Maths For Physics 2 Lecture Notes

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These are my notes for the *maths for physics 2* course from the University of Edinburgh as part of the first year of the theoretical physics degree. When I took this course in the 2018/19 academic year it was taught by Dr Kristel Torokoff[1]. These notes are based on the lectures delivered as part of this course, and the notes provided as part of this course. The content within is correct to the best of my knowledge but if you find a mistake or just disagree with something or think it could be improved please let me know.

These notes were produced using LaTeX[2]. Diagrams were drawn with tikz[3], or by hand.

This is version 1.0 of these notes, which is up to date as of 04/01/2021.

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[1] `https://www.ph.ed.ac.uk/people/kristel-torokoff`
[2] `https://www.latex-project.org/`
[3] `https://www.ctan.org/pkg/pgf`

Contents

Part I

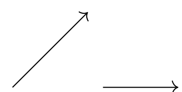
Vectors

1 Vector Operations

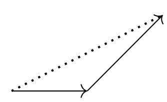
Vectors are geometric entities that carry information about magnitude and direction. Vectors can be represented geometrically or algebraically. If represented geometrically then a vector is an arrow with length $\propto$ magnitude and the same direction as the vector.The vector is labled with a letter (ie v) which is marked as a vector by one of a various number of methods including $\underline{v}$, $\mathbf{v}$, $\overrightarrow{v}$and$\bar{v}$. The magnitude of the vector is represented by $|\boldsymbol{v}|$ which is not a vector but a length. We often shorten this to $v \triangleq |\boldsymbol{v}|$. The direction of a vector is given as a unit vector $\hat{\boldsymbol{v}}$ in the direction of the original vector, a unit vector has magnitude 1, $|\hat{\boldsymbol{v}}| \triangleq 1$. From this we know $\boldsymbol{v} \equiv v\hat{\boldsymbol{v}}$ and that a unit vector is $\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{v}$

Operations On Vectors

1. The position of a vector doesn't matter as long as it has the same direction and magnitude it is the same vector. This is called parallel transport (||-transport).
2. Vector addition

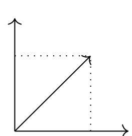
   These two vectors can be added together by using ||-transport to join them tip to tail and then drawing a new vector between the unconnected ends of the vector, the new vector is the sum of the original vectors.

   Vector addition is commutative ($\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$) and associative ($\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}$)
3. Multiplication by a scalar c:

Value of scalar c	What happens to the magnitude?	What happens to the direction?
$c > 1$	Magnitude increases	Direction is unchanged
$0 < c < 1$	Magnitude decreases	Direction is unchanged
$c = -1$	Magnitude is unchanged	Direction is reversed

4. By combining multiplication by -1 and vector addition we can define vector subtraction as $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$
5. Projection:

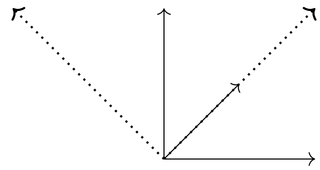
   The length of $\boldsymbol{v}$ projected on x is v_x, likewise the length of $\boldsymbol{v}$ projected on y is v_y. If the angle between $\boldsymbol{v}$ and the x axis is ϑ then v_x and v_y can be calculated as:

   $$v_x = v\cos\vartheta \quad \& \quad v_y = v\sin\vartheta$$

   It can be seen from the diagram that $\boldsymbol{v} = v_x\hat{\boldsymbol{x}} + v_y\hat{\boldsymbol{y}}$

2 More Vector operations

By choosing a different coordinate system one of the components can be 0:



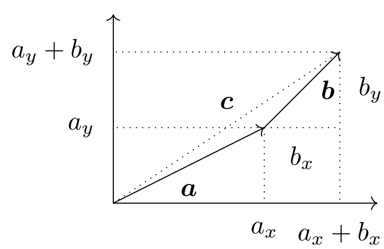
If we call the new coordinate system x' and y' then it is possible to express the vector $\boldsymbol{v}$ in terms of $\hat{\boldsymbol{x}}'$ and $\hat{\boldsymbol{y}}'$:

$$\boldsymbol{v} = v_{x'}\hat{\boldsymbol{x}}' + v_{y'}\hat{\boldsymbol{y}}' = v_{x'}\hat{\boldsymbol{x}}' = v\hat{\boldsymbol{x}}' \quad \text{as } \hat{\boldsymbol{y}}' = \boldsymbol{0}$$

This shows that the magnitude is not dependant on the coordinate system used, because of this we call it a scalar. The components of the vector are not scalars as they depend on the coordinate system used.

Vector operations revisited

- Vector addition:



$$\begin{aligned}
\boldsymbol{a} &= a_x\hat{\boldsymbol{x}} + a_y\hat{\boldsymbol{y}} \\
\boldsymbol{b} &= b_x\hat{\boldsymbol{x}} + b_y\hat{\boldsymbol{y}} \\
\boldsymbol{c} &= c_x\hat{\boldsymbol{x}} + c_y\hat{\boldsymbol{y}} \\
\boldsymbol{c} &= \boldsymbol{a} + \boldsymbol{b} \\
c_x &= a_x + b_x \\
c_y &= a_y + b_y \\
\boldsymbol{c} &= (a_x + b_y)\hat{\boldsymbol{x}} + (a_y + b_y)\hat{\boldsymbol{y}} \\
\boldsymbol{a} + \boldsymbol{b} &= (a_x + b_x)\hat{\boldsymbol{x}} + (a_y + b_y)\hat{\boldsymbol{y}}
\end{aligned}$$

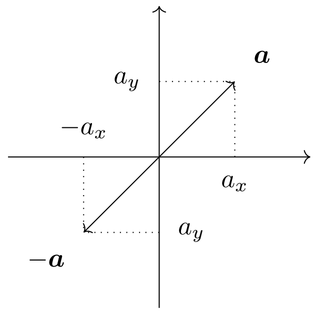
  This works in 3 dimensions as well. If $\boldsymbol{v}$ and $\boldsymbol{u}$ are 3-dimensional vectors then:

$$\boldsymbol{v} + \boldsymbol{u} = (v_x + u_x)\hat{\boldsymbol{x}} + (v_y + u_y)\hat{\boldsymbol{y}} + (v_z + u_z)\hat{\boldsymbol{z}}$$

- Scalar multiplication:

$$\begin{aligned}
\boldsymbol{a} &= a_x\hat{\boldsymbol{x}} + a_y\hat{\boldsymbol{y}} \\
c\boldsymbol{a} &= c(a_x\hat{\boldsymbol{x}} + a_y\hat{\boldsymbol{y}}) \\
c\boldsymbol{a} &= ca_x\hat{\boldsymbol{x}} + ca_y\hat{\boldsymbol{y}}
\end{aligned}$$

  Multiplication by -1



$$\boldsymbol{a} = a_x\hat{\boldsymbol{x}} + a_y\hat{\boldsymbol{y}}$$
$$-\boldsymbol{a} = -(a_x\hat{\boldsymbol{x}} + a_y\hat{\boldsymbol{y}})$$
$$-\boldsymbol{a} = -a_x\hat{\boldsymbol{x}} - a_y\hat{\boldsymbol{y}}$$

The negative signs don't mean that the length is negative but that they go in the opposite direction to the direction defined as positive.

Calculaing the magnitude: If $\boldsymbol{v}$ is a 2-dimensional vector then $v = |\boldsymbol{v}| = \sqrt{v_x^2 + v_y^2}$ If $\boldsymbol{w}$ is a 3-dimensional vector then $w = |\boldsymbol{w}| = \sqrt{w_x^2 + w_y^2 + w_z^2}$

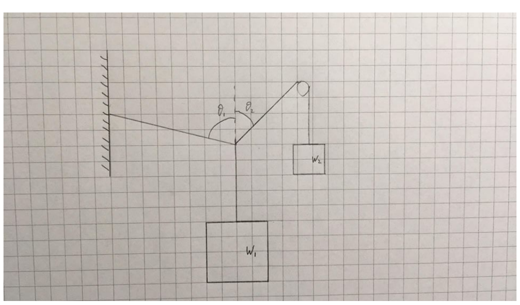
Notation

If $\boldsymbol{v}$ is an n-dimensional vector then as we run out of letters for cartesian coordinates it is necessary to use different notation, we call each direction e_i instead of x, y, z etc.:

$$\begin{aligned}\boldsymbol{v} &= v_1\hat{\boldsymbol{e_1}} + v_2\hat{\boldsymbol{e_2}} + v_3\hat{\boldsymbol{e_3}} + \cdots + v_n\hat{\boldsymbol{e_n}}\\ &= \sum_{i=1}^{n} v_i\hat{\boldsymbol{e_i}}\\ &= v_i\hat{\boldsymbol{e_i}}\end{aligned}$$

The last notation ($v_i\hat{\boldsymbol{e_i}}$) is called Einstein's summation convention. The repeated subscript shows it is a sum.

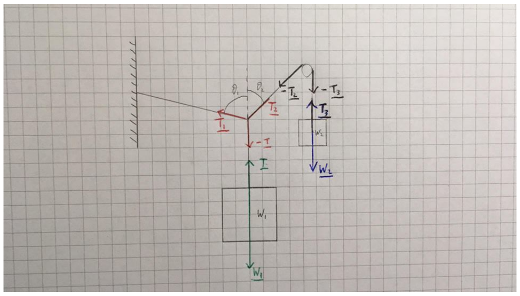
3 General Method For Vector Mechanics



Find $\boldsymbol{w_1}$ in terms of ϑ_1, ϑ_2 and $\boldsymbol{w_1}$

<u>method:</u>

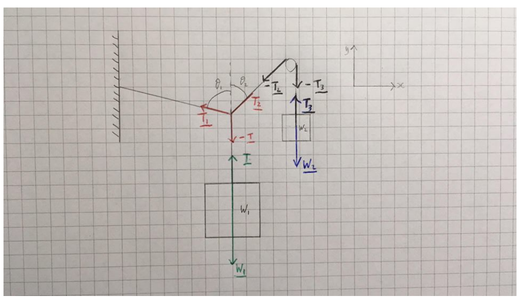
1. Sketch a diagram and draw vectors:



2. Identify any relevant vector equations. In this case we have Newton's 2nd law (NII) and the nature of a pulley to give the equations:

$$\boldsymbol{w_1} + \boldsymbol{T} = \boldsymbol{0} \tag{2.1}$$
$$\boldsymbol{w_2} + \boldsymbol{T_3} = \boldsymbol{0} \tag{2.2}$$
$$\boldsymbol{T_1} + \boldsymbol{T_2} + (-\boldsymbol{T}) = \boldsymbol{0} \tag{2.3}$$
$$|-\boldsymbol{T_2}| = |-\boldsymbol{T_3}| \tag{2.4}$$

3. Choose a coordinate frame and draw it on the diagram:



4. Decompose vectors into this coordinate system:

$$\begin{aligned}
\boldsymbol{w_1} &= -w_1\hat{\boldsymbol{y}} \\
\boldsymbol{w_2} &= -w_2\hat{\boldsymbol{y}} \\
\boldsymbol{T} &= T\hat{\boldsymbol{y}} \\
\boldsymbol{T_1} &= -T_1\hat{\boldsymbol{x}}\sin\vartheta_1 + T_1\hat{\boldsymbol{y}}\cos\vartheta_1 \\
\boldsymbol{T_2} &= T_2\hat{\boldsymbol{x}}\sin\vartheta_2 + T_2\hat{\boldsymbol{y}}\cos\vartheta_2 \\
\boldsymbol{T_3} &= T_3\hat{\boldsymbol{y}}
\end{aligned}$$

5. Extract the number equations (or component equations) from the vector equations:

$$(2.1) \implies w_1 = T \tag{2.5}$$
$$(2.2) \implies w_2 = T_3 \tag{2.6}$$
$$(2.3) \implies T_1\sin\vartheta_1 + T_2\sin\vartheta_2 = 0 \tag{2.7}$$
$$(2.3) \implies -T + T_1\cos\vartheta_1 + T_2\cos\vartheta_2 = 0 \tag{2.8}$$
$$(2.4) \implies T_2 = T_3 \tag{2.9}$$

6. Solve:

$$(2.7) \implies T_1 = T_2\frac{\sin\vartheta_2}{\sin\vartheta_1}$$

Substituting T_1 in (2.8)

$$T = T_2\left(\frac{\sin\vartheta_2}{\tan\vartheta_1} + \cos\vartheta_2\right)$$

(2.6) and (2.9)

$$T_2 = T_3 = w_2$$

Therefore

$$w_1 = w_2\left[\frac{\sin\vartheta_2}{\tan\vartheta_1} + \cos\vartheta_2\right]$$

is the magnitude of w_1 and $\boldsymbol{w_1} = -w_1\hat{\boldsymbol{y}}$ in the chosen coordinate frame.

4 Scalar and Vector Products

Defining new vector operations

Vectors are invarient under coordinate transformation. To define more vector operations we start by constructing other quantities that share this property. $|\boldsymbol{a}||\boldsymbol{b}|$ is invarient and so is ϑ the angle between $\boldsymbol{a}$ and $\boldsymbol{a}$. Combining these it is possible to get:

$$(3.1)\quad ab\cos\vartheta \qquad \& \qquad (3.2)\quad ab\sin\vartheta$$

(3.1) can be interpreted as the magnitude of the projection of $\boldsymbol{a}$ on $\boldsymbol{b}$ times the magnitude of $\boldsymbol{b}$ or the magnitude of the projection of $\boldsymbol{b}$ on $\boldsymbol{a}$ times the magnitude of $\boldsymbol{a}$. We define this as the scalar (dot) product of $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\vartheta$$

Note that the dot product is commutative:

$$\boldsymbol{a}\cdot\boldsymbol{b} \equiv \boldsymbol{b}\cdot\boldsymbol{a}$$

(3.2) can be interpreted as the area of the parallelogram formed by $\boldsymbol{a}$ and $\boldsymbol{b}$. Surfaces can be represented by a vector giving the area and which side of the surface. We introduce the normal vector $\hat{\boldsymbol{n}}$ which is perpendicular to the surface at all points and points out of the side that we are interested in. We define this as the vector (cross) product of $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{a}\times\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\hat{\boldsymbol{n}}\sin\vartheta$$

Note that the cross product is not commutative:

$$\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}$$

Algebraic representation of the scalar product

$$\boldsymbol{a} = a_1\hat{\boldsymbol{e_1}} + a_2\hat{\boldsymbol{e_2}} \quad \& \quad \boldsymbol{b} = b_1\hat{\boldsymbol{e_1}} + b_2\hat{\boldsymbol{e_2}}$$

$$\boldsymbol{a}\cdot\boldsymbol{b} = (a_1\hat{\boldsymbol{e_1}} + a_2\hat{\boldsymbol{e_2}})\cdot(b_1\hat{\boldsymbol{e_1}} + b_2\hat{\boldsymbol{e_2}})$$

Since a_1, a_2, b_1 and b_2 are scalars the dot product turns into normal multiplication for them. Expanding the brackets gives:

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1(\hat{\boldsymbol{e}}_\mathbf{1}\cdot\hat{\boldsymbol{e}}_\mathbf{1}) + a_1b_2(\hat{\boldsymbol{e}}_\mathbf{1}\cdot\hat{\boldsymbol{e}}_\mathbf{2}) + a_2b_1(\hat{\boldsymbol{e}}_\mathbf{2}\cdot\hat{\boldsymbol{e}}_\mathbf{1}) + a_2b_2(\hat{\boldsymbol{e}}_\mathbf{2}\cdot\hat{\boldsymbol{e}}_\mathbf{2})$$

$$e_1 = e_2 = 1 \quad \& \quad \vartheta = \pi$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{1}} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{1}} = \hat{\boldsymbol{e}}_{\boldsymbol{2}} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{2}} = 1 \cdot 1 \cos 0 = 1$$
$$\hat{\boldsymbol{e}}_{\boldsymbol{1}} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{2}} = \hat{\boldsymbol{e}}_{\boldsymbol{2}} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{1}} = 1 \cdot 1 \cos \pi = 0$$
$$\therefore \boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

This result is true in all dimensions. In general:

$$\boxed{a_i \hat{\boldsymbol{e}}_{\boldsymbol{i}} \cdot b_i \hat{\boldsymbol{e}}_{\boldsymbol{i}} = a_i b_i = ab \cos \vartheta}$$

Several useful results come from this:

- The angle between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be calculated using the following formula:

$$\vartheta = \arccos \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{ab}$$

- If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ then either a and/or b is $\boldsymbol{0}$ or $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.

5 Properties of Scalar and Vector Products

Distributive law of the dot product:

$$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) \equiv \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$$

Vector product

Distributive law of the cross product:

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) \equiv \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$$

Algebraic representation of the vector product

Let $\boldsymbol{a} = a_1 \hat{\boldsymbol{e}}_{\boldsymbol{1}} + a_2 \hat{\boldsymbol{e}}_{\boldsymbol{2}} + a_3 \hat{\boldsymbol{e}}_{\boldsymbol{3}}$ and $\boldsymbol{b} = b_1 \hat{\boldsymbol{e}}_{\boldsymbol{1}} + b_2 \hat{\boldsymbol{e}}_{\boldsymbol{2}} + b_3 \hat{\boldsymbol{e}}_{\boldsymbol{3}}$

$$\boldsymbol{a} \times \boldsymbol{b} = (a_1 \hat{\boldsymbol{e}}_{\boldsymbol{1}} + a_2 \hat{\boldsymbol{e}}_{\boldsymbol{2}} + a_3 \hat{\boldsymbol{e}}_{\boldsymbol{3}}) \times (b_1 \hat{\boldsymbol{e}}_{\boldsymbol{1}} + b_2 \hat{\boldsymbol{e}}_{\boldsymbol{2}} + b_3 \hat{\boldsymbol{e}}_{\boldsymbol{3}})$$

$$\begin{aligned} = a_1 b_1 (\hat{\boldsymbol{e}}_{\boldsymbol{1}} \times \hat{\boldsymbol{e}}_{\boldsymbol{1}}) + a_1 b_2 (\hat{\boldsymbol{e}}_{\boldsymbol{1}} \times \hat{\boldsymbol{e}}_{\boldsymbol{2}}) + a_1 b_3 (\hat{\boldsymbol{e}}_{\boldsymbol{1}} \times \hat{\boldsymbol{e}}_{\boldsymbol{3}}) + a_2 b_1 (\hat{\boldsymbol{e}}_{\boldsymbol{2}} \times \hat{\boldsymbol{e}}_{\boldsymbol{1}}) + a_2 b_2 (\hat{\boldsymbol{e}}_{\boldsymbol{2}} \times \hat{\boldsymbol{e}}_{\boldsymbol{2}}) + a_2 b_3 (\hat{\boldsymbol{e}}_{\boldsymbol{2}} \times \hat{\boldsymbol{e}}_{\boldsymbol{3}}) \\ + a_3 b_1 (\hat{\boldsymbol{e}}_{\boldsymbol{3}} \times \hat{\boldsymbol{e}}_{\boldsymbol{1}}) + a_3 b_2 (\hat{\boldsymbol{e}}_{\boldsymbol{3}} \times \hat{\boldsymbol{e}}_{\boldsymbol{2}}) + a_3 b_3 (\hat{\boldsymbol{e}}_{\boldsymbol{3}} \times \hat{\boldsymbol{e}}_{\boldsymbol{3}}) \end{aligned}$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{1}} \times \hat{\boldsymbol{e}}_{\boldsymbol{1}} = \hat{\boldsymbol{e}}_{\boldsymbol{2}} \times \hat{\boldsymbol{e}}_{\boldsymbol{2}} = \hat{\boldsymbol{e}}_{\boldsymbol{3}} \times \hat{\boldsymbol{e}}_{\boldsymbol{3}} = 1 \cdot 1 \sin \frac{\pi}{2} \hat{\boldsymbol{n}} = 0$$
$$\hat{\boldsymbol{e}}_{\boldsymbol{1}} \times \hat{\boldsymbol{e}}_{\boldsymbol{2}} = \hat{\boldsymbol{e}}_{\boldsymbol{3}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{1}} \times \hat{\boldsymbol{e}}_{\boldsymbol{3}} = -\hat{\boldsymbol{e}}_{\boldsymbol{2}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{2}} \times \hat{\boldsymbol{e}}_{\boldsymbol{1}} = -\hat{\boldsymbol{e}}_{\boldsymbol{3}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{2}} \times \hat{\boldsymbol{e}}_{\boldsymbol{3}} = \hat{\boldsymbol{e}}_{\boldsymbol{1}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{3}} \times \hat{\boldsymbol{e}}_{\boldsymbol{1}} = \hat{\boldsymbol{e}}_{\boldsymbol{2}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{3}} \times \hat{\boldsymbol{e}}_{\boldsymbol{2}} = -\hat{\boldsymbol{e}}_{\boldsymbol{1}}$$
$$\boldsymbol{a} \times \boldsymbol{b} = a_1 b_2 \hat{\boldsymbol{e}}_{\boldsymbol{3}} - a_1 b_3 \hat{\boldsymbol{e}}_{\boldsymbol{2}} - a_2 b_1 \hat{\boldsymbol{e}}_{\boldsymbol{3}} + a_2 b_3 \hat{\boldsymbol{e}}_{\boldsymbol{1}} + a_3 b_1 \hat{\boldsymbol{e}}_{\boldsymbol{2}} - a_3 b_2 \hat{\boldsymbol{e}}_{\boldsymbol{1}}$$
$$= \hat{\boldsymbol{e}}_{\boldsymbol{1}} (a_2 b_3 - a_3 b_2) + \hat{\boldsymbol{e}}_{\boldsymbol{2}} (a_3 b_1 - a_1 b_3) + \hat{\boldsymbol{e}}_{\boldsymbol{3}} (a_1 b_2 - a_2 b_1)$$

The following is not on the mfp2 course:

For some $i, j, k \in \{1, 2, 3\}$:

$$\hat{\boldsymbol{e}}_{\boldsymbol{i}} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{j}} = \delta_{ij}$$

δ_{ij} is called Kroneker's delta and has the property that:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{i}} \times \hat{\boldsymbol{e}}_{\boldsymbol{j}} = \hat{\boldsymbol{e}}_{\boldsymbol{k}} \varepsilon_{ijk}$$

ε_{ijk} *is called the Levi-Civita symbol and has the property that:*

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{for an even permutation} \\ -1 & \text{for an odd permutation} \\ 0 & \text{for all other cases} \end{cases}$$

An even permutation is one where that is formed from cycling through 1,2,3 in that order and an odd permutation cycles through 3,2,1.

If $\boldsymbol{a} \times \boldsymbol{b} = 0$ then either $\boldsymbol{a}$ or $\boldsymbol{b} = \boldsymbol{0}$ or they are parallel.

$$\boldsymbol{a} \cdot \boldsymbol{a} = a^2 \cos 0 = a^2$$

We can use this to 'remove' all vectors and equate coefficients since if $\boldsymbol{a} = a_1\hat{\boldsymbol{e}}_{\boldsymbol{1}} + a_2\hat{\boldsymbol{e}}_{\boldsymbol{2}} + a_3\hat{\boldsymbol{e}}_{\boldsymbol{3}}$ then $\boldsymbol{a} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{1}} = a_1$.

Vector identities

- Scalar triple product:

  $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) =$ Volume of the parallelipiped with CSA $|\boldsymbol{b} \times \boldsymbol{c}|$ and side length a

- Vector triple product (bac-cab rule):

  $$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot c) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$$

  Since for cartesian coordinates all directions are indistinguishable then this result can be proved to be true for the $\hat{\boldsymbol{e}}_{\boldsymbol{1}}$ component and that implies it is true for all components. To do this we dot product both sides with $\hat{\boldsymbol{e}}_{\boldsymbol{1}}$:

  $$\hat{\boldsymbol{e}}_{\boldsymbol{1}} \cdot (\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})) = \hat{\boldsymbol{e}}_{\boldsymbol{1}} \cdot (\boldsymbol{b}(\boldsymbol{a} \cdot c) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b}))$$

Part II

Determinants

6 Vector Product as a Determinant

$$\boldsymbol{a} \times \boldsymbol{b} = \hat{\boldsymbol{e}}_{\boldsymbol{1}}(a_2b_3 - a_3b_2) + \hat{\boldsymbol{e}}_{\boldsymbol{2}}(a_3b_1 - a_1b_3) + \hat{\boldsymbol{e}}_{\boldsymbol{3}}(a_1b_2 - a_2b_1) = \begin{vmatrix} \hat{\boldsymbol{e}}_{\boldsymbol{1}} & \hat{\boldsymbol{e}}_{\boldsymbol{2}} & \hat{\boldsymbol{e}}_{\boldsymbol{3}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

We call this the determinant of the matrix $\begin{bmatrix} \hat{\boldsymbol{e}}_{\boldsymbol{1}} & \hat{\boldsymbol{e}}_{\boldsymbol{2}} & \hat{\boldsymbol{e}}_{\boldsymbol{3}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$. We define the determinant of a 2×2 matrix as:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \triangleq ad - bc$$

3×3 determinant by Pierre Sarrus' rule:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

Note that we have expanded by the first row here, taking a_i times a determinant of what remains after deleting the row and column containing a_i, as well as a sign according to

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}.$$

It is also possible to expand by any other row, or indeed any column, by taking the values in that row/column, deleting one by one, and multiplying them by the determinant of the matrix we get deleting the row and column of that element and the sign according to the table above.

The following is known as an "augmented" matrix:

$$A = \left|\begin{array}{ccc|cc} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{array}\right|$$

Take the product of every diagonal that has an a_i, b_i and c_i From both top left to bottom right and top right to bottom left. If it is right to left it gets a positive sign and if it is left to right it gets a negative sign. The sum of these producets is $\det A$.

Generalisation to $n \times n$ matrix ($n \in \mathbb{N}$) by Pierre Laplace:

Let A be a matrix size $n \times n$ where A is composed of elements a_{ij} where a_{ij} is in row i and column j:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Then the determinant of A is given as:

$$\det A = \sum_{i=1}^{n} a_{ij}C_{ij}$$

In this C_{ij} is the cofactor of the element a_{ij} and it is defined as:

$$C_{ij} = (-1)^{i+j}M_{ij}$$

Where M_{ij} is the minor of element a_{ij} which is the determinant of the $(n-1) \times (n-1)$ matrix made by removing row i and column j.

This formula allows any value of j to be picked and then is computed for all values of i. The formula also holds if you pick a value of i and then replace all is in the formula with js and compute it that way. We say that the matrix is expanded/reduced by row i or column j

Example 6.1

Compute the following determinant by Lapalce expansion about the first row:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Scalar triple product

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$$
$$= (a_1\hat{\boldsymbol{e}}_{\boldsymbol{1}} + a_2\hat{\boldsymbol{e}}_{\boldsymbol{2}} + a_3\hat{\boldsymbol{e}}_{\boldsymbol{3}})\cdot\left[\hat{\boldsymbol{e}}_{\boldsymbol{1}}\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \hat{\boldsymbol{e}}_{\boldsymbol{2}}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \hat{\boldsymbol{e}}_{\boldsymbol{3}}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\right]$$
$$a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

7 General Properties of Determinants

Illustrated by the 3×3 matrix A:

$$\det A = \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

1. Writing rows as columns leaves the value of the determinant unchanged:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$
$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$
$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - b_2a_3) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \det A$$

2. If each element of a row or column of a determinant is multiplied by a number λ then the value of the determinant is multiplied by lambda:

$$\begin{vmatrix} \lambda a_1 & \lambda a_2 & \lambda a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (\lambda\boldsymbol{a})\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \lambda(\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})) = \lambda\det A$$

3. The determinant is zero when:
   - Any one row or column has only zero as elements:

$$\begin{vmatrix} 0 & 0 & 0 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \boldsymbol{0}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = 0$$

   - Any of the rows or columns are identcal:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{b}) = 0$$

   - Any row or column is a multiple of another row or column:

$$\begin{vmatrix} a_1 & a_1 & a_2 \\ b_1 & b_2 & b_3 \\ \lambda b_1 & \lambda b_2 & \lambda b_3 \end{vmatrix} = \boldsymbol{a}\cdot(\boldsymbol{b}\times(\lambda\boldsymbol{b})) = \lambda(\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{b})) = 0$$

4. If two consecutive rows or columns are interchanged then the sign reverses:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = -\boldsymbol{a}\cdot(\boldsymbol{c}\times\boldsymbol{b}) = -\boldsymbol{b}\cdot(\boldsymbol{a}\times\boldsymbol{c}) = \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

5. The value of the determinant is unchanged if, to each element in a row or column, you add λ times the corresponding element of another row or column:

$$\begin{vmatrix} a_1+\lambda b_1 & a_2+\lambda b_2 & a_3+\lambda b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (\boldsymbol{a}+\lambda\boldsymbol{b})\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) + \lambda\boldsymbol{b}\cdot(\boldsymbol{b}\times\boldsymbol{c}) =$$

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) + \lambda\boldsymbol{c}\cdot(\boldsymbol{b}\times\boldsymbol{b}) = \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vectors

In 3D three vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are linearly independant if, for some constants α, β and γ the equation $\alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} = \boldsymbol{0}$ has only one solution which is $\alpha = \beta = \gamma = 0$

This is the same as saying $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \det A \neq 0$.

If $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are linearly independant then they can be combined linearly to express any 3D vector.

Example 7.1

Express $\boldsymbol{w} = w_1\hat{\boldsymbol{e}}_{\boldsymbol{1}} + w_2\hat{\boldsymbol{e}}_{\boldsymbol{2}} + w_3\hat{\boldsymbol{e}}_{\boldsymbol{3}}$ as a linear combination of $\boldsymbol{a} = (1,1,1), \boldsymbol{b} = (1,1,0)$ and $\boldsymbol{c} = (1,0,0)$

First check that they are linearly independant:

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1 \neq 0$$

so they are linearly independant.

$$\boldsymbol{w} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c}$$

$$w_1 = \alpha + \beta + \gamma \tag{6.1}$$
$$w_2 = \alpha + \beta \tag{6.2}$$
$$w_3 = \alpha \tag{6.3}$$

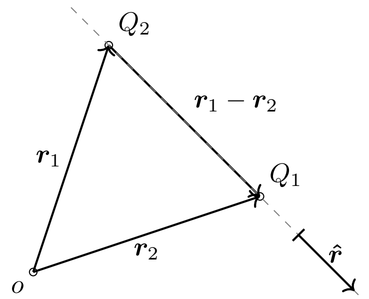
$$(6.3) \implies \alpha = w_3$$
$$(6.2) - (6.3) \implies \beta = w_2 - w_3$$
$$(6.1) - (6.2) \implies \gamma = w_1 - w_2$$
$$\boldsymbol{w} = w_3\boldsymbol{a} + (w_2 - w_3)\boldsymbol{b} + (w_1 - w_2)\boldsymbol{c}$$

8 Physics Applications

Electrostatics

Coulomb's law:

$$\boldsymbol{F}_{12} = k\frac{Q_1Q_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3}(\boldsymbol{r}_1 - \boldsymbol{r}_2)$$

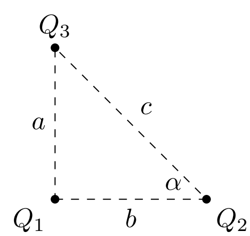


$$\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|}$$

$$\boldsymbol{F}_{12} = k\frac{Q_1Q_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2}\hat{\boldsymbol{r}}$$

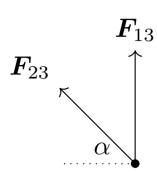
Example 8.1

Three positive charges Q_1, Q_2 and Q_3 are intially in the corners of a right angle triangle as shown:



Find the resultant force $\boldsymbol{R}$ on Q_3:

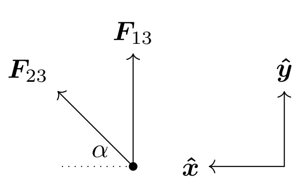
Step one - Draw the vectors:



Step 2 - Identify vector equations:

$$\boldsymbol{R} = \boldsymbol{F}_{13} + \boldsymbol{F}_{23}$$

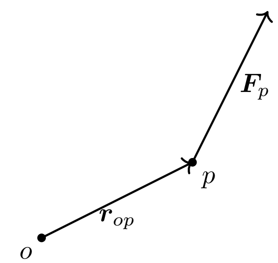
Step 3 - Choose a coordinate system:



Step 4 - Decompose into coordinates:

$$\boldsymbol{F}_{13} = F_{13}\hat{\boldsymbol{y}} = k\frac{Q_1Q_3}{a^2}\hat{\boldsymbol{y}}$$
$$\boldsymbol{F}_{23} = F_{23}\cos\alpha\hat{\boldsymbol{x}} + F_{23}\sin\alpha\hat{\boldsymbol{y}} = k\frac{Q_2Q_3}{c^2}(\cos\alpha\hat{\boldsymbol{x}} + \sin\alpha\hat{\boldsymbol{y}})$$
$$\boldsymbol{R} = kQ_3\left[\frac{Q_2}{c^2}\cos\alpha\hat{\boldsymbol{x}} + \left(\frac{Q_1}{a^2} + \frac{Q_2}{c^2}\sin\alpha\right)\hat{\boldsymbol{y}}\right]$$

Torque $\boldsymbol{\tau}$ or moment $\boldsymbol{M}_o$



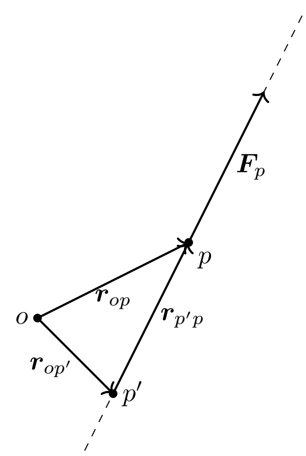
$\boldsymbol{F}_p$ is a force acting through point p. The moment about o is given by:

$$\boldsymbol{M}_o = \boldsymbol{r}_{op} \times \boldsymbol{F}_p$$

$\boldsymbol{M}_o$ is perpendicular to both $\boldsymbol{r}_{op}$ and $\boldsymbol{F}_p$.

If $\boldsymbol{M}_o = \boldsymbol{0}$ then there is no rotation and, $\boldsymbol{r}_{op}$ and $\boldsymbol{F}_p$ are colinear.

But what if, instead of p, we picked a different point on the line of action of the force?



From the diagram it can be seen that:

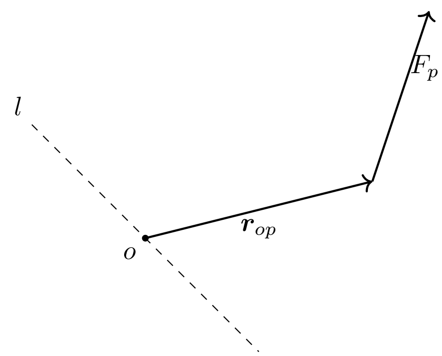
$$\boldsymbol{r}_{op} = \boldsymbol{r}_{op'} + \boldsymbol{r}_{p'p}$$
$$\boldsymbol{M}_o = \boldsymbol{r}_{op} \times \boldsymbol{F}_p = (\boldsymbol{r}_{op'} + \boldsymbol{r}_{p'p}) \times \boldsymbol{F}_p = \boldsymbol{r}_{op'} \times \boldsymbol{F}_p + \boldsymbol{r}_{p'p} \times \boldsymbol{F}_p$$

However from the diagram we know that $\boldsymbol{r}_{p'p}$ and $\boldsymbol{F}_p$ are colinear so $\boldsymbol{r}_{p'p} \times \boldsymbol{F}_p = \boldsymbol{0}$ therefore the moment about o is given by:

$$\boldsymbol{r}_{op'} \times \boldsymbol{F}_p$$

So as long as point p is on the line of action of the force the formula is the same.

If instead we want to find the moment $\boldsymbol{M}_l$ about some line l:



First find $\boldsymbol{M}_o$ where o is a point on line l:

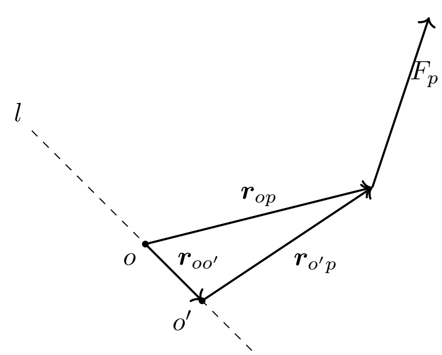
$$\boldsymbol{M}_o = \boldsymbol{r}op \times \boldsymbol{F}_p$$

Next find the projection of $\boldsymbol{M}_o$ along line l:

$$\boldsymbol{M}_o \cdot \hat{\boldsymbol{l}} = M_l$$

$$\boldsymbol{M}_l = M_l\hat{\boldsymbol{l}} = (M_o \cdot \hat{\boldsymbol{l}})\hat{\boldsymbol{l}}$$

But what if we had chosen a different point o' on l?



$$\boldsymbol{M}_l = \hat{\boldsymbol{l}} \cdot \boldsymbol{M}_o = \hat{\boldsymbol{l}} \cdot (\boldsymbol{r}_{op} \times \boldsymbol{F}_p) = \hat{\boldsymbol{l}}\,[(\boldsymbol{r}_{oo'} + \boldsymbol{r}_{o'p}) \times \boldsymbol{F}_p] = \hat{\boldsymbol{l}} \cdot (\boldsymbol{r}_{oo'} \times \boldsymbol{F}_p) + \hat{\boldsymbol{l}} \cdot (\boldsymbol{r}_{o'p} \times \boldsymbol{F}_p)$$

$(\boldsymbol{r}_{oo'} \times \boldsymbol{F}_p)$ is perpendicular to $\hat{\boldsymbol{l}}$ as $\boldsymbol{r}_{oo'}$ is parallel to $\hat{\boldsymbol{l}}$ so $\hat{\boldsymbol{l}} \cdot (\boldsymbol{r}_{oo'} \times \boldsymbol{F}_p) = 0$. Therefore the moment about line l is given by:

$$\boldsymbol{M}_l = \hat{\boldsymbol{l}} \cdot (\boldsymbol{r}_{op} \times \boldsymbol{F}_p) = \hat{\boldsymbol{l}} \cdot (\boldsymbol{r}_{o'p} \times \boldsymbol{F}_p)$$

So the point on the line doesn't matter.

Part III

Matrices

9 Matrices introduction

We will introduce a way of representing the cartesian components of $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

We call this a matrix. A matrix is an ordered list of elements. A marix is of order rows×columns or $m \times n$. The example above is 3×3. A matrix is denoted by a letter eg. $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Sometimes it is double underlined $\underline{\underline{M}}$.

In general a matrix of order $m \times n$ with elements a_{ij} where i is row number and j is column number $i, j \in \mathbb{N}$ and $i \leq m, j \leq n$ will look like this:

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

1. Matrices have no algebraic value
2. We define the zero matrix as an $m \times n$ matrix with all elements equal to 0. That is $a_{ij} = 0 \,\forall\, i, j$:
$$\begin{pmatrix} 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
3. We define a square matrix as a matrix order $n \times n$. Square matrices have several special properties, eg they ahave a determinant.
4. If a square matrix has all elemets as 0 except the elements along the leading diagonal then it is called a diagonal matrix. That is a diagonal matrix has elements $a_{ij} = 0$ if $i \neq j$:
$$\begin{pmatrix} 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$
5. The identity matrix is a diagonal matrix with all elements on the leading diagonal equal to 1. It is denoted $\mathbb{1}$ or I. A subscript number is sometimes added to show the number of rows:
$$I_1 = \begin{pmatrix} 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
6. Matrix equations - Two matrices are only equal if they are identical element by element (that is $a_ij = b_ij \,\forall\, i, j$. As a result of this both matrices must be of the same order to be compared. This means that two matrices being equal encodes a lot of information:
$$\begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \iff a = 1, b = 2, c = 3, x = 4, y = 5, z = 6$$
7. Multiplying every element of a matrix by a constant is the same as multiplying the whole matrix by that constant:
$$\lambda \begin{pmatrix} a & b & c \\ x & y & z \\ \alpha & \beta & \gamma \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b & \lambda c \\ \lambda x & \lambda y & \lambda z \\ \lambda\alpha & \lambda\beta & \lambda\gamma \end{pmatrix}$$

10 Matrix Properteis

8. To add two matrices they must be of the same order and then just add the corresponding elements. That is the new matrix will have elemnts $a_{ij} + b_{ij}$:
$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} + \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \zeta \\ \eta & \vartheta & \iota \end{pmatrix} = \begin{pmatrix} a+\alpha & b+\beta & c+\gamma \\ d+\delta & e+\varepsilon & f+\zeta \\ g+\eta & h+\vartheta & i+\iota \end{pmatrix}$$

9. If we interchange rows and columns of matrix M (so that $a_{ij} \longrightarrow a_{ji}$) then we get the transposed matrix M^T:

$$M = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \iff M^T = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$$

If we have a vector $\boldsymbol{v} = v_1\hat{\boldsymbol{e}}_{\boldsymbol{1}} + v_2\hat{\boldsymbol{e}}_{\boldsymbol{2}} + v_3\hat{\boldsymbol{e}}_{\boldsymbol{3}}$ then we can write it as a column matrix and by transposing a row matrix:

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad v^T = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$$

10. Matrix product - The row by column matrix product of A and B is AB

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}, \qquad B = \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix}$$

This can be viewed as two row and two column vectors.

$$AB = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{a} \cdot \boldsymbol{c} & \boldsymbol{a} \cdot \boldsymbol{d} \\ \boldsymbol{b} \cdot \boldsymbol{c} & \boldsymbol{b} \cdot \boldsymbol{d} \end{pmatrix} = \begin{pmatrix} a_1c_1 + a_2c_2 & a_1d_1 + a_2d_2 \\ b_1c_1 + b_2c_2 & b_1d_1 + b_2d_2 \end{pmatrix}$$

For the product AB to exist matrices A and B must have orders $m \times p$ and $p \times n$ respectively and the resulting matrix will be order $m \times n$.

Matrix multiplication is not commutative ($AB \neq BA$) but it is associative ($A(BC) = (AB)C$) and distributive over addition ($A(B + C) = AB + AC \neq BA + CA = (B + C)A$)

11. For a square matrix M if $AM = MA = I$ then we say that A is the inverse of M denoted M^{-1}. If $\det M = 0$ then M^{-1} doesn't exist. This is useful for solving matrix equations as it allows us to get rid of matrices without matrix division:

$$\boldsymbol{r} = (x, y, z)$$
$$M\boldsymbol{r} = \boldsymbol{k}$$

Where $\boldsymbol{k}$ is a constant vector eg $\boldsymbol{k} = (a, b, c)$

$$M^{-1}M\boldsymbol{r} = M^{-1}\boldsymbol{k}$$
$$I\boldsymbol{r} = M^{-1}\boldsymbol{k}$$
$$\boldsymbol{r} = M^{-1}\boldsymbol{k}$$

A square matrix A of order $n \times n$ can be denoted A_n

Every square matrix has an associated determinant $\det A_n$

Characteristic polynomial

Every square matrix A_n has a characteristic polynomial $p(\lambda)$ which is defined as

$$p(\lambda) \triangleq \det(\lambda I_n - A_n)$$

Example 10.1

$$A_n = \begin{pmatrix} 4 & 3 \\ 5 & 2 \end{pmatrix}$$

$$\begin{aligned} p(\lambda) &= \det\left[\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 3 \\ 5 & 2 \end{pmatrix}\right] = \det\left[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 4 & 3 \\ 5 & 2 \end{pmatrix}\right] \\ &= \begin{vmatrix} \lambda - 4 & -3 \\ -5 & \lambda - 2 \end{vmatrix} = (\lambda - 4)(\lambda - 2) - 15 = \lambda^2 - 6\lambda - 7 \end{aligned}$$

Any order $n \times n$ matrix will have an n^{th} order polynomial.

The Cayley-Hamilton theorem states that any square matrix satisfies its own characteristic polynomial if all constants are multiplied by I_n. That is $p(A_n) = 0$ so from the example above:

$$p(A) = A^2 - 6A - 7I = 0 \implies A^2 = 6A - 7I$$

This is very useful for large order matrices.

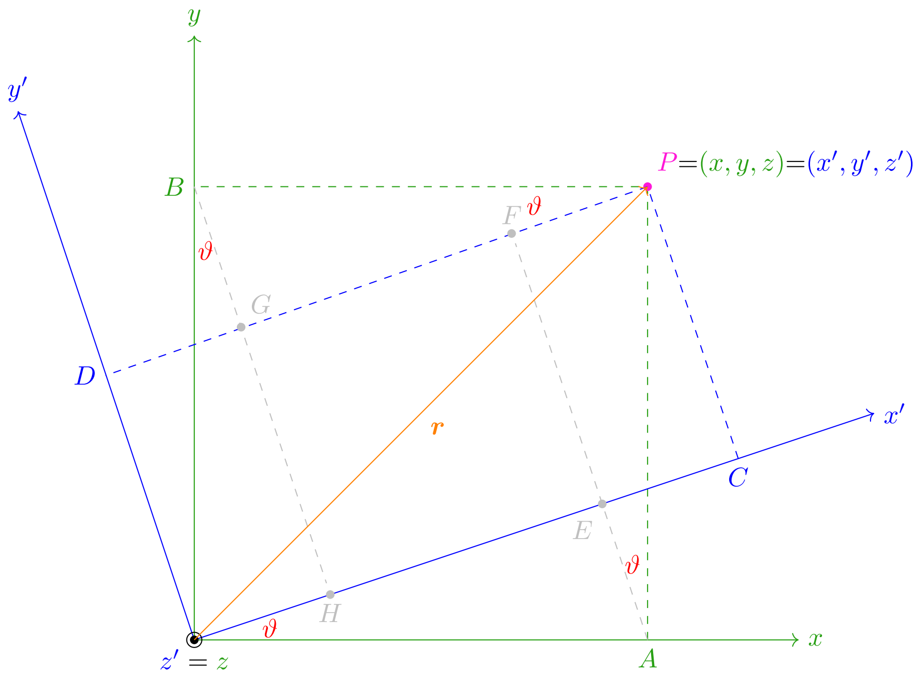
$p(A)$ is a matrix valued polynomial. We can do the same with other functions using their power series expansions:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \mathcal{O}(x^4) \\ e^A &= 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \mathcal{O}(A^4) \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \mathcal{O}(x^7) \\ \sin A &= A - \frac{A^3}{3!} + \frac{A^5}{5!} + \mathcal{O}(A^7) \end{aligned}$$

Note that raising matrices to powers is defined only for positive integer powers and square matrices as:

$$A^n = \underbrace{AA \cdots A}_{n \text{ times}}$$

11 Rotation Matrices



We want to find $x' = f(x, y, z)$ and $y' = g(x, y, z)$. We already know that $z' = z$

$$x' = OE + EC = OA\cos\vartheta + FP = OA\cos\vartheta + AP\sin\vartheta = OA\cos\vartheta + OB\sin\vartheta = x\cos\vartheta + y\sin\vartheta$$

$$x' = x\cos\vartheta + y\sin\vartheta$$

$$y' = OD = GH = HB - GB = OB\cos\vartheta - BP\sin\vartheta = OB\cos\vartheta - OA\sin\vartheta = -x\sin\vartheta + y\cos\vartheta$$

$$y' = -x\sin\vartheta + y\cos\vartheta$$

$$\begin{aligned} x' &= x\cos\vartheta + y\sin\vartheta \\ y' &= -x\sin\vartheta + y\cos\vartheta \\ z' &= z \end{aligned}$$

These simultaneous equations describe how coordinates transform under rotation of coordinate axis about the z axis by angle ϑ

$$\begin{aligned} &\begin{cases} x' = x\cos\vartheta + y\sin\vartheta \\ y' = -x\sin\vartheta + y\cos\vartheta \\ z' = z \end{cases} \\ \iff &\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x\cos\vartheta + y\sin\vartheta \\ -x\sin\vartheta + y\cos\vartheta \\ z \end{pmatrix} \\ \iff &\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{Rotation matrix}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{aligned}$$

The rotation matrix is denoted by a letter, often R for rotation about an axis a subscript is used to denot the axis, R_z and the angle of rotation, ϑ, is sometimes also given, $R_z(\vartheta)$.

$$\boldsymbol{r} = OP = (x, y, z) = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$$

This shows a 1 to 1 correspondance between the coordinates of a point and the coefficents of the position vector. Hence matrix tranformations can be applied to vectors as well:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R_z(\vartheta) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \iff \boldsymbol{r}' = R_z(\vartheta)\boldsymbol{r}$$

$\vartheta > 0$ for a counter clockwise rotation. Therefore a clockwise rotation is negative. A clockwise rotation by ϑ is the same as a counter clockwise rotation by $-\vartheta$:

$$R_z(-\vartheta) = \begin{pmatrix} \cos(-\vartheta) & \sin(-\vartheta) & 0 \\ -\sin(-\vartheta) & \cos(-\vartheta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} = R_z^T(\vartheta)$$

Rotating a coordinate frame by an angle ϑ is the same as rotating all vectors by an angle $-\vartheta$.

If A and B are both rotation matrices and we do rotation A and then B we get:

$$\boldsymbol{r}' = A\boldsymbol{r} \qquad \& \qquad \boldsymbol{r}'' = B\boldsymbol{r}' \implies \boldsymbol{r}'' = BA\boldsymbol{r}$$

It is obvious by the nature of rotations that if we do two succesive rotations about the same axis then the total angle of rotation is the sum of the angles of rotation of the two individual rotations:

$$R_z(\vartheta)R_z(\varphi) = R_z(\vartheta + \varphi)$$

From this we can derive the trig addition formulae: In 2D the rotation matrix about the origin is

$$\begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$$

So for two rotation by angles ϑ and φ:

$$\begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \cos(\vartheta+\varphi) & \sin(\vartheta+\varphi) \\ -\sin(\vartheta+\varphi) & \cos(\vartheta+\varphi) \end{pmatrix}$$

$$\begin{pmatrix} \cos\vartheta\cos\varphi - \sin\vartheta\sin\varphi & \cos\vartheta\sin\varphi + \sin\vartheta\cos\varphi \\ -\sin\vartheta\cos\varphi - \cos\vartheta\sin\varphi & -\sin\vartheta\sin\varphi + \cos\vartheta\cos\varphi \end{pmatrix} = \begin{pmatrix} \cos(\vartheta+\varphi) & \sin(\vartheta+\varphi) \\ -\sin(\vartheta+\varphi) & \cos(\vartheta+\varphi) \end{pmatrix}$$

$$\implies \begin{cases} \cos\vartheta\cos\varphi - \sin\vartheta\sin\varphi = \cos(\vartheta+\varphi) \\ \cos\vartheta\sin\varphi + \sin\vartheta\cos\varphi = \sin(\vartheta+\varphi) \end{cases}$$

12 Matrix Transformations

A general matrix transformation takes the form:

$$\boldsymbol{a}' = M\boldsymbol{a}$$

An orthogonal matrix is a matrix who's rows are orthogonal unit vectors. Rotation matrices are orthogonal matrices. One consequence is that the determinant of an orthogonal matrix is 1. Considering the rotation matrix $R(\vartheta)$ it can be seen that the inverse of this transformation is a rotation through angle $-\vartheta$. As we showed last lecture this is equivalent to a transformation by matrix $R^T(\vartheta)$. This can be shown by rotation of vector $\boldsymbol{v}$:

$$R\boldsymbol{v} = \boldsymbol{v}', \quad R^T\boldsymbol{v}' = \boldsymbol{v}''$$

Combining these gives:

$$R^T R\boldsymbol{v} = \boldsymbol{v}'$$

But since these are rotations we know that $\boldsymbol{v} = \boldsymbol{v}''$:

$$R^T R\boldsymbol{v} = \boldsymbol{v} \implies R^T R = I \implies R^T = R^{-1}$$

This result holds for all orthogonal matrices.

There is another way to derive rotation matrices. First we draw the cartesian unit vectors so that the the axis of rotation points out of the page. Then we draw the transposed unit vectors. We then decompose the transposed vectors into the original vectors. This gives the rotation matrices:

$$R_x(\vartheta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{pmatrix}, \quad R_y(\vartheta) = \begin{pmatrix} \cos\vartheta & 0 & -\sin\vartheta \\ 0 & 1 & 0 \\ \sin\vartheta & 0 & \cos\vartheta \end{pmatrix}, \quad R_z(\vartheta) = \begin{pmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can now define a vector as a quantity that is tranformed like a vector

Consider the point $P = (x, y, z) = (x(t), y(t)z(t)) \implies \boldsymbol{r}(t) = x(t)\hat{\boldsymbol{x}} + y(t)\hat{\boldsymbol{y}} + z(t)\hat{\boldsymbol{z}}$ where $\boldsymbol{r}(t)$ is the position vector of the point at time t. Differentiating $\boldsymbol{r}$ with respect to time:

$$\frac{d}{dt}\boldsymbol{r}(t) = \frac{d}{dt}(x(t)\hat{\boldsymbol{x}}) + \frac{d}{dt}(x(t)\hat{\boldsymbol{x}}) + \frac{d}{dt}(y(t)\hat{\boldsymbol{y}}) + \frac{d}{dt}(z(t)\hat{\boldsymbol{z}}) \tag{10.1}$$

The cartesian unit vectors are time independant so their derivative with respect to time is 0. If we apply the chain rule to each term in (10.1) then we get:

$$\begin{aligned} \frac{d\boldsymbol{r}}{dt} &= \frac{dx}{dt}\hat{\boldsymbol{x}} + \frac{d\hat{\boldsymbol{x}}}{dt}x + \frac{dy}{dt}\hat{\boldsymbol{y}} + \frac{d\hat{\boldsymbol{y}}}{dt}y + \frac{dz}{dt}\hat{\boldsymbol{z}} + \frac{d\hat{\boldsymbol{z}}}{dt}x \\ &= \frac{dx}{dt}\hat{\boldsymbol{x}} + \frac{dy}{dt}\hat{\boldsymbol{y}} + \frac{dz}{dt}\hat{\boldsymbol{z}} \\ &= \dot{x}\hat{\boldsymbol{x}} + \dot{y}\hat{\boldsymbol{y}} + \dot{z}\hat{\boldsymbol{z}} \end{aligned}$$

Where $\dot{x}$ is the derivative of x with respect to time.

If instead we want the derivative of $\boldsymbol{r}'$:

$$\frac{d}{dt}\boldsymbol{r}' = \frac{d}{dt}(R_z(\vartheta)\boldsymbol{r}) = \frac{dR_z(\vartheta)}{dt}\boldsymbol{r} + R_z(\vartheta)\frac{d\boldsymbol{r}}{dt}$$

$R_z(\vartheta)$ is time independant so its derivative with respect to time is 0

$$\frac{d}{dt}\boldsymbol{r}' = R_z(\vartheta)\frac{d\boldsymbol{r}}{dt} = R_z(\vartheta)(\dot{x}\hat{\boldsymbol{x}} + \dot{y}\hat{\boldsymbol{y}} + \dot{z}\hat{\boldsymbol{z}})$$

The derivative with respect to time of the position vector is velocity vector and the second derivative with respect to time is the acceleration vector:

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v}, \qquad \frac{d^2\boldsymbol{r}}{dt^2} = \frac{d\boldsymbol{v}}{dt} = \boldsymbol{a}$$

If $\boldsymbol{r}(0)$ is a constant vector wich gives the position at time $t = 0$ and $\boldsymbol{r}(t)$ is a variable vector giving the position at time t then we define the displacement vector $\boldsymbol{s}$ as $\boldsymbol{s}(t) = \boldsymbol{r}(t) - \boldsymbol{r}(0)$.

$$\frac{d\boldsymbol{s}}{dt} = \frac{d}{dt}(\boldsymbol{r}(t) - \boldsymbol{r}(0)) = \frac{d\boldsymbol{r}}{dt} = \boldsymbol{r}$$

If we split the path that the point takes into infinitesimal vectors $\delta\boldsymbol{s}$ then we get:

$$\sum_{i=0}^{t} \delta\boldsymbol{s}_i = \boldsymbol{s}(t)$$

Taking the limit as $\delta \boldsymbol{s} \to \mathbf{0}$:

$$\int_0^t d\boldsymbol{s} = \boldsymbol{s}(t)$$

We can also calculate $\delta \boldsymbol{s}$ in terms of $\boldsymbol{v}$ and δt

$$\delta s = \boldsymbol{v}\delta t$$
$$\int_0^t \boldsymbol{v}\, dt = \boldsymbol{s}(t)$$

We can calculate path length s as:

$$s = \int_0^t |d\boldsymbol{s}| = \int_0^t |\boldsymbol{v}|\, dt = \int_0^t v\, dt$$

$$\boldsymbol{a} = \frac{d^2\boldsymbol{r}}{dt^2} = \frac{d\boldsymbol{v}}{dt} = \ddot{x}(t)\hat{\boldsymbol{x}} + \ddot{y}(t)\hat{\boldsymbol{y}} + \ddot{z}(t)\hat{\boldsymbol{z}}$$

Since these are linear functions:

$$\boldsymbol{s}(t) = \int \boldsymbol{v}\, dt = \hat{\boldsymbol{x}} \int v_x(t)\, dt + \hat{\boldsymbol{y}} \int v_y(t)\, dt + \hat{\boldsymbol{x}} \int v_z(t)\, dt$$

Part IV

Differential equations

13 Seperable Differential Equations

A differential equation (DE) is an equation that contains derivatives.

Ordinary differential equations (ODE) contain ordinary derivatives: $\frac{d}{dx}f(x)$, $\frac{d}{dt}g(x)$, $\frac{d^2}{dt^2}\boldsymbol{s}(t)$

Partial differential equations (PDE) contain partial derivatives $\partial_x = \frac{\partial}{\partial x}f(x, y)$. They are not a part of MFP2.

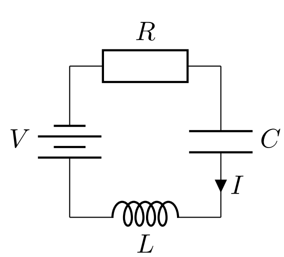
A linear ODE (LODE) is an ODE that is linear in $y(x)$ and in all of its derivatives. It can therefore be written in the form:

$$a_0 y(x) + a_1\frac{dy}{dx} + a_2\frac{d^2y}{dx^2} + \cdots + a_n\frac{d^ny}{dx^n} = b$$

Where a_i and b are functions of x. If it can't be written in this form it is non-linear.

Example 13.1

RCL circuit:



The current in this circuit is described as a function of time which satisfies:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = V(t)$$

This is a second order LODE

$\dfrac{dx}{dy} = \cos y$ this is a first order non-linear ODE

$yy' = 1$ this is a first order non-linear ODE

The solution to a DE is a function such that when substituted into the equation the equal sign holds true

Example 13.2

$$\frac{dy}{dx} = \cos x \implies y(x) = \sin x + c$$

This is the general solution to the DE. It has constants of integration. A DE will have as many constants of integration as its order. The paticular solution is the general solution with specific values calculated from boundary conditions for the constants of integration.

A DE is seperable if it is possible to seperate different variables to either side of the equal sign:

$$\frac{dy}{dx} = f(x)g(y) \implies \frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

Seperable DE can be solved by integration:

$$\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx$$

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

The uniueness theorem states that there are as many unique solutions to a DE as the order of the DE.

Example 13.3

Radioavtive decay: $N(t)$ is the number of atoms in the sample at time t. λ is the decay constant. At $t = 0$, $N(0) = N_0$. The system is modeled by the DE:

$$\begin{aligned} \frac{dN}{dt} &= -\lambda N(t) \\ \iff \frac{1}{N(t)}\frac{dN}{dt} &= -\lambda \end{aligned}$$

There are two methods to solve this from this point on, method one is to solve then apply the boundary conditions:

$$\begin{aligned} \iff \int \frac{1}{N(t)}\frac{dN}{dt}\,dt &= -\lambda \int dt \\ \iff \int \frac{dN}{N(t)} &= -\lambda \int dt \\ \iff \ln N(t) &= -\lambda t + c \\ \iff N(t) &= e^{-\lambda t + c} \\ \iff N(t) &= e^c e^{-\lambda t} \end{aligned}$$

e^c is just a constant so we can rename it

$$\begin{aligned}
&\iff N(t) = \gamma e^{-\lambda t} \\
&\implies N(0) = \gamma e^{0} \\
&\implies N_0 = \gamma \\
&\implies N(t) = N_0 e^{-\lambda t}
\end{aligned}$$

The second method is to apply the boundary conditions in the limits of the integrals:

$$\begin{aligned}
&\iff \int_{N_0}^{N(t)} \frac{1}{N(t)} \frac{dN}{dt}\, dt = -\lambda \int_0^t dt \\
&\iff \int_{N_0}^{N(t)} \frac{dN}{N(t)} = -\lambda \int_0^t dt \\
&\iff [\ln N(t)]_{N_0}^{N(t)} = -\lambda [t]_0^t \\
&\iff \ln N(t) - \ln N_0 = -\lambda t \\
&\iff \ln \frac{N(t)}{N_0} = -\lambda t \\
&\iff N(t) = N_0 e^{-\lambda t}
\end{aligned}$$

Example 13.4

Seperable first order ODE based on examples from

$$\begin{cases} \boldsymbol{a} = \dfrac{d\boldsymbol{v}}{dt} \\ \boldsymbol{v} = \dfrac{d\boldsymbol{s}}{dt} \end{cases}$$

All of these examples refer to straight line motion where $\boldsymbol{v} = v\hat{\boldsymbol{e}}_{\boldsymbol{t}}$ where $\hat{\boldsymbol{e}}_{\boldsymbol{t}}$ is the unit vector tangental to the direction of motion. By differentiating we get $\boldsymbol{a} = \hat{\boldsymbol{e}}_{\boldsymbol{t}} \dfrac{dv}{dt}$ and $\boldsymbol{v} = \hat{\boldsymbol{e}}_{\boldsymbol{t}} \dfrac{dv}{dt}$

$$\begin{cases} \boldsymbol{a} = \hat{\boldsymbol{e}}_{\boldsymbol{t}} \dfrac{dv}{dt} \\ \boldsymbol{v} = \hat{\boldsymbol{e}}_{\boldsymbol{t}} \dfrac{ds}{dt} \end{cases}$$

Case 1

$a = a(t)$ is given find $v(t)$:

$$\begin{aligned}
a(t) &= \frac{dv}{dt} \\
\int a(t)\, dt &= \int dv \\
\int a(t) &= v(t) + c
\end{aligned}$$

Case 2

$a = a(v)$ is given find $v(t)$:

$$a(v) = \frac{dv}{dt}$$

$$
\begin{aligned}
1 &= \frac{dv}{dt}\frac{1}{a(v)} \\
\int dt &= \int \frac{dv}{a(v)} \\
t &= \int \frac{dv}{a(v)} + c
\end{aligned}
$$

Case 3

$a = a(s)$ is given find $v(s)$:

$$
\begin{aligned}
a(s) &= \frac{dv}{dt} \\
a(s) &= \frac{dv}{ds}\cdot\frac{ds}{dt} \\
a(s) &= \frac{dv}{ds}v \\
\int a(s)\,ds &= \int v\,dv \\
\int a(s)\,ds &= \frac{1}{2}v^2 + c
\end{aligned}
$$

14 Linear Differential Equations

For the first order ODE (1^{st} ODE) (13.1):

$$\frac{dy}{dx} = F(x,y) \tag{13.1}$$

the method of solving depends on the form that $F(x,y)$ takes. If $F(x,y)$ takes the form $f(x)g(y)$ then as we have already seen the equation is seperable so rearranging and integrating with respect to x will solve it. If instead $F(x,y) = Q(x) - P(x)y$ then the equation won't be seperable but it is linear.

This means that (13.1) can be expressed as:

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{13.2}$$

To solve this we need to find a way to express (13.2) in the form:

$$\frac{d}{dx}[f(x,y)] = \tilde{Q}(x)$$

If instead we consider the homogeneous form:

$$\frac{dy}{dx} + P(x)y = 0$$

This is now a seperable ODE:

$$
\begin{aligned}
\frac{dy}{dx} &= -P(x)y \\
\frac{1}{y}\frac{dy}{dx} &= -P(x) \\
\int \frac{dy}{y} &= -\int P(x)\,dx \\
\ln y &= -\int P(x)\,dx + c
\end{aligned}
$$

$$
\begin{aligned}
y &= e^{-\int P(x)\,dx + c} \\
y &= c_1 e^{-\int P(x)\,dx}
\end{aligned}
$$

Let $\int P(x)\,dx = I(x)$:

$$
\begin{aligned}
y &= c_1 e^{-I} \\
ye^I &= c_1
\end{aligned}
$$

We can differentiate this to show it is of the desired form

$$
\begin{aligned}
\frac{d}{dx}[e^{I(x)}y(x)] &= y'(x)e^{I(x)} + I'(x)y(x)e^{I(x)} \\
&= \frac{dy}{dx}e^{I(x)} + \frac{dI}{dx}y(x)e^{I(x)} \\
&= \frac{dy}{dx}e^{I(x)} + P(x)y(x)e^{I(x)} \\
&= e^{I(x)}\left[\frac{dy}{dx} + P(x)y(x)\right] \\
&= \frac{d}{dx}c_1 \\
&= 0
\end{aligned}
$$

As expected

$e^{I(x)}$ is called the integrating factor.

By multiplying (13.2) through by the integrating factor we get:

$$
e^{I(x)}\left[\frac{dy}{dx} + P(x)y\right] = e^{I(x)}Q(x)
$$

$$
\iff \frac{d}{dx}\left[e^{I(x)}y\right] = e^{I(x)}Q(x)
$$

This is now a seperable 1^{st} ODE:

$$
\begin{aligned}
\int \frac{d}{dx}\left[e^{I(x)}y(x)\right]\,dx &= \int e^{I(x)}Q(x)\,dx \\
\implies e^{I(x)}y(x) &= \int e^{I(x)}Q(x)\,dx + c_2 \\
\implies y(x) &= e^{-I(x)}\int e^{I(x)}Q(x)\,dx + c_2e^{-I(x)}
\end{aligned}
$$

Since $P(x)$ and $Q(x)$ are known this gives a general solution to 1^{st} ODE of the form:

$$
\frac{dy}{dx} + P(x)y = Q(x)
$$

Example 14.1

$xy' + 3y = x^3$ and $y = \frac{1}{6}$ when $x = 1$. Find $y = y(x)$.

There are two ways to solve this, we could compare to the general solution or we could derive the integrating factor seperately.

Consider $y' = -\frac{3}{x}y$. This is a seperable 1$^{\text{st}}$ ODE:

$$\begin{aligned}
\frac{y'}{y} &= -\frac{3}{x} \\
\int \frac{dy}{y} &= -\int \frac{3}{x}\,dx \\
\ln y &= -3\ln x + c \\
\ln y &= \ln \frac{1}{x^{-3}} + \ln c_1 \\
\ln y &= \ln \frac{c_1}{x^{-3}} \\
y &= \frac{c_1}{x^3} \\
x^3 y &= c_1
\end{aligned}$$

Check:

$$\begin{aligned}
\frac{d}{dx}[x^3 y] &= 3x^2 y + x^3 \frac{dy}{dx} \\
&= x^3 \left[\frac{dy}{dx} + \frac{3}{x}y\right]
\end{aligned}$$

So our integrating factor is x^3

$$\begin{aligned}
\frac{d}{dx}[x^3 y] &= x^3 x^2 = x^5 \\
\int \frac{d}{dx}[x^3 y] &= \int x^5\,dx \\
x^3 y &= \frac{1}{6}x^6 + c_2 \\
y &= \frac{x^3}{6} + \frac{c_2}{x^3} \\
y(1) &= \frac{1}{6} + c_2 \\
\implies c &= 0 \\
\implies y(x) &= \frac{x^3}{6}
\end{aligned}$$

Example 14.2

$$dx + (x - e^y)dy = 0$$

This is written in terms of differentials, we can write it by dividing through by dx:

$$\begin{aligned}
1 + (x - e^y)\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= \frac{1}{e^y - x}
\end{aligned}$$

This is not linear for $y = y(x)$ or seperable

$$\frac{dx}{dy} = e^y - x$$

This is linear for $x = x(y)$

$$\frac{dx}{dy} + x = e^y \tag{13.3}$$

Consider the homogeneous form

$$\begin{aligned}
\frac{dx}{dy} + x &= 0\\
\frac{dx}{dy} &= -x\\
\frac{1}{x}\frac{dx}{dy} &= -1\\
\int \frac{dx}{x} &= -\int dy\\
\ln x &= -y + c\\
x &= e^{c-y}\\
x &= c_1 e^{-y}
\end{aligned}$$

Check

$$\begin{aligned}
\frac{d}{dy}[x(y)e^y] &= \frac{dx}{dy}e^y + x(y)e^y\\
&= e^y\left[\frac{dx}{dy} + x\right]
\end{aligned}$$

The integrating factor is e^y

Hence we can multiply (13.3) through by e^y to create a seperable equation:

$$\begin{aligned}
e^y\frac{dx}{dy} + xe^y &= e^{2y}\\
\frac{d}{dy}[xe^y] &= e^{2y}\\
xe^y &= \int e^{2y}\,dy\\
xe^y &= \frac{e^{2y}}{2} + c_2\\
x(y) &= \frac{e^y}{2} + \frac{c_2}{e^y}
\end{aligned}$$

Example 14.3

$$xy' = y + 1$$

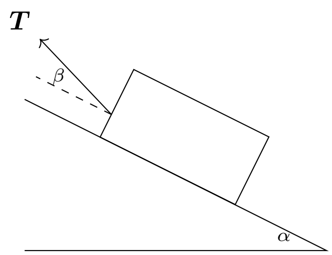
This is both sepereable and linear but much easier to solve by seperation

$$\begin{aligned}
xy' &= y + 1\\
y' &= \frac{y+1}{x}\\
\frac{y'}{y+1} &= \frac{1}{x}
\end{aligned}$$

$$\int \frac{dy}{y+1} = \int \frac{dx}{x}$$
$$\ln|y+1| = \ln x + c$$
$$y+1 = e^{\ln x + c}$$
$$y(x) = c_1 x - 1$$

Part V
Dynamics

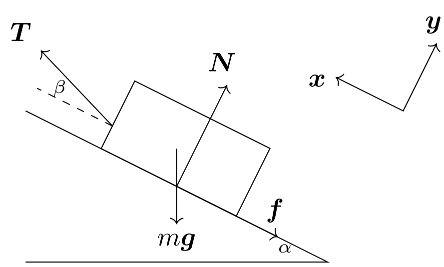
15 Ship on a Slope



A ship rests is on a slope held stationary by a light rope. What is the maximum tension T of the rope before the ship moves up the slope?

Method

1. First draw a vector diagram:



   Note friction is downhill as we want the maximum value T can take.

2. Identify vector equations and equations of motion (EOM):

   Newton's second law:
$$m\boldsymbol{a} = m\boldsymbol{g} + \boldsymbol{T} + \boldsymbol{N} + \boldsymbol{f}$$
   The ship is stationary so $\boldsymbol{a} = \boldsymbol{0}$

3. Choose a coordinate system and decompose vectors:
$$\boldsymbol{N} = N\boldsymbol{y}$$
$$\boldsymbol{f} = -f\boldsymbol{x}$$
$$m\boldsymbol{g} = mg(-\sin\alpha\boldsymbol{x} - \cos\alpha\boldsymbol{y})$$
$$\boldsymbol{T} = T(\cos\beta\boldsymbol{x} + \beta\boldsymbol{y})$$

4. Extract component equations

$$\boldsymbol{x}: \quad -f - mg\sin\alpha + T\cos\beta = 0 \tag{14.1}$$
$$\boldsymbol{y}: \quad T\sin\beta - mg\cos\alpha + N = 0 \tag{14.2}$$

   Also $f \le \mu N$. Since we want the maximum T we take $f = \mu N$(14.3)

5. Solve

   Substitute (14.3) in (14.1)

$$0 = T\cos\beta - mg\sin\alpha - \mu N \tag{14.4}$$

   $\mu \cdot (14.2) + (14.4)$

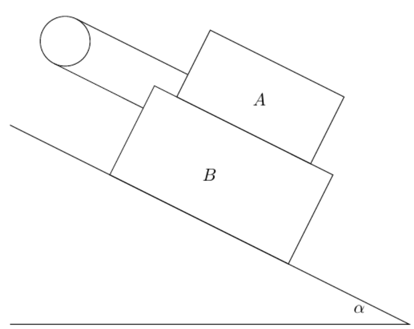
$$0 = T(\mu\sin\beta + \cos\beta) - mg(\mu\cos\alpha + \sin\alpha)$$
$$T = mg\frac{\mu\cos\alpha + \sin\alpha}{\mu\sin\beta + \cos\beta}$$

6. Check

$$[\mathrm{N}] = [\mathrm{kg}][\mathrm{m\,s^{-2}}]$$
$$[\mathrm{kg\,m\,s^{-2}}] = [\mathrm{kg\,m\,s^{-2}}] \quad \checkmark$$

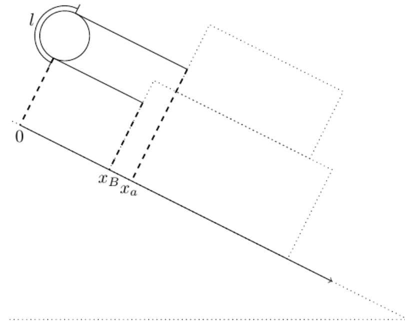
$$\begin{array}{lll}
m \to 0 & T \to 0 & \checkmark \\
g \to 0 & T \to 0 & \checkmark \\
m \to \infty & T \to \infty & \checkmark \\
g \to \infty & T \to \infty & \checkmark \\
\alpha \to 0 \text{ and } \beta \to \frac{\pi}{2} & T \to mg & \checkmark \\
\alpha \to 0 \text{ and } \beta \to 0 & T \to mg\mu = \mu N = f & \checkmark
\end{array}$$

16 Boxes and Pulleys on a Slope



Two boxes, box A mass m and box B mass M where $m \ll M$. The boxes are connected by a light inextendable string which passes over a light, smooth pulley. All surfaces are made of the same material and the coefficient of kinetic friction is μ

It is possible to show that the acceleration of box A has the same magnitude and opposite direction to the acceleration of box B by considering the length of the string L.



The length of the string is a constant and the length of the string around the pulley is also a constant l. The total length of the string is given by

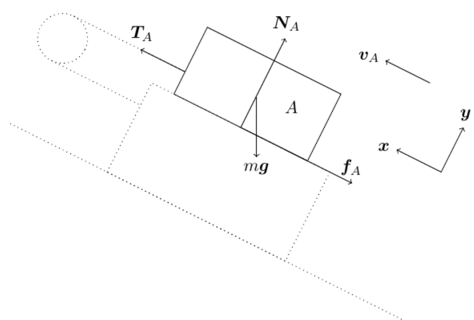
$$L = x_A + x_B + l$$

Taking the second derivative with respect to time gives

$$\underbrace{\ddot{L}}_{=0} = \ddot{x}_A + \ddot{x}_B + \underbrace{\ddot{l}}_{=0}$$

Hence $\ddot{x}_A = -\ddot{x}_B$

We have three seperable systems. The first is box A



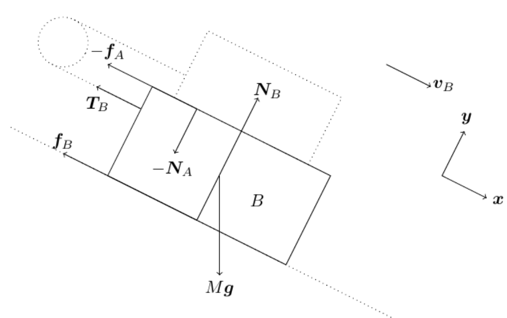
$$\text{NII:} \qquad m\boldsymbol{a} = m\boldsymbol{g} + \boldsymbol{T}_A + \boldsymbol{N}_A + \boldsymbol{f}_A$$

$$\boldsymbol{a} = \ddot{x}\hat{\boldsymbol{x}}$$

$$\implies \begin{cases} \boldsymbol{x}: & m\ddot{x} = -mg\sin\alpha + T_A - f_A \\ \boldsymbol{y}: & 0 = -mg\cos\alpha + N_A \end{cases}$$

Also $f_A = \mu N_A$

Considering box B



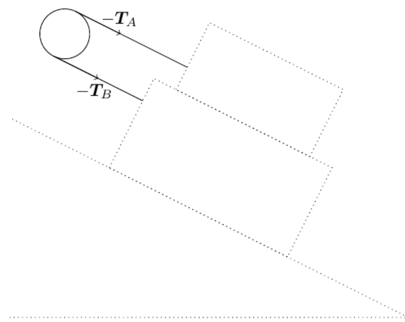
$$\text{NII:} \qquad M\boldsymbol{a} = M\boldsymbol{g} + \boldsymbol{T}_B + \boldsymbol{N}_B + \boldsymbol{f}_B - \boldsymbol{N}_A - \boldsymbol{f}_A$$

$$\boldsymbol{a} = \ddot{x}\hat{\boldsymbol{x}}$$

$$\implies \begin{cases} \boldsymbol{x} : M\ddot{x} = Mg\sin\alpha - T_B - f_B - f_A \\ \boldsymbol{y} : 0 = -Mg\cos\alpha + N_B - N_A \end{cases}$$

Also $f_B = \mu N_B$

Considering the pulley



$$\text{NII:} \qquad \boldsymbol{T}_A + \boldsymbol{T}_B = 0$$

$$\implies T_A = T_B = T$$

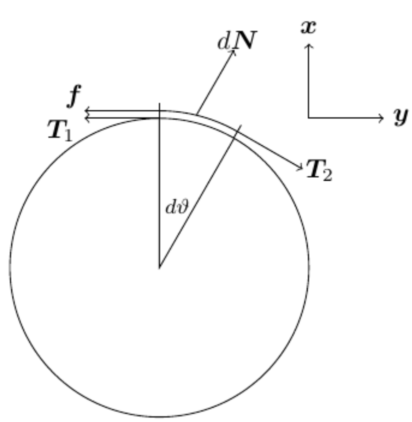
Acceleration is constant. By eliminating contact forces we get

$$\implies \begin{cases} m\ddot{x} = T - mg(\sin\alpha + \mu\cos\alpha) \\ M\ddot{x} = Mg\sin\alpha - \mu(m+M)g\cos\alpha - \mu mg\cos\alpha - T \end{cases}$$

It is then possible to solve these DE for $\ddot{x}$ and x.

What if the pulley is rough?

If the pulley is rough then there will be friction between the rope and the pulley. How much friction can be calculated by considering a small length of rope.



In the case where $T_2 > T_1$ it is possible to write $T_2 = T_1 + dT$ for some small amount $0 < dT \ll 1$.

$$\boldsymbol{T}_1 = T\hat{\boldsymbol{x}}$$

$$\boldsymbol{T}_2 = (T + dT)(\cos d\vartheta\hat{\boldsymbol{x}} - \sin d\vartheta\hat{\boldsymbol{y}}$$

For a rough enough pulley $\ddot{x} = 0$

$$\text{NII: } \mathbf{0} = \boldsymbol{T}_1 + \boldsymbol{T}_2 + d\boldsymbol{N} + d\boldsymbol{f}$$

$$\implies \begin{cases} \boldsymbol{x}: & 0 = (T + dT)\cos d\vartheta - df - T \\ \boldsymbol{y}: & 0 = dN - (T + dT)\sin d\vartheta \end{cases}$$

$$df \leq \mu dN$$

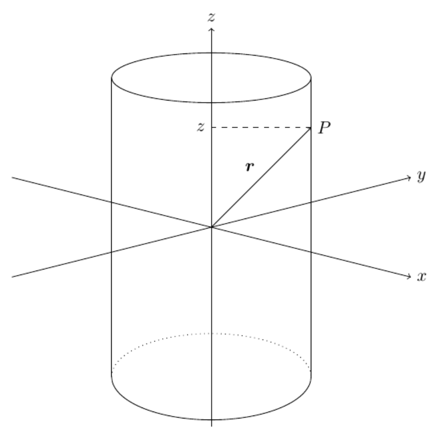
From the x components we get $dt = df$ and from the y components we get $dN = Td\vartheta$ by using the series expansions of sine. Combining all equations gives $dN = \mu Td\vartheta$. This gives us the DE $\frac{1}{T}\frac{dT}{d\vartheta} = \mu$. Solving this gives $T = T_0e^{\mu\vartheta}$ and since $dT = df$ $f = f_0e^{\mu\vartheta}$

Part VI

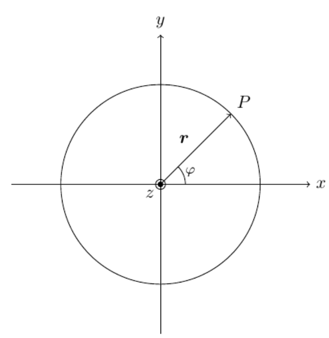
Curvilinear motion

17 Polar Coordinates

Cylindrical Polar Coordinates



To describe a point on the cylinder shown we need its z coordinate, the radial distance from the z axis ϱ and the angle round the z axis φ. By convention φ is measured from the positive x axis and increases counterclockwise. as shown below.

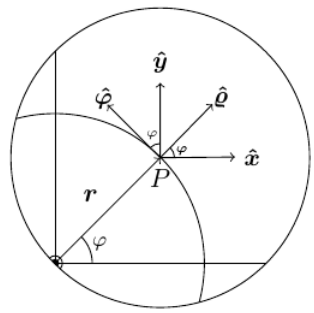


The natural domains of z, ϱ and φ are $z, \varrho \in [0, \infty)$ and $\varphi \in [0, 2\pi)$.

We need two sets of transformation equations, one for the coordinate axis and one for the unit vectors. It can be seen from the diagram that the coordinates are given by

$$\begin{cases} x = \varrho\cos\varphi \\ y = \varrho\sin\varphi \\ z = z \end{cases} \iff \begin{cases} \varrho = \sqrt{x^2+y^2} \\ \varphi = \arctan\frac{y}{x} \\ z = z \end{cases}$$

The cylindrical polar coordinates unit vectors are $\hat{\boldsymbol{z}}$, $\hat{\boldsymbol{\varrho}}$ which is radial to the z axis and starts at the point, and $\hat{\boldsymbol{\varphi}}$ which is normal to $\hat{\boldsymbol{\varrho}}$ and tangental to the cylindar. It points in the direction in which φ increases.



It can be seen that the cylindrical polar unit vectors are just a rotation of the cartesian unit vectors about the z axis through an angle φ.

$$\begin{cases} \hat{\boldsymbol{x}} = \cos\varphi\hat{\boldsymbol{\varrho}} - \sin\varphi\hat{\boldsymbol{\varphi}} \\ \hat{\boldsymbol{y}} = \sin\varphi\hat{\boldsymbol{\varrho}} + \cos\varphi\hat{\boldsymbol{\varphi}} \\ \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \end{cases} \iff \begin{cases} \hat{\boldsymbol{\varrho}} = \cos\varphi\hat{\boldsymbol{x}} + \sin\varphi\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\boldsymbol{x}} + \cos\varphi\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \end{cases}$$

$$\begin{pmatrix} \hat{\boldsymbol{\varrho}} \\ \hat{\boldsymbol{\varphi}} \\ \hat{\boldsymbol{z}} \end{pmatrix} = R_z(\varphi) \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{pmatrix} \iff \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{pmatrix} = R_z^T(\varphi) \begin{pmatrix} \hat{\boldsymbol{\varrho}} \\ \hat{\boldsymbol{\varphi}} \\ \hat{\boldsymbol{z}} \end{pmatrix}$$

For a general point with position vector $\boldsymbol{r}$

$$\begin{aligned} \boldsymbol{r} &= x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}} \\ &= \rho\cos\varphi(\cos\varphi\hat{\boldsymbol{\rho}} - \sin\varphi\hat{\boldsymbol{\varphi}}) + \varrho\sin\varphi(\sin\varphi\hat{\boldsymbol{\varrho}} + \cos\varphi\hat{\boldsymbol{\varphi}}) + z\hat{\boldsymbol{z}} \\ &= \rho(\cos^2\varphi + \sin^2\varphi)\hat{\boldsymbol{\varrho}} + \varrho(-\cos\varphi\sin\varphi + \sin\varphi\cos\varphi)\hat{\boldsymbol{\varphi}} + z\hat{\boldsymbol{z}} \\ &= \varrho\hat{\boldsymbol{\varrho}} + z\hat{\boldsymbol{z}} \end{aligned}$$

This shows that the choice of $\varphi = 0$ and hence the x axis is unimportant due to symmetry. This has the advantage of making a lot of problems easier but does mean we can't pin down the exact position of the point just give a circle that we know it is on.

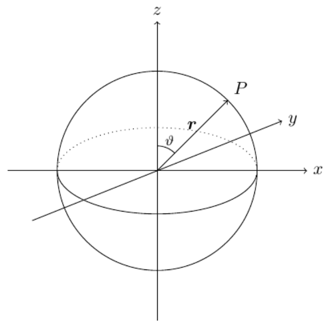
Comments

- $\hat{\boldsymbol{\varrho}} \times \hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{z}}$ defines the right hand rule for cylindrical polar coordinates.
- $\hat{\boldsymbol{\varrho}}, \hat{\boldsymbol{\varphi}}$ and $\hat{\boldsymbol{z}}$ form a orthonormal base (ortho - all orthogonal, normal - all have length 1)
- ϱ and z have dimension length
- φ is dimensionless
- $\hat{\boldsymbol{\varrho}}$ and $\hat{\boldsymbol{\varphi}}$ are time dependant as they change as the point moves

Spherical Polar Coordinates

To describe a point on the surface of the sphere we need the radial distance from the origin r, the "latitude" from the the positive z axis ϑ and the "longitude" φ, defined the same as with cylindrical polar coordinates.

Typical domains for these values are $r \in [0, \infty), \vartheta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$.

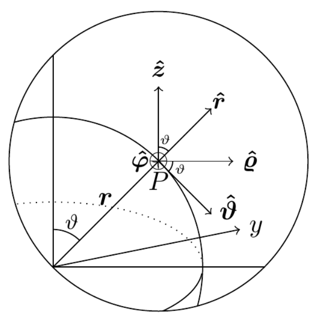


It can be seen from the diagram that $\varrho = r\sin\vartheta$. By substitiuting cartesian coordinates for cylindrical and then again for spherical we can get the coordinate transform equations

$$\begin{cases} x &= \varrho\cos\varphi &= r\sin\vartheta\cos\varphi \\ y &= \varrho\sin\varphi &= r\sin\vartheta\sin\varphi \\ z &= z &= r\cos\vartheta \end{cases}$$

18 Derivatives in Polar Coordinates

The spherical coordinate unit vectors are $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\vartheta}}$ and $\hat{\boldsymbol{\varphi}}$. $\hat{\boldsymbol{r}}$ extends radialy from the origin. $\hat{\boldsymbol{\vartheta}}$ is tangental to the sphere and points from the z axis to the x, y plane. $\hat{\boldsymbol{\varphi}}$ is the same as in cylindrical polar coordinates. $\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\vartheta}} = \hat{\boldsymbol{\varphi}}$ defines the right hand rule for this system.



It can be seen that the spherical polar unit vectors are a rotation of the sylindrical polar unit vectors through an angle of ϑ. It is a positive rotation but in the diagram $\hat{\boldsymbol{\varphi}}$ is into the page so it looks like a clockwise rotation. The unit vectors are given as

$$\begin{cases} \hat{\boldsymbol{r}} = \cos\vartheta\hat{\boldsymbol{z}} + \sin\vartheta\hat{\boldsymbol{\varrho}} \\ \hat{\boldsymbol{\vartheta}} = -\sin\vartheta\hat{\boldsymbol{z}} + \cos\vartheta\hat{\boldsymbol{\varrho}} \\ \hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}} \end{cases} \iff \begin{cases} \hat{\boldsymbol{z}} = \cos\vartheta\hat{\boldsymbol{r}} - \sin\vartheta\hat{\boldsymbol{\vartheta}} \\ \hat{\boldsymbol{\varrho}} = \sin\vartheta\hat{\boldsymbol{r}} + \cos\vartheta\hat{\boldsymbol{\vartheta}} \\ \hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}} \end{cases}$$

$$R_\varphi(\vartheta)\begin{pmatrix}\hat{\boldsymbol{z}}\\ \hat{\boldsymbol{\varrho}}\\ \hat{\boldsymbol{\varphi}}\end{pmatrix} = \begin{pmatrix}\cos\vartheta & \sin\vartheta & 0\\ -\sin\vartheta & \cos\vartheta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\hat{\boldsymbol{z}}\\ \hat{\boldsymbol{\varrho}}\\ \hat{\boldsymbol{\varphi}}\end{pmatrix} = \begin{pmatrix}\hat{\boldsymbol{r}}\\ \hat{\boldsymbol{\vartheta}}\\ \hat{\boldsymbol{\varphi}}\end{pmatrix}$$

$$R_z(\varphi)\begin{pmatrix}\hat{\boldsymbol{x}}\\ \hat{\boldsymbol{y}}\\ \hat{\boldsymbol{z}}\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\\ \cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\end{pmatrix}\begin{pmatrix}\hat{\boldsymbol{x}}\\ \hat{\boldsymbol{y}}\\ \hat{\boldsymbol{z}}\end{pmatrix} = \begin{pmatrix}\hat{\boldsymbol{\varrho}}\\ \hat{\boldsymbol{\varphi}}\\ \hat{\boldsymbol{z}}\end{pmatrix}$$

Let $R = R_\varphi(\vartheta)R_z(\varphi)$

$$R\begin{pmatrix}\hat{\boldsymbol{x}}\\ \hat{\boldsymbol{y}}\\ \hat{\boldsymbol{z}}\end{pmatrix} = \begin{pmatrix}\sin\vartheta\cos\varphi & \sin\vartheta\sin\varphi & \cos\vartheta\\ \cos\vartheta\cos\varphi & \cos\vartheta\sin\varphi & -\sin\vartheta\\ -\sin\varphi & \cos\varphi & 0\end{pmatrix}\begin{pmatrix}\hat{\boldsymbol{x}}\\ \hat{\boldsymbol{y}}\\ \hat{\boldsymbol{z}}\end{pmatrix} = \begin{pmatrix}\hat{\boldsymbol{r}}\\ \hat{\boldsymbol{\vartheta}}\\ \hat{\boldsymbol{\varphi}}\end{pmatrix}$$

Since R is a rotation matrix it is orthogonal so its inverse is R^T We can use this to give the spherical polar unit vectors in terms of cartesian unit vectors

$$\begin{cases}\hat{\boldsymbol{r}} = \sin\vartheta\cos\varphi\hat{\boldsymbol{x}} + \sin\vartheta\sin\varphi\hat{\boldsymbol{y}} + \cos\vartheta\hat{\boldsymbol{z}}\\ \hat{\boldsymbol{\vartheta}} = \cos\vartheta\cos\varphi\hat{\boldsymbol{x}} + \cos\vartheta\sin\varphi\hat{\boldsymbol{y}} - \sin\vartheta\hat{\boldsymbol{z}}\\ \hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\boldsymbol{x}} + \cos\varphi y + 0\hat{\boldsymbol{z}}\end{cases} \iff \begin{cases}\hat{\boldsymbol{x}} = \sin\vartheta\cos\varphi\hat{\boldsymbol{r}} + \cos\vartheta\cos\varphi\hat{\boldsymbol{\vartheta}} - \sin\varphi\hat{\boldsymbol{\varphi}}\\ \hat{\boldsymbol{y}} = \sin\vartheta\sin\varphi\hat{\boldsymbol{r}} + \cos\vartheta\sin\varphi\hat{\boldsymbol{\vartheta}} + \cos\varphi\hat{\boldsymbol{\varphi}}\\ \hat{\boldsymbol{z}} = \cos\vartheta\hat{\boldsymbol{r}} - \sin\vartheta\hat{\boldsymbol{\vartheta}} + 0\hat{\boldsymbol{\varphi}}\end{cases}$$

Let $\boldsymbol{r}$ be a position vector

$$\begin{aligned}\boldsymbol{r} &= x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}\\ &= r\sin\vartheta\cos\varphi(\sin\vartheta\cos\varphi\hat{\boldsymbol{r}} + \cos\vartheta\cos\varphi\hat{\boldsymbol{\vartheta}} - \sin\varphi\hat{\boldsymbol{\varphi}})\\ &\quad + r\sin\vartheta\sin\varphi(\sin\vartheta\sin\varphi\hat{\boldsymbol{r}} + \cos\vartheta\sin\varphi\hat{\boldsymbol{\vartheta}} + \cos\varphi\hat{\boldsymbol{\varphi}})\\ &\quad + r\cos\vartheta(\cos\vartheta\hat{\boldsymbol{r}} - \sin\vartheta\hat{\boldsymbol{\vartheta}})\\ &= (r\sin^2\vartheta\cos^2\varphi + r\sin^2\vartheta\sin^2\varphi + r\cos^2\vartheta)\hat{\boldsymbol{r}}\\ &\quad + (r\sin\vartheta\cos\vartheta\cos^2\varphi + r\sin\vartheta\cos\vartheta\sin^2\varphi - r\cos\vartheta\sin\vartheta)\hat{\boldsymbol{\vartheta}}\\ &\quad + (-r\sin\vartheta\cos\varphi\sin\varphi + r\sin\vartheta\sin\varphi\cos\varphi + r\cos\vartheta)\hat{\boldsymbol{\varphi}}\\ &= r\hat{\boldsymbol{r}}\end{aligned}$$

This shows that the choice of x and z axes is unimportant. However, we can't exactly specify a position in sperical polar coordinates since we would need to know $\hat{\boldsymbol{r}}$ and it is time dependant.

Time derivateives

In cylinindrical coordinates (ϱ, φ, z) a general position vector is given by

$$\boldsymbol{r} = \varrho\hat{\boldsymbol{\varrho}} + z\hat{\boldsymbol{z}}$$

The velocity vector is given by

$$\begin{aligned}\boldsymbol{v} &= \frac{d\hat{\boldsymbol{r}}}{dt}\\ &= \frac{d}{dt}(\varrho\hat{\boldsymbol{\varrho}}) + \frac{d}{dt}(z\hat{\boldsymbol{z}})\\ &= \frac{d\varrho}{dt}\hat{\boldsymbol{\varrho}} + \varrho\frac{d\hat{\boldsymbol{\varrho}}}{dt} + \dot{z}\hat{\boldsymbol{z}}\end{aligned}$$

We can calculate $\dot{\hat{\boldsymbol{\varrho}}}$ by substituting in the cartesian coordinates

$$\begin{aligned}\frac{d\hat{\boldsymbol{\varrho}}}{dt} &= \frac{d}{dt}(\cos\varphi\hat{\boldsymbol{x}} + \sin\varphi\hat{\boldsymbol{y}})\\ &= \frac{d\cos\varphi}{dt}\hat{\boldsymbol{x}} + \frac{d\sin\varphi}{dt}\hat{\boldsymbol{y}}\\ &= \frac{d\cos\varphi}{d\varphi}\frac{d\varphi}{dt} + \frac{d\sin\varphi}{d\varphi}\frac{d\varphi}{dt}\\ &= -\dot{\varphi}\sin\varphi\hat{\boldsymbol{x}} + \dot{\varphi}\cos\varphi\hat{\boldsymbol{y}}\\ &= \dot{\varphi}(-\sin\varphi\hat{\boldsymbol{x}} + \cos\varphi\hat{\boldsymbol{y}})\end{aligned}$$

$$= \dot{\varphi}\hat{\boldsymbol{\varphi}}$$

$\dot{\hat{\boldsymbol{\varrho}}}$ has units of s^{-1} so we interpret it as angular velocity

$$\begin{aligned}\boldsymbol{v} &= \frac{d\varrho}{dt}\hat{\boldsymbol{\varrho}} + \varrho\frac{d\varphi}{dt}\hat{\boldsymbol{\varphi}} + \frac{dz}{dt}\hat{\boldsymbol{z}} \\ &= \dot{\varrho}\hat{\boldsymbol{\varrho}} + \varrho\dot{\varphi}\hat{\boldsymbol{\varphi}} + \dot{z}\hat{\boldsymbol{z}}\end{aligned}$$

It is also possible to find the velocity vector in cylindrical coordinates

$$\begin{aligned}\boldsymbol{a} &= \frac{d\boldsymbol{v}}{dt} \\ &= \frac{d}{dt}(\dot{\varrho}\hat{\boldsymbol{\varrho}} + \varrho\dot{\varphi}\hat{\boldsymbol{\varphi}} + \dot{z}\hat{\boldsymbol{z}}) \\ &= \ddot{\varrho}\hat{\boldsymbol{\varrho}} + \dot{\varrho}\dot{\hat{\boldsymbol{\varrho}}} + \dot{\varrho}\dot{\varphi}\hat{\boldsymbol{\varphi}} + \varrho\ddot{\varphi}\hat{\boldsymbol{\varphi}} + \varrho\dot{\varphi}\dot{\hat{\boldsymbol{\varphi}}} + \ddot{z}\hat{\boldsymbol{z}} \\ \dot{\hat{\boldsymbol{\varphi}}} &= \frac{d}{dt}(-\sin\varphi\hat{\boldsymbol{x}} + \cos\varphi\hat{\boldsymbol{y}}) \\ &= -\dot{\varphi}\cos\varphi\hat{\boldsymbol{x}} - \dot{\varphi}\sin\varphi\hat{\boldsymbol{y}} \\ &= -\dot{\varphi}\hat{\boldsymbol{\varrho}} \\ \boldsymbol{a} &= (\ddot{\varrho} - \varrho\dot{\varphi}^2)\hat{\boldsymbol{\varrho}} + (\varrho\ddot{\varphi} + 2\dot{\varrho}\dot{\varphi})\hat{\boldsymbol{\varphi}} + \ddot{z}\hat{\boldsymbol{z}}\end{aligned}$$

Circular motion

Consider circular motion on the x, y plane with constant angular velocity, for a particle travelling on a circular path radius R. This means that $z = \text{constant} \implies \dot{z} = \ddot{z} = 0$, $\varrho = R = \text{constant} \implies \dot{\varrho} = \ddot{\varrho} = 0$ and $\omega = \dot{\varphi} = \text{constant} \implies \ddot{\varphi} = \alpha = 0$.

$$\boldsymbol{v} = \varrho\dot{\varphi}\hat{\boldsymbol{\varphi}} = R\omega\hat{\boldsymbol{\varphi}}$$

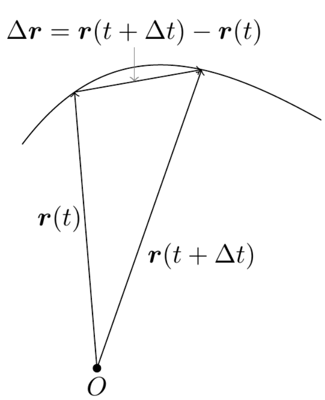
This is tangental to the path and has constant magnitude as expected.

$$\boldsymbol{a} = -\varrho\dot{\varphi}^2\hat{\boldsymbol{\varrho}} = -R\omega^2\hat{\boldsymbol{\varrho}}$$

This is towards the center as expected.

Note that while using cylindrical and spherical polar coordinates cylindrical polar coordinates are (ϱ, φ, z). When not using spherical polar coordinates it is common to use (r, ϑ, z) instead for cylindrical polar coordinates.

19 Normal and Tangental Coordinates



This shows a path s in a plane. $\boldsymbol{r}(t)$ is the position vectorof a particle travelling on the path at time t. The velocity $\boldsymbol{v}$ is given by

$$\boldsymbol{v} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)}{\Delta t} = \frac{d\boldsymbol{r}}{dt}$$

A small element of the path $d\boldsymbol{r}$ is given by

$$d\boldsymbol{r} = ds\hat{\boldsymbol{e}}_{\boldsymbol{t}}$$

Where ds is a small segment of the path and $\hat{\boldsymbol{e}}_{\boldsymbol{t}}$ is a unit vector tangental to the path. Sometimes $\hat{\boldsymbol{e}}_{\boldsymbol{t}}$ is denoted $\hat{\boldsymbol{t}}$. This gives us the relation

$$\hat{\boldsymbol{e}}_{\boldsymbol{t}} = \frac{d\boldsymbol{r}}{ds}$$

$$\implies \boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{ds}\frac{ds}{dt} = v\hat{\boldsymbol{e}}_{\boldsymbol{t}}$$

$$\implies \boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d}{dt}(v\hat{\boldsymbol{e}}_{\boldsymbol{t}}) = \frac{dv}{dt}\hat{\boldsymbol{e}}_{\boldsymbol{t}} + \frac{d\hat{\boldsymbol{e}}_{\boldsymbol{t}}}{dt}v$$

$\dfrac{dv}{dt} = a_t$ is the magnitude of the tangental component of the acceleration vector and describes the rate of change of speed.

$$\hat{\boldsymbol{e}}_{\boldsymbol{t}} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{t}} = 1$$

$$\frac{d}{dt}(\hat{\boldsymbol{e}}_{\boldsymbol{t}} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{t}} = \frac{d}{dt}1 = 0$$

Applying the product rule gives

$$2\hat{\boldsymbol{e}}_{\boldsymbol{t}} \cdot \frac{d\hat{\boldsymbol{e}}_{\boldsymbol{t}}}{dt} = 0$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{t}} \cdot \frac{d\hat{\boldsymbol{e}}_{\boldsymbol{t}}}{dt} = 0$$

This means that $\hat{\boldsymbol{e}}_{\boldsymbol{t}}$ is normal to $\frac{d\hat{\boldsymbol{e}}_{\boldsymbol{t}}}{dt}$.

$$\frac{d\hat{\boldsymbol{e}}_{\boldsymbol{t}}}{dt} = \frac{d\hat{\boldsymbol{e}}_{\boldsymbol{t}}}{dds}\frac{ds}{dt} = v\frac{d\hat{\boldsymbol{e}}_{\boldsymbol{t}}}{ds}$$

Let $\hat{\boldsymbol{e}}_{\boldsymbol{n}} = \hat{\boldsymbol{n}}$ be a unit vector normal to the path

$$\hat{\boldsymbol{e}}_{\boldsymbol{n}} = \frac{\frac{d\hat{\boldsymbol{e}}_{\boldsymbol{t}}}{ds}}{\left|\frac{d\hat{\boldsymbol{e}}_{\boldsymbol{t}}}{ds}\right|}$$

$$\left|\frac{d\hat{\boldsymbol{e}}_{\boldsymbol{t}}}{ds}\right| = \kappa$$

κ is the curvature of the path

$$\frac{d\hat{\boldsymbol{e}}_{\boldsymbol{t}}}{ds} = \kappa\hat{\boldsymbol{e}}_{\boldsymbol{n}}$$

For a straight line $\kappa = 0$ Let ϱ be the instantaneous radius of curvature.

$$\varrho = \frac{1}{\kappa}$$

For a straight line $\varrho \to \infty$

$$\hat{\boldsymbol{e}}_{\boldsymbol{n}} = \varrho\frac{d\hat{\boldsymbol{e}}_{\boldsymbol{t}}}{ds}$$

$$\boldsymbol{a} = \frac{dv}{dt}\hat{\boldsymbol{e}}_{\boldsymbol{t}} + \frac{v^2}{\varrho}\hat{\boldsymbol{e}}_{\boldsymbol{n}}$$

The normal term describes the rate of change of direction of $\boldsymbol{v}$.

By convention $\hat{\boldsymbol{e}}_{\boldsymbol{n}}$ points in towards the center of the curve.

For a small arc of the cuve with arc length ds, angle ϑ and radius R we get

$$ds = Rd\vartheta$$

$$s = R\vartheta$$

$$s = \varrho\vartheta$$

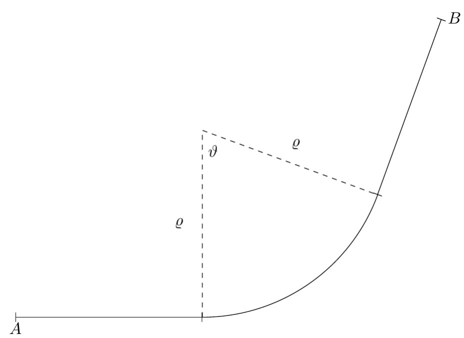
$$\implies v = \frac{ds}{dt} = \frac{d}{dt}(\varrho\vartheta) = \varrho\frac{d\vartheta}{dt} = \varrho\omega$$

This result comes from the product rule and the fact that $\dot{\varrho} = 0$. This is the result that we would expect for circular motion which is approximated by a curved path.

$$a_n = \frac{v^2}{\varrho} = \frac{1}{\varrho}(\varrho\dot{\vartheta})^2 = \varrho\omega^2$$

Which is the centripetal acceleration from circular motion, which is what we would expect since centripetal acceleration is tangental to the circle and inwards.

Example 19.1



This path is made of two straight sections of length D and one arc angle ϑ and radius ϱ. A car travels along this path. At A it has speed v_A and at B it has speed v_B. Its speed increases at a constant rate. Find $|\boldsymbol{a}(t)|$.

1. Speed increases at a constant rate so $a_t = \frac{dv}{dt}$ is a constant.

   Total path length $s = 2D + \varrho\vartheta$

$$\boldsymbol{a} = \frac{dv}{dt}\hat{\boldsymbol{e}}_{\boldsymbol{t}} + a_n\hat{\boldsymbol{e}}_{\boldsymbol{n}}$$

$$\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds}$$

$$\implies \int_A^B a_t\,ds = \int_{v_A}^{v_B} v\,dv = \left[\frac{v^2}{2}\right]_{v_A}^{v_B}$$

$$a_t = \frac{v_B^2 - v_A^2}{2(2D + \varrho\vartheta)}$$

2. Find $s(t)$

$$a_t = \frac{dv}{dt} \implies v(t) = v_A + a_t t$$

   Let $t = 0$ at A

$$v = \frac{ds}{dt} \implies \int_0^t v_A + a_t t\,dt = \int_{s_A}^{s_B} ds$$

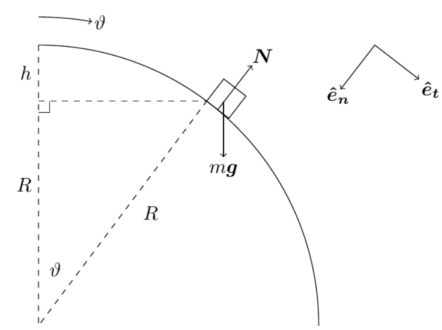
$$s(t) = v_A t + \frac{1}{2}a_t\frac{t^2}{2}$$

3. Let the car be at the end of the first straight at t_1 and at the end of the curve at t$_2$. $s(t_1) < D \implies |\boldsymbol{a}| = a_t$. $D \le s(t_2) < D + \varrho\vartheta \implies \boldsymbol{a} = a_t\hat{\boldsymbol{e}}_{\boldsymbol{t}} + \frac{v^2}{\varrho}\hat{\boldsymbol{e}}_{\boldsymbol{n}}$ where $v(t_1) = v_A + a_t t$.

$$|\boldsymbol{a}| = \sqrt{a_t^2 + a_n^2}$$

$$D + \varrho\vartheta < s(t_1) < s\vartheta \implies \boldsymbol{a} = a_t\hat{\boldsymbol{e}}_{\boldsymbol{t}}$$

20 Ant on a Pipe



An ant of mass m sits on top of a wet cylindrical pipe radius R. Suddenly the ant starts sliding down the pipe. Find the speed of the ant and the point at which it loses contact with the pipe.

The pipe is wet so we can ignore friction.

$$\begin{aligned}\text{NII: } m\boldsymbol{a} &= m\boldsymbol{g} + \boldsymbol{N}\\ \boldsymbol{N} &= -N\hat{\boldsymbol{e}}_{\boldsymbol{n}}\\ m\boldsymbol{g} &= mg(\sin\vartheta\hat{\boldsymbol{e}}_{\boldsymbol{t}} + \cos\vartheta\hat{\boldsymbol{e}}_{\boldsymbol{n}})\\ \boldsymbol{a} &= \frac{dv}{dt}\hat{\boldsymbol{e}}_{\boldsymbol{t}} + \frac{v^2}{R}\hat{\boldsymbol{e}}_{\boldsymbol{n}}\end{aligned}$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{t}}: \quad m\frac{dv}{dt} = mg\sin\vartheta \tag{19.1}$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{n}}: \quad m\frac{v^2}{R} = mg\cos\vartheta - N \tag{19.2}$$

Find $v(\vartheta)$

$$\begin{aligned}(19.2) \implies \frac{dv}{dt} &= g\sin\vartheta\\ v &= \frac{ds}{dt}, \quad s = R\vartheta\\ v &= \frac{ds}{dt} = R\frac{d\vartheta}{dt} = R\dot{\vartheta}\\ v^2 &= R^2\dot{\vartheta}^2\\ \frac{dv}{dt} &= \frac{d}{dt}(R\dot{\vartheta}) = R\frac{d\dot{\vartheta}}{dt} = R\frac{d\dot{\vartheta}}{d\vartheta}\frac{d\vartheta}{dt}\\ \frac{dv}{dt} &= R\dot{\vartheta}\frac{d\dot{\vartheta}}{dt} = g\sin\vartheta\\ \dot{\vartheta}\frac{d\dot{\vartheta}}{dt} &= \frac{g}{R}\sin\vartheta\\ \int_{\vartheta(0)}^{\vartheta(t)}\dot{\vartheta}\,d\dot{\vartheta} &= \frac{g}{R}\int_{\vartheta(0)}^{\vartheta(t)}\sin\vartheta\,d\vartheta\end{aligned}$$

Where $\vartheta(t) < \vartheta_c$, which is the point at which the ant loses contact with the cylinder, and $\vartheta(0) = 0$

$$\begin{aligned}
\frac{\dot{\vartheta}^2}{2} &= \frac{g}{R}[-\cos\vartheta]_0^\vartheta = \frac{g}{R}(-\cos\vartheta) \\
\implies v^2 &= 2gR(1-\cos\vartheta) \\
\cos\vartheta &= \frac{R-H}{R} = 1 - \frac{h}{R} \\
\implies v^2 &= 2gR\left(1 - 1 + \frac{h}{R}\right) = 2gh \\
K &= \frac{mv^2}{2} = U = mgh \\
\implies v^2 &= 2gh
\end{aligned}$$

Energy and forces agree, this is a good sign.

Consider motion in a straight line along the x axis. This means $\boldsymbol{F} = m\boldsymbol{a}$, $\boldsymbol{v} = v\hat{\boldsymbol{x}}$ and $\boldsymbol{a} = \frac{dv}{dt}\hat{\boldsymbol{x}}$. Let $\boldsymbol{F}$ be a force that is independant of t and $\boldsymbol{v}$.

$$\begin{aligned}
m\frac{dv}{dt} &= F(x) \\
m\frac{dv}{dx}\frac{dx}{dt} &= F(x) \\
m\frac{dv}{dx}v &= F(x) \\
m\int_{v_1}^{v_2} v\,dv &= \int_{x_1}^{x_2} F(x)\,dx \\
m\left[\frac{v^2}{2}\right]_{v_1}^{v_2} &= \int_{x_1}^{x_2} F(x)\,dx \\
\frac{mv_2^2}{2} - \frac{mv_1^2}{2} &= \int_{x_1}^{x_2} F(x)\,dx \\
\Delta K &= W
\end{aligned}$$

For more general motion

$$\underbrace{\frac{mv_2^2}{2}}_{=T_2} - \underbrace{\frac{mv_1^2}{2}}_{=T_1} = \underbrace{\int \boldsymbol{F}\cdot d\boldsymbol{r}}_{=W}$$

Define $U(x)$ such that $F(x) = -\frac{dU}{dx}$

$$\begin{aligned}
\int_{x_1}^{x_2} F(x)\,dx &= \int_{x_1}^{x_2} -\frac{dU}{dx}\,dx = -U_2 + U_1 \\
T_2 - T_1 &= U_1 - U_2 \\
U_1 + T_1 &= U_2 + T_2 = E
\end{aligned}$$

E is the total mechanical energy. All forces are conservative so energy at the start is equal to energy at the end.

Alternate solution using energy instead of forces, apply the energy principal of word and kinetic energy

$$\frac{mv_2^2}{2} - \underbrace{\frac{mv_1^2}{2}}_{=0} = \int \boldsymbol{F}\cdot d\boldsymbol{r}$$

$$\begin{aligned}
\frac{mv^2}{2} &= \int \boldsymbol{F}\cdot d\boldsymbol{r}\\
\boldsymbol{F} &= (mg\cos\vartheta - N)\hat{\boldsymbol{e}}_{\boldsymbol{n}} + mg\sin\vartheta\hat{\boldsymbol{e}}_{\boldsymbol{t}}\\
d\boldsymbol{r} &= ds\hat{\boldsymbol{e}}_{\boldsymbol{t}}\\
\boldsymbol{F}\cdot d\boldsymbol{r} &= mg\sin\vartheta ds\\
&= mgR\sin\vartheta d\vartheta\\
\frac{mv^2}{2} &= \int_{\vartheta_1}^{\vartheta_2}\sin\vartheta\, d\vartheta
\end{aligned}$$

The same result follows that $v^2 = 2gh$

Find the critical angle ϑ_c where the ant loses contact with the cylinder.

$$\begin{aligned}
N(\vartheta_c) &= 0\\
(19.1) \implies N(\vartheta_c) &= mg\cos\vartheta - \frac{mv^2}{R} = 0\\
g\cos\vartheta_c &= \frac{v^2}{R} = 2g(1-\cos\vartheta_c)\\
\cos\vartheta_c &= 2(1-\cos\vartheta_c)\\
\cos\vartheta_c &= \frac{2}{3}\\
\vartheta_c &= \arccos\frac{2}{3}
\end{aligned}$$

Part VII

Second order ODE

21 Simple Harmonic Motion

Consider a mass m attached to a spring k, unextended length L where $m, k, L \in \mathbb{R}$ and $m, k, L > 0$. If the mass is allowed to drop slowly it will reach an equilibrium point x_0 where the spring holds it still.

$$\text{NII}: \quad m\boldsymbol{a} = m\boldsymbol{g} + \boldsymbol{F}_k$$

If we define x direction as vertical downwards then decomposing the vectors gives us

$$\begin{aligned}
m\boldsymbol{g} &= mg\hat{\boldsymbol{x}}\\
\boldsymbol{F}_k &= -F_k\hat{\boldsymbol{x}}\\
\boldsymbol{a} &= \ddot{x}\hat{\boldsymbol{x}}
\end{aligned}$$

At x_0 $\boldsymbol{a} = \boldsymbol{0}$ since it is at equilibrium.

$$\begin{aligned}
0 &= mg - kx_0\\
\frac{mg}{k} &= x_0
\end{aligned}$$

If we consider small oscillations about x_0 then x ranges across $x_0 \pm \Delta x$ where $\Delta x \ll 1$. Since x_0 is a constant its time derivative is 0.

$$\begin{aligned}
\ddot{x} &= \ddot{\Delta x} \\
m\ddot{\Delta x} &= mg - k(x_0 + \Delta x) \\
&= mg - \frac{mg}{k}kx_0 - k\Delta x \\
\ddot{\Delta x} &= -\frac{k}{m}\Delta x
\end{aligned}$$

If we choose $x_0 = 0$ since $x = \Delta x - x_0$ we get $x = \Delta x \implies \ddot{x} = -\frac{k}{m}x$ where $x \ll 1$.

Let $\frac{k}{m} = \omega^2 \implies \ddot{x} = -\omega^2 x$.

This gives a linear, second order ODE which is solved by $x(t)$ where $\ddot{x}(t) = -\omega^2 x(t)$. For $A, B \in \mathbb{C}$ one solution is

$$\begin{aligned}
x(t) &= A\sin\omega t + B\cos\omega t \\
\dot{x}(t) &= \omega A\cos\omega t - \omega B\sin\omega t \\
\ddot{x}(t) &= -\omega^2 A\sin\omega t - \omega^2 B\cos\omega t \\
&= -\omega^2(A\sin\omega t + B\cos\omega t) \\
&= -\omega^2 x(t)
\end{aligned}$$

The units of ω are s^{-1} so we interpret it as angular frequency.

The period of oscillation T is given by $\omega T = 2\pi \implies T = 2\pi\sqrt{\frac{m}{k}}$

Let $x(t) = A\sin\omega t$. $\boldsymbol{F}(x) = -kx\hat{\boldsymbol{x}}$, $d\boldsymbol{r} = dx\hat{\boldsymbol{x}}$

$$U(x) = -\int \boldsymbol{F}\cdot d\boldsymbol{r} = -\int -kx\hat{\boldsymbol{x}}\cdot dx\hat{\boldsymbol{x}} = k\int x\,dx = \frac{kx^2}{2} + u_0$$

u_0 is a constant of integration. Usually boundary conditions are chosen such that $u_0 = 0$. Plotting the function of U against x we get a parabola, the minimum of the parabola is at x_0 which is 0 if $u_0 = 0$. Using this we can say that any system will oscillate if there is a linear restoring force $F = -kx$ where $k > 0$ and teh potential takes the form $\frac{kx^2}{2} + u_0$. All other results follow from this.

If we consider a system with potential $U(x)$ if $U(x_0)$ is a minimum then the system will oscillate around this point for sufficently small Δx. It is possible to approximate $U(x)$ about $x = x_0$ using a taylor series:

$$U(x) = U(x_0) + U'(x_0)(x - x_0) + \frac{U''(x_0)}{2}(x - x_0)^2 + \mathcal{O}(x^3)$$

It is possible to choose what point to call $U = 0$ so that $U(x_0) = 0$. Since we are expanding about a stationary point we know $U(x_0) = 0$. This leaves us with

$$U(x) \approx \frac{U''(x_0)}{2}x^2$$

Since it is a minimum we know that $U''(x_0) > 0$. Let $k = U''(x_0) \implies U(x) \approx \frac{kx^2}{2} \implies m\ddot{x} = -kx \implies x = A\sin\omega t + B\cos\omega t$. Where $\omega^2 = \frac{U''(x_0)}{m}$ and $T = 2\pi\sqrt{\frac{m}{U''(x_0)}}$

A minimum is a stable position of equilibrium as after a small displacement Δx it will return to the minimum. At a maximum the above expansion still holds but now $U''(x_0) < 0 \implies k < 0 \implies m\ddot{x} = kx$. There are no oscillations.

22 Damped Motion

In this course we will consider only second order, linear, homogeneous ODE with constant coefficents, such as

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + ax = 0 \tag{21.1}$$

where a and b are constants and $x = x(t)$. This is linear in x and all its derivatives. If $b = 0$ then this is the same as simple harmonic motion for $a \geq 0$ and exponential for $a < 0$.

The solution $x(t)$ is a function. This means that it must be (for this course) a combination of exponetiation, logarithms, sine/cosine and polynomials. Exponentials, logarithms and sine/cosine can be expressed as $e^{\lambda t}$ for $\lambda \in \mathbb{C}$. If $\lambda \in \mathbb{C}\backslash\mathbb{R}$ then it is a combination of sine and cosine. If $\lambda \in \mathbb{R}$ then it is a combination of exponentials and logarithms.

We make the anzatz that $x(t) = Ce^{\lambda t}$ where C and λ are constants.

$$\dot{x}(t) = C\lambda e^{\lambda t} = \lambda x(t)$$
$$\ddot{x}(t) = C\lambda^2 e^{\lambda t} = \lambda^2 x(t)$$

If we substitute this into (21.1) we get

$$x(t)(\lambda^2 + 2b\lambda + a) = 0$$

$x(t) = 0$ is a trivial solution. If $x(t) \neq 0$ then we get the non trivial solutions given by

$$\lambda^2 + 2b\lambda + a = 0$$
$$\implies \lambda = -b \pm \sqrt{b^2 - a}$$
$$\implies \begin{cases} \lambda_1 = -b + \sqrt{b^2 - a} \\ \lambda_2 = -b - \sqrt{b^2 - a} \end{cases}$$

If $b^2 \neq a$ then

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

where C_1, C_2 are constants and λ_1, λ_2 are as above.

If $b^2 < a$ then $\sqrt{b^2 - a} = \sqrt{-(a - b^2)} = i\sqrt{a - b^2} = i\omega$ where $\omega = \sqrt{a - b^2} \in \mathbb{R}$. This means that the solutions are

$$x(t) = e^{-bt}[C_1 e^{i\omega t} + C_2 e^{-i\omega t}] = e^{-bt}[A\sin\omega t + B\cos\omega t]$$

where A, B are constants. Oscillations occur here but they are damped by the e^{-bt} term which causes the amplitude to decay to 0. This is called underdamped motion.

If $b^2 > a$ then $\sqrt{b^2 - a} = \gamma \in \mathbb{R}$. This means the solutions are

$$x(t) = e^{-bt}[C_1 e^{\gamma t} + C_2 e^{-\gamma t}] = e^{-bt}[A\sinh\gamma t + B\cosh\gamma t]$$

There are no oscillations, instead x just tends towards 0. This is called overdamped motion.

If $b^2 = a$ then $x(t) = Ce^{-bt}$. There is only one solution. We have to modify our anzatz to contain a polynomial $p(t)$ to get all solutions. The modified anzatz is

$$x(t) = p(t)e^{-bt}$$

$$\dot{x}(t) = \dot{p}e^{-bt} + p(-b)e^{-bt} = e^{-bt}[\dot{p} - bp]$$
$$\ddot{x}(t) = -be^{-bt}(\dot{p} - bp) + e^{-bt}(\ddot{p} - b\dot{p}) = e^{-bt}[\ddot{p} - 2b\dot{p} + b^2 p]$$

Substituting this into (21.1) gives

$$e^{-bt}[(\ddot{p} - 2b\dot{p} + b^2p) + 2b(\dot{p} - bp) + ap] = 0$$
$$e^{-bt}[\ddot{p} + p\underbrace{(a - b^2)}_{=0}] = 0$$
$$\underbrace{e^{-bt}}_{>0}\ddot{p} = 0$$
$$\implies \ddot{p} = 0$$

This can only happen if all terms in p of order 3 or higher are 0.

$$\frac{d^2p}{dt^2} = 0$$
$$\int \frac{d^2p}{dt^2}\,dt = \int 0\,dt$$
$$\frac{dp}{dt} = 0t + C_2$$
$$= C_2$$
$$\int \frac{dp}{dt}\,dt = \int C_2\,dt$$
$$p = C_2t + C_1$$

So an anzatz of $x(t) = (C_1 + C_2t)e^{\lambda t}$ will give us the solutions

$$x(t) = C_1e^{-bt} + C_2te^{-bt}$$

This gives no oscillations and decays quickly to 0. This is called criticaly damped motion.