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Theoretical Physics

Gauge Theories in Particle Physics

January 16, 2023

COURSE NOTES

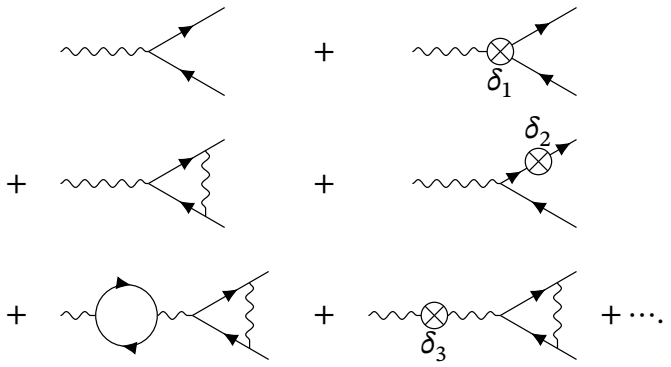
Gauge Theories in Particle Physics

Willoughby Seago

January 16, 2023

These are my notes from the course gauge theories in particle physics. I took this course as a part of the theoretical physics degree at the University of Edinburgh.

These notes were last updated at 13:23 on April 13, 2023. For notes on other topics see https://github.com/WilloughbySeago/Uni-Notes.



Chapters

Page

Contents

Page

List of Figures

Page

One

Introduction

1.1 Other Relevant Courses

This course follows on directly from the *Quantum Field Theory* course, so much of the relevant background is contained in those notes. A lot of the notation is also taken from this course. It is also expected that students on this course will have taken *Symmetries of Particles and Fields*, and the prerequisite for that course, *Symmetries of Quantum Mechanics*, so for any group theory related topics see the notes of one of these courses. Other relevant courses will be flagged throughout the notes.

1.2 Conventions

- We will mostly work in natural units, where $c = \hbar = 1$.
- We use the mostly-minuses metric, $(+---)$, or $(+-\cdots-)$ in D spacetime dimensions.
- We will use the Einstein summation convention where repeated indices are summed over. In $d+1$ dimensions Greek letters, $\mu, \nu, \rho, \ldots$, run from 0 to d and Latin letters, $i, j, k, \ldots$, run from 1 to d.
- We use the Fourier transform

$$\tilde{f}(p) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{ipx} f(x), \qquad \text{and} \qquad f(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \mathrm{e}^{-ipx} \tilde{f}(p). \tag{1.2.1}$$

- Feynman diagrams are drawn with time increasing to the right.
- The electric charge, e, is taken to be positive, so an electron has charge $-e$.

1.3 Dirac Algebra

Recall from *Quantum Field Theory* that the **gamma matrices**, γ^μ, are defined such that

$$\{\gamma^\mu, \gamma^\nu\} \coloneqq 2\eta^{\mu\nu} \tag{1.3.1}$$

where $\{A, B\} \coloneqq AB + BA$ is the **anticommutator**. This means that

$$(\gamma^0)^2 = 1, \qquad \text{and} \qquad (\gamma^i)^2 = -1. \tag{1.3.2}$$

We can also impose that

$$(\gamma^0)^\dagger = \gamma^0, \qquad \text{and} \qquad (\gamma^i)^\dagger = \gamma^0\gamma^i\gamma^0 = -\gamma^i, \tag{1.3.3}$$

but this comes with a choice of normalisation.

The **Dirac adjoint** of a spinor, ψ, is defined to be $\bar{\psi} \coloneqq \psi^\dagger\gamma^0$.

We also define

$$\gamma^5 \coloneqq i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{1.3.4}$$

which is such that

$$\{\gamma^5, \gamma^\mu\} = 0, \qquad (\gamma^5)^\dagger = \gamma^5, \qquad \text{and} \qquad (\gamma^5)^2 = 1. \tag{1.3.5}$$

When contracting a gamma matrix and a four vector we use **slash notation**, writing $\not{a} \coloneqq a_\mu\gamma^\mu$.

The following identities hold in $D = 4$ dimensions:

$$\gamma^\mu\gamma_\mu = 4, \tag{1.3.6}$$

$$\gamma^\mu\not{a}\gamma_\mu = -2\not{a}, \qquad \gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu, \tag{1.3.7}$$

$$\gamma^\mu\not{a}\not{b}\gamma_\mu = 4a\cdot b, \qquad \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4\eta^{\nu\rho}, \tag{1.3.8}$$

$$\gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a}, \qquad \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu. \tag{1.3.9}$$

The following identities hold in D dimensions:

$$\gamma^\mu\gamma_\mu = D, \tag{1.3.10}$$

$$\gamma^\mu\not{a}\gamma_\mu = (2-D)\not{a}, \qquad \gamma^\mu\gamma^\nu\gamma_\mu = (2-D)\gamma^\nu. \tag{1.3.11}$$

The following identities hold for traces in $D = 4$ dimensions:

$$\text{tr}(\text{odd number of gamma matrices}) = 0, \tag{1.3.12}$$

$$\text{tr}(\text{odd number of gamma matrices times } \gamma^5) = 0, \tag{1.3.13}$$

$$\text{tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}, \tag{1.3.14}$$

$$\text{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\sigma}), \tag{1.3.15}$$

$$\text{tr}(\gamma^5) = \text{tr}(\gamma^\mu\gamma^\nu\gamma^5) = 0, \tag{1.3.16}$$

$$\text{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\varepsilon^{\mu\nu\rho\sigma}. \tag{1.3.17}$$

Remember that the trace is cyclic, so terms may appear in different orders.

Part I

Quantum Electrodynamics

Two

Classical Electrodynamics

Quantum electrodynamics (QED) is the quantum field theory (QFT) of electromagnetism. It supersedes classical electrodynamics (CED) as a theory for predicting what happens to electric charges, as well as electromagnetic radiation, light. We'll start this section by discussing CED and some of its short comings which necessitate the development of QED. For more details on CED see the *Classical Electrodynamics* course.

2.1 Classical Action

Classical electrodynamics follows from the action

$$S = \int \mathrm{d}^4x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - J_\mu A^\mu \right] \tag{2.1.1}$$

where A_μ is the electromagnetic field (also called the electromagnetic potential), J_μ is the current, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength (confusingly also called the electromagnetic field in some contexts).

The first term in the action tells us how the electromagnetic field evolves and the second term encodes interactions. One common case is a current made of particles of charge q_i. The current due to the ith particle is $J^\mu = q_i u_i^\mu = q_i \mathrm{d}x_i^\mu / \mathrm{d}\tau_i$ (no sum on i). In this case we can write the interaction term as a sum over particles and the action from each particle is given by an integral over the particle's world line. Thus the action can be written as

$$S = \int \mathrm{d}^4x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right] - \sum_i \int \mathrm{d}\tau_i \, q_i A_\mu(x_i(\tau_i)) \frac{\mathrm{d}x_i^\mu(\tau_i)}{\mathrm{d}\tau_i} + \cdots . \tag{2.1.2}$$

The "$\cdots$" here accounts for other effects not already covered, such as spin corrections. These terms are suppressed by factors of $1/m_i$ or more, m_i being the mass of the ith particle.

2.2 Problems with CED

The theory of point-like electric charges developed as above is not entirely satisfactory. Recall that the Lorentz force on a particle with charge q_i and four-velocity u_i^μ in an electromagnetic field with field strength tensor $F^{\mu\nu}$ is

$$\frac{\mathrm{d}p_i^\mu}{\mathrm{d}\tau} = q_i F^{\mu\nu}(x_i(\tau)) u_{i\nu}(x_i(\tau)) \tag{2.2.1}$$

where $p_i = m_i u_i = m_i \mathrm{d}x_i/\mathrm{d}\tau$ is the four-momentum of the particle, with m_i being the mass of the ith particle. This law follows from conservation of energy, so it's pretty fundamental.

Now consider the Coulomb field of particle 1 in particle 1's rest frame. At a point x a distance r from particle 1, which has position x_1, the Coulomb field is

$$A^0(x) = \frac{q_i}{4\pi r} = \frac{q_i}{4\pi} \frac{1}{\sqrt{(x^1 - x_1^1)^2 + (x^2 - x_1^2)^2 + (x^3 - x_1^3)^2}}. \tag{2.2.2}$$

If we can write this in a covariant way then it will apply in all frames. We can do this using $u_1 = (1, \mathbf{0})$, which is the four-velocity of the particle in it's rest frame in units where $c = 1$. We then have

$$[u_1 \cdot (x - x_1)]^2 - (x - x_1)^2 = (x^0 - x_1^0)^2 - (x^0 - x_1^0)^2 + (x^i - x_1^i)^2. \tag{2.2.3}$$

Using u_1^μ to get a μ index we have

$$A^\mu(x) = \frac{q_i}{4\pi} \frac{u_1^\mu}{\sqrt{[u_1 \cdot (x - x_1)]^2 - (x - x_1)^2}}. \tag{2.2.4}$$

This is covariant and so holds in any inertial frame.

Now suppose that particles 1 and 2 interact. The force on particle one is

$$\frac{\mathrm{d}p_1^\mu}{\mathrm{d}\tau} = q_1 F^{\mu\nu}(x_1(\tau)) u_{1\nu}(\tau). \tag{2.2.5}$$

The $F^{\mu\nu}$ appearing here is the total electromagnetic field strength. This proves to be a problem because part of $F^{\mu\nu}$ is the Coulomb field of particle 1, A^μ. We can see from the expression above that this is singular when $x = x_1$, which is exactly the case when we try to compute the force above. So we get a divergent result which we have to somehow make sense of.

The standard solution in CED to avoid this problem is to just use the electromagnetic field strength due to particle 2, and ignore the field from particle 1. Then, so long as the two particles can't have the same position, $F^{\mu\nu}$ is nonsingular and we avoid the problem with infinity. This workaround works well at low energies (small velocities).

The problem is that this workaround is not consistent with conservation of energy. For example, consider a classical atom, formed from a nucleus of charge Ze, which we take to have infinite mass, and an electron of charge $-e$. The Lorentz force applied to the electron, calculated using the electric field only from the nucleus, is consistent with stable circular orbits. The problem is that as the electron is orbiting it is changing direction, and so accelerating, and therefore must radiate. There is only one possible source for this energy, the potential, and so the orbit must decay.

Since the orbit decays there must be some force we have not accounted for causing this decay. It is possible to account for this force only if we include the electron's own field in the calculation of the force upon the electron. This means we have to account for interactions of particles with their own fields. The correction that we get when doing so is suppressed by inverse powers of c, which is why removing the electron's electromagnetic field works well enough for many purposes at low energies.

Once we accept that particles interact with their own fields there is another problem. The Coulomb field of a point-like particle contains an infinite electrostatic energy. The energy density of an electromagnetic field is $(E^2 + B^2)/2$, meaning that the energy contained in the Coulomb field outside of a spherical region of radius r_{min} centred on the electron is given by

$$\int \mathrm{d}^3x \, \frac{1}{2}E^2 = \frac{1}{2}\frac{e^2}{(4\pi)^2}\int_{S^2}\mathrm{d}\Omega \int_{r_{\text{min}}}^{\infty}\mathrm{d}r\, r^2 \frac{1}{r^4} = \frac{1}{8\pi}\frac{1}{r_{\text{min}}}. \tag{2.2.6}$$

Taking $r_{\text{min}} \to \infty$ the energy diverges.

The classical physics solution to this is to say that the electron isn't point-like. Then if we choose r_{min} to be the size of the electron so long as

$$mc^2 \gtrsim \frac{1}{2}\frac{e^2}{4\pi r_{\text{min}}}, \tag{2.2.7}$$

so the energy of the field is less than the total energy available from the electron, things are fine. Setting $c = 1$ again gives

$$r_{\text{min}} \gtrsim \frac{1}{2}\frac{e^2}{4\pi}\frac{1}{m} = \alpha \ell_{\text{Compton}} \tag{2.2.8}$$

where $\alpha = e^2/(4\pi)$ is the fine structure constant and ℓ_{Compton} is the Compton wavelength, which is the wavelength of a photon with the same energy as the rest mass of the particle. What this tells us is that scales at which quantum mechanical effects become important become important before the size of the electron becomes a problem classically, so if we're only interested in classical computations we don't need to worry about treating the electron as point-like.

However, we know from experiments that the electron is point-like at least up to the 1 TeV scale, which is about one millionth of the limit above. These point-like particles lead to divergences in classical theories, and also in quantum theories, like the ones above. Interestingly the divergence is actually not as bad in the quantum theory, being log divergent instead of going as $1/r_{\text{min}}$. In the quantum theory we can deal with these infinities by absorbing them into a finite number of measured parameters, such as e and m, and we get a very powerful and predictive theory, QED.

Three

QFT Recap

It's part of my job to give you problems

Donal O'Connell

3.1 QED Lagrangian

In QFT, in particular in QED, we replace the classical action with the **QED action**:

$$S = \int \mathrm{d}^D x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \right]. \tag{3.1.1}$$

Here $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ as before. The change comes in the interaction term, where we now have the spinor field ψ, which is acted on by the covariant derivative, which in QED is given by $D_\mu := \partial_\mu - ieA_\mu$. Recall that $\not{a} := \gamma^\mu a_\mu$ where γ^μ are the Dirac gamma matrices. Note that we leave the dimension as a variable, in preparation for dimensional regularisation later.

Notice that there are no world lines appearing in the action now. This reflects the fact that we've replaced the particles with definite position with fields, which aren't localised in the same way. One of the biggest changes upon moving to a *quantum* field theory is that we have a the new phenomenon of pair production. This creates new world lines, which is part of the reason we have to move away from actions involving sums over world lines.

We use the Dirac Lagrangian in QED since we are mostly interested in electrons, which are spin 1/2 particles. If instead we have a spin 0 particle with complex scalar field Φ then we can use the **scalar QED** Lagrangian

$$S = \int \mathrm{d}^D x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + (D_\mu \Phi)^\dagger (D^\mu \Phi) - m^2 \Phi^\dagger \Phi - V(\Phi) \right] \tag{3.1.2}$$

where V is some potential.

If we neglect spin corrections in QED, often a valid thing to do since the spin is on the order of $\hbar$, then we get similar results in both normal and scalar QED, and these results are similar to those we get in classical electrodynamics. In this way scalar QED is more similar to classical electrodynamics, and we will make use of scalar QED as an example while focusing on normal QED for applications.

In some ways normal QED is actually simpler than scalar QED, there is no potential in normal QED and in normal QED the largest number of fields comes from the $-ie\bar{\psi}\not{A}\psi$ term, with three fields, whereas scalar QED has a four field term, $-\bar{\psi}A_\mu^\dagger A_\mu\psi$.

The main objects of physical interest in QFT are scattering amplitudes. We will make use of both canonical quantisation and path integral methods to compute these and other quantities.

3.2 Canonical Quantisation

We'll compute one term of a tree-level amplitude using the canonical quantisation approach. For more details and similar calculations see the first half of the *Quantum Field Theory* course. We'll consider a real scalar field with a cubic interaction

$$\mathcal{L}_{\text{int}} = -\frac{g}{3!}\varphi^3. \tag{3.2.1}$$

[1] two changes here from *Quantum Field Theory*, in that course we called the amplitude $\mathcal{M}$ and the factor of i was missing, this phase doesn't effect the final physics which always depend on $|\mathcal{A}|^2$

The two-to-two amplitude[1], $\mathcal{A}$, is given by

$$i\mathcal{A} = \langle p_1', p_2'|S|p_1, p_2\rangle \tag{3.2.2}$$

where p_i are the momenta of the incoming particles and p_i' the momenta of the outgoing particles. The S-matrix, S, is given by the Dyson expansion:

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \mathrm{T} \int \mathrm{d}t_1 \cdots \int \mathrm{d}t_n \, H_{\text{int}}(t_1) \cdots H_{\text{int}}(t_n) && (3.2.3)\\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \mathrm{T} \int \mathrm{d}^D x_1 \cdots \int \mathrm{d}x_n \, \mathcal{L}_{\text{int}}(x_1) \cdots \mathcal{L}_{\text{int}}(x_n) && (3.2.4)\\ &= \mathrm{T} \exp\left\{ i \int \mathrm{d}^D x \, \mathcal{L}_{\text{int}} \right\} && (3.2.5)\\ &= \mathrm{T} \exp\{iS_{\text{int}}\} && (3.2.6) \end{aligned}$$

where T is the time ordering operator, acting on everything to its right, H_{int} is the interaction Hamiltonian, related to the interaction Lagrangian by

$$\int \mathrm{d}^{D-1} x \, \mathcal{L}_{\text{int}} = -H_{\text{int}} \tag{3.2.7}$$

and S_{int} is the interaction action, given by

$$S_{\text{int}} \coloneqq \int \mathrm{d}^D x \, \mathcal{L}_{\text{int}}. \tag{3.2.8}$$

Note that the exponential is just a short hand for the expansion above based on the similarity with the Taylor series of the exponential.

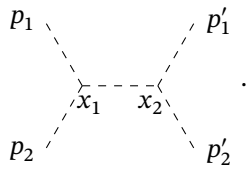
Suppose we are interested in $i\mathcal{A}$ at order g^2. Then we consider the following term:

$$i\mathcal{A}^{(2)} = \langle p_1', p_2'| \frac{i^2}{2} \left(-\frac{g}{3!}\right)^2 \int \mathrm{d}^D x_1 \int \mathrm{d}^D x_2 \; \mathrm{T}\, \varphi(x_1)\varphi(x_1)\varphi(x_1)\varphi(x_2)\varphi(x_2)\varphi(x_2) |p_1, p_2\rangle.$$

We compute this using contractions. One particular set of contractions is

$$\langle p_1', p_2'| \frac{i^2}{2}\left(-\frac{g}{3!}\right)^2 \int \mathrm{d}^D x_1 \int \mathrm{d}^D x_2 \; \mathrm{T}\,\varphi(x_1)\varphi(x_1)\varphi(x_1)\varphi(x_2)\varphi(x_2)\varphi(x_2)\varphi(x_2)|p_1, p_2\rangle.$$

We can interpret this in terms of creation and annihilation of particles. The fields at x_2 contracted with the incoming particles annihilate them and then the fields at x_1 contracted with the outgoing particles create the outgoing particles. The contraction between the fields at x_1 and x_2 gives a propagator between these points. This is best seen in a Feynman diagram,



(3.2.9)

The initial particles enter on the left, annihilate at x_1, where there is a propagator to x_2, where two new particles are created.

3.3 Path Integral

In the path integral formulation we don't compute amplitudes directly. Instead we compute correlators, which we can extract the amplitude from later. For more details and similar calculations see the second half of the *Quantum Field Theory* course. The starting point for using the path integral formalism is to define the generating functional, which for QED is

$$Z[J] = \int \mathcal{D}\varphi \exp\left\{i \int \mathrm{d}^D x \left[\mathcal{L}(\varphi) + J(x)\varphi(x)\right]\right\}. \tag{3.3.1}$$

We then define the n point **correlator**

$$\begin{aligned} G^{(n)}(x_1, \ldots, x_n) &\coloneqq \langle 0|\,\mathrm{T}\,\varphi(x_1)\cdots\varphi(x_n)|0\rangle && (3.3.2)\\ &=: \langle\varphi(x_1)\cdots\varphi(x_n)\rangle && (3.3.3)\\ &= \frac{1}{Z[0]}\left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\cdots\left(\frac{1}{i}\frac{\delta}{\delta J(x_n)}\right) Z[J]\Big|_{J=0} && (3.3.4)\\ &= \int \mathcal{D}\varphi\, \varphi(x_1)\cdots\varphi(x_n) \exp\left\{i\int \mathrm{d}^D x\,(\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}})\right\}. \end{aligned}$$

The factor of $1/Z[0]$ is just normalisation and we typically don't worry about it. The source is just there to allow us to pull down factors of φ by differentiating, which is why we set it to zero at the end.

As the first example of the path integral we'll compute the three-point correlator to first order in g. We'll choose our normalisation such that $Z[0] = 1$. Then

$$G^{(3)}(x_1, x_2, x_3) = \int \mathcal{D}\varphi\, \varphi(x_1)\varphi(x_2)\varphi(x_3) \mathrm{e}^{iS_{\text{free}} + iS_{\text{int}}}. \tag{3.3.5}$$

Expanding the interaction exponential to first order we get

$$G^{(3)}(x_1,x_2,x_3) \approx \int \mathcal{D}\varphi\, \varphi(x_1)\varphi(x_2)\varphi(x_3)\left[1-\frac{ig}{3!}\int \mathrm{d}^D x \varphi(x)^3\right] \mathrm{e}^{iS_{\text{free}}}. \quad (3.3.6)$$

We have now reduced this to a Gaussian path integral which can be computed with contractions. One set of contractions gives the result

$$C(x_1,x_2,x_3) = -ig\int \mathcal{D}\varphi\, \overbrace{\varphi(x_1)\varphi(x_2)\varphi(x_3)\int \mathrm{d}^D x\, \varphi(x)\varphi(x)\varphi(x)}\mathrm{e}^{iS_{\text{free}}}. \quad (3.3.7)$$

The contraction of two fields is given by the **Feynman propagator**:

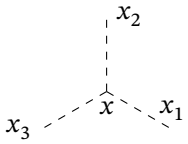
$$\overbrace{\varphi(x)\varphi(y)} = i\Delta(x-y) = \int \frac{\mathrm{d}^D p}{(2\pi)}\frac{i}{p^2-m^2+i\varepsilon}\mathrm{e}^{ip\cdot(x_1-x_2)}. \quad (3.3.8)$$

Here ε is a small positive real number included to make expressions converge. Note that $\Delta(x-y)=\Delta(y-x)$. At this point we introduce the short hand notation $\hat{\mathrm{d}}p = \mathrm{d}p/(2\pi)$.

Using this result the contraction above can be calculated as

$$C(x_1,x_2,x_3) = -ig\int \mathrm{d}^D x\, i\Delta(x_1-x)\, i\Delta(x_2-x)\, i\Delta(x_3-x). \quad (3.3.9)$$

We can again summarise this in a diagram



(3.3.10)

In fact, given a diagram we can read off the corresponding expression for a correlator through the following prescription:

- Each line, x ----- y , is a factor of $i\Delta(x-y)$.
- Each vertex,  , is a factor of $-ig\int \mathrm{d}^D x$.
- Conserve momentum at each vertex.

There are other possible contractions, one such contraction contributing to $\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\rangle$ is

$$-ig\int \mathcal{D}\varphi\, \overbrace{\varphi(x_1)\overbrace{\varphi(x_2)}\varphi(x_3)\int \mathrm{d}^D x\, \varphi(x)\overbrace{\varphi(x)}\varphi(x)}\mathrm{e}^{iS_{\text{free}}} \quad (3.3.11)$$

$$= -ig\int \mathrm{d}^D x\, i\Delta(x_1-x)\, i\Delta(x-x)\, i\Delta(x_2-x_3) \quad (3.3.12)$$

$$= \quad (3.3.13)$$

This diagram is disconnected. Often we are only interested in correlators involving connected diagrams, since these are the only diagrams that contribute to quantities such as $\log(Z[J])$, as we saw in *Quantum Field Theory*.

For another example consider the four-point correlator

$$G^{(4)}(y_1, y_2, z_1, z_2) = \langle 0|\,\mathrm{T}\,\varphi(y_1)\varphi(y_2)\varphi(z_1)\varphi(z_2)|0\rangle \tag{3.3.14}$$

$$= \int \mathcal{D}\varphi\,\varphi(y_1)\varphi(y_2)\varphi(z_1)\varphi(z_2)\exp\left\{i\int \mathrm{d}^D x(\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}})\right\}. \tag{3.3.15}$$

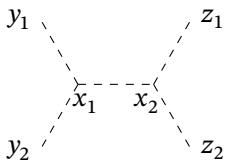
If we want to evaluate the order g^2 contribution to this correlator then we can do so by considering the quadratic term after expanding $\exp\{iS_{\text{int}}\}$:

$$\int \mathcal{D}\varphi\varphi(y_1)\varphi(y_2)\varphi(z_1)\varphi(z_2)$$
$$\times\left[\frac{1}{2}\left(-\frac{ig}{3!}\right)^2\int \mathrm{d}^D x_1\int \mathrm{d}^D x_2\,\varphi(x_1)\varphi(x_1)\varphi(x_1)\varphi(x_2)\varphi(x_2)\varphi(x_2)\right]\mathrm{e}^{iS_{\text{free}}}$$

Again this is a Gaussian integral and can be computed using contractions. One particular contraction we may want to consider is

$$\overbrace{\varphi(y_1)\varphi(y_2)\varphi(z_1)\varphi(z_2)\,\varphi(x_1)\varphi(x_1)\varphi(x_1)\varphi(x_2)\varphi(x_2)\varphi(x_2)}\tag{3.3.16}$$

where we've only written the fields, not any of the integrals, constants, or exponentials, to fit it all on one line. This corresponds to the diagram

$D =$ 

. (3.3.17)

There are multiple different contractions which all give the same diagram, and so the same contribution to the correlator. We combine these into one, including a **symmetry factor** counting the number of such diagrams. The factor of $1/3!$ has been chosen to cancel out this symmetry factor, in this case because we can permute the three $\varphi(x_1)$ fields and the three $\varphi(x_2)$ fields without changing anything, giving $(3!)^2$ as a symmetry factor, which cancels the $(3!)^2$ from expanding the exponential at second order. The result of evaluating all contractions giving diagram D is

$$D = \int \mathrm{d}^D x_1\int \mathrm{d}^D x_2\,(-ig)^2\,i\Delta(y_1-x_2)\,i\Delta(y_2-x_3)\,i\Delta(x_2-x_1)\,i\Delta(x_1-z_1)\,iD(x_1-z_2).$$

Usually we prefer to work in momentum space. The simplest way to move to momentum space here is to replace each propagator with the inverse Fourier transform of the Fourier transform:

$$\Delta(x-y) = \int \hat{\mathrm{d}}p\mathrm{e}^{ip\cdot(x-y)}\underbrace{\frac{1}{p^2-m^2+i\varepsilon}}_{=\widetilde{\Delta}(p)}. \tag{3.3.18}$$

We can then manipulate the result until it is of the form $D = \mathcal{F}^{-1}\{\widetilde{D}\}$ and then identify $\widetilde{D} = \mathcal{F}\{D\}$. Making this replacement of propagators we get the somewhat unwieldy

$$\begin{aligned} D = (-ig)^2 \int \mathrm{d}^D x_1 \, \mathrm{d}^D x_2 \int \hat{\mathrm{d}} p_1 \, \hat{\mathrm{d}} p_2 \, \hat{\mathrm{d}} p_2 \, \hat{\mathrm{d}} p_3 \, \hat{\mathrm{d}} p_4 \, \hat{\mathrm{d}} q \\ \times i\mathrm{e}^{ip_1\cdot(y_1-x_2)} i\mathrm{e}^{ip_2\cdot(y_2-x_2)} i\mathrm{e}^{ip_3\cdot(x_1-z_1)} i\mathrm{e}^{ip_4\cdot(x_1-z_2)} i\mathrm{e}^{iq\cdot(x_2-x_1)} \\ \times \frac{1}{q^2-m^2+i\varepsilon} \prod_{j=1}^{4} \frac{1}{p_j^2-m^2+i\varepsilon}. \end{aligned} \tag{3.3.19}$$

We use q for the momentum of the internal propagator to distinguish it from the external propagators. We can perform the integrals over x_i using the identity

$$\int \mathrm{d}^D x \, \mathrm{e}^{ip\cdot x} = (2\pi)^D \delta(p). \tag{3.3.20}$$

Rewriting the exponentials slightly we get

$$\mathrm{e}^{-i(p_1+p_2-q)\cdot x_2} i\mathrm{e}^{ip_1\cdot y_1} i\mathrm{e}^{ip_2\cdot y_2} \mathrm{e}^{i(p_3+p_4-q)\cdot x_1} i\mathrm{e}^{-ip_3\cdot z_1} i\mathrm{e}^{-ip_4\cdot z_2} i \tag{3.3.21}$$

so we'll get two Dirac deltas:

$$\begin{aligned} D = (-ig)^2 \int \hat{\mathrm{d}} p_1 \, \hat{\mathrm{d}} p_2 \, \hat{\mathrm{d}} p_2 \, \hat{\mathrm{d}} p_3 \, \hat{\mathrm{d}} p_4 \, \hat{\mathrm{d}} q \, (2\pi)^D \delta(p_1+p_2-q) \\ \times (2\pi)^D \delta(p_3+p_4-q) \mathrm{e}^{ip_1\cdot y_1} \mathrm{e}^{ip_2\cdot y_2} \mathrm{e}^{-ip_3\cdot z_1} \mathrm{e}^{-ip_4\cdot z_2} \\ \times \frac{i}{q^2-m^2+i\varepsilon} \prod_{j=1}^{4} \frac{i}{p_j^2-m^2+i\varepsilon}. \end{aligned} \tag{3.3.22}$$

We can then perform the q integral using the second Dirac delta to set $q = p_3 + p_4$, giving

$$\begin{aligned} D = (-ig)^2 \int \hat{\mathrm{d}} p_1 \, \hat{\mathrm{d}} p_2 \, \hat{\mathrm{d}} p_2 \, \hat{\mathrm{d}} p_3 \, \hat{\mathrm{d}} p_4 \, \hat{\mathrm{d}} q \, (2\pi)^D \delta(p_1+p_2-p_3-p_4) \\ \times \mathrm{e}^{ip_1\cdot y_1} \mathrm{e}^{ip_2\cdot y_2} \mathrm{e}^{-ip_3\cdot z_1} \mathrm{e}^{-ip_4\cdot z_2} \\ \times \frac{i}{(p_3+p_4)^2-m^2+i\varepsilon} \prod_{j=1}^{4} \frac{i}{p_j^2-m^2+i\varepsilon}. \end{aligned} \tag{3.3.23}$$

Note that the factor of $(2\pi)^D$ in front of the Dirac delta cancels with the hidden factor of $1/(2\pi)^D$ in $\hat{\mathrm{d}} p$. This is now of the form $D = \mathcal{F}^{-1}\{\widetilde{D}\}$, so we can identify $\widetilde{D} = \mathcal{F}\{D\}$ as

$$\widetilde{D}(p_1,p_2,p_3,p_4) = (2\pi)^D \delta(p_1+p_2-p_3-p_4) \frac{i}{(p_3+p_4)^2-m^2+i\varepsilon} \prod_{j=1}^{4} \frac{i}{p_j^2-m^2+i\varepsilon}. \tag{3.3.24}$$

Notice that the signs of p_i in the Dirac delta reflect a sign choice where p_1 and p_2 are incoming momenta and p_3 and p_4 are outgoing momenta. If all momenta are chosen to be incoming, as is sometimes the case, then all of the signs would be $+$. This Dirac delta is simply telling us that the total momentum is conserved, so it's not that interesting and is not considered to be part of the amplitude.

Similarly, the external line factors, the product above, don't carry any information, beyond the number of external lines, and aren't present in the amplitude. For this reason we consider **amputated correlators**, which are given by omitting this term in momentum space. In position space it's slightly more work to amputate a correlator, but it can be done by using

$$(\partial_x^2 + m^2)\, i\Delta(x-y) = i \int \hat{\mathrm{d}}^D p(-p^2 + m^2) \frac{\mathrm{e}^{-ip\cdot(x-y)}}{p^2 - m^2 + i\varepsilon} = -i\delta(x-y), \quad (3.3.25)$$

where ∂_x^2 is the d'Alembert operator with respect to x, rather than $\partial/\partial x$ squared. This statement is simply that the propagator is a Green's function of the Klein–Gordon operator, which should be familiar from *Quantum Field Theory*. We also see from this that the "inverse" of $\partial^2 + m^2$ is $1/(p^2 - m^2 + i\varepsilon)$, another fact we saw in the previous course. Using this we can amputate a correlator in position space by acting on it with

$$\prod_{j=1}^{n} (+i)(\partial_j^2 + m_j^2) \quad (3.3.26)$$

where ∂_j^2 is the d'Alembert with respect to x_j and m_j is the mass of the jth particle.

Four

Quantum Electrodynamics

If you don't use dim reg you'll be shot.

Donal O'Connell

4.1 The QED Lagrangian

The QED Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \tag{4.1.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu - ieA_\mu$. One important property of this Lagrangian is the presence of a U(1) gauge symmetry, given by

$$\psi(x) \mapsto \psi(x)e^{i\alpha(x)}, \quad \bar{\psi}(x) \mapsto \bar{\psi}(x)e^{-i\alpha(x)}, \quad \text{and} \quad A_\mu(x) \mapsto A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x)$$

where α is some function of spacetime taking values in $[0, 2\pi)$ (with continuous second derivatives). We can see that this leaves the Lagrangian invariant by considering how each term transforms. First,

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu && (4.1.2)\\ &\mapsto \partial_\mu\left(A_\nu + \frac{1}{e}\partial_\nu\alpha\right) - \partial_\nu\left(A_\mu + \frac{1}{e}\partial_\mu\alpha\right) && (4.1.3)\\ &= \partial_\mu A_\nu + \frac{1}{e}\partial_\mu\partial_\nu\alpha - \partial_\nu A_\mu - \frac{1}{e}\partial_\mu\alpha && (4.1.4)\\ &= F_{\mu\nu}. && (4.1.5) \end{aligned}$$

Second,

$$\begin{aligned} \bar{\psi}(i\not{D} - m)\psi &= \bar{\psi}(i\not{\partial} + e\not{A})\psi && (4.1.6)\\ &\mapsto \bar{\psi}e^{-i\alpha}\left(i\not{\partial} + e\left(\not{A} + \frac{1}{e}\not{\partial}\alpha\right) - m\right)\psi e^{i\alpha} && (4.1.7)\\ &= \bar{\psi}e^{-i\alpha}(i\not{\partial} + e\not{A} + \not{\partial}\alpha - m)\psi e^{i\alpha} && (4.1.8)\\ &= \bar{\psi}e^{-i\alpha}e^{i\alpha}(i\not{\partial}(i\alpha) + i\not{\partial} + e\not{A} + \not{\partial}\alpha - m)\psi && (4.1.9)\\ &= \bar{\psi}(\not{\partial} + e\not{A} - m)\psi && (4.1.10)\\ &= \bar{\psi}(i\not{D} - m)\psi. && (4.1.11) \end{aligned}$$

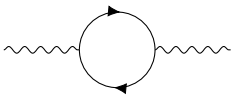
4.2 Divergences

Personally I detest dim reg. It's weird, but it's easy. It's detestable.

Donal O'Connell

Our focus in the first part of this course will be on renormalisation of divergences in QED. We will mainly look at correlators and how we can extract physics from renormalised correlators. We will do this through the process of renormalised perturbation theory.

Loop diagrams, such as



(4.2.1)

are often divergent. We can absorb these divergences into a finite (in QED) set of measured parameters.

To do this we distinguish between the fields and parameters appearing in the original Lagrangian, which we call **bare** fields and parameters, and the renormalised fields and parameters. Add a label B to each bare quantity so the Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mathrm{B}\mu\nu}F_{\mathrm{B}}^{\mu\nu} + \bar{\psi}_{\mathrm{B}}(i\not{D}_{\mathrm{B}} - m_{\mathrm{B}})\psi_{\mathrm{B}} \tag{4.2.2}$$

where $F_{\mathrm{B}\mu\nu} = \partial_\mu A_{\mathrm{B}\nu} - \partial_\nu A_{\mathrm{B}\mu}$ and $D_{\mathrm{B}\mu} = \partial_\mu - ie_{\mathrm{B}}A_{\mathrm{B}\mu}$. We can think of the original Lagrangian and the bare quantities as being "true", corresponding to some high energy theory. Then the renormalised quantities are from the low energy limit of this theory. We define the following renormalised quantities in terms of the bare quantities:

$$A_{\mathrm{B}\mu} = \sqrt{Z_3}A_\mu, \tag{4.2.3}$$
$$\psi_{\mathrm{B}} = \sqrt{Z_2}\psi, \tag{4.2.4}$$
$$m_{\mathrm{B}} = m + \delta m, \tag{4.2.5}$$
$$e_{\mathrm{B}} = Z_e e. \tag{4.2.6}$$

We fix the values of the unknown parameters introduced in these definitions by requiring that correlators of renormalised fields are finite and choosing these unknown parameters in such a way that this is enforced.

In *Quantum Field Theory* we didn't give a special notation for the bare quantities, and instead labelled the renormalised quantities, for example, φ_{R} was the renormalised scalar field and φ was the bare scalar field. This was because we didn't cover renormalisation until the end of the course, so the extra B labels would have been a nuisance. In this course we start with renormalisation, so we will work with the renormalised quantities more, so we don't give them a special label and instead label the bare quantities.

The parameters Z_2 and Z_3 are called the **wave function renormalisation constants**. The labels 2 and 3 are convention, and we'll introduce Z_1 shortly. The square roots are chosen as these fields appear squared in the Lagrangian.

The definition of the renormalised mass above doesn't fit the pattern. We've chosen to think of the renormalised mass, m, as simply being shifted by δm from the bare mass, m_{B}. We could have followed the pattern and written $m_{\mathrm{B}} = Z_m m$ with $Z_m = 1 + \delta m/m$. This is nice in QED where if $m_{\mathrm{B}} = 0$ then $m = 0$. However in other theories, such as scalar QED this isn't the case, it is possible for the bare field to be massless but the renormalised field has a mass. This can work with $m_{\mathrm{B}} = Z_m m$, we just have to choose Z_m so that it diverges when $m_{\mathrm{B}} = 0$. The reason that we don't define A_μ and ψ in the same way, i.e. $A_{\mathrm{B}\mu} = A_\mu + \delta A_\mu$ and $\psi_{\mathrm{B}} = \psi + \delta\psi$, is because we don't have any other vectors or spinors in our theory to give us δA_μ or $\delta\psi$. The reason we don't define e in this way is that when $e_{\mathrm{B}} = 0$ the theory is non-interacting, and thus we should also have $e = 0$. Later we will expand Z_i as $1 + \delta_i$, which corresponds to $\delta_m = \delta m/m$.

To make the Z_i and δm well defined we need to pick a regulator. We'll use **dimensional regularisation**, or **dim reg**. We also need to pick a renormalisation scheme. We'll use **modified minimal subtraction**, or $\overline{\mathbf{MS}}$.

In dim reg it is actually better to set

$$e_{\mathrm{B}} = Z_e e \mu^\varepsilon \tag{4.2.7}$$

since e_{B} is dimensionless in $D = 4$, but in $D = 4 - 2\varepsilon$ dimensions e_{B} has mass dimension ε. We choose μ to be a mass scale so that e is dimensionless in $D = 4-2\varepsilon$ dimensions. Importantly μ is not a parameter of the bare theory. This means that no physics can depend on μ so μ must cancel out in any computation giving a measurable result. The choice of μ can effect how quickly perturbation theory converges, for terms of the form $\log(m/\mu)$ are common, and if we choose $\mu \approx m$ then this value will be small, whereas if $\mu \gg m$ we'll get large logs, which we usually want to avoid. The parameter μ is called the **renormalisation point** or occasionally the **'t Hooft scale**.

The Lagrangian can the be rewritten in terms of the renormalised quantities. First,

$$F_{\mathrm{B}\mu\nu} = \partial_\mu A_{\mathrm{B}\nu} - \partial_\nu A_{\mathrm{B}\mu} = \partial_\mu(\sqrt{Z_3}A_\mu) - \partial_\nu(\sqrt{Z_3}A_\nu) = \sqrt{Z_3}F_{\mu\nu}, \tag{4.2.8}$$

and so

$$F_{\mathrm{B}\mu\nu}F_{\mathrm{B}}^{\mu\nu} = Z_3 F_{\mu\nu}F^{\mu\nu}. \tag{4.2.9}$$

We also have

$$i\not{D}_{\mathrm{B}} = i\not{\partial} + e_{\mathrm{B}}\not{A}_{\mathrm{B}} = i\not{\partial} + Z_e\sqrt{Z_3}e\mu^\varepsilon\not{A} \tag{4.2.10}$$

so $\bar\psi_{\mathrm{B}} i\not{D}_{\mathrm{B}}\psi_{\mathrm{B}} = Z_e Z_2\sqrt{Z_3}$. We define $Z_1 = Z_e Z_2\sqrt{Z_3}$ for notational compactness. One imagines that this process was followed in the reverse when Z_i were named. The mass term gives

$$\bar\psi_{\mathrm{B}} m_{\mathrm{B}}\psi_{\mathrm{B}} = Z_2 m\bar\psi\psi + Z_2\delta m\bar\psi\psi. \tag{4.2.11}$$

So the Lagrangian in terms of the renormalised quantities is

$$\mathcal{L} = -\frac{1}{4}Z_3 F^{\mu\nu}F_{\mu\nu} + Z_2\bar\psi(i\not{\partial} - m)\psi + Z_1 e\mu^\varepsilon\bar\psi\not{A}\psi - Z_2\delta m\bar\psi\psi. \tag{4.2.12}$$

It is not immediately clear that this Lagrangian has a gauge symmetry, but of course it does, since it inherits the gauge symmetry of the bare theory. This will

be made more clear later when we show that $Z_1 = Z_2$ (Section 8.4), which allows us to write this with a covariant derivative again.

To make sense of this we define $Z_i = 1 + \delta_i$ and then the Lagrangian is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi \\ &\quad - \frac{1}{4}\delta_3 F^{\mu\nu}F_{\mu\nu} + \delta_2\bar{\psi}(i\partial\!\!\!/ - m)\psi + \delta_1 e\mu^\varepsilon\bar{\psi}A\!\!\!/\psi - \delta m\bar{\psi}\psi - \delta m\delta_2\bar{\psi}\psi. \end{aligned} \tag{4.2.13}$$

This is of the form

$$\mathcal{L} = \mathcal{L}_{\text{classical}} + \mathcal{L}_{\text{ct}} \tag{4.2.14}$$

where

$$\mathcal{L}_{\text{classical}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{4.2.15}$$

is the "classical" Lagrangian, being of the same form as the Lagrangian in terms of the bare parameters and

$$\mathcal{L}_{\text{ct}} = -\frac{1}{4}\delta_3 F^{\mu\nu}F_{\mu\nu} + \delta_2\bar{\psi}(iD\!\!\!/ - m)\psi + \delta_1 e\mu^\varepsilon\bar{\psi}A\!\!\!/\psi - \delta m\bar{\psi}\psi - \delta m\delta_2\bar{\psi}\psi \tag{4.2.16}$$

are the **counterterms**, which we choose in such a way that the divergences cancel out.

Note that in this expression $D_\mu = \partial_\mu - ie\mu^\varepsilon A_\mu$, with the factor of μ^ε. It is common to miss out writing in μ^ε both here and in the Lagrangian since it can always be inserted by dimensional analysis and disappears in final results. For tree diagrams we can take $\varepsilon \to 0$ anyway since there are no divergences, making it even more common to leave μ^ε out.

In QED divergences first appear at one loop, and since a single loop with external particles has at least two vertices these divergences are $\mathcal{O}(e^2)$. Since we choose the renormalisation parameters to cancel these divergences they must be of the form $Z_i = 1 + \mathcal{O}(e^2)$, with the 1 giving us the bare theory and the $\mathcal{O}(e^2)$ cancelling the divergences. This means that $\delta m\delta_2$ is $\mathcal{O}(e^4)$, so it is only important if we are doing a next-to-next-to leading order (NNLO) calculation or working with two or more loops. We'll only be doing leading order computations with one loop, so we'll neglect the final term of the Lagrangian.

4.3 Feynman Rules

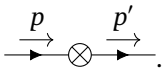
The Feynman rules tell us how a diagram translates into an equation. For a given term involving a product of fields, such as $\bar{\psi}A\!\!\!/\psi$, the Feynman rules can be derived by considering the tree level correlator $\langle\bar{\psi}A\!\!\!/\psi\rangle$. Since this is at tree level the terms just add linearly allowing us to consider them one at a time. We then amputate the correlator, working in momentum space, and drop the overall momentum conservation Dirac delta. What is left is the Feynman rule for this term.

4.3.1 $\bar{\psi}\psi$ Term

First we'll consider the $\bar{\psi}\psi$ terms in the Lagrangian. These correspond to an interaction with the action

$$S_{\text{int}} = \int \mathrm{d}^D x \left[i\delta_2\bar{\psi}\partial\!\!\!/\psi - (\delta_2 m + \delta m)\bar{\psi}\psi\right]. \tag{4.3.1}$$

Diagrammatically this corresponds to the correlator



(4.3.2)

The symbol $\otimes$ is used to signify a counter term, since these are often 1-to-1 scattering processes which would otherwise look like just a propagator.

Recall that in the canonical quantisation formalism we can expand ψ in terms of the electron annihilation operator, $a_s(p)$, and the positron creation operator, $b_s^\dagger(p)$:

$$\psi(x) = \sum_s \int \frac{\hat{\mathrm{d}}p}{2E_p} \left[a_s(p) u(p,s) \mathrm{e}^{-ip\cdot x} + b_s^\dagger(p) v(p,s) \mathrm{e}^{ip\cdot x} \right]. \tag{4.3.3}$$

Then we have

$$\not{\partial}\psi(x) = \sum_s \int \frac{\hat{\mathrm{d}}p}{2E_p} \left[-ip\!\!\!/ a_s(p) u(p,s) \mathrm{e}^{-ip\cdot x} + ip\!\!\!/ b_s^\dagger(p) v(p,s) \mathrm{e}^{ip\cdot x} \right]. \tag{4.3.4}$$

We want to calculate $\langle \bar{\psi} \not{A} \psi \rangle$ to tree level. To do this we expand $\mathrm{e}^{iS} = 1 + iS + \cdots$ to first order and consider the first order term, which corresponds to tree level processes (zeroth order corresponds to no interaction occurring). So for an incoming particle with momentum p and outgoing particle with momentum p' we need to calculate

$$\langle p' | i \int \mathrm{d}^D x \left[i\delta_2 \bar{\psi} \not{\partial} \psi - (\delta_2 m + \delta m) \bar{\psi} \psi \right] | p \rangle. \tag{4.3.5}$$

To do this we need to contract the fields. We have an incoming and outgoing electron, which need to be created and destroyed and this can only be done one way. The field ψ can annihilate the incoming electron and the adjoint $\bar{\psi}$ can create the outgoing electron. Thus we must contract as follows:

$$\langle \overbrace{p' | i \int \mathrm{d}^D x \, i\delta_2 \bar{\psi} \not{\partial} \psi}^{} \overbrace{| p \rangle}^{} - \langle \overbrace{p' | i \int \mathrm{d}^D x \, (\delta_2 m + \delta m) \bar{\psi} \psi}^{} \overbrace{| p \rangle}^{}. \tag{4.3.6}$$

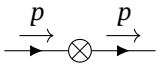
Completing this contraction, and including the factor of $ip\!\!\!/$ we get from acting with the derivative we get the result

$$i \int \mathrm{d}^D x \, [\delta_2 (p\!\!\!/ - m) - \delta m] \, \bar{u} u \mathrm{e}^{-i(p-p')\cdot x}. \tag{4.3.7}$$

Performing this integral we get

$$i(2\pi)^D \delta(p - p') \, [\delta_2 (p\!\!\!/ - m) - \delta m] \, \bar{u} u. \tag{4.3.8}$$

To get the Feynman rule we strip off the factors corresponding to external legs, since these are dealt with by other Feynman rules, so we remove $\bar{u}u$, and we strip off the overall momentum conserving Dirac delta $(2\pi)^D \delta(p - p')$, since this is enforced by conserving momentum at each vertex, which in this case just corresponds to setting $p = p'$. The resulting Feynman rule is



$= i[\delta_2 (p\!\!\!/ - m) - \delta m].$ (4.3.9)

4.3.2 $F^{\mu\nu}F_{\mu\nu}$ Term

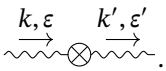
Well you can't stop me. No one said we were going to use sane notation, just consistent notation.

Donal O'Connell

Now consider the $F^{\mu\nu}F_{\mu\nu}$ counterterm, which corresponds to the interaction action

$$S_{\text{int}} = \int \mathrm{d}^D x \left[-\frac{1}{4}\delta_3 F^{\mu\nu}F_{\mu\nu} \right]. \tag{4.3.10}$$

To calculate the Feynman rule associated with this interaction consider the diagram



. (4.3.11)

Here k and k' are momenta and ε and ε' are polarisation vectors. To first order this diagram gives

$$\langle k', \varepsilon' | i \int \mathrm{d}^D x \left[-\frac{1}{4}\delta_3 F^{\mu\nu}F_{\mu\nu} \right] |k, \varepsilon\rangle. \tag{4.3.12}$$

There are two possible ways to perform contractions on this:

$$\langle \overbrace{k', \varepsilon' | i \int \mathrm{d}^D x \left[-\frac{1}{4}\delta_3 F^{\mu\nu}} \overbrace{F_{\mu\nu}} \right] |k, \varepsilon\rangle, \tag{4.3.13}$$

$$\langle \overbrace{k', \varepsilon' | i \int \mathrm{d}^D x \left[-\frac{1}{4}\delta_3 \overbrace{F^{\mu\nu}} F_{\mu\nu} \right]} |k, \varepsilon\rangle. \tag{4.3.14}$$

Since we can freely commute F with itself and raise and lower the paired indices these two contractions are actually exactly the same. So we include a symmetry factor of 2 and only consider one of these contractions. We'll take the first contraction and compute

$$-i\frac{\delta_3}{2}\langle \overbrace{k', \varepsilon' | \int \mathrm{d}^D x F^{\mu\nu}} \overbrace{F_{\mu\nu}} | k, \varepsilon\rangle. \tag{4.3.15}$$

The next simplification is that for any two index tensor X we have

$$(X^{\mu\nu} - X^{\nu\mu})(X_{\mu\nu} - X_{\nu\mu}) = X^{\mu\nu}(X_{\mu\nu} - X_{\nu\mu}) - X^{\nu\mu}(X_{\mu\nu} - X_{\nu\mu}) \tag{4.3.16}$$

exchanging μ and ν in the second term

$$(X^{\mu\nu} - X^{\nu\mu})(X_{\mu\nu} - X_{\nu\mu}) = X^{\mu\nu}(X_{\mu\nu} - X_{\nu\mu}) - X^{\mu\nu}(X_{\nu\mu} - X_{\mu\nu}) \tag{4.3.17}$$

$$= 2X^{\mu\nu}(X_{\mu\nu} - X_{\nu\mu}) \tag{4.3.18}$$

so, taking $X^{\mu\nu} = \partial^\mu A^\nu$, we have

$$F^{\mu\nu}F_{\mu\nu} = 2(\partial^\mu A^\nu)(\partial_\mu A_\nu - \partial_\nu A_\mu). \tag{4.3.19}$$

Hence the correlator is

$$-i\delta_3\langle k',\varepsilon'|\int \mathrm{d}^D x\,(\partial^\mu A^\nu)\overbrace{(\partial_\mu A_\nu - \partial_\nu A_\mu)}|k,\varepsilon\rangle. \tag{4.3.20}$$

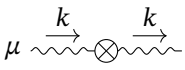
As with the ψ case the derivatives act on the mode expansion to bring down a factor of $\pm ik_\mu$ from terms like $\varepsilon^\mu(k)a_s(k)\mathrm{e}^{-ik\cdot x}$. The result of performing the contractions is

$$-i\delta_3\int \mathrm{d}^D x\,k'^\mu\varepsilon'^\nu(k_\mu\varepsilon_\nu - k_\nu\varepsilon_\mu)\mathrm{e}^{-ik\cdot x}\mathrm{e}^{ik'\cdot x}. \tag{4.3.21}$$

Performing the integral gives a Dirac delta, and we'll also expand the bracket in terms of inner products while we're at it

$$-i\delta_3(2\pi)^D\delta(k-k')[(k\cdot k')(\varepsilon\cdot\varepsilon') - (k\cdot\varepsilon)(k'\cdot\varepsilon')]. \tag{4.3.22}$$

As before to get the Feynman rule we strip of terms corresponding to external legs and the momentum conservation Dirac delta, setting $k = k'$, to get the Feynman rule



$$= -i\delta_3(k^2\eta_{\mu\nu} - k_\mu k_\nu). \tag{4.3.23}$$

Notice that we have to include the metric from $\varepsilon'\cdot\varepsilon = \eta_{\mu\nu}\varepsilon'^\mu\varepsilon^\nu$ in order for the indices to match up.

4.3.3 $\bar{\psi}\not{A}\psi$ Term

The final counterterm to compute is the one from the $\bar{\psi}\not{A}\psi$ term. The corresponding action is

$$S_{\text{int}} = \int \mathrm{d}^D x[e\mu^\varepsilon\delta_1\bar{\psi}\not{A}\psi]. \tag{4.3.24}$$

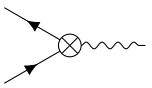
This has three fields, so contributes as a three point vertex. To compute the counterterm in this case we consider the correlator $\langle\bar{\psi}(x_1)\psi(x_2)A^\mu(x_3)\rangle$. Expanding the exponential to first order in the interaction we want to compute

$$\int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\bar{\psi}(x_1)\psi(x_2)A^\mu(x_3)\int \mathrm{d}^D x\,e\mu^\varepsilon\delta_1\bar{\psi}(x)\not{A}^\nu(x)\psi(x)\mathrm{e}^{iS[\psi,\bar{\psi},A]} \tag{4.3.25}$$

where $S[\psi,\bar{\psi},A]$ is the free action. There is only one contraction not giving zero:

$$\int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\overbrace{\bar{\psi}(x_1)\psi(x_2)A^\mu(x_3)\int \mathrm{d}^D x\,e\mu^\varepsilon\delta_1\bar{\psi}(x)\not{A}^\nu(x)\psi(x)}\mathrm{e}^{iS[\psi,\bar{\psi},A]}. \tag{4.3.26}$$

Computing the contractions we are left with external propagators, which we can drop, and then we get the amputated correlator $ie\mu^\varepsilon\delta_1\gamma^\mu$. This gives the Feynman rule

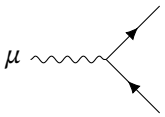


$$= ie\delta_1\gamma^\mu\mu^\varepsilon \tag{4.3.27}$$

4.4 Vacuum Polarisation

In this section we will compute the vacuum polarisation, which is a correction to an internal photon propagator. Before we proceed we'll recap the Feynman rules for QED, including the two counterterms we've just computed.

- The **QED vertex**:



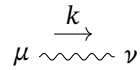
$$= ie\mu^{\varepsilon}\gamma^{\mu}. \tag{4.4.1}$$

  Note the factor of μ^{ε} which we had not previously included as at tree level we were taking $\varepsilon = 0$.

- The QED propagators:
  - **Electron propagator**:

$$= \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon}. \tag{4.4.2}$$

  - **Photon propagator**:



$$= \frac{-i}{k^2 + i\varepsilon}\left(\eta^{\mu\nu} - (1-\xi)\frac{k^{\mu}k^{\nu}}{k^2}\right). \tag{4.4.3}$$

    Here ξ is a gauge parameter. The choice of $\xi = 1$ is the **Feynman gauge**, giving the propagator

$$\frac{-i\eta_{\mu\nu}}{k^2 + i\varepsilon}, \tag{4.4.4}$$

    which is nice since the second term vanishes. The choice of $\xi = 0$ is the **Landau gauge** or **Lorenz gauge**, giving the propagator

$$\frac{i}{k^2 + i\varepsilon}\left(\eta^{\mu\nu} - \frac{k^{\mu}k^{\nu}}{k^2}\right), \tag{4.4.5}$$

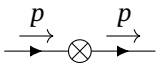
    which is nice because when we multiply by k_{μ} this vanishes.

- Conserve momentum at each vertex.
- Integrate over internal momenta which aren't fixed by momentum conservation, giving an integral

$$\int \hat{\mathrm{d}}^D \ell = \int \frac{\mathrm{d}^D \ell}{(2\pi)}. \tag{4.4.6}$$

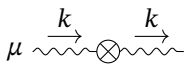
- Each closed fermion loop gives a factor of -1.
- Counterterms:

– electron counter term:



$$= i(\delta_2(\not{p} - m) - \delta m). \tag{4.4.7}$$

– photon counter term:



$$= -i\delta_3(k^2\eta_{\mu\nu} - k_\mu k_\nu). \tag{4.4.8}$$

- For an amputated correlator each external line simply gives a factor of 1.

Consider the correlator

$$\langle A_\mu(x)A_\nu(y)\rangle = \int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,A_\mu(x)A_\nu(y)\mathrm{e}^{iS}. \tag{4.4.9}$$

At order e^0 there is no interaction so the photon just propagates freely giving

$$\langle A_\mu(x)A_\nu(y)\rangle_{\mathcal{O}(e^0)} = iD_{\mu\nu}(x-y) \tag{4.4.10}$$

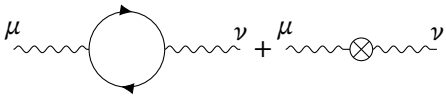
where $D_{\mu\nu}$ is the free photon propagator.

At order e^1 if we expand e^{iS} to get $1+\mathcal{O}(e)$ we get three A fields and two ψ fields with no way to contract them all. This means that there is no $\mathcal{O}(e)$ contribution.

At order e^2 expanding e^{iS} we get

$$\int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,A_\mu(x)A_\nu(y)\frac{(ie)^2}{2}\int \mathrm{d}^D x_1\,\mathrm{d}^D x_2\,(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2} + \text{counterterm} \tag{4.4.11}$$

where $(-)_x$ means that all the fields in the brackets are evaluated at x. Diagrammatically this is



(4.4.12)

The interesting part of the correlator is the loop, so define

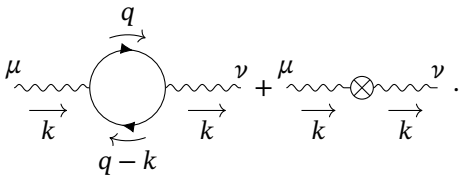
$$i\Pi_{\mu\nu}(x-y) \coloneqq \langle A_\mu(x)A_\nu(y)\rangle|_{1\text{ loop, amputated}} \tag{4.4.13}$$

to be just the loop without the counter term or external photon propagators. We want to work in momentum space so define

$$\widetilde{\Pi}_{\mu\nu}(k) = \int \mathrm{d}^D x\,\mathrm{e}^{ik\cdot x}\Pi_{\mu\nu}(x). \tag{4.4.14}$$

Then, enforcing momentum conservation, we have

$$i\widetilde{\Pi}_{\mu\nu}(k) =$$



(4.4.15)

To evaluate this we use the Feynman rules, which give

- A factor of -1 from the fermion loop.
- An integral $\int \hat{\mathrm{d}}^D q$ for the undetermined momentum.
- Entering on the left we first come to a photon propagator, but this is external and we're considering an amputated correlator so it gives a factor of 1.
- Next we reach a QED vertex giving a factor of $ie\mu^\varepsilon \gamma^\mu$.
- We then proceed *backwards* along the fermion line, which is an electron propagator with momentum $q-k$, giving a factor of

$$\frac{i(\not{q}-\not{k}+m)}{(q-k)^2-m^2+i\varepsilon}. \tag{4.4.16}$$

- Another QED vertex giving a factor of $ie\mu^\varepsilon \gamma^\nu$.
- Continuing backwards along the fermion propagator with momentum q we get

$$\frac{i(\not{q}+m)}{q^2-m^2+i\varepsilon}. \tag{4.4.17}$$

Call these electron propagators $S(q-k)$ and $S(q)$ and write in the spinor indices. We then have a factor of

$$\begin{aligned}
\gamma^\mu_{ab} S_{bc}(q-k)\gamma^\nu_{cd} S_{da}(q) &= S_{da}(q)\gamma^\mu_{ab} S_{bc}(q-k)\gamma^\nu_{cd} && (4.4.18)\\
&= \mathrm{tr}(S(q)\gamma^\mu S(q-k)\gamma^\nu) && (4.4.19)\\
&= \mathrm{tr}(\gamma^\mu S(q-k)\gamma^\nu S(q)). && (4.4.20)
\end{aligned}$$

Putting this all together we get

$$-\int \hat{\mathrm{d}}^D q (ie\mu^\varepsilon)^2 \,\mathrm{tr}\left[\gamma^\mu \frac{i(\not{q}-\not{k}+m)}{(q-k)^2-m^2+i\varepsilon}\gamma^\nu \frac{i(\not{q}+m)}{q^2-m^2+i\varepsilon}\right] - i\delta_3(k^2\eta^{\mu\nu}-k^\mu k^\nu).$$

Five

Evaluating Loop Integrals

One loop integrals can be computed using the following algorithm[1]

[1] for examples using this method see the second half of *Quantum Field Theory*

1. Use Feynman parametrisation to rewrite the denominators. The simple case is

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \, \frac{1}{[xA + (1-x)B]^2} \tag{5.0.1}$$

and the more general case is

$$\frac{1}{A_1^{\alpha_1} \cdots A_n^{\alpha_n}} = \int_0^1 \mathrm{d}x_1 \, x^{\alpha_1 - 1} \cdots \int_0^1 \mathrm{d}x_n \, x^{\alpha_n - 1} \times \frac{\delta(1 - \sum_i \alpha_i)}{[x_1 A_1 + \cdots + x_n A_n]^{\sum_i \alpha_i}} \frac{\Gamma(\sum_i \alpha_i)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)}.$$

2. Shift the loop momentum to get the integral in the form

$$\int \hat{\mathrm{d}}^D \ell \frac{N}{(\ell^2 - \Delta)^n}. \tag{5.0.2}$$

3. Simplify the numerator. This step is new to QED since for scalar φ^3 theory as seen in *Quantum Field Theory* the numerator is always one. We'll see several tricks for performing this simplification later.

4. Wick rotate defining $\ell^0 = i\ell_{\mathrm{E}}^0$ and $\ell^2 = -(\ell_{\mathrm{E}})^2 - \ell^2 = -\ell_{\mathrm{E}}^2$ where ℓ_{E}^2 is the Euclidean inner product, $\ell_{\mathrm{E}}^2 = \ell_{\mathrm{E}}^2 + \ell^2$.

5. Use the identity

$$\int \hat{\mathrm{d}}^D \ell_{\mathrm{E}} \frac{(\ell_{\mathrm{E}}^2)^p}{(\ell_{\mathrm{E}} + \Delta)^n} = \frac{\Gamma(n - p - D/2)\Gamma(p + D/2)}{(4\pi)^{D/2}\Gamma(n)\Gamma(D/2)} \Delta^{D/2+p-n}. \tag{5.0.3}$$

We can combine the fourth and fifth steps into one to get the identity

$$\int \hat{\mathrm{d}}^D \ell \frac{(\ell^2)^p}{(\ell^2 - \Delta)^n} = \frac{i}{(4\pi)^{D/2}} \frac{(-1)^{p+n}\Gamma(n - p - D/2)\Gamma(p + D/2)}{\Gamma(n)\Gamma(D/2)} \Delta^{D/2+p-n}.$$

Note that the Δ term can be found from dimensional analysis. We know from the $\ell^2 - \Delta$ term that[2] $[\Delta] = 2$. Then looking at the integral on the left we have

[2] $[X]$ is the mass dimension of X, that is the power of mass (or energy or momentum) appearing in the dimensions

$[\hat{\mathrm{d}}^D\ell] = D$, $[(\ell^2)^p] = 2p$, and $[(\ell^2 - \Delta)^n] = 2n$, so the integral has mass dimension $D + 2p - 2n$. Dimensions can only enter the right hand side as Δ^x, since we are integrating out ℓ, so we must have $[\Delta^x] = 2x = D + 2p - 2n$, so $x = D/2 + p - n$.

In QED we often encounter **tensor integrals** such as

$$I^{\mu_1\cdots\mu_r} \coloneqq \int \hat{\mathrm{d}}^D\ell \frac{\ell^{\mu_1}\ldots\ell^{\mu_r}}{(\ell^2-\Delta)^n}. \tag{5.0.4}$$

First consider the $r = 1$ case:

$$I^\mu = \int \hat{\mathrm{d}}^D\ell \frac{\ell^\mu}{(\ell^2-\Delta)^n}. \tag{5.0.5}$$

A change of variables from ℓ to $-\ell$ gives an overall negative from $\mathrm{d}\ell \to -\mathrm{d}\ell$ as well as exchanging the limits from $(-\infty,\infty)$ to $(\infty,-\infty)$. We can change the limits back at the cost of another overall negative. We also get an overall negative from $\ell^\mu \to \ell^\mu$ so

$$I^\mu = \int \hat{\mathrm{d}}^D\ell \frac{-\ell^\mu}{(\ell^2-\Delta)^n} = -I^\mu \tag{5.0.6}$$

and so we must have $I^\mu = 0$. This same logic can be applied whenever r is odd, so $I^{\mu_1\cdots\mu_r} = 0$ for all odd r.

Now consider the $r = 2$ case:

$$I^{\mu\nu} = \int \hat{\mathrm{d}}^D\ell \frac{\ell^\mu\ell^\nu}{(\ell^2-\Delta)^n}. \tag{5.0.7}$$

The result must also be a rank two tensor. Since we are integrating over ℓ the result can only depend on Δ, which is just a scalar. The only other dependence in this integral is on the metric, $\eta^{\mu\nu}$, so this must be where the indices come from. We make the ansatz that

$$I^{\mu\nu} = \eta^{\mu\nu}X(\Delta) \tag{5.0.8}$$

where X is some function to be determined. Now notice that

$$\eta_{\mu\nu}I^{\mu\nu} = I^\mu{}_\mu = \eta_{\mu\nu}\eta^{\mu\nu}X(\Delta) = DX(\Delta) = \int \hat{\mathrm{d}}^D\ell \frac{\ell^2}{(\ell^2-\Delta)^n}, \tag{5.0.9}$$

and we know what this result integral is so we have

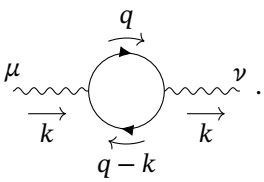
$$I^{\mu\nu} = \frac{1}{D}\int \hat{\mathrm{d}}^D\ell \frac{\eta^{\mu\nu}\ell^2}{(\ell^2-\Delta)^n}. \tag{5.0.10}$$

To conclude, in a loop integral we can replace $\ell^\mu\ell^\nu$ with $\eta^{\mu\nu}\ell^2/D$ so long as the denominator is of the form $(\ell^2 - \Delta)^n$. Other even r cases can be worked out in a similar manor.

5.1 Vacuum Polarisation

Let's return to the problem of computing the vacuum polarisation. Define a function corresponding to just the loop part of the vacuum polarisation:

$$i\widetilde{\Pi}^{\mu\nu}_{\mathrm{L}}(k) = \tag{5.1.1}$$



As we computed before using the Feynman rules gives the result

$$i\widetilde{\Pi}^{\mu\nu}_{\mathrm{L}}(k) = -(ie\mu^{\varepsilon})^2 \int \hat{\mathrm{d}}^D q \frac{\mathrm{tr}[\gamma^\mu i(\not{q} - \not{k} + m)\gamma^\nu i(\not{q} + m)]}{[(q-k)^2 - m^2 + i\varepsilon][q^2 - m^2 + i\varepsilon]}. \tag{5.1.2}$$

The first step of the algorithm is to use Feynman parametrisation. In this case we have two denominators, $(q-k)^2 - m^2 + i\varepsilon$ and $q^2 - m^2 + i\varepsilon$. We have to decide which one goes with x and which goes with $1-x$ in

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \, \frac{1}{[xA + (1-x)B]^2}. \tag{5.1.3}$$

A good idea is to put the more complicated factor with x, so that we don't increase the complexity. This gives

$$A = x[(q-k)^2 - m^2 + i\varepsilon] = xq^2 - 2xq\cdot k + xk^2 - xm^2 + ix\varepsilon, \tag{5.1.4}$$
$$B = (1-x)[q^2 - m^2 + i\varepsilon] = q^2 - xq^2 - m^2 + xm^2 + i\varepsilon - ix\varepsilon. \tag{5.1.5}$$

Hence, we have

$$xA + (1-x)B = q^2 - 2xq\cdot k + xk^2 - m^2 + i\varepsilon. \tag{5.1.6}$$

This allows us to rewrite the loop integral as

$$i\widetilde{\Pi}^{\mu\nu}_{\mathrm{L}}(k) = e^2\mu^{2\varepsilon} \int_0^1 \mathrm{d}x \int \hat{\mathrm{d}}^D q \frac{\mathrm{tr}[(\gamma^\mu(\not{q} - \not{k} + m)\gamma^\nu(\not{q} + m))]}{q^2 - 2xq\cdot k + xk^2 - m^2 + i\varepsilon}. \tag{5.1.7}$$

The next step of the algorithm is to shift the integration variable to get a denominator of the form $(\ell^2 - \Delta)^n$. To do this we choose $\ell = q - xk$, then we have $\ell^2 = q^2 - 2xq\cdot k + x^2k^2$ and so

$$q^2 - 2xq\cdot k + xk^2 - m^2 + i\varepsilon = \ell^2 - m^2 + x(1-x)k^2 + i\varepsilon = \ell^2 - \Delta \tag{5.1.8}$$

with

$$\Delta \coloneqq m^2 - x(1-x)k^2 - i\varepsilon. \tag{5.1.9}$$

Then the loop integral is

$$i\widetilde{\Pi}^{\mu\nu}_{\mathrm{L}}(k) = -e^2\mu^{2\varepsilon} \int_0^1 \mathrm{d}x \int \hat{\mathrm{d}}^D \ell \frac{N^{\mu\nu}}{(\ell^2 - \Delta)^2} \tag{5.1.10}$$

with

$$N^{\mu\nu} = \mathrm{tr}[\gamma^\mu(\not{q} - \not{k} + m)\gamma^\nu(\not{q} + m)]. \tag{5.1.11}$$

The next step in the algorithm is to simplify the numerator. The first step is to write the numerator in terms of ℓ which can be done by inverting $\ell = q - xk$ to get $q = \ell + xk$. We also have $q - k = \ell + xk - k = \ell - (1-x)k$. Thus the numerator is

$$N^{\mu\nu} = \mathrm{tr}[\gamma^\mu(\not{\ell} - (1-x)\not{k} + m)\gamma^\nu(\not{\ell} + x\not{k} + m)]. \tag{5.1.12}$$

We can now expand the product in the trace. In doing so we keep only terms with an even power of gamma matrices, since the trace of an odd number of gamma matrices vanishes. We also keep only terms with an even power of ℓ, since under the integral any odd power of ℓ vanishes. Note that this is only true under the integral, not in general, so rather than using equality we'll use the symbol $\rightsquigarrow$:

$$N^{\mu\nu} \rightsquigarrow \mathrm{tr}[\gamma^\mu \not{\ell} \gamma^\nu \not{\ell} - x(1-x)\gamma^\mu \not{k} \gamma^\nu \not{k} + m^2 \gamma^\mu \gamma^\nu]. \tag{5.1.13}$$

To proceed we need the following two identities for traces of gamma matrices:

$$\mathrm{tr}[\gamma^\mu \gamma^\nu] = 4\eta^{\mu\nu}, \tag{5.1.14}$$

$$\mathrm{tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = 4(\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\sigma\mu}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\sigma}). \tag{5.1.15}$$

Note that the factor of 4 is the dimension of the spinors. It is possible to define the spinors in such a way that they are four component objects even in dimensional regularisation where the spacetime dimension is $D = 4 - 2\varepsilon$. Consider the first term in the trace, we have

$$\begin{aligned}
\mathrm{tr}[\gamma^\mu \not{\ell} \gamma^\nu \not{\ell}] &= \ell_\rho \ell_\sigma \,\mathrm{tr}[\gamma^\mu \gamma^\rho \gamma^\nu \gamma^\sigma] && (5.1.16)\\
&= 4\ell_\rho \ell_\sigma [\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\rho\nu}\eta^{\sigma\mu} - \eta^{\mu\nu}\eta^{\rho\sigma}] && (5.1.17)\\
&= 4[\ell^\mu \ell^\nu + \ell^\nu \ell^\mu - \eta^{\mu\nu}\ell^2] && (5.1.18)\\
&= 4[2\ell^\mu \ell^\nu - \eta^{\mu\nu}\ell^2]. && (5.1.19)
\end{aligned}$$

In general in traces like this if we have $\gamma^\mu \not{\ell}$ then we the index μ in the identity for traces of $\gamma^\mu \gamma^\nu$ will be attached to the ℓ in the final result. The second term in the numerator is the same but with ks in place of the ℓs, and an extra scalar factor. The last term in the trace is simply $4m^2\eta^{\mu\nu}$. Combined we have

$$N^{\mu\nu} \rightsquigarrow 4[2\ell^\mu \ell^\nu - \eta^{\mu\nu}\ell^2 - x(1-x)(2k^\mu k^\nu - \eta^{\mu\nu}k^2) + m^2\eta^{\mu\nu}]. \tag{5.1.20}$$

We can further simplify this by using $\Delta = m^2 - x(1-x)k^2 - i\varepsilon$ and so $m^2 = \Delta + x(1-x)k^2 + i\varepsilon$ giving

$$N^{\mu\nu} \rightsquigarrow 4[2\ell^\mu \ell^\nu - \eta^{\mu\nu}\ell^2 - 2x(1-x)(k^\mu k^\nu - \eta^{\mu\nu}k^2) + \Delta\eta^{\mu\nu}]. \tag{5.1.21}$$

The final simplification we can make is to use our earlier discovery that we can replace $\ell^\mu \ell^\nu$ with $\eta^{\mu\nu}\ell^2/D$ under a one loop integral with denominator $(\ell^2 - \Delta)^n$. This gives

$$\begin{aligned}
N^{\mu\nu} &\rightsquigarrow 4\left[\frac{2}{D}\eta^{\mu\nu}\ell^2 - \eta^{\mu\nu}\ell^2 - 2x(1-x)(k^\mu k^\nu - \eta^{\mu\nu}k^2) + \Delta\eta^{\mu\nu}\right] && (5.1.22)\\
&= 4\left[\left(\frac{2}{D} - 1\right)\eta^{\mu\nu}\ell^2 - 2x(1-x)(k^\mu k^\nu - \eta^{\mu\nu}) + \Delta\eta^{\mu\nu}\right]. && (5.1.23)
\end{aligned}$$

We can now use the two following integrals, first

$$\int \hat{\mathrm{d}}^D \ell \frac{1}{(\ell^2 - \Delta)^2} = \frac{i}{(4\pi)^{D/2}} \Delta^{D/2-2} \Gamma(2 - D/2) \tag{5.1.24}$$

which follows by setting $p = 0$ and $n = 2$ in the general formula and using $\Gamma(1) = 1$. Second, we need

$$\int \hat{\mathrm{d}}^D \ell \frac{\ell^2}{(\ell^2 - \Delta)^2} = -\frac{i}{(4\pi)^{D/2}} \Delta^{D/2-1} \frac{\Gamma(1 - D/2)\Gamma(1 + D/2)}{\Gamma(D/2)}. \tag{5.1.25}$$

This can be simplified using the identity $\Gamma(1+z) = z\Gamma(z)$, which is just the factorial recursive definition extended to the Γ function. With this we have

$$\Gamma(1 + D/2) = \frac{D}{2}\Gamma(D/2) \tag{5.1.26}$$

and so

$$\int \hat{\mathrm{d}}^D \ell \frac{\ell^2}{(\ell^2 - \Delta)^2} = -\frac{i}{(4\pi)^{D/2}} \Delta^{D/2-1} \frac{D}{2}\Gamma(1 - D/2). \tag{5.1.27}$$

Using these integrals for each term of the full integral we get the result

$$\begin{aligned} i\widetilde{\Pi}^{\mu\nu}_{\mathrm{L}}(k) = -4e^2\mu^{2\varepsilon} \int_0^1 \mathrm{d}x \frac{i}{(4\pi)^{D/2}} \\ \times \Bigg[\frac{D}{2}\left(\frac{2}{D} - 1\right)\Gamma(1 - D/2)\Delta^{D/2-1}\eta^{\mu\nu} \\ - 2x(1 - x)\Gamma(2 - D/2)\Delta^{D/2-2}(k^\mu k^\nu - \eta^{\mu\nu}k^2) \\ + \Gamma(2 - D/2)\Delta^{D/2-1}\eta^{\mu\nu} \Bigg]. \end{aligned} \tag{5.1.28}$$

Now we can use

$$\frac{D}{2}\left(\frac{2}{D} - 1\right)\Gamma(1 - D/2) = \left(1 - \frac{D}{2}\right)\Gamma(1 - D/2) = \Gamma(2 - D/2), \tag{5.1.29}$$

which is just another application of $z\Gamma(z) = \Gamma(1 + z)$. Doing this the first and last term above are the same and so cancel out giving the final result

$$i\widetilde{\Pi}^{\mu\nu}_{\mathrm{L}}(k) = \frac{8e^2\mu^{2\varepsilon}}{(4\pi)^{D/2}}(k^\mu k^\nu - \eta^{\mu\nu}k^2)\Gamma(2 - D/2) \int_0^1 \mathrm{d}x\, x(1 - x)\Delta^{D/2-2}. \tag{5.1.30}$$

Note that Δ depends on x so we can't pull it outside of the integral.

5.1.1 Regularisation

We will use dimensional regularisation, setting the number of dimensions to $D = 4 - 2\varepsilon$ for some $\varepsilon \in \mathbb{R}$ which we will later take to zero. This means that we can neglect terms on the order of ε, although these terms may be important for two loop diagrams. We keep the poles, terms like $1/\varepsilon$, and finite[3] terms, $\mathcal{O}(\varepsilon^0)$.

[3] finite here meaning finite and nonzero

The gamma function appearing in the vacuum polarisation loop becomes

$$\Gamma(2 - D/2) = \Gamma(\varepsilon). \tag{5.1.31}$$

We can then use the expansion of the gamma function about zero:

$$\Gamma(\varepsilon) \approx \frac{1}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon) \tag{5.1.32}$$

where

$$\gamma \approx 0.57721566 \tag{5.1.33}$$

is the **Euler–Mascheroni constant**. It is actually more useful to write this expansion as

$$\frac{1}{\varepsilon}\mathrm{e}^{-\varepsilon\gamma} + \mathcal{O}(\varepsilon) \approx \frac{1}{\varepsilon}(1 - \varepsilon\gamma) + \mathcal{O}(\varepsilon) = \frac{1}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon) \approx \Gamma(\varepsilon). \tag{5.1.34}$$

Using this, and factoring out a minus sign from the tensor structure, we can write the vacuum polarisation loop as

$$i\widetilde{\Pi}^{\mu\nu}_{\mathrm{L}}(k) = -\frac{8ie^2}{16\pi^2}(4\pi\mu^2\mathrm{e}^{-\gamma})^{\varepsilon}(k^2\eta^{\mu\nu} - k^\mu k^\nu)\frac{1}{\varepsilon}\int_0^1 \mathrm{d}x\, x(1-x)\Delta^{-\varepsilon}. \quad (5.1.35)$$

The integral here is over a finite region, $[0,1]$, and so diverges only if the integrand diverges. In dimensional regularisation we always get factors of $4\pi\mu^2\mathrm{e}^{-\gamma}$, so it helps to define

$$\tilde{\mu}^2 = 4\pi\mu^2\mathrm{e}^{-\gamma} \approx 7.0555\mu^2. \quad (5.1.36)$$

Now write

$$\begin{aligned}\tilde{\mu}^{2\varepsilon}\Delta^{-\varepsilon} &= \left(\frac{\Delta}{\tilde{\mu}^2}\right)^{-\varepsilon} && (5.1.37)\\ &= \exp\left\{\log\left(\frac{\Delta}{\tilde{\mu}^2}\right)^{-\varepsilon}\right\} && (5.1.38)\\ &= \exp\left\{-\varepsilon\log\frac{\Delta}{\tilde{\mu}^2}\right\} && (5.1.39)\\ &\approx 1 - \varepsilon\log\frac{\Delta}{\tilde{\mu}^2} + \mathcal{O}(\varepsilon^2). && (5.1.40)\end{aligned}$$

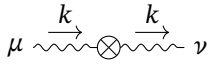
Then we have

$$i\widetilde{\Pi}^{\mu\nu}_{\mathrm{L}}(k) = -\frac{ie^2}{2\pi^2}(k^2\eta^{\mu\nu} - k^\mu k^\nu)\int_0^1 \mathrm{d}x\, x(1-x)\left[\frac{1}{\varepsilon} - \log\frac{\Delta}{\tilde{\mu}^2}\right]. \quad (5.1.41)$$

This is now perfectly finite so long as $\varepsilon \neq 0$ (and $\Delta/\tilde{\mu}^2 > 0$).

5.1.2 Renormalisation

We require that correlators of renormalised fields are finite. Recall that the counter term appearing in the vacuum polarisation is

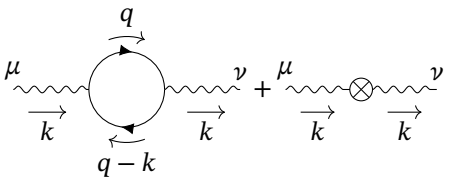


$$= -i\delta_3(k^2\eta^{\mu\nu} - k^\mu k^\nu). \quad (5.1.42)$$

Fortunately this has the same tensor structure, $k^2\eta^{\mu\nu} - k^\mu k^\nu$, as the loop part of the vacuum polarisation. This is important as it allows us to choose δ_3 to cancel the infinities which occur in the loop integral to get a finite result. This is not always the case. For example, if we use an ultraviolet cut-off, Λ, then the cancellation earlier in the calculation doesn't occur and we don't have matching tensor structures. The reason for this is that dimensional regularisation respects gauge symmetry, but an ultraviolet cut-off doesn't, since, for example, a gauge transformation $p_\mu \mapsto p_\mu + eA_\mu$ could result in a momentum with $p^2 > \Lambda^2$.

We can write the full one-loop vacuum polarisation as

$$\widetilde{\Pi}^{\mu\nu}(k) = \quad (5.1.43)$$



$$= (k^2\eta^{\mu\nu} - k^\mu k^\nu)\left[-\delta_3 - \frac{e^2}{2\pi^2}\int_0^1 \mathrm{d}x\, x(1-x)\left[\frac{1}{\varepsilon} - \log\frac{\Delta}{\tilde{\mu}^2}\right]\right].$$

We can compute the first term in the integral,

$$\frac{1}{\varepsilon}\int_0^1 \mathrm{d}x\, x(1-x) = \frac{1}{6\varepsilon}, \quad (5.1.44)$$

giving

$$\begin{aligned}\widetilde{\Pi}^{\mu\nu}(k) &= (k^2\eta^{\mu\nu} - k^\mu k^\nu)\left[-\delta_3 - \frac{e^2}{12\pi^2\varepsilon} - \frac{e^2}{2\pi^2}\int_0^1 \mathrm{d}x\, x(1-x)\log\frac{\Delta}{\tilde{\mu}^2}\right] \\ &= (k^2\eta^{\mu\nu} - k^\mu k^\nu)\Pi(k^2) \quad (5.1.45)\end{aligned}$$

where we define $\Pi(k^2)$ to contain the scalar structure of the vacuum polarisation.

We demand that this scalar structure is finite, however currently we have a pole at $\varepsilon = 0$ from the $1/\varepsilon$ term. The solution is to choose δ_3 to be of the form

$$\delta_3 = -\frac{e^2}{12\pi^2\varepsilon} + \text{finite terms} \quad (5.1.46)$$

so that the $1/\varepsilon$ terms cancel out and we are left only with the finite terms.

5.1.2.1 Choice of Renormalisation Scheme

Clearly in the above prescription for δ_3 we have some freedom in the choice of finite terms. In general we can add terms $\mathcal{O}(\varepsilon^0)$ to δ_3 and the result will still be finite. This freedom is the freedom in choosing a **renormalisation scheme**. We'll discuss two renormalisation schemes here.

$\overline{MS}$ Scheme. The **$\overline{\text{MS}}$** scheme, standing for **modified minimal subtraction**, is defined by two requirements:

- We use
$$\tilde{\mu}^2 = 4\pi\mu^2 e^{-\gamma} \quad (5.1.47)$$
as a parameter. This is the "modified" part of modified minimal subtraction, and simply reduces the number of 4πs and $\mathrm{e}^{-\gamma}$s that we have to deal with.
- We require counterterms to contain only poles, so no finite terms, this is the "minimal subtraction" part of modified minimal subtraction. This means that all counterterms are of the form
$$\delta_i = \sum_{n=1}^{\infty}\frac{a_n}{\varepsilon^n} \quad (5.1.48)$$
for some finite quantities a_n.

Thus in $\overline{\text{MS}}$ the counterterm for the vacuum polarisation term is

$$\delta_3 = \frac{e^2}{12\pi^2\varepsilon} + \mathcal{O}\left(\frac{e^4}{\varepsilon^2}\right). \tag{5.1.49}$$

The factor in front of any higher order term is just a number, it has no dependence on momentum, since it can be traced back to the Lagrangian where all momenta are explicitly accounted for.

We will almost entirely use the $\overline{\text{MS}}$ scheme, but it isn't the only choice. For the rest of this section we'll briefly discuss an alternative.

On-shell Scheme. The **on-shell scheme** is imposed by the normalisation condition $\Pi(0) = 0$. If we write out $\Pi(k^2)$ in full[4],

$$\Pi(k^2) = -\left[\delta_3 + \frac{e^2}{12\pi^2\varepsilon} - \frac{e^2}{2\pi^2}\int_0^1 \mathrm{d}x\, x(1-x)\log\left(\frac{m^2 - x(1-x)k^2 - i\epsilon}{\tilde{\mu}^2}\right)\right] \tag{5.1.50}$$

[4] we use ε here for the parameter of dimensional regularisation and ϵ for the parameter of the $i\epsilon$ prescription.

then we see that at $k^2 = 0$ we can take $\epsilon \to 0$ since $m^2 > 0$. The integral is then easy to do, again giving $1/6$, and so

$$\Pi(0) = -\left[\delta_3 + \frac{e^2}{12\pi^2\varepsilon} - \frac{e^2}{12\pi^2}\log\frac{m^2}{\tilde{\mu}^2}\right]. \tag{5.1.51}$$

Demanding $\Pi(0) = 0$ we get

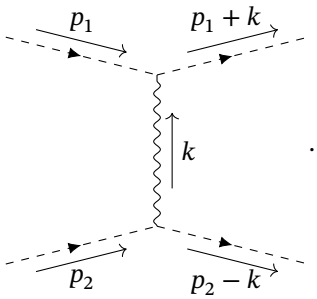
$$\delta_3 = -\frac{e^2}{12\pi^2\varepsilon} + \frac{e^2}{12\pi^2}\log\frac{m^2}{\tilde{\mu}^2}. \tag{5.1.52}$$

A sensible question is why we impose $\Pi(0) = 0$ as a condition. There are at least two reasons for this. First, doing so ensures that the photon propagator takes on its tree form,

$$\frac{i\eta^{\mu\nu}}{k^2 + i\epsilon}. \tag{5.1.53}$$

Essentially, this means that the photon remains massless, since the propagator has a pole at the physical mass.

The second reason can be demonstrated by considering $2 \to 2$ scattering. For simplicity we'll consider scalars scattering, which just allows us to drop the spinors that would normally occur. Consider the process



. (5.1.54)

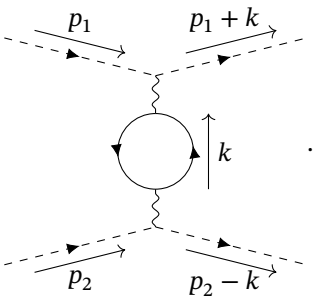
The amplitude for this at tree level is

$$i\mathcal{A}_{\text{tree}} = \frac{ie^2}{k^2}(4p_1 \cdot p_2 - k^2). \tag{5.1.55}$$

At large distances k^2 is small, so

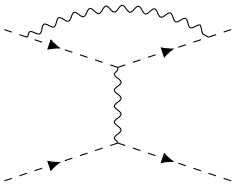
$$i\mathcal{A}_{\text{tree}} \approx \frac{ie^2}{k^2} 4p_1 \cdot p_2. \tag{5.1.56}$$

At one loop we encounter diagrams like



. (5.1.57)

In fact, we can carefully choose our theory so that this is the dominant diagram at one-loop. Diagrams such as



(5.1.58)

are suppressed by powers of $1/m$, with m the mass of the external particles. By choosing a theory with lots of fermions we can also make the fermion loop the dominant loop.

Notice that the diagram contains the loop diagram in the vacuum polarisation as a subdiagram. This means that we can just replace the normal photon propagator with this term, which we've already computed. This gives

$$i\mathcal{A}_{\text{loop}} = (ie)^2(2p_1 + k)^\mu(2p_2 - k)^\nu \left(-\frac{i}{k^2}\right)(k^2\eta_{\mu\nu} - k_\mu k_\nu)\Pi(k^2) \tag{5.1.59}$$

$$= (ie)^2 4(k^2(p_1 \cdot p_2) - (p_1 \cdot k)(p_2 \cdot k))\frac{1}{k^4}\Pi(k^2) \tag{5.1.60}$$

where in the second line we've used $k^\mu(k^2\eta_{\mu\nu} - k_\mu k_\nu) = 0$. At large distances we then have

$$i(\mathcal{A}_{\text{tree}} + \mathcal{A}_{\text{loop}}) \approx \frac{i}{k^2} 4p_1 \cdot p_2 e^2(1 + \Pi(k^2)). \tag{5.1.61}$$

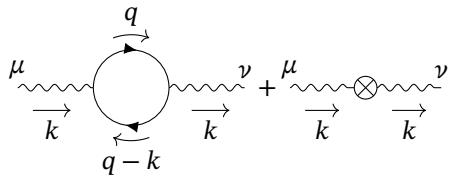
One way to measure the charge of a particle is to perform scattering experiments like this one. These are typically done at low momentum. What we actually measure in these experiments is the factor $e^2(1+\Pi(k^2))$, plus any higher order corrections. Choosing $\Pi(0) = 0$ then means that the value we are measuring is the renormalised charge, which is the charge appearing in the Lagrangian after regularisation.

Six

Interpreting The Vacuum Polarisation

Recall that in the last chapter we computed the vacuum polarisation

$$\Pi^{\mu\nu}(k) = \quad + \tag{6.0.1}$$



$$= (k^\mu k^\nu - \eta k^2)\Pi(k^2) \tag{6.0.2}$$

where

$$\Pi(k^2) = \begin{cases} \dfrac{e^2}{2\pi^2}\displaystyle\int_0^1 \mathrm{d}x\, x(1-x)\log\frac{\Delta}{\tilde{\mu}^2} & \overline{\text{MS}}, \\ \dfrac{e^2}{2\pi^2}\displaystyle\int_0^1 \mathrm{d}x\, x(1-x)\log\frac{\Delta}{m^2} & \text{on-shell}, \end{cases} \tag{6.0.3}$$

where

$$\Delta = m^2 - x(1-x)k^2 - i\epsilon. \tag{6.0.4}$$

Now we're going to try to extract some physics from this result and discuss the effects of the two different renormalisation scales.

6.1 Scale Dependence of Charge

Previously (Equation (5.1.61)) we saw that the amplitude for $2 \to 2$ scattering process between scalar particles exchanging a photon has

$$i\mathcal{A} = \frac{4i}{k^2} p_1 \cdot p_2 e^2(1 + \Pi(k^2)). \tag{6.1.1}$$

This includes the tree level process and a single fermion loop in the photon propagator, and it is assumed that k^2 is small.

We can use this process to measure the charge of the particle in scattering process with zero, or very little, momentum. We find that measured charge squared,

at one loop accuracy, is then

$$e^2_{\text{meas}} = e^2(1 + \Pi(0)) \tag{6.1.2}$$

$$= \begin{cases} e^2\left(1 + \dfrac{e^2}{12\pi^2}\log\dfrac{m^2}{\tilde{\mu}^2}\right) & \overline{\text{MS}}, \\ e^2 & \text{on-shell}. \end{cases} \tag{6.1.3}$$

We can see that the relation between the Lagrangian parameters and the physical observables is scheme dependent. This isn't too much of a problem because in QED we can make a finite number of measurements to determine these parameters and from there we can use them to make scheme independent predictions.

For example, at nonzero momentum there is a correction to the measured charge squared:

$$e^2_{\text{meas}}(k^2) - e^2_{\text{meas}}(0) = e^2(1 + \Pi(k^2)) - e^2(1 + \Pi(0)) \tag{6.1.4}$$

$$= e^2(\Pi(k^2) - \Pi(0)) \tag{6.1.5}$$

$$= \frac{e^4}{2\pi^2}\int_0^1 \mathrm{d}x\, x(1-x)\log\frac{\Delta}{m^2} \tag{6.1.6}$$

in the $\overline{\text{MS}}$ scheme. This $\mathcal{O}(e^4)$ correction to the measured charge is a physical prediction of QED.

6.2 Non-Relativistic Limit

We can take the non-relativistic limit in QFT to compare results to results from non-relativistic quantum mechanics. Recall that in quantum mechanics the Born approximation gives the scattering amplitude

$$\mathcal{A} = -4m_1m_2\int \mathrm{d}^3x\, \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}V(\boldsymbol{x}) \tag{6.2.1}$$

where m_i are the masses of the two particles scattering and $V(\boldsymbol{x})$ is the potential they are scattering in, with $\boldsymbol{x}$ being the vector between the two particles.

Note that this differs by a factor of $2m$ from the amplitudes we have been calculating. This is due to a difference in normalisation, where in quantum mechanics

$$\langle \boldsymbol{p}|\boldsymbol{p}'\rangle = \delta(\boldsymbol{p} - \boldsymbol{p}') \tag{6.2.2}$$

and in QFT

$$\langle p|p'\rangle = \frac{1}{2E_p}\delta(p - p') \tag{6.2.3}$$

with $2E_p \approx 2m$ in the non-relativistic limit.

Consider Coulomb scattering, the amplitude calculated in quantum mechanics is then

$$\mathcal{A} = -4m_1m_2\int \mathrm{d}^3x\, \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{e_1e_2}{4\pi|\boldsymbol{x}|} \tag{6.2.4}$$

$$= -4m_1m_2\frac{e_1e_2}{|\boldsymbol{k}|^2}. \tag{6.2.5}$$

For the non-relativistic QFT result we take k^2 to be small. Note that $p_1 \cdot p_2 \approx m_1 m_2$. There is an incoming (and hence on-shell) particle with momentum p_1, so $p_1^2 = m_1^2$. There is an outgoing (and hence on-shell) particle with momentum $p_1 + k$, so

$$(p_1 + k)^2 = m_1^2 = m_1^2 + 2p_1 \cdot k + k^2. \tag{6.2.6}$$

From this, neglecting k^2 terms, we see that $2p_1 \cdot k \approx 0$. Since $p_1^0 \approx m_1$, which can be large, we must have $k^0 \approx 0$, and so $k^2 \approx -\boldsymbol{k}^2$. Then the amplitude calculated in QFT is

$$\mathcal{A} = -\frac{4m_1 m_2}{|\boldsymbol{k}|^2} e_1 e_2 (1 + \Pi(-\boldsymbol{k}^2)). \tag{6.2.7}$$

We can interpret this as having a one-loop correction to the potential, given in momentum space by

$$\Delta\tilde{V}(\boldsymbol{k}) = \frac{e_1 e_2}{\boldsymbol{k}^2} \Pi(-\boldsymbol{k}^2) \tag{6.2.8}$$

and in real space by

$$\Delta V(\boldsymbol{r}) = e_1 e_2 \int \hat{\mathrm{d}}^3 k \, \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \frac{\Pi(-\boldsymbol{k}^2)}{\boldsymbol{k}^2}. \tag{6.2.9}$$

This corrected potential is called the **Uehling potential**.

In the on-shell scheme for $k^2 \ll m^2$ we can expand

$$\begin{aligned} \Pi(k^2) &= \frac{e^2}{2\pi^2} \int_0^1 \mathrm{d}x \, x(1-x) \log\left(1 - x(1-x)\frac{k^2}{m^2}\right) && (6.2.10)\\ &\approx \frac{e^2}{2\pi^2} \int_0^1 \mathrm{d}x \, x(1-x) \left[-x(1-x)\frac{k^2}{m^2}\right] && (6.2.11)\\ &= \frac{e^2}{60\pi^2} \frac{\boldsymbol{k}^2}{m^2}. && (6.2.12) \end{aligned}$$

Here we've used the expansion $\log(1 + \alpha) \approx \alpha$ and we've set $\epsilon = 0$ since the term in the logarithm is positive as $k^2 < m^2$ and $0 < x(1-x) < 1/4$.

Hence, the correction to the potential is

$$\Delta V(\boldsymbol{r}) = \frac{e_1 e_2 e^2}{60\pi^2 m^2} \delta(\boldsymbol{r}). \tag{6.2.13}$$

If we consider one of the particles to be a nucleus with Z protons then $e_1 = -Ze$, taking the other particle to be an electron, $e_2 = e$, we can interpret this as a shift in the energy levels of the atom. Note that since we're considering scalar particles scattering we are essentially ignoring spin-orbit effects, however these are subdominant terms since there is no direct spin-dependence in the potential. The energy shift is given by

$$\Delta E = \int \mathrm{d}^3 x \, \psi^*(\boldsymbol{x}) \Delta V(\boldsymbol{x}) \psi(\boldsymbol{x}) \tag{6.2.14}$$

where ψ is the wave function of the electron in the atom. Since we have a factor of $\delta(\boldsymbol{r})$ this shift only applies to wave functions which are nonzero at the origin, meaning it only applies to s states. The energy shift to these states is

$$\Delta E = -|\psi(0)|^2 \frac{4Z\alpha^2}{15m^2} \tag{6.2.15}$$

with $\alpha = e^2/(4\pi)$ being the **fine structure constant**.

This shift is one contribution to the Lamb shift, the difference between the energy levels $^2\mathrm{S}_{1/2}$ and $^2\mathrm{P}_{1/2}$ which is not predicted by the Dirac equation alone. It is not the dominant contribution to the Lamb shift. This shift is more important for muonic atoms, where the electron is replaced by a muon, since heavier external particles make the one-fermion-loop diagram more dominant, as opposed to diagrams with, say, photon loops, since these are suppressed by factors of $1/m$, with m the mass of the external particles.

6.3 Effective Charge Distribution

We can write the corrected potential as

$$V(\boldsymbol{r}) = e_1 e_2 \int \mathrm{d}^3x\, \mathrm{d}^3y\, \frac{\eta(\boldsymbol{x})\eta(\boldsymbol{y})}{4\pi|\boldsymbol{x}-\boldsymbol{y}+\boldsymbol{r}|} \tag{6.3.1}$$

where

$$\eta(\boldsymbol{r}) = \delta(\boldsymbol{r}) + \frac{1}{2}\hat{\mathrm{d}}^3k\, \Pi(-\boldsymbol{k}^2)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{6.3.2}$$

Note that we are implicitly dropping the Π^2 term appearing in the product $\eta(\boldsymbol{x})\eta(\boldsymbol{y})$ since this term is $\mathcal{O}(e^6)$, which is of a higher order than the one loop calculations we have been doing, and so is not physically meaningful if we don't also include two loop calculations. To see how this works expand this form of the potential, dropping the Π^2 term, giving

$$\begin{aligned} V(\boldsymbol{r}) &= e_1 e_2 \int \mathrm{d}^3x\, \mathrm{d}^3y\, \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}+\boldsymbol{r}|}\left[\delta(\boldsymbol{x}) + \frac{1}{2}\int \hat{\mathrm{d}}^3k\, \Pi(-\boldsymbol{k}^2)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\right] \\ &\qquad \times \left[\delta(\boldsymbol{y}) + \frac{1}{2}\int \hat{\mathrm{d}}^3k\, \Pi(-\boldsymbol{k}^2)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{y}}\right] && (6.3.3) \\ &= e_1 e_2 \int \mathrm{d}^3x\, \mathrm{d}^3y \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}+\boldsymbol{r}|}\Big[\delta(\boldsymbol{x})\delta(\boldsymbol{y}) && (6.3.4) \\ &\qquad + \frac{1}{2}\delta(\boldsymbol{x})\int \hat{\mathrm{d}}^3k\, \Pi(-\boldsymbol{k}^2)\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{y}} + \frac{1}{2}\delta(\boldsymbol{y})\int \hat{\mathrm{d}}^3k\, \Pi(-\boldsymbol{k}^2)\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\Big] \\ &= \frac{e_1e_2}{4\pi|\boldsymbol{r}|} + \frac{1}{2}\int \mathrm{d}^3y \frac{e_1e_2}{4\pi|-\boldsymbol{y}+\boldsymbol{r}|}\int \hat{\mathrm{d}}^3k\, \Pi(-\boldsymbol{k}^2)\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{y}} \\ &\qquad + \frac{1}{2}\int \mathrm{d}^3y \frac{e_1e_2}{4\pi|\boldsymbol{x}+\boldsymbol{r}|}\int \hat{\mathrm{d}}^3k\, \Pi(-\boldsymbol{k}^2)\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}} && (6.3.5) \\ &= \frac{e_1e_2}{4\pi|\boldsymbol{r}|} + \int \mathrm{d}^3x \frac{e_1e_2}{4\pi|\boldsymbol{x}+\boldsymbol{r}|}\int \hat{\mathrm{d}}^3k\, \Pi(-\boldsymbol{k}^2)\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}. && (6.3.6) \end{aligned}$$

In the last step we make the transformation of variables $\boldsymbol{y} \mapsto -\boldsymbol{x}$ and identify that both integrals are the same. Now recognise that we can write the first factor in the

integral as the inverse Fourier transform of it's Fourier transform:

$$\frac{1}{4\pi|\boldsymbol{x}+\boldsymbol{r}|} = \int \hat{\mathrm{d}}^3 k \, \frac{1}{|\boldsymbol{k}|^2} \mathrm{e}^{-i\boldsymbol{k}\cdot(\boldsymbol{x}+\boldsymbol{r})} \tag{6.3.7}$$

and so we get

$$V(\boldsymbol{r}) = \frac{e_1 e_2}{4\pi|\boldsymbol{r}|} + \int \hat{\mathrm{d}}^3 k \, \frac{1}{|\boldsymbol{k}|^2} \Pi(-\boldsymbol{k}^2) \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{6.3.8}$$

This takes the form we had in the previous section of a Coulomb potential plus the correction from one loop diagrams.

The interpretation of this rewriting of the potential is that the electron's charge distribution is a superposition of a point charge, given by the $1/|\boldsymbol{r}|$ Coulomb part of the potential, but also has some spatial distribution coming from this extra one loop term. The intuition is that this is similar to a point charge in a medium causing that medium to become polarised around it. The electron polarises the sea of virtual particles around it. This is why we call this term the vacuum polarisation, since it occurs outside of any medium.

6.4 Large Logs

In the on-shell scheme we have

$$\Pi(k^2) = \frac{e^2}{2\pi^2} \int_0^1 \mathrm{d}x \, x(1-x) \log\left[1 - \frac{x(1-x)k^2 + i\epsilon}{m^2}.\right] \tag{6.4.1}$$

In perturbation theory we expect that the one loop diagrams are small corrections to the tree level amplitude. Since $e^2 = 4\pi\alpha \approx 4\pi/137 \approx 0.09$ is small many one loop corrections are small. However this isn't enough on its own. If k^2 is large enough then the logarithm can become large. This is the problem of **large logs**.

This is a very real worry, for example, the LHC operates at $\sqrt{s} = 13\,\mathrm{TeV}$, meaning that $k^2 \approx s = (13\,\mathrm{TeV})^2$, so an electron created in this process will have $k^2/m^2 \sim 10^{14}$, so $\log(k^2/m^2) \sim 34$, which is large enough to require higher order terms to achieve the desired accuracy. We also run into this issue if we try to take the $m \to 0$ limit, which can be a useful thing to do.

This problem of large logs is worse in QCD where $\alpha_S \approx 1/10$, so the coupling doesn't provide as much suppression of higher order terms.

The way we get around this is to instead work in the $\overline{\mathrm{MS}}$ scheme, where this can still be an issue. We know that $\tilde{\mu}$ is non-physical, and so can't appear in an amplitude calculated to all orders, however $\tilde{\mu}$ does appear in truncated calculations. By carefully choosing the value of $\tilde{\mu}$ we can avoid large logs. This leads to the somewhat unsettling observation that while $\tilde{\mu}$ is non-physical it does affect the rate of "convergence" of our perturbative series, which certainly feels like it is a physical result.

Seven

All Orders Properties of QED

7.1 Ward Identities

We've seen that $i\Pi^{\mu\nu}(k) \propto (k^\mu k^\nu - \eta^{\mu\nu}k^2)$, and this means that $k_\mu \Pi^{\mu\nu}(k) = 0$ at one loop. This is reminiscent of the Ward identities discussed in *Quantum Field Theory*. For processes with an external photon the amplitude contains a factor of $\varepsilon^\mu(k)$, so we can define $\mathcal{A}_\mu$ to be such that $\mathcal{A} = \varepsilon^\mu(k)\mathcal{A}_\mu$. It can then be shown that $k^\mu \mathcal{A}_\mu = 0$, this is called the **Ward identity**, and is required for the gauge transformation of $\varepsilon^\mu(k)$ to leave the physics invariant, since this transformation takes the form $\varepsilon^\mu(k) \mapsto \varepsilon^\mu(k) + \alpha p^\mu$ under the gauge transformation $\psi \mapsto \mathrm{e}^{i\alpha}\psi$. We will generalise this identity to correlators in QED and any loop level.

7.2 Rewriting the Action

The photon action, that is the terms in the action involving only the four-potential, is given in terms of the bare fields by

$$S_\gamma = \int \mathrm{d}^D x \left[-\frac{1}{4} F_{\mathrm{B}}^{\mu\nu} F_{\mathrm{B}\mu\nu} - \frac{1}{2}\xi(\partial_\mu A_{\mathrm{B}}^\mu)^2 \right]. \tag{7.2.1}$$

If we work in the Feynman gauge, which we will for the rest of this chapter, then $\xi = 1$ and we have

$$\begin{aligned}
S_\gamma &= \int \mathrm{d}^D x \left[-\frac{1}{4}(\partial^\mu A_{\mathrm{B}}^\nu - \partial^\nu A_{\mathrm{B}}^\mu)(\partial_\mu A_{\mathrm{B}\nu} - \partial_\nu A_{\mathrm{B}\mu}) - \frac{1}{2}(\partial_\mu A_{\mathrm{B}}^\mu)^2 \right] && (7.2.2)\\
&= \int \mathrm{d}^D x \Big[-\frac{1}{4}\{(\partial^\mu A_{\mathrm{B}}^\nu)(\partial_\mu A_{\mathrm{B}\nu}) - (\partial^\mu A_{\mathrm{B}}^\nu)(\partial_\nu A_{\mathrm{B}\mu}) \\
&\qquad - (\partial^\nu A_{\mathrm{B}}^\mu)(\partial_\nu A_{\mathrm{B}\mu}) + (\partial^\nu A_{\mathrm{B}}^\mu)(\partial_\nu A_{\mathrm{B}\mu})\} - \frac{1}{2}(\partial_\mu A_{\mathrm{B}}^\mu)^2 \Big] && (7.2.3)\\
&= \int \mathrm{d}^D x \left[-\frac{1}{2}\{(\partial^\mu A_{\mathrm{B}}^\nu)(\partial_\mu A_{\mathrm{B}\nu}) - (\partial^\mu A_{\mathrm{B}}^\nu)(\partial_\nu A_{\mathrm{B}\mu})\} - \frac{1}{2}(\partial_\mu A_{\mathrm{B}}^\mu)^2 \right] && (7.2.4)\\
&= \int \mathrm{d}^D x \left[-\frac{1}{2}(\partial^\mu A_{\mathrm{B}}^\nu)\{\partial_\mu A_{\mathrm{B}\nu} - \partial_\nu A_{\mathrm{B}\mu}\} - \frac{1}{2}(\partial_\mu A_{\mathrm{B}}^\mu)^2 \right], && (7.2.5)
\end{aligned}$$

note that we could have gotten to this result using

$$F_{\mathrm{B}}^{\mu\nu} F_{\mathrm{B}\mu\nu} = (\partial^\mu A_{\mathrm{B}}^\nu)(\partial_\mu A_{\mathrm{B}\nu} - \partial_\nu A_{\mathrm{B}\mu}), \tag{7.2.6}$$

which follows using the antisymmetry of $F_{\mathrm{B}\mu\nu}$ to write $F_{\mathrm{B}}^{\mu\nu}$ as $\partial^\mu A_{\mathrm{B}}^\nu$, which has $F_{\mathrm{B}}^{\mu\nu}$ as its antisymmetric part. Now we have

$$\partial_\mu A_{\mathrm{B}}^\mu = \partial^\mu A_{\mathrm{B}}^\nu \eta_{\mu\nu}, \tag{7.2.7}$$

We can then write the action as

$$S_\gamma = \int \mathrm{d}^D x \left[-\frac{1}{2} (\partial^\mu A_{\mathrm{B}}^\nu)(\partial_\mu A_{\mathrm{B}\nu} - \partial_\nu A_{\mathrm{B}\mu} + \eta_{\mu\nu}\partial_\rho A_{\mathrm{B}}^\rho) \right]. \tag{7.2.8}$$

We can now integrate by parts:

$$\begin{aligned} S_\gamma = & -\frac{1}{2} \int \mathrm{d}^D x \, \partial^\mu \{ A_{\mathrm{B}}^\nu (\partial_\mu A_{\mathrm{B}\nu} - \partial_\nu A_{\mathrm{B}\mu} + \eta_{\mu\nu}\partial_\rho A_{\mathrm{B}}^\rho) \} \\ & + \frac{1}{2} \int \mathrm{d}^D x [A_{\mathrm{B}}^\nu \partial^\mu (\partial_\mu A_{\mathrm{B}\nu} - \partial_\nu A_{\mathrm{B}\mu} + \eta_{\mu\nu}\partial_\rho A_{\mathrm{B}}^\rho)]. \end{aligned} \tag{7.2.9}$$

As usual we use the divergence theorem to turn the first term into a surface integral, which we assume vanishes, leaving us with

$$S_\gamma = \frac{1}{2} \int \mathrm{d}^D x [A_{\mathrm{B}}^\nu \partial^\mu (\partial_\mu A_{\mathrm{B}\nu} - \partial_\nu A_{\mathrm{B}\mu} + \eta_{\mu\nu}\partial_\rho A_{\mathrm{B}}^\rho)] \tag{7.2.10}$$

$$= \frac{1}{2} \int \mathrm{d}^D x [A_{\mathrm{B}}^\nu (\partial^2 A_{\mathrm{B}\nu} - \partial^\mu \partial_\nu A_{\mathrm{B}\mu} + \partial_\nu \partial_\rho A_{\mathrm{B}}^\rho)]. \tag{7.2.11}$$

After renaming $\rho \to \mu$ and swapping raised and lowered indices in the last term the second and third term cancel, leaving us with

$$S_\gamma = \frac{1}{2} \int \mathrm{d}^D x \, A_{\mathrm{B}}^\nu \partial^2 A_{\mathrm{B}\nu}. \tag{7.2.12}$$

One way we can tell that this is correct is that in the Feynman gauge the photon propagator goes as $1/k^2$, and $1/k^2$ is the inverse of ∂^2 in Fourier space, and the propagator should always be the inverse of the kernel in Fourier space.

After regularisation this term becomes

$$S_\gamma = \frac{1}{2} \int \mathrm{d}^D x \, Z_3 A_\mu \partial^2 A^\mu. \tag{7.2.13}$$

The full action after regularising can then be written, in the Feynman gauge, as

$$S[A, \psi, \bar{\psi}] = \int \mathrm{d}^D x \left[\frac{1}{2} Z_3 A_\mu \partial^2 A^\mu + Z_2 \bar{\psi} (i\partial\!\!\!/ - m - \delta m)\psi + Z_1 e \bar{\psi} A\!\!\!/ \psi \right]. \tag{7.2.14}$$

We obtain the classical equations of motion by varying this action with respect to A_μ. The change in the action when $A_\mu \mapsto A_\mu + \delta A_\mu$ is

$$\delta S = S[A + \delta A, \psi, \bar{\psi}] - S[A, \psi, \bar{\psi}] \tag{7.2.15}$$

$$\begin{aligned} = & \int \mathrm{d}^D x \Big[\frac{1}{2} Z_3 (A_\mu + \delta A_\mu) \partial^2 (A^\mu + \delta A^\mu) + Z_2 (\bar{\psi} - m - \delta m)\psi \\ & + Z_1 e \bar{\psi} \gamma^\mu (A_\mu + \delta A_\mu)\psi \Big] - S[A, \psi, \bar{\psi}] \end{aligned} \tag{7.2.16}$$

$$= \int \mathrm{d}^D x \left[\frac{1}{2} Z_3 \delta A_\mu \partial^2 A^\mu + \frac{1}{2} Z_3 A_\mu \partial^2 \delta A^\mu + Z_1 e \bar{\psi} \gamma^\mu \delta A_\mu \psi \right]. \tag{7.2.17}$$

Integrating the second term by parts twice, and again assuming that surface terms vanish, we move both derivatives from the δA^μ term to the A_μ term, and pick up two minus signs, leaving us with exactly the first term, giving

$$\delta S = \int \mathrm{d}^D x \left[Z_3 \partial^2 A^\mu + Z_1 e \bar{\psi} \gamma^\mu \psi \right] \delta A_\mu. \tag{7.2.18}$$

This must then vanish for all δA_μ so we have

$$Z_3 \partial^2 A^\mu + Z_1 e \bar{\psi} \gamma^\mu \psi = 0. \tag{7.2.19}$$

Defining the current to be

$$j_\mu(x) \coloneqq \bar{\psi}(x) \gamma_\mu \psi(x) \tag{7.2.20}$$

the equations of motion become

$$Z_3 \partial^2 A^\mu = -e Z_1 j^\mu, \tag{7.2.21}$$

which is just Maxwell's equations in terms of the renormalised fields, and factoring the charge out of the current.

In the rest of this section we'll use the action in the form derived here to derive some identities which apply to all orders in QED.

7.3 Ward–Takahashi Identity

The QED Lagrangian has a symmetry given by

$$\psi \mapsto \mathrm{e}^{i\alpha} \psi, \qquad \text{and} \qquad \bar{\psi} \mapsto \mathrm{e}^{-i\alpha} \bar{\psi}. \tag{7.3.1}$$

Lets explore what happens if we promote α to be a function of spacetime without changing the photon, A_μ. This ceases to be a symmetry of the Lagrangian, we need the extra term from transforming A_μ to cancel the derivative of α which appears. The change in the action comes entirely from the term with derivatives of ψ, and is

$$\begin{aligned}
\delta S &= \int \mathrm{d}^D x \left[Z_2 \mathrm{e}^{-i\alpha} \bar{\psi} i \partial\!\!\!/ (\mathrm{e}^{i\alpha} \psi) - Z_2 \bar{\psi} i \partial\!\!\!/ \psi \right] && (7.3.2)\\
&= \int \mathrm{d}^D x \left[Z_2 \mathrm{e}^{-i\alpha} \mathrm{e}^{i\alpha} \bar{\psi} (i \partial\!\!\!/ \psi - (\partial\!\!\!/ \alpha) \psi) - Z_2 \bar{\psi} i \partial\!\!\!/ \psi \right] && (7.3.3)\\
&= \int \mathrm{d}^D x \left[Z_2 \bar{\psi} (-\partial\!\!\!/ \alpha) \psi \right]. && (7.3.4)
\end{aligned}$$

Continuing on we integrate by parts, and assume the surface term vanishes

$$\begin{aligned}
\delta S &= -\int \mathrm{d}^D x \left[Z_2 \bar{\psi} (\gamma^\mu \partial_\mu \alpha) \psi \right] && (7.3.5)\\
&= -\int \mathrm{d}^D x \left[Z_2 \partial_\mu (\bar{\psi} \gamma^\mu \alpha \psi) - Z_2 \alpha \partial_\mu (\bar{\psi} \gamma^\mu \psi) \right] && (7.3.6)\\
&= \int \mathrm{d}^D x \left[Z_2 \alpha \partial_\mu (\bar{\psi} \gamma^\mu \psi) \right] && (7.3.7)\\
&= \int \mathrm{d}^D x \left[Z_2 \alpha \partial^\mu j_\mu \right]. && (7.3.8)
\end{aligned}$$

We are free to choose α, and choosing it to be a nonzero constant implies that $\partial^\mu j_\mu = 0$, since despite not being a symmetry we clearly have $\delta S = 0$ as $\partial\!\!\!/\alpha = 0$ for constant α so before integrating by parts we had zero. So j_μ is a conserved current.

Now consider the correlator $\langle\psi(x_1)\bar\psi(x_2)\rangle$. This can be calculated using the path integral

$$\langle\psi(x_1)\bar\psi(x_2)\rangle = \int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar\psi\,\psi(x_1)\bar\psi(x_2)\mathrm{e}^{iS[A,\psi,\bar\psi]}. \tag{7.3.9}$$

The fields ψ and $\bar\psi$ appear as integration variables, so we can consider a change of variables

$$\psi(x) \mapsto \psi'(x) = (1+i\alpha(x))\psi(x), \quad \text{and} \quad \bar\psi(x) \mapsto \bar\psi'(x) = (1-i\alpha(x))\bar\psi(x), \tag{7.3.10}$$

which is just the linearised version of the transformation $\psi \mapsto \mathrm{e}^{i\alpha}\psi$. It is reasonable to assume, but tricky to prove since path integrals aren't very well defined, that the integration measure is invariant under this, after all, we would expect that the measure transforms as

$$\mathcal{D}\psi\,\mathcal{D}\bar\psi \mapsto \mathrm{e}^{i\alpha}\,\mathcal{D}\psi\,\mathrm{e}^{-i\alpha}\,\mathcal{D}\bar\psi = \mathcal{D}\psi\,\mathcal{D}\psi. \tag{7.3.11}$$

We will assume that this is the case. Then we have

$$\begin{aligned}\langle\psi(x_1)\bar\psi(x_2)\rangle &= \int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar\psi\,\psi(x_1)\bar\psi(x_2)\mathrm{e}^{iS[A,\psi,\bar\psi]} &(7.3.12)\\ &= \int \mathcal{D}A\,\mathcal{D}\psi'\,\mathcal{D}\bar\psi'\,\psi'(x_1)\bar\psi'(x_2)\mathrm{e}^{iS[A,\psi',\bar\psi']} &(7.3.13)\\ &= \int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar\psi\,\psi'(x_1)\bar\psi'(x_2)\mathrm{e}^{iS[A,\psi',\bar\psi']}. &(7.3.14)\end{aligned}$$

Now consider what we get if we subtract the last line here from the first. On the one hand we clearly get zero, since both lines are equal, on the other hand we can expand the third line as

$$\int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar\psi\,(1+i\alpha(x_1))\psi(x_1)(1-i\alpha(x_2))\bar\psi(x_2)\mathrm{e}^{iS[A,\psi',\bar\psi]}. \tag{7.3.15}$$

Notice that since we're working to linear order we have

$$\mathrm{e}^{iS[A,\psi',\bar\psi']} = \mathrm{e}^{iS[A,\psi,\bar\psi]+i\delta S} \approx \left(1 + i\int \mathrm{d}^D x\alpha(x)Z_2\partial_\mu j^\mu(x)\right)\mathrm{e}^{iS[A,\psi,\bar\psi]}. \tag{7.3.16}$$

Computing the difference then gives us

$$\begin{aligned}0 = \int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar\psi\Big[&(i\alpha(x_1) - i\alpha(x_2))\psi(x_1)\bar\psi(x_2)\\ &+ \psi(x_1)\bar\psi(x_2)i\int \mathrm{d}^D x\,\alpha(x)Z_2\partial_\mu j^\mu(x)\Big]\mathrm{e}^{iS[A,\psi,\bar\psi]}. \qquad (7.3.17)\end{aligned}$$

We can now drop the overall factor of i, use some Dirac deltas to write $\alpha(x_1)$ as $\int \mathrm{d}^D x\, \alpha(x)\delta(x-x_1)$. This gives us

$$\begin{aligned} 0 &= \int \mathcal{D}A\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi} \int \mathrm{d}^D x[(\alpha(x)\delta(x-x_1) - \alpha(x)\delta(x-x_2))\psi(x_1)\bar{\psi}(x_2) \\ &\qquad + \psi(x_1)\bar{\psi}(x_2)\alpha(x)Z_2\partial_\mu j^\mu(x)]\mathrm{e}^{iS[A,\psi,\bar{\psi}]} \qquad (7.3.18) \\ &= \int \mathrm{d}^D x\, \alpha(x)[(\delta(x-x_1) - \delta(x-x_2))\langle\psi(x_1)\bar{\psi}(x_2)\rangle \\ &\qquad + Z_2\partial_x^\mu\langle j_\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle]. \qquad (7.3.19) \end{aligned}$$

Note that the order of ψ, $\bar{\psi}$, and j^μ in the path integral only matters up to the sign. Within a correlator the order doesn't matter at all, since we are time ordering. However, we have to be careful with the derivative, since the time ordering introduces factors of $\theta(t-t')$, which the derivative acts on, so it's best to keep ∂_x^μ out of the correlator. We could expand the correlator in terms of the time orderings and acting on the Heaviside step functions with the derivative would give Dirac deltas, and would just lead to us rederiving this result.

Since this result must hold for all α we can assume $\alpha \neq 0$ and so we have

$$Z_2\partial_x^\mu\langle j_\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle = (\delta(x-x_2) - \delta(x-x_1))\langle\psi(x_1)\bar{\psi}(x_2)\rangle. \qquad (7.3.20)$$

This is the **Ward–Takahashi identity**.

Classically one would expect that terms of the form $\partial^\mu j_\mu$ vanish, and this identity tells us that this is almost the case, unless x and x_1 or x_2 happen to coincide, for this reason the term on the right is called a **contact term**.

We can generalise this result to include more fermions and gauge fields. The gauge fields are just bystanders in this, and don't change anything. Each pair of fermion fields picks up a Dirac delta term in exactly the same way, so the **generalised Ward–Takahashi identity** is

$$\begin{aligned} &Z_2\partial_x^\mu\left\langle j_\mu(x)\prod_{i=1}^n \psi(y_i)\bar{\psi}(z_i)\prod_{j=1}^n A_{\mu_j}(w_j)\right\rangle \\ &\qquad = \sum_{k=1}^n (\delta(x-z_k) - \delta(x-y_k))\left\langle\prod_{i=1}^n \psi(y_i)\bar{\psi}(z_i)\prod_{j=1}^n A_{\mu_j}(w_j)\right\rangle. \qquad (7.3.21) \end{aligned}$$

7.4 Schwinger–Dyson Equations

The Ward–Takahashi identity arose from classical symmetries. The Schwinger–Dyson equations arise similarly from the classical equations of motion. Consider the correlator

$$\langle A_\nu(x_1)\rangle = \int \mathcal{D}A\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, A_\nu(x_1)\mathrm{e}^{iS[A,\psi,\bar{\psi}]}. \qquad (7.4.1)$$

Consider the change of variables $A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) + \zeta_\mu(x)$ where ζ_μ is some arbitrary vector field, not a gauge field. We think of ζ_μ as a variation on A_μ, hence the relation to the classical equations of motion.

The change in the action from this transformation comes from the two terms in the action in which A_μ appears, and using the result found for the classical

equations of motion (Equation (7.2.18)) we have

$$\delta S = \int \mathrm{d}^D x \left[Z_3 \partial^2 A^\mu(x) + Z_1 e j^\mu(x) \right] \zeta_\mu. \tag{7.4.2}$$

We assume that the integration measure is invariant with respect to this change of variables, so $\mathcal{D}A = \mathcal{D}A'$, or $\mathcal{D}\zeta = 0$, which seems reasonable. Then we have

$$\langle A_\nu(x_1) \rangle = \int \mathcal{D}A\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi} A_\nu(x_1) \mathrm{e}^{iS[A,\psi,\bar{\psi}]} \tag{7.4.3}$$

$$= \int \mathcal{D}A'\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi} A'_\nu(x_1) \mathrm{e}^{iS[A',\psi,\bar{\psi}]} \tag{7.4.4}$$

$$= \int \mathcal{D}A\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi} A'_\nu(x_1) \mathrm{e}^{iS[A',\psi,\bar{\psi}]}. \tag{7.4.5}$$

We now consider the first line minus the third line. On the one hand clearly this gives zero, on the other hand we can expand the third line as

$$\int \mathcal{D}A\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi} (A_\nu(x_1) + \zeta_\nu(x_1)) \mathrm{e}^{iS[A,\psi,\bar{\psi}] + i\delta S} \tag{7.4.6}$$

which gives

$$\int \mathcal{D}A\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi} (A_\nu(x_1) + \zeta_\nu(x_1))(1 + i\delta S) \mathrm{e}^{iS[A,\psi,\bar{\psi}]}. \tag{7.4.7}$$

Expanding the brackets we get $A_\nu(x_1) + \zeta_\nu(x_1) + A_\nu(x_1) i\delta S + \zeta_\nu(x_1) i\delta S$. The $A_\nu(x_1)$ term cancels with the first line and we're left with

$$0 = \int \mathcal{D}A\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi} (\zeta_\nu(x_1) + A_\nu(x_1) i\delta S + \zeta_\nu(x_1) i\delta S) \mathrm{e}^{iS[A,\psi,\bar{\psi}]} \tag{7.4.8}$$

$$= \int \mathcal{D}A\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi} \Big[\zeta_\nu(x_1) + i \int \mathrm{d}^D x \{ Z_3 \partial^2 A^\mu(x) + Z_1 e j^\mu(x) \} \zeta_\mu(x) (A_\nu(x_1) + \zeta_\nu(x_1)) \Big] \mathrm{e}^{iS[A,\psi,\bar{\psi}]}. \tag{7.4.9}$$

Choosing ζ_μ to be small and neglecting the quadratic term in ζ_μ we get

$$0 = \int \mathcal{D}A\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi} \Big[\zeta_\nu(x_1) + i \int \mathrm{d}^D x \{ Z_3 \partial^2 A^\mu(x) + Z_1 e j^\mu(x) \} \zeta_\mu(x) A_\nu(x_1) \Big] \mathrm{e}^{iS[A,\psi,\bar{\psi}]}.$$

We can then write $\zeta_\nu(x_1) = \int \mathrm{d}^D x\, \delta(x - x_1) \zeta_\nu(x)$ and $\zeta_\nu(x) = \eta_{\mu\nu} \zeta^\mu(x)$ to give

$$0 = \int \mathrm{d}^D x \int \mathcal{D}A\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi} \left[\delta(x - x_1) \eta_{\mu\nu} + i \{ Z_3 \partial^2 A_\mu(x) + Z_1 e j_\mu(x) \} A_\nu(x_1) \right] \zeta^\mu(x) \mathrm{e}^{iS[A,\psi,\bar{\psi}]}. \tag{7.4.10}$$

Finally we can rewrite this in terms of correlators:

$$0 = \int \mathrm{d}^D x \left[\delta(x - x_1) \eta_{\mu\nu} + i \{ Z_3 \partial^2 \langle A_\mu(x) A_\nu(x_1) \rangle + e Z_1 \langle j_\mu(x) A_\nu(x_1) \rangle \} \right] \zeta^\mu(x).$$

(7.4.11)

Since we can choose ζ_μ arbitrarily we must have that

$$\delta(x-x_1)\eta_{\mu\nu} + i\{Z_3\partial_x^2\langle A_\mu(x)A_\nu(x_1)\rangle + eZ_1\langle j_\mu(x)A_\nu(x_1)\rangle\} = 0, \tag{7.4.12}$$

which tells us that

$$Z_3 i\partial_x^2\langle A_\mu(x)A_\nu(x_1)\rangle = -eZ_1 i\langle j_\mu(x)A_\nu(x_1)\rangle - \eta_{\mu\nu}\delta(x-x_1). \tag{7.4.13}$$

This is the **Schwinger–Dyson equation**. Compare this to the classical equations of motion,

$$Z_3\partial^2 A^\mu = -eZ_1 j^\mu, \tag{7.4.14}$$

and we see that the Schwinger–Dyson equation is simply the classical equations of motion plus a contact term. Intuitively this contact term only comes into effect when $x = x_1$, so $A_\mu(x)$ and $A_\nu(x_1)$ are evaluated at the same point.

We can generalise this process to arbitrary correlators of fermion and gauge fields. The fermion fields have no effect, since they aren't modified by the change of variables, so we'll just consider some arbitrary product $\psi\cdots\bar\psi\cdots$, in which every time we write this each fermion field is evaluated at some point which is the same each time we write this. Then we have the **generalised Schwinger–Dyson equation**

$$\begin{aligned} &Z_3 i\partial_x^2\langle A_\mu(x)A_{\nu_1}(x_1)\cdots A_{\nu_n}(x_n)\psi\cdots\bar\psi\cdots\rangle \\ &= -ieZ_1\langle j_\mu(x)A_{\nu_1}(x_1)\cdots A_{\nu_n}(x_n)\psi\cdots\bar\psi\cdots\rangle \\ &\qquad - \sum_{i=1}^{n}\eta_{\mu\nu_i}\delta(x-x_i)\langle A_{\nu_1}(x_1)\cdots\widehat{A_{\nu_i}(x_i)}\cdots A_{\nu_n}(x_n)\rangle \end{aligned} \tag{7.4.15}$$

where the notation $\hat{x}$ means that x is left out of the product, so, for example,

$$5\cdot\hat{4}\cdot 3\cdot 2\cdot 1 = 5\cdot 3\cdot 2\cdot 1 = 30. \tag{7.4.16}$$

7.4.1 Schwinger–Dyson in Perturbation Theory

7.4.1.1 Order e^0

At $\mathcal{O}(e^0)$ the Schwinger–Dyson equation gives

$$Z_3 i\partial_x^2\langle A_\mu(x)A_\nu(y)\rangle = -\eta_{\mu\nu}\delta(x-y). \tag{7.4.17}$$

This is just the free propagator. Note that $Z_3 = 1$ at order e^0.

For a photon, since $m = 0$, we amputate a correlator by acting on it with $i\partial^2$, so we can interpret this as saying that if we amputate the photon propagator then we simply get a Dirac delta, which represents a point. So amputating a propagator, a line between points, gives a single point.

$$i\partial_x^2(x,\mu\sim\sim y,\nu) = -\eta_{\mu\nu}\delta(x-y). \tag{7.4.18}$$

7.4.1.2 Order e^1

At order e^1 the Schwinger–Dyson equation gives

$$Z_3 i\partial^2 \langle A_\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle = -e\langle j_\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle, \tag{7.4.19}$$

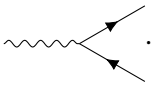
we don't include the $\eta_{\mu\nu}\delta(x)$ term since this is $\mathcal{O}(e^0)$. Expanding j_μ in terms of the fermion fields we get

$$\langle j_\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle = \langle \bar{\psi}(x)\gamma_\mu\psi(x)\psi(x_1)\bar{\psi}(x_2)\rangle. \tag{7.4.20}$$

The only nonzero contraction, since we must contract each ψ with a $\bar{\psi}$ and contracting fields at the same point gives zero, is

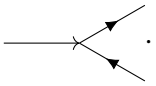
$$\bar{\psi}(x)\gamma_\mu\psi(x)\psi(x_1)\bar{\psi}(x_2). \tag{7.4.21}$$

The correlator $\langle A_\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle$ corresponds to the QED vertex



(7.4.22)

We are amputating the photon by acting with $i\partial^2$, which removes the photon, but still results in the injection of momentum from the photon. We draw this as

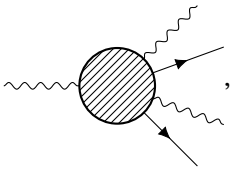


(7.4.23)

We can think of this as replacing the photon, A_μ, with a current, j_μ, of electrons and positrons, carrying momentum but no net charge.

7.4.1.3 General Order

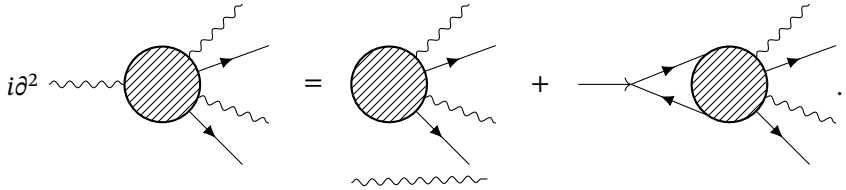
From the zeroth and first order analysis we can now deduce what happens at arbitrary order. Consider some diagram



(7.4.24)

and focus on the incoming photon in particular, note that the other particles may be incoming or outgoing, and there may be any number of them.

Acting on this with $i\partial^2$ amputates this photon, and there are two things that can happen, corresponding to the two terms on the right hand side of the Schwinger–Dyson equation. Either we get a propagator alone, which corresponds to the contact term, or we replace the photon with the amputated QED vertex. Diagrammatically this looks like



Eight

Consequences

8.1 Recap

In the last chapter we derived the Schwinger–Dyson equation:

$$Z_3\partial^2\langle A_\mu(x)\cdots\rangle = -eZ_1\langle j_\mu(x)\cdots\rangle + \text{contact terms}, \tag{8.1.1}$$

and the Ward–Takahashi identity:

$$Z_2\partial_\mu\langle j^\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle = (\delta(x-x_2)-\delta(x-x_1))\langle\psi(x_1)\bar{\psi}(x_2)\rangle. \tag{8.1.2}$$

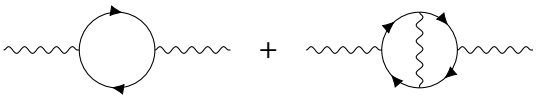
These results hold to all orders in QED. We will now derive and discuss several consequences of these results.

8.2 $k_\mu\Pi^{\mu\nu}(k) = 0$ to All Orders

We can generalise the definition of $\Pi^{\mu\nu}(k)$ from just the vacuum polarisation to an all orders object

$$i\Pi^{\mu\nu}(x-y) = \langle A^\mu(x)A^\nu(y)\rangle_{\text{amputated, connected}}. \tag{8.2.1}$$

That is,

$i\Pi^{\mu\nu}(x-y) =$  $+\cdots$, (8.2.2)

whereas before $i\Pi^{\mu\nu}(x-y)$ was just the first term in this expansion. Here the diagrams stand for the amputated correlators, so really we are acting on each photon with $i\partial^2$.

Since these diagrams are amputated it follows from the Schwinger–Dyson equation, applied twice, once to each external photon, that

$$\Pi^{\mu\nu}(x-y) \propto \langle A^\mu(x)A^\nu(y)\rangle_{\text{amputated, connected}} \propto \langle j^\mu(x)j^\nu(y)\rangle. \tag{8.2.3}$$

Then we have

$$\begin{aligned} ik_\mu\Pi^{\mu\nu}(k) &= ik_\nu\int \mathrm{d}^D x\, \mathrm{e}^{-ik\cdot x}\Pi^{\mu\nu}(x) && (8.2.4)\\ &= \int \mathrm{d}^D x\, \mathrm{e}^{-ik\cdot x}\partial_\mu\Pi^{\mu\nu}(x) && (8.2.5)\\ &\propto \int \mathrm{d}^D x\, \mathrm{e}^{-ik\cdot x}\partial_\mu\langle j^\mu(x)j^\nu(0)\rangle && (8.2.6)\\ &= 0. && (8.2.7) \end{aligned}$$

Here we've used the fact that $\mathcal{F}\{\partial_\mu\} = ik_\mu$. The last step follows using the Ward–Takahashi identity recognising that $j^\nu(0) = \bar{\psi}(0)\gamma^\nu\psi(0)$ and so we get $\delta(x)-\delta(x) = 0$, at least when we consider the Dirac deltas as distributions under an integral. So, we have

$$k_\mu \Pi^{\mu\nu}(k) = 0 \tag{8.2.8}$$

to all orders in QED.

We can go further in the case where the only four-vector is k^μ, then it must be that $\Pi^{\mu\nu}(k) = \Pi_1(k^2)\eta^{\mu\nu} + \Pi_2(k^2)k^\mu k^\nu$ for some scalar functions Π_1 and Π_2. Using $k_\mu \Pi^{\mu\nu}(k) = 0$ we must then have $\Pi_1(k^2)k_\nu + \Pi_2(k^2)k^2 k_\nu = 0$, and for $k_\nu \neq 0$ we must then have $\Pi_1(k^2) = -\Pi_2(k^2)k^2$, so

$$\Pi^{\mu\nu}(k) = (k^\mu k^\nu - \eta^{\mu\nu}k^2)\Pi(k^2) \tag{8.2.9}$$

with $\Pi = \Pi_2$.

It is possible to introduce another four-vector quantity through a choice of non-Lorentz invariant gauge fixing, and this four-vector can then appear in the tensor structure above, but we won't do this.

8.3 Ward Identities

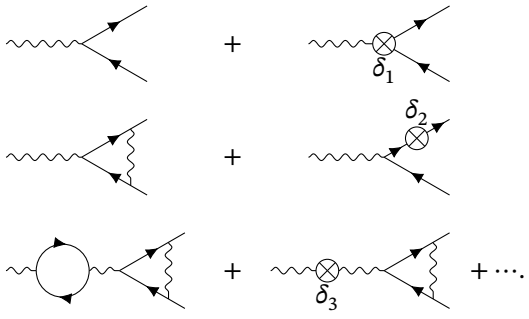
Any non-connected amplitude can be factored into connected amplitudes, so we'll only consider connected amplitudes. Given some amplitude, $\mathcal{A}(k)$, with k the momentum of an external photon we can write the amplitude as $\mathcal{A}(k) = \varepsilon^\mu(k)\mathcal{A}_\mu(k)$ where ε^μ is the polarisation vector of the photon and this equation defines $\mathcal{A}_\mu$. Then the **Ward identity** is

$$k^\mu \mathcal{A}_\mu(k) = 0. \tag{8.3.1}$$

This is important as it means we have only two polarisation states and gauge invariance, both basic requirements of QED. This Ward identity follows from the Ward–Takahashi identity since $\mathcal{A}_\mu$ is proportional to the Fourier transform of $\langle j_\mu(x)\cdots\rangle$, and the Ward–Takahashi identity tells us that $\partial^\mu\langle j_\mu(x)\cdots\rangle = 0$ for connected, amputated correlators. Taking the Fourier transform then turns the derivative into a k^μ proving the identity. This proof shows that the Ward identity holds to all orders in perturbation theory, not just the tree level at which we proved in in *Quantum Field Theory*.

8.4 Charge Renormalisation

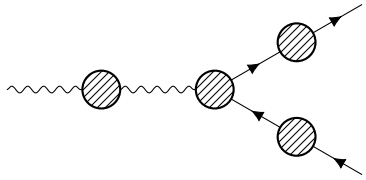
Consider the correlator $\langle A_\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle$. This is in position space, but we can identify it with it's Fourier transform. Then we can expand this as

$\langle A_\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle =$ 

(8.4.1)

We can then write this more compactly as

$$\langle A_\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle = \tag{8.4.2}$$



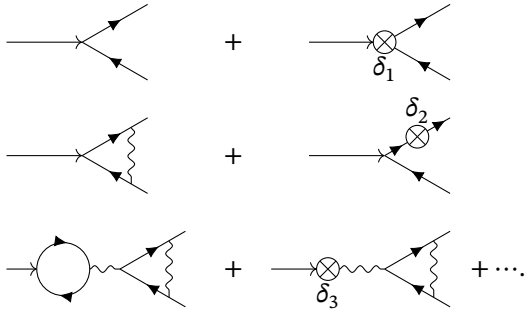
Here each circle represents a sum over all one particle irreducible (1PI) diagrams with the matching external states.

Now we can apply the Schwinger–Dyson equation:

$$Z_3\partial^2\langle A_\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle = -eZ_1\langle j_\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle. \tag{8.4.3}$$

Acting with ∂^2 amputates the incoming photon, reducing this series to

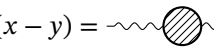
$$i\partial^2\langle A_\mu(x)\psi(x_1)\bar{\psi}(x_2) = \tag{8.4.4}$$



We can write the first two terms here as $1 + \delta_1$ times the first first term, since the only difference between the QED vertex and its counter term is a factor of δ_1.

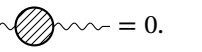
This correlator is infinite. We know that all photon-to-photon 1PI diagrams are exactly the diagrams contributing to

$$i\Pi^{\mu\nu}(x-y) = \tag{8.4.5}$$



We also know that $\partial_\mu\Pi^{\mu\nu}(x-y) = 0$ to all orders, so ∂^μ kills all 1PI diagrams on a photon, that is

$$\partial^\mu \tag{8.4.6}$$



$= 0.$

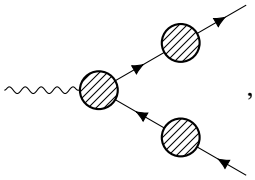
Everything else appearing in the expansion has been renormalised, and so is finite. As a consequence we know that

$$\partial^\mu Z_3\partial^2\langle A_\mu(x)\psi(x_1)\bar{\psi}(x_3)\rangle = \text{finite}. \tag{8.4.7}$$

Using the Schwinger–Dyson equation we can rewrite this as

$$Z_1\partial^\mu\langle j_\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle = \text{finite}. \tag{8.4.8}$$

This correlator corresponds to the diagram



, (8.4.9)

which must then also be finite. Recalling that $Z_1 = 1 + \delta_1$ is the counter term for the QED vertex it is not that surprising that this is finite, assuming that the two electron propagators are properly regulated.

We can also apply the

$$Z_2 \partial^\mu \langle j_\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle = \text{finite} \tag{8.4.10}$$

by the Ward–Takahashi identity this becomes

$$\langle \psi(x_1)\bar{\psi}(x_2)\rangle = \text{finite}, \tag{8.4.11}$$

which corresponds to a fermion propagator. This is surprising, by regulating the QED vertex with Z_2 we've also regulated the fermion propagator.

In the $\overline{\text{MS}}$ scheme we have

$$Z_i = 1 + \sum_{n=1}^{\infty} \frac{a_i}{\varepsilon^n}, \tag{8.4.12}$$

and so what we have shown here is that, in the $\overline{\text{MS}}$ scheme $Z_1 = Z_2$ to all orders in perturbation theory. More generally the divergent parts of Z_1 and Z_2 are equal to all orders in perturbation theory, but they may differ by finite terms.

Recall that the Fermion Lagrangian involves the term

$$\bar{\psi}_{\text{B}}(i\partial\!\!\!/)\psi_{\text{B}} + e_{\text{B}}\bar{\psi}_{\text{B}}A\!\!\!/_{\text{B}}\psi_{\text{B}} = Z_2\bar{\psi}(i\partial\!\!\!/)\psi + e\mu^\varepsilon \underbrace{Z_e Z_2\sqrt{Z_3}}_{Z_1}\bar{\psi}A\!\!\!/\psi. \tag{8.4.13}$$

This factorises nicely if $Z_1 = Z_2$ allowing us to rewrite this term using the covariant derivative as

$$Z_1\bar{\psi}(iD\!\!\!/)\psi \tag{8.4.14}$$

for $D_\mu = \partial_\mu - ie\mu^\varepsilon A_\mu$. This is nice since it allows us to make gauge invariance more obvious.

We previously found that the electric charge becomes scale dependent, with the result that

$$e^2_{\text{meas}}(k^2) = e^2_{\text{meas}}(0) + \frac{e^4}{2\pi^2}\int_0^1 \mathrm{d}x\, x(1-x)\log\left(\frac{x(1-x)k^2}{\tilde{\mu}^2}\right) \tag{8.4.15}$$

in the $\overline{\text{MS}}$ scheme, neglecting the m^2 term since we are interested in high energies. We found this by studying the vacuum polarisation of the electron, that is, one loop corrections to the photon propagator. This is a bit odd since it is the QED vertex which involves the charge, e, and is used to renormalise the interaction, which is where the charge is important. The link between the vacuum polarisation and the QED vertex is that $Z_1 = Z_2$, so the counter terms of these two diagrams are related and we can study either one to learn things about the other.

Nine

The QED β Function

9.1 What is the QED β Function?

The bare charge, e_B, is independent of μ, since we only introduce μ in the regularisation process to hold the dimensions of e away from $D = 4$. Thus we have

$$\mu \frac{de_B}{d\mu} = 0. \tag{9.1.1}$$

Note that it is common to work with the log-derivative,

$$\mu \frac{d}{d\mu} = \frac{d}{d(\log \mu)}, \tag{9.1.2}$$

in this context, rather than just $d/d\mu$.

On the other hand after regularising we have $e_B = e\mu^\varepsilon Z_3$, which clearly has μ dependence. The solution is that eZ_3 must also have μ dependence, and this dependence must be such as to exactly cancel the dependence of the μ^ε term. We also have

$$Z_1 = Z_e Z_2 \sqrt{Z_3}, \tag{9.1.3}$$

which combined with our recent derivation that $Z_1 = Z_2$ to all orders in the $\overline{\text{MS}}$ scheme gives

$$Z_e = \frac{1}{\sqrt{Z_3}}. \tag{9.1.4}$$

Hence, we are looking to solve

$$0 = \mu \frac{d}{d\mu}(e\mu^\varepsilon Z_e) \tag{9.1.5}$$

$$= \mu \frac{d}{d\mu}\left(\frac{e\mu^\varepsilon}{\sqrt{Z_3}}\right) \tag{9.1.6}$$

$$= \mu \frac{de}{d\mu}\mu^\varepsilon \frac{1}{\sqrt{Z_3}} + e\varepsilon\mu^\varepsilon \frac{1}{\sqrt{Z_3}} - \frac{1}{2} e\mu^\varepsilon \frac{1}{Z_3^{3/2}} \mu \frac{dZ_3}{d\mu}. \tag{9.1.7}$$

Cancelling a common factor of $\mu^\varepsilon/\sqrt{Z_3}$ we get

$$0 = \mu \frac{de}{d\mu} + e\varepsilon - \frac{1}{2}\frac{e}{Z_3}\mu \frac{dZ_3}{d\mu}. \tag{9.1.8}$$

Recall that

$$Z_3 = 1 + \frac{e^2}{12\pi^2\varepsilon}, \tag{9.1.9}$$

which allows us to write this all in terms of $\mathrm{d}e/\mathrm{d}\mu$:

$$0 = \mu\frac{\mathrm{d}e}{\mathrm{d}\mu} + e\varepsilon - \frac{1}{2}\frac{e}{Z_3}\mu\frac{2e}{12\pi^2\varepsilon}\frac{\mathrm{d}e}{\mathrm{d}\mu} \tag{9.1.10}$$

so we have

$$0 = e\varepsilon + \left(1 - \frac{1}{Z_3}\frac{e^2}{12\pi^2\varepsilon}\right)\mu\frac{\mathrm{d}e}{\mathrm{d}\mu} \tag{9.1.11}$$

$$\approx e\varepsilon + \left(1 - \frac{e^2}{12\pi^2\varepsilon}\right)\mu\frac{\mathrm{d}e}{\mathrm{d}\mu} \tag{9.1.12}$$

where we've used

$$\frac{1}{Z_3} \approx 1 - \frac{e^2}{12\pi^2\varepsilon} \tag{9.1.13}$$

and dropped any terms of order e^3 or higher. We then have

$$\mu\frac{\mathrm{d}e}{\mathrm{d}\mu} \approx -\frac{e\varepsilon}{1 - e^2/(12\pi^2\varepsilon)} \tag{9.1.14}$$

$$\approx -e\varepsilon\left(1 + \frac{e^2}{12\pi^2\varepsilon}\right) \tag{9.1.15}$$

$$= -e\varepsilon + \frac{e^3}{12\pi^2}. \tag{9.1.16}$$

Finally, take the $\varepsilon \to 0$ limit and define the **beta function**:

$$\beta(e) \coloneqq \mu\frac{\mathrm{d}e}{\mathrm{d}\mu} = \frac{e^3}{12\pi^2}. \tag{9.1.17}$$

This is a **renormalisation group equation**.

9.2 Solving the Renormalisation Group Equation

Set $t = \log\mu$, then we have

$$\beta(e) = \frac{\mathrm{d}e}{\mathrm{d}t} = \frac{e^3}{12\pi^2}. \tag{9.2.1}$$

Multiplying through by[1] $\mathrm{d}t$ to get

$$\frac{\mathrm{d}e}{e^3} = \frac{\mathrm{d}t}{12\pi^2} \tag{9.2.2}$$

and recognising that

$$\frac{\mathrm{d}(1/e^2)}{\mathrm{d}e} = -2\frac{1}{e^3} \tag{9.2.3}$$

[1] sorry to any mathematicians reading, although I doubt any have made it this far since nothing converges and we've been computing divergent integrals for about 35 pages

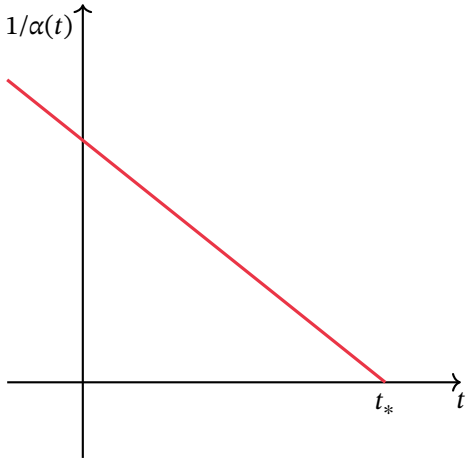


Figure 9.1: A plot of $1/\alpha(t)$ as a function of t. Notice that at some value t_* we have $1/\alpha(t_*) = 0$, meaning $\alpha(t_*)$ must be infinite.

we have

$$\frac{de}{e^3} = -\frac{1}{2}d\left(\frac{1}{e^2}\right) \tag{9.2.4}$$

and so

$$d\left(\frac{1}{e^2}\right) = -\frac{dt}{6\pi^2}. \tag{9.2.5}$$

Integrating from some t_0 to t we have

$$\int_{t_0}^{t} d(\frac{1}{e^2}) = -\frac{1}{6\pi^2}\int_{t_0}^{t} dt \tag{9.2.6}$$

which gives

$$\frac{1}{e^2(t)} - \frac{1}{e^2(t_0)} = -\frac{1}{6\pi^2}(t - t_0). \tag{9.2.7}$$

Usually we write this in terms of the fine structure constant, $\alpha = e^2/(4\pi)$, in which case we have

$$\frac{1}{4\pi\alpha(t)} - \frac{1}{4\pi\alpha(t_0)} = -\frac{1}{6\pi^2}(t - t_0) \implies \frac{1}{\alpha(t)} - \frac{1}{\alpha(t_0)} = -\frac{2}{3\pi}(t - t_0). \tag{9.2.8}$$

So, for fixed t_0, we have found a linear relationship between $1/\alpha(t)$ and t, which is logarithmic in the energy. We can plot the result as in Figure 9.1. From this we see that as t, and hence the energy μ, grows $1/\alpha(t)$ becomes smaller until it vanishes at some value t_*. That is, $\alpha(t)$ becomes larger and larger and eventually at $t = t_*$ it takes on an infinite value. We call this a **Landau pole**.

The modern understanding of this result is that QED is not well defined as a quantum theory to arbitrarily high energy. At some energy scale, $\Lambda_{\text{QED}} = e^{t_*}$ the theory breaks down and stops working. This isn't really too much of a worry for two reasons:

- the approximation that $e(\mu)$ is small breaks far below this energy scale;
- Λ_{QED} is truly massive, using $t_0 = \log m_Z$, where measurements give $\alpha(t_0) \approx 1/127$, we find that $\Lambda_{\text{QED}} \sim 10^{270}$ eV.

We can rearrange this equation:

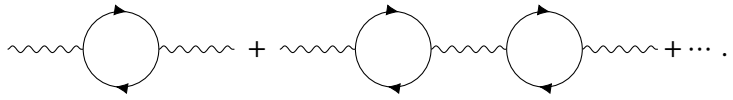
$$\frac{1}{\alpha(t)} = \frac{1}{\alpha(t_0)} - \frac{2}{3\pi}(t - t_0) \tag{9.2.9}$$

$$\implies \alpha(t) = \frac{1}{1/\alpha(t_0) - 2(t-t_0)/(3\pi)} = \frac{\alpha(t_0)}{2\alpha(t_0)(t-t_0)/(3\pi)}. \tag{9.2.10}$$

We can then expand this to get

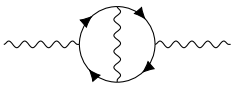
$$\alpha(t) = \alpha(t_0) + \frac{2}{3\pi}\alpha^2(t_0)(t-t_0) + \frac{4}{9\pi}\alpha^3(t_0)(t-t_0)^2 + \cdots \tag{9.2.11}$$

$$= \alpha(t_0) + \frac{2}{3\pi}\alpha^2(t_0)\log\frac{\mu}{\mu_0} + \frac{4}{9\pi}\alpha^3(t_0)\left(\log\frac{\mu}{\mu_0}\right)^2 + \cdots \tag{9.2.12}$$

$=$  $+ \cdots$.

Here we interpret this series as a series of diagrams, each order of α giving an extra loop, which each come with two vertices, accounting for the two extra powers of e. So we see that the divergence of the coupling constant can be understood as the result of resuming an infinite class of diagrams.

There are other two loop corrections to this, such as

 $\sim \alpha^4(t_0)\log\left(\frac{\mu}{\mu_0}\right)$, (9.2.13)

but if $\log(\mu/\mu_0)$ is large then the class of diagrams already accounted for is the most important. This is called resummation of large logs.

Ten

Magnetic Moment

This is a story where the 2s matter.
I'm not going to get the 2s right.

Donal O'Connell

10.1 Goal

Consider a particle in an external magnetic field, $\boldsymbol{B}$. The contribution to the Hamiltonian from this is

$$\delta H = -\boldsymbol{\mu} \cdot \boldsymbol{B}, \tag{10.1.1}$$

where $\boldsymbol{\mu}$ is the particle's **magnetic moment**. For a particle with spin $\boldsymbol{S}$ and charge q the magnetic moment is

$$\boldsymbol{\mu} = \frac{q}{2m} g \boldsymbol{S} \tag{10.1.2}$$

where g, known as the **Landé g factor** is some value which we will compute in this chapter.

For an electron, of charge $-e$, or a muon, also of charge $-e$ but a different mass, the magnetic moment can be written as

$$\boldsymbol{\mu} = -\frac{e}{2m} g \boldsymbol{S} = -\mu_{\mathrm{B}} g \boldsymbol{S} \tag{10.1.3}$$

where

$$\mu_{\mathrm{B}} \coloneqq \frac{e}{2m} \tag{10.1.4}$$

is the **Bohr magneton**.

A calculation of g not accounting for spin will give $g = 1$. A calculation of g using the non-relativistic limit of the Dirac equation, which accounts for spin, will give $g = 2$, we'll do this calculation in the next section. A calculation in QED predicts g as a power series, giving $g = 2 + \mathcal{O}(\alpha)$. We will do a one loop QED calculation of g. Calculations up to five loops have been performed which predict g for an electron to many significant figures, and align well with experimental results.

10.2 Non-Relativistic Calculation of the Landé g Factor

In this section we will compute the magnetic moment of an electron using the non-relativistic limit of the Dirac equation. This is valid for energies small compared to the electron mass where there can be no pair production. Define the **wave function** of the electron to be

$$\psi_{\text{WF}}(x) = \langle 0|\psi(x)|\text{single particle state}\rangle \tag{10.2.1}$$

where ψ is the normal quantum fermion field. The motivation here is that a single particle state is of the form

$$\int \frac{\hat{\mathrm{d}}^3 p}{2E_p}\varphi(p)|p\rangle \tag{10.2.2}$$

for some function φ. The annihilation operator in ψ will then annihilate the single particle state, and then the creation operator acts as its conjugate on the vacuum state, so we only get the overlap between single particles, which is what we would expect in the non-relativistic limit.

It follows from this definition that ψ_{WF} also satisfies the Dirac equation, since the only x dependence comes from the ψ field. That is,

$$(i\not{D} - m)\psi_{\text{WF}} = 0. \tag{10.2.3}$$

Applying $(-i\not{D} + m)$ to this we must still get zero,

$$-(i\not{D} + m)(i\not{D} - m)\psi_{\text{WF}} = 0. \tag{10.2.4}$$

Expanding this operator we get

$$-(i\not{D} + m)(i\not{D} - m) = (\not{D}\not{D} + m^2). \tag{10.2.5}$$

So we need to compute $\not{D}\not{D}$. To start off write this as

$$\not{D}\not{D} = D_\mu D_\nu \gamma^\mu \gamma^\nu \tag{10.2.6}$$

$$= \frac{1}{2}\{D_\mu, D_\nu\}\gamma^\mu\gamma^\nu + \frac{1}{2}[D_\mu, D_\nu]\gamma^\mu\gamma^\nu \tag{10.2.7}$$

where the last step is simply writing $D_\mu D_\nu$ as a sum of its symmetric and antisymmetric part, expanding the (anti)commutators it should be clear that this holds. We can then use the fact that $\{D_\mu, D_\nu\}$ is symmetric to replace $\gamma^\mu\gamma^\nu$ with its symmetric part, $\{\gamma^\mu, \gamma^\nu\}/2$, and similarly the antisymmetry of $[D_\mu, D_\nu]$ allows us to replace $\gamma^\mu\gamma^\nu$ with its antisymmetric part, $[\gamma^\mu, \gamma^\nu]/2$. Doing so we get

$$\not{D}\not{D} = \frac{1}{4}\{D_\mu, D_\mu\}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{4}[D_\mu, D_\nu][\gamma^\mu, \gamma^\nu] \tag{10.2.8}$$

$$= \frac{1}{4}\{D_\mu, D_\nu\}2\eta^{\mu\nu} + \frac{1}{4}\left(-ieF_{\mu\nu}\right)\left(-2i\sigma^{\mu\nu}\right) \tag{10.2.9}$$

$$= \frac{1}{2}\{D_\mu, D^\mu\} - \frac{1}{2}eF_{\mu\nu}\sigma^{\mu\nu} \tag{10.2.10}$$

$$= D^2 - \frac{1}{2}eF_{\mu\nu}\sigma^{\mu\nu}. \tag{10.2.11}$$

Here we've used the commutators

$$[D_\mu, D_\nu] = -ieF_{\mu\nu}, \qquad \text{and} \qquad \sigma^{\mu\nu} \coloneqq \frac{i}{2}[\gamma^\mu, \gamma^\nu], \tag{10.2.12}$$

the definition of the gamma matrices, $\{\gamma^\mu, \gamma^\nu\} \coloneqq 2\eta^{\mu\nu}$, and

$$\{D_\mu, D^\mu\} = D_\mu D^\mu + D^\mu D_\mu = 2D^2. \tag{10.2.13}$$

Hence, the wave function must satisfy

$$\left[D^2 - \frac{1}{2}eF_{\mu\nu}\sigma^{\mu\nu} + m^2\right]\psi_{\text{WF}} = 0. \tag{10.2.14}$$

Now consider a magnetic field, $\boldsymbol{B}$, in the z direction. Then $F^{12} = -F^{21} = -B$, and all other components of $F^{\mu\nu}$ vanish. With this we have

$$F^{\mu\nu}\sigma_{\mu\nu} = F^{12}\sigma_{12} + F^{21}\sigma_{21} = -B\sigma_{12} + B\sigma_{21} = 2B\sigma_{21}. \tag{10.2.15}$$

In the Dirac representation we have

$$\sigma_{21} = \frac{i}{2}[\gamma_2, \gamma_1] \tag{10.2.16}$$

$$= \frac{i}{2}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} - \frac{i}{2}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \tag{10.2.17}$$

$$= -\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}. \tag{10.2.18}$$

Define

$$\boldsymbol{S} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \tag{10.2.19}$$

[1] we've also called this $\boldsymbol{\Sigma}$

which we can recognise as the spin operator[1]. and we can write this result as

$$F^{\mu\nu}\sigma_{\mu\nu} = 4BS^3 = 4\boldsymbol{B}\cdot\boldsymbol{S}, \tag{10.2.20}$$

having used the fact that $\boldsymbol{B} = (0, 0, B)$. Now, since we picked the z direction arbitrarily this result must hold for any uniform magnetic field, that is

$$-\frac{e}{2}F^{\mu\nu}\sigma_{\mu\nu} = 2e\boldsymbol{B}\cdot\boldsymbol{S}. \tag{10.2.21}$$

Thus we have

$$\left[D^2 + 2e\boldsymbol{B}\cdot\boldsymbol{S} + m^2\right]\psi_{\text{WF}} = 0. \tag{10.2.22}$$

To pass to the non-relativistic limit consider plane wave solutions

$$\exp\{-ip\cdot x\} = \exp\{-iEt + i\boldsymbol{p}\cdot\boldsymbol{x}\}. \tag{10.2.23}$$

Then using

$$E = \sqrt{\boldsymbol{p}^2 + m^2} \approx m + \frac{\boldsymbol{p}^2}{2m} \tag{10.2.24}$$

we get

$$\mathrm{e}\{-ip\cdot x\}\approx\exp\{-imt\}\exp\left\{-i\frac{\boldsymbol{p}^2}{2m}t+i\boldsymbol{p}\cdot\boldsymbol{x}\right\}. \tag{10.2.25}$$

Recognising that $\boldsymbol{p}^2/(2m)$ is the kinetic energy we can see that the second exponential is the plane wave we would make as an ansatz in a non-relativistic setting. This suggests that we should look for solutions of the form

$$\psi_{\mathrm{WF}}(x)=\mathrm{e}^{-imt}\psi_{\mathrm{NR}}(x) \tag{10.2.26}$$

where ψ_{NR} is the non-relativistic wave function.

Now consider the D^2 term, we have

$$\begin{aligned}
D^2f&=(\partial_\mu-ieA_\mu)(\partial^\mu-ieA^\mu)f && (10.2.27)\\
&=\partial^2f-\partial_\mu(ieA^\mu f)-ieA_\mu\partial^\mu f-e^2A_\mu A^\mu f && (10.2.28)\\
&=\partial^2f-ie(\partial_\mu A^\mu)f-ieA^\mu\partial_\mu f-ieA_\mu\partial^\mu f-e^2A_\mu A^\mu f && (10.2.29)\\
&=\partial^2f-ie(\partial_\mu A^\mu)f-2ieA^\mu\partial_\mu f-e^2A_\mu A^\mu f. && (10.2.30)
\end{aligned}$$

We can drop the e^2 term, since it will be small in the non-relativistic limit. If $\boldsymbol{E}=\boldsymbol{0}$ then we can take $A^0=0$. Taking $f=\psi_{\mathrm{WF}}=\mathrm{e}^{-imt}\psi_{\mathrm{NR}}$ we get

$$\begin{aligned}
D^2\psi_{\mathrm{WF}}&=\partial^2\psi_{\mathrm{WF}}-ie(\partial_\mu A^\mu)\psi_{\mathrm{WF}}-2ieA^\mu\partial_\mu(\mathrm{e}^{-imt}\psi_{\mathrm{NR}}) && (10.2.31)\\
&=\partial^2\psi_{\mathrm{WF}}-ie(\partial_\mu A^\mu)\psi_{\mathrm{WF}}\\
&\quad-2ie[A^0\partial_0(\mathrm{e}^{-imt}\psi_{\mathrm{NR}})-\boldsymbol{A}\cdot\nabla(\mathrm{e}^{-imt}\psi_{\mathrm{NR}})] && (10.2.32)\\
&=\partial^2\psi_{\mathrm{WF}}-ie(\partial_\mu A^\mu)\psi_{\mathrm{WF}} && (10.2.33)\\
&\quad-2ie[-imA^0\mathrm{e}^{-imt}\psi_{\mathrm{NR}}+A^0\mathrm{e}^{-imt}\partial_t\psi_{\mathrm{NR}}-\mathrm{e}^{-imt}\boldsymbol{A}\cdot\nabla\psi_{\mathrm{NR}}].
\end{aligned}$$

Now suppose that we have a uniform, constant magnetic field, and no electric field, so $\partial_\mu A^\mu=0$. Reinstating factors of c we have $A^\mu=(\varphi/c,\boldsymbol{A})$, so we can neglect A^0 in the non-relativistic limit. Thus,

$$D^2\psi_{\mathrm{WF}}=\partial^2\psi_{\mathrm{WF}}+2ie\mathrm{e}^{-imt}\boldsymbol{A}\cdot\nabla\psi_{\mathrm{NR}}. \tag{10.2.34}$$

We can expand the $\partial^2\psi_{\mathrm{WF}}$ term as

$$\begin{aligned}
\partial^2\psi_{\mathrm{WF}}&=(\partial_t^2-\nabla^2)(\mathrm{e}^{-imt}\psi_{\mathrm{NR}}) && (10.2.35)\\
&=\partial_t(-im\mathrm{e}^{-imt}\psi_{\mathrm{NR}}-\mathrm{e}^{-imt}\partial_t\psi_{\mathrm{NR}})-\mathrm{e}^{-imt}\nabla^2\psi_{\mathrm{NR}} && (10.2.36)\\
&=(-im)^2\mathrm{e}^{-imt}\psi_{\mathrm{NR}}-im\mathrm{e}^{-imt}\partial_t\psi_{\mathrm{NR}}-im\mathrm{e}^{-imt}\partial_t\psi_{\mathrm{NR}}\\
&\quad-\mathrm{e}^{-imt}\partial_t^2\psi_{\mathrm{NR}}-\mathrm{e}^{-imt}\nabla^2\psi_{\mathrm{NR}} && (10.2.37)\\
&=\mathrm{e}^{-imt}\left[-m^2\psi_{\mathrm{NR}}-2im\partial_t\psi_{\mathrm{NR}}+\partial_t^2\psi_{\mathrm{NR}}-\nabla^2\psi_{\mathrm{NR}}\right]. && (10.2.38)
\end{aligned}$$

Notice that the first term cancels with the term coming from m^2 in Equation (10.2.14). Thus we have

$$\begin{aligned}
0&=\left[D^2+2e\boldsymbol{B}\cdot\boldsymbol{S}+m^2\right]\psi_{\mathrm{WF}} && (10.2.39)\\
&=\mathrm{e}^{-imt}[-2im\partial_t+\partial_t^2-\nabla^2+2ie\mathrm{e}^{-imt}\boldsymbol{A}\cdot\nabla+2e\boldsymbol{B}\cdot\boldsymbol{S}]\psi_{\mathrm{NR}}. && (10.2.40)
\end{aligned}$$

Dividing through by $2me^{-imt}$ we get

$$0 = \left[-i\partial_t + \frac{1}{2m}\partial_t^2 - \frac{1}{2m}\nabla^2 + \frac{e}{m}\boldsymbol{A}\cdot\nabla + \frac{e}{2m}2\boldsymbol{B}\cdot\boldsymbol{S}\right]\psi_{\text{NR}}. \tag{10.2.41}$$

We can neglect the ∂_t^2 term as reinstating factors of c we see it is suppressed by $1/c^2$. Similarly, we drop the $\boldsymbol{A}\cdot\nabla$ term, leaving us with

$$\left[-\frac{1}{2m}\nabla^2 + \frac{e}{2m}2\boldsymbol{B}\cdot\boldsymbol{S}\right]\psi_{\text{NR}} = -i\frac{\partial}{\partial t}\psi_{\text{NR}}. \tag{10.2.42}$$

This is simply the Schrödinger equation.

From this we can read off the change to the Hamiltonian due to the magnetic field, it is

$$\delta H = \frac{e}{2m}2\boldsymbol{B}\cdot\boldsymbol{S}. \tag{10.2.43}$$

Comparing this with the definition of the magnetic moment, $\delta H = -\boldsymbol{\mu}\cdot\boldsymbol{B}$ we see that

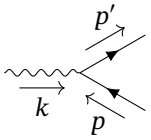
$$\boldsymbol{\mu} = -\frac{e}{2m}2\boldsymbol{S}, \tag{10.2.44}$$

that is,

$$g = 2. \tag{10.2.45}$$

10.3 As an Effective Field Theory

We can see this same result in QED by considering the three point tree amplitude



$$= ie\bar{u}(p')\gamma^\mu u(p)\varepsilon_\mu(k) = \mathcal{A}_3. \tag{10.3.1}$$

Note that $k = p' - p$. We might think of this as probing an electron with a photon. We'll take the electron to be on-shell and the photon to be either on-shell or at least have k^2 be negligible.

Consider the following expression with $k = p' - p$ and $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$:

$$\bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}k_\nu}{2m}\right]u(p) \tag{10.3.2}$$

$$= \frac{\bar{u}(p')}{2m}\left[p'^\mu + p^\mu - \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(p'_\nu - p_\nu)\right]u(p) \tag{10.3.3}$$

$$= \frac{\bar{u}(p')}{2m}\left[p'^\mu + p^\mu - \frac{1}{2}(\gamma^\mu \not{p}' - \gamma^\mu \not{p} - \not{p}'\gamma^\mu + \not{p}\gamma^\mu)\right]u(p) \tag{10.3.4}$$

$$= \frac{\bar{u}(p')}{2m}\left[p'^\mu + p^\mu - \frac{1}{2}(2\eta^{\mu\nu}p'_\nu - \not{p}'\gamma^\mu - \gamma^\mu\not{p} - \not{p}'\gamma^\mu + 2\eta^{\mu\nu}p_\nu - \gamma^\mu\not{p})\right]u(p)$$

$$= \frac{\bar{u}(p')}{2m}\left[p'^\mu + p^\mu - (p'^\mu - \not{p}'\gamma^\mu - \gamma^\mu\not{p} + p^\mu)\right]u(p) \tag{10.3.5}$$

$$= \frac{\bar{u}(p')}{2m}\left[p'^{\mu} + p^{\mu} - (p'^{\mu} - \not{p}'\gamma^{\mu} - \gamma^{\mu}\not{p} + p^{\mu})\right]u(p) \tag{10.3.6}$$

$$= \frac{\bar{u}(p')}{2m}\left[p'^{\mu} + p^{\mu} - (p'^{\mu} - m\gamma^{\mu} - \gamma^{\mu}m + p^{\mu})\right]u(p) \tag{10.3.7}$$

$$= \frac{\bar{u}(p')}{2m}\left[2m\gamma^{\mu}\right]u(p) \tag{10.3.8}$$

$$= \bar{u}(p')\gamma^{\mu}u(p). \tag{10.3.9}$$

Here we've made use of the definition of $u(p)$ as the momentum space solution to the Dirac equation, meaning $(\not{p} - m)u(p) = 0$, so $\not{p}u(p) = mu(p)$, and the adjoint, $\bar{u}(p)\not{p} = m$. This gives us the **Gordon identity**:

$$\bar{u}(p')\gamma^{\mu}u(p) = \bar{u}(p')\left[\frac{p'^{\mu} + p^{\mu}}{2m} + \frac{i\sigma^{\mu\nu}k_{\nu}}{2m}\right]u(p). \tag{10.3.10}$$

Using this we can rewrite the three point amplitude as

$$\mathcal{A}_3 = ie\bar{u}(p')\left[\frac{p'^{\mu} + p^{\mu}}{2m} + \frac{i\sigma^{\mu\nu}k_{\nu}}{2m}\right]u(p)\varepsilon_{\mu}(k). \tag{10.3.11}$$

We can think of the first term as being the result from scalar QED, the vertex term here being $ie(p'^{\mu} + p^{\mu})\varepsilon_{\mu}(k)$, and the second term being a spin effect. Taking this spin correction we can imagine that it has a prefactor of 1, and we'll see that this term corresponds to $g/2$, meaning if this prefactor is changed to $1 + a$ then g will change from 2 to $2 + 2a$, this is essentially what we will compute. This quantity, $a = (g - 2)/2$, is called the **anomalous magnetic moment**.

We can see how this relates to classical physics by comparing world lines. The action in this case is given by

$$S = -m\int \mathrm{d}\tau - \int \mathrm{d}\tau\, eu_{\mu}A^{\mu}(r(\tau)) + \int \mathrm{d}\tau\, \frac{e}{2m}g\frac{1}{4}S_{\mu\nu}F^{\mu\nu}(r(\tau)). \tag{10.3.12}$$

The first two terms here are the normal action[2] for a charged particle with four-velocity u^{μ} moving in an external field, A^{μ}. The third term contains the spin tensor, $S_{\mu\nu}$, which is needed for a particle with spin, this term is included as it is Lorentz invariant, and there is no reason to exclude it. The spin tensor generalises the spin vector, s^{μ}, to $S^{\mu\nu} \propto s^{\mu}u^{\nu}$, where u is the four-velocity of the particle. It is just a neat way to encode the required information about spin.

[2] derived in *Classical Electrodynamics*

In QFT we are interested in allowing for high energies where we have pair production. The action written down here is for an effective field theory at low energies which doesn't allow for pair production. We have to perform a matching process to compare this effective field theory to QED. To do this we have to consider some scenario where we have a reasonable idea of what the outcome will be.

Consider the case of a particle moving at a constant speed starting at the origin at time $\tau = 0$. It's position is given by $r^{\mu}(\tau) = u^{\mu}\tau$. The interaction action in the effective field theory, simply the action of this effective theory without the first term, is then given by

$$S_{\text{int}} = \int \mathrm{d}\tau\left[-eu_{\mu}A^{\mu}(u\tau) + \frac{e}{2m}g\frac{1}{4}S_{\mu\nu}F^{\mu\nu}(u\tau)\right]. \tag{10.3.13}$$

Now we can write this as the inverse Fourier transform of the Fourier transform, defining $\varepsilon^{\mu}(k) = \mathcal{F}\{A^{\mu}(u\tau)\}$. Note that this isn't a polarisation vector, but we pick

the notation ε^μ as it is *like* a polarisation vector. Then the action is

$$S_{\text{int}} = \int \hat{\mathrm{d}}^4 k \int \mathrm{d}\tau\, \mathrm{e}^{-ik\cdot u\tau}\left[-eu_\mu \varepsilon^\mu(k) - i\mu_{\text{B}} g \frac{1}{4} S_{\mu\nu}(k^\mu \varepsilon^\nu - k^\nu \varepsilon^\mu)\right], \quad (10.3.14)$$

where derivatives in $F^{\mu\nu}$ have turned into ks and As have turned into εs. The spin tensor, $S_{\mu\nu}$, is antisymmetric, so we have

$$S_{\mu\nu} k^\mu \varepsilon^\nu = S_{\mu\nu} k^{(\mu} \varepsilon^{\nu)} = \frac{1}{2} S_{\mu\nu}(k^\mu \varepsilon^\nu - k^\nu \varepsilon^\mu) \quad (10.3.15)$$

allowing us to write the interaction action as

$$S_{\text{int}} = \int \hat{\mathrm{d}}^4 k \int \mathrm{d}\tau\, \mathrm{e}^{-ik\cdot u\tau}\left[-eu_\mu \varepsilon^\mu(k) - i\mu_{\text{B}} g \frac{1}{2} S_{\mu\nu} k^\mu \varepsilon^\nu(k)\right]. \quad (10.3.16)$$

We can also compute the τ integral, since the only τ dependence is in the exponential. Doing this, and assuming that the interaction is switched on at $\tau = 0$, we get

$$\int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-ik\cdot u\tau} = \left[\frac{i}{k\cdot u}\mathrm{e}^{-ik\cdot u\tau}\right]_{\tau=0}^{\tau=\infty}. \quad (10.3.17)$$

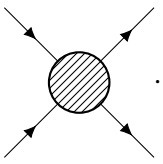
This doesn't converge. However, if we modify it slightly so that $k\cdot u \mapsto k\cdot u \mapsto k\cdot u - i\epsilon$ with $\epsilon > 0$ then we get a term $\mathrm{e}^{-\epsilon k\cdot u\tau}$, which goes to zero as $\tau \to \infty$. We then have convergence and we get

$$\int \mathrm{d}\tau_0^\infty\, \mathrm{e}^{-i(k\cdot u - i\epsilon)\tau} = \left[\frac{i}{k\cdot u - i\epsilon}\mathrm{e}^{-i(k\cdot u - i\epsilon)\tau}\right]_{\tau=0}^{\tau=\infty} = \frac{-i}{k\cdot u - i\epsilon}, \quad (10.3.18)$$

and we are then free to take $\epsilon \to 0$. Then we have

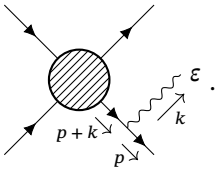
$$S_{\text{int}} = \int \hat{\mathrm{d}}^4 k\, \frac{-i}{k\cdot u}\left[-eu\cdot\varepsilon - i\mu_{\text{B}} g \frac{1}{2} S^{\mu\nu} k_\mu \varepsilon_\nu\right]. \quad (10.3.19)$$

Consider a four point process occurring via this interaction,



. (10.3.20)

At time $\tau = 0$ we can probe one of the particles emitted in this interaction with a photon,



. (10.3.21)

On the QFT side we can consider this interaction followed by probing. Computing the amputated correlator of this probing part, where the only particle not

amputated is the electron before probing, after throwing away factors like $u(p)$ which aren't interesting we get

$$ie\frac{i}{(p+k)^2+i\epsilon}\left(\frac{2p\cdot\varepsilon}{2m}+i2S^{\mu\nu}k_\mu\varepsilon_\mu\right) \tag{10.3.22}$$

where we've used the Gordon identity on the $\bar{u}(p)\gamma^\mu u(p+k)$ term before throwing away the spinor factors. We've also used $k\cdot\varepsilon=0$.

We can rewrite this as

$$-\frac{e}{u\cdot k}\frac{1}{2m}(2u\cdot\varepsilon+i\mu_{\mathrm{B}}S^{\mu\nu}\varepsilon_\mu k_\nu). \tag{10.3.23}$$

This can then be compared with Equation (10.3.19) and we see that we must have

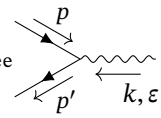
$$g=2. \tag{10.3.24}$$

Eleven

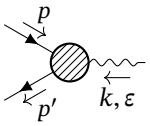
Anomalous Magnetic Moment

11.1 Setup

The derivation towards the end of the last chapter suggests that we should get a more accurate value of g if we compute loop corrections to the QED vertex. At tree level we have

$$i\mathcal{A}_{\text{tree}} \quad \text{  } \quad = ie\bar{u}(p')\gamma^\mu u(p)\varepsilon_\mu(k). \tag{11.1.1}$$

To arbitrary order this becomes

$$i\mathcal{A} = \quad \text{  } \quad = ie\bar{u}(p')\Gamma^\mu(p', p)u(p)\varepsilon_\mu(k). \tag{11.1.2}$$

Here $\Gamma^\mu(p', p)$ generalises the gamma matrix, γ^μ, to include loop corrections. Our goal will be to compute Γ^μ.

11.2 The Structure of Γ^μ

Fortunately the form of Γ^μ is rather limited. It must be a four-vector, meaning it can only be formed from other four-vectors and Lorentz invariant scalars. The only four-vectors we have are p, p', and $k = p - p'$, as well as γ^μ. This means we only have two independent variable four vectors, p and p', but we can change these to $p + p'$ and $p - p'$, which are also independent. From this we see that Γ^μ must be a linear combination of γ^μ, $p^\mu + p'^\mu$, and $p^\mu - p'^\mu$. The coefficients must be Lorentz scalars. We assume that the electrons are on shell, so $p^2 = p'^2 = m^2$, which is just a constant. Since the photon is being used to probe the electron it must have been produced in some other process, so it is a virtual particle, but we assume it is almost on-shell, meaning that $k^2 \approx 0$, but k^2 is not necessarily exactly zero. This means that the only variable Lorentz scalar we have is k^2, so the coefficients must be functions of k^2. We also know that we can't have products of gamma matrices other than γ^μ on its own, since either these reduce to Lorentz scalars, such as $\gamma^\nu\gamma_\nu = D$, or we can use $\not{p}u(p) = mu(p)$ to remove them, or they reduce to something which can written in terms of γ^5, but this is not parity invariant, which is something that we require of QED.

Combing this we see that Γ^μ must be of the form

$$\Gamma^\mu(p', p) = A(k^2)\gamma^\mu + B(k^2)(p^\mu + p'^\mu) + C(k^2)(p^\mu - p'^\mu) \tag{11.2.1}$$

where A, B, and C are scalar functions to be determined.

The first observation we can make about this is that $p^\mu - p'^\mu = k^\mu$, and so the final term is $C(k^2)k^\mu$. However, the Ward identity means that we must have $k_\mu\Gamma^\mu(p', p) = 0$, meaning we must have $C(k^2) = 0$ so that $C(k^2)k^2$ vanishes identically, not just when the photon is on-shell.

Notation 11.2.2 We are about to make a lot of steps which don't quite leave the result the same. We will use the symbol $\rightsquigarrow$, such as $A \rightsquigarrow B$ to denote that for our purposes A and B are equivalent, if not equal. For example, since we know that our final result has no k^μ term we have $ak^\mu \rightsquigarrow 0$, since any k^μ factor must eventually cancel out.

We can then rewrite Γ^μ as

$$\Gamma^\mu(p', p) \rightsquigarrow \gamma^\mu F_1(k^2) + \frac{i\sigma^{\mu\nu}k_\nu}{2m}F_2(k^2). \tag{11.2.3}$$

This is equivalent by again applying the Gordon identity since $\Gamma^\mu(p', p)$ appears sandwiched between $\bar{u}(p')$ and $u(p)$. This allows us to replace $p'^\mu + p^\mu$ with $2m\gamma^\mu - i\sigma^{\mu\nu}k_\nu$. We then absorb various constants into the definitions of F_1 and F_2 to get the result above. This form of $\Gamma^\mu(p', p)$ is the one which people work with, however we'll now invert what we've done and use the Gordon identity to write Γ^μ as

$$\Gamma^\mu(p', p) = \frac{p'^\mu + p^\mu}{2m}F_1(k^2) + \frac{i\sigma^{\mu\nu}k_\nu}{2m}(F_1(k^2) + F_2(k^2)). \tag{11.2.4}$$

Finally, we have to renormalise this, and as a renormalisation condition we use $F_1(0) = 1$. This is a nice condition because it enforces $e_{\text{meas}} = e$. We could use another condition and get a result in terms of e which we then eliminate in preference of e_{meas}, but this is just more work.

Comparing this result to Equation (10.3.19), and working with $k^2 \approx 0$, we see that the Landé g factor is

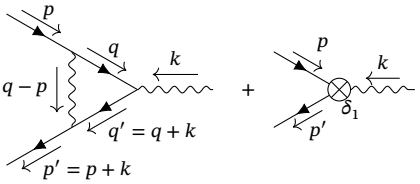
$$g = 2(F_1(0) + F_2(0)) = 2 + 2F_2(0). \tag{11.2.5}$$

We can see this as the Dirac equation result, 2, plus a higher order correction, $2F_2(0)$. So, we now just need to compute $F_2(0)$.

11.3 One Loop Calculation of $F_2(0)$

The one-loop correction to the amplitude is

$$i\mathcal{A}_{\text{one loop}} = \quad + \tag{11.3.1}$$



$$= ie\bar{u}(p')\delta\Gamma^\mu(p', p)u(p)\varepsilon_\mu(k) \tag{11.3.2}$$

where

$$\Gamma^\mu(p',p) = \gamma^\mu + \delta\Gamma^\mu(p',p) + \cdots \tag{11.3.3}$$

with the one loop correction being $\delta\Gamma^\mu(p',p)$. Then $\delta\Gamma^\mu(p',p)$ is given by evaluating the above Feynman diagrams and stripping off the external states, so the spinors and the polarisation vector, as well as a factor of ie.

Applying the Feynman rules, stripping the external states and a factor of ie gives

$$\delta\Gamma^\mu(p',p) = (ie)^2\mu^{3\varepsilon}\int \hat{\mathrm{d}}^D q \frac{\gamma^\nu i(\not{q}' + m)\gamma^\mu i(\not{q}+m)\gamma_\nu}{(q'^2 - m^2 + i\epsilon)(q^2-m^2+i\epsilon)}\frac{-i}{(q-p)^2+i\epsilon} + ie\mu^\varepsilon\delta_1\gamma^\mu. \tag{11.3.4}$$

We now need to compute this integral.

Lots of terms in this integral will only contribute to F_1, and we aren't interested in this term, so we will drop these terms as we go. These terms will be the ones which are proportional to γ^μ, including the counter term. Since there is no contribution of the counter term to F_2 and we have regularised so everything is finite we know that this integral should be convergent, which allows us to drop the $i\epsilon$s. We can also drop the ε of dimensional regularisation, setting $D = 4$, this means that $\mu^{3\varepsilon} \rightsquigarrow 1$. We can also throw away k^μ terms, since these must all cancel before we get to the final result. Note that this applies only to k^μ, with a free μ index, not to any summed over k_ν as this appears in $\sigma^{\mu\nu}k_\nu$. We can also use the Gordon identity and the fact that Γ^μ is sandwiched by $\bar{u}(p')$ and $u(p)$ to replace $p'^\mu + p^\mu$ with $-i\sigma^{\mu\nu}k_\nu$ wherever this term appears.

Using these simplifications we have

$$\delta\Gamma^\mu(p',p) \rightsquigarrow -ie^2\int \hat{\mathrm{d}}^4 D \frac{\gamma^\nu(\not{q}'+m)\gamma^\mu(\not{q}+m)\gamma_\nu}{(q^2-m^2)(q'^2-m^2)(q-p)^2}. \tag{11.3.5}$$

Note that this integral is still divergent, but we will drop any divergent terms as they arise since they don't contribute to F_2.

Now we have to use Feynman parametrisation, which for three variables takes the form

$$\frac{1}{ABC} = \int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\int_0^1 \mathrm{d}z \frac{\delta(1-x-y-z)}{(xA+bY+cZ)^3}\frac{\Gamma(3)}{\Gamma(1)\Gamma(1)\Gamma(1)} \tag{11.3.6}$$

$$= 2\int_{[0,1]^3}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\delta(1-x-y-z)}{(xA+bY+cZ)^3}. \tag{11.3.7}$$

So we want to compute

$$\delta\Gamma^\mu(p',p) \rightsquigarrow -2ie^2\int_{[0,1]^3}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\delta(1-x-y-z)\int \mathrm{d}^4\ell\frac{N^\mu}{d^3} \tag{11.3.8}$$

where N^μ is the numerator and denominator of Equation (11.3.5) r,eparametrised to be in terms of ℓ and

$$d = x(q^2-m^2) + y(q'^2-m^2) + z(q-p)^2. \tag{11.3.9}$$

At this point it's helpful to list things we can do to simplify the calculation:

- drop any γ^μ term as these contribute only to F_1;
- drop any k^μ term as these must cancel in the final result;
- drop any k^2 terms as we are assuming $k^2 \approx 0$;
- $p'^\mu + p^\mu \rightsquigarrow -i\sigma^{\mu\nu}k_\nu$, by the Gordon identity and dropping γ^μ terms;
- use $\delta(1-x-y-z)$ at any time to replace, for example, $x+y$ with $1-z$.

We can use $q' = q + k$ to simplify d. Expanding d we get

$$d = xq^2 + yq^2 + yk^2 + 2yq\cdot k - (x+y)m^2 + zq^2 + zp^2 - 2zq\cdot p. \quad (11.3.10)$$

Now we can take $k^2 \approx 0$ and $p^2 = m^2$ giving

$$d = (x+y+z)q^2 + 2yq\cdot k - (x+y+z)m^2 - 2zq\cdot p. \quad (11.3.11)$$

Now using the Dirac delta we can replace $x+y+z \rightsquigarrow 1$ and $x+y \rightsquigarrow 1-z$, giving

$$d = q^2 + 2q\cdot(yk - zp) - (1-2z)m^2. \quad (11.3.12)$$

Looking at this we see that if we set $\ell = q + yk - zp$ then $\ell^2 = q^2 + 2q\cdot(yk - zp) + (yk-zp)^2$ contains the first two terms here, so let's do that. The last term here is

$$(yk-zp)^2 = y^2k^2 + z^2p^2 - 2yzk\cdot p \rightsquigarrow z^2m^2 \quad (11.3.13)$$

having used $k^2 \approx 0$, $p^2 = m^2$, and

$$p' = p+k \implies p'^2 = m^2 = p^2 + k^2 + 2p\cdot k \approx m^2 + 2p\cdot k \implies p\cdot k \approx 0. \quad (11.3.14)$$

This gives

$$\begin{aligned} d &= q^2 + 2q\cdot(yk-zp) - (1-2z)m^2 && (11.3.15)\\ &= q^2 + 2q\cdot(yk-zp) + z^2m^2 - (1-2z)m^2 - z^2m^2 && (11.3.16)\\ &= \ell^2 - (1-2z)m^2 - z^2m^2 && (11.3.17)\\ &= \ell^2 - (z^2 - 2z + 1)m^2 && (11.3.18)\\ &= \ell^2 - (1-z)^2m^2 && (11.3.19)\\ &= \ell^2 - \Delta && (11.3.20) \end{aligned}$$

with $\Delta = (1-z)^2m^2$.

This takes us to

$$\delta\Gamma^\mu(p',p) \rightsquigarrow -2ie^2 \int_{[0,1]^3} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\delta(1-x-y-z)\int \hat{\mathrm{d}}^4\ell\,\frac{N^\mu}{(\ell^2-\Delta)^3}. \quad (11.3.21)$$

Now we need to simplify the numerator. For this we'll use the following identities which are valid in $D = 4$ dimensions:

$$\begin{aligned} \gamma^\nu \not{a}\gamma_\nu &= -2\not{a}, && (11.3.22)\\ \gamma^\nu \not{a}\not{b}\gamma_\nu &= 4a\cdot b, && (11.3.23)\\ \gamma^\nu \not{a}\not{b}\not{c}\gamma_\nu &= -2\not{c}\not{b}\not{a}. && (11.3.24) \end{aligned}$$

The numerator is

$$N^\mu = \gamma^\nu(\not{q}' + m)\gamma^\mu(\not{q} + m)\gamma_\nu \tag{11.3.25}$$
$$= \gamma^\nu \not{q}'\gamma^\mu \not{q}\gamma_\nu + m\gamma^\nu \not{q}'\gamma^\mu\gamma_\nu + m\gamma^\nu\gamma^\mu \not{q}\gamma_\nu + m^2\gamma^\nu\gamma^\mu\gamma_\nu \tag{11.3.26}$$
$$= \not{q}\gamma^\mu\not{q}' + 4mq^\mu + 4mq'^\mu - 2m^2\gamma^\mu \tag{11.3.27}$$
$$\rightsquigarrow \not{q}\gamma^\mu\not{q}' + 4m(q^\mu + q'^\mu) \tag{11.3.28}$$

having dropped the γ^μ term as it only contributes to F_1.

Using $\ell = q + yk - zp$ we have $q = \ell - yk + zp$, and then using $q' = q + k$ we have $q' = \ell + (1-y)k + zp$. This gives

$$N^\mu \rightsquigarrow -2[(\not{\ell}-y\not{k}+z\not{p})\gamma^\mu(\not{\ell}+(1-y)\not{k}+z\not{p})-2m(2\ell^\mu+(1-2y)k^\mu+2zp^\mu)]. \tag{11.3.29}$$

This can be simplified by dropping the ℓ^μ term, since it vanishes in the integral by symmetry, and the k^μ term, since it must cancel in the final result, and dropping the $\not{\ell}$ terms since if we have just one of them in the product then the integral of that part vanishes by symmetry and if we have two of them then this gives a divergent integral, which doesn't contribute to F_2. Applying these simplifications, as well as using $p = p' - k$ we get

$$N^\mu \rightsquigarrow -2[(z\not{p}' - (y+z)\not{k})\gamma^\mu((1-y)\not{k} + z\not{p}) - 4zmp^\mu]. \tag{11.3.30}$$

One further simplification is to use the Dirac delta to replace $y + z$ with $1 - x$, which just slightly increases the symmetry between the two factors and is slightly simpler:

$$N^\mu \rightsquigarrow -2[(z\not{p}' - (1-x)\not{k})\gamma^\mu((1-y)\not{k} + z\not{p}) - 4zmp^\mu]. \tag{11.3.31}$$

Since $u(p)$ is a solution to the Dirac equation in momentum space we have $(\not{p} + m)u(p) = 0$ and $\bar{u}(p)(\not{p} + m) = 0$. This gives $\not{p}u(p) = mu(p)$ and $\bar{u}(p)\not{p} = m$. Using the fact that $\delta\Gamma^\mu$ is sandwiched between $\bar{u}(p')$ and $u(p)$ we get

$$N^\mu \rightsquigarrow -2[(zm - (1-x)\not{k})\gamma^\mu((1-y)\not{k} + zm) - 4zmp^\mu]. \tag{11.3.32}$$

Now we can use

$$\not{k}\gamma^\mu\not{k} = \not{k}\not{k}\gamma^\mu + 2\not{k}k^\mu \rightsquigarrow 0 \tag{11.3.33}$$

since $\not{k}\not{k} \propto k^2 \approx 0$ and k^μ terms can be dropped as they must all cancel in the final result. Using this we can expand the numerator to get

$$N^\mu \rightsquigarrow -2[zm\gamma^\mu(1-y)\not{k} - (1-x)\not{k}\gamma^\mu zm - 4zmp^\mu] \tag{11.3.34}$$
$$= -2[zm((1-y)\gamma^\mu\not{k} - (1-x)\not{k}\gamma^\mu) - 4zmp^\mu]. \tag{11.3.35}$$

We can replace the terms with two gamma matrices with $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$ using

$$\gamma^\mu\not{k} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]k_\nu + \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}k_\nu \tag{11.3.36}$$
$$= -i\sigma^{\mu\nu}k_\nu + k^\mu \tag{11.3.37}$$
$$\rightsquigarrow -i\sigma^{\mu\nu}k_\nu \tag{11.3.38}$$

having used $\{\gamma^\mu, \gamma^\nu\} \coloneqq 2\eta^{\mu\nu}$ and dropped the k^μ term.

We have

$$p^\mu = \frac{1}{2}(p^\mu + p'^\mu) - \frac{1}{2}k^\mu \rightsquigarrow -\frac{1}{2}i\sigma^{\mu\nu}k_\nu \tag{11.3.39}$$

also which follows as previously mentioned by Gordon's identity.

Thus we have

$$\begin{aligned} N^\mu &\rightsquigarrow -2[zm(2-x-y)(-i\sigma^{\mu\nu}k_\nu) - 2zm(-i\sigma^{\mu\nu}k_\nu)] && (11.3.40)\\ &= -2zm(x+y)(i\sigma^{\mu\nu}k_\nu) && (11.3.41)\\ &\rightsquigarrow -2imz(1-z)\sigma^{\mu\nu} && (11.3.42)\\ &= -\frac{i\sigma^{\mu\nu}k_\nu}{2m}4m^2z(1-z). && (11.3.43) \end{aligned}$$

This is the end of step 3: simplify the numerator, in computing loop integrals, now we need to do steps 4 and 5: do the integral. We want to evaluate

$$\begin{aligned} &\delta\Gamma^\mu(p',p) \rightsquigarrow \\ &-2ie^2\int_{[0,1]^3}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\delta(1-x-y-z)\int\hat{\mathrm{d}}^4\ell\,\frac{1}{(\ell^2-\Delta)^3}\left(-\frac{i\sigma^{\mu\nu}k_\nu}{2m}4m^2z(1-z)\right). \end{aligned} \tag{11.3.44}$$

The momentum integral is particularly simple since we can take $D = 4$, and using our formula then gives

$$\int\hat{\mathrm{d}}^4\ell\,\frac{1}{(\ell^2-\Delta)^3} = \frac{-i}{32\pi^2}\Delta^{-1}. \tag{11.3.45}$$

Putting in $\Delta = (1-z)^2m^2$ we see that the m^2 cancels with one in the integral and one $1-z$ cancels from $(1-z)^2$ giving

$$\delta\Gamma^\mu(p',p) \rightsquigarrow ie^2\int_{[0,1]^3}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\delta(1-x-y-z)\frac{\sigma^{\mu\nu}k_\nu}{2m}\frac{z}{4\pi^2(1-z)}. \tag{11.3.46}$$

We can perform the y integral, which just gives a factor of 1, but we have to be careful because the Dirac delta adds the restriction that x and z must then be such that $1-x-z$ can vanish, so $x < 1-z$, giving

$$\delta\Gamma^\mu(p',p) \rightsquigarrow \frac{e^2}{4\pi^2}\frac{i\sigma^{\mu\nu}k_\nu}{2m}\int_0^1\mathrm{d}z\int_0^{1-z}\mathrm{d}z\,\frac{z}{1-z}. \tag{11.3.47}$$

Doing the z x integral gives a factor of $1-z$, so we get

$$\delta\Gamma^\mu(p',p) \rightsquigarrow \frac{e^2}{4\pi^2}\frac{i\sigma^{\mu\nu}k_\nu}{2m}\int_0^1\mathrm{d}z\mathrm{d}zz. \tag{11.3.48}$$

Finally, we can do the z integral giving $1/2$ for the final result

$$\delta\Gamma^\mu(p',p) \rightsquigarrow \frac{e^2}{8\pi^2}\frac{i\sigma^{\mu\nu}k_\nu}{2m}. \tag{11.3.49}$$

Using this we see that

$$F_2(0) = \frac{e^2}{8\pi^2} = \frac{\alpha}{2\pi}. \tag{11.3.50}$$

This gives

$$g - 2F_2(0) = \frac{\alpha}{\pi}. \tag{11.3.51}$$

The correction to g then is quite small,

$$\frac{\alpha}{\pi} \approx 0.002323. \tag{11.3.52}$$

So we now have

$$g = 2.002323. \tag{11.3.53}$$

Each extra term has an added factor of $\alpha/\pi \sim 1/400$, so the next term is a correction on the order of $1/400^2 = 1/160000$. This is born out in how close this result is to experimental and higher order theoretical results.

Recent experimental results give

$$g = 2.002\,319\,304\,361\,46(56). \tag{11.3.54}$$

Precision calculations up to fifth order give

$$g = 2.002\,319\,304\,363\,286(1528). \tag{11.3.55}$$

These agree up to 12 significant figures At this level of precision the calculations need to include other effects, such as non-perturbative QCD from quark-antiquark pairs which can produced in loops. This is even more the case for the calculation of the muon magnetic moment.

Now days this calculation is used to define α, and hence to define e. This means that it's no longer that interesting to compare the two results, since there's some circular logic, but the level of agreement is a large success of QED. A similar calculation and measurement can be performed for the muon, and disagreement between these caused some excitement in 2021 when it was suggested that experiment and theory weren't in perfect agreement.

Part II

Quantum Chromodynamics

Twelve

Yang–Mills Theory

In this part of the course we will attempt to understand quantum chromodynamics (QCD). To do so we actually start with a more general theory, which encompasses QED, QCD, and electroweak theory. This theory was first developed by Yang Chen-Ning and Robert Mills in 1954. Their aim was to generalise QED, as a U(1) gauge theory, to SU(N). We will follow the same steps as they did to develop Yang–Mills theory, but we shall at stages be more general, since they originally focused on developing the theory for the proton and neutron.

12.1 Non-Interacting Yang–Mills Theory

Consider a theory with two non-interacting fermions of equal mass, m. We'll assume these are the proton and neutron, which have approximately equal mass. The Lagrangian for this theory is simply the Dirac Lagrangian for each particle:

$$\mathcal{L} = \bar{\mathrm{n}}(i\partial\!\!\!/ - m)\mathrm{n} + \mathrm{p}(i\partial\!\!\!/ - m)\mathrm{p} = \sum_{i=1}^{2} \mathrm{n}_i(i\partial\!\!\!/ - m)\mathrm{n}_i \tag{12.1.1}$$

where $\mathrm{n}_1 = \mathrm{n}$ and $\mathrm{n}_2 = \mathrm{p}$. Here we are using the symbols for particles to represent their spinor wave functions. Note that $\bar{\mathrm{n}} = \mathrm{n}^\dagger\gamma^0$ here is the Dirac adjoint of n, not the antineutron.

If we package the neutron and proton up into a single object,

$$N = \begin{pmatrix} \mathrm{n} \\ \mathrm{p} \end{pmatrix}, \tag{12.1.2}$$

so that

$$\mathcal{L} = \bar{N}(i\partial\!\!\!/ - m)N, \tag{12.1.3}$$

then it becomes clear that this theory is invariant under rotations of these fields into each other. By this we mean that if we have $U \in \mathrm{SU}(2)$ then N and UN both result in the same Lagrangian. That is, we have a symmetry

$$\psi \mapsto U\psi. \tag{12.1.4}$$

This symmetry also gives

$$\bar{\psi} = \psi^\dagger\gamma^0 \mapsto (U\psi)^\dagger\gamma^0 = \psi^\dagger U^\dagger\gamma^0 = \psi^\dagger\gamma^0 U^\dagger = \bar{\psi}U^\dagger = \bar{\psi}U^{-1}. \tag{12.1.5}$$

Note that U and γ^0 commute since they act on different spaces, U is acting on the vector of spinors N, whereas γ^0 is acting on each spinor individually. Also recall that $U^\dagger = U^{-1}$ as part of the definition of SU(2).

Note that we use SU(2) rather than SO(2) to allow for complex "rotations". We use SU(2) instead of U(2) since U(2) can be decomposed into SU(2) and U(1), and U(1) isn't interesting right now, its just more QED.

We want to generalise this theory to the case of N fermions[1] of equal mass, m, which will be non-interacting for the moment. The Lagrangian is then

[1] here N is just some positive integer, not the combined neutron and proton.

$$\mathcal{L} = \sum_{i=1}^{N} \bar{\mathrm{q}}_i (i\partial\!\!\!/ - m)\mathrm{q}_i = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{12.1.6}$$

where q_i are the individual particle spinor wave functions, we call them q as we will think of them as quarks later, and

$$\psi = \begin{pmatrix} \mathrm{q}_1 \\ \vdots \\ \mathrm{q}_N \end{pmatrix}. \tag{12.1.7}$$

We will then consider SU(N) transformations applied to SU(N).

12.2 Gauging Yang–Mills Theory

Yang and Mills didn't like that the SU(N) symmetry was global and sought to find a local SU(N) theory, such that the Lagrangian was invariant under the action of some $U(x) \in \mathrm{SU}(N)$, now depending on position. This poses a problem, due to the derivative term, since we have

$$\begin{aligned} \bar{\psi}(x)(i\partial\!\!\!/)\psi(x) &\mapsto (\bar{\psi}U^\dagger(x))(i\partial\!\!\!/)(U(x)\psi(x)) \\ &= \bar{\psi}U^\dagger(x)(iU(x)\partial\!\!\!/\psi(x) + i(\partial\!\!\!/U(x))\psi(x)) \neq \bar{\psi}(x)(i\partial\!\!\!/)\psi(x). \end{aligned} \tag{12.2.1}$$

The mass term is fine, since we have

$$\begin{aligned} m\bar{\psi}(x)\psi(x) &\mapsto \bar{\psi}(x)U^\dagger(x)U(x)\psi(x) \\ &= \bar{\psi}(x)U^{-1}(x)U(x)\psi(x) = \bar{\psi}(x)\psi(x). \end{aligned} \tag{12.2.2}$$

Inspired by QED we introduce a covariant derivative, D_μ, and hence a gauge field, A_μ,

$$D_\mu \coloneqq \partial_\mu - igA_\mu. \tag{12.2.3}$$

Here g is a coupling constant, generalising e in the QED case. Our goal will be to figure out the properties of the gauge field from symmetry.

By definition we want our covariant derivative to transform covariantly, by this we mean that $D_\mu\psi$ should transform exactly as ψ does, so we demand that under this SU(N) local symmetry

$$D_\mu\psi(x) \mapsto D'_\mu\psi'(x) = U(x)D_\mu\psi(x). \tag{12.2.4}$$

We know that $\psi'(x) = U(x)\psi(x)$, so we just need to work out what we need for D'_μ to make this statement true.

To find the transformation law for the derivative we can expand the definitions, assuming that $D'_\mu = \partial_\mu - igA'_\mu$, that is the only part of the covariant derivative that transforms under a gauge transformation is the gauge field, which seems right. Then we have

$$D'_\mu\psi'(x) = (\partial_\mu - igA'_\mu(x))U(x)\psi(x) \tag{12.2.5}$$
$$= (\partial_\mu U(x))\psi(x) + U(x)\partial_\mu\psi(x) - igA'_\mu(x)U(x)\psi(x). \tag{12.2.6}$$

We also have

$$D'_\mu\psi'(x) = U(x)D_\mu\psi(x) = U(x)(\partial_\mu - igA_\mu(x))\psi(x) = U(x)\partial_\mu - igU(x)A_\mu(x), \tag{12.2.7}$$

note that we are being careful not to commute anything here, since we're not sure exactly what commutativity properties our new fields, ψ and A_μ, will have. We want these quantities to be equal for all ψ, so we can strip off the factors of ψ and demand that

$$(\partial_\mu U(x)) + U(x)\partial_\mu - igA'_\mu(x)U(x) = U(x)\partial_\mu - igU(x)A_\mu(x). \tag{12.2.8}$$

Cancelling the $U(x)\partial_\mu$ terms we have

$$\partial_\mu U(x) - igA'_\mu(x)U(x) = -igU(x)A_\mu(x). \tag{12.2.9}$$

Solving for $A'_\mu(x)$ we get

$$A'_\mu(x)U(x) = U(x)A_\mu(x)U^{-1}(x) - \frac{i}{g}(\partial_\mu U(x))U^{-1}(x). \tag{12.2.10}$$

Note that for QED we have $g = 3$ and $U(x) \in \mathrm{U}(1)$, so $U(x) = \mathrm{e}^{i\alpha(x)}$, then we have $\partial_\mu U(x) = i(\partial_\mu\alpha(x))\mathrm{e}^{i\alpha(x)}$ and $U^{-1}(x) = \mathrm{e}^{-i\alpha(x)}$, and both of these commute with $A_\mu(x)$ so we get

$$A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x), \tag{12.2.11}$$

as expected.

[2]see *Symmetries of Quantum Mechanics* or *Symmetries of Particles and Fields*

Recall that[2] that given some Lie group, G, with Lie algebra $\mathfrak{g}$ we can write $g \in G$ as[3]

$$g = \mathrm{e}^{i\vartheta^a t_a} \tag{12.2.12}$$

where $t_a \in \mathfrak{g}$ are the **generators** of $\mathfrak{g}$ and there is a sum over $a = 1, \dots, \dim\mathfrak{g}$.

[3]Conventions vary here, often the factor of i is left out by mathematicians or people working with real groups, but we like it here as it makes t^a Hermitian for $\mathrm{SU}(N)$.

We can write $U(x) \in \mathrm{SU}(N)$ as[4]

$$U(x) = \mathrm{e}^{i\vartheta^a(x)t^a}. \tag{12.2.13}$$

[4]Note that it doesn't matter for $\mathrm{SU}(N)$ if the indices are raised or lowered, since we can impose the Kronecker delta as the metric, see *Symmetries of Particles and Fields*.

For SU(2) we have $t^a = \sigma^a/2$ with $a = 1, 2, 3$, and for SU(3) we have $t^a = \lambda^a/2$ with $a = 1, \dots, 8$ where λ^a are the Gell-Mann matrices. For $\mathrm{SU}(N)$ there are $N^2 - 1$ generators, so $a = 1, \dots, N^2 - 1$.

Consider a small gauge transformation, by this we mean that $\vartheta^a(x)$ is small, or rather that we can parametrise things such that $U(x) = \exp\{i\varepsilon\vartheta^a(x)t^a\}$ and ε is small. Then we can expand to linear order in ε, this means that terms like

$\varepsilon\vartheta^a(x)\partial_\mu\varepsilon\vartheta^b(x)$ are small, since they are quadratic in ε. However, we already have so many εs floating around that we absorb ε into the definition of ϑ and simply remember that terms like $\vartheta^a(x)\partial_\mu\vartheta^b(x)$ are of quadratic order and so are small.

For a small gauge transformation we have

$$U(x) = \mathrm{e}^{i\vartheta^a(x)t^a} \approx 1 + i\vartheta^a(x)t^a, \tag{12.2.14}$$

$$U^{-1}(x) = \mathrm{e}^{-i\vartheta^a(x)t^a} \approx 1 - i\vartheta^a(x)t^a. \tag{12.2.15}$$

Note that $\vartheta^a(x)$ is just a number so commutes with everything except position derivatives, whereas $t^a \in \mathfrak{su}(N)$ so we cannot assume it commutes with anything except position derivatives, since t^a is constant in spacetime. For these small gauge transformations the gauge field transforms as

$$\begin{aligned} A'_\mu(x) &= (1 + i\vartheta^a(x)t^a)A_\mu(x)(1 - i\vartheta^b(x)t^b) \\ &\qquad - \frac{i}{g}(\partial_\mu(1 + i\vartheta^a(x)t^a))(1 - i\vartheta^b(x)t^b) && (12.2.16) \\ &= A_\mu(x) + i\vartheta^a(x)t^aA_\mu(x) - i\underbrace{\vartheta^b(x)A_\mu(x)t^b}_{b\to a} + \underbrace{\vartheta^a(x)\vartheta^b(x)}_{\to 0}t^at^b \\ &\qquad - \frac{i}{g}it^a(\partial_\mu\vartheta^a(x))(1 - i\vartheta^b(x)t^b) && (12.2.17) \\ &= A_\mu(x) + i\vartheta^a(x)(t^aA_\mu(x) - A_\mu(x)t^a) && (12.2.18) \\ &\qquad + \frac{1}{g}(\partial_\mu\vartheta^a(x))t^a - \frac{1}{g}\underbrace{(\partial_\mu\vartheta^a(x))t^a\vartheta^b(x)}_{\to 0}t^b && (12.2.19) \\ &= A_\mu(x) + i\vartheta^a(x)[t^a, A_\mu(x)] + \frac{1}{g}(\partial_\mu\vartheta^a(x))t^a. && (12.2.20) \end{aligned}$$

From this we see that $A_\mu(x)$ is a linear combination of the generators, t^a. This means that $A_\mu(x)$ is Lie algebra valued, and so does not, in general, commute with t^a. In QED we happen to have an Abelian gauge group and so $[t^a, A_\mu(x)] = 0$, and in fact there is only one generator, $t = 1$, so we have

$$A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x), \tag{12.2.21}$$

again recovering the expected transformation law.

12.3 Gauge Field Lagrangian

In order to fully specify Yang–Mills theory we also need a Lagrangian for the gauge fields. We look to generalise

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{12.3.1}$$

from QED.

To do so we need to write $F^{\mu\nu}$ in such a way that its transformation is understood with the Yang–Mills theory we have developed so far. The QED option of $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is not appropriate here, since it isn't gauge invariant and the transformation of A_μ is not simple. Instead, we use the fact that in QED we have

$$[D_\mu, D_\nu] = -ieF_{\mu\nu} \tag{12.3.2}$$

to generalise our definition of $F_{\mu\nu}$.

Define $F_{\mu\nu}$ in Yang–Mills theory to be such that

$$[D_\mu, D_\nu]\psi(x) = -igF_{\mu\nu}\psi(x) \tag{12.3.3}$$

where ψ is a spinor valued test function to aid in computing the commutator, and we assume that ψ has everywhere continuous second partial derivatives, so that $\partial_\mu\partial_\nu\psi = \partial_\nu\partial_\mu\psi$. Expanding this all out we get

$$\begin{aligned}
[D_\mu, D_\nu] &= [\partial_\mu - igA_\mu, \partial_\nu - igA_\nu]\psi && (12.3.4)\\
&= \underbrace{[\partial_\mu, \partial_\nu]\psi}_{=0} - ig[\partial_\mu, A_\nu]\psi - ig[A_\mu, \partial_\nu]\psi + (ig)^2[A_\mu, A_\nu]\psi && (12.3.5)\\
&= -ig(\partial_\mu(A_\nu\psi) - A_\nu\partial_\mu\psi)\\
&\qquad - ig(A_\mu\partial_\nu\psi - \partial_\nu(A_\mu\psi)) + (ig)^2[A_\mu, A_\nu]\psi && (12.3.6)\\
&= -ig[(\partial_\mu A_\nu)\psi + A_\nu\partial_\mu\psi - A_\nu\partial_\mu\psi]\\
&\qquad - ig[A_\mu\partial_\nu\psi - (\partial_\nu A_\mu)\psi - A_\mu\partial_\nu\psi] + (ig)^2[A_\mu, A_\nu] && (12.3.7)\\
&= -ig(\partial_\mu A_\nu)\psi + ig(\partial_\nu A_\mu)\psi + (ig)^2[A_\mu, A_\nu] && (12.3.8)\\
&= -ig[\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]]\psi. && (12.3.9)
\end{aligned}$$

So, we can define $F_{\mu\nu}$ according to

$$F_{\mu\nu}(x) \coloneqq \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ig[A_\mu(x), A_\nu(x)]. \tag{12.3.10}$$

Notice that in QED we have $[A_\mu, A_\nu] = 0$ so this reduces to the usual $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Notice also that $F_{\mu\nu}$ is still antisymmetric, since the commutator is antisymmetric, $[x, y] = -[y, x]$.

Since A_μ is Lie algebra valued we can define A^a_μ to be such that

$$A_\mu(x) = A^a_\mu(x)t^a. \tag{12.3.11}$$

Then we have

$$[A_\mu, A_\nu] = [A^a_\mu t^a, A^b_\nu t^b] = A^a_\mu A^b_\nu [t^a, t^b] = iA^a_\mu A^b_\nu f^{cba} t^c \tag{12.3.12}$$

where f^{abc} are the **structure constants** defined by[5]

$$[t^a, t^b] = if^{cba}t^c. \tag{12.3.13}$$

[5] conventions vary here, commonly mathematicians, and those working with real groups, leave out the factor of i, but it is due to our inclusion of i in $U = \exp\{i\vartheta^a t^a\}$, which we like as it makes t^a Hermitian for SU(N). Conventions also vary over which of the indices of the structure constant is summed over in the definition.

Similarly, we can define $F^a_{\mu\nu}$ to be such that

$$F_{\mu\nu} = F^a_{\mu\nu}t^a. \tag{12.3.14}$$

Using this we have

$$F^a_{\mu\nu}(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + gf^{abc}A^b_\mu A^c_\nu. \tag{12.3.15}$$

We can now explore how $F_{\mu\nu}$ transforms under a gauge transformation by studying the gauge transformation

$$F_{\mu\nu}\psi \mapsto F'_{\mu\nu}\psi'. \tag{12.3.16}$$

Covariance of the covariant derivative means that

$$[D_\mu, D_\nu]\psi \mapsto U(x)[D_\mu, D_\nu]\psi, \tag{12.3.17}$$

and so we have

$$F'_{\mu\nu}\psi' = -ig[D'_\mu, D'_\nu]\psi' = -igU(x)[D_\mu, D_\nu]\psi = U(x)F_{\mu\nu}\psi. \tag{12.3.18}$$

On the other hand, we have $\psi(x) \mapsto U(x)\psi(x)$, so

$$F'_{\mu\nu}\psi' = F'_{\mu\nu}U(x)\psi. \tag{12.3.19}$$

Comparing these two results we can see that

$$U(x)F_{\mu\nu} = F'_{\mu\nu}U(x) \implies F'_{\mu\nu} = U(x)F_{\mu\nu}U^{-1}(x). \tag{12.3.20}$$

In group theory terms we see that F is a rank two tensor, since it transforms as $T \mapsto UTU^{-1}$, or $T'_{ij} = U_{ik}U^*_{lj}T_{kl}$. Fundamentally, this is because $A_\mu \in \mathfrak{su}(2)$, so A_μ, and hence $F_{\mu\nu}$, transform in the adjoint representation of $\mathfrak{su}(2)$.

We are now in a position to write down a Lagrangian for the gauge field. We want this Lagrangian to be invariant under both Lorentz transformations and gauge transformations. The Lorentz invariance is taken care of by contracting the Lorentz indices. The gauge invariance can be taken care of using the trace, since we have

$$\begin{aligned}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu}) \mapsto \operatorname{tr}(F'_{\mu\nu}F'^{\mu\nu}) &= \operatorname{tr}(UF_{\mu\nu}U^{-1}UF^{\mu\nu}U) \\ &= \operatorname{tr}(UF_{\mu\nu}F^{\mu\nu}U^{-1}) = \operatorname{tr}(F_{\mu\nu}F^{\mu\nu}U^{-1}U) = \operatorname{tr}(F_{\mu\nu}F^{\mu\nu}),\end{aligned} \tag{12.3.21}$$

using the cyclic property of the trace. Note that the trace here is taken over $\mathfrak{su}(N)$, since $F_{\mu\nu}$ is Lie algebra valued. We can define $F^a_{\mu\nu}$ to be such that

$$F_{\mu\nu} = F^a_{\mu\nu}t^a, \tag{12.3.22}$$

then we have

$$\begin{aligned}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu}) = \operatorname{tr}(F^a_{\mu\nu}t^aF^{b\mu\nu}t^b) &= F^a_{\mu\nu}F^{b\mu\nu}\operatorname{tr}(t^at^b) \\ &= F^a_{\mu\nu}F^{b\mu\nu}\frac{1}{2}\delta^{ab} = F^a_{\mu\nu}F^{a\mu\nu}.\end{aligned} \tag{12.3.23}$$

Here we have used

$$\operatorname{tr}(t^at^b) = \frac{1}{2}\delta^{ab}, \tag{12.3.24}$$

which is the Killing form of $\mathfrak{su}(N)$.

Putting this together we define the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} = -\frac{1}{2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu}). \tag{12.3.25}$$

We can then define the full Lagrangian for Yang–Mills theory:

$$\mathcal{L} = -\frac{1}{2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu}) + \bar\psi(i\not{D} - m)\psi. \tag{12.3.26}$$

Studying this for a while we see that it generates all the same terms as QED, such as $m\bar\psi\psi$, $(\partial_\mu A_\nu)(\partial^\nu A^\mu)$, and $\bar\psi A_\mu\psi$, but it also generates new terms, such as $\partial_\mu(A^a_\nu)f^{abc}A^b_\mu A^c_\nu$, which corresponds to a three-point interaction between gauge bosons, and $(A^b_\mu A^c_\nu)(A^{b\mu}A^{c\nu})$, which corresponds to a four-point interaction between gauge bosons. Interactions between gauge bosons is not something we see in QED, which is fundamentally due to $\mathrm{U}(1)$ being Abelian and so these terms cancel out.

Thirteen

Group Theory

For more details see *Symmetries of Quantum Mechanics* and *Symmetries of Particles and Fields*.

13.1 Basics of SU(N)

Yang–Mills theory is defined by the existence of a gauge symmetry. We will restrict ourselves to SU(N) gauge symmetries, since this is the case for QCD, which is the SU(3) case.

Our first definition is SU(N) itself, this is the group of $N \times N$ unitary matrices with unit determinant. Strictly this is only one representation of SU(N), called the defining or fundamental representation. Any $U \in \mathrm{SU}(N)$ can be written as[1] $U = \mathrm{e}^{i\vartheta^a t^a}$ for $\vartheta^a \in \mathbb{C}$ where t^a are the **generators** of $\mathfrak{su}(N)$, the Lie algebra of SU(N). In the defining representation t^a are $N \times N$ matrices over $\mathbb{C}$. The conditions upon SU(N) enforce conditions at the level of the Lie algebra:

[1] mathematicians follow a different convention and write $U = \mathrm{e}^{\vartheta^a t^a}$

- Unitarity, $U^\dagger = U^{-1}$ forces the generators to be Hermitian[2], $(t^a)^\dagger = t^a$, to see this expand $(\mathrm{e}^{i\vartheta^a t^a})^\dagger = \mathrm{e}^{-i\vartheta^a t^{a\dagger}} = \mathrm{e}^{-i\vartheta^a t^a} = (\mathrm{e}^{-i\vartheta^a t^a})^{-1}$ to linear order.
- Unit determinant, $\det U = 1$, enforces $\operatorname{tr} t^a = 0$, through the identity $\det(\mathrm{e}^A) = \mathrm{e}^{\operatorname{tr} A}$.

[2] in the mathematics convention the generators are instead anti-Hermitian

There are $2N^2$ linearly independent complex $N \times N$ matrices, since there are N^2 entries each with two degrees of freedom. Imposing Hermiticity restricts the entries on the diagonal to be real, removing N degrees of freedom, and fixes the entries below the diagonal to be complex conjugates of those above the diagonal, fixing $N(N-1)/2$ complex values, so removing $N(N-1)$ degrees of freedom. We are therefore left with $2N^2 - N - N(N-1) = N^2$ degrees of freedom. So there are N^2 linearly independent Hermitian $N \times N$ matrices. Imposing tracelessness fixes one more degree of freedom, since given all but the first element in the diagonal we can deduce this value by requiring that the diagonal sums to zero, and recalling that all values on the diagonal are real for a Hermitian matrix. Thus there are $N^2 - 1$ generators of $\mathfrak{su}(N)$. This means that $\dim \mathrm{SU}(N) = N^2 - 1$ as a manifold also.

The generators of $\mathfrak{su}(2)$ are the Pauli matrices, of which there are $2^2 - 1 = 3$. The generators of $\mathfrak{su}(3)$ are the Gell-Mann matrices, of which there are $3^2 - 1 = 8$. The standard choice in both cases is actually to include an extra factor of $1/2$, for reasons we'll see later.

13.2 Structure Constants

The **structure constants**, f^{abc}, are defined by[3]

$$[t^a, t^b] = if^{abc}t^c. \tag{13.2.1}$$

[3] the i here is only present in the physics convention of $U = e^{i\vartheta^a t^a}$.

Here $[-,-]$ is the commutator, or more generally a Lie bracket. It is simple to show that the following holds for commutators, and it holds by definition for a Lie bracket:

$$[[t^a, t^b], t^c] + [[t^b, t^c], t^a] + [[t^c, t^a], t^b] = 0. \tag{13.2.2}$$

From this it is possible to show that

$$f^{abd}f^{dce} + f^{bcd}f^{dae} + f^{cad}f^{dbe} = 0. \tag{13.2.3}$$

Both of these results are known as the **Jacobi identity**.

13.3 Other Representations

So far we have thought of $\mathrm{SU}(N)$ as acting on $\mathbb{C}^N$ through the group action

$$U \,.\, v = Uv \tag{13.3.1}$$

for $U \in \mathrm{SU}(N)$ in the fundamental representation and $v \in \mathbb{C}^N$.

Other representations act on different representation spaces. We'll often work with an arbitrary representation, R. In this representation we'll denote the generators by t^a_R. Note that the structure constants are representation independent, since the Lie bracket is representation independent as part of the definition of a Lie group. So we have

$$[t^a_R, t^b_R] = if^{abc}t^c_R \tag{13.3.2}$$

in any representation. It is common to label a representation by its dimension, so the fundamental representation is N.

One particularly important representation is the **adjoint representation** in which $\mathfrak{su}(N)$ acts on itself by taking the generators, t^b_{adj}, to have components

$$(t^b_{\mathrm{adj}})^{ac} = if^{abc}. \tag{13.3.3}$$

In an arbitrary representation, R, the group action on some vector v is

$$g \,.\, v = e^{i\vartheta^a t^a_R}v = (1 + i\vartheta^a t^a_R)v \tag{13.3.4}$$

for infinitesimal ϑ^a. We can define the conjugate representation, $\overline{R}$, through the action

$$g \,.\, v^* = (1 - i\vartheta^a (t^a_R)^*)v^*, \tag{13.3.5}$$

so $t^a_{\overline{R}} = -(t^a_R)^*$.

Two relations are **equivalent** if they are related by a change of basis of the representation space. That is, two representation, R_1 and R_2, are equivalent if

$$t^a_{R_1} = X t^a_{R_2} X^\dagger \tag{13.3.6}$$

for some fixed matrix X. A representation which is equivalent to its conjugate is called a **real representation**. The adjoint representation is always real. So is the fundamental representation of $\mathrm{SU}(2)$, but the fundamental representation of $\mathrm{SU}(3)$ is not.

13.4 Casimirs

In the fundamental representation, writing t^a for the generators, we are free to choose the normalisation of the generators such that

$$\mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}. \tag{13.4.1}$$

In a general representation, we can define the trace, $T(R)$, through

$$\mathrm{tr}(t^a_R t^b_R) = T(R)\delta^{ab}. \tag{13.4.2}$$

We cannot in general take $T(R) = 1/2$, since this is in conflict with $[t^a_R, t^b_R] = if^{abc}t^c_R$, where the normalisation of the structure constants is fixed by taking $T(N) = 1/2$.

Now consider the operator $t^a_R t^a_R$. This commutes with all of the generators, which we can show by commuting each t^a_R with the generator individually, picking up a commutator term, and then using the structure constants:

$$\begin{aligned} t^a_R t^a_R t^b_R &= t^a_R t^b_R t^a_R + t^a_R[t^a_R, t^b_R] = t^b_R t^a_R t^a_R + [t^a_R, t^b_R]t^a_R + t^a_R[t^a_R, t^b_R] \\ &= t^b_R t^a_R t^a_R + if^{abc}t^c_R t^a_R + if^{abc}t^a_R t^c_R = t^b_R t^a_R t^a_R + if^{abc}(t^c_R t^a_R + t^a_R t^c_R) = t^b_R t^a_R t^a_R \end{aligned} \tag{13.4.3}$$

having used the fact that f^{abc} is totally antisymmetric and $(t^c_R t^a_R + t^a_R t^c_R)$ is symmetric in a and c, so their product vanishes.

In group theory language we say that $t^a_R t^a_R$ is a **Casimir operator**, it commutes with all generators, and hence all elements of $\mathfrak{su}(N)$. More specifically, $t^a_R t^a_R$ is the **quadratic Casimir**, since it is quadratic in the generators, but it is common to refer to it simply as *the* Casimir, since we aren't interested in any of the other Casimirs, apart from trivial things like scalars which commute with everything. A basic result of Lie theory, Schur's lemma, says that all Casimir operators are simply a (representation dependent) scalar times the identity, so we have

$$(t^a_R)_{ik}(t^a_R)_{kj} = C_2(R)\delta_{ij} \iff t^a_R t^a_R = C_2(R)1 \tag{13.4.4}$$

where the i and j indices run from 1 to $d(R)$, where $d(R)$ is the dimension of representation R. Note that a runs from 1 to $N^2 - 1$, regardless of the representation. Taking traces we have

$$\mathrm{tr}_R(t^a_R t^a_R) = C_2(R)\delta_{ii} = C_2(R)d(R) = T(R)\delta^{aa} = T(R)\dim G. \tag{13.4.5}$$

Using $T(N) = 1/2$ we have

$$C_2(N)N = \frac{1}{2}(N^2 - 1) \implies C_2(N) = \frac{1}{2N}(N^2 - 1). \tag{13.4.6}$$

We denote the adjoint representation by A, and it can be shown that $T(\mathrm{A}) = C_2(\mathrm{A}) = N$.

13.5 Representations and Yang–Mills Theory

Recall that the gauge transformation of A_μ is

$$A'^a_\mu = A^a_\mu + i\vartheta^a[t^a, t^b]A^b_\mu + \frac{1}{g}(\partial_\mu \vartheta^a)t^a. \tag{13.5.1}$$

If we consider a global transformation, where ϑ^a is constant, then the derivative vanishes and we are left with

$$A'^a_\mu = A^a_\mu + i\vartheta^a[t^a, t^b]A^b_\mu \tag{13.5.2}$$
$$= A^a_\mu - f^{abc}\vartheta^b A^c_\mu \tag{13.5.3}$$
$$= (\delta^{ca} - f^{abc}\vartheta^b)A^c_\mu. \tag{13.5.4}$$

This is the condition for the gauge fields to transform in the adjoint representation for global transformations. In turn this means that the gauge fields are elements of the Lie algebra, $\mathfrak{su}(N)$, or "Lie algebra valued". This is quite hand wavy and will annoy mathematicians, who will insist on talking about fibre bundles here.

Under a global transformation we get an extra term from the derivative called an **affine transformation**. This extra term cancels in $F^{a\mu\nu}$, so $F^{a\mu\nu}$ is always in the adjoint representation for both global and local transformations.

Recall that for matter we have the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi \tag{13.5.5}$$

where now ψ and $\bar{\psi}$ transform under some representation, R, of SU(N). Note that

$$D_\mu = \partial_\mu - igA^a_\mu t^a_R \tag{13.5.6}$$

is representation dependent. Note that here t^a_R are operators on spinor space. If we introduce spinor indices, i and j, then we have

$$(D_\mu\psi)_i = (\delta_{ij}\partial_\mu - igA^a_\mu(t^a_R)_{ij})\psi_j. \tag{13.5.7}$$

Including the gauge fields we get the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\operatorname{tr}(F^{\mu\nu}F_{\mu\nu}) + \bar{\psi}(i\not{D} - m)\psi = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}(i\not{D} - m)\psi. \tag{13.5.8}$$

We can also add in scalar matter, again transforming under an arbitrary, potentially different, representation. We arrange these scalars into a column matrix, φ, of size $d(R)$. The most general renormalisable Lagrangian including a single fermion and and a single scalar, without interactions between them, is

$$\mathcal{L} = -\frac{1}{2}\operatorname{tr}(F^{\mu\nu}F_{\mu\nu}) + \bar{\psi}(i\not{D} - m)\psi + (D_\mu\varphi)^\dagger(D^\mu\varphi) - M^2\varphi^\dagger\varphi + \frac{\lambda}{4}(\varphi^\dagger\varphi)^2. \tag{13.5.9}$$

A Yang–Mills theory is completely specified by the following information:

- the gauge group, G;
- the matter fields, ψ and/or φ;
- the interactions of the matter fields, terms like $\lambda(\varphi^\dagger\varphi)^2/4$ or $g\varphi\bar{\psi}\psi$;
- the masses of the particles;

QCD is a Yang–Mills theory with the gauge group $G = \mathrm{SU}(3)$. There are no scalar fields and six fermion fields, the quarks, q_f with $f = 1, \ldots, 6$, which we take to all be the same mass. The Lagrangian is then

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \sum_{f=1}^{6}\bar{\mathrm{q}}_f(i\not{D} - m)\mathrm{q}_f. \tag{13.5.10}$$

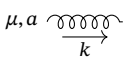
13.6 Feynman Rules for Yang–Mills Theory

Given how similar the Yang–Mills Lagrangian is to the QED Lagrangian it should not be too surprising that the Feynman rules for Yang–Mills theory are similar to the Feynman rules for QED. We won't derive most of them here, the process is very similar to the process for deriving the QED Feynman rules. We'll state the rules, and derive one example.

First, we have the fermion propagator, this is exactly the same as in QED, although now the mass is a quark mass:

$$= \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}. \tag{13.6.1}$$

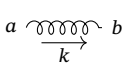
Next, we have the gluon propagator. This is very similar to the photon propagator, but we have these extra a indices hanging about. Since the gluon doesn't change colour when propagating the colour at either end of a gluon propagator must be the same, so we just have an extra δ^{ab} to include:



$$= -\frac{i\delta^{ab}}{k^2 + i\epsilon}\left(\eta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right). \tag{13.6.2}$$

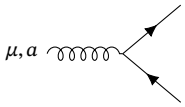
Here, as in QED, ξ is a gauge parameter, which we'll often take to be 1.

We'll see at the end of the next chapter that in the gauge fixing process we are forced to introduce another particle, and that these are massless scalars with a colour index, a, which we call Faddeev–Popov ghosts. They enter the Lagrangian through the term $-\bar{c}^a \partial_\mu D^\mu c^a$. The propagator for these ghosts, coming from the $-\bar{c}^a \partial_\mu \partial^\mu c^a$ term, is very similar to the usual scalar propagator, but with an extra δ^{ab}, since the ghosts don't change colour as they propagate. We then have the Feynman rule



$$= \frac{i\delta^{ab}}{k^2 + i\epsilon}. \tag{13.6.3}$$

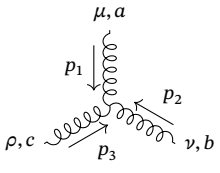
Now we get to the interactions. The most familiar is the three point quark and gluon vertex, which is very similar to the QED vertex, and indeed differs only by a colour factor, t^a_R:



$$= ig\mu^\varepsilon \gamma^\mu t^a_R. \tag{13.6.4}$$

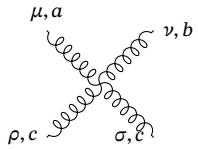
Note that this depends on the representation under which our fermions transform. The generators included through this vertex usually reduce to $T(R)$ and $C(R)$ or $T(\mathrm{A})$ and $C(\mathrm{A})$. These are known as **colour factors**.

As well as this familiar interaction we also have an interaction coming from the cubic and quartic terms in A^a_μ contained in $F^a_{\mu\nu}F^{a\mu\nu}$. These give three-point and four-point interactions between gluons, for which the Feynman rules are



$$\begin{aligned} = gf^{abc}[&(p_{1\rho} - p_{2\rho})\eta_{\mu\nu} \\ &+ (p_{2\mu} - p_{3\mu})\eta_{\nu\rho} \\ &+ (p_{3\nu} - p_{1\nu})\eta_{\rho\mu}], \end{aligned} \tag{13.6.5}$$

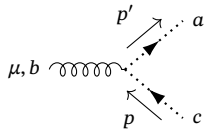
and



$$= -ig^2[f^{abe}f^{cde}(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}) + f^{ace}f^{dbe}(\eta_{\mu\sigma}\eta_{\rho\nu} - \eta_{\mu\nu}\eta_{\rho\sigma}) + f^{ade}f^{bce}(\eta_{\mu\nu}\eta_{\sigma\rho} - \eta_{\mu\rho}\eta_{\sigma\nu})]. \tag{13.6.6}$$

The final interaction is the interaction of gluons with the ghosts. This arises from the $-\bar{c}^a\partial_\mu(igA^{b\mu}t^a_R c^a)$ term, which is very similar to the $\varphi A^\mu\varphi$ term appearing in scalar QED[4] and the Feynman rule for this term is

[4] see tutorials



$$= -gf^{abc}p'_\mu. \tag{13.6.7}$$

Note that it is the momentum of the outgoing ghost that appears on the right hand side.

13.6.1 Derivation of the Three Gluon Interaction

Consider the $F^a_{\mu\nu}F^{a\mu\nu}$ term in the Lagrangian. Expanding this we have

$$-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu)(\partial^\mu A^{a\nu} - \partial^\mu A^{a\mu} + gf^{ade}A^{d\mu}A^{e\nu}).$$

Partially expanding this one of the terms we get is

$$-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)gf^{ade}A^{d\mu}A^{e\nu}. \tag{13.6.8}$$

Notice that $f^{ade}A^{d\mu}A^{e\nu}$ is antisymmetric in μ and ν, since we can swap μ and ν to get

$$f^{ade}A^{d\nu}A^{e\mu} = f^{ade}A^{e\mu}A^{d\nu} = -f^{aed}A^{e\mu}A^{d\nu} = -f^{ade}A^{d\mu}A^{e\nu}, \tag{13.6.9}$$

where the last step is just renaming the indices $d \leftrightarrow e$. We can therefore use

$$\partial_\mu A^a_\nu - \partial_\nu A^a_\mu = 2\partial_{(\mu}A^a_{\nu)} \tag{13.6.10}$$

to replace $\partial_\mu A^a_\nu - \partial_\nu A^a_\mu$ with $\partial_\mu A^a_\nu$, which when antisymmetrised over μ and ν reproduces $\partial_\mu A^a_\nu - \partial_\nu A^a_\mu$, and this antisymmetrising is enforced by the antisymmetry of $f^{ade}A^{e\mu}A^{d\nu}$, which kills the symmetric part of $\partial_\mu A^a_\nu$. Therefore we have the term

$$-\frac{1}{2}gf^{abc}(\partial_\mu A^a_\nu)A^{b\mu}A^{c\nu}, \tag{13.6.11}$$

having renamed the colour indices $d \to b$ and $e \to c$.

The Feynman rule corresponding to this, which is cubic in the gauge field so corresponds to a three-point gluon interaction, is given by considering an amputated correlator for three incoming particles, which expanded to first order gives

$$\langle 0|i\int \mathrm{d}^4x\,(-g)f^{ABC}(\partial_\mu A^A_\nu)A^{B\mu}A^{C\nu}|p_1\,\alpha\,a,\,p_2\,\beta\,b,\,p_3\,\gamma\,c\rangle. \tag{13.6.12}$$

The other factor of 2 is removed by the antisymmetry of exchanging the two gauge fields which we aren't taking the derivative of. Note that in the initial state we place the momenta, colour indices, and also the Lorentz indices which are attached to the external polarisation states. This means that a contraction between, for example, $A^{B\mu}$ and $p_1\,\alpha\,a$ gives $\varepsilon^\mu\varepsilon^\alpha$, and a contraction between $\partial_\mu A^A_\nu$ and $p_1\,\alpha\,A$ gives $-ip_1\varepsilon_\nu\varepsilon^\alpha$, as the derivative brings down a factor of $-i$ from the plane wave in the mode expansion.

One particular contraction which we have to consider is

$$\langle 0|i\int \mathrm{d}^4x\,(-g)f^{ABC}(\partial_\mu A^A_\nu)A^{B\mu}A^{C\nu}|p_1\,\alpha\,a,\,p_2\,\beta\,b,\,p_3\,\gamma\,c\rangle. \tag{13.6.13}$$

The contractions are the same as in QED, but with an extra Kronecker delta on the colour indices, so this contraction sets $A = a$, $B = b$, and $C = c$. The derivative brings down an $-ip_{1\mu}$, but then since the μ index is contracted (as in indices match across) with $A^{B\mu}$ and $A^{B\mu}$ is contracted (as in Wick contractions) with $p_2\,\beta\,b$, so this μ becomes a β. The other Lorentz indices are contracted together and there is only really one reasonable result, $\eta_{\gamma\alpha}$. Overall this contraction gives the term

$$(-ig)f^{abc}(-ip_{1\beta}\eta_{\gamma\alpha}) \tag{13.6.14}$$

There are then two other contractions which preserve the cyclic order of f^{ABC}, and these are just cyclic permutations of the above. There are then three terms which change the cyclic order of f^{ABC}, these are just the contraction we've considered swapping which of the two contractions goes into $p_2\,\beta\,b$ and which goes into $p_3\,\gamma\,c$, so they're quite similar, and these can then be made negative to return to the same cyclic order of f^{ABC}. Overall this gives us six terms which give the result quoted above.

Fourteen

Faddeev–Popov Gauge Fixing

In this chapter we discuss gauge fixing through the Faddeev–Popov procedure. We followed this same procedure in *Quantum Field Theory* for QED, here we follow it for the more general Yang–Mills theory, but the idea is the same.

14.1 Setup and the Problem

The pure Yang–Mills action is

$$S_{\text{YM}} = \int \mathrm{d}^D x \left[-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \right] \tag{14.1.1}$$

where

$$F^a_{\mu\nu}(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + g f^{abc} A^b_\mu(x) A^c_\nu(x). \tag{14.1.2}$$

The gauge transformation law for the gauge field under an infinitesimal transformation is

$$A'^a_\mu(x) = A^a_\mu(x) + f^{abc} A^b_\mu(x) \vartheta^c(x) + \frac{1}{g} \partial_\mu \vartheta^a(x) \tag{14.1.3}$$

$$= A^a_\mu(x) + \frac{1}{g} D_\mu \vartheta^a(x). \tag{14.1.4}$$

The problem we face is that the naive Yang–Mills path integral with this action is undefined. This is because it includes an integral over all gauge fields, which includes an integral over A^a_μ and also every ${}^{\vartheta}A^a{}_\mu$ related to A^a_μ by a gauge transformation parametrised by $\vartheta^a(x)$.

In the language of group theory we are integrating over all gauge fields, and also the result of any SU(N) group action on a gauge field. Given some gauge field A^a_μ and some $g \in \text{SU}(N)$ we can construct the **orbit** of A^a_μ to be

$$\{ g^n \,.\, A^a_\mu \mid n \in \mathbb{N} \} \tag{14.1.5}$$

which consists of all fields related to A^a_μ by the gauge transformation g. The solution is to somehow limit our integral to include only one gauge field from each orbit.

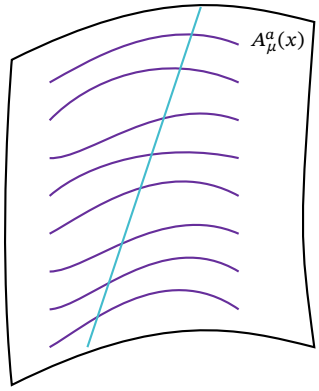


Figure 14.1: All possible values of $A^a_\mu(x)$ related by gauge transformations at fixed x, a, and μ. The purple lines represent orbits, all points on the line are given by acting on one point of the line with a fixed gauge transformation. The other gauge transformations will move points between lines. The blue line is a gauge fixing condition, we assume that it crosses each orbit once, and we choose the point it crosses the orbit to be the one included in the path integral.

The problem is evident on the level of the free action, with $g = 0$, where we have

$$\begin{aligned}
\frac{1}{2}F^a_{\mu\nu}F^a_{\mu\nu} &= (\partial_\mu A^a_\nu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) && (14.1.6)\\
&= (\partial_\mu A^a_\nu)(\partial^\mu A^{a\nu}) - (\partial_\mu A^a_\nu)(\partial^\nu A^{a\mu}) && (14.1.7)\\
&= \partial_\mu(A^a_\nu \partial^\mu A^{a\nu}) - A^a_\nu \partial_\mu \partial^\mu A^{a\nu} - \partial_\mu(A^a_\nu \partial^\nu A^{a\mu}) + A^a_\nu(\partial_\mu \partial^\nu A^{a\mu})\\
&= \partial_\mu(A^a_\nu \partial^\mu A^{a\nu} - A^a_\nu \partial^\nu A^{a\mu}) - A^a_\nu \partial^2 A^{a\nu} + A^a_\nu \partial_\mu \partial^\nu A^{a\mu} && (14.1.8)\\
&= \partial_\mu(A^a_\nu \partial^\mu A^{a\nu} - A^a_\nu \partial^\nu A^{a\mu}) - A^a_\mu[\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu]A^a_\mu. && (14.1.9)
\end{aligned}$$

The first term here is just a surface term, so doesn't contribute to the action. So the free action is

$$S_0 = \int \mathrm{d}^D x \, \frac{1}{2} A^a_\mu(x)[\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu]A^a_\nu(x). \tag{14.1.10}$$

The problem comes when trying to compute path integrals of $\mathrm{e}^{iS_0[A]}$, since the general answer requires the inverse of the kernel, which here is $\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu$, but this isn't invertible.

Geometrically, if we fix some value of x then we can consider a volume defined by all possible values of $A^a_\mu(x)$ for fixed a and μ. For simplicity we'll assume that this defines a two-dimensional surface. The orbits are then lines on this surface generated by acting with some fixed gauge transformation on A^a_μ repeatedly. We want to impose a gauge fixing constraint which picks out one value of A^a_μ for each orbit, that is one point on each line. These orbit lines are dense in the surface, and so the collection of points chosen after gauge fixing define another line. We assume that there is a single solution to the gauge fixing condition for each orbit.

14.2 The Faddeev–Popov Procedure

We use the gauge fixing constraint

$$G^a(A) = \partial_\mu {}^\vartheta A^{a\mu}(x) - \omega^a(x) = 0 \tag{14.2.1}$$

for some function ω^a. This is a generalisation of the Lorenz gauge condition, $\partial_\mu A^{a\mu} = 0$, and indeed choosing $\omega^a(x) = 0$ recovers the Lorenz gauge.

Now consider the correlator of some gauge invariant observable $\mathcal{O}$. This is given by

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}A\, \mathcal{O} e^{iS[A]}}{\int \mathcal{D}A\, e^{iS[A]}}. \tag{14.2.2}$$

The denominator is just the normalisation, which normally we leave off, but we want it here as it will simplify things later.

As with QED in *Quantum Field Theory* we aim to get this in the form

$$\frac{\int \mathcal{D}A\, \delta(G(A)) \mathcal{O} e^{iS[A]}}{\int \mathcal{D}A\, \delta(G(A)) e^{iS[A]}}. \tag{14.2.3}$$

We won't quite succeed but we'll get close enough. Here δ is a functional Dirac delta given by

$$\delta(G(A)) = \prod_{x,a} \delta(G(A^a_\mu(x))) \tag{14.2.4}$$

where on the right δ is just a normal Dirac delta. We use this to impose that $\partial^\mu A^a_\mu(x) = \omega^a(x)$ for all a and all x.

To proceed we insert unity. First notice that for a function g with $g(\vartheta) > 0$ we have

$$1 = \int \mathrm{d}g\, \delta(g) = \int \mathrm{d}\vartheta\, \frac{\partial g}{\partial \vartheta} \delta(g(\vartheta)). \tag{14.2.5}$$

If $g(\vartheta) < 0$ we instead get -1, but the sign will cancel out in the normalisation, so it doesn't really matter. We can generalise this to multiple dimensions with

$$1 = \int \mathrm{d}^n g\, \delta(g) = \int \mathrm{d}^n \vartheta\, \det\left(\frac{\partial g^i}{\partial \vartheta^j}\right) \delta(g(\vartheta)). \tag{14.2.6}$$

Further generalising to a functional identity we get

$$1 = \int \mathcal{D}\vartheta\, \det\left(\frac{\delta G}{\delta \vartheta}\right) \delta(G(\vartheta)). \tag{14.2.7}$$

We will use

$$G(A^a(x)) = \partial_\mu {}^\vartheta A^{a\mu}(x) - \omega^a(x) \tag{14.2.8}$$

$$= \partial_\mu A^{a\mu}(x) + \frac{1}{g} \partial_\mu D^\mu \vartheta^a(x) - \omega^a(x). \tag{14.2.9}$$

We then have

$$\frac{\delta G(A^a(x))}{\delta \vartheta^b(y)} = \frac{1}{g} \partial_\mu D^\mu \delta^{ab} \delta(x - y). \tag{14.2.10}$$

Importantly this value depends on $A^{a\mu}$ through D^μ. This wasn't the case in QED. Expanding the definition of D^μ we have

$$\frac{\delta G(A^a(x))}{\delta \vartheta^b(y)} = \frac{1}{g}\partial_\mu(\partial^\mu \delta^{ab} - g f^{abc} A^{c\mu}(x))\delta(x-y) =: \mathcal{F}[A^{a\mu}(x)]. \tag{14.2.11}$$

Intuitively, the difference between Yang–Mills theory and QED in this respect is that in QED the orbits are all parallel, in the sense that if a gauge transformation takes you between two orbits then it doesn't matter where you start on one of those orbits, you'll always move to the same orbit. In Yang–Mills this is not the case, to move from orbit one to orbit two at one point requires a different gauge transformation than moving between orbit one and orbit two at another point.

We can proceed to insert unity into the path integral:

$$\int \mathcal{D}A\, 1 \mathcal{O} e^{iS[A]} = \int \mathcal{D}A \int \mathcal{D}\vartheta\, \det(\mathcal{F}[A])\delta(G({}^\vartheta A))\mathcal{O}e^{iS[A]} \tag{14.2.12}$$

$$= \int \mathcal{D}\vartheta \int \mathcal{D}A\, \det(\mathcal{F}[A])\delta(G({}^\vartheta A))\mathcal{O}e^{iS[A]}. \tag{14.2.13}$$

We can swap the integrals because we aren't worrying about convergence and both ϑ and A are bosonic, so commute.

Most gauge fields in this integral don't contribute, because of the $\delta(G({}^\vartheta A))$ term. A given gauge field only contributes if it satisfies $G({}^\vartheta A) = 0$, that is if

$$\partial_\mu A^{a\mu}(x) + \frac{1}{g}\partial_\mu D^\mu \vartheta^a(x) - \omega^a(x) = 0. \tag{14.2.14}$$

Now define A' to be

$$A'^{a\mu}(x) = A^{a\mu}(x) + \frac{1}{g} D^\mu \vartheta^a(x). \tag{14.2.15}$$

We can change A to A' in the integral, assuming that $\mathcal{D}A = \mathcal{D}A'$, and using the gauge invariance of the observable, $\mathcal{O}$, and the action. We then have

$$\int \mathcal{D}A\, \mathcal{O} e^{iS[A]} = \int \mathcal{D}\vartheta \int \mathcal{D}A'\, \det(\mathcal{F}[A'])\delta(G(A'))\mathcal{O}e^{iS[A']}. \tag{14.2.16}$$

Now the only ϑ dependence is in the $\int \mathcal{D}\vartheta$ part. Doing exactly the same with the denominator of $\langle \mathcal{O} \rangle$ we can cancel these two integrals with ϑ dependence. Dropping the primes we then have

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}A\, \det(\mathcal{F}[A])\delta(G(A))e^{iS[A]}}{\int \mathcal{D}A\, \det(\mathcal{F}[A])\delta(G(A))e^{iS[A]}}. \tag{14.2.17}$$

In QED $\mathcal{F}[A]$ is actually independent of A, and so we can bring it outside of the integral and it cancels. This isn't the case in Yang–Mills, so we won't quite get the desired form, but we continue anyway.

Now use

$$1 = \frac{1}{N}\int \mathcal{D}\omega\, \exp\left\{-i\int \mathrm{d}^D x \frac{1}{2\xi}\omega^a(x)\omega^a(x)\right\} \tag{14.2.18}$$

where N is a normalisation factor chosen to make this true. This is a Gaussian integral. We can use it, as well as $G(A^a(x)) = \partial_\mu A^{a\mu}(x) - \omega^a(x)$, to rewrite our result as

$$\int \mathcal{D}A \, \det(\mathcal{F}[A])\delta(\partial_\mu A^{a\mu}(x) - \omega^a(x))\mathcal{O}e^{iS[A]} \tag{14.2.19}$$

$$= \frac{1}{N}\int \mathcal{D}A \int \mathcal{D}\omega \, \det(\mathcal{F}[A])\delta(\partial_\mu A^{a\mu}(x) - \omega^a(x)) \times \exp\left\{-i\int \mathrm{d}^D x \frac{1}{2\xi}\omega^b(x)\omega^b(x)\right\}\mathcal{O}e^{iS[A]} \tag{14.2.20}$$

$$= \frac{1}{N}\int \mathcal{D}A \, \det(\mathcal{F}[A])\mathcal{O}\exp\left\{iS[A] - i\int \mathrm{d}^D x \frac{1}{2\xi}(\partial_\mu A^{b\mu})(\partial_\nu A^{b\nu})\right\}$$

where in the last step we've used the δ to perform the ω integral.

We can interpret this as modifying the original Lagrangian to contain a gauge fixing term

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})(\partial_\nu A^{a\nu}). \tag{14.2.21}$$

What this tells us is that the gluon propagator is exactly the photon propagator with an extra δ^{ab} factor coming from this term.

We still have to deal with the $\det(\mathcal{F}[A])$ term. To do this we can recognise that determinants can arise from Gaussian path integrals. In particular, we know that if M is an operator on spinors we have

$$\int \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, \exp\left\{-i\int \mathrm{d}^D x \, \bar{\psi}M\psi\right\} = N' \det M \tag{14.2.22}$$

where N' is another not-important normalisation constant which will cancel in the ratio. Note that we need anticommuting variables in order to get the correct power of the determinant. We can thus get an appearance of the desired determinant by taking

$$M = \frac{1}{g}\partial_\mu D^\mu. \tag{14.2.23}$$

Then the Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})(\partial_\nu A^{a\nu}) - \bar{c}^a\partial_\mu D^\mu c^a. \tag{14.2.24}$$

Here c replaces ψ in the above. First note that we just absorbed the $1/g$ factor into the two cs, changing the normalisation N' appropriately, but N' will cancel in the ratio so we don't worry about it. Next note that cs must carry an adjoint index, since $\partial_\mu D^\mu$ is an operator in the adjoint representation. Now notice that the cs don't carry a Lorentz index, since the Lorentz index in the operator is already contracted. This means that the cs are scalars. But they anticommute. This seems to violate the spin-statistics theorem, which says that fermionic states are the anticommuting ones. The solution is that the cs aren't physical states. In fact it is possible to show that there is no process starting without cs which can produces cs in the final state. For this reason the cs are called **Faddeev–Popov ghosts**.

We can't see the ghosts in external states, but have to allow for ghost loops in our expansions. The purpose of the ghosts is to remove a degree of freedom, since for each a we have two helicities, but four degrees of freedom in A^a_μ. In QED we fixed two degrees of freedom by requiring that $\varepsilon \cdot k = 0$ and the gauge symmetry $\varepsilon \mapsto \varepsilon' = \varepsilon + \alpha k$, which satisfies $\varepsilon' \cdot k = \varepsilon \cdot k + \alpha k^2 = 0 + \alpha k^2$, and $k^2 = 0$ is just the Fourier transformed equation of motion, $\partial^2 A^\mu = 0$. In Yang–Mills the equation of motion is more complicated, so we can't fix the degrees of freedom as simply and are forced to introduce these ghost particles.

Fifteen

Renormalising Yang–Mills

As with QED Yang–Mills theory at one loops is plagued by infinities. We will remove these following an almost identical procedure to QED. In particular, we will use

- dimensional regularisation;
- the $\overline{\text{MS}}$ scheme;
- renormalised perturbation theory,

and we will make use of results from QED, as many results in Yang–Mills theory differ only by factors of the SU(N) generators, t^a_R, which we call **colour factors**, taking this name from QCD and applying it to general Yang–Mills theory.

Our ultimate goal will be to compute the beta function of Yang–Mills theory:

$$\beta(g) \coloneqq \mu \frac{\mathrm{d}g}{\mathrm{d}\mu} \tag{15.0.1}$$

which is defined when $D = 4$, so after taking $\varepsilon \to 0$.

15.1 Regulating the Lagrangian

The Yang–Mills Lagrangian, now in terms of bare fields, is

$$\mathcal{L} = -\frac{1}{4} F^a_{\mathrm{B}\mu\nu} F^{a\mu\nu}_{\mathrm{B}} + \bar{\psi}_{\mathrm{B}}(i\not{D}_{\mathrm{B}} - m_{\mathrm{B}})\psi_{\mathrm{B}} - \bar{c}^a_{\mathrm{B}} \partial_\mu D^\mu_{\mathrm{B}} c^a. \tag{15.1.1}$$

where

$$F^{\mu\nu}_{\mathrm{B}} = \partial^\mu A^\nu_{\mathrm{B}} - \partial^\nu A^\mu_{\mathrm{B}} + g_{\mathrm{B}} f^{abc} A^{b\mu}_{\mathrm{B}} A^{c\nu}_{\mathrm{B}} \tag{15.1.2}$$

and

$$D^\mu_{\mathrm{B}} = \partial^\mu - i g_{\mathrm{B}} t^a_R A^{a\mu}_{\mathrm{B}}. \tag{15.1.3}$$

We can now introduce renormalised fields by defining

$$A^a_{\mathrm{B}\mu} = \sqrt{Z_3} A^a_\mu, \qquad \psi_{\mathrm{B}} = \sqrt{Z_2}\psi, \qquad c^a_{\mathrm{B}} = \sqrt{Z_{3c}} c^a, \tag{15.1.4}$$

$$g_{\mathrm{B}} = Z_g \mu^\varepsilon g, \qquad \text{and} \qquad m_{\mathrm{B}} = Z_m m. \tag{15.1.5}$$

Note that Z_i are dimensionless, and μ carries dimensions of energy, with ε being the ε of dimensional regularisation, $D = 4 - 2\varepsilon$.

Substituting these into the Lagrangian we get

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}Z_3(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu})(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + Z_2\bar\psi(i\partial\!\!\!/ - Z_m m)\psi \\ &- Z_{3c}\bar c^a \partial^2 c^a - \underbrace{Z_g Z_3^{3/2}}_{Z_{A^3}} g\mu^\varepsilon f^{abc}(\partial^\mu A^{a\nu})A^b_\mu A^c_\nu \\ &- \frac{1}{4}\underbrace{Z_g^2 Z_3^2}_{Z_{A^4}} g^2\mu^{2\varepsilon} f^{abc}f^{ade}A^b_\mu A^c_\nu A^{d\mu}A^{e\nu} + \underbrace{Z_2\sqrt{Z_3}Z_g}_{Z_1} g\mu^\varepsilon \bar\psi A\!\!\!/^a t^a_R \psi \\ &+ \underbrace{Z_g Z_{3c}\sqrt{Z_3}}_{Z_{1c}} g\mu^\varepsilon(\partial_\mu \bar c^a) f^{abc}A^{b\mu}c^c.\end{aligned} \tag{15.1.6}$$

This is getting out of hand, so we'll focus only on the fermions for simplicity:

$$\mathcal{L} = \bar\psi_B(i\not{D}_B - m_B)\psi_B \tag{15.1.7}$$

$$= Z_2\bar\psi(i\partial\!\!\!/ - Z_m m)\psi + Z_1\mu^\varepsilon g\bar\psi A\!\!\!/^a t^a_R\psi. \tag{15.1.8}$$

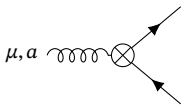
Note that in QED we had $Z_1 = Z_2$, this is not the case in Yang–Mills theory. Now for each Z_i write $Z_i = 1 + \delta_i$, then we have

$$\begin{aligned}\mathcal{L} = &\bar\psi(i\partial\!\!\!/ - m)\psi + g\mu^\varepsilon\bar\psi A\!\!\!/^a t^a_R\psi \\ &+ \delta_2\bar\psi(i\partial\!\!\!/ + m)\psi - \delta_m m\bar\psi\psi + \delta_1 g\mu^\varepsilon\bar\psi A\!\!\!/^a t^a_R\psi - \delta_2\delta_m m\bar\psi\psi.\end{aligned} \tag{15.1.9}$$

As with QED the $\delta_2\delta_m$ term only contributes at lowest order if we have two or more loops, so we neglect it.

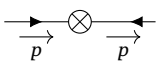
15.2 Feynman Rules for Counterterms

We have three counterterms, and so three new, albeit familiar, Feynman rules. The first corresponds to the δ_1 counterterm. This has the form of the QCD gluon-gluon-fermion vertex, but with an extra factor of δ_1, so we have



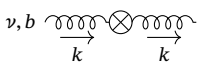
$$= i\delta_1 g\mu^\varepsilon\gamma^\mu t^a_R. \tag{15.2.1}$$

The δ_2 term is very similar to the QED result:



$$= i\delta_2(\not{p} - m) - i\delta_m m. \tag{15.2.2}$$

The δ_3 term is also very similar to the QED result, but as each gluon propagator carries a δ^{ab} this factor is carried through to the counterterm:

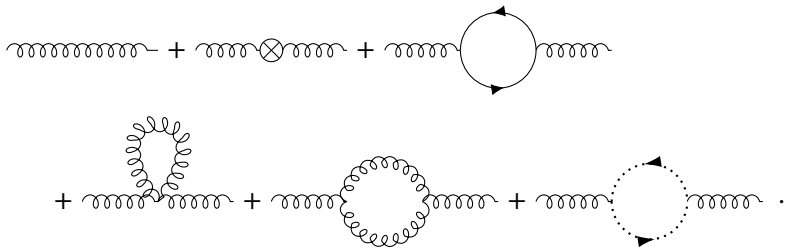


$$= i\delta_3\delta^{ab}(k^\mu k^\nu - \eta^{\mu\nu}k^2). \tag{15.2.3}$$

15.3 Computing The δ_3 Counterterm

To compute the beta function we will need to find the values of δ_1, δ_2, and δ_3. We'll start by computing δ_3. We do so in a theory of N_f fermions of mass m, all in some representation, R, of SU(N). If we want to add in scalars or fermions in some other representation this can be done by following the same process and then just adding all the contributions to δ_3 together.

Recall that the $\overline{\text{MS}}$ scheme is defined by taking the counterterm to contain only divergent terms, as well as rescaling μ. For this reason we need only keep divergent terms in our calculations. To compute δ_3 we consider one loop corrections to the gluon propagator, which is itself finite. Thus, we need to consider the following

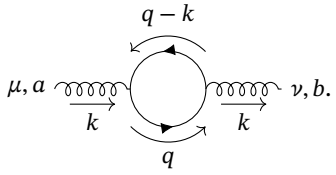


(15.3.1)

We demand that this sum is finite, and define δ_3 to make this true. We can neglect the first diagram, since it is finite. We'll compute the rest of the diagrams one at a time, then combine the results to find δ_3.

15.3.1 Fermion Loop

Consider the diagram



(15.3.2)

This is straightforward to turn into an integral:

- The closed fermion loop gives a factor of -1.
- The left most vertex gives a factor of

$$ig\mu^{\epsilon}\gamma^{\mu}t^{a}_{R}. \tag{15.3.3}$$

- Following the fermion propagator backwards we have a fermion of momentum $q-k$, contributing

$$\frac{i(\not{q}-\not{k}+m)}{(q-k)^2-m^2+i\epsilon}. \tag{15.3.4}$$

- Next we reach the other vertex, giving a factor of

$$ig\mu^{\epsilon}\gamma^{\nu}t^{b}_{R}. \tag{15.3.5}$$

- Then we have the second fermion propagator,

$$\frac{i(\not{q}+m)}{q^2-m^2+i\epsilon}. \tag{15.3.6}$$

- We have to integrate over the undetermined momentum,

$$\int \hat{\mathrm{d}}^D q. \tag{15.3.7}$$

The generators of the $\mathfrak{su}(N)$ algebra, t_R^a, and the gamma matrices, γ^μ, commute, since they're in different spaces, and act on different spaces. Just as there is a trace over gamma matrices we have a trace over the $\mathfrak{su}(N)$ generators. Thus, this diagram corresponds to

$$-(ig\mu^\varepsilon)^2 \operatorname{tr}(t_R^a t_R^b) \int \hat{\mathrm{d}}^D q \frac{\operatorname{tr}[\gamma^\mu i(\not{q}-\not{k}+m)\gamma^\nu i(\not{q}+m)]}{[(q-k)^2-m^2+i\epsilon][q^2-m^2+i\epsilon]}. \tag{15.3.8}$$

Fortunately, we've already done most of the work to compute this in QED in Section 5.1. The only new part in Yang–Mills is

$$\operatorname{tr}(t_R^a t_R^b) = T(R)\delta^{ab}. \tag{15.3.9}$$

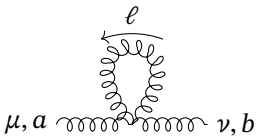
Using the result of Equation (5.1.46) we can evaluate this integral, and then drop the terms which are finite as $\varepsilon \to 0$, and we find that the divergent contribution is

$$\frac{ig^2}{16\pi^2\varepsilon}\frac{4}{3}T(R)\delta^{ab}(p^\mu p^\nu - \eta^{\mu\nu}p^2). \tag{15.3.10}$$

We choose to include the factor of 4/3, as we will later see that all of the terms come with a factor of $ig^2/(16\pi^2\varepsilon)$. Notice that this has the same tensor structure as the counterterm, a good sign.

15.3.2 Single Gluon Loop

Now consider the diagram



. (15.3.11)

This integral has the form

$$\int \hat{\mathrm{d}}^D \ell \frac{N^{ab\mu\nu}}{\ell^2}, \tag{15.3.12}$$

where the numerator is independent of the loop momenta since the four-gluon vertex is independent of momentum. Using the general loop formula,

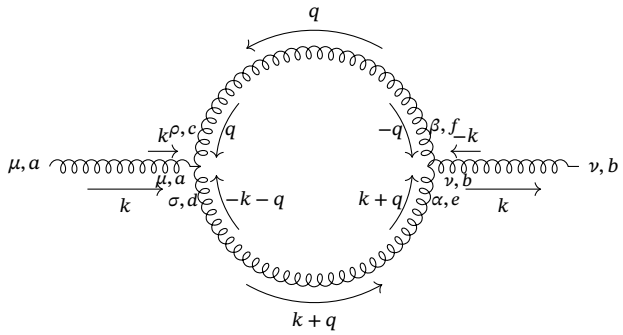
$$\int \hat{\mathrm{d}}^D \ell \, \frac{1}{\ell^2-\Delta} = -\frac{i}{(4\pi)^{D/2}}\Gamma(1-D/2)\Delta^{D/2-1}, \tag{15.3.13}$$

with $\Delta = 0$ taking ε to be small we have $D/2-1 > 0$, and so this vanishes. This is known as **Veltman's formula**.

This integral vanishing is a property of dimensional regularisation. If instead we worked with a momentum cut-off then we would find that the integral is quadratically divergent. This quadratic divergence would cancel in the final sum of diagrams, since there is no counterterm to absorb it and the result must be finite, and we're only interested in the divergent part, so we can neglect this diagram.

15.3.3 Double Gluon Loop

Now consider the diagram



(15.3.14)

Here we label each end of the internal gluons with different colour and Lorentz indices. Usually people don't do this, since the gluon propagators carry factors of δ^{ab} and $\eta^{\mu\nu}$ (in the Feynman gauge, $\xi = 1$) and so in practice we can treat the indices at each end as being the same, and just make sure that all indices are contracted over and upper/lower Lorentz indices match. For this diagram, and this diagram only, we'll do the full calculation without using this.

This diagram has a symmetry factor of 2, we can swap the upper and lower gluon. this means we need to include an extra factor of 1/2. For the vertex terms the Feynman rule assumes all momenta are incoming, so we've labelled each vertex with incoming momenta. The cyclic nature of the Feynman rule means it doesn't matter which gluon we start with, but we must always go in the same direction around the vertex. In this case we'll go clockwise. The first vertex gives us a factor of

$$\begin{aligned} &gf^{dac}[(-k-q-k)_\rho\eta_{\sigma\mu} + (k-q)_\sigma\eta_{\mu\rho} + (q-(-k-q))_\mu\eta_{\rho\sigma}] \\ &\quad = gf^{dac}[(-2k-q)_\rho\eta_{\sigma\mu} + (k-q)_\sigma\eta_{\mu\rho} + (k+2q)_\mu\eta_{\rho\sigma}]. \end{aligned} \tag{15.3.15}$$

The second vertex gives us a factor of

$$\begin{aligned} &gf^{fbe}[(-q-(-k))_\alpha\eta_{\beta\nu} + (-k-(k+q))_\beta\eta_{\nu\alpha} + (k+q-(-q))_\nu\eta_{\alpha\beta}] \\ &\quad = gf^{fbe}[(-q-k)_\alpha\eta_{\beta\nu} + (-2k+q)_\beta\eta_{\nu\alpha} + (k+2q)_\nu\eta_{\alpha\beta}]. \end{aligned} \tag{15.3.16}$$

We also have the two propagators,

$$\frac{-i}{p^2+i\epsilon}\delta^{cf}\eta^{\rho\beta}, \quad \text{and} \quad \frac{-i}{(k+q)^2+i\epsilon}\delta^{de}\eta^{\sigma\alpha}. \tag{15.3.17}$$

The diagram is then given by integrating the product of these terms over q. Considering just the colour indices we have

$$\begin{aligned} f^{dac}f^{fbe}\delta^{cf}\delta^{de} = f^{dac}f^{cbd} &= -(t^a_{\text{adj}})^{dc}(t^b_{\text{adj}})^{cd} \\ &= -(t^a_{\text{adj}}t^b_{\text{adj}})^{cc} = -\operatorname{tr}(t^a_{\text{adj}}t^b_{\text{adj}}) = -T(\mathrm{A})\delta^{ab}. \end{aligned} \tag{15.3.18}$$

Now considering the Lorentz structure, we have

$$[(-2k-q)_\rho\eta_{\sigma\mu}+(k-q)_\sigma\eta_{\mu\rho}+(k+2q)_\mu\eta_{\rho\sigma}] \tag{15.3.19}$$
$$\times[(-q-k)_\alpha\eta_{\beta\nu}+(-2k+q)_\beta\eta_{\nu\alpha}+(k+2q)_\nu\eta_{\alpha\beta}]\eta^{\rho\beta}\eta^{\sigma\alpha} \tag{15.3.20}$$
$$=[(-2k-q)_\rho\eta_{\sigma\mu}+(k-q)_\sigma\eta_{\mu\rho}+(k+2q)_\mu\eta_{\rho\sigma}] \tag{15.3.21}$$
$$\times[(-q-k)^\sigma\delta^\rho_\nu+(-2k+q)^\rho\delta^\sigma_\nu+(k+2q)_\nu\eta^{\rho\sigma}]. \tag{15.3.22}$$

We could have gone straight to this result if we chose to just label both ends of the propagator with the same indices and ensured that all indices were summed over properly. Expanding this out we get

$$\begin{aligned}&(-2k-q)_\nu(-q-k)_\mu+(-2k-q)\cdot(-2k+q)\eta_{\mu\nu}+(-2k-q)_\mu(k+2q)_\nu\\&+(k-q)\cdot(-q-k)\eta_{\mu\nu}+(k-q)_\nu(-2k+q)_\mu+(k-q)_\mu(k+2q)_\nu\\&+(k+2q)_\mu(-q-k)_\nu+(k+2q)_\mu(-2k+q)_\nu+(k+2q)_\mu(k+2q)_\nu 4.\end{aligned} \tag{15.3.23}$$

Here we've used $\eta^{\rho\sigma}\eta_{\rho\sigma}=D\rightsquigarrow 4$ dropping the ε term as we're only interested in divergent terms when $\varepsilon\to 0$. Further expanding this we get

$$-2k^\mu k^\nu+10q^\mu q^\nu+5(k^\mu q^\nu+k^\nu q^\mu)+\eta^{\mu\nu}[(q-k)^2+(q+2k)^2]=N^{\mu\nu}. \tag{15.3.24}$$

Putting this together we can write the term corresponding to this diagram as

$$\frac{1}{2}g^2T(\mathrm{A})\delta^{ab}\int\hat{\mathrm{d}}q\frac{N^{\mu\nu}}{q^2(k+q)^2}. \tag{15.3.25}$$

We can now use Feynman parametrisation to write this as

$$\frac{1}{2}g^2T(\mathrm{A})\delta^{ab}\int_0^1\mathrm{d}x\int\hat{\mathrm{d}}q\frac{N^{\mu\nu}}{[q^2+x(k+q)^2]^2}. \tag{15.3.26}$$

Expanding the denominator we have

$$q^2+xk^2+2xk\cdot q+x^2q^2=q^2+2xk\cdot q+x^2q^2 \tag{15.3.27}$$

since the external gluons are on-shell, so $k^2=0$.

We can then shift the loop momentum, defining $\ell=q+xk$, and the result is very similar to Equation (5.1.2), but with $m=0$, replacing q with k and $-k$ with q. We then have the denominator $(\ell^2-\Delta)^2$ with $\Delta=-x(1-x)k^2$.

The next step is to simplify the numerator using $q=\ell-xk$. We can then drop any terms which are linear in ℓ, since these vanish in the integral. We can also replace $\ell^\mu\ell^\nu$ with $\ell^2\eta^{\mu\nu}/D\rightsquigarrow\ell^2\eta^{\mu\nu}/4$. Thus,

$$\begin{aligned}N^{\mu\nu}=&-2k^\mu k^\nu+10(\ell^\mu-xk^\mu)(\ell^\nu-xk^\nu)+5(k^\mu(\ell^\nu-xk^\nu)\\&+k^\nu(\ell^\mu-xk^\mu))+\eta^{\mu\nu}[(\ell-(x-1)k)^2+(\ell-(2-x)k)^2]\end{aligned} \tag{15.3.28}$$
$$\begin{aligned}\rightsquigarrow&-2k^\mu k^\nu+\frac{5}{2}\ell^2\eta^{\mu\nu}+10x^2k^\mu k^\nu-10xk^\mu k^\nu\\&+\eta^{\mu\nu}[2\ell^2+(x-1)^2k^2+(2-x)^2k^2]\end{aligned} \tag{15.3.29}$$
$$=(10x^2-10x-2)k^\mu k^\nu+\eta^{\mu\nu}\left[\frac{9}{2}\ell^2+(5-2x+2x^2)k^2\right]. \tag{15.3.30}$$

This is, unfortunately, as simple as it gets.

Next we have to perform the integrals. To do this we use

$$\int \hat{\mathrm{d}}^D\ell \frac{1}{(\ell^2-\Delta)^2} = \frac{i}{(4\pi)^{D/2}}\Gamma(2-D/2)\Delta^{D/2-2} = \frac{i}{16\pi^2\varepsilon} + \text{finite},$$
$$\int \hat{\mathrm{d}}^D\ell \frac{\ell^2}{(\ell^2-\Delta)^2} = \frac{-i}{(4\pi)^{D/2}}\frac{D}{2}\Gamma(1-\Delta/2)\Delta^{D/2-2} = \frac{i}{16\pi^2\varepsilon}2\Delta + \text{finite}.$$

We get these by setting $D = 4-2\varepsilon$, expanding the gamma functions, then keeping only the divergent terms as $\varepsilon \to 0$.

This gives the result

$$\frac{ig^2}{16\pi^2\varepsilon}\frac{1}{2}T(\mathrm{A})\delta^{ab}\int_0^1 \mathrm{d}x\,[\eta^{\mu\nu}[(5-2x+2x+2x^2)-9x(1-x)]k^2 + (10x^2-10x-2)k^\mu k^\nu]. \tag{15.3.31}$$

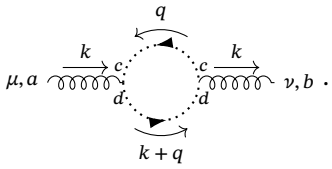
Performing the x integral we get the final result

$$\frac{ig^2}{16\pi^2\varepsilon}T(\mathrm{A})\delta^{ab}\left[\frac{19}{12}k^2\eta^{\mu\nu} - \frac{11}{6}k^\mu k^\nu\right]. \tag{15.3.32}$$

This doesn't have the correct tensor structure to give the counterterm. The solution to this problem is that we need to include the ghosts, only when combined do we get the required tensor structure.

15.3.4 Ghost Loop

Finally, we consider the diagram



(15.3.33)

We now label both ends of the propagators with the same colour indices, since both the ghost and gluon propagators act as the identity on colour space. We can easily write down what this diagram represents:

- The closed ghost loop gives a factor of -1, just like a closed fermion loop, since the ghosts are anticommuting.
- each gluon-ghost vertex gives a factor of $-gf^{abc}p^\mu$ where p is the momentum of the outgoing ghost, which also carries colour index a, and then we go around the vertex in a consistent direction, which we've chosen to be clockwise. This means we get factors of

$$-gf^{dac}(q+k)^\mu, \qquad \text{and} \qquad -gf^{cdb}q^\nu. \tag{15.3.34}$$

As before

$$f^{dac}f^{cbd} = -T(\mathrm{A})\delta^{ab}. \tag{15.3.35}$$

- The two ghost propagators give factors of

$$\frac{i}{q^2+i\epsilon}, \qquad \text{and} \qquad \frac{i}{(q+k)^2+i\epsilon}, \tag{15.3.36}$$

  having already used the δ^{ab} terms to set the indices at either end of the propagator to the same value.

- We then integrate over the undetermined loop momentum.

All together, this gives

$$-g^2T(\mathrm{A})\delta^{ab}\int \hat{\mathrm{d}}^D q\frac{(q^\mu+k^\mu)q^\nu}{q^2(q+k)^2}. \tag{15.3.37}$$

Note that the overall sign comes from three minuses, one from the ghost loop, one from the i^2 from the propagators, and one from $f^{cad}f^{dbc} = -T(\mathrm{A})\delta^{ab}$.

As usual we proceed using Feynman parametrisation:

$$g^2T(\mathrm{A})\delta^{ab}\int_0^1 \mathrm{d}x\int \hat{\mathrm{d}}^D q\frac{(k^\mu+q^\mu)q^\nu}{(q^2+x(k+q)^2)^2}. \tag{15.3.38}$$

The denominator is the same as the double gluon loop, so we can set $\ell = q + xk$ as we did there and we get

$$g^2T(\mathrm{A})\delta^{ab}\int_0^1 \mathrm{d}x\hat{\mathrm{d}}q\frac{(q^\mu+k^\mu)q^\nu}{(\ell^2-\Delta)^2} \tag{15.3.39}$$

with $\Delta = -x(1-x)k^2$. Then the numerator simplifies as

$$(q^\mu+k^\mu)q^\nu = (\ell^\mu - xk^\mu + k^\mu)(\ell^\nu - xk^\nu) \rightsquigarrow \frac{1}{4}\ell^2\eta^{\mu\nu} + x(x-1)k^\mu k^\nu, \tag{15.3.40}$$

having dropped terms with a single ℓ and used $\ell^\mu\ell^\nu \rightsquigarrow \ell^2\eta^{\mu\nu}/D \rightsquigarrow \ell^2\eta^{\mu\nu}/4$.

So we are looking to compute

$$g^2T(\mathrm{A})\delta^{ab}\int_0^1 \mathrm{d}x\hat{\mathrm{d}}\ell\frac{\ell^2\eta^{\mu\nu}/4 + x(x-1)k^\mu k^\nu}{(\ell^2-\Delta)^2}. \tag{15.3.41}$$

Again, we can make use of the standard integral formula and keep only the divergent terms, to get

$$\frac{i}{16\pi^2\varepsilon}g^2T(\mathrm{A})\delta^{ab}\int_0^1 \mathrm{d}x[2x(x-1)k^2+x(x-1)k^\mu k^\nu] = \frac{i}{16\pi^2\varepsilon}g^2T(\mathrm{A})\delta^{ab}\left[+\frac{1}{12}\eta^{\mu\nu}+\frac{1}{6}k^\mu k^\nu\right]. \tag{15.3.42}$$

15.3.5 Combining Results

We can start by considering a Yang–Mills theory with no fermions, just gluons and the ghosts. Then we can neglect the term with the fermion loop and combine the results of the last three sections where we find that summing all of the one-loop contributions gives

$$-\frac{ig^2}{16\pi^2\varepsilon}T(\mathrm{A})\delta^{ab}\frac{5}{3}[k^\mu k^\nu - \eta^{\mu\nu}k^2], \tag{15.3.43}$$

where the tensors structures of the double-gluon and ghost loops combine to give the required counterterm tensor structure.

Now consider a Yang–Mills theory with N_f fermions all transforming in some representation R of SU(N). Each fermion contributes one loop diagram, and so we just multiply the fermion loop diagram by N_f. If we had fermions in some other representation, R', then we would have to sum these and we would have $T(R)$ and $T(R')$ terms. As it is, summing the one-loop contributions gives

$$\frac{ig^2}{16\pi^2\varepsilon}\delta^{ab}[k^\mu k^\nu - \eta^{\mu\nu}k^2]\left[\frac{4}{3}N_\mathrm{f}T(R) - \frac{5}{3}T(\mathrm{A})\right]. \tag{15.3.44}$$

Demanding that the gluon self energy, given by

$$i\delta^{ab}(k^\mu k^\nu - \eta^{\mu\nu}k^2)\left[\delta_3 + \frac{g^2}{16\pi^2\varepsilon}\left(\frac{4}{3}N_\mathrm{f}T(R) - \frac{5}{3}T(\mathrm{A})\right)\right], \tag{15.3.45}$$

is finite we find that, in the $\overline{\mathrm{MS}}$ scheme, the counterterm is given by

$$\delta_3 = \frac{g^2}{16\pi^2\varepsilon}\left[\frac{5}{3}T(\mathrm{A}) - \frac{4}{3}N_\mathrm{f}T(R)\right]. \tag{15.3.46}$$

Notice that in QED we have $T(R) = 1$, $N_\mathrm{f} = 1$, and no contribution from a photon loop, since photons are neutral, meaning we don't get the $T(\mathrm{A})$ term, and then this result reduces to the QED result of

$$\delta_3^{\mathrm{QED}} = -\frac{e^2}{12\pi^2\varepsilon}. \tag{15.3.47}$$

Sixteen

Yang–Mills Beta Function

The **beta function** is defined as

$$\beta(g) = \mu \frac{dg}{d\mu}, \tag{16.0.1}$$

after taking $\varepsilon \to 0$, with ε here being the ε of dimensional regularisation. There are various ways to calculate this in Yang–Mills theory, and we'll discuss two of them and then proceed with one.

16.1 Expressing the Beta Function in Terms of Counterterms

In renormalised Yang–Mills theory the bare coupling is given by

$$g_B = g\mu^\varepsilon Z_g. \tag{16.1.1}$$

Taking the derivative we get

$$\frac{dg_B}{d\mu} = \frac{dg}{d\mu}\mu^\varepsilon Z_g + g\varepsilon\mu^{\varepsilon-1}Z_g + g\mu^\varepsilon \frac{dZ_g}{d\mu}. \tag{16.1.2}$$

The bare coupling, g_B, must be independent of the scale μ, and so the left hand side vanishes. We also have $Z_g = 1 + \delta_g$, so $dZ_g/d\mu = d\delta_g/d\mu$. We're looking for the beta function,, $\mu dg/d\mu$, so we multiply through by $\mu/(\mu^\varepsilon Z_g)$ to get this term on its own:

$$0 = \mu \frac{dg}{d\mu} + g\varepsilon + g\frac{\mu}{Z_g}\frac{d\delta_g}{d\mu}. \tag{16.1.3}$$

Solving for the beta function we find

$$\beta(g) = -g\varepsilon - g\frac{\mu}{Z_g}\frac{d\delta_g}{d\mu}. \tag{16.1.4}$$

The quantity δ_g is of order g^2, so we can write

$$g\frac{1}{Z_g} = g\frac{1}{1+\delta_g} \approx g(1 - \delta_g + \mathcal{O}(g^4)) \approx g. \tag{16.1.5}$$

Thus,

$$\mu \frac{dg}{d\mu} = -g\varepsilon - g\mu\frac{d\delta_g}{d\mu}, \tag{16.1.6}$$

which is $\mathcal{O}(g^2)$ since $d\delta_g/d\mu$ will be $\mathcal{O}(g)$.

We can now evaluate this in a variety of ways. First, we have

$$Z_g = \frac{Z_1}{Z_2\sqrt{Z_3}} \tag{16.1.7}$$

$$= \frac{1+\delta_1}{(1+\delta_2)(1+\delta_3)^{1/2}} \tag{16.1.8}$$

$$\approx (1+\delta_1)(1-\delta_2)\left(1-\frac{1}{2}\delta^3\right)+\mathcal{O}(g^4) \tag{16.1.9}$$

$$\approx 1+\delta_1-\delta_2-\frac{1}{2}\delta_3+\mathcal{O}(g^4), \tag{16.1.10}$$

so $\delta_g = \delta_1 - \delta_2 - \delta_3/2$. Hence,

$$\mu\frac{dg}{d\mu} = -g\varepsilon - g\mu\frac{d}{d\mu}\left(\delta_1-\delta_2-\frac{1}{2}\delta_3\right). \tag{16.1.11}$$

Note that in QED $Z_1 = Z_2$, so $\delta_1 = \delta_2$, and this depends only on δ_3.

We also have

$$Z_g = \frac{Z_{1c}}{Z_{3c}\sqrt{Z_3}} \approx 1+\delta_{1c}-\delta_{3c}-\frac{1}{2}\delta_3+\mathcal{O}(g^4). \tag{16.1.12}$$

So,

$$\mu\frac{dg}{d\mu} = -g\varepsilon - g\mu\frac{d}{d\mu}\left(\delta_{1c}-\delta_{3c}-\frac{1}{2}\delta_3\right). \tag{16.1.13}$$

We'll use the first result here to compute the beta function.

16.2 Computing the Beta Function

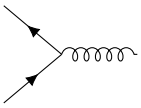
16.2.1 Computing δ_1

The δ_1 counter term is associated with the term

$$\delta_1 g\mu^\varepsilon \bar{\psi}\not{A}^a t_R^a \psi. \tag{16.2.1}$$

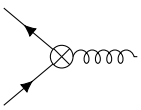
This corresponds to a two-fermion-one-gluon vertex, and so we look at this term to one loop.

The vertex is given by



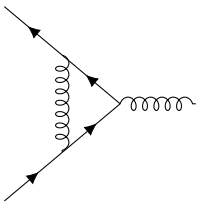
$$= ig\mu^\varepsilon \gamma^\mu t_R^a. \tag{16.2.2}$$

The counterterm vertex is similar, but there's an extra factor of δ_1 in the Lagrangian term which propagates through to give



$$= ig\mu^\varepsilon \gamma^\mu t_R^a \delta_1. \tag{16.2.3}$$

There are two one-loop contributions to this term, the first one we'll consider is



$$= igt^a_R \gamma_\mu \frac{g^2}{16\pi^2\varepsilon}\left[C(R) - \frac{1}{2}T(A)\right]. \tag{16.2.4}$$

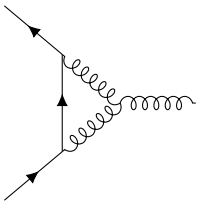
This can be computed by considering the equivalent diagram in QED and then working out the extra colour factors. We did the QED calculation in Section 11.3. For the colour factors suppose that the external gluon carries the colour label a and the internal gluon carries the colour label b. Then when writing out the integral the colour factor is found by following backwards along the fermion line, starting at the top in this diagram we first come a cross a vertex with the internal gluon, giving a factor of t^b_R, then the external gluon, t^a_R, and then the internal gluon again, t^b_R, giving an overall colour factor of $t^b_R t^a_R t^b_R$.

We can use some algebra to get this colour factor in the form above:

$$\begin{aligned}
t^b_R t^a_R t^b_R &= t^b_R t^b_R t^a_R + t^b_R [t^a_R, t^b_R] && (16.2.5)\\
&= C(R) t^a_R + i f^{abc} t^b_R t^c_R && (16.2.6)\\
&= C(R) t^a_R + \frac{1}{2} i f^{abc} [t^b_R, t^c_R] \qquad \text{antisymmetry of } f^{abc} && (16.2.7)\\
&= C(R) t^a_R + \frac{1}{2} i f^{abc} i f^{bcd} t^d_R && (16.2.8)\\
&= C(R) t^a_R - \frac{1}{2} i f^{cab} i f^{bdc} t^d_R && (16.2.9)\\
&= C(R) t^a_R - \frac{1}{2} (t^a_{\text{adj}})^{cb} (t^d_{\text{adj}})^{bc} t^d_R && (16.2.10)\\
&= C(R) t^a_R - \frac{1}{2} (t^a_{\text{adj}} t^d_{\text{adj}})^{cc} t^d_R && (16.2.11)\\
&= C(R) t^a_R - \frac{1}{2} \operatorname{tr}(t^a_{\text{adj}} t^d_{\text{adj}}) t^d_R && (16.2.12)\\
&= C(R) t^a_R - \frac{1}{2} T(\mathrm{A}) \delta^{ad} t^d_R && (16.2.13)\\
&= C(R) t^a_R - \frac{1}{2} T(\mathrm{A}) t^a_R. && (16.2.14)
\end{aligned}$$

This gives the result above.

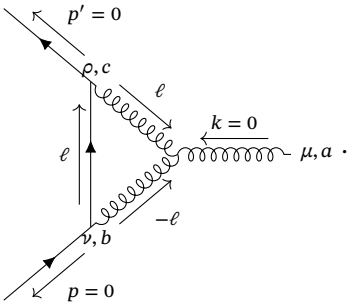
The second diagram contributing at one loop is



. (16.2.15)

To aid with computing this diagram notice that the relevant counterterm doesn't depend on the external momenta, so we are free to choose a value for these momenta, and in this case taking all of the external momenta to be zero simplifies

things. So consider



(16.2.16)

We can now compute the amplitude. This time we'll implicitly use the Kronecker deltas which come with the gluon propagators to set the colour indices on both ends of the propagator to be the same.

- We have two fermion-gluon vertices, giving an overall factor of $(ig)^2 t^c_R t^b_R$, going backwards along the fermion line to get the order of the generators correct. There are also factors of γ^μ and γ^ν from these two vertices.

- The three gluon vertex gives the term

$$\begin{aligned} g f^{abc}[(0+\ell_\rho)\eta_{\mu\nu} + (-\ell_\mu - \ell_\mu)\eta_{\nu\rho} + (\ell_\nu - 0)\eta_{\rho\mu}] \\ = g f^{abc}[\ell_\rho \eta_{\mu\nu} - 2\ell_\mu \eta_{\nu\rho} + \ell_\nu \eta_{\rho\mu}] \end{aligned} \quad (16.2.17)$$

- The fermion propagator comes with a factor of $i(\not{\ell} + m)/(\ell^2 + m^2)$.

- The gluon propagators come with a factor of $1/\ell^2$.

- We have to integrate over ℓ.

Putting this together we get

$$(ig)^2 g f^{abc} t^c_R t^b_R \int \hat{\mathrm{d}}^D \ell \frac{\gamma^\rho i(\not{\ell} + m)\gamma^\nu}{(\ell^2)^2(\ell^2 + m^2)}[\ell_\rho \eta_{\mu\nu} - 2\ell_\mu \eta_{\nu\rho} + \ell_\nu \eta_{\rho\mu}]. \quad (16.2.18)$$

Using the antisymmetry of the structure constants we have

$$\begin{aligned} f^{abc} t^c_R t^b_R &= \frac{1}{2} f^{abc}[t^c_R, t^b_R] = \frac{1}{2} f^{abc} f^{cbd} t^d_R = -\frac{1}{2} f^{cab} f^{bdc} t^d_R \\ &= \frac{1}{2}(t^a_{\mathrm{adj}})^{cb}(t^d_{\mathrm{adj}})^{bc} t^d_R = \frac{1}{2}\,\mathrm{tr}(t^a_{\mathrm{adj}} t^d_{\mathrm{adj}}) t^d_R = \frac{1}{2} T(\mathrm{A}) \delta^{ad} t^d_R = \frac{1}{2} T(\mathrm{A}) t^a_R. \end{aligned} \quad (16.2.19)$$

We can then use Feynman parameterisation,

$$\frac{1}{AB^2} = \int_0^1 \mathrm{d}x \frac{2(1-x)}{[xA + (1-x)B]^3}. \quad (16.2.20)$$

Simplifying the numerator we have

$$N_\mu = i\gamma^\rho(\not{\ell} + m)\gamma^\nu[\ell_\rho\eta_{\mu\nu} - 2\ell_\mu\eta_{\nu\rho} + \ell_\nu\eta_{\rho\mu}] \tag{16.2.21}$$
$$\rightsquigarrow i\gamma^\rho\not{\ell}\gamma^\nu[\ell_\rho\eta_{\mu\nu} - 2\ell_\mu\eta_{\nu\rho} + \ell_\nu\eta_{\rho\mu}] \tag{16.2.22}$$
$$= i[\not{\ell}\not{\ell}\gamma_\mu - 2\ell_\mu\gamma_\nu\not{\ell}\gamma^\nu + \gamma_\mu\not{\ell}\not{\ell}] \tag{16.2.23}$$
$$= i[\ell^2\gamma_\mu + 4\ell_\mu\not{\ell} + i\gamma_\mu\ell^2] \tag{16.2.24}$$
$$= i[2\ell^2\gamma_\mu + 4\ell_\mu\not{\ell}] \tag{16.2.25}$$
$$= 3i\ell^2\gamma_\mu \tag{16.2.26}$$

These steps follow from

- Keeping only even powers of ℓ, odd powers vanish in the integral by symmetry.
- $\gamma_\nu\not{\ell}\gamma^\nu = -2\not{\ell} + \mathcal{O}(\varepsilon)$.
- $\not{\ell}\not{\ell} = \ell^2$.
- $4\ell_\mu\not{\ell} = 4\ell_\mu\ell_\nu\gamma^\nu \rightsquigarrow \ell^2\eta_{\mu\nu}\gamma^\nu = \ell^2\gamma_\mu$, using the rule that we can replace $\ell^\mu\ell^\nu$ with $\ell^2\eta^{\mu\nu}/D$ in the integral, and dropping the part that vanishes when $\varepsilon \to 0$.

The divergent part of the diagram is then

$$\frac{3}{2}g^3T(\mathrm{A})t_R^a\gamma^\mu\int_0^1 \mathrm{d}x\,(1-x)\int \hat{\mathrm{d}}^D\ell\,\frac{\ell^2}{(\ell^2-\Delta)^3} \tag{16.2.27}$$

where $\Delta = xm^2$. Using

$$\int \hat{\mathrm{d}}^D\ell\,\frac{\ell^2}{(\ell^2-\Delta)^3} = \int \hat{\mathrm{d}}^D\ell\left[\frac{\ell^2-\Delta}{(\ell^2-\Delta)} + \frac{\Delta}{(\ell^2-\Delta)^3}\right] \tag{16.2.28}$$

and identifying that the second term here is finite we can use the result that the divergent part of the first term is

$$\int \hat{\mathrm{d}}^D\ell\,\frac{1}{(\ell^2-\Delta)^2} = \frac{i}{16\pi^2\varepsilon} \tag{16.2.29}$$

we find that the divergent part of the diagram is

$$\frac{ig^3}{16\pi^2\varepsilon}\frac{3}{2}T(\mathrm{A})t_R^a\gamma^\mu. \tag{16.2.30}$$

We can now compute δ_1 by considering the sum of all diagrams considered in this section, which gives

$$ig\gamma^\mu t_R^a\left[\delta_1 + \frac{g^2}{16\pi^2\varepsilon}\left(C(R) - \frac{1}{2}T(\mathrm{A}) + \frac{3}{2}T(\mathrm{A})\right)\right]. \tag{16.2.31}$$

As usual we work in the $\overline{\mathrm{MS}}$ scheme and require that this quantity is finite, so we must have

$$\delta_1 = -\frac{g^2}{16\pi^2\varepsilon}[C(R) + T(\mathrm{A})]. \tag{16.2.32}$$

The first term, $C(R)$, is QED-like, the Casimir there simply being 1, as this is the sole generator of $\mathfrak{u}(1)$. The second term, $T(\mathrm{A})$, is new.

16.2.2 Computing δ_2

The δ_2 counter term is the same as the QED δ_2 counter term, but with an extra factor of $C(R)$:

$$\delta_2 = -\frac{g^2}{16\pi^2\varepsilon}C(R). \tag{16.2.33}$$

16.2.3 Computing the Beta Function

We are now in a position to compute the beta function, which we'll do using Equation (16.1.11). To do so we need to compute

$$\delta_1 - \delta_2 - \frac{1}{2}\delta_3 = -\frac{g^2}{16\pi^2\varepsilon}\left[C(R) + T(\mathrm{A}) - C(R) - \frac{2}{3}N_\mathrm{f}T(R) + \frac{5}{6}T(\mathrm{A})\right] \tag{16.2.34}$$

$$= \frac{g^2}{16\pi^2\varepsilon}\left[\frac{11}{6}T(\mathrm{A}) - \frac{2}{3}N_\mathrm{f}T(R)\right]. \tag{16.2.35}$$

The cancelling of the $C(R)$ terms is a vestige of the relation $\delta_1 = \delta_2$ in QED.

The only μ dependence here enters through g, so the derivative is easy to compute, we have

$$\mu\frac{\mathrm{d}g}{\mathrm{d}\mu} = -g\varepsilon - g\mu\frac{1}{16\pi^2\varepsilon}\left[\frac{11}{6}T(\mathrm{A}) - \frac{2}{3}N_\mathrm{f}T(R)\right]\frac{\mathrm{d}g^2}{\mathrm{d}\mu} \tag{16.2.36}$$

$$= -g\varepsilon - g\mu\frac{1}{16\pi^2\varepsilon}\left[\frac{11}{6}T(\mathrm{A}) - \frac{2}{3}N_\mathrm{f}T(R)\right]2g\frac{\mathrm{d}g}{\mathrm{d}\mu} \tag{16.2.37}$$

Rearranging this gives

$$\left[1 - \frac{g^2}{8\pi^2\varepsilon}\left(\frac{11}{6}T(\mathrm{A}) - \frac{2}{3}N_\mathrm{f}T(R)\right)\right]\mu\frac{\mathrm{d}g}{\mathrm{d}\mu} = -\varepsilon g. \tag{16.2.38}$$

To compute the beta function we need the derivative on its own, so we can write this as

$$\mu\frac{\mathrm{d}g}{\mathrm{d}\mu} = -\varepsilon g\left[1 - \frac{g^2}{8\pi^2\varepsilon}\left(\frac{11}{6}T(\mathrm{A}) - \frac{2}{3}N_\mathrm{f}T(R)\right)\right]^{-1} \tag{16.2.39}$$

$$\approx -\varepsilon g\left[1 + \frac{g^2}{8\pi^2\varepsilon}\left(\frac{11}{6}T(\mathrm{A}) - \frac{2}{3}N_\mathrm{f}T(R)\right)\right] \tag{16.2.40}$$

$$= -\varepsilon g + \frac{g^3}{8\pi^2}\left[\frac{11}{6}T(\mathrm{A}) - \frac{2}{3}N_\mathrm{f}T(R)\right]. \tag{16.2.41}$$

Taking the limit $\varepsilon \to 0$ gives us the beta function:

$$\beta(g) = \mu\frac{\mathrm{d}g}{\mathrm{d}\mu}\bigg|_{\varepsilon\to 0} = -\frac{g^3}{8\pi^2}\left[\frac{11}{6}T(\mathrm{A}) - \frac{2}{3}N_\mathrm{f}T(R)\right]. \tag{16.2.42}$$

It is common to write this as

$$\beta(g) = -\frac{g^3 b_0}{8\pi^2} \tag{16.2.43}$$

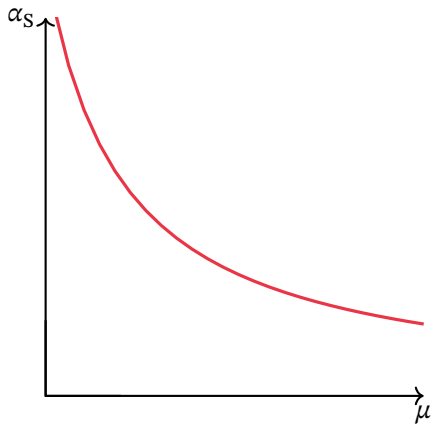


Figure 16.1: Plot of α_S as a function of the energy μ.

defining

$$b_0 = \frac{11}{6}T(\mathrm{A}) - \frac{2}{3}N_f T(R). \tag{16.2.44}$$

The index 0 here denoting that this is just the first coefficient in a perturbative series for the true value of the beta function.

In pure QCD, as we shall see in the next chapter, $N_f = 6$, and all of these fermions are in the fundamental representation, so $T(R) = T(N) = 1/2$, and $T(\mathrm{A}) = 3$ in SU(N). This gives

$$\beta(g) = -\frac{7g^3}{16\pi^2} < 0, \tag{16.2.45}$$

that is, $b_0 = 7$. The beta function being negative is new in Yang–Mills, in QED the beta function was positive. It is possible to solve the differential equation $\mu \mathrm{d}g/\mathrm{d}\mu = \beta(g)$ for $\alpha_S = g^2/(4\pi)$ and we find

$$\alpha_S(Q^2) = \frac{\alpha_S(M^2)}{1 + \frac{b_0\alpha_S(M^2)}{4\pi}\log\frac{Q^2}{M^2}} \tag{16.2.46}$$

where M^2 is a energy scale at which α_S is known, and $\alpha_S(Q^2)$ is the value of α_S at some other momentum scale Q. Plotting this we get Figure 16.1.

Notice that at low energies the coupling is large, so perturbation theory doesn't work at low energies, this is the opposite of what we see in QED. This means that at low energies we don't see individual gluons and quarks, but instead see a complicated mix of bound states, such as pions and kaons. This is the scale of nuclear physics, where the pions are often thought of as the particles mediating the strong force.

At large energies the coupling tends to zero, and so we get only very weak interactions, we call this **asymptotic freedom**.

Seventeen

QCD

We've now spent quite a long time looking at general Yang–Mills theory. In order to perform calculations for processes, say in the LHC, we need to be more specific about which Yang–Mills theory we are considering, and of course the correct theory to consider is QCD. To understand QCD as a Yang–Mills theory we need to know the gauge group, the particles that can occur, and what representations they transform under. This will be the focus of this section.

17.1 QCD as a Yang–Mills Theory

Quantum chromodynamics, or **QCD**, is a Yang–Mills theory with an SU(3) gauge group. There are six flavours of fermion in QCD, which we call the **quarks**. These are all in the fundamental representation, **3**.

This is a three dimensional representation, so quarks must have some three-dimensional property that this representation encapsulates, this is what we call **colour**. Roughly, we allow quarks to have three colours, red (Pr), green (g), and blue (b). A state will in general be a linear combination of these colours.

Gluons transform in the adjoint representation, which for SU(3) has dimension $3^2 - 1 = 8$, so they're in the **8** representation. Note that since

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \square \otimes \begin{array}{|c|}\hline\\\hline\end{array} = \begin{array}{|c|}\hline\\\hline\\\hline\end{array} \oplus \begin{array}{|c|c|}\hline\\\hline\end{array} = \mathbf{1} \oplus \mathbf{8} \tag{17.1.1}$$

we can think of gluons as carrying both colour (**3** indices) and anticolour ($\overline{\mathbf{3}}$ indices). See *Symmetries of Particles and Fields* for group theory details.

The QCD Lagrangian is then

$$\mathcal{L} = -\frac{1}{4}\sum_{a=1}^{8} F^{a\mu\nu}F^{a}_{\mu\nu} + \sum_{f=1}^{6} \bar{\mathrm{q}}_f(i\not{D} - m_f)\mathrm{q} \tag{17.1.2}$$

where

$$D_\mu = \partial_\mu - igA^a_\mu t^a_{\text{adj}}. \tag{17.1.3}$$

As discussed at the end of the last section the beta function in QCD is

$$\beta(g) = -\frac{7g^3}{16\pi^2}. \tag{17.1.4}$$

This means that we have non-perturbative behaviour at low energies, the coupling is large, this is where the "strong" in strong force comes from. This leads to **colour confinement**, states are colour singlets, so we don't see colour charge outside of a state, and hence don't see the strong force outside of a state. There is no mathematical proof of colour confinement, but there is a lot of experimental evidence.

17.2 Quark Model

QCD states are colourless combinations of quarks and gluons. Quarks come in six flavours, up, down, strange, charm, top, and bottom, which can be arranged as

$$\begin{array}{ccc} \mathrm{u} & \mathrm{c} & \mathrm{t} \\ \mathrm{d} & \mathrm{s} & \mathrm{b}. \end{array} \tag{17.2.1}$$

Quarks in the top row, or up-type quarks, are positive, with electric charge 2/3, whereas quarks in the bottom row are negative, with electric charge $-1/3$. These are arranged in increasing mass left to right, with quarks in the same column having approximately the same mass.

We can define Λ_{QCD} as the energy scale below which perturbation theory breaks down for QCD, so $\alpha_{\mathrm{S}}(\Lambda_{\mathrm{QCD}}) \approx 1$. The exact value of Λ_{QCD} depends on exactly how we define it, but it is on the order of a few hundred mega electronvolts. This partitions the quarks into three mass regions, less than Λ_{QCD}, on the order of Λ_{QCD}, and greater than Λ_{QCD}. This splits the quarks as follows:

$$\begin{aligned} m_{\mathrm{u}} = 2.16\,\mathrm{MeV},\ m_{\mathrm{d}} = 4.67\,\mathrm{MeV}, && (< \Lambda_{\mathrm{QCD}}) \\ m_{\mathrm{s}} = 93.4\,\mathrm{MeV}, && (\sim \Lambda_{\mathrm{QCD}}) \\ m_{\mathrm{c}} = 1.27\,\mathrm{GeV},\ m_{\mathrm{b}} = 4.18\,\mathrm{GeV},\ m_{\mathrm{t}} = 172.69\,\mathrm{GeV}. && (> \Lambda_{\mathrm{QCD}}) \end{aligned}$$

We call QCD states **hadrons**, examples are

$$\begin{array}{lll} \text{protons and neutrons} & \mathrm{p}\ \&\ \mathrm{n} & \sim 1\,\mathrm{GeV}, \\ \text{pions} & \pi^+,\ \pi^-,\ \&\ \pi^0 & \sim 140\,\mathrm{MeV}, \\ \text{kaons} & \mathrm{K}^+,\ \pi^-,\ \&\ \mathrm{K}^0 & \sim 500\,\mathrm{MeV}. \end{array} \tag{17.2.2}$$

The constituents of hadrons, which are mostly quarks and gluons, but also include other particles like electrons and photons, which we can think of as appearing from loops, are called **partons**. The most naive understanding of QCD states is the **quark model**. In this model hadrons are colourless bound states of quarks, there are no gluons or other particles, which arise in more accurate models through pair production and as exchange particles.

There are two ways[1] to get a colourless state. First, we can consider **mesons**

$$\bar{\mathrm{q}}_1^i \mathrm{q}_{2i} \tag{17.2.3}$$

where q_1 and q_2 are quarks of potentially different flavours, the bar denoting an antiquark. The index i runs from 1 to 3 over all colours, which we may also think of as taking on the values Pr, g, and b. Summing over all three colours like this makes the final state colourless, which is just a fancy way of saying it doesn't change under SU(3), compare this to a Lorentz invariant, like $x_\mu y^\mu$ where we sum over all Lorentz indices and so the quantity is invariant under SO(1, 3). There are two

[1] there are also more exotic states, like tetraquarks and pentaquarks, but we won't discuss these

fermions here in a product, so the overall state is symmetric, and is a boson. Consider the product of the following $\mathfrak{su}(2)$ representations:

$$\mathbf{2} \otimes \mathbf{2} = \square \otimes \square = \square\square \oplus \begin{array}{|c|}\hline\\\hline\\\hline\end{array} = \mathbf{3} \oplus \mathbf{1}, \tag{17.2.4}$$

so mesons can be spin 1 ($2 \cdot 1 + 1 = 3$), or spin 0 ($2 \cdot 0 + 1 = 1$).

Second, we can consider **baryons**

$$\varepsilon^{ijk} q_{1i} q_{2j} q_{3k} \tag{17.2.5}$$

where q_1, q_2, and q_3 are quarks of potentially different flavours. We can also consider antibaryons

$$\varepsilon^{ijk} \bar{q}_{1i} \bar{q}_{2j} \bar{q}_{3k}. \tag{17.2.6}$$

Again, all the colour indices are summed so this state is colourless. This works because ε^{ijk} is an SU(3) invariant tensor. There are three fermions, so this state is antisymmetric under exchange of particles, so it is a fermion. Consider the product of the following $\mathfrak{su}(2)$ representations:

$$\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \left(\begin{array}{|c|}\hline\\\hline\\\hline\end{array} \oplus \square\square \right) \otimes \square \tag{17.2.7}$$

$$= \begin{array}{|c|c|}\hline\\\hline\\\hline\end{array} \oplus \begin{array}{|c|}\hline\\\hline\\\hline\end{array} \oplus \square\square\square \oplus \begin{array}{|c|c|}\hline\\\hline\end{array} \tag{17.2.8}$$

$$= \mathbf{2} \oplus \mathbf{4} \oplus \bar{\mathbf{2}} \tag{17.2.9}$$

so baryons can be spin 1/2 ($2 \cdot 1/2 + 1 = 2$) or spin 3/2 ($2 \cdot 3/2 + 1 = 4$). Note that diagrams in $\mathfrak{su}(2)$ can have at most 2 rows.

Mesons tend to be lighter than baryons, for the simple reason that 2 is less than 3. That said, the masses do depend on exactly which quarks the particle is made from.

The prototypical baryons are **protons** and **neutrons**, collectively **nucleons**. The proton, p, has electric charge 1 and quark content uud. The neutron, n, is electrically neutral and has quark content udd. Both of these have spin 1/2.

The prototypical mesons are pions, which come in three types, π^+, π^-, and π^0, with electrical charge $+1$, 0, and -1 respectively, and quark contents $u\bar{d}$, $d\bar{u}$, and $u\bar{u} + d\bar{d}$, this third quark content being a superposition of the two ways to make a neutral meson from up and down quarks, ignoring any normalisation factor. Pions are all scalars with spin 0.

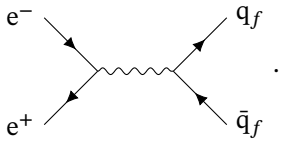
A more unusual example of a baryon is the Δ^{++}, with electric charge $+2$ and quark content uuu. This has mass 1.2 GeV and spin 3/2. Notice that we have three quarks, all with the same flavour, and with aligned spins. This seems to violate Pauli's exclusion principle. The solution is that all three quarks must have different colours, which is the case due to the ε^{ijk} factor in the baryon. This conundrum originally motivated the need for a new quantum number which lead to the introduction of colour.

One feature of the quark model, and of QCD more broadly, is **confinement**. We can't get quarks on their own. If we try we need to put in lots of energy to separate them and overcome the strong force. This energy will eventually result in pair production, and then the new quarks will pair with the old quarks and we'll have produced new hadrons. This means that for most accelerator processes resulting in quarks we don't see individual quarks, but instead **jets** of hadrons, emitted in roughly the direction the the quarks produced in the action were emitted. This means that the final state of a QCD process is usually quite complex, so rather than looking at specific processes, such as $e^-e^+ \to u\bar{u}$ we look at processes like $e^-e^+ \to$ hadrons.

17.3 $e^-e^+ \to$ hadrons

Consider the process $e^-e^+ \to$ hadrons, where "hadrons" means that the final state is formed from jets of hadrons. At tree level this process is[2]

[2] Assuming that this happens at low enough energies that Z production is negligible.



. (17.3.1)

This is a QED process, despite involving quarks, the exchange particle is the photon. The QCD stuff takes place after the quarks are created, but don't need it to compute the amplitude for this process. In fact, the amplitude for this process is almost identical to the result we get for $e^-e^+ \to \mu^-\mu^+$, the only difference is that the quarks don't have charge $\pm e$, instead they have charge $\pm Q_f e$, where $Q_f = 1/3$ or $Q_f = 2/3$, depending on the flavour of the quarks. This affects only the vertex with the quarks, so we get an additional factor of Q_f, the sign isn't important for any observables. Thus, the amplitude is, up to a sign, given by

$$\mathcal{A}(e^-e^+ \to q_f\bar{q}_f) = Q_f\mathcal{A}(e^-e^+ \to \mu^-\mu^+). \tag{17.3.2}$$

This suggests that we can define an observable

$$R = \frac{\sigma(e^-e^+ \to \text{hadrons})}{\sigma(e^-e^+ \to \mu^-\mu^+)}, \tag{17.3.3}$$

where we account for all possible pairs of quarks. Since the only difference between the amplitudes is the charge everything else cancels from the cross sections in this ratio, and so, recalling that we have to square the amplitude to get the cross section, we have

$$R = \sum_f Q_f^2, \tag{17.3.4}$$

It is important in this sum that we consider the kinematics of this process, and only sum over final states, quarks, which are accessible at the energy of the interaction. It is also important that as far as QED processes care, a red up quark and a blue up quark are two completely different particles, which just happen to have the same charge and mass. This means that at low energies, where up, down, and strange quarks are the only ones accessible, we have

$$R = N_c(Q_u^2 + Q_d^2 + Q_s^2) = 3\left(\frac{4}{9} + \frac{1}{9} + \frac{1}{9}\right) = 2, \tag{17.3.5}$$

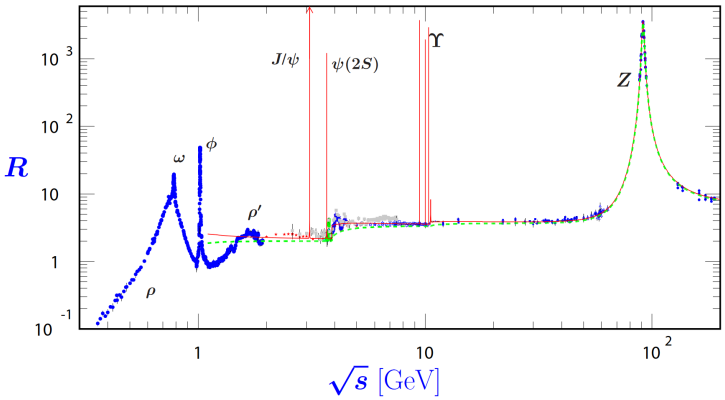


Figure 17.1: Plot of both experimental and theoretical values of R. The dashed green line is the result we have calculated using the naive quark model, but with the transition regions where new quarks become available treated correctly. The red line is the result of three loop perturbative QCD calculations. The points are data from various experiments. Notice that the two theoretical results align better, ignoring the spikes, at higher energies when the QCD coupling constant is smaller. The spikes correspond to QCD resonances, unstable bound states of quarks. The peak at the end occurs at the mass of the Z boson, where production of a nearly on-shell Z boson in place of the photon increases the amplitude.

where $N_c = 3$ is the number of colours.

At slightly higher energies the charm quark becomes available, and so we have

$$R = 2 + N_c Q_c^2 = 2 + 3 \cdot \frac{4}{9} = \frac{10}{3}. \tag{17.3.6}$$

At even higher energies the bottom quark is also accessible and so

$$R = \frac{10}{3} + N_c Q_b^2 = \frac{10}{3} + 3 \cdot \frac{1}{9} = \frac{11}{3}. \tag{17.3.7}$$

One may think that at yet higher energies we get top quarks, but the top quarks are so massive that at energy scales where they become accessible our first order approximations here break down.

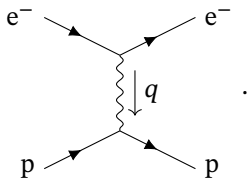
These predictions have been compared against data and provide good experimental verification of both the number of quarks below about the Z boson mass, and the number of colours. See Figure 17.1.

Eighteen

Parton Model

18.1 Low Energies

We want to discuss electron-proton scattering. How we think about the proton depends on the energy scale of the interaction. At low energies, $q^2 \ll \Lambda_{\text{QCD}}$, we can treat the proton as a point particle, and consider QED scattering, which at tree level is

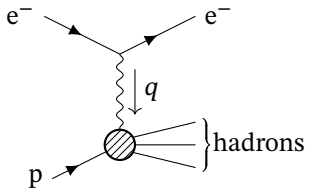


. (18.1.1)

This scattering is elastic, the particles before and after the interaction are the same, and so the kinetic energy is the same, although how its shared between the particles may change. Intuitively this is an appropriate description when the wavelength of the photon is much larger than the size of the proton.

18.2 Medium Energies

At intermediate energies, $q^2 \sim \Lambda_{\text{QCD}}$, we have to consider the proton as a composite particle and the interaction looks more like

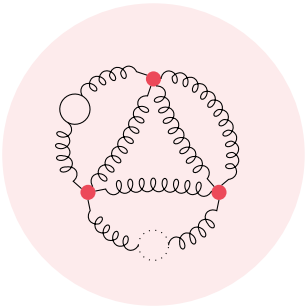


. (18.2.1)

In this regime the interactions are non-perturbative. Hadrons are being created in a complicated process. This makes the process inelastic. It is generally hard to work in this regime and there are very few theoretical tools for these sorts of computations.

18.3 Deep Inelastic Scattering

At high energies, $q \gg \Lambda_{\text{QCD}}$, lots of particles are created and annihilate in the proton, we call these **partons**. The proton looks a bit like



(18.3.1)

where the three dots are the "valence" quarks, the ones that we think of as making up the proton, and the other particles are virtual. Intuitively, this regime is important when the wavelength of the photon is much less than the size of the proton. We call this **deep inelastic scattering** (DIS), the deep referring to the energies allowing us to probe "deep" into the proton.

The problem is that while the interaction may occur at high enough energies for perturbation theory the initial state, the proton, is still non-perturbative. The solution is to use some physical reasoning. The momentum of the partons in the rest frame of the proton will be relatively low. This means that for sufficiently high momentum protons the momentum of the partons mostly aligns with the momentum of the proton. Hence the momentum of a parton can be expressed as

$$p^\mu = xP^\mu \tag{18.3.2}$$

where P^μ is the momentum of the proton and x is the **momentum fraction** of the parton, the fraction of the momentum that single parton carries. Since the momentum is carried entirely by the partons we must have

$$P^\mu = \sum_i p_i^\mu = \sum_i x_i P^\mu = P_\mu \sum_i x_i \implies \sum_i x_i = 1 \tag{18.3.3}$$

where p_i is the momentum carried by parton i, which has momentum fraction x_i. Strictly the partons momentum is only mostly aligned with the protons momentum, so actually we find that

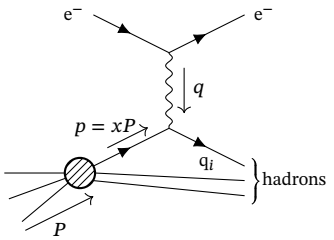
$$\sum_i x_i = 1 + \mathcal{O}(\Lambda_{\text{QCD}}), \tag{18.3.4}$$

since the momentum of the partons in the rest frame of the proton is on the order of Λ_{QCD}.

In order to utilise these momentum fractions we introduce the **parton distribution functions** (PDF), f_i, which is defined such that $f_i(x_i)$ is the probability density for finding parton i with momentum fraction x_i.

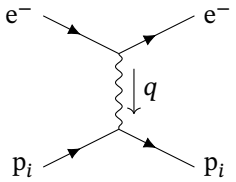
The other tool that we have is asymptotic freedom. This suggests that we can understand electron proton scattering at high energies as the interaction of a single

parton with the electron. This looks a bit like



. (18.3.5)

We can define σ to be the total cross section for this interaction, and then the **parton level cross section**, $\hat{\sigma}_i$, to be the cross section for the interaction



(18.3.6)

The total cross section is simply given by summing over all possible partons which could take part in the interaction, and integrating over all possible momentum fractions that they can have, including the parton distribution function, f_i, for each parton to properly account for the probability of it having a given share of the momentum. That is,

$$\sigma = \sum_i \int_0^1 \mathrm{d}x\, f_i(x)\hat{\sigma}_i(x). \tag{18.3.7}$$

This is the **factorisation formula**. It splits the physics into two parts:

- The parton distribution function contains the non-perturbative physics associated with the initial state, the proton. This is a low energy quantity. We can measure the PDF once and then it will be the same for all processes, it is, in this sense, universal. It does depend on the type of particle, so the PDF will be different for a pion or a Δ^{++}.
- The parton level cross section contains all of the perturbative physics. It is calculated in perturbative QCD and QED. It is a high energy quantity, so we can use asymptotic freedom. It is specific to a given process.

Feynman originally introduced the parton model, and the factorisation formula, following much the same logic as we have here, but impressively did it before quarks and gluons were understood to be the (majority of) partons. Perhaps surprisingly this formula treats the scattering of partons as being incoherent, we don't add the amplitudes and the square to get the cross section, instead we find the individual parton level cross sections and simply add these. This is justified because scattering occurs on a very short time scale, so we can essentially ignore the particles not directly involved in the scattering and there is no interference.

We can extend the hat notation and parton level quantities to define

$$s = (P + k)^2, \qquad \text{and} \qquad \hat{s} = (p + k)^2 = (xP + k)^2 \tag{18.3.8}$$

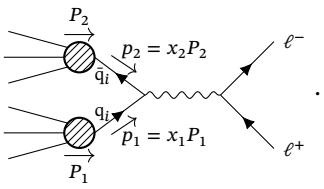
as the hadron and parton level s Mandelstam variable respectively, where k is the momentum of the incoming electron. This extends to other variables, such as $\hat{t}$ and $\hat{u}$ in the obvious way:

$$\hat{t} = (k - k')^2 = q^2, \qquad \text{and} \qquad \hat{u} = (k' - p)^2 = (k' - xP)^2 \tag{18.3.9}$$

where k' is the momentum of the outgoing electron.

18.4 Drell–Yan Process

The **Drell–Yan process** is scattering of two protons into leptons and hadrons, $pp \to \ell^+\ell^- +$ hadrons. We can picture this as



. (18.4.1)

This process involves two hadrons, so we have two momentum fractions, and two parton distribution functions, other than that the result is exactly the same:

$$\sigma = \sum_{i,\bar{\imath}} \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \, f_i(x_1) f_{\bar{\imath}}(x_2) \hat{\sigma}_{i\bar{\imath}} \tag{18.4.2}$$

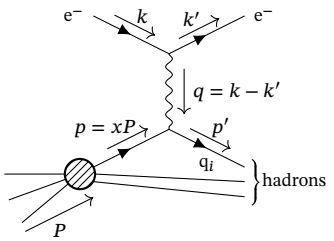
where i is summed over particles and $\bar{\imath}$ over antiparticles, and $\hat{\sigma}_{i\bar{\imath}}$ is the parton level cross section for the process $\mathrm{q}_i\bar{\mathrm{q}}_i \to \ell^-\ell^+$.

Nineteen

Deep Inelastic Scattering

19.1 The Process

Consider the process $pe^- \to e^- + \text{hadrons}$:



(19.1.1)

We can calculate the parton level cross section for this process, it's just a QED interaction between the electron and a quark. The problem comes when we try to compute the total cross section. Ideally we would do the kinematics in the parton centre of mass frame, since this makes things a lot simpler when we also neglect the electron mass. However, we don't know what the parton centre of mass frame is, since we don't know the momentum of the parton.

We will assume that the masses of the particles involved in this interaction are negligible, but also that we are at low enough energies that we don't have to worry about Z bosons as exchange particles. We can easily compute the parton level amplitude, and then the amplitude squared, averaged over initial states and summed over final states. The result is

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{A}_i|^2 = 2Q_i^2 e^4 \frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}. \tag{19.1.2}$$

This is just the t-channel electron positron or electron muon scattering process, but with an extra factor of Q_i in the amplitude and using parton level values, $\hat{t} = q^2$, $\hat{s} = (k+p)^2 = (k+xP)^2$, and $\hat{u} = (k'-p)^2 = (k'-xP)^2$.

In the parton centre of mass frame the cross section is, as usual, given by

$$\frac{\mathrm{d}\hat{\sigma}_i}{\mathrm{d}\Omega} = \frac{1}{64\pi^2\hat{s}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{A}_i| = \frac{1}{64\pi^2\hat{s}}2Q_i^2e^4\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}. \tag{19.1.3}$$

This can be rewritten as

$$\frac{\mathrm{d}\hat{\sigma}_i}{\mathrm{d}(\cos\vartheta)} = \frac{1}{32\pi\hat{s}}2Q_i^2e^4\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}. \tag{19.1.4}$$

19.2 Experimentally Accessible Variables

We'll first consider $\hat{t}$. As the square of a four-vector this is a Lorentz invariant, so we can calculate it in any frame. Choose the frame where the initial electron is moving along the x-axis. Further, neglecting the electron mass we have $k^2 = 0$, and so we can write $k^\mu = (E, E, 0, 0)$. The momentum of the outgoing electron is then $k'^\mu = (E, E\hat{\boldsymbol{n}})$, where $\hat{\boldsymbol{n}}$ is a unit three-vector. In this frame we have

$$\hat{t} = (k - k')^2 = -2k \cdot k' = -2E^2(1 - \cos\vartheta) = -\frac{\hat{s}^2}{2}(1 - \cos\vartheta), \tag{19.2.1}$$

where we've used $\hat{s} = (2E)^2$, this is just the usual result that the s Mandelstam invariant gives the energy coming into the process squared, assuming both the parton and electron have the same energy, which is usually true in scattering experiments.

For the differential cross section we can use $\hat{t}$ as the differential quantity, using

$$\mathrm{d}\hat{t} = \frac{\hat{s}}{2}\,\mathrm{d}(\cos\vartheta). \tag{19.2.2}$$

Here we've used the fact that the three Mandelstam invariants are pairwise independent, so differentials in $\mathrm{d}\hat{t}$ don't need to change $\hat{s}$. Using this we have

$$\frac{\mathrm{d}}{\mathrm{d}(\cos\vartheta)} = \frac{\hat{s}}{2}\frac{\mathrm{d}}{\mathrm{d}\hat{t}}. \tag{19.2.3}$$

This allows us to write

$$\frac{\mathrm{d}\hat{\sigma}_i}{\mathrm{d}\hat{t}} = \frac{2}{\hat{s}}\frac{\mathrm{d}\hat{\sigma}_i}{\mathrm{d}(\cos\vartheta)} = 2\pi\alpha^2 Q_i^2 \frac{\hat{s}^2 + (\hat{t} + \hat{s})^2}{\hat{s}^2\hat{t}^2} = 2\pi\alpha^2 Q_i^2 \frac{1 + (1 + \hat{t}/\hat{s})^2}{\hat{t}^2}. \tag{19.2.4}$$

Here we've used $\hat{u} = -\hat{t} - \hat{s}$ to replace the $\hat{u}$ Mandelstam invariant with $\hat{s}$ and $\hat{t}$ which we have a better handle on.

There is a problem with this result though. While we may know the energy of the electron and proton we don't know what share of this energy any given parton has. The quantities which are experimentally accessible are

- Anything to do with the electron, including its momenta, k and k', and the Mandelstam invariant $\hat{t} = (k - k')^2$.
- Anything to do with the incoming proton, such as its momentum, P, and the proton level Mandelstam invariant $s = (k + P)^2 \approx 2k \cdot P$.

Parton level data is generally inaccessible unless it is related to the above data through, for example, momentum conservation. For example, the parton level Mandelstam invariant $\hat{s}$ is actually accessible since $\hat{s} = (k + p)^2 = 2k \cdot p = 2xk \cdot P$ and it turns out that, somewhat surprisingly, x is actually observable under the assumption that the incoming and outgoing partons are on-shell. This is a safe assumption if the partons travel for a long time relative to the interaction time, or equivalently if the interaction energy is high compared to Λ_{QCD}. This is the case generally, for example in the LHC where $\sqrt{s} = 13.6\,\mathrm{TeV}$. This means that the interaction time goes as

$$\hbar/(13.6\,\mathrm{TeV}) \ll \hbar/\Lambda_{\mathrm{QCD}} \sim 10^{-23}\,\mathrm{s}. \tag{19.2.5}$$

Corrections for the partons not really being on-shell then are suppressed by factors of $\Lambda^2_{\mathrm{QCD}}/q^2$.

Assuming that the parton mass is also negligible, so $p'^2 \approx 0$, we have $(p+q)^2 \approx 2p\cdot q + q^2$. Thus, in terms of the proton momentum we have $2xP\cdot q = -q^2$. Rearranging this gives us the **Bjorken formula**

$$x = \frac{-q^2}{2P\cdot q} = \frac{\hat{t}}{2P\cdot q}. \tag{19.2.6}$$

The right hand side consists of experimentally measurable quantities. The quantity x is sometimes called the **Bjorken** x.

Notice that $\hat{t} \leq 0$ since

$$\hat{t} = -\frac{\hat{s}}{2}(1-\cos\vartheta) \tag{19.2.7}$$

and $|\cos\vartheta| \leq 1$ and $\hat{s} = (2E)^2 \geq 0$. We therefore define a positive measure of energy transfer, $Q^2 \coloneqq -\hat{t}$. Then

$$x = \frac{Q^2}{2P\cdot q}. \tag{19.2.8}$$

In terms of this quantity we have

$$\frac{\mathrm{d}\hat{\sigma}_i}{\mathrm{d}Q^2} = -\frac{\mathrm{d}\hat{\sigma}_i}{\mathrm{d}\hat{t}} = 2\pi\alpha^2 Q_i^2 \frac{1+(1-Q^2/(xs))^2}{Q^4}. \tag{19.2.9}$$

Both $\hat{t}$ and x are experimentally accessible, so we can make use of a doubly differential cross section at the hadron level:

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}Q^2} = \sum_i f_i(x) 2\pi\alpha^2 Q_i^2 \frac{1+(1-Q^2/(xs))^2}{Q^4}. \tag{19.2.10}$$

Hence the total cross section is

$$\sigma = \int_0^1 \mathrm{d}x \int \mathrm{d}Q^2 \frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}Q^2}. \tag{19.2.11}$$

It is possible to give a more general form for the differential cross section:

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}Q^2} = \frac{2\pi\alpha^2}{xQ^4}\left[\left(1+\left[1-\frac{Q^2}{xs}\right]^2\right)F_2(x,Q^2) - \left(\frac{Q^2}{xs}\right)F_L(x,Q^2)\right], \tag{19.2.12}$$

The two functions F_2 and F_L parametrise a more general interaction, we've considered a QED interaction here, but there could also be weak interactions. Our result is just the special case where

$$F_2(x,Q^2) = \sum_i x f_i(x), \qquad \text{and} \qquad F_L(x,Q^2) = 0. \tag{19.2.13}$$

It turns out that F_L vanishes for fermionic partons. This is evidence that quarks have spin 1/2, which we need since we can't see quarks on their own.

Part III

Infrared Divergences

Twenty

Infrared Divergences

This new material in this part of the course is not examinable.

We have now spent a great deal of time studying divergences as momenta go to infinity. These are called "ultraviolet divergences". It turns out that since lots of quantities go as one over momentum to some power we also get divergences when momenta go to zero. These are called **infrared divergences**. In this part of the course we will look at how to deal with these divergences. Infrared divergences occur in most quantum field theories, so for simplicity we will mostly focus on QED, although we'll start with QCD as the motivation.

Part IV

Electroweak

Twenty-One

Overview

There are four fundamental forces,

- electromagnetism;
- the strong force;
- the weak force;
- gravity.

In this course we are interested in the first three. To each force we can associate a type of radiation,

- gamma radiation;
- alpha radiation;
- beta radiation;
- gravitational waves.

These forces all have different ranges, reflecting the different mechanisms behind the difference forces, at long ranges these forces scale as

- $1/r$;
- $e^{-\mu r}/r$ with $\mu \approx 1\,\text{GeV}$;
- $e^{-\mu r}$ with $\mu \approx 100\,\text{GeV}$;
- $1/r$.

Each of these forces can be described as a gauge theory with a different gauge group,

- U(1);
- SU(3);
- SU(2);
- diffeomorphisms of spacetime[1].

[1] some don't consider this to be a gauge theory, for one thing the group of diffeomorphisms is non-compact, and the theory doesn't work in quite the same way as the other examples.

No one is quite sure why these particular gauge groups are the ones which apply in each case, and the choice of gauge group pretty much completely determines the physics. For example, the fact that SU(3) and SU(2) are non-Abelian is responsible for the short range interactions of the strong and weak force, which is ultimately due to the self interactions of exchange particles, which doesn't occur in the Abelian U(1) case.

One of the main tools in our arsenal is perturbation theory, although this doesn't work in all cases

- perturbation theory works at sufficiently low energies, including most cases of real world interest;
- perturbation theory works at high energies, but not low energies;
- perturbation theory works but is "weird" and "interesting";
- perturbation theory works for sufficiently small perturbations.

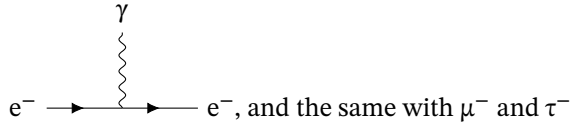
Symmetry is important in physics, these forces posses, or don't posses, various symmetries. One symmetry being preserving flavour, that is the type of fermion involved in interactions,

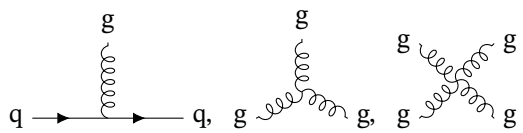
- flavour stays the same;
- flavour stays the same;
- flavours can change;
- flavours stay the same.

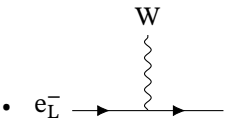
Another set of symmetries is parity (P), charge conjugation (C), and time reversal (T), although often due to the CPT theorem, which states that the combination of all three of these symmetries is respected, we replace time reversal, which is weird involving anti-unitary operations, with charge conjugation and parity transformation (CP).

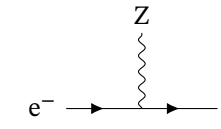
- P, C, and T are all respected;
- P, C, and T are all respected;
- P and C are maximally violated, CP is violated a very small amount;
- GR respects these symmetries, other theories of gravity do not.

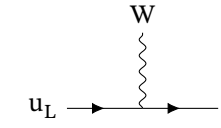
Each force allows for different interactions,

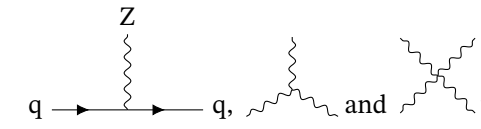
- 

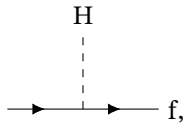
- 

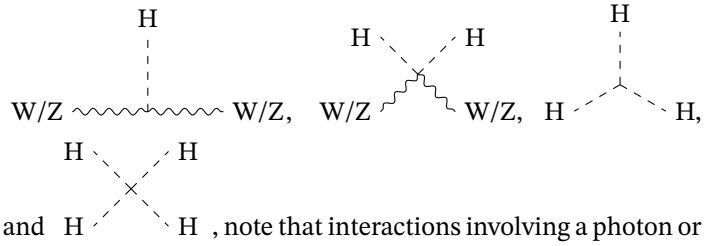
- 

, and the same with μ_L^- and $\nu_{\mu L}$, and τ_L^- and $\nu_{\tau L}$,


, and the same with μ^- and τ^-,


, and the same with c_L and s_L, and t_L and b_L,


with the particles being any of W, Z, or γ, so long as charge is conserved,




, note that interactions involving a photon or Higgs boson are due to the electroweak interaction, rather than just the weak interaction;

- depends on your theory of gravity.

21.1 The Weak Force

As we saw in the previous section the weak force is quite different from electromagnetism and the strong force, which really only differ due to U(1) being Abelian and SU(2) being non-Abelian. The weak force allows for flavour change, and includes many more interactions than either electromagnetism or the strong interaction.

Examples of processes mediated by the weak force are **beta decay**,

$$n \to pe^-\bar{\nu}_e, \qquad udd \to uude^-\bar{\nu}_e, \tag{21.1.1}$$

pion decays, such as

$$\pi^- \to \mu^-\bar{\nu}_\mu, \qquad \pi^- \to e^-\bar{\nu}_e, \qquad \pi^0 \to \gamma\gamma, \qquad \pi^0 \to \gamma e^-e^+, \tag{21.1.2}$$

and **kaon decays**, such as

$$K^- \to \pi^-\pi^0, \qquad K^- \to \pi^-\pi^-\pi^+. \tag{21.1.3}$$

The violation of symmetries and the ability of the weak force to change the flavour of particles makes the world a much more interesting place. Without out this the universe would be static, up to uninteresting dynamical changes, with lepton and quark number remaining constant. The fact that the weak force allows for decays also means that if we reverse time, despite the symmetry breaking, the weak force allows for creation of particles as well, and this is one of the major avenues of study for early universe physics.

In this course we will follow a historical approach, piecing together first the theory of the weak interaction, and then combining this with electromagnetism to get electroweak theory. As such there will be points where we study theories which later turned out to be wrong, or at least not the complete picture.

The theory of weak interactions begins with Pauli in 1930 who was studying the kinematics of beta decay and realised that seemingly missing energy could be explained if the decay was producing another particle, the neutrino. The neutrino's existence was the confirmed experimentally in 1956 by Frederick Reines and Clyde Cowan[2]. In the time between positing the existence of the neutrino and its experimental confirmation the theory of the weak interaction had progressed a lot. In the same year Chien-Shiung Wu[3] observed parity violation for the first time, which was a big surprise, since the prevailing wisdom was that parity was always conserved.

[2]Nobel prize for (electro)weak theory count: 1

[3]Nobel prize for (electro)weak theory count: 2

Before we can discus the weak force further we need to recap chirality, helicity, and parity.

Twenty-Two

Left and Right

22.1 Chirality

Recall that the fifth gamma matrix is defined as

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{22.1.1}$$

and is such that $(\gamma^5)^2 = 1$, $(\gamma^5)^\dagger = \gamma^5$, and $\{\gamma^5, \gamma^\mu\} = 0$. Using this we can define a complete set of Hermitian, orthogonal projection operators:

$$P_\text{L} \coloneqq \frac{1}{2}(1-\gamma^5), \qquad \text{and} \qquad P_\text{R} \coloneqq \frac{1}{2}(1+\gamma^5). \tag{22.1.2}$$

We call these the left and right projection operators respectively, we'll justify this name later. Breaking it down "complete set of Hermitian, orthogonal projection operators" tells us the following:

- Projectors: $P_\text{L}^2 = (1-\gamma^5)^2/4 = (1-2\gamma^5+1)/4 = (1-\gamma^5)/2 = P_\text{L}$, and similarly $P_\text{R}^2 = P_\text{R}$;
- Orthogonal: $P_\text{L}P_\text{R} = (1-\gamma^5)(1+\gamma^5)/2 = (1-(\gamma^5)^2)/2 = (1-1)/2 = 0$, and similarly $P_\text{R}P_\text{L} = 0$;
- Complete: $P_\text{L} + P_\text{R} = (1-\gamma^5)/2 + (1+\gamma_5)/2 = 1$;
- Hermitian: $P_\text{L}^\dagger = [(1-\gamma^5)/2]^\dagger = (1-(\gamma^5)^\dagger)/2 = (1-\gamma^5)/2 = P_\text{L}$, and similarly $P_\text{R}^\dagger = P_\text{R}$.

Note that we have $P_\text{R} - P_\text{L} = (1+\gamma^5)/2 - (1-\gamma^5)/2 = \gamma^5$. Another useful property is that commuting a γ^μ with a projector turns left into right and vice versa, so $\gamma^\mu P_\text{L} = \gamma^\mu(1-\gamma^5)/2 = (\gamma^\mu - \gamma^\mu\gamma^5)/2 = (\gamma^\mu + \gamma^5\gamma^\mu)/2 = (1+\gamma^5)\gamma^\mu/2 = P_\text{R}\gamma^\mu$, and similarly $\gamma^\mu P_\text{R} = P_\text{L}\gamma^\mu$.

Given a (Dirac) spinor, ψ, we can decompose it into left and right components using

$$\psi = 1\psi = (P_\text{L} + P_\text{R})\psi = P_\text{L}\psi + P_\text{R}\psi = \psi_\text{L} + \psi_\text{R}, \tag{22.1.3}$$

where

$$\psi_\text{L} \coloneqq P_\text{L}\psi, \qquad \text{and} \qquad \psi_\text{R} \coloneqq P_\text{R}\psi \tag{22.1.4}$$

are the left and right components of ψ.

For the adjoint spinor left and right are swapped around, that is

$$\bar{\psi}P_{\rm L} = \psi^\dagger\gamma^0 P_{\rm L} = \psi^\dagger P_{\rm R}\gamma^0 = (P_{\rm R}\psi)^\dagger\gamma^0 = \psi_{\rm R}^\dagger\gamma^0 = \overline{\psi_{\rm R}}. \tag{22.1.5}$$

Note that this is the adjoint of the right component of ψ, rather than the right component of the adjoint, which we might write as $\bar{\psi}_{\rm R}$. However, this notation is confusing and so we assume that R or L is always applied to the spinor before we take the adjoint, and write this as $\bar{\psi}_{\rm R}$. Some brackets may help here, we have

$$\bar{\psi}P_{\rm L} = (\bar{\psi})_{\rm L} = \bar{\psi}_{\rm R}. \tag{22.1.6}$$

Similarly, we have $\bar{\psi}P_{\rm R} = (\bar{\psi})_{\rm R} = \bar{\psi}_{\rm L}$.

Now consider the mass term in the Dirac Lagrangian, $\bar{\psi}\psi$. We have

$$\begin{aligned}
\bar{\psi}\psi &= \bar{\psi}(\psi_{\rm R} + \psi_{\rm L}) && \text{completeness} && (22.1.7)\\
&= \bar{\psi}\psi_{\rm R} + \bar{\psi}\psi_{\rm L} && && (22.1.8)\\
&= \bar{\psi}P_{\rm R}\psi + \bar{\psi}P_{\rm L}\psi && && (22.1.9)\\
&= \bar{\psi}P_{\rm R}^2\psi + \bar{\psi}P_{\rm L}^2\psi && \text{idempotency} && (22.1.10)\\
&= \bar{\psi}_{\rm L}\psi_{\rm R} + \bar{\psi}_{\rm R}\psi_{\rm L}. && && (22.1.11)
\end{aligned}$$

From this we can see that the mass term mixes left and right spinors. We'll see that this is a general property of massive particles, they have some mixing of parities.

Consider a fermion current, $\bar{\psi}\gamma^\mu\psi$, we have

$$\begin{aligned}
\bar{\psi}\gamma^\mu\psi &= \bar{\psi}\gamma^\mu(\psi_{\rm R} + \psi_{\rm L}) && \text{completeness} && (22.1.12)\\
&= \bar{\psi}\gamma^\mu P_{\rm R}\psi + \bar{\psi}\gamma^\mu P_{\rm L}\psi && && (22.1.13)\\
&= \bar{\psi}\gamma^\mu P_{\rm R}^2\psi + \bar{\psi}\gamma^\mu P_{\rm L}^2\psi && \text{idempotency} && (22.1.14)\\
&= \bar{\psi}P_{\rm L}\gamma^\mu P_{\rm R}\psi + \bar{\psi}P_{\rm R}\gamma^\mu P_{\rm L}\psi && \gamma^\mu P_{\rm L} = P_{\rm R}\gamma^\mu \text{ and } \gamma^\mu P_{\rm R} = P_{\rm L}\gamma^\mu && (22.1.15)\\
&= \bar{\psi}_{\rm R}\gamma^\mu\psi_{\rm R} + \psi_{\rm L}\gamma^\mu\psi_{\rm L}. && && (22.1.16)
\end{aligned}$$

From this we see that the current doesn't mix left and right.

The Dirac Lagrangian is then

$$\begin{aligned}
\mathcal{L}_{\rm D} &= \bar{\psi}(i\partial\!\!\!/ - m)\psi && (22.1.17)\\
&= \bar{\psi}_{\rm R}i\partial\!\!\!/\psi_{\rm R} + m\bar{\psi}_{\rm L}\psi_{\rm R} + \bar{\psi}_{\rm L}i\partial\!\!\!/\psi_{\rm L} + m\bar{\psi}_{\rm R}\psi_{\rm L}. && (22.1.18)
\end{aligned}$$

Again, we see that the mass term mixes the left and right. The Dirac Lagrangian gives rise to the Dirac equation, $(i\partial\!\!\!/ - m)\psi$. For a massless particle this gives two independent equations for the left and right components,

$$i\partial\!\!\!/\psi_{\rm L} = 0, \qquad \text{and} \qquad i\partial\!\!\!/\psi_{\rm R} = 0. \tag{22.1.19}$$

These are known as the **Weyl equations**. Importantly for massless particles the left and right components evolve separately.

Currently we are treating $\psi_{\rm L}$ and $\psi_{\rm R}$ as four component Dirac spinors projected onto a two-dimensional subspace. Instead we can treat them as two component Weyl spinors. We won't do this.

The operators $P_{\rm L}$ and $P_{\rm R}$ define what we call the **chirality** of the particle, for example, a right handed chiral state is an eigenstate of $P_{\rm R}$ with eigenvalue 1.

22.2 Helicity

For a massive fermion the **spin operator** is[1]

$$\Sigma^i := \frac{i}{4}\varepsilon^{ijk}[\gamma^j,\gamma^k] \tag{22.2.1}$$
$$= \frac{i}{4}\varepsilon^{ijk}(\gamma^j\gamma^k - \gamma^k\gamma^j) \tag{22.2.2}$$
$$= \frac{i}{4}(\varepsilon^{ijk}\gamma^j\gamma^k - \varepsilon^{ijk}\gamma^k\gamma^j) \tag{22.2.3}$$
$$= \frac{i}{4}(\varepsilon^{ijk}\gamma^j\gamma^k - \varepsilon^{ikj}\gamma^j\gamma^k) \tag{22.2.4}$$
$$= \frac{i}{4}(\varepsilon^{ijk}\gamma^j\gamma^k + \varepsilon^{ijk}\gamma^j\gamma^k) \tag{22.2.5}$$
$$= \frac{i}{2}\varepsilon^{ijk}\gamma^j\gamma^k. \tag{22.2.6}$$

[1] note that this differs by a sign from the definition in *Quantum Field Theory*, by a factor of 2 from the definition in the notes, and a factor of 4 from the value given in the lecture, I've checked it works as written here in the Dirac representation.

Alternatively we could have used the antisymmetry of ε^{ijk} to replace $[\gamma^j,\gamma^k]$ with $[\gamma^j,\gamma^k]+\{\gamma^j,\gamma^k\} = 2\gamma^j\gamma^k$. Either way, $\Sigma^1 = i\varepsilon^{1jk}\gamma^j\gamma^k/2 = (i\gamma^2\gamma^3 - i\gamma^3\gamma^2)/2 = i\gamma^2\gamma^3$, and similarly $\Sigma^2 = i\gamma^3\gamma^1$ and $\Sigma^3 = i\gamma^1\gamma^2$. We can write all of this as $\Sigma^i = \gamma^5\gamma^0\gamma^i$, which follows by expanding γ^5:

$$\gamma^5\gamma^0\gamma^i = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^i = -i(\gamma^0)^2\gamma^1\gamma^2\gamma^3\gamma^i = -i\gamma^1\gamma^2\gamma^3\gamma^i, \tag{22.2.7}$$

and so we have $\gamma^5\gamma^0\gamma^1 = -i\gamma^1\gamma^2\gamma^3\gamma^1 = -i(\gamma^1)^2\gamma^2\gamma^3 = i\gamma^2\gamma^3$, and similarly $\gamma^5\gamma^0\gamma^2 = -i\gamma^1\gamma^3$ and $\gamma^5\gamma^0\gamma^3 = i\gamma^1\gamma^2$, which shows that

$$\Sigma^i = \frac{i}{4}\varepsilon^{ijk}[\gamma^j,\gamma^k] = \gamma^5\gamma^0\gamma^i. \tag{22.2.8}$$

Now consider the commutator

$$[P_L,\Sigma^i] = \frac{1}{2}([1,\Sigma^i] - [\gamma^5,\Sigma^i]) \tag{22.2.9}$$
$$= -\frac{1}{2}[\gamma^5,\gamma^5\gamma^0\gamma^i] \tag{22.2.10}$$
$$= -\frac{1}{2}((\gamma^5)^2\gamma^0\gamma^i - \gamma^5\gamma^0\gamma^i\gamma^5) \tag{22.2.11}$$
$$= -\frac{1}{2}(\gamma^0\gamma^i - (\gamma^5)^2\gamma^0\gamma^i) \tag{22.2.12}$$
$$= -\frac{1}{2}(\gamma^0\gamma^i - \gamma^0\gamma^i) \tag{22.2.13}$$
$$= 0. \tag{22.2.14}$$

Similarly, $[P_R,\Sigma^i] = 0$. This means that spin and chirality commute, and so we can measure both at once.

For massless particles instead of spin we have **helicity**, given by the helicity operator[2]

$$h = \frac{\boldsymbol{\Sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|}. \tag{22.2.15}$$

[2] note that this differs from the value given in the notes by a factor of 2.

This has eigenvalues ±1, which can be shown by considering the case where $\boldsymbol{p} = (0,0,p)$, in which case $h = \Sigma^3 = \mathrm{diag}(1,-1,1,-1)$, and so the eigenvalues are ±1,

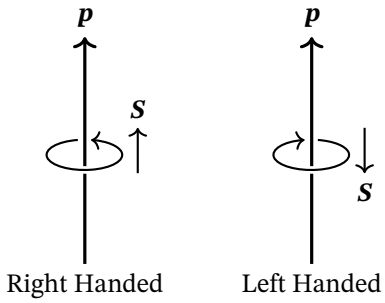


Figure 22.1: Left and right helicities showing the relation between spin and momentum.

each with multiplicity 2. States with helicity +1 are called **right handed**, because their spin aligns with their momentum. Interpreting the spin as the angular momentum of the particle this means that the particle "spins" around the momentum axis in accordance with the right hand grip rule. Similarly states with helicity −1 are called **left handed**, as their spin goes in the opposite direction around the momentum, so is instead given by a left hand grip rule. This is shown in Figure 22.1. Importantly if momentum and spin are aligned then it is a right handed helicity state, and if they are antialigned it is a left handed helicity state.

Now consider a spinor, $u^{\pm}(p)$, in momentum space which is a positive energy solution to the Dirac equation, that is

$$(\not{p} - m)u^{\pm}(p) = 0 \implies \not{p}u^{\pm}(p) = mu^{\pm}(p), \tag{22.2.16}$$

with helicity ± 1, that is $hu^{\pm 1}(p) = \pm u^{\pm}(p)$.

Note that we have

$$\not{p} = \gamma^0 E - \boldsymbol{\gamma}\cdot\boldsymbol{p} \implies \boldsymbol{\gamma}\cdot\boldsymbol{p} = \gamma^0 E - \not{p}, \tag{22.2.17}$$

and so we have

$$h = \frac{1}{|\boldsymbol{p}|}\Sigma^i p^i = \frac{1}{|\boldsymbol{p}|}\gamma^5\gamma^0\gamma^i p^i = \frac{1}{|\boldsymbol{p}|}\gamma^5\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{p} = \frac{\gamma^5\gamma^0}{|\boldsymbol{p}|}(\gamma^0 E - \not{p}). \tag{22.2.18}$$

Multiplying through the factor of γ^0 and using $(\gamma^0)^2 = 1$ this becomes

$$h = \frac{\gamma^5}{|\boldsymbol{p}|}(E - \gamma^0\not{p}). \tag{22.2.19}$$

Acting with this on $u^{\pm}(p)$ on the one hand should give $\pm u^{\pm}(p)/|\boldsymbol{p}|$, and on the other hand gives

$$\frac{\gamma^5}{|\boldsymbol{p}|}(E - \gamma^0\not{p})u^{\pm}(p) = \frac{\gamma^5}{|\boldsymbol{p}|}(-E - \gamma^0 m)u^{\pm}(p). \tag{22.2.20}$$

That is,

$$\gamma^5(E - \gamma^0 m)u^{\pm}(p) = \pm|\boldsymbol{p}|u^{\pm}(p). \tag{22.2.21}$$

Now use $\gamma^5 = P_R - P_L$ and insert a factor of $1 = P_R + P_L$ on the right to get

$$(P_R - P_L)(E - \gamma^0 m)u^\pm(p) = \pm|\boldsymbol{p}|(P_R + P_L)u^\pm(p). \tag{22.2.22}$$

Since the projectors are orthogonal we can read off the left and right components from this separately, for example taking just the right handed component gives

$$P_R(E - \gamma^0 m)u^\pm(p) = \pm|\boldsymbol{p}|P_R u^\pm(p). \tag{22.2.23}$$

Commuting the P_R through γ^0 turns it into P_L and then acting on the spinor we have

$$Eu_R^\pm(p) - \gamma^0 m u_L^\pm(p) = \pm|\boldsymbol{p}|u_R^\pm(p). \tag{22.2.24}$$

Similarly, reading off the left handed component we have

$$-P_L(E - \gamma^0 m)u^\pm(p) = \pm|\boldsymbol{p}|P_L u^\pm(p), \tag{22.2.25}$$

which gives

$$-Eu_L^\pm(p) + \gamma^0 m u_R^\pm(p) = \pm|\boldsymbol{p}|u_L^\pm(p). \tag{22.2.26}$$

Rearranging these two results gives

$$(E \mp |\boldsymbol{p}|)u_R^\pm(p) = m\gamma^0 u_L^\pm(p), \tag{22.2.27}$$
$$(E \pm |\boldsymbol{p}|)u_L^\pm(p) = m\gamma^0 u_R^\pm(p). \tag{22.2.28}$$

Again, we see that if $m \neq 0$ then we get mixing of left and right handed states.

Now consider the $m \to 0$ limit, then we have

$$E^2 = |\boldsymbol{p}|^2 + m^2 \implies |\boldsymbol{p}| = \sqrt{E^2 - m^2} \approx E + \mathcal{O}\left(\frac{m^2}{E^2}\right). \tag{22.2.29}$$

So when we choose + in the sum on the left hand side of these two equations we get $2E$, and the right hand side vanishes as $m \to 0$, giving

$$2Eu_R^-(p) = 0, \qquad \text{and} \qquad 2Eu_L^+ = 0. \tag{22.2.30}$$

That is, $u_R^- = u_L^+ = 0$.

This means that the spinor $u_R = u_R^+ + u_R^- = u_R^+$ has helicity +1, so is right handed. Similarly, $u_L = u_L^+ + u_L^- = u_L^-$ has helicity −1, so is left handed. This justifies our choice to call the chirality states left and right, it's because in the case of massless particles left and right chiral states coincide with left and right helicity states.

For $m \neq 0$ but still $m \ll E$ we have

$$u_R^- = \frac{m\gamma^0}{E + |\boldsymbol{p}|}u_L^- \approx \frac{m\gamma^0}{2E}u_L^-, \tag{22.2.31}$$

which means that the right handed chiral component of a left handed helicity eigenstate, that is u_R^-, is small. Similarly the left handed chiral component of a right handed helicity eigenstate, that is u_L^+, is small.

22.3 Discrete Symmetries

22.3.1 Parity

Consider the parity transformation, P, defined on a spinor as

$$\psi \overset{P}{\mapsto} \psi_P \coloneqq \gamma^0 \psi. \tag{22.3.1}$$

Thus, we have

$$\psi_{\mathrm{L}} \overset{P}{\mapsto} \gamma^0 \psi_{\mathrm{L}} = \gamma^0 P_{\mathrm{L}} \psi = P_{\mathrm{R}} \gamma^0 \psi = (\psi_P)_{\mathrm{R}}. \tag{22.3.2}$$

Similarly,

$$\psi_{\mathrm{R}} \overset{P}{\mapsto} (\psi_P)_{\mathrm{L}}. \tag{22.3.3}$$

That is, $(\psi_{\mathrm{L}})_P = (\psi_P)_{\mathrm{R}}$ and $(\psi_{\mathrm{R}})_P = (\psi_{\mathrm{L}})_P$. This shows that parity swaps left and right, further justifying us naming these projectors left and right in the first place.

Since

$$[\gamma^0, \Sigma^i] = \gamma^0 \gamma^5 \gamma^0 \gamma^i - \gamma^5 \gamma^0 \gamma^i \gamma^0 = -\gamma^5 (\gamma^0)^2 \gamma^i + \gamma^5 (\gamma^0)^2 \gamma^i = 0 \tag{22.3.4}$$

parity doesn't change spin. Under parity the three-momentum is reversed,

$$\boldsymbol{p} \overset{P}{\mapsto} -\boldsymbol{p}. \tag{22.3.5}$$

Thus, the helicity is reversed,

$$h = \frac{\boldsymbol{\Sigma} \cdot \boldsymbol{p}}{|\boldsymbol{p}|} \overset{P}{\mapsto} \frac{\boldsymbol{\Sigma} \cdot (-\boldsymbol{p})}{|-\boldsymbol{p}|} = -\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{p}}{|\boldsymbol{p}|} = -h. \tag{22.3.6}$$

So under a parity transformation

$$u^+_{\mathrm{R}} \overset{P}{\mapsto} u^-_{\mathrm{L}}. \tag{22.3.7}$$

That is, parity transformation swap left and right chiralities and left and right helicities, all while leaving spin unchanged.

22.3.2 Charge Conjugation

Consider charge conjugation, C, defined on a spinor as

$$\psi \overset{C}{\mapsto} \psi_C \coloneqq C \bar{\psi}^{\mathsf{T}} \tag{22.3.8}$$

where $C = i\gamma^2 \gamma^0$. We then have

$$P_{\mathrm{L}} C = i P_{\mathrm{L}} \gamma^2 \gamma^0 = i \gamma^2 P_{\mathrm{R}} \gamma^0 = i \gamma^2 \gamma^0 P_{\mathrm{L}} = C P_{\mathrm{L}}. \tag{22.3.9}$$

Therefore

$$\psi_{\mathrm{L}} \overset{C}{\mapsto} C (\bar{\psi})^{\mathsf{T}}_{\mathrm{L}}. \tag{22.3.10}$$

So charge conjugation leaves chirality unchanged. Similarly, the spin and momentum, and hence momentum, are not changed by charge conjugation. The only thing that changes is particles become antiparticles and vice versa.

22.3.3 Time Reversal

Consider time reversal, T, defined on a spinor as

$$\psi \overset{T}{\mapsto} \psi_T = T\psi \tag{22.3.11}$$

with $T = i\gamma^1\gamma^3 = -i\gamma^5 C$ and $T^\dagger = T = T^{-1}$. We then again have $P_\text{L} T = T P_\text{L}$, and so

$$\psi_\text{L} \overset{T}{\mapsto} T\psi_\text{L}. \tag{22.3.12}$$

Time reversal reverses momentum, and also spin, since both are dynamical and so reverse if we do things backwards. This means helicity is left unchanged, since both spin and momentum are reversed and the two minus signs cancel.

22.4 Chiral Representation

In the **chiral representation** the gamma matrices are

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \qquad \text{and} \qquad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{22.4.1}$$

Then the projection operators are

$$P_\text{R} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad P_\text{L} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{22.4.2}$$

These act on the spinor

$$\psi = \begin{pmatrix} \psi_\text{R} \\ \psi_\text{L} \end{pmatrix}. \tag{22.4.3}$$

The spin operator is

$$\Sigma^i = \gamma^5\gamma^0\gamma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \tag{22.4.4}$$

which gives the four helicity eigenstates

$$\begin{pmatrix} \psi_\text{R} \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} 0 \\ \psi_\text{L}^\pm \end{pmatrix}. \tag{22.4.5}$$

Consider the positive and negative energy solutions to the momentum space Dirac equation, $(\not{p} - m)u = 0$, these are such that

$$\begin{pmatrix} -m & E + \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ E - \boldsymbol{\sigma}\cdot\boldsymbol{p} & -m \end{pmatrix} \begin{pmatrix} u_\text{R} \\ u_\text{L} \end{pmatrix} = 0. \tag{22.4.6}$$

We can then use $(\boldsymbol{\sigma}\cdot\boldsymbol{p})u_{\text{L,R}} = \pm p u^\pm_{\text{L,R}}$, so $(E \pm p)u^\pm_\text{L} = m u^\pm_\text{R}$. Since $E^2 = p^2 + m^2$ we have

$$(E \pm p)u^\pm_\text{L} = \sqrt{(E+p)(E-p)}\,u^\pm_\text{R} \tag{22.4.7}$$

which gives

$$\sqrt{E \pm p}\,u^\pm_\text{L} = \sqrt{E \mp p}\,u^\pm_\text{R}. \tag{22.4.8}$$

So we can write

$$u^{\pm} = \begin{pmatrix} \sqrt{E \pm p}\xi^{\pm} \\ \sqrt{E \mp p}\xi^{\pm} \end{pmatrix} \tag{22.4.9}$$

where

$$\xi^{+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \xi^{0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{22.4.10}$$

These are such that $\bar{u}u = 2m$ and $u^{\dagger}u = 2E$ whenever all spinors are evaluated at the same momentum.

Similarly, for the negative energy solutions, $(\not{p} + m)v = 0$, we have $(\boldsymbol{\sigma} \cdot \boldsymbol{p})v^{\pm} = \mp p v^{\pm}$ and so

$$(E \mp p)v_{\mathrm{L}}^{\pm} = -m v_{\mathrm{R}}^{\pm} \tag{22.4.11}$$

which gives

$$\sqrt{E \mp p}v_{\mathrm{L}}^{\pm} = -\sqrt{E \pm p}v_{\mathrm{R}}^{\pm}. \tag{22.4.12}$$

Then we can write

$$v^{\pm} = \begin{pmatrix} \sqrt{E \mp p}\xi^{\pm} \\ -\sqrt{E \pm p}\xi^{\pm} \end{pmatrix}. \tag{22.4.13}$$

Recall that we used

$$\chi^{+} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \text{and} \qquad \chi^{-} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \tag{22.4.14}$$

as a basis for the negative energy solutions in the Dirac representation. This is related to $\xi^{\pm}$ by charge conjugation, using $v^{\pm} = i\gamma^2 u^{\pm *}$.

Twenty-Three

Charged Current Weak Interactions

23.1 Fermi Theory

After Pauli theorised the existence of the neutrino to explain seeming violations of conservation of energy and momentum a new theory was needed to explain an interaction involving four particles. The first attempt was **Fermi theory**. Fermi hypothesised a four point interaction, which we now call the four-Fermi interaction, involving four fermions. The assumption was that this interaction must be similar to QED.

For the case of beta decay, $\mathrm{n} \to \mathrm{p}\mathrm{e}^{-}\bar{\nu}_{\mathrm{e}}$, the four-Fermi interaction takes the form

$$G_{\mathrm{F}}(\bar{\mathrm{n}}\gamma_{\mu}\mathrm{p})(\bar{\nu}\gamma^{\mu}\mathrm{e}) \tag{23.1.1}$$

where we are using the symbols for the particles to represent their spinors, so e is a spinor corresponding to the electron, $\bar{\mathrm{n}}$ is the adjoint of a spinor corresponding to the neutron and so on. This helps to declutter the notation, and avoids confusing subscripts like ψ_{μ} vs. ψ_{μ}, the first being a spinor for a muon, the second a four–vector with index μ. Here G_{F} is a constant called the **Fermi constant**. It is a coupling constant.

We can generalise this to an interaction

$$G_{\mathrm{F}}J_{\mu}^{\dagger}J^{\mu} \tag{23.1.2}$$

where J_{μ} is the **weak current** given by

$$J_{\mu} \coloneqq \bar{\nu}\gamma_{\mu}\mathrm{e} + \bar{\mathrm{p}}\gamma_{\mu}\mathrm{n} + \cdots . \tag{23.1.3}$$

Here the first term corresponds to a current of electrons and electron neutrinos, the second to a current of protons and neutrons, and subsequent terms correspond to currents of other similar pairs of particles. Note that we can split J_{μ} into a leptonic term and a hadronic term.

Notice that J_{μ} has $\Delta Q = 1$, by this we mean that if we interpret a term in the current, for example $\bar{\nu}\gamma_{\mu}\mathrm{e}$, as an incoming electron and outgoing neutrino then the change in charge is $\Delta Q = 1$. Since the Hermitian conjugate reverses the order of factors we have terms like $\bar{\mathrm{e}}\gamma_{\mu}\nu$ appearing in $J_{\mu}^{\dagger}$, so $J_{\mu}^{\dagger}$ has $\Delta Q = -1$. This means that overall the interaction conserves charge.

Write $[X]$ for the mass dimension of X, so for a mass $[m] = 1$, for the charge $[e] = 0$, and so on. We know that the action is dimensionless, $[S] = 0$, and working in four dimensions we have that $[\mathrm{d}^4x] = -4$. So, we must have $[\mathcal{L}] = 4$. Hence all terms appearing in the Lagrangian have mass dimension 4, then $[m\bar{\psi}\psi] = 4$, but $[m] = 1$, so we have $[m\bar{\psi}\psi] = 1+[\bar{\psi}\psi]$, so $[\bar{\psi}\psi] = 3$, and since the gamma matrices are dimensionless we have $[\bar{\psi}] = [\psi] = 3/2$. This means that $[J_\mu] = [\bar{\nu}\mathrm{e}] = 3$, and so $[J_\mu^\dagger J^\mu] = 6$. We need to have $[G_\mathrm{F} J_\mu^\dagger J^\mu] = [G_\mathrm{F}] + 6 = 4$, so $[G_\mathrm{F}] = -2$. This means that the coupling is *not* dimensionless, in contrast to QED where e is dimensionless.

We therefore expect that the Fermi constant is of the form $1/\mathrm{mass}^2$ for some mass scale. It turns out that the mass scale is on the order of 300 GeV. This makes the interaction very weak, for comparison the proton and neutron have masses of less than 1 GeV.

As stated here this interaction conserves parity, as well as C and CP, for exactly the same reasons that QED does. But we know that this is wrong, having been disproven in 1956 by Reins and Cowan, and experimentally by Wu. In charged current weak interactions, that is weak interactions where the flavour and charge of the fermions changes, both parity and charge conjugation conservation are violated. Later on we'll see that even CP is not a good symmetry. For now though, we just look for a theory which allows parity violations.

23.2 $V-A$ Currents

In 1957 Robert Marshak and George Sudarshan formulated a theory, later popularised by Richard Feynman[1] and Murray Gell-Mann[2], which allowed for parity violations.

[1] Nobel prize for (electro)weak theory count: 3

[2] Nobel prize for (electro)weak theory count: 4

We want the interaction term to be a Lorentz invariant and to depend on the spinors for our fermions. To this end we consider Dirac bilinears. There are only five such types of object:

- scalars;
- pseudoscalars;
- vectors;
- axial vectors; and
- tensors.

We can just consider combinations of these until we find one that we works. Fortunately this has been done already and the solution is a combination of a vector current,

$$V_\mu = \bar{\nu}\gamma_\mu \mathrm{e} + \bar{\mathrm{p}}\gamma_\mu \mathrm{n} + \cdots , \tag{23.2.1}$$

and an axial current,

$$A_\mu = \bar{\nu}\gamma_\mu\gamma^5 \mathrm{e} + \bar{p}\gamma_\mu\gamma^5 \mathrm{n} + \cdots . \tag{23.2.2}$$

Combining these we get what is called a $V - A$ **current**,

$$\frac{1}{2}J_\mu = \frac{1}{2}(V_\mu - A_\mu) \tag{23.2.3}$$

$$= \bar{\nu}\gamma_\mu \frac{1}{2}(1-\gamma^5)\mathrm{e} + \bar{\mathrm{p}}\gamma_\mu \frac{1}{2}(1-\gamma^5)\mathrm{n} + \cdots \tag{23.2.4}$$

$$= \bar{\nu}\gamma_\mu P_\mathrm{L}\mathrm{e} + \bar{\mathrm{p}}\gamma_\mu P_\mathrm{L}\mathrm{n} + \cdots \tag{23.2.5}$$

$$= \bar{\nu}_\mathrm{L}\gamma_\mu \mathrm{e}_\mathrm{L} + \bar{\mathrm{p}}_\mathrm{L}\gamma_\mu \mathrm{n}_\mathrm{L} + \cdots . \tag{23.2.6}$$

We see that using this current the weak interaction involves only left handed fields. This means that not only is parity violated, but it is maximally violated.

Under a parity transformation

$$V^\mu \overset{P}{\mapsto} V_\mu, \qquad \text{and} \qquad A^\mu \overset{P}{\mapsto} -A_\mu \tag{23.2.7}$$

so

$$V^\mu - A^\mu \overset{P}{\mapsto} V_\mu + A_\mu. \tag{23.2.8}$$

Under charge conjugation

$$V^\mu \overset{C}{\mapsto} -V_\mu, \qquad \text{and} \qquad A^\mu \overset{C}{\mapsto} -A_\mu \tag{23.2.9}$$

so

$$V^\mu - A^\mu \overset{P}{\mapsto} -V_\mu - A_\mu. \tag{23.2.10}$$

This means that charge conjugation symmetry is also violated. Combining these under CP transformation

$$V^\mu \overset{CP}{\longmapsto} -V^\mu, \qquad \text{and} \qquad A^\mu \overset{CP}{\longmapsto} A^\mu \tag{23.2.11}$$

so

$$V^\mu - A^\mu \overset{CP}{\longmapsto} -V_\mu + A_\mu = -(V_\mu - A_\mu). \tag{23.2.12}$$

This means that

$$(V^\mu - A^\mu)^\dagger (V_\mu - A_\mu) \tag{23.2.13}$$

is invariant under CP, and hence under time reversal.

Since neutrinos only interact weakly (ignoring gravity) right handed neutrinos don't act at all, so it's as if they don't exist[3]. If the neutrino is massless then left and right handed neutrinos are independent so we don't need to consider right handed neutrinos.

[3] not ignoring gravity, right handed neutrinos are a candidate for dark matter

From first being theorised in 1930 all the way to 1998 it was assumed that neutrinos were massless. Evidence since then shows that this isn't the case. While the exact mass of the neutrinos is not known it is known that it is nonzero, but less than 0.120 eV, for comparison the electron, the next lightest massive particle, has a mass of about 511 keV, more than a million times larger.

We will assume that neutrinos are massless for two reasons

- their mass is negligible for kinematics;

- it makes things significantly easier.

This means that all neutrinos will be left handed, and all antineutrinos are right handed, which we will also assume and so don't bother with left and right labels, which is good as it makes space for the flavour labels like e on the electron neutrino ν_e, which would otherwise be ν_{eL}.

So, in the $V-A$ theory we have the four-Fermi interaction

$$\mathcal{L}_{4F} = -\frac{G_F}{\sqrt{2}} J^\dagger_\mu J^\mu. \tag{23.2.14}$$

Here the minus sign is a matter of convention, the factor of $\sqrt{2}$ is due to historical reasons and arises when we go from Fermi theory to $V-A$ theory, and the current splits as

$$J_\mu = J^l_\mu + J^h_\mu \tag{23.2.15}$$

where

$$\frac{1}{2} J^l_\mu = \bar{\nu}_e \gamma_\mu e_L + \bar{\nu}_\mu \gamma_\mu \mu_L + \bar{\nu}_\tau \gamma_\mu \tau_L \tag{23.2.16}$$

is the leptonic part of the current, now including all three generations, and

$$\frac{1}{2} J^h_\mu = \bar{u}_L \gamma_\mu d'_L + \bar{c}_L \gamma_\mu s'_L + \bar{t}_L \gamma_\mu b'_L \tag{23.2.17}$$

is the hadronic part of the current. Note that the down type quarks are given a prime here for reasons we will discuss later, for now just ignore the prime.

The **lepton number** is defined for electrons as

$$L_e = N_{e^-} - N_{e^+} + N_\nu - N_{\bar{\nu}_e} \tag{23.2.18}$$

where N_p is the number of particles p, and similarly for muons and tau. The lepton number is conserved[4].

The **baryon number** is defined as

$$B = \sum_{\text{generations}} (N_u - N_{\bar{u}} + N_d - N_{\bar{d}}). \tag{23.2.19}$$

This is conserved, but individual baryon numbers for each generation are not conserved.

There are lots of cross terms which all give different interactions. There is one simplifying feature, which is that there is only one coupling constant, G_F. This is called **universality** and is an important feature meaning that all interactions are the same strength regardless of the particles involved[5].

There are three types of charged current weak interaction, they are as follows.

- **leptonic** processes involving only muons, such as muon decay: $\mu^- \to e^- \nu_\mu \bar{\nu}_e$. These are typically the easiest processes to work with.
- **semi-leptonic** processes occur due to the cross terms and involve a lepton and a quark, for example, beta decay, $d \to u e^- \bar{\nu}_e$, or pion decay $d\bar{u} \to \mu^- \bar{\nu}_\mu$. These processes are harder to work with due to the strongly interacting particles.

[4] assuming that neutrinos are massless, if they aren't then much like quarks we have to put primes on all of the neutrinos and the reason for this leads to non-conservation of lepton number, in which case $L_e + L_\mu + L_\tau$ is still conserved

[5] kinematics and the primes placed on the down-type quarks make a difference though

- **non-leptonic** processes involving only quarks, such as kaon decay $s\bar{u} \to u\bar{d}u\bar{d}$. These are very hard to work with, and we won't do any calculations with them, since these calculations require us to consider lots of strong effects which cannot easily be separated from the from the weak effects.

All of these violate parity and charge conjugation symmetry, but leave CP invariant[6] Only left handed particles (and right handed antiparticles) are involved in any of these decays. Since the coupling is the same in all cases these interactions are all the same if we ignore QCD effects.

[6] except that non-leptonic interactions don't *quite* leave CP invariant, again this is to do with the primes.

Consider a decay of the positive pion, π^+, which has quark content $u\bar{d}$, into an antimuon and a muon neutrino,

$$\pi^+ \approx u\bar{d} \to \mu^+ \nu_\mu. \tag{23.2.20}$$

We use $\approx$ here to denote the quark content, rather than $=$, as the quark content is not quite the whole picture of what makes up a pion, but for our purposes right now it's enough. Working in the rest frame of the pion the antimuon and neutrino must go in opposite directions to conserve momentum. The neutrino must also be left handed, so its spin is antialigned with its momentum. We can depict this as

$$\nu_\mu \xleftarrow[s]{} \bullet \longrightarrow \mu^+ \ . \tag{23.2.21}$$

Now apply a parity transformation to this. The particles momenta will be reversed, so they go in the other direction. The particles spins will be unchanged. We can write the final state as

$$\mu^+ \longleftarrow \bullet \xrightarrow[s]{} \nu_\mu \ . \tag{23.2.22}$$

From this we can see that the neutrino spin and momentum are aligned, the neutrino is right handed. This is not allowed, so the parity inverted process cannot occur.

Instead consider charge conjugation. This doesn't change the momentum or spin, it just changes the antimuon into a muon and the neutrino into an antineutrino. We can depict this as

$$\bar{\nu}_\mu \xleftarrow[s]{} \bullet \longrightarrow \mu^- \ . \tag{23.2.23}$$

From this we can see that the antineutrino spin and momentum are antialigned, the antineutrino is left handed. This is not allowed, so the charge conjugated process cannot occur.

If we apply both charge conjugation and a parity transformation then the momenta reverse and the particles become antiparticles and vice versa, giving

$$\mu^- \longleftarrow \bullet \xrightarrow[s]{} \bar{\nu}_\mu \ . \tag{23.2.24}$$

From this we can see that the antineutrino spin and momentum are aligned, the antineutrino is right handed, which is allowed, so this the charge conjugated *and* parity transformed version of this process can occur. This is summarised in Figure 23.1.

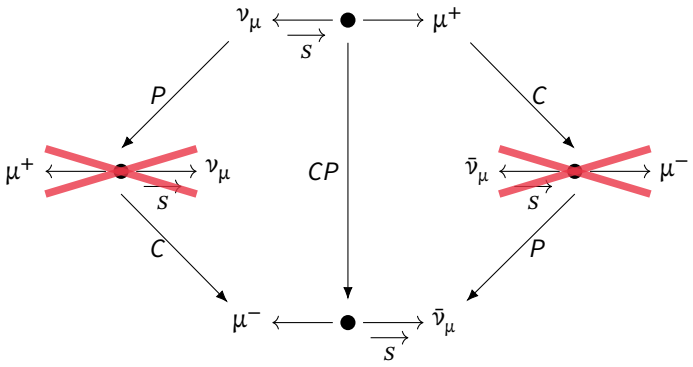


Figure 23.1: While charge conjugation and parity transformations don't yield valid process when applied to pion decay when combined they do.

Twenty-Four

Leptonic Decay

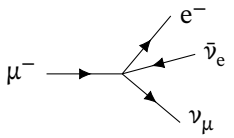
24.1 Setup

Consider the decay of a muon into an electron:

$$\mu^- \to e^- \bar{\nu}_e \mu^-. \tag{24.1.1}$$

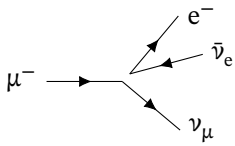
This is the first example of a decay we've seen, since decays can only occur through the weak interaction as they require flavours to change.

We can draw this decay as



. (24.1.2)

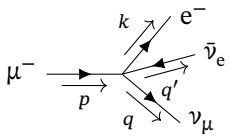
It can be helpful to separate out the two currents, instead drawing



. (24.1.3)

Note that we have a crossing symmetry. If we cross the muon neutrino over to become an antimuon neutrino this same calculation can be applied to the scattering $\mu^- \bar{\nu}_\mu \to e^- \bar{\nu}_e$.

Adding in labels for momenta we get



. (24.1.4)

The amplitude for this process at tree level is given by the matrix element of $i\mathcal{L}_{\text{int}} = i\mathcal{L}_{4F}$, which means that

$$\mathcal{M} = \langle e^-(k), \nu_e(q'), \nu_\mu(q) | \mathcal{L}_{4F} | \mu^-(p) \rangle. \tag{24.1.5}$$

To proceed we need the Feynman rule for the four fermion vertex. This can be found through the usual method, consider the vertex and compute the amputated correlator stripping off external states. Doing so one finds that the Feynman rule gives the four fermion vertex

$$-i\frac{G_F}{\sqrt{2}}[\gamma_\mu(1-\gamma_5)]_{ab}[\gamma^\mu(1-\gamma^5)]_{cd} \tag{24.1.6}$$

where a, b, c, and d are spinor indices. In particular, notice that these two spinor matrices are not multiplied together, and so amplitudes with the four fermion vertex factorise.

Applying this to our case

$$\mathcal{M} = -i\frac{G_F}{\sqrt{2}}[\bar{u}(k)\gamma^\mu(1-\gamma^5)v(q')][\bar{u}(q)\gamma_\mu(1-\gamma^5)u(p)] \tag{24.1.7}$$

where $\bar{u}(k)$, $v(q')$, $\bar{u}(q)$, and $u(p)$ are refer to the electron, antielectron neutrino, muon neutrino, and muon respectively.

We will follow an approach of using Dirac spinors for everything and enforcing left handed interactions through factors of $1-\gamma^5$. This allows us to use the same machinery we developed for QED. It is also possible to use two-component Weyl spinors to automatically enforce the left handed interactions, but then we need to develop new machinery.

24.1.1 Spin Averaged Amplitude

As with QED we now want to compute the spin averaged amplitude squared, so we want to compute $\mathcal{M}^\dagger\mathcal{M}$. To do this we need to compute $\overline{\gamma^\mu(1-\gamma^5)}$. This is the adjoint of a spinor matrix, not a spinor, and so it is slightly different. Given a matrix, A, we can define its Dirac adjoint to be $\bar{A} = \gamma^0 M^\dagger \gamma^0$, which is compatible with the definition of the Dirac adjoint of a spinor, since

$$\overline{A\psi} = (A\psi)^\dagger\gamma^0 = \psi^\dagger A^\dagger\gamma^0 = \psi^\dagger\gamma^0\gamma^0 A^\dagger\gamma^0 = \bar{\psi}\bar{A}. \tag{24.1.8}$$

Then we have

$$\bar{\gamma}^\mu = \gamma^0(\gamma^\mu)^\dagger\gamma^0. \tag{24.1.9}$$

If $\mu = 0$ then $(\gamma^0)^\dagger = \gamma^0$, and so we have $(\gamma^0)^3 = \gamma^0$ as $(\gamma^0)^2 = 1$. If $\mu = i$ then $(\gamma^i)^\dagger = -\gamma^i$, and so we have $-\gamma^0\gamma^i\gamma^0 = (\gamma^0)^2\gamma^i = \gamma^i$. Either way, $\bar{\gamma}^\mu = \gamma^\mu$. This definition respects the usual rule that the adjoint of a product is the reversed product of the adjoints. Thus

$$\overline{\gamma^\mu(1-\gamma^5)} = \overline{(1-\gamma^5)}\bar{\gamma}^\mu = \gamma^0(1-\gamma^5)^\dagger\gamma^0\gamma^\mu = (\gamma^0)^2(1+\gamma^5)\gamma^\mu = \gamma^\mu(1-\gamma^5).$$

Here we've used $(\gamma^5)^\dagger = \gamma^5$ and $\{\gamma^\mu,\gamma^5\} = 0$. The two spinor matrices in the vertex, and hence two independent terms of the form $\bar{u}(\gamma)u$ both give traces, just like in QED. Then terms like $u\bar{u}$ become $\not{p} + m$, for appropriate momenta and masses. We need to average over the initial spins, which there are two of since the muon is the only particle in the initial state. So we have

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{4}G_F^2\,\text{tr}[(\not{k}+m_e)\gamma^\mu(1-\gamma^5)\not{q}'\gamma^\nu(1-\gamma^5)] \times \text{tr}[\not{q}\gamma_\mu(1-\gamma^5)(\not{p}+m_\mu)\gamma_\nu(1-\gamma^5)]. \tag{24.1.10}$$

Note that we've already assumed that the neutrino is massless here since we haven't included its mass in the $\not{q}$ or $\not{q}'$ terms.

Now consider the first trace

$$\mathcal{M}_{\mu\nu}(q,p) = \mathrm{tr}[\not{q}\gamma_\mu(1-\gamma^5)(\not{p}+m_\mu)\gamma_\nu(1-\gamma^5)]. \tag{24.1.11}$$

Often the first useful thing to do is use the idempotency of the projectors to reduce the number of γ^5 matrices. Commuting the first $1-\gamma^5$ factor past both γ_μ and the gamma matrix implicit in $\not{q}$ changes the sign and changes it back, so we can bring this factor to the front

$$\mathcal{M}_{\mu\nu}(p,q) = \mathrm{tr}[(1-\gamma^5)\not{q}\gamma_\mu(\not{p}+m_\mu)\gamma_\nu(1-\gamma^5)]. \tag{24.1.12}$$

Now we can use the cyclic property of the trace and $[(1-\gamma^5)/2]^2 = (1-\gamma^5)/2$ to get

$$\mathcal{M}_{\mu\nu}(p,q) = \mathrm{tr}[\not{q}\gamma_\mu(\not{p}+m_\mu)\gamma_\nu(1-\gamma^5)^2] = 2\,\mathrm{tr}[\not{q}\gamma_\mu(\not{p}+m_\mu)\gamma_\nu(1-\gamma^5)]. \tag{24.1.13}$$

Expanding the brackets inside the trace we can throw away any term with an odd number of gamma matrices in the product, since these all vanish in the trace. Note that $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is an even number of gamma matrices, so a product of an odd number of gamma matrices and a γ^5 can also be thrown out. This gives

$$\mathcal{M}_{\mu\nu}(p,q) = 2\,\mathrm{tr}[\not{q}\gamma_\mu\not{p}\gamma_\nu(1-\gamma^5)], \tag{24.1.14}$$

since the term with the muon mass includes an odd number of gamma matrices.

Now we can use the identity

$$\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\rho\nu}) \tag{24.1.15}$$

to write the term with 1 as

$$\begin{aligned}\mathrm{tr}[\not{q}\gamma_\mu\not{p}\gamma_\nu] &= q^\rho p^\sigma\,\mathrm{tr}[\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\sigma] &(24.1.16)\\ &= q^\rho p^\sigma 4(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\sigma}\eta_{\rho\nu}) &(24.1.17)\\ &= 4(q_\mu p_\nu - \eta_{\mu\nu}(q\cdot p) + p_\mu q_\nu). &(24.1.18)\end{aligned}$$

We also have the identity

$$\mathrm{tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta] = 4i\varepsilon^{\mu\nu\alpha\beta}. \tag{24.1.19}$$

Using this

$$\mathrm{tr}[\not{q}\gamma_\mu\not{p}\gamma_\nu\gamma^5] = q^\alpha p^\beta\,\mathrm{tr}[\gamma^5\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu] = 4iq^\alpha p^\beta\varepsilon_{\alpha\mu\beta\nu} = -4iq^\alpha p^\beta\varepsilon_{\mu\nu\alpha\beta}. \tag{24.1.20}$$

Combining these we have

$$\mathcal{M}_{\mu\nu}(p,q) = 8[q_\mu p_\nu - \eta_{\mu\nu}(q\cdot p) + p_\mu q_\nu + i\varepsilon_{\mu\nu\alpha\beta}q^\alpha p^\beta]. \tag{24.1.21}$$

Since the mass term didn't contribute the other trace works exactly the same and so we have

$$\begin{aligned}\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = 16G_F^2(k^\mu q'^\nu + q'^\mu k^\nu - \eta^{\mu\nu}(k\cdot q') - i\varepsilon^{\mu\nu\alpha\beta}q'_\alpha k_\beta)\\ \times(q_\mu p_\nu + p_\mu q_\nu - \eta_{\mu\nu}(p\cdot q) + i\varepsilon_{\mu\nu\rho\sigma}q^\rho p^\sigma).\end{aligned} \tag{24.1.22}$$

If we expand these brackets then we get

$$(k\cdot q)(q'\cdot p)+(k\cdot p)(q'\cdot q)-(k\cdot q')(p\cdot q) \tag{24.1.23}$$
$$+(q'\cdot q)(k\cdot p)+(q'\cdot p)(k\cdot q)-(q'\cdot k)(p\cdot q) \tag{24.1.24}$$
$$-(q\cdot p)(k\cdot q')-(p\cdot q)(k\cdot q')+4(k\cdot q')(p\cdot q) \tag{24.1.25}$$
$$+i\varepsilon_{\mu\nu\rho\sigma}(k^\mu q'^\nu+q'^\mu k^\nu)i\varepsilon^{\mu\nu\alpha\beta}q'_\alpha k_\beta(q_\mu p_\nu+p_\mu q_\nu) \tag{24.1.26}$$
$$+\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\rho\sigma}q'_\alpha k_\beta q^\rho p^\sigma. \tag{24.1.27}$$

The first three lines of terms come from the first three terms in each bracket, the term with a 4 coming from $\eta^{\mu\nu}\eta_{\mu\nu}=4$. The single Levi-Civita terms come from the first two terms in one bracket and the Levi-Civita in the other. Products which would contain $\eta^{\mu\nu}\varepsilon_{\mu\nu\rho\sigma}$ or $\eta_{\mu\nu}\varepsilon^{\mu\nu\alpha\beta}$ vanish as the metric is symmetric and the Levi-Civita is antisymmetric. Now notice that $k^\mu q'^\nu+q'^\mu k^\nu$ is symmetric in μ and ν, so its product with $\varepsilon_{\mu\nu\rho\sigma}$ vanishes. The other single Levi-Civita term vanishes for the same reason. Thus, we are left with

$$2(k\cdot q)(q'\cdot p)+2(k\cdot p)(q'\cdot q)+\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\rho\sigma}q'_\alpha k_\beta q^\rho p^\sigma. \tag{24.1.28}$$

We can further simplify this with the identity

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\rho\sigma}=-2(\delta^\alpha_\rho\delta^\beta_\sigma-\delta^\alpha_\sigma\delta^\beta_\rho). \tag{24.1.29}$$

We have

$$\begin{aligned}\varepsilon_{\mu\nu\rho\sigma}q'_\alpha k_\beta q^\rho p^\sigma &= -2(\delta^\alpha_\rho\delta^\beta_\sigma-\delta^\alpha_\sigma\delta^\beta_\rho)q'_\alpha k_\beta q^\rho p^\sigma &(24.1.30)\\ &= -2(q'_\rho k_\sigma q^\rho p^\sigma-q'_\sigma k_\rho q^\rho p^\sigma) &(24.1.31)\\ &= -2((k\cdot p)(q'\cdot q)-(k\cdot q)(q'\cdot p)). &(24.1.32)\end{aligned}$$

Then we are left with just

$$4(k\cdot q)(q'\cdot p). \tag{24.1.33}$$

Giving

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2=64G_F^2(k\cdot q)(p\cdot q'). \tag{24.1.34}$$

24.1.2 Decay Rate

No one knows how to do phase space integrals, ..., you just keep trying until it works.

Richard Ball

The differential decay rate in the lab frame of the muon is given by

$$\mathrm{d}\Gamma=\frac{1}{2m_\mu}\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2(\mathrm{dPS})_3. \tag{24.1.35}$$

Computing the phase space measure, $(\mathrm{dPS})_3$, is no easy task. Expanding the definition of the phase space measure we are looking to compute

$$\mathrm{d}\Gamma = \frac{1}{2m_\mu}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2k^0}\frac{\mathrm{d}^3q}{(2\pi)^3 2q^0}\frac{\mathrm{d}^3q'}{(2\pi)^3 2q'^0} \tag{24.1.36}$$

$$\times (2\pi)^4\delta(p-k-q-q')\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 \tag{24.1.37}$$

$$= \frac{1}{8m_\mu\pi^5}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2k^0}\frac{\mathrm{d}^3q}{(2\pi)^3 2q^0}\frac{\mathrm{d}^3q'}{(2\pi)^3 2q'^0} \tag{24.1.38}$$

$$(2\pi)^4\delta(p-k-q-q')(k\cdot q)(p\cdot q'). \tag{24.1.39}$$

The first thing we have to do is decide which variable to integrate out first. There is an argument, of questionable validity as it doesn't hold for beta decay, that we should start with the momenta of the neutrinos, since we don't usually see the neutrinos in the final state, we just deduce their existence from kinematics. This happens to work in this case, so we'll start by integrating out the neutrino momenta q and q'.

Set $Q = q + q' = p - k$. Then we have

$$Q^2 = q^2 + q'^2 + 2q\cdot q' \approx 2q\cdot q' \tag{24.1.40}$$

since the neutrinos are on shell so $q^2 = q'^2 = m_\nu^2 \approx 0$. We can write this as

$$Q^2 = 2(q^0 q'^0 - \boldsymbol{q}\cdot\boldsymbol{q}') \geq 0 \tag{24.1.41}$$

since $q^0 = |\boldsymbol{q}|$ and $q'^0 = |\boldsymbol{q}'|$ for massless (or essentially massless) particles and the dot product comes with a factor of $\cos\vartheta$ and $|\cos\vartheta| \leq 1$.

Pulling out just the q and q' integrals we want to compute

$$I_{\mu\nu}(Q) = \int \frac{\mathrm{d}^3q}{q^0}\frac{\mathrm{d}^3q'}{q'^0}\delta(Q-q-q')q_\mu q'_\nu. \tag{24.1.42}$$

This isn't straightforward on its own. Fortunately, we know that the decay rate is Lorentz invariant, so this must be a Lorentz tensor, since the decay rate is proportional to

$$\int \frac{\mathrm{d}^3k}{k^0} I_{\mu\nu}p^\mu k^\nu \tag{24.1.43}$$

which must be a Lorentz scalar. Using this requirement we notice that we can write

$$I_{\mu\nu}(Q) = aQ_\mu Q_\nu + b\eta_{\mu\nu}Q^2 \tag{24.1.44}$$

since this is the only tensor we can form with the given data.

All that is left is to find the values of a and b. To do this we can contract this result with $\eta^{\mu\nu}$, giving

$$I_{\mu\nu}(Q)\eta^{\mu\nu} = \int \frac{\mathrm{d}^3q}{q^0}\frac{\mathrm{d}^3q'}{q'^0}\delta(Q-q-q')\underbrace{(q\cdot q')}_{Q^2/2} = aQ^2 + 4bQ^2, \tag{24.1.45}$$

so

$$a + 4b = \frac{1}{2}I \tag{24.1.46}$$

where

$$I = \int \frac{\mathrm{d}^3q}{q^0}\frac{\mathrm{d}^3q'}{q'^0}\delta(Q - q - q'). \tag{24.1.47}$$

Contracting with $Q^\mu Q^\nu$ gives

$$I_{\mu\nu}(Q)Q^\mu Q^\nu = \int \frac{\mathrm{d}^3q}{q^0}\frac{\mathrm{d}^3q'}{q'^0}\delta(Q - q - q')q_\mu q'_\nu Q^\mu Q^\nu = aQ^4 + bQ^4, \tag{24.1.48}$$

so

$$a + b = \frac{1}{4}I \tag{24.1.49}$$

having used

$$\begin{aligned} q_\mu q'_\nu Q^\mu Q^\mu &= q_\mu q'_\nu (q^\mu + q'^\mu)(q^\nu + q'^\nu) && (24.1.50)\\ &= (q^2 q'_\nu + (q\cdot q')q'_\nu)(q^\nu + q'^\nu) && (24.1.51)\\ &= \underbrace{q^2}_{=0}(q'\cdot q) + \underbrace{q^2 q'^2}_{=0} + \underbrace{(q\cdot q')^2}_{=Q^4/4} + (q\cdot q')\underbrace{q'^2}_{=0}. && (24.1.52)\\ & && (24.1.53) \end{aligned}$$

To compute I, which is a Lorentz scalar, we can simply pick a frame. An easy one to do the computation in is the frame in which $Q^\mu = (Q^0, \mathbf{0})$, so $\boldsymbol{q} = -\boldsymbol{q}'$ and $q^0 = q'^0 = |\boldsymbol{q}| = |\boldsymbol{q}'|$. We then have the fairly easy integral

$$I = \int \frac{\mathrm{d}^3q}{|\boldsymbol{q}|^2}\delta(Q - 2|\boldsymbol{q}|) = 4\pi\int_0^\infty \mathrm{d}q\,\delta(Q - 2q) \tag{24.1.54}$$

having transformed to spherical polar coordinates and performed the angular integrals. Note that the $|\boldsymbol{q}|^2$ in the denominator cancels with the q^2 in the Jacobian. To move on we need the following identity for Dirac deltas, which holds under an integral,

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|} \tag{24.1.55}$$

where x_0 are the zeros of f. In our case we have $f(q) = Q - 2q$, which has a single zero at $q = Q/2$, and $f'(q) = -2$, giving

$$I = 4\pi\int_0^\infty \mathrm{d}q\,\frac{1}{2}\delta(q - Q/2) = 2\pi. \tag{24.1.56}$$

Solving the simultaneous equations then gives us

$$a = \frac{\pi}{3}, \qquad \text{and} \qquad b = \frac{\pi}{6}. \tag{24.1.57}$$

Using this we kahve

$$p^\mu k^\nu I_{\mu\nu}(Q) = \frac{\pi}{6}(2(Q\cdot p)(Q\cdot k) + (p\cdot k)Q^2). \tag{24.1.58}$$

We can simplify this by substituting in $Q = p - k$, giving

$$\begin{aligned} p^\mu k^\nu I_{\mu\nu}(Q) &= \frac{\pi}{6}(2(p^2 - p\cdot k)(p\cdot k - k^2) + (p\cdot k)(p^2 + k^2 - 2p\cdot k)) \\ &= \frac{\pi}{6}(2(p^2(p\cdot k) - p^2k^2 - (p\cdot k)^2 + k^2(p\cdot k)) && (24.1.59) \\ &\qquad + p^2(p\cdot k) + k^2(p\cdot k) - 2(p\cdot k)^2) && (24.1.60) \\ &= \frac{\pi}{6}(-2p^2k^2 + 3(p^2 + k^2)(p\cdot k) - 4(p\cdot k)^2) && (24.1.61) \\ &= \frac{\pi}{6}(-2m_\mu^2 m_e^2 + 3(m_\mu^2 + m_e^2)(p\cdot k) - 4(p\cdot k)^2). && (24.1.62) \end{aligned}$$

In the rest frame of the muon $p^\mu = (m_\mu, \mathbf{0})$ and $k^\mu = (E, \boldsymbol{k})$ with $E = k^0$ and $E^2 = m_e + \boldsymbol{k}^2$. Then $p\cdot k = m_\mu E$. We also have $E\,\mathrm{d}E = |\boldsymbol{k}|\,\mathrm{d}|\boldsymbol{k}|$. We can transform to spherical coordinates for the k integral, and perform the angular integrals. This allows us to substitute

$$\frac{\mathrm{d}^3k}{k^0} \to 4\pi\frac{|\boldsymbol{k}^2|\,\mathrm{d}|\boldsymbol{k}|}{E} = 4\pi|\boldsymbol{k}|\,\mathrm{d}E. \tag{24.1.63}$$

Since we're looking for the differential decay rate it's fine to stop here with the integrals, and just give the differential cross section as a differential in $\mathrm{d}E$.

The electron is much lighter than the muon, so we have $m_e \ll E$, meaning that $|\boldsymbol{k}| \approx E$. Note that this approximation is not valid for beta decay as up and down quarks don't have this large difference in masses. After neglecting the m_e terms we finally have the differential decay rate

$$\mathrm{d}\Gamma = \frac{2G_F^2}{3(2\pi)^3}m_\mu E^2(2m_\mu - 4E)\,\mathrm{d}E. \tag{24.1.64}$$

Since $Q^2 \geq 0$ we must have $m_\mu^2 - 2p\cdot k = m_\mu(m_\mu - 2E) \geq 0$, so $E \leq m_\mu/2$. This means that the neutrinos have to carry away at least half the energy in order to conserve momentum. The differential cross section rises from 0 at $E = 0$ to a maximum at $E = m_\mu/2$. Above this it drops off again but this region is not physical as we need $E \leq m_\mu/2$.

The interpretation for why the differential cross section is smaller at lower energies comes from considering the spins of the neutrinos. If E is small then the neutrinos must come out almost back to back to conserve momentum. Both neutrinos when they're produced must be left handed. This means that there spins are antialigned with their momenta, and hence their spins are antialigned with each other. This makes it "hard" to conserve angular momentum so decays like this are suppressed.

Computing the total cross section we then have

$$\Gamma = \frac{G_F^2}{12\pi^3}m_\mu\int_0^{m_\mu/2}\mathrm{d}E\,(3m_\mu E^2 - 4E^3) = \frac{G_F^2 m_\mu^5}{192\pi^3}. \tag{24.1.65}$$

The first thing to notice is that $\Gamma \propto m_\mu^5$, so the decay rate is very sensitive to the mass.

We can use this equation to determine the value of G_F. It is known that the lifetime of a muon is $\tau_\mu = 2.2\,\mu s$. This gives $\Gamma_\mu = \hbar/\tau_\mu = 3.0 \times 10^{-19}$ GeV. So the value of Fermi's constant is $G_F \approx 1.2 \times 10^{-5}\,\text{GeV}^{-2}$. Note that we are neglecting other decay modes of the muon here, which are possible, for example we can add in extra neutrinos, but these are much less likely.

The exact same calculation works for the tau decays $\tau^- \to e^- \bar{\nu}_e \nu_\tau$ and $\tau^- \to \mu^- \bar{\nu}_\mu \nu_\tau$. However, there are multiple non-negligible decay modes for tau, and it turns out that both of these occur about 18 % of the time so the branching ratio is $B = 0.18$, we then expect that

$$\Gamma(\tau^- \to e^- \bar{\nu}_e \nu_\tau) = \Gamma(\tau^- \to \mu^- \bar{\nu}_\mu \nu_\tau) = 0.18 = B\Gamma_{\tau^-} = \frac{m_\tau^5}{m_\mu^5}\Gamma_{\mu^-} \tag{24.1.66}$$

where Γ_τ is the total decay width for the tau. Measurements of this ratio give

$$\frac{m_\tau^5}{m_\mu^5}\Gamma_{\mu^-} \approx 4 \times 10^{-13}\,\text{GeV} \tag{24.1.67}$$

and measurements of the masses give

$$\frac{m_\tau^5}{m_\mu^5} \approx 1.4 \times 10^6. \tag{24.1.68}$$

Combining these gives

$$\tau_\tau = \frac{\hbar}{\Gamma_\tau} = \frac{Bm_\tau^5\hbar}{m_\mu^5\Gamma_\mu} = 2.8 \times 10^{-13}\,\text{s} \tag{24.1.69}$$

which compare well to the measured tau lifetime of $\tau_\tau = 2.9 \times 10^{-13}$ s. This is evidence for the universality of the weak interaction, it wouldn't work if the coupling was different for different particles, since then we couldn't cancel out G_F like we have here.

Twenty-Five

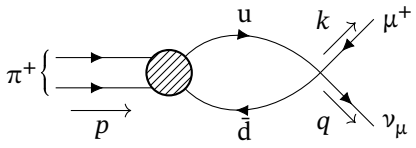
Semileptonic Decays

25.1 Pion Decay

Consider the example of a charged pion decay, such as

$$\pi^+ \to \mu^+ \nu_\mu, \qquad \text{or} \qquad \pi^- \to \mu^- \bar{\nu}_\mu \tag{25.1.1}$$

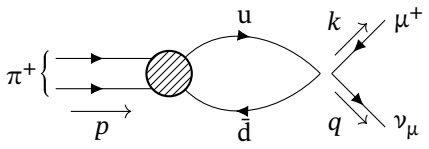
The case of the neutral pion is slightly different as the dominant decay mode is $\pi^0 \to 2\gamma$, which will require us to discuss combined electromagnetic and weak interactions, which we aren't ready for yet. For simplicity we'll just consider the π^+ case. The quark content of the π^+ is $u\bar{d}$. We can draw this decay as



. (25.1.2)

Here the blob represents some strong interactions which we won't deal with, resulting in a quark content of $u\bar{d}$ which then interacts through the weak interaction.

As with the last example we can separate the two currents and draw this as



. (25.1.3)

The amplitude then factors into two parts, one for each current:

$$\mathcal{M} = \langle \mu^+(k)\nu_\mu(q)|\mathcal{L}_{4F}|\pi^+(p)\rangle \tag{25.1.4}$$

$$= -i\frac{G_F}{\sqrt{2}}\bar{u}(q)\gamma^\mu(1-\gamma^5)v(k)\langle 0|\bar{d}\gamma_\mu(1-\gamma^5)u|\pi^+(p)\rangle. \tag{25.1.5}$$

Here $\bar{u}(q)$ corresponds to the neutrino and $v(k)$ to the antimuon. We have then accounted for all of the particles in the final state, so we're left with just the vacuum, $|0\rangle$. We leave the strong part of the interaction as a $V-A$ current with the up quark, antidown quark, and pion.

We can use the fact that the pion is a pseudoscalar, so has negative parity, to impose that $\langle 0|V_\mu\pi^+ = 0$ for a vector V_μ, since this product is an axial vector and

we need a vector quantity for $\mathcal{M}$ to be a scalar. We are then left with $-\langle 0|A_\mu|\pi^+\rangle = if_\pi p_\mu$, where f_π is some constant to be determined from experiments or difficult non-perturbative QCD calculations, and the p_μ dependence is because this result must be a vector, and this is the only vector it could possibly be. Thus the amplitude is

$$\mathcal{M} = \frac{G_F f_\pi}{\sqrt{2}} \bar{u}(q) \not{p} (1-\gamma^5) v(k). \tag{25.1.6}$$

Now we can use momentum conservation to impose that $p = k + q$, and so we have

$$\bar{u}(q)(\not{q}+\not{k})(1-\gamma^5)v(k) = \bar{u}(q)\not{q}(1-\gamma^5)v(k) + \bar{u}(q)(1+\gamma^5)\not{k}v(k) = -m_\mu \bar{u}(q)(1+\gamma^5)v(k) \tag{25.1.7}$$

having used the fact that u and v are solutions to the Dirac equation with positive and negative energy respectively, meaning that $\not{q}u(q) = 0$, for massless neutrinos, and $(\not{k} + m_\mu)v(k) = 0$.

We want to evaluate the spin-averaged matrix element. Since the pion has spin zero there are no initial spins to average over, so we just sum over final spins and we get

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{G_F^2 f_\pi^2}{2} m_\mu^2 \,\text{tr}[\not{q}(1+\gamma^5)\not{k}(1-\gamma^5)] \tag{25.1.8}$$

where we've used the orthogonality of the projectors, $(1+\gamma^5)(1-\gamma^5) = 0$, and done the usual manipulations of spin indices to get a trace. Commuting the $\not{k}$ through the $1-\gamma^5$ we get $(1+\gamma^5)\not{k}$, and then we can use $(1+\gamma^5)^2 = 2(1+\gamma^5)$ to simplify this result:

$$\sum_{\text{spins}} |\mathcal{M}|^2 = G_F^2 f_\pi^2 m_\mu^2 \,\text{tr}[\not{q}(1+\gamma^5)\not{k}]. \tag{25.1.9}$$

We can then use the identities

$$\text{tr}[\not{q}\not{k}] = 4q\cdot k, \qquad \text{and} \qquad \text{tr}[\not{q}\gamma^5\not{k}] = 0 \tag{25.1.10}$$

to get

$$\sum_{\text{spins}} |\mathcal{M}|^2 = 4G_F^2 f_\pi^2 m_\mu^2 (q\cdot k). \tag{25.1.11}$$

Consider $p^2 = m_\pi^2$ and $p^2 = (q+k)^2 = q^2 + k^2 + 2q\cdot k \approx m_\mu^2 + 2q\cdot k$ having used $q^2 = m_\nu^2 \approx 0$. We therefore have $k\cdot q = (m_\pi^2 - m_\mu^2)/2$, giving

$$\sum_{\text{spins}} |\mathcal{M}|^2 = 2G_F^2 f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu). \tag{25.1.12}$$

The differential decay rate in the lab frame is

$$d\Gamma = \frac{1}{2m_\pi} \sum_{\text{spins}} |\mathcal{M}|^2 (\text{dPS})_2. \tag{25.1.13}$$

In *Quantum Field Theory* we computed the phase space measure for two particles to be

$$(\mathrm{dPS})_2 = \frac{1}{(4\pi)^2}\frac{|\boldsymbol{k}|}{m_\pi}\int \mathrm{d}\Omega = \frac{1}{4\pi}|\boldsymbol{k}|. \tag{25.1.14}$$

Working in the rest frame of the pion we have $p^\mu = (m_\pi, \mathbf{0})$, $k^\mu = (E_\mu, \boldsymbol{k})$, and $q^\mu = (E_\nu, \boldsymbol{q})$. Using $q = p - k$ we get $q^2 = 0 = p^2 + k^2 - 2p\cdot k = m_\pi^2 + m_\mu^2 - 2m_\pi E_\mu$. Setting $E_\mu = \sqrt{m_\mu^2 + |\boldsymbol{k}|^2}$, and enforcing $|\boldsymbol{k}| > 0$ we get

$$|\boldsymbol{k}| = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}. \tag{25.1.15}$$

Thus we have

$$\Gamma = \frac{1}{8\pi}G_\mathrm{F}^2 f_\pi^2 \frac{m_\mu^2}{m_\pi^3}(m_\pi^2 - m_\mu^2)^2. \tag{25.1.16}$$

Note that $\mathrm{d}\Gamma$ is differential in $\mathrm{d}\Omega$ here, and we've already performed this integral since there is no angular dependence.

The measured pion lifetime, $\tau_\pi = 2.6 \times 10^{-8}\,\mathrm{s}$, can be used to calculate $f_\pi \approx 130\,\mathrm{MeV}$.

We can use the same calculation to compute the decay rate for $\pi^+ \to \mathrm{e}^+\nu_\mathrm{e}$. Universality tells us that the result should be the same, apart from masses, so if we take the ratio most of the constants should cancel and we should find that

$$\frac{\Gamma(\pi^+ \to \mathrm{e}^+\nu_\mathrm{e})}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} = \frac{m_\mathrm{e}^2(m_\pi^2 - m_\mathrm{e}^2)^2}{m_\mu^2(m_\pi^2 - m_\mu^2)^2} \approx 10^{-4} \tag{25.1.17}$$

and this is the same order of magnitude as measurements.

Notice that as $m_\mathrm{e} \to 0$ this result vanishes. This is due to helicity conservation. A neutrino must be left handed, and in the rest frame of the pion the neutrino and positron emerge back to back. Since the pion has no spin the positron and neutrino must have opposite spins, and so the positron must also be left handed. If the positron was massless then this would be impossible, just as it is for antineutrinos. As it is the decay must produce a right handed positron, since this is a weak interaction, so the positron must almost immediately switch to being left handed, which is only possible if it is massive.

25.2 Kaon Decay

By 1963 kaons, which contain strange quarks, had been discovered. The quark content of a positive kaon is, approximately, $\mathrm{u\bar{s}}$. We can consider the decay $\mathrm{K}^+ \to \mu^+\mu^-$. The weak part of this interaction is completely the same as the decay of the π^+, since the quantum numbers of $\bar{\mathrm{s}}$ and $\bar{\mathrm{d}}$ relevant for the weak interaction are all the same. The strong stuff does change, so we just replace f_π with f_K.

Experiments show that

$$\frac{\Gamma(\mathrm{K}^+ \to \mu^+\nu_\mu)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} \approx 1.3, \tag{25.2.1}$$

so the kaon and pion aren't that different.

The problem is that this interaction is not possible with the four fermion interaction we have been working with, since there is no way to get a current of up quarks and strange quarks, there is no $\bar{u}_L \gamma_\mu s_L$ term. Suppose that there was. Then we would expect that

$$\frac{\Gamma(K^+ \to \mu^+ \nu_\mu)}{\Gamma(\pi^+ \to \mu^+ \nu_\mu)} = \frac{f_K^2}{F_\pi^2} \frac{m_\pi^3 (m_K^2 - m_\mu^2)^2}{m_K^3 (m_\pi^2 - m_\mu^2)^2} \approx 24 \tag{25.2.2}$$

and experimental measurements put $f_K/f_\pi \approx 1.2$, so this is off by an order of magnitude.

The solution to this problem is due to Nicola Cabbibo, and finally explains the primes in the current. He posited that the correct hadronic current was instead

$$\frac{1}{2} J_\mu^h = \bar{u}_L \gamma_\mu (d_L \cos\vartheta_c + s_L \sin\vartheta_c) + \cdots . \tag{25.2.3}$$

We allowed for this before by including primes, so we have

$$d'_L = \cos\vartheta_c d_L + \sin\vartheta_c s_L. \tag{25.2.4}$$

The interpretation is that the down quark which interacts through weak interactions is a mixture of down and strange quarks interacting through other means, this is called **quark mixing**. The exact mix is controlled by a dimensionless parameter, ϑ_c, called the **Cabibbo angle**, which is $\vartheta_c \approx 13°$, meaning $\cos\vartheta_c = 0.97$ and $\sin\vartheta_c = 0.22$. Since $\cos\vartheta_c$ is so close to one this tells us that d'_L is mostly formed from d_L and only a small amount of s_L.

Including this we see that

$$\Gamma(\pi^+ \to \mu^+ \nu_\mu) = \frac{G_F^2 f_{\pi^2}}{4\pi} \frac{m_\mu^2}{m_\pi^3} (m_\pi^2 - m_\mu)^2 \cos^2\vartheta_c, \tag{25.2.5}$$

which is 0.94 times our previous result, so the effects of quark mixing is pretty negligible here. On the other hand

$$\Gamma(\pi^+ \to \mu^+ \nu_\mu) = \frac{G_F^2 f_{K^2}}{4\pi} \frac{m_\mu^2}{m_K^3} (m_K^2 - m_\mu)^2 \sin^2\vartheta_c, \tag{25.2.6}$$

which is $0.045 \approx 1/20$ times the previous result, which brings the theoretical result inline with the experimental result.

Note that in 1963 only two generations of quarks were known, this has since been extended to include all three generations. We'll come back to this later. If we have massive neutrinos we also get a similar effect.

25.3 Beta Decay

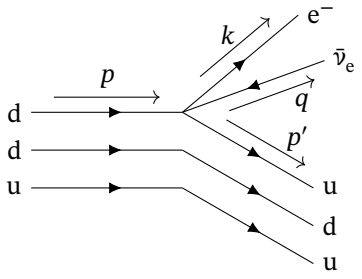
Consider the example of beta decay,

$$n \to p e^- \bar{\nu}_e, \tag{25.3.1}$$

or on the quark level

$$d \to u e^- \bar{\nu}_e, \tag{25.3.2}$$

with the two other quarks making up the neutron as spectators, not contributing to the weak interaction, although they are important for the strong interactions occurring. We can draw this as



. (25.3.3)

As with pion decay we can factorise the matrix element into a weak interaction and some strong interaction, giving

$$\mathcal{M} = \langle \mathrm{p}(p')\mathrm{e}^-(k)\bar{\nu}_\mathrm{e}(q)|\mathcal{L}_{4\mathrm{F}}|\mathrm{n}(p)\rangle \tag{25.3.4}$$

$$= -i\frac{G_\mathrm{F}}{\sqrt{2}}\bar{u}(k)\gamma^\mu(1-\gamma^5)\upsilon(q)\langle \mathrm{p}(p')|\bar{\mathrm{d}}\gamma_\mu(1-\gamma^5)\mathrm{u}|\mathrm{n}(p)\rangle\cos\vartheta_\mathrm{c}. \tag{25.3.5}$$

Here $\bar{u}(k)$ and $\upsilon(p)$ represent the electron and neutrino.

Now consider the hadronic part, which is a $V-A$ current, so takes the form

$$\langle \mathrm{p}(p')|(V_\mu - A_\mu)|\mathrm{n}(p)\rangle = \cos(\vartheta_\mathrm{c})\bar{u}(p')\gamma_\mu(g_V - g_A\gamma^5)u(p) \tag{25.3.6}$$

for some constants g_V and g_A. Here $\bar{u}(p')$ and $u(p)$ represent the proton and neutron respectively. Conservation of the vector current implies that $g_V = 1$, whereas the axial current is only partially conserved and experimental measurements give $g_A \approx 1.25$.

We can then compute the spin-averaged matrix element squared,

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{4}G_\mathrm{F}^2\cos^2(\vartheta_\mathrm{c})\,\mathrm{tr}[(\not{k}+m_\mathrm{e})\gamma^\mu(1-\gamma^5)\not{q}\gamma^\mu(1-\gamma^5)] \tag{25.3.7}$$

$$\times\,\mathrm{tr}[(\not{p}'+m_\mathrm{p})\gamma_\mu(1-g_A\gamma^5)(\not{p}+m_\mathrm{n})\gamma_\nu(1-g_A\gamma^5)]$$

Consider the second trace:

$$\mathcal{M}_{\mu\nu} = \mathrm{tr}[(\not{p}'+m_\mathrm{p})\gamma_\mu(1-g_A\gamma^5)(\not{p}+m_\mathrm{n})\gamma_\nu(1-g_A\gamma^5)] \tag{25.3.8}$$

$$= \mathrm{tr}[(\not{p}'+m_\mathrm{p})\gamma_\mu(1-g_A\gamma^5)\not{p}\gamma_\nu(1-g_A\gamma^5)] \tag{25.3.9}$$

$$+ m_\mathrm{n}\,\mathrm{tr}[(\not{p}'+m_\mathrm{p})\gamma_\mu(1-g_A\gamma^5)\gamma_\nu(1-g_A\gamma^5)] \tag{25.3.10}$$

$$= \mathrm{tr}[(\not{p}'+m_\mathrm{p})\gamma_\mu\not{p}\gamma_\nu(1-g_A\gamma^5)^2] \tag{25.3.11}$$

$$m_\mathrm{n}\,\mathrm{tr}[(\not{p}'+m_\mathrm{p})\gamma_\mu\gamma_\nu(1+g_A\gamma^5)(1-g_A\gamma^5)] \tag{25.3.12}$$

having anticommuted $1-g_A\gamma^5$ through $\not{p}\gamma_\nu$ and γ_ν, changing the sign as appropriate. We then can use

$$(1-g_A\gamma^5)^2 = 1+g_A^2(\gamma^5)^2+2g_A\gamma^5 = 1+g_A^2+2g_A\gamma^5, \tag{25.3.13}$$

$$(1+g_A\gamma^5)(1-g_A\gamma^5) = 1-g_A^2(\gamma^5)^2 = 1-g_A^2. \tag{25.3.14}$$

So we get

$$\mathcal{M}_{\mu\nu} = \text{tr}[(\not{p}' + m_\text{p})\gamma_\mu \not{p}\gamma_\nu(1 + g_A^2 + 2g_A\gamma^5)] \tag{25.3.15}$$
$$+ m_\text{n}(1 - g_A^2)\,\text{tr}[(\not{p}' + m_\text{p})\gamma_\mu\gamma_\nu]. \tag{25.3.16}$$

We can now expand the arguments of the trace, and we need only keep terms with an even number of gamma matrices, remembering that γ^5 is formed from a product of four gamma matrices. We then have

$$\mathcal{M}_{\mu\nu} = 2g_A\,\text{tr}[\not{p}'\gamma_\mu\not{p}\gamma_\nu\gamma^5] + (1 + g_A^2)\,\text{tr}[\not{p}'\gamma_\mu\not{p}\gamma_\nu] + m_\text{n}m_\text{p}(1 - g_A^2)\,\text{tr}[\gamma_\mu\gamma_\nu].$$

Now we can apply the identities

$$\text{tr}(\gamma_\mu\gamma_\nu) = 4\eta_{\mu\nu}, \tag{25.3.17}$$
$$\text{tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 4(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}), \tag{25.3.18}$$
$$\text{tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma^5) = 4i\varepsilon_{\mu\nu\rho\sigma}. \tag{25.3.19}$$

The first trace then gives us

$$\text{tr}[\not{p}'\gamma_\mu\not{p}\gamma_\nu\gamma^5] = p'^\rho p^\sigma\,\text{tr}[\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu\gamma^5] \tag{25.3.20}$$
$$= p'^\rho p^\sigma 4i\varepsilon_{\rho\mu\sigma\nu}. \tag{25.3.21}$$

The second trace gives

$$\text{tr}[\not{p}'\gamma_\mu\not{p}\gamma_\nu] = p'^\rho p^\sigma\,\text{tr}[\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu] \tag{25.3.22}$$
$$= p'^\rho p^\sigma 4(\eta_{\rho\mu}\eta_{\sigma\nu} - \eta_{\rho\sigma}\eta_{\mu\nu} + \eta_{\rho\nu}\eta_{\mu\sigma}) \tag{25.3.23}$$
$$= 4(p'_\mu p_\nu - (p'\cdot p)\eta_{\mu\nu} + p'_\nu p_\mu). \tag{25.3.24}$$

Hence,

$$\mathcal{M}_{\mu\nu} = 8ig_A\varepsilon_{\rho\mu\sigma\nu}p'^\rho p^\sigma + 4(1 + g_A^2)(p'_\mu p_\nu - (p'\cdot p)\eta_{\mu\nu} + p'_\nu p_\mu)$$
$$+ 4m_\text{n}m_\text{p}(1 - g_A^2)\eta_{\mu\nu}. \tag{25.3.25}$$

The other trace is almost identical, just replace p' with k, p with q, m_p with m_e, and m_n with $m_\nu = 0$, and swap which indices are raised and lowered. The result is

$$\text{tr}[(\not{k} + m_{\text{e}^-})\gamma^\mu(1 - \gamma^5)\not{q}\gamma^\mu(1 - \gamma^5)]$$
$$= 8i\varepsilon^{\rho\mu\sigma\nu}k_\rho p_\sigma + 4(k^\mu q^\nu - (k\cdot q)\eta^{\mu\nu} + k^\nu q^\mu). \tag{25.3.26}$$

The product required for the spin-averaged amplitude is quite large so we'll break it down. First,

$$(k^\mu q^\nu - (k\cdot q)\eta^{\mu\nu} + k^\nu q^\mu)(p'_\mu p_\nu - (p'\cdot p)\eta_{\mu\nu} + p'_\nu p_\mu)$$
$$= (k\cdot p')(q\cdot p) - (k\cdot q)(p'\cdot p) + (k\cdot p)(q\cdot p') \tag{25.3.27}$$
$$- (k\cdot q)(p\cdot p' - 4p\cdot p' + p\cdot p') - 2p\cdot p'$$
$$+ (k\cdot p)(1\cdot p') - (k\cdot q)(p'\cdot p) + (k\cdot p')(q\cdot p)$$
$$= 2[(k\cdot p')(q\cdot p) + (k\cdot p)(q\cdot p')]. \tag{25.3.28}$$

Second, the terms with a single Levi-Civita vanish by symmetry, for example taking the Levi-Civita from the first trace the single Levi-Civita term we get is

$$\varepsilon^{\alpha\mu\beta\nu}k_\alpha q_\beta(p'_\mu p_\nu - (p'\cdot p)\eta_{\mu\nu} + p'_\nu p_\mu) = 0 \tag{25.3.29}$$

as the term in brackets is symmetric in μ and ν, while $\varepsilon^{\alpha\mu\beta\nu}$ is antisymmetric in μ and ν. The double Levi-Civita terms can be simplified using the identity

$$\varepsilon^{\alpha\mu\beta\nu}\varepsilon_{\rho\mu\sigma\nu} = \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\rho\sigma} = -2(\delta^\mu_\rho\delta^\beta_\sigma - \delta^\alpha_\rho\delta^\beta_\rho). \tag{25.3.30}$$

We then get

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = 16G_\text{F}^2\cos^2(\vartheta_\text{c})[(1-g_A^2)(p\cdot k)(p'\cdot q) + (1+g_A^2)(p\cdot q)(p'\cdot k) - (1-g_A^2)m_\text{n}m_\text{p}(k\cdot q)]. \tag{25.3.31}$$

The decay rate is then given by

$$\mathrm{d}\Gamma = \frac{1}{2m_\text{n}}\int\frac{\mathrm{d}^3k}{(2\pi)^3 2k^0}\int\frac{\mathrm{d}^3q}{(2\pi)^3 2q^0}\int\frac{\mathrm{d}^3p'}{(2\pi)^3 2p'^0} \tag{25.3.32}$$

$$\times(2\pi)^4\delta(p-p'-k-q)\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 \tag{25.3.33}$$

$$= \frac{1}{16m_\text{n}(2\pi)^5}\int\frac{\mathrm{d}^3k}{k^0}\int\frac{\mathrm{d}^3q}{q^0}\frac{1}{p'^0}\delta(p^0-p'^0-k^0-q^0)\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2, \tag{25.3.34}$$

having used three of the Dirac deltas to perform the p' integral. We can now write $p^0 = E_\text{n}$, $p'^0 = E_\text{p}$, $k^0 = E_\text{e}$, and $q^0 = E_\nu$, as well as changing to spherical coordinates, so $\mathrm{d}^3k = p_\text{e}^2\,\mathrm{d}p_\text{e}\,\mathrm{d}\Omega_\text{e} = p_\text{e}E_\text{e}\,\mathrm{d}E_\text{e}\,\mathrm{d}\Omega_\text{e}$, having used $p_\text{e}^2 = E_\text{e}^2 - m_\text{e}^2$ so $p_\text{e}\,\mathrm{d}p_\text{e} = E_\text{e}\,\mathrm{d}E_\text{e}$, and $\mathrm{d}^3q = E_\nu^2\,\mathrm{d}E_\nu\,\mathrm{d}\Omega_\nu$, having used $p_\nu = E_\nu$ for massless neutrinos. This gives

$$\mathrm{d}\Gamma = \frac{1}{16m_\text{n}(2\pi)^5}\int\mathrm{d}E_\text{e}\,\mathrm{d}\Omega_\text{e}\,\mathrm{d}E_\nu\,\mathrm{d}\Omega_\nu\,\frac{p_\text{e}E_\nu}{E_\text{p}}\delta(E_\text{n}-E_\text{p}-E_\text{e}-E_\nu)\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2.$$

Using the Dirac delta to perform the E_ν integral, fixing $E_\nu = E_\text{n} - E_\text{p} - E_\text{e}$, we get

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}E_\text{e}} = \frac{1}{16m_\text{n}(2\pi)^5}\int\mathrm{d}\Omega_\text{e}\,\mathrm{d}\Omega_\nu\,\frac{p_\text{e}E_\nu}{E_\text{p}}\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2. \tag{25.3.35}$$

In the lab frame $E_\text{n} = m_\text{n}$, and $\boldsymbol{p}_\text{n} = \boldsymbol{0}$. We can define $\Delta \coloneqq m_\text{n} - m_\text{p} = 1.29\,\text{MeV} \ll m_\text{n}$. The Born–Oppenheimer approximation, that nuclear recoil is negligible, so $\boldsymbol{p}_\text{p} \approx \boldsymbol{0}$ and $E_\text{p} \approx m_\text{p}$, giving $\Delta \approx E_\text{n} - E_\text{p} = E_\text{e} + E_\nu$. Notice that Δ is of the same order as $m_\text{e} = 0.51\,\text{MeV}$, so we must keep both. Then we have

$$(p\cdot k)(p'\cdot q) \approx m_\text{n}E_\text{e}m_\text{p}E_\nu, \tag{25.3.36}$$

$$(p\cdot q)(p'\cdot k) \approx m_\text{n}E_\nu m_\text{p}E_\text{e}, \tag{25.3.37}$$

$$(k\cdot q) = E_\text{e}E_\nu - \boldsymbol{p}_\text{e}\cdot\boldsymbol{p}_\nu. \tag{25.3.38}$$

Giving

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}E_\mathrm{e}} = \frac{G_\mathrm{F}^2\cos^2\vartheta_\mathrm{c}}{m_\mathrm{n}(2\pi)^5}\int \mathrm{d}\Omega_\mathrm{e}\,\mathrm{d}\Omega_\nu\,\frac{p_\mathrm{e}E_\nu}{E_\mathrm{p}}m_\mathrm{n}m_\mathrm{p} \tag{25.3.39}$$

$$\times\left[E_\mathrm{e}E_\nu(1-g_A^2)+E_\nu E_\mathrm{e}(1+g_A^2)-E_\mathrm{e}E_\nu(1-g_A^2)\left(1-\frac{\boldsymbol{p}_\mathrm{e}\cdot\boldsymbol{p}_\nu}{E_\mathrm{e}E_\nu}\right)\right]$$

$$= \frac{G_\mathrm{F}^2\cos^2\vartheta_\mathrm{c}}{2\pi^3}(1+3g_A^2)p_\mathrm{e}E_\mathrm{e}(\Delta-E_\mathrm{e})^2 \tag{25.3.40}$$

with $m_\mathrm{e} < E_\mathrm{e} < \Delta$. Note that the $\boldsymbol{p}_\mathrm{e}\cdot\boldsymbol{p}_\nu$ term integrates to zero by symmetry, and we use $p_\mathrm{e} = \sqrt{E_\mathrm{e}^2 - m_\mathrm{e}^2}$.

Integrating over E_e gives the total decay rate, we won't do this integral, but the result is

$$\Gamma \propto \frac{G_\mathrm{F}^2\cos^2\vartheta_\mathrm{c}}{2\pi^3}(1+3g_A^2)\Delta^5 \tag{25.3.41}$$

with the constant of proportionality turning out to be around 0.47.

25.4 Other Semileptonic Decays

Many other semileptonic decays are very similar kinematically to beta decay. For example,

$$\pi^+ \to \pi^0\mathrm{e}^+\nu_\mathrm{e}, \tag{25.4.1}$$

which has $\Delta S = 0$, that is strangeness is conserved, and

$$\mathrm{K}^+ \to \pi^0\mu^+\nu_\mu, \qquad \text{and} \qquad \mathrm{K}^0 \to \pi^-\mu^+\nu_\mu, \tag{25.4.2}$$

which both have $\Delta S = 1$, strangeness is not conserved as a strange quark becomes some other type of quark. Note that these last two examples are Cabibbo suppressed, since they come with a factor of $\sin\vartheta_\mathrm{c}$.

Other semileptonic decays which one might want to consider are hyperon decays of $\Sigma^p \approx$ uus, $\Lambda^0 \approx$ uds, $\Xi^0 \approx$ uss, and $\Omega^- \approx$ sss. Some examples of these decaying are the strangeness conserving

$$\Sigma^p \to \Lambda^0\mathrm{e}^-\bar{\nu}_\mathrm{e}, \tag{25.4.3}$$

and the strangeness changing

$$\Lambda^0 \to \mathrm{p}\mathrm{e}^-\nu, \quad \Sigma^p \to \mathrm{n}\mathrm{e}^-\nu, \quad \Xi^0 \to \Sigma^p\mathrm{e}^-\bar{\nu}_\mathrm{e}, \quad \text{and} \quad \Omega^- \to \Xi^0\mathrm{e}^-\bar{\nu}_\mathrm{e}. \tag{25.4.4}$$

All of which have $\Delta S = 1$.

25.5 Non-Leptonic Decays

Non-leptonic deecays, such as

$$\mathrm{K}^+ \to \pi^+\pi^0, \quad \mathrm{K}^+ \to \pi^+\pi^+\pi^-, \quad \Lambda^0 \to \mathrm{p}\pi^-, \quad \text{and} \quad \Omega^- \to \Xi^0\pi^-, \tag{25.5.1}$$

are much harder. The factorisation of the amplitude no longer works, everything interacts strongly. Generally we have to resort to non-perturbative methods, such as lattice QCD. We won't do much with these types of decays.

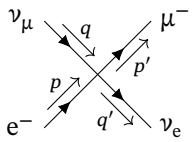
Twenty-Six

Problems with Fermi Theory

Fermi theory works well at low energies, but there are problems when we consider high energy processes. We'll investigate a few of these problems before introducing the fix in the next chapter.

26.1 Divergent Cross Sections

Consider the interaction $\nu_\mu e^- \to \mu^- \nu_e$, known as **inverse muon decay**:



(26.1.1)

The matrix element for this is

$$\mathcal{M} = -i\frac{G_F}{2}\bar{v}(q')\gamma_\mu(1-\gamma^5)u(p)\bar{u}(p')\gamma^\mu(1-\gamma^5)u(q). \tag{26.1.2}$$

To compute the spin-averaged matrix element we need only average over the electrons spin, since the neutrinos spin is fixed by the requirement that it is left handed. So we have

$$\begin{aligned}\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{1}{4}G_F^2\,\text{tr}[\not{q}'(1-\gamma^5)(\not{p}+m_e)\gamma_\nu(1-\gamma^5)] && (26.1.3)\\ &\quad\times \text{tr}[(\not{p}'+m_\mu)\gamma^\mu(1-\gamma^5)\not{q}\gamma^\nu(1-\gamma^5)] && (26.1.4)\\ &= 64G_F^2(p\cdot q)(p'\cdot q') && (26.1.5)\\ &= 16G_F^2(s-m_e^2)(s-m_\mu^2) && (26.1.6)\end{aligned}$$

where $s = (p+q)^2 = (p'+q')^2 = m_e^2 + 2p\cdot q = m_\mu + 2p'\cdot q'$. We were able to write this down so quickly since we already computed these traces for the case of muon decay, $\mu^- \to e^-\bar{\nu}_e\nu_\mu$.

In the centre of mass frame at high energies, so masses are negligible, we have

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CoM}} = \frac{1}{64\pi^2 s}\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{4\pi}G_F^2 s. \tag{26.1.7}$$

Now using $s = 4E^2$ and $\mathrm{d}\Omega = 2\pi\,\mathrm{d}(\cos\vartheta)$ we ahve

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\vartheta} = \frac{2G_\mathrm{F}^2}{\pi}E^2. \tag{26.1.8}$$

Contrast this with the QED process $\mathrm{e}^-\mathrm{e}^+ \to \mu^-\mu^+$:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\vartheta} = \frac{\alpha^2}{4\pi E^2}\left(\cos^4\frac{\vartheta}{2} + \sin^4\frac{\vartheta}{2}\right). \tag{26.1.9}$$

The QED cross section scales as $1/E^2$, so vanishes as $E \to \infty$. On the other hand the Fermi theory cross-section scales as E^2, so diverges as $E \to \infty$. This is a problem since the probability of an interaction occurring is proportional to the cross section, so it must be bounded.

More generally any cross section can be decomposed into a sum of partial waves,

$$\sigma = \frac{4\pi}{E^2}\sum_j (2j+1)|f_j|^2, \tag{26.1.10}$$

where f_j is the partial amplitude for some subprocess. Unitarity of the S-matrix, $S^\dagger S = 1$, requires that $|f_j|^2 \leq 1$. This means that the partial wave contributions to the cross-section must satisfy

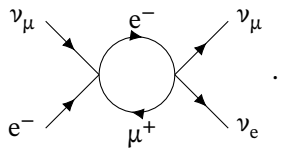
$$\sigma_j \leq \frac{4\pi}{E^2}(2j+1), \tag{26.1.11}$$

where σ_j is the contribution of the jth term in the sum above.

Such divergent quantities are inevitable from dimensional analysis. The mass dimension of the Fermi constant is $[G_\mathrm{F}] = -2$, and similarly the cross section is an area, so $[\sigma] = -2$. Interactions in Fermi-theory come with G_F^2, and since at high energies the only energy scale is E we must have $\sigma \propto G_\mathrm{F}^2E^2$ to get the correct dimensions. Clearly this will diverge as $E \to \infty$. This argument means that we can only satisfy the above unitarity conditions for couplings with non-negative mass dimension. Ideally, couplings would be dimensionless, as is the case with e in QED.

26.2 Renormalisability

Consider the process of neutrino scattering, $\nu_\mathrm{e}\bar{\nu}_\mu \to \nu_\mathrm{e}\bar{\nu}_\mu$. When computing the full amplitude we must include all possible Feynman diagrams, including



. (26.2.1)

Suppose that the loop momentum of this diagram is k. That is, the electron has momentum k and the muon momentum $p-k$, where p is the combined momentum of both the incoming neutrinos, Then when computing the amplitude we come across divergent integrals:

$$\mathcal{M} \sim \int \mathrm{d}^4k \frac{\not{k}}{k^2}\frac{\not{k}}{k^2} \sim \int \frac{\mathrm{d}^4k}{k^2}. \tag{26.2.2}$$

This is a problem since the Lagrangian for Fermi theory does not contain an appropriate $\bar{\nu}\nu\bar{\nu}\nu$ counterterm to cancel this divergence.

26.2.1 Power Counting in QED

Consider QED as a simple example. A Feynman diagram is just a graph with L loops, I_F internal fermion lines, and I_B internal boson lines. The **superficial degree of divergence** is

$$D = 4L - I_\mathrm{F} - 2I_\mathrm{B}. \tag{26.2.3}$$

The logic being that a diagram with superficial degree of divergence D is given by a momentum integral with D powers of momentum, which we expect to be divergent if $D \geq 0$. The $4L$ term comes from counting the number of loop integrals, each of which comes with four powers of momentum, d^4k. The i_F term comes from the $\not{k}/k^2 \sim k^{-1}$ terms from fermion propagators, and the I_B term from counting the $1/k^2$ terms from boson propagators.

A result from graph theory tells us that a graph with L loops, I internal lines, and V vertices satisfies

$$L = I - V + 1. \tag{26.2.4}$$

Hence we have

$$D = 2I_\mathrm{B} + 3I_\mathrm{F} - 4V + 4, \tag{26.2.5}$$

where the number of vertices now counts the number of interactions plus the number of external particles.

Since in QED we have only one type of interaction, involving two fermion lines and one photon, we can make some more deductions, including

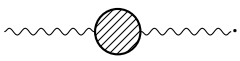
$$2I_\mathrm{B} + E_\mathrm{B} = V, \qquad \text{and} \qquad 2I_\mathrm{F} + E_\mathrm{F} = 2V, \tag{26.2.6}$$

where E_B and E_F are the number of external boson and fermion lines respectively. Using this we have

$$D = V - E_\mathrm{B} + \frac{3}{2}(2V - E_\mathrm{F}) - 4V + 4 = 4 - E_\mathrm{B} - \frac{3}{2}E_\mathrm{F}. \tag{26.2.7}$$

This means that V is independent of V. The only divergent graphs in QED are ones with appropriate counterterms to regulate the divergences. We can list all potentially divergent graphs in QED, based solely on external particles, and they either turn out to be finite by symmetry or are regulated.

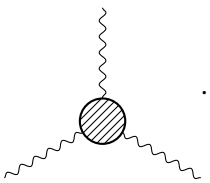
The first potentially divergent graph is



. (26.2.8)

This diagram comes from $F_{\mu\nu}F^{\mu\nu}$ and is regulated by Z_3.

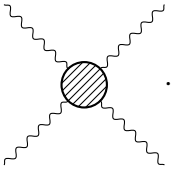
The second potentially divergent graph is



. (26.2.9)

This diagram with three external photons picks up a minus sign under charge conjugation, since the photon is odd under charge conjugation, and since QED respects charge conjugation as a symmetry diagrams like this must always vanish.

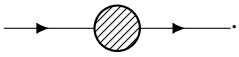
The third potentially divergent graph is



(26.2.10)

This turns out to actually be finite due to gauge invariance.

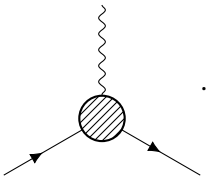
The fourth potentially divergent diagram is



(26.2.11)

This comes from the term $\bar{\psi}(\partial\!\!\!/ + m)\psi$, which is regulated by Z_2 and Z_m.

The fifth potentially divergent diagram is



(26.2.12)

This comes from the term $\bar{\psi}A\!\!\!/\psi$, which is regulated by Z_1.

Since the Lagrangian contains all the necessary counterterms we say that QED is power-counting renormalisable.

26.2.2 Power Counting in Fermi Theory

Now consider our four fermion interaction in Fermi theory. We have

$$2I_{\mathrm{F}} + E_{\mathrm{F}} = 4V, \tag{26.2.13}$$

since every internal fermion goes between two vertices and every external fermion goes into/comes out of one vertex, and each vertex has four fermions. In an interaction with only fermions we then have

$$D = \frac{3}{2}(4V - E_{\mathrm{F}}) - 4V + 4 = 4 + 2V - \frac{3}{2}E_{\mathrm{F}}. \tag{26.2.14}$$

This means that every time we add a vertex the graph becomes more divergent. This means that processes with any number of external fermions will eventually diverge at high enough order. For example, if we have six external fermions, then we have divergences at third order in G_{F}, when $V = 3$.

26.2.3 Power-Counting Renormalisable Interactions

Theorem 26.2.15. A theory is power-counting renormalisable if all the couplings have non-negative mass dimension.

Proof. Consider a vertex, with b boson lines, f fermion lines, and p derivatives, suppose we have V_{bf}^p such vertices. The corresponding term in the Lagrangian is, in full generality, $g\varphi^b\psi^f\partial^p$, where φ are bosons (scalar or vector, or even tensor), ψ are fermions, and include adjoints like $\bar{\psi}$ as required, and the derivatives act on some combination of the fields. The mass dimension of the coupling is then

$$d_{bf}^p = 4 - b - \frac{3}{2}f - p, \tag{26.2.16}$$

which we know since $[\mathcal{L}] = 4$, $[\varphi] = [A] = 1$, $[\psi] = 3/2$, and $[\partial] = 1$. Counting the ends of the lines in any given diagram we have

$$2I_\mathrm{B} + E_\mathrm{B} = \sum bV_{bf}^p, \qquad \text{and} \qquad 2I_\mathrm{F} + E_\mathrm{F} = \sum fV_{bf}^p. \tag{26.2.17}$$

Hence, the superficial degree of divergence is

$$D = \sum bV_{bf}^p - E_\mathrm{B} + \frac{3}{2}\left(\sum fV_{bf}^p - E_\mathrm{F}\right) - 4\sum V_{bf}^p + 4 + \sum pV_{bf}^p \tag{26.2.18}$$

$$= \sum\left(b + \frac{3}{2}f + p - 4\right)V_{bf}^p - E_\mathrm{B} - \frac{3}{2}E_\mathrm{F} + 4 \tag{26.2.19}$$

$$= -\sum d_{bf}^p V_{bf}^p - E_\mathrm{B} - \frac{3}{2}E_\mathrm{F} + 4. \tag{26.2.20}$$

Thus, for renormalisability we require that $d_{bf}^p \geq 0$ in order for D to not increase as the number of vertices increases. □

This theorem, combined with Lorentz invariance, is actually very restricting. In fact, between this course and *Quantum Field Theory*, we have already seen examples of every possible interaction this theorem allows.

Corollary 26.2.21 For a theory with scalar, fermion, and vector fields the only vertices allowed in a power-counting renormalisable Lorentz invariant Lagrangian are

- φ^3, φ^4 — pure scalars
- $\partial_\mu A_\nu A^\mu A^\nu$, $A_\mu A_\nu A^\mu A^\nu$ — Yang–Mills
- $\varphi A_\mu A^\mu$, $\varphi^2 A_\mu A^\mu$, $\varphi(\partial_\mu\varphi)A^\mu$ — gauge-scalar interactions
- $\varphi\bar{\psi}\psi$ — Yukawa interaction
- $\bar{\psi}\not{A}\psi$ — gauge-fermion interactions

Proof. We need at least three fields for an interaction. Lorentz invariance requires that if we have ψ we also have $\bar{\psi}$. One can check that adding an extra field to any of the above results in a coupling with negative mass dimension and/or breaks Lorentz invariance. The only other terms with appropriate couplings such as, $A_\mu A^\mu A^\nu$, $\varphi^2\partial_\mu\varphi$, and $\varphi^2 A_\mu$, are not Lorentz invariant. □

Not only are these the only allowed interactions, they are also exactly the interactions needed to formulate a consistent theory of weak interactions. It seems like if an interaction is possible then it occurs.

Twenty-Seven

Intermediate Vector Bosons

27.1 New Bosons

The arguments of the previous section suggest that we need a new theory which reduces to Fermi theory in particular, low energy, cases. Such a theory must be renormalisable, and so must contain only interactions of the form of those listed in Corollary 26.2.21. Further, this interaction must act only on left handed fermions. This rules out Yukawa interactions, since $\varphi\bar{\psi}\psi = \varphi(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$ necessarily includes right handed fermions.

We also know that we have two charged currents, since we have

$$J_\mu = 2(\bar{\nu}_{eL}\gamma_\mu e_L + \cdots + \bar{u}_L\gamma_\mu d'_L + \cdots), \tag{27.1.1}$$

which has $\Delta Q = 1$, following the fermion line through a vertex increases the charge by 1, and we also have its adjoint

$$J^\dagger_\mu = 2(\bar{e}_L\gamma_\mu\nu_{eL} + \cdots + \bar{d}'_L\gamma_\mu u_L + \cdots), \tag{27.1.2}$$

which has $\Delta Q = -1$. Contrast this with QED, where we have a single current, $\bar{\psi}\gamma_\mu\psi$, with $\Delta Q = 0$.

Because of this we need two bosons, and they need to be charged. That's the physics, translated to maths we need to introduce two complex vector fields, and we call them

$$W^\mu_\pm = \frac{1}{\sqrt{2}}(W^\mu_1 \pm iW^\mu_2), \tag{27.1.3}$$

where W^μ_1 and W^μ_2 are real vector fields. Note that we can equivalently define a single complex vector field, W^μ, and then take $W^\mu_+ = W^\mu$ and $W^\mu_- = W^{\mu\dagger}$, which is analogous to our treatment of charged scalars and fermions.

We can construct a Lagrangian for a charged current interaction:

$$\mathcal{L}_{CC} = \frac{g}{2\sqrt{2}}(J^\dagger_\mu W^\mu + W^\dagger_\mu J^\mu) \tag{27.1.4}$$

where the factor of $1/2$ is the usual factor of 2 for complex fields, and the $\sqrt{2}$ is convention so that our results align with Fermi theory. The coupling constant, g, is dimensionless, and is the same for every term appearing in J_μ, another example of universality.

The new bosons, W^μ, differ from A^μ in electromagnetism in two important ways:

- W^μ is charged, and so complex;
- W^μ is massive.

An argument for the second point is that charged current interactions are short range, and so the exchange boson must be massive, but this isn't particularly compelling as QCD interactions are also often short range even though gluons are massive, the short range comes from gluons interacting with each other.

For a full theory it is not enough to specify the interactions, we also need a Lagrangian for the new, massive, W^μ field. The appropriate choice is the **Proca Lagrangian**, which modifies the QED-like $F^{\mu\nu}F_{\mu\nu}$ to allow for massive bosons:

$$\mathcal{L}_W = -\frac{1}{2}(\partial_\mu W_\nu - \partial_\nu W_\mu)^\dagger(\partial^\mu W^\nu - \partial^\nu W^\mu) + m_\mathrm{W}^2 W_\mu^\dagger W^\mu \tag{27.1.5}$$

$$= -\frac{1}{2}W_{\mu\nu}^\dagger W^{\mu\nu} + M_\mathrm{W}^2 W_\mu^\dagger W^\mu. \tag{27.1.6}$$

Here

$$W_{\mu\nu} \coloneqq \partial_\mu W_\nu - \partial_\nu W_\mu \tag{27.1.7}$$

generalises the field strength tensor, making use of the same symbol and distinguishing it simply by the number of indices. Notice that the typical factor of $1/4$ becomes $1/2$ in this case because of the reduced symmetry of having a Hermitian conjugate of one term. If we compared the mass term to the mass term of a scalar we would also see that the sign is different, in a charged scalar theory we might have $-\frac{1}{2}m\varphi^\dagger\varphi$. This is because the physical degrees of freedom of a massive vector field are the three spatial components, and in our metric convention these carry an extra minus sign.

The equations of motion which one derives from this are a modification on the Klein–Gordon equation:

$$(\partial^2 + M_\mathrm{W}^2)W^\mu - \partial^\mu\partial^\nu W_\nu = 0. \tag{27.1.8}$$

Notice that if we assume that W^μ also satisfies the Klein–Gordon equation, $(\partial^2 + M_\mathrm{W}^2)W^\mu = 0$, and $M_\mathrm{W} \neq 0$ then we must have $\partial^\nu\partial^\mu W_\mu = 0$, which we can assume means $\partial^\mu W_\mu = 0$.

As usual we look for plane wave solutions, with some polarisation vector, ε^μ, to give the correct Lorentz transformation of our field:

$$W^\mu = \varepsilon^\mu \mathrm{e}^{-ik\cdot x}. \tag{27.1.9}$$

Substituting this into the equations of motion give

$$(-k^2 + M_\mathrm{W}^2)\varepsilon^\mu \mathrm{e}^{-ik\cdot x} - k^\mu k^\nu \varepsilon_\nu \mathrm{e}^{-ik\cdot x} = 0 \tag{27.1.10}$$

which implies that

$$M_\mathrm{W}^2 k_\mu \varepsilon^\mu = 0, \tag{27.1.11}$$

meaning that on-shell solutions, where $M_\mathrm{W} \neq 0$, must have $k \cdot \varepsilon = 0$.

Since W^μ is massive it has three spin states. In the rest frame of the boson we can take these to be

$$\varepsilon_1^\mu = (0,1,0,0), \quad \varepsilon_2^\mu = (0,0,1,0), \quad \text{and} \quad \varepsilon_3^\mu = (0,0,0,1). \tag{27.1.12}$$

Note that we can also swap ε_1 and ε_2 for the circular polarisations $\varepsilon_\pm = (0, 1, \pm i, 0)/\sqrt{2}$.

For momentum in the z-direction, $k^\mu = (E, 0, 0, k)$, we have $\varepsilon_3^\mu = (k, 0, 0, E)/M_\mathrm{W}$. This means that at high energies, when $E \gg M_\mathrm{W}$, we have

$$\varepsilon_3^\mu \approx \frac{k^\mu}{M_\mathrm{W}} + \mathcal{O}\left(\frac{M_\mathrm{W}}{E}\right). \tag{27.1.13}$$

This is somewhat strange, $\varepsilon_3 \cdot k = 0$, so ε_3 is perpendicular to the momentum, but at high energies ε_3 is parallel to the momentum to leading order.

As with the massless case we have polarisation sums. We can define $\varepsilon_0^\mu = k^\mu/m_\mathrm{W}$, so that in the rest frame $\varepsilon_0^\mu = (1, 0, 0, 0)$, which is the time-like, non-physical degree of freedom. Completeness of these states as a basis for Minkowski space tells us that

$$\sum_{r=0}^{3} \eta^{rr} \varepsilon_r^\mu \varepsilon_r^{\nu *} = \eta^{\mu\nu}, \tag{27.1.14}$$

where the complex conjugate allows for exchanging the 1 and 2 states for the circular polarisations. Summing over only the physical modes, and using $\eta^{ii} = -1$ (no sum no i) we then have

$$\sum_{r=1}^{3} \varepsilon_r^\mu \varepsilon_r^{\nu *} = -\eta^{\mu\nu} + \varepsilon_0^\mu \varepsilon_0^\nu = -\eta^{\mu\nu} + \frac{k^\mu k^\nu}{m_\mathrm{W}^2}. \tag{27.1.15}$$

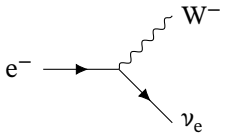
The result of this is that the tensor structure of the propagator is modified to

$$\Delta_\mathrm{W}^{\mu\nu} = \frac{i}{k^2 - M_{\mathrm{W}^2} + i\epsilon}\left(-\eta^{\mu\nu} + \frac{k^\mu k^\nu}{M_\mathrm{W}^2}\right). \tag{27.1.16}$$

We can check that this is correct by recalling that the propagator is the inverse in Fourier space to the equations of motion, and expanding everything shows that

$$[(-k^2 + M_\mathrm{W}^2)\eta_{\mu\nu} + k_\mu k_\nu]\Delta_\mathrm{W}^{\nu\rho} = -i\delta_\mu^\rho. \tag{27.1.17}$$

We also have a new interaction, a three point vertex involving an electron, electron neutrino, and a W boson. The Feynman rule for this follows just like the QED vertex Feynman rule, but with an extra factor of $(1-\gamma^5)/2$, and a different coupling:

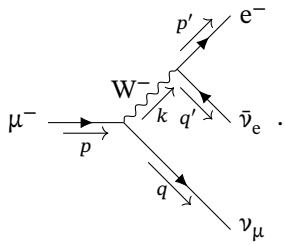


$$= i\frac{g}{\sqrt{2}}\gamma_\mu \frac{1}{2}(1 - \gamma^5) \tag{27.1.18}$$

where W^- is the boson associated with W_-^μ. The Feynman rule for interactions involving W^+, the boson associated with W_+^μ, is exactly the same.

27.2 Muon Decay

Consider muon decay, $\mu^- \to e^- \nu_\mu \bar{\nu}_e$. At tree level this process is



(27.2.1)

The amplitude for this is

$$\mathcal{M} = \frac{(ig)^2}{8}\bar{u}(p')\gamma_\mu(1-\gamma^5)v(q')\frac{i(-\eta^{\mu\nu}+k^\mu k^\nu/M_W^2)}{k^2-M_W^2+i\epsilon}\bar{u}(q)\gamma_\nu(1-\gamma^5)u(p). \quad (27.2.2)$$

Here the spinors $u(p')$, $v(q')$, $u(q)$ and $u(p)$ correspond to the electron, antielectron neutrino, muon neutrino, and muon respectively.

Since the incoming energy is simply $p^2 = m_\mu^2 < M_W^2$ it is not possible for the W^- boson to go on-shell, so we can drop the $i\epsilon$ term. We have $k = p - q$ so $k^2 = p^2 + q^2 - 2p\cdot q = m_\mu^2 - 2p\cdot q < m_{\mu^-}^2 \ll m_W^2$ for massless neutrinos and $p\cdot q = m_\mu E > 0$ having performed this calculation in the muon rest frame.

Using the Dirac equation in Fourier space, $(\not{p}-m)u(p) = 0$ and $\bar{u}(p)(\not{p}-m) = 0$, we have

$$\begin{aligned}
\bar{u}(q)\not{k}(1-\gamma^5)u(p) &= \bar{u}(q)(\not{p}-\not{q})(1-\gamma^5)u(p) && (27.2.3)\\
&= \bar{u}(q)(1+\gamma^5)\not{p}u(p) - \bar{u}(q)\not{q}(1-\gamma^5)u(p) && (27.2.4)\\
&= (1+\gamma^5)m_\mu u(p) - 0, && (27.2.5)
\end{aligned}$$

for massless neutrinos. Similarly, we also have $(\not{p}+m)v(p) = 0$, so

$$\begin{aligned}
\bar{u}(p')\not{k}(1-\gamma^5)v(q') &= \bar{u}(p')(\not{p}'+\not{q}')v(q') && (27.2.6)\\
&= \bar{u}(p')\not{p}'(1-\gamma^5)v(q') + \bar{u}(p')(1+\gamma^5)\not{q}'v(q') && (27.2.7)\\
&= \bar{u}(p')m_e(1-\gamma^5)v(q') + 0 && (27.2.8)
\end{aligned}$$

again, for massless neutrinos. Combining these we can replace the second term in the propagator with $m_e m_\mu/M_W^2$, and measurements tell us that this is about 10^{-8}, so this term is negligible.

Using these simplifications we can square the amplitude, which works the same way as the muon decay with a four point interaction as in Chapter 24. The result is then

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{2g^4}{M_W^4}(p\cdot q')(p'\cdot q). \quad (27.2.9)$$

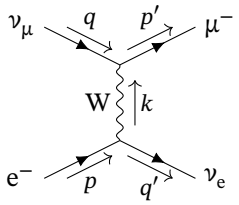
This agrees with the result we got in $V-A$ Fermi theory provided that

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}. \quad (27.2.10)$$

Intuitively the weak force is weak, that is G_F is small, because M_W is large.

27.3 Scattering

Consider the process $e^- \nu_\mu \to \mu^- \nu_e$. At tree level the diagram for this process is



. (27.3.1)

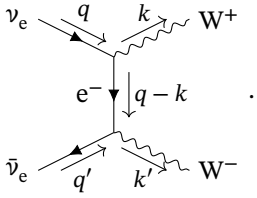
The differential cross section for this process is

$$\frac{d\sigma}{d\cos\vartheta} = \frac{g^4}{128\pi^2}\frac{s}{(t-M_W^2)^2} \sim \frac{1}{E^2}. \tag{27.3.2}$$

When we considered this process in Fermi theory (Section 26.1) we saw that the differential cross section instead went as E^2 and diverged. So the intermediate vector boson model is an improvement over Fermi theory in this sense.

27.4 Problems with the Intermediate Vector Boson Model

Consider the process $\nu_e \bar{\nu}_e \to W^+W^-$. This is similar to example's we've seen in *Quantum Field Theory* with photons in the final state. There we saw that there were t-channel and u-channel processes at tree level and only in considering both did we get a finite result. Here since the W bosons produced have opposite charge they are distinguishable, so there is no u-channel diagram to consider. The only tree level contribution to this process is



. (27.4.1)

The amplitude for this process is

$$\mathcal{M} = \frac{(ig)^2}{8}\bar{v}(q')\gamma_\mu(1-\gamma^5)\frac{i(\not{q}-\not{k}+m_e)}{(q-k)^2-m_e^2}\gamma_\nu(1-\gamma^5)u(q)\varepsilon^\mu_{r'}(k')\varepsilon^\nu_r(k). \tag{27.4.2}$$

Here $\bar{v}(q')$ and $u(q)$ is the spinor for the incoming antineutrino and neutrino respectively. The W boson polarisation vectors are $\varepsilon^\mu_{r'}(k')$ and $\varepsilon^\nu_r(k)$ for W^- and W^+ respectively. Note that it is impossible for the electron to go on-shell, so we can drop the $i\epsilon$.

At high energies we have seen that the polarisation vectors can be substituted for k^μ/M_W. We can also neglect the electron mass at high energies. Thus the amplitude at high energy simplifies to

$$\mathcal{M} \approx -\frac{ig^2}{4}\frac{1}{M_W^2(q-k)^2}\bar{v}(q')\not{k}'(\not{q}-\not{k})\not{k}(1-\gamma^5)u(q). \tag{27.4.3}$$

The differential cross section is then

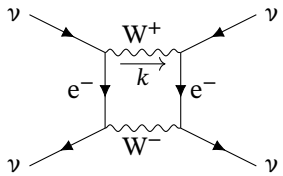
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{g^4}{64\pi^2 M_{\mathrm{W}}^4}\frac{ut - M_{\mathrm{W}}^4}{4s} \sim E^2 \tag{27.4.4}$$

and the total cross section is

$$\sigma = \frac{g^4}{96\pi}\frac{E^2}{M_{\mathrm{W}}^4}. \tag{27.4.5}$$

This shows that we still have divergent diagrams, so we haven't completely fixed our problems with unitarity by introducing these intermediate vector bosons. We should have predicted this since we don't get the same cancelling as we did with the electrically neutral photons.

The solution is also not as simple as excluding external massive vector bosons, for example, if we consider one-loop contributions to the $\nu\nu \to \nu\nu$ process then we must consider the diagram



$$\sim \int \mathrm{d}^4k \left(\frac{k^2/M_{\mathrm{W}}^2}{k^2 - m_{\mathrm{W}}^2}\right)\left(\frac{\not{k}}{k^2}\right)^2 \sim \int \frac{\mathrm{d}^4k}{M_{\mathrm{W}}^2 k^2}. \tag{27.4.6}$$

This is quadratically divergent. There is also no counter term to cancel out this divergence.

The reason for the failure of unitarity and renormalisability is that our intermediate gauge bosons have no gauge invariance. The gauge invariance fixes constraints and removes non-physical modes, such as the longitudinal mode of massless gauge bosons, and the temporal mode which is non-physical for all vector bosons. To fix this then we must look to extend our theory of intermediate vector bosons to a theory with a gauge symmetry.

Twenty-Eight

$\mathrm{SU}(2)_\mathrm{L} \otimes \mathrm{U}(1)_Y$ Gauge Symmetry

28.1 Electroweak Unification

We need a gauge theory for the W bosons. We know *a priori* that this gauge theory must

- be non-Abelian since the gauge bosons are charged,
- have massive gauge bosons.

We'll come back to the massive-gauge-bosons bit later. For now we focus on the non-Abelian bit. This is justified as unitarity and renormalisability are primarily of interest at high energies, when masses can be neglected.

The weak current is given by

$$\frac{1}{2}J_\mu = \nu_{\mathrm{eL}}\gamma_\mu \mathrm{e}_\mathrm{L} + \nu_{\mu\mathrm{L}}\gamma_\mu \mu_\mathrm{L} + \nu_{\tau\mathrm{L}}\gamma_\mu \tau_\mathrm{L}. \tag{28.1.1}$$

We neglect quarks for now. The particles appear as pairs, ν_e and e^-, ν_μ and μ^-, and so on. Motivated by this we introduce doublets of left handed particles:

$$\psi_\mathrm{L} = \begin{pmatrix}\nu_{\mathrm{eL}} \\ \mathrm{e}_\mathrm{L}\end{pmatrix}, \begin{pmatrix}\nu_{\mu\mathrm{L}} \\ \mu_\mathrm{L}\end{pmatrix}, \begin{pmatrix}\nu_{\tau\mathrm{L}} \\ \tau_\mathrm{L}\end{pmatrix}. \tag{28.1.2}$$

The right handed particles remain as singlets,

$$\psi_\mathrm{R} = \mathrm{e}_\mathrm{R}, \mu_\mathrm{R}, \tau_\mathrm{R}. \tag{28.1.3}$$

We don't include any right handed neutrinos since if we assume neutrinos to be massless then there is no neutrino mixing so we don't need to have right handed neutrinos and we don't see right handed neutrinos since they don't interact via the weak force, so they may as well not exist. We will now drop the L label on left handed neutrinos and assume that all neutrinos are left handed.

These doublets lead us to one obvious candidate for the gauge group, $\mathrm{SU}(2)_\mathrm{L}$, the L standing for left, since we might assume that the left handed doublets transform under the fundamental representation of SU(2), whereas the right handed singlets transform under the trivial representation of SU(2), which is to say they don't transform at all. In this context SU(2) is called **weak isospin**, spin because SU(2) also explains particle spin. The transformation laws for the particles under this gauge group are then

$$\psi_\mathrm{L} \mapsto U\psi_\mathrm{L}, \qquad \text{and} \qquad \psi_\mathrm{R} \mapsto \psi_\mathrm{R} \tag{28.1.4}$$

where $U \in \mathrm{SU}(2)$ is a 2×2 unitary matrix with determinant 1.

The generators of SU(2) are the Pauli matrices (divided by 2), which in this context are often denoted τ^i rather than the more common σ^i. Forming a three-vector of generators, $\boldsymbol{\tau} = (\tau^1, \tau^2, \tau^3)$, we can write an arbitrary element of SU(2) as

$$U = \mathrm{e}^{i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}} \tag{28.1.5}$$

where $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3) \in \mathbb{R}^3$ parametrises U.

Notice that $\dim(\mathrm{SU}(2)) = 3$, and there are no non-Abelian Lie groups of dimension 1 or 2, so in a sense this is the most simple choice we could make. There is a problem though, the dimension of a gauge group is exactly the number of gauge bosons. The hope initially was that these three gauge bosons would unify the two $\mathrm{W}^\pm$ bosons with the photon. This turns out to almost be the case, but it's not quite as straight forward as this.

Notice that we can use these doublets to write a single term in the current in terms of matrices:

$$\bar{\nu}_\mathrm{e}\gamma_\mu \mathrm{e}_\mathrm{L} + \bar{\mathrm{e}}_\mathrm{L}\gamma_\mu \nu_\mathrm{e} = \begin{pmatrix}\nu_\mathrm{e} & \mathrm{e}_\mathrm{L}\end{pmatrix} \underbrace{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}}_{=\tau^1} \gamma_\mu \begin{pmatrix}\nu_\mathrm{e}\\ \mathrm{e}_\mathrm{L}\end{pmatrix}. \tag{28.1.6}$$

Notice that the γ_μ matrix is acting on a different space so it doesn't matter if we write the $\tau^1\gamma_\mu$ or $\gamma_\mu\tau^1$, but it does matter that both of these factors appear between the doublet and its adjoint. Further, we can get the term appearing in the current, up to factors of 2, by writing

$$\bar{\nu}_\mathrm{e}\gamma_\mu \mathrm{e}_\mathrm{L} = \begin{pmatrix}\nu_\mathrm{e} & \mathrm{e}_\mathrm{L}\end{pmatrix} \underbrace{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}}_{T^+} \begin{pmatrix}\nu_\mathrm{e}\\ \mathrm{e}_\mathrm{L}\end{pmatrix} \tag{28.1.7}$$

where we define the ladder operator

$$T^+ \coloneqq \frac{1}{2}(\tau^1 + i\tau^2) = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}. \tag{28.1.8}$$

Similarly, we can define

$$T^- \coloneqq \frac{1}{2}(\tau^1 - i\tau^2) = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}. \tag{28.1.9}$$

Then we can write the current and its adjoint as

$$J_\mu = 2\bar{\psi}_\mathrm{L} T^+ \gamma_\mu \psi_\mathrm{L}, \qquad \text{and} \qquad J_\mu^\dagger = 2\bar{\psi}_\mathrm{L} T^- \gamma_\mu \psi_\mathrm{L}. \tag{28.1.10}$$

Here the adjoint is on the representation space, not the spinor space, so the γ_μ is unaffected.

Recall that we have the $\mathfrak{su}(2)$ commutation relations[1]

$$[T^+, T^-] = 2T^3, \qquad \text{and} \qquad [T^3, T^\pm] = \pm T^\pm \tag{28.1.11}$$

where $T^3 = \tau^3/2$. These suggest that we can define a third Noether current,

$$J_\mu^3 \coloneqq 2\bar{\psi}_\mathrm{L} T^3 \gamma_\mu \psi_\mathrm{L}. \tag{28.1.12}$$

[1] see *Symmetries of Quantum Mechanics* or *Symmetries of Particles and Fields*

With each of these currents we get a charge operator, call them $Q^\pm$ and Q^3 respectively. These also satisfy the $\mathfrak{su}(2)$ Lie algebra:

$$[Q^+, Q^-] = 2Q^3, \qquad \text{and} \qquad [Q^3, Q^\pm] = \pm Q^\pm. \tag{28.1.13}$$

We can combine these three currents into one three-vector,

$$\boldsymbol{J}_\mu = \bar{\psi}_\mathrm{L} \boldsymbol{T} \gamma_\mu \psi_\mathrm{L}, \tag{28.1.14}$$

where $\boldsymbol{T} = \boldsymbol{\tau}/2$. It is worth taking a second to reflect on all of the different spaces here:

- The three-dimensional space of the three currents, represented by bold three-vectors.
- The four-dimensional space of spacetime, represented by Greek indices, μ.
- The three-dimensional space of the $\mathfrak{su}(2)$ Lie algebra, represented by the presence of the generators T^i, and the doublets, ψ_L.
- The four-dimensional spinor space, represented by the presence of the gamma matrices, γ_μ, and the use of spinors $\bar{\psi}_\mathrm{L}$.

We can define

$$J^\pm_\mu = \frac{1}{2}(J^1_\mu \pm i J^2_\mu) \tag{28.1.15}$$

and identify $J_\mu = J^+_\mu$ and $J^\dagger_\mu = J^-_\mu$. Then $\boldsymbol{J}_\mu$ transforms under the adjoint representation of SU(2), which is exactly the fundamental representation of SO(3), which is what justifies us treating it as a three-vector.

In terms of the spinors the third current is

$$J^3_\mu = \frac{1}{2}(\bar{\nu}_\mathrm{e} \gamma_\mu \nu_\mathrm{e} - \bar{\mathrm{e}}_\mathrm{L} \gamma_\mu \mathrm{e}_\mathrm{L} + \cdots) \tag{28.1.16}$$

where we've left out the later generations. This follows since

$$T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{28.1.17}$$

Notice that this is a neutral current, since there is no charge difference between the two particles in each term, meaning that this current cannot change the charge of a particle, $\Delta Q = 0$. This is very similar QED, but not quite identical since we have neutrinos participating in this current. This means that the third gauge boson is not the photon as we had hoped.

Writing Q for the electric charge operator then we see that Q acts on ψ_L as

$$Q\psi_\mathrm{L} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \Psi_\mathrm{L}. \tag{28.1.18}$$

This is because if we take a pure neutrino state we get

$$Q\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{28.1.19}$$

and if we take a pure electron state we get

$$Q\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -1\begin{pmatrix}0\\1\end{pmatrix}, \tag{28.1.20}$$

so we do get the charge as the eigenvalue.

The charge operator acts on right handed singlets as $Q\psi_{\mathrm{R}} = -\psi_{\mathrm{R}}$ since all of the right handed leptons (again, ignoring right handed neutrinos) are negative.

We can write the charge operator in terms of the generator T^3 and some quantity, Y, which we call the **weak hypercharge**. In particular, writing

$$Q = T^3 + Y \tag{28.1.21}$$

we get the correct charge operator if

$$Y = \begin{cases} -\frac{1}{2} & \text{for left handed leptons,}\\ -1 & \text{for right handed leptons.}\end{cases} \tag{28.1.22}$$

Then for left handed particles $T^3 + Y = Q$ becomes

$$\begin{pmatrix}\frac{1}{2} & 0\\0 & -\frac{1}{2}\end{pmatrix} + \begin{pmatrix}-\frac{1}{2} & 0\\0 & -\frac{1}{2}\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, \tag{28.1.23}$$

and for right handed particles

$$0 + -1 = -1. \tag{28.1.24}$$

This Y is the generator of a new group of symmetries. Since Y is proportional to the identity in this two dimensional representation this group is Abelian and one-dimensional, and hence is $\mathrm{U}(1)$. To distinguish from the $\mathrm{U}(1)$ of QED we write this new $\mathrm{U}(1)$ as $\mathrm{U}(1)_Y$ and the $\mathrm{U}(1)$ of QED as either $\mathrm{U}(1)_Q$, since it is generated by the charge, or $\mathrm{U}(1)_{\mathrm{EM}}$.

Both the doublets and singlets transform under this new group, since both carry non-zero weak hypercharge. The transformation laws are

$$\psi_{\mathrm{L}} \mapsto \mathrm{e}^{-i\beta/2}\psi_{\mathrm{L}}, \qquad \text{and} \qquad \psi_{\mathrm{R}} \mapsto \mathrm{e}^{-i\beta}\psi_{\mathrm{R}} \tag{28.1.25}$$

where $\beta \in \mathbb{R}$ parametrises this transformation. From this we can see that $\mathrm{U}(1)_Y$ is not the same as QED's $\mathrm{U}(1)_Q$, since $\mathrm{U}(1)_Y$ acts differently on left and right handed states, so is chiral, and we know this isn't the case for QED.

Associated with this new symmetry is a new Noether current, the **hypercharge current**,

$$J^Y_\mu = j_\mu - J^3_\mu = -\frac{1}{2}(\bar{\nu}_{\mathrm{e}}\gamma_\mu\nu_{\mathrm{e}} + \bar{\mathrm{e}}_{\mathrm{L}}\gamma_\mu\mathrm{e}_{\mathrm{L}}) - (\bar{\mathrm{e}}_{\mathrm{R}}\gamma_\mu\mathrm{e}_{\mathrm{R}}) + \cdots \tag{28.1.26}$$

where j_μ is the QED current and we have once again left out the higher generations.

The generators of $\mathrm{SU}(2)_{\mathrm{L}}$ and $\mathrm{U}(1)_Y$ commute, this means that we can form the full symmetry group as a direct product, $\mathrm{SU}(2)_{\mathrm{L}} \otimes \mathrm{U}(1)_Y$. If the generators didn't commute we would instead need a semidirect product. The Lie algebra of this Lie group is $\mathfrak{su}(2)_{\mathrm{L}} \oplus \mathfrak{u}(1)_Y$, and its elements are of the form $i\boldsymbol{\alpha}\cdot\boldsymbol{T} + i\beta Y$, meaning an arbitrary element of $\mathrm{SU}(2)_{\mathrm{L}} \otimes \mathrm{U}(1)_Y$ is of the form

$$\exp\{i\boldsymbol{\alpha}\cdot\boldsymbol{T} + i\beta Y\} \tag{28.1.27}$$

for some $\alpha^i, \beta \in \mathbb{R}$.

28.2 Gauge Theory of $\mathrm{SU}(2)_\mathrm{L} \otimes \mathrm{U}(1)_Y$

The Lie algebra $\mathfrak{su}(2)_\mathrm{L} \oplus \mathfrak{u}(1)_Y$ has four generators, T^i and Y. As such the Lie group $\mathrm{SU}(2)_\mathrm{L} \otimes \mathrm{U}(1)_Y$ is four-dimensional. This means we have four gauge bosons, three arising from the SU(2) symmetry and one from the U(1) symmetry.

Following what we did for QCD we look for a Yang–Mills theory of this gauge group. To do so we introduce Lie-algebra valued gauge fields. We can do so independently for each group SU(2) and U(1). For SU(2) introduce the gauge fields[2]

[2] cf. $A^a_\mu t^a_R$ in QCD

$$\mathcal{W}_\mu \coloneqq \boldsymbol{W}_\mu \cdot \boldsymbol{T} = W^i_\mu T^i \tag{28.2.1}$$

where $\boldsymbol{W}_\mu = (W^1_\mu, W^2_\mu, W^3_\mu)$ consists of three complex fields, and $\boldsymbol{T} = (T^1, T^2, T^3)$ are the generators of SU(2). Associated with these three gauge fields is a coupling, g. Note that we may raise and lower the 1, 2, and 3 labels on W and T since these correspond to three-vector indices, so we shall choose upper and lower indices so as to avoid too many indices in the same place.

Similarly, introduce a gauge field B_μ, and a coupling g', for the U(1) symmetry. This is significantly simpler as the generators of U(1) are just scalars, or for higher dimensional reps, multiples of the identity.

The gauge fields transform under the adjoint representation, which means that we have the transformation laws

$$\mathcal{W}_\mu \mapsto U\mathcal{W}_\mu U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger \tag{28.2.2}$$

$$B_\mu \mapsto B_\mu + \frac{i}{g'}\partial_\mu \beta \tag{28.2.3}$$

where $U = \exp\{i\boldsymbol{\alpha} \cdot \boldsymbol{T}\} \in \mathrm{SU}(2)$ and $\exp\{i\beta Y\} \in \mathrm{U}(1)$ are the transformations occurring here.

We can then introduce a covariant derivative for these transformations. This derivative will act differently on left and right handed particles since they are in different representations:

$$D_\mu \psi_\mathrm{L} = \left(\partial_\mu - ig\mathcal{W}_\mu + ig'\frac{1}{2}B_\mu\right)\psi_\mathrm{L}, \tag{28.2.4}$$

$$D_\mu \psi_\mathrm{R} = (\partial_\mu + ig' B_\mu)\psi_\mathrm{R}. \tag{28.2.5}$$

Notice that there is no $\mathcal{W}_\mu$ term when the covariant derivative acts on a right handed state since for right handed particles $\mathcal{W}_\mu$ transforms trivially under $\mathrm{SU}(2)_\mathrm{L}$, and so there is no need to adjust the derivative to account for this transformation.

Using this we can write down the Dirac Lagrangian for our fields, again ignoring mass for now:

$$\mathcal{L}_\mathrm{D} = \bar{\psi}_\mathrm{L} i\not{D}\psi_\mathrm{L} + \bar{\psi}_\mathrm{R} i\not{D}\psi_\mathrm{R} \tag{28.2.6}$$

$$= \bar{\psi} i\not{\partial}\psi + g\bar{\psi}_\mathrm{L}\mathcal{W}^\mu\gamma_\mu - g'B^\mu\left(\frac{1}{2}\bar{\psi}_\mathrm{L}\gamma_\mu\psi_\mathrm{L} + \bar{\psi}_\mathrm{R}\gamma_\mu\psi_\mathrm{R}\right) \tag{28.2.7}$$

$$= \bar{\psi} i\not{\partial}\psi + g\boldsymbol{W}^\mu \cdot \boldsymbol{J}_\mu + g'B^\mu J^Y_\mu \tag{28.2.8}$$

where $\bar{\psi}\not{\partial}\psi = \bar{\psi}_\mathrm{L}\not{\partial}\psi_\mathrm{L} + \bar{\psi}_\mathrm{R}\not{\partial}\psi_\mathrm{R}$.

We can write the complex W^μ field in terms of two real fields, W_1^μ and W_2^μ:

$$W^\mu = \frac{1}{\sqrt{2}}(W_1^\mu + iW_2^\mu), \qquad \text{and} \qquad W^{\mu\dagger} = \frac{1}{\sqrt{2}}(W_1^\mu - iW_2^\mu). \tag{28.2.9}$$

Thus we can compute $\boldsymbol{W}^\mu \cdot \boldsymbol{T}$:

$$\begin{aligned}\boldsymbol{W}_\mu \cdot \boldsymbol{T} &= T^1 W_\mu^1 + T^2 W_\mu^2 + T^3 W_\mu^3 &(28.2.10)\\ &= \frac{1}{2}(T^1 + iT^2)(W_\mu^1 - iW_\mu^2) + \frac{1}{2}(T^1 - iT^2)(W_\mu^1 - iW_\mu^2) + T^3 W_\mu^3 &(28.2.11)\\ &= \frac{1}{\sqrt{2}}(T^+ W_\mu^\dagger + T^- W_\mu) + T^3 W_\mu^3. &(28.2.12)\end{aligned}$$

Thus, we get the following term in the Lagrangian:

$$g\boldsymbol{W}^\mu \cdot \boldsymbol{J}_\mu = \frac{1}{2\sqrt{2}} g(W^{\mu\dagger} J_\mu + J_\mu^\dagger W^\mu) + g J_\mu^3 W_3^\mu. \tag{28.2.13}$$

The first term is just the charged current interaction we have been using already. The $gJ_\mu^3 W_3^\mu$ term is new, representing a new interaction that we have not seen. Since this term contains terms like $\bar{\mathrm{e}}_\mathrm{L} T^3 \gamma_\mu \mathrm{e}_\mathrm{L} W_3^\mu$ we can see that this new interaction will be between left handed fermions and the W_3^μ boson.

This is not the only new term in the Lagrangian, we also have $g' B^\mu J_\mu^Y$. This is an interaction between fermions and the B^μ boson. Neither of these terms is of the form $e j_\mu A^\mu$ needed for electromagnetism, so clearly W_3^μ and B^μ are not the photon.

We can always choose a different basis for our bosons, in which the new basis bosons, Z^μ and A^μ, are related to W_3^μ and B^μ through

$$\begin{pmatrix} W_3^\mu \\ B^\mu \end{pmatrix} = \begin{pmatrix} \cos\vartheta_\mathrm{w} & \sin\vartheta_\mathrm{w} \\ -\sin\vartheta_\mathrm{w} & \cos\vartheta_\mathrm{w} \end{pmatrix} \begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix}. \tag{28.2.14}$$

Here ϑ_w is the **Weinberg angle**, or **weak mixing angle**, and this process is known as weak mixing.

If we substitute this into the Lagrangian then the relevant terms become

$$\begin{aligned} g J_\mu^3 W_3^\mu + g'(j_\mu J_\mu^3) B^\mu &= \left[(g\sin\vartheta_\mathrm{w} - g'\cos\vartheta_\mathrm{w}) J_\mu^3 + g'\cos(\vartheta_\mathrm{w}) j_\mu\right] A^\mu \\ &\quad + \left[(g\cos\vartheta_\mathrm{w} + g'\sin\vartheta_\mathrm{w}) J_\mu^3 - g'\sin(\vartheta_\mathrm{w}) j_\mu\right] Z^\mu \\ &= e j_\mu A^\mu + \frac{g}{\cos\vartheta_\mathrm{w}} (J_\mu^3 - \sin^2(\vartheta_\mathrm{w}) j_\mu) Z^\mu \end{aligned} \tag{28.2.15}$$

where we have identified

$$e = g'\cos\vartheta_\mathrm{w} = g\sin\vartheta_\mathrm{w} \tag{28.2.16}$$

as the electric charge.

Notice that since $\cos\vartheta$ and $\sin\vartheta$ are on order of unity g and g' are the same order as e. Some basic algebra gives us the relations

$$\frac{1}{e^2} = \frac{1}{g^2} + \frac{1}{g'^2}, \qquad \text{and} \qquad e = \frac{gg'}{\sqrt{g^2 + g'^2}}. \tag{28.2.17}$$

The key point here is that by considering a gauge theory for weak interactions electromagnetism naturally drops out as well. In terms of the gauge groups we have $\mathrm{U}(1)_Q \subset \mathrm{SU}(2)_\mathrm{L} \otimes \mathrm{U}(1)_Y$, there is a second copy of $\mathrm{U}(1)$ within the product $\mathrm{SU}(2) \otimes \mathrm{U}(1)$ which we can identify with QED. We call this combined theory **electroweak theory**. Putting all the interactions together we have

$$e j_\mu A^\mu + \frac{1}{2\sqrt{2}} g(W^{\mu\dagger} J_\mu + J_\mu^\dagger W^\mu) + \frac{g}{2\cos\vartheta_\mathrm{w}} J_\mu^\mathrm{NC} Z^\mu \tag{28.2.18}$$

where the first term is the electromagnetic interaction, QED, the second is the charged current interaction that we have been developing in this part of the course, and the third term is a new **neutral current weak interaction**. We call it neutral because, like QED, it leaves the electric charge of the particles unchanged. The neutral current written out in full is

$$\begin{aligned} J_\mu^\mathrm{NC} &= 2(J_\mu^3 - \sin^2(\vartheta_\mathrm{w}) j_\mu) && (28.2.19)\\ &= 2\left[\frac{1}{2}(\bar\nu_\mathrm{L}\gamma_\mu\nu_\mathrm{L} - \bar\mathrm{e}_\mathrm{L}\gamma_\mu\mathrm{e}_\mathrm{L}) + \sin^2(\vartheta_\mathrm{w})\bar\mathrm{e}\gamma_\mu\mathrm{e}\right] && (28.2.20)\\ &= \bar\nu_\mathrm{L}\gamma_\mu\nu_\mathrm{L} + \bar\mathrm{e}\gamma_\mu\left[\left(2\sin^2\vartheta_\mathrm{w} - \frac{1}{2}\right) + \frac{1}{2}\gamma^5\right]\mathrm{e} && (28.2.21)\\ &= \bar\nu_\mathrm{L}\gamma_\mu\nu_\mathrm{L} + \bar\mathrm{e}\gamma_\mu(c_\mathrm{V} - c_\mathrm{A}\gamma^5)\mathrm{e} && (28.2.22) \end{aligned}$$

where $c_\mathrm{V} \coloneqq 2\sin^2\vartheta_\mathrm{w} - 1/2$ and $c_\mathrm{A} \coloneqq -1/2$. This shows that the neutral current is chiral, treating left and right handed fermions differently, but it is also not purely a $V-A$ current like the charged current, since it also couples to both left and right handed fermions.

Note that it is common to write

$$c_\mathrm{V} - c_\mathrm{A}\gamma^5 = c_\mathrm{L} P_\mathrm{L} + c_\mathrm{R} P_\mathrm{R} \tag{28.2.23}$$

which gives

$$c_\mathrm{L} = c_\mathrm{V} + c_\mathrm{A} = 2\sin^2\vartheta_\mathrm{w} - 1, \qquad \text{and} \qquad c_\mathrm{R} = c_\mathrm{V} - c_\mathrm{A} = 2\sin^2\vartheta_\mathrm{w}. \tag{28.2.24}$$

28.3 Feynman Rules

We have new interactions, so we also get new Feynman rules. The easiest to start with is the neutral current vertex. At this point we can just compare this term to the QED vertex and make the appropriate substitutions. In QED the term $e j_\mu A^\mu$ gives the vertex term $ie\gamma^\mu$, and so similarly the neutral current term $g J_\mu^\mathrm{NC} Z^\mu/(2\cos\vartheta_\mathrm{w})$ gives the vertex term

$$\frac{ig\gamma^\mu}{2\cos\vartheta_\mathrm{w}}(c_\mathrm{V} - c_\mathrm{A}\gamma^5) = \frac{ie}{\sin(2\vartheta_\mathrm{w})}\gamma_\mu(c_\mathrm{V} - c_\mathrm{A}\gamma^5). \tag{28.3.1}$$

For the other interactions we need to consider the kinetic terms. We do so by writing down the Yang–Mills Lagrangian

$$\mathcal{L}_\mathrm{YM} = -\frac{1}{2}\operatorname{tr}(\mathcal{W}^{\mu\nu}\mathcal{W}_{\mu\nu}) - \frac{1}{4}B^{\mu\nu}B_{\mu\nu} + \text{interactions} \tag{28.3.2}$$

where

$$\mathcal{W}_{\mu\nu} \coloneqq \partial_\mu \mathcal{W}_\nu - \partial_\nu \mathcal{W}_\mu - ig[\mathcal{W}_\mu, \mathcal{W}_\nu] \tag{28.3.3}$$

and

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{28.3.4}$$

Expanding this in terms of the generators we have

$$\mathcal{W}_\mu = \frac{1}{\sqrt{2}}(W_\mu^\dagger T^+ + W_\mu T^-) + W_\mu^3 T^3 \tag{28.3.5}$$

so

$$\mathcal{W}_{\mu\nu} = \frac{1}{\sqrt{2}}(W_{\mu\nu}^\dagger T^+ + W_{\mu\nu} T^-) + W_{\mu\nu}^3 \tag{28.3.6}$$

and using $[T^+, T^-] = 2T^3$ and $[T^3, T^\pm] = \pm T^\pm$ we can equate terms proportional to each generator and we find that

$$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - ig(W_\mu^3 W_\nu - W_\nu^3 W_\mu), \tag{28.3.7}$$

$$W_{\mu\nu}^3 = \partial_\mu W_\nu^3 - \partial_\nu W_\mu^3 - ig(W_\mu^\dagger W_\nu - W_\nu^\dagger W_\mu). \tag{28.3.8}$$

Next we can use $\mathrm{tr}(T_1^2 + T_2^2) = \mathrm{tr}(T^- T^+) = 1$ and $\mathrm{tr}(T_3^2) = 1/2$ to write the Yang–Mills Lagrangian as

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2} W_{\mu\nu}^\dagger W^{\mu\nu} - \frac{1}{4} W_{\mu\nu}^3 W_3^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \tag{28.3.9}$$

$$= -\frac{1}{2}(\partial_\mu W_\nu^\dagger - \partial_\nu W_\mu^\dagger)(\partial^\mu W^\nu - \partial^\nu W^\mu) \tag{28.3.10}$$

$$- \frac{1}{2}(\partial_\mu Z_\nu - \partial_\nu Z_\mu)(\partial^\mu Z^\nu - \partial^\nu Z^\mu) \tag{28.3.11}$$

$$- \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{28.3.12}$$

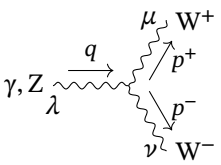
$$+ \mathcal{L}_{\mathrm{YM}}^3 + \mathcal{L}_{\mathrm{YM}}^4 \tag{28.3.13}$$

where $\mathcal{L}_{\mathrm{YM}}^3$ and $\mathcal{L}_{\mathrm{YM}}^4$ contain the three and four point interactions:

$$\mathcal{L}_{\mathrm{YM}}^3 = ig\big[(\partial_\mu W_\nu^\dagger - \partial_\nu W_\mu^\dagger) W_3^\mu W^\nu - W_3^\mu W^{\nu\dagger}(\partial_\mu W_\nu - \partial_\nu W_\mu) + (\partial_\mu W_\nu^3 - \partial_\nu W_\mu^3) W^{\mu\dagger} W^\nu\big], \tag{28.3.14}$$

$$\mathcal{L}_{\mathrm{YM}}^4 = g^2\big[\frac{1}{4}(W_\mu^\dagger W_\nu - W_\nu^\dagger W_\mu)(W^{\mu\dagger} W^\nu - W^{\nu\dagger} W^\mu) - (W_\mu^3 W_\nu^\dagger - W_\nu^3 W_\mu^\dagger) W_3^\mu W^\nu\big]. \tag{28.3.15}$$

Since the weak mixing matrix is orthogonal it preserves lengths, which in this case means that $W_3^2 + B^2 = Z^2 + A^2$, and so we get the usual kinetic terms for the W and Z bosons and the photon. Using the relations $W_\mu^3 = \cos(\vartheta_{\mathrm{w}}) Z^\mu + \sin(\vartheta_{\mathrm{w}}) A^\mu$ and $g = e/\sin\vartheta_{\mathrm{w}}$ we get the following Feynman rules for the three and four point interactions:

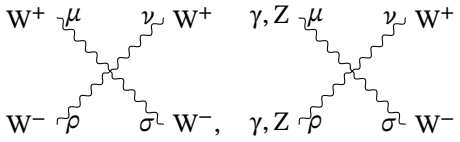


$$= iec_3[\eta_{\mu\nu}(p_\lambda^- - p_\lambda^+) - \eta_{\nu\lambda}(q_\mu + p_\mu^-) + \eta_{\lambda\mu}(q_\nu + p_\nu^+)] \tag{28.3.16}$$

where

$$c_3 = \begin{cases} 1 & \text{for photons,} \\ \cot \vartheta_\mathrm{w} & \text{for Z bosons,} \end{cases} \tag{28.3.17}$$

and



$$= ie^2 c_4 [2\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\sigma}] \tag{28.3.18}$$

where

$$c_4 = \begin{cases} \mathrm{cosec}^2 \vartheta_\mathrm{w} & \text{for WWWW,} \\ -1 & \text{for WW}\gamma\gamma\text{,} \\ -\cot \vartheta_\mathrm{w} & \text{for WWZ}\gamma\text{,} \\ -\cot^2 \vartheta_\mathrm{w} & \text{for WWZZ.} \end{cases} \tag{28.3.19}$$

Notice that the photon couples directly to the $\mathrm{W}^\pm$ through their charge, there are no couplings between the photon and the Z boson, since both are electrically neutral and the photon only couples to particles with non-zero electric charge. Notice that if we ignore the Z boson then the Lagrangian is invariant under $\mathrm{U}(1)_Q$, clearly the photon part is, since this is how we constructed it for QED, and the W parts all appear in conjugate pairs, so any phase transformation is going to cancel out.

Twenty-Nine

Neutral Current Interactions

29.1 Z Boson Mass

For most of us 2×2 matrices are about all we can cope with.

Richard Ball

The Z boson must be massive, otherwise neutral current interactions would be long ranged, which we don't see. We can manually insert a mass term for the Z boson. In doing so we must be careful to preserve the *global* $\mathrm{SU}(2)_\mathrm{L} \otimes \mathrm{U}(1)_Y$ symmetry. A mass term like this will always break gauge invariance, so we don't worry about the local symmetry. The most general mass term we can introduce which respects this global symmetry is

$$\mathcal{L}_M = M_\mathrm{W}^2 W_\mu^\dagger W^\mu + \frac{1}{2} M_\mathrm{W}^2 W_\mu^3 W_3^\mu + \frac{1}{2} M_B^2 B_\mu B^\mu. \tag{29.1.1}$$

The first two terms are formed from the fields which transform under the global SU(2) symmetry, and the last transforms under the global U(1) symmetry, although by construction each term is actually invariant here. Notice that W_3^μ and B^μ are real fields, and so we include the conventional factor of 1/2 which differs between real and complex fields.

The problem is that we cannot make the photon massless and the Z boson massive in this way. The solution is to make use of the fact that W_3^μ and B^μ mix, and introduce this mixing in the mass term using a mass matrix modifying this term by adding a mixing term at the end:

$$\mathcal{L}_M = M_\mathrm{W}^2 W_\mu^\dagger W^\mu + \frac{1}{2} M_\mathrm{W}^2 W_\mu^3 W_3^\mu + \frac{1}{2} M_B^2 B_\mu B^\mu - M_X^2 W_\mu^3 B^\mu. \tag{29.1.2}$$

Then we can express the mass terms for W_3^μ and B^μ as

$$\mathcal{L}_M = M_\mathrm{W}^2 M_\mu^\dagger W^\mu + \frac{1}{2} \begin{pmatrix} W_3^\mu & B^\mu \end{pmatrix} \begin{pmatrix} M_\mathrm{W}^2 & -M_X^2 \\ -M_X^2 & M_B^2 \end{pmatrix} \begin{pmatrix} W_\mu^3 & B_\mu \end{pmatrix}. \tag{29.1.3}$$

Since the photon is massless we know that the mass matrix,

$$M^2 = \begin{pmatrix} M_\mathrm{W}^2 & -M_X^2 \\ -M_X^2 & M_B^2 \end{pmatrix}, \tag{29.1.4}$$

must have a zero eigenvalue. This follows as the A^μ and Z^μ states are the mass eigenstates, that is the states with definite, unmixed, masses, and so their masses are eigenvalues of the mass matrix. This is what is meant by A^μ and Z^μ being the physical states, as opposed to W_3^μ and B^μ, which aren't seen in experiments. We also know that the eigenvector corresponding to this zero eigenvalue is the photon, $A^\mu = \sin(\vartheta_\mathrm{w})W_3^\mu + \cos(\vartheta_\mathrm{w})B^\mu$.

An eigenvalue of zero means that $\det(M) = 0$, and so we have $M_X^2 = M_\mathrm{W} M_B$. Thus, we have the eigenvalue equation

$$\begin{pmatrix} M_\mathrm{W}^2 & -M_\mathrm{W} M_B \\ -M_\mathrm{W} M_B & M_B^2 \end{pmatrix} \begin{pmatrix} \sin\vartheta_\mathrm{w} \\ \cos\vartheta_\mathrm{w} \end{pmatrix} = 0 \tag{29.1.5}$$

which we can solve to give

$$M_B \cos\vartheta_\mathrm{w} = M_\mathrm{W} \sin\vartheta_\mathrm{w}. \tag{29.1.6}$$

We can calculate the other eigenvalue using the fact that the trace is the sum of the eigenvalues, and since one of the eigenvalues is zero the trace is just the other eigenvalue. This second eigenvalue is M_Z^2, so

$$M_Z^2 = M_\mathrm{W}^2 + M_B^2 = M_\mathrm{W}^2(1 + \tan^2\vartheta_\mathrm{w}) = \frac{M_\mathrm{W}^2}{\cos\vartheta_\mathrm{w}}. \tag{29.1.7}$$

Notice that this is greater than M_W^2, so the Z boson is more massive than the $\mathrm{W}^\pm$ bosons.

Part V

Lattice Field Theory

Thirty

Lattices

Is it convenient? Not at all.

Antonin Portelli

30.1 Motivation

Normally we consider quantum field theories to be defined on Minkowski space, $\mathbb{R}^{1,3}$. That is, our fields are functions of four vectors, x, which have four real, continuous, coordinates in a chosen basis. In lattice field theory we instead define our fields on a lattice, a discrete grid of points. There are numerous reasons to do this, which will be the focus of this first section.

High momenta cause problems in quantum field theory through UV divergences. Through the Fourier transform high momenta are related to small distances. By working on a lattice we introduce a cut-off for how small a distance can be, and so how large a momentum can be. In this way lattice field theory is a regulation scheme for UV divergences.

As a regulation scheme lattice field theory is more complex than dimensional regularisation, which has the advantage of only capturing logarithmic divergences. The problems with dimensional regularisation are that it doesn't work in non-perturbative physics and has no physical meaning, it's just a mathematical trick. Lattice field theory on the other hand works non-perturbatively and can be understood physically. The trade off is that lattice field theory calculations are harder and in practice we resort to numerical methods.

Lattice field theory was first introduced by Kenneth Wilson in 1974, who was attempting to demonstrate quark confinement. Wilson also introduced the renormalisation group to physics.

The need for lattice field theory is essentially to avoid perturbation theory. Suppose we have a function, X, of some coupling g, then a perturbative calculation of X would give us

$$X(g) = X_0 + X_1 g^2 + X_2 g^4 + \mathcal{O}(g^6) \tag{30.1.1}$$

where we expand in $\alpha \propto g^2$. This can fail for two reasons, first perturbation theory assumes that α is small. If this is not the case then this series will not converge. This is an issue in QCD where at low energies the coupling is large.

The second problem is that not all functions are analytic. Further, a function can be finite and not analytic, it's not just about poles which we are used to working

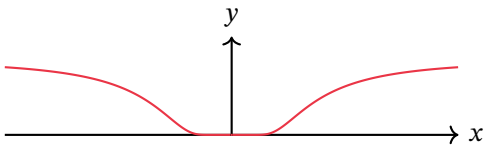


Figure 30.1: The function $f(x) = \exp\{-1/x^2\}$ is not analytic at zero since all derivative vanish at zero but the function vanishes only at zero, so its Taylor series at zero is identically zero, which does not converge to f.

around. For example, the function given by $f(x) = \exp\{-1/x^2\}$ is not analytic at 0 since every derivative is proportional to $\exp\{-1/x^2\}$ which vanishes as $x \to 0$, but the function vanishes only at 0. So the series $f(0) + f'(0)x + f''(x)/2 + \cdots = 0$ which does not converge to f. The problem here is basically that $\exp\{-1/x^2\}$ is really flat at the origin, so flat that all of its derivatives vanish (see Figure 30.1). This may cause problems if X is of the form

$$X(g) = \mathrm{e}^{-M/g^2}(X_0 + X_1 g^2 + X_2 g^4 + \mathcal{O}(g^6)) \tag{30.1.2}$$

then a perturbative expansion about $\alpha = 0$ will fail to capture the behaviour of $\exp\{-M/g^2\}$, no matter how many terms we include. This isn't the only time this could cause issues, the function $X(g) = g$ also can't be dealt with perturbatively in α, since derivatives of $\sqrt{\alpha}$ vanish at $\alpha = 0$.

30.2 Euclidean vs Minkowski

The whole method is simple. Making it work is difficult.

Antonin Portelli

For relativity we want to work in Minkowski space, but we have to work on a lattice in Euclidean space. This means we need to translate familiar quantities into Euclidean space. The difference between Minkowski space and Euclidean space is the metric. In Minkowski space we have

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1). \tag{30.2.1}$$

In Euclidean space we instead have

$$\delta_{\mu\nu} = \mathrm{diag}(1, 1, 1, 1). \tag{30.2.2}$$

Suppose we have a four-vector $x = (x_0, \boldsymbol{x})$. Then the square of this quantity in Minkowski space is

$$(x \cdot x)_{\mathrm{M}} = x^\mu x^\nu \eta_{\mu\nu} = x_0^2 - \boldsymbol{x}^2. \tag{30.2.3}$$

The square of this quantity in Euclidean space is

$$(x \cdot x)_{\mathrm{E}} = x_\mu x_\nu \delta_{\mu\nu} = x_0^2 + \boldsymbol{x}^2. \tag{30.2.4}$$

Notice that in Euclidean space we are free to lower all indices.

Now make the substitution $x_0 \mapsto ix_0$, so $x \mapsto \bar{x} = (ix_0, \boldsymbol{x})$. Then the square of this quantity in Minkowski space is

$$(\bar{x} \cdot \bar{x})_{\mathrm{M}} = (ix_0)^2 - \boldsymbol{x}^2 = -x_0^2 - \boldsymbol{x}^2 = -(x \cdot x)_{\mathrm{E}}. \tag{30.2.5}$$

So we get the Euclidean metric by analytically continuing the time variable to take on purely imaginary values. This is known as a **Wick rotation**, since multiplication by i is the same as a rotation by $\pi/2$ anticlockwise in the complex plane.

This Wick rotation is slightly different to Wick rotations in loop integrals since those are just a change of variable to a variable which happens to be complex, but here we are not integrating anything. We have to be careful because there is a sense in which Wick rotating and integrating don't rotate. It is not always the case that a change of variables to imaginary time in an integral will give the same result as starting off with a theory in Euclidean spacetime, although often the result is the same.

Take as an example φ^4 theory. Given a scalar field φ the Lagrangian in Minkowski space is

$$\mathcal{L}_{\mathrm{M}}[\varphi] = \frac{1}{2}[(\partial_\mu \varphi(x))(\partial^\mu \varphi(x))]_{\mathrm{M}} - \frac{1}{2}m^2\varphi(x)^2 - \frac{\lambda}{4!}\varphi(x)^4. \tag{30.2.6}$$

The action is then

$$S_{\mathrm{M}}[\varphi] = \int \mathrm{d}^4x\, \mathcal{L}_{\mathrm{M}}[\varphi]. \tag{30.2.7}$$

The expectation value of an observable O is

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}f\, O[\varphi] \exp\{iS_{\mathrm{M}}[\varphi]\}. \tag{30.2.8}$$

Here $\mathcal{Z}$ is the partition function, chosen such that $\langle 1 \rangle = 1$, so

$$\mathcal{Z} = \int \mathcal{D}\varphi\, \exp\{iS_{\mathrm{M}}[\varphi]\}. \tag{30.2.9}$$

A change of variables to imaginary time gives an overall factor of i in the exponential coming from the $\mathrm{d}t$ measure, and we are left with

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\varphi\, O[\varphi] \exp\{-S_{\mathrm{E}}[\varphi]\} \tag{30.2.10}$$

where the Euclidean action is

$$S_{\mathrm{E}}[\varphi] = \int \mathrm{d}^4x\, \mathcal{L}_{\mathrm{E}}[\varphi] \tag{30.2.11}$$

and the Euclidean Lagrangian is

$$\mathcal{L}_{\mathrm{E}}[\varphi] = \frac{1}{2}[(\partial_\mu \varphi(x))(\partial_\mu \varphi(x))]_{\mathrm{E}} + \frac{1}{2}m^2\varphi(x)^2 + \frac{\lambda}{4!}\varphi(x)^4. \tag{30.2.12}$$

Notice that the Euclidean action is non-negative. This means that the $\exp\{-S_{\mathrm{E}}[\varphi]\}$ factor makes the integral converge, well, as much as a path integral can converge. If this was a normal integral we could apply the saddle point method[1] and we

[1] See *Methods of Mathematical Physics*.

would find that the integral is dominated by the region where $\delta S_{\mathrm{E}} = 0$, which is to say we recover the principle of least action and classical physics.

On the other hand for the Minkowski path integral we have a factor of $\exp\{iS_\eta[\varphi]\}$ which is oscillatory. We then have to apply the slightly more complex method of stationary phase[2]. The result however is the same, the $\delta S_{\mathrm{E}} = 0$ region dominates and we recover the principle of least action and classical physics.

[2] Again, see *Methods of Mathematical Physics*

The difference is that of absolute convergence vs semi-convergence. A convergent integral,

$$I = \int f(x)\,\mathrm{d}x, \tag{30.2.13}$$

is absolutely convergent if

$$\int |f(x)|\,\mathrm{d}x \tag{30.2.14}$$

is convergent, and semi-convergent otherwise. Intuitively an absolutely convergent integral converges because as we go to infinity the function vanishes sufficiently quickly. A semi-convergent integral on the other hand converges because as we go to infinity the oscillations cancel. This means that for the purposes of numerical methods absolutely convergent integrals converge faster than semi-convergent integrals. The Euclidean path integral mimics absolute convergence, whereas the Minkowski path integral only mimics semi-convergence.

To proceed with a lattice field theory calculation from this point the procedure is fairly simple. We have a probability density $\exp\{-S_{\mathrm{E}}[\varphi]\}/\mathcal{Z}$ from which we can draw fields. So long as the lattice is finite we can then compute O for each field we draw, and take an average to find $\langle O\rangle$. This gives a Monte Carlo approximation for $\langle O\rangle$ and we can extract pretty much any physics of interest in this way.

From now on we will work exclusively with Euclidean quantities, unless we specify otherwise. This means we will drop the E label. We will lower indices. We also *do not use* the Einstein summation convention because we will end up writing terms like x_μ^2 to mean the μth component of x squared, not $\sum_\mu x_\mu x_\mu = x^2$. This is because in the discrete case we consider a different symmetry group to the Lorentz group.

! From now on all quantities are assumed to be Euclidean, and the Einstein summation convention is not used.

30.3 What is a Lattice

R For more on lattices see *Introduction to Condensed Matter Physics*.

Our ideal lattice is infinite, extending in all four dimensions. We take the simplest geometry possible, a cubic, or rather hypercubic, lattice. This means that a point on our lattice is of the form $x = an$ where a is some fixed length and $n \in \mathbb{Z}^4$ is a four-vector with integer components. Thus, the set of points making up our lattice is

$$\Lambda^4(a) \coloneqq a\mathbb{Z}^4 = \{(an_1, an_2, an_3, an_4) \mid n_i \in \mathbb{Z}\}. \tag{30.3.1}$$

A two-dimensional slice of this lattice is drawn in Figure 30.2.

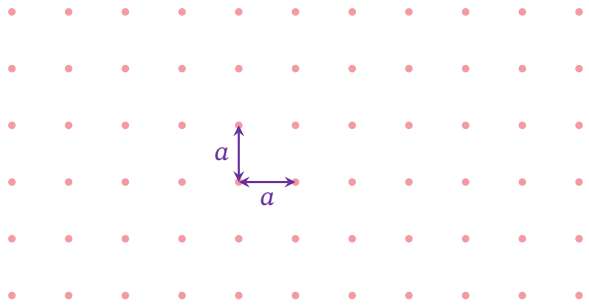


Figure 30.2: A slice through the lattice $\Lambda^4(a)$ in a plane with two components of the lattice vector n fixed to integer values.

The length a is called the **lattice spacing**, as it is the shortest distance between lattice points. We want to take $a \to 0$ to recover quantum field theory on a continuos space. This is called the **continuum limit**. Note that this limit is defined for functions on $\Lambda^4(a)$, not for the set $\Lambda^4(a)$ itself.

We equip our lattice with the Euclidean geometry it inherits as a subspace of $\mathbb{R}^4$. The Lorentz group, $\mathrm{O}(1,3)$, is the group of symmetries we are interested in in Minkowski space, $\mathbb{R}^{1,3}$. In Euclidean space, $\mathbb{R}^4$, we are instead interested in $\mathrm{O}(4)$. On our lattice we are interested in the subgroup of $\mathrm{O}(4)$ which maps the lattice onto itself. This rules out most transformations, for example a rotation by $\pi/3$ would not map the lattice onto itself, but a rotation by $\pi/2$ would. The resulting set of symmetries are those which are in $\mathrm{O}(4)$ and are also symmetries of the hypercube. This is the finite automorphism group of the hypercube, which is of order $2^4 4! = 384$.

The consequence of reducing the number of symmetries is that we have more independent covariant tensors. For example, under this group if we have a four vector, x, then the vector x_μ^n given by the μth component of x raised to the nth power for $n \in \mathbb{N}\setminus\{0,1\}$ is hypercubic covariant, even though such an object would not be Lorentz covariant.

Continuous symmetries are very strong constraints, and so don't leave us with many invariant tensors. On the other hand a finite symmetry group has an infinite number of independent invariants, in this case k^2, k^4, k^6 and so on are all independent invariants. Note that $k^n = \sum_\mu k_\mu^n \neq \|k\|^n = (k^2)^{n/2}$.

Intuitively this breaking of symmetry to a finite symmetry group leaves us with higher order terms in our expressions, so we end up with an expansion in a about the $a = 0$ continuum limit. The extra terms should not be too surprising, we introduce a dimensionful parameter, a, and this lets us produce the dimensionless quantity ak_μ, and we can then perform expansions in this, and there is no reason why the higher order terms need to vanish.

30.3.1 Non-Ideal Lattices

In a numerical simulation we can't have an infinite lattice, instead we make do with a finite lattice,

$$\Lambda^4(a;N) \coloneqq a([\![0,N_0-1]\!]\times[\![0,N_1-1]\!]\times[\![0,N_2-1]\!]\times[\![0,N_3-1]\!]) \quad (30.3.2)$$

where $N \in \mathbb{Z}^4$ is a four-vector with integer components and

$$[\![n, m]\!] \coloneqq [n, m] \cap \mathbb{Z} = \{k \in \mathbb{Z} \mid n \leq n \leq m\} = \{n, n+1, \dots, m\} \tag{30.3.3}$$

denotes the set of all integers between n and m.

The integer N_μ tells us the number of lattice sites in the μth direction. We can then define the spacetime extent of the lattice in the μth direction, $L_\mu \coloneqq aN_\mu$. The spacetime volume of the lattice is then $V_4 = L_0 L_1 L_2 L_3$, and the spatial volume is $V_3 = L_1 L_2 L_3$.

If N_i are not all equal then the group of symmetries is further reduced to a subgroup of the hypercube group. In a typical calculation the spatial extents are all the same but the temporal extent differs. This means rotations taking spatial axes to the time axis are no longer symmetries and the symmetry group is reduced to that of the three-dimensional cube, which is of order $2^3 3! = 48$. We might then denote the lattice $\Lambda^4(a; N_t, N_x)$ where $T = L_0 = aN_t$ is the temporal extent and $L = L_1 = L_2 = L_3 = N_x a$ is the spatial extent, equal in all directions.

If we want to compare this finite lattice to the continuum we have to define a finite volume continuous spacetime. For this equal-spacetime-extent case this can be done by defining

$$\mathrm{T}^4(T, L) \coloneqq [0, T) \times [-L/2, L/2)^3. \tag{30.3.4}$$

Sometimes we also want to have the time extent be infinite, and the spatial extent finite, in this case we may work on the space

$$\mathrm{C}^4(L) \coloneqq \mathbb{R} \times \mathrm{T}^3(L), \qquad \text{where} \qquad \mathrm{T}^3(L) \coloneqq [-L/2, L/2)^3. \tag{30.3.5}$$

As both of these finite spaces have boundaries we need boundary conditions in order to solve any equations. The most common assumption being periodic boundary conditions. That is, if we have a field φ then we have

$$\varphi(x + L_\mu \hat{\mu}) = \varphi(x) \tag{30.3.6}$$

for all $\mu = 0, \dots, 3$ where $\hat{\mu}$ is a unit vector in the μth direction.

For most of our work however we shall assume infinite, ideal, hypercubic lattices.

Thirty-One

Calculus on a Lattice

Most concepts in calculus are defined on continuous spaces, so we need to modify them to work on a discrete lattice. For this we turn to the fundamental definitions, in terms of limits, and simply replace a limit to zero with the smallest non-zero distance in our space, the lattice spacing. This allows us to define derivatives, integrals, Fourier transforms, and pretty much anything else we're interested in.

31.1 Derivatives

31.1.1 First Derivatives

R In this section we define the lattice derivatives. In the context of numerically solving differential equations these quantities are known as finite differences. For their use in this context see *Modelling and Visualisation in Physics*.

Consider the limit definition of the derivative of a differentiable function $f : \mathbb{R}^{1,3} \to \mathbb{R}$:

$$\partial_\mu f(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon\hat{\mu}) - f(x)}{\varepsilon} \tag{31.1.1}$$

where $\hat{\mu}$ is a unit vector in the μth direction. We can discretise this by simply replacing the limit to zero with the smallest non-zero distance possible on a lattice, the lattice spacing, a. Then we define the **forward derivative** for a function $f : \Lambda^4(a) \to \mathbb{R}$:[1]

$$\delta_\mu f(x) \coloneqq \frac{f(x + a\hat{\mu}) - f(x)}{a}. \tag{31.1.2}$$

This has the property that in the continuum limit, $a \to 0$, $\delta_\mu f(x) \to \partial_\mu f(x)$.

We call this the forward derivative because it involves taking a step "forward" in the μth direction and computing the ratio of how the function changes to the step size. However, the derivative can be more generally defined for a differentiable function $f : \mathbb{R}^{1,3} \to \mathbb{R}$ through

$$\partial_\mu f(x) \coloneqq \lim_{y \to x} \frac{f(y) - f(x)}{y - x}. \tag{31.1.3}$$

From this we see that the first limit definition was simply the case where $y = x + \varepsilon\hat{\mu}$. If instead we take $y = x - \varepsilon\hat{\mu}$ then we get

$$\partial_\mu f(x) = \lim_{\varepsilon \to 0} \frac{f(x) - f(x - \varepsilon\hat{\mu})}{\varepsilon}. \tag{31.1.4}$$

[1] often δ, and the other first derivatives we're about to define, are simply denoted ∂_μ, but this is obviously confusing when we are starting out and want to go back and forth between the lattice and the continuous case.

We can then define the **backward derivative** for a function $f : \Lambda^4(a) \to \mathbb{R}$

$$\delta^*_\mu f(x) \coloneqq \frac{f(x) - f(x - a\hat{\mu})}{a}. \tag{31.1.5}$$

This also has the property that in the continuum limit $\delta^*_\mu f(x) \to \partial_\mu f(x)$.

If we assume that f is analytic and expand $f(x + a\hat{\mu})$ around x then we find

$$f(x + a\hat{\mu}) = f(x) + \partial_\mu f(x) + \frac{a^2}{2}\partial^2_\mu f(x) + \mathcal{O}(a^3). \tag{31.1.6}$$

Remember that we aren't using the Einstein summation convention here. Moving $f(x)$ over to the left hand side and dividing by a we find that

$$\delta_\mu f(x) = \frac{f(x + a\hat{\mu}) - f(x)}{a} = \partial_\mu f(x) + \frac{a}{2}\partial^2_\mu f(x) + \mathcal{O}(a^2). \tag{31.1.7}$$

Hence, in the $a \to 0$ limit our forward derivative does indeed coincide with the normal derivative. In the discrete case however we have next to leading order effects from the a term. These are known as **discretisation effects**. Similarly, for the backward derivative expanding $f(x - a\hat{\mu})$ about x gives

$$\delta^*_\mu f(x) = \partial_\mu f(x) - \frac{a}{2}\partial^2_\mu f(x) + \mathcal{O}(a^2). \tag{31.1.8}$$

So we again have next to leading order effects, in fact the same size of effect, but in the opposite direction. This means that the forward and backward derivative are both equally good approximations of the derivative.

We can combine the forward and backward derivatives in such a way that these next to leading orders cancel to leave us with next to next to leading order corrections, and then define the **central derivative**:

$$\bar{\delta}_\mu f(x) \coloneqq \frac{1}{2}[\delta_\mu f(x) + \delta^*_\mu f(x)]. \tag{31.1.9}$$

Combining the expansions above we see that

$$\bar{\delta}_\mu f(x) = \partial_\mu f(x) + \mathcal{O}(a^2). \tag{31.1.10}$$

This means that the central derivative has better convergence than the forward or backward derivatives.

We can write the derivatives in terms of translation operators,

$$\tau_\mu f(x) \coloneqq f(x + a\hat{\mu}), \qquad \text{and} \qquad \tau_{-\mu} f(x) \coloneqq f(x - a\hat{\mu}). \tag{31.1.11}$$

Then

$$\delta_\mu = \frac{1}{a}(\tau_\mu - 1), \quad \delta^*_\mu = \frac{1}{a}(1 - \tau_{-\mu}), \quad \text{and} \quad \bar{\delta}_\mu = \frac{1}{2a}(\tau_\mu - \tau_{-\mu}). \tag{31.1.12}$$

It is actually possible to get arbitrarily good precision in our estimates for the derivative. By this we mean we can define $\delta^{(j)}_\mu$ for $j \in \mathbb{N}$ such that $\delta^{(j)}_\mu f(x) = \partial_\mu f(x) + \mathcal{O}(a^j)$. For example, we can take the third order forward derivative

$$\delta^{(3)}_\mu = \frac{1}{a}\left(-\frac{11}{6} + 3\tau_\mu - \frac{3}{2}\tau^2_\mu + \frac{1}{3}\tau^3_\mu\right), \tag{31.1.13}$$

$$\delta^{(3)}_\mu f(x) = \partial_\mu f(x) + \mathcal{O}(a^3), \tag{31.1.14}$$

or the fourth order central derivative

$$\bar{\delta}^{(4)}_{\mu} = \frac{1}{a}\left(\frac{1}{12}\tau^2_{-\mu} - \frac{2}{3}\tau_{-\mu} + \frac{2}{3}\tau_{\mu} - \frac{1}{12}\tau^2_{\mu}\right), \tag{31.1.15}$$

$$\bar{\delta}^{(4)}_{\mu} f(x) = \partial_{\mu} f(x) + \mathcal{O}(a^4). \tag{31.1.16}$$

While these are more accurate they are also more complicated to compute, so there is a trade-off. The derivatives also start to depend on lattice sites further and further away, so become non-local.

31.1.1.1 Identities

We can derive several identities for the discrete derivatives. Some of these are simply the discrete versions of familiar identities, and some are new. For example,

$$\delta_{\mu} - \delta^*_{\mu} = \frac{1}{a}(1 - \tau_{\mu} - 1 - \tau_{-\mu}) = \frac{1}{a}(-\tau_{\mu} - \tau_{-\mu}) \tag{31.1.17}$$

and

$$\delta_{\mu}\delta^*_{\mu} = \frac{1}{a^2}(1 - \tau_{\mu})(1 - \tau_{-\mu}) = \frac{1}{a^2}(1 - \tau_{-\mu} - \tau_{\mu} - \tau_{\mu}\tau_{-\mu}) \tag{31.1.18}$$

and clearly $\tau_{\mu}\tau_{-\mu} = 1$ since

$$\tau_{\mu}\tau_{-\mu} f(x) = \tau_{\mu} f(x - a\hat{\mu}) = f(x - a\hat{\mu} + a\hat{\mu}) = f(x) \tag{31.1.19}$$

so

$$\delta_{\mu}\delta^*_{\mu} = \frac{1}{a^2}(1 - \tau_{-\mu} - \tau_{\mu} - 1) = \frac{1}{a^2}(-\tau_{-\mu} - \tau_{\mu}) \tag{31.1.20}$$

and so

$$\delta_{\mu} - \delta^*_{\mu} = a\delta_{\mu}\delta^*_{\mu}. \tag{31.1.21}$$

Another identity is

$$\delta_{\mu}\tau_{-\mu} = \tau_{-\mu}\partial_{\mu} = \delta^*_{\mu} \tag{31.1.22}$$

which can be shown by expanding the left hand side

$$\delta_{\mu}\tau_{-\mu} = \frac{1}{a}(\tau_{\mu} - 1)\tau_{-\mu} = \frac{1}{a}(1 - \tau_{-\mu}) = \delta^*_{\mu}. \tag{31.1.23}$$

We now define a, slightly pedantic, notion which should allow for less ambiguity in the following derivations. If g is a function then we denote by $\underline{g}$ the operator defined by

$$(\underline{g} f)(x) = g(x) f(x). \tag{31.1.24}$$

This notation reduces the number of brackets we need for unambiguous expressions. For example, the Leibniz rule, also known as the product rule, normally written as

$$\partial_{\mu}(fg) = (\partial_{\mu} f)g + f(\partial_{\mu} g) \tag{31.1.25}$$

can be expressed as

$$[\partial_\mu, \underline{f}] = \underline{\partial_\mu f}. \tag{31.1.26}$$

To see this we act on a test function, g:

$$[\partial_\mu, \underline{f}]g = \partial_\mu \underline{f} g - \underline{f}\partial_\mu g \tag{31.1.27}$$
$$= \partial_\mu(fg) - f\partial_\mu g \tag{31.1.28}$$

and on the other side

$$\underline{\partial_\mu f} g = (\partial_\mu f)g. \tag{31.1.29}$$

Setting these equal and rearranging we get the normal form of the Leibniz rule.

The Leibniz rule does not hold in general for our discrete derivatives, but is instead affected by discretisation effects. In particular, we find that

$$\delta_\mu(fg) = (\delta_\mu f)\tau_\mu g + f(\delta_\mu g) \tag{31.1.30}$$
$$= (\delta_\mu f)g + f(\delta_\mu g) + a(\delta_\mu f)(\delta_\mu g), \tag{31.1.31}$$
$$\delta^*_\mu(fg) = (\delta^*_\mu f)\tau_{-\mu} g + f(\delta^*_\mu g) \tag{31.1.32}$$
$$= (\delta^*_\mu f)g + f(\delta^*_\mu g) - a(\delta^*_\mu f)(\delta^*_\mu g), \tag{31.1.33}$$
$$\bar{\delta}_\mu(fg) = (\bar{\delta}_\mu f)g + f(\bar{\delta}_\mu g) + \frac{a}{2}[(\delta_\mu f)(\delta_\mu g) - (\delta^*_\mu f)(\delta^*_\mu g)] \tag{31.1.34}$$
$$= (\bar{\delta}_\mu f)g + f(\bar{\delta}_\mu g) + \frac{a^2}{2}[(\delta_\mu f)(\delta_\mu \delta^*_\mu g) + (\delta_\mu \delta^*_\mu f)(\delta^*_\mu g)]. \tag{31.1.35}$$

All of these can be written in terms of commutators:

$$[\delta_\mu, \underline{f}] = \underline{\delta_\mu f \tau_\mu} \quad = \underline{\delta_\mu f} + a\underline{\delta_\mu f}\delta_\mu \tag{31.1.36}$$
$$[\delta^*_\mu, \underline{f}] = \underline{\delta^*_\mu f \tau_{-\mu}} = \underline{\delta^*_\mu f} - a\underline{\delta^* f}\delta^*_\mu \tag{31.1.37}$$
$$[\bar{\delta}_\mu, \underline{f}] = \underline{\bar{\delta}_\mu f} + \frac{a^2}{2}[\underline{\delta_\mu f}\delta_\mu\delta^*_\mu + \underline{\delta_\mu \delta^*_\mu f}\delta^*_\mu]. \tag{31.1.38}$$

The Lie bracket also has the Leibniz rule, as a rewriting of the Jacobi identity. In particular, fix some $x \in \mathfrak{g}$ and define $D(y) = [x, y]$. Then the Jacobi identity tells us that

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0. \tag{31.1.39}$$

Using the anticommutativity of the Lie bracket we can write $[z, x] = -[x, z] = -D(z)$ and so

$$D([y, z]) - [y, D(z)] + [z, D(y)] = 0. \tag{31.1.40}$$

Rearranging this we get

$$D([y, z]) = [y, D(z)] + [D(y), z] \tag{31.1.41}$$

where we have used antisymmetry to absorb a minus sign. If we now write $[a, b] = ab$ this becomes

$$D(yz) = yD(z) + D(y)z \tag{31.1.42}$$

which is clearly the Leibniz rule if y and z are functions and D is a derivative.

Like the Leibniz rule for derivatives the Leibniz rule for Lie brackets also suffers discretisation effects. Worse than this, the entire concept of a Lie algebra breaks down since on the lattice there is no notion of an infinitesimal transformation. One of the main effects of this is that we no longer have continuous translational symmetries, and so the energy-momentum tensor is no longer a conserved current.

We also have

$$(\tau_\mu \underline{f} g)(x) = \tau_\mu(f(x)g(x)) = f(x + a\hat{\mu})g(x + a\hat{\mu}) \tag{31.1.43}$$

and

$$(\underline{\tau_\mu f} \tau_\mu g)(x) = \underline{\tau_\mu f} g(x + a\hat{\mu}) = f(x + a\hat{\mu})g(x + a\hat{\mu}), \tag{31.1.44}$$

so

$$\tau_\mu \underline{f} = \underline{\tau_\mu f} \tau_\mu. \tag{31.1.45}$$

31.1.2 Second Derivatives

We can combine the derivatives we have already found to for second derivatives. The most interesting case being the Laplacian, or in Minkowski space, the d'Alembertian. The naive definition is simply to square one of our existing derivatives and sum over dimensions, defining the forward, backward, and central Laplacians:

$$\delta^2 \coloneqq \sum_\mu \delta_\mu^2, \qquad \delta^{*2} \coloneqq \sum_\mu \delta_\mu^{*2}, \qquad \text{and} \qquad \bar{\delta}^2 \coloneqq \sum_\mu \bar{\delta}_\mu^2. \tag{31.1.46}$$

However, we shall see that often the following mixed Laplacian has nicer properties:

$$\tilde{\delta}^2 \coloneqq \sum_\mu \delta_\mu^* \delta_\mu. \tag{31.1.47}$$

31.2 Integrals

Recall that the integral of some function $f : \mathbb{R} \to \mathbb{R}$ can be defined as a Riemann sum, approximating the area under the curve as the sum of the areas of rectangles of some fixed width Δx and height $f(x_i)$ where x_i is the value of x at some point along the bottom edge of the rectangle. Then, assuming this limit converges, the integral is defined to be

$$\int f(x)\,\mathrm{d}x \coloneqq \lim_{\substack{\Delta x \to 0 \\ N \to \infty}} \sum_{i=1}^{N} f(x_i)\Delta x. \tag{31.2.1}$$

This generalises to higher dimensions and complex functions, although the geometric notion of area doesn't work so well for complex functions.

Motivated by this definition of the integral we define the discrete integral, I, of $f : \Lambda^4(a) \to \mathbb{R}$ as

$$I \coloneqq a^4 \sum_{x \in \Lambda^4} f(x) \tag{31.2.2}$$

where a^4 plays the role of the infinitesimal spacetime volume, $\mathrm{d}V = \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ in the discrete case. Sometimes sums like this one will be written simply as $\sum_x$, rather than $\sum_{x \in \Lambda^4}$.

31.2.1 Discrete Dirac Delta

The Dirac delta arises from taking some space of functions and asking, is there a function g such that for all functions f we have

$$\int f(x)g(x)\mathrm{d}x = f(0)? \tag{31.2.3}$$

The answer is no, there is no such *function* g, since it can be shown that such a function must diverge at $x = 0$ which makes this integral ill-defined. However, if we relax the requirement that g be a function and allow what we call distributions and build a theory of integrals which works with distributions then the answer becomes yes, and $g(x) = \delta(x)$ defines the Dirac delta. Of course, in practice in physics one rarely makes such a distinction.

We don't need to make this function-distribution distinction in the finite case, since we don't need to assign the value of infinity to discrete Dirac delta. The discrete version of the Dirac delta, $\delta(x)$, is, unsurprisingly, the Kronecker delta, $a^{-4}\delta_{x0}$, where the factor of a^{-4} ensures we get the correct continuum limit where "$\delta(0) = \infty$". We then have the expected sifting property

$$a^4 \sum_{x\in\Lambda^4} f(x)[a^{-4}\delta_{x0}] = f(0) \tag{31.2.4}$$

More generally, $\delta(x - y)$ is given by $a^{-4}\delta_{xy}$.

31.2.2 Inner Product

We can use the discrete integral to define an inner product on the space of square integrable lattice fields, which for now we take to be the space, $L^2(\Lambda^4(a))$, of functions $f : \Lambda^4(a) \to \mathbb{C}$ such that

$$a^4 \sum_{x\in\Lambda^4} f(x)^* f(x) \tag{31.2.5}$$

converges. The inner product, $\langle -|-\rangle$, on this space is then

$$\langle f|g\rangle = a^4 \sum_{x\in\Lambda^4} f(x)^* g(x). \tag{31.2.6}$$

Recall that the Hermitian adjoint, $T^\dagger$, of an operator, T, is defined, with respect to a given inner product $\langle -|-\rangle$, by the equality

$$\langle f|Tg\rangle = \langle T^\dagger f|g\rangle. \tag{31.2.7}$$

Using this, we see that

$$\begin{aligned}\langle f|\tau_\mu g\rangle &= a^4 \sum_{x\in\Lambda^4} f(x)^* \tau_\mu g(x) = a^4 \sum_{x\in\Lambda^4} f(x)^* g(x + a\hat{\mu}) \\ &= a^4 \sum_{x\in\Lambda^4} f(x - a\hat{\mu})^* g(x) = a^4 \sum_{x\in\Lambda^4} [\tau_{-\mu} f(x)]^* g(x) = \langle \tau_{-\mu} f|g\rangle\end{aligned} \tag{31.2.8}$$

where we get to the second line by performing a change of integration variable, $x \to x - a\hat{\mu}$, which doesn't change the integration limits on an infinite lattice. Thus, the Hermitian adjoint of the translation is the translation in the opposite

direction, so $\tau_\mu^\dagger = \tau_{-\mu}$. We also know that $\tau_{-\mu} = \tau_\mu^{-1}$, and so we see that τ_μ is a unitary operator.

Using this, and the usual rules for computing adjoints of combined operators we see that

$$\delta_\mu^\dagger = \frac{1}{a}(\tau_\mu - 1)^\dagger = \frac{1}{a}(\tau_{-\mu} - 1) = -\delta_\mu^* \tag{31.2.9}$$

and

$$\bar{\delta}_\mu^\dagger = \frac{1}{2a}(\tau_\mu - \tau_{-\mu})^\dagger = \frac{1}{2a}(\tau_{-\mu} - \tau_\mu) = -\bar{\delta}_\mu. \tag{31.2.10}$$

This shouldn't be too surprising, after all, the usual derivative must be anti-Hermitian, $(\mathrm{d}/\mathrm{d}x)^\dagger = -\mathrm{d}/\mathrm{d}x$, so that $p = i\mathrm{d}/\mathrm{d}x$ is a Hermitian operator.

Putting these relations into the definition of the Hermitian adjoint we have

$$\langle \delta_\mu f | g \rangle = a^4 \sum_{x \in \Lambda^4} [\delta_\mu f(x)]^* g(x) = -a^4 \sum_{x \in \Lambda^4} f(x)^* [\delta_\mu^* g(x)] = \langle f | \delta_\mu^{*\dagger} g \rangle \tag{31.2.11}$$

and

$$\langle \bar{\delta}_\mu f | g \rangle = a^4 \sum_{x \in \Lambda^4} [\bar{\delta}_\mu f(x)]^* g(x) = -a^4 \sum_{x \in \Lambda^4} f(x)^* [\bar{\delta}_\mu g(x)] = \langle f | \bar{\delta}_\mu^\dagger g \rangle. \tag{31.2.12}$$

We can then interpret these as discrete versions of integration by parts, where the surface term vanishes.

The Laplacian operators $\tilde{\delta}^2$ and $\bar{\delta}^2$ are both Hermitian operators, which can be seen from their definitions as the negatives cancel and the order of the operators is reversed by the adjoint:

$$(\tilde{\delta}^2)^\dagger = \left(\sum_\mu \delta_\mu^* \delta_\mu \right)^\dagger = \sum_\mu \delta_\mu^\dagger \delta_\mu^{*\dagger} = \sum_\mu \delta_\mu^* \delta_\mu = \tilde{\delta}^2 \tag{31.2.13}$$

and similarly for $\bar{\delta}^2$. This means we should use $\tilde{\delta}^2$ or $\bar{\delta}^2$ to define the kinetic energy of a particle, since it will then be positive definite. Of course, in the continuum limit the forward and backward Laplacian also become Hermitian, since all of these operators converge to the Laplacian.

31.3 Functional Derivatives

For a function, f, defined on a continuous spacetime we can define the functional derivative of some functional F by

$$\frac{\delta}{\delta f(x)} F[f] \coloneqq \lim_{\sigma \to 0} \lim_{\varepsilon \to 0} \frac{F[f + \varepsilon \varphi_{y,\sigma}] - F[f]}{\varepsilon} \tag{31.3.1}$$

assuming this converges with $\varphi_{y,\sigma}$ a sequence of smooth functions converging, in the sense of distributions, to $\delta(x-y)$ as $\sigma \to 0$. In this definition it is important to take the $\varepsilon \to 0$ limit first, since otherwise we end up with Dirac deltas on the right hand side despite not being in an integral, which isn't well-defined.

We don't usually compute explicit functional derivatives, instead we make use of properties like

$$\frac{\delta}{\delta f(x)} f(y) = \delta(x - y), \tag{31.3.2}$$

which is the functional version of

$$\frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu{}_\nu, \tag{31.3.3}$$

and

$$\frac{\delta}{\delta f(x)} g(f(y)) = g'(f(y))\delta(x-y), \tag{31.3.4}$$

which is simply the chain rule.

On the lattice the definition of the functional derivative is actually simpler, since we don't have to worry about distributions, and so we simply replace the Dirac delta with the Kronecker delta. We can do away with the sequence of smooth functions converging to the Dirac delta and simply place the Kronecker delta in immediately, giving the discrete functional derivative

$$\frac{\delta}{\delta f(x)} F[f] \coloneqq \lim_{\varepsilon \to 0} \frac{F[f + \varepsilon a^{-4}\delta_y] - F[f]}{\varepsilon} \tag{31.3.5}$$

where $\delta_y : \Lambda^4(a) \to \{0,1\}$ is the function defined by $\delta_y(x) = \delta_{xy}$.

This definition gives us the lattice version of the useful properties of the functional derivative

$$\frac{\delta}{\delta f(x)} f(y) = a^{-4}\delta_{xy}, \qquad \text{and} \qquad \frac{\delta}{\delta f(x)} g(f(y)) = g'(f(x))a^{-4}\delta_{xy}. \tag{31.3.6}$$

31.4 Fourier Transform

Using our definition of discrete integration we can define the discrete Fourier transform of some function, f:

$$(\mathcal{F}f)(k) = \hat{f}(k) \coloneqq a^4 \sum_{x \in \Lambda^4} f(x) \mathrm{e}^{-ik\cdot x}. \tag{31.4.1}$$

Notice that $k \cdot x = \sum_\mu k_\mu x_\mu$ is the Euclidean dot product. The continuum limit of this is then

$$\int \mathrm{d}^4x\, f(x) \mathrm{e}^{-ik\cdot x}, \tag{31.4.2}$$

which is indeed the Fourier transform. With this convention of Fourier transforms the Fourier transform in Minkowski space has the kernel $\mathrm{e}^{ik\cdot x}$, where $k \cdot x = \sum_\mu k^\mu x_\mu$ is the Minkowski dot product.

Notice that while x is discrete there is no reason why k must be discrete. Since $x = an$ for some integer four-vector $n \in \mathbb{Z}^4$ the function $\hat{f}$ defined here has period $2\pi/a$. We can then interpret this Fourier transform as simply being the Fourier series of a periodic function. From this we know that the inverse transform is

$$f(x) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4k}{(2\pi)^4} \hat{f}(k) \mathrm{e}^{ik\cdot x} \tag{31.4.3}$$

where we use the shorthand

$$\int_a^b \mathrm{d}^4k\, g(k) \coloneqq \int_{[a,b]^4} \mathrm{d}^4k\, g(k). \tag{31.4.4}$$

What we see here is that momentum space is bounded, in the sense that all momenta can be mapped into the volume $[-\pi/a, \pi/a]^4$, so that the magnitude of any momenta is bounded above by $2\pi/a$. In terms of the physics this means that using a lattice introduces an automatic momentum cut-off which regulates the UV divergences. The continuum limit then corresponds to removing the regulator.

This compactification of momentum space leading to periodic functions is familiar in lattices, where it gives rise to the notion of Brillouin zones[2]. The idea is that once momentum leaves one side of the Brillouin zone it re-enters on the opposite side.

[2] See *Introduction to Condensed Matter Physics*

We can also perform Fourier transforms on our finite continuous spacetime $\mathrm{T}^4(T, L)$. If we have periodic boundary conditions then we can define the Fourier transform and its inverse through

$$\hat{f}(k) = \int_{\mathrm{T}^4(T,L)} \mathrm{d}^4x\, f(x)\mathrm{e}^{-ik\cdot x}, \tag{31.4.5}$$

$$f(x) = \frac{1}{V_4} \sum_{k\in\mathcal{F}\mathrm{T}^4(T,L)} \hat{f}(k)\mathrm{e}^{ik\cdot x} \tag{31.4.6}$$

where the periodic boundary conditions allow us to interpret the inverse as a Fourier series. The space $\mathcal{F}\mathrm{T}^4(T, L)$ is the orthorhombic lattice with lattice points

$$\left(\frac{2\pi}{T}m_0, \frac{2\pi}{L}\boldsymbol{m}\right) \tag{31.4.7}$$

where $m = (m_0, \boldsymbol{m}) \in \mathbb{Z}^4$ is an integer four-vector. We can extend this to the infinite time case, $\mathrm{C}^4(L)$, by

$$\hat{f}(k) = \int \mathrm{d}x_0 \int_{\mathrm{T}^3(L)} \mathrm{d}^3\boldsymbol{x}\, f(x)\mathrm{e}^{-ik\cdot x}, \tag{31.4.8}$$

$$f(x) = \int \frac{\mathrm{d}k_0}{2\pi} \frac{1}{V_3} \sum_{\boldsymbol{k}\in\mathcal{F}\mathrm{T}^3(L)} \hat{f}(k)\mathrm{e}^{ik\cdot x} \tag{31.4.9}$$

where $\mathcal{F}\mathrm{T}^3(L)$ is the cubic lattice with spacing $2\pi/L$.

The discretisation of momentum occurring here is identical to that happening in quantum mechanics when we consider, for example, a particle in a box. Apart from the zero mode, $p = 0$, all momenta are bounded below by the positive value $\min\{2\pi/L, 2\pi/T\}$ and in the infinite time case by $2\pi/L$. Thus finite volumes act as an IR cut-off.

31.4.1 Diagonalising Lattice Derivatives

The reason that the Fourier transform is so useful is it diagonalises differential operators. In continuous space we know that lane waves are the eigenfunctions of differential operators, and their eigenvalues are their momenta:

$$\partial_\mu \mathrm{e}^{ik\cdot x} = ik_\mu \mathrm{e}^{ik\cdot x}. \tag{31.4.10}$$

Thus using plane waves as a basis for functions renders the derivative diagonal. Something similar holds on the lattice, but the expressions are slightly more complicated.

Define the plane wave of momentum k by $e_k(x) = \mathrm{e}^{ik\cdot x}$. Clearly this is an eigenvector of the translation operator:

$$\tau_\mu e_k(x) = \mathrm{e}^{ik\cdot(x+a\hat{\mu})} = \mathrm{e}^{ik\cdot x}\mathrm{e}^{iak_\mu} = \mathrm{e}^{iak_\mu}e_k(x). \tag{31.4.11}$$

Thus, the plane waves are also eigenvectors of the finite derivatives:

$$\begin{aligned}\delta_\mu e_k(x) &= \frac{1}{a}(\tau_\mu - 1)e_k(x) = \frac{1}{a}(\mathrm{e}^{iak_\mu} - 1)e_k(x) \\ &= \frac{1}{a}\mathrm{e}^{iak_\mu/2}(\mathrm{e}^{iak_\mu/2} - \mathrm{e}^{-iak_\mu/2})e_k(x) = \frac{1}{a}\mathrm{e}^{iak_\mu/2}2i\sin\left(\frac{ak_\mu}{2}\right)e_k(x)\end{aligned} \tag{31.4.12}$$

and similarly for the other derivatives, giving

$$\delta_\mu e_k = i\mathrm{e}^{iak_\mu/2}\hat{k}_\mu e_k, \tag{31.4.13}$$

$$\delta_\mu^* e_k = i\mathrm{e}^{-iak_\mu/2}\hat{k}_\mu e_k, \tag{31.4.14}$$

$$\bar{\delta}_\mu e_k = i\bar{k}_\mu e_k \tag{31.4.15}$$

where

$$\hat{k}_\mu \coloneqq \frac{2}{a}\sin\left(\frac{ak_\mu}{2}\right), \qquad \text{and} \qquad \bar{k}_\mu \coloneqq \frac{1}{a}\sin(ak_\mu). \tag{31.4.16}$$

We define hat and bar as such for any quantity with dimensions of energy.

For the two Hermitian Laplacians we have

$$\hat{\delta}^2 e_k = -\hat{k}^2 e_k, \qquad \text{and} \qquad \bar{\delta}^2 e_k = -\bar{k}^2 e_k \tag{31.4.17}$$

where

$$\hat{k}^2 = \sum_\mu \hat{k}_\mu^2 = \frac{4}{a^2}\sum_\mu \sin\left(\frac{ak_\mu}{2}\right), \qquad \text{and} \qquad \bar{k}^1 = \sum_\mu \bar{k}_\mu^2 = \frac{1}{a^2}\sum_\mu \sin(ak_\mu). \tag{31.4.18}$$

We can then define the functions

$$E_\mu(k) \coloneqq \mathrm{e}^{iak_\mu}, \quad \hat{P}_\mu(k) \coloneqq \mathrm{e}^{iak_\mu/2}\hat{k}_\mu, \quad \bar{P}_\mu(k) \coloneqq \bar{k}_\mu, \quad \text{and} \quad \bar{P}^2(k) \coloneqq \hat{k}^2. \tag{31.4.19}$$

These are the diagonal forms of the finite difference operators, in the sense that they are all proportional to the identity operator on the lattice, and we have

$$\tau_\mu = i\mathcal{F}^{-1}\underline{E_\mu}\mathcal{F}, \qquad \delta_\mu = i\mathcal{F}^{-1}\underline{\hat{P}_\mu}\mathcal{F}, \qquad \delta_\mu^* = i\mathcal{F}^{-1}\underline{P_\mu^*}\mathcal{F}, \tag{31.4.20}$$

$$\bar{\delta}_\mu = i\mathcal{F}^{-1}\underline{\bar{P}_\mu}\mathcal{F}, \qquad \hat{\delta}_\mu^2 = -\mathcal{F}^{-1}\underline{\hat{P}^2}\mathcal{F}, \qquad \bar{\delta}_\mu^2 = -\mathcal{F}^{-1}\underline{\bar{P}^2}\mathcal{F}. \tag{31.4.21}$$

These all follow by expanding out and changing variables:

$$(i\mathcal{F}^{-1}\underline{E_\mu}\mathcal{F})f(x) = a^4\sum_{x'\in\Lambda^4} i\mathcal{F}^{-1}\underline{E_\mu}f(x')\mathrm{e}^{-ik\cdot x'} \tag{31.4.22}$$

$$= a^4\sum_{x'\in\Lambda^4} i\mathcal{F}^{-1}\mathrm{e}^{iak_\mu}f(x')\mathrm{e}^{-ik\cdot x'} \tag{31.4.23}$$

$$= a^4\sum_{x'\in\Lambda^4} i\mathcal{F}^{-1}f(x')\mathrm{e}^{-ik\cdot(x'-a\hat{\mu})} \tag{31.4.24}$$

$$= a^4 \sum_{x' \in \Lambda^4} i\mathcal{F}^{-1} f(x' + a\hat{\mu}) e^{-ik\cdot x'} \tag{31.4.25}$$

$$= ia^4 \int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4} \sum_{x' \in \Lambda^4} f(x' + a\hat{\mu}) e^{-ik\cdot x'} e^{ik\cdot x} \tag{31.4.26}$$

$$= e^{i\pi/2} a^4 \int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4} \sum_{x' \in \Lambda^4} f(x' + a\hat{\mu}) e^{ik\cdot(x-x')} \tag{31.4.27}$$

$$= a^4 \int_{-\pi/2}^{\pi/a} \frac{d^4k}{(2\pi)} \sum_{x' \in \Lambda^4} f(x' + a\hat{\mu}) e^{ik\cdot(x-x')} e^{i\pi/2} \tag{31.4.28}$$

$$= a^4 \sum_{x' \in \Lambda^4} f(x' + a\hat{\mu}) e^{i\pi/2} a^{-4} \delta_{xx'} \tag{31.4.29}$$

$$= f(x + a\hat{\mu}) \tag{31.4.30}$$

$$= \tau_\mu f(x) \tag{31.4.31}$$

what happens with the factor of i? where we've swapped the order of the sum and the integral and used

$$\int_{\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} = a^{-4} \delta_{xx'} \tag{31.4.32}$$

which is the lattice version of the usual integral representation of the Dirac delta.

Thirty-Two

Scalar Field Theory

32.1 Lattice Action

The Euclidean action on a continuous spacetime for a real scalar field theory with a quartic interaction is

$$S[\varphi] = \int \mathrm{d}^4x \left[\frac{1}{2}\sum_\mu (\partial_\mu \varphi(x))^2 + \frac{1}{2}m^2\varphi(x)^2 + \frac{1}{4!}\lambda\varphi(x)^4\right]. \tag{32.1.1}$$

This can naively be discretised by replacing the integral and derivatives with the discrete integral and derivative:

$$S[\varphi] = a^4 \sum_{x\in\Lambda^4} \left[\frac{1}{2}\sum_\mu (\delta_\mu \varphi(x))^2 + \frac{1}{2}m^2\varphi(x)^2 + \frac{1}{4!}\lambda\varphi(x)^2\right]. \tag{32.1.2}$$

Here we chose the forward derivative, but any discrete derivative would give the correct continuum limit. The reason we chose the forward derivative, as opposed to the central derivative, is that it ensures that the equations of motion are given by a Hermitian operator. This works by integrating by parts:

$$\sum_{x\in\Lambda^4} (\delta_\mu\varphi(x))(\delta_\mu\varphi(x)) = -\sum_{x\in\Lambda^4} \varphi(x)\delta_\mu^*\delta_\mu\varphi(x) \tag{32.1.3}$$

and in the sum over μ this derivative becomes $\tilde{\delta}^2$, which is indeed Hermitian. This is important because it gives positive definite energies.

We can then rewrite the action as

$$S[\varphi] = a^4 \sum_{x\in\Lambda^4} \left[\frac{1}{2}\sum_\mu \varphi(x)(-\tilde{\delta}^2 + m^2)\varphi(x) + \frac{1}{4!}\lambda\varphi(x)^4\right]. \tag{32.1.4}$$

If we then impose the principle of least action, $\delta S = 0$, we get the discrete Euclidean Klein–Gordon equation with a source as the equation of motion:

$$(-\tilde{\delta}^2 + m^2)\varphi(x) = -\frac{1}{3!}\lambda\varphi(x)^3. \tag{32.1.5}$$

Given an observable, O, its expectation value is then given by the path integral

$$\langle O\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}\varphi\, O[\varphi]\mathrm{e}^{-S[\varphi]}. \tag{32.1.6}$$

Here the constant $\mathcal{Z}$ is defined such that $\langle 1 \rangle = 1$. We can then define the generating functional by introducing a source, η:

$$\mathcal{Z}[\eta] = \int \mathcal{D}\varphi \exp\left\{-S[\varphi] + a^4 \sum_{x\in\Lambda^4} \eta(x)\varphi(x)\right\} \tag{32.1.7}$$

which is such that $\mathcal{Z}[0] = \mathcal{Z}$. Correlators can then be expressed through functional derivatives acting on the generating functional:

$$\langle \varphi(x_1)\cdots\varphi(x_n)\rangle = \frac{\delta}{\delta\eta(x_1)}\cdots\frac{\delta}{\delta\eta(x_n)}\log(\mathcal{Z}[\eta])\Big|_{\eta=0}. \tag{32.1.8}$$

Taking the log before taking derivatives simply takes care of the normalisation, we could have simply defined the generating functional with an extra factor of $1/\mathcal{Z}$ to start with taken derivatives without the log. The derivatives are straightforward to compute:

$$\frac{\delta}{\delta\eta(x)}\mathcal{Z}[\eta] = \int \mathcal{D}\varphi\,\varphi(x)\exp\left\{-S[\varphi] + a^4 \sum_{x'\in\Lambda^4} \eta(x')\varphi(x')\right\} \tag{32.1.9}$$

with the a^4 that we bring down in the derivative cancelling with the a^{-4} appearing in the discrete Dirac delta from

$$\frac{\delta}{\delta\eta(x)}\eta(x) = a^{-4}\delta_{xx'}. \tag{32.1.10}$$

Upon setting $\eta = 0$ we are left with

$$\int \mathcal{D}\varphi\,\varphi(x)\mathrm{e}^{-S[\varphi]}. \tag{32.1.11}$$

Hence,

$$\frac{\delta}{\delta\eta(x)}\log(\mathcal{Z}[\eta])\Big|_{\eta=0} = \frac{1}{\mathcal{Z}[\eta]}\frac{\delta}{\delta\eta(x)}\mathcal{Z}[\eta]\Big|_{\eta=0}. \tag{32.1.12}$$

32.2 Free Theory

If we consider the free theory, setting $\lambda = 0$, then the action is

$$S_0[\varphi] = \frac{a^4}{2}\sum_{x\in\Lambda^4}\varphi(x)\Delta\varphi(x) \tag{32.2.1}$$

where the operator Δ is defined as $\Delta = -\tilde{\delta}^2 + m^2$. Completing the square we have

$$a^4 \sum_{x\in\Lambda^4}\left[-\frac{1}{2}\varphi(x)\Delta\varphi(x) + \eta(x)\varphi(x)\right] = \tag{32.2.2}$$
$$a^4 \sum_{x\in\Lambda^4}\left(-\frac{1}{2}[\varphi(x) - (\Delta^{-1}\eta)(x)]\Delta[\varphi(x) - (\Delta^{-1}\eta)(x)] + \frac{1}{2}\eta(x)(\Delta^{-1}\eta)(x)\right)$$

where Δ^{-1} is the inverse of the operator Δ. This formula is evidently true upon expanding the brackets on the right hand side.

If we perform a unitary change of variables, $\varphi \mapsto \varphi + \Delta^{-1}\eta$, and assume that the path integral measure is unchanged then we find that the free generating functional is

$$\mathcal{Z}[\eta] = \int \mathcal{D}\varphi \, \exp\left\{-S_0[\varphi] + a^4 \sum_{x\in\Lambda^4} \eta(x)\varphi(x)\right\} \tag{32.2.3}$$

$$= \int \mathcal{D}\varphi \, \exp\left\{a^4 \sum_{x\in\Lambda^4} \left[-\frac{1}{2}\varphi(x)\Delta\varphi(x) + \eta(x)\varphi(x)\right]\right\} \tag{32.2.4}$$

$$= \int \mathcal{D}\varphi \, \exp\left\{a^4 \sum_{x\in\Lambda^4} \left[-\frac{1}{2}\varphi(x)\Delta\varphi(x) + \frac{1}{2}\eta(x)(\Delta^{-1}\eta)(x)\right]\right\} \tag{32.2.5}$$

$$= \exp\left\{\frac{a^4}{2} \sum_{x\in\Lambda^4} \eta(x)(\Delta^{-1}\eta)(x)\right\} \int \mathcal{D}\varphi \, \exp\left\{-\frac{a^4}{2} \sum_{x\in\Lambda^4} \varphi(x)\Delta\varphi(x)\right\}$$

$$= \exp\left\{\frac{a^4}{2} \sum_{x\in\Lambda^4} \eta(x)(\Delta^{-1}\eta)(x)\right\} \int \mathcal{D}\varphi \, \mathrm{e}^{-S_0[\varphi]} \tag{32.2.6}$$

$$= \mathcal{Z} \exp\left\{a^4 \sum_{x\in\Lambda^4} \eta(x)(\Delta^{-1}\eta)(x)\right\}. \tag{32.2.7}$$

32.2.1 Propagator

We can now compute the two point correlator for the free theory:

$$D_0(x,y) \coloneqq \langle\varphi(x)\varphi(y)\rangle_0 \tag{32.2.8}$$

$$= \frac{\delta}{\delta\eta(x)} \frac{\delta}{\delta\eta(y)} \log(\mathcal{Z}[\eta])\Big|_{\eta=0} \tag{32.2.9}$$

$$= \frac{\delta}{\delta\eta(x)} \frac{\delta}{\delta\eta(y)} \left[\log\mathcal{Z} + \frac{a^4}{2} \sum_{z\in\Lambda^4} \eta(z)(\Delta^{-1}\eta)(z)\right]\Big|_{\eta=0} \tag{32.2.10}$$

$$= \frac{\delta}{\delta\eta(x)} a^4 \sum_{z\in\Lambda^4} a^{-4}\delta_{yz}(\Delta^{-1}\eta)(z)\Big|_{\eta=0} \tag{32.2.11}$$

$$= \frac{\delta}{\delta\eta(x)} (\Delta^{-1}\eta)(y)\Big|_{\eta=0} \tag{32.2.12}$$

$$= a^{-4}(\Delta^{-1}\delta_x)(y) \tag{32.2.13}$$

where the factor of $1/2$ vanishes due to the two different η with which the first derivative can act, bot of which give the same result. This can be seen by expanding out Δ and integrating by parts the functional derivative with the discrete derivatives and the m^2 term.

We can diagonalise the operator Δ in momentum space giving

$$D_0(x,y) = a^{-4}[\mathcal{F}^{-1}(\tilde{P}^2 + m^2)^{-1}\mathcal{F}\delta_x](y) \tag{32.2.14}$$

$$= \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4k}{(2\pi)^4} D_0(\hat{k}^2)\mathrm{e}^{ik\cdot(x-y)} \tag{32.2.15}$$

$$= \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{e}^{ik\cdot(x-y)}}{\hat{k}^2 + m^2}. \tag{32.2.16}$$

Compare this result to the continuum case in Minkowski space:

$$D_{0,\mathrm{c,M}}(x,y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} D_{0,\mathrm{c,M}}(k^2)\mathrm{e}^{-ik\cdot(x-y)} = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{e}^{-ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon} \tag{32.2.17}$$

where $D_{0,\mathrm{c,M}}(k^2)$ denotes the Fourier transform of $D_{0,\mathrm{c,M}}(x,y)$, and the continuum case in Euclidean space:

$$D_{0,\mathrm{c,M}}(x,y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} D_{0,\mathrm{c,M}}(k^2)\mathrm{e}^{ik\cdot(x-y)} = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{e}^{ik\cdot(x-y)}}{k^2 + m^2}. \tag{32.2.18}$$

Wick's theorem states that n point correlators can be computed by considering pairs of 2 point correlators, in particular,

$$\langle \varphi(x_1)\cdots\varphi(x_n)\rangle_0 = \sum_{\text{pair partitions of } [\![1,n]\!]} D_0(x_{i_1},x_{i_2})\cdots D_0(x_{i_{n-1}},x_{i_n}). \tag{32.2.19}$$

32.2.2 Energy

A free particle has energy corresponding to the poles in the momentum space propagator. For the continuum Minkowski space there are poles in the momentum space propagator whenever $k^2 = m^2$, leading to the famous $E^2 = \boldsymbol{p}^2 + m^2$, that is the energy is

$$k_0 = \sqrt{\boldsymbol{k}^2 + m^2}. \tag{32.2.20}$$

We take the positive root as we want to have positive energies, studying the negative root leads to the idea of antimatter. For the continuum Euclidean case the poles come at

$$k_0 = \pm i\sqrt{\boldsymbol{k}^2 + m^2} \tag{32.2.21}$$

where, since this value is purely imaginary, there is no preferred sign.

On the lattice the idea is the same, but things are more complex as we have an additional sin to deal with. In general, we have a pole when $\hat{k}^2 + m^2$ vanishes, which means

$$\hat{k}_0^2 = \frac{4}{a^2}\sin^2\left(\frac{ak_0}{2}\right) = -\hat{\boldsymbol{k}}^2 - m^2 \tag{32.2.22}$$

so

$$k_0 = \pm\frac{2}{a}\arcsin\left(i\frac{a}{2}\sqrt{\hat{\boldsymbol{k}}^2 + m^2}\right) = \pm i\frac{2}{a}\operatorname{arcsinh}\left(\frac{a}{2}\sqrt{\hat{\boldsymbol{k}}^2 + m^2}\right) := \pm i\omega(\boldsymbol{k}).$$

32.2.3 Continuum Limit

We can expand in a to study the convergence of these values in the continuum limit. First, we have

$$\hat{k}_\mu = \frac{2}{a}\sin\left(\frac{ak_\mu}{2}\right) = k_\mu - \frac{k_\mu^3}{24}a^2 + \mathcal{O}(a^4). \tag{32.2.23}$$

Clearly this converges to k_μ in the continuum limit. We then have

$$\hat{k}^2 = k^2 - \frac{k^4}{12}a^2 + \mathcal{O}(a^4), \tag{32.2.24}$$

where we define

$$k^n \coloneqq \sum_\mu k_\mu^n \neq \|k\|^n = (k^2)^{n/2}. \tag{32.2.25}$$

So we see that both terms suffer quadratic discretisation effects. Notice also that both expressions are hypercube covariant, depending on k_μ^n, but only become Lorentz covariant in the limit of $a \to 0$.

Using this we can expand the energy on the lattice about the continuum energy $\omega_c(\boldsymbol{k}) \coloneqq \sqrt{\boldsymbol{k}^2 + m^2}$. The result is

$$\omega(\boldsymbol{k}) = \omega_c(\boldsymbol{k}) - \frac{\boldsymbol{k}^4 + \omega_c(\boldsymbol{k})^4}{24\omega_c(\boldsymbol{k})} a^2 + \mathcal{O}(a^4). \tag{32.2.26}$$

Again, this depends on the term $\boldsymbol{k}^4$ which is not Lorentz invariant. The next to leading order corrections scale with energy. This makes sense since a large energy corresponds to a small scale, which means that the lattice spacing is comparably larger and so we would expect larger discretisation effects.

Finally, we can expand the momentum space propagator in terms of the continuum Euclidean propagator $D_{0,c}(k^2) = i/(k^2 + m^2)$:

$$D_0(\hat{k}^2) = D_{0,c}(k^2) + \frac{1}{12} k^4 D_{0,c}(p^2)^2 a^2 + \mathcal{O}(a^4). \tag{32.2.27}$$

Thirty-Three

Perturbation Theory

The pain increases factorially as we go through the series.

Antonin Portelli

Perturbation theory on the lattice works almost identically to the continuum case. Here we will briefly recap the basics. The main difference is not from working on a lattice but from working with the Euclidean metric, which means that most quantities differ by a sign or a factor of i.

33.1 Basic Idea

We start with the action, $S[\varphi]$ written in terms of the free action, $S_0[\varphi]$, and a perturbation corresponding to the interaction term, $\lambda\varphi^4/4!$: Then we look at the Boltzmann weight $\exp\{-S[\varphi]\}$ and expand about the free action:

$$\mathrm{e}^{-S[\varphi]} = \mathrm{e}^{-S_0[\varphi]} \exp\left\{-a^4 \sum_{x\in\Lambda^4} \frac{\lambda}{4!}\varphi(x)^4\right\} \tag{33.1.1}$$

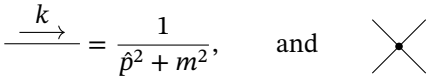
$$= \mathrm{e}^{-S_0[\varphi]} \left(1 - a^4 \sum_{x\in\Lambda^4} \frac{\lambda}{4!}\varphi(x)^4 + \mathcal{O}(\lambda^2)\right). \tag{33.1.2}$$

If we substitute this into both the partition function, $\mathcal{Z}$, and the path integral for the generating functional, $\mathcal{Z}[\eta]$, then we find that

$$\langle\varphi(x_1)\cdots\varphi(x_n)\rangle = \langle\varphi(x_1)\cdots\varphi(x_n)\rangle_0 - a^4\frac{\lambda}{4!}\sum_{x\in\Lambda^4}\langle\varphi(x)^4\varphi(x_1)\cdots\varphi(x_n)\rangle_{0,\text{conn.}} + \mathcal{O}(\lambda^2) \tag{33.1.3}$$

where conn. means we consider only fully connected contractions in the Wick expansion, that is we disallow terms which don't contract any of the $\varphi(x)$ with the $\varphi(x_i)$, so don't contain terms like $D_0(x, x_i)$ and only contain $D_0(x, x)$ and $D_0(x_i, x_j)$.

Of course, this becomes more clear when we introduce Feynman diagrams, and connected correlators correspond exactly to sums over connected diagrams. Let



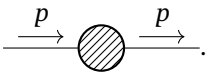
$$= \frac{1}{\hat{p}^2 + m^2}, \qquad \text{and} \qquad = -\lambda. \tag{33.1.4}$$

Notice the missing factor of i in the propagator and $-\lambda$ instead of $i\lambda$ compared to the values in Minkowski space.

33.2 Two Point Function

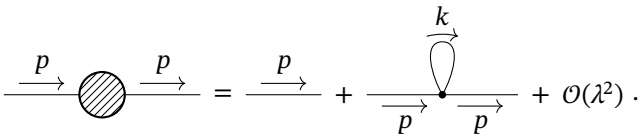
Consider the two point function, which is the full, non-perturbative, Fourier transform of the two point correlator in position space. Diagrammatically we represent this as

$$\int \mathrm{d}^4x\,\langle\varphi(x)\varphi(0)\rangle \mathrm{e}^{ip\cdot x} = \tag{33.2.1}$$



Here we've chosen coordinates such that the second point is at the origin. This works since we assume translation invariance, so the result depends only on the distance between the two points, if we started with $\langle\varphi(x)\varphi(y)\rangle$ we could simply shift all coordinates by $-y$ to get $\langle\varphi(x-y)\varphi(0)\rangle$ and then rename $x-y$ to x changing $\mathrm{e}^{ip\cdot x}$ to $\mathrm{e}^{ip\cdot(x+y)}$.

We can expand the two point function in terms of all connected diagrams, doing so up to first order in λ, so having at most one vertex, we have



(33.2.2)

These diagrams can then be turned into maths in the usual way except the momentum is bounded, and we find that

$$\int \mathrm{d}^4x\,\langle\varphi(x)\varphi(0)\rangle \mathrm{e}^{ip\cdot x} = \frac{1}{\hat{p}^2+m^2} - \frac{\lambda}{2}\frac{1}{(\hat{p}^2+m^2)^2}\int_{-\pi/a}^{\pi/a}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{\hat{k}^2+m^2}. \tag{33.2.3}$$

The factor of $1/2$ in the second term comes from the symmetry factor of 2 which this diagram has as a correlator. Another way to compute this is to notice that in the expansion we have the correlator $\langle\varphi(z)^4\varphi(x)\varphi(0)\rangle_{0,\text{conn.}}$ which comes with a factor of $1/4!$, and if we demand one of the $\varphi(z)$ terms is contracted with each of the $\varphi(x)$ and $\varphi(0)$ terms, which is required for a connected diagram, then there are $12 = 4\cdot 3$ ways to make these connections, so we have an overall factor of $12/4! = 1/2$.

The integral

$$\int_{-\pi/a}^{\pi/a}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{\hat{k}^2+m^2} \tag{33.2.4}$$

diverges in the continuum limit. To proceed set $\ell_\mu = ak_\mu$, and notice that this makes ℓ_μ a dimensionless quantity, so $[\ell_\mu] = 0$. Then we have $\mathrm{d}^4\ell = a^4\,\mathrm{d}^4k$, and

$$a\hat{k} = a\frac{2}{a}\sin\left(\frac{ak_\mu}{2}\right) = 2\sin\left(\frac{\ell_\mu}{2}\right) =: \hat{\ell}_\mu \tag{33.2.5}$$

and we make this definition of $\hat{\ell}_\mu$ for any dimensionless four-vector ℓ. Finally, we have

$$\frac{1}{\hat{k}^2+m^2} = a^2\frac{1}{a^2\hat{k}^2+(am)^2} = a^2\frac{1}{\hat{\ell}^2+(am)^2}. \tag{33.2.6}$$

Thus, the integral becomes

$$\frac{1}{a^2}\int_{-\pi}^{\pi}\frac{\mathrm{d}^4\ell}{(2\pi)^4}\frac{1}{\hat{\ell}^2+(am)^2}. \tag{33.2.7}$$

where the actual integral is dimensionless. While there is no closed form for this integral it can be computed numerically, and we find

$$Z_0 \coloneqq \int_{-\pi}^{\pi}\frac{\mathrm{d}^4\ell}{(2\pi)^4}\frac{1}{\hat{\ell}^2} \approx 0.15493339\ldots \tag{33.2.8}$$

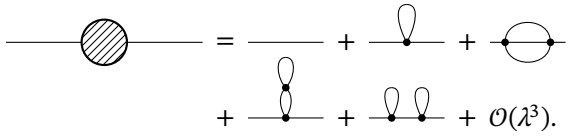
We'll come back to this later.

Notice that in the continuum limit this diverges as a^{-2}, so it's quadratically divergent. We don't see this in dimensional regularisation, since in dim reg only logarithmic divergences show as a result of using a dimensionless regulator.

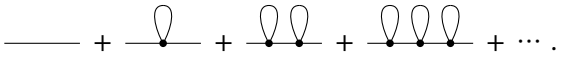
33.3 Renormalisation

We can measure the mass of a particle. The question is how to relate this to the parameter m appearing in the Lagrangian. This leads to the process of renormalisation. We can see that the Lagrangian parameter m corresponds to poles in the propagator, so we look to compute the classical limit, $\hbar \to 0$, and compare the measured mass and the location of the poles.

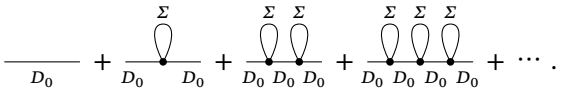
Consider the two point function again, but expanded to second order in λ:

$D(p) =$  (33.3.1)

The first two two loop diagrams are "true" two loop diagrams, and are hard to compute. The last two loop diagram however is not 1PI and is just the product of two one loop diagrams, which we've already computed. The solution to these hard diagrams is just to ignore them, and instead consider the quantity

 (33.3.2)

While this quantity is not equal to $D(p)$ they do agree to first order in λ, which is all that we care about. If we write $D_0(p)$ for the propagator and $\Sigma(p)$ for the loop then we can label these diagrams:

 (33.3.3)

We can then write this quantity as

$$D_0(p)+D_0(p)^2\Sigma(p)+D_0(p)^3\Sigma(p)^2+D_0(p)^4\Sigma(p)^3+\cdots = D_0(p)\sum_{n=0}^{\infty}[D_0(p)\Sigma(p)]^n.$$

Formally, we can then recognise this as a geometric series and write it as

$$D_0(p)\sum_{n=0}^{\infty}[D_0(p)\Sigma(p)]^n = D_0(p)\frac{1}{1-D_0(p)\Sigma(p)} \tag{33.3.4}$$

$$= \frac{1}{D_0(p)^{-1}-\Sigma(p)} \tag{33.3.5}$$

$$= \frac{1}{\hat{p}^2+m^2-\Sigma(p)} \tag{33.3.6}$$

where we've recognised that $D_0(p) = 1/(\hat{p}^2+m^2)$ so $D_0(p)^{-1} = \hat{p}^2+m^2$.

The quantity $\Sigma(p)$ is finite as we have regulated it with the lattice spacing. Therefore the only poles arise from $D_0(p)$. These poles are exactly those which appear when the energy is

$$p_0 = \pm i\frac{2}{a}\operatorname{arcsinh}\left(\frac{a}{2}\sqrt{\hat{\boldsymbol{p}}^2+m^2}\right). \tag{33.3.7}$$

Now suppose that $\Sigma(p)$ is actually constant with respect to p. This is the case at one loop where that constant is related to the value Z_0. Then $D(p)$ has poles at

$$p_0 = \pm i\frac{2}{a}\operatorname{arcsinh}\left(\frac{a}{2}\sqrt{\hat{\boldsymbol{p}}^2+m^2-\Sigma}\right). \tag{33.3.8}$$

This means that what we see in the real world as the mass squared is not the m^2 appearing in the Lagrangian, but the renormalised mass $m_{\mathrm{R}} = m^2-\Sigma$.

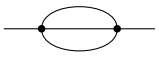
In the continuum limit $\Sigma \sim 1/a^2$. Suppose that m^2 also goes as $1/a^2$ in such a way that $m^2-\Sigma$ converges to the finite quantity m_{R}^2. There is a problem with this, since we performing the sum $\infty-\infty$, which can give any value we like, and we are choosing m_{R} as the result. This means that this process doesn't make a prediction of the measured mass, but rather gives us a way to relate measured quantities to parameters in the Lagrangian.

What we have done here is implement a renormalisation scheme. Renormalisation schemes come in two pieces, a choice of regulator, which we take to be the lattice spacing, which makes loop integrals finite, and a choice of renormalisation condition, which relates a measured value to the Lagrangian parameters.

33.4 Higher Correlators

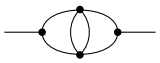
The four point, six point, etc. functions are also divergent. Fortunately, in four dimensions φ^4 theory is renormalisable, meaning that we can remove all of the infinities by relating a finite number of parameters to measured values. In the case of φ^4 theory we choose the mass and coupling, λ as the two parameters we need to renormalise everything.

A superrenormalisable theory, which φ^4 isn't, is one in which we stop getting new divergent diagrams at some order, although diagrams may still diverge due to lower order divergent subdiagrams. Since φ^4 is not superrenormalisable we expect to get new divergent diagrams at all orders, for example, at two loop the diagram



(33.4.1)

is divergent, and at three loops



(33.4.2)

is divergent.

The important thing to notice is that in order for the continuum limit to be defined a QFT must be renormalisable. In practice however, we often don't actually care if a a theory is renormalisable or the continuum limit exists. For example, if we only care about describing physics at the LHC then if we take a such that $1/a > 100\,\text{TeV}$ then this should be more than good enough to describe the relevant physics. It is possible that the universe itself doesn't even have a true continuum but instead is a lattice with lattice spacing given by the Planck length, $\ell_{\text{P}} = 1.616 \times 10^{-35}\,\text{m}$ in which case it doesn't even make sense to consider a continuum limit.

Supposing that the universe isn't a lattice, but working with a non-zero lattice spacing gives us an effective field theory for the physics of interest, and effective field theories are in general not renormalisable, but this really doesn't matter. It is not known if the standard model is renormalisable, if it isn't then it can be viewed as an effective field theory of some beyond the standard model theory. We can use this to guess at what this beyond the standard model theory is.

In reality all known physics breaks down long before we reach the $a = 0$ limit anyway, since energies are so high, so it really doesn't matter if a theory actually has a continuum limit.

Thirty-Four

Renormalisation

34.1 Counterterms

The action, now in terms of bare quantities, is

$$S[\varphi_0] = a^4 \sum_{x\in\Lambda^4} \left[\frac{1}{2} \sum_\mu (\delta_\mu \varphi_0(x))^2 + \frac{1}{2} m_0 \varphi_0(x)^2 + \frac{1}{4!} \lambda_0 \varphi_0(x)^4 \right]. \tag{34.1.1}$$

We have already seen that this action leads to divergent terms in the continuum limit. To fix this we introduce counterterms, which we do by introducing renormalised quantities

$$\varphi_0 = \sqrt{Z}\varphi, \qquad m^2 = Z m_0^2 - \delta_{m^2}, \qquad \text{and} \qquad \lambda = Z^2 \lambda_0 - \delta_\lambda. \tag{34.1.2}$$

Here for $x = m^2$ or $x = \lambda$ we define $\delta_x := x - x_0$, and $\delta_Z := Z - 1$, so that $\delta_x = 0$ for all $x \in \{m^2, \lambda, Z\}$ if $Z = 1$.

Substituting these into the action we see that we get the regular action, in terms of the new fields and parameters, plus new terms with δ_x in them, which are the counterterms:

$$S[\varphi] = a^4 \sum_{x\in\Lambda^4} \left[\frac{1}{2} \sum_\mu (\delta_\mu \varphi(x))^2 + \frac{1}{2} m^2 \varphi(x)^2 + \frac{1}{4!} \lambda \varphi(x)^4 \right. \tag{34.1.3}$$

$$\left. + \frac{\delta_Z}{2} \sum_\mu (\delta_\mu \varphi(x))^2 + \frac{1}{2} \delta_{m^2} \varphi(x)^2 + \frac{1}{4!} \delta_\lambda \varphi(x)^4 \right]. \tag{34.1.4}$$

The counterterms are all $\mathcal{O}(\lambda)$ since the free theory doesn't have any divergences to cancel.

At the LHC the only things we measure are energies and the number of events. The energy corresponds to the poles in the correlators and the counts give the probability of a given event. To compute the energy we therefore care about the residue at the poles, but rescaling the fields changes these residues. For consistent normalisation we require that Z also depends on a in such a way that the rescaled values correspond in some way to measurable quantities. The new parameters are just affine transformations of the original variables. It is possible to renormalise a theory without introducing new parameters, but the maths isn't as clean, so no one does. The choice made here is particularly nice because it ensures that each counterterm scales equally with Z.

To proceed we identify that the first two terms of the Lagrangian are the Lagrangian of the free theory, and treat everything else, the $\lambda\varphi^4/4!$ term and counterterms, as perturbations. This introduces two new types of "interactions" compared

to the last chapter. We have a two point interaction from the φ^2 terms, of which there are two,

$$\frac{\delta_Z}{2}\sum_\mu(\delta_\mu\varphi(x))^2+\frac{1}{2}\delta_{m^2}\varphi(x)^2. \tag{34.1.5}$$

We want to Fourier transform this to compute the Feynman rule. To do so we integrate by parts:

$$\sum_\mu(\delta_\mu\varphi(x))^2=\sum_\mu\varphi(x)(-\delta_\mu^*\delta_\mu)\varphi(x)=\varphi(x)(-\tilde{\delta}^2)\varphi(x). \tag{34.1.6}$$

Taking the Fourier transform $\tilde{\delta}^2$ becomes $-\hat{p}^2$, and so the Feynman rule is

$$= \hat{p}^2\delta_Z-\delta_{m^2} \tag{34.1.7}$$

where the overall minus sign is from the minus in $\exp\{-S[\varphi]\}$ in Euclidean space and as usual the numerical prefactor of $1/2$ cancels with the combinatoric factor from symmetry. There is also a four point interaction from the φ^4 term,

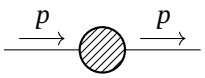
$$\frac{1}{4!}\delta_\lambda\varphi(x)^4 \tag{34.1.8}$$

which is identical to the existing four point term except with δ_λ in the place of λ, giving the Feynman rule

$$= -\delta_\lambda. \tag{34.1.9}$$

34.2 Renormalisation Conditions

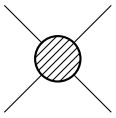
In order to implement a renormalisation scheme we need to give renormalisation conditions which fix how the parameters correspond to physical variables. To do so here we will implement two renormalisation conditions, first for the full two point function, now with counterterms, we demand that



$$= \frac{1}{\hat{p}^2+m^2}+\mathcal{O}(\hat{p}^2+m^2) \qquad \text{when } \hat{p}^2\approx -m^2. \tag{34.2.1}$$

So the two point function is given by the propagator plus terms which vanishes at the pole. This is really two renormalisation conditions in one. First, we require that there is a pole at $\hat{p}^2=-m^2$, and second that the residue at that pole is 1. The first condition ensures that there is a pole at the measured mass, and the second is just to keep things simple, since any other value we choose for the pole just appears as an overall factor, which can just be reabsorbed into the definition of the field, so we may as well choose 1 from the start.

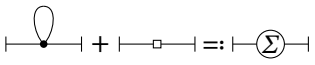
The second renormalisation condition we make is somewhat non-standard, it corresponds to fixing the full four point function, again with counterterms, but rather than fixing it for on-shell particles, the more common choice, we will find it simpler for our purposes to fix it so that when all external momenta vanish



$$= -\lambda \qquad \text{when all external momenta vanish.} \tag{34.2.2}$$

This renormalisation choice is motivated by noticing that the frequency of seeing an event with an interaction is proportional to the coupling constant, λ, assuming that events with more than one interaction are negligible. Our choice here then fixes this constant of proportionality to unity.

We can again resum the two-point function. The self energy is more complicated this time, since it now the new two point interaction, so

$\Sigma(\hat{p}^2) =$  (34.2.3)

where the bars on the ends of the lines denote that we aren't considering the external propagators, just the internal propagator in the loop and the two vertices. We don't include the loop with the new four point interaction since we will see that δ_λ turns out to be $\mathcal{O}(\lambda)^2$. Then, accurate to one loop, we can express the two point function as

$D(\hat{p}^2) \approx$  (34.2.4)

This can then be resummed as in Section 33.3 to give

$$D(\hat{p}^2) = D_0(\hat{p}^2) \sum_{n=0}^{\infty} [\Sigma(\hat{p}^2) D_0(\hat{p}^2)]^n = \frac{1}{\hat{p}^2 + m^2 - \Sigma(\hat{p}^2)}. \tag{34.2.5}$$

Now suppose that $\hat{p}^2 \approx -m^2$, then suppose that $\Sigma(\hat{p}^2)$ is small, then expanding in $\Sigma(\hat{p}^2)$ we get

$$D(\hat{p}^2) = \frac{1}{\hat{p}^2 + m} + \frac{\Sigma(\hat{p}^2)}{(\hat{p}^2 + m^2)^2} + \mathcal{O}(\Sigma(\hat{p}^2)^2) \tag{34.2.6}$$

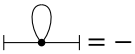
and so we can see that in order for the first renormalisation condition to hold, and the higher order terms here to vanish at $\hat{p}^2 = -m^2$, we need to have $\Sigma(-m^2) = 0$. In order for the residue to be one we also need

$$\frac{\mathrm{d}}{\mathrm{d}\hat{p}^2} \Sigma(\hat{p}^2)\Big|_{\hat{p}^2 = -m^2} = 0. \tag{34.2.7}$$

This allows us to write this renormalisation in terms of the self energy evaluated when $\hat{p}^2 = -m^2$, this is the on-shell renormalisation scheme.

34.3 Two Point Function Renormalisation

The new two point interaction corresponds to two counterterms, one for δ_Z and one for δ_{m^2}. We also have two renormalisation conditions corresponding to the two point function, so we can use these to fix δ_Z and δ_{m^2}. We saw previously that

 $= -\frac{\lambda}{2} \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{\hat{k}^2 + m^2} = -\frac{\lambda}{2a^2} \int_{-\pi}^{\pi} \frac{\mathrm{d}^4 \ell}{(2\pi)^4} \frac{1}{\hat{\ell}^2 + (am)^2}$ (34.3.1)

and that as $a \to 0$ this integral goes as $\lambda Z_0/(2a^2)$ where

$$Z_0 = \int_{-\pi}^{\pi} \frac{\mathrm{d}^4 \ell}{(2\pi)^4} \frac{1}{\hat{\ell}^2} = 0.15493339\ldots\ . \tag{34.3.2}$$

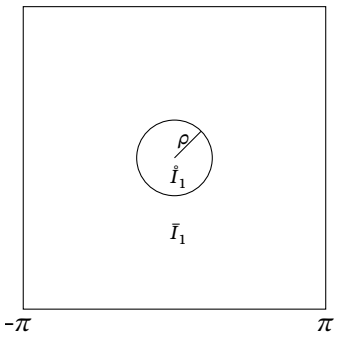


Figure 34.1: The two regions of integration we use to isolate the divergent part of the integral in two dimensions.

The important thing here is that at leading order this diagram diverges as $1/a^2$. It is possible however that a higher order correction might cancel this divergence and so we would be missing information here. We don't have to worry about this with dim reg as we only have logarithmic divergence there.

To proceed we define

$$I_j(\xi^2) \coloneqq \int_{-\pi}^{\pi} \frac{\mathrm{d}^4\ell}{(2\pi)^4} \frac{1}{(\hat{\ell}^2 + \xi^2)^j} \tag{34.3.3}$$

so that $Z_0 = I_1(0)$. We want to consider I_1 for small ξ. We expect that $I_j(\xi^2) = Z_0 + \mathcal{O}(\xi^2)$.

First we might try expanding the integrand, giving

$$I_1(\xi) = Z_0 - \xi^2 \int_{-\pi}^{\pi} \frac{\mathrm{d}^4\ell}{(2\pi)^4} \frac{1}{(\hat{\ell})^2} + \mathcal{O}(\xi^4). \tag{34.3.4}$$

However, the integral appearing here is logarithmically divergent, and the next term will quadratically divergent. This problem here is that I_1 is not analytic in ξ due to the $\ell = 0$ pole. This means we can't expand in ξ, instead we need to look for an expansion which allows for terms like $\xi \log \xi$ which capture these divergences.

Integrals of this form have been fully classified, and it has been shown that in d dimensions they can be parametrised fully by $d-1$ dimensionless numbers. In four dimensions we need three numbers, these numbers are usually taken as Z_0, Z_1, and F_0, which are all defined, but we'll only use Z_0 here, and another number C, which can be rewritten in terms of these numbers, but we won't.

In order to deal with the divergent integral we split up the integration domain into two regions. One, is the region with $\hat{\ell}^2 < \rho^2$ for ρ, and the other is the region with $\hat{\ell}^2 > \rho^2$. Thus we split momemtum space with a sphere of radius ρ. Then we define

$$I_1(\xi^2) = \mathring{I}_1(\xi^2, \rho) + \bar{I}_1(\xi^2, \rho) \tag{34.3.5}$$

where the first term is the integral inside this sphere, which is divergent, and the second term is the integral outside of this sphere, which is convergent. This is shown in two dimensions in Figure 34.1.

The divergent part of this integral can then be processed by first recognising that at low momenta $\hat{\ell}_\mu \approx \ell_\mu$, so

$$\mathring{I}_1(\xi^2,\rho) = \int_{|\hat{\ell}|<\rho} \frac{\mathrm{d}^4\ell}{(2\pi)^4}\frac{1}{\hat{\ell}^2+\xi^2} \approx \int_{|\ell|<\rho} \frac{\mathrm{d}^4\ell}{(2\pi)^4}\frac{1}{\ell^2+\xi^2}. \tag{34.3.6}$$

We can compute this integral by going to spherical coordinates. The measure becomes $\mathrm{d}^4\ell = r^3\,\mathrm{d}r\,\mathrm{d}\Omega$ before performing the angular integrals, which just give the surface area of the three-sphere, $2\pi^2$, because the integrand is rotationally invariant. Hence, we have

$$\mathring{I}_1(\xi^2,\rho) \approx \frac{1}{8\pi^2}\int_0^\rho \mathrm{d}r\,\frac{r^3}{r^2+\xi^2} \tag{34.3.7}$$

$$= \frac{1}{16\pi^2}[\rho^2 + \xi^2\log\xi^2 - \xi^2\log(\xi^2+\rho^2)] \tag{34.3.8}$$

$$= \frac{1}{16\pi^2}\left[\rho^2 + \xi^2\log\frac{\xi^2}{\rho^2} + \mathcal{O}(\xi^4)\right]. \tag{34.3.9}$$

It is not possible to compute $\bar{I}_1$. If it was then we would have been able to compute I_1. Instead we can expand the integrand now since it is analytic on the region of integration. Doing so we can write

$$\bar{I}_1(\xi^2,\rho) = C_1(\rho) - \xi^2 C_2(\rho) + \mathcal{O}(\xi^4) \tag{34.3.10}$$

where

$$C_j(\rho) = \int_{|\ell|<\rho}\frac{\mathrm{d}^4\ell}{(2\pi)^4}\frac{1}{(\hat{\ell}^2)^j}. \tag{34.3.11}$$

This diverges as $\rho \to 0$, but is finite for non-zero ρ. Then we can write the full integral as

$$I_1(\xi^2) = Z_0 + \frac{1}{16\pi^2}\xi^2\log\xi^2 - C\xi^2 + \mathcal{O}(\xi^4) \tag{34.3.12}$$

where

$$C = \lim_{\rho\to 0}\left[C_2(\rho) + \frac{1}{16\pi^2}\log\rho^2\right] = 0.030345\ldots \tag{34.3.13}$$

has no known simple closed form. It is possible to relate C to one of the numbers, F_0, which can be used to parametrise integrals of this form.

We therefore have

$$= -\frac{\lambda}{2a^2}I_1(a^2m^2) \tag{34.3.14}$$

$$= -\frac{\lambda}{2}\left[\frac{Z_0}{a^2} + \frac{1}{16\pi^2}m^2\log(a^2m^2) + Cm^2 + \mathcal{O}(a^2)\right]. \tag{34.3.15}$$

The quadratic divergence Z_0/a^2 is new here and doesn't appear in dim reg, the logarithmic divergence is what we see in dim reg.

We can now evaluate the self energy,

$$\Sigma(\hat{p}^2) = I_1(a^2\hat{p}^2) + \hat{p}^2\delta_Z - \delta_{m^2}. \tag{34.3.16}$$

The derivative of this is

$$\frac{\mathrm{d}}{\mathrm{d}\hat{p}^2}\Sigma(\hat{p}^2) = \delta_Z \tag{34.3.17}$$

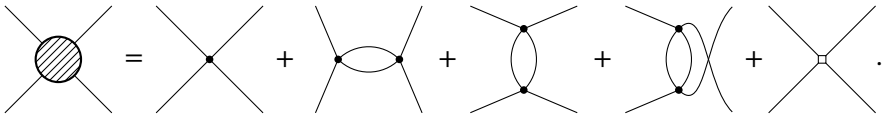
and so if we want this to vanish at $\hat{p}^2 = -m^2$ we must have $\delta_Z = 0$ to one loop. Evaluating the self energy at $\hat{p}^2 = -m^2$ demanding the result vanishes we find we must have

$$\delta_{m^2} = -\frac{\lambda}{2}\left[\frac{Z_0}{a^2} + \frac{1}{16\pi^2}m^2\log(a^2m^2) + Cm^2\right]. \tag{34.3.18}$$

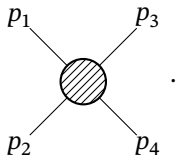
We've dropped the $\mathcal{O}(a^2)$ here since we only require the renormalisation conditions be true in the continuum limit, and we are therefore free to ignore any discretisation effects.

34.4 Four Point Function Renormalisation

To one loop we can expand the four point function as



In each diagram label the momenta of the particles according to



. (34.4.1)

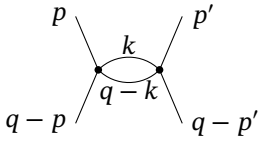
Notice that the three one loop diagrams are all the same up to relabelling of momenta, so we can write the four point function as

$$-\lambda + \frac{\lambda^2}{2}[V(p_1+p_2) + V(p_1-p_3) + V(p_1-p_4)] - \delta_\lambda \tag{34.4.2}$$

where

$$V(q) = \int_{-\pi/a}^{\pi/a}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(\hat{k}^2+m^2)(\widehat{[q-k]}^2+m^2)} \tag{34.4.3}$$

corresponds to the integral from the diagram



(34.4.4)

with total incoming momentum q.

The usual approach to loop diagrams, Feynman parametrisation, doesn't work here since we have $\hat{k}^2$, rather than k^2. Instead, we use appeal to the fact that the renormalisation condition is based on $p_i = 0$ and so we can take zero incoming momentum and just compute

$$V(0) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{(\hat{k}^2 + m^2)^2} =: I_2(a^2 m^2) \tag{34.4.5}$$

where we again write $\ell = ak$. We can recognise that I_2 is related to I_1 through

$$I_2(\xi^2) = -\frac{\mathrm{d}}{\mathrm{d}\xi^2} I_1(\xi^2) \tag{34.4.6}$$

and for small ξ this gives

$$I_2(\xi^2) = -\frac{1}{16\pi^2} \log(\xi^2) - \frac{1}{16\pi^2} + C + \mathcal{O}(\xi^2). \tag{34.4.7}$$

Requiring that the four point function at zero momentum is $-\lambda$ we then find that

$$\delta_\lambda = \frac{3}{2}\lambda^2 \left[-\frac{1}{16\pi^2} \log(a^2 m^2) - \frac{1}{16\pi^2} + C \right]. \tag{34.4.8}$$

With these assignments for δ_Z, δ_{m^2}, and δ_λ we have successfully renormalised φ^4 theory to one loop. We will get new infinities at higher loop orders, but these can also be dealt with. It is possible that φ^4 theory is only renormalisable perturbatively, this would mean that at high energies φ^4 theory would break down. In the real world φ^4 theory is important as it arises in the study of the Higgs mechanism. It is possible that φ^4 theory for the Higgs breaks down at approximately 10 TeV, which is just above the energy of the LHC, which runs at 13.6 TeV but the energy of an individual parton is closer to 1 TeV. A possible solution to this problem is that the Higgs boson is not a fundamental particle, but is instead composite. This is a fix because UV divergences are less consequential for composite particles.

34.5 Renormalisation Group Equation

Consider the full non-perturbative two and three point functions

$$\Gamma^{(2)}(a, m_0(a), \lambda_0(a)) = \frac{1}{\hat{p}^2 + m^2} + \mathcal{O}(\hat{p}^2 + m^2), \qquad \text{where} \qquad \hat{p}^2 \approx -m^2, \tag{34.5.1}$$

$$\Gamma^{(4)}(a, m_0(a), \lambda_0(a)) = -\lambda, \qquad \text{where} \qquad p_i = 0. \tag{34.5.2}$$

These are the renormalisation conditions we have been working with. Since the right hand sides are independent of a these renormalisation conditions necessarily imply that the correlators are constant with respect to a:

$$a\frac{\mathrm{d}}{\mathrm{d}a}\Gamma^{(2)}(a, m_0(a), \lambda_0(a)) = 0, \qquad \text{and} \qquad a\frac{\mathrm{d}}{\mathrm{d}a}\Gamma^{(4)}(a, m_0(a), \lambda_0(a)). \tag{34.5.3}$$

This is known as the condition of constant physics. The physics that we want to describe cannot be allowed to change with the lattice spacing we choose.

We cannot solve these differential equations exactly, but we can work perturbatively. Taking the four point function and writing out the total derivative we have

$$a\frac{\mathrm{d}}{\mathrm{d}a}\Gamma^{(4)} = a\frac{\partial\Gamma^{(4)}}{\partial a} + a\frac{\mathrm{d}\lambda_0}{\mathrm{d}a}\frac{\partial\Gamma^{(4)}}{\partial\lambda_0} + a\frac{\mathrm{d}m_0}{\mathrm{d}a}\frac{\partial\Gamma^{(4)}}{\partial m_0} = 0. \tag{34.5.4}$$

This is known as the **renormalisation group equation**, or the **Callan–Symanzik equation**. This separates the tuning of the parameters with respect to a into derivatives such as $\mathrm{d}m_0/\mathrm{d}a$ and the effect on $\Gamma^{(4)}$ into derivatives such as $\partial\Gamma^{(4)}/\partial m_0$.

It is possible to show that

$$\frac{\partial\Gamma^{(4)}}{\partial a} = -\frac{1}{a}\frac{3}{16\pi^2}\lambda_0^2 + \mathcal{O}(\lambda^2), \tag{34.5.5}$$

where the factor of three comes from the three diagrams, in the s, t, and u-channels with the double propagator, which contribute to the four point function. We know that to tree level the four point function is given by $-\lambda$, which becomes exact at zero external momentum, so we have

$$\frac{\partial\Gamma^{(4)}}{\partial\lambda_0} = -1 + \mathcal{O}(\lambda). \tag{34.5.6}$$

Finally, it can be shown that the tree level expression for $\Gamma^{(4)}$ is independent of mass, so

$$\frac{\partial\Gamma^{(4)}}{\partial m_0} = \mathcal{O}(\lambda^2). \tag{34.5.7}$$

We can also look at how the parameters scale with a and we find that

$$\frac{\mathrm{d}\Gamma^{(4)}}{\mathrm{d}m_0} = \mathcal{O}(\lambda^2), \qquad \text{and} \qquad \frac{\mathrm{d}m_0}{\mathrm{d}a} = \mathcal{O}(\lambda). \tag{34.5.8}$$

This follows by looking at the counterterms.

If we consider the renormalisation group equation at order λ then we get the equation

$$a\frac{\mathrm{d}\lambda_0}{\mathrm{d}a} = -\frac{3}{16\pi^2}\lambda_0^2 = \beta(a). \tag{34.5.9}$$

Here we define the beta function for φ^4. Notice that the sign differs compared to the beta function computed using dim reg since there we take derivatives with respect to an energy scale, μ, which can be taken as $1/a$. This differential equation can be solved, and we find that

$$\lambda_0(a) = \frac{\lambda_\mathrm{i}}{1 + \frac{3}{16\pi^2}\lambda_\mathrm{i}\log(a/a_\mathrm{i})} \tag{34.5.10}$$

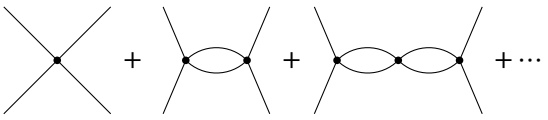
where $\lambda_\mathrm{i} \coloneqq \lambda_0(a_\mathrm{i})$ is the predetermined value of the coupling at some fixed, but arbitrary, initial value of a which we call a_i.

This result has a pole when

$$\frac{3}{16\pi^2}\lambda_\mathrm{i}\log\frac{a}{a_\mathrm{i}} = -1, \tag{34.5.11}$$

which is known as the **Landau pole**. Suppose that this pole occurs at some value $a = a_*$. The existence of this pole means that if we start with $a > a_*$ we cannot say anything about $a < a_*$. This means that there is no continuum limit for this theory. This isn't really an issue since we can expand in small λ_i and the pole goes away, another example of Taylor series messing up the analytic structure of a function, but we can then only work perturbatively. It isn't clear if we can renormalise φ^4 non-perturbatively.

It is also possible to achieve this result by resumming the four point function, just like we did with the two point function, considering the sum



(34.5.12)

which agrees with the four point function to tree level.

QED also has a Landau pole, meaning it too has no continuum limit and may only be renormalisable perturbatively. Fortunately for QED this pole occurs at about 10^{286} eV, a scale so ridiculously large that it will never be reached and any worry about non-perturbative renormalisability is purely academic and cannot possibly affect any real physics, which must all break down far below this scale.

Thirty-Five

Gauge Fields

In this section we will look at how we can treat gauge fields on the lattice. For simplicity, and since we haven't yet discussed fermions on the lattice, we'll look at a gauge theory of a scalar field. We will then look at Yang–Mills theory, for a scalar matter fields, on the lattice.

35.1 Continuous Gauge Theory

If we perform a naive discretisation of the Yang–Mills Lagrangian, replacing continuum derivatives with some discrete derivative, then gauge symmetry is broken for a non-Abelian gauge theory. In order to fix this we first need to study the continuous case of a gauge theory.

Suppose we have N scalar fields, which we can package up into some vector of scalar fields $\varphi(x) \in \mathbb{C}^N$. Then under an SU($N$) gauge transformation this field transforms as

$$\varphi(x) \overset{\omega}{\mapsto} V_\omega(x)\varphi(x) \tag{35.1.1}$$

where the transformation is parametrised by an $\mathfrak{su}(N)$ valued function of spacetime ω which is such that the transformation matrix is given by

$$V_\omega(x) \coloneqq \mathrm{e}^{i\omega(x)} \in \mathrm{SU}(N). \tag{35.1.2}$$

As is now familiar the Klein–Gordon equation,

$$(-\partial^2 + m^2)\varphi = 0, \tag{35.1.3}$$

is not gauge invariant, since we get extra derivatives of ω after a gauge transformation. The solution is of course to introduce the covariant derivative, defined by

$$D_\mu\varphi(x) \coloneqq \lim_{\varepsilon\to 0} \frac{U(x, x+\varepsilon\hat{\mu})\varphi(x+\varepsilon\hat{\mu}) - \varphi(x)}{\varepsilon}. \tag{35.1.4}$$

This is almost the same as the normal derivative,

$$\partial_\mu\varphi(x) \coloneqq \lim_{\varepsilon\to 0} \frac{\varphi(x+\varepsilon\hat{\mu}) - \varphi(x)}{\varepsilon}, \tag{35.1.5}$$

except we have this extra term, $U(x, x+\varepsilon\hat{\mu})$, which is called the **gauge link**. The interpretation of this extra term is that it fixes the failure of parallel transport[1] which

[1] for more details on the failure of parallel transport requiring the introduction of covariant derivatives see *General Relativity*. GR can be phrased as a classical gauge theory on a curved spacetime with the symmetry group being all affine diffeomorphisms, or in the physics lingo, invariance of physics under a change of reference frames.

occurs in a curved space, the curved space here being the manifold SU(N). Thus, the gauge link fixes the problem of having a different "phase", $e^{i\omega(x)}$, at each point. For this to actually be a fix we require that the gauge link transforms according to

$$U(x,y) \overset{\omega}{\mapsto} V_\omega(x)U(x,y)V_\omega(y)^\dagger. \tag{35.1.6}$$

Then, since SU(N) is made of unitary matrices, we have

$$D_\mu\varphi(x) = \lim_{\varepsilon\to 0}\frac{U(x,x+\varepsilon\hat{\mu})\varphi(x+\varepsilon\hat{\mu})-\varphi(x)}{\varepsilon} \tag{35.1.7}$$

$$\overset{\omega}{\mapsto} \lim_{\varepsilon\to 0}\frac{V_\omega(x)U(x,x+\varepsilon\hat{\mu})\overbrace{V_\omega(x+\varepsilon\hat{\mu})^\dagger V_\omega(x+\varepsilon\hat{\mu})}^{=1}\varphi(x+\varepsilon\hat{\mu})-V_\omega(x)\varphi(x)}{\varepsilon}$$

$$= V_\omega(x)\lim_{\varepsilon\to 0}\frac{U(x,x+\varepsilon\hat{\mu})\varphi(x+\varepsilon\hat{\mu})-\varphi(x)}{\varepsilon} \tag{35.1.8}$$

$$= V_\omega(x)D_\mu\varphi(x). \tag{35.1.9}$$

Which is the expected transformation law and means that the modified Klein–Gordon equation

$$(-D^2+m^2)\varphi = 0, \qquad \text{where} \qquad D^2 \coloneqq \sum_\mu D_\mu^\dagger D_\mu, \tag{35.1.10}$$

is gauge invariant.

In addition to this transformation law we also impose $U(x,x) = 1$, which is to say that if there is no translation then there is no need to correct for relative "phase". Using this we can expand $U(x,x+\varepsilon\hat{\mu})$ for small ε and we find

$$U(x,x+\varepsilon\hat{\mu}) = 1 + ig\varepsilon A_\mu(x) + \mathcal{O}(\varepsilon^2) \tag{35.1.11}$$

where the factors of i and g are conventional, the factor of i being due to the physicist convention that $\mathfrak{su}(N)$ consists of Hermitian matrices, and $A_\mu(x) = \partial_\mu U(x,x)$. As the derivative of an element of SU(N) at the identity this is an element of the tangent space $T_e\,\mathrm{SU}(N)$, which is to say an element of the Lie algebra, $\mathfrak{su}(N)$. Alternatively, we can see this by identifying that up to first order in ε we have

$$U(x,x+\varepsilon\hat{\mu}) = e^{ig\varepsilon A_\mu(x)} \tag{35.1.12}$$

and so we must have $A_\mu(x) \in \mathfrak{su}(N)$.

The gauge field, A_μ, must transform according to

$$A_\mu(x) \overset{\omega}{\mapsto} A_\mu - \frac{1}{g}\partial_\mu\omega + [A_\mu,\omega] \tag{35.1.13}$$

since this gives the correct transformation law for U. Using this gauge field we can then write the covariant derivative as

$$D_\mu\varphi(x) = \lim_{\varepsilon\to 0}\frac{[1+ig\varepsilon A_\mu(x)+\mathcal{O}(\varepsilon^2)]\varphi(x+\varepsilon\hat{\mu}-\varphi(x))}{\varepsilon} \tag{35.1.14}$$

$$= \lim_{\varepsilon\to 0}\frac{\varphi(x+\varepsilon\hat{\mu})-\varphi(x)}{\varepsilon} + \lim_{\varepsilon\to 0}[igA_\mu(x)\varphi(x+\varepsilon\hat{\mu})+\mathcal{O}(\varepsilon)] \tag{35.1.15}$$

$$= \partial_\mu\varphi(x) + igA_\mu(x)\varphi(x). \tag{35.1.16}$$

The most general gauge invariant, CP invariant, action with a single dimensionless coupling is the Yang–Mills action

$$S[A_\mu] = \frac{1}{2}\int \mathrm{d}^4x \sum_{\mu,\nu} \mathrm{tr}[F_{\mu\nu}(x)^2] \tag{35.1.17}$$

where

$$F_{\mu\nu} \coloneqq -\frac{i}{g}[D_\mu, D_\nu]1 = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \tag{35.1.18}$$

with 1 in the second term representing the constant function sending all x to 1. The factor of $1/2$ is motivated by the desire to recover the QED action if the gauge group is $\mathrm{U}(1)$, the factor of $1/4$ appearing in the QED action is in turn motivated by requiring that the Maxwell equations are the equations of motion.

If a theory is not CP invariant then it is possible to write down an action containing the term $\vartheta F_{\mu\nu}\tilde{F}_{\mu\nu}$ where ϑ is a dimensionless coupling and

$$\tilde{F}_{\mu\nu} = -\frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}F_{\mu\nu} \tag{35.1.19}$$

is the dual field strength tensor. We won't worry about this term since it has been verified experimentally that if QCD has CP violation then this effect is less tiny and $\vartheta \sim 10^{-7}$ is negligible.

Notice that the definition $F_{\mu\nu} \propto [D_\mu, D_\nu]$ reflects the definition $R^\mu{}_{\nu\rho\sigma}V^\nu = [\nabla_\rho, \nabla_\sigma]V^\mu$ in general relativity where $R^\mu{}_{\nu\rho\sigma}$ is the Riemann tensor which tells us about the curvature of spacetime. Thus, we can extend the point we made earlier that the non-trivial nature of gauge transformations is due to the curvature of the manifold $\mathrm{SU}(N)$.

35.2 Discretising Gauge Fields

We can discretise a gauge theory by introducing discrete versions of the covariant derivative. To do so we take the definition

$$D_\mu\varphi(x) \coloneqq \lim_{\varepsilon\to 0}\frac{U(x, x+\varepsilon\hat{\mu})\varphi(x+\varepsilon\hat{\mu}) - \varphi(x)}{\varepsilon} \tag{35.2.1}$$

and replace the limit to zero with the smallest non-zero distance on the lattice, the lattice spacing, a. Then we define the forward discrete covariant derivative for a function $f : \Lambda^4(a) \to \mathbb{C}$ by

$$\Delta_\mu\varphi(x) \coloneqq \frac{U_\mu(x)f(x+a\hat{\mu}) - f(x)}{a} = \frac{1}{a}(\underline{U_\mu}\tau_\mu - 1)f(x) \tag{35.2.2}$$

where we define the gauge link for a step along the lattice, linking x and $x + a\hat{\mu}$:

$$U_\mu(x) \coloneqq U(x, x+a\hat{\mu}). \tag{35.2.3}$$

We can then interpret $U_\mu\tau_\mu$ as doing parallel transport, τ_μ, and correcting for the relative "phase", U_μ. We can also interpret the conjugate gauge link, $U_\mu(x)^\dagger$, as the gauge link between $x + a\hat{\mu}$ and x, so that we can define

$$U_{-\mu}(x) \coloneqq U_\mu(x - a\hat{\mu})^\dagger = U(x - a\hat{\mu}, x)^\dagger, \tag{35.2.4}$$

In a similar way we can define the backward and central discrete covariant derivatives:

$$\Delta^*_\mu f(x) \coloneqq \frac{f(x) - U_{-\mu}(x) f(x - a\hat{\mu})}{a} = \frac{1}{a}(1 - \tau_{-\mu}\underline{U^\dagger_\mu}) f(x) \tag{35.2.5}$$

$$\bar{\Delta}_\mu f(x) \coloneqq \frac{U_\mu(x) f(x + a\hat{\mu}) - U_{-\mu}(x) f(x - a\hat{\mu})}{2a} = \frac{1}{2}(\Delta_\mu + \Delta^*_\mu) f(x). \tag{35.2.6}$$

These satisfy the same conjugacy relations as the normal discrete derivatives,

$$\Delta^\dagger_\mu = -\Delta^*_\mu, \qquad \text{and} \qquad \bar{\Delta}^\dagger_\mu = -\bar{\Delta}_\mu. \tag{35.2.7}$$

We can also define the covariant Laplacian

$$\bar{\Delta}^2 \coloneqq \sum_\mu \Delta^*_\mu \Delta_\mu. \tag{35.2.8}$$

The gauge link must then transform according to

$$U_\mu(x) \overset{V}{\mapsto} V(x) U_\mu(x) V(x + a\hat{\mu})^\dagger \tag{35.2.9}$$

where V is an SU(N) valued function. We have to use SU(N) for transformations of the gauge link, as opposed to $\mathfrak{su}(N)$, if we want to maintain gauge invariance in the discrete case. Ultimately the notion of an infinitesimal transformation which motivates the definition of a Lie algebra doesn't really work on a discrete lattice anyway, so it makes sense that we work with the Lie group instead. This also means that we won't expand $U_\mu(x)$ in the way that we expanded $U(x, x + \varepsilon\hat{\mu})$ to find the gauge field A_μ, and instead work with U_μ as the gauge field.

If we want to transport a field over a distance greater than a we can do so by defining a **Wilson line**

$$W^{(\varepsilon_1\mu_1,\dots,\varepsilon_n\mu_n)}(x) \coloneqq U_{\varepsilon_1\mu_1}(x) U_{\varepsilon_2\mu_2}(x + a\varepsilon_1\hat{\mu}_1) \cdots U_{\varepsilon_n\mu_n}(x + a\varepsilon_1\hat{\mu} + \cdots + a\varepsilon_{n-1}\hat{\mu}_{n-1}) \tag{35.2.10}$$

where ε_i are signs. This represents transporting a quantity along n steps across the lattice with the direction of the ith step given by $\varepsilon_i\hat{\mu}_i$. If our Wilson line starts and ends at the same position it is called a **Wilson loop**, in which case we must have $\sum_{i=1}^n \varepsilon_i\hat{\mu}_i = 0$. Wilson lines transform as

$$W^{(\varepsilon_1\mu_1,\dots,\varepsilon_n\mu_n)}(x) \overset{V}{\mapsto} V(x) W^{(\varepsilon_1\mu_1,\dots,\varepsilon_n\mu_n)}(x) V\left(x + a\textstyle\sum_{i=1}^n \varepsilon_i\hat{\mu}_i\right)^\dagger, \tag{35.2.11}$$

since each gauge link transforms according to Equation (35.2.9) and the internal factors of V come in conjugate pairs, and so just reduce to the identity as they are elements of SU(N). A Wilson loop then transforms as

$$W^{(\varepsilon_1\mu_1,\dots,\varepsilon_n\mu_n)}(x) \overset{V}{\mapsto} V(x) W^{(\varepsilon_1\mu_1,\dots,\varepsilon_n\mu_n)}(x) V(x)^\dagger. \tag{35.2.12}$$

The cyclic nature of the trace means that the trace of a Wilson loop is gauge invariant:

$$\operatorname{tr}(W^{(\varepsilon_1\mu_1,\dots,\varepsilon_n\mu_n)}(x)) \overset{V}{\mapsto} \operatorname{tr}(V(x) W^{(\varepsilon_1\mu_1,\dots,\varepsilon_n\mu_n)}(x) V(x)^\dagger) \tag{35.2.13}$$

$$= \operatorname{tr}(V(x)^\dagger V(x) W^{(\varepsilon_1\mu_1,\dots,\varepsilon_n\mu_n)}(x)) \tag{35.2.14}$$

$$= \operatorname{tr}(W^{(\varepsilon_1\mu_1,\dots,\varepsilon_n\mu_n)}(x)). \tag{35.2.15}$$

The simplest, non-trivial, Wilson loop goes in a square of side length a, and is called the **plaquette**:

$$P_{\mu\nu}(x) \coloneqq W^{+\mu,+\nu,-\mu,-\nu}(x) = U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu(x+a\hat{\nu})^\dagger U)\nu(x)^\dagger. \quad (35.2.16)$$

This quantity can be used to define the **Wilson action**

$$S\text{Wilson}[U_\mu] \coloneqq \frac{\beta}{N}\sum_{x\in\Lambda^4}\sum_{\mu>\nu}\text{Re}(\text{tr}[1-P_{\mu\nu}(x)]). \quad (35.2.17)$$

It can be shown that this converges to the Yang–Mills action in the continuum limit. First, define the gauge field, A_μ, according to

$$U_\mu(x) = \mathrm{e}^{igaA_\mu(x)}, \quad (35.2.18)$$

and then if we expand in a we find that

$$P_{\mu\nu}(x) = 1 + ia^2gF_{\mu\nu}(x) - \frac{a^4g^2}{2}F_{\mu\nu}(x)^2 + \mathcal{O}(a^6) \quad (35.2.19)$$

so to leading order $1 - P_{\mu\nu}$ is the field strength tensor and we find that

$$S[U_\mu] = \frac{\beta g^2}{4N}\int \mathrm{d}^4x \sum_{\mu,\nu}\text{tr}[F_{\mu\nu}(x)^2] + \mathcal{O}(a^2) \quad (35.2.20)$$

which gives the Yang–Mills action if we set $\beta = 2N/g^2$. We can interpret β as the inverse coupling, the notation inspired by the use of β as the inverse temperature in statistical mechanics.

We can then proceed as usual, defining the expectation value of an operator, O, to be

$$\langle O\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}U_\mu\, O[U_\mu]\mathrm{e}^{-S_{\text{Wilson}[U_\mu]}}. \quad (35.2.21)$$

This integral is slightly more complicated than in the continuum case because we are now integrating over a Lie group, rather than a Lie algebra as we would when computing $\int \mathcal{D}A_\mu$, and the Lie group is not flat, and we need to use the Haar measure to perform the integral. One nice thing about this integral though is that the the Lie group SU(N) is compact (in particular it is bounded) whereas the Lie algebra isn't, which is why we need gauge fixing in the continuum case, but any gauge fixing term will cancel between $\mathcal{Z}$ and the integral here so we don't need to fix the gauge.

35.3 Continuum Limit

To compute the continuum limit we need to fix some renormalisation condition. The condition we choose is that when the lattice spacing is a the Yang–Mills coupling, $g_0(a)$, should be such that the observable $X(a, g_0(a))$ gives the measured physical value X^φ. This should work for all values of a, so $X(a, g_0(a))$ should be constant with respect to a, meaning

$$a\frac{\mathrm{d}}{\mathrm{d}a}X(a,g_0(a)) = \left(a\frac{\partial}{\partial a} + a\frac{\mathrm{d}g_0}{\mathrm{d}a}\frac{\partial}{\partial g_0}\right)X(a,g_0(a)) = 0. \quad (35.3.1)$$

We can identify the beta function for this theory as

$$\beta(g_0) = a\frac{\mathrm{d}g_0}{\mathrm{d}a}. \tag{35.3.2}$$

This is known up to $\mathcal{O}(g_0^7)$ corrections and doesn't depend on the UV regulator, a. To one loop we have

$$\beta(g_0) = \frac{11N}{48\pi^2}g_0^3 + \mathcal{O}(g_0^5). \tag{35.3.3}$$

Solving this differential equation we find that

$$\alpha_0(a) \coloneqq \frac{g_0(a)^2}{4\pi} = \frac{\alpha_\mathrm{i}}{1 - \frac{11N}{6\pi}\log(a/a_\mathrm{i})} \tag{35.3.4}$$

where again a_i is some initial lattice spacing at which $\alpha_\mathrm{i} = \alpha_0(a_i)$ is known. Notice that $\alpha_0(a) \to 0$ in the continuum limit, this is asymptotic freedom. We can also express the lattice spacing in terms of the coupling:

$$a = a_\mathrm{i}\exp\left\{-\frac{11N}{6\pi}\left(\frac{1}{\alpha_0} - \frac{1}{\alpha_\mathrm{i}}\right)\right\}. \tag{35.3.5}$$

The continuum limit is then reached by taking $\alpha_0 \to 0$ while keeping the physics constant. For the Wilson action this corresponds to taking $\beta_0 \to 0$.

35.4 Abelian Gauge Theories

For an Abelian gauge theory we actually don't need to introduce a covariant derivative. For example, consider $\mathrm{U}(1)$. The continuum Euclidean action is

$$S[A_\mu] = \frac{1}{4}\int \mathrm{d}^4x \sum_{\mu,\nu}(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x))^2 \tag{35.4.1}$$

where A_μ is a real vector field (that is, a $\mathrm{u}(1)$ valued vector field). The naive discretisation does work in the Abelian case and we can define the discrete action

$$S[A_\mu] = \frac{a^4}{4}\sum_{x\in\Lambda^4}\sum_{\mu,\nu}(\delta_\mu A_\nu(x) - \delta_\nu A_\mu(x))^2. \tag{35.4.2}$$

This action is invariant under the $\mathrm{U}(1)$ gauge action

$$A_\mu \overset{\omega}{\mapsto} A_\mu - \delta_\mu\omega \tag{35.4.3}$$

where ω is a real valued function.

This action is non-compact, since it is a functional of the unbounded field A_μ, rather than a bounded $\mathrm{U}(1)$ gauge link, which has magnitude 1 by definition. Alternatively, if we restrict $\mathrm{u}(1)$ to $[0, 2\pi)$ then this set is not closed, so still non-compact.

We can proceed as usual and define the path integral

$$\langle O\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}A_\mu\, O[A_\mu]\mathrm{e}^{-S[A_\mu]}. \tag{35.4.4}$$

By integrating by parts we can write the action in a quadratic form as

$$S[A_\mu] = \frac{a^4}{2} \sum_{x\in\Lambda^4} \sum_{\mu,\nu} A_\mu(x) \Omega_{\mu\nu} A_\nu(x) \tag{35.4.5}$$

with the operator

$$\Omega_{\mu\nu} \coloneqq -\delta_{\mu\nu}\tilde{\delta}^2 + \delta^*_\mu \delta_\nu. \tag{35.4.6}$$

Since there is no interaction here the only relevant correlator is the two point correlator

$$D_{\mu\nu}(x,y) = \langle A_\mu(x) A_\nu(y) \rangle. \tag{35.4.7}$$

This simply corresponds to the matrix elements of $\Omega_{\mu\nu}$. The problem is that this operator is not invertible, a problem familiar from the continuum version of this operator. This is usually where we introduce Faddeev–Popov gauge fixing, which extends to the lattice in a straightforward manner. However, we'll just look at a particular gauge for simplicity. In the Feynman gauge the action is

$$S_{\text{Feyn}}[A_\mu] = S[A_\mu] + \frac{a^4}{2} \sum_{x\in\Lambda^4} \sum_{\mu} (\delta_\mu A_\mu)^2. \tag{35.4.8}$$

It can be shown that this doesn't change the physical transverse part of the field A_μ. Integrating by parts we get

$$S_{\text{Feyn}}[A_\mu] = \frac{a^4}{2} \sum_{x\in\Lambda^4} \sum_{\mu,\nu} A_\mu(x) \Omega^{\text{Feyn}}_{\mu\nu} A_\nu(x) \tag{35.4.9}$$

where

$$\Omega^{\text{Feyn}}_{\mu\nu} = \Omega_{\mu\nu} - \delta^*_\mu \delta_\nu. \tag{35.4.10}$$

This is simply the Klein–Gordon operator for a massless particle, with an extra factor of $\delta_{\mu\nu}$. If we compute the propagator then we find the Feynman gauge propagator is

$$D_{\mu\nu}(x,y) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\delta_{\mu\nu}}{\hat{k}^2} \mathrm{e}^{ik\cdot(x-y)}. \tag{35.4.11}$$

So, there are two ways to approach Abelian gauge theories. We can naively discretise the action, leading to a theory very similar to the continuum case, or we can use the Wilson action, leading to a theory more akin to Yang–Mills. There are advantages and drawbacks to both. The naive discretisation doesn't introduce extra terms or interactions, but does require gauge fixing which we don't need for the Wilson action.

Index

X