# Physics 1B Lecture Notes

Willoughby Seago

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# Part I

# Waves

## Lecture 3

### Propagation of errors

If we measure a value $x$ then we call the uncertainty in this measurement $\delta x$. We give the result as $x \pm \delta x$. If $y = f(x)$ then by plotting the function evaluating at $x, x + \delta x$ and $x - \delta x$ it can be shown that an there is an error range either side of the calculated value of $y$. We assume that the error on both sides of $y$ is the same and we call it $\delta y$. We assume that the error is small enough that the function can be approximated as a straight line

$$\therefore \frac{\delta y}{\delta x} = \text{gradient} = \frac{dy}{dx}$$

$$\implies \delta y = \left.\frac{dy}{dx}\right|_x \delta x$$

$$\implies y \pm \delta y = f(x) \pm \left.\frac{dy}{dx}\right|_x \delta x$$

For a multivariable function such as $w = f(x, y, z)$ where $x, y$ and $z$ have uncorrelated errors (errors that don't effect eachother) we write the contribution of $x$ to the error of $w$ as $\delta w_x$. It is calculated as:

$$\delta w_x = \left.\frac{\partial w}{\partial x}\right|_x \delta x$$

We combine the errors by adding errors in quadrature. That is each error is squared and then added together and squarerooted:

$$\delta w = \sqrt{\delta w_x^2 + \delta w_y^2 + \delta w_z^2}$$

## Lecture 4

A wave is anything that satisfies the wave equation. It is a function of poition in space $x$ and time $t$. The displacement of part of the medium distance $x$ from the source at time $t$ is given by $y(x, t)$ where function $y$ is a solution to the wave equation:

$$\boxed{\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}}$$

This is an idealised equation. In reality only light in a vacuum follows this. One solution to the wave equation is:

$$y(x, t) = f(x \pm vt)$$

A particularly common form is:

$$y(x, t) = A sin(kx \pm \omega t + \varphi)$$

The $\pm$ represents the two directions the waves can go in. If it is a $+$ then the wave travels in the negative $x$ direction. If it is a $-$ then the wave travels in the positive $x$ direction. $A$ is the amplitude or maximum displacement of the wave. $k$ is how much the wave changes with space and $\omega$ is how much the wave changes with time. $\varphi$ is the phase difference. Some common wave properties can be calculated as follows:

$$\lambda = \frac{2\pi}{k}, \quad T = \frac{2\pi}{\omega} \quad \text{and} \quad f = \frac{\omega}{2\pi} = \frac{1}{T}$$

The units of $f$ and $\omega$ are $\mathrm{s}^{-1}$ and $\mathrm{rad\,s}^{-1}$ respectively, both of these are the same as $\mathrm{s}^{-1}$ and Hz. The units of $k$ are $\mathrm{rad\,m}^{-1}$ which is the same as $\mathrm{m}^{-1}$. The units of $x$ and $y$ are m and the units of $t$ are s.

A node is anywhere where $y(x,t)=0$. For the example with the sine function since $\sin t = 0 \,\forall\, t \in \{n\pi | n \in \mathbb{Z}\}$. If the $n^{\text{th}}$ node is at $x_n$ then:

$$\begin{aligned} kx_n - \omega t &= n\pi \\ kx_0 - \omega t &= 0 \\ kx_0 &= \omega t \\ \frac{x_0}{t} &= \frac{\omega}{k} = v \\ v &= \frac{2\pi f}{\frac{2\pi}{\lambda}} = f\lambda \end{aligned}$$

The wave speed depends on the medium not the source:

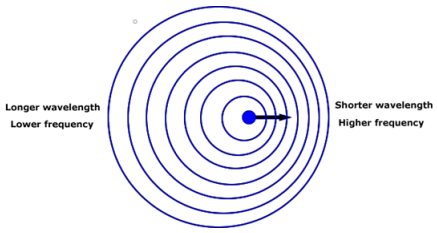
$$v^2 = \frac{\text{stiffness}}{\text{inertia}}$$

The wave speed can also depend on frequency. If frequency doesn't effect wavespeed then we say the wave is dispersionless.

The speed that any one part of the medium is moving $u$, is given by:

$$u = \frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(A\sin(kx - \omega t + \varphi)) = -\omega A\cos(kx - \omega t + \varphi)$$

## Doppler shift




The wavespeed is $v$ and the source is moving at speed $v_s$. In this equation $n'$ means $n$ after the doppler shift is taken into account. At $t=0$ the first peak of the wave is emitted. At time $t=T$ the second peak is emitted. In the time between peaks the first peak has moved distance $x_1 = vT$ and the source has moved distance $x_2 = v_sT$. The distance between the two peaks is:

$$\lambda' = x_1 - x_2 = vT - v_sT = T(v - v_s)$$

The original wavelength $\lambda$:

$$\lambda = vT \implies T = \frac{\lambda}{v} \implies \lambda' = \lambda\frac{v - v_s}{v}$$

Finally substituting for $f$:

$$v = f\lambda \implies \lambda = \frac{v}{f} \implies f' = f\frac{v}{v - v_s}$$

As $v_s \to v$ we get a sonic boom. If both the source and detector are moving and the detector speed is $v_d$ then the same logic can lead to the equation:

$$f' = f\left(\frac{v \pm v_d}{v \pm v_s}\right)$$

The $\pm$ is just dependant on the relative directions of the speeds, just think about whether the frequency should increase (source and detector coming together) or decrease (source and detector seperating).

# Lecture 5

Linearity is a property of some systems. If $x$ and $y$ are solutions to the system then $x + y$ is also a solution to the system. Differentiation is a linear operator, that is:

$$\frac{d^2}{dx^2}(f(x) + g(x) = f'(x) + g'(x)$$

The wave equation is also linear so if $y_1(x,t)$ and $y_2(x,t)$ are solutions to the wave equation so is $y_1 + y_2$

If $n$ waves are present then the resulting wave is given by:

$$y(x,t) = y_1(x,t) + y_2(x,t) + \cdots y_n(x,t) = \sum_{i=1}^{n} y_i(x,t)$$

You must take into account the phase difference when you do this calculation.

## Example 5.1

Two waves that are the same but out of phase:

$$y_1(x,t) = A\sin(kx - \omega t), \quad \& \quad y_2 = A\sin(kx - \omega t + \varphi)$$

$$\begin{aligned} y &= y_1 + y_2 \\ &= A\sin(kx - \omega t) + A\sin(kx - \omega t + \varphi) \\ &= A(\sin(kx - \omega t) + \sin(kx - \omega t + \varphi) \end{aligned}$$

There is a useful trig identity for this: $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$$\begin{aligned} y &= 2A\sin\left(\frac{kx - \omega t + kx - \omega t + \varphi}{2}\right)\cos\left(\frac{kx - \omega t - kx + \omega t - \varphi}{2}\right) \\ &= 2A\sin\left(kx - \omega t + \frac{\varphi}{2}\right)\cos\left(\frac{\varphi}{2}\right) \\ &= 2A\cos\left(\frac{\varphi}{2}\right)\sin\left(kx - \omega t + \frac{\varphi}{2}\right) \end{aligned}$$

The new wave that is formed now has amplitude $2A\cos\left(\frac{\varphi}{2}\right)$ and is $\frac{\varphi}{2}$ rad out of phase.

## Example 5.2

Two waves that are the same but traveling in opposite direction:

$$y_1(x,t) = A\sin(kx - \omega t), \quad \& \quad y_2 = A\sin(kx + \omega t)$$

$$\begin{aligned} y &= y_1 + y_2 \\ &= A\sin(kx - \omega t) + A\sin(kx + \omega t) \\ &= A(\sin(kx - \omega t) + \sin(kx + \omega t) \\ &= 2A\sin\left(\frac{kx - \omega t + kx_\omega t}{2}\right)\cos\left(\frac{kx - \omega t - kx + \omega t}{2}\right) \\ &= 2A\sin(kx)\cos(\omega t) \end{aligned}$$

$\sin(kx)$ is spatial oscillation. $cos(\omega t)$ is time oscillation.

This is what happens to produce standing waves. A standing wave doesn't move, that means that no energy is transfered. A node is anywhere where $y(x,t) = 0 \forall\, x, t$, nodes don't move.

To solve a differential equation you must know the boundary conditions eg. position or speed of the wave at a given time or what is causing the wave.

### Example 5.3

A standing wave is set up in a string length $L$ fixed at both ends. What are the boundary conditions of the system?

Since the ends of the string are at positions $x = 0, L$ and they are fixed ($y = 0$) we know that $y(0,t) = y(L,t) = 0 \forall\, t$.

We know that $y(x,t) = 2A\sin(kx)\cos(\omega t)$. We need $\sin(kx) = 0$ for $x = 0, L \implies kx = n\pi\, n \in \mathbb{Z}$, $n \neq 0 \implies KL = n\pi$. Substituting in $k = \frac{2\pi}{\lambda}$ gives $\lambda = \frac{2L}{n}$

The longest wavelength ($n = 1$) is the "fundamental". All shorter wavelengths are "harmonics". Physical systems usually give a mix of harmonics. Any pattern on the string can be made from a combination of sine nodes.

### Example 5.4

Two waves with different frequencies:

When this occurs the beats phenomonem can be observed, this is the reason that two slightly out of tune notes have a slightly oscillating pitch.

$$y_1(x,t) = A\sin(kx - \omega_1 t), \quad \& \quad y_2 = A\sin(kx + \omega\_2t)$$

$$\begin{aligned}
y &= y_1 + y_2 \\
&= A\sin(kx - \omega_1 t) + A\sin(kx + \omega_2t) \\
&= A(\sin(kx - \omega_1 t) + \sin(kx - \omega_2 t)) \\
&= 2A\sin\left(\frac{kx - \omega_1 t + kx - \omega_2 t}{2}\right)\cos\left(\frac{kx - \omega_1 t - kx + \omega_2 t}{2}\right) \\
&= 2A\sin\left(kx - \frac{\omega_1 + \omega_2}{2}\right)\cos\left(\frac{\omega_2 - \omega_1}{2}t\right) \\
&= 2A\sin(kx - \bar{\omega})\cos\left(\frac{\Delta\omega}{2}t\right)
\end{aligned}$$

Where the mean angualr frequency $\bar{\omega} = \frac{\omega_1 + \omega_2}{2}$ and the difference in angular frequencies $\Delta\omega = \omega_2 - \omega_1$

## Lecture 6

James Clerk Maxwell showed that light was a wave that followed the following vector calculus:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\boldsymbol{B}$$

$$\nabla \times \boldsymbol{B} = \mu_0(\boldsymbol{J} + \varepsilon_0)$$

From this we can derive wave equations for $\boldsymbol{E}$ and $\boldsymbol{B}$. In 1D the equations are:

$$\frac{\partial^2 E}{\partial t^2} = \frac{1}{\mu_0\varepsilon_0}\frac{\partial^2 E}{\partial x^2}$$

$$\frac{\partial^2 B}{\partial t^2} = \frac{1}{\mu_0\varepsilon_0}\frac{\partial^2 B}{\partial x^2}$$

From this equation we get:

$$v = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 299\,792\,458\,\mathrm{m\,s^{-1}}$$

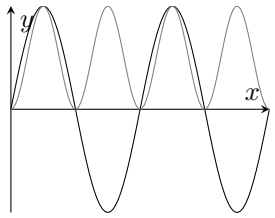
A light wave is made of perpendicular waves in the electromagnetic field. The light can be polarised and it can rotate along its axis of travel but the waves always remain perpendicular.

Thomas Young's double slit experiment showed light interfering with itself which is wave behaviour.

Light is not a wave or a particle. It is a quantum object with properties of a wave and a particle. There is no intuitive picture analagous to something that we can see.

We can't see the oscillations of light with time as our eyes don't react fast enough. What we see is the time average of the square of the wave amplitude. That is the time average of the energy of the wave ($E \propto A^2$).

## Light intensity of a sinusoidal wave




This graph shows $y = \sin x$ and $y = \sin^2 x$.

The intensity $I$ is given by $I = \langle |y|^2 \rangle_t$ there $\langle p \rangle_q$ is the average of $p$ as $q$ varies.

$$I = \frac{1}{t} \int_0^T y^2 \, dt$$

Here the integral is like taking the mean of a continuous function rather than the discrete data you would usually take the mean of.

$$y = A \sin(\omega t)$$

$$y^2 = A^2 \sin^2(\omega t)$$

$$I = \frac{1}{T} \in_0^T A^2 \sin^2(\omega t) \, dt$$

$$I = \frac{A^2}{T} \left[ \frac{t}{2} - \frac{1}{4} \sin(2\omega t) \right]_0^T$$

At $t = nT$ for $n \in \mathbb{Z}$ $\sin(2\omega n T) = \sin(2m\pi) = 0$ for $m \in \mathbb{Z}$

$$I = \frac{A^2}{2}$$

$$I \propto A^2$$

A prisim can split white light showing that it is composed of all other frequencies of visible (and some non-visible) light.

Real instruments (including our eyes) are sensitive to specific wavelength ranges

$$R = \int s(\lambda) I(\lambda) \, d\lambda$$

Where $R$ is the recieved signal, $s(\lambda)$ is the sensitivity function and $I(\lambda)$ is the incoming intensity.

### Dispersion

A pulse of light is made of lots of frequencies of wave together. In free space (a vacuum) all of these travel at the same speed. In any other medium they all travel at different speeds.

The peaks of each frequency travel at a different speed to the whole pacet. We call the speed of the individual frequencies the phase velocity, $v_p$ and the speed of the whole pacet the qroup velocity, $v_g$.

In free space light travels at $c$. In a mediu with refractive index $n$ light travels at $\frac{c}{n}$ $(n > 1)$. In this case $\omega = ck$ isn't always true. Media have a dispersion relation $\omega(k)$:

$$v_p = \frac{\omega}{k} \quad \& \quad v_g = \frac{\partial \omega}{\partial k} = \frac{\partial \omega(k)}{\partial k}$$

Often but not always for light $v_p v_g = c^2$ and $v_p \geq c, v_g \leq c$. It is ok for $v_p > c$ as no information is being transferred.

In a plasma (ionised gas) free electrons interact with the electromagnetic field giving:

$$\omega = \frac{ck}{\sqrt{1 - \frac{\omega_0^2}{\omega^2}}}$$

$$v_p = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \frac{\omega_0^2}{\omega^2}}}$$

Note that, for some values of $\omega$, $v_p \notin \mathbb{R}$ This is what happens in opaque objects.

## Lecture 7

$$v_g = \frac{\partial \omega}{\partial k}$$

$$\omega = \frac{ck}{\sqrt{1 - \frac{\omega_0^2}{\omega^2}}}$$

$$\omega^2 = \frac{c^2 k^2}{1 - \frac{\omega_0^2}{\omega^2}}$$

$$\omega^2 \left(1 - \frac{\omega_0^2}{\omega^2}\right) = c^2 k^2$$

$$\omega^2 - \omega_0^2 = c^2 k^2$$

$$\omega^2 = \omega_0^2 + c^2 k^2$$

$$\omega = \sqrt{\omega_0^2 + c^2 k^2}$$

$$v_g = \frac{\partial}{\partial k} \left(\omega_0^2 + c^2 k^2\right)^{-\frac{1}{2}}$$

$$= \frac{c^2 k}{\sqrt{\omega_0^2 + c^2 k^2}}$$

### Geometric optics

Geometric optics is an approximation of light to linear rays travelling in the same direction as the wave.

When light enters a medium it changes speed so that (group) speed is equal to $v = \frac{c}{n}$ where $n$ is the refractive index of the material. $n > 1$ for all media. $n = 1$ for a vacuum and $n \approx 1$ for air. When light travels across a boundary it can either reflect or refract. Usually it does a combination of both. It depends on the properties of the medium, the light path and the frequency as to what occurs.

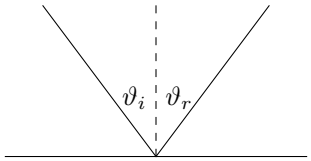
## Reflection mechanis

When a light ray is incident on a surface it causes free electrons accelerate which causes them to emit EM radiation which cancels with some waves and not others resulting in only the reflected wave leaving the medium. If the new medium has a higher refractive index then the reflected wave has a phase shift of $\pi$ rad. If the new medium has a lower refractive index then there is no phase change.

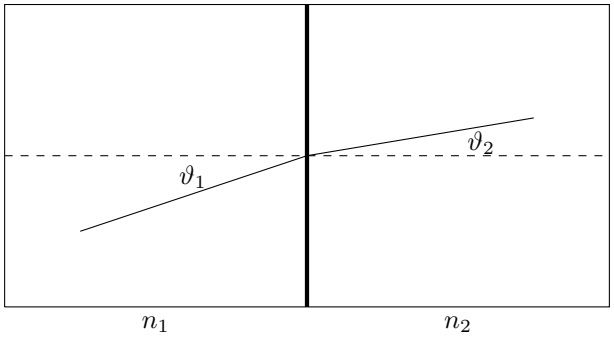
Specular reflection occurs on smooth surfaces with lots of free electrons (metals). Specular reflections preserve the image. Disperse reflections occur in all other cases, usually because at a small enough scale the surface is not flat.

## Law of reflection

- The incident ray, reflected ray and surface normal are all in the same plane.
- The angle of incident is the same as the angle of refelection $\vartheta_i = \vartheta_r$




## Refraction




When light passes from an optically less dense to optically more dense material it bends towards the normal and vice versa.

## Snell's law

$$n_1 \sin \vartheta_1 = n_2 \sin \vartheta_2$$

## Fermat's principle

The path taken by light minimise the travel time. (Note that it is a local minimum not a global minimum)

If we define path length $l = nd$ where $d$ is the distance travelled and $n$ is the refractive index then if we have several materials we get:

$$l = \sum_i d_i n_i$$

and for a continuously changing material:

$$l = \int_{\text{path}} n \, ds$$

where $ds$ is an element of the path.

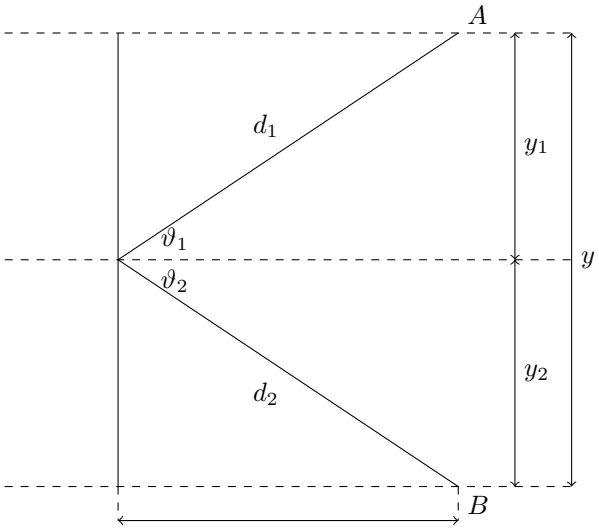
The speed of light is given by $v = \frac{c}{n}$ so the time taken to travel $t$ is given by:

$$t = \frac{d}{v} = \frac{d}{\frac{c}{n}} = \frac{nd}{c} = \frac{l}{c}$$

From this we can see that $t \propto l$ so Fermat's principle can be restated as light takes the shortest local path.

Note that for both definitions the paths are local minima and make physical sense.

## Derivation of the law of reflection




There are two local minima in this. One has path length $y$ and is going from $A$ straight to $B$ the other is to reflect off of the surface. To calculate the two angles $\vartheta_1$ and $\vartheta_2$ we must find the minimum distance for the light to travel:

Total distance $D = d_1 + d_2$ By pythagoras we get:

$$d_1^2 = y_1^2 + x^2 \quad \& \quad d_2^2 = y_2^2 + x^2$$

At the local minimum $\frac{dD}{dy_1} = 0$:

$$\frac{dD}{dy_1} = \frac{1}{2} \cdot 2y_1(y_1^2 + x^2)^{\frac{1}{2}} + \frac{1}{2} \cdot 2y_2(y_2^2 + x^2)^{\frac{1}{2}} \cdot \frac{dy_2}{dy_1}$$

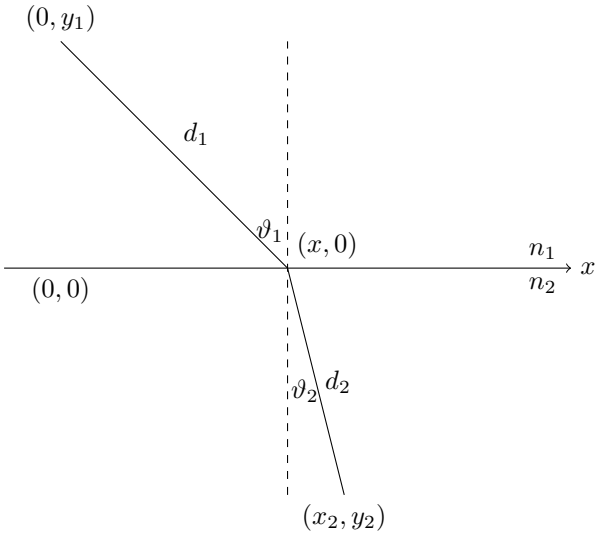
$$y = y_1 + y_2 \implies y_2 = y - y_1$$

$$\frac{dy_2}{dy_1} = -1$$

$$\frac{dD}{dy_1} = \frac{y_1}{\sqrt{y_1^2 + x^2}} - \frac{y - y_1}{\sqrt{(y - y_1)^2 + x^2}} = 0$$

This is true when $y_1 = y - y_1 \implies y_1 = \frac{1}{2}y \implies y_1 = y_2$ Since the lengths are equal and all angles are less than $2\pi$ the angles $\vartheta_1$ and $\vartheta_2$ must be the same.

### Derivation of Snell's law




First by the definition of sin we get:

$$\sin\vartheta_1 = \frac{x}{\sqrt{x^2+y_1^2}} \quad \& \quad \sin\vartheta_2 = \frac{x_2-x}{\sqrt{(x-x_2)^2+y_2^2}}$$

$$l = n_1 d_1 + n_2 d_2$$

$$= n_1\sqrt{(x^2+y_1^2)} + n_2\sqrt{(x-x_2)^2+y_2^2}$$

$$\frac{dl}{dx} = 0 = \frac{2xn_1}{\sqrt{x^2+y_1^2}} + \frac{2(x-x_2)n_2}{\sqrt{(x-x_2)^2+y_2^2}}$$

$$0 = n_1\sin\vartheta_1 - n_2\sin\vartheta_2$$

$$n_1\sin\vartheta_1 = n_2\sin\vartheta_2$$

Fermat's principle can be derived from Huygen's principle that each wave front acts as a new source of waves in all directions.

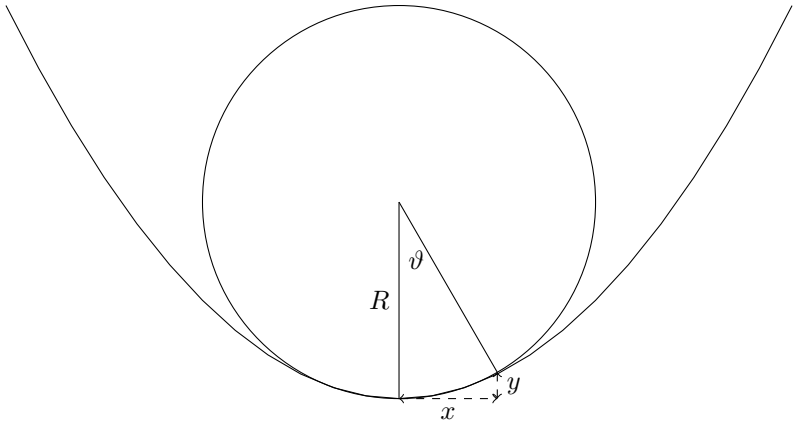
## Lecture 8

The radius of curvature of a curve is an approximation of how curved it is given by fitting a circle to part of the curve and taking the radius of the circle.

Expand the function as a taylor series:

$$y = y_0 + a(x-x_0) + b(x-x_0)^2 + \mathcal{O}(x^3)$$

The $x^0$ term gives position, the $x^1$ term gives orientation and the $x^2$ term gives the curvature.




$$\sin\vartheta \approx \vartheta \quad \& \quad \cos\vartheta \approx 1 - \vartheta^2$$

$$x = R\sin\vartheta \approx R\vartheta \quad \& \quad y = R(1-\cos\vartheta) \approx R(1-(1-\vartheta^2)) = R\vartheta^2$$

$$x^2 \approx R^2\vartheta^2 \implies \vartheta^2 \approx \frac{x^2}{R^2} \implies y = \frac{x^2}{R}$$

Equating coefficients of $x^2$ with the taylor expansion:

$$bx^2 \approx \frac{1}{R}x^2 \implies b \approx \frac{1}{R} \implies R \approx \frac{1}{b}$$

The same can be done in 3D with a sphere to get the radius of curvature of a surface. The radius of curvature has a sign which defines which side of the line the circle is on.

## Ray diagrams

It is convention to have the rays go from left to right and call this direction the positive $z$ direction, the $z$ axis is then the same as the optical axis. A lens has two boundaries that a ray must cross with two radii of curvature. The first surface the ray meets is $R_1$ and the second surface is $R_2$. By convention a surface that looks like "(" has positive radius of curvature and a surface that looks like ")" has negative radius of curvature.

## Focal length

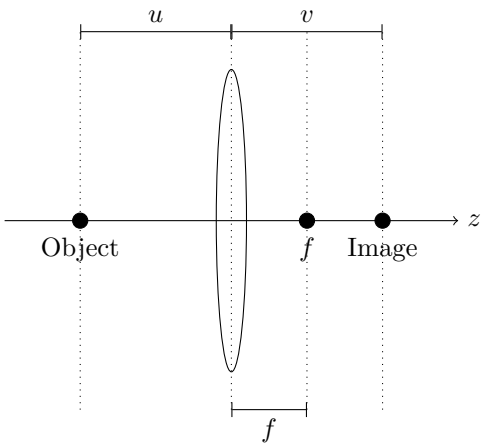
- Converging lenses have focal length $f$ which is the displacement from the center of the lens to the plane upon which rays that started parallel will come together. $f > 0$
- Diverging lenses have focal length $f$ which is the displacement from the center of the lens to the plane from where it looks like rays that started parallel emerged when looked at through the lens. $f < 0$

The stronger a lens is the smaller $|f|$ is. We define power of a lens as $p = \frac{1}{f}$

The lens maker's formula:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

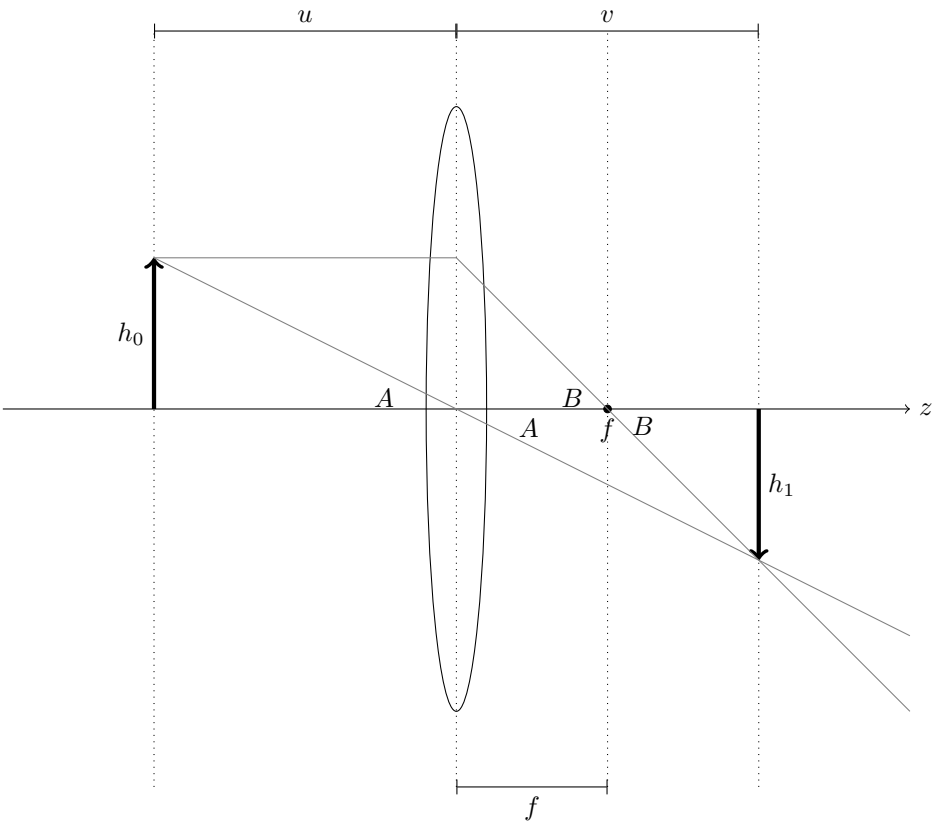
A real image is one that can be projected onto a screen and sensed by a detector, it occurs where the rays converge. A virtual image can't be projected on a screen but is where it looks like the light comes from if you view it after it went through the lens.




By convention when they are the side of the lens that they are in the diagram $f, u, v > 0$

Rules for thin lens, paraxial (small angle) approximation:

- Rays through the centre of the lens don't change direction
- Rays that come in parallel end on the focal plane
- Rays from the focal plane end up in parallel




By convention an inverted image has negative height

$$\frac{h_0}{u} = \tan A = \frac{-h_1}{v}$$

Magnification is the ratio of the image height to the object height:

$$M = \frac{h_1}{h_2} = -\frac{v}{u}$$

$$\begin{aligned}
\frac{h_0}{f} = \tan B = \frac{-h_1}{v-f} \\
\implies -\frac{h_1}{h_0} = \frac{v-f}{f} \\
\implies \frac{v}{u} = \frac{v-f}{f} \\
= \frac{v}{f} - \frac{f}{f} \\
= \frac{v}{f} - 1 \\
\implies \frac{v}{u} = \frac{v}{f} - 1 \\
\implies \frac{1}{u} = \frac{1}{f} - \frac{1}{v} \\
\implies \frac{1}{f} = \frac{1}{u} + \frac{1}{v}
\end{aligned}$$

This is known as the Gaussian lens equation.

If $v > 0$ then the image is real. If $v < 0$ then the image is virtual. For $v > 0$ we need $f > 0$ and $u > f$. Since when this occurs $u, v > 0$ $M = -\frac{v}{u} < 0$ so the image is inverted. For $v < 0$ either $f > 0$ and $u < f$, so $M > 1$, so the image is upright and magnified, or $f < 0$ so $M > 0$ so the image is upright.

# Lecture 9

## Multiple lenses

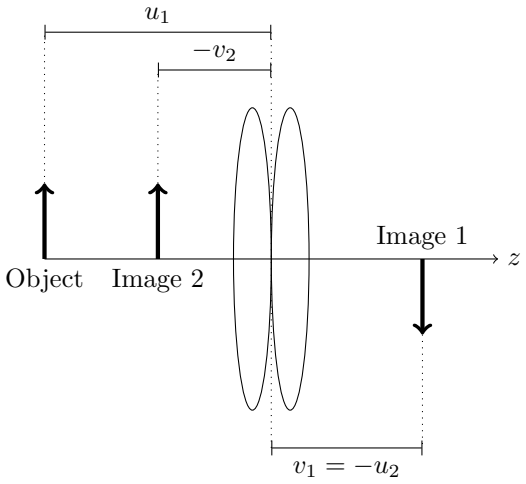
The image of the first lens becomes the object for the second lens. This rule still applies for virtual images, negative lenses and images on the opposite side of the lens.

From the last lecture we have $M = -\frac{v}{u}$. The height of the 1st image is $h_1 = M_1 h_0$. The height of the 2nd image is $h_2 = M_2 h_1$. The total magnification is:

$$M_{tot} = \frac{h_2}{h_0} = \frac{M_2 h_1}{\frac{h_1}{M_1}} = M_1 M_2$$

This is true for all $M$ including $M < 0$ and $|M| < 1$.

## Lenses in contact




The gaussian lens equation:

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

Applying the gaussian lens equation for lens 1:

$$\frac{1}{u_1} + \frac{1}{v_1} = \frac{1}{f_1}$$

$$\implies \frac{1}{v_1} = \frac{1}{f_1} - \frac{1}{u_1} \tag{9.1}$$

Applying the gaussian lens equation for lens 2:

$$\frac{1}{u_2} + \frac{1}{v_2} = \frac{1}{f_2}$$

$$\implies \frac{1}{v_2} = \frac{1}{f_2} - \frac{1}{u_2}$$

$$v_1 = -u_2 \implies \frac{1}{v_2} = \frac{1}{f_2} + \frac{1}{v_1}$$

Substituting in (9.1) gives:

$$\frac{1}{v_2} = \frac{1}{f_2} + \frac{1}{f_1} - \frac{1}{u_1}$$

$$\frac{1}{v_2} + \frac{1}{u_1} = \frac{1}{f_2} + \frac{1}{f_1}$$

Let $F$ be the effective focus distance of the system:

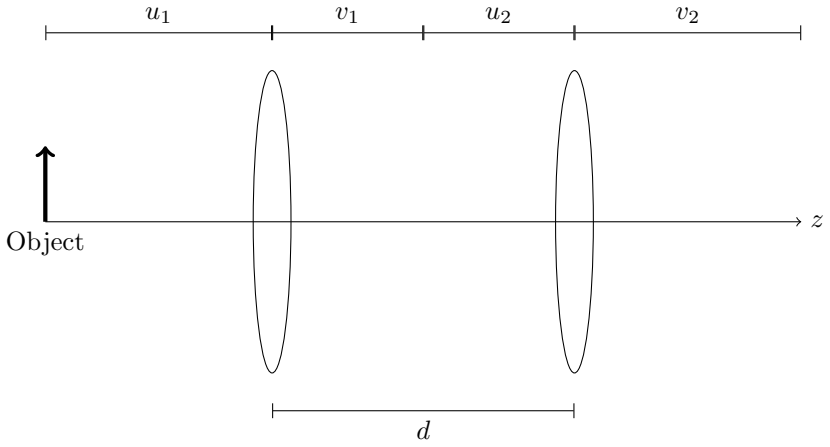
$$\frac{1}{F} = \frac{1}{v} + \frac{1}{u}$$

$$\implies \frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{v_2} + \frac{1}{u_1}$$

So the whole system is the same as a single lens with effective focal length $F$, object distance $u_1$ and image distance $v_2$. This shows that focal lengths add in parralel for lenses in contact. This can also be written as powers adding in series as $P = p_1 + p_2$

## Compound microscopes

Compound microscopes are made from two converging lenses. Image 1 is real and between the two lenses. Image 2 is virtual and further away than the object. You can see the virtual image as you eye is another lens that makes the image real again. The total magnification is given by $M = M_1 M_2$.




Applying the gaussian lens equation to both lenses:

$$\frac{1}{v_1} = \frac{1}{f_1} - \frac{1}{u_1} \qquad \frac{1}{v_2} = \frac{1}{f_2} - \frac{1}{u_2} = \frac{1}{f_2} - \frac{1}{d - v_1}$$

We want the rays from the second image to be parallel as this makes the eye relax so the microscope is easier to use. This means that $v_2 \to \infty$:

$$\lim_{v_2 \to \infty} \frac{1}{u_2} + \frac{1}{v_2} = \frac{1}{u_2} = \frac{1}{f_2} \implies u_2 = f_2 \tag{9.2}$$

It can be seen from the diagram that:

$$u_2 + v_1 = d \implies v_1 = d - u_2$$

If we subsititute (9.2) into this we get:

$$v_1 = d - f_2 \tag{9.3}$$

If we substitute (9.3) into the gaussian lens equation for lens 1 we get:

$$\frac{1}{u_1} = \frac{f_1}{-} \frac{d - f_2}{=} \frac{1}{f_1} + \frac{1}{f_2 - d}$$

Since we want the waves to end up parallel then we want the object at the focal distance from the lens so $u_1 = F$ so we get that:

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2 - d}$$

If $d = 0$ then we get the equation for lenses in contact.

## Telescopes

When and object is viewed through a telescope the object is far enough away that we can assume the light rays are parallel. This means that $u_1 \to \infty$. We also want the rays to come out parallel so $v_2 \to \infty$

The simplest telescope is the gallilean telescope. It is made of a converging and a diverging lens which are set up such that they have there focal point in the same spot. The distance between the lenses is $d$. The gaussian lens equation for the first lens as $u_1 \to \infty$:

$$\lim_{u_1 \to \infty} \frac{1}{u_1} + \frac{1}{v_1} = \frac{1}{v_1} = \frac{f_1}{\implies} v_1 = f_1$$

The gaussian lens equation for the second lens as $v_2 \to \infty$:

$$\lim_{v_2 \to \infty} \frac{1}{u_2} + \frac{1}{v_2} = \frac{1}{u_2} = \frac{f_2}{\implies} u_2 = f_2$$

The magnification is given by:

$$M = -\frac{v_2}{u_1} = \lim_{u_1, v_2 \to \infty} = -\frac{\infty}{\infty}$$

This is not a valid answer ☹

# Lecture 10

If instead we consider the angle $\vartheta_o$ between the optical axis and the incoming ray which goes through the center of the objective lens. This angle is also made by the ray and the optical axis on the other side of the lens. This forms a right angle triangle between the center of the lens, and the top and bottom of the image. If the image height is $h$ and the focal length is $f_o$ then we get the relationship:

$$\tan \vartheta_o = \frac{h}{f_o} \approx \vartheta_o \tag{10.1}$$

If we now consider the angle $\vartheta_e$ between the optical axis and the ray passing through the center of the eyepiece lens. This angle is part of the right angle triangle fomed between the center of the eyepiece lens, and the top and bottom of the object of the eyepiece lens (which is the image of the objective lens). We get the relationship:

$$\tan \vartheta_e = \frac{h}{f_e} \approx \vartheta_e \tag{10.2}$$

Combining (10.1) and (10.2) we get:

$$\frac{\vartheta_o}{\vartheta_e} \approx \frac{f_e}{f_o}$$

## Diffraction

Diffraction and inteference are a set of phenomona that occur when waves encounter objects about the size of their wavelenght.

In 1D with waves with phase $\varphi$:

$$y_1 + y_2 = 2A \cos\left(\frac{\varphi}{2}\right) \sin\left(kx - \omega t + \frac{\varphi}{t}\right)$$

$\cos\left(\frac{\varphi}{2}\right)$ is the phase factor when $\cos\left(\frac{\varphi}{2}\right) = 0$ the waves cancel. In 2D the phase factor is a function of position.

⁀ These waves are antiphase, they will cancel out, this is called destructive inteference.

⁀ These waves are in phase, they will add up, this is called constructive inteference.

The difference in path lengths determins how they intefere. For two waves one of which has traveled a distance $x_1$ and the other $x_2$ the sum of the two waves is:

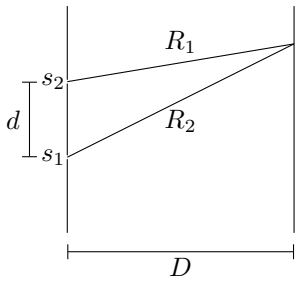
$$y_1 + y_2 = 2A \sin\left(k\frac{x_1 + x_2}{2} - \omega t\right) \cos\left(k\frac{x_1 - x_2}{2}\right)$$

The first term oscilates with time. The second term effects amplitude at different points.

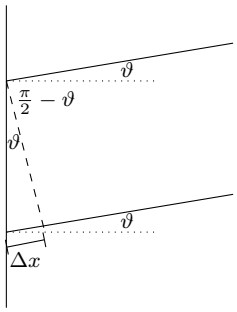
$$I \propto \cos^2\left(k\frac{x_1 - x_2}{2}\right) = \cos^2\left(\pi\frac{x_1 - x_2}{\lambda}\right)$$

So if the distances differ by an integer number of wavelengths then the waves will be in phase. If it differs by a half integer number of wavelengths then the waves will be out of phase.

## Double slit




In general this problem is quite hard to solve but it is simpler when $D \gg d$ as rays $R_1$ and $R_2$ are basically parallel:




The path difference is $\Delta x = d \sin \vartheta$. There are maxima at $\Delta x = m\lambda$ for $m \in \mathbb{Z}$ therefore there are maxima at:

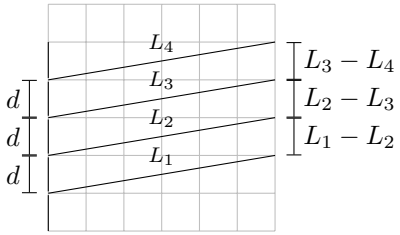
$$m\lambda = d \sin \vartheta$$

Minima are at:

$$\left(m + \frac{1}{2}\right) \lambda = d \sin \vartheta$$

## Diffraction gratings

A diffraction grating is a regularly spaced grid of slits, usually engraved on glass.




Making the same assumption as for single slit that $D \gg d$ so the rays are basically parallel

$$L_1 - L_2 = L_2 - L_3 = L_3 - L_4 = \cdots = d \sin \vartheta$$

$$L_1 - L_3 = (L_1 - L_2) + (L_2 - L_3) = d \sin \vartheta + d \sin \vartheta = 2d \sin \vartheta$$

In general:

$$L_1 - L_{n+1} = nd \sin \vartheta$$

This is generally quite complicated but we know that any two neighbouring rays are in phase as if they where in a double slit experiment. This means that if $L_1$ and $L_2$ are in phase then $L_2$ and $L_3$ are in phase so by strong induction we know that if $L_1$ and $L_2$ are in phase then $L_1$ is in phase with all other rays. This means that there are maxima at $m\lambda = d \sin \vartheta$.

The pattern of light depends on $k$ since $k = \frac{2\pi}{\lambda}$

$$m\lambda = d\sin\vartheta$$

For the first fringe:

$$\begin{aligned}\lambda &= d\sin\vartheta\\ \vartheta &\approx \sin\vartheta\\ \lambda &\approx d\vartheta\\ \frac{\lambda}{d} &\approx \vartheta\end{aligned}$$

This means light will be split into its wavelengths with red light further out than blue. The central fringe will be white.

# Lecture 11

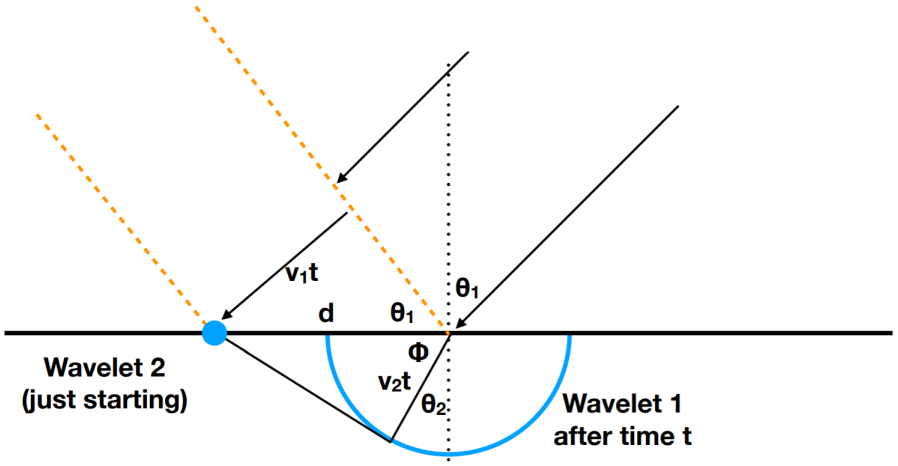
Wavefronts are points/lines/surfaces of constant phase. The exact choice of phase is arbitrary. In 1D it is a series of points traveling with the wave. In 2D it is a line. Wavefronts travel perpendicularly to the energy.

## Huygen's principal (HP)

> Each point of the wavefront acts as a new source of hemispherical "wavelets" moving in the forward direction

This isn't a fundamental science law but a model. It arrises from a method of solving the underlying wave equation using Green's functions. We can use it to understand diffraction. Each point of the wavefront is creating new wavelets which then create a new wavefront so when a block is discovered the wavefront will create new wavelets which can go around it. The number of wavelets considered isn't important as long as you use enough to get a good picture of what is happening.

It is also possible to derive Snell's law from Huygen's principal:



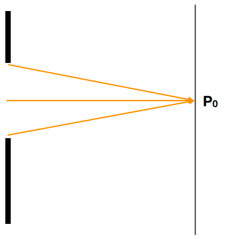

If the wave starts in a medium with refractive index $n_1$ and travels with velocity $v_1$ then moves into a medium with refractive index $n_2$ where it travels with velocity $v_2$. At time $t = 0$ the first part of the wavefront enters the second medium and creates a wavelet. and at time $t$ the second ray enters the new medium. By comparing expressions for $d$ we get Snell's law:

$$\sin\vartheta_1 = \frac{v_1 t}{d}$$

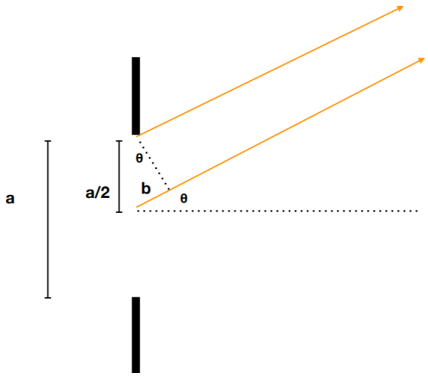
$$\varphi = \frac{\pi}{2} - \vartheta_2$$
$$\cos\varphi = \frac{v_2 t}{d} = \sin\vartheta_2$$

$$\begin{aligned} d &= \frac{v_1 t}{\sin\vartheta_1} = \frac{v_2 t}{\sin\vartheta_2} \\ \frac{\sin\vartheta_1}{v_1} &= \frac{\sin\vartheta_2}{v_2} \\ \frac{c\sin\vartheta_1}{v_1} &= \frac{c\sin\vartheta_2}{v_2} \\ n_1\sin\vartheta_1 &= n_2\sin\vartheta_2 \end{aligned}$$

It is also possible to derive the double slit inteference pattern by considering two slits that are thin enough that only one wavelet can get through. Since this is a physicaly impossible system we can consider one slit inteference:




There is a bright fringe at $P_0$ because if the distance to the screen is much bigger than the slit width then the rays are approximately parallel. Each pair of rays at equal distance apart has the same path difference so it is only necessary to consider one pair:




As with the double slit cancelation occurs when $b = \frac{a}{2}\sin\vartheta$. First order cancelation $\frac{\lambda}{2} = b = \frac{a}{2}\sin\vartheta$ So there are minima at $\lambda = a\sin\vartheta$. So for the complete single slit:

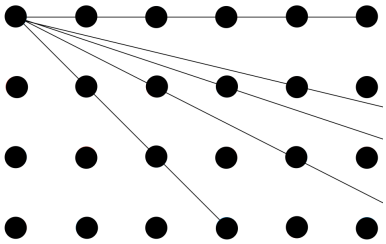
$$b = x\sin\vartheta$$

$$\begin{aligned} A_x &= A_0\sin(k(D-b) - \omega t) \\ &= A_0\sin(kD - kx\sin\vartheta - \omega t) \\ &= A_0[\sin(kD-\omega t)cos(kx\sin\vartheta) - \cos(kD-\omega t)\sin(kx\sin\vartheta)] \end{aligned}$$

Where $x$ is the distance from the center of the slit.

# Lecture 12

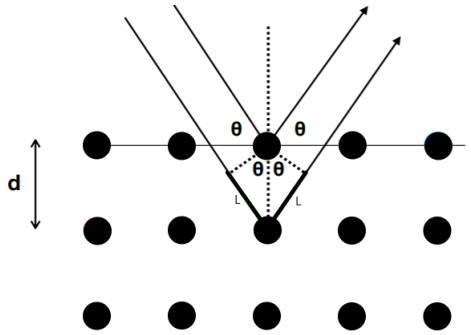
A crystal is a solid with a repeating molecular structure. It has planes of molecules:



The distance between planes is approximately $10^{-10}$ which is about the same as the wavelength of x-rays therefore x-rays will defract in a crystal. The different layers of the crystal act as a diffraction grating.

## X-ray crystalography - single crystal

When an x-ray is incident on a crystal if it hits a molecule a diffuse reflection occurs. Most of the reflected rays will cancel leaving rays at only specific angles with perfecly constructive inteference. If we consider only the rays that don't cancel:




The two rays leaving are close enough to intefere. The path difference $\Delta$ is given by:

$$\Delta = 2L = 2d\sin\vartheta$$

So the inteference is constructive if:

$$m\lambda = 2d\sin\vartheta \text{ for } m \in \mathbb{Z}$$

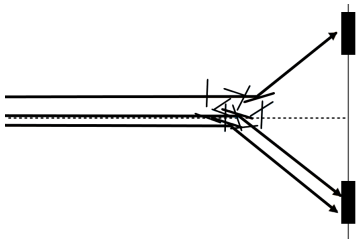
Note that $\vartheta$ is not the angle of incidence $\vartheta_i$ but rathter $\vartheta = \frac{\pi}{2} - \vartheta_i$.

This is Bragg's law

Each plane has different spacing so there are different angles of reflection depending on the angle the light is shone.

## X-ray crystalography - powdered crystal

In reality a powederd crystal is used. This results in a circular diffraction pattern.



### Thin Film Inteference

A thin film will cause light reflecting off of the two edges of the film to intefere with itself. We imagine light normal to the surface of the film. The path difference length is affected by three factors:

- Physical length difference
- Refractive index ($n_{\text{film}} > n_{\text{air}}$)
- Phase shift on reflection from surface of higher refractive index by $\pi$rad

This results in a phase difference $\Delta\varphi$ for a film thickness $L$

$$\Delta\varphi = 2kL - \pi$$

The first term $2kL$ is due to the physical length difference. The second term $\pi$ is due to the reflection. $k$ is the wavenumber in the film. $k = k_{\text{film}} = \frac{2\pi}{\lambda_{\text{film}}}$. $\omega$ is constant and $v$ changes.

$$\lambda_{\text{film}} = \frac{\lambda_{\text{air}}}{n_{\text{film}}}$$

$$\implies \Delta\varphi = \frac{4\pi nL}{\lambda} - \pi$$

Constructive inteference occurs if $\Delta\varphi = 2m\pi$ hence $2L = \left(m + \frac{1}{2}\right)\frac{\lambda}{n}$ and destructive inteference occurs if $\Delta\varphi = (2m+1)\pi$ hence $2L = \frac{m\lambda}{n}$ for $m \in \mathbb{Z}$

### Uses of interference

- Interferometer - A genereal class of tools using path difference for measurement
- Radio interferometry - Multiple radiotelescopes aproximately 10 km apart with correlated ouputs that act as a 10 km wide telescope. It can only see small stuff as $\sin\vartheta \propto \frac{\lambda}{d}$
- The Michelson-Morley experiment (which shows there is no aether)
- LIGO (which discovered gravitational waves)

# Part II

# Quantum Physics

## Lecture 13

The photoelectric effect implies that light has particle nature. When a light is shon on a photo emmitter it causes electrons to be emitted following the following equation:

$$eV_s = hf - \varphi$$

Where $e$ is the charge of an electron, $V_s$ is the stopping voltage required to stop current flow, $h$ is plank's constant $h = 6.626 \times 10^{-34}$, $f$ is the frequency of the incident light and $\varphi$ is the workfunction which is the value $hf_0$ such that $hf - hf_0 = 0$ where $f_0$ is the minimum frequency.

When shone on a slit electrons will diffract, this shows that they have wave like nature.

## Wave function

The wave function $\psi$ is a multivariable function $\psi(x, y, z, t)$ where $x, y$ and $z$ are the three spatial dimensions and $t$ is time. For this course we will only look at one dimensional functions which are time independant (time doesn't effect the potential energy of the particle). These take the form $\psi(x)$. An example would be $\psi_+$:

$$\psi_+(x) = Ae^{ikx}$$

where $A$ and $k$ are constants. There is another solution closely linked to this one and also both can be written in a different form:

$$\psi_+(x) = Ae^{ikx} = A(\cos kx + i \sin kx) \qquad \psi_-(x) = Ae^{-ikx} = A(\cos kx - i \sin kx)$$

These are waves. It is possible to work out the probability of the wave being found in a particular space using the probability function $|\psi|^2 = \psi\overline{\psi}$. For the example $\psi = Ae^{ikx}$:

$$\psi = Ae^{ikx} \implies \overline{\psi} = Ae^{-ikx}$$

$$|\psi|^2 = \psi\overline{\psi} = Ae^{ikx}Ae^{-ikx} = A^2e^{ikx-ikx} = A^2$$

This is independant of $x$ and therefore the particle is equally likely to be at any point in space.

## The Schrödinger equation

$$\boxed{\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}[E - U(x)]\psi = 0}$$

In this equation $m$ is the mass of the particle, $E$ is the total energy of the system and $U(x)$ is the potential energy of the system. For the case where $U(x)$ is independant of $x$ since the point at wich $U(x) = 0$ is arbitrary we can just set it equal to 0 so we get:

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}E\psi = 0$$

It is possible to show that our example from before satisfies the equation under certain conditions:

$$\psi = Ae^{ikx}$$

$$\frac{d\psi}{dx} = Aike^{ikx}$$

$$\frac{d^2\psi}{dx^2} = -Ak^2e^{ikx}$$

Substituting into the Schrödinger equation:

$$-Ak^2e^{ikx} + \frac{8\pi^2 m}{h^2}EAe^{ikx} = 0$$

$$-k^2 + \frac{8\pi^2 m}{h^2}E = 0$$

So $\psi = Ae^{ikx}$ is a solution if:

$$E = \frac{h^2k^2}{8\pi^2 m} \iff k^2 = \frac{8\pi^2 mE}{h^2}$$

The reduced plank's constant $\bar{h}$ is defined as $\bar{h} \triangleq \frac{h}{2\pi}$. Using this we can write the conditions as:

$$E = \frac{\hbar^2k^2}{2m} \iff k^2 = \frac{mE}{\hbar^2}$$

Note that this only applies for a free particle (a particle with a boudary will have $U(x)$ that depends on $x$). The whole Schrödinger equation can be written as:

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}[E - U(x)]\psi = 0$$

# Lecture 14

Using $\hbar$ we get $E = \hbar\omega$

From last lecture we saw that if $\psi_+ = Ae^{ikx}$ is a solution then $k^2 = \frac{8\pi^2 m}{h^2}E$ If this is true then the Scrödinger equation can be rewritten as:

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0$$

This is the wave equation.

$k$ is the wave number in this equation. It is the number of waves per unit distance.

$$k^2 = \frac{8\pi^2 m}{h^2}E \implies E = \frac{k^2h^2}{8\pi^2 m}$$

For non relativistic particles $E = \frac{p^2}{2m} = \frac{m^2v^2}{2m} = \frac{1}{2}mv^2$

$$E = \frac{h^2k^2}{8\pi^2 m} = \frac{p^2}{2m}$$

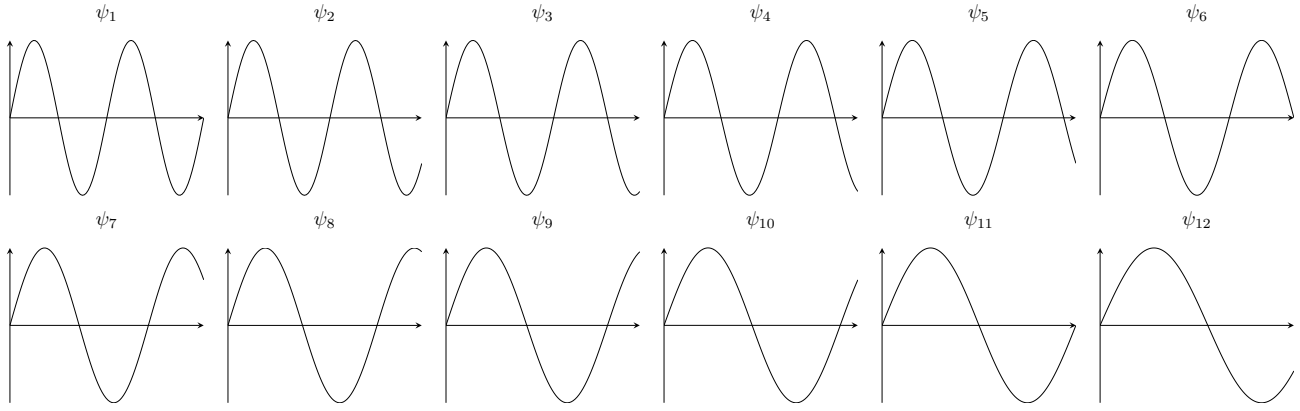
$$p^2 = \frac{h^2k^2}{4\pi^2}$$

$$p = \frac{hk}{4\pi^2} = \hbar k$$

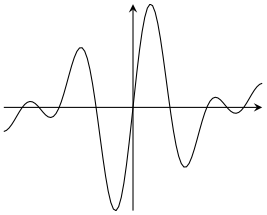
The momentum $p$ of a solution to the free Scrödinger equation is given by $p = \hbar k$

A more realistic solution to the Schrödinger equation is the sum of sinusoidal terms of different frequencies:

$$\psi(x) = \sum_j Ae^{ik_j x}$$




$\sum_{j=1}^{12} \psi_j$



This superposition of different wave frequencies is known as a wave packet. The wave packet is also a solution to the Schrödinger equation. The particle isn't equally likely to be anywhere, it is now localised.

Each frequency has an associated wave number $k_j$. The momentum is $p = \hbar k$ so the momentum is now no longer known precisely. We have an uncertainty of $\Delta p$. The smaller the wave packet the more localised the particle is and the greater the range of frequencies needed. This gives rise to Heisenberg's uncertainty principle:

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}$$

Where $\Delta x$ is uncertainty in position and $\Delta p_x$ is uncertainty in momentum in the $x$ direction. This fundamentally limits how well we can measure $p_x$ and $x$. A similar relationship between energy and time exists:

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

# Lecture 15

If we introduce a potential $U(x)$ then only certain energies $E$ are allowed. The energy becomes quantised.

$\psi_+(x) = Ae^{ikx}$ is a particle moving in the positive $x$ direction. $\psi_-(x) = Ae^{-ikx}$ is a particle moving in the negative $x$ direction. If we sum them together we get a standing wave:

$$\psi = \psi_+ + \psi_- = Ae^{ikx} + Ae^{-ikx} = A[\cos kx + i \sin kx + \cos kx - i \sin kx] = 2A \cos kx$$

A more general solution to the wave equation is given by $\psi_+ = Ae^{i(kx+\varphi)}$:

$$\frac{d}{dx}(Ae^{i(kx+\varphi)}) = Aike^{i(kx+\varphi)}$$

$$\frac{d^2}{dx^2}(Ae^{i(kx+\varphi)}) = -Ak^2e^{i(kx+\varphi)}$$

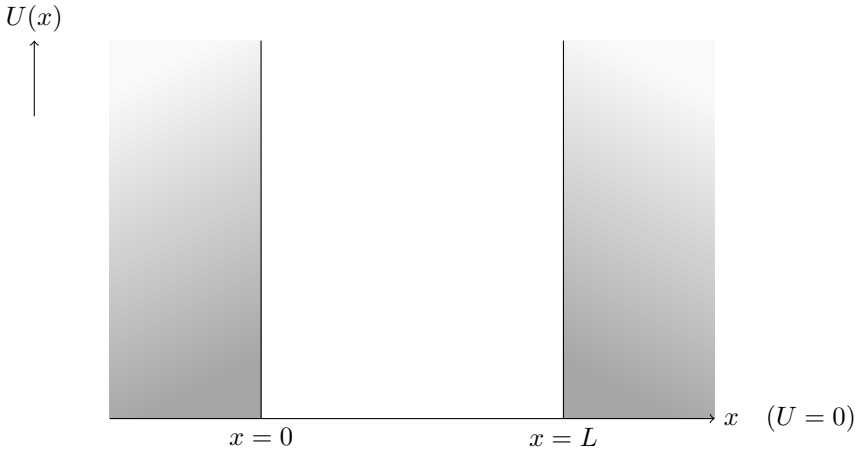
$$-Ak^2e^{i(kx+\varphi)} + \frac{8\pi^2 m}{h^2}[E - u(x)]Ae^{i(kx+\varphi)} = 0$$

So given the condition $k^2 = \frac{8\pi^2 m}{h^2}$ this is a solution. The standing wave formed with this solution is:

$$\psi = \psi_+ + \psi_- = 2A \cos(kx + \varphi) = 2A \sin\left(kx + \varphi - \frac{\pi}{2}\right)$$

This shows that the standing waves which satisfy the Schrödinger equation are sinusoidal.

## Infinite square potential

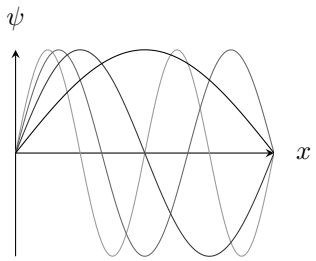


Now particles can be at $x$ values outside of the range $x \in (0, L)$. This means that the probability density function (p.d.f.) is 0 outside of this range:

$$|\psi(0)|^2 = |\psi(L)|^2 = 0$$

$$\implies \psi(0) = \psi(L) = 0$$

A standing wave with nodes at $x = 0$ and $x = L$ is therfore a valid solution to the Scrödinger equation in this case:




The solution has wavelength $\lambda$ which can take the values:

$$\lambda = 2L, L, \frac{2}{3}L, \frac{1}{2}L, \cdots$$

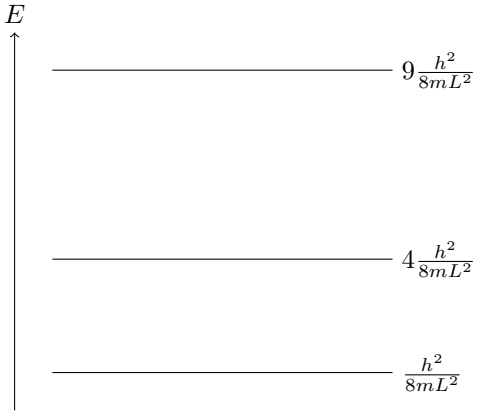
Since$k = \frac{2\pi}{\lambda}$ then $k$ can take the values:

$$k = \frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L}, \frac{4\pi}{L}, \cdots = \frac{n\pi}{L} \quad n \in \mathbb{N}$$

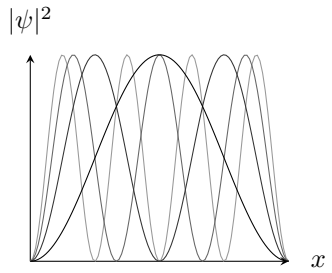
Here $n$ is the quantum number $n = 1, 2, 3, 4 \cdots$

In the region between $x = 0$ and $x = L$ the particle is free so we can use:

$$E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2mL^2} = \frac{h^2}{8mL^2} n^2$$

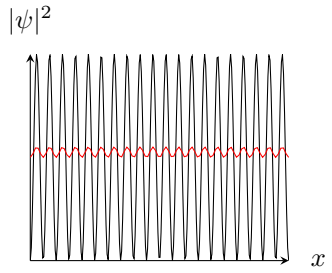



The energy levels are getting further and further appart. Note that the lowest possible energy level is not 0 as $n \neq 0$




This shows the p.d.f. for $n = 1, 2, 3, 4$

We can only measure the probability to find a particle in a region of width $\Delta x$. To find if the particle is close to $\frac{L}{2}$ we measure the probability to find the particle in $x \in \left(\frac{L}{2} - \frac{\Delta x}{2}, \frac{L}{2} + \frac{\Delta x}{2}\right)$ This means that for $n = 20$ we measure the red line when the p.d.f. looks like the black:




This agrees better with what classical mechanics would predict. This is an example of the correspondance principle:

> For sufficently high quantum numbers the results predicted by quantum mechanics are indistinguishable from the results predicted by classical maechanics.

This can be seen if we consider an airhockey puck on a airhockey table. This idealised system behaves as an infinite square potential. For a puck mass $m = 50\,\mathrm{g}$, velocity $v = 5\,\mathrm{m\,s^{-1}}$ and a table of length $L = 2\,\mathrm{m}$ the energy of the puck is given by:

$$E = \frac{1}{2}mv^2 = \frac{1}{2}0.05 \cdot 5^2 = \frac{5}{8}$$

$$E_n = \frac{h^2 n^2}{8mL^2}$$

$$n = \sqrt{\frac{8mL^2E_n}{h^2}}$$

$$n = 1.5 \times 10^{33}$$

$n$ is very big so quantum predictions will agree with classical predictions.

# Lecture 16

The lowest allowed energy is called the zero-point energy

$$\frac{d^2\psi(x)}{dx^2} + \frac{8\pi^2 m}{h^2}[E - U(x)]\psi(x) = 0$$

We will consider the potential energy function

$$U(x) = \begin{cases} 0 & \text{for } x < L \\ U_1 & \text{for } x \geq L \end{cases}$$

Where $U_1 > E$. A solution to this is $\psi(x) = Be^{-\alpha x}$ for $\alpha \in \mathbb{R}$

$$\frac{d\psi}{dx} = -\alpha B e^{-\alpha x}, \qquad \frac{d^2\psi}{dx^2} = \alpha^2 B e^{-\alpha x}$$

$$\alpha^2 B e^{-\alpha x} + \frac{8\pi^2 m}{h^2} \underbrace{[E - U_1]}_{<0} B e^{-\alpha x} = 0$$

$$\alpha^2 + \frac{8\pi^2 m}{h^2}[E - U_1] = 0$$

$$\alpha^2 = \frac{8\pi^2 m}{h^2} \underbrace{[U_1 - E]}_{>0}$$

$$\alpha = \sqrt{\frac{8\pi^2 m}{h^2}[U_1 - E]}$$

So depending on the values of $\alpha$ and $x$ $\psi$ is either increasing or decreasing exponetially in the region where $x \geq L$. Classicaly the particle wouldn't be allowed in this region but in quantum mechanics it is allowed. In the region where $U(x) = 0$ $\psi$ is a standing wave caused by partial reflection at the boundary. In the region where $U(x) = U_1$ $\psi$ decays exponetially. Importantly:

- $\psi \neq 0$ in the clasicaly forbidden region
- The wavefunction transitions smoothly at the boundary
- $\psi \to 0$ as$x \to \infty$

## Harmonic Potential well

$U(x) = \frac{1}{2} k_{\text{spr}} x^2$ where $k_{\text{spr}}$ is th espring constant not the wavenumber.

We expect sinusoidal standing waves in the classicaly allowed region and $\psi \to 0$ as $x \to \pm\infty$

$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$ where $\omega = \sqrt{\frac{k_{\text{spr}}}{m}}$ and $n = 0, 1, 2 \ldots$

zero-point energy $E_0 = \frac{1}{2}\hbar\omega$

This creates evenly spaced energy levels. Inside the clasical boundaries $\psi$ is a standing wave, outside it decays exponentially.

## Quantum tunneling

If instead we consider a potential

$$U(x) = \begin{cases} 0 & \text{for } L_1 > x \\ U_1 & \text{for } L_1 \leq x < L_2 \\ 0 & \text{for } L_2 \leq x \end{cases}$$

Before the increased potential $\psi$ is a large amplitude standing wave, between $L_1$ and $L_2$ $\psi$ decays exponentially and after $L_2$ $\psi$ is a small amplitude standing wave.

# Lecture 17

## The Bohr Model of H

The Bohr model is based on four unjustified postulates:

- Electrons orbit the nucleus at a fixed radius due to the Coulomb force
- The angular momentum of the electron is quantised $l = n\hbar$ where $n = 1, 2, 3, \ldots$
- The electron doesn't radiate photons when it is in one of these orbits
- Emission/absorption occurs due to transitions between these orbits

Potential energy of an electron in H:

$$PE = U(r) = \frac{q_1 q_2}{4\pi\varepsilon_0} = -\frac{e^2}{4\pi\varepsilon_0}$$

Electronstatic and centripetal forces balance:

$$\frac{mv^2}{r} = \frac{e^2}{4\pi\varepsilon_0} \implies KE = \frac{1}{2}mv^2 = \frac{e^2}{8\pi\varepsilon_0 r}$$

Total energy:

$$E = KE + PE = -\frac{e^2}{8\pi\varepsilon_0 r} \tag{17.1}$$

The total energy is negative as the electron is bound to the atom and needs to gain energy to leave. Note that $E \propto \frac{1}{r}$

$$l = mvr = n\hbar \qquad n = 1, 2, 3, \ldots \tag{17.2}$$

Balancing forces:

$$\frac{mv^2}{r} = \frac{e^2}{4\pi\varepsilon_0 r^2} \tag{17.3}$$

Rearranging (17.1) gives

$$v = \frac{n\hbar}{mr}$$

Substituting this into (17.3) gives

$$\frac{m}{r}\left(\frac{n\hbar}{mr}\right)^2 = \frac{e^2}{4\pi\varepsilon_0 r^2}$$

$$r = \frac{4\pi\varepsilon_0\hbar^2}{me^2}n^2 = \underbrace{\frac{\varepsilon_0 h^2}{\pi me^2}}_{\text{constant}} n^2$$

Note that $r \propto n^2$

The smalles possible radius occurs at $n = 1$:

$$r_1 = \frac{\varepsilon_0 h^2}{\pi me^2} = 5.29 \times 10^{-11}\,\text{m} = a$$

$a$ is the Bohr radius. If we substitute this into (17.1) we get

$$E = -\frac{m^2e^4}{8\varepsilon_0^2 h^2 n^2} = -\frac{13.6}{n^2}\,\text{eV}$$

### Quantum Model

The Bohr model misses several predictions that the quantum model sucssesfully makes. The quantum model comes from solving the 3D time invariant Schrödinger equation for $\psi(x, y, z)$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + + \frac{8\pi^2 m}{h^2}[E - U(x, y, z)]\psi = 0$$

$$U(x, y, z) = -\frac{e^2}{4\pi\varepsilon_0 r} \quad \text{where} \quad r = \sqrt{x^2 + y^2 + z^2}$$

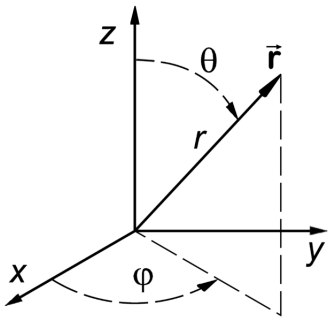
Using spherical polar coordinates $(r, \vartheta, \varphi)$, as the system is sphericaly symmetrical, the solution can be written as a product of functions dependant on quantum numbers $n, l$ and $m_l$

$$\psi(r, \vartheta, \varphi) = R_{n,l}(r)\Theta_{l,m_l}(\vartheta)\Phi_{m_l}(\varphi)$$

This gives

$$E_n = -\frac{m^2e^4}{8\varepsilon_0^2h^2n^2} = -\frac{13.6}{n^2}\,\text{eV}$$

## Lecture 18




We have to apply boundary conditions for the 3D Schrödinger equation that $\psi \to 0$ as $x, y, z \to \infty$

$n$ is the principal quantum number. It determines the energy and radial extent of the wave function.

$n = 1, 2, 3, \ldots$ $n \in \mathbb{N}$

$l$ is the orbital angular momentum quantum number. It determines the magnitude of angular momentum.

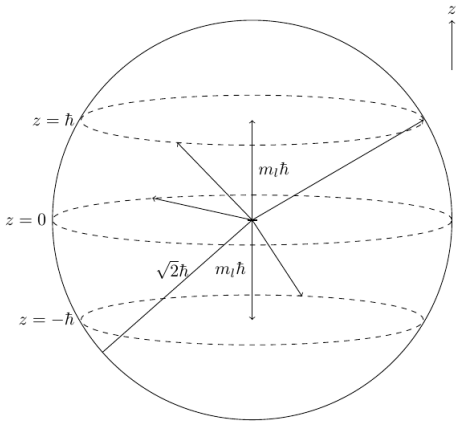
$l = 1, 2, \ldots, n-1$ $l \in \mathbb{N}$ and $l < n$

$m_l$ is the magnetic quantum number. It determines the direction of angular momentum.

$m_l = -l, -l+1, \ldots, 0, \ldots l-1, l$ $m_l \in \mathbb{Z}$ and $|m_l| \leq l$

Suppose that $n = 2$, this means that $l$ can take two values, 0 or 1. When $l = 0$ $m_l = 1$, when $l = 1$ $m_l = -1, 0, 1$.

The angular momentum is given by $|\boldsymbol{L}| = \sqrt{l(l+1)}\hbar$. For $l = 1$ this means the angular momentum is $|\boldsymbol{L}| = \sqrt{2}\hbar$. Using $m_l$ we can measure the components of $\boldsymbol{L}$ in a direction. By convention we measure it for the $z$ direction $L_z = m_l\hbar$. This gives us possible anugular momentums $\boldsymbol{L}$ as shown in the diagram




This shows some of the possible angular momentum vectors $\boldsymbol{L}$ for $n = 2$. The vectors are all from the origin to some point on the three dotted circles which lie in the planes $z = -\hbar$, $z = 0$ and $z = \hbar$. The radius of the sphere is $\sqrt{2}\hbar$.

If we proceed to measure a different component of the angular momentum we no longer know the component in the previously measured direction.

For $n = 1$

$$R_{n=1}(r) = \frac{1}{\sqrt{\pi}a^{\frac{3}{2}}}e^{-\frac{r}{a}}$$

Where $a = 5.3 \times 10^{-11}$ is the Bohr radius

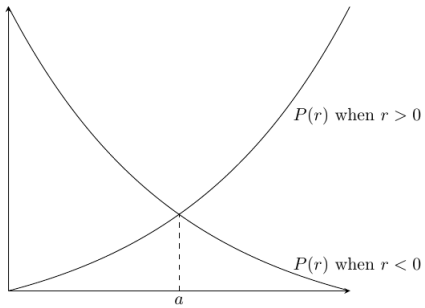
For a randomly chosen point in space the probability of it being a given distance $r$ from the origin is proportional to the surface area of the sphere radius $r$ which is $4\pi r^2$

When using $\psi(r)$ instead of $\psi(x$ the radial probability density is

$$P(r) = 4\pi r^2|\psi(r)|^2$$

For $n = 1$

$$P_{n=1}(r) = 4\pi r^2[R_{n=1}(r)]^2 = \frac{4}{a^3}r^2e^{-2ra}$$




The most likely radial distance is $a$.

This gives rise to the orbitals. Each orbital is denoted by $n$ and a letter representing $l$:

| $l$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| letter | s | p | d | f | g | h |

For example 1s is $n = 1$, $l = 0$ and 3d is $n = 3$, $l = 2$

# Lecture 19

Electrons also have spin quantum numbers $s$ and $m_s$. SPin is predicted when relativity is added to quantum mechanics. The magnitude of total spin $|\boldsymbol{S}|$ is given by

$$|\boldsymbol{S}| = \sqrt{s(s+1)}\hbar$$

The component of spin in the $z$ direction is given by

$$S_z = m_s\hbar$$

For electrons $s = \frac{1}{2}$. $m_2 \in \{-s, -s+1, \ldots, s-1, s\}$ so for an electron $m_s = \pm\frac{1}{2}$. Electrons are described by the four quantum numbers $n, l, m_l$ and $m_s$.

$$|\boldsymbol{S}| = \sqrt{s(s+1)}\hbar = \frac{\sqrt{3}}{2}$$

$$S_z = m_s\hbar = \pm\frac{1}{2}\hbar$$

For hydrogen the Schr$\acute{}$odinger equation can be solved exactly. For multi-electron systems all of the electrons repel eachother. This means it isn't possible to solve the Schr$\acute{}$odinger equation exactly, however, numerical solutions can be found. In multi-electron systems the electrons may have the same quantum numbers as a single electron system but the energies will be different. In a multi-electron system the energy depends mostly on $n$ and $l$.

The Pauli exclusion principal states

> No two fermions in a system may have the same quantum state

A fermion is any particle with $s = \frac{2m+1}{2}$ for $m \in \mathbb{Z}$. This includes electrons, protons, neutrons, muons and neutrinos. Particles with $s = m$ for $m \in \mathbb{Z}$. This includes photons, the Higg's boson and $^2$He.

Spin up is used to refer to $s = \frac{1}{2}$ and spin down refers to $s = -\frac{1}{2}$.

A consequence of the Pauli exclusion principal is that each orbital can hold two electrons for each value of $l$ (spin up and spin down). This means that each s orbital can hold two electrons, p orbitals can hold 6 , d orbitals can hold 10 and f orbitals can hold 14.

The probability per unit time of a transition between energy levels is not the same for all transitions. If $\Delta l = 1$ the transition is likely to occur and is said to be an allowed transition. If an electron can deexcite by such a transition it will and quickly. If $\Delta l \neq 1$ then the transition is less likely and is said to be "forbiden" (but can still occur). If there are no other transitions possible an electron will deexcite by this transition but it will take a long time.

Flourescence is emission of light during illumination as the light excites electrons which then fall back to the ground state in several lower frequency stages emitting multiple photon along the way.

Phosphorecence is emission of light after illumination as the light excites electrons into forbiden transitions which then don't fall back to ground state for a long time, possibly after the illumination has been removed.

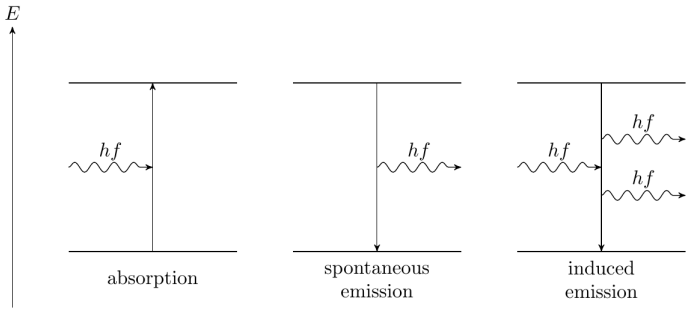
# Lecture 20

Fast (allowed) transitions have small $\Delta t$ and therefore large error in their energy $\Delta E$.

Slow (forbidden) transitions have large $\Delta t$ and therefore small error in their energy $\Delta E$.

$hf = E_1 - E_2$ has some variance since $\Delta E \neq 0$. THis means we don't get one frequency $f$ but a range of frequencies. This means instead of a sharp line we get a brad line. The range of frequencies is known as the line width.

## Induced emission




Induced (stimulated) emission ocurs when an electron can only decay by a forbidden transistion. A photon energy $hf = E_1 - E_0$ will induce the electron to decay and emit two photons both with energy $hf$ in identical quantum states.

$N_0$ is the number of atoms in ground state $E_0$

$N_1$ is the number of atoms in excited state $E_1$

If we introduce a photon $hf = E_1 - E_0$

- if $N_0 > N_1$ absorption is more likely
- if $N_1 > N_0$ induced emission is more

So for stimulated emission to occur on a large scale we need $N_1 > N_0$ When this is true the number of photons increases. This is called photon amplification.

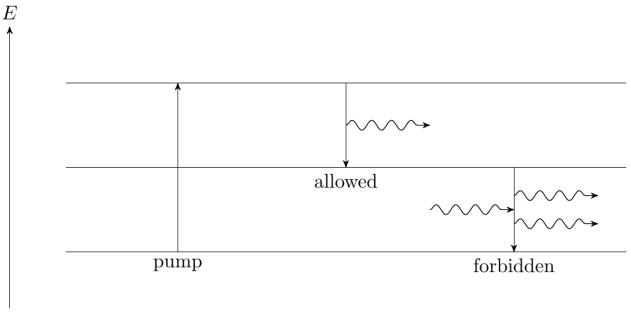
## Boltzmann Distribution

The Boltzmann distribution is that when in thermal equilibrium the ratio of $N_1$ and $N_0$ is

$$\frac{N_1}{N_0} = e^{-\frac{\Delta E}{kT}}$$

Where $\Delta E = E_1 - E_0$, $k$ is the Boltzmann constant $k = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$ and $T$ is the temperature in $K$.

For a simple two level energy system $\frac{N_1}{N_0} < 1 \,\forall\, \Delta E$. However for photon amplification we want $\frac{N_1}{N_0} > 1$. This is possible in a three or more energy level system. The electrons are excited to the highest energy level from the ground state by the "pump". They quickly decay to a middle energy level by an allowed transition and they collect here until $N_1 > N_0$ where they decay by stimulated emission down a forbidden energy level.




The pump can be other photons of a higher energy, electricity, heat or any other energy source. The pump stage is very quick so atoms are moved out of the ground state quickly.

In this case the forbidden transition is called the lasing transition.

Efficency can be improved by surrounding the material being used to emit light in mirrors with a semi silvered mirror where we want the light to leave. This increases efficency by reflecting back photons of necessary energy to create more stimulated electrons and perpetuate the photon amplification.

# Lecture 21

Electrons are responsible for bonding. Generally the highest energy (highest $n$ and $l$) electrons take part in bonding. THe electrons arrange to get the lowest possible energy. In a covalent bond electrons are shared and both nuclei are attracted to the elctrons. In an ionic bond electrons are donoted from one atom and accepeted by another leading to a difference in charge and hence electrostatic attraction. A shell of electrons is all electrons with the same value of $n$. A subshell of electrons is all the electorns with the same values of $n$ and$l$.

## Formation of $H_2$

As the hydrogen nuclei come together the electrostatic potentials overlap lowering the energy of each energy level. Eventually the pdfs of the electrons overlap and the most likely place to find the electrons is between the two nuclei and the electrons can move between the two atoms. The energy to seperate the two nuclei is approximately 4.5 eV which is a lot less than the ionisation energy of the electrons 13.6 eV. In general it is much easier to break bonds than to ionise an atom.

## Ionic Bonding

$$\mathrm{Na(Z=11)1s^2 2s^2 2p^6 3s^1 \longrightarrow Na^+ 1s^2 2s^2 2p^6}$$
$$\mathrm{Cl(Z=17)1s^2 2s^2 2p^6 3s^2 3p^5 \longrightarrow Cl^- 1s^2 2s^2 2p^6 3s^2 3p^6}$$

Both have full valence shells. This arrangement is lower energy as Na goes from 3s energy to 2p energy and Cl stays at 3p energy.

## Properties

Typical properties of ionic and covalent substances at stp

| Property | Ionic | Covalent |
|---|---|---|
| Phase | Solid | Gas |
| Density | High | Low |
| Melting point | High | Low |
| Boiling point | High | Low |
| Electrical properties | Conducts when molten/in solution | Insulator |

Resistivity is an intrinsic property of a material. It is related to resistance $R$

$$R = \rho \frac{L}{A}$$

Where $L$ is lenght and $A$ is cross sectional area. The units of resistivity are $\Omega$m. Conductors have a low resestivity ($\rho \approx 10^{-8}\,\Omega$m) and insulators have relatively high resistivity ($\rho \approx 10^{17}\,\Omega$m). Resistivity isn't constant with temperature. We quantify how resistivity changes by $\alpha$ the temperature coefficent of resistivity

$$\alpha = \frac{1}{\rho}\frac{d\rho}{dT}$$

Where $T$ is the temperature in K. The units of $\alpha$ are $\mathrm{K}^{-1}$. The larger $\alpha$ is the more resistivity changes with respect to temperature. This gives us the approximation $\Delta\rho = \alpha \Delta T \rho$