Proofs and Problem Solving Lecture Notes

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These are my notes for the *proofs and problem solving* course from the University of Edinburgh School of Mathematics, which I took as an optional course outside of the School of Physics. When I took this course in the 2018/19 academic year it was taught by Dr Jonas Azzam[1]. These notes are based on the lectures delivered as part of this course, and the notes provided as part of this course, as well as the textbook for the course 'A Concise Introduction to Pure Mathematics'[2]. The content within is correct to the best of my knowledge but if you find a mistake or just disagree with something or think it could be improved please let me know.

These notes were produced using LaTeX[3]. Diagrams were drawn with tikz[4]. Some images were taken from the lecture notes provided.

This is version 1.0 of these notes, which is up to date as of 04/01/2021.

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[1] https://www.maths.ed.ac.uk/school-of-mathematics/people/a-z?person=516
[2] Liebeck, M *A Concise Introduction To Pure Mathematics*, fourth edition (CRC Press, Boca Raton, 2016)
[3] https://www.latex-project.org/
[4] https://www.ctan.org/pkg/pgf

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1 Proofs

What is a proof?

A claim/proposition/theorem/lemma is an assertion that some statement follows from:

1. An assumption (eg. assuming that n is odd ...)
2. Some definition (eg. from the definition of integers ...)
3. Axioms (eg. the operations $+, -, \cdot$)

A proof is:

1. An explanation
2. Written in complete sentances
3. Comprehensible to any intelligent reader
4. Demonstrates how to deduce the claim

A conjecture is a claim without proof but backed up by some evidence

Why are proofs so important?

- Any falsehood propogates through any further mathematics
- It isn't always obvious as to the validity of a claim
- "Obvious" statements are sometimes false
- The proof can be more useful than the statement

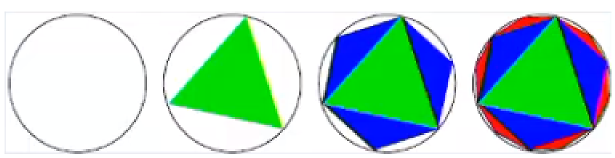
Examples with π

We define π as the area of the unit circle

Claim The number π is also the ratio of a circles area A and the square of its radius r. That is $\pi = \frac{A}{r^2}$.

Proof If T is a triangle and T' is a congruent triangle that is r-times the size of T then the area of T' is r^2-times the area of T.

If S is the unit circle, write it as the union of triangles $T_1, T_2, T_3, \ldots$

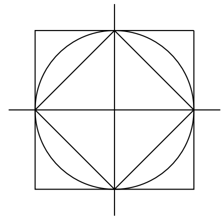


If we increase the size of S by a factor of r then we also increase the size of the triangles by a factor of r. Therefore thee area of the triangles increases by a factor of r^2 so the area of the circle increases by a factor of r^2. Thus $A = \pi r^2 \implies \pi = \frac{A}{r^2}$. ■

Knowing that something exists is not the same as knowing it.

Claim $2 \le \pi \le 4$

Proof Consider the following unit circle with area π by definition and the two squares circumscribed and inscribed.



The circumscribed square has an area larger than the area of the unit circle and the inscribed square has an area less than the area of the unit circle.

The circumscribed square has side length equal to two times the radius of the circle which means that it has side length 2 so area 4.

The inscribed square has diagonal equal to two times the radius of the circle which means that it has diagonal length 2 so by pythagoras the square has side length $\sqrt{2}$.

From the area of the three shapes and the knowledge of which is larger it follows that $2 \leq \pi \leq 4$ ■

The following is an example of a bad (but technically correct) proof:

Claim Every rational number is the sum of three cubes of rational numbers.

Proof Let a be any rational number. Then a direct computation yields:

$$a = \left(\frac{a^3 - 3^5}{3^2a^2 + 3^4a + 3^6}\right)^3 + \left(\frac{-a^3 + 3^5a + 3^6}{3^2a^2 + 3^4a + 3^6}\right)^3 + \left(\frac{a^2 + 3^4a}{3^2a^2 + 3^4a + 3^6}\right)^3 \blacksquare$$

This is a correct but not very good proof for a few reasons:

- It doesn't show the nature of the claim
- It can't be repeated
- It doesn't generalize
- There exists a much better proof

Some plausible conjectors turn out to be false:

The prime number theorem:

Claim If $\pi(n)$ is the number of primes less than n and $Li(n) = \int_0^n \frac{dx}{\log x}$, then:

$$\lim_{n\to\infty} \frac{\pi(n)}{Li(n)}$$

- Mathemeticians observed $\pi(n) < Li(n)$ for many values of n (all values below 10^{19} have been checked)
- Gauss and Riesmann conjectured that $\pi(n) < Li(n)$ for all n
- But, Littlewood (1914) showed $\pi(n) \geq Li(n)$ infinitley often.
- All we know is that this first happens in the range $n \in [10^{19}, 10^{317}]$
- For comparison there are only 10^{80} protons in the universe

2 Logic

Logic

If A and B are statements then:

- If A and B are statements then $A \implies B$ means "A implies B" or "if A is true then B is true"
- $\bar{A}$ is the negation of statement A
- If A is true then $\bar{A}$ is false and vice versa
- $A \iff B$ means "A and B are equivalent", that is $A \implies B$ and $B \implies A$. From this it follows that either A and B are true or A and B are false.
- $\bar{\bar{A}} \iff A$
- A or B means either A or B or both are true
- A and B means A and B are both true
- Statements about statements are allowed eg. $\overline{(A \text{ and } B)} \iff (\bar{A} \text{ or } \bar{B})$, $\overline{(A \text{ or } B)} \iff (\bar{A} \text{ and } \bar{B})$

$$(A \implies B) \iff (\bar{B} \implies \bar{A}) \iff (\bar{A} \text{ or } B)$$

The above statements are equivalent to each other. ($\bar{B} \implies \bar{A}$ is called the contrapositive of $A \implies B$)

Claim If A and B are statements then $(A \implies B) \iff (\bar{B} \implies \bar{A})$

Proof Let us assume that $A \implies B$. Then, if $\bar{B}$ is true $\bar{A}$ must also be true as if it were false then A would be true which implies B is true which it isn't. The same logic can be used when assuming the second statement and double negating to get the first statement. ■

The following is a proof by cases, every possible case is proved individualy which means that the statement is true.

Claim If A and B are statements then $(A \implies B) \iff (\bar{A} \text{ or } B)$.

Proof We must show 1. if $A \implies B$ then $\bar{A}$ or B is true and 2. if $\bar{A}$ or B is true then $A \implies B$ is true.

1. Suppose $A \implies B$ then we need to show that either $\bar{A}$ or B is true. We do this by splitting into two further cases a. A is true and b. A is false.
   (a) A is true and since we have assumed that $A \implies B$ it follows that B is true so $\bar{A}$ or B is true.
   (b) A is false so $\bar{A}$ is true so $\bar{A}$ or B is true

   Since all cases give $\bar{A}$ or B is true given $A \implies B$ it follows that $\bar{A}$ or B is true if$A \implies B$
2. Suppose $\bar{A}$ or B is true. We need to show $A \implies B$ is true. Assume that A is true. Since we assume A is true $\bar{A}$ must be false. Since we are also assuming $\bar{A}$ or B is true we must have B is true. Since B is true whenever A is true it follows that $A \implies B$.

Since we have shown that $(A \implies B) \implies (\bar{A} \text{ or } B)$ and $(\bar{A} \text{ or } B) \implies (A \implies B)$ then $(A \implies B) \iff (\bar{A} \text{ or } B)$. ■

Claim Let A and B be statements, then, $\overline{(A \implies B)}$ is equivalent to $(A \text{ and } B)$.

Proof By the previous claim $(A \implies B) \iff (\bar{A} \text{ or } B)$. Thus $\overline{(A \implies B)} \iff \overline{(\bar{A} \text{ or } B)} \iff (\bar{\bar{A}} \text{ or } \bar{B}) \iff (A \text{ and } \bar{B})$ ■

Proving $A \implies B$ by contradiction means assuming $\overline{A \implies B}$ is true and showing that a logical contradiction occurs.

Claim If m^2 is even then m is even.

Proof Assume m^2 is even but m is odd. Then $m = 2n+1$ for some $n \in \mathbb{Z}$. Hence $m^2 = (2n+1)^2 = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$ which is odd, but we assumed that m^2 was even so this is a contradiction so m must be even. ■

Claim Any false statement can imply anyother statement. That is if A and B are statements an A is false then $A \implies B$

Proof Since A is false $\bar{A}$ is true so $\bar{A}$ or B is true which is equivalent to $A \implies B$ ■

corollary If unicorns are real then $1 + 1 = 3$

Sets

Sets are a collection of things called elements eg. $\{1, 2, 3\}$ is the set of 1, 2 and 3.

Let S and T be two sets:

1. If S and T have the same elements then $S = T$
2. If x is an element of S then we write $x \in S$
3. If x is not an element of S then we write $x \notin S$
4. T is a subset of S if $(t \in T) \implies (t \in S)$
5. If T is a subset of S we write $T \subseteq S$
6. If T is not a subset of S we write $t \nsubseteq S$
7. $\{\}$ and $\emptyset$ both represent the empty set (Nb $\emptyset$ is a subset of all sets as there is no item in the empty set that isn't in any other set so it always follows the condition for subsets)

Question How many subsets does the set $\{1, 2, 3\}$ have?

Answer It has 8 subsets:

- $\{1, 2, 3\}$
- $\{1, 2\}, \{1, 3\}, \{2, 3\}$
- $\{1\}, \{2\}, \{3\}$
- $\emptyset$

3 Sets and Quantifiers

Sets can be elements of other sets. For example $S = \{a, b, \{c, d\}\}$ is the set containing elements a, b and $\{c, d\}$

There are different ways to denote a set. The following sets are all equivalent:

- The circle in $\mathbb{R}^2$ with radius 1 and centre $(0, 0)$

- The set of points (x, y) in $\mathbb{R}^2$ such that $x^2 + y^2 = 1$
- $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$

Quantifiers

- $\forall$ means "for all"
- $\exists$ means "there exists"
- s.t. means "such that"
- ! means negation ($!P \iff \bar{P}$)

To negate a statement start by negating the entire statement and then move the ! from left to right inverting quantifiers according to the following:

- $!!P \iff P$
- $!\forall \longrightarrow \exists!$
- $!\exists \longrightarrow \forall!$
- $= \longleftrightarrow \neq$
- $\geq \longleftrightarrow <$
- $\leq \longleftrightarrow >$
- Negate the conclusion e.g $!(x \text{ is odd}) \longrightarrow x$ is even.

$$\begin{aligned} &!(\forall n \in \mathbb{Z}, \exists m \in \mathbb{R}\, st\, m^2 = n) \\ \iff &\exists n \in \mathbb{Z},\, st\, !(\exists m \in \mathbb{R}\, st\, m^2 = n) \\ \iff &\exists n \in \mathbb{Z},\, st\, \forall m \in \mathbb{R}\, !(m^2 = n) \\ \iff &\exists n \in \mathbb{Z},\, st\, \forall m \in \mathbb{R}\, m^2 \neq n \end{aligned}$$

To prove a statement like $\forall x \in S$, $P(x)$ is true begin by letting x be any element in S.

To prove a statement like $\exists x \in S$, $P(x)$ is true just find one x in S such that $P(x)$ is true.

Claim The statement "$A = \forall n \in \mathbb{Z}\ \exists m \in \mathbb{R}\, st\, m^2 = n$" is false

Proof We will show that the negation is true. The negation is:

$$!A = \exists n \in \mathbb{Z}\, st\, \forall m \in \mathbb{R}, m^2 \neq n$$

Let $n = -1$. Now we will show that $\forall m \in \mathbb{R} m^2 \neq -1$ which will prove $!A$.

Let $m \in \mathbb{R}$. Then $m^2 \geq 0 > -1 \implies m^2 \neq -1$.

This proves $!A$ is true so A is false. ■

Claim $\forall n \in \mathbb{Z} \exists m \in \mathbb{Z}\, st\, m > n$

Proof Let $n \in Z$. We need to show that $\exists m \in \mathbb{Z}\, st\, m > n$.

Let $m = n + 1 > n$ ■

Claim $\sqrt{2} - \sqrt{n} \notin \mathbb{Q}$ for some $n \in \mathbb{Q}$.

Proof Proof by contradiction.

Assume that $\sqrt{2}-\sqrt{n}\in\mathbb{Q}$.

By (3) we know that if $a\in\mathbb{Q}$, $b\notin\mathbb{Q}$ then $a+b\notin\mathbb{Q}$.

By (2) we know that if $a,b\in\mathbb{Q}$ then $a+b\in\mathbb{Q}$.

$$(\sqrt{2}+\sqrt{n})(\sqrt{2}-\sqrt{n})=2-n$$

By our initial assumption both $\sqrt{2}+\sqrt{n}$ and $2-n$ are rational. Since $ab\in\mathbb{Q}$ if both a and b are rational or both irrational this means that $\sqrt{2}-\sqrt{n}$ must be rational.

$$\sqrt{2}-\sqrt{n}=-(\sqrt{n}-\sqrt{2})=\sqrt{n}+\sqrt{2}-2\sqrt{2}$$

From our initial assumption $\sqrt{n}+\sqrt{2}$ is rational and $-2\sqrt{2}$ is irrational so it follows that $\sqrt{2}-\sqrt{n}$ is irrational. This is a contradiction so $\sqrt{2}-\sqrt{n}\notin\mathbb{Q}$ ■

Claim $\forall\, a,b\in\mathbb{R}$ if $a<b$ then $\exists\, r\in\mathbb{Q}\, st\, a<r<b$

Proof Let n be such that $\frac{1}{n}<b-a \implies n>\frac{1}{b-a}$

Let j be the largest integers such that $\frac{j}{n}\leq a$

Since j is the maximum $\frac{j+1}{n}>a$

$\frac{j+1}{n}=\frac{j}{n}+\frac{1}{n}<\frac{j}{n}+b-a\leq a+b-a=b$

Thus $a<\frac{j+1}{n}<b$ and $\frac{j+1}{n}\in\mathbb{Q}$ ■

4 Decimals

Geometric series

For $x\in\mathbb{R}$:

$$\sum_{n=1}^{k}x^k=x+x^2+\cdots+x^k=\frac{x-x^{k+1}}{1-x}$$

For $|x|<1$:

$$\sum_{n=1}^{\infty}x^k=x+x^2+x^3+\cdots=\frac{x}{1-x}$$

For $|x|>1$

$$\sum_{n=1}^{\infty}(\frac{1}{x})^k=\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\cdots=\frac{\frac{1}{x}}{1-\frac{1}{x}}=\frac{1}{x-1}$$

Claim $0.\overline{12}=\frac{4}{33}$

Proof We will write $0.\bar{1}\bar{2}$ as a geometric series

$$\begin{aligned}0.\overline{12}&=\frac{1}{10}+\frac{2}{10^2}+\frac{1}{10^3}+\frac{2}{10^4}+\cdots\\&=\frac{10}{10^2}+\frac{2}{10^2}+\frac{10}{10^4}+\frac{2}{10^4}+\cdots\\&=\frac{12}{10^2}+\frac{12}{10^4}+\cdots\end{aligned}$$

$$\begin{aligned}
&= 12\left(\frac{1}{10^2} + \frac{1}{10^4} + \cdots\right) \\
&= 12\left(\frac{1}{10^2} + \left(\frac{1}{10^2}\right)^2 + \cdots\right) \\
&= 12\left(\frac{1}{10^2 - 1}\right) \\
&= \frac{4}{33} \quad \blacksquare
\end{aligned}$$

A decimal can be written as $a_0.a_1a_2a_3\ldots$ where $a_n \in \mathbb{Z}$ and $0 \le a_n \le 9$. There is only one way to represent most numbers unless they end with a string on 9s:

$$a_0.a_1a_2\ldots a_{k-1}a_k9\ldots 9 \equiv a_0.a_1a_2\ldots a_{k-1}(a_k+1)0\ldots 0$$

Formaly two numbers $a_0.a_1a_2\ldots$ and $b_0.b_1b_2\ldots$ are equal if and only if there exists $k \in N$ such that:

- For $j < k$ we have $a_j = b_j$
- For $j = k$ we have $a_j = b_j + 1$
- For $j > k$ we have $a_j = 0$, $b_j = 9$

Claim Every real number $x \in \mathbb{R}$ has a decimal expansion $x = a_0.a_1a_2\ldots$

Proof Let $x \in \mathbb{R}$

Pick $a_0 \in \mathbb{Z}$ such that $a_0 \le x < a_0 + 1$

Pick $a_0 \in \mathbb{Z}$ such that $a_0 + \frac{a_1}{10} \le x < a_0 + \frac{a_1+1}{10}$

$0 \le a_0 < 10$, if $a_1 \ge 10$ then $a_0 + 1 \le a_0 + \frac{a_1}{10} \le x < a_0 + 1$ which is a contradiction so $0 \le a_0 < 10$.

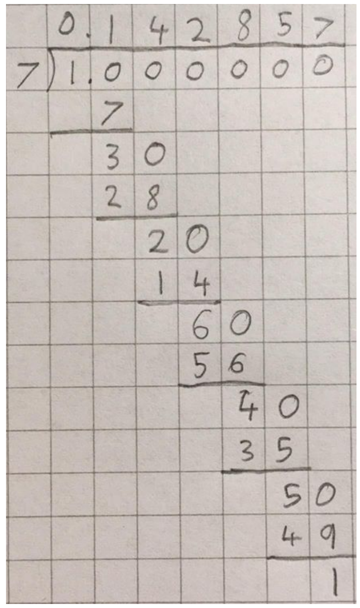
Continue for a_n. As $n \to \infty$, $|x - a_0.a_1a_2\ldots a_n| = 0$ so $x = a_0.a_1a_2\ldots$ $\blacksquare$

Pigeon hole principle

Suppose $k > n$ and we have $a_1, \ldots, a_k \in S$ with $|S| = n$. Then there exists $i \ne j$ such that $a_i = a_j$

Claim $\frac{1}{7}$ has a repeating decimal

Proof Using the division algorithm:



The remainder is again one which will result in a repeat of the expansion from the last time the remainder was one. This shows that $\frac{1}{7} = 0.\overline{142857}$ which is a repeating decimal. ■

Claim A real number $x \in \mathbb{R}$ has a periodic decimal expansion $\iff x \in \mathbb{Q}$

Proof We need to prove the implication both ways:

- A real number $x \in \mathbb{R}$ has a periodic decimal expansion $\implies x \in \mathbb{Q}$:

  Let $x = a_0.a_1a_2\ldots a_k\overline{b_1\ldots b_l}$ Let $a = a_1a_2\ldots a_k$ and $b = b_1\ldots b_l$ Then:

  $$x = a_0 + \frac{a}{10^k} + \frac{1}{10^k}\cdot\frac{b}{10^l - 1}$$

  Therefore x has a periodic decimal expansion.

- $x \in \mathbb{Q} \implies$ A real number $x \in \mathbb{R}$ has a periodic decimal expansion:

  Each step of the long division algorithm $q\,\overline{)p.000\ldots}$ returns a remainder in $1, \ldots, q$. By the pigeon hole princile ome remainder k must occur twice. Because $p.000\ldots$ has trailing zeroes the algorithm repeats at that point. The period of $\frac{p}{q}$ is at most $q - 1$.

The implication is true in both directions so the claim is true. ■

All of this works in different bases:

For $n > 1$ we say x has base n expansion

$$b_jb_{j-1}\ldots b_0.a_1a_2a_3\ldots$$

For integers $b_i, a_i \in \{0, 1, 2, \ldots, n\}$ if

$$x = n^jb_j + n^{j-1}b_{j-1} + \cdots + nb_1 + b_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \frac{a_3}{n^3}\cdots$$

Example 4.1

If $x = 0.\overline{012}$ in base 3 find x as a fraction in base 10

$$\begin{aligned}
x &= 0 + \frac{0}{3} + \frac{1}{3^2} + \frac{2}{3^3} + \frac{0}{3^4} + \frac{1}{3^5} + \frac{2}{3^6} + \cdots \\
&= \frac{3 \cdot 0 + 1 \cdot 3^2 + 2 \cdot 3^3}{3^3} + \frac{3 \cdot 0 + 1 \cdot 3^2 + 2 \cdot 3^3}{3^6} + \cdots \\
&= \frac{5}{3^3} + \frac{5}{3^6} + \cdots \\
&= 5\left(\frac{1}{3^3} + \frac{1}{3^6}\right) \\
&= 5\left(\frac{1}{3^3} + \left(\frac{1}{3^3}\right)^2\right) \\
&= 5 \cdot \frac{1}{3^3 - 1} \\
&= \frac{5}{26}
\end{aligned}$$

5 Inequalities

Defining $<$ / $>$

Given $x, y \in \mathbb{R}$ we may write $x < y$ which we pronounce "x is less than y" or $y > x$ which we pronounce "y is greater than x". The symbols $<$ and $>$ satisfy the the following axioms:

1. Positive/negative - If $x \in \mathbb{R}$ then exactly one of the following is true:
   - $x < 0$
   - $x = 0$
   - $x > 0$
2. Negations reverse - If $x > y$ then $-x < -y$
3. Adding constants - If $x > y$ and $c \in \mathbb{R}$ then $x + c > y + c$
4. Positive multiplication - If $x > 0$ and $y > 0$ then $xy > 0$
5. Transitive property - If $x > y$ and, for $z \in \mathbb{R}$, $y > z$ then $x > z$

This is a minimum set of axioms. No axiom can be deduced from the others.

This captures the essence of "$<$ / $>$" but we must prove some things.

All apart from the first axiom apply to $\leq$ / $\geq$

Claim For any $x, y \in \mathbb{R}$ exactly one of the following is true: $x > y$, $x = y$ or $x < y$

Proof Let $x, y \in \mathbb{R}$. By axiom 1 exactly one of the following is true:

$$x - y > 0, \quad x - y = 0 \quad \text{or} \quad x - y < 0$$

By axiom 3 with $c = y$

$$x - y + y > 0 + y, \quad x - y + y = 0 + y \quad \text{or} \quad x - y + y < 0 + y$$

Simplifying:

$$x > y, \quad x = y \quad \text{or} \quad x < y \quad \blacksquare$$

Claim $\sqrt{xy} \leq \frac{x+y}{2}$

Proof By Liebeck example 5.2

$$\begin{aligned} & 0 \leq (a-b)^2 \\ \implies & 0 \leq a^2 + b^2 - 2ab \end{aligned}$$

By axiom 3 with $c = 2ab$

$$\implies 2ab \leq a^2 + b^2 \quad (5.1)$$

By axiom 4 if $u > v > 0$ and $c > 0$ then $cu > cv$, applying this to (5.1) with $c = \frac{1}{2}$

$$\implies ab \leq \frac{a^2 + b^2}{2}$$

Let $a = \sqrt{x}$ and $b = \sqrt{y}$

$$\begin{aligned} \implies & \sqrt{x}\sqrt{y} \leq \frac{\sqrt{x}^2 \sqrt{y}^2}{2} \\ \implies & \sqrt{xy} \leq \frac{x+y}{2} \quad \blacksquare \end{aligned}$$

Claim If $x > y > 0$ then $\frac{1}{x} < \frac{1}{y}$

Proof We will prove the contrapositve:

For $x, y > 0$ if $\frac{1}{x} \geq \frac{1}{y}$ then $x \leq y$

Suppose that $\frac{1}{x} \geq \frac{1}{y}$. By axiom 4:

$$\begin{aligned} & \frac{1}{x} \geq \frac{1}{y} \\ \implies & x \cdot \frac{1}{x} \geq x \cdot \frac{1}{y} \\ \implies & 1 \geq \frac{x}{y} \\ \implies & y \cdot 1 \geq y \cdot \frac{x}{y} \\ \implies & y \geq x \\ \implies & x \leq y \quad \blacksquare \end{aligned}$$

Claim Let $u, v > 0$ then $u > v \iff u^2 > v^2$

Proof Suppose that $u, v > 0$.

We need to prove the implication in both directions:

- $u > v \implies u^2 > v^2$

  Assume $u > v$ we need to show $u^2 > v^2$. By axiom 3:

$$\begin{aligned} & u > v \\ \implies & u - v > v - v \end{aligned}$$

$$\implies u - v > 0$$

Since $u, v > 0$ by axiom 3:

$$\begin{aligned}
&\implies u > 0 \\
&\implies u + v > 0 + v \\
&\implies u + v > v > 0
\end{aligned}$$

Since $u + v > 0$ and $u - v > 0$ then by axiom4:

$$\begin{aligned}
&\implies (u + v)(u - v) > 0 \\
&\implies u^2 - v^2 > 0 \\
&\implies u^2 > v^2
\end{aligned}$$

- $u^2 > v^2 \implies u > v$ Take the contrapositive:

  $$u \leq v \implies u > v$$

  Then since we didn't use axiom 1 by the same process as above but with $\leq$ instead of $>$ we get $u^2 > v^2 \implies u > v$

We have proved the implication in both directions so the claim is true. ■

6 Exponentiation

Claim For $u, v, x \in \mathbb{R}$, $x > 0$ if $u > v$ then $xu > xv$

Proof Let $u > v$ and $x > 0$ for all $u, v, x \in \mathbb{R}$

By axiom 3 with $c = -v$:

$$u > v \implies u - v > v - v = 0$$

By axiom 4:

$$x(u - v) > 0 \implies xu - xv > 0$$

By axiom 3 with $c = xv$:

$$xu - xv > 0 \implies xu - xv + xv > xv \implies xu > xv \quad ■$$

Roots

Proposition $\forall x > 0$ and $n \in \mathbb{N}$ $\exists$ a unique number $y > 0$ such thant $y^n = x$.

We denote this number as $y = x^{\frac{1}{n}}$.

For $m, n \in \mathbb{N}$ if $m, n > 0$ then we define $x^{\frac{m}{n}} = (x^m)^{\frac{1}{n}}$

Claim If $x > 0$ and $\frac{m}{n} \in \mathbb{Q}$ then $x^{\frac{m}{n}} = (x^m)^{\frac{1}{n}}$

Proof Since the number $y > 0$ such that $y^n = x^m$ is unique $\left(\text{and equal to } (x^m)^{\frac{1}{n}}\right)$ then we just need to show that $\left(x^{\frac{m}{n}}\right)^n = x^m$.

$$\left(x^{\frac{m}{n}}\right)^n = \left(\left(x^{\frac{1}{n}}\right)^m\right)^n = \left(x^{\frac{1}{n}}\right)^{mn} = \left(\left(x^{\frac{1}{n}}\right)^n\right)^m = \left(x^{\frac{n}{n}}\right)^m = x^m \quad ■$$

Claim If $x > 0$ and $m, n \in \mathbb{N}$ then $\left(x^{\frac{1}{n}}\right)^{\frac{1}{m}} = x^{\frac{1}{mn}}$

Proof Since the $(mn)^{\text{th}}$ root is unique we just need to show that $\left[\left(x^{\frac{1}{n}}\right)^{\frac{1}{m}}\right]^{mn} = x$:

$$\left[\left(x^{\frac{1}{n}}\right)^{\frac{1}{m}}\right]^{mn} = \left[\left(\left(x^{\frac{1}{n}}\right)^{\frac{1}{m}}\right)^{m}\right]^{n} = \left[x^{\frac{1}{n}}\right]^{n} = x \quad \blacksquare$$

Claim $100^{10000} > 10000^{100}$

Proof

$$100 = 10^2, \quad 10000 = 10^4$$
$$100^{10000} = (10^2)^{10000} = 10^{20000}$$
$$10000^{100} = (10^4)^{100} = 10^{400}$$

Since both bases are the same and $20000 > 400$ then $100^{10000} > 10000^{100}$ $\blacksquare$

Claim $\sqrt[3]{3} > \sqrt{2}$

Proof

$$\sqrt[3]{3} > \sqrt{2}$$
$$\iff 3^{\frac{1}{3}} > 2^{\frac{1}{2}}$$

By Liebeck example 5.4:

$$\iff \left(3^{\frac{1}{3}}\right)^2 > \left(2^{\frac{1}{2}}\right)^2$$
$$\iff 3^{\frac{2}{3}} > 2$$

By homework 2 problem 4:

$$\iff \left(3^{\frac{2}{3}}\right)^3 > 2^3$$
$$\iff 3^2 > 2^3$$
$$\iff 9 > 8$$

Since this is true and the implication goes both ways then $\sqrt[3]{2} > \sqrt{2}$ $\blacksquare$

Claim For $x_1, x_2, y_1, y_2 \in \mathbb{R}$:

$$x_1y_1 + x_2y_2 \le \sqrt{x_1^2 + x_2^2} \cdot \sqrt{y_1^2 + y_2^2}$$

Proof Assume $x_1, x_2, y_1, y_2 > 0$ as if they are negative then the LHS would be smaller without the RHS decreasing.

Assume $\sqrt{x_1^2 + x_2^2} = \sqrt{y_1^2 + y_2^2} = 1$ Recall from previous proof that $ab \le \frac{a^2+b^2}{2}$. Hence:

$$x_1y_1 + x_2y_2 \le \frac{x_1^2 + y_1^2}{2} + \frac{x_2^2 + y_2^2}{2} = \frac{x_1^2 + x_2^2}{2} + \frac{y_1^2 + y_2^2}{2}$$

Since $\sqrt{x_1^2+x_2^2}=\sqrt{y_1^2+y_2^2}=1$ then $x_1^2+x_2^2=y_1^2+y_2^2=1$:

$$x_1y_1+x_2y_2\leq\frac{1}{2}+\frac{1}{2}=1=1\cdot 1=\sqrt{x_1^2+x_2^2}\cdot\sqrt{y_1^2+y_2^2}$$

Now for the general case assume that $x_1,x_2,y_1,y_2\in\mathbb{R}$ and let:

$$a_1=\frac{x_1}{\sqrt{x_1^2+x_2^2}}\quad\&\quad a_2=\frac{x_2}{\sqrt{x_1^2+x_2^2}}$$

$$b_1=\frac{y_1}{\sqrt{y_1^2+y_2^2}}\quad\&\quad b_2=\frac{y_2}{\sqrt{y_1^2+y_2^2}}$$

$$\sqrt{a_1^2+a_2^2}=\left[\frac{x_1^2}{x_1^2+x_2^2}+\frac{x_2^2}{x_1^2+x_2^2}\right]^{\frac{1}{2}}=\left[\frac{x_1^2+x_2^2}{x_1^2+x_2^2}\right]^{\frac{1}{2}}=[1]^{\frac{1}{2}}=1$$

By the same logic $\sqrt{b_1^2+b_2^2}=1$ So by applying the previous case we get:

$$1\geq a_1b_1+a_2b_2=\frac{x_1}{\sqrt{x_1^2+x_2^2}}\cdot\frac{y_1}{\sqrt{y_1^2+y_2^2}}+\frac{x_2}{\sqrt{x_1^2+x_2^2}}\cdot\frac{y_2}{\sqrt{y_1^2+y_2^2}}$$

$$1\geq\frac{x_1y_1}{\sqrt{x_1^2+x_2^2}\cdot\sqrt{y_1^2+y_2^2}}+\frac{x_2y_2}{\sqrt{x_1^2+x_2^2}\cdot\sqrt{y_1^2+y_2^2}}$$

$$1\geq\frac{x_1y_1+x_2y_2}{\sqrt{x_1^2+x_2^2}\cdot\sqrt{y_1^2+y_2^2}}$$

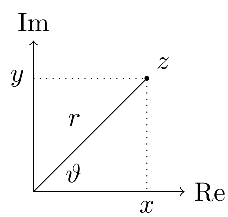
$$\frac{1}{x_1y_1+x_2y_2}\geq\frac{1}{\sqrt{x_1^2+x_2^2}\cdot\sqrt{y_1^2+y_2^2}}$$

By a previous proof if $x\geq y>0$ then $\frac{1}{y}\geq\frac{1}{x}$:

$$x_1y_1+x_2y_2\leq\sqrt{x_1^2+x_2^2}\cdot\sqrt{y_1^2+y_2^2}\quad\blacksquare$$

7 Complex Numbers

A number z is in the set of complex numbers $\mathbb{C}$ if $z=x+iy$, $x,y\in\mathbb{R}$, $i^2=-1$.



$$x=\mathrm{Re}(z),\quad y=\mathrm{Im}(z),\quad \vartheta=\arg(z)\quad\&\quad r=|z|=\sqrt{x^2+y^2}$$
$$z=x+iy=r(\cos\vartheta+i\sin\vartheta)=re^{i\vartheta}$$

The complex conjugate of z is denoted $\bar{z}$ It is a reflection in the real axis. $\bar{z}=x-iy$

$$\bar{z}=x-iy=r(\cos\vartheta-i\sin\vartheta)=r(\cos(-\vartheta)+i\sin(-\vartheta))=re^{-i\vartheta}$$

If $z=re^{i\vartheta}$ and $w=se^{i\varphi}$ then zw is given by:

$$zw=rse^{i(\vartheta+\varphi)}$$

From this we can see:

$$|zw|=|z||w|\quad\&\quad\arg(zw)=\arg(z)+\arg(w)$$

8 More Complex Numbers

Example 8.1

What is the locus of points such that $\frac{1}{2} = |z-1-i|$?

$|z|$ is the distance from the origin to z.

$|z-w|$ is the distance from z to w.

Let $z = x + yi$ for some $x, y \in \mathbb{R}$

$$|z-1-i| = |x+yi-1-i| = |(x-1)+(y-1)i| = \sqrt{(x-1)^2+(y-1)^2} = \frac{1}{2}$$

This is a circle center $1+i$ radius $\frac{1}{2}$

Example 8.2

What is the locus of points such that $|z-1| = |z+1|$?

Using the fact that $|w|^2 = w\bar{w}$:

$$\begin{aligned} |z-1|^2 &= |z+1|^2 \\ \implies (z-1)\overline{(z+1)} &= (z+1)\overline{(z+1)} \end{aligned}$$

Using the fact that $\overline{(a+b)} = \bar{a} + \bar{b}$:

$$\begin{aligned} \implies (z-1)(\bar{z}-1) &= (z+1)(\bar{z}+1) \\ \implies z\bar{z} - z - \bar{z} + 1 &= z\bar{z} + z + \bar{z} + 1 \\ \implies -z - \bar{z} &= z\bar{z} \\ \implies 0 &= 2(z+\bar{z}) \end{aligned}$$

Using the fact that $z + \bar{z} = 2\operatorname{Re}(z)$:

$$\begin{aligned} \implies 0 &= 2(2\operatorname{Re}(z)) \\ \implies \operatorname{Re}(z) &= 0 \end{aligned}$$

So the set of points is given by:

$$\{z : |z-1| = |z+1|\} = \text{Imaginary axis} = \{yi : y \in \mathbb{R}\}$$

Roots of unity

We want to find solutions to $z^n = 1$. One obvious solution is $z = 1$. The funamental theorem of algebra tells us that a degree n polynomial has n roots (including multiplicity). If we let w be another root to the equation then:

$$w^n = 1 \implies w^n = \left(e^{\frac{2\pi i}{n}}\right)^n = e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1 + 0i = 1$$

Claim $1, w, w^2, \ldots, w^{n-1}$ are n distinct solutions to the equation $z^n = 1$

Proof First we will prove that they are solutions:

Let $j \in \mathbb{R}$ such that $0 \le j \le n-1$ and w be a root such that $w^n = 1$:

$$\left(w^j\right)^n = w^{jn} = (w^n)^j = 1^j = 1$$

So w^j is a root.

Now we must show that they are distinct solutions, we will do this by contradiction:

Let $j, k \in \mathbb{R}$ such that $0 \le j < k \le n-1$

Assume that $w^j = w^k$

$$\implies 1 = w^{j-k} = \left(e^{\frac{2\pi i}{n}}\right)^{j-k} = e^{\frac{2\pi(j-k)i}{n}} = \cos\frac{\frac{2\pi(j-k)i}{n}}{+} i \sin\frac{2\pi(j-k)i}{n} = 1$$

This only happens when $\frac{2\pi(j-k)i}{n}$ is an integer multiple of 2π

$$\implies \frac{j-k}{n} \in \mathbb{Z}$$

This is not posible as $0 \le j < k \le n-1 \implies 0 \le k-j < n$ thus $w^j \ne w^k$ so the roots are distinct ■

Example 8.3

Find the solutions to $z^2 = 1$

$$e^{\frac{2\pi i}{2}} = e^{\pi i} = -1$$

Also 1 is an obvious root as $1^2 = 1$

Claim If one solution to $z^n = p$ for some $p \in \mathbb{C}$ is $z = q$ then the roots are given by $q, wq, w^2q, \dots, w^{n-1}q$ where $w = e^{\frac{2\pi i}{n}}$

Proof $\left(w^j q\right)^n = w^{jn}q^n = 1p = p$ Since $w, w^2, \dots, w^{n-1}$ are distinct then $q, wq, w^2q, \dots, w^{n-1}q$ are distinct. ■

Example 8.4

Find the solutions to $z^2 = -1$. $z = i$ is an obvious solution. The second roots of unity are 1 and -1 so the solutions are i and $-i$

9 Polynomials

An n-degree polynomial has the form:

$$p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

Where $a_n \ne 0$ and $a_i \in \mathbb{C}\ \forall i$. We say that α is a root of $p(z)$ if $p(\alpha) = 0$. A complex polynomial has coefficients in $\mathbb{C}$, a real polynomial only has coefficients (but not necessarily roots) in $\mathbb{R}$. Any complex polynomial has at least one root in $\mathbb{C}$. If α is a root of $p(z)$ then we can write $p(z) = (z-\alpha)q(z)$ where $q(z)$ is another polynomial. Repeated application of this shows that all polynomials can be factored into the product of linear terms:

$$p(z) = a(z-r)(z-r_2)\cdots(z-r_n)$$

Where r_i is a root such that $p(r_i) = 0\ \forall i$. Some roots may repeat. If a root occurs m times in $r_1, r_2, \dots, r_n$ then we say it has multiplicity m.

Claim If p is a real polynomial and r is a root then $\bar{r}$ is a root.

Proof If $p(x) = a_0 + a_1x + \cdots + a_nx^n$ with $a_i \in \mathbb{R}$ and $p(r) = 0$ then:

$$\begin{aligned}
0 &= p(r) \\
&= \overline{p(r)} \qquad \text{(Using the fact that if } c \in \mathbb{R} \text{ then } \bar{c} = c) \\
&= \overline{a_0 + a_1r + \cdots + a_nr^n} \\
&= \overline{a_0} + \overline{a_1r} + \cdots + \overline{a_nr^n} \qquad \text{(Using the fact that } \overline{z+w} = \bar{z} + \overline{w}) \\
&= \overline{a_0} + \overline{a_1}\overline{r} + \cdots + \overline{a_n}\overline{r^n} \qquad \text{(Using the fact that } \overline{zw} = \bar{z}\overline{w}) \\
&= a_0 + a_1\overline{r} + \cdots + a_n\overline{r^n} \\
&= a_0 + a_1\overline{r} + \cdots + a_n(\overline{r})^n \qquad \text{(Repeated application of } \overline{zw} = \bar{z}\overline{w}) \\
&= p(\overline{r})
\end{aligned}$$

So $p(\overline{r}) = 0$ so $\overline{r}$ is a root ■

If $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ has roots $r_1, r_2, \ldots, r_n$ then:

$$r_1 + r_2 + \cdots + r_n = \sum_i r_i = -a_{n-1}$$

$$r_1r_2\cdots r_n = \prod_i r_i = (-1)^n a_0$$

Claim If p and q are n degree polynomials and they share $n+1$ distinct roots then $p = q$.

Proof Let $h = p - q$ then h is a degree n polynomial with $n+1$ distinct roots $x_1, x_2, \ldots, x_{n+1}$ by Liebeck theorem 7.2 we know:

$$h(x) = a(x - x_1)(x - x_2)\cdots(x - x_n)$$

Since they are distinct we know that $x_{n+1} \neq x_i$ for $i = 1, 2, \ldots, n$. We also know that by Liebeck theorem 7.2:

$$0 = h(x_{n+1}) = a(x_{n+1} - x_1)(x_{n+1} - x_2)\cdots(x_{n+1} - x_n)$$

None of the linear terms can be zero so $a = 0$ which means that $h = 0$ that is $p - q = 0$ and this can only happen if $p = q$ ■

10 Induction

Summation notation

$$\sum_{k=m}^{n} = a_m + a_{m+1} + \cdots + a_n$$

It is possible to do a change of variables:

$$\sum_{k=m}^{n} a_{k+p} = a_{m+p} + a_{m+p+1} + \cdots + a_{n+p} = \sum_{k=m+p}^{n+p} a_k$$

Product notation

$$\prod_{k=m}^{n} a_k = a_k a_{k+1} \cdots a_n$$

Induction

Let k be a known integer

Let $P(n)$ be a statement that makes sense for any integer $n \geq k$.

The principle of induction is then that if:

- $P(k)$ is true
- $P(n) \implies P(n+1)$ for any integer $n \geq k$

then by mathematical induction $P(n)$ is true for all integers $n \geq k$.

Claim $n^3 + 2n$ is divisible by 3 for all integers $n \geq 1$

Proof We will use proof by induction

Let $P(n)$ be the statement "$n^3 + 2n$ is divisible by 3".

Base case:

If $n = 1$ then $n^3 + 2n = 1^3 + 2(1) = 3$ which is divisible by 3

Induction step:

Assume $P(n)$ is true for some integer $n \geq 1$

We will show $P(n+1)$ is true:

$$
\begin{aligned}
&(n+1)^3 + 2(n+1) \\
=&n^3 + 3n^2 + 3n + 1 + 2n + 2 \\
=&n^3 + 2n + 3n^2 + 3n + 3 \\
=&\underbrace{n^3 + 2n}_{\text{divisible by 3 by assumption}} + \underbrace{3(n^2 + n + 1)}_{\text{divisible by 3 as has factor of 3}} \\
\implies &(n+1)^3 + 2(n+1) \text{ is divisible by } 3 = P(n+1)
\end{aligned}
$$

So $P(n) \implies P(n+1)$ and the base case is true so the claim is true ∎

Not all statements naturally start at $n = 1$

For which n do we have $2^n > n + 3$?

Let $P(n)$ be the statement "$2^n > n + 3$"

$$
\begin{aligned}
n = 1: &\quad 2^1 < 1 + 3 \implies P(1) \text{ is false} \\
n = 2: &\quad 2^2 < 2 + 3 \implies P(2) \text{ is false} \\
n = 3: &\quad 2^3 > 3 + 3 \implies P(3) \text{ is true} \\
n = 4: &\quad 2^4 > 4 + 3 \implies P(4) \text{ is true}
\end{aligned}
$$

It looks like $P(n)$ is true for integers $n \geq 3$

Claim $2^n > n + 3 \; \forall n \geq 3$

Proof We will use proof by induction

Let $P(n)$ be the claim "$2^n > n + 3$"

Base case:

$P(3) \iff 2^3 > 3 + 3 \iff 8 > 6$ this is true so $P(3)$ is true

Induction step:

Assume $P(n)$ is true

We want to show $2^{n+1} > n + 1 + 3 = n + 4$

$$2^{n+1} = 2 \cdot 2^n > 2(n + 3) = 2n + 6 > n + 4$$

So $P(n) \implies P(n + 1)$ and thebase case is true so the by induction the claim is true ■

The union of two sets is a new set that contains all of the elements that are in at least one of the two sets. The union of sets A and B is written $A \cup B$. The intersection of two sets is a new set that contains all of the elements that are found in both of the sets. The intersection of sets A and B is written $A \cap B$

Claim A set of size n has 2^n subsets $\forall\, n \geq 0$

Proof We will use proof by induction

Let $P(n)$ be the statement "A set of size n has 2^n subsets"

Base case:

When $n = 0$ the set is $\emptyset$ which has one subset ($\emptyset \subseteq T\ \forall T$ by definition). $2^n = 2^0 = 1$ so $P(0)$ is true.

Induction step:

Assume that $P(n)$ is true for some $n \geq 0$

We will show $P(n + 1)$

Let S be a set with $n + 1$ elements.

Let $a \in S$ and $A = \{x \in S | x \neq a\}$ (That is A is the set S with element a removed). From this it can be seen that $A \subseteq S$ and $|A| = |S| - 1 = n + 1 - 1 = n$.

There are two cases for subsets of S:

1. Case 1 Let E be a set such that $E \subseteq A$ this implies that $E \subseteq S$. There are by assumption 2^n subsets of A.
2. Case 2 Every subset of S that is not a subset of A must contain a thus everyone of these subsets is of the form $E \cup \{a\} = \{x \in E \text{ or } x = a\}$ so by the assumption there are 2^n such sets.

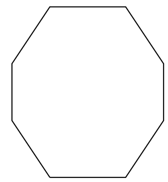
These two cases encompass all subsets of S so S has $2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$ subsets so $P(n + 1)$ is true if $P(n$ is true so by induction $P(n)$ is true $\forall\, n \geq 0$

11 Strong induction

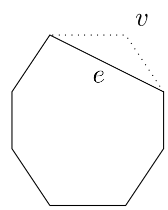
Given a convex polyhedron with n vertices how many diagonals does it have?

n	Shape	Number of diagonals
3	Triangle	0
4	Quadrilateral	2
5	Pentagon	5
6	Hexagon	9

Let $f(n)$ be the number of diagonals that a convex n-gon has.



This is shape P it is a convex n-gon. From P we construct a new shape by adding an edge between two vertices:



This is shape P' it is a convex $(n-1)$-gon. P' has $f(n-1$ diagonals. P has the following diagonals that P' doesn't:

1. $n-3$ diagonals from v (v can't make diagonals with its self or with its direct neighbours)
2. The diagonal where e is

Thus P has

$$f(n) = \underbrace{f(n-1)}_{\text{From } P'} + \underbrace{n-3}_{\text{From (1)}} + \underbrace{1}_{\text{from (2)}} = f(n-1) + n - 2$$

From this it is possible to work out $f(n)$ as a sum:

$$\begin{aligned} f(n) &= f(n-1) + n - 2 \\ &= f(n-2) + n - 3 + n - 2 \\ &= f(n-3) + n - 4 + n - 3 + n - 2 \\ &\vdots \\ &= \underbrace{f(3)}_{=0} + 2 + 3 + \cdots + n - 2 \\ &= \sum_{k=2}^{n-2} k \end{aligned}$$

Claim If $f(3) = 0$ and $f(n) = f(n-1) + n - 2$ then, $\forall\, n \geq 4$:

$$f(n) = \sum_{k=2}^{n-2} k$$

Proof

$$\sum_{k=2}^{n-2} k = \sum_{k=1}^{n-2} k - 1 = \frac{1}{2}(n-2)(n-1) - 1$$

The last step is using Liebeck example 8.6

We wil use proof by induction

Base case:

$$f(4) = \sum_{k=2}^{4-2} k = 2$$

This shows that $P(4)$ is true.

Induction step:

Assume $f(n) = \sum_{k=2}^{n-2} k$

We want to show $f(n+1) = \sum_{k=2}^{n-1} k$

$$f(n+1) = f(n) + (n+1) - 2 = f(n) + n - 1 = \sum_{k=2}^{n-2} k + n - 1 = \sum_{k=2}^{n-1} \quad \blacksquare$$

Strong induction

Let $k \in \mathbb{Z}$ and $P(k), P(k+1), \ldots$ be statements. Suppose:

- $P(k)$ is true
- For any integer $n \geq k$:
$$P(k), P(k+1), \ldots, P(n) \implies P(n+1)$$

Then $P(n)$ is true $\forall\, n \geq k$

If $a_1 = 1$, $a_2 = 3$ and $a_n = 2a_{n-1} - a_{n-2}$ find an expression for a_n in terms of n

$$a_1 = 1,\ a_2 = 3,\ a_3 = 5,\ a_4 = 7,\ a_5 = 9$$

Claim $a_n = 2n - 1\ \forall\, n \geq 1$

Scratch work:

What does the induction step look like?

$$\begin{aligned} a_{n+1} &= 2a_n - a_{n-1} \\ &= 2(2n-1) - (2(n-1) - 1) \\ &= 4n - 2 - 2n + 2 + 1 \\ &= 2n + 1 \\ &= 2(n+1) - 1 \end{aligned}$$

To make this work we needed the formula to hold for a_n and a_{n-1} so we need to use strong induction. We need two previous terms so we need two base cases.

Proof We will use proof by strong induction

Base cases:

$$a_1 = 1 = 2(1) - 1 \quad \& \quad a_2 = 3 = 2(2) - 1$$

Induction step:

Assume $a_k = 2k - 1\ \forall\, k \leq n$

We will show $a_{n+1} = 2k + 1$:

$$\begin{aligned} a_{n+1} &= 2a_n - a_{n-1} \\ &= 2(2n-1) - (2(n-1) - 1) \\ &= 4n - 2 - 2n + 2 + 1 \\ &= 2n + 1 \\ &= 2(n+1) - 1 \\ &= 2n + 1 \quad \blacksquare \end{aligned}$$

Claim $\forall\, n \geq 12$ n can be written as a sum of 4s and 5s.

$P(n)$ is the statement "n can be written as the sum of 4s and 5s"

Scratch work:

Suppose I want to show $P(n+1)$ is true, that is $n+1$ can be written as a sum of 4s and 5s.

$n + 1 - 4 = n - 3$ so if $n - 3$ is a sum of 4s and 5s so is $n + 1$. So since the smallest $n - 3$ can be is 12 we know that $n \geq 12 + 3 = 15$ so we need base cases 12, 13, 14 and 15

Proof We will use strong induction

Base cases:

$$\begin{aligned} 12 &= 4 + 4 + 4 \\ 13 &= 4 + 4 + 5 \\ 14 &= 4 + 5 + 5 \\ 15 &= 5 + 5 + 5 \end{aligned}$$

Induction step:

Assume $n \geq 15$ and $P(12), P(13), \dots, P(n)$ are true.

We will prove $P(n+1)$

$n + 1 - 4 = n - 3$ and since $n - 3$ is a sum of 4s and 5s so is $n + 1$ as $n + 1 = n - 3 + 4$ $\blacksquare$

Claim If $f_1 = 1$, $f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ then:

$$f_n \geq \left(\frac{3}{2}\right)^{n-2} \tag{10.1}$$

$\forall\, n \geq 1$

Proof We will use strong induction

Base cases:

$$\begin{aligned} f_1 = 1 &> \left(\frac{3}{2}\right)^{1-2} \\ f_2 = 1 &= \left(\frac{3}{2}\right)^{2-2} \end{aligned}$$

Induction step:

Assume that (10.1) holds for $n = 1, 2, 3, \dots, k$

We will show it holds for $n = k + 1$

$$\begin{aligned}
f_{k+1} &= f_k + f_{k-1} \\
&\geq \left(\frac{3}{2}\right)^{k-2} + \left(\frac{3}{2}\right)^{k-3} \\
&= \left(\frac{3}{2}\right)^{k-2} \left(1 + \left(\frac{3}{2}\right)^{-1}\right) \\
&= \left(\frac{3}{2}\right)^{k-2} \left(1 + \frac{2}{3}\right) \\
&= \left(\frac{3}{2}\right)^{k-2} \left(\frac{5}{3}\right) \\
&> \left(\frac{3}{2}\right)^{k-1} \qquad \blacksquare
\end{aligned}$$

12 Graphs

A graph is a set of edges and verices so that each edge begins and ends on a vertex.

A graph is simple if there are no loops or double edges

A graph is connected if you can move from any vertex to another by following a sequence of connected vertices.

A plane graph can be drawn on a plane without edges crossing.

A polyhedron is a union of polygon faces so that any side of each face is the side of another face. The vertices are the corners of each face and edges are the sides of each face. A polyhedron P is conves if for any two points in P the line between the points lies entirely in P.

Euler's identity

If a convex polyhedron has E edges, V vertices and F faces then Euler's identity states:

$$V - E + F = 2$$

An icosidodecahedron is a convex polyhedron built from P regular pentagons and T equilateral triangles such that every pentagon can only share an edge with a triangle and vice versa. Around every vertex is a triangle-pentagon-triangle-pentagon in that order. Find P and T.

We do this by finding formulas relating edges vertices and faces. To do this we count edge-vertex, edge-face, and vertex-face pairs.

If we consider the set Ω of pentagon-edge pairs:

$$\Omega = \{(p, e) | e \text{ is an edge on pentagonal face } p\}$$

How many pairs are there in terms of P? Since every pentagon has 5 edges $5P = |\Omega|$. How many pairs are there in terms of the number of edges E? Since every edge belongs to one exactly pentagon $E = |\Omega|$. This gives us our first equation:

$$5P = E = |\Omega| \tag{12.1}$$

Consider the set Θ of triangle-edge pairs:

$$\Theta = \{(t, e) | e \text{ is an edge on triangular face } t\}$$

How many pairs are there in terms of T? Since every triangle has 3 edges $3T = |\Theta|$. How many pairs are there in terms of E? Since every edge belongs to exactly one triangle $E = |\Theta|$. This gives us our second equation:

$$3T = E = |\Theta| \tag{12.2}$$

Consider the set Φ of triangle-vertex pairs:

$$\Phi = \{(v,t)|v \text{ is a vertex of triangle }\}$$

How many pairs are there in terms of T? Since every triangle has 3 vertices $3T = |\Phi|$. How many pairs are there in terms of the number of vertices V? Since every vertex has 2 triangles $2V = |\Phi|$. This gives us our third equation:

$$2V = 3T = |\Phi| \tag{12.3}$$

Euler's formula gives us:

$$2 = V - E + F$$

The number of faces is equal to the number of triangles and pentagons

$$= V - E + P + T$$

Using (12.2) E=3T

$$\begin{aligned} &= V - 3T + P + T \\ &= V + P - 2T \end{aligned}$$

Using (12.1) and (12.2) $P = \frac{3}{5}T$

$$\begin{aligned} &= V + \frac{3}{5}T - 2t \\ &= V - \frac{7}{5}T \end{aligned}$$

Using (12.3) $V = \frac{3}{2}T$

$$\begin{aligned} &= \frac{3}{2}T - \frac{7}{5}T \\ &= \frac{1}{10}T \\ \implies &T = 20 \\ \implies &P = 12 \end{aligned}$$

13 Divisibility

We say d divides n and write $d|n$ if there is an integer k such that $n = kd$

The Easy Lemma

Claim If $d|a$ and $d|b$ then $d|(ma + nb)$ for some integers m and n.

Proof Let a, b and d be integers such that $d|a$ and $d|b$. That is $a = dk$ and $b = dr$ for some integers k and r. From this it is clear that $ma + nb = mdk + ndr = d(mk + nr)$ so it is clear that d divides $ma + nb$ ∎

Claim $\forall\, n \geq 1$, $2n+3$ and $n+1$ have no common factor bigger than 1

Proof

$$2n+3-2(n+1)=1$$

By the easy lemma if $d>0$ and divides both $2n+3$ and $n+1$ then $d|1$. This is only true if $d=1$ therefore there are no common factors bigger than 1 ■

The highest common factor of a and b, $\text{hcf}(a,b)$ is the largest positive integer which divides a and b.

If $\text{hcf}(a,b)=1$ then we say a and b are coprime.

If p is prime and n isn't then:

$$\text{hcf}(p,n)=\begin{cases} 1 & \text{if } p \text{ doesn't divide } n \\ p & \text{if } p \text{ divides } n \end{cases}$$

Given $n \in \mathbb{N}$ what is $\text{hcf}(4n^2+9n+4, n+2)$?

$$4n^2+9n+4=(n+2)(4n+1)+2$$
$$2=4n^2+9n+4=(n+2)(4n+1)$$

By the easy lemma since $\text{hcf}\,|(4n^2+9n+4$ and $\text{hcf}\,|(n+2)$ then $\text{hcf}\,|2$.

If n is even then $4n^2+9n+4$ and $n+2$ are even so $2|\,\text{hcf}$, since$\text{hcf}\,|2$ then $\text{hcf}=1$ or $\text{hcf}=2$ but 2 also divides hcf so $\text{hcf}=2$ ■

If n is odd then $4n^2+9n+4$ and $n+2$ are odd so 2 doesn't divide hcf so hcf$=1$

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$\forall\, n \in \mathbb{N}$ what is $\text{hcf}(4n+1, 3n+1)$?

Method 1:

By the easy lemma $\text{hcf}\,|(4n+1-(3n+1)=n$ therefore $\text{hcf}<n$.

$\text{hcf}\,|(3n+1)$ and $\text{hcf}\,|n$ so by the easy lemma $\text{hcf}\,|(3n+1-3n)=1$ so $hcf(4n+1,3n+1)=1$

Method 2:

By the easy lemma $\text{hcf}\,|(4(3n+1)-3(4n+1))=1$ so $\text{hcf}(4n+1,3n+1)=1$

$\exists!$ means there exists a unique...

Claim If $a>0$ $\exists! q, r \in \mathbb{Z}$ such that $b=qa+r$ and $0 \leq r < a$

Proof Let q be the greatest integer such that $qa \leq b$

$$r=b-qa$$

$$b \geq qa \implies r \geq 0$$
$$r \geq a \implies b-qa \geq a \implies b \geq (q+1)a$$

However q is the largest integer such that $qa \geq b$ but here $q+1$ is the largest integer. This is a contradiction so $r<a$. Why are q and r unique?

Suppose that there exists $q_1, q_2, r_1 r_2$ such that $q_1 \neq q_2$, $r_1 \neq r_2$, $0 \leq r_1, r_2 < a$ and $q_1 a + r_1 = q_2 a + r_2 = b$

$$\implies (q_1-q_2)a = r_2 - r_1$$

$$\begin{aligned}
&\implies |(q_1 - q_2)a| = |r_2 - r_1 \\
&\implies |r_2 - r_1| < a - 1 \\
&\quad\;\; |q_1 - q_2| \leq 1 \\
&\quad\;\; |(q_1 - q_2)a| \leq a \\
&\quad\;\; a \leq |(q_1 - q_2)a| = |r_1 - r_2| < a - 1 \\
&\implies a < a - 1
\end{aligned}$$

This is a contrandiction so q and r must be unique ■

Claim $\text{hcf}(a, b) = \text{hcf}(a, r)$ if $b = qa + r$

Proof Let $c = \text{hcf}(a, b)$ and $d = \text{hcf}(a, r)$

$c|a$ and $c|b \implies c|(b - qa) = r \implies c|r \implies c|d$

$d|a$ and $d|r \implies d|(qa + r) = b \implies d|b \implies d|c$

If $d|c$ and $c|d$ then $c = d$ ■

Euclidean algorithm

The Euclidean algorith uses the easy lemma to propose smaller and smaller upper bounds on the hcf until the upper bound is the hcf.

Example 13.1

$$\begin{aligned}
91 &= 5 \cdot 17 + 6 \\
17 &= 2 \cdot 6 + 5 \\
6 &= 1 \cdot 5 + 1
\end{aligned}$$

Hence $\text{hcf}(17, 91) \leq 1$, this means $\text{hcf} = 1$ but if this gave us a bound other that 1 we prove this by reversing the steps:

$$\begin{aligned}
1 &= 6 - 1 \cdot 5 \\
&= 6 - (17 - 2 \cdot 6) \\
&= -17 + 3 \cdot 6 \\
&= -17 + 3(91 - 5 \cdot 17) \\
&= 3 \cdot 91 - 16 \cdot 17 \\
\implies \text{hcf}\, &|(3 \cdot 91 - 16 \cdot 17) \\
\implies \text{hcf}\, &|1 \\
\implies \text{hcf} &= 1
\end{aligned}$$

14 Fundamental Theorem of Arithmetic

What is the strongest statement we can make about $\text{hcf}(2n + 4, 4n + 6)$?

Easy lemma $\implies d|(2(2n + 4) - 4n + 6) = 2$ therefore $d|2$ so $d = 1$ or $d = 2$. $2n + 4$ and $4n + 6$ are even for all n so $2|(2n + 4)$ and $2|4n + 6$ so $2|d$ and $d|2$ so $d = 2$

Claim If $c|ab$ and $\text{hcf}(a, c) = 1$ then $c|b$

Proof If $\text{hcf}(a, c) = 1$ then $ma + nc = 1$ for some m, n. Thus $b = mab + mcb$ clearly $c|mcb$ and by assumption $c|mab$ so the easy lemma implies $c|mab + ncb = b \implies c|b$ ■

Claim The $\text{hcf}(a, b)$ is divisible by any common factor of a and b

Proof Suppose $d|a$ and $d|b$. Since $hcf(a, b) = ma + nb$ for some m, n the easy lemma implies that $d|ma + nb = hcf(a, b)$ so $d|hcf(a, b)$ ■

Claim If $a, b \in \mathbb{Z}$ are coprine, p is prime and $p|ab$ then either $p|a$ or $p|b$ or both.

Proof This is a special case of the earlier claim. If c is coprime to a but divides ab then $c|b$ ■

Claim If $n = p_1 \cdots p_k$ and p_i is prime $\forall\, i$ and $p|n$ then $p = p_i$ for some $i = 1, \ldots, k$

Proof Let $P(k)$ be the claim.

We will use proof by induction.

Basis case:

$P(1)$ has $k = 1$ hence $n = 2$ the only divisors of 2 are 1 and 2. 1 isn't prime so 2's only prime divisor is 2. Which is p_1. The basis case is true.

Induction step:

Let us assume $P(k)$

Suppose $p|p_1 \cdots p_{k+1}$. By the previous claim either $p = p_{k+1}$ or $p|p_1 \cdots p_k$. By induction $p = p_i$ for some $i = 1, \ldots, k + 1$

How many $x \in \mathbb{Q}$ solve $x^3 - 10x + 1 = 0$?

Let $x = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ are coprime.

If x solves the equation then :

$$\begin{aligned}
0 &= x^3 - 10x + 1 \\
&= \frac{p^3}{q^3} - 10\frac{p}{q} + 1 \\
\implies 0 &= p^3 - 10pq^2 + q^3 \\
\implies q^3 &= 10pq^2 - p^3 \\
&= p(10q^2 - p^2) \\
\implies p|q^3 &= qqq
\end{aligned}$$

This contradicts the claim: $c|ab$ and $\text{hcf}(a, c) = 1$ then $c|b$. This would imply $p|q$. This could only happen if $p = 1$ so $x = \frac{1}{q}$:

$$\begin{aligned}
0 &= \frac{1}{q^3} - \frac{10}{q} + 1 \\
\implies 0 &= 1 - 10q^2 + q^3 \\
\implies 1 &= 10q^2 - q^3 \\
\implies 1 &= q^2(10 - q)
\end{aligned}$$

This impossible for an integer q so $\frac{p}{q}$ is not a solution so there are no rational solutions.

What are all the integer solutions to $2x = 5y$?

2 and 5 are coprime and $2|5y \implies 2|y \implies y = 2z$ for some integer z

$2x = 5(2z) = 10z \implies x = 5z \implies 5|x \implies x = 5w$ for some integer w

$5w = x = 5z \implies w = z \implies x = 5z$ and $y = 2z$ are solutions for $z \in \mathbb{Z}$.

All solutions are of the form $(x, y) = (5z, 2z)$

Fundamental theorem of arithmetic

All integers have a unique prime factorization as a product of prime numbers, formally:

Let n be an integer larger than 2

1. Existence:

   n can be written as $n = p_1p_2 \cdots p_k$ for prime p_i where $p_1 \leq p_2 \leq \cdots \leq p_k$

2. Uniqueness:

   The factorisation is unique. If we have:

   $$p_1p_2 \cdots p_k = n = q_1q_2 \cdots q_l$$

   Where p_i, q_i are prime then $l = k$ and $p_i = q_i$ for all i

Proof

1. Existence:

   Let $P(n)$ be the claim that n has a prime factorisation.

   Basis case:

   $P(2)$ is that 2 has a prime factorisation which it does since 2 is prime.

   $P(3)$ is that 3 has a prime factorisation which it does since 3 is prime.

   Induction step:

   Suppose that $P(n)$ holds for $n \leq k$. If $k+1$ is prime then it has prime factorisation k. If k isn't prime then it can be wirtten as $k = a \cdot b$ where $a, b < k$ hence a and b have prime factorisations by assumption. Therefore the prime factorisation of k is the product of the prime factorisations of a and b.

2. Uniqueness:

   Suppose that n has two prime factorisations. That is $p_1 \cdots p_k = n = q_1 \cdots q_l$. By cancelling common factors we can assume that there are no p_1 equal to q_j. If there are p_i and q_j remaining after canelation the earlier claim implies that each p_i divides some q_j. Since p_i and q_j are prime this is only possible if $p_i = q_j$ which is a contradiction $\blacksquare$

Consequences

If we know the prime decomposition of m and n then it is easy to tell whether $m|n$:

$$m|n \iff \text{all primes in } m \text{ occur to less or equal powers in } n$$

Claim Let $n = p_1^{a1} \cdots p_k^{a_k}$ be a prime decomposition (That is $p_1 \leq \cdots \leq p_k$ and $a_i > 0$) Then $m|n$ if and only if:

$$m = p_1^{b_1} \cdots p_k^{b_k} \text{ and } 0 \leq b_i \leq a_i$$

Proof If the second part of the claim is true then by taking out a common factor of the prime factorisation of m from n then it can be seen that $m|n$.

Let $n = mc$ then:

$$\underbrace{p_1^{a_1}\cdots p_k^{a_k}}_{n} = \underbrace{q_1^{c_1}\cdots q_l^{c_l}}_{m}\underbrace{r_1^{d_1}\cdots r_s^{d_s}}_{c}$$

Then FTA $\implies$ each q_i and r_i equals to some p_j and each power $c_i + d_i$ equals to a_j ■

Find all integer solutions to $x^2 = y^5$

Note: If (x, y) is a solution so is $(-x, y)$ so we can assume $x \geq 0$ without losing solutions by applying this after our solutions are found.

If $x^2 = y^5$ and $x \geq 0$ then $y \geq 0$.

$x = p_1^{m_1}\cdots p_k^{m_k}$ and $y = q_1^{n_1}\cdots q_l^{n_l}$

$\implies x^2 = p_1^{2m_1}\cdots p_k^{2m_k} = q_1^{5n_1}\cdots q_l^{5n_l} = y^5$.

By the fundamental theorem of arithmetic these two prime factorisations are identical, that is $k = l$, $p_i = q_i$ and $2m_i = 5n_i$ for all i. By earlier work we know that this means $(m_i, n_i) = (5z_i, 2z_i)$ for some integer z_i:

$$\implies \begin{cases} x = p_1^{m_1}\cdots p_k^{m_k} = p_1^{5z_1}\cdots p_k^{5z_k} = (p_1^{z_1}\cdots p_k^{z_k})^5 \\ y = q_1^{n_1}\cdots q_l^{n_l} = p_1^{2z_1}\cdots p_k^{2z_l} = (p_1^{z_1}\cdots p_l^{z_l})^2 \end{cases}$$

$\implies$ solutions $(x, y) = (\pm z^5, z^2)$ for $z \in \mathbb{Z}$

15 Diophantine Equations

The hightest factor of the numbers is the product of all common prime factos

The least common multiple $\text{lcm}(a, b)$ of positive integers a and b is the smallest positive integer divisible by a and b.

Let a and b be numbers with prime factorisations:

$$a = p_1^{r_1}\cdots p_n^{r_n}, \qquad b = q_1^{s_1}\cdots q_n^{s_n}$$

Where p_is are distinct and $r_i, s_i \in \{\mathbb{N}, 0\}$

- $\text{hcf}(a, b) = p_1^{\min(r_1,s_1)}\cdots p_m^{\min(r_m,s_m)}$
- $\text{lcm}(a, b) = p_1^{\max(r_1,s_1)}\cdots p_m^{\max(r_m,s_m)}$
- $\text{lcm}(a, b) = \dfrac{ab}{\text{hcf}(a, b)}$

What is the smallest possible integer I can get from writing $375a + 147b$ for $a, b \in \mathbb{Z}$?

$$147 = 3 \cdot 7^2$$
$$375 = 3 \cdot 5^3$$
$$\implies \text{hcf}(147, 375) = 3$$
$$\implies \exists\, a, b \text{ such that } 375a + 147b = 3 \text{ by the easy lemma}$$

Claim If $n \in \mathbb{Z}_+$ then $\sqrt{n} \in \mathbb{Q} \iff n$ is a perfect square

Proof

$$\sqrt{n} = \frac{r}{s} \implies s^2 n = r^2$$

Both s^2 and r^2 have even powers for each prime in their prime factorisation so n must also have even powers so that when the power of p_i in n is added to the power of p_i in s^2 it will give an even power of p_i in r^2. This means that it is possible to factor a power of 2 out of n so n is a perfect square.

If instead n is a perfect square then $n = m^2$ so $\sqrt{n} = m \in \mathbb{Z}_+$ ■

Claim If a and b are coprime and ab is an n^{th} power then so are a and b

Proof If $a = p_1^{a_1} \cdots p_k^{a_k}$ and $b = q_1^{b_1} \cdots q_l^{b_l}$ and $c^n = ab$ where $c = r_1^{c_1} \cdots r_m^{c_m}$ then $r_1^{nc_1} \cdots r_m^{nc_m} = p_1^{a_1} \cdots p_k^{a_k} q_1^{b_1} \cdots q_l^{b_l}$

$\implies n|a_i$ and $n|b_i$

Hence $a_i = nA_i$ and $b_i = nB_i$

$$\implies \begin{cases} a = p_1^{nA_1} \cdots p_k^{nA_k} = (p_1^{A_1} \cdots p_k^{A_k})^n \\ b = q_1^{nB_1} \cdots q_l^{nB_l} = (q_1^{B_1} \cdots q_l^{B_l})^n \end{cases} \quad ■$$

What are the integer coprime solutions to $x^2 + y^2 = z^2$?

1. If (x, y, z) is a solution so is $(\pm x, \pm y, \pm x)$
    - This means we can assume $x, y, z \geq 0$ without loss of generality
2. If (x, y, x) is a solution so is (y, x, z)
3. If (x, y, z) is a solution so is $(2x, 2y, 2z)$
    - This means we can assume that at least one of x, y and z is odd
    - This means that we can assume that x or y is odd since if both are even so is z
    - We will assume x is odd without loss of generality
4. $x^2 = z^2 - y^2 = (z + y)(z - y)$

    Claim $z \pm y$ are coprime

    Proof Let q be prime.

    Suppose $q|z$ and $q|y$

    Then $2z = z + y + z - y \implies q|2z$

    Similarly $2y = z + y - (x - y) \implies q|2y$

    So ($q|2$ or $q|y$) and ($q|2$ or $q|z$

    q can't divide z and y since the are coprime

    So $q|z$ and $q|2 \implies q = 2 \implies 2|x^2$ so x is even but this is a contradiction ■
5. $z \pm y$ are perfect squares:
    $$z + y = s^2, \qquad z - y = t^2$$
    for some $s \geq t \geq 0$
    $$\implies x^2 = (z - y)(z + y) = s^2 t^2$$
    $$\implies x = st$$

6. By solving the simultaneous equations for y and z in terms of s and t we get that valid solutions have the form:
$$\left(st, \frac{s^2-t^2}{2}, \frac{s^2+t^2}{2}\right)$$

Aug 2012

Solve $x^2 = y^4 - 15$

If (x, y) is a solution so is $(\pm x, \pm y)$

So we can assume $x, y \geq 0$

$15 = y^4 - x^2 = (y^2 - x)(y^2 + x)$

$3 \cdot 5 = (y^2 - x)(y^2 + x)$

Therefore the prime factorisation of the RHS must be $3 \cdot 5$

Either $y^2 + x = 15$ and $y^2 - x = 1$ (1)

or $y^2 + x = 5$ and $y^2 - x = 3$ (2)

1.
$$y^2 + x + y^2 - x = 15 + 1$$
$$2y^2 = 16$$
$$y^2 = 8$$
$$y \notin \mathbb{Z}$$

2.
$$y^2 + x + y^2 - x = 5 + 3$$
$$2y^2 = 8$$
$$y^2 = 4$$
$$y = 2$$
$$y^2 + x = 5$$
$$x = 1$$

Thus the solutions are $(1, 2), (1, -2), (-1, 2)$ and $(-1, -2)$

16 Modular Arithmetic

For $m \in \mathbb{N}$ and $a, b \in \mathbb{Z}$ we say $a \equiv b \mod m$ if one of the following holds:

- $m|(a - b)$
- a and b have the same remainder when divided by m
- $b = qm + a$ for some $q \in \mathbb{Z}$

Example 16.1

$32 \equiv 2 \mod 10$ as $32 = 3 \cdot 10 + 2$

$82 \equiv 1 \mod 3$ as $3|81 = 82 - 1$

Suppose $a \equiv x \mod m$ and $b \equiv y \mod m$ then:

- $a + b \equiv x + y \mod m$
- $ab \equiv xy \mod m$
- $a^k \equiv x^k \mod m$

Claim The 1's place of the number 6^n is always 6

Proof We will use proof by induction

The claim is equivalent to the statement $6^n \equiv 6 \mod 10$

Let $P(n)$ be the statement "$6^n \equiv 6 \mod 10$"

Basis case:

$P(1)$ $6^1 \equiv 6 \mod 10$

Induction step

Assume $P(n)$ is true. That is $6^n \equiv 6 \mod 10$. Hence

$$6^{n+1} = 6^n \cdot 6 \equiv 6 \cdot 6 = 36 = 3 \cdot 10 + 6 \equiv 6 \mod 10$$

$\implies 6^{n+1} \equiv 6 \mod 10$ hence the claim is true $\forall\, n \in \mathbb{N}$ $\blacksquare$

Find $r \in \{0, 1, \dots, 10\}$ such that $7^{38} = r \mod 11$

$$\begin{aligned}
7^2 &= 49 \equiv 5 \mod 11 \\
7^4 &= (7^2)^2 \equiv 25 \equiv 3 \mod 11 \\
7^8 &= (7^4)^2 \equiv 9 \mod 11 \\
7^{16} &= (7^8)^2 \equiv 81 \equiv 4 \mod 11 \\
7^{32} &= (7^{16})^2 \equiv 16 \equiv 5 \mod 11
\end{aligned}$$

$$7^{38} = 7^{32} \cdot 7^4 \cdot 7^2 \equiv 5 \cdot 3 \cdot 5 = 75 \equiv 9 \mod 11$$
$$r = 9$$

Find the remainder of $\frac{4^7}{17}$

This is the same as $4^7 \mod 17$

$$\begin{aligned}
4^2 &= 16 \equiv -1 \mod 17 \\
4^4 &= (4^2)^2 \equiv 1 \mod 17
\end{aligned}$$

$$4^7 = 4^4 \cdot 4^2 \cdot 4 \equiv 1 \cdot (-1) \cdot 4 = -4 \equiv 13 \mod 17$$

Find the remainder of $\frac{5^{100}}{6}$

$$5^2 = 25 \equiv 1 \mod 6$$

Since $5^2 \equiv 1 \mod 6$ $5^{2n} \equiv 1 \mod 6$ for $n \in \mathbb{N}$ hence $5^{100} \equiv 1 \mod 6$

The equation $ax \equiv b \mod m$ has a solution if and only if $\text{hcf}(a, m) | b$

How do we solve $ax \equiv b \mod m$ when $h = hcf(a, m)$ and $h | b$?

1. Assume $h = b = 1$

$$\begin{aligned} &ax \equiv 1 \mod m \\ \iff &ax = 1 + qm \text{ for some} q \in \mathbb{Z} \\ \iff &1 = ax - qm \end{aligned}$$

   Since $h = \text{hcf}(a, m) = 1$ this has a solution via the Euclidean algorithm.

2. Assume $h = 1$ and $b \neq 1$

   First solve $ax \equiv 1 \mod m$ by case 1. This implies $b(ax) \equiv b \mod m$ which in turn implies $a(bx) \equiv b \mod m$ so bx is the solution where x is the solution from case 1.

3. Assume $\text{hcf}(a, m) \neq 1$

   $\text{hcf}\left(\frac{a}{h}, \frac{m}{h}\right) = 1$ so by case 2 $\frac{a}{h}x \equiv \frac{b}{h} \mod \frac{m}{h} \iff \frac{a}{h}x = \frac{b}{h} + q\frac{m}{h}$ for some $q \in \mathbb{Z}$ hence by multiplying by h we get $ax \equiv b \mod m$ so we solve $\frac{a}{h}x \equiv \frac{b}{h} \mod m$ by case 2 and hence solve $ax \equiv b \mod m$.

17 Modular equations

Find a solution to $5x \equiv 4 \mod 7$ where $x \in \{0, 1, \ldots, 6\}$

$\text{hcf}(5, 7) = 1$ and $1|4$ so there is a solution.

Solve $5x \equiv 1 \mod 7$

$$\begin{aligned} \iff &5x = 1 + 7q \text{ for some } q \in \mathbb{Z} \\ \iff &1 = 5x - 7q \text{ use Euclidean algorithm} \\ &7 = 5 + 2 \\ &5 = 2 \cdot 2 + 1 \\ \implies &1 = 5 - 2 \cdot 2 \\ &\quad = 5 - 2(7 - 5) \\ &\quad = 5 - 2 \cdot 7 + 2 \cdot 5 \\ &\quad = 3 \cdot 5 - 2 \cdot 7 \\ \implies &x = 3 \text{ and } q = 2 \end{aligned}$$

Multiply both sides by 4

$$\begin{aligned} \implies &5(3 \cdot 4) \equiv 4 \mod 7 \\ \implies &x = 3 \cdot 4 = 12 \end{aligned}$$

So 12 solves $5x \equiv 4 \mod 7$

$$\begin{aligned} &12 \notin \{0, 1, \ldots, 6\} \\ &12 = 5 + 7 \equiv 5 \mod 7 \\ \therefore\ &x = 5 \end{aligned}$$

If $\text{hcf}(a, m)|b$ then $ax \equiv b \mod m$ has $h = \text{hcf}(a, m)$ distinct solutions in $\{0, 1, \ldots, m - 1\}$. If x is one solution the other soltions y are of the form $x \equiv y + j\frac{m}{h} \mod m$ for $j = 1, 2, \ldots, hcf(a, m)$.

Suppose x and y are two solutions

$$\begin{aligned} &\implies ax \equiv b \equiv ay \mod m \\ &\implies 0 \equiv ax - ay = a(x - y) \mod m \end{aligned}$$

$$\implies a(x-y) = mq \text{ for some } q \in \mathbb{Z}$$
$$\implies \frac{a}{h}(x-y) = \frac{m}{h}q$$

Since $\text{hcf}(\frac{a}{h}, \frac{m}{h}) = 1 \implies \frac{m}{h}|(x-y)$

$$\implies x - 7 = \frac{m}{h}j \text{ for some } j \in \mathbb{Z}$$
$$\implies x \equiv y + j\frac{m}{h} \mod m$$

Note that if $x \in \{0, 1, \ldots, m-1\}$ then there are $\frac{m}{\frac{m}{h}} = h$ many numbers in $\{0, 1, \ldots, m-1\}$ of the form $x + \frac{m}{h}j$

Find all solutions in $\{0, 1, \ldots, 13\}$ to $6x \equiv 4 \mod 14$

$hcf(6, 14) = 2$ and $2|4$. There are 2 solutions.

First solve $\frac{6}{2}x \equiv \frac{4}{2} \mod \frac{14}{2} \implies 3x \equiv 2 \mod 7$. One solution is $x = 3$. The other solution is of the form $3 + j\frac{14}{2} = 3 + 7j$ for some $j \in \mathbb{Z}$ One that is in$\{0, 1, \ldots, 13\}$ is $3 + 1 \cdot 7 = 10$. Therefore the solutions are $x = 3$ and $x = 10$.

We define $\mathbb{Z}_m = \{\bar{0}, \bar{1}, \ldots, \overline{m-1}\}$ The bar shows we are working in $\mathbb{Z}_m$ but sometimes we drop the bar.

we define addition as $\bar{x} + \bar{y} = \bar{z}$ if $\bar{z} \in \mathbb{Z}_m$ such that $x + y \equiv z \mod m$

we define multiplication as $\bar{x} \cdot \bar{y} = \bar{z}$ if $\bar{z} \in \mathbb{Z}_m$ such that $xy \equiv z \mod m$

Example 17.1

In $\mathbb{Z}_5$:

$$\bar{2} + \bar{3} = \bar{0} \text{ since } 2 + 3 = 5 \equiv 0 \mod 5$$
$$\bar{2} \cdot \bar{3} = \bar{1} \text{ since } 2 \cdot 3 = 6 \equiv 1 \mod 5$$

What is the cube root of $\bar{2}$ in $\mathbb{Z}_m$?

$$\begin{array}{cc} 0^3 = 0 \implies \bar{0}^3 = \bar{0} & 1^3 = 1 \implies \bar{1}^3 = \bar{1} \\ 2^3 = 8 \implies \bar{2}^3 = \bar{3} & 3^3 = 27 \implies \bar{3}^3 = \bar{2} \\ 4^3 = 64 \implies \bar{4}^3 = \bar{4} & \end{array}$$

Therefore the cube root of $\bar{2}$ in $\mathbb{Z}_5$ is $\bar{3}$ since $\bar{3}^3 = \bar{2}$

18 Fermat's Little Theorem

How many $a \in \mathbb{Z}_{81}$ are inversable? That is how many $a, b \in \mathbb{Z}$ such that $ab \equiv 1 \mod 81$?

$ab \equiv 1 \mod 81$ has a solution $\iff \text{hcf}(a, 81)|1 \iff a$ and 81 are coprime. Therefore a and 81 don't share any prime factors. $81 = 3^4$ therefore $3 \nmid a$. The number of $a \in \mathbb{Z}_{81}$ such that $3|a$ is $\frac{81}{3} = 27$ so there are $80 - 27 = 54$ distinct as such that $a \in \mathbb{Z}$ has an inverse.

Fermat's Little Theorem

If p is prime and $p \nmid a \in \mathbb{Z}$ then $a^{p-1} \equiv 1 \mod p$

This allows s to solve some equations of the form $x^n \equiv b \mod p$. This only works if n and $p-1$ are coprime and $p \nmid b$.

First use the Euclidean algorithm to find $s, t > 0$ such that $sn - t(p-1) = 1$

Then $b \equiv x^n \mod p \implies b^s \equiv (x^n)^s = s^{ns} = x^{t(p-1)+1} = x(x^{p-1})^t \mod p$

Therefore by Fermat's little theorem $b^s \equiv x(1)^t = x \mod p$. Hence if $x = b^s \mod p$ then x is a solution. It is unique $\mod p$. That is

- If x and y are solutions then $y \equiv x \mod p$
- There is only one solution in $\{0, 1, \ldots, p-1\} = \mathbb{Z}_p$

Example 18.1

Find a solution to $x^7 \equiv 13 \mod 17$ for $x \in \mathbb{Z}_{17}$

First check 7 and $17 - 1 = 16$ are coprime. 7 is prime and 16 is not a multiple of 7 so 16 and 7 are coprime.

$$\begin{aligned} 16 &= 2 \cdot 7 + 2 \\ 7 &= 3 \cdot 2 + 1 \\ \implies 1 &= 7 - 3 \cdot 2 \\ &= 7 - 3(16 - 2 \cdot 7) \\ &= 7 \cdot 7 - 3 \cdot 16 \end{aligned}$$

Hence $s = 7$ and $4^7(\mod p)$ is a solution

$$4 \equiv 7 \implies 4^7 \equiv x^{7 \cdot 7} = x^{1+3\cdot16} = x(x^{16})^3 \equiv x(1)^3 = x \mod 17$$

$$x^7 \equiv 4^7 = 4^2 \cdot 4^2 \cdot 4^2 \cdot 4 = 16 \cdot 16 \cdot 16 \cdot 4 \equiv (-1)(-1)(-1)4 = -4 \equiv 13 \mod 17$$

Example 18.2

Find a solution in $\mathbb{Z}_{11}$ to $x^{22} \equiv 3 \mod 11$

$\text{hcf}(11, 22) = 11 \neq 1$

By Fermat's little theorem

$$x^{22} = x^{2\cdot10+2} = x^2(x^{10})^2 \equiv x^2(1)^2 = x^2 \mod 11$$

Hence by inspection $x = 5$ is a solution since

$$5^2 = 25 \equiv 25 - 2 \cdot 11 = 3 \mod 11$$

and $x = 6$ is a solution since

$$6^2 = 36 \equiv 36 - 3 \cdot 11 = 3 \mod 11$$

and there are no other solutions.

Example 18.3

What are all possible integer soutions to $9x^2 + 9x + 2 = y^4$?

If (x, y) is a solution then it is a solution to the equation mod 3.

$$y^4 = 9x^2 + 9x + 2 \equiv 0x^2 + 0x + 2 = 0 \mod 3$$

$$y^4 \equiv 2 \mod 3$$

All possible solutions are in$\mathbb{Z}_3$ $0^4 = 0 \not\equiv 2 \mod 3$ so 0 isn't a solution, $1^4 = 1 \not\equiv 2 \mod 3$ so 1 isn't a solution, $2^4 = 16 \equiv 16 - 5 \cdot 3 = 1 \not\equiv 2 \mod 3$. Therefore there are no integer solutions.

19 Counting

$\binom{n}{k}$ is by definition the number of ways to pick k things from a set of n things.

Example 19.1

How many subsets of size 2 are there of a set size 23?

$$\binom{23}{2} = \frac{23!}{2!(23-2)!} = \frac{23!}{2! \cdot 21!} = \frac{23 \cdot 22 \cdot 21 \cdot 20 \cdots 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 21 \cdot 20 \cdots 3 \cdot 2 \cdot 1} = \frac{23 \cdot 22}{2} = 253$$

Multiplication principal

The multiplication principal states

> If p is a process consisting of n stages and the k^{th} stage has a_k ways to proceed then there are $a_1 a_2 \cdots a_n$ possible outcomes

Example 19.2

How many subsets of size n are there? (We proved this using indurtion in lecture 10)

Let $S = \{s_1, s_2, \dots, s_n\}$. The process of picking a subset starts at s_1, we must decide whether or not to include it. We then move on to s_2 and continue until s_n. This gives us n stages with two possiblities at each stage. By the multiplication principal we have $2 \cdot 2 \cdots 2 = 2^n$ possible subsets.

Claim 120 has 16 factors

Proof $120 = 2^3 \cdot 3 \cdot 5$ therefore any factor of 120 is of the form $2^i \cdot 3^j \cdot 5^k$ where $i = 0, 1, 2, 3$ and $j, k = 0, 1$. There are four options for i and two for j and k. By the multiplication principal there are $4 \cdot 2 \cdot 2 = 16$ options hence 120 has 16 factors ■

Example 19.3

How many ways are there to

(a) pick three different objects from a set of size 6?

(b) pick three objects, from a set of size 6, 1 by 1?

(c) pick three objects, from a set of size 6, with repetition?

(a) $\binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{6!}{3! \cdot 3!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 5 \cdot 4 = 20$

(b) There are 6 objects for the first pick, 5 for the second pick and 4 for the final pick giving $6 \cdot 5 \cdot 4 = 120$ options.

(c) There will be 6 objects for all three picks giving $6 \cdot 6 \cdot 6 = 216$ options.

Example 19.4

How many ways are there to distribute

(a) 6 distinct objects to 4 people?

(b) 6 identical objects to 4 people?

(a) For each object there are 4 options for where it can go so the total number of ways to distribute them is $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^6 = 4096$.

(b) Since the objects are identical what we are really counting is the number of ways for 4 people to have onjects sucj that the total number of objects is 6. This is the same as picking integers x_1, x_2, x_3 and x_4 where $x_i \geq 0 \; \forall \, i$ and $x_1 + x_2 + x_3 + x_4 = 6$. This in turn is the same as counting the number of ways of splitting 6 1s into 4 groups, which we can do by placing 3 0s between them, eg 110100111 is a group of 2, a grop of 1, a group of 0 and a group of 3. Since there are 9 symbols and all we can do is choose the different positions for the 0s there are

$$\binom{9}{3} = \frac{9!}{3!(9-3)!} = \frac{9!}{3! \cdot 6!} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 3 \cdot 4 \cdot 7 = 84$$

20 Partitions

Binomial theorem

The binomial theorem is that for $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

What is the coefficent of x^4 in $(2x^2 - x^{-2})^8$?

$$\begin{aligned}(2x^2 - x^{-2})^8 &= \sum_{k=0}^{n} \binom{8}{k} (2x^2)^k (-x^{-2})^{8-k} \\ &= \sum_{k=0}^{8} \binom{8}{k} 2^k (-1)^{8-k} x^{2k-16+2k}\end{aligned}$$

$(-1)^{8-k} = (-1)^8(-1)^{-k} = (-1)^{-k} = (-1)^k$ since k is even if, and only if, $-k$ is even.

$$= \sum_{k=0}^{8} \binom{8}{k} (-2)^k x^{4k-16}$$

We want $4k - 16 = 4 \implies k = 5$, hence the coefficent of x^4 is

$$\binom{8}{5}(-2)^5 = \frac{8!}{5!(8-5)!}(-32) = \frac{8!}{5! \cdot 3!} = \frac{8 \cdot 7 \cdot 6 \cdots 1}{5 \cdot 4 \cdot 3 \cdots 3 \cdot 2}(-32) = 8 \cdot 7(-32) = -1792$$

Partitions

A partition of a set S is a collection of sets $\{A_1, A_2, \dots, A_k\}$ such that each $x \in S$ is in exactly one A_i.

An ordered partition of a set S is an ordered sequence of sets $(A_1, A_2, \dots, A_k)$ such that each $x \in S$ is in exactly one A_i.

Given $r_1, \dots, r_k \geq 0$, $r_1, \dots, r_k \in \mathbb{Z}$, such that $r_1 + r_2 + \dots + r_k = |S| = n$, the number of ordered partitions is

$$\binom{n}{r_1, r_2, \dots, r_k} = \frac{n!}{r_1! \cdot r_2! \cdots r_k!}$$

Example 20.1

The number of ordered partitions (A_1, A_2, A_3) of $S = \{1, 2, \dots, 9\}$ such that $|A_1| = 4$, $|A_2| = 3$ and $|A_3| = 2$ is

$$\binom{9}{4, 3, 2} = \frac{9!}{4! \cdot 3! \cdot 2!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{4 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 2} = 7 \cdot 6 \cdot 5 \cdot 3 \cdot 2 = 1260$$

Overcounting

Overcounting occurs when we try to count a set but count some elements more than once. This can be useful if we can't directly count the set but we can count a larger set and the compensate for the overcounting to give the size of the smaller set.

Example 20.2

How many partitions are there of 7 objects into a set of size 3 and two sets of size 2?

There are $\binom{7}{3,2,2}$ ordered partitions (A_1, A_2, A_3) such that $|A_1| = 3$ and $|A_2| = |A_3| = 2$. If $\{A_1, A_2, A_3\}$ is a partition that meets these criteria then this corresponds to two ordered partitions (A_1, A_2, A_3) and (A_1, A_3, A_2). This means that we have overcounted by a factor of 2 so we need to divide by 2 to get the number of partitions

$$\binom{7}{3,2,2}/2 = \frac{7!}{3! \cdot 2! \cdot 2! \cdot 2} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 2 \cdot 2 \cdot 2} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{6 \cdot 4 \cdot 2} = 7 \cdot 5 \cdot 3 = 105$$

Example 20.3

How many ways are there to partition 9 bjects into 3 sets of size 3?

The number of ordered partitions (A_1, A_2, A_3) such that $|A_1| = |A_2| = |A_3| = 3$ is $\binom{9}{3,3,3}$. Every partition $\{A_1, A_2, A_3\}$ has 3! arrangements, hence there are

$$\binom{9}{3,3,3}/3! = \frac{9!}{3! \cdot 3! \cdot 3! \cdot 3!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot} = 8 \cdot 7 \cdot 5 = 280$$

Sometimes it is possible to prove a fomula $A = B$ by showing A and B count the same number of objects and hence are the same.

Example 20.4

Show $2^n = \sum_{k=0}^{n} \binom{n}{k}$

As shown earlier 2^n is the number of subsets of a set size n. $\binom{n}{k}$ is the number of subsets of size k so the sum of all of these for all possible values of k is the total number of subsets is 2^n so $2^n = \sum_{k=0}^{n} \binom{n}{k}$

21 Counting Sets

Set notation

For sets A and B the set difference is denoted and defined as

$$A - B = A \backslash B = \{x \in A | x \notin B\}$$

The union of the two sets is denoted and defined

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

The intersection of the two sets is denoted and defined

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

Inclusion-exclusion principal

For $n \in \mathbb{N}$ if $A_1, \ldots, A_n$ are finite sets then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = c_1 - c_2 + \cdots + (-1)^n c_n$$

Where, for $1 \le i \le n$, c_i is the sum of the size of the intersections of the sets taken i at a time. For example for $n = 3$

$$\begin{aligned} c_1 &= |A_1| + |A_2| + |A_3| \\ c_2 &= |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| \\ c_3 &= |A_1 \cap A_2 \cap A_3| \end{aligned}$$

Example 21.1

How many rearrangements of $1, 2, \ldots, 6$ such that either the first digit is a 1, the second digit is a 2 or the third digit is a 3? Let

$$A = \{a_1, a_2, \ldots, a_6 | a_i \ne a_j \text{ when } i \ne j \text{ and } a_1 = 1\}$$

$$B = \{a_1, a_2, \ldots, a_6 | a_i \ne a_j \text{ when } i \ne j \text{ and } a_2 = 2\}$$

$$C = \{a_1, a_2, \ldots, a_6 | a_i \ne a_j \text{ when } i \ne j \text{ and } a_3 = 3\}$$

We want to know how many elements are in at least one of these sets, that is $|A \cup B \cup C|$. We can do this by the inclusion-exclusion principal. $|A| = |B| = |C| = 5!$ since each has the first value set and the others can be any permutation. $|A \cap B| = |A \cap C| = |B \cap C| = 4!$ since each has two set values and the rest can be any permutation. $|A \cap B \cap C| = 3!$ since there are three set values and the rest can be any permutation. By the inclusion-exclusion principal

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| + |A \cap B| + |A \cap C| + |B \cap C| + |A \cap B \cap C| \\ &= 5! + 5! + 5! + 4! + 4! + 4! + 3! \\ &= 294 \end{aligned}$$

Example 21.2

How many numbers in $G = \{1, \ldots, 60\}$ are coprime to 60?

x is coprime to 60 if, and only if, x and 60 don't share any prime factors. $60 = 2^3 \cdot 3 \cdot 5$. Hence for $\operatorname{hcf}(x, 60) = 1$ we need $2 \not| x$, $3 \not| x$ and $5 \not| x$. However it is much easier to count the size of the sets such that $2|x$, $3|x$ or $5|x$. Let $A = \{x | x \in G \text{ and } 2|x\}$, $B = \{x | x \in G \text{ and } 3|x\}$ and $C = \{x | x \in G \text{ and } 5|x\}$.

$$|A| = \frac{60}{2} = 30, \qquad |B| = \frac{60}{3} = 20, \qquad |C| = \frac{60}{5} = 12,$$

$$|A \cap B| = \frac{60}{2 \cdot 3} = 10, \qquad |A \cap C| = \frac{60}{2 \cdot 5} = 6, \qquad |B \cap C| = \frac{60}{3 \cdot 5} = 4 \qquad \text{and} \qquad |A \cap B \cap C| = \frac{60}{2 \cdot 3 \cdot 5} = 2$$

By the inclusion-exclusion principal there are

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| + |A \cap B| + |A \cap C| + |B \cap C| + |A \cap B \cap C| \\ &= 30 + 20 + 12 - 10 - 6 - 4 + 2 \\ &= 44 \end{aligned}$$

integers coprime to 60 in G. Hence there are $60 - 44 = 16$ integers under 60 which are coprime to 60.

22 Relations

Cartesian products

The Cartesian product of two sets A, B is denoted $A \times B$. It is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. Another way of writing this is $A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$. Two ordered pairs (a, b) and (a', b') are equal if, and only if, $a = a'$ and $b = b'$. $A \times A$ is often denoted as A^2. For multiple sets $A_1, A_2, \dots, A_n$ the Cartesian product is the set of ordered tuples $(a_1, a_2, \dots, a_n)$ where $a_i \in A_i$.

Relations

Given a set S a relation R is a subset of $S^2 = \{(a, b) | a, b \in S\}$. We declare a and b to be related and write $a \sim b$ if $(a, b) \in R$.

For example "$=$", "$<$" and "$\equiv \mod m$" are all relations.

The number of relations on a set S is the number of possible subsets of S^2 which is $2^{|S^2|}$.

Equivalence relations

An equivalence relation is a relation that is:

- Reflexive, $\forall\, a \in S, a \sim a$
- Symmetric, $\forall\, a, b \in S, a \sim b \implies b \sim a$
- Transitive, $\forall\, a, b, c \in S, a \sim b$ and $b \sim c \implies a \sim c$

Example 22.1

Let $A \sim B$ if $A \cap B \neq \emptyset$, for $A, B \subseteq \mathbb{R}$. Is this an equivalence relation?

Check all three properties:

- Reflexivity: If $A \neq \emptyset$ then $A \cap A = A \neq \emptyset$ so $A \sim A$
- Symmetry: Suppose $A \sim B \implies A \cap B \neq \emptyset \implies B \cap A \neq \emptyset \implies B \sim A$
- Transitivity: To show transitivity is false we need to show the negation: $\exists\, A, B, C \subseteq \mathbb{R}$ such that $A \sim B$ and $B \sim C$ but $A \not\sim C$

  Let $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{3, 4\}$

  $A \cap B = \{2\} \neq \emptyset \implies A \sim B$

  $B \cap C = \{3\} \neq \emptyset \implies B \sim C$

  $A \cap C = \emptyset \implies A \not\sim C$

  So the relation is not transitive

The relation doesn't have all three properties so isn't an equivalence relation

Example 22.2

Let $\sim$ be defined on $\mathbb{Z}$ so that $a \sim b$ if $a^2 \equiv b^3 \mod 3$. Which conditions of an equivalence relation does this have?

$\forall\, a \in \mathbb{Z}, a(\mod 3) \in \mathbb{Z}_3$ so we only need to check the properties on $0, 1$ and 2.

When is $a \sim b$?

$\sim$	0	1	2
0	✓	×	×
1	×	✓	×
2	×	✓	×

- Reflexivity: $2^2 \not\equiv 2^3 \mod 3 \implies \sim$is not reflexive
- Symmetry: $2^2 \equiv 1^3$ and $1^2 \not\equiv 2^3 \implies \sim$ is not symmtetric
- Transitivity: Assume $a \sim b$ and $b \sim c$. Does this imply $a \sim c$?

  Case 1: If a, b or c is 0 then since 0 is only related to itself this means that all of a, b and c are 0 so $a \sim c$ since $0^2 \equiv 0^3 \mod 3$.

  Case 2: Suppose $a, b, c \neq 0$

  $\implies a, b, c \in \{1, 2\} \implies a = b, b = c$ or $a = c$

  1. $a = b \implies a \sim a$ and $a \sim c \implies a \sim c$
  2. $b = c \implies a \sim c$ and $c \sim c \implies a \sim c$
  3. $a = c \implies a \sim b$ and $b \sim a \implies a = 0$, which is a contradiction to our assumption, or $a = 1$ which means $a = b = c \implies a \sim c$

  In all of these transitivity holds

Example 22.3

Define a relation on $\mathbb{N}^2$ that $(x_1, y_1) \sim (x_2, y_2)$ if $x_1y_2 = x_2y_1$. Which properties of an equivalence relation hold?

- Reflexivity $\iff (x_1, y_1) \sim (x_1y_1) \iff x_1y_1 = x_1y_1$ so it is reflexive
- Symmetry $\iff (x_1, y_1) \sim (x_2, y_2) \iff x_1y_2 = x_2y_1 \iff x_2y_1 = x_1y_2 \iff (x_2, y_2) \sim (x_1, y_1)$ so it is symmetric
- Transitivity: Suppose $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$

$$\begin{aligned}
\implies x_1y_2 &= x_2y_1 \\
x_2y_3 &= x_3y_2 \\
\implies x_1 \underbrace{y_2x_3}_{=y_3x_2} &= x_2y_1x_3 \\
\implies x_1y_3x_2 &= x_2y_1x_3 \\
\implies x_1y_3 &= x_3y_1 \\
\implies (x_1, y_1) &\sim (x_3, y_3)
\end{aligned}$$

  So it is transitive

This is an equivalnce relation as it satisfies all three criterea.

23 Functions

Given an equivalence relation $\sim$ of S and $x \in S$, the equivalence class for x is

$$[x] = \{y \in S | y \sim x\}$$

The sets $\{[x] | x \in S\}$ partition S, that is every $x \in S$ is in exactly one equivalence class. If $z \in [x]$ then

$$[x] = \{y | y \sim x\} = \{y | y \sim z\} = [z]$$

Example 23.1

Define a relation $\sim$ on $\mathbb{Z}$ such that $a \sim b$ if $a \equiv b \mod 3$.

$$[0] = \{x \in \mathbb{Z} | x \equiv 0 \mod 3\}$$

$$[1] = \{x \in \mathbb{Z} | x \equiv 1 \mod 3\}$$

$$[2] = \{x \in \mathbb{Z} | x \equiv 2 \mod 3\}$$

$$[3] = \{x \in \mathbb{Z} | x \equiv 3 \mod 3\} = \{x \in \mathbb{Z} | x \equiv 0 \mod 3\} = [0]$$

There are only three distinct equivalence classes $[0], [1]$ and $[2]$.

Example 23.2

For $z, w \in \mathbb{C}\backslash\{0\}$ if $\arg z = \arg w$ then $z \sim w$. What are the equivalence classess?

$$[z] = \{w \in \mathbb{C}\backslash\{0\} | z \sim w\}$$
$$[z] = \{w \in \mathbb{C}\backslash\{0\} | \arg z = \arg w\}$$

Let $z = re^{i\vartheta}$ for $r > 0$ and $\vartheta \in [0, 2\pi)$

$$\begin{aligned}[z] &= \{w \in \mathbb{C}\backslash\{0\} | \vartheta = \arg w\} \\ &= \{w \in \mathbb{C}\backslash\{0\} | 2 = se^{i\vartheta}, s > 0\} \\ &= \{te^{i\vartheta} | t > 0\} \\ &= L_\vartheta\end{aligned}$$

Thus all equivalence classes are of the form $L_\vartheta = \{te^{i\vartheta} | t > 0\}$ for some $\vartheta \in [0, 2\pi)$

Example 23.3

Let $z \sim w$ where $x, w \in \mathbb{C}$ if $\operatorname{Re}(z) = \operatorname{Re}(w)$. What are the equivalence classes?

Let $z, w \in \mathbb{C}$ and $z = x + iy$ for $x, y \in \mathbb{R}$.

$$\begin{aligned}[z] &= \{w \in \mathbb{C} | w \sim z\} \\ &= \{w \in \mathbb{C} | \operatorname{Re}(w) = \operatorname{Re}(z)\} \\ &= \{w \in \mathbb{C} | \operatorname{Re}(w) = x\} \\ &= \{w_1 + iw_2 | w_1, w_2 \in \mathbb{R}, w_1 = x\} \\ &= \{x + iw_2 | w_2 \in \mathbb{R}\} \\ &= L(x)\end{aligned}$$

Thus all equivalence classes are of the form $L(x) = \{x + iy | y \in \mathbb{R}\}$. They are all vertical lines. When we write the equivalence classes for $x \in \mathbb{R}$ there is no repetition, that is for some $x, x' \in \mathbb{R}$ such that $x \neq x'$ then $L(x) \neq L(x')$.

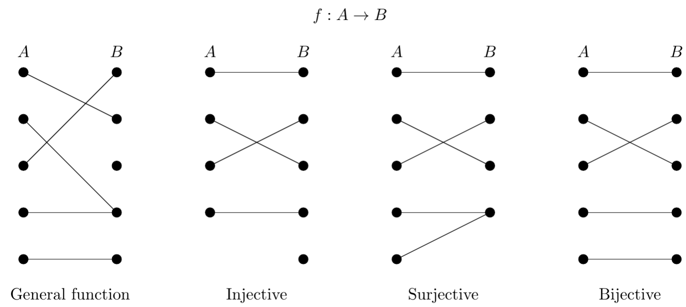
Functions

Given two sets X, Y a function $f : X \to Y$ is a relation pairing each $x \in X$ with exactly one element in Y denoted $f(x)$.

f is surjective if $\forall\, y \in Y\, \exists\, x \in X$ such that $f(x) = y$

f is injective if $\forall\, x, y \in X\, f(x) = f(y) \implies x = y$

f is bijective if it is both surjective and injective



Example 23.4

Let $f : \mathbb{N} \to \mathbb{N}$ be $f(x) = x^2$

Claim f is injective

Proof Let $x, y \in \mathbb{N}$. SUppose $f(x) = f(y)$, we need to show $x = y$.

$$\begin{aligned} f(x) &= f(y) \\ x^2 &= y^2 \\ x^2 - y^2 &= 0 \\ (x-y)(x+y) &= 0 \end{aligned}$$

$$x - y = 0 \text{ or } x + y = 0$$

$$x + y > 0 \; \forall \, x, y \in \mathbb{N}$$

$$\therefore x - y = 0 \implies x = y$$

Since $f(x) = f(y) \implies x = y$ by definition f is injective. ■

Claim f is not surjective

Proof We need to show the negation of surjectivity. That is $\exists \, y \in \mathbb{N}$ such that $\forall \, x \in \mathbb{N} f(x) \neq y$.
If $y = 2$ then $x^2 \neq 2 \; \forall \, x \in \mathbb{N}$ since $\sqrt{2} \notin \mathbb{N}$. Hence f is not surjective. ■

Example 23.4

Let $f : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ be $f(\bar{x}) = 2\bar{x}$

Claim f isn't injective

Proof We need to show the negation of surjectinity. That is $\exists \, \bar{x}, \bar{y} \in \mathbb{Z}_{10}$ such that $f(\bar{x}) = f(\bar{y})$ but $\bar{x} \neq \bar{y}$

$$f(\bar{0}) = 2 \cdot \bar{0} = \bar{0}$$

$$f(\bar{5}) = 2 \cdot \bar{5} = \bar{0} \text{ since } 2 \cdot 5 = 10 \equiv 0 \mod 10$$

Hence $f(\bar{0}) = f(\bar{5})$ but $\bar{0} \neq \bar{5}$ so f is not injective. ■

Claim f isn't surjective

Proof We need to show $\exists\, \bar{y} \in \mathbb{Z}_{10}$ such that $f(\bar{x}) = 2\bar{x} \neq \bar{y}\, \forall\, x$

$$\begin{aligned} \exists\, \bar{x} \text{ such that } 2\bar{x} = \bar{y} &\iff \exists\, x \text{ such that } 2x \equiv y \mod 10 \\ &\iff \text{hcf}(2,10) | y \\ &\iff 2|y \\ &\implies 2\bar{x} = \bar{3} \text{ has no solutions} \end{aligned}$$

So f isn't surjective. ■

Example 23.5

Let $f : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ be $f(\bar{x}) = 3\bar{x}$

Claim f is bijective

Proof We need to show that f is injective and surjective:

Injectivity:

Suppose $f(\bar{x}) = f(\bar{y})$

$$\begin{aligned} &\implies 3\bar{x} = 3\bar{y} \\ &\implies 3x \equiv 3y \mod 10 \\ &\implies 10|3x - 3y \\ &\implies 10|3(x - y) \\ &\implies 10|x - y \\ &\implies x \equiv y \mod 10 \\ &\implies \bar{x} = \bar{y} \end{aligned}$$

So f is injective

Surjectivity

Let $\bar{y} \in \mathbb{Z}$

Then $f(\bar{x} = 3\bar{x}$ has a solution $\iff 3x \equiv y \mod 10 \iff \text{hcf}(3,10)|y \iff 1|y$ which is true $\forall\, y$. Therefore this always has a solution so f is surjective.

f is infective and surjective so f is bijective. ■

24 Inverse Functions

If $f : X \to Y$ is bijective then the inverse of f, denoted $f^{-1} : Y \to X$ is the function such that

$$f^{-1}(y) = x \iff f(x) = y$$

If $f : X \to Y$ is bijective then you can solve for $f^{-1}(y)$ by solving$f(x) = y$ for x and setting $f^{-1}(y) = x$. This definition of inverses gives rise to the following

$$f^{-1}(f(x)) = x\; \forall\, x \in X \text{ and } f(f^{-1}(y)) = y\; \forall\, y \in Y$$

Let $f : \mathbb{Z} \to \mathbb{Z}$ be $f(n) = n + 1$. This is bijective so has an inverse $f^{-1}(x) = x - 1$.

Let $g : \mathbb{N} \to \mathbb{N}$ be $f(n) = n + 1$. This is not bijective so has no inverse.

Function Composition

Let S, T, U be sets, and let $f : S \to T$ and $g : T \to U$ be functions. The composition of f and g is the function $f \circ g : S \to U$ which is defined by the rule

$$(g \circ f)(s) = g(f(s))\ \forall\, s \in S$$

Example 24.1

Claim If $f : S \to T$ and $g : T \to R$ are bijective then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Proof We need to verify $f^{-1} \circ g^{-1}(r) = s \iff r = g \circ f(s)$ for $s \in S$ and $r \in R$.

$$\begin{aligned}
f^{-1} \circ g^{-1}(r) = s &\iff f^{-1}(g^{-1}(r)) = s \\
&\iff f(f^{-1}(g^{-1}(r))) = f(s) \text{ since } f \text{ is injective} \\
&\iff g^{-1}(r) = f(s) \\
&\iff g(g^{-1}(r)) = g(f(s)) \text{ since } g \text{ is injective} \\
&\iff r = g(f(s)) = g \circ f(s) \\
&\implies (g \circ f)^{-1} = f^{-1} \circ g^{-1} \quad \blacksquare
\end{aligned}$$

If $f : X \to Y$ and $A \subseteq X$ the image of A under f is

$$f(A) = \{f(x) | x \in A\} \subseteq Y$$

If $B \subseteq Y$ the preimage of B under f is

$$f^{-1}(B) = \{x \in X | f(x) \in B\} \subseteq X$$

Note that f does not have to be bijective for this.

Claim If $f : X \to Y$ then f is injective $\iff f^{-1}(f(A)) = A\ \forall \subseteq X$

Proof Suppose $f^{-1}(f(A)) = A\ \forall\, A \subseteq X$

Suppose $f(x) = f(y) = z$. Then $f^{-1}(f(\{x\})) = \{x\}$ by assumption.

$$f^{-1}(f(\{x\})) = \{a \in \{x\} | f(a) \in f(\{x\})\} = \{z\} \ni y \implies y \in \{x\} \implies y = x$$

Hence f is injective

Suppose f is injective. Let $A \subseteq X$. We want to show $f^{-1}(f(A)) = A$.

$$f^{-1}(f(A)) = \{x \in X | f(x) \in f(a)\} \supseteq A$$

So now we need to show $f^{-1}(f(A)) \subseteq A$. Suppose $\exists\, x \in f^{-1}(f(A)) \setminus A$

$$\begin{aligned}
x \in f^{-1}(f(A)) &\implies f(x) \in f(A) = \{f(y) | y \in A\} \\
&\implies \exists\, y \in A \text{ such that } f(y) = f(x) \\
&\implies y = x \text{ since } f \text{ is injective}
\end{aligned}$$

However $x \notin A$ and $y \in A$, therefore $x \neq y$. This is a contradiction, therefore $f^{-1}(f(A)) \subseteq A$ and $A \subseteq f^{-1}(f(A))$ This can only be true if $f^{-1}(f(A)) = A$. $\blacksquare$

Likewise $f^{-1}(f(B)) = B \iff f$ is surjective

It isn't always the case that $A_1 \cap A_2 = \emptyset \implies f(A_1) \cap f(A_2) = \emptyset$. If f isn't injective then it fails. This is because $\exists\, f(x) = f(y)$ such that $x \neq y$. Hence $\{x\} \cap \{y\} = \emptyset$ but $f(\{x\}) \cap f(\{y\}) = \{f(x)\} \cap \{f(y)\} = \{f(x)\} \cap \{f(x)\} = \{f(x)\} \neq \emptyset$

Claim If $B_1 \cap B_2 = \emptyset$ then $f^{-1}(B_1) \cap f^{-1}(B_2) = \emptyset$

Proof We will prove the contrapositive that if $f^{-1}(B_1) \cap f^{-1}(B_2) \neq \emptyset$ then $B_1 \cap B_2 \neq \emptyset$

$$\begin{aligned} \exists\, x \in f^{-1}(B_1) \cap f^{-1}(B_2) &\implies \begin{cases} x \in f^{-1}(B_1) \implies f(x) \in B_1 \\ x \in f^{-1}(B_2) \implies f(x) \in B_2 \end{cases} \\ &\implies f(x) \in B_1 \cap B_2 \neq \emptyset \quad \blacksquare \end{aligned}$$

For two sets A, B $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

25 Permutations

Let S_n be the set of bijections $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$. That is the set of all permutations of $\{1, 2, \dots, n\}$.

We often denote a permutation f as

$$f = \begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix}$$

When composing two functions f and g we often write $fg = f \circ g$ for convenience. Similarly if we compose f with itself we write $f^2 = ff = f \circ f$

Example 25.1

If f is defined as below what is f^2?

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

It is often useful to use a table of all elements of $\{1, 2, \dots, n\}$ as columns and each row being the result of a permutation:

$$\begin{array}{c|cccc} f^2 & 1 & 2 & 3 & 4 \\ \hline f & 2 & 1 & 4 & 3 \\ f & 1 & 2 & 3 & 4 \end{array} \implies f^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

The identity permutation is denoted ι and defined as $f\iota = \iota f = f$. In example 25.1 f^2 is the identity element. This means that $\iota \in S_n$ and has the property that $\iota(x) = x\ \forall\, x$.

S_n forms a group under function composition $(S_n, \circ)$. This means that the four group properties hold:

- Closure: $f, g \in S_n \implies fg \in S_n$
- Identity: There exists a unique element ι such that $f \in S_n \implies f\iota = \iota f = f$
- Inverse: For each $f \in S_n$ there exists a unique element f^{-1} such that $ff^{-1} = f^{-1}f = \iota$
- Associativity: For $f, g, h \in S_n$ $f(gh) = (fg)h$

To find the inverse f^{-1} of f just swap the two rows and then rearrange the columns such that the top row is in order.

Example 25.2

Find the inverse of f where

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}$$

$$f^{-1} = \begin{pmatrix} 5 & 4 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}$$

Suppose $f : X \to Y$ and $g : Y \to X$ and $gf(x) = x \, \forall x \in X \not\Longrightarrow g = f^{-1}$. We also need to show that $fg(y) = y \, \forall y \in Y$ as well before we can say $g = f^{-1}$. This only works if gf is a bijection. By closure we know that $gf \in S_n$ so we know that gf is a bjection for $f, g \in S_n$.

It is easy to show that for a non-bijection this property doesn't hold. Let $f : \{1\} \to \{1, 2\}$ and $g : \{1, 2\} \to \{1\}$ be functions defined as $f(1) = 2$ and $g(1) = g(2) = 1$.

$$gf(1) = g(f(1)) = g(2) = 1 \implies gf(x) = x \, \forall x \in \{1\}$$

however $g \neq f^{-1}$ since f isn't a bijection so doesn't have an inverse.

Claim If $f, g \in S_n$ and $gf = \iota$ then $g = f^{-1}$

Proof By definition fo an inverse:

$$\begin{aligned} g = f^{-1} &\iff (g(y) = x \iff y = f(x)) \\ y = f(x) &\iff g(y) = g(f(x)) \\ &\iff g(y) = x \quad \blacksquare \end{aligned}$$

Because this is true for $f, g \in S_n$ we can check if $g = f^{-1}$ by checking that $gf = \iota$.

Example 25.3

Verify that, for $f, g \in S_n$), $(fg)^{-1} = g^{-1}f^{-1}$.

If this is true the $g^{-1}f^{-1}fg = \iota$

$$g^{-1}f^{-1}fg = g^{-1}\iota g = g^{-1}g = \iota$$

so $(fg)^{-1} = g^{-1}f^{-1}$

There is one way in which S_n is not nicely multiplicative, it is not commutative. This means that $fg = h \not\Longrightarrow gf = h$.

Example 25.4

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 5 & 1 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix}$$

$$\begin{array}{c|ccccc} fg & 1 & 2 & 3 & 4 & 5 \\ \hline g & 2 & 1 & 3 & 5 & 4 \\ f & 2 & 3 & 4 & 1 & 5 \end{array} \implies fg \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 5 \end{pmatrix}$$

$$\begin{array}{c|ccccc} gf & 1 & 2 & 3 & 4 & 5 \\ \hline f & 3 & 2 & 4 & 5 & 1 \\ g & 3 & 1 & 5 & 4 & 2 \end{array} \implies gf \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 4 & 2 \end{pmatrix}$$

It can be seen that $fg \neq gf$

Cycles

An r-cycle in S_n is a permutation of f such that for some distinct $a_1, a_2, \dots, a_r$ such that $a_1 \xrightarrow{f} a_2 \xrightarrow{f} \cdots \xrightarrow{f} a_{r-1} \xrightarrow{f} a_r \xrightarrow{f} a_1$ and $f(x) = x \, \forall x \notin \{a_1, a_2, \dots, a_r\}$.

We can denote a cycle as above as $(a_1 \, a_2 \, \dots \, a_r)$.

Example 25.5

What cycles are in f when f is defined as below?

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 5 & 1 \end{pmatrix}$$

$$1 \xrightarrow{f} 3 \xrightarrow{f} 4 \xrightarrow{f} 5 \xrightarrow{f} 1, \qquad 2 \xrightarrow{f} 2$$

So f contains the cycles $(1\,3\,4\,5)$ and (2). We can write f as a product of these cycles $f = (1\,3\,4\,5)(2)$. Note that since $(2) \implies f(x) = x\ \forall\, x$ then $(2) = \iota$ since it doesn't change anything. Any 1-cycle is the identity. So we can write $f = (1\,3\,4\,5)$ and it means the same thing.

Example 25.6

How many possible 5-cycles are there of $\{1, 2, 3, 4, 5\}$?

Let f be a 5-cycle $(b_1\, b_2\, \dots\, b_5) = (b_2\, b_3 \dots b_5\, b_1)$. By repeatedly applying this all f can be written as $(1\, b_1\, \dots\, b_4)$. This means that there are 4 choices for b_1, which leaves 3 choices for b_2 etc. By the multiplication principal there are $4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$ options so there are 24 5-cylces of $\{1, 2, 3, 4, 5\}$

In general there are $(n-1)!$ different n-cycles of $\{1, 2, \dots, n\}$.

26 Cycle decompositions

To find the inverse of a cycle we invert the cycle

$$(a_1\, a_2\, \dots\, a_r)^{-1} = (a_r\, a_r - 1\, \dots\, a_1)$$

If $a = (a_1\, a_2\, \dots\, a_r)$ and $b = (b_1\, b_2\, \dots\, b_s)$ are disjoint cycles (that is $a_i \neq b_j\ \forall\, i, j$) then the composition of both is commutative

$$a \circ b = b \circ a$$

Cycle decompositions

Proposition 20.3 proves that all $f \in S_n$ can be decomposed into disjoint cycles, that is $f = \alpha_1 \alpha_2 \cdots \alpha_k$ where α_i is disjoint from α_j if $i \neq j$.

Given a permutation f how do we work out the cycle decomposition?

We start with a number x and look at $x, f(x), f^2(x) \dots$ until we get $f^k(x) = x$ and this gives us the first cycle $(x\, f(x)\, \dots\, f^k(x))$. We then repeat with a number not in this cycle until there are no numbers left not in a cycle.

Example 26.1

What is the cycle decomposition of $f = (1\,2\,3)(2\,3\,4)(5\,2\,1)$?

$$\begin{array}{c|ccccc} f & 1 & 2 & 3 & 4 & 5 \\ \hline (5\,2\,1) & 5 & 1 & 3 & 4 & 2 \\ (2\,3\,4) & 5 & 1 & 4 & 2 & 3 \\ (1\,2\,3) & 5 & 2 & 4 & 3 & 1 \end{array} \implies f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 3 & 1 \end{pmatrix}$$

$$\begin{array}{ccc} 1 \to 5 \to 1 & \implies & (1\,5) \\ 2 \to 2 & \implies & (2) \\ 3 \to 4 \to 3 & \implies & (3\,4) \end{array}$$

$$\implies f = (1\,5)(3\,4)$$

The order of $f \in S_n$ is the smallest value of k such that $f^k = \iota$.

Proposition 20.4 proves that if $f = \alpha_1 \cdots \alpha_l$ where α_i are disjoint cycles then the order of f is the least common multiple of the lengths of the cycles.

Example 26.2

What is the order of the following permutations?

(a) $f = (1\,2)(\,3\,4\,5)$

(b) $g = (1\,4\,2)(2\,3\,5)$

(a) The order of f is $\mathrm{lcm}(2, 3) = 6$

(b)

$$\begin{array}{c|ccccc} f & 1 & 2 & 3 & 4 & 5 \\ \hline (1\,4\,2) & 4 & 1 & 3 & 2 & 5 \\ (2\,3\,5) & 4 & 1 & 5 & 3 & 2 \end{array} \implies f = (1\,4\,2\,3\,5) \implies \text{order of } f \text{ is } 5$$

27 Sign function

Example 27.1

Suppose I shuffle 16 cards according to the permutation

$$f = \left(\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 9 & 2 & 10 & 3 & 11 & 4 & 12 & 5 & 13 & 6 & 14 & 7 & 15 & 8 & 16 \end{pmatrix}\right)$$

How many times do I have to do this to get back to the original order?

This is the same as asking what the order of the f is.

$$\begin{array}{cccccc} 1 \to 1 & 2 \to 9 \to 5 \to 3 \to 2 & 4 \to 10 \to 13 \to 7 \to 4 & 6 \to 11 \to 6 & 8 \to 12 \to 14 \to 15 \to 8 & 16 \to 16 \\ (1) & (2\,9\,5\,3) & (4\,10\,13\,7) & (6\,11) & (8\,12\,14\,15) & (16) \end{array}$$

$$f = (2\,9\,5\,3)(4\,10\,13\,7)(6\,11)(8\,12\,14\,15)$$

These are cycles of length 2 or 4 so the number of shuffles needed is $\mathrm{lcm}(2, 4) = 4$.

Example 27.2

How many permutations of order 6 are there in S_5?

The only way $f \in S_5$ can have order 6 is if it is a 2-cylce and a 3-cycle. There are $\binom{5}{3} = 10$ different ways to select 3 elements from a set of 5, to form a 3-cycle, and the other two elements can form a 2-cycle. This gives 10 different arrangements of elements into each cycle. Each cycle then has $(3-1)! = 2$ or $(2-1)! = 1$ possible arrangement. Therefore by the multiplicative principal there are $10 \cdot 2 \cdot 1 = 20$ different cycles of order 6 in S_5.

Sign function

The sign function sgn is a function $\mathrm{sgn} : S_n \to \{\pm 1\}$ such that for $f, g, \iota \in S_n$ where ι is the identity permutation

- $\mathrm{sgn}(f) = (-1)^{r-1}$ if f is an r-cycle
- $\mathrm{sgn}(\iota) = 1$
- $\mathrm{sgn}(fg) = \mathrm{sgn}(f)\,\mathrm{sgn}(g)$
- $\mathrm{sgn}(f^{-1}) = \mathrm{sgn}(f)$

Example 27.3

What is the sign of f where

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 3 & 4 & 1 & 6 & 7 & 8 & 9 & 5 & 11 & 10 \end{pmatrix}$$

$$\begin{array}{ccc} 1 \to 2 \to 3 \to 4 \to 1 & 5 \to 6 \to 7 \to 8 \to 9 \to 5 & 10 \to 11 \to 10 \\ (1\,2\,3\,4) & (5\,6\,7\,8\,9) & (10\,11) \end{array}$$

So $f = (1\,2\,3\,4)(5\,6\,7\,8\,9)(10\,11)$. This means that $\operatorname{sgn}(f)$ is given by:

$$\begin{aligned} \operatorname{sgn}(f) &= \operatorname{sgn}((1\,2\,3\,4)(5\,6\,7\,8\,9)(10\,11)) \\ &= \operatorname{sgn}(1\,2\,3\,4)\operatorname{sgn}(5\,6\,7\,8\,9)\operatorname{sgn}(10\,11) \\ &= (-1)^{4-1}(-1)^{5-1}(-1^{2-1} \\ &= (-1)^3(-1)^4(-1)^1 \\ &= (-1)(1)(-1) \\ &= 1 \end{aligned}$$

If $\operatorname{sgn}(f) = 1$ we say f is even. If $\operatorname{sgn}(f) = -1$ we say f is odd.

We say $S \subseteq S_n$ generates S_n if every $f \in S_n$ can be written as a product of functions in S, that is for $f \in S_n$ $f = \alpha_1 \cdots \alpha_k$ where $\alpha_i \in S$.

Example 27.4

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$$

Can f be written as a product of 2-cycles or a product of 3-cycles?

By proposition 20.6 all permutations can be written as a product of 2-cycles. $f = (1\,2\,3)(4\,5) \implies \operatorname{sgn}(f) = -1$. If α is a 3-cycle then $\operatorname{sgn}(\alpha) = 1$ so $\prod_i \alpha_i \neq -1$ for any α, i, hence there is no way to write f as a product of 3-cycles.

28 Infinity

How do we compare the size of infinite sets?

If S and T are finite then $|S| = |T| \iff f : S \to T$ is a bijection. If $|S| = n$ then we can list the elements of S as $\{s_1, s_2, \ldots, s_n\}$ so $f : \{1, 2, \ldots, n\} \to S$ is a bijection and $f(i) = s_i$. We use this same idea as a motivation for comparing the size of infinite sets.

If A and B are sets we say they have the same cardinality if $\exists\, f : A \to B$ which is a bijection. We write $|A| = |B|$ or $A \sim B$.

If $A \sim \mathbb{N}$ we say A is countable. If A is infinite and $A \not\sim \mathbb{N}$ we say A is uncountable.

Claim If S is countable and T is finite then $S \backslash T$ is countable

Proof Let us assume $S \backslash T$ is finite

We can write $S = (S \backslash T) \cup T$. This means that S is the union of two finite sets which means S is finite. This is a contradiction so $S \backslash T$ must be infinite. By proposition 21.2 every infinite subset of $\mathbb{N}$ is countable. If S is countable there is a bijection $f : S \to \mathbb{N} \implies f(A) \subseteq \mathbb{N}$ where $f(A)$ is an infinite set. This means $A \sim f(A) \sim \mathbb{N}$. $\sim$ is an equivalence relation so $A \sim \mathbb{N}$ which means that A is countable if it is an infinite subset of S. This means that $S \backslash T$ is countable. ■

If S and $T \subseteq S$ are both countable we can't conclude anything about the size of $S \backslash T$

- Let $S = \mathbb{N}$ and $T = 2n$ where $n \in \mathbb{N}$. This means that T is the set of even numbers so $S \backslash T$ is the set of odd numbers. This is an infinite subset of $\mathbb{N}$ so $S \backslash T$ is countable
- Let $S = T$ then $S \backslash T = \emptyset$ is empty
- Let $T = S \backslash s_1$ this means $S \backslash T = s_1$ has finite size

So $S \backslash T$ can be countable, finite or empty

Example 28.1

Let $f : S \to \mathbb{N}$ and $g : \mathbb{N} \to S$ be functions where S is infinite. Which of the following properties imply that S is countable?

- f is injective – By proposition 21.4 (the injection lemma) if S is infinite and $f : S \to \mathbb{N}$ is injective then $S \sim \mathbb{N}$.
- f is surjective – This doesn't mean that S is countable. A counter example is $f : \mathbb{R}_+ \to \mathbb{N}$ and $f(x) = \lfloor x \rfloor$ is surjective but $\mathbb{R}_+$ is uncountable.
- g is injective – This doesn't mean that S is countable. A counter example is $g : \mathbb{N} \to \mathbb{R}$ and $g(n) = n$ is injective byt $\mathbb{R}$ is uncountable.
- g is surjective – This does mean that S is countable by the surjection lemma

Surjection Lemma

Claim If $g : \mathbb{N} \to S$ is surjective and S is infinite then S is countable.

Proof We need to show that there is an injective function $h : S \to \mathbb{N}$ then by the injective lemma $S \sim \mathbb{N}$ so S is countable.

Since g is surjective $\implies \forall\, x \in S\, \exists\, y \in \mathbb{N}$ such that $g(y) = x$

Set $h(x) = y \implies g(h(x)) = g(y) = x$

Suppose $h(x) = h(y)$ we need to show that this means $x = y$ so h is injective

By our coice of h we know $g(h(x)) = x \implies g(h(y)) = x \implies x = y$ so h is injective so by the injection lemma S is countable. ■

Better Injection Lemma

Claim If A is infinite and B is countable and there is an injection $f : A \to B$ then A is countable.

Proof Since $B \sim \mathbb{N}\, \exists\, g : B \to \mathbb{N}$ which is a bijection $\implies g \circ f : A \to \mathbb{N}$ is injective so by the injection lemma $A \sim \mathbb{N}$. ■

Example 28.2

Claim $\mathbb{N} \times \mathbb{N}$ is countable

Proof We need to show that there exists $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

One such function is $f(m, n) = 2^m 3^n$. Since prime factorisations are unique this is injective. Also $\mathbb{N} \times \mathbb{N}$ is infinite as there are a countably infinite number of options for n in $(1, n)$. So $\mathbb{N} \times \mathbb{N}$ is infinite and there is an injection to $\mathbb{N}$. This means that $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$. ■

This example extends using n distinct primes for $\mathbb{N}^n$

Example 28.3

Claim $\mathbb{Q} \sim \mathbb{N}$

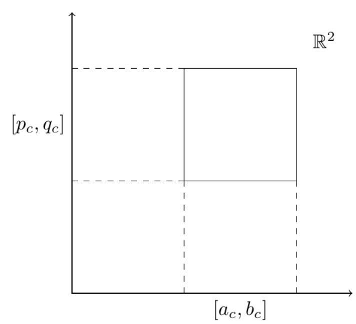
Proof We can write every element in $\mathbb{Q}$ uniquely as $\frac{m}{n}$ where $m, n \in \mathbb{Z}$, $n > 0$ and $\text{hcf}(m, n) = 1$. This means that there is an injection $f : \left(\frac{m}{n}\right) \to (m, n) \in \mathbb{N} \times \mathbb{N}$ so by the better injection lemma $\mathbb{Q} \sim \mathbb{N} \times \mathbb{N} \implies \mathbb{Q} \sim \mathbb{N}$. ■

29 Larger infinities

How big can a collection $\mathcal{C}$ of non-overlapping (disjoint) squares in $\mathbb{R}^2$ be?

Claim If $\mathcal{C}$ is infinite then it is countable

Proof Let $c \in \mathcal{C}$



$\forall\, c\, \exists\, a_c, b_c, p_c, q_c \in \mathbb{R}$ such that $c = [a_c, b_c] \times [p_c, q_c]$

We know $\exists\, r_c \in [a_c, b_c] \cap \mathbb{Q}$ and $\exists\, s_c \in [p_c, q_c] \cap \mathbb{Q}$

Let $f : \mathcal{C} \to \mathbb{Q}^2$ be $f(c) = (r_c, s_c)$

f is injective since if c and c' are distinct squares then $c \cap c = \emptyset$

So $f(c) \in c \implies f(c') \notin c' \implies f(c) \neq f(c')$

By the better injection lemman since $\mathbb{Q}^2$ is countable and $f : \mathcal{C} \to \mathbb{Q}^2$ is injective $\mathcal{C}$ is countable. ■

Claim If S is uncountable and T is countable then $S \backslash T$ is uncountable.

Lemma If $A \sim \mathbb{N} \sim B$ then $A \cup B \sim \mathbb{N}$

Proof $A \sim \mathbb{N} \implies \exists$ a bijection $f : A \to \mathbb{N}$

$B \sim \mathbb{N} \implies \exists$ a bijection $g : B \to \mathbb{N}$

Define $h : A \cup B \to \mathbb{N}^2$ as

$$h(x) = \begin{cases} (f(x), 1) & \text{if } x \in A \\ (g(x), 2) & \text{if } x \in B \backslash A \end{cases}$$

We need to show that h is injective and then by the better injection lemma $A \cup B \sim \mathbb{N}$.

Suppose $h(x) = h(y) \implies$ the right coordinates are the same. This means either $x, y \in A$ or $x, y \in B \backslash A$.
Case 1: $x, y \in A$

$$\implies h(x) = (f(x), 1) = (f(y), 1) = h(y) \implies f(x) = f(y) \implies x = y$$

since f is bijective.

Case 2: $x, y \in B \backslash A$

$$\implies h(x) = (g(x), 2) = (g(y), 2) = h(y) \implies g(x) = g(y) \implies x = y$$

since g is bijective.

h is injective in all cases so h is injective. Therefore by the better injection lemma $A \cup B \sim \mathbb{N}$.

Since $S = (S \backslash T) \cup T$ if $S \backslash T$ is countable then S is countable. This is a contradiction so $S \backslash T$ must be uncountable. ■

The power set of a set S is denoted $\mathcal{P}(S)$ and is the set of all subsets of S including $\emptyset$ and S.

$$\mathcal{P}(S) = \{A | A \subseteq S\}$$

The size of $\mathcal{P}(S)$ is $2^{|S|}$. If S is infinite then $|\mathcal{P}(S)|$ is a larger infinity by proposition 21.5. This means that repeatedly taking the power set results in an infinite number of different sized infinities, if S is countable then

$$|\mathbb{N}| \sim |S| < |\mathcal{P}(S)| < |\mathcal{P}(\mathcal{P}(S))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(S)))| < \cdots$$

How do we prove a set S is uncountable?

One way to do this is avoidance. Assume S is countable and as such can be listed as $S = \{s_1, s_2, \dots\}$ and then construct $s \in S$ such that $s \neq s_i \, \forall \, i$

Example 29.1

Let $S = \{(a_1, a_2, \dots) | a_i \in \{0, 1\}\}$

Claim $|S| > |N|$

Proof 1 Suppose $|S| = |\mathbb{N}|$, then

$$S = \{s_1, s_2, \dots\}$$
$$s_i = (a_1^i, a_2^i, \dots) \text{ where } a_j^i \in \{0, 1\}$$

Now define $s = (b_1, b_2, \dots)$ where

$$b_i = \begin{cases} 0 & \text{if } a_i^i = 1 \\ 1 & \text{if } a_i^i = 0 \end{cases}$$

This means that $s \neq s_i \, \forall \, i$ as if $\exists \, i$ such that $s = s_i$ then $b_i = a_i^i$ and this is a contradiction to the definition of s. This means $s \notin \{s_1, s_2, \dots\} = S$ this is a contradiction since $s \in S$ so $|S| > |\mathbb{N}|$. ■

Another method is to find a bijection $f : S \to T$ where $|T| > |\mathbb{N}|$

Proof 2 Let $\mathcal{P}(\mathbb{N}) = \{A \subseteq \mathbb{N}\}$ and let $f : \mathcal{P}(\mathbb{N}) \to S$ where $f(A) = (a_1^A, a_2^A, \dots)$ where

$$a_i^A = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases}$$

Since all A are unique this means that $A, B \in \mathcal{P}(N)$ must have at least one element different and therefore this corresponds to a difference in one of the a_i^A and b_i^B. This means f is injective. Since every subset of $\mathbb{N}$ constructs an countably long tuple we must get every possible infinitely long tuple under the constraints. This means $f : \mathcal{P}(\mathbb{N}) \to S$ is a bijection therefore $|S| = |\mathcal{P}(A)| > |\mathbb{N}|$. ■

A final method is to find an injection from an uncountable set to S

Proof 3 If there is an injection $f : A \to S$ then $|A| \leq |S| \implies$ if $|A| > |\mathbb{N}|$ then $|S| > |\mathbb{N}|$. ■

Claim $|\mathbb{R}| = |(0, \infty)|$

Proof Construct a bijection $f : (0, \infty) \to \mathbb{R}$

$$f(x) = x - \frac{1}{x}$$

is a bijection from $(0, \infty) \to \mathbb{R}$ hence $|(0, \infty)| = |\mathbb{R}|$. ■

30 Bounds

A set $A \subseteq \mathbb{R}$ is

- bounded above if $\exists\, M$ such that $\forall\, x \in Ax \leq M$
- bounded below if $\exists\, m$ such that $\forall\, x \in Ax \geq m$
- bounded if it is bounded above and below

Example 30.1

Claim $S = \{x|x^2 - x - 2 < 0\}$ is bounded

Proof Let $x \in S$ then $0 > x^2 - x - 2 = (x + 1)(x - 2)$

Exactly one of these must be negative and since $x + 1 > x - 2$ this means

$$x - 2 < 0 < x + 1 \implies -1 < x < 2\,\forall\, x \in S$$

so S is bounded. ■

Example 30.2

Claim $A = \{n + (-1)^n n|n \in \mathbb{N}\}$ is bounded below

Proof If $n + (-1)^n n \in A$ then $n + (-1)^n n \geq n - n = 0 \implies x \geq 0\,\forall\, x \in A \implies A$ is bounded below. ■

Claim $A = \{n + (-1)^n n|n \in \mathbb{N}\}$ is unbounded below

Proof We need to show the negation of the condition for bounded above

$$\forall\, M\, \exists\, x \in A \text{ such that } x > M$$

Let $M \in \mathbb{R}$. We need to show that there exists $x \in A$ such that $x > M$

Case 1 $M \leq 0$ then $x \geq 0 \geq M\,\forall\, x \in A$

Case 2 Suppose $M > 0$

We need to find $n + (-1)^n n > M$

Let $n = 2m$ then $n + (-1)^n n = 2m + 2m = 4m$

Let $m > \frac{M}{4}$ then $n + (-1)^n n = 4m > M$. ■

Least Upper Bound

If $A \subseteq \mathbb{R}$ is bounded above then the least upper bound of A (LUB(A)) is the number such that

- $\forall\, x \in Ax \leq \text{LUB}(A)$ (ie LUB(A) is an upper bound)
- If $y < \text{LUB}(A)$ then $\exists\, x \in A$ such that $x > y$ (ie y is not an upper bound)

Example 30.3

Claim If $A = \{x \in \mathbb{Q} | x < \sqrt{2}\}$ then $\mathrm{LUB}(A) = \sqrt{2}$

Proof $x < \sqrt{2} \,\forall\, x \in A$ by definition. This means that $\sqrt{2}$ is an upper bound.

Let $y < \sqrt{2}$. We have previously shown that $\forall\, a < b \,\exists\, r \in \mathbb{Q} \cap (a, b)$

This means $\exists\, x \in \mathbb{Q} \cap (y, \sqrt{2}) \implies x \in A$ and $x > y$ hence the upper bound can't be less than $\sqrt{2}$. This means that $\mathrm{LUB}(A) = \sqrt{2}$. ■

Example 30.4

Claim If $A = \{1 - n^{(-1)^n} | n \in \mathbb{N}$ then $\mathrm{LUB}(A) = 1$

Proof $1 - n^{(-1)^n} < 1 \,\forall\, n > 0$ therefore 1 is an upper bound.

Let $y < 1$, we need to show $\exists\, x \in A$ such that $x > y$

Let $n = 2m + 1$ then $1 - n^{(-1)^n} = 1 - \frac{1}{2m+1}$

So we need to find m such that

$$\begin{aligned} 1 - \frac{1}{2m+1} > y &\iff \frac{1}{2m+1} < 1 - y \\ &\iff 2m + 1 > \frac{1}{1-y} \\ &\iff m > \left(\frac{1}{1-y} - 1\right)/2 \end{aligned}$$

Let $m = \left\lceil \left(\frac{1}{y-1} - 1\right)/2 \right\rceil$

Then $A \ni 1 - (2m+1)^{(-1)^{2m+1}} > y$ so $1 = \mathrm{LUB}(A)$. ■

Which of the following are true?

1. If $\alpha > 0$ and $\alpha A = \{\alpha x | x \in A\}$ then $\mathrm{LUB}(\alpha A) = \alpha \,\mathrm{LUB}(A)$
2. If $-A = \{-x | x \in A\}$ then $\mathrm{LUB}(-A) = -\mathrm{LUB}(A)$
3. $\mathrm{LUB}(A) = \max(A)$
4. If $c \in \mathbb{R}$ and $c + A = \{c + x | x \in A\}$ then $\mathrm{LUB}(c + A) = c + \mathrm{LUB}(A)$

1. This is true. To show $\mathrm{LUB}(\alpha A) = \alpha \,\mathrm{LUB}(A)$ we need to show
   (a) $x \leq \alpha \,\mathrm{LUB}(A) \,\forall\, x \in \alpha A$

   Let $x \in \alpha A \implies x = \alpha a$ for some $a \in A \implies a \leq \mathrm{LUB}(A) \implies x = \alpha a \leq \alpha \,\mathrm{LUB}(A)$

   (b) $\forall\, y < \alpha \,\mathrm{LUB}(A) E x \in \alpha A$ such that $x > y$

   Let $y < \alpha \,\mathrm{LUB}(A)$

$$\begin{aligned} &\implies \frac{y}{\alpha} < \mathrm{LUB}(A) \\ &\implies \exists\, a \in A \text{ such that } a > \frac{y}{\alpha} \\ &\implies y < \frac{\alpha a}{x} \in \alpha A \qquad ■ \end{aligned}$$

2. This is false. This can be shown by proof by counterexample

   Let $A = (-1, 2) \implies \mathrm{LUB}(A) = 2$

   $-A = (-2, 1) \implies \mathrm{LUB}(A) = 1 \neq -2$ as we would expect if $\mathrm{LUB}(-A) = -\mathrm{LUB}(A)$

3. This is false. This can be shown by proof by counterexample

   For example $\mathrm{LUB}(0, 1) = 1$ and $\max(0, 1)$ doesn't exist.

   Claim $\max(0, 1)$ doesn't exist

   Proof Suppose $m = \max(0, 1) \in (0, 1)$

   $\implies m < 1$ but as we have shown there is always a rational between any two rationals, in this case $m < \frac{m+1}{2} \in (0, 1)$. This is a contradiction since $m = \max(0, 1)$ so $\max(0, 1)$ doesn't exist. ■

4. This is true. To show $\mathrm{LUB}(c + A) = c + \mathrm{LUB}(A)$ we need to show

   (a) $x \leq c + \mathrm{LUB}(A) \, \forall \, x \in c + A$

   Let $x \in c + A \implies x = c + a$ for some $a \in A \implies a \leq \mathrm{LUB}(A) \implies x = c + a \leq c + \mathrm{LUB}(A)$

   (b) $\forall \, y < c + \mathrm{LUB}(A) \, \exists \, x \in c + A$ such that $x > y$

   Let $y < c + \mathrm{LUB}(A)$

$$
\begin{aligned}
&\implies y - c < \mathrm{LUB}(A) \\
&\implies \exists \, a \in A \text{ such that } a > y - c \\
&\implies y < \underset{x}{\underline{a + c}} \in A + c \qquad \blacksquare
\end{aligned}
$$

Completeness axion

The completeness axiom states

> Every non-empty set of real numbers with an upper bound must have a least upper bound in $\mathbb{R}$