real axis.

– The third path we consider has $x \in (-\infty, -1)$.

Above the axis in the limit $\delta \to 0$ we have $\vartheta^{\pm} = \pi$. Hence

$$\lim_{\delta \to 0} w(x + i\delta) = \sqrt{R^+R^-} \exp\left[i\frac{1}{2}(\pi + \pi)\right] = e^{i\pi}\sqrt{R^+R^-} = \sqrt{R^+R^-}.$$

Below the axis in the limit $\delta \to 0$ we have $\vartheta^+ = -\pi$ and $\vartheta^- = \pi$. Hence

$$\lim_{\delta \to 0} w(x - i\delta) = \sqrt{R^+R^-} \exp\left[i\frac{1}{2}(-\pi + \pi)\right] = \sqrt{R^+R^-}.$$

These are not equal so there is a discontinuity in the $(-\infty, 1)$ interval of the real axis.

We conclude that with the choice $\vartheta^+ \in (-\pi, \pi]$ and $\vartheta^- \in [0, 2\pi)$ there is a discontinuity crossing the real axis in either $(-\infty, -1)$ or $(1, \infty)$.

This demonstrates the importance of choosing the argument. In general it is possible to move discontinuities by changing the argument but we cannot get rid of them. There is no 'right answer', it is best to make a choice and write it down and be careful to stick with it.

Definition 29: Branch cut

A **branch cut** is an arbitrary path that joins two branch points and as a result of which the function in question becomes single valued.

Figure 6.2 shows a Riemann surface, which is a way of plotting complex functions. This one shows $\sqrt{z}$ which we can see is multi valued. We can think of branch points as the points where multiple sheets coincide. Figure 6.3 shows the same Riemann surface after a branch cut along the negative real axis, which joins 0 and ∞, the two branch points of $\sqrt{z}$. The function is now single valued and the discontinuity at the branch cut is fairly obvious.

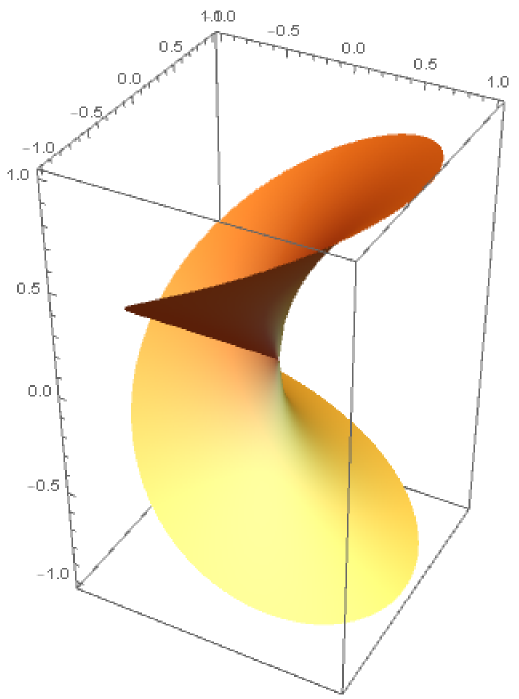


Figure 6.2: The Riemann surface of $\sqrt{z}$ showing the real and imaginary parts in the bottom plane and $\text{Re}(\sqrt{z})$ in the vertical axis.

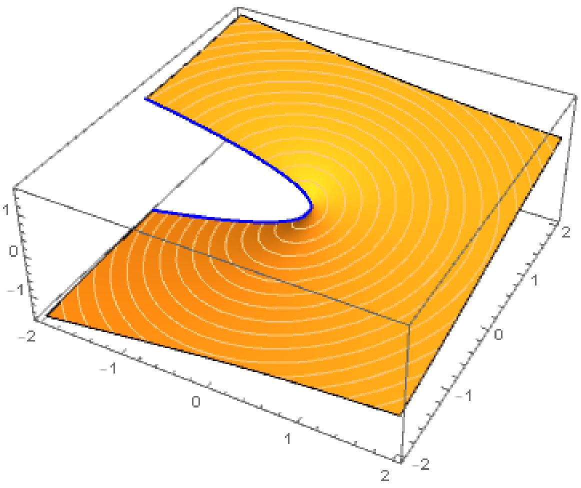


Figure 6.3: The Riemann surface of $\sqrt{z}$ with a branch cut along the negative real axis in order to make the function single valued. The branch cut is shown in blue.

Part III

Integrals

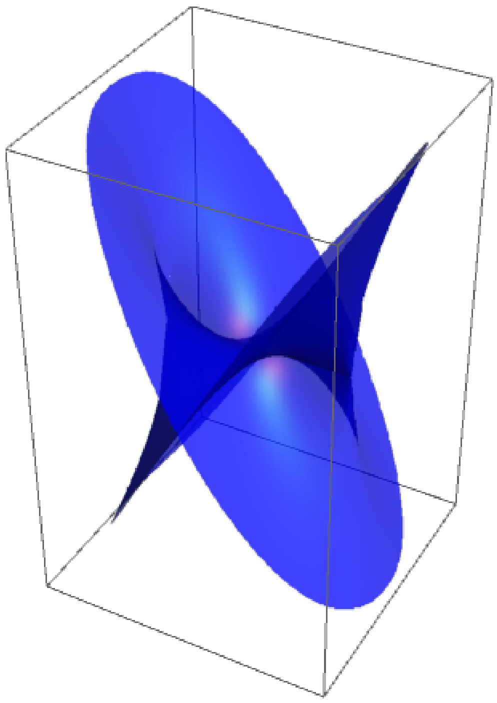


Figure 6.4: The Riemann surface of $\sqrt{z^2-1}$ with two branch points at $z = \pm 1$.

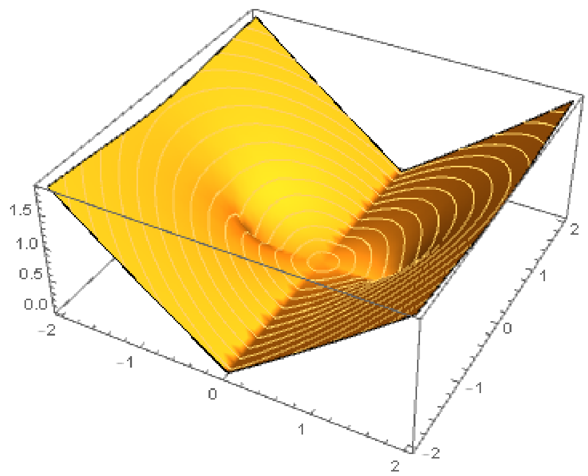


Figure 6.5: The Riemann surface of $\sqrt{z^2-1}$ with a branch point from $z = \infty$ to $z = \pm 1$. Essentially this cuts the bottom half off to make the function single valued.

7 Complex Integration

7.1 Integrating Functions From $\mathbb{R}$ to $\mathbb{C}$

Before we define integration in the complex plane we consider a simpler case. Consider a function, $w\colon \mathbb{R} \to \mathbb{C}$. We can write this function as

$$w(t) = u(t) + iv(t)$$

where $u, v\colon \mathbb{R} \to \mathbb{R}$. If u and v are both piecewise continuous (that is contain a countable number of discontinuities) then we can integrate w the same way we would integrate a real integral. For example integrating over (a, b) we have

$$\int_a^b w(t)\,\mathrm{d}t = \int_a^b u(t)\,\mathrm{d}t + i\int_a^b v(t)\,\mathrm{d}t = [U(t) + iV(t)]_a^b$$

where U and V are the antiderivatives of u and v. This simplified case of integrating only along $\mathbb{R}$ actually turns out to be very close to the more common case of a function defined on a particular path in the complex plane. We simply need to parametrise the path with a real parameter and then complex integrals become remarkably similar to their real counterpart. For example the integral as defined above is linear, that is for $\alpha, \beta \in \mathbb{C}$ and piecewise continuous functions $f, g\colon \mathbb{R} \to \mathbb{C}$

$$\int_a^b [\alpha f(t) + \beta g(t)]\,\mathrm{d}t = \alpha\int_a^b f(t)\,\mathrm{d}t + \beta\int_a^b g(t)\,\mathrm{d}t\,.$$

7.2 Arcs and Contours

The main difference between integrating a function defined on $\mathbb{R}$ and a function defined on $\mathbb{C}$ is that in the second case there are multiple paths we can take to get from a to b. We have to define the path we are integrating along and in general it makes a difference which path we choose. This should be familiar from line integrals from multivariable calculus. In the complex plane we define a contour to integrate along. There are a few restraints on what paths we can choose as a contour.

Definition 30: Smooth arc

The set of points

$$\{\gamma(t) = x(t) + iy(t) \mid x, y\colon \mathbb{R} \to \mathbb{R} \wedge t \in (a, b) \subseteq \mathbb{R}\}$$

is a **smooth arc** with end points $\{a, b\}$ if both $x(t)$ and $y(t)$ are continuous and differentiable and $\gamma'(t) = x'(t) + iy'(t) \neq 0$ for all $t \in (a, b)$.

Notation

We denote by $C^k(X)$ the set of all functions defined on X with at least k continuous derivatives (on some domain of interest). Often we write C^k when the domain is obvious. For example C^0 is the set of continuous functions which may or may not be differentiable and C^1 is the set of differentiable functions with continuous derivatives. Note that since we define $f \in C^k$ to have at least k continuous derivatives we also know that $f \in C^{k'}$ for all $k' \leq k$, for example the function defined by $f(z) = e^z$ is in C^∞ but it is also in C^k fo all $k \in \mathbb{N}$.

Definition 31: Contour

A **contour**, γ, is a continuous chain of a finite number of smooth arcs. We say that γ is piecewise smooth.

The condition that the number of arcs is finite is important.

Example 7.1. Consider the path defined by

$$\gamma(t) = z_0 + te^{i\vartheta}, \qquad t \in (0, 1)$$

for some constant ϑ. To see what this arc looks like consider $\gamma(0) = z_0$ and $\gamma(1) = z_0 + e^{i\vartheta}$. This arc corresponds to a straight line between these two points as can be seen in figure 7.1a. Notice that $\gamma'(t) = e^{i\vartheta}$ is non-zero for all finite ϑ so γ is a smooth arc.

Consider the path defined by

$$\gamma(t) = z_0 + Re^{it}, \qquad t \in (0, \pi/2)$$

for some constant R. To see what this arc looks like consider $\gamma(0) = z_0 + R$ and $\gamma(1) = z_0 + iR$. This arc corresponds to an arc (in the part of a circle sense) centred on z_0 with radius R starting at 0 and going round to $\pi/2$ as can be seen in figure 7.1b. Notice that $\gamma'(t) = iRe^{i}t$ is non-zero for all finite t so γ is a smooth arc.

Consider the path defined by

$$\gamma(t) = t + ie^{-it}, \qquad t \in (0, 3\pi).$$

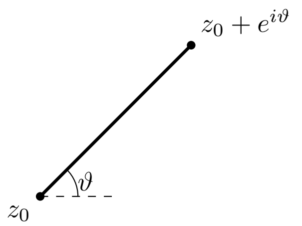
Notice that

$$\mathrm{Re}(z) = t + \cos t, \qquad \text{and} \qquad \mathrm{Im}(z) = 1 - \sin t$$

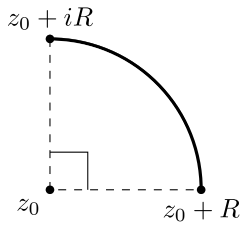
so

$$z'(t) = 1 - \sin t - i\cos t.$$

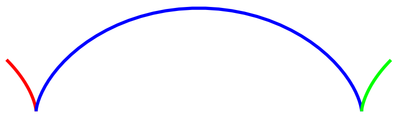
This is zero for $t = \pi/2 + 2n\pi$, $n \in \mathbb{Z}$. Therefore this is *not* a smooth arc. However as we see in figure 7.1c we can decompose it into three smooth arcs which makes it a valid contour.



(a) The contour defined by $\gamma(t) = z_0 + te^{i\vartheta}$ for $t \in (0, 1)$.



(b) The contour defined by $\gamma(t) = z_0 + Re^{it}$ for $t \in (0, \pi/2)$.



(c) The contour defined by $\gamma(t) = t + i + e^{-it}$ for $t \in (0, 3\pi)$. Different colours are used to plot the three smooth arcs that make up the contour. The smooth arcs start and end at the points where $\gamma'(t) = 0$ which is $\pi/2 + 2n\pi$ for $n \in \mathbb{Z}$.

Figure 7.1: Various examples of contours.

7.3 The Integral

Definition 32: Integral

Let γ be a contour. We split γ into n sections and define $\{z_\alpha\}$ as the set of $n+1$ end points of each section. Let $w\colon \mathbb{C} \to \mathbb{C}$ be a piecewise continuous complex function. Define the sum

$$S_n = \sum_{\alpha=1}^{n} w(z_\alpha)(z_\alpha - z_{\alpha-1}).$$

Now let $n \to \infty$ and let the length of each segment $|z_\alpha - z_{\alpha-1}| \to 0$. If this limit exists we say that w is **integrable** on γ. In this case the limit is called the **integral** of w on γ, or in the normal notation

$$\lim_{n\to\infty} S_n = \int_\gamma w(z)\,\mathrm{d}z\,.$$

Lemma 9: Linearity of the integral

Let γ be a contour, let f and g be piecewise continuous functions on γ, and let $\alpha, \beta \in \mathbb{C}$ be constants. Then

$$\int_\gamma (\alpha f(z) + \beta g(z)) \,\mathrm{d}z = \alpha \int_\gamma f(z) \,\mathrm{d}z + \beta \int_\gamma g(z) \,\mathrm{d}z \,.$$

Proof. Split the contour γ, as in the definition of the integral, into n sections with end points z_i and length $|z_i - z_{i-1}|$ such that the of each interval vanishes as $n \to \infty$. Then

$$\begin{aligned}\int_\gamma (\alpha f(z) + \beta g(z)) \,\mathrm{d}z &= \lim_{n\to\infty} [(\alpha f(z_i)\beta g(z_i))(z_i - z_{i-1})] \\ &= \lim_{n\to\infty} [\alpha f(z_i)(z_i - z_{i-1})\beta g(z_i)(z_i - z_{i-1})] \\ &= \alpha \lim_{n\to\infty} f(z_i)(z_i - z_{i-1})\beta \lim_{n\to\infty} g(z_i)(z_i - z_{i-1}) \\ &= \alpha \int_\gamma f(z) \,\mathrm{d}z + \beta \int_\gamma g(z) \,\mathrm{d}z \,.\end{aligned}$$

Here we have used the fact that

$$\lim_{n\to\infty} [aA_n + bB_n] = a \lim_{n\to\infty} A_n + b \lim_{n\to\infty} B_n$$

for some convergent complex sequences (A_n) and (B_n) and complex constants a and b. □

Definition 33: Closed contour

Let γ be a contour with endpoints a and b. If $a = b$ we say that the contour is **closed**.

Note that this sort of closed is not the same as the topological sort of closed (i.e. that the set contains its boundary).

Notation

If γ is a closed contour and w is integrable on γ then we denote the integral by

$$\oint_\gamma w(z) \,\mathrm{d}z \,.$$

As a matter of convention when integrating along a closed contour we go anticlockwise.

7.4 Evaluating Integrals

Now that we have defined an integral we can start actually integrating things. Let $\{\gamma_k\}$ be smooth arcs which together form a contour, γ. Let γ_k be parametrised by some real variable $t \in (a, b)$. Under this parametrisation the points on γ_k can be written as $z = z[\gamma_k(t)] = x(t) + iy(t)$ where $x, y \colon \mathbb{R} \to \mathbb{R}$. Differentiating we have

$$\frac{\mathrm{d}z}{\mathrm{d}t} = x'(t) + iy'(t) = \gamma_k'(t) \implies \mathrm{d}z = [x'(t) + iy(t)] \,\mathrm{d}t = \gamma_k'(t) \,\mathrm{d}t \,.$$

Now let $w \colon \mathbb{C} \to \mathbb{C}$ be integrable on γ and let $w(\gamma(t)) = u(t) + iv(t)$ where $u, v \colon \mathbb{R} \to \mathbb{R}$. Then the substitution $z \to \gamma_k$ gives us

$$\begin{aligned}\int_{\gamma_k} w(z) \,\mathrm{d}z &= \int_a^b w[\gamma_k(t)]\gamma_k'(t) \,\mathrm{d}t \\ &= \int_a^b [u(t) + iv(t)][x'(t) + iy'(t)] \,\mathrm{d}t \\ &= \int_a^b [u(t)x'(t) - v(t)y'(t)] \,\mathrm{d}t + i \int_a^b [v(t)x'(t) + u(t)y'(t)] \,\mathrm{d}t \,.\end{aligned}$$

These integrals can then be easily evaluated using techniques from real integral calculus.

Lemma 10: Integral properties

Let γ be a piecewise smooth contour and let f be a continuous function on γ. Then the following hold.

1. Inverting the contour inverts the integral:

$$\int_{-\gamma} f(z)\,\mathrm{d}z = -\int_{\gamma} f(z)\,\mathrm{d}z\,.$$

2. If γ can be decomposed into two smooth contours γ_1 and γ_2 then

$$\int_{\gamma} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z\,.$$

3. Two different parametrisations of γ will yield the same result. That is if γ_1 and γ_2 are smooth arcs with endpoints $\{a, b\}$ and $\{a', b'\}$ respectively and $g\colon (a, b) \to (a', b')$ such that $g(\gamma_1) = \gamma_2$ then

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z\,.$$

Proof.

1. Consider the definition of the integral over γ

$$\begin{aligned}
-\int_{\gamma} f(z)\,\mathrm{d}z &= -\lim_{n\to\infty}\sum_{\alpha=1}^{n} f(z_\alpha)(z_\alpha - z_{\alpha-1})\\
&= \lim_{n\to\infty}\sum_{\alpha=1}^{n} f(z_\alpha)(z_{\alpha-1} - z_\alpha)\\
&= \int_{-\gamma} f(z)\,\mathrm{d}z\,.
\end{aligned}$$

This proves the first statement.

An alternative proof of this statement can be given by parametrising the contour. Consider a parametrisation of γ by some real variable $t \in (0, 1)$ and notice that $-\gamma$ is then parametrised by the same parametrisation with $1 - t'$ where $t' \in (0, 1)$. The end points are inverted but it is the same set of points. Hence

$$\begin{aligned}
\int_{-\gamma} f(z)\,\mathrm{d}z &= -\int_{t'=0}^{t'=1} f[\gamma(1-t')]\gamma'(1-t')\,\mathrm{d}t'\\
&= \int_{t'=1}^{t'=0} f[\gamma(1-t')]\gamma'(1-t')\,\mathrm{d}t'
\end{aligned}$$

nows substituting $1 - t' = t$

$$\begin{aligned}
&= -\int_{t=0}^{t=1} f[\gamma(t)]\gamma'(t)\,\mathrm{d}t\\
&= -\int_{\gamma} f(z)\,\mathrm{d}z\,.
\end{aligned}$$

This proves the property also.

2. Let γ be parametrised by a real variable $t \in [a, b]$. Let c be such that $\gamma(t)$ corresponds to a point on γ_1 if $t \in (a, c)$ and a point on γ_2 if $t \in (c, b)$. The integrals of the real and imaginary parts of f behave like real integrals and we know for a real integral of some integrable function h on (a, b) that

$$\int_a^b h(t)\,\mathrm{d}t = \int_a^c h(t)\,\mathrm{d}t + \int_c^b h(t)\,\mathrm{d}t$$

where $a \le c \le b$. Applying this to the real and complex parts of the integral of f we have

$$\int_a^b f[\gamma(t)]\gamma'(t) = \int_a^c f[\gamma_1(t)]\gamma_1'(t)\,\mathrm{d}t + \int_c^b f[\gamma_1(t)]\gamma_2'(t)\,\mathrm{d}t\,.$$

Hence

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z\,.$$

This proves the second property.

3. For the final property we have

$$\begin{aligned}\int_{\gamma_2} f(z)\,\mathrm{d}z &= \int_{a'}^{b'} f[\gamma_2(t)]\gamma_2'(t)\,\mathrm{d}t \\ &= \int_{a'}^{b'} f\{g[\gamma_1(t)]\}\frac{\mathrm{d}g[\gamma_1(t)]}{\mathrm{d}t}\,\mathrm{d}t \\ &= \int_{a'}^{b'} f\{g[\gamma_1(t)]\}\,\mathrm{d}g[\gamma_1(t)]\end{aligned}$$

making the substitution $g[\gamma_2(t)] \to \gamma_1(t)$ we have $\mathrm{d}t' = \mathrm{d}g[\gamma_2]$ so we get

$$\begin{aligned}&= \int_a^b f[\gamma_1(t')]\gamma_1'(t)\,\mathrm{d}t' \\ &= \int_{\gamma_1} f(z)\,\mathrm{d}z\,.\end{aligned}$$

This proves the third property.

□

7.4.1 Evaluating Integrals Examples

With these properties we are finally ready for some examples.

Example 7.2. Let $f(z) = z^2$. Find the integral

$$\int_\gamma z^2\,\mathrm{d}z$$

where γ is one of the following contours:

1. *a straight line segment from 0 to $1+i$,*
2. *two straight line segments from 0 to 1 and from 1 to $1+i$.*

1. We can parametrise the first contour as $\gamma(t) = t(1+i)$ with $t \in (0,1)$. Then $\gamma'(t) = 1+i$, which is non-zero as required. The integral is then

$$\int_\gamma z^2\,\mathrm{d}z = \int_0^1 [t(1+i)]^2(1+i)\,\mathrm{d}t = (1+i)^3\int_0^1 t^2\,\mathrm{d}t = (1+i)^3\left[\frac{1}{3}t^3\right]_0^1 = (1+i)^3\frac{1}{3} = \frac{-2+2i}{3}.$$

2. The second contour can be parametrised in two parts by $\gamma_1(t) = t$ for $t \in (0,1)$ and $\gamma_2(t) = 1+it$ for $t \in (0,1)$. The derivatives of the contours are $\gamma_1'(t) = 1$ and $\gamma_2'(t) = i$ which are both non-zero as required. Integrating over these contours we have

$$\int_{\gamma_1} z^2\,\mathrm{d}z = \int_0^1 t^2\,\mathrm{d}t = \frac{1}{3}$$

and

$$\int_{\gamma_2} z^2\,\mathrm{d}z = \int_0^1 (1+it)^2 i\,\mathrm{d}t = i\int_0^t (1-t^2+2it)\,\mathrm{d}t = -1+\frac{2i}{3}.$$

Thus the integral along the whole contour is

$$\int_{\gamma_1} z^2\,\mathrm{d}z + \int_{\gamma_2} z^2\,\mathrm{d}z = \frac{-2+2i}{3}.$$

Notice that both contours gave the same result. This is not the case in general as we will see in the next example.

Example 7.3. *Let $f(z) = 1/z$. Find the integral*

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z$$

where γ is one of the following contours

1. γ_1, the upper semicircle from -1 to 1,

2. γ_2, the lower semicircle from -1 to 1.

1. γ_1 can be parametrised as $\gamma_1(t) = e^{it}$ where $t \in (0, \pi)$. Note that for the direction to be from -1 to 1 we need to start at $t = \pi$. The derivative of this contour parametrisation is $\gamma'(t) = ie^{it}$ which is non-zero on $(0, \pi)$ as required. The integral is then

$$\int_{\gamma_1} \frac{1}{z}\,\mathrm{d}z = \int_\pi^0 \frac{1}{e^{it}} ie^{it}\,\mathrm{d}t = i \in_\pi^0 \mathrm{d}t = -i\pi.$$

2. γ_2 can be parametrised as $\gamma_2(t) = e^{it}$ where $t \in (-\pi, 0)$. The derivative of this contour parametrisation is $\gamma'(t) = ie^{it}$ which is non-zero on $(-\pi, 0)$ as required. The integral is then

$$\int_{\gamma_2} \frac{1}{z}\,\mathrm{d}z = \int_{-\pi}^0 \frac{1}{e^{it}} ie^{it}\,\mathrm{d}t = i \in_{-\pi}^0 \mathrm{d}t = i\pi.$$

We see that the value of the integral depends on the contour chosen.

Example 7.4. *Let $f(z) = |z|^2$. Find the integral*

$$\int_\gamma f(z)\,\mathrm{d}z$$

where γ is one of the following contours

1. γ_1, a straight segment from -1 to i,

2. γ_2, a $\pi/2$ circular arc of radius 1 from -1 to i.

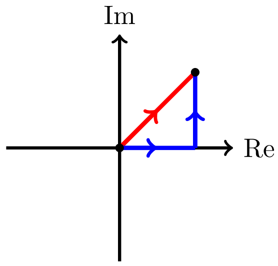
1. γ_1 can be parametrised as $\gamma_1(t) = -1 + (1 + i)t$ with $t \in (0, 1)$. The derivative of this contour parametrisation is $\gamma_1'(t) = 1 + i$ which is non-zero as required. The integral is then

$$\int_{\gamma_1} |z|^2 = \int_0^1 |t - 1 + it|(1 + i)\,\mathrm{d}t = (1 + i)\int_0^1 [(t - 1)^2 + t^2]\,\mathrm{d}t = \frac{2}{3}(1 + i).$$

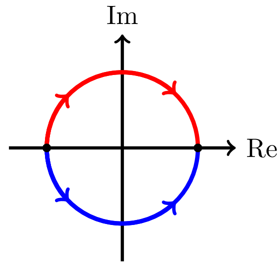
2. γ_2 can be parametrised as $\gamma_2(t) = e^{it}$ where $t \in (\pi/2, \pi)$. Note that for the direction to be from -1 to i we need to start at $t = \pi$. The derivative of this contour parametrisation is $\gamma'(t) = ie^{it}$ which is non-zero on $(\pi/2, \pi)$ as required. The integral is then

$$\int_{\gamma_2} |z|^2\,\mathrm{d}z = \int_\pi^{\pi/2} \left|e^{it}\right|^2 ie^{it}\,\mathrm{d}t = \int_\pi^{\pi/2} ie^{it}\,\mathrm{d}t = i + 1.$$

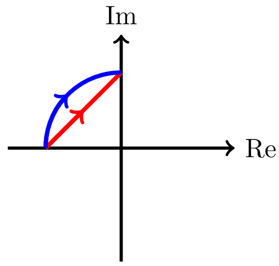
Again we see that the result depends on the contour.



(a) The contours in example 7.2



(b) The contours in example 7.3



(c) The contours in example 7.3

8 Cauchy's Integral Theorem

8.1 Bounding Integrals

Lemma 11: Integral bound

Let γ be a contour and let f be a piecewise continuous function on γ. Then

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \le \int_\gamma |f(z)\,\mathrm{d}z| = \int_\gamma |f(z)||\mathrm{d}z|.$$

Proof. Let

$$I = \int_\gamma f(z)\,\mathrm{d}z\,.$$

Since this is a complex integral in general $I \in \mathbb{C}$. Therefore we can write I as $I = Re^{i\vartheta}$. This lemma is interested in providing a bound on $R = |I|$. Note that $e^{-i\vartheta}I = e^{-i\vartheta}Re^{i\vartheta} = R$ and that $\left|e^{i\vartheta}\right| = 1$. Then

$$\begin{aligned}\left|\int_\gamma f(z)\,\mathrm{d}z\right| &= e^{i\vartheta}\int_\gamma f(z)\,\mathrm{d}z\\ &= \int_\gamma e^{-i\vartheta} f(z)\,\mathrm{d}z\,.\end{aligned}$$

The integral in the last step is real since it is equal to the absolute value of another integral. This means we can take the real part without changing anything so

$$\begin{aligned}\left|\int_\gamma f(z)\,\mathrm{d}z\right| &= \mathrm{Re}\left[\int_\gamma e^{-i\vartheta} f(z)\,\mathrm{d}z\right]\\ &= \int_\gamma \mathrm{Re}[e^{-i\vartheta} f(z)\,\mathrm{d}z]\end{aligned}$$

Now we use $\mathrm{Re}(z) \le |z|$ from the first point of theorem 3 and we get

$$\begin{aligned}\left|\int_\gamma f(z)\,\mathrm{d}z\right| &\le \int_\gamma |e^{-i\vartheta} f(z)\,\mathrm{d}z|\\ &= \int_\gamma |f(z)\,\mathrm{d}z|\end{aligned}$$

$$= \int_\gamma |f(z)||\mathrm{d}z|.$$

□

Note that we have to take the absolute value of the measure, $\mathrm{d}z$, for this inequality to hold. This will be important in the next lemma. Consider

$$\int_\gamma |\mathrm{d}z|$$

for some contour γ. Appealing to the definition of the integral this is simply

$$\int_\gamma |\mathrm{d}z| = \lim_{N\to\infty} \sum_{i=1}^{N} |z_i - z_{i-1}|.$$

That is the sum of the length of all of the segments of the contour, which when summed just gives the length of the contour, L.

Lemma 12: ML lemma

Let γ be a contour on $\mathbb{C}$ with length L. Let f be a function that is piecewise continuous on γ. Let M be a real constant that bounds $|f(z)|$ on γ. That is for all $z \in \gamma$ $M \geq |f(z)|$. Then

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leq ML.$$

Proof. By lemma 11 we know that

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leq \int_\gamma |f(z)||\mathrm{d}z|.$$

Clearly if we replace $|f(z)|$ with M the value of the integral can only increase. Therefore

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leq \int_\gamma M|\mathrm{d}z| = M\int_\gamma |\mathrm{d}z| = ML.$$

□

Example 8.1. *Let $\alpha \in \mathbb{R}$. Show, using integral bounds, that $|\exp(2\pi\alpha i) - 1| \leq 2\pi|\alpha|$.*

If $\alpha = 0$ then equality holds. Therefore assume $\alpha \neq 0$. We consider

$$\int_\gamma e^{i\alpha z}\,\mathrm{d}z$$

where γ is a straight line from 0 to 2π. Applying a bound to the integral we have

$$\left|\int_\gamma e^{i\alpha z}\,\mathrm{d}z\right| \leq \int_\gamma \left|e^{-i\alpha z}\,\mathrm{d}z\right| = \int_0^{2\pi} \left|e^{i\alpha t}\right|\mathrm{d}t = \int_0^{2\pi} \mathrm{d}t = 2\pi.$$

Evaluating the integral instead gives

$$\left|\int_\gamma e^{i\alpha z}\,\mathrm{d}z\right| = \left|\int_0^{2\pi} e^{i\alpha t}\,\mathrm{d}t\right| = \left|\left[\frac{1}{i\alpha}e^{i\alpha z}\right]_0^{2\pi}\right| = \left|\frac{1}{\alpha}\right|\left|e^{2\pi i\alpha} - 1\right|.$$

Comparing these we have

$$\left|e^{2i\alpha} - 1\right| \leq 2\pi|\alpha|.$$

Example 8.2. Let γ be a circle of radius R. Estimate the following limits:

$$\lim_{R\to\infty}\oint_\gamma \frac{z^2}{z^4+1}\,\mathrm{d}z\,, \qquad \text{and} \qquad \lim_{R\to 0}\oint_\gamma \frac{z^2}{z^4+1}\,\mathrm{d}z\,.$$

We can parametrise γ as $\gamma(t) = Re^{it}$ for $t \in (0, 2\pi)$. Therefore $\gamma'(t) = iRe^{it}$ which is non-zero as required. We will also need another result that we can derive from the third point of theorem 3 which states that for $\alpha, \beta \in \mathbb{C}$

$$|\alpha - \beta| \geq ||\alpha| - |\beta||.$$

Inverting this we have

$$\frac{1}{|\alpha-\beta|} \leq \frac{1}{||\alpha| - |\beta||}.$$

We can now bound the integrals.

$$\begin{aligned}
\left|\oint_\gamma \frac{z^2}{z^4+1}\,\mathrm{d}z\right| &\leq \oint_\gamma \left|\frac{z^2}{z^4+1}\,\mathrm{d}z\right| \\
&= \oint_\gamma \frac{|z^2|}{|z^4+1|}|\mathrm{d}z| \\
&= \int_0^{2\pi} \frac{R^2e^{2it}}{|R^4e^{4it}+1|}\left|Re^it\,\mathrm{d}t\right| \\
&= \int_0^{2\pi} \frac{R^3}{|R^4e^{4it}+1|}\,\mathrm{d}t\,.
\end{aligned}$$

Considering the specific case $R \to \infty$ we have

$$\begin{aligned}
\lim_{R\to\infty}\left|\oint_\gamma \frac{z^2}{z^4+1}\,\mathrm{d}z\right| &\leq \lim_{R\to\infty}\int_0^{2\pi} \frac{R^3}{|R^4e^{4it}+1|}\,\mathrm{d}t \\
&\leq \lim_{R\to\infty}\int_0^{2\pi} \frac{R^3}{|R^4e^{4it}| - |1|}\,\mathrm{d}t \\
&= \lim_{R\to\infty}\frac{R^3}{R^4-1}\int_0^{2\pi}\mathrm{d}t \\
&= \lim_{R\to\infty}\frac{2\pi R^3}{R^4-1} \\
&= 0.
\end{aligned}$$

Here we have used $1 = -(-1)$ and the inequality we derived at the start of the example. Similarly considering $R \to 0$ we have

$$\begin{aligned}
\lim_{R\to 0}\left|\oint_\gamma \frac{z^2}{z^4+1}\,\mathrm{d}z\right| &\leq \lim_{R\to 0}\int_0^{2\pi} \frac{R^3}{|R^4e^{4it}+1|}\,\mathrm{d}t \\
&\leq \lim_{R\to 0}\int_0^{2\pi} \frac{R^3}{|1| - |R^4e^{4it}|}\,\mathrm{d}t \\
&= \lim_{R\to 0}\frac{R^3}{1-R^3}\int_0^{2\pi}\mathrm{d}t \\
&= \lim_{R\to 0}\frac{2\pi R^3}{1-R^4} \\
&= 0.
\end{aligned}$$

8.2 The Fundamental Theorem of Calculus

Previously we saw that the value of an integral depends on the contour. We have also seen an example where this wasn't the case. In this section we will establish conditions on a function such that its integral between two points is independent of the contour.

Definition 34: Antiderivative

Let S be a region of $\mathbb{C}$ and let f be a continuous function on S. We say that $F\colon S \to \mathbb{C}$ is the **antiderivative** of f if for all $z \in S$ $F'(z) = f(z)$.

Notice that this means that F must be differentiable on S which in turn means that F must be analytic inside S but not necessarily on the border.

For example $f(z) = z^2$ has as an antiderivative $F(z) = z^3/3$. The function $F(z) = \log z$ is in general *not* the antiderivative of $f(z) = 1/z$ as to make F single valued and therefore a valid function we have to make a branch cut. This means that F is only an antiderivative of $1/z$ on regions that don't surround the origin.

Theorem 6: Fundamental theorem of calculus

Let S be an open region in $\mathbb{C}$ and let $f\colon S \to \mathbb{C}$ be continuous. Then the following statements are equivalent:

1. f has an antiderivative, F, on S.
2. Let γ and γ' be arbitrary contours in S such that both have the same endpoints. Then
$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\gamma'} f(z)\,\mathrm{d}z\,.$$
3. For any given closed contour γ in S
$$\oint_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof. We will start by proving that the first statement implies the second. Suppose there exists a function, F, such that F is the antiderivative of f. Let γ be a contour parametrised by some real variable, t and let γ have endpoints t_i and t_f. Applying the chain rule we have

$$[F(\gamma(t))]' = F'[\gamma(t)]\gamma'(t) = f(\gamma(t))\gamma'(t)$$

having used the definition $F'(z) = f(z)$. We can write $F(\gamma(t)) = U(t) + iV(t)$ where $U, V\colon \mathbb{R} \to \mathbb{R}$ and satisfy the real fundamental theorem of calculus. Therefore

$$\begin{aligned}\int_\gamma f(z)\,\mathrm{d}z &= \int_{t_i}^{t_f} f[\gamma(t)]\gamma'(t)\,\mathrm{d}t\\ &= \int_{t_i}^{t_f} [F(\gamma(t))]'\,\mathrm{d}t\\ &= \int_{t_i}^{t_f} [U + iV(t)]'\,\mathrm{d}t\\ &= [U(t) + iV(T)]_{t_i}^{t_f}\\ &= F[\gamma(t_f)] - F[\gamma(t_f)].\end{aligned}$$

This depends only on the endpoints of γ proving that the existence of the antiderivative implies the equality of integrals over different contours.

Next we will prove that the second statement proves the third. Combining this with the first part of this proof this also proves that the first statement implies the third by transitivity of implication. Let γ_1 and γ_2 be contours joining points a and b in S. Then if we start at a and first traverse γ_1 we get to b and the traversing γ_2 backwards takes us back to a so $\gamma = \gamma_1 - \gamma_2$ is a closed contour. Assuming that the integral over any contour from a to b is the same we have

$$\begin{aligned}0 &= \int_{\gamma_1} f(z)\,\mathrm{d}z - \int_{\gamma_2} f(z)\,\mathrm{d}z\\ &= \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{-\gamma_2} f(z)\,\mathrm{d}z\end{aligned}$$

$$= \oint_{\gamma} f(z)\,\mathrm{d}z\,.$$

Hence independence of contours leads to all integrals over a closed contour being zero.

Next we will show that the third statement implies the second. Let γ be a closed contour and choose $a, b \in \gamma$ such that γ is split into two contours, γ_1 and $-\gamma_2$. Then by hypothesis the integral over γ is zero so

$$\begin{aligned} 0 &= \oint_{\gamma} f(z)\,\mathrm{d}z \\ &= \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{-\gamma_2} f(z)\,\mathrm{d}z \\ &= \int_{\gamma_1} f(z)\,\mathrm{d}z - \int_{\gamma_2} f(z)\,\mathrm{d}z \end{aligned}$$

so

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2 f(z)} \mathrm{d}z\,.$$

Finally we need to prove that either of the second two statements implies the first. Let γ be some contour with endpoints z_0 and z. Parametrise γ with some real variable ζ. Assuming that the integral is independent of the contour chosen the following definition of a function, F, is well defined as it depends only on the endpoints of the contour.

$$F(z) = \int_{z_0}^{z} f(\zeta)\,\mathrm{d}\zeta\,.$$

To show the first statement holds we need to show $F'(z) = f(z)$ for all $z \in S$. To do this we appeal to the definition of the derivative. First we write

$$\begin{aligned} F(z+\Delta z) - F(z) &= \int_{z_0}^{z+\Delta z} f(\zeta)\,\mathrm{d}\zeta - \int_{z_0}^{z} f(\zeta)\,\mathrm{d}\zeta \\ &= \int_{z}^{z+\Delta z} f(\zeta)\,\mathrm{d}\zeta\,. \end{aligned}$$

We now use a trick of adding zero to get

$$\int_{z}^{z+\Delta z} f(\zeta)\,\mathrm{d}\zeta = \int_{z}^{z+\Delta z} f(z) - f(z) + f(\zeta)\,\mathrm{d}\zeta\,.$$

We now divide by Δz and get

$$\begin{aligned} \frac{F(z+\Delta z) - F(z)}{\Delta z} &= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z) - f(z) + f(\zeta)\,\mathrm{d}\zeta \\ &= \frac{f(z)}{\Delta z}\int_{z}^{z+\Delta z} \mathrm{d}\zeta + \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\zeta) - f(z)\,\mathrm{d}\zeta \\ &= \frac{f(z)}{\Delta z}\Delta z + \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\zeta) - f(z)\,\mathrm{d}\zeta \\ &= f(z) + \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\zeta) - f(z)\,\mathrm{d}\zeta\,. \end{aligned}$$

We now take a limit as $\Delta z \to 0$ and we have

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z) + \lim_{\Delta z \to 0} \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\zeta) - f(z)\,\mathrm{d}\zeta\,.$$

We see that if the limit on the right hand side vanishes then we will be left with $F'(z) = f(z)$ which is what we want. To show that this integral does indeed vanish we use the ML lemma (lemma 12). Since S is open and $\Delta z \to 0$ we are safe to assume that $z + \Delta z \in D(z;\delta)$ for some $\delta > 0$ and that $D \subseteq S$. We are free to choose any contour as by hypothesis the integral does not depend on the contour. Therefore we choose a straight line from z to $z + \Delta z$. The length of this contour is $L = |\Delta z|$.

Since f is continuous in S, and therefore continuous in D, by definition of continuity if $|z - z_0| < \delta$ for some $\delta > 0$ then there exists $\varepsilon > 0$ such that $|f(z) - f(z_0)| < \varepsilon$. Specifically if δ is allowed to be arbitrarily small then ε can become arbitrarily small. We choose a δ which is slightly bigger than $|\Delta z|$ and this means that $|f(\zeta) - f(z)| < \varepsilon$ for some ε. Let $M = \varepsilon$ as clearly this bounds $|f(\zeta) - f(z)|$. Then by the ML lemma

$$\left| \lim_{\Delta z \to 0} \frac{1}{\Delta z} \int_z^{z+\Delta z} f(\zeta) - f(z) \, \mathrm{d}z \right| \le \lim_{\Delta z \to 0} \frac{\varepsilon |\Delta z|}{\Delta z} = \lim_{\Delta z \to 0} \varepsilon = 0.$$

This proves that

$$f(z) = \lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = F'(z).$$

□

8.3 Cauchy's Integral Theorem

The fundamental theorem of calculus (theorem 6) tells us that if $f\colon S \subseteq \mathbb{C} \to \mathbb{C}$ is continuous then it has an antiderivative if and only if all of its closed contour integrals vanish. However in general there are an infinite number closed contours and checking them all is not possible. Fortunately there is another theorem that we will prove in this section which tells us exactly when this is the case.

Definition 35: Simply connected region

Let $S \subseteq \mathbb{C}$ be an open region. We say that S is **simply connected** if any path between two points in S can be continuously transformed into any other path with the same endpoints without leaving S. Equivalently S is **simply connected** if any closed path inside S can be contracted into a point without leaving S.

Intuitively S is simply connected if it connected has no holes. This is because for us to deform a path to a path on the other side of a hole we would have to either jump over the hole, breaking the continuity condition, or leave S.

Definition 36: Simple contour

A contour, γ, is a **simple contour** if it does not intersect itself.

We are now ready for perhaps the most important theorem of this course.

Theorem 7: Cauchy's integral theorem

Let S be an open, simply connected region. Let $f\colon S \to \mathbb{C}$ be analytic in S and let $f'(z)$ be continuous in S. Then for any given closed contour γ

$$\oint_\gamma f(z) \, \mathrm{d}z = 0.$$

Proof. We assume without loss of generality that γ is a simple contour. If this is not the case and γ has a finite number of self intersections then we can split γ at each self intersection into two separate contours and the following applies to each sub-contour individually. If γ intersects itself an infinite number of times then this theorem can be proven using an auxiliary contour that only intersects the original at its endpoints. This produces two simple contours and then by the fundamental theorem of calculus (theorem 6) it is possible to prove that Cauchy's theorem also holds for the original contour.

Any simple closed contour divides $\mathbb{C}$ into two simply connected components[9]. Let S' be the interior of this curve. For this proof we will use the $\mathbb{R}^2$ version of Green's theorem which states that if $P, Q\colon \mathbb{R}^2 \to \mathbb{R}$

[9] while stated here as an obvious fact this is actually known as the Jordan curve theorem and is actually not very easy to prove.

are C^1 functions on a simply connected region, $\tilde{S}$ then on a simple closed contour, $\partial\tilde{S}$, bounding $\tilde{S}$ we have

$$\begin{aligned}\oint_{\partial\tilde{S}}(P(\boldsymbol{r})\hat{\boldsymbol{x}}+Q(\boldsymbol{r})\hat{\boldsymbol{y}})\cdot\mathrm{d}r &= \oint_{\partial\tilde{S}}[P(x,y)\,\mathrm{d}x+Q(x,y)\,\mathrm{d}y]\\ &= \iint_{\tilde{S}}\left[\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right]\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

Since f' is continuous by hypothesis its real and imaginary parts, $u, v\colon \mathbb{R}^2\to\mathbb{R}$, individually are C^1 and satisfy the conditions for Green's theorem to apply. Thus

$$\begin{aligned}\oint_\gamma f(z)\,\mathrm{d}z &= \oint_\gamma [u(x,y)+iv(x,y)](\mathrm{d}x+i\,\mathrm{d}y)\\ &= \oint_\gamma [u(x,y)\,\mathrm{d}x - v(x,y)\,\mathrm{d}y] + i\oint_\gamma [u(x,y)\,\mathrm{d}y+v(x,y)\,\mathrm{d}x].\end{aligned}$$

Applying Green's theorem to the real part of the integral and also the Cauchy–Riemann relations we have

$$\begin{aligned}\oint_\gamma [u(x,y)\,\mathrm{d}x - v(x,y)\,\mathrm{d}y] &= \iint_{S'}\left[-\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right]\mathrm{d}x\,\mathrm{d}y\\ &= \iint_{S'}[-v_x-u_y]\,\mathrm{d}x\,\mathrm{d}y\\ &= \iint_{S'}[u_y-u_y]\,\mathrm{d}x\,\mathrm{d}y\\ &= 0.\end{aligned}$$

Applying Green's theorem to the imaginary part of the integral and also the Cauchy–Riemann relations we have

$$\begin{aligned}\oint_\gamma [v(x,y)\,\mathrm{d}x + u(x,y)\,\mathrm{d}y] &= \iint_{S'}\left[\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right]\mathrm{d}x\,\mathrm{d}y\\ &= \iint_{S'}[u_x-v_y]\,\mathrm{d}x\,\mathrm{d}y\\ &= \iint_{S'}[u_x-u_x]\,\mathrm{d}x\,\mathrm{d}y\\ &= 0.\end{aligned}$$

So we find that

$$\oint_\gamma f(Z)\,\mathrm{d}z = 0.$$

□

This result is useful but the condition that f' is continuous is actually be dropped as analyticity is such a strict condition. The theorem then requires a different proof as we can no longer use Green's theorem directly. Instead the approach to proving this theorem is similar to the proof of Green's theorem. This version of the theorem is often called the Cauchy–Goursat theorem.

Theorem 8: Cauchy–Goursat theorem

Let S be an open simply connected region and let $f\colon S\to\mathbb{C}$ be analytic in S. Then for any given closed contour γ

$$\oint_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof. Again we assume without loss of generality that γ is a simple contour. We also assume that it is oriented anticlockwise. Let S' be the region bounded by γ. We split S into a grid of smaller square regions each with a contour, γ_i, as a border. Where one of these squares intersects γ we take γ_i as the square border inside S' and the contour γ as the border where the square would leave S'.

We now consider the sum of the integrals over each γ_i. We use the fact that every edge of a square is traversed in both directions and so the contributions cancel. Since the number of squares is finite we have

$$\sum_i \oint_{\gamma_i} f(z)\,\mathrm{d}z = \oint_{\gamma} f(z)\,\mathrm{d}z\,.$$

For each square contour we now define the following function

$$\delta_i(z) = \begin{cases} [f(z) - f(z_i)]/[z - z_i] - f'(z_i), & z \neq z_i \\ 0, & z = z_i \end{cases}$$

where z_i is an arbitrary fixed point in γ_i. In the limit $z \to z_i$ the fraction in the definition of δ_i approaches $f'(z)$. Since f is analytic f' exists not only at z_i but in a neighbourhood of z_i. Therefore we can choose our square contours in such a way that

$$\left|\frac{f(z) - f(z_i)}{z - z_i} - f'(z_i)\right| \le \varepsilon$$

for some arbitrarily small $\varepsilon > 0$ simply by choosing sufficiently small squares. From this we can write

$$f(z) = f(z_i) + f'(z_i)(z - z_i) + (z - z_i)\delta_i(z)$$

for some $z \in \gamma_i$. Hence

$$\begin{aligned} \oint_{\gamma_i} f(z)\,\mathrm{d}z &= \oint_{\gamma_i} f(z_i)\,\mathrm{d}z + \oint_{\gamma_i} f'(z_i)(z - z_i)\,\mathrm{d}z + \oint_{\gamma_i} (z - z_i)\delta_i(z)\,\mathrm{d}z \\ &= f(z_i) \oint \gamma_i\,\mathrm{d}z + f'(z_i) \oint_{\gamma_i} (z - z_i)\,\mathrm{d}z + \oint (z - z_i)\delta_i(z)\,\mathrm{d}z\,. \end{aligned}$$

The first two integrals vanish as the integrands satisfy the conditions of Cauchy's integral theorem (theorem 7). The third integral can be bounded. If we take the side of one of the squares to be l_i then γ_i is at most as long as $4l_i$. Also $|z - z_i| < \sqrt{2}l_i$ as the furthest two points in a square can be is diagonally opposite. The length of γ_i is then less that $4l_i + L_i$ where L_i is the length of the portion of γ that is contained in γ_i for squares on the boundary. Therefore by the ML lemma.

$$\left|\oint_{\gamma_i} (z - z_i)\delta_i(z)\,\mathrm{d}z\right| \le \sqrt{2}l_i\varepsilon(4l_i + L_i).$$

This means that

$$\left|\oint_{\gamma} f(z)\,\mathrm{d}z\right| = \left|\sum_i \oint_{\gamma_i} f(z)\,\mathrm{d}z\right|$$

applying the triangle inequality for multiple summands this becomes

$$\begin{aligned} &\le \sum_i \left|\oint_{\gamma_i} f(z)\,\mathrm{d}z\right| \\ &\le \sum_i \sqrt{2}l_i\varepsilon(4l_i + L_i). \end{aligned}$$

Now notice that

$$\sum_i l_i^2 = A$$

is the total area of the internal squares and that

$$\sum_i L_i = L$$

is the length of γ. Since L_i is zero for internal squares we can perform this sum only for external squares. Therefore

$$\sum_i l_i(4l_i + L_i) = 4\sum_{\text{int } i} l_i^2 + \sum_{\text{ext } i} l_i L_i$$

$$= 4A + L\sqrt{A}.$$

Since we can make ε arbitrarily small and the length of a smooth contour is finite we have

$$\left|\oint_\gamma f(z)\,\mathrm{d}z\right| \le \sqrt{2}\varepsilon(4A + L\sqrt{A}) \to 0.$$

And since the absolute value must be non-negative we have

$$\oint_\gamma f(z)\,\mathrm{d}z = 0.$$

□

This proof worked because analyticity is such a strict condition. While we lost the condition that f' be continuous we were able to use analyticity to construct δ_i which is continuous and plays the role of f' in this proof.

9 Consequence's of the Cauchy–Goursat Theorem

9.1 Deformation Theorem

Theorem 9: Deformation theorem

Let S be an open region and let γ_1 and γ_2 be positively oriented closed contours in S. Let $f\colon S \to \mathbb{C}$ be analytic in the region between the two contours. Then

$$\oint_{\gamma_1} f(z)\,\mathrm{d}z = \oint_{\gamma_2} \mathrm{d}z\,.$$

Proof. Without loss of generality we can assume that the contours do not intersect. Consider the contours shown in figure 9.1. Choose two points, z_1 on γ_1 and z_2 on γ_2. Take an arbitrary contour $\overline{z_1 z_2}$ from z_1 to z_2 such that this contour remains entirely in the region where f is analytic. Construct the auxiliary

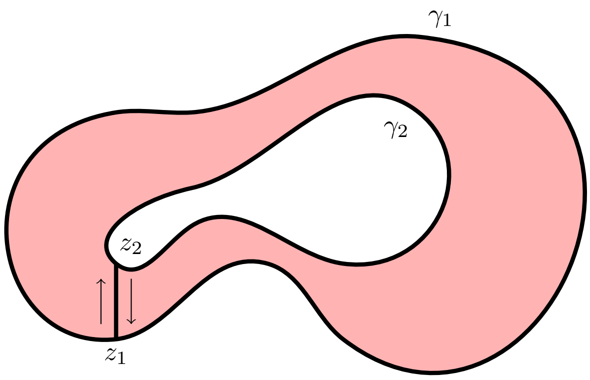


Figure 9.1: Two closed, non-intersecting contours with f analytic on the red region.

contour

$$\gamma = \gamma_1 + \overline{z_1 z_2} - \gamma_2 + \overline{z_2 z_1}.$$

Since γ encloses a region where f is analytic the integral along γ must be zero by the Cauchy–Goursat theorem (theorem 8). Hence

$$0 = \oint_\gamma f(z)\,\mathrm{d}z$$

$$
\begin{aligned}
&= \oint_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\overline{z_1 z_2}} f(z)\,\mathrm{d}z + \oint_{-\gamma_2} f(z)\,\mathrm{d}z + \int_{\overline{z_2 z_1}} f(z)\,\mathrm{d}z \\
&= \oint_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\overline{z_1 z_2}} - \oint_{\gamma_2} f(z)\,\mathrm{d}z - \int_{\overline{z_1 z_2}} f(z)\,\mathrm{d}z \\
&= \oint_{\gamma_1} f(z)\,\mathrm{d}z - \oint_{\gamma_2} f(z)\,\mathrm{d}z\,.
\end{aligned}
$$

So

$$\oint_{\gamma_1} f(z)\,\mathrm{d}z = \oint_{\gamma_2} f(z)\,\mathrm{d}z\,.$$

□

Example 9.1. Let $f(z) = 1/z$. Let γ_1 be the positively oriented unit circle, $|z| = 1$, and let γ_2 be any arbitrary contour containing the origin. Combining the two contours in example 7.3 we see that

$$\oint_{\gamma_1} \frac{1}{z}\,\mathrm{d}z = 2\pi i.$$

Using the deformation theorem (theorem 9) since $1/z$ is analytic for all $z \neq 0$ and 0 is not between the two contours we have

$$\oint_{\gamma_2} \frac{1}{z}\,\mathrm{d}z = \oint_{\gamma_1} \frac{1}{z}\,\mathrm{d}z = 2\pi i.$$

9.2 Cauchy's Integral Formula

Theorem 10: Cauchy's Integral Formula

Let γ be a closed, positively oriented contour. Let S be an open, simply connected region containing γ. Let $f\colon S \to \mathbb{C}$ be analytic on S. Let z_0 be any point inside γ. Then

$$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - z_0}\,\mathrm{d}z\,.$$

Proof. Since f and $z - z_0$ are analytic on S the only point where $f(z)/(z - z_0)$ is not analytic is $z = z_0$. Let γ_ε be the contour consisting of the circle centred on z_0 with radius ε, that is $|z - z_0| = \varepsilon$. Since f is analytic in the region between γ and γ_ε we have

$$
\begin{aligned}
\int_{\gamma} \frac{f(z)}{z - z_0}\,\mathrm{d}z &= \oint_{\gamma_\varepsilon} \frac{f(z)}{z - z_0}\,\mathrm{d}z \\
&= \oint_{\gamma_\varepsilon} \frac{f(z_0) + f(z) - f(z_0)}{z - z_0}\,\mathrm{d}z \\
&= \oint_{\gamma_\varepsilon} \frac{f(z_0)}{z - z_0}\,\mathrm{d}z + \oint_{\gamma_\varepsilon} \frac{f(z) - f(z_0)}{z - z_0}\,\mathrm{d}z\,.
\end{aligned}
$$

Since $f(z_0)$ is a constant the first integral is

$$
\begin{aligned}
\oint_{\gamma_\varepsilon} \frac{f(z_0)}{z - z_0}\,\mathrm{d}z &= f(z_0) \oint_{\gamma_\varepsilon} \frac{1}{z - z_0}\,\mathrm{d}z \\
&= f(z_0) 2\pi i.
\end{aligned}
$$

We can bound the second integral. Consider the parametrisation $\gamma_\varepsilon(\vartheta) = z_0 + \varepsilon e^{i\vartheta}$ for $\vartheta \in [0, 2\pi]$. $\gamma_\varepsilon'(\vartheta) = i\varepsilon e^{i\vartheta} \neq 0$, therefore

$$
\begin{aligned}
\left| \oint_{\gamma_\varepsilon} \frac{f(z) - f(z_0)}{z - z_0}\,\mathrm{d}z \right| &\leq \oint_{\gamma_\varepsilon} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| |\mathrm{d}z| \\
&= \int_0^{2\pi} \left| \frac{f(z) - f(z_0)}{(z_0 + \varepsilon e^{i\vartheta}) - z_0} \right| \left| i\varepsilon e^{i\vartheta} \right| \mathrm{d}\vartheta
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{2\pi} \left| \frac{f(z) - f(z_0)}{\varepsilon e^{i\vartheta}} \right| \left| i\varepsilon e^{i\vartheta} \right| \mathrm{d}\vartheta \\
&= \int_0^{2\pi} |f(z) - f(z_0)||i| \,\mathrm{d}\vartheta \\
&= \int_0^{2\pi} |f(z) - f(z_0)| \,\mathrm{d}\vartheta\,.
\end{aligned}
$$

Since f is analytic it is also continuous which means there exists arbitrarily small $\delta > 0$ such that by choosing a sufficiently small ε we have $|f(z) - f(z_0)| < \delta$. Thus

$$\left| \oint_{\gamma_\varepsilon} \frac{f(z) - f(z_0)}{z - z_0} \,\mathrm{d}z \right| \le \int_0^{2\pi} |f(z) - f(z_0)| \,\mathrm{d}\vartheta \le 2\pi\delta \to 0 \qquad \text{as} \qquad \varepsilon \to 0.$$

Therefore

$$\oint_\gamma \frac{f(z)}{z - z_0} \,\mathrm{d}z = 2\pi i f(z_0).$$

□

This is a very strong result. It tells us that the values of an analytic function inside a simply connected region are completely determined by the values of the function on the boundary no matter how large the region. We can use this result to compute many integrals without needing to integrate properly. The general process for this is as follows:

1. Plot the contour.
2. Find the singularities of f inside the contour.
3. Split the contour into smaller pieces each containing at most one singularity.
4. Evaluate each sub-contour separately by re-expressing $f(z)$ as

$$f(z) = \frac{g(z)}{z - z_0}.$$

5. Add the values of all contours being careful to consider directions so that the required parts cancel.

This will be demonstrated with the next few examples.

Example 9.2. *Let* $f(z) = \cos(z)/z$. *Evaluate*

$$\oint_\gamma f(z) \,\mathrm{d}z$$

where γ *is the circle* $|z - 4| = 5$.

The only singularity that f has is $z = 0$ which is contained in γ. We can write f as

$$f(z) = \frac{\cos z}{z} = \frac{g(z)}{z}$$

where $g(z) = \cos(z)$. We can directly apply Cauchy's integral formula then:

$$\oint_\gamma \frac{\cos z}{z - 0} \,\mathrm{d}z = 2\pi_i g(0) = 2\pi i \cos(0) = 2\pi i.$$

Example 9.3. *Let* $f(z) = z^2/(z^2 + 1)$. *Evaluate*

$$\oint_\gamma f(z) \,\mathrm{d}z$$

where γ *is the circle* $|z - i| = 1$.

First notice that

$$f(z) = \frac{z^2}{(z+i)(z-i)}$$

so $z = \pm i$ are the only singularities of f. Of these singularities $z = i$ is the only singularity inside the contour. Therefore

$$g(z) = \frac{z^2}{z+i}$$

is analytic inside γ. We can then directly apply Cauchy's integral formula:

$$\begin{aligned}\oint_\gamma \frac{z^2}{z^2+1}\,\mathrm{d}z &= \oint_\gamma \frac{g(z)}{z-i}\,\mathrm{d}z \\ &= 2\pi i g(i) \\ &= 2\pi i \frac{i^2}{i+i} \\ &= -\pi.\end{aligned}$$

Example 9.4. *Let* $f(z) = \cos(\pi z)/(z^2-1)$. *Evaluate*

$$\oint_\gamma f(z)\,\mathrm{d}z$$

where γ *is the circle* $|z| = 2$.

f has two singularities, $z = \pm 1$. Both of these lie inside γ. We split γ into two parts, γ_1 and γ_2 such that $\gamma_1 + \gamma_2 = \gamma$. One way to do this is shown in figure 9.2. Notice that the part of the contour that

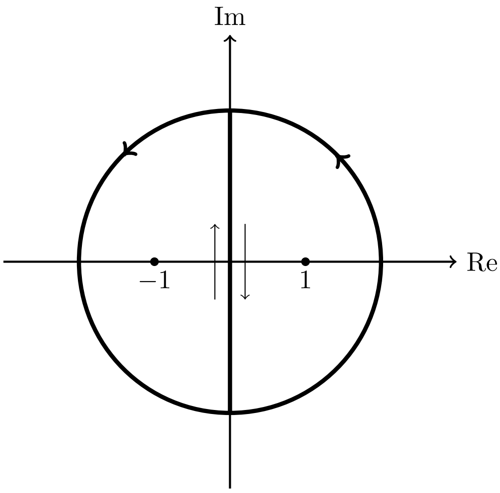


Figure 9.2: One possible way to split the contour into two parts.

appears in both sub-contours is in opposite directions and so that part of the integral cancels. We can now write

$$f(z) = \frac{\cos(\pi z)}{z^2-1} = \frac{\cos(\pi z)}{(z+1)(z-1)}.$$

Considering first the sub-contour, γ_+, containing $z = 1$ we can write

$$f(z) = \frac{g_+(z)}{z-1}, \qquad \text{where} \qquad g_+(z) = \frac{\cos(\pi z)}{z+1}$$

and so using Cauchy's integral formula

$$\oint_{\gamma_+} \frac{\cos(\pi z)}{z^2-1}\,\mathrm{d}z = \oint_{\gamma_+} \frac{g_+(z)}{z-1}\,\mathrm{d}z = 2\pi i g_+(1) = 2\pi i \frac{\cos(\pi)}{1+1} = -\pi i.$$

Similarly considering the sub-contour, γ_-, containing $z = -1$ we can write

$$f(z) = \frac{g_-(z)}{z+1}, \qquad \text{where} \qquad g_-(z) = \frac{\cos(\pi z)}{z-1}.$$

Using Cauchy's integral formula we have

$$\oint_{\gamma_-} \frac{\cos(\pi z)}{z^2-1}\,\mathrm{d}z = \oint_{\gamma_-} \frac{g_-(z)}{z+1}\,\mathrm{d}z = 2\pi i g_-(-1) = 2\pi i \frac{\cos(-\pi)}{-1-1} = i\pi.$$

Hence

$$\oint_\gamma f(z)\,\mathrm{d}z = \oint_{\gamma_+} f(z)\,\mathrm{d}z + \oint_{\gamma_-} f(z)\,\mathrm{d}z = -i\pi + i\pi = 0.$$

---

10 More Consequences of the Cauchy–Goursat Theorem

10.1 Generalised Cauchy's Integral Formula

Theorem 11: Generalised Cauchy's Integral Formula

Let γ be a closed, positively oriented contour. Let S be an open, simply connected region containing γ. Let $f\colon S \to \mathbb{C}$ be analytic on S. Let z_0 be any point inside γ. Then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z\,.$$

Proof. First consider the case of $n = 1$. Applying Cauchy's integral formula to the definition of the derivative we have

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \\
&= \frac{1}{2\pi i} \lim_{\Delta z\to 0} \frac{1}{\Delta z} \left[\oint_\gamma \frac{f(z)}{z-(z_0+\Delta z)}\,\mathrm{d}z - \oint_\gamma \frac{f(z)}{z-z_0}\,\mathrm{d}z \right] \\
&= \frac{1}{2\pi i} \lim_{\Delta z\to 0} \frac{1}{\Delta z} \oint_\gamma \left[\frac{f(z)}{z-(z_0+\Delta z)} - \frac{f(z)}{z-z_0} \right] \mathrm{d}z \\
&= \frac{1}{2\pi i} \lim_{\Delta z\to 0} \frac{1}{\Delta z} \oint_\gamma \frac{[z-z_0]f(z) - [z-(z_0+\Delta z)]f(z)}{(z-(z_0+\Delta z))(z-z_0)}\,\mathrm{d}z \\
&= \frac{1}{2\pi i} \lim_{\Delta z\to 0} \frac{1}{\Delta z} \oint_\gamma \frac{f(z)\Delta z}{(z-(z_0+\Delta z))(z-z_0)}\,\mathrm{d}z \\
&= \frac{1}{2\pi i} \lim_{\Delta z\to 0} \oint_\gamma \frac{f(z)}{(z-(z_0+\Delta z))(z-z_0)}\,\mathrm{d}z \\
&= \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z\,.
\end{aligned}$$

So Cauchy's generalised integral formula holds for $n = 1$.

This can be extended inductively to prove the theorem. □

Example 10.1. *Let γ be a positively oriented closed contour and let a be a point inside γ. Then for $n \in \mathbb{Z}$ evaluate*

$$\oint_\gamma (z-a)^{-n-1}\,\mathrm{d}z\,.$$

If $n < 0$ then the integrand is analytic and so the integral vanishes by the Cauchy–Goursat theorem (theorem 8). If $n \geq 0$ we can use the generalised Cauchy integral formula for the nth derivative:

$$\oint_\gamma \frac{1}{(z-a)^{n+1}} = \frac{2\pi i}{n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n}(1)\bigg|_{z=a} = \begin{cases} 2\pi i, & \text{if} \quad n = 0, \\ 0, & \text{if} \quad n \neq 0. \end{cases}$$

Cauchy's integral formula leads to the following corollary:

Corollary 3

Let S be an open region and let $f\colon S \to \mathbb{C}$ be analytic on S. Then for all $z \in S$ the all derivatives at z exist and are analytic. We say that f is **smooth** or C^∞.

Proof. This follows immediately from Cauchy's integral formula for derivatives which we can apply since f is analytic. Since S is open for any $z \in S$ there exists a simple closed contour satisfying the necessary conditions for Cauchy's integral formula to apply. By the hypothesis since f is analytic f' exists. We can then use Cauchy's integral formula for the second derivative to construct f'' in that neighbourhood. Therefore f' is analytic. Assume now that $f^{(k)}$ is analytic. Then this means that $f^{(k+1)}$ exists. Cauchy's integral formula then allows us to construct $f^{(k+2)}$ so $f^{(k+1)}$ is analytic. Thus by mathematical induction $f^{(n)}$ is analytic for all $n \in \mathbb{N}$ which means that f is smooth. We can also conclude that all partial derivatives of $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$ exists and are continuous. $\square$

10.2 Liouville's Theorem

Theorem 12: Liouville's Theorem

Let $f\colon \mathbb{C} \to \mathbb{C}$ be a bounded, entire function. Then f is constant

Proof. By hypothesis there exists $M \in \mathbb{R}_{>0}$ such that $|f(z)| \le M$ for all $z \in \mathbb{C}$. We can use Cauchy's integral formula on a circular contour, γ, of radius R to bound f':

$$\begin{aligned}
|f'(z)| &= \left|\frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta\right| \\
&= \frac{1}{2\pi}\left|\oint_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta\right| \\
&\le \frac{1}{2\pi}\oint_\gamma \left|\frac{f(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta\right| \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{\left|f(Re^{i\vartheta})\right|}{\left|Re^{i\vartheta} - z\right|^2} R\,\mathrm{d}\vartheta \\
&\le \frac{1}{2\pi}\int_0^{2\pi} \frac{MR}{(R - |z|^2)^2}\,\mathrm{d}\vartheta \\
&= \frac{MR}{(R - |z|)^2} \\
&\sim \frac{1}{R}
\end{aligned}$$

This vanishes independently of z since we can take R to be arbitrarily large. Hence $f'(z) = 0$ which means $f(z)$ is constant. $\square$

The contrapositive[10] of this theorem is often more useful:

Theorem 12: Liouville's Theorem (Contrapositive)

If f is non-constant, entire function then there exists $z \in \mathbb{C}^*$ such that $f(z)$ diverges.

We can use this to prove the fundamental theorem of algebra.

[10] If $P \implies Q$ then $\neg Q \implies \neg P$ is a logically equivalent statement called the contrapositive.

Theorem 13: Fundamental Theorem of Algebra

Let P_n be a polynomial of degree $n \geq 1$ with complex coefficients ($P_n \in \mathbb{C}[z]$). Then P_n has n roots, z_i, including repeated roots, for which $P_n(z_i) = 0$.

Proof. Suppose that P_n has no roots. Then $P_n(z) \neq 0$ for all $z \in \mathbb{C}$. Hence $1/P_n(z)$ is entire since P_n is entire. Let $\zeta \in \mathbb{C}$ be the minimum of P_n. Then $|1/P_n(z)| \leq |1/P_n(\zeta)|$ for all $z \in \mathbb{C}$. Hence by Liouville's theorem (theorem 12) $1/P_n$ is constant which can only be the case if P_n is constant which it clearly isn't. Thus P_n has at least one root. Call this root z_1. Then we can factorise P_n as

$$P_n(z) = (1 - z_1)Q_{n-1}(z)$$

where $Q_{n-1} \in \mathbb{C}[z]$ is a polynomial of order $n - 1$.

Suppose now that $P_k \in \mathbb{C}[z]$ is a kth order polynomial for some $k \in \mathbb{N}$ and has k roots . Hence $P_{k+1} \in \mathbb{C}[z]$ is a polynomial of order $k + 1$ and has at least k roots. Consider

$$f(z) = \frac{\prod_{i=1}^{k}(z - z_i)}{P_n(z)}$$

where z_i are the the k roots of P_k assumed to exist. The function above is analytic almost everywhere except at the point $z = a_{k+1}$ where $f(z) \sim (z - a_{k+1})^{-1} \to 0$ as $z \to \infty$. The other singularities cancel with the numerator. Since $a_{k+1} \neq 0$ (as P_{k+1} must have a z^{k+1} term to be of order $k + 1$) we must have that P_{k+1} has another root meaning it has $k + 1$ roots. □

Theorem 14: Morera's Theorem

Let S be an open region and let $f\colon S \to \mathbb{C}$ be continuous on S. Suppose that

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

for all contours γ contained in S. Then f is analytic.

Proof. The fundamental theorem of calculus (theorem 6) means that

$$\oint_\gamma f(z)\,\mathrm{d}z = 0$$

for all γ is equivalent to f having an antiderivative, F, such that $f(z) = F'(z) \forall z \in S$. This is true for all z in S so F is analytic. By corollary 3 the derivative of an analytic function is also analytic so $f = F'$ is analytic. □

This theorem is the converse of the Cauchy–Goursat theorem.

Part IV

Series

11 Sequences and Series

Definition 37: Sequence

A **sequence** is a function whose domain is an interval of integers, i.e. $a\colon [\alpha, \beta] \cap \mathbb{Z} \to X$.

This is a very formal and broad definition. For our use we are mostly interested in (one-sided) infinite sequences of complex numbers. In this case a sequence is a function $a\colon \mathbb{Z}_{\geq 0} \to \mathbb{C}$.

Notation

If $a\colon \mathbb{Z}_{\geq 0} \to \mathbb{C}$ is an infinite complex sequence (henceforth a sequence) then we write $a_n = a(n)$ and denote the whole sequence as $\{a_n\} = \{a_0, a_1, \dots\} = \{a_n\}_{n=0}^{\infty}$.

Sequences are naturally ordered by n and the order is important, this means $\{1, 2, 3, \dots\} \neq \{2, 1, 3, \dots\}$. There are two common ways to define a sequence. We could give an explicit formula as a function of n, for example $a_n = n$ corresponds to the sequence $\{0, 1, 2, \dots\}$. Or we could give a recursive definition such as $a_n = a_{n-1} + 1$ and $a_0 = 0$, this also corresponds to $\{0, 1, 2, \dots\}$.

Definition 38: Convergence (Sequences)

A sequence, $\{a_n\}$, **converges** to a value $a \in \mathbb{C}$ if for all $\varepsilon > 0$ there exists $N \in \mathbb{Z}_{\geq 0}$ such that for all $n \geq N$

$$|a_n - a| < \varepsilon.$$

Notation

If $\{a_n\}$ is a sequence and converges to a then we write

$$a = \lim_{n\to\infty} a_n, \qquad \text{or} \qquad \{a_n\} \to a.$$

Example 11.1. Consider the sequence defined by $a_n = (n+1)^{-1}$. This converges to 0. To show this we need to find a value of N, which is in general a function of ε such that the convergence definition is satisfied for a given ε. The convergence criteria is

$$|a_n - a| < \varepsilon \implies \left|\frac{1}{n+1} - 0\right| = \frac{1}{n+1} < \varepsilon.$$

Rearranging we have

$$n > \frac{1}{\varepsilon} - 1.$$

Thus we choose N to be the smallest integer satisfying this, that is $N = \lceil 1/\varepsilon \rceil - 1$. Having found N we now check that the convergence criteria are satisfied. Let $n \geq N = \lceil 1/\varepsilon \rceil - 1$. Then

$$|a_n - 0| = \left|\frac{1}{n+1}\right| < \left|\frac{1}{(1/\varepsilon - 1) + 1}\right| = \left|\frac{1}{1/\varepsilon}\right| = |\varepsilon| = \varepsilon.$$

So the convergence definition is satisfied and

$$\lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Not being mathematicians we aren't that interested in proving convergence and often simply assume convergence.

Definition 39: Limit Point (Sequence)

Let $\{a_n\}$ be a sequence. We say that l is a **limit point** of $\{a_n\}$ if for all $\varepsilon > 0$ there exists at least one $n \in \mathbb{Z}_{\geq 0}$ such that $a_n \in D'(l; \varepsilon)$.

Note that the disc is punctured. This means that terms of a sequence aren't automatically limit points. Consider the sequence defined by $a_n = n$. A punctured disc of radius 0.1 placed at 1 doesn't contain any points in $\{a_n\}$ and so 1 is not a limit point of $\{a_n\}$. Even sequences which don't converge can have limit points. For example the sequence defined by

$$a_n = (-1)^n \frac{n}{n+1}$$

oscillates between ± 1 so doesn't converge but both ± 1 are limit points.

Definition 40: Series

A **series**, S, is the formal sum of all the elements of a sequence.

Definition 41: Partial Sum

The nth **partial sum** of the sequence $\{a_n\}$ is

$$S_n = \sum_{j=1}^{n} a_j = a_0 + a_1 + \cdots + a_n.$$

Definition 42: Convergence (Series)

The series

$$\sum_{j=0}^{\infty} a_j$$

converges to S if the sequence $\{S_n\}$ of partial sums,

$$\sum_{j=0}^{n} a_n,$$

converges to S.

Definition 43: Absolute Convergence

A complex series,

$$\sum_{j=0}^{\infty} a_n,$$

is **absolutely convergent** if the real series,

$$\sum_{j=0}^{\infty} |a_n|,$$

converges.

If a complex series is absolutely convergent it is also convergent.[11]

Definition 44: Uniform Convergence

Let $\{a_n\}$ be a sequence depending on z. Let $\{S_n\}$ be the sequence of partial sums of $\{a_n\}$. Let $S = \sum_{i=1}^{\infty} a_n$ be a convergent series. Then S is a **uniformly convergent** series if for arbitrary $\varepsilon > 0$ there exists N such that for all $n \geq N$ we have $|S_n - S| < \varepsilon$. That is S converges independent of the value of z.

Intuitively the terms of a uniformly convergent series get closer and closer to the value the series converges to. Uniform convergence is an important condition as it allows us to do things like exchange integrals and infinite sums or integrals and derivatives without changing anything. We often assume this is ok in physics but if the objects involved aren't uniformly convergent then it may not be.

11.1 Convergence Tests

Proving convergence is often not that easy but there are convergence tests that can sometimes be helpful. In the following take $\{a_n\}$ to be a convergent sequence.

[11] The property of a metric space that all absolutely convergent series are convergent is called completeness and is a key property of a Hilbert space.

- If
$$\lim_{n\to\infty} a_n \neq 0$$
then $\sum a_n$ diverges.
- *Comparison Test:* If $\sum |b_n|$ converges and there exists $N \in \mathbb{Z}_{\geq 0}$ such that for all $n > N$ we have $|a_n| < |b_n|$ then $\sum |a_n|$, and hence $\sum a_n$, converges.
- *Ratio Test:* Let
$$l = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$
If $l < 1$ then $\sum a_n$ converges absolutely. If $l > 1$ then $\sum a_n$ diverges. If $l = 1$ then no conclusion can be drawn.
- *nth Root Test:* Let
$$l = \lim_{n\to\infty} \sqrt[n]{a_n}.$$
If $l < 1$ then $\sum a_n$ converges absolutely. If $l > 1$ then $\sum a_n$ diverges. If $l = 1$ then no conclusion can be drawn.
- *Weierstrass M-Test:* If a_n depends on z and there exists a sequence of non-negative, real numbers, $\{M_n\}$, such that for all $n \geq 1$ and for all $z \in R$ we have $|a_n(z)| \leq M_n$ and
$$\sum_{n=1}^{\infty} M_n$$
converges then $\sum a_n$ converges absolutely and uniformly on the region R.

11.2 Geometric Series

A geometric series is a series defined by $a_n = z^n$ for some $z \in \mathbb{C}$. Consider

$$\begin{aligned}
S_n &= 1 + z + z^2 + \ldots + z^{n-1} + z^n \\
zS_n &= \quad z + z^2 + z^3 + \ldots + z^n + + z^{n+1} \\
(1-z)S_n &= 1 + z^{n+1} \\
\implies S_n &= \frac{1 + z^{n+1}}{1 - z}.
\end{aligned}$$

If $|z| > 1$ then this series diverges. If $|z| < 1$ then this series converges. The partial sums converge uniformly to $1/(1-z)$ so

$$S = \sum_{n=0}^{\infty} z^n = \frac{1}{1-c}.$$

11.3 Taylor Series

Theorem 15: Taylor's Theorem

Let S be an open region and let $f\colon S \to \mathbb{C}$ be analytic. Let $z_0 \in S$ and let $R \in [0,\infty)$ be the largest value such that the open disc, $D(z_0; R)$, centred on z_0 and of radius R, is contained entirely in S. Then there exists a unique sequence, $\{c_n\}$, such that

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n.$$

Further this series is uniformly convergent on the disc $D(z_0; R)$ and

$$c_n = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z = \frac{f^{(n)}(z_0)}{n!}$$

where γ is any positively oriented contour inside $D(z_0; R)$ and contains z_0.

Proof. For a given $z \in D'(z_0; R)$ we can choose a contour γ which is a a circle of radius r centred on z_0 such that $|z - z_0| < r < R$. Since $f(z)$ is analytic in this region we can apply Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w - z} \, \mathrm{d}w \,.$$

Consider

$$\frac{1}{w - z} = \frac{1}{w - z_0} \frac{1}{1 - \frac{z - z_0}{w - z_0}}$$

since $|w - z_0| > |z - z_0|$ we have $|(z - z_0)/(w - z_0)| < 1$ and so we recognise the second fraction as the value of a geometric series giving us

$$\frac{1}{w - z} = \frac{1}{w - z_0} \sum_{n=0} \left(\frac{z - z_0}{w - z_0} \right)^n .$$

Since the geometric series is uniformly convergent we could just substitute it directly into the integral and safely exchange the integral and sum. However we can also split the sum in two and after some manipulations we don't need to rely on uniform convergence.

$$\begin{aligned}
\frac{1}{w - z} &= \frac{1}{w - z_0} \left[\sum_{n=0}^{N-1} \left(\frac{z - z_0}{w - z_0} \right) + \sum_{n=N}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n \right] \\
&= \frac{1}{w - z_0} \left[\sum_{n=0}^{N-1} \left(\frac{z - z_0}{w - z_0} \right) + \left(\frac{z - z_0}{w - z_0} \right)^N \sum_{n=N}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^{n-N} \right] \\
&= \frac{1}{w - z_0} \left[\sum_{n=0}^{N-1} \left(\frac{z - z_0}{w - z_0} \right) + \left(\frac{z - z_0}{w - z_0} \right)^N \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n \right] \\
&= \frac{1}{w - z_0} \left[\sum_{n=0}^{N-1} \left(\frac{z - z_0}{w - z_0} \right) + \left(\frac{z - z_0}{w - z_0} \right)^N \frac{1}{1 - \frac{z - z_0}{w - z_0}} \right] \\
&= \frac{1}{w - z_0} \left[\sum_{n=0}^{N-1} \alpha^n + \frac{\alpha^N}{1 - \alpha} \right]
\end{aligned}$$

where $\alpha = (z - z_0)/(w - z_0)$. We can now substitute this into Cauchy's integral formula. Since the only sums involved are now finite then exchanging the integral and sum is valid as the integral is linear.

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w - z} \, \mathrm{d}w \\
&= \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w - z_0} \left[\sum_{n=0}^{N-1} \alpha^n + \frac{\alpha^N}{1 - \alpha} \right] \mathrm{d}w \\
&= \frac{1}{2\pi i} \sum_{n=0}^{N-1} \oint_\gamma \frac{f(w)\alpha^n}{w - z_0} \, \mathrm{d}w + \frac{1}{2\pi i} \oint_\gamma \frac{f(w)\alpha^N}{(w - z_0)(1 - \alpha)} \, \mathrm{d}w \,.
\end{aligned}$$

We now take the limit of $N \to \infty$ and we have

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_\gamma \frac{f(w)\alpha^n}{w - z_0} \, \mathrm{d}w \,.$$

The second integral vanishes in the limit as $|\alpha| < 1$. We can further simplify this first integral:

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_\gamma \frac{f(w)\alpha^n}{w - z_0} \, \mathrm{d}w \\
&= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_\gamma \frac{f(w)(z - z_0)^n}{(w - z_0)^{n+1}} \, \mathrm{d}w
\end{aligned}$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_\gamma \frac{f(w)}{(w - z_0)^{n+1}} \, \mathrm{d}w$$

we can now apply Cauchy's integral formula for the nth derivative to this and we find

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{(w - z_0)^{n+1}} \, \mathrm{d}w = \sum_{n=0}^{\infty} \frac{f^n}{n!} (z - z_0)^n.$$

Uniform convergence follows as the geometric series is uniformly convergent. □

Taylor series in complex analysis are very similar to the real Taylor series we are used to. That all analytic functions can be expressed as a Taylor series is sometimes taken as the definition of analyticity and then our definition of analyticity is referred to as being holomorphic.

There are many properties we can now exploit to find Taylor series without having to compute terms individually. For example we can exploit the uniform convergence to find the Taylor series for $f(z) = \log(1 + z)$. First note that

$$f'(z) = \frac{1}{1 + z} = \frac{1}{1 - (-z)} = \sum_{n=0}^{\infty} (-z)^n.$$

Integrating again and using uniform convergence to change the order of operations we have

$$f(z) = \int \sum_{n=0}^{\infty} (-1)^n z^n \, \mathrm{d}z = \sum_{n=0}^{\infty} (-1)^n \int z^n \, \mathrm{d}z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1}.$$

Note that since z^n is a polynomial it is entire and therefore by the fundamental theorem of complex calculus has an antiderivative and the contour along which we integrate is therefore unimportant as the result depends only on its end points.

Since $\{c_n\}$ is unique for a given function and point z_0 if we are expanding a function about a point in $\mathbb{R}$, most commonly 0, we can simply use the real Taylor series as we have defined functions such that restricting them to the real axis recovers the real definition.

The radius of convergence, that is R as defined in the theorem, of a function which is analytic everywhere apart from at a set of singularities is simply the distance to the nearest singularity. For example

$$f(z) = \frac{1}{1 + z^2}$$

has two singularities at $z = \pm i$. Therefore if we wish to expand this at $z = 0$ then the radius of convergence is 1 as this is the distance to the nearest singularity. We can again exploit the geometric series and spot that

$$f(z) = \frac{1}{1 - (-z^2)} = \sum_{n=0}^{\infty} (-z^2)^n$$

which converges when $\left|z^2\right| < 1$ which means $|z| < 1$.

12 Laurent Series

12.1 Isolated Singularities

Recall the definition of an isolated singularity:

Definition: Isolated singular point

Let $f \colon S \to \mathbb{C}$ be a function which is singular at $z_0 \in S$. We say z_0 is an isolated singularity if there exists $r > 0$ such that f is analytic on $D'(z_0; r)$.

For example:

- $f(z) = 1/z$ has an isolated singularity at $z = 0$ as f is analytic on $\mathbb{C} \setminus \{0\}$.

- $f(z) = \operatorname{cosec}(z)$ has isolated singularities at $z = n\pi$ for $n \in \mathbb{Z}$.
- $f(z) = 1/\sin(1/z)$ has a singularity at $z = 0$ but this is not an isolated singularity. $1/\sin(1/z)$ is singular whenever $\sin(1/z) = 0$ which happens when $1/z = n\pi$ for $n \in \mathbb{Z}$. That is $z = 1/2\pi n$. However this can be arbitrarily close to zero simply by choosing sufficiently large n. This means that any disc centred on zero will also contain an infinite number of points of the form $z = 1/2\pi n$.
- $f(z) = \log z$ is multivalued. If we choose the principle branch $(-\pi \le \arg z \le \pi)$ then $z = 0$ is not an isolated singularity as any disc centred on the origin will contain part of the branch cut.

Theorem 16: Laurent Series

Let $f\colon D \to \mathbb{C}$ be an analytic function on the open disc $D(z_0;\rho)$ except possibly at $z = z_0$. Then we can expand $f(z)$ as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n$$

where

$$c_n = \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{(w - z_0)^{n+1}}\,\mathrm{d}w$$

for some positively orientated contour, γ, contained in D such that the point z_0 is inside γ. The resulting series is called a Laurent series.

Proof. The proof is similar to that of Taylor's theorem however we can't directly apply Cauchy's integral formula as we don't know if f is analytic at z_0. Since $D(z_0;\rho)$ for $z_0 \ne z \in D$ we can choose $r, R \in (0, \rho)$ such that

$$0 < r < |z - z_0| < R < \rho.$$

Let γ_+, γ_-, γ_R, and γ_r be the contours shown in figure 12.1. By construction f is analytic inside γ_+ and γ_-. This allows us to apply Cauchy's integral formula to γ_+ which gives us

$$f(z) = \frac{1}{2\pi i}\oint_{\gamma_+} \frac{f(w)}{w - z}\,\mathrm{d}w\,.$$

Also f is analytic in γ_- and z is outside of γ_- so $f(w)/(w - z)$ is analytic on γ_-. Thus by Cauchy–Goursat's theorem

$$0 = \frac{1}{2\pi i}\oint_{\gamma_-} \frac{f(w)}{w - z}\,\mathrm{d}w\,.$$

Hence

$$f(z) = \frac{1}{2\pi i}\oint_{\gamma_+} \frac{f(w)}{w - z}\,\mathrm{d}w + \frac{1}{2\pi i}\oint_{\gamma_-} \frac{f(w)}{w - z}\,\mathrm{d}z\,.$$

Now carefully considering these integrals and the contours they are along we see that the straight sections of the contours cancel out and we are left with an integral over the two circles. Notice that the integral over the inner circle is negatively orientated. Hence

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(w)}{w - z}\,\mathrm{d}w - \int_{\gamma_r} \frac{f(w)}{w - z}\,\mathrm{d}w\,.$$

The contour integral around γ_R is the same as the integral that appears in the proof of Taylor's theorem. For this integral we have $|w - z_0| < |z - z_0|$ and we get the following series:

$$\frac{1}{2\pi i}\oint_{\gamma_R} \frac{f(w)}{w - z}\,\mathrm{d}w = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{2\pi i}\oint_{\gamma_R} \frac{f(w)}{w - z}\,\mathrm{d}w\,.$$

On γ_r we can't do this as $|w - z_0| < |z - z_0|$ so the geometric series that we used in the proof of Taylor's theorem doesn't converge. Instead we use the following which is valid when $|z| > |w|$:

$$\frac{1}{w - z} = \frac{1}{z(w/z - 1)}$$

$$
\begin{aligned}
&= -\frac{1}{z}\frac{1}{1 - w/z} \\
&= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{w}{z}\right)^n \\
&= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{w}{z}\right)^{-n} \\
&= -\frac{1}{z}\sum_{n=-\infty}^{0}\left(\frac{z}{w}\right)^n \\
&= -\sum_{n=-\infty}^{0}\frac{z^{n-1}}{w^n} \\
&= -\sum_{n=-\infty}^{-1}\frac{z^n}{w^{n+1}}.
\end{aligned}
$$

This is still valid if we substitute $z - z_0$ for z and $w - z_0$ for w provided $|z - z_0| > |w - z_0|$, as is the case on the contour γ_r. Hence we have

$$
\frac{1}{2\pi i}\oint_{\gamma_r}\frac{f(w)}{w - z}\,\mathrm{d}w = -\oint_{\gamma_r}\sum_{n=-\infty}^{-1}\frac{(z - z_0)^n}{2\pi i}\frac{f(w)}{(w - z_0)^{n+1}}\,\mathrm{d}w\,.
$$

Since the sum is a convergent geometric series it is uniformly convergent and so we can swap the sum and the integral and we have

$$
f(z) = \sum_{n=0}^{\infty}\frac{(z - z_0)^n}{2\pi i}\oint_{\gamma_r}\frac{f(w)}{(w - z_0)^{n+1}}\,\mathrm{d}w + \sum_{n=-\infty}^{-1}\oint_{\gamma_r}\frac{f(w)}{(w - z)^{n+1}}\,\mathrm{d}w\,.
$$

By hypothesis the only possible singularity in these integrands is at z_0 which is inside both contours. Therefore between the contours the integrands are perfectly analytic and we can use the deformation theorem to turn the integrals over γ_r and γ_R into integrals over some arbitrary contour, γ, inside D and containing z_0. Hence

$$
f(z) = \sum_{n=-\infty}^{\infty}\frac{(z - z_0)^n}{2\pi i}\oint_{\gamma}\frac{f(w)}{(w - z)^{n+1}}\,\mathrm{d}w\,.
$$

□

This proof, and hence Laurent series, can be extended to an annulus of radii ρ_1, ρ_2, where f is analytic on $\rho_1 < |z - z_0| < \rho_2$ and the singularities inside $|z - z_0| < \rho_1$ are isolated.

12.2 Properties of Laurent Series

First notice that for $n \geq 0$ the Laurent series coefficients are identical to the Taylor series coefficients. As well as this if f is analytic then $c_n = 0$ for all $n < 0$ since the integrand becomes $f(w)(w - z)^{-n+1}$ which is analytic ($n < 0 \implies -n > 0$). Thus if f is analytic its Laurent series is the same as its Taylor series.

Definition 45: Principle Part

If f is a function with the Laurent series

$$
f(z) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n
$$

then the **principle part** of f is

$$
\sum_{n=-\infty}^{-1} c_n(z - z_0)^n.
$$

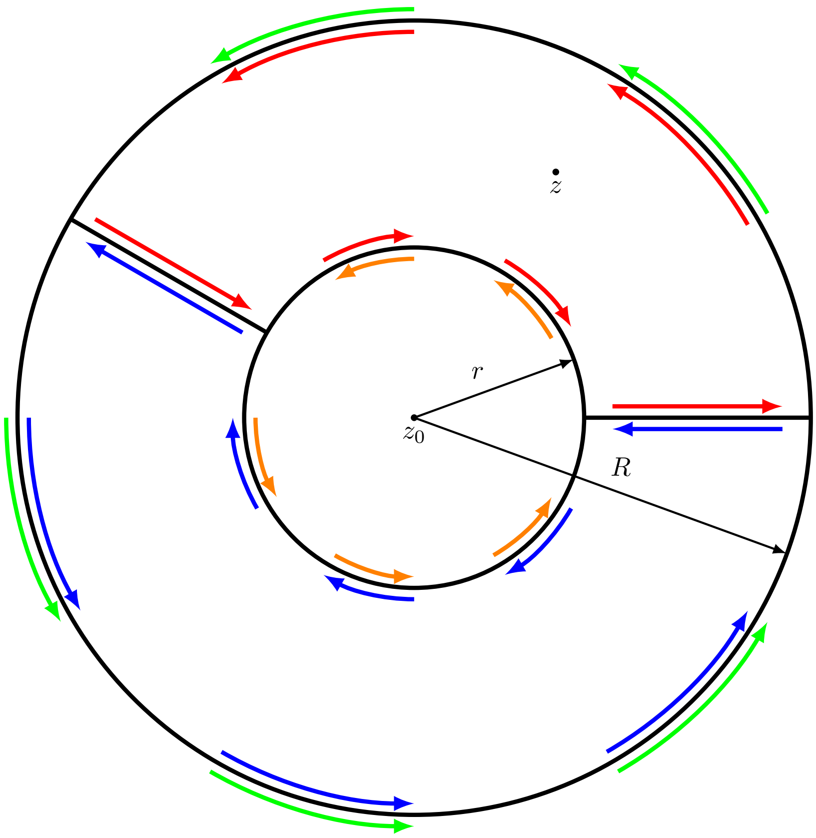


Figure 12.1: The contours used to prove the validity of a Laurent series.

The principle part dominates near $z = z_0$. In particular very close to z_0 if the Laurent series has a finite number of non-zero negative power terms, i.e.

$$f(z) = \sum_{n=-m}^{\infty} c_n(z - z_0)^n$$

for some $m \in \mathbb{N}$ then the c_{-m} term dominates near $z = z_0$. As with Taylor series we avoid actually using the integral definition of the coefficients wherever possible preferring to use known series and other techniques.

Example 12.1. *Find the Laurent series of $f(z) = e^z/z^2$ at $z = 0$ and state its radius of convergence.*

We start with the Taylor series for e^z:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Hence

$$\begin{aligned} f(z) &= \frac{1}{z^2} e^z \\ &= \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{z^{n-2}}{n!} \\ &= \sum_{n=-2}^{\infty} \frac{z^n}{(n+2)!}. \end{aligned}$$

Since the Taylor series of e^z converges for all $z \in \mathbb{C}$ this Laurent series converges for all $z \in \mathbb{C} \setminus \{0\}$.

Example 12.2. *Find the Laurent series of $f(z) = 1/z(z-1)^2$ at $z = 1$ and state its radius of convergence.*

There are two singularities now at $z = 0, 1$. This divides $\mathbb{C}$ into two regions: $0 < |z-1| < 1$ and $|z-1| > 1$. For $0 < |z-1| < 1$ we have

$$\begin{aligned}
f(z) &= \frac{1}{z(z-1)^2} \\
&= \frac{1}{1+(z-1)} \frac{1}{(z-1)^2} \\
&= \frac{1}{(z-1)^2} \sum_{n=0}^{\infty} (-1)^n (z-1)^n \\
&= \sum_{n=-2}^{\infty} (-1)^n (z-1)^n
\end{aligned}$$

where we have used the geometric series:

$$\frac{1}{1+\alpha} = \frac{1}{1-(-\alpha)} = \sum_{n=0}^{\infty} (-\alpha)^n = \sum_{n=0}^{\infty} (-1)^n \alpha^n.$$

For $|z-1| > 1$ we have

$$\begin{aligned}
f(z) &= \frac{1}{z(z-1)^2} \\
&= \frac{1}{(z-1)^3} \frac{1}{1 + \frac{1}{1-z}} \\
&= \frac{1}{(z-1)^3} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(z-1)^n} \\
&= \sum_{n=-3}^{\infty} (-1)^{n+1} (z-1)^n
\end{aligned}$$

where we have used $(-1)^{n+3} = (-1)^{n+1}$ as it is only the parity of the power that is important here.

The first of these Laurent series converges for $|z| < 1$ and the second for $|z| > 1$.

Example 12.3. *Compute the first few terms of the Laurent series of $f(z) = \cot z$ at $z = 0$ to first order in z. State its radius of convergence.*

To do this we use the Taylor series of sin and cos:

$$\begin{aligned}
f(z) &= \cot z \\
&= \frac{1}{\tan z} \\
&= \frac{\cos z}{\sin z} \\
&= \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots} \\
&= \frac{1}{z} \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots} \\
&= \frac{1}{z} \left(1 - \frac{z^2}{2!} + \frac{z^4}{4} - \cdots\right) \left(1 + \frac{z^2}{3!} - \cdots\right) \\
&= \frac{1}{z} \left(1 + z^2 \left(\frac{1}{6} - \frac{1}{2}\right) + \cdots\right)
\end{aligned}$$

$$= \frac{1}{z} - \frac{1}{3}z.$$

Here we used the Binomial expansion to first order to turn the fraction of series into the product of series. For convergence we have two considerations. The first is f has singularities at $z = n\pi$ for $n \in \mathbb{Z}$, hence the maximum radius of convergence is $|z| < \pi$. The binomial expansion of $(1 + w)^-1$ is also only valid for $|w| < 1$, in this case we need $\left|z^2/3!\right| < 1$ which means $\left|z^2\right| < 3! = 6$. Hence $|z| < \sqrt{6} \approx 2.44 < \pi$ so the radius of convergence is $|z| < \sqrt{6}$.

Example 12.4. *Obtain the Laurent series of* $f(z) = \exp(1/z)$ *at* $z = 0$.

Here we use the fact that if g is entire (i.e., its Taylor series converges everywhere on $\mathbb{C}$) then the Taylor series for $g(1/z)$ converges for all $z \neq 0$. Hence

$$f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^n} = \sum_{-\infty}^{0} \frac{z^n}{(-n)!}.$$

Note that this method is not always valid. For example since the series for cot does not converge everywhere we can't simply substitute $1/z$ into the series to find $\cot(1/z)$. We would have to redo the analysis in the previous section.

Example 12.5. *Obtain the Laurent series for* $f(z) = 1/[z^2(z-1)]$ *at* $z = \infty$.

In a similar way to branch points at infinity we consider series at infinity through the substitution $z \to 1/w$ and then take $w \to 0$. Hence

$$\begin{aligned} f(z) &= f(1/w) \\ &= \frac{1}{w^{-2}(1/w - 1)} \\ &= \frac{w^3}{1-w} \\ &= w^3 \sum_{n=0}^{\infty} w^n \\ &= \sum_{n=3}^{\infty} w^n. \end{aligned}$$

Here we have used the fact that for $w \to 0$ surely $|w| < 1$ and so the geometric series expansion is valid. Converting back to z we get the final answer

$$f(z) = \sum_{n=3}^{\infty} \frac{1}{z^n} = \sum_{n=-\infty}^{-3} z^n.$$

---

13 Zeros and Singularities

13.1 Classifying Singularities

The behaviour of many functions is almost entirely described by their behaviour at singularities and zeros. For this reason it is important to be able to classify these regions.

Definition 46: Zero of Order N

If $f(z_0) = 0$ then we can write

$$f(z) = \sum_{n=N}^{\infty} c_n(z-z_0)^n = (z-z_0)^N \sum_{n=0}^{\infty} c_{n+N}(z-z_0)^n = (z-z_0)^N g(z)$$

where g is analytic at z_0 and $c_N = g(z_0) \neq 0$. We call z_0 a **zero of order N**.

Intuitively a zero of order N allows us to factor out N copies of $z - z_0$ before we get a function that is non-zero at z_0. For a polynomial the order of a root is simply its multiplicity. For example, trivially $z(z-1)^2$ has two zeros, a zero of order one at $z = 0$ and a zero of order 2 at $z = 1$.

Definition 47: Pole of order N

If f is singular at z_0 and its Laurent series has a finite number of negative terms then we can write

$$f(z) = \sum_{n=-N}^{\infty} c_n(z - z_0)^n = \frac{1}{(z - z_0)^N} \sum_{n=0}^{\infty} c_{n-N}(z - z_0)^n = \frac{g(z)}{(z - z_0)^N}$$

where g is analytic at z_0 and $c_{-N} = g(z_0) \neq 0$. We call z_0 a **pole of order N**. In the special case of $N = 1$ we call z_0 a **simple pole**.

A pole of order N is similar to a zero of order N except we can factor out N copies of $z - z_0)^{-1}$ before we have a function that is non-singular at z_0.

Definition 48: Removable Singularity

Suppose that f is singular at z_0. It may be possible to assign a value to f at z_0 such that the resulting function is analytic. This can be done by defining f to be equal to it's Taylor series at z_0. In this case the principle part of the Laurent series must be zero. If this can be done we call z_0 a **removable singularity**.

Intuitively a removable singularity is a point where the function is formally undefined but there is a logical value. For example $f(z) = \sin(z)/z$ is indeterminate at $z = 0$ but by defining f to be 1 at $z = 0$ we recover a nicely behaved function. A removable singularity is also a pole of order zero.

Definition 49: Essential Singularity

If f is singular at z_0 and the Laurent series of f has an infinite number of negative terms then we say z_0 is an **essential singularity**.

13.2 Picard's Theorems

This section is non-examinable.

The following two functions are important in complex analysis but are entirely beyond the scope of this course and are given for completeness. They are non-examinable and the proofs are far to involved to get into.

Theorem: Picard's Little Theorem

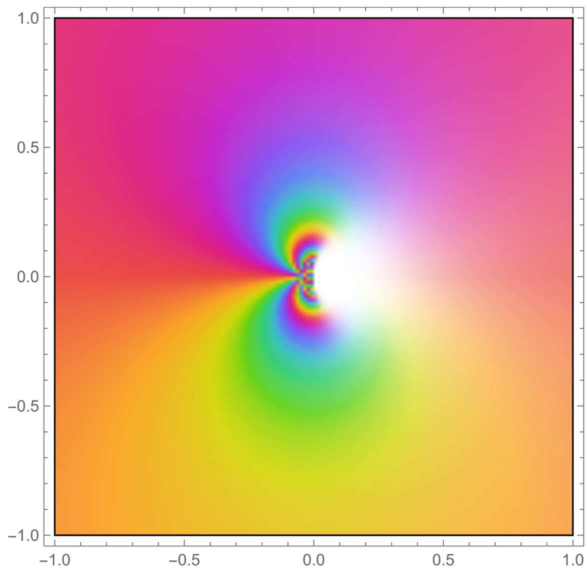
This theorem is non-examinable.
If $f\colon \mathbb{C} \to \mathbb{C}$ is entire it is either constant or its image is either the entire complex plane or the plane minus a single point.

For example e^z is entire and its image is $\mathbb{C} \setminus \{0\}$.

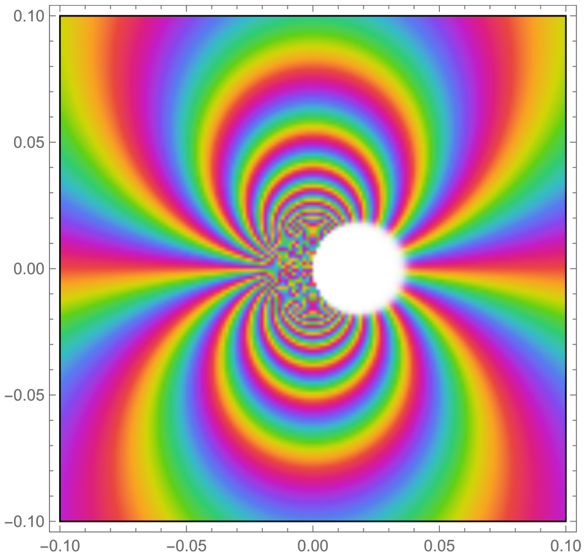
Theorem: Picard's Great Theorem

This theorem is non-examinable.
If $f\colon \mathbb{C} \to \mathbb{C}$ is analytic and has an essential singularity at z_0 then on any punctured neighbourhood of z_0 the image of f is either the entire complex plane or the plane minus a single point.

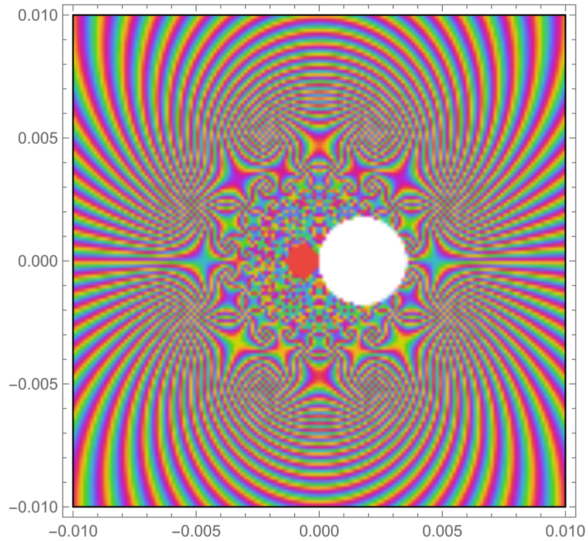
For example $e^{1/z}$ achieves all values apart from 0 in a neighbourhood of $z = 0$. This leads to the interesting plots seen in figure 13.1. Notice how as we zoom in on $z = 0$ the colour (representing the phase) chances rapidly.



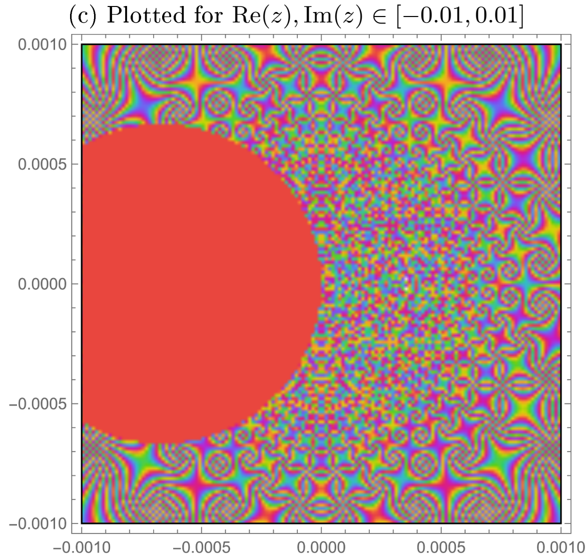
(a) Plotted for $\mathrm{Re}(z), \mathrm{Im}(z) \in [-1, 1]$



(b) Plotted for $\mathrm{Re}(z), \mathrm{Im}(z) \in [-0.1, 0.1]$



(c) Plotted for $\mathrm{Re}(z), \mathrm{Im}(z) \in [-0.01, 0.01]$



(d) Plotted for $\mathrm{Re}(z), \mathrm{Im}(z) \in [-0.001, 0.001]$

Figure 13.1: $f(z) = e^{1/z}$ with different levels of zoom. Brightness gives absolute value and colour gives argument. The large red regions near the centre of the two most zoomed in plots are simply a plotting artefact due to a lack of precision with floating points.

13.3 Zeros and Singularities of Composite Functions

Often the easiest way to classify the zeros and singularities of a function is to break it apart into functions where we already know the answer. For this to be useful we need to know how the zeros and singularities of a function change when we combine them.

Lemma 13: Combining Zeros and Singularities

Let $f, g\colon S \to \mathbb{C}$ be analytic on S. Let $z_0 \in S$ be a zero of order p for f and of order q for g. Then

- fg has a zero of order $p + q$ at $z = z_0$,
- $1/f$ has a pole of order p at z_0,
- f/g has
  - a zero of order $p - q$ at z_0 if $p > q$,
  - a zero of order $q - p$ at z_0 if $p < q$.

Proof. Since f and g are analytic with zeros of order p and q respectively they have Taylor series

$$f(z) = (z - z_0)^p \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad \text{and} \qquad g(z) = (z - z_0)^q \sum_{n=0}^{\infty} b_n (z - z_0)^n.$$

Hence

$$\begin{aligned} (fg)(z) &= \left[(z - z_0)^p \sum_{n=0}^{\infty} a_n (z - z_0)^n\right] \left[(z - z_0)^q \sum_{n=0}^{\infty} b_n (z - z_0)^n\right] \\ &= (z - z_0)^{p+q} \left[\sum_{n=0}^{\infty} a_n (z - z_0)^n\right] \left[\sum_{n=0}^{\infty} b_n (z - z_0)^n\right]. \qquad = (z - z_0)^{p+q} \sum_{n=0}^{\infty} c_n (z - z_0)^n \\ &= (z - z_0)^{p+q} h(z) \end{aligned}$$

where h is an analytic function with Taylor series

$$h(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n \sum_{n=0}^{\infty} b_n (z - z_0)^n.$$

Hence fg has a pole of order $p + q$.

Now consider

$$\begin{aligned} \frac{1}{g} &= \left[(z - z_0)^p \sum_{n=0}^{\infty} a_n (z - z_0)^n\right]^{-1} \\ &= \frac{1}{(z - z_0)^p} \frac{1}{\sum_{n=0}^{\infty} a_n (z - z_0)^n} \\ &= \frac{1}{(z - z_0)^p} \frac{1}{a_0} \frac{1}{\sum_{n=0}^{\infty} \frac{a_n}{a_0} (z - z_0)^n} \\ &= \frac{1}{(z - z_0)^p} \frac{1}{a_0} \frac{1}{1 + \frac{a_1}{a_0}(z - z_0) + \mathcal{O}((z - z_0)^2)} \\ &= \frac{1}{(z - z_0)^p} \frac{1}{a_0} \left[1 - \frac{a_1}{a_0}(z - z_0) + \mathcal{O}\big((z - z_0)^2\big)\right] \\ &= \frac{1}{(z - z_0)^q} \sum_{n=0}^{\infty} d_n (z - z_0) \\ &= \frac{1}{(z - z_0)^q} \tilde{h}(z) \end{aligned}$$

where $\tilde{h}$ is an analytic function with Taylor series

$$\tilde{h}(z) = \sum_{n=0}^{\infty} d_n (z - z_0)^n = \frac{1}{a_0} \left[1 - \frac{a_1}{a_0}(z - z_0) + \mathcal{O}\big((z - z_0)^2\big)\right]$$

here we have used the fact that near z surely $|z - z_0| < 1$ and so the binomial expansion is valid. A mixture of the two arguments above suffices to prove the final point. □

Example 13.1. *Characterise the zeros and singularities of the following functions:*

1. $f(z) = z(z-1)$,
2. $f(z) = (z-3)^5 \sin z$,
3. $f(z) = \frac{\sin z}{z(z-1)}$,
4. $f(z) = z\cos^2\left(\frac{\pi}{2z}\right)$.

1. The zeros of this function are $z = 0, 1$, both of which are order 1.
2. $z = 3$ is a zero of this function with order 5. $z = \pi n$ for $n \in \mathbb{Z}$ are zeros of order 1.
3. $z = n\pi$ for $n \in \mathbb{Z} \setminus \{0\}$ are zeros of order 1 for this function. $z = 0$ is a removable singularity and $z = 1$ is a simple pole.
4. The cosine part is zero when $\frac{\pi}{2z} = \frac{(2n+1)\pi}{2}$ for $n \in \mathbb{Z} \setminus \{0\}$, hence $z = 2n+1$ are zeros of order 2 (2 because cos is squared). $z = 0$ is an essential singularity. We can see this by considering the Laurent series. Since cos is entire the Laurent series is simply given by inserting $\pi/2z$ into the Taylor series for cos:

$$z\cos^2\left(\frac{\pi}{2z}\right) = z\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(-\frac{\pi}{2z}\right)^n\right]^2 = z\left[\sum_{n=-\infty}^{0} \frac{z^n}{(-2n)!}\left(\frac{\pi}{2}\right)^{-n}\right]^2.$$

This clearly has an infinite number of negative terms.

---

14 Meromorphic Functions

Definition 50: Meromorphic Function

A function, f, is called **meromorphic** if it is analytic in $\mathbb{C}$ except for at countably many isolated poles, that is f is analytic almost everywhere.

Recall that countable includes a finite number of poles or infinitely many with cardinality $\aleph_0 = |\mathbb{N}|$[12].

Example 14.1. The function $f(z) = e^{-z^2}$ is a composition of the entire functions e^z and $-z^2$. Therefore f is entire and thus by definition f is meromorphic.

Let P_n and Q_m be polynomials of order n and m. Then $f(z) = P_n(z)/Q_m(z)$ has singularities only where $Q_m(z) = 0$ and since this happens m times by the fundamental theorem of algebra f is meromorphic.

Let $f(z) = \tan z$. Then f is singular for $z = (n+1/2)\pi$ for $n \in \mathbb{Z}$. Since $\mathbb{Z}$ is countable f is meromorphic.

Let $f(z) = \sec(1/z)$. Then $1/z$ is a non-isolated singularity so f is not meromorphic.

Let $f(z) = \log(z)$ with a branch cut from 0 to negative infinity. Then the set of poles is $\mathbb{R}_{\le 0} = (-\infty, 0]$ and this has the cardinality of the continuum so is uncountable meaning that f is not meromorphic.

Meromorphic functions have a Laurent series at every point in the complex plane and we can think of them as being the Laurent series analogue of an analytic function which has a Taylor series at every point in the complex plane.

Theorem 17

Let f be analytic. Then all of its zeros are isolated or it is identically zero.

[12]The following are all countable: any finite set, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, the set of algebraic numbers (roots of non-zero complex polynomials with rational coefficients), and the set of all constructible numbers

Proof. Let f be an analytic function which is not identically zero. Let z_0 be a zero fo order N. Then we can write

$$f(z) = (z - z_0)^N g(z)$$

for some analytic function g where $g(z_0) \neq 0$. Since g is analytic it is also continuous. This means that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|z - z_0| < \delta \implies |g(z) - g(z_0)| < \varepsilon.$$

Specifically we can find δ such that this is true for $\varepsilon = |g(z_0)|/2$. Suppose that f has non-isolated zeros. Then no matter how small δ is we can find $z' \in D'(z_0; \delta)$ such that $f(z') = 0$. However $z' \neq z_0$ so for this to be the case we have

$$0 = f(z') = (z' - z_0)^N g(z')$$

and so $g(z') = 0$. However this means that

$$|g(z') - g(z_0)| = |0 - g(z_0)| > \frac{|g(z_0)|}{2}.$$

So we have a contradiction which means that our assumption that non-isolated zeros exist is false and the theorem is proven. □

Corollary 4

Let f and g be analytic in a region S. Then if $f = g$ in a smooth arc, γ, then both functions are identical in S.

Proof. Let f and g be analytic functions on S which are equal on γ. Then $f - g$ is also an analytic function. Along the arc γ $(f - g)(z) = 0$ for all $z \in \gamma$. Since γ is smooth we can pick two points on γ which are arbitrarily close. $f - g$ is zero at both of these points and therefore the zeros of $f - g$ are not isolated. This means that $f - g$ must be identically zero and hence $f = g$ on all of S. □

Part V

Residues

15 Cauchy's Residue Theorem

We now have methods for computing many types of integrals. In example 10.1 we have seen that

$$\oint_\gamma \frac{1}{(z-a)^{n+1}} \,\mathrm{d}z = \begin{cases} 2\pi i, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases} \tag{15.1}$$

From the Cauchy–Goursat theorem we know that

$$\oint_\gamma f(z) \,\mathrm{d}z = 0$$

for any analytic function, f. From Cauchy's integral formula we know that if f is analytic then

$$f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - a} \,\mathrm{d}z$$

and from Cauchy's generalised integral formula we have

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-a)^{n+1}} \,\mathrm{d}z\,.$$

We will now use these results to find a way to integrate *any* meromorphic function over a closed contour.

15.1 One Isolated Singularity

Let $f\colon S \to \mathbb{C}$ be analytic on $S \setminus \{z_0\}$ and let z_0 be an isolated singularity of f. Let γ be a positively orientated closed contour inside S with z_0 inside γ.

Since f only has isolated singularities we know that it has a Laurent series at every point in S. Thus

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n.$$

Hence the integral of f over γ is

$$I = \oint_\gamma f(z)\, \mathrm{d}z = \oint_\gamma \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n\, \mathrm{d}z\,.$$

Since Laurent series are uniformly convergent we can exchange the integral and summation so

$$I = \sum_{n=-\infty}^{\infty} c_n \oint_\gamma (z - z_0)^n\, \mathrm{d}z\,.$$

We now use equation 15.1 and we see that all terms of the sum when $n \neq -1$ vanish leaving us with

$$I = \int_\gamma f(z)\, \mathrm{d}z = c_{-1} \oint_\gamma \frac{1}{za}\, \mathrm{d}z = 2\pi i c_{-1}.$$

This result holds for any isolated singularity, z_0, even an essential singularity.

15.2 Cauchy's Residue Theorem

We saw in the previous section that the integral,

$$\oint_\gamma f(z)\, \mathrm{d}z\,,$$

is entirely determined by the $n = -1$ coefficient of the Laurent series. This is such a useful result that we give this coefficient a special name.

Definition 51: Residue

Let $f\colon S \to \mathbb{C}$ be meromorphic on S and let $z_0 \in S$ be a singularity of f. The **residue** of f at z_0 is the c_{-1} coefficient of the Laurent series of f expanded around z_0, which we denote

$$c_{-1} = \mathrm{Res}(f, z_0).$$

We are now ready for one of the most useful theorems of this course which will allow us to compute many integrals, both real and complex, by simply performing a summation.

Theorem 18: Cauchy's Residue Theorem

Let $f\colon S \to \mathbb{C}$ be meromorphic and let γ be a positively orientated closed contour in S. Then

$$\oint_\gamma f(z)\, \mathrm{d}z = 2\pi i \sum_i \mathrm{Res}(f, z_i)$$

where $\{z_i\}$ is the set of isolated singularities inside γ.

Proof. The singularities, z_i, are isolated which allows us to create contours, γ_i, such that each contour contains exactly one singularity, z_i, and the boundaries of said contours are such that the sum of the contours is γ. See figure 15.1 Then

$$\oint_\gamma f(z)\, \mathrm{d}z = \sum_i \oint_{\gamma_i} f(z)\, \mathrm{d}z\,.$$

Since each contour, γ_i, contains only a single isolated singularity we already saw in the previous section that

$$\oint_{\gamma_i} f(z)\,\mathrm{d}z = 2\pi i c_{-1} = 2\pi i\,\mathrm{Res}(f, z_i)$$

and so

$$\oint_{\gamma} f(z)\,\mathrm{d}z = 2\pi i \sum_i \mathrm{Res}(f, z_i).$$

□

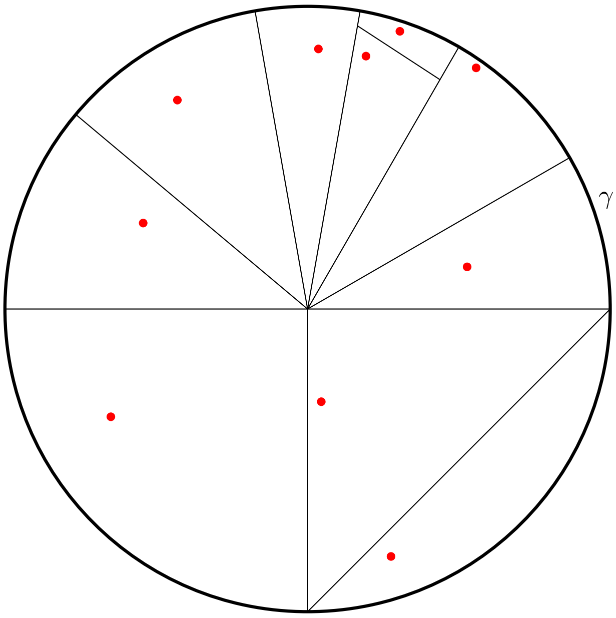


Figure 15.1: An example of contours used to prove Cauchy's residue theorem. Notice if each closed contour is integrated around in an anticlockwise direction then each internal line is integrated along twice, once in each direction, so the net result is an integral over the external contour.

Cauchy's residue theorem gives us a way to integrate any meromorphic function. Simply follow these steps:

1. Find all singularities of the integrand.
2. Find which of these singularities fall within the contour we are integrating.
3. Compute the residues of these singularities and sum.

15.3 Computing Residues

For Cauchy's residue theorem to be useful we have to be good at computing residues. Fortunately this is often surprisingly easy once we have identified the type of singularity. Here we will discuss computation of residues for different types of singularities starting with the simplest case and progressing on to harder cases. The goal is to find a way to compute

$$\mathrm{Res}(f, a) = \oint_{\gamma} f(z)\,\mathrm{d}z$$

where $z = a$ is a singularity of f.

15.3.1 Polynomial, Simple Poles

If f has a simple pole at a then we can write

$$f(z) = \frac{g(z)}{z - a}$$

where g is analytic at $z = a$. The residue of f at $z = a$ is then

$$\operatorname{Res}(f, a) = \oint_\gamma \frac{g(z)}{z - a} \, \mathrm{d}z = g(a)$$

where we have used Cauchy's integral formula for the final equality.

15.3.2 Pole of Order n

If f has a pole of order n at a then we can write

$$f(z) = \frac{g(z)}{(z - a)^n}$$

where g is analytic at $z = a$. The residue of f at $z = a$ is then

$$\operatorname{Res}(f, a) = \oint_\gamma \frac{g(z)}{(z - a)^n} \, \mathrm{d}z = \frac{1}{(n-1)!} g^{(n-1)}(a)$$

where we have used Cauchy's generalised integral formula for the final equality.

15.3.3 General, Simple Pole

If f has a simple pole at a then we can write

$$f(z) = \frac{g(z)}{h(z)}$$

where f and h are analytic at a. We can write f as a Laurent series:

$$f(z) = \frac{c_{-1}}{z - a} + c_0 + c_1(z - a) + \cdots .$$

Then

$$(z - a)f(z) = c_-1 + c_0(z - a) + c_1(z - a)^2 + \cdots .$$

This is a Taylor series so $(z - a)f(z)$ is analytic. Evaluating this Taylor series at $z = a$ we expect to get c_{-1} as this is the only term that won't be zero. However care has to be taken since f is not analytic at a so we take limits:

$$\begin{aligned}
\operatorname{Res}(f, a) &= \lim_{z \to a} (z - a)f(z) \\
&= \lim_{z \to a} \frac{(z - a)g(z)}{h(z)} \\
&= \lim_{z \to a} \frac{\frac{\mathrm{d}}{\mathrm{d}z}[(z - a)g(z)]}{\frac{\mathrm{d}}{\mathrm{d}z} h(z)} \\
&= \lim_{z \to a} \frac{g(z) + (z - a)g'(z)}{h'(z)} \\
&= \frac{g(a)}{h'(a)}.
\end{aligned}$$

Here we have used L'Hôpital's rule, which applies to complex functions just as it does real functions. In the last step we assume that $g(a)/h'(a)$ is not indeterminate, if it is then we may need to apply L'Hôpital's rule again.

15.3.4 None of the Previous Cases

If none of the previous situations apply, say a is an essential singularity of f, then often we can find the residue by simply computing the Laurent series from known series. For example if $f(z) = e^{-2z}/z^3$ then since e^z is entire we have

$$f(z) = \frac{1}{z^3} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^{2n}} = \frac{1}{z^3} - \frac{2}{z^2} + \frac{2}{z} - \frac{4}{3} + \cdots$$

and so we identify $\operatorname{Res}(f, 0) = 2$.

15.3.5 General, Non-Essential Singularity

If f has a non-essential singularity of order n at a then expanding f as a Laurent series we have

$$f(z) = \frac{c_{-n}}{(z-a)^n} + \frac{c_{-n+1}}{(z-a)^{n-1}} + \dots + \frac{c_{-1}}{(z-a)^n} + c_0 + \cdots .$$

Hence

$$(z-a)^n f(z) = c_{-n} + c_{-n+1}(z-a) + \dots + c_{-1}(z-a)^{n-1} + c_0(z-a)^n + \cdots$$

so $(z-a)^n f(z)$ is analytic. We can then compute c_{-1} as the $(n-1)$th term of the Taylor series for $(z-a)^n f(z)$. That is

$$\operatorname{Res}(f, a) = \frac{1}{(n-1)!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}[(z-a)^{n-1}f(z)]\bigg|_{z=a}.$$

However due to being awful to compute this method should be reserved for when all other methods have failed.

Example 15.1. *Find the residue of $f(z) = e^{ika}/(z-a)$ at $z = a$.*

We can write

$$f(z) = \frac{g(z)}{z-a}$$

where $g(z) = e^{ika}$. f has a simple pole at $z = a$ so we have

$$\operatorname{Res}(f, a) = g(a) = e^{ika}.$$

Example 15.2. *Find the residue of $f(z) = e^{ika}/(z-a)^2$ at $z = a$.*

We can write

$$f(z) = \frac{g(z)}{(z-a)^2}.$$

f has a pole of order 2 at $z = a$ so we have

$$\operatorname{Res}(f, a) = \frac{1}{(2-1)!}g^{(2-1)}(a) = g'(a) = ike^{ika}.$$

Example 15.3. *Find the residues of $f(z) = \pi\cot(\pi z)$ at all singularities.*

We can write

$$f(z) = \pi\frac{\cos(\pi z)}{\sin(\pi z)} = \pi\frac{g(z)}{h(z)}$$

where $g(z) = \cos(\pi z)$ and $h(z) = \sin(\pi z)$. This has simple poles at $z = n \in \mathbb{Z}$. Hence the residue is

$$\begin{aligned}
\operatorname{Res}(f, n) &= \lim_{z\to n}\pi\frac{(z-n)g(z)}{h(z)} \\
&= \pi\lim_{z\to n}\frac{(z-n)g'(z) + g(z)}{h'(z)} \\
&= \pi\frac{g(n)}{h'(n)} \\
&= \pi\frac{\cos(\pi n)}{\pi\cos(\pi n)} \\
&= 1.
\end{aligned}$$

Example 15.4. *Find the residues of $f(z) = e^{1/z}$ at $z = 0$.*

Here f has an essential singularity at $z = 0$. Since e^z is entire we have

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^n} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots$$

so we see that

$$\operatorname{Res}(z, 0) = 1.$$

Example 15.5. *Find the singularities of the following function and state their order. Compute the residues.*

$$f(z) = \frac{\cos z}{z^2(z-\pi)^3}.$$

This function has singularities at $z = 2$ (2nd order) and $z = \pi$ (3rd order). The denominator is a polynomial so we can evaluate the residues as follows:

$$\begin{aligned}
\operatorname{Res}(f, 0) &= \frac{\mathrm{d}}{\mathrm{d}z}\frac{\cos z}{(z-\pi)^3}\bigg|_{z=0} \\
&= -\frac{\sin z}{(z-\pi)^3}\bigg|_{z=0} - 3\frac{\cos z}{(z-\pi)^4}\bigg|_{z=-} \\
&= \frac{-3}{\pi^4}. \\
\operatorname{Res}(f, \pi) &= \frac{1}{2!}\frac{\mathrm{d}}{\mathrm{d}z}\frac{\cos z}{z^2}\bigg|_{z=\pi} \\
&= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\frac{\sin z}{z^2}\bigg|_{z=\pi} - \frac{\mathrm{d}}{\mathrm{d}z}\frac{\cos z}{z^3}\bigg|_{z=\pi} \\
&= -\frac{1}{2}\frac{\cos z}{z^2}\bigg|_{z=\pi} + \frac{3}{2}\frac{\sin z}{z^3}\bigg|_{z=\pi} + \frac{\sin z}{z^3}\bigg|_{z=\pi} + 3\frac{\cos z}{z^4}\bigg|_{z=\pi} \\
&= \frac{\pi^2 - 6}{2\pi^4}
\end{aligned}$$

Example 15.6. *Compute the residue of $f(z) = 1/(z^{27} - 1)$ at $z = 1$.*

This is an example to show that case 1 is not always the best method for computing residues, even though it could be applied here. We would need to compute all 27 roots of $z^27 = 1$ which is not too bad but is much slower than using L'Hôpital's rule[13]:

$$\begin{aligned}
\operatorname{Res}(f, 1) &= \lim_{z\to 1}\frac{z-1}{z^{27}-1} \\
&= \lim_{z\to 1}\frac{\frac{\mathrm{d}}{\mathrm{d}z}(z-1)}{\frac{\mathrm{d}}{\mathrm{d}z}(z^{27}-1)} \\
&= \lim_{z\to 1}\frac{1}{27z^{26}} \\
&= \frac{1}{27}.
\end{aligned}$$

---

16 Applications of Cauchy's Residue Theorem

16.1 Improper Integrals

One of the most common applications of Cauchy's residue theorem is to evaluate real, improper integrals. An improper integral is one where one (or both) of the limits is $\pm\infty$. Formally this is simply short hand

[13] note that although z is raised to the 27th power $z^{27} - 1$ is only raised to the first power and therefore $z = 1$ is a simple pole.

for a limit. Let $a \in \mathbb{R}$ then for an integrable function f we have

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_a^R f(x)\,\mathrm{d}x\,,$$
$$\int_{-\infty}^a f(x)\,\mathrm{d}x = \lim_{\rho\to-\infty} \int_\rho^a f(x)\,\mathrm{d}x\,,$$
$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \lim_{\rho,R\to\infty} \int_{-\rho}^R f(x)\,\mathrm{d}x\,.$$

Often these integrals can be solved using complex analysis by the following procedure:

1. Choose a suitable meromorphic function on $\mathbb{C}$. Often simply extending the domain of the integrand to $\mathbb{C}$ is sufficient.
2. Choose a suitable contour such that one arc of the contour reduces to the integral we want to know. Often semicircles or rectangles with one side coinciding with $\mathbb{R}$ are good choices.
3. Evaluate the integral over the entire contour using various methods from complex analysis.
4. Evaluate the integral over each arc.
5. Hope that the extra arcs either vanish (in the limit) or are easily computable.
6. Rearrange to find the integral we want.

Example 16.1. *Evaluate the following integral*[14]*:*

$$I = \int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x$$

Let $f(z) = 1/(1+z^2)$ define a complex function. Consider the semicircle of radius R centred on $z = 0$ with its flat side along $\mathbb{R}$. In the limit $R \to \infty$ the integral along the flat side of the semicircle will be our desired integral. That is

$$\lim_{R\to\infty} \int_\gamma f(z)\,\mathrm{d}z = \lim_{R\to\infty} \int_{-R}^R f(z)\,\mathrm{d}z + \lim_{R\to\infty} \int_{C_R} f(z)\,\mathrm{d}z = 2I + \lim_{R\to\infty} \int_{C_R} f(z)\,\mathrm{d}z$$

where γ is the entire semicircle and C_R is the curved part of the semicircle. We have used here that f is an even function so the integral over $[-R, R]$ is twice the integral over $[0, R]$. We simply need to compute the integral over γ and C_R to find I.

We will start with the integral over γ. f has two singularities, at $z = \pm i$. Only $z = i$ is inside γ. Hence

$$\oint_\gamma f(z)\,\mathrm{d}z = \oint_\gamma \frac{1}{(z+i)(z-i)}\,\mathrm{d}z = 2\pi i \frac{1}{z+i}\bigg|_{z=i} = \pi.$$

Here we used Cauchy's integral formula.

Next we bound the integral over C_R. To do this we parametrise this arc as $C_R(\vartheta) = Re^{i\vartheta}$ and so $C_R'(\vartheta) = iRe^{i\vartheta} \neq 0$. Thus

$$\begin{aligned}
\lim_{R\to\infty} \left| \int_{C_R} \frac{1}{1+z^2}\,\mathrm{d}z \right| &\le \lim_{R\to\infty} \int_{C_R} \left| \frac{1}{1+z^2}\,\mathrm{d}z \right| \\
&= \lim_{R\to\infty} \int_0^\pi \left| \frac{iRe^{i\vartheta}}{1+R^2e^{2i\vartheta}} \right| \mathrm{d}\vartheta \\
&= \lim_{R\to\infty} \int_0^\pi \frac{R}{|1+R^2e^{2i\vartheta}|}\,\mathrm{d}\vartheta \\
&\le \lim_{R\to\infty} \int_0^\pi \frac{R}{|R^2e^{2i\vartheta}|-1}\,\mathrm{d}\vartheta \\
&= \lim_{R\to\infty} \int_0^\pi \frac{R}{R^2-1}\,\mathrm{d}\vartheta
\end{aligned}$$

[14] this is just arctan but knowing that is not really in the spirit of the example.

$$\leq \lim_{R\to\infty} \frac{\pi R}{R^2 - 1}$$
$$= 0$$

So we have

$$\lim_{R\to\infty} \int_{C_R} f(z)\,\mathrm{d}z = 0.$$

This means that

$$\oint_\gamma f(z)\,\mathrm{d}z = 2I + 0 \implies I = \frac{1}{2}\oint_\gamma f(z)\,\mathrm{d}z = \frac{\pi}{2}.$$

This is the same result as we get by identifying

$$\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x = [\arctan x]_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

16.2 Trigonometric Integrals

An integral of the form

$$I = \int_0^{2\pi} f(\cos\vartheta, \sin\vartheta)\,\mathrm{d}\vartheta$$

can often be computed with the substitution

$$\cos\vartheta = \mathrm{Re}(z) = \frac{1}{2}\left(z + \frac{1}{z}\right), \qquad \text{and} \qquad \mathrm{Im}(z) = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

We integrate along the unit circle, $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, then $\mathrm{d}z = ie^{i\vartheta}\,\mathrm{d}\vartheta = iz\,\mathrm{d}\vartheta$. We then find that

$$\int_0^{2\pi} f(\cos\vartheta, \sin\vartheta)\,\mathrm{d}\vartheta = \oint_{S^1} f\left(\frac{1}{2}\left[z + \frac{1}{z}\right], \frac{1}{2}\left[z - \frac{1}{z}\right]\right)\frac{1}{iz}\,\mathrm{d}z\,.$$

This is simply a change of variables to a complex variable. Since this integral is over a closed contour it is often easy to evaluate. It does only work if we integrate all the way around the circle but it is sometimes possible to rearrange the integrand so that this is the case.

Example 16.2. *Evaluate*

$$I = \int_0^{2\pi} \frac{1}{\frac{5}{4} + \sin\vartheta}\,\mathrm{d}\vartheta\,.$$

Making the suggested variable change we have

$$\begin{aligned} I &= \int_0^{2\pi} \frac{1}{\frac{5}{4} + \sin\vartheta}\,\mathrm{d}\vartheta \\ &= \oint_{S^1} \frac{1}{\frac{5}{4} + \frac{1}{2i}\left(z - \frac{1}{z}\right)} \\ &= 2\oint_{S^1} \frac{1}{(z+2i)\left(z + \frac{i}{2}\right)}\,\mathrm{d}z\,. \end{aligned}$$

The integrand has two poles and only $z = -i/2$ is inside the unit circle. Therefore using Cauchy's integral formula

$$I = 2\pi i 2 \frac{1}{z+2i}\bigg|_{z=-i/2} = \frac{8\pi}{3}.$$

---

17 Estimating Arc Integrals

In this section we are interested in bounding integrals over part of a circle. We will introduce a lemma bounding integrals over infinite circles and the switch directions and bound integrals over infinitesimal circles.

17.1 Jordan's Lemma

Lemma 14

Let $f\colon \mathbb{C} \to \mathbb{C}$. Suppose that far from the origin this function goes as $|f(z)| \sim M/R^k$ for $k \in \mathbb{R}$, $k > 1$. Formally what this means is that there exists $r \in \mathbb{R}$ such that for all z satisfying $|z| > r$ we have $|f(z)| \le M/R^k$. Then if C_R is a segment of a circle from ϑ_1 to ϑ_2 with radius R we have

$$\lim_{R\to\infty} \int_{C_R} f(z)\,\mathrm{d}z = 0.$$

Proof. Let f be a function with the specified behaviour far from the origin. Then we can parametrise the arc C_R as $C_R(\vartheta) = Re^{i\vartheta}$ so $C_R'(\vartheta) = iRe^{i\vartheta}$. Hence

$$\begin{aligned}
\lim_{R\to\infty} \left| \int_{C_R} f(z)\,\mathrm{d}z \right| &\le \lim_{R\to\infty} \int_{C_R} |f(z)\,\mathrm{d}z| \\
&\le \lim_{R\to\infty} \int_{C_R} \frac{M}{R^k} |\mathrm{d}z| \\
&= \lim_{R\to\infty} \int_0^\pi \frac{MR}{R^k}\,\mathrm{d}\vartheta \\
&\le \lim_{R\to\infty} \frac{M\pi}{R^{k-1}} \\
&= 0.
\end{aligned}$$

□

This result isn't particularly useful on its own but it allows us to prove the next lemma which is.

Lemma 15: Jordan's Lemma

Let $f\colon \mathbb{C} \to \mathbb{C}$ be meromorphic. Let $|f(z)| \le M/R^k$ for all $|z| > R_0$ for some $R_0 \in \mathbb{R}$. Let C_R be a semicircular arc of radius $R > R_0$ from 0 to π in the upper half plane. Let m be some positive real number. Then

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{imz}\,\mathrm{d}z = 0.$$

Proof. We start by bounding the integral. To do this we will use the same parametrisation as the last proof.

$$\begin{aligned}
\lim_{R\to\infty} \left| \int_{C_R} f(z)e^{imz}\,\mathrm{d}z \right| &\le \lim_{R\to\infty} \int_{C_R} \left| f(z)e^{imz}\,\mathrm{d}z \right| \\
&= \lim_{R\to\infty} \int_{C_R} |f(z)||\exp[imR\cos\vartheta - mR\sin\vartheta]||\mathrm{d}z| \\
&\le \lim_{R\to\infty} \int_0^\pi \frac{M}{R^k} \exp[-mR\sin\vartheta] R\,\mathrm{d}\vartheta \\
&= \lim_{R\to\infty} \frac{2M}{R^{k-1}} \int_0^{\pi/2} e^{-mR\sin\vartheta}\,\mathrm{d}\vartheta\,.
\end{aligned}$$

In this last step we have used the fact that $\sin\vartheta = \sin(\pi - \vartheta)$ and so

$$\int_0^\pi e^{-mR\sin\vartheta}\,\mathrm{d}\vartheta = 2\int_0^{\pi/2} e^{-mR\sin\vartheta}\,\mathrm{d}\vartheta$$

since sin is even on the interval $[0, \pi]$.

sin is a monotonically increasing convex function on $[0, \pi/2]$. At $\vartheta = 0$ we have $\sin 0 = 0$ and at $\vartheta = \pi/2$ we have $\sin(\pi/2) = 1$. Hence at the points $\vartheta = 0, \pi/2$ the equality $\sin\vartheta = 2\vartheta/\pi$ holds. Since between these points sin is monotonically increasing and convex it the equality between the points doesn't hold but becomes an inequality:

$$\sin\vartheta \geq \frac{2}{\pi}\vartheta, \qquad \text{for } \vartheta \in [0, \pi/2].$$

This is known as **Jordan's inequality**. We can use it to further bound our integral:

$$\begin{aligned}\int_0^{\pi/2} e^{-mR\sin\vartheta}\,\mathrm{d}\vartheta &\leq \int_0^{\pi/2} e^{-mR2\vartheta/\pi}\,\mathrm{d}\vartheta\\ &= -\frac{\pi}{2mR}[e^{-mR2\vartheta/\pi}]_0^{\pi/2}\\ &= \frac{\pi}{2mR}(1 - e^{-mR}).\end{aligned}$$

Applying this to our bound we have

$$\lim_{R\to\infty}\left|\int_{C_R} f(z)e^{imz}\,\mathrm{d}z\right| \leq \lim_{R\to\infty}\frac{\pi M}{mR^k}(1 - e^{-mR}) = 0$$

since both $k, m > 0$. □

The methods we have seen previously for bounding integrals would still work here except we would find a polynomial limit and would need $f(z)/R^{k-1} \to 0$ instead with Jordan's lemma we only need $f(z)/R^k \to 0$ which allows an extra factor of R in f. There is an analogous limit when $m < 0$ and the proof is similar but we integrate over a semicircle in the lower half plane. In the case of $m = 0$ we can't do better than the R^{-k+1} bounding.

17.2 Indentation Lemma

Suppose we have a contour but at some point on the contour the function we wish to integrate is singular. We can sometimes get around this by deviating slightly from the contour along a circular arc. We then take the radius of this arc to zero and hope that the contribution from the arc vanishes.

Lemma 16: Indentation Lemma

Let $f\colon D'(z_0; r) \to \mathbb{C}$ be analytic and let z_0 be a simple pole of f. We define an indentation around z_0 as the contour parametrised by $C_\rho = z_0 + \rho e^{i\vartheta}$ where $\rho \in (0, r)$ and $\vartheta \in [\vartheta_1, \vartheta_2]$. Then

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)\,\mathrm{d}z = (\vartheta_2 - \vartheta_1)i\operatorname{Res}(f, z_0).$$

Proof. Since f has a simple pole at z_0 the residue is

$$\operatorname{Res}(f, z_0) = \lim_{z\to z_0}(z - z_0)f(z).$$

Then by the definition of the limit for a given $\varepsilon > 0$ there exists δ such that $|(z - z_0)f(z) - \operatorname{Res}(f, z_0)| < \delta$. On the arc, C_ρ, we have $\mathrm{d}z = i\rho e^{i\vartheta}\,\mathrm{d}\vartheta = i(z - z_0)\,\mathrm{d}\vartheta$. It is actually simpler to consider the difference between the two sides of the equation and show that the difference is zero:

$$\begin{aligned}\left|\int_{C_\varepsilon} f(z)\,\mathrm{d}z - (\vartheta_2 - \vartheta_1)i\operatorname{Res}(f, z_0)\right| &= \left|\int_{C_\varepsilon} f(z)\,\mathrm{d}z - \int_{\vartheta_1}^{\vartheta_2} i\operatorname{Res}(f, z_0)\,\mathrm{d}\vartheta\right|\\ &= \left|\int_{\vartheta_1}^{\vartheta_2} i[(z - z_0)f(z) - \operatorname{Res}(f, z_0)]\,\mathrm{d}\vartheta\right|\\ &\leq \delta|\vartheta_2 - \vartheta_1|\end{aligned}$$

By choosing sufficiently small ε we can make δ arbitrarily small and so

$$\int_{C_\varepsilon} f(z)\,\mathrm{d}z - (\vartheta_2 - \vartheta_1)i\operatorname{Res}(f, z_0) = 0$$

which proves the theorem. □

17.3 Integral of sinc

The un-normalised sinc function is defined as

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

It is common to require the integral

$$I = \int_{-\infty}^{\infty} \operatorname{sinc} x \, \mathrm{d}x \,.$$

We can use the lemmas proved in this section to compute this value.

We cannot easily integrate $\operatorname{sinc} z$ for complex z as this is unbounded, in particular

$$\lim_{y \to \infty} |\operatorname{sinc}(iy)| = \infty$$

Instead we consider $f(z) = e^{iz}/z$. We integrate this over the contour shown in figure 17.1 Let C_R be

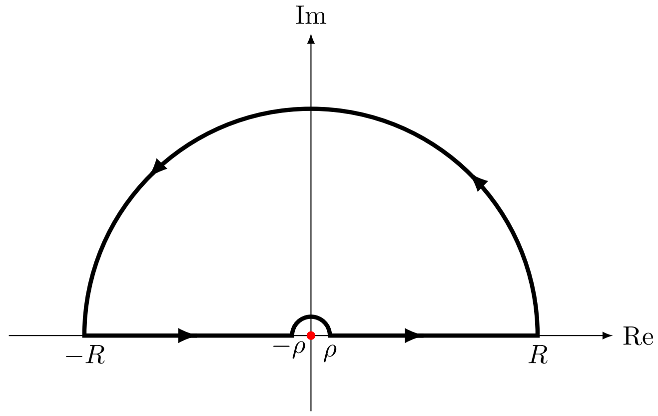


Figure 17.1: The contour used to find the integral of $\operatorname{sinc} x$.

the outer semicircle of radius R and C_ρ be the inner semicircle of radius ρ. Let γ be the entire contour. Then we see that

$$\oint_\gamma f(z) \, \mathrm{d}z = \int_{\gamma_R} f(z) \, \mathrm{d}z + \int_{\gamma_\rho} f(z) \, \mathrm{d}z + \int_\rho^R f(z) \, \mathrm{d}z + \int_{-R}^{-\rho} f(z) \, \mathrm{d}z \,.$$

Considering first the integrals along the real axis we can compute these together:

$$\begin{aligned}
\int_{-R}^{-\rho} f(z) \, \mathrm{d}z + \int_\rho^R f(z) \, \mathrm{d}z &= \int_{-R}^{-\rho} \frac{e^{iz}}{z} \, \mathrm{d}z + \int_\rho^R \frac{e^{iz}}{z} \, \mathrm{d}z \\
&= \int_{-R}^{-\rho} \frac{\cos x}{x} \, \mathrm{d}x + \int_\rho^R \frac{\cos x}{x} \, \mathrm{d}x + i \left[\int_{-R}^{-\rho} \frac{\sin x}{x} \, \mathrm{d}x + \int_\rho^R \frac{\sin x}{x} \, \mathrm{d}x \right] \\
&= -\int_{-\rho}^{-R} \frac{\cos x}{x} \, \mathrm{d}x + \int_\rho^R \frac{\cos x}{x} \, \mathrm{d}x + i \left[-\int_{-\rho}^{-R} \frac{\sin x}{x} \, \mathrm{d}x + \int_\rho^R \frac{\sin x}{x} \, \mathrm{d}x \right] \\
&= i \left[\int_\rho^R \frac{\sin x}{x} \, \mathrm{d}x + \int_\rho^R \frac{\sin x}{x} \, \mathrm{d}x \right] \\
&= 2i \int_\rho^R \frac{\sin x}{x} \, \mathrm{d}x
\end{aligned}$$

Here we have used the symmetry of cos and sin. Since cos is even the integral over $(-R, -\rho)$ and ρ, R are equal and so the integral cancel. Similarly for the sin integrals the integrals over these intervals differ only by a sign and therefore combine together. Further

$$\lim_{\rho \to 0} \lim_{R \to \infty} \left[\int_{-R}^{-\rho} f(z) \, \mathrm{d}z + \int_\rho^R f(z) \, \mathrm{d}z \right] = 2i \int_0^R \frac{\sin x}{x} \, \mathrm{d}x = \int_{-\infty}^{\infty} \frac{\sin x}{x} \, \mathrm{d}x = iI.$$

Here we have used the fact that sin and $1/x$ are odd functions so $\sin(x)/x$ is even.

For the integral over C_R we can use Jordan's Lemma which readily implies

$$\lim_{R\to\infty}\int_{C_R} f(z)\,\mathrm{d}z = \lim_{R\to\infty}\int_{C_R} \frac{e^{iz}}{z}\,\mathrm{d}z = 0.$$

For the integral over C_ρ we can use the indentation lemma as the pole at $z=0$ is simple. We transverse the small semicircle starting at $\vartheta_1=\pi$ and ending at $\vartheta_2=0$. It is also simple to compute the residue:

$$\operatorname{Res}(f,0) = e^{iz}|_{z=0} = 1$$

and so

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)\,\mathrm{d}z = i(\vartheta_2-\vartheta_1)\operatorname{Res}(f,0) = i(0-\pi)1 = -i\pi.$$

Since there are no singularities inside the contour we have

$$\oint_\gamma f(z)\,\mathrm{d}z = 0$$

by Cauchy's theorem. Hence

$$\oint_\gamma f(z)\,\mathrm{d}z = 0 = 0 - i\pi + iI \implies I = \int_{-\infty}^{\infty}\frac{\sin x}{x} = \pi.$$

18 Integral Reinforcement and Series

18.1 Integral Reinforcement

Most of the examples we have seen so far of real integrals computed with complex analysis have been done by finding a contour which coincides with the interval we are integrating and hoping that all other contributions to the integral are either zero or easy to compute. This isn't always the case. Sometimes the best we can do is find a contour which we can split into sections each of which results in an integral proportional to the integral we want. This is best shown with an example.

Evaluate

$$I = \int_0^\infty \frac{1}{x^5+a^5}\,\mathrm{d}x$$

for some positive real number a. To do this we consider

$$f(z) = \frac{1}{z^5+a^5}$$

integrated along the contour shown in figure 18.1. This contour has three sections. The horizontal section corresponds to the integral we want. The function f has singularities whenever $z_n = ae^{i\pi/5+2n\pi/5}$ for $n=0,1,2,3,4$. These are all simple poles as we can factorise the denominator and each only appears once. Hence

$$\operatorname{Res}(f,z_n) = \lim_{z\to z_n}\frac{z-z_n}{z^5+a^5} = \lim_{z\to z_n}\frac{1}{5z^4} = \frac{1}{5a^4e^{i\pi/5+2i\pi n/5}}.$$

Only one of these singularities is actually inside the contour and hence

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi i\operatorname{Res}(f,z_0) = \frac{2\pi i}{5a^4e^{4i\pi/5}}.$$

By Jordan's lemma the integral over the circular arc vanishes as $R\to\infty$. We need to consider the integral over the line from $z=Re^{2i\pi i/5}$ to $z=0$. We can parametrise this part of the contour as $z=xe^{2i\pi/5}$ for $x\in(0,R)$. We then have $\mathrm{d}z = e^{2i\pi/5}\,\mathrm{d}x$ and so the integral over this line is

$$\int_R^0 \frac{e^{2i\pi/5}}{x^5e^{2i\pi}+a^5}\,\mathrm{d}x = -e^{2i\pi/5}\int_0^R\frac{1}{x^5+a^5}\,\mathrm{d}x \to -e^{2i\pi/5}I.$$

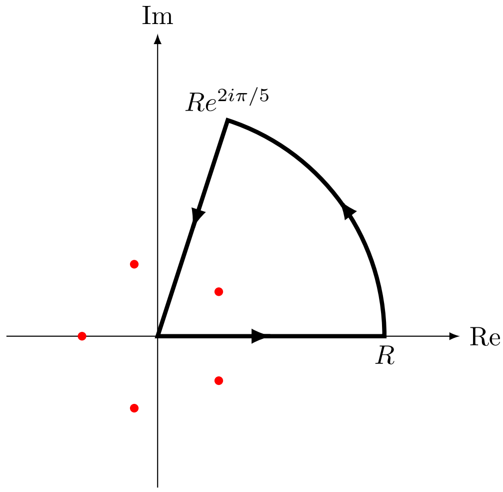


Figure 18.1: The contour used to integrate $1/(x^5 + a^5)$.

Hence in the limit of $R \to \infty$ we have

$$\begin{aligned}\oint_\gamma f(z)\,\mathrm{d}z &= \frac{2\pi i}{5a^4 e^{4i\pi/5}} \\ &= (1 - e^{2i\pi/5})I \\ \implies I = \int_0^\infty \frac{1}{x^5 + a^5}\,\mathrm{d}x \\ &= \frac{\pi}{5a^4 e^{4i\pi/4}} \frac{2i}{1 - e^{2\pi i/5}} \\ &= \frac{\pi}{5a^4(e^{i\pi/5} - e^{-i\pi/5})} \\ &= \frac{\pi}{5a^4 \sin(\pi/5)}.\end{aligned}$$

18.2 Series

We have seen that summing residues can give us the value of an integral but we can turn this on its head and use an integral to give us the value of a series. We simply need to find a function, f, such that the residues of f in some contour of our choosing correspond to the terms of the sum (up to a factor of $2\pi i$) and then we hope that we can compute the integral of f. This is best demonstrated with an example.

Show that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

The first thing we need to do is find a function with residues equal to $1/n^2$. We saw in example 15.3 that $f(z) = \pi\cot(\pi z)$ has residues at $z = n \in \mathbb{Z}$ and that $\mathrm{Res}(f, n) = 1$. Consider a general function f which has a simple pole at z_0 which has residue a. Now consider a second function, g, which is analytic at z_0 and $g(z_0) \neq 0$. We can write the product, fg, using the Laurent series of f:

$$(fg)(z) = f(z)g(z) = g(z)\left[\frac{a}{z - z_0} + \sum_{n=0}^{\infty} c_n(z - z_0)^n\right]$$

and so

$$\mathrm{Res}(fg, z_0) = g(z)(z - z_0)\left[\frac{a}{z - z_0} + \sum_{n=0}^{\infty} c_n(z - z_0)^n\right]\Bigg|_{z=z_0} = ag(z_0).$$

Hence for the specific case of $f(z) = \pi\cot(\pi z)$ we have $\mathrm{Res}(fg, n) = g(z_0)$. So we simply need to choose a function such that $g(n) = 1/n^2$ for all $n \in \mathbb{Z}$. The analytic function satisfying this is $g(z) = 1/z^2$. Hence for $n \neq 0$ we have

$$\mathrm{Res}(fg, n) = \frac{1}{n^2}.$$

For $n = 0$ we have to do a little more work. Expanding fg as a Laurent series at the origin we have

$$\begin{aligned}(fg)(z) &= \frac{\pi}{z^2}\cot(\pi z)\\ &= \frac{\pi}{z^2}\frac{1 - \frac{(\pi z)^2}{2!} + \frac{(\pi z)^4}{4!} + \cdots}{\pi z - \frac{(\pi z)^3}{3!} + \frac{(\pi z)^5}{5!} + \cdots}\\ &= \frac{1}{z^3}\left[1 + (\pi z)^2\left(\frac{1}{6} - \frac{1}{2}\right) + \cdots\right]\end{aligned}$$

so

$$\operatorname{Res}(fg, 0) = -\frac{\pi^3}{3}.$$

We now have a function that, for $n \in \mathbb{Z}_{>0}$, has the required residue $\operatorname{Res}(fg, n) = 1/n^2$. We now need to choose a contour that includes all of these residues and is also easy to integrate over. The contour of choice turns out to be a rectangle, shown in figure 18.2. This contour contains all $n \in \mathbb{Z}_{>0}$ but also

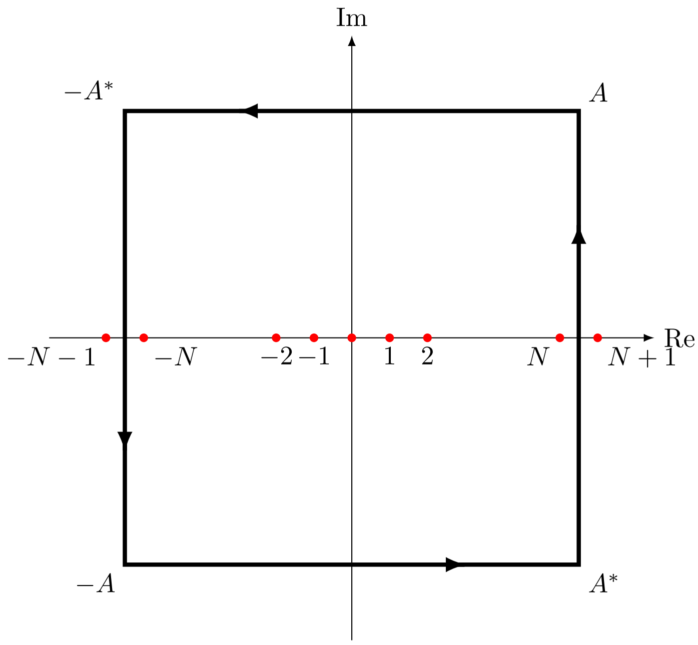


Figure 18.2: Contour used to evaluate $\zeta(2)$. The point A is $A = (N + 1/2)(1 + i)$.

all $n \in \mathbb{Z}_{\le 0}$. This integral vanishes in the limit of $N \to \infty$, which is fairly easy to show using the ML lemma:

$$\begin{aligned}\left|\oint_\gamma f(z)\,\mathrm{d}z\right| &\le 8\left(N + \frac{1}{2}\right)\pi \max_\gamma \left|\frac{\cot(\pi z)}{z^2}\right|\\ &\le \frac{8\pi}{N + \frac{1}{2}}\max_\gamma |\cot \pi z|.\end{aligned}$$

Here we have used $N + 1/2 > 1$ and on γ we have $|z| \le N + 1/2$ so replacing $\left|1/z^2\right|$ with $(N + 1/2)^{-2}$ gives us the result above. The integral will then vanish as long as $\max_\gamma |\cot(\pi z)|$ is finite. We can show that this is the case. First consider the horizontal lines which we can parametrise as $z = x \pm i(N + 1/2)$ giving

$$\begin{aligned}|\cot(\pi z)| &= \left|\frac{\cos(\pi z)}{\sin(\pi z)}\right|\\ &= \left|\frac{\exp\left[i\pi x \mp \pi\left(N + \frac{1}{2}\right)\right] + \exp\left[-i\pi x \pm \pi\left(N + \frac{1}{2}\right)\right]}{\exp\left[i\pi x \mp \pi\left(N + \frac{1}{2}\right)\right] - \exp\left[-i\pi x \pm \pi\left(N + \frac{1}{2}\right)\right]}\right|\\ &\le \left|\frac{\exp\left[\mp\pi\left(N + \frac{1}{2}\right)\right] + \exp\left[\pm\pi\left(N + \frac{1}{2}\right)\right]}{\exp\left[\mp\pi\left(N + \frac{1}{2}\right)\right] - \exp\left[\pm\pi\left(N + \frac{1}{2}\right)\right]}\right|\end{aligned}$$

$$= \coth\left(\pi\left(N+\frac{1}{2}\right)\right)$$

this is finite as $\lim_{x\to\infty}\coth x = 1$. For the vertical lines we can parametrise $z = \pm\pi(N+1/2)+iy$ which, after using trig addition formulae, gives

$$\begin{aligned}
|\cot(\pi z)| &= \left|\frac{\cos(\pm\pi(N+1/2)+i\pi y)}{\sin(\pm\pi(N+1/2)+i\pi y)}\right| \\
&= \left|\frac{(-1)^N\sin(i\pi y)}{(-1)^N\cos(i\pi y)}\right| \\
&= |\tanh(\pi y)| \\
&\le 1.
\end{aligned}$$

So $|\cot(\pi z)|$ is bounded on γ and therefore the integral vanishes. We then have

$$\begin{aligned}
0 &= \lim_{N\to\infty}\oint_\gamma (fg)(z)\,\mathrm{d}z \\
&= \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{n^2} \\
&= \lim_{N\to\infty} 2\pi i\left[\operatorname{Res}(fg,0)+\sum_{n=1}^{N}\operatorname{Res}(fg,n)+\sum_{n=-N}^{-1}\operatorname{Res}(fg,n)\right] \\
&= \lim_{N\to\infty} 2\pi i\left[\operatorname{Res}(fg,0)+\sum_{n=1}^{N}\frac{1}{n^2}+\sum_{n=-N}^{-1}\frac{1}{n^2}\right] \\
&= \lim_{N\to\infty} 2\pi i\left[-\frac{\pi^2}{3}+\sum_{n=1}^{N}\frac{1}{n^2}+\sum_{n=1}^{N}\frac{1}{n^2}\right] \\
&= \lim_{N\to\infty} 2\pi i\left[-\frac{\pi^2}{3}+2\sum_{n=1}^{N}\frac{1}{n^2}\right] \\
&= 2\pi i\left[-\frac{\pi^2}{3}+2\sum_{n=1}^{\infty}\frac{1}{n^2}\right]
\end{aligned}$$

$$\implies \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

The exact choice of the function to integrate depends on the series we wish to sum but there are a few common choices:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} g(n) &\longleftrightarrow \oint g(z)\pi\cot(\pi z)\,\mathrm{d}z \\
\sum_{n=-\infty}^{\infty} (-1)^n g(n) &\longleftrightarrow \oint g(z)\pi\operatorname{cosec}(\pi z)\,\mathrm{d}z \\
\sum_{n=-\infty}^{\infty} g(n+1/2) &\longleftrightarrow \oint g(z)\pi\tan(\pi z)\,\mathrm{d}z \\
\sum_{n=-\infty}^{\infty} (-1)^n g(n+1/2) &\longleftrightarrow \oint g(z)\pi\sec(\pi z)\,\mathrm{d}z\,.
\end{aligned}$$

All of these will commonly be integrated over a square but make sure to choose the corners such that the square doesn't intersect any of the singularities. This means in the first two cases choose the vertex $A = (N+1/2)(1+i)$ and in the second two $A = N(1+i)$.

19 Principle Values and Multivalued Functions

19.1 Principle Value

Often it is possible to compute an integral over some interval even if parts of the integrand diverge over those limits. For example consider

$$\lim_{a\to\infty}\int_0^a x\,\mathrm{d}x = \infty.$$

However since x is an odd function we have

$$\lim_{a\to\infty}\int_{-a}^a x\,\mathrm{d}x = 0.$$

Normally we could split the integral into two parts over $(-a, 0)$ and $(0, a)$ but that is not the case here as these two individual integrals diverge in such a way that they just happen to cancel out. When this is the case we say that this is a **principle value integral** or a Cauchy principle value integral[15]. If we are aware that an integral cannot be split in this way we denote it with one of the following notations:

$$\text{P.V.}\int_a^b f(x)\,\mathrm{d}x\,,\qquad\text{or}\qquad ⨍_a^b f(x)\,\mathrm{d}x\,.$$

An example may be for some positive real numbers a and b

$$⨍_{-a}^b \frac{1}{x}\,\mathrm{d}x = \log b - \log a = \log\frac{b}{a}$$

even though log is undefined at $0 \in (-a, b)$. The way to deal with these integrals formally is by missing out the problematic points with a limit. For example if f has a singularity at $x_0 \in (a, b)$ then

$$⨍_a^b f(x)\,\mathrm{d}x = \lim_{\varepsilon\to 0^+}\left[\int_a^{x_0-\varepsilon} f(x)\,\mathrm{d}x + \int_{x_0+\varepsilon}^b f(x)\,\mathrm{d}x\right].$$

This notation suggests that the correct way to deal with these sorts of integrals in the complex plane is with and indentation of radius ε, and this is indeed the case.

Example 19.1. *Evaluate the following principle value integral:*

$$⨍_{-\infty}^{\infty}\frac{1}{x^2-4}\,\mathrm{d}x\,.$$

The obvious choice is to integrate $f(z) = 1/(z^2 - 4)$ over a semicircle in the upper half plane. The problem with this is that f is singular at $z = \pm 2$. We can avoid these problem points with indentations. If we choose these indentations to be semicircles in the upper half plane then there are no singularities in the contour and so the integral over the closed contour is zero. Hence we have

$$\begin{aligned}0 &= \oint_\gamma f(z)\,\mathrm{d}z\\ &= \int_{-R}^{-2-\varepsilon}\frac{1}{z^2-4}\,\mathrm{d}z + \int_{-2+\varepsilon}^{2-\varepsilon}\frac{1}{z^2-4}\,\mathrm{d}z + \int_{2+\varepsilon}^{R}\frac{1}{z^2-4}\,\mathrm{d}z + \int_{C_R}\frac{1}{z^2-4}\,\mathrm{d}z + \int_{\gamma_{\varepsilon-}}\frac{1}{z^2-4}\,\mathrm{d}z + \int_{\gamma_{\varepsilon+}}\frac{1}{z^2-4}\,\mathrm{d}z\,.\end{aligned}$$

In the limit of $R \to 0$ the first three integrals become the principle value integral we wish to compute, the integral over γ_R vanishes by Jordan's lemma, and the integrals over $\gamma_{\varepsilon\pm}$ can be evaluated by the indentation lemma:

$$\begin{aligned}0 &= i(0-\pi)[\mathrm{Res}(f,-2) + \mathrm{Res}(f,2)]⨍_{-\infty}^{\infty}\frac{1}{x^2-4}\,\mathrm{d}x\\ &= -i\pi\left(\frac{1}{-2-2} + \frac{1}{2+2}\right) + ⨍_{-\infty}^{\infty}\frac{1}{x^2-4}\,\mathrm{d}x\\ \implies ⨍_{-\infty}^{\infty}&\frac{1}{x^2-4}\,\mathrm{d}x = 0.\end{aligned}$$

---

[15]because what we really need at this point in the course is something else named after Cauchy.

19.2 Multivalued Functions

We saw in sections 5 and 6 that functions involving logarithms or roots lead to multivalued functions and that to have a properly defined function we need to make a branch cut. A branch cut joins two branch points in such a way that the function becomes singular. This always results in a discontinuity which means that most of the theorems we have developed so far won't work if our contour crosses or contains a branch cut/point. This is because branch cuts/points are non-isolated singularities and therefore do not admit a Laurent series at this point. The easiest way round this is simply to ensure that we never have a branch cut/point inside our contour.

Example 19.2. *For $\alpha \in (0,1)$ evaluate*

$$I = \int_0^\infty \frac{x^{\alpha-1}}{x+1} \, \mathrm{d}x \, .$$

We consider the function

$$f(z) = \frac{z^{\alpha-1}}{z+1}.$$

This has two branch points, at $z = 0, \infty$. We choose a branch cut to join these two points along the positive real axis. This corresponds to the choice $0 < \arg z < 2\pi$. The contour that we use for this problem is shown in figure 19.1. This is commonly called a **keyhole contour**. This contour, γ, can be

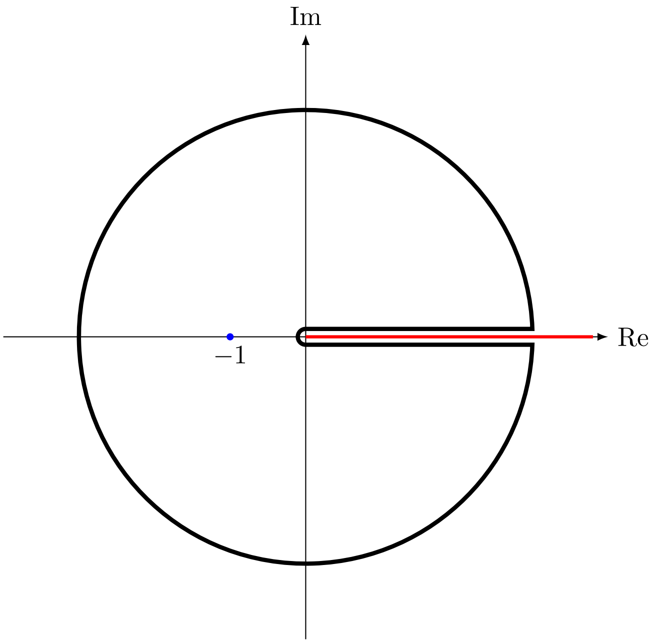


Figure 19.1: A keyhole contour for a branch cut from 0 to ∞ along the positive real axis.

split into four parts. First the outer circle, γ_R, second the inner indentation, γ_ε, which as drawn goes from $3\pi/2$ to $\pi/2$ but any angle that takes us around the origin works, and the final two sections are the straight lines just above and just below the real axis.

The function f has a one singularity at $z = -1$ and the residue at this point is

$$\operatorname{Res}(f, -1) = z^{\alpha-1}|_{z=-1} = (-1)^{\alpha-1} = e^{(\alpha-1)\pi i}.$$

Hence

$$\oint_\gamma f(z) \, \mathrm{d}z = 2\pi i e^{\pi i(\alpha-1)}$$

For the integral around the outer circle we have

$$\lim_{R\to\infty} \left| \int_{\gamma_R} f(z) \, \mathrm{d}z \right| \le \lim_{R\to\infty} \int_{\gamma_R} |f(z) \, \mathrm{d}z|$$

$$
\begin{aligned}
&= \lim_{R\to\infty} \int_{\gamma_R} \left|\frac{z^{\alpha-1}}{z+1}\right| |\mathrm{d}z| \\
&= \lim_{R\to\infty} \int_0^{2\pi} \frac{\left|R^{(\alpha-1)}e^{i\vartheta(\alpha-1)}\right|}{|Re^{i\vartheta}+1|} \left|iRe^{i\vartheta}\right| \mathrm{d}\vartheta \\
&\le \lim_{R\to\infty} \int_0^{2\pi} \frac{R^\alpha}{R-1} \,\mathrm{d}\vartheta \\
&= \lim_{R\to\infty} 2\pi \frac{R^\alpha}{R-1} \\
&= 0
\end{aligned}
$$

where we have used the fact that $\alpha < 1$ so $R^\alpha/(R-1) \sim R^{\alpha-1}$ vanishes as does the integral around the outer circle. A similar treatment of the indentation gives

$$
\begin{aligned}
\lim_{\varepsilon\to 0} \left|\int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z\right| &= \lim_{\varepsilon\to 0} \int_{\gamma_\varepsilon} |f(z)\,\mathrm{d}z| \\
&= \lim_{\varepsilon\to 0} \int_{\gamma_\varepsilon} \frac{\left|z^{\alpha-1}\right|}{|z-1|} |\mathrm{d}z| \\
&= \lim_{\varepsilon\to 0} \int_{\pi/2}^{3\pi/2} \frac{\left|\varepsilon e^{i\vartheta(\alpha-1)}\right|}{|\varepsilon e^{i\vartheta}+1|} \left|i\varepsilon e^{i\vartheta}\right| \mathrm{d}\vartheta \\
&\le \lim_{\varepsilon\to 0} \int_{\pi/2}^{3\pi/2} \frac{\varepsilon^\alpha}{1-\varepsilon} \,\mathrm{d}\vartheta \\
&= \lim_{\varepsilon\to 0} \pi \frac{\varepsilon^\alpha}{1-\varepsilon} \\
&= 0
\end{aligned}
$$

Note that we *cannot* use the indentation lemma here as $z = 0$ is a branch point and therefore *not* a simple pole as the indentation lemma requires (it doesn't even make sense to try to compute a residue at $z = 0$ as there is no Laurent series there).

The integral over the line just above the real axis reduces to the target integral. For the integral just below the real axis we can parametrise this line as $z = x - i\varepsilon \approx xe^{2\pi i}$ (technically this should be $z = x^{i\varphi}$ and then take the limit $\varphi \to 2\pi$ from below). Then we have

$$
\begin{aligned}
\lim_{R\to\infty} \int_{R-i\varepsilon}^{i\varepsilon} \frac{z^{\alpha-1}}{z+1} \,\mathrm{d}z &= \int_\infty^0 \frac{x^{\alpha-1}e^{2\pi i(\alpha-1)}}{x+1} \,\mathrm{d}x \\
&= -e^{2\pi i(\alpha-1)} \int_0^\infty \frac{x^{\alpha-1}}{x+1} \,\mathrm{d}x \\
&= -e^{2\pi i(\alpha-1)} I.
\end{aligned}
$$

Combining all of these integrals we have

$$
\oint_\gamma f(z)\,\mathrm{d}z = 2\pi i e^{\pi i(\alpha-1)} = I(1 - e^{2\pi i(\alpha-1)})
$$

and so

$$
I = \frac{2\pi i e^{\pi i(\alpha-1)}}{1 - e^{2\pi i(\alpha-1)}} = \frac{2\pi i}{e^{-\pi i(\alpha-1)} + e^{\pi i(\alpha-1)}} = \frac{\pi}{-\sin[\pi(\alpha-1)]} = \frac{\pi}{\sin[\pi(1-\alpha)]}.
$$

As a sanity check since $x^{\alpha-1}/(x+1) \ge 0$ for all $x \in (0, \infty)$ we expect that the integral is positive and real, which is indeed the case since $\pi(1-\alpha) \in (0, 1\pi)$ and sin is positive on this interval.

Example 19.3. *For $a > 0$ evaluate*

$$
I = \int_0^\infty \frac{\log x}{x^2 + a^2} \,\mathrm{d}x\,.
$$

We do this by considering the function

$$
f(z) = \frac{\log z}{z^2 + a^2}.
$$

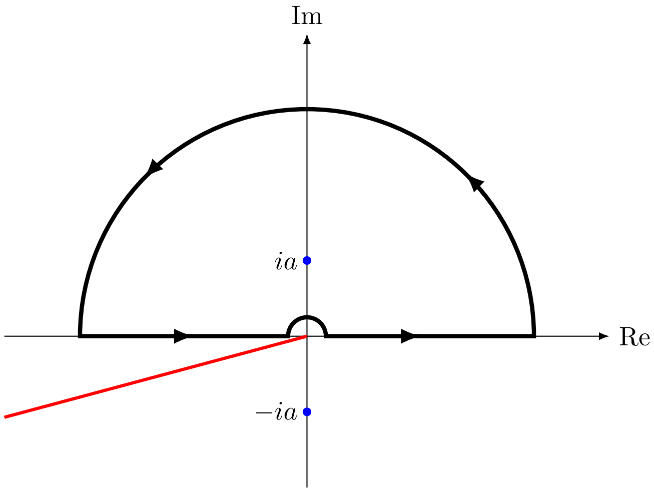


Figure 19.2: The contour used to evaluate the integral of $\log(x)/(x^2+a^2)$.

This has two simple poles, at $z = \pm ia$. Further f has two branch points at $z = 0, \infty$. The contour that we will use to evaluate this function is shown in figure 19.2. Here we choose the branch cut to be the half line from 0 to ∞ at angle $\pi + 1/4$. There is nothing particularly special about this choice, any value from $(\pi, 2\pi)$ would do. There is only one singularity, $z = ia$, inside the contour so

$$\begin{aligned}\oint_{\gamma f(z)} \mathrm{d}z &= 2\pi i \operatorname{Res}(f, ia) \\ &= 2\pi i \frac{\log z}{z + ia}\bigg|_{z=ia}\end{aligned}$$

We can also split the integral into four sections, the outer semicircle of radius R, C_R, the indentation of radius ρ, C_ρ, and the two straight segments. The integrals over the two semicircles vanish:

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{C_R} \frac{\log z}{z^2+a^2}\,\mathrm{d}z\right| &\le \lim_{R\to\infty}\int_{C_R}\left|\frac{\log z}{z^2+a^2}\right||\mathrm{d}z| \\ &= \lim_{R\to\infty}\int_0^\pi \frac{\left|\log\left(Re^{i\vartheta}\right)\right|}{\left|R^2e^{2i\vartheta}+a^2\right|}\left|iRe^{i\vartheta}\right|\mathrm{d}\vartheta \\ &\le \lim_{R\to\infty}\int_0^\pi \frac{R(\log R + \vartheta)}{R^2-a^2}\,\mathrm{d}\vartheta \\ &= \lim_{R\to\infty}\frac{R\log R}{R^2-a^2}\int_0^\pi \mathrm{d}\vartheta + \lim_{R\to\infty}\frac{R}{R^2-a^2}\int_0^\pi \vartheta\,\mathrm{d}\vartheta \\ &= \lim_{R\to\infty}\frac{\pi R\log R}{R^2-a^2} + \lim_{R\to\infty}\frac{\pi^2 R}{2(R^2-a^2)} \\ &= 0,\end{aligned}$$

here we have used $\log\left(Re^{i\vartheta}\right) = \log R + i\vartheta$ and $|\log R + i\vartheta| \le \log R + \vartheta$ for positive ϑ.

$$\begin{aligned}\lim_{\rho\to 0}\left|\int_{C_\rho} \frac{\log z}{z^2+a^2}\,\mathrm{d}z\right| &\le \lim_{\rho\to 0}\int_{C_\rho}\left|\frac{\log z}{z^2+a^2}\right||\mathrm{d}z| \\ &= \lim_{\rho\to 0}\int_\pi^0 \frac{\left|\log\left(\rho e^{i\vartheta}\right)\right|}{\left|\rho^2e^{2i\vartheta}+a^2\right|}\left|i\rho e^{i\vartheta}\right|\mathrm{d}\vartheta \\ &\le \lim_{\rho\to 0}\int_\pi^0 \frac{\rho(\log \rho + \vartheta)}{a^2-\rho^2}\,\mathrm{d}\vartheta \\ &= \lim_{\rho\to 0}\frac{\rho\log \rho}{a^2-\rho^2}\int_\pi^0 \mathrm{d}\vartheta + \lim_{\rho\to 0}\frac{\rho}{a^2-\rho^2}\int_\pi^0 \mathrm{d}\vartheta\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\rho\to 0} \frac{\rho\log\rho}{a^2-\rho^2}\pi + \lim_{\rho\to 0}\frac{\rho}{a^2-\rho^2}\frac{\pi^2}{2} \\
&= 0.
\end{aligned}
$$

Hence the integrals along the semicircles vanish.

The integral along the positive real axis gives us the target integral. The integral along the negative real axis can be found using $z = e^{i\pi} = -x$ and $\mathrm{d}z = e^{i\pi}\,\mathrm{d}x = -\,\mathrm{d}x$ so

$$
\begin{aligned}
\int_{-\infty}^{0} &\frac{\log z}{z^2+a^2}\,\mathrm{d}z \\
&= -\int_{\infty}^{0} \frac{\log\left(xe^{i\pi}\right)}{x^2e^{2\pi i}+a^2}\,\mathrm{d}x \\
&= \int_0^\infty \frac{\log x + i\pi}{x^2+a^2}\,\mathrm{d}x \\
&= \int_0^\infty \frac{\log x}{x^2+a^2}\,\mathrm{d}x + i\pi\int_0^\infty \frac{1}{x^2+a^2}\,\mathrm{d}x \\
&= I + i\pi\int_0^\infty \frac{1}{x^2+a^2}\,\mathrm{d}x\,.
\end{aligned}
$$

Hence we have

$$
\oint_\gamma f(z)\,\mathrm{d}z = 2I + i\pi\int_0^\infty \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{\pi}{a}\left(\log a + i\frac{\pi}{2}\right).
$$

Equating the real parts we get the desired integral

$$
I = \int_0^\infty \frac{\log x}{x^2+a^2}\,\mathrm{d}x = \frac{\pi}{a}\log a.
$$

By equating the imaginary parts we get for free the integral

$$
\int_0^\infty \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{\pi}{2a}.
$$

Part VI

Conformal Mapping

This part is non-examinable.

20 Kramers–Kronig Relations

Consider light passing through a dielectric material. In general the refractive index, n, and susceptibility, ε, are both complex numbers. The real part of ε corresponds to propagation through the material and the imaginary part determines absorption. Conservation of energy means that light that is absorbed cannot also propagate through the material. This implies a connection between the real and imaginary parts of ε. In fact this relationship turns out to be incredibly general and applies to any linear causal system.

Consider a system with no time dependence other than the input. For example a pendulum driven by some time dependent force but with all other terms only depending on time due to this force. This system can be completely described by the Green's function, $G(t-t')$ for some input $f(t')$. The Greens function is related to the solution for the system by

$$
u(t) = \int_{-\infty}^{\infty} G(t-t')f(t')\,\mathrm{d}t'\,.
$$

For the system to be causal we must have that $G(t-t') > 0$ for $t - t' < 0$.

We can solve ordinary differential equations with a Fourier transform. The inverse Fourier transform of the Greens function is

$$G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{G}(\omega) e^{-i\omega t} \, \mathrm{d}\omega \, .$$

It turns out that the conditions for the Fourier transform to exist, in particular vanishing at infinity, imply that Jordan's lemma applies. This means that we can sometime compute the Fourier transforms by considering a semicircle in the upper half plane. We then use $f(z) = \tilde{G}(z)e^{-izt}$ and since exp is entire the only singularities are those of $\tilde{G}$ so

$$\begin{aligned}
\lim_{R\to\infty} \oint_\gamma f(z) \, \mathrm{d}z &= 2\pi i \sum_n \operatorname{Res}\left(\tilde{G}, \omega_n\right) \\
&= \lim_{R\to\infty} \int_{C_R} f(z) \, \mathrm{d}z + \lim_{R\to\infty} \int_{-R}^{R} \tilde{G} e^{-i\omega t} \, \mathrm{d}\omega \\
&= 0 + G(t).
\end{aligned}$$

We assume G has no singularities on the real axis as these would make the response non-local. For $t < 0$, and hence $G(t) = 0$, the only way the above formula can hold is if either $\sum_n \operatorname{Res}\left(\tilde{G}, \omega_n\right) = 0$ or $\tilde{G}$ is analytic in the upper half plane. In general it is unlikely that the residues happen to cancel perfectly and so most of the time G is analytic in the upper half plane.

Now consider this integral over the same contour, γ:

$$\oint_\gamma \frac{\tilde{G}(\omega')}{\omega' - \omega} \, \mathrm{d}\omega'$$

where $\omega \in \mathbb{R}$. The integrand has a single simple pole at $\omega' = \omega$. Assuming that G is analytic in the upper half plane we know by Cauchy–Goursat's theorem that

$$\lim_{R\to\infty} \oint_\gamma f(z) \, \mathrm{d}z = 0$$

but we now need to split the integral over the real axis in two and treat it as a principle value integral:

$$\lim_{R\to\infty} \oint_\gamma f(z) \, \mathrm{d}z = \lim_{R\to\infty} \left[⨍_{-R}^{R} f(z) \, \mathrm{d}z + \int_{C_\rho} f(z) \, \mathrm{d}z + \int_{C_R} f(z) \, \mathrm{d}z \right]$$

where C_ρ is a semicircle in the upper half plane of radius ρ at ω such that ω is outside of the resulting contour, γ.

For the Fourier transform to converge we must have $\left|\tilde{G}(\omega)\right| \sim 1/\omega^k$ for $k > 0$ and so

$$\begin{aligned}
\lim_{R\to\infty} \left| \int_{C_R} f(z) \, \mathrm{d}z \right| &\le \lim_{R\to\infty} \int_{C_R} |f(z)||\mathrm{d}z| \\
&\sim \lim_{R\to\infty} \int_0^\pi \frac{R}{R - \omega} \frac{1}{R^k} \, \mathrm{d}\vartheta \\
&= \lim_{R\to\infty} \frac{R\pi}{R - \omega} \frac{1}{R^k} \\
&= 0.
\end{aligned}$$

For the integral over C_ρ we can use the indentation lemma as it is a simple pole:

$$\int_{C_\rho} f(z) \, \mathrm{d}z = i(0 - \pi) \operatorname{Res}\left(\frac{\tilde{G}(\omega')}{\omega - \omega'}\right) = -i\pi\tilde{G}(\omega).$$

Splitting the principle value integral into real and imaginary parts we have

$$0 = \oint_\gamma f(z) \, \mathrm{d}z$$

$$= \operatorname{Re}\left[\fint_{-\infty}^{\infty} \frac{\tilde{G}(\omega')}{\omega - \omega'} \,\mathrm{d}\omega\right] + i \operatorname{Im}\left[\fint_{-\infty}^{\infty} \frac{\tilde{G}(\omega')}{\omega - \omega'} \,\mathrm{d}\omega\right] - i\pi\tilde{G}(\omega).$$

Equating the real and imaginary parts of the principle value integrals and $i\pi\tilde{G}(\omega)$ we find that

$$\operatorname{Re}[\tilde{G}(\omega)] = \frac{1}{\pi}\fint_{-\infty}^{\infty} \frac{\operatorname{Im}[\tilde{G}(\omega)]}{\omega - \omega'} \,\mathrm{d}\omega'$$
$$\operatorname{Im}[\tilde{G}(\omega)] = \frac{1}{\pi}\fint_{-\infty}^{\infty} \frac{\operatorname{Re}[\tilde{G}(\omega)]}{\omega - \omega'} \,\mathrm{d}\omega'$$

These are the **Kramer–Kronig relations**. They hold for all linear causal systems and show the relationship between absorption and dispersion.

21 Conformal Mapping

We have seen that we can view complex functions as maps from the complex plane to itself. Most of the functions we have considered have been analytic meaning the derivative exists at a given point. This allows us to find more properties of the map. Let f be analytic at $z_0 \in \mathbb{C}$. Then by the definition of the derivative we have

$$f(z) - f(z_0) = f'(z_0)(z - z_0) + \mathcal{O}\big((z - z_0)^2\big). \tag{21.1}$$

We then take the limit of $z \to z_0$. If $f'(z_0) \neq 0$ then we have

$$|f(z) - f(z_0)| = |f'(z_0)||z - z_0|$$

and

$$\arg(f(z) - f(z_0)) = \arg(f'(z_0)) + \arg(z - z_0).$$

This tells us how a neighbourhood of z_0 transforms. The distance from z_0 is scaled by $|f'(z_0)|$, in particular the disc $D\,(z_0; \varepsilon)$ is transformed into the disc $f[D\,(z_0; \varepsilon)] = D\,(f(z_0); |f'(z_0)|\varepsilon)$.

Consider now some smooth curves passing through z_0. In a small region each curve can be approximated as a straight line which can be parametrised as $\gamma_i = z_0 + C_i t e^{i\vartheta_i} + \mathcal{O}\big(t^2\big)$ where C_i and ϑ_i are some real constants and t is the real parametric variable. The variable ϑ_i represents the slope of the line. After the map, f, is applied the same line segment has slope ϑ_i' given by

$$\vartheta_i' = \arg(f'(z_0)) + \arg(z - z_0) = \arg(f'(z_0)) + \vartheta_i.$$

Suppose that the angle between two curves meeting at z_0 was α before the map. Then after the map the angle is

$$\alpha' = \vartheta_2' - \vartheta_1' = \vartheta_2 - \vartheta_1 = \alpha.$$

So the angle at which smooth curves cross is invariant under this map as long as $f'(z_0) \neq 0$. What this means is that, locally, shapes are preserved, they are simply stretched by $|f'(z_0)|$. However this is only a local effect. Globally shapes will change as the scaling factor, $|f'(z_0)|$, needn't be the same at all points. Also if $f'(z_0) = 0$ then equation 21.1 simply says that there is some second order term that we can't ignore. If this is the case we call z_0 a **critical point** of the transformation.

Definition 52: Conformal Map

Let $S, R \subseteq \mathbb{C}$ be open regions. Then the function $f\colon S \to R$ is called **conformal** at some point $z_0 \in S$ if it preserves the angle between lines passing through z_0 and the orientation. Equivalently f is **conformal** if and only if it is analytic at z_0 and $f'(z_0) \neq 0$.

If we consider $f\colon \mathbb{C} \to \mathbb{C}$ to be $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ then the Jacobian of f is given by

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$
$$= u_x v_y - u_y v_x$$

$$= u_x^2 + v_x^2$$
$$= |f'(z)|^2$$

so we see that the magnitude of the derivative is exactly the Jacobian. This should make sense as the Jacobian of a transformation gives the scaling factor. We also know that a map is invertible if and only if the Jacobian is non=zero and the function and its derivatives are continuous. If f is analytic then certainly it, and its derivatives, are continuous and so if $f'(z) \neq 0$ then it is invertible. So conformal maps have a local inverse.

21.1 Mapping Regions

It is often important to know how a given region transforms. This will be demonstrated with several examples all of which follow the following procedure:

1. Identify easy to map points at the boundary.
2. Parametrise the boundary of the region and map it.
3. Determine if the inside of the region maps to the inside or outside of the new region.
4. Find any other points of interest.

21.1.1 Linear Mapping

The simplest conformal map is

$$f(z) = \alpha z + \beta$$

for some $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$. We have $f'(z) = \alpha$ and so any region is scaled by $|f'(z)| = |\alpha|$. The region is also rotated by $\arg(\alpha)$ and finally shifted by β. An example of this map mapping a square region is shown in figure 21.1. Notice that straight lines map to straight lines. To see why this is true consider

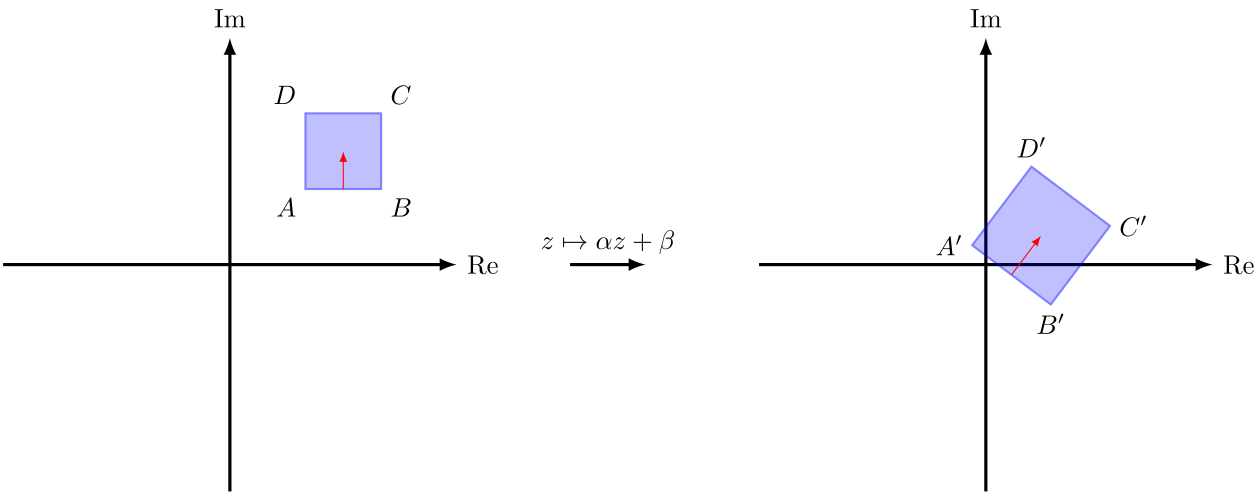


Figure 21.1: The map $z \mapsto \alpha z + \beta$. In particular $|\alpha| = 1.3$, $\arg(\alpha) = -0.645$, and $\beta = -2$.

the straight line AB. We can parametrise this as

$$\gamma(t) = (B - A)t + A$$

for $t \in [0, 1]$. After the map we have

$$f(\gamma(t)) = \alpha(B - A)t + \alpha A + B$$

which is again a straight line. Finally since angles are preserved the red arrow starts pointing in and must end pointing in to preserve the angles and so the inside maps to the inside.

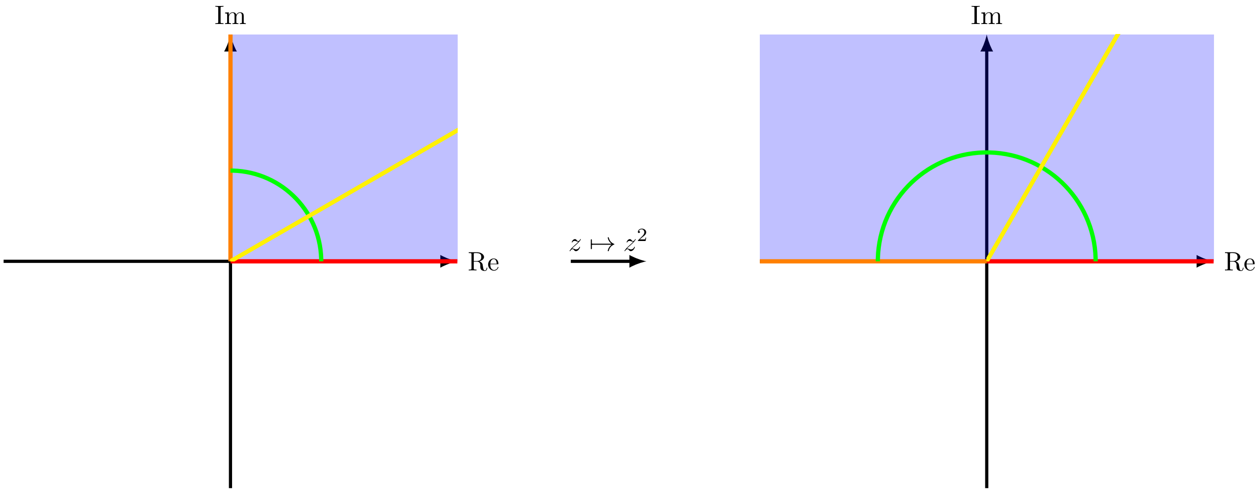


Figure 21.2: The map $z \mapsto z^2$ showing a region defined by $\arg(z) \in (0, \pi/2)$, the boundaries of this region, and two interesting curves.

21.1.2 Quadratic Mapping

Consider the map $f\colon z \mapsto z^2$. Since $f'(z) = 2z$ this map is conformal with critical point $z = 0$. Figure 21.2 shows the result of this mapping. The blue region is defined by $\arg(z) \in (0, \pi/2)$. After the map this becomes

$$\arg(z) \in (0, \pi/2) \mapsto \arg(z) \in (0, \pi)$$

since squaring a complex number doubles its argument. The boundaries of this shape are the lines $z_1 = x$ for $x \in (0, \infty)$ and $z_2 = iy$ for $y \in (0, \infty)$. After the mapping these regions map to

$$z_1 = x \in (0, \infty) \mapsto x^2 \in (0, \infty)$$
$$z_2 = iy \in i(0, \infty) \mapsto -y^2 \in (-\infty, 0).$$

Now consider an arbitrary line through the origin. This can be parametrised as $z_3 = te^{i\vartheta}$ for $t \in (0, \infty)$. This line then maps to

$$z_3 = te^{i\vartheta} \mapsto t^2 e^{2i\vartheta}$$

so a line through the origin with the argument doubled. Finally consider the quarter circle of radius R parametrised by $z_4 = Re^{it}$ for $t \in (0, \pi/2)$. This maps to

$$z_4 = Re^{it} \mapsto R^2 e^{2it}.$$

So a semicircle of radius R^2. Notice, for example, that the yellow line and green arc in figure 21.2 meet at right angles both before and after the mapping. However the orange and red lines meet at $\pi/2$ before the mapping and π after the mapping. This is because they meet at $z = 0$ which is a critical point of this mapping.

21.1.3 Exponential Mapping

Consider the mapping $f\colon z \mapsto e^z$, Since $f'(z) \neq 0$ this map is conformal for all finite z. We saw in figure 5.2 how stripes map under the exponential function. Instead consider the square shown in figure 21.3. The unit square maps to an annulus. We can see this by parametrising the four sides:

$z_1 = R,$	$R \in (0, 1)$	$\mapsto$	$e^R,$	line segment: $(1, e)$
$z_2 = i\vartheta + 1,$	$\vartheta \in (0, 2\pi)$	$\mapsto$	$e^{i\vartheta+1} = ee^{i\vartheta},$	circle radius e
$z_3 = R + 2\pi i,$	$R \in (0, 1)$	$\mapsto$	$e^{R+2\pi i} = e^R,$	line segment: $(1, e)$
$z_4 = i\vartheta,$	$\vartheta \in (0, 2\pi)$	$\mapsto$	$e^{i\vartheta},$	circle radius 1
$z_4 = 0.5 + i\vartheta,$	$\vartheta \in (0, 2\pi)$	$\mapsto$	$e^{0.5+i\vartheta} = e^{0.5}e^{i\vartheta},$	circle radius $e^{0.5}$

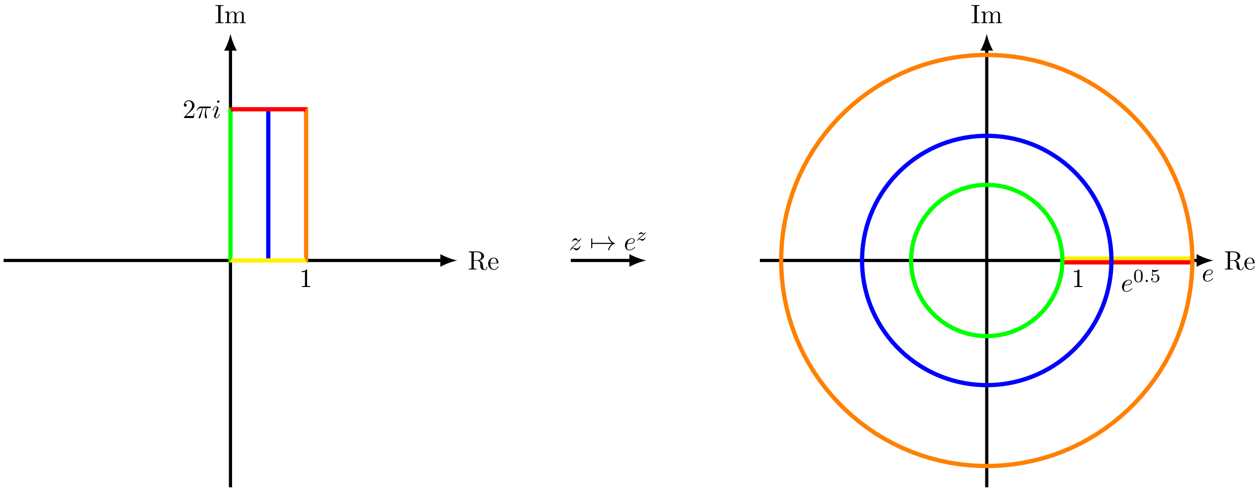


Figure 21.3: The map $z \mapsto e^z$ showing the mapping of a rectangle of side lengths 1 and 2π to an annulus of radii 1 and e. Note that the real and imaginary axes before the mapping do not have the same scale.

21.1.4 Trigonometric Mapping

Consider the map $f\colon z \mapsto \sin z$. This is shown in figure 21.4 This is best understood by expressing sin

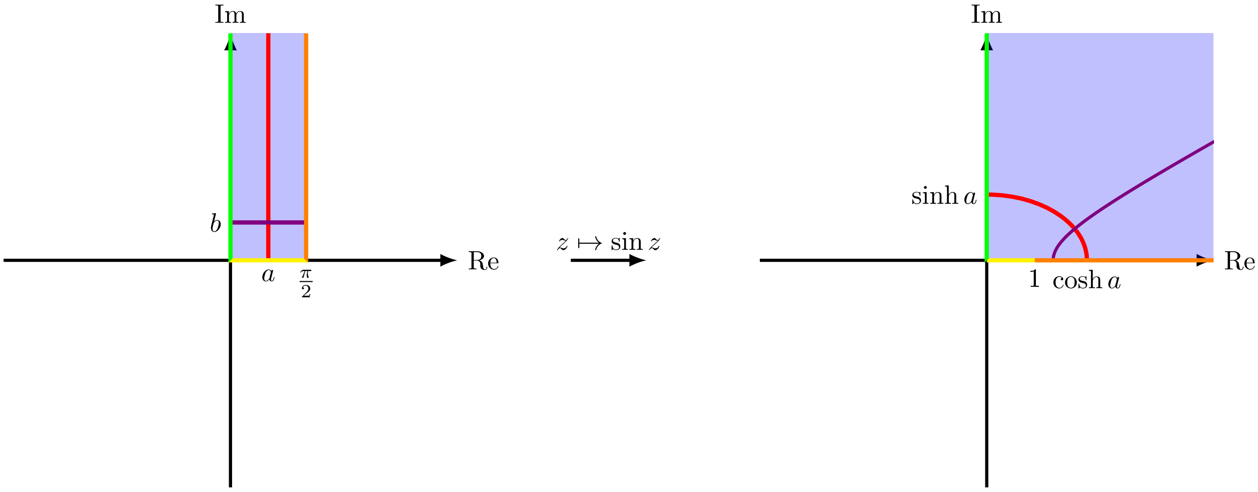


Figure 21.4: The map $z \mapsto \sin z$ showing the region $\operatorname{Re} z \in (0, \pi/2)$ and $\operatorname{Im} z > 0$.

in terms of real and imaginary parts:

$$f(z) = \sin z = \frac{e^{iz} - e^{-iz}}{2i} = \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$

We see that line segment, —, which is the interval $(0, \pi/2)$, maps to the interval $(0, 1)$ since $y = 0$ and $\sinh 0 = 0$, $\cosh 0 = 1$, and $\sin x$ maps onto $(0, 1)$ for $x \in (0, \pi/2)$. The positive imaginary axis, —, maps to itself since $x = 0$ and $\sin 0 = 0$, $\cos 0 = 1$ and $\sinh y$ maps onto $(0, \infty)$ for $y \in (0, \infty)$. The line, —, given by $z = \pi/2 + iy$ maps to $(1, \infty)$ since for this line f reduces to $\cosh y$ and $\cosh y$ maps onto $(1, \infty)$ for $y \in (0, \infty)$. Notice that the angle at $\pi/2$ isn't conserved as $f'(\pi/2) = \cos(\pi/2) = 0$.

Now consider the line, —, which can be parametrised as $z = t + ia$. This line then maps to

$$f(z) = u(t) + iv(t) = \sin(t)\cosh(a) + i\cos(t)\sinh(a).$$

Square the real and imaginary parts and divide by $\cosh^2 a$ and $\sinh^2 a$ respectively and we get

$$\frac{u^2(t)}{\cosh^2 a} + \frac{v^2(t)}{\sinh^2 a} = \sin^2 t + \cos^2 t = 1.$$

This is an ellipse with semimajor axis $\cosh a$ and semiminor axis $\sinh a$. Further for $a \in [0, \infty)$ we have $0 < \sinh a \le \cosh a < \infty$ and so the ellipse covers the whole of the first first quadrant.

Now consider the line, —, which can be parametrised by $z = t + ib$ for $t \in (0, \pi/2)$. The map reduces for this line to

$$f(z) = u(t) + iv(t) = \sin(b)\cosh(t) - \cos(b)\sinh(t).$$

Squaring the real and imaginary parts and dividing by $\sin^2 b$ and $\cos^2 b$ respectively we get

$$\frac{u^2(t)}{\sin^2 b} - \frac{v^2(t)}{\cos^2 b} = \cosh^2 t - \sinh^2 t = 1$$

where we have used the identity

$$\cosh^2 x - \sinh^2 x = 1.$$

We see that the result is the formula for a hyperbola.

21.2 Möbius Transformations

A **Möbius transformation**, also known as a **bilinear map** is a map of the form

$$f(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$$

for $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. These are conformal provided that $\alpha\delta - \beta\gamma \ne 0$, which is equivalent to the statement that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{U}(2).$$

These transformations are interesting because they map straight lines to either another straight line or a circle and circles to either another circle or a straight line. The set of all Möbius transformations forms a group under function composition called the Möbius group. This group acts on the Riemann sphere, $\mathbb{C}^*$, as the automorphism group of the Riemann sphere (i.e. the group is the set of isomorphisms of the Riemann sphere into itself) and so the Möbius group is sometimes denoted $\mathrm{Aut}(\mathbb{C}^*)$. This is a Lie group.

21.3 Boundary Conditions

Now that we've spent all this time discussing conformal maps, why do we care? The most useful property of conformal maps is that they have a local inverse. This means that we can take a geometrically complicated problem and then map it to some less complicated problem, solve it there, and map the solution back to the original problem.

This is possible for two reasons:

1. A constant valued boundary maps to the same value at a new boundary.
2. If the derivative normal to a boundary vanishes then the derivative normal to the new boundary vanishes.

Example 21.1. *Find a real function, $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$, which is harmonic on the wedge between the positive real axis and half line from* 0 *to infinity defined by $y = x$ and $x > 0$ such that the following are satisfied:*

- *On upper boundary of the wedge, $\varphi(x, x) = 2$.*
- *On the real axis $\varphi_y(x, 0) = 0$.*

This problem may arise in physics, for example, as finding the equilibrium temperature on wedge with one edge at a fixed temperature and the other edge in contact with insulating material.

The first thing we need to do is to find a conformal map that transforms the problem into one with simpler geometry. The map $w(z) = z^2$ does exactly this, transforming the wedge into a quadrant as shown in figure 21.5. Let $(u, v) = w(x, y)$.

Recall that an analytic function has harmonic real and imaginary parts so we should look for an analytic function, f, which has real or imaginary parts satisfying the new boundary conditions, $f_v(u) = 0$ and $f(iv) = 2$. Typical candidates for this are logarithms and harmonic polynomials. In this case $\varphi(u, v) =$

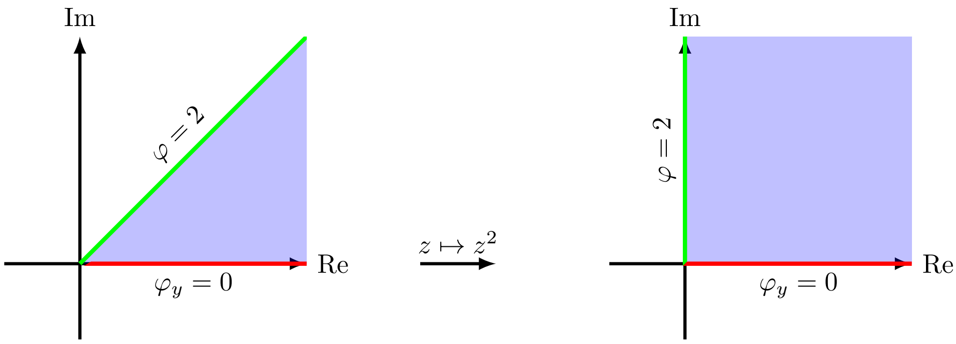


Figure 21.5: The map $z \mapsto z^2$ showing how the wedge maps to a quadrant.

$u+2$ satisfies these conditions. We could find the harmonic conjugate the long way or we can just notice that $f(w) = w + 2$ has our solution as a real part and as a polynomial is entire. Hence the solution to the original problem is

$$\varphi(x, y) = \mathrm{Re}[w(x, y) + 2] = x^2 - y^2 + 2$$

Example 21.2. *Find a function, $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ which fulfils the following conditions:*

- *The function satisfies Laplace's equation in the region of interest, that is $\nabla^2\varphi = 0$.*
- $\varphi(0, y) = 0$.
- $\varphi(x, 0) = 1$ *for all* $x > 1$.
- *For* $x \in (0, 1)$ $\varphi_y(x, 0) = 0$.

This problem may arise in physics when finding the equilibrium temperature distribution of an infinite plate with two edges kept at fixed temperature, the $x = 0$ edge held at $T_y = 0$ and the $x > 1$ and $y = 0$ edge held at $T_{x>1} = 1$.

Again the first step is to find a conformal map that simplifies the boundary conditions. Consider for this purpose the function $w(z) = \arcsin z$. This is the inverse of the sin transformations discussed in section 21.1.4. The quadrant $x, y \geq 0$ is mapped to a half-infinite strip, $S = \{(u, v) \in \mathbb{R}^2 \mid u \in [0, \pi/2] \wedge v > 0\} = [0, \pi/2] \times \mathbb{R}_{>0}$. The boundary conditions then become:

$$\begin{aligned}
\nabla^2\varphi(u, v) &= 0 \,\forall\, u, v \in S,\\
\varphi_v(u, 0) &= 0 \,\forall\, u \in [0, \pi/2],\\
\varphi(0, v) &= 0 \,\forall\, v \in \mathbb{R}_{>0},
\end{aligned}$$

and

$$\varphi(\pi/2, v) = 1 \,\forall\, v \in \mathbb{R}_{>0}.$$

These are satisfied by a simple function of w. In particular $\varphi(u, v) = 2u/\pi$ satisfies these boundary conditions. So the solution is then

$$\varphi(x, y) = \frac{2}{\pi}u = \frac{2}{\pi}\,\mathrm{Re}\, w = \frac{2}{\pi}\,\mathrm{Re}(\arcsin z) = \frac{2}{\pi}\,\mathrm{Re}(\arcsin(x + iy)).$$

We aren't quite finished as this isn't a real function since we have to be careful about branch cuts with arcsin. We could use a general formula for the inversion but then we would have to be careful, instead we will invert the formula explicitly:

$$w = \arcsin z \implies z = \sin w = \sin(u)\cosh(v) + i\cos(u)\sinh(v) = x + iy$$

and following the same procedure as we did in section 21.1.4 when considering the mapping $z \mapsto \sin z$ we find

$$\frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1.$$

Multiplying by $\sin^2(u)\cos^2(u) = \sin^2(u)[1 - \sin^2(u)]$ we end up with a quadratic equation in $\sin^2 u$:

$$\sin^2 u(1 - \sin^2 u) = x^2(1 - \sin^2 u) - y^2 \sin^2 u \implies \sin^4 u - [1 + x^2 + y^2]\sin^2 u + x^2 = 0$$

Solving this we find

$$\sin^2 u = \frac{1}{2}[(1 + x^2 + y^2) \pm \sqrt{(1 + x^2 + y^2)^2 - 4x^2}].$$

For a bounded solution we must take the negative square root. Completing the square a few times and doing some algebra this becomes

$$\sin^2 u = \frac{1}{4}\left(\sqrt{(1 + x)^2 + y^2} - \sqrt{(1 - x)^2 + y^2}\right)^2.$$

Finally square rooting and taking arcsin on both sides we have

$$u = \arcsin\left(\frac{1}{2}\left(\sqrt{(1 + x)^2 + y^2} - \sqrt{(1 - x)^2 + y^2}\right)\right).$$

The boundary conditions force us to take the positive square root. Hence the solution is

$$\varphi(x, y) = \frac{2}{\pi}u = \frac{2}{\pi}\arcsin\left(\frac{1}{2}\left(\sqrt{(1 + x)^2 + y^2} - \sqrt{(1 - x)^2 + y^2}\right)\right).$$

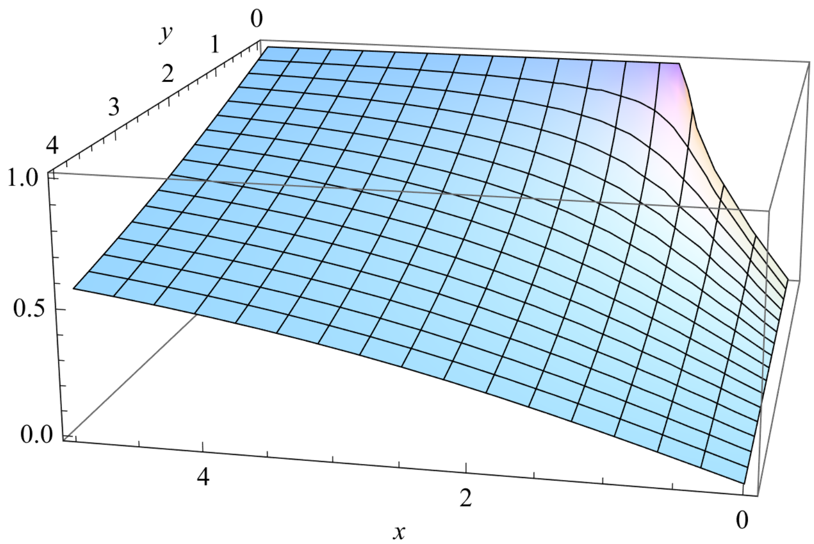


Figure 21.6: The solution, $\varphi = 2\,\mathrm{Re}(\arcsin(z))/\pi$.

This last example was pretty complicated with some nasty algebra but the method is the same for all of these problems:

1. Find a conformal map, $w = u + iv$, which simplifies the boundary conditions.
2. Translate the boundary conditions to be in terms of u and v.
3. Come up with a solution in terms of w or u and v.
4. Invert the conformal map to get a solution in terms of x and y.

An important point is that we need w to be conformal and for all our theorems to apply, which means we need an analytic solution which means that the ansatz should be a harmonic function for u or v and then we can find the harmonic conjugate. Typical choices for the solution to the simplified problem are harmonic polynomials, logarithms, $\arg w$ (which is just the imaginary part of the logarithm), and the real or imaginary parts of elementary analytic functions such as exponentials or (hyperbolic) trig.

This method can be applied to many physics problems including, but not limited to

- Gravitational potentials in vacuum (i.e. $\rho = 0$):

$$\nabla^2 \varphi = 0.$$

- Electrostatic potentials in vacuum (i.e. $\rho = 0$):

$$\nabla^2 V = -\nabla \cdot \boldsymbol{E} = 0$$

- Stationary waves:

$$\nabla^2 f(x, y, t) = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} = 0.$$

- Equilibrium solutions to the heat equation:

$$\nabla^2 f(x, y, t) = \frac{1}{k} \frac{\partial f}{\partial t} = 0.$$

- Irrotational, incompressible flow, in a steady state:

$$\rho \nabla \cdot \boldsymbol{v} = -\frac{\partial \rho}{\partial t} = 0.$$

  Since if this holds and $\rho \neq 0$ then necessarily $\nabla^2 \boldsymbol{v} = \mathbf{0}$.

Riemann showed that any non-empty, simply connected, open subset of $\mathbb{C}$ can be mapped to the unit disc, $D(0; 1)$. Further this can always be done by an invertible analytic function. This implies that it is always possible to map such a region to the upper half plane (since we can always go via the unit disc). There is also a general solution, in terms of integrals, for any Dirichlet problem (fixed value boundary conditions) on the unit disc or the upper half plane. Therefore it is always possible to apply this method to solve Dirichlet problems (assuming a solution exists). However it is often non-trivial and finding the correct map can be difficult and there is no algorithmic way to do it.