Willoughby Seago

Theoretical Physics

Quantum Field Theory

September 20th, 2022

COURSE NOTES

Quantum Field Theory

Willoughby Seago

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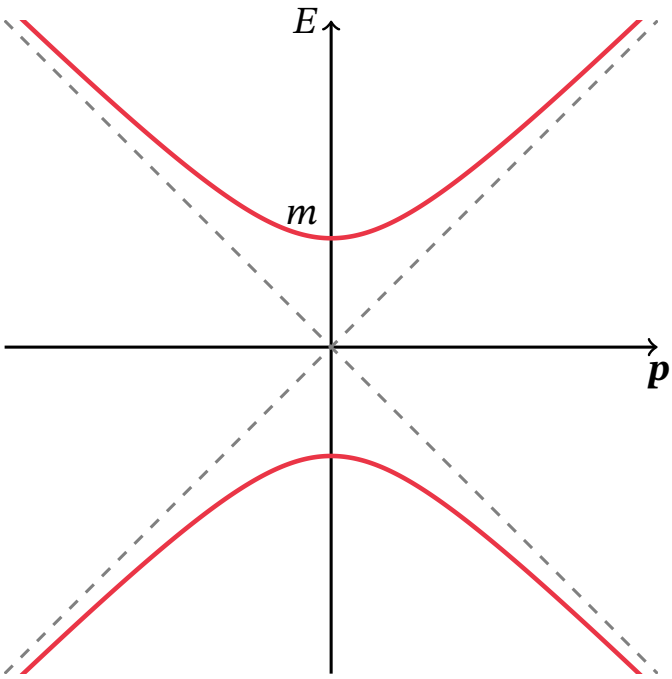
These are my notes from the course quantum field theory. I took this course as a part of the theoretical physics degree at the University of Edinburgh.

These notes were last updated at 21:46 on January 8, 2023. For notes on other topics see https://github.com/WilloughbySeago/Uni-Notes.



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Introduction

1.1 Course Overview

Quantum field theory (QFT) is combination of relativity and quantum mechanics. It will be our topic of study in this course. We will focus mostly on a single quantum field theory, namely **quantum electrodynamics (QED)**, which is the result of combining electrodynamics, which is inherently relativistic, and quantum mechanics.

There are various courses leading into this course, some of the important ones for which I have notes are

- *Principles of Quantum Mechanics*;
- *Quantum Theory*;
- *Classical Electrodynamics*; and
- *Symmetries of Quantum Mechanics*.

This course has a companion course, *Symmetries of Particles and Fields*, which focuses on the mathematical abstraction of symmetry. Ideas from this companion course, and the related course *Symmetries of Quantum Mechanics* will occur throughout this course. Another related course, focused more on the experimental side of this field, is *Particle Physics*. I have notes for both *Symmetries of Particles and Fields* and *Particle Physics*.

This course, and its companion, lead naturally into the course *Gauge Theories in Particle Physics*, for which I also have notes. Many ideas in this course will be expanded upon in the gauge theories course. The gauge theories course will also treat other quantum field theories, such as **electroweak** interactions (a combination of electromagnetism and the weak interactions), and **quantum chromodynamics (QCD)** (the quantum field theory of the strong force). The course also briefly touches on lattice gauge theory, a particular way of doing calculations in nonperturbative QCD.

This course is roughly divided into four sections:

- Canonical QFT: Here we deal with operators and quantisation, familiar from traditional quantum mechanics, such as in *Principles of Quantum Mechanics*.
- QED: We will use this as an example, to allow us to work in a concrete setting rather than in the abstract. This is in many ways the simplest of quantum

field theories, we study mostly interactions of photons and electrons, and since photons aren't charged the interactions are about as simple as they get. Mathematically this "simplicity" is due to the underlying U(1) symmetry, which is an Abelian gauge group, and is significantly easier to work with than, say, the SU(3) symmetry of QCD.

- Path integral formalism: This is a more abstract, but often more powerful, yet equivalent formulation of quantum field theory. This formalism leads itself to gauge theories, and so will be used heavily in *Gauge Theories in Particle Physics*. Nonrelativistic path integrals were treated in *Quantum Theory*.

- Renormalisation: When we do QFT calculations we often get infinite results. Renormalisation is the process of removing these infinities and extracting meaningful results. The renormalisation of QED will be treated in more detail in *Gauge Theories*.

There is some concurrent teaching of these topics, but in these notes I separate them out, so if something's not making sense maybe check to see if its been explained in more detail in a different section.

1.2 Conventions

Quantum field theory is full of conventions. One convention that almost everyone follows is that we work in natural units, where $c = \hbar = 1$. This makes the formulas look much simpler and less cluttered, and we can put c and $\hbar$ back in by dimensional analysis.

The second convention is unfortunately much more varied, its the choice of metric. We will use the $(+---)$ metric,

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.2.1}$$

The most common alternative to this is $(-+++)$, where the diagonal has a single negative 1 and the rest is made of positive 1s. There are some even rarer choices to use an imaginary metric, $(i+++)$, which still gives the same relative minus sign when we square each element in the inner product.

Throughout the course, unless specified otherwise, we will use the Einstein summation convention. Specifically, if an index appears exactly twice in a term, once in a raised position and once in a lowered position, then we sum over all values of that index. We also follow the convention where Greek indices, such as μ and ν, run from 0 to the number of dimensions minus one, most commonly meaning $\mu = 0, 1, 2, 3$. On the other hand, Latin indices, such as i and j, run from 1 to the number of dimensions minus one, most commonly, $i = 1, 2, 3$. This means that Greek indices are summed over all components, whereas Latin indices are summed only over spatial components.

We follow the convention that the electromagnetic field strength tensor is

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{1.2.2}$$

An alternative convention is

$$F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu, \tag{1.2.3}$$

which differs from our choice by an overall minus sign. For example, this results in Maxwell's equations being written as $\partial_\mu F^{\mu\nu} = -j^\nu$, instead of $\partial_\mu F^{\mu\nu} = j^\nu$, which is what we'll be using.

Part I

Canonical Quantisation

Two

Classical Fields

2.1 Relativity Basics

In order to understand quantum field theory we will need to quantise fields. In order to quantise fields we should be sure that we understand classical fields, so this is where we start the course. We will briefly recap ideas mostly from the *Classical Electrodynamics* and *Quantum Theory* courses, so check the notes for this course for more details.

2.1.1 Four-Vectors

We can consider a generic four-vector,

$$a^\mu = (a^0, \boldsymbol{a}) = (a^0, a^1, a^2, a^3). \tag{2.1.1}$$

Using the metric, $\eta_{\mu\nu} \coloneqq \text{diag}(1, -1, -1, -1)$, we can lower the indices to get a covariant four-vector:

$$a_\nu \coloneqq \eta_{\mu\nu} a^\mu = (a^0, -\boldsymbol{a}) = (a^0, -a^1, -a^2, -a^3) = (a_0, a_1, a_2, a_3). \tag{2.1.2}$$

Given two four vectors, $a^\mu = (a^0, \boldsymbol{a})$ and $b = (b^0, \boldsymbol{b})$, we can take the inner product,

$$a \cdot b \coloneqq a^\mu b_\mu = a_\mu b^\mu = \eta_{\mu\nu} a^\mu b^\nu = a^0 b^0 - \boldsymbol{a} \cdot \boldsymbol{b}. \tag{2.1.3}$$

Here $\boldsymbol{a} \cdot \boldsymbol{b} = a^i b^i = a^1 b^1 + a^2 b^2 + a^3 b^3$ is the standard dot product for three-vectors.

For the specific case of the inner product of a four-vector with itself we use the shorthand

$$a^2 = a \cdot a = a^0 a^0 - \boldsymbol{a} \cdot \boldsymbol{a} = (a^0)^2 - \boldsymbol{a}^2. \tag{2.1.4}$$

Four vectors transform under Lorentz transformations. The broadest class of such transformations is the **Lorentz group**, $\text{O}(1, 3)$. This includes all boosts and rotations, including those that invert the directions of space or time. In this course we only consider **proper orthochronous Lorentz transformations**, that is Lorentz transformations preserving the orientation of space (proper) and time (orthochronous). These form the **proper orthochronous Lorentz group** $\text{SO}^+(1, 3)$. From now on if we say Lorentz transformation we mean *proper orthochronous* Lorentz transformation.

2.1.2 Specific-Four Vectors

The position is a four-vector, $x^\mu = (t, \boldsymbol{x})$, where t is the time coordinate and $\boldsymbol{x}$ is the spatial position. Note that in SI units this would be $x^\mu = (ct, \boldsymbol{x})$, since the components of a four-vector must have the same dimensions. We can also construct a covariant position four-vector, $x_\mu = (t, -\boldsymbol{x})$.

The momentum is a four-vector, $p^\mu = (E, \boldsymbol{p})$, where E is the energy, and $\boldsymbol{p}$ is the relativistic three-momentum. Note that in SI units this would be $p^\mu = (E/c, \boldsymbol{p})$. We can also construct a covariant momentum four-vector, $p_\mu = (E, -\boldsymbol{p})$.

[1] as in, not a virtual particle

For a (real[1]) particle of mass m the four-momentum squares to the mass squared, that is

$$m^2 = p^2 = E^2 - \boldsymbol{p}^2. \tag{2.1.5}$$

Rearranging this gives us

$$E^2 = \boldsymbol{p}^2 + m^2. \tag{2.1.6}$$

This is the relativistic **energy-momentum relation**, it's perhaps more familiar if we reinstate the factors of c:

$$E^2 = \boldsymbol{p}^2 c^2 + m^2 c^4, \tag{2.1.7}$$

and is most famous in the case where $\boldsymbol{p} = \boldsymbol{0}$:

$$E = mc^2. \tag{2.1.8}$$

We can construct a four-vector derivative by defining the derivative with respect to x^μ:

$$\partial_\mu \coloneqq \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla\right). \tag{2.1.9}$$

Note that this is a covariant vector, since the contravariant quantity, x^μ, appears in the denominator, so the derivative transforms in the opposite way to how it would if x^μ were in the numerator. We can similarly define a contravariant operator,

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\nabla\right). \tag{2.1.10}$$

The square of these operators is common enough to be given it's own name, it's called the **d'Alembert operator**,

$$\partial^2 = \partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \nabla^2. \tag{2.1.11}$$

This same quantity is also denoted $\Box^2$, as a sort of four-dimensional (note four sides) Lorentzian-manifold analogue of the Laplacian, ∇^2, and also confusingly sometimes denoted $\Box$, without the superscript 2. We also call this the wave operator, since if f is a field satisfying $\partial^2 f = 0$ then expanding and rearranging this we have

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}, \tag{2.1.12}$$

which is the equation of a wave travelling at the speed of light.

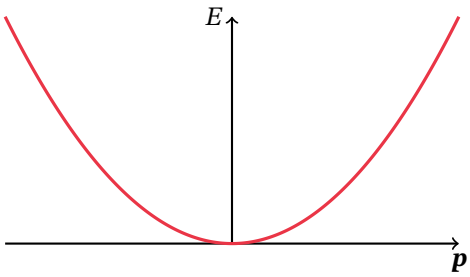


Figure 2.1: The nonrelativistic energy-momentum relation assigns a single energy to each three-momentum. Note that this is really just a slice through a four-dimensional plot, but all that really matters is the magnitude of the three-momentum.

2.2 Nonrelativistic Particle and Wave

A nonrelativistic particle of mass m has the energy-momentum relation

$$E = \frac{\boldsymbol{p}^2}{2m} = \frac{1}{2}m\boldsymbol{v}^2. \tag{2.2.1}$$

To get an equation for this particle we substitute in the usual operators,

$$E \to i\frac{\partial}{\partial t}, \quad \text{and} \quad \boldsymbol{p} \to -i\nabla. \tag{2.2.2}$$

Acting on an arbitrary state, ψ, we then get

$$i\frac{\partial\psi}{\partial t} = \frac{(-i\nabla)^2}{2m}\psi = -\frac{1}{2m}\nabla^2\psi. \tag{2.2.3}$$

This is exactly the **Schrödinger equation** for a free particle.

One solution to this is a plane wave,

$$\psi(x) = \mathrm{e}^{-iEt+i\boldsymbol{p}+\boldsymbol{x}} = \mathrm{e}^{-ip\cdot x}. \tag{2.2.4}$$

This is only a solution if $E^2 = \boldsymbol{p}^2/2m$. Notice that while we can write the exponent with the relativistic four-vectors this is still nonrelativistic since the energy-momentum is not relativistic. In particular, we have a single energy value for each possible momentum, as seen in Figure 2.1.

We know that solutions to the Schrödinger equation can be interpreted as amplitudes, which we then take the modulus square of, $|\psi|^2$, to get a probability density function. We call a function with this property a **wave function**.

2.3 Relativistic Particle and Wave

A relativistic particle of mass m has the energy-momentum relation

$$E^2 = \boldsymbol{p}^2 + m^2. \tag{2.3.1}$$

Making the operator substitutions and acting on an arbitrary state, φ, we get

$$\left(i\frac{\partial}{\partial t}\right)^2\varphi = (-i\nabla)^2\varphi + m^2\varphi. \tag{2.3.2}$$

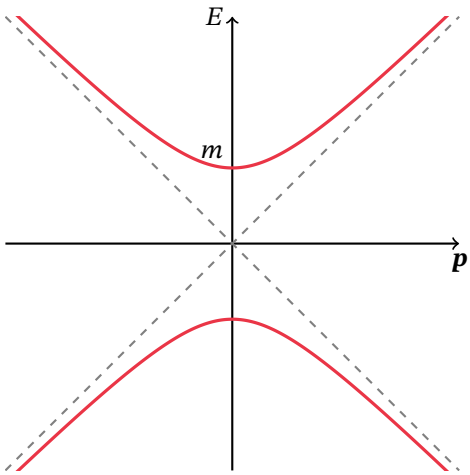


Figure 2.2: The relativistic energy momentum assigns two energies to a each three-momentum. The curve is a hyperbola. Notice that at zero three-momentum there we don't have zero energy, instead we have energy $\pm m$, since we are counting the mass towards the energy in the relativistic formulation. The hyperbola here are the "shell" referred to in "on-shell".

Rearranging this we get

$$(\partial^2 + m^2)\varphi = 0. \tag{2.3.3}$$

This is the **Klein–Gordon equation**.

One solution to this is a plane wave,

$$\varphi(x) = \mathrm{e}^{-ip\cdot x}. \tag{2.3.4}$$

This is a solution provided that $p^2 = m^2$. If $p^2 = m^2$ then we say that p^μ is **on-shell**.

If $e^{-ip\cdot x}$ is a solution then so is

$$\varphi^* = \mathrm{e}^{ip\cdot x}. \tag{2.3.5}$$

We see that we get pairs of solutions. These correspond to the pairs of possible energy values, $E = \pm\sqrt{\boldsymbol{p}^2 + m^2}$, as shown in Figure 2.2.

The negative energy solutions to the Klein–Gordon equation pose a problem, in particular, it is possible for $\varphi^*\varphi$ to be negative, which means there is no probabilistic interpretation of φ, and so φ is *not* a wave function. Instead, we just call φ a field.

States of a system obeying the Klein–Gordon equation correspond to relativistic free particles (plural).

2.4 Maxwell Fields

The most familiar fields are electromagnetic fields. In this section we will briefly recap a relativistic treatment of these fields. For more details see the *Classical Electrodynamics* course.

The electromagnetic "potential" is

$$A^\mu(x) = (\varphi(x), \boldsymbol{A}(x)), \tag{2.4.1}$$

where φ is the electric potential and $\boldsymbol{A}$ is the electromagnetic vector potential. The electric "field" is then

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{2.4.2}$$

An alternative convention,

$$F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu, \tag{2.4.3}$$

differs by a sign.

The electromagnetic current density is

$$j^\mu = (\rho, \boldsymbol{j}). \tag{2.4.4}$$

This current is conserved, mathematically this is expressed as its four-divergence vanishing,

$$\partial_\mu j^\mu = \frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0. \tag{2.4.5}$$

This is a continuity equation.

Maxwell's equation is

$$\partial_\mu F^{\mu\nu} = j^\nu. \tag{2.4.6}$$

Writing $F^{\mu\nu}$ in terms of A^μ we get

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = \partial^2 A^\nu - \partial^\nu \partial_\mu A^\mu. \tag{2.4.7}$$

The **Lorenz gauge** condition is to choose A^μ such that its four-divergence vanishes, $\partial_\mu A^\mu$. Then we have $\partial_\mu F^{\mu\nu} = \partial^2 A^\nu$, so we can write Maxwell's equation as

$$\partial^2 A^\nu = j^\nu. \tag{2.4.8}$$

In QFT we call A^μ the **electromagnetic field** and we call $F^{\mu\nu}$ the **electromagnetic field strength**. This is partly an annoying historical artefact and partly because A^μ is the field appearing in our equations in an analogous way to φ in the Klein–Gordon equation.

In free space, that is when $j^\mu = 0$, Maxwell's equation is

$$\partial^2 A^\mu = 0. \tag{2.4.9}$$

Notice that this is similar to the Klein–Gordon equation with massless particles, $m = 0$. This is good, because we would like electromagnetism to work with photons, which are massless. The one difference from the Klein–Gordon equation is that A^μ is a four-vector, whereas φ is a scalar. The only change in the solutions is that we, in theory, get a different solution for each component of A^μ. The way

that this manifests in the solutions is we get an extra vector out front called the **polarisation vector**, ε^μ:

$$A^\mu = \varepsilon^\mu \mathrm{e}^{-ip\cdot x}. \tag{2.4.10}$$

Assuming the momentum is on-shell, that is $p^2 = 0$ in this massless case. This means the momentum is light-like, more good news for electromagnetism explaining photons.

It can be shown that[2] $\varepsilon_\mu p^\mu = 0$. We are also free to choose ε^μ such that $\varepsilon^0 = 0$, then the condition $\varepsilon_\mu p^\mu = 0$ becomes $\boldsymbol{\varepsilon} \cdot \boldsymbol{p} = 0$. This means that electromagnetic waves are **transverse**, since their polarisation vector, $\boldsymbol{\varepsilon}$, is perpendicular to their direction of travel, $\boldsymbol{p}$.

Since the mathematics of the vector field A^μ is so similar to the mathematics of the scalar field φ we will mostly deal with scalar fields in this course to develop our theory, then work with other fields once we have a good understanding off the maths. Each type of field corresponds to a different spin. Scalar fields describe spin zero particles, like the Higgs boson, and vector fields describe spin one particles, like the photon. Spin 1/2 particles are described by spinors.

[2] See the *Classical Electrodynamics* course, note that there we work with $k^\mu = \hbar p^\mu$.

Three

Lagrangians and Hamiltonians

3.1 Lagrangian Dynamics of a Particle

That's all there is to a Lagrangian dynamics course, the rest is just examples.

Richard Ball

R See the *Lagrangian Dynamics* course for more details.

Consider a particle of mass m moving in one dimension in a conservative force field. We can define its position with a **generalised coordinate**, q, which gives us a **generalised velocity**, $\dot{q} \coloneqq \mathrm{d}q/\mathrm{d}t$. We can then define the **Lagrangian**:

$$L(q, \dot{q}) = T(\dot{q}) - V(q). \tag{3.1.1}$$

Here $T(\dot{q})$ is the kinetic energy of the particle, which we assume depends only on the velocity for simplicity, and $V(q)$ is the potential energy of the particle, which must depend only on the position. The Lagrangian is a function of the generalised position and velocity.

The **action** of the particle between the times t_1 and t_2 is defined to be

$$S[q(t)] \coloneqq \int_{t_1}^{t_2} \mathrm{d}t\, L(q, \dot{q}). \tag{3.1.2}$$

Hamilton's principle states that the path the particle takes, $q(t)$, is such that the action is extremised, that is the variation, δS, given by varying $q \to q+\delta q$ vanishes. We can compute the variation in a general action, S, by computing the variation in the Lagrangian:

$$\delta L = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}. \tag{3.1.3}$$

For a smooth variation, δq, we have

$$\delta \dot{q} = \frac{\partial}{\partial t}(\delta q), \tag{3.1.4}$$

and so

$$\delta L = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{\mathrm{d}}{\mathrm{d}t}(\delta q) \tag{3.1.5}$$

Integrating this the variation in the action is

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}} \frac{\mathrm{d}}{\mathrm{d}t}(\delta q) \right]. \tag{3.1.6}$$

The product rule tells us that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) = \frac{\partial L}{\partial \dot{q}} \frac{\mathrm{d}}{\mathrm{d}t}(\delta \dot{q}) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q. \tag{3.1.7}$$

Rearranging this we can rewrite the second term in δS as a total derivative minus a term:

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t \left[\frac{\partial L}{\partial q}\delta q + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q \right] \tag{3.1.8}$$

$$= \int_{t_1}^{t_2} \mathrm{d}t \left[\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) \right]\delta q + \int_{t_1}^{t_2} \mathrm{d}t \, \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) \tag{3.1.9}$$

$$= \int_{t_1}^{t_2} \mathrm{d}t \left[\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) \right]\delta q + \left[\frac{\partial L}{\partial \dot{q}}\delta q \right]_{t_1}^{t_2}. \tag{3.1.10}$$

Now suppose that δq vanishes at t_1 and t_2, so $\delta q(t_1) = \delta q(t_2) = 0$, then the last term above vanishes and we have

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t \left[\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) \right]\delta q. \tag{3.1.11}$$

If δS is to vanish for all variations δq then we must have

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0. \tag{3.1.12}$$

This is the **Euler–Lagrange equation** for a single particle in one dimension.

This generalises to n-dimensional space by replacing q with n coordinates q_i and $\dot{q}$ with $\dot{q}_i$. The Lagrangian is then

$$L(q_i, \dot{q}_i) = T(\dot{q}_i) - V(q_i), \tag{3.1.13}$$

and the action is

$$S[q_i(t)] = \int_{t_1}^{t_2} \mathrm{d}t \, L(q_i, \dot{q}_i). \tag{3.1.14}$$

We then get n Euler–Lagrange equations:

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0. \tag{3.1.15}$$

Example 3.1.16 The Lagrangian for a simple harmonic oscillator is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2. \tag{3.1.17}$$

We have

$$\frac{\partial L}{\partial x} = -m\omega^2 x, \quad \text{and} \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x} \implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}. \tag{3.1.18}$$

Hence,

$$m\ddot{x} = -m\omega^2 x \implies \ddot{x} = -\omega^2 x \tag{3.1.19}$$

is the equation of motion for the simple harmonic oscillator, which is exactly what we would expect.

3.2 Hamiltonian Dynamics of a Particle

Let's go back to the one-dimensional case. We can define the **canonical momentum** of the particle as

$$p \coloneqq \frac{\partial L}{\partial \dot{q}}. \tag{3.2.1}$$

Note that, in general, this is *not* the normal momentum.

We then define the **Hamiltonian**, H, as a function of the generalised position and canonical momentum:

$$H(q, p) \coloneqq p\dot{q} - L(q, \dot{q}). \tag{3.2.2}$$

Note that we must eliminate any $\dot{q}$ remaining in the expression for the Hamiltonian, this can be done by solving the defining relation $p = \mathrm{d}L/\mathrm{d}\dot{q}$ for $\dot{q}$ and then substituting in the result.

Now consider what happens when we vary H. Varying the left hand side we get

$$\mathrm{d}H = \frac{\partial H}{\partial q}\,\mathrm{d}q + \frac{\partial H}{\partial p}\,\mathrm{d}p. \tag{3.2.3}$$

Varying the right hand side we have

$$\mathrm{d}H = p\,\mathrm{d}q + q\,\mathrm{d}p - \frac{\partial L}{\partial q}\,\mathrm{d}q - \frac{\partial L}{\partial \dot{q}}\,\mathrm{d}\dot{q}. \tag{3.2.4}$$

We can rewrite the last term in terms of the canonical momentum:

$$\mathrm{d}H = p\,\mathrm{d}q + q\,\mathrm{d}p - \frac{\partial L}{\partial q}\,\mathrm{d}q - p\,\mathrm{d}\dot{q} = q\,\mathrm{d}p - \frac{\partial L}{\partial q}\,\mathrm{d}q. \tag{3.2.5}$$

Using the Euler–Lagrange equations the second term can be rewritten to give

$$\mathrm{d}H = q\,\mathrm{d}p - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\mathrm{d}q = q\,\mathrm{d}p - \dot{p}\,\mathrm{d}q. \tag{3.2.6}$$

Comparing this with the result for the left hand side variation we can read off

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \text{and} \quad \dot{p} = -\frac{\partial H}{\partial q}. \tag{3.2.7}$$

These are **Hamilton's equations**.

This all generalises to n dimensions. First replace q with q_i, and p with $p_i := \partial L/\partial q_i$. Then the Hamiltonian is

$$H(q_i, p_i) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i). \tag{3.2.8}$$

Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{3.2.9}$$

It can be useful when doing Hamiltonian mechanics to define the **Poisson bracket** of two functions, A and B, depending on position, q_i, and momentum, p_i:

$$\{A, B, :=\} \sum_i \left(\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} - \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} \right). \tag{3.2.10}$$

This has the advantage of allowing us to express Hamilton's equations in the more symmetric form

$$\dot{q}_i = \{H, q_i\}, \quad \text{and} \quad \dot{p}_i = \{H, p_i\}. \tag{3.2.11}$$

Note the similarity to Heisenberg's operator equation of motion for a time independent operator, A:

$$\dot{A}(t) = i[H, A]. \tag{3.2.12}$$

Replacing Poisson brackets with i times the commutator can get us quite a long way in quantum mechanics.

The Hamiltonian is conserved. To show this consider the time derivative:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial q}\dot{q} = \dot{q}\dot{p} - \dot{p}\dot{q} = 0. \tag{3.2.13}$$

We can often identify the Hamiltonian with the total energy when this is also conserved.

Example 3.2.14 The canonical momentum for a simple harmonic oscillator is

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \tag{3.2.15}$$

which is just the normal momentum in this case. The Hamiltonian is then

$$H = p\dot{q} - L = m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2, \tag{3.2.16}$$

which is exactly the energy of a simple harmonic oscillator.

3.3 Lagrangian Dynamics of a Field

Lagrangian dynamics doesn't change that much when we work with fields. We start by replacing the generalised coordinate $q(t)$ with the field, $\varphi(x)$, which now depends on a position in spacetime, rather than just a time. Instead of a Lagrangian we work with a **Lagrangian density**, $\mathcal{L}(\varphi, \partial_\mu \varphi)$, replacing the time derivative of the coordinate with a four-vector derivative. We then have to integrate over a region of spacetime to get the action:

$$S[\varphi(x)] \coloneqq \int \mathrm{d}^4 x \, \mathcal{L}(\varphi, \partial_\mu \varphi). \tag{3.3.1}$$

The integral should be taken over some arbitrary region of spacetime. Note that we can write this as

$$S[\varphi(x)] = \int \mathrm{d}t \int \mathrm{d}^3 \boldsymbol{x} \, \mathcal{L}(\varphi, \partial_\mu \varphi) = \int \mathrm{d}t \, L(\varphi, \partial_\mu \varphi) \tag{3.3.2}$$

where

$$L(\varphi, \partial_\mu \varphi) \coloneqq \int \mathrm{d}^3 \boldsymbol{x} \, \mathcal{L}(\varphi, \partial_\mu \varphi). \tag{3.3.3}$$

So, integrating $\mathcal{L}$ over space gives L, which we call the Lagrangian, which explains the interpretation of $\mathcal{L}$ as a Lagrangian *density*. While this distinction is important the Lagrangian doesn't actually appear that much in QFT, so people often just call $\mathcal{L}$ the Lagrangian, leaving the density part implicit.

The Euler–Lagrange equations for the Lagrangian density don't change much, and can be derived in a very similar way. We start by varying the field, $\varphi \to \varphi + \delta\varphi$, and looking at the resulting variation in the Lagrangian density:

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta(\partial_\mu \varphi). \tag{3.3.4}$$

For a smooth variation we have

$$\delta(\partial_\mu \varphi) = \partial_\mu(\delta\varphi) \tag{3.3.5}$$

and so

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu(\delta\varphi). \tag{3.3.6}$$

We can use the product rule,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \right) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu(\delta\varphi), \tag{3.3.7}$$

to write the last term as a total derivative minus a term:

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta\varphi. \tag{3.3.8}$$

Integrating over some spacetime region, Ω, to get the variation in the action we have

$$\delta S = \int_\Omega \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\delta\varphi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta\varphi \right] \tag{3.3.9}$$

$$= \int_\Omega \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta\varphi \right] + \int_\Omega \mathrm{d}^4x\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\delta\varphi \right). \tag{3.3.10}$$

Applying the divergence theorem to the second integral we can replace it with an integral of $(\partial\mathcal{L}/\partial(\partial_\mu\varphi))\delta\varphi$ over the boundary, $\partial\Omega$. Choosing a variation, $\delta\varphi$, which vanishes on the boundary this term will vanish, and we will be left with

$$\delta S = \int_\Omega \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta\varphi \right]. \tag{3.3.11}$$

For this to vanish for all variations of the field we require that

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = 0. \tag{3.3.12}$$

These are the **Euler–Lagrange equations** for a scalar field.

3.4 Hamiltonian Dynamics of a Field

Following the same logic as for a single particle we can define the **canonical momentum**,

$$\pi(x) \coloneqq \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}. \tag{3.4.1}$$

This is *not* the same as the actual momentum. Notice that this definition treats time differently to position, this will result in the Hamiltonian formulation not being Lorentz invariant, although the results we get still are, the steps between are just frame dependent. This corresponds to the non-Lorentz invariance of the energy, the energy of a particle of mass m is γm.

We then define the Hamiltonian, H, to be

$$H \coloneqq \int \pi\dot{\varphi}\,\mathrm{d}^4x - L = \int \mathrm{d}^4x\,(\pi\dot{\varphi} - \mathcal{L}) = \int \mathrm{d}^3\boldsymbol{x}\,\mathcal{H} \tag{3.4.2}$$

where

$$\mathcal{H} \coloneqq \pi\dot{\varphi} - \mathcal{L} \tag{3.4.3}$$

is the **Hamiltonian density**. Both the Hamiltonian and Hamiltonian density must be functions of the field, φ, and the canonical momentum, π, which is just another field.

The same derivation as before works without modification, so we won't repeat it here, the result is **Hamilton's equations** for a field:

$$\dot{\varphi} = \frac{\partial \mathcal{H}}{\partial \pi}, \quad \text{and} \quad \dot{\pi} = -\frac{\partial \mathcal{H}}{\partial \varphi}. \tag{3.4.4}$$

As before, the Hamiltonian density is conserved:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \int \mathrm{d}^3\boldsymbol{x}\left(\frac{\partial\mathcal{H}}{\partial\varphi}\dot{\varphi} + \frac{\partial\mathcal{H}}{\partial\pi}\dot{\pi}\right) \tag{3.4.5}$$

$$= \int \mathrm{d}^3\boldsymbol{x}(-\dot{\pi}\dot{\varphi} + \dot{\varphi}\dot{\pi}) \tag{3.4.6}$$

$$= 0. \tag{3.4.7}$$

Again, when energy is conserved we can often identify it with the Hamiltonian.

3.5 Klein–Gordon Equation

We have to learn to run before we walk.

Richard Ball

The Lagrangian for a scalar field, φ, is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}m^2\varphi^2. \tag{3.5.1}$$

Clearly we have

$$\frac{\partial\mathcal{L}}{\partial\varphi} = -m^2\varphi. \tag{3.5.2}$$

We can also compute the derivative with respect to $\partial_\mu\varphi$, note that $(\partial_\mu\varphi)(\partial^\mu\varphi)$ is just $(\partial\varphi)^2$, so we can treat this just like the derivative of a quantity squared, giving

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \partial^\mu\varphi. \tag{3.5.3}$$

We end with an upper index, since the left hand side has a lower index in the denominator, which is an upper index in the numerator. We then have

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\varphi}\right) = \partial_\mu\partial^\mu\varphi = \partial^2\varphi. \tag{3.5.4}$$

Using the Euler–Lagrange equations we then have

$$\partial^2\varphi + m^2\varphi = 0, \tag{3.5.5}$$

which is just the Klein–Gordon equation.

The canonical momentum associated with this Lagrangian is

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)} = \partial^0\varphi = \dot{\varphi}. \tag{3.5.6}$$

The Hamiltonian is then

$$\mathcal{H} = \pi\dot{\varphi} - \mathcal{L} = \frac{1}{2}(\pi^2 + (\nabla\varphi)^2 + m^2\varphi^2). \tag{3.5.7}$$

The $\nabla\varphi$ term comes from

$$\pi^2 - \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) = \pi^2 - \frac{1}{2}[(\partial_0\varphi)(\partial^0\varphi) - (\partial_i\varphi)(\partial^i\varphi)] \tag{3.5.8}$$

$$= \pi^2 - \frac{1}{2}[\dot{\varphi}^2 - (\nabla\varphi)^2] = \frac{1}{2}[\pi^2 - (\nabla\varphi)^2] \tag{3.5.9}$$

where we've used $\dot{\varphi} = \pi$.

Compare the Hamiltonian density for the Klein–Gordon equation with the Hamiltonian for the harmonic oscillator:

$$H = \frac{1}{2m}(p^2 + m^2\omega^2x^2). \tag{3.5.10}$$

Up to a factor of ω^2 and $1/m$ this is pretty much the same as the Klein–Gordon Hamiltonian density, minus the $(\nabla\varphi)^2$ term. Similarly the Klein–Gordon Lagrangian density is very similar to the harmonic oscillator Lagrangian. This suggests that we should study the harmonic oscillator, and indeed we shall soon.

Example 3.5.11 — Electromagnetism The Lagrangian density for the electromagnetic field, A^μ, is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{3.5.12}$$

We can insert the definition of $F^{\mu\nu}$ and expand this out to get

$$\mathcal{L} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{3.5.13}$$

$$= -\frac{1}{4}[(\partial^\mu A^\nu)(\partial_\mu A_\nu) - (\partial^\nu A^\mu)(\partial_\mu A_\nu) \tag{3.5.14}$$

$$- (\partial^\mu A^\nu)(\partial_\nu A_\mu) + (\partial^\nu A^\mu)(\partial_\nu A_\mu)] \tag{3.5.15}$$

$$= -\frac{1}{2}(\partial^\mu A^\nu)(\partial_\mu A_\nu) + \frac{1}{2}(\partial^\mu A^\nu)(\partial_\nu A_\mu). \tag{3.5.16}$$

We treat each component of A_ν as an independent field. We then have

$$\frac{\partial\mathcal{L}}{\partial A_\nu} = 0, \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \partial^\nu A^\mu = -F^{\mu\nu}. \tag{3.5.17}$$

Hence, the Euler–Lagrange equations give

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = -\partial_\mu\partial^\mu A^\nu + \partial^\nu\partial_\mu A^\mu = 0. \tag{3.5.18}$$

That is,

$$\partial^2 A^\nu - \partial^\nu\partial_\mu A^\mu = 0 \iff -\partial_\mu F^{\mu\nu} = 0 \tag{3.5.19}$$

This is Maxwell's equation in a vacuum in an arbitrary gauge.
The canonical momentum is

$$\pi^\mu(x) = \frac{\partial\mathcal{L}}{\partial\dot{A}_\mu} = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_\mu)} = -\partial^\mu A^0 + \partial^0 A^\mu = -F^{0\mu}. \tag{3.5.20}$$

Antisymmetry of $F^{\mu\nu}$ implies that $\pi^\mu(x) = 0$, and so $\pi^\mu(x) = (0, -\boldsymbol{E}(x))$, where $\boldsymbol{E}(x)$ is the electric field.
The fact that $\pi^0(x) = 0$ will cause problems later when we try to quantise the electromagnetic field.
In the presence of sources the Lagrangian density is instead

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu. \tag{3.5.21}$$

This doesn't change the term given by differentiating with respect to $\partial_\mu A_\nu$, all that changes is we now have

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = -J^\mu, \tag{3.5.22}$$

and so the result of applying the Euler–Lagrange equations is now

$$\partial^2 A^\nu - \partial^\nu \partial_\mu A^\mu = J^\nu, \tag{3.5.23}$$

which is Maxwell's equation in the presence of a source.

3.6 Symmetries

3.6.1 Discrete Symmetries

There are three discrete symmetries of particular interest in quantum field theory, they are

- **parity**: $\mathcal{P}: \boldsymbol{x} \mapsto -\boldsymbol{x}$, $\mathcal{P}: x^\mu = (x^0, x^i) \mapsto (x^0, -x^i) = (x_0, x_i) = x_\mu$, so parity acts to swap raised and lowered indices of positions;
- **time reversal**: $\mathcal{T}: t \mapsto t$, $\mathcal{T}: x^\mu = (x^0, x^i) \mapsto (-x^0, x^i) = -(x^0, -x^i) = -(x_0, x_i) = -x_\mu$, so time reversal acts to swap raised and lowered indices and negate positions;
- **charge conjugation**: $\mathcal{C}: e \mapsto -e$, where e is the charge of the particle.

Consider, for example, the Lagrangian of the Klein–Gordon equation:

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \varphi)(\partial_\mu \varphi) - \frac{1}{2}m^2\varphi^2. \tag{3.6.1}$$

Under parity the scalar field is unchanged, and $\partial^\mu \leftrightarrow \partial^\mu$, so the Lagrangian is unchanged under parity. Under time reversal the scalar field is unchanged, and $\partial^\mu \leftrightarrow -\partial_\mu$, so the Lagrangian is unchanged under time reversal. Since this Lagrangian has no charges involved it is trivially invariant under charge conjugation.

Now consider the Electromagnetic Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu. \tag{3.6.2}$$

Under parity transformations $F^{\mu\nu} \leftrightarrow F_{\mu\nu}$, $J^\mu \to J_\mu$, and $A_\mu \to A^\mu$, so

$$\mathcal{P}\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu = \mathcal{L}, \tag{3.6.3}$$

so the Lagrangian is unchanged by parity. Under time reversal transformations $F^{\mu\nu} \leftrightarrow -F_{\mu\nu}$ since the derivatives in the definition of $F^{\mu\nu}$ transform as $\partial^\mu \to -\partial_\mu$ under time reversal. On the other hand, $J^\mu \to J^\mu$, and $A_\mu \to A_\mu$, since time reversal doesn't effect the current or potential. Hence,

$$\mathcal{T}\mathcal{L} = -\frac{1}{4}(-F_{\mu\nu})(-F^{\mu\nu}) - J^\mu A_\mu = \mathcal{L}, \tag{3.6.4}$$

so the Lagrangian is unchanged under time reversal. Under charge conjugation $F^{\mu\nu} \leftrightarrow -F_{\mu\nu}$, $J^\mu \to -J_\mu$, and $A^\mu \to -J_\mu$, so

$$\mathcal{C}\mathcal{L} = -\frac{1}{4}(-F_{\mu\nu})(-F^{\mu\nu}) - (-J_\mu)(-A^\mu) = \mathcal{L}, \tag{3.6.5}$$

so the Lagrangian is invariant under charge conjugation.

Invariance under each of these symmetries means that both of these Lagrangians are invariant under the combined $\mathcal{CPT}$ symmetry, given by $x^\mu \mapsto -x^\mu$ and $e \mapsto -e$.

3.6.2 Continuous Symmetries

R For more details see the *Symmetries of Particles and Fields* course.

If the action is invariant under a group of continuous symmetries then we have a conserved quantity. This is the essence of **Noether's theorem**. The most common examples being

- invariance under translations in time leading to energy conservation;
- invariance under spatial translations leading to momentum conservation;
- invariance under rotation leading to angular momentum conservation.

The simplest case is, perhaps, when the Lagrangian is invariant under translations, then the action is also necessarily translation invariant. Suppose we have a translation

$$\varphi(x) \to \varphi'(x) = \varphi(x) + \delta\varphi(x). \tag{3.6.6}$$

Note that the translation is allowed to depend on x. The variation in $\mathcal{L}$ is

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu)}\delta(\partial_\mu\varphi) \tag{3.6.7}$$

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi) \tag{3.6.8}$$

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi\right). \tag{3.6.9}$$

The second equality is simply an application of the Euler–Lagrange equations to rewrite the first term. The last equality is just recognising the product rule.

Imposing that $\mathcal{L}$ is invariant under translations like this we have $\delta\mathcal{L} = 0$, which gives us the continuity equation

$$\partial_\mu J^\mu(x) = 0, \tag{3.6.10}$$

where

$$J^\mu(x) \coloneqq \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\delta\varphi \tag{3.6.11}$$

is the conserved current, a generalisation of the normal current in electrodynamics. There is also a corresponding conserved "charge":

$$Q = \int \mathrm{d}^3\boldsymbol{x}\, J^0(x). \tag{3.6.12}$$

To see that this quantity is conserved consider

$$\dot{Q} = \int \mathrm{d}^3\boldsymbol{x}\, \partial_0 J^0(x) = -\int \mathrm{d}^3\boldsymbol{x}\, \partial_i J^i(x) = 0 \tag{3.6.13}$$

where in the last step we've used the divergence theorem to rewrite the last term as a surface integral, then take this surface to be at infinity, and made the usual assumption that fields vanish sufficiently quickly at infinity.

Example 3.6.14 — Probability Current Consider the Lagrangian

$$\mathcal{L} = i\psi^*\partial_t\psi - \frac{1}{2m}(\nabla\psi^*)\cdot(\nabla\psi) - V(\boldsymbol{x},t)\psi^*\psi. \tag{3.6.15}$$

If we treat ψ and ψ^* as separate fields then varying one of them gives the Schrödinger equation. This Lagrangian is invariant under the transformation

$$\psi \to \mathrm{e}^{i\alpha}\psi, \quad \text{and} \quad \psi^* \to \mathrm{e}^{-i\alpha}\psi^* \tag{3.6.16}$$

where α is a constant. Linearising we get the transformation

$$\psi \to \psi(1+i\alpha), \quad \text{and} \quad \psi^* \to \psi^*(1-i\alpha). \tag{3.6.17}$$

Since we have two fields, ψ and ψ^*, the conserved current is a sum over field:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi^*)}\delta\psi^*. \tag{3.6.18}$$

Computing the derivatives we have

$$\frac{\partial \mathcal{L}}{\partial(\partial_t\psi)} = i\psi^*, \qquad \frac{\partial \mathcal{L}}{\partial(\partial_i\psi)} = -\frac{1}{2m}\partial^i\psi^*, \tag{3.6.19}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_t\psi^*)} = 0, \qquad \frac{\partial \mathcal{L}}{\partial(\partial_i\psi^*)} = -\frac{1}{2m}\partial^i\psi. \tag{3.6.20}$$

Hence, using the variations

$$\delta\psi = i\alpha\psi, \quad \text{and} \quad \delta\psi^* = -i\alpha\psi^* \tag{3.6.21}$$

we have the time component of the conserved current:

$$J^0 = \rho = -\alpha\psi^*\psi, \tag{3.6.22}$$

and the position components:

$$J^i = \frac{i\alpha}{2m}(\psi^*\partial^i\psi - \psi\partial^i\psi^*) \tag{3.6.23}$$

$$\boldsymbol{J} = \frac{i\alpha}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi). \tag{3.6.24}$$

We can then identify J^μ as, up to a constant, the conserved probability current of Section A.3.2.

It can be shown that the Euler–Lagrange equations are invariant under the addition of a four-divergence,

$$\mathcal{L} \to \mathcal{L} + \partial_\mu \Lambda^\mu \tag{3.6.25}$$

for some four-vector field Λ^μ with a sufficiently smooth derivative. The same derivation as before, but setting $\delta\mathcal{L} = \partial_\mu\Lambda^\mu$ instead of zero, then gives a conserved current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi - \Lambda^\mu. \tag{3.6.26}$$

As well as changing the Lagrangian directly we can instead act on spacetime, for example, by translation:

$$x^\mu \to x'^\mu = x^\mu + a^\mu \tag{3.6.27}$$

for some infinitesimal four-vector a^μ. This induces a change in the scalar field, given by Taylor expanding:

$$\varphi(x) \to \varphi(x+a) = \varphi(x) + a^\mu\partial_\mu\varphi(x). \tag{3.6.28}$$

Since the Lagrangian density is also a scalar it must transform the same way:

$$\mathcal{L} \to \mathcal{L} + a^\mu\partial_\mu\mathcal{L} = \mathcal{L} + a^\nu\partial_\mu(\delta^\mu{}_\nu\mathcal{L}). \tag{3.6.29}$$

Comparing this to the transformation $\mathcal{L} \to \mathcal{L} + \partial_\mu\Lambda^\mu$ we can identify

$$\partial_\mu\Lambda^\mu = a^\nu\partial_\mu(\partial^\mu_\nu\mathcal{L}). \tag{3.6.30}$$

We then have four conserved currents, one for each value of ν, such as

$$J^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}a^0\partial_0\varphi - a^0\delta^\mu{}_0\mathcal{L}, \tag{3.6.31}$$

which comes from setting $\nu = 0$. We can combine these four currents into a single rank 2 tensor, scaling out the a^ν parameter, to get

$$T^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\mathcal{L})}\partial_\nu\varphi - \mathcal{L}\delta^\mu{}_\nu. \tag{3.6.32}$$

Raising the index we get

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial^\nu\varphi - \mathcal{L}\eta^{\mu\nu}. \tag{3.6.33}$$

This is the **energy-momentum tensor**, also called the **stress-energy tensor**[1].

The conserved "charge" associated with the $\nu = 0$ component is

$$\int \mathrm{d}^3\boldsymbol{x}\, T^{00} = \int \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi)} \partial^0 \varphi - \mathcal{L} \right) = \int \mathrm{d}^3\boldsymbol{x}\, \mathcal{H} = H, \tag{3.6.34}$$

so this is an expression of energy conservation. This shouldn't be surprising as translations by a^0 correspond to time translations.

Similarly the conserved charges associated with the $\nu = i$ components are

$$P^i = \int \mathrm{d}^3\boldsymbol{x}\, T^{0i} = \int \mathrm{d}^3\boldsymbol{x}\, \pi \partial^i \varphi. \tag{3.6.35}$$

Since we expect these three components to form a four-vector with time component given by H we interpret them as the actual momentum of the particle, as opposed to the conjugate momentum, π. We can then interpret $\pi \partial_i \varphi$ as the momentum density.

[1] Strictly this is the energy-momentum density tensor, but very rarely does anyone bother to make that distinction.

Four

Quantised Fields

Quantum field theory is a quantum theory of fields. We've seen classical fields, now we need some quantum mechanics before we can start quantising fields. So, in this section we'll give a quick recap of quantum mechanics. For more details see the *Principles of Quantum Mechanics* and *Quantum Theory* courses.

4.1 Quantum Mechanics

> *It's slightly disturbing that people believe it.*
>
> Richard Ball, on quantum mechanics

4.1.1 The Basics

In quantum mechanics we start with coordinates q_i, and a Lagrangian, $L(q_i, \dot{q}_i)$. We then define canonical momenta,

$$p_i \coloneqq \frac{\partial L}{\partial q_i}. \tag{4.1.1}$$

From this we can find a Hamiltonian,

$$H = \sum_i p + i\dot{q}_i - L. \tag{4.1.2}$$

So far this could all be classical. The quantum mechanics begins when we interpret q_i, p_i, and H as **Hermitian operators**[1] acting on a Hilbert space of states. The Hermitian requirement is so that these operators have real eigenvalues, which we can then interpret as the results of measurements. For example, the average energy is given by

$$E = \langle \psi, t | H | \psi, t \rangle, \tag{4.1.3}$$

where $|\psi, t\rangle$ is the state of the particle, that is a vector in the Hilbert space, and $\langle \psi, t| = |\psi, t\rangle^\dagger$ is the corresponding vector in the dual space.

The next step is to realise that operators don't necessarily commute, and so we impose the **canonical commutation relation (CCR)**:

$$[q_i, p_i] \coloneqq i\delta_{ij}, \tag{4.1.4}$$

[1] An operator, A, is Hermitian if $A^\dagger = A$, where $A^\dagger$ is the **Hermitian conjugate**, defined such that $\langle \psi | A | \varphi \rangle = \langle \varphi | A^\dagger | \psi \rangle^*$.

where $[A, B] \coloneqq AB - BA$ is the **commutator**. As well as this we impose that the positions commute, and the momenta commute, so

$$[q_i, q_j] = [p_i, p_j] = 0. \tag{4.1.5}$$

4.1.2 Time Dependence

The time evolution of a state, $|\psi, t\rangle$, under a system with Hamiltonian H is given by the **Schrödinger equation**

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\psi, t\rangle = H|\psi, t\rangle. \tag{4.1.6}$$

This is a fairly simple differential equation, if we ignore the fact that H is an operator and $|\psi, t\rangle$ is a vector, the solution is just

$$|\psi, t\rangle = \mathrm{e}^{-iHt}|\psi\rangle \tag{4.1.7}$$

where $|\psi\rangle = |\psi, 0\rangle$ is the initial state of the system. Fortunately this is perfectly valid, even though H *is* an operator and $|\psi, t\rangle$ *is* vector, we just have to interpret the exponential through its power series:

$$\mathrm{e}^{-iHt} \coloneqq \sum_{n=0}^{\infty} \frac{1}{n!}(-iHt)^n. \tag{4.1.8}$$

This scheme with which we have worked so far, where operators are time independent and states are time dependent, is called the **Schrödinger picture**. It turns out that its actually easier to do QFT by interpreting operators to be time dependent and states to be time independent, called the **Heisenberg picture**. We distinguish between these two pictures by either including t or not, as appropriate in our operators and states. A state, $|\psi\rangle$, in the Heisenberg picture is related to the state $|\psi, t\rangle$ in the Schrödinger picture by

$$|\psi\rangle = \mathrm{e}^{iHt}|\psi, t\rangle, \tag{4.1.9}$$

which is just what we get rearranging Equation (4.1.7). An operator, $A(t)$, in the Heisenberg picture is related to the operator A in the Schrödinger picture by

$$A(t) = \mathrm{e}^{iHt}A\mathrm{e}^{-iHt}. \tag{4.1.10}$$

Importantly, expectation values are the same in both pictures, and so both pictures describe the same physics:

$$\begin{aligned}
\langle A\rangle_{\mathrm{S}} &= \langle\psi, t|A|\psi, t\rangle && (4.1.11)\\
&= \langle\psi|\mathrm{e}^{iHt}A\mathrm{e}^{-iHt} && (4.1.12)\\
&= \langle\psi|A(t)|\psi\rangle && (4.1.13)\\
&= \langle A\rangle_{\mathrm{H}}, && (4.1.14)
\end{aligned}$$

where the subscripts refer to the picture in which we interpret the expectation value.

This picture changing is the same for all operators, so in particular, the position and momentum in the Heisenberg picture are given by

$$q_i(t) = \mathrm{e}^{iHt}q_i\mathrm{e}^{-iHt}, \qquad \text{and} \qquad p_i(t) = \mathrm{e}^{iHt}p_i\mathrm{e}^{-iHt}. \tag{4.1.15}$$

We also have

$$H(t) = \mathrm{e}^{iHt} H \mathrm{e}^{-iHt} = \mathrm{e}^{iHt}\mathrm{e}^{-iHt} H = H, \tag{4.1.16}$$

which works since H commutes with itself, and the exponentials are just power series in H, so also commute with H. This shows that the Hamiltonian is time independent in both pictures, which is a statement of energy conservation.

Consider two operators, $A(t)$ and $B(t)$, in the Heisenberg picture. If we know their commutator, $[A, B]$, in the Schrödinger picture then we can also compute it in the Heisenberg picture:

$$\begin{aligned}
[A(t), B(t)] &= \langle \mathrm{e}^{iHt} A \mathrm{e}^{-iHt} | \mathrm{e}^{iHt} B \mathrm{e}^{-Ht} \rangle && (4.1.17)\\
&= \mathrm{e}^{iHt} A \mathrm{e}^{-iHt} \mathrm{e}^{iHt} B \mathrm{e}^{-iHt} - \mathrm{e}^{iHt} B \mathrm{e}^{-iHt} \mathrm{e}^{iHt} A \mathrm{e}^{iHt} && (4.1.18)\\
&= \mathrm{e}^{iHt} AB \mathrm{e}^{-iHt} - \mathrm{e}^{iHt} BA \mathrm{e}^{-iHt} && (4.1.19)\\
&= \mathrm{e}^{iHt} (AB - BA) \mathrm{e}^{-iHt} && (4.1.20)\\
&= \mathrm{e}^{iHt} [A, B] \mathrm{e}^{iHt}. && (4.1.21)
\end{aligned}$$

In particular,

$$[q_i(t), p_j(t)] = \mathrm{e}^{iHt} [q_i, p_j] \mathrm{e}^{-iHt} = \mathrm{e}^{iHt} i\delta_{ij} \mathrm{e}^{-iHt} = i\delta_{ij} \mathrm{e}^{iHt} \mathrm{e}^{-iHt} = i\delta_{ij}, \tag{4.1.22}$$

so the canonical commutation relations hold if both operators are evaluated at the same time. Similarly,

$$[q_i(t), q_j(t)] = [p_i(t), p_j(t)] = 0. \tag{4.1.23}$$

In the Heisenberg picture instead of the Schrödinger equation we have the **Heisenberg equation**, which we can derive by considering the time derivative of a generic operator, $A(t)$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} A(t) &= \frac{\mathrm{d}}{\mathrm{d}t} (\mathrm{e}^{iHt} A \mathrm{e}^{-iHt}) && (4.1.24)\\
&= iH \mathrm{e}^{iHt} A \mathrm{e}^{-iHt} + \mathrm{e}^{iHt} A (-iH) \mathrm{e}^{-iHt} && (4.1.25)\\
&= i\mathrm{e}^{iHt} HA \mathrm{e}^{-iHt} - i\mathrm{e}^{iHt} AH \mathrm{e}^{-iHt} && (4.1.26)\\
&= i\mathrm{e}^{iHt} [H, A] \mathrm{e}^{-iHt} && (4.1.27)\\
&= i[H, A(t)]. && (4.1.28)
\end{aligned}$$

That is,

$$\frac{\mathrm{d}}{\mathrm{d}t} A(t) = i[H, A(t)]. \tag{4.1.29}$$

The Heisenberg equation applied to the position and momentum gives

$$\frac{\mathrm{d}}{\mathrm{d}t} q(t) = i[H, q(t)], \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} p(t) = i[H, p(t)]. \tag{4.1.30}$$

Note the similarity to Equation (3.2.11). In general when we work in the Heisenberg picture things can look pretty similar to classical mechanics, but with i times the commutator in place of Poisson brackets.

Example 4.1.31 Consider the Lagrangian for a particle of mass m in a position dependent potential, V, in one dimension:

$$L = \frac{1}{2}m\dot{q}^2 - V(q). \tag{4.1.32}$$

The Hamiltonian for this system is

$$H = \frac{p^2}{2m} + V(q). \tag{4.1.33}$$

We then have

$$\dot{q} = i[H, q(t)] = \frac{i}{2m}[p^2, q]. \tag{4.1.34}$$

Now, consider three operators, A, B, and C. We have

$$\begin{aligned}[AB, C] &= ABC - CBA &(4.1.35)\\ &= ABC - ACB + ACB - CBA &(4.1.36)\\ &= A[B, C] + [A, C]B. &(4.1.37)\end{aligned}$$

Using this with $A = B = p$ and $C = q$ gives

$$[p^2, q] = p[p, q] + [p, q]p = -2ip, \tag{4.1.38}$$

where the negative comes from the antisymmetry of the commutator, $[p, q] = -[q, p] = -i$. Hence,

$$\dot{q} = \frac{p}{2m}. \tag{4.1.39}$$

Classically we would have $p = mv = m\dot{q}$, and this is just the quantum analogue of the classical nonrelativistic momentum.
We also have

$$\dot{p} = i[H, p(t)] = i[V(q), p]. \tag{4.1.40}$$

We can expand $V(q)$ as a power series in q, we also have the identity

$$[q^n, p] = inq^{n-1}. \tag{4.1.41}$$

This can be proven with induction on n. First, take $n = 0$, then

$$[q^0, p] = [\mathbb{1}, p] = 0. \tag{4.1.42}$$

Now suppose that

$$[q^k, p] = ikq^{k-1} \tag{4.1.43}$$

for some nonnegative integer k. Then

$$\begin{aligned}[q^{k+1}, p] = [qq^k, p] &= q[q^k, p] + [q, p]q^k\\ &= qikq^{k-1} + iq^k = i(k+1)q^k, &(4.1.44)\end{aligned}$$

and so by induction the identity holds for all natural numbers, n.
Expanding V as a Taylor series we have

$$V(q) = \sum_{n=0}^{\infty} \frac{q^n}{n!} \frac{\partial^n V}{\partial q^n}\bigg|_{q=0}. \tag{4.1.45}$$

Then we have

$$[V(q), p] = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n V}{\partial q^n}\bigg|_{q=0} [q^n, p] \tag{4.1.46}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n V}{\partial q^n}\bigg|_{q=0} inq^{n-1} \tag{4.1.47}$$

$$= \sum_{n=1}^{\infty} \frac{iq^{n-1}}{(n-1)!} \frac{\partial^n V}{\partial q^n}\bigg|_{q=0}. \tag{4.1.48}$$

To first order we then have

$$[V(q), p] = i \frac{\partial^n V}{\partial q^n}\bigg|_{q=0} \tag{4.1.49}$$

and so

$$\dot{p} = i[V(q), p] = -\frac{\partial V}{\partial q}. \tag{4.1.50}$$

Compare this to the classical nonrelativistic equation

$$F = -\frac{\partial V}{\partial x}, \tag{4.1.51}$$

expressing the force due to a potential.
Both of these two analogies, known as **Ehrenfest's theorem**, between classical mechanics and quantum mechanics are really due to the similarity between the classical and quantum equations of motion in the Heisenberg picture.

4.2 Quantum Fields

Fundamentally silly

Richard Ball

The procedure for quantising fields is almost identical to the procedure for quantising position and momentum. Instead of the position and momentum we have the fields φ and π. We interpret these as operators in the Heisenberg picture. Since in field theory we deal with densities instead to be integrated over, such as $\mathcal{L}$ instead of L, we replace δ_{ij} in the canonical commutation relations with $\delta^3(\boldsymbol{x}-\boldsymbol{x}')$, note that this treats time differently to space. We postulate that the at some time,

t, we have

$$[\varphi(t,\boldsymbol{x}),\pi(t,\boldsymbol{x}')] = i\delta^3(\boldsymbol{x}-\boldsymbol{x}') \tag{4.2.1}$$

and the fields commute with themselves at all positions:

$$[\varphi(t,\boldsymbol{x}),\varphi(t,\boldsymbol{x}')] = [\pi(t,\boldsymbol{x}),\pi(t,\boldsymbol{x}')] = 0. \tag{4.2.2}$$

This is called the **equal time commutation relation (ETCR)**.

In quantum mechanics as done in the previous section we treat t as a parameter and $\boldsymbol{x}$ as an operator. This is clearly not Lorentz covariant. There are also other problems, such as having only a single $\boldsymbol{x}$ operator which only allows us to talk of a single particle. In quantum field theory we interpret both t and $\boldsymbol{x}$ as parameters and φ as an operator. We can then make this Lorentz covariant and treat systems with multiple particles.

4.2.1 Equations of Motion

The Heisenberg equation,

$$\dot{A}(t,\boldsymbol{x}) = i[H,A(t,\boldsymbol{x})], \tag{4.2.3}$$

applies to any operator, A, and hence applies to φ. We then have

$$\dot{\varphi}(t,\boldsymbol{x}) = i[H,\varphi(t,\boldsymbol{x})] = i\int \mathrm{d}^3\boldsymbol{x}'\,[\mathcal{H}(t,\boldsymbol{x}'),\varphi(t,\boldsymbol{x})] \tag{4.2.4}$$

We've replaced the Hamiltonian with an integral over the Hamiltonian density. We'll now introduce a shorthand notation where primes denote quantities evaluated at $(t,\boldsymbol{x}')$ and an absence of a prime is a quantity evaluated at $(t,\boldsymbol{x})$, so $\varphi' = \varphi(t,\boldsymbol{x}')$ and $\varphi = \varphi(t,\boldsymbol{x})$. The Hamiltonian density is

$$\mathcal{H}' = \frac{1}{2}(\pi'^2 + (\nabla'\varphi')^2 + m^2\varphi'^2). \tag{4.2.5}$$

The commutator then becomes

$$[\mathcal{H}',\varphi] = \frac{1}{2}\left([\pi'^2,\varphi] + [(\nabla'\varphi')^2,\varphi] + m^2[\varphi'^2,\varphi]\right). \tag{4.2.6}$$

The last two terms vanish, since the fields commute with themselves at all positions. The first term isn't too bad if we use

$$[AB,C] = A[B,C] + [A,C]B, \tag{4.2.7}$$

we have

$$\begin{aligned}
\frac{1}{2}[\pi'^2,\varphi] &= \frac{1}{2}\pi'[\pi',\varphi] + \frac{1}{2}[\pi',\varphi]\pi' && (4.2.8)\\
&= -\frac{1}{2}\pi'[\varphi,\pi'] - \frac{1}{2}[\varphi,\pi']\pi' && (4.2.9)\\
&= -\frac{1}{2}\pi' i\delta^3(\boldsymbol{x}-\boldsymbol{x}') - \frac{1}{2}i\delta^3(\boldsymbol{x}-\boldsymbol{x}')\pi' && (4.2.10)\\
&= -i\pi'\delta^3(\boldsymbol{x}-\boldsymbol{x}'). && (4.2.11)
\end{aligned}$$

Hence, we have the equation of motion

$$\dot{\varphi}(t,\boldsymbol{x}) = i\int \mathrm{d}^3\boldsymbol{x}'[\mathcal{H}(t,\boldsymbol{x}'),\varphi(t,\boldsymbol{x})] \tag{4.2.12}$$

$$= i\int \mathrm{d}^3\boldsymbol{x}'(-i\pi(t,\boldsymbol{x}')\delta^3(\boldsymbol{x}-\boldsymbol{x}')) \tag{4.2.13}$$

$$= \pi(t,\boldsymbol{x}). \tag{4.2.14}$$

So, $\dot{\varphi} = \pi$. This should be no surprise since we found this same relationship for φ satisfying the Klein–Gordon equation before we started quantising.

Similarly we can find the equation of motion for π:

$$\dot{\pi}(t,\boldsymbol{x}) = i[H,\pi(t,\boldsymbol{x})] = i\int \mathrm{d}^3\boldsymbol{x}'[\mathcal{H}(t,\boldsymbol{x}'),\pi(t,\boldsymbol{x})]. \tag{4.2.15}$$

The commutator in this case is

$$[\mathcal{H}',\pi] = \frac{1}{2}([\pi'^2,\pi] + [(\nabla'\varphi')^2,\pi] + m^2[\varphi'^2,\pi]). \tag{4.2.16}$$

The first term vanishes this time. The last term can be computed easily:

$$\frac{1}{2}m^2[\varphi'^2,\pi] = \frac{1}{2}m^2\varphi'^2[\varphi',\pi] + \frac{1}{2}m^2[\varphi',\pi]\varphi' \tag{4.2.17}$$

$$= \frac{1}{2}m^2 i\delta^3(\boldsymbol{x}-\boldsymbol{x}') + \frac{1}{2}m^2\varphi' i\delta^3(\boldsymbol{x}-\boldsymbol{x}') \tag{4.2.18}$$

$$= im^2\delta^3(\boldsymbol{x}-\boldsymbol{x}'). \tag{4.2.19}$$

The second term isn't too bad, we just have to notice that ∇' acts on $\boldsymbol{x}'$ and not on $\boldsymbol{x}$, allowing us to pull it outside of commutators like $[\nabla'\varphi',\pi] = \nabla'[\varphi',\pi]$. Proceeding as before we then have

$$\frac{1}{2}[(\nabla'\varphi')^2,\pi] = \frac{1}{2}(\nabla'\varphi')\cdot[\nabla'\varphi',\pi] + \frac{1}{2}[\nabla'\varphi',\pi]\cdot(\nabla'\varphi') \tag{4.2.20}$$

$$= \frac{1}{2}(\nabla'\varphi')\cdot(\nabla'[\varphi',\pi]) + \frac{1}{2}(\nabla'[\varphi',\pi])\cdot(\nabla'\varphi') \tag{4.2.21}$$

$$= \frac{1}{2}(\nabla'\varphi')\cdot(\nabla' i\delta^3(\boldsymbol{x}-\boldsymbol{x}')) + \frac{1}{2}(\nabla' i\delta^3(\boldsymbol{x}-\boldsymbol{x}'))\cdot(\nabla'\varphi')$$

$$= i(\nabla'\varphi')\cdot(\nabla'\delta^3(\boldsymbol{x}-\boldsymbol{x}')). \tag{4.2.22}$$

We can deal with the gradient of the delta distribution by integrating by parts, and assuming that fields vanish sufficiently quickly at infinity:

$$\int \mathrm{d}^3\boldsymbol{x}'(\nabla'\varphi)\cdot(\nabla'\delta^3(\boldsymbol{x}-\boldsymbol{x}')) = \int \mathrm{d}^3\boldsymbol{x}'\,\nabla'\cdot[\delta^3(\boldsymbol{x}-\boldsymbol{x}')(\nabla'\varphi')] \tag{4.2.23}$$

$$- \int \mathrm{d}^3\boldsymbol{x}'\,(\nabla'^2\varphi')\delta^3(\boldsymbol{x}-\boldsymbol{x}') \tag{4.2.24}$$

$$= \int_{\partial V}[\delta^3(\boldsymbol{x}-\boldsymbol{x}')\nabla'\varphi'] \tag{4.2.25}$$

$$- \int \mathrm{d}^3\boldsymbol{x}'\,(\nabla'^2\varphi')\delta^3(\boldsymbol{x}-\boldsymbol{x}') \tag{4.2.26}$$

$$= -\int \mathrm{d}^3\boldsymbol{x}'\,(\nabla'^2\varphi')\delta^3(\boldsymbol{x}-\boldsymbol{x}') \tag{4.2.27}$$

where as usual the boundary term vanishes. This means that, as distributions,

$$i(\nabla'\varphi')\cdot(\nabla'\delta^3(\boldsymbol{x}-\boldsymbol{x}')) = -i(\nabla'^2\varphi')\delta^3(\boldsymbol{x}-\boldsymbol{x}'). \tag{4.2.28}$$

Hence, we have

$$\dot{\pi}(t,\boldsymbol{x}) = i\int \mathrm{d}^3\boldsymbol{x}'[\mathcal{H}(t,\boldsymbol{x}'),\varphi(t,\boldsymbol{x})] \tag{4.2.29}$$

$$= i\int \mathrm{d}^3\boldsymbol{x}'\left[-i(\nabla'^2\varphi')\delta^3(\boldsymbol{x}-\boldsymbol{x}') + im^2\delta^3(\boldsymbol{x}-\boldsymbol{x}')\right] \tag{4.2.30}$$

$$= \nabla^2\varphi - m^2\varphi. \tag{4.2.31}$$

Now, we have $\pi = \dot{\varphi}$, so $\dot{\pi} = \ddot{\varphi} = \partial_0\partial^0\varphi$. Hence,

$$\partial_0\partial^0\varphi = \nabla^2\varphi - m^2\varphi = \partial_i\partial^i\varphi - m^2\varphi. \tag{4.2.32}$$

Rearranging this we get

$$0 = (\partial_0\partial^0 - \partial_i\partial^i)\varphi + m^2\varphi = (\partial^2 + m^2)\varphi, \tag{4.2.33}$$

so the quantised fields satisfy the Klein–Gordon equation. This means that the Heisenberg equations for field operators look a lost like classical field equations with operators replacing the fields and i times the commutator replacing Poisson brackets.

Even though the Hamiltonian formulation and the equal time commutation relations aren't Lorentz covariant the final result is, and that's all that we care about. The steps in the derivation, however, are frame dependent. This is an inescapable problem with canonical quantisation.

Five

Quantum Harmonic Oscillator

By understanding the harmonic oscillator you understand everything! Well, except from all the things we can't solve.

Richard ball

Surely the hydrogen atom isn't a harmonic oscillator? Well it is!

Richard Ball

The quantum harmonic oscillator is one of the most important systems in quantum mechanics. It's everywhere, we use it to model atoms and phonons, molecules, and pretty much anything else. Even other systems we study in quantum mechanics, such as the hydrogen atom or pairs of particles, can be interpreted as a harmonic oscillator by a change of coordinates. After all, any potential in physics can always be Taylor expanded to second order at a minimum to get a harmonic potential. Part of the reason why we study the harmonic oscillator is that its very nice. We can solve it exactly. The results are also nice, evenly spaced energy levels, bounded below.

The harmonic oscillator is the system with the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2. \tag{5.0.1}$$

We have previously remarked on the similarity of this and the Klein–Gordon Hamiltonian density, and suggested that study of the harmonic oscillator could be enlightening for the Klein–Gordon equation. In this section we'll study the harmonic oscillator, computing the spectrum of the Hamiltonian, and discussing various interpretations. In the next chapter we'll go back through the same steps but for fields obeying the Klein–Gordon and the Hamiltonian density.

5.1 Solution

There are two ways to solve the harmonic oscillator: the analytic way, involving Hermite polynomials, which is frankly, horrible, and the algebraic way, which is

maybe more mysterious but is much more inline with the quantum field theory interpretation, so this is the way we'll proceed.

Start by defining a new operators from the position and momentum operators:

$$a \coloneqq \sqrt{\frac{m\omega}{2}}q + \frac{i}{\sqrt{2m\omega}}p. \tag{5.1.1}$$

The factors are chosen so that our final statements have a simple form, note that a is dimensionless. The Hermitian conjugate of this is

$$a^\dagger = \sqrt{\frac{m\omega}{2}}q - \frac{i}{\sqrt{2m\omega}}p. \tag{5.1.2}$$

Note that a is not Hermitian, it's not an observable, for now we can just think of it as a mathematical trick. We'll see later that combinations of a and $a^\dagger$ can be treated as observables with meaningful values. The motivation for this definition is that if a and $a^\dagger$ commuted (they don't, as we'll see soon) we could factorise the Hamiltonian as

$$\omega a a^\dagger. \tag{5.1.3}$$

A bit of algebra allows us to rewrite q and p in terms of a and $a^\dagger$:

$$q = \frac{1}{2m\omega}(a + a^\dagger), \qquad \text{and} \qquad p = -i\sqrt{\frac{m\omega}{2}}(a - a^\dagger). \tag{5.1.4}$$

We can now compute the commutation relations for a and $a^\dagger$ from the canonical commutation relation:

$$\begin{aligned} i &= [q, p] && (5.1.5)\\ &= -\frac{i}{2}[a + a^\dagger, a - a^\dagger] && (5.1.6)\\ &= -\frac{i}{2}\left([a, a] - [a, a^\dagger] + [a^\dagger, a] - [a^\dagger, a^\dagger]\right) && (5.1.7)\\ &= i[a, a^\dagger]. && (5.1.8) \end{aligned}$$

Hence, we have that

$$[a, a^\dagger] = 1. \tag{5.1.9}$$

An alternative approach to quantum mechanics is to start with a and $a^\dagger$, motivating them starting with the interpretation we'll see later, and then show that the canonical commutation relations are what they are as a consequence of this choice.

We can write the Hamiltonian in terms of a and $a^\dagger$:

$$\begin{aligned} H &= -\frac{\omega}{4}(a - a^\dagger)^2 + \frac{\omega}{4}(a + a^\dagger)^2 && (5.1.10)\\ &= -\frac{\omega}{4}(a^2 - aa^\dagger - a^\dagger a + (a^\dagger)^2) + \frac{\omega}{4}(a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2) && (5.1.11)\\ &= \frac{\omega}{2}(aa^\dagger + a^\dagger a). && (5.1.12) \end{aligned}$$

Now use

$$1 = [a, a^\dagger] = aa^\dagger - a^\dagger a \implies aa^\dagger = a^\dagger a + 1 \tag{5.1.13}$$

and we get

$$H = \frac{\omega}{2}(aa^\dagger + a^\dagger a) = \frac{\omega}{2}(a^\dagger a + 1 + a^\dagger a) = \omega\left(a^\dagger a + \frac{1}{2}\right). \tag{5.1.14}$$

To compute the spectrum of the Hamiltonian we work with the operator $N \coloneqq a^\dagger a$. This *is* a Hermitian operator, $N^\dagger = (a^\dagger a) = a^\dagger (a^\dagger)^\dagger = a^\dagger a = N$. We can then write the Hamiltonian as

$$H = \omega\left(N + \frac{1}{2}\right), \tag{5.1.15}$$

if you remember what the energy eigenvalues of the harmonic oscillator are you may see where this is going.

Let $|n\rangle$ be an eigenstate of N with eigenvalue n, so $N|n\rangle = n|n\rangle$. For now all we know is that $n \in \mathbb{R}$, since N has real eigenvalues as a Hermitian operator. We choose to normalise these states so that

$$\langle n|m\rangle = \delta_{nm} \tag{5.1.16}$$

The use of n and m as labels, as well as the Dirac delta hints at something we will show later, that n is an integer, but for now we just treat n as a real number.

The expectation value of N in the state $|n\rangle$ is

$$\langle N\rangle = \langle n|N|n\rangle = n\langle n|n\rangle = n. \tag{5.1.17}$$

Using $N = a^\dagger a$ we can calculate this expectation value a different way:

$$n = \langle n|N|n\rangle = \langle n|a^\dagger a|n\rangle = \|a|n\rangle\|^2, \tag{5.1.18}$$

and since $\|a|n\rangle\|^2 \geq 0$ we have that $n \geq 0$.

Now consider commutators of N with $a^\dagger$ and a:

$$[N, a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger [a, a^\dagger] + [a^\dagger, a^\dagger]a = a^\dagger. \tag{5.1.19}$$

Similarly,

$$[N, a] = [a^\dagger a, a] = a^\dagger [a, a] + [a^\dagger, a]a = -a. \tag{5.1.20}$$

Suppose we have an eigenstate, $|n\rangle$. What happens when we act on this with N? Well, it turns out that we get another eigenstate:

$$\begin{aligned} Na^\dagger|n\rangle &= ([N, a^\dagger] + a^\dagger N)|n\rangle && (5.1.21)\\ &= (a^\dagger + a^\dagger N)|n\rangle && (5.1.22)\\ &= a^\dagger|n\rangle + a^\dagger N|n\rangle && (5.1.23)\\ &= a^\dagger|n\rangle + a^\dagger n|n\rangle && (5.1.24)\\ &= (n+1)a^\dagger|n\rangle. && (5.1.25) \end{aligned}$$

So, $a^\dagger|n\rangle$ is an eigenstate of N with eigenvalue $n+1$. That is, $a^\dagger|n\rangle \propto |n+1\rangle$. We can fairly easily figure out the constant of proportionality by considering $\|a^\dagger|n\rangle\|^2$:

$$\begin{aligned}\|a^\dagger|n\rangle\|^2 &= \langle n|aa^\dagger|n\rangle && (5.1.26)\\ &= \langle n|(a^\dagger a + [a,a^\dagger])|n\rangle && (5.1.27)\\ &= \langle n|(a^\dagger a + 1)|n\rangle && (5.1.28)\\ &= \langle n|a^\dagger a|n\rangle + \langle n|n\rangle && (5.1.29)\\ &= \langle n|N|n\rangle + 1 && (5.1.30)\\ &= n+1. && (5.1.31)\end{aligned}$$

Since the overall phase of a state has no physical significance we are free to choose the phase of the proportionality constant, so we make the sensible choice that its real and positive. That is, we choose the positive square root, $\|a^\dagger|n\rangle\| = \sqrt{n+1}$, and we then have

$$a^\dagger|n\rangle = \sqrt{n+1}|n\rangle. \tag{5.1.32}$$

We can proceed similarly for $a|n\rangle$, we have

$$\begin{aligned}Na|n\rangle &= ([N,a] + aN)|n\rangle && (5.1.33)\\ &= (-a + aN)|n\rangle && (5.1.34)\\ &= (n-1)a|n\rangle. && (5.1.35)\end{aligned}$$

So $a|n\rangle$ is an eigenstate of N with eigenvalue $n-1$, so $a|n\rangle \propto |n-1\rangle$. We can, again, work out the constant of proportionality:

$$\|a|n\rangle\|^2 = \langle n|a^\dagger a|n\rangle = \langle n|N|n\rangle = n. \tag{5.1.36}$$

So, again choosing a real, positive, solution we have

$$a|n\rangle = \begin{cases}\sqrt{n}|n-1\rangle & n>0,\\ 0 & n=0,\end{cases} \tag{5.1.37}$$

the reason for the careful distinction when $n=0$ is that the state $|0-1\rangle = |-1\rangle$ is not defined, as the eigenvalues of N are nonnegative. The reason we don't have to exclude, say, $n=0.5$, which would have $|0.5-1\rangle = |-0.5\rangle$, is that, as we will now show, n is a nonnegative integer, that is $n \in \mathbb{N} = \{0,1,2,\dots\}$.

Suppose that $n \notin \mathbb{N}$. Consider some $m \in \mathbb{N}\setminus\{0\}$. Then we can apply a m times to $|n\rangle$:

$$\begin{aligned}a^m|n\rangle = \sqrt{n}a^{m-1}|n\rangle &= \sqrt{n}\sqrt{n-1}a^{m-2}|n\rangle\\ &= \cdots = \sqrt{n}\sqrt{n-1}\cdots\sqrt{n-m+1}|n-m\rangle. \quad (5.1.38)\end{aligned}$$

For sufficiently large m we will have $n-m<0$. However, $n-m$ is an eigenvalue and all eigenvalues of N are nonnegative. This is a contradiction and so we must have $n \in m$. In this case one of the square roots will vanish before we get to negative eigenvalues and we'll avoid the contradiction.

Since $H = \omega(N+1/2)$ we can see that the eigenvalues of the Hamiltonian are

$$\left(n+\frac{1}{2}\right)\omega \qquad \text{where} \qquad n \in \mathbb{N}. \tag{5.1.39}$$

The eigenvectors of the Hamiltonian can all be written as

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \tag{5.1.40}$$

where $|0\rangle$ is such that when a acts on it we get zero:

$$a|0\rangle \coloneqq 0. \tag{5.1.41}$$

Note that $|0\rangle$ is an eigenstate with eigenvalue 0, and 0 is the zero vector, these are not the same. The factor of $1/\sqrt{n!}$ is just normalisation, if $|0\rangle$ is normalised then $a^\dagger|0\rangle = |1\rangle = \sqrt{1}|1\rangle$, then $(a^\dagger)^2|0\rangle = \sqrt{1}a^\dagger|1\rangle = \sqrt{1}\sqrt{2}|2\rangle$, and so on, so we can get rid of all these factors with $1/\sqrt{n!}$, and we're left with the normalised states $|n\rangle$.

The most important thing about the harmonic oscillator is that the energy eigenvalues are equally spaced. This makes harmonic oscillators very nice. Another nice property is that the energy eigenvalues are nonnegative. This means that the energy is bounded below, which is good as if it wasn't then systems would quickly decay to having negative infinite energies. The equally spaced eigenvalues is what makes all of quantum field theory possible.

5.2 Interpretations

It's fine, but in a sense, it's just wrong.

Richard Ball

There are two ways to interpret the harmonic oscillator. Historically the first we discuss here was Schrödinger's interpretation, and actually came a year after the second interpretation we'll discuss, which is due to Heisenberg. However, for a long time Schrödinger's interpretation was preferred, mostly because it uses more familiar mathematics, like differential equations, and also because it aligns more similarly with the notion of a classical harmonic oscillator.

5.2.1 Wave Function Interpretation

This is the way you were taught the harmonic oscillator in kindergarten I guess.

Richard Ball

The quantum harmonic oscillator is a quantised classical harmonic oscillator. We take the normal harmonic oscillator Hamiltonian, replace the position and momentum with operators, and solve the equations of motion. The picture that people then have in their head is a harmonic potential with equally spaced energy levels, as seen in Figure 5.1.

The physical variables are q and p, with $q = x$ the position operator, and p chosen to satisfy the canonical commutation relations, which it can be shown is the case if $p = -i\mathrm{d}/\mathrm{d}x$. We solve the equation $a|0\rangle = 0$ to find the vacuum state $|0\rangle$, and it's wave function $\psi_0(x) = \langle x|0\rangle$:

$$\psi_0(x) = \sqrt{\frac{m\omega}{2\pi}}\exp\left[-\frac{1}{2}m\omega x^2\right]. \tag{5.2.1}$$

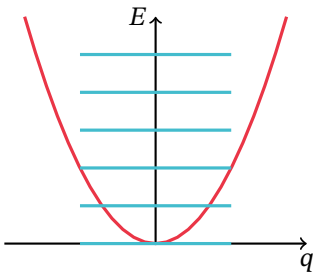


Figure 5.1: The harmonic potential and the evenly spaced energy levels it produces.

We then find the wave equation for the nth excited state, $\sqrt{n!}\psi_n(x) = \langle x|(a^\dagger)^n|0\rangle$:

$$\psi_n(x) = \frac{1}{\sqrt{n!}}\left(0\frac{i}{\sqrt{m\omega}}\frac{\mathrm{d}}{\mathrm{d}x} + i\sqrt{\frac{m\omega}{2}}x\right)^n \psi_0(x) \tag{5.2.2}$$

where the expression in brackets is just $a^\dagger = i/\sqrt{2m\omega}p + \sqrt{m\omega/2}q$ with the operators substituted in. After working through lots of derivatives of Gaussians we will find that the solutions are Gaussians times a Hermite polynomial, H_n:

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi}\right)^{1/4} \exp\left[-\frac{1}{2}m\omega x^2\right] H_n\left(\sqrt{m\omega}x\right). \tag{5.2.3}$$

This interpretation is called **first quantisation**, presumably because it is the first way that most people are taught to interpret the harmonic oscillator.

5.2.2 Quanta

The energy eigenvalues of the harmonic oscillator are evenly spaced. This allows us to interpret the state $|n\rangle$ as consisting of n identical quanta, all of the same energy, ω. The state $|0\rangle$ is then the vacuum state, where there are no quanta, and we have the vacuum energy $\omega/2$.

Rather than try to interpret what a harmonic oscillator is we focus on what it does. Think of it as a black box which can absorb a quanta of energy, so it has $n+1$ quanta, or it can emit a quanta, so it has $n-1$ quanta. Clearly these occurrences align with the action of $|a\rangle^\dagger$ and $|a\rangle$ respectively. This leads to us interpreting $a^\dagger$ as a **creation operator**, producing a quanta of energy in the black box, and a as a **annihilation operator**, destroying a quanta of energy in the black box. Of course this energy must come from somewhere and go somewhere, but just within the black box it may as well be appearing and vanishing.

This interpretation, while a bit odd at first, is actually closer to what we do in experiments. We scatter particles, essentially adding them to the black box, and see what comes out, that is, what leaves the black box. This interpretation also works well when considering atomic spectra, which occur in much the same way.

This interpretation is intrinsically multiparticle, which is good for quantum field theory. This interpretation is called **second quantisation**, presumably because it is the second way that most people are taught to interpret the harmonic oscillator.

5.3 Harmonic Oscillator with Time Dependence

If we include time dependence in the Harmonic oscillator the, working in the Heisenberg picture, this corresponds to replacing q and p with time dependent operators, which in term means introducing a time dependence to a and $a^\dagger$. This time dependence doesn't change the first step of solving the Harmonic oscillator, which is finding q and p in terms of a and $a^\dagger$, which gives us

$$q(t) = \frac{1}{\sqrt{2m\omega}}[a(t) + a^\dagger(t)], \tag{5.3.1}$$

$$p(t) = i\sqrt{\frac{m\omega}{2}}[a(t) - a^\dagger(t)]. \tag{5.3.2}$$

The commutation relations for $a(t)$, $a^\dagger(t)$, and H are $[a(t), a^\dagger(t)] = 1$, $[H, a(t)] = -\omega a(t)$, and $[H, a^\dagger(t)] = \omega a^\dagger(t)$, which are the same as if we replace H with N, since $H = \omega(N + 1/2)$. The time dependence doesn't change the commutators, which we can check quite easily:

$$\begin{aligned}
[a(t), a^\dagger(t)] &= [e^{-iHt} a e^{iHt}, e^{-iHt} a^\dagger e^{iHt}] && (5.3.3)\\
&= e^{-iHt} a e^{iHt} e^{-iHt} a^\dagger e^{iHt} - e^{-iHt} a^\dagger e^{iHt} e^{-iHt} a e^{iHt} && (5.3.4)\\
&= e^{-iHt} a a^\dagger e^{iHt} - e^{-iHt} a^\dagger a e^{iHt} && (5.3.5)\\
&= e^{-iHt} [a, a^\dagger] e^{iHt} && (5.3.6)\\
&= e^{-iHt} 1 e^{iHt} && (5.3.7)\\
&= 1. && (5.3.8)
\end{aligned}$$

Likewise, the commutators with the Hamiltonian are unchanged:

$$\begin{aligned}
[H, a^\dagger(t)] &= [H, e^{-iHt} a^\dagger e^{iHt}] && (5.3.9)\\
&= H e^{-iHt} a^\dagger e^{iHt} - e^{-iHt} a^\dagger e^{iHt} H && (5.3.10)\\
&= e^{-iHt} H a^\dagger e^{iHt} - e^{-iHt} a^\dagger H e^{iHt} && (5.3.11)\\
&= e^{-iHt} [H, a^\dagger] e^{iHt} && (5.3.12)\\
&= e^{-iHt} \omega a^\dagger e^{iHt} && (5.3.13)\\
&= \omega a^\dagger(t). && (5.3.14)
\end{aligned}$$

An similarly for $[H, a(t)]$.

We can then solve the Heisenberg equation for the creation and annihilation operators:

$$\dot{a}(t) = i[H, a(t)] = -i\omega a(t). \tag{5.3.15}$$

This very simple differential equation, $\dot{a}(t) = -i\omega a(t)$, has a plane wave solution,

$$a(t) = a e^{-i\omega t}, \tag{5.3.16}$$

where $a = a(0)$ is the annihilation operator in the Schrödinger picture. Taking the Hermitian conjugate of this we get

$$\dot{a}^\dagger(t) = a^\dagger e^{i\omega t}. \tag{5.3.17}$$

The position and momentum operators are then

$$q(t) = \frac{1}{\sqrt{2m\omega}}[a\mathrm{e}^{-i\omega t} + a^{\dagger}\mathrm{e}^{i\omega t}], \tag{5.3.18}$$

$$p(t) = i\sqrt{\frac{m\omega}{2}}[a\mathrm{e}^{-i\omega t} - a^{\dagger}\mathrm{e}^{i\omega t}]. \tag{5.3.19}$$

Six

Mode Expansions

6.1 Expanding the Field

The field $\varphi(x)$ satisfies the Klein–Gordon equations. We've seen that this allows plane wave solutions, $e^{\pm ip\cdot x}$. This suggests that we can write a general solution as a superposition of wave solutions, which is just a Fourier transform. Since φ is a function of x this is really an inverse Fourier transform. For comparison consider the inverse transform of a single variable, real, one-dimensional function, f:

$$f(x) = \int \frac{\mathrm{d}p}{2\pi}(\tilde{f}(p)e^{-ipx} + \tilde{f}^*(p)e^{ipx}). \tag{6.1.1}$$

Here $\tilde{f}(p)$ is the Fourier transformed function, which acts as a Fourier coefficient here. We need both the e^{ipx} and conjugate e^{-ipx} terms to get a real function. The factor of 2π is part of the normalisation, but exactly where it appears between the forward and inverse is just convention.

We want our field to have on-shell momentum, that is $p^2 = m^2$. To enforce this we include a factor of $2\pi\delta(p^2 - m^2)$, where the 2π is factor is from the integral representation of the Dirac delta, as the Fourier transform of one:

$$\int \mathrm{d}x\, e^{-ip\cdot x} = 2\pi\delta(p). \tag{6.1.2}$$

We also want to restrict ourselves to positive energy solutions, so $p_0 > 0$, we can enforce this with a factor of $\theta(p_0)$, where θ is the **Heaviside step function** defined by

$$\theta(u) = \begin{cases} 1 & u > 0, \\ 1/2 & u = 0, \\ 0 & u < 0. \end{cases} \tag{6.1.3}$$

The **mode expansion** of φ is then

$$\varphi(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} 2\pi\delta(p^2 - m^2)\theta(p_0)[a(\boldsymbol{p})e^{-ip\cdot x} + a^\dagger(\boldsymbol{p})e^{ip\cdot x}]. \tag{6.1.4}$$

Here $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$ are Fourier coefficients, we'll see later that they are the quantum field theory analogues of the creation and annihilation operators for the harmonic oscillator. They depend only on $\boldsymbol{p}$, rather than the four-vector p, since we can take $\delta(p^2 - m^2)$ as fixing the value of p_0.

Define $\omega(\boldsymbol{p}) \coloneqq +\sqrt{\boldsymbol{p}^2 + m^2}$, which is always nonnegative. This allows us to rewrite the argument of the Dirac delta:

$$(p_0 - \omega(\boldsymbol{p}))(p_0 + \omega(\boldsymbol{p})) = p_0 p_0 - \omega(\boldsymbol{p})^2 = p_0 p_0 - \boldsymbol{p}^2 - m^2 = p^2 - m^2. \quad (6.1.5)$$

So, now we have $\delta(\omega(\boldsymbol{p}))$ in our integral. Recall the distribution identity

$$\delta(f(x)) = \sum_i \frac{\delta(x - a_i)}{|f'(a_i)|} \quad (6.1.6)$$

where a_i are all the zeros of f lying in the range of integration. Since in the integral we are enforcing $p_0 \geq 0$ the only time $p^2 - m^2 = (p_0 - \omega(\boldsymbol{p}))(p_0 + \omega(\boldsymbol{p}))$ vanishes is when $p_0 = \omega(\boldsymbol{p})$, and

$$\frac{\partial}{\partial p}(p^2 - m^2)\Big|_{p_0 = \omega(\boldsymbol{p})} = 2\omega(\boldsymbol{p}) \quad (6.1.7)$$

so, as distributions, we have

$$\delta(p^2 - m^2)\theta(p_0) = \frac{1}{2\omega(\boldsymbol{p})}\delta(p^0 - \omega(\boldsymbol{p})). \quad (6.1.8)$$

This lets us write $\varphi(x)$ as

$$\varphi(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^3} \frac{1}{2\omega(\boldsymbol{p})} \delta(p^0 - \omega(\boldsymbol{p}))[a(\boldsymbol{p})\mathrm{e}^{-ip\cdot x} + a^\dagger(\boldsymbol{p})\mathrm{e}^{ip\cdot x}]. \quad (6.1.9)$$

Performing the p^0 integral using the sifting property of the Dirac delta we have

$$\varphi(x) = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \frac{1}{2\omega(\boldsymbol{p})} [a(\boldsymbol{p})\mathrm{e}^{-ip\cdot x} + a^\dagger(\boldsymbol{p})\mathrm{e}^{ip\cdot x}]\Big|_{p_0 = \omega(\boldsymbol{p})}. \quad (6.1.10)$$

At this point we make some remarks on this expansion:

- The expansion is manifestly Lorentz invariant, while the presence of p^0 may be concerning orthochronous Lorentz transformations (the only type we consider in this course) don't change the sign of p^0, and so $\theta(p^0)$ is Lorentz invariant.
- Since $p^0 = \omega(\boldsymbol{p})$ the energy is always positive, and so is bounded below.
- The integral over $\delta(p^0 - \omega(\boldsymbol{p}))$ fixes $p^0 = \omega(\boldsymbol{p})$, and from now on this will be implicit in the integrals, instead of writing $|_{p_0 = \omega(\boldsymbol{p})}$.
- The operator $\varphi(x)$ is in the Heisenberg picture, but the time dependence comes from the exponential factors, both $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$ are time independent operators.
- We need both the a and $a^\dagger$ terms for φ to be Hermitian.
- The measure

  $$\frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \frac{1}{2\omega(\boldsymbol{p})} = \frac{\mathrm{d}^4 p}{(2\pi)^4} 2\pi\delta(p^2 - m^2)\theta(p^0) \quad (6.1.11)$$

  is Lorentz invariant, but takes a lot of writing, we will use the shorthand notation

  $$đp = \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \frac{1}{2\omega(\boldsymbol{p})} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} 2\pi\delta(p^2 - m^2)\theta(p^0). \quad (6.1.12)$$

Now that we've expanded the field φ it is easy to expand the conjugate field, π, using $\pi = \dot{\varphi}$:

$$\pi(x) = \dot{\varphi}(x) = -\frac{i}{2}\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}[a(\boldsymbol{p})\mathrm{e}^{-ip\cdot x} - a^\dagger(\boldsymbol{p})\mathrm{e}^{ip\cdot x}] \tag{6.1.13}$$

$$= -\frac{i}{2}\int \text{đ}p[a(\boldsymbol{p})\mathrm{e}^{-ip\cdot x} - a^\dagger(\boldsymbol{p})\mathrm{e}^{ip\cdot x}]. \tag{6.1.14}$$

Here we've used

$$\frac{\partial}{\partial x^0}\mathrm{e}^{\pm ip\cdot x} = \frac{\partial}{\partial x^0}\exp[\pm ip^0x_0 \mp p^ix_i]$$

$$= \pm ip^0\exp[\pm ip^0x_0 \mp p^ix_i] = \pm ip^0\exp[\pm ip\cdot x] \tag{6.1.15}$$

and then set $p^0 = \omega(\boldsymbol{p})$.

6.2 The Fourier Coefficients

As with the Harmonic oscillator we now want to invert these relations to find expressions for $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$. Since these are inverse Fourier transformations the inversion happens through Fourier transformation. For a single variable, real, one-dimensional function, f, the Fourier transform is

$$\tilde{f}(p) = \int \mathrm{d}x\, f(x)\mathrm{e}^{ip\cdot x}. \tag{6.2.1}$$

Using this, but in three dimensions, we have

$$\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{e}^{ip'\cdot x}\varphi(x) = \int \text{đ}p\frac{1}{2\omega(\boldsymbol{p})}\int \mathrm{d}^3\boldsymbol{x}\left[a(\boldsymbol{p})\mathrm{e}^{i(p'-p)\cdot x} + a^\dagger\mathrm{e}^{i(p'+p)\cdot x}\right]$$

$$= \int \text{đ}p\,(2\pi)^3\big[a(\boldsymbol{p})\delta^3(\boldsymbol{p}'-\boldsymbol{p})\mathrm{e}^{i(p'^0-p^0)t}$$

$$+ a^\dagger(\boldsymbol{p})\delta^3(\boldsymbol{p}'+\boldsymbol{p})\mathrm{e}^{i(p'^0+p^0)t}\big]$$

$$= \int \frac{\mathrm{d}^3\boldsymbol{p}}{2\omega(\boldsymbol{p})}\big[a(\boldsymbol{p})\delta^3(\boldsymbol{p}'-\boldsymbol{p})\mathrm{e}^{i(p'^0-p^0)t}$$

$$+ a^\dagger(\boldsymbol{p})\delta^3(\boldsymbol{p}'+\boldsymbol{p})\mathrm{e}^{i(p'^0+p^0)t}\big].$$

Here we've recognised the integral representation of the Dirac delta:

$$\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{e}^{-ip\cdot x} = (2\pi)^3\delta^3(\boldsymbol{x})\mathrm{e}^{ip^0t}, \tag{6.2.2}$$

and made the usual assumption that everything is sufficiently convergent for us to be able to swap the order of integrals. Performing the integrals now using the Dirac deltas, and the fact that $\omega(-\boldsymbol{p}) = \omega(\boldsymbol{p})$, we have

$$\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{e}^{ip'\cdot x}\varphi(x) = \frac{1}{2\omega(\boldsymbol{p}')}a(\boldsymbol{p}') + \frac{1}{2\omega(-\boldsymbol{p}')}a^\dagger(-\boldsymbol{p}')\mathrm{e}^{2ip'^0t} \tag{6.2.3}$$

$$= \frac{1}{2\omega(\boldsymbol{p}')}\big[a(\boldsymbol{p}') + a^\dagger(-\boldsymbol{p}')\mathrm{e}^{2i\omega(\boldsymbol{p}')t}\big]. \tag{6.2.4}$$

Here we've used the Dirac delta's to set $\boldsymbol{p} = \pm\boldsymbol{p}'$ in each term as appropriate. Restricting ourselves to on-shell, positive energy, solutions this then fixes the value of p^0 to $\sqrt{\boldsymbol{p}'^2 + m^2} = \omega(\boldsymbol{p}')$.

We can do exactly the same for π, the only difference is signs and a factor of $1/\omega(\boldsymbol{p})$, so we get

$$\int \mathrm{d}^3\boldsymbol{x}\, \mathrm{e}^{ip'\cdot x}\pi(x) = -\frac{i}{2}[a(\boldsymbol{p}') - a(-\boldsymbol{p}')^\dagger \mathrm{e}^{2i\omega(\boldsymbol{p}')t}]. \tag{6.2.5}$$

Adding i times this to $\omega(\boldsymbol{p})$ times the result for φ we get

$$a(\boldsymbol{p}) = \int \mathrm{d}^3\boldsymbol{x}\, \mathrm{e}^{ip\cdot x}[\omega(\boldsymbol{p})\varphi(t,\boldsymbol{x}) + i\pi(t,\boldsymbol{x})]. \tag{6.2.6}$$

Taking the conjugate we get

$$a^\dagger(\boldsymbol{p}) = \int \mathrm{d}^3\boldsymbol{x}\, \mathrm{e}^{-ip\cdot x}[\omega(\boldsymbol{p})\varphi(t,\boldsymbol{x}) - i\pi(t,\boldsymbol{x})]. \tag{6.2.7}$$

Now we have expressions for $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$ we can compute their commutator:

$$\begin{aligned}[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] = \int \mathrm{d}^3\boldsymbol{x} \int \mathrm{d}^3\boldsymbol{x}'\, \mathrm{e}^{ip\cdot x - ip'\cdot x'} \\ [\omega(\boldsymbol{p})\varphi(t,\boldsymbol{x}) + i\pi(t,\boldsymbol{x}), \omega(\boldsymbol{p}')\varphi(t,\boldsymbol{x}') - i\pi(t,\boldsymbol{x}')].\end{aligned} \tag{6.2.8}$$

Using the equal time commutation relations only the cross terms don't vanish, and so this becomes

$$\begin{aligned}[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] = \int \mathrm{d}^3\boldsymbol{x} \int \mathrm{d}^3\boldsymbol{x}'\, \mathrm{e}^{ip\cdot x - ip'\cdot x'}\big(- i\omega(\boldsymbol{p})[\varphi(t,\boldsymbol{x}), \pi(t,\boldsymbol{x}')] \\ + i\omega(\boldsymbol{p}')[\pi(t,\boldsymbol{x}), \varphi(t,\boldsymbol{x}')]\big).\end{aligned} \tag{6.2.9}$$

$$= \int \mathrm{d}^3\boldsymbol{x} \int \mathrm{d}^3\boldsymbol{x}'\, \mathrm{e}^{ip\cdot x - ip'\cdot x'}(\omega(\boldsymbol{p})\delta^3(\boldsymbol{x} - \boldsymbol{x}') + \omega(\boldsymbol{p}')\delta^3(\boldsymbol{x} - \boldsymbol{x}')) \tag{6.2.10}$$

$$= \int \mathrm{d}^3\boldsymbol{x}\, \mathrm{e}^{i(p-p')\cdot x}(\omega(\boldsymbol{p}) + \omega(\boldsymbol{p}')) \tag{6.2.11}$$

$$= (2\pi)^3\delta(\boldsymbol{p} - \boldsymbol{p}')2\omega(\boldsymbol{p}) \tag{6.2.12}$$

where again we've recognised the integral representation of the Dirac delta, and made use of the resulting Dirac delta to set $\boldsymbol{p} = \pm\boldsymbol{p}'$, and hence $p^0 = p'^0$ choosing to stay on-shell and positive energy. We introduce the shorthand notation,

$$(2\pi)^3\delta(\boldsymbol{p} - \boldsymbol{p}')2\omega(\boldsymbol{p}) = \text{đ}(\boldsymbol{p} - \boldsymbol{p}'). \tag{6.2.13}$$

This allows us to write

$$[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] = \text{đ}(\boldsymbol{p} - \boldsymbol{p}'). \tag{6.2.14}$$

Compare this to $[a, a^\dagger] = 1$ for the harmonic oscillator. We also have

$$[a(\boldsymbol{p}), a(\boldsymbol{p}')] = [a^\dagger(\boldsymbol{p}'), a^\dagger(\boldsymbol{p}')] = 0. \tag{6.2.15}$$

Note that $\text{đ}(\boldsymbol{p} - \boldsymbol{p}')$ is just a number, not an operator. This notation is nice, since if we combine the invariant measure, $\text{đ}p$, and $\text{đ}(\boldsymbol{p} - \boldsymbol{p}')$ then we can just pretend we have $\mathrm{d}p$ and $\delta^3(\boldsymbol{p} - \boldsymbol{p})$:

$$\int \text{đ}p\, f(\boldsymbol{p})\text{đ}(\boldsymbol{p} - \boldsymbol{p}') = f(\boldsymbol{p}'). \tag{6.2.16}$$

6.3 The Hamiltonian

To complete the analogy with the harmonic oscillator we need to find the Hamiltonian in terms of $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$. Starting with the hamiltonian

$$H = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\left[\pi(x)^2 + (\nabla\varphi(x))^2 + m^2\varphi(x)^2\right] \tag{6.3.1}$$

we can split this into three components,

$$H_1 = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\,\pi(x)^2, \tag{6.3.2}$$

$$H_2 = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\,(\nabla\varphi(x))^2, \tag{6.3.3}$$

$$H_3 = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\,m^2\varphi(x)^2, \tag{6.3.4}$$

$$\tag{6.3.5}$$

We'll then treat each of these separately.

Start with the first term. Substituting in the expansion for $\pi(x)$, being careful to use different integration variables for each factor of π, we get

$$H_1 = -\frac{1}{8}\int \mathrm{d}^3\boldsymbol{x}\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\left[a(\boldsymbol{p})\mathrm{e}^{-ip\cdot x} - a^\dagger(\boldsymbol{p})\mathrm{e}^{ip\cdot x}\right] \times\int\frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi)^3}\left[a(\boldsymbol{p}')\mathrm{e}^{-ip'\cdot x} - a^\dagger(\boldsymbol{p}')\mathrm{e}^{ip'\cdot x}\right]. \tag{6.3.6}$$

Expanding this all the integrand becomes

$$a(\boldsymbol{p})a(\boldsymbol{p}')\mathrm{e}^{-i(p+p')\cdot x} - a(\boldsymbol{p})a^\dagger(\boldsymbol{p}')\mathrm{e}^{-i(p-p')\cdot x} - a^\dagger(\boldsymbol{p})a(\boldsymbol{p}')\mathrm{e}^{i(p-p')\cdot x} + a^\dagger(\boldsymbol{p})a^\dagger(\boldsymbol{p}')\mathrm{e}^{i(p+p')\cdot x}. \tag{6.3.7}$$

We can perform the integration over position first, and, recalling that $p^0 = \omega(\boldsymbol{p})$, we get two types of terms, time dependent and time independent, these come from recognising the integral representation of the Dirac delta:

$$\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{e}^{\pm i(p+p')\cdot x} = (2\pi)^3\delta(\boldsymbol{p}+\boldsymbol{p}')\mathrm{e}^{\pm 2i\omega(\boldsymbol{p})t}, \tag{6.3.8}$$

$$\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{e}^{\pm i(p-p')\cdot x} = (2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{p}'). \tag{6.3.9}$$

We should expect that the time dependent terms in H will all cancel, since H is time independent.

After performing the integral over positions we are left with the following integrand

$$(2\pi)^3 a(\boldsymbol{p})a(\boldsymbol{p}')\delta(\boldsymbol{p}-\boldsymbol{p}')\mathrm{e}^{-2i\omega(\boldsymbol{p})t} - (2\pi)^3 a(\boldsymbol{p})a^\dagger(\boldsymbol{p}')\delta(\boldsymbol{p}-\boldsymbol{p}') - (2\pi)^3 a^\dagger(\boldsymbol{p})a(\boldsymbol{p}')\delta(\boldsymbol{p}-\boldsymbol{p}') + (2\pi)^3 a^\dagger(\boldsymbol{p})a^\dagger(\boldsymbol{p}')\delta(\boldsymbol{p}+\boldsymbol{p}')\mathrm{e}^{2i\omega(\boldsymbol{p})t}.$$

Performing the integral over $\boldsymbol{p}'$, and cancelling one factor of $(2\pi)^3$ we are left with

$$H_1 = -\frac{1}{8}\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\left[a(\boldsymbol{p})a(\boldsymbol{p})\mathrm{e}^{-2i\omega(\boldsymbol{p})t} - a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) - a^\dagger(\boldsymbol{p})a(\boldsymbol{p}) + a^\dagger(\boldsymbol{p})a^\dagger(\boldsymbol{p})\mathrm{e}^{2i\omega(\boldsymbol{p})t}\right]. \tag{6.3.10}$$

Keeping only the time independent terms, since the time independent terms will cancel with terms in H_2 and H_3, we get the contribution from the first term:

$$H_1 \to \frac{1}{8}\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\big[a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + a^\dagger(\boldsymbol{p})a(\boldsymbol{p})\big]. \tag{6.3.11}$$

Doing the same for the other two terms we get their time independent contributions:

$$H_2 \to \frac{1}{8}\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{\boldsymbol{p}^2}{\omega(\boldsymbol{p})^2}\big[a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + a^\dagger(\boldsymbol{p})a(\boldsymbol{p})\big], \tag{6.3.12}$$

$$H_3 \to \frac{1}{8}\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{m^2}{\omega(\boldsymbol{p})^2}\big[a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + a^\dagger(\boldsymbol{p})a(\boldsymbol{p})\big]. \tag{6.3.13}$$

See Chapter B for the full calculation. Noticing that $\boldsymbol{p}^2 + m^2$ cancels with $1/\omega(\boldsymbol{p})^2$ in the full expression we get

$$H = \frac{1}{4}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\big[a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + a(\boldsymbol{p})a^\dagger(\boldsymbol{p})\big]. \tag{6.3.14}$$

Compare this to the Harmonic oscillator result,

$$H = \frac{1}{2}\omega(aa^\dagger + a^\dagger a). \tag{6.3.15}$$

We can now compute the commutators of H with $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$:

$$\begin{aligned}
[H, a^\dagger(\boldsymbol{p})] &= \frac{1}{4}\int \frac{\mathrm{d}^3 p'}{(2\pi)^3}[a(\boldsymbol{p}')a^\dagger(\boldsymbol{p}') + a^\dagger(\boldsymbol{p}')a(\boldsymbol{p}'), a^\dagger(\boldsymbol{p})] && (6.3.16)\\
&= \frac{1}{4}\int \frac{\mathrm{d}^3 p'}{(2\pi)^3}\big([a(\boldsymbol{p}')a^\dagger(\boldsymbol{p}'), a^\dagger(\boldsymbol{p})] + [a^\dagger(\boldsymbol{p}')a(\boldsymbol{p}'), a^\dagger(\boldsymbol{p})]\big)\\
&= \frac{1}{4}\int \frac{\mathrm{d}^3 p'}{(2\pi)^3}\big(a(\boldsymbol{p}')[a^\dagger(\boldsymbol{p}'), a^\dagger(\boldsymbol{p})] + [a(\boldsymbol{p}'), a^\dagger(\boldsymbol{p})]a^\dagger(\boldsymbol{p}')\\
&\qquad + a^\dagger(\boldsymbol{p}')[a(\boldsymbol{p}'), a^\dagger(\boldsymbol{p})] + [a^\dagger(\boldsymbol{p}'), a^\dagger(\boldsymbol{p})]\big) && (6.3.17)\\
&= \frac{1}{4}\int \mathrm{d}^3 p'\, 2\omega(\boldsymbol{p})[\delta(\boldsymbol{p}-\boldsymbol{p}')a^\dagger(\boldsymbol{p}') + a^\dagger(\boldsymbol{p}')\delta(\boldsymbol{p}'-\boldsymbol{p})] && (6.3.18)\\
&= \omega(\boldsymbol{p})a^\dagger(\boldsymbol{p}). && (6.3.19)
\end{aligned}$$

Similarly, with $a(\boldsymbol{p})$ we get

$$[H, a(\boldsymbol{p})] = -\omega(\boldsymbol{p})a(\boldsymbol{p}). \tag{6.3.20}$$

Compare these to the harmonic oscillator results, $[H, a^\dagger] = \omega a^\dagger$ and $[H, a] = -\omega a$.

We can view the quantum fields as being a quantum harmonic oscillator of energy $\omega(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m^2}$ for every Fourier mode, $\boldsymbol{p}$.

6.4 Particle Interpretation

Consider an energy eigenstate, $|E\rangle$, with eigenvalue E, so $H|E\rangle = E|E\rangle$. Then the state $a^\dagger(\boldsymbol{p})|E\rangle$ is also an energy eigenstate, with energy $E + \omega(\boldsymbol{p})$:

$$\begin{aligned}
Ha^\dagger(\boldsymbol{p})|E\rangle &= \big([H, a^\dagger(\boldsymbol{p})] + a^\dagger(\boldsymbol{p})H\big)|E\rangle && (6.4.1)\\
&= (\omega(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + a^\dagger(\boldsymbol{p})E) && (6.4.2)\\
&= (\omega(\boldsymbol{p}) + E)a^\dagger(\boldsymbol{p})|E\rangle. && (6.4.3)
\end{aligned}$$

Similarly, $a(\boldsymbol{p})|E\rangle$ is an energy eigenstate with energy $E - \omega(\boldsymbol{p})$.

This suggests that, following the interpretation of the harmonic oscillator, $a^\dagger(\boldsymbol{p})$ is a **creation operator**, creating a quantum with energy $\omega(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m^2}$, and $a(\boldsymbol{p})$ is an **annihilation operator**, destroying a quantum with energy $\omega(\boldsymbol{p})$. Note that all quanta in this interpretation have positive energy. There is another interpretation, which we'll discuss later, where $a^\dagger(\boldsymbol{p})$ creates a quantum with energy $\omega(\boldsymbol{p})$ and $a(\boldsymbol{p})$ creates a quantum with energy $-\omega(\boldsymbol{p})$.

Since the Hamiltonian is nonnegative, its defined as a sum of squares of real quantities, there must be some lowest energy state, $|0\rangle$, which we call the **vacuum state**. This state is such that any annihilation operator acting on it gives the zero vector:

$$a(\boldsymbol{p})|0\rangle = 0 \quad \forall \boldsymbol{p}. \tag{6.4.4}$$

Starting with $|0\rangle$ we can act with creation operators to make new states. Each creation operator, $a^\dagger(\boldsymbol{p})$, makes a new particle with energy $\omega(\boldsymbol{p})$ and momentum $\boldsymbol{p}$. For example,

$$|\psi\rangle = a^\dagger(\boldsymbol{p})a^\dagger(\boldsymbol{q})a^\dagger(\boldsymbol{r})|0\rangle \tag{6.4.5}$$

is a state with three particles with total energy $\omega(\boldsymbol{p}) + \omega(\boldsymbol{q}) + \omega(\boldsymbol{r})$, and total momentum $\boldsymbol{p} + \boldsymbol{q} + \boldsymbol{r}$.

Since all $a^\dagger(\boldsymbol{p})$ commute the state

$$a^\dagger(\boldsymbol{p})a^\dagger(\boldsymbol{q})a^\dagger(\boldsymbol{r})|0\rangle \tag{6.4.6}$$

is the same as $|\psi\rangle$. We interpret this as meaning that the particles are bosons, swapping any two particles leaves the state unchanged. This means the particles follow Bose–Einstein statistics.

The space of all states which can be constructed in this way, called **Fock space**, is the space given by all complex-linear combinations of

$$a^\dagger(\boldsymbol{p_1}) \cdots a^\dagger(\boldsymbol{p_n})|0\rangle \tag{6.4.7}$$

for any values of $n \in \mathbb{N}$.

Formally, the Fock space (for bosons) is defined to be

$$F_+(\mathbb{H}) = \overline{\bigoplus_{n=0}^{\infty} S\mathbb{H}^{\otimes n}} = \overline{\mathbb{C} \oplus \mathbb{H} \oplus S(\mathbb{H} \otimes \mathbb{H}) \oplus S(\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H}) \oplus \cdots} \tag{6.4.8}$$

where $\mathbb{H}$ is the single particle Hilbert space,

$$\mathbb{H}^{\otimes n} \coloneqq \bigotimes_{i=1}^{n} \mathbb{H} = \underbrace{\mathbb{H} \otimes \cdots \otimes \mathbb{H}}_{n \text{ times}}, \tag{6.4.9}$$

the operator S symmetrises all tensors, the direct sum, $\oplus$, runs from $n = 0$, for states with no particles, to infinity, with each term representing an n-particle state space, and the line represents that we complete the space to form a Hilbert space, this entails adding in the values of all absolutely convergent series.

We make the normalisation choice that $\langle 0|0\rangle = 1$, then, if $|\boldsymbol{p}\rangle = a^\dagger(\boldsymbol{p})|\boldsymbol{p}\rangle$ is a single particle state we have

$$\langle \boldsymbol{p}|\boldsymbol{q}\rangle = \langle 0|a(\boldsymbol{p})a^\dagger(\boldsymbol{q})|0\rangle \tag{6.4.10}$$

$$= \langle 0|[a(\boldsymbol{p}), a^\dagger(\boldsymbol{q})]|0\rangle + \langle 0|a^\dagger(\boldsymbol{q})a(\boldsymbol{p})|0\rangle \tag{6.4.11}$$

$$= \delta(\boldsymbol{p} - \boldsymbol{q}). \tag{6.4.12}$$

Note that $a(\boldsymbol{p})|0\rangle$ and $\langle 0|a^\dagger(\boldsymbol{q}) = [a(\boldsymbol{q})|0\rangle]^\dagger$ both vanish, since they are annihilators acting on the vacuum state.

As with the harmonic oscillator we can define a **number density operator**, $N(\boldsymbol{p}) \coloneqq a^\dagger(\boldsymbol{p})a(\boldsymbol{p})$, the eigenvalues of which integrate to give the number of particles in the state with momentum $\boldsymbol{p}$.

6.5 Ground State Energy

We can rewrite the Hamiltonian in terms of the number density operator:

$$H = \frac{1}{4}\int đp\, \omega(\boldsymbol{p})\big(a^\dagger(\boldsymbol{p})a(\boldsymbol{p}) + a(\boldsymbol{p})a^\dagger(\boldsymbol{p})\big) \tag{6.5.1}$$

$$= \frac{1}{2}\int đp\, \omega(\boldsymbol{p})\left(a^\dagger(\boldsymbol{p})a(\boldsymbol{p}) + \frac{1}{2}[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p})]\right) \tag{6.5.2}$$

$$= \frac{1}{2}\int đp\, \omega(\boldsymbol{p})\big(N(\boldsymbol{p}) + \omega(\boldsymbol{p})(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{p})\big) \tag{6.5.3}$$

$$= \frac{1}{2}\int đp\, \omega(\boldsymbol{p})\big(N(\boldsymbol{p}) + \omega(\boldsymbol{p})(2\pi)^3\delta^3(\boldsymbol{0})\big). \tag{6.5.4}$$

Clearly we have $\langle 0|N(\boldsymbol{p})|0\rangle = \langle 0|a^\dagger(\boldsymbol{p})a(\boldsymbol{p})|0\rangle = 0$, and so

$$\langle 0|H|0\rangle = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{p}\, \omega(\boldsymbol{p})\delta^3(\boldsymbol{0}). \tag{6.5.5}$$

This is infinite. In fact, its *very* infinite. Each $\delta(0)$ is infinite, and we have three of them multiplied together, then we integrate over all momentum space, which is infinite, to make matters worse $\omega(\boldsymbol{p})$ grows arbitrarily large with $|\boldsymbol{p}|$.

The solution, as with so many similar problems in physics, is just not to think about it. All that we can measure is the energy of particles absorbed and emitted by the system. We cannot measure the ground state energy, so it doesn't matter that our theory gives a nonsensical answer when we try to compute the expected ground state energy.

6.5.1 Normal Ordering

One solution to the infinity arising in the ground state energy is to be careful about how we order $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$. If we keep all $a^\dagger(\boldsymbol{p})$ on the left of all $a(\boldsymbol{p})$ then, when we take expectation values, one of our $a(\boldsymbol{p})$ will act on $|0\rangle$ giving the zero vector, and one of our $a^\dagger(\boldsymbol{p})$ will act on $\langle 0|$, also giving the zero vector.

Mathematically what happens is that when we change to order of operators to have all $a^\dagger(\boldsymbol{p})$ on the left we'll get commutators of $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p}')$, which give rise to Dirac deltas which cancel with the existing ones, to leave a finite result.

This idea that careful ordering of operators can avoid infinities goes beyond the expectation value of the ground state energy. Consider our field, $\varphi(x)$. We can split this into parts,

$$\varphi(x) = \varphi^+(x) + \varphi^-(x) \tag{6.5.6}$$

where

$$\varphi^+(x) = \int đp\, a(\boldsymbol{p})\mathrm{e}^{-ip\cdot x}, \tag{6.5.7}$$

and

$$\varphi^-(x) = \int đp\, a^\dagger(\boldsymbol{p}) e^{ip\cdot x} = (\varphi^+)^\dagger. \tag{6.5.8}$$

If these were single particle states in quantum mechanics then they would correspond to positive and negative energy states respectively, but they aren't, although we'll see an interpretation along these lines later.

Now suppose we want to consider the product $\varphi(x)\varphi(y)$ for two arbitrary spacetime points, x and y. We can expand this as follows:

$$\begin{aligned}\varphi(x)\varphi(y) &= [\varphi^+(x) + \varphi^-(x)][\varphi^+(y) + \varphi^-(y)] &(6.5.9)\\ &= \varphi^+(x)\varphi^+(y) + \varphi^+(x)\varphi^-(y) + \varphi^-(x)\varphi^+(y) + \varphi^-(x)\varphi^-(y). &(6.5.10)\end{aligned}$$

The **normal ordering** of $\varphi(x)\varphi(y)$, denoted $:\varphi(x)\varphi(y):$, is defined as the result of taking this product, but swapping terms so that all $a^\dagger(\boldsymbol{p})$ appear on the left of all $a(\boldsymbol{p}')$, that is all φ^+ appear on the left of all φ^-, so

$$:\varphi(x)\varphi(y): = \varphi^+(x)\varphi^+(y) + \varphi^+(x)\varphi^-(y) + \varphi^+(y)\varphi^-(x) + \varphi^-(x)\varphi^-(y). \tag{6.5.11}$$

Then we have

$$\langle 0|:\varphi(x)\varphi(y):|0\rangle = 0. \tag{6.5.12}$$

[1] That is, X is not just a scalar, in which case $\langle 0|:X:|0\rangle = X\langle 0|0\rangle = X$.

In general, if X is any nontrivial[1] polynomial in $a^\dagger(\boldsymbol{p})$ and $a(\boldsymbol{p})$ we'll have

$$\langle 0|:X:|0\rangle = 0. \tag{6.5.13}$$

Subtracting the zero point energy of H from H is equivalent to normal ordering

$$:H: = H - \langle 0|H|0\rangle, \tag{6.5.14}$$

since by normal ordering H we swap exactly those terms which give infinity, and all others give zero, so we can think of normal ordering as subtracting them off.

From now on we'll assume that H and $\boldsymbol{p}$ are normal ordered, often without writing it.

6.6 Ground State Momentum

The physical momentum operator is given by

$$\boldsymbol{P} = \int \mathrm{d}^3\boldsymbol{x}\, \pi(x)\nabla\varphi(x). \tag{6.6.1}$$

Recall that this comes from the T^{i0} component of the energy-momentum tensor. Expressing the fields in terms of $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$ we get

$$\boldsymbol{P} = \frac{1}{2} đp\, \boldsymbol{p}(a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + a^\dagger(\boldsymbol{p})a(\boldsymbol{p})) \tag{6.6.2}$$

where the factor of $\boldsymbol{p}$ comes from computing $\nabla\varphi$. Rewriting this using the commutator of $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$ we get

$$\boldsymbol{P} = \int đp\, \boldsymbol{p}(N(\boldsymbol{p}) + \omega(\boldsymbol{p})(2\pi)^3\delta(\boldsymbol{0})). \tag{6.6.3}$$

Unlike with the ground state energy the $\delta(\mathbf{0})$ factor doesn't cause a problem. Integrating over all modes in the modes with momenta $\boldsymbol{p}$ and $-\boldsymbol{p}$ cancel, since $\omega(\boldsymbol{p}) = \omega(-\boldsymbol{p})$. So, we have

$$\langle 0|\boldsymbol{P}|0\rangle = \langle 0|:\boldsymbol{P}:|0\rangle = 0, \tag{6.6.4}$$

that is, the vacuum state has no momentum, which really ought to be the case.

Seven

Covariant Commutators

7.1 What are They

The equal time commutation relations treat time specially. This is generally undesirable in relativity. So, in this section we derive a commutator, $[\varphi(x), \varphi(y)]$ at two arbitrary spacetime points, x and y. We start by writing

$$\varphi(x) = \varphi^+(x) + \varphi^-(x), \tag{7.1.1}$$

where φ^+ and φ^- are as in Equations (6.5.7) and (6.5.8). Since φ^+ only consists of annihilation operators and φ^- only consists of creation operators we have

$$[\varphi^+(x), \varphi^+(y)] = [\varphi^-(x), \varphi^-(y)] = 0. \tag{7.1.2}$$

Hence, expanding out the commutator we have

$$[\varphi(x), \varphi(y)] = [\varphi^-(x), \varphi^+(y)] + [\varphi^-(x), \varphi^+(y)]. \tag{7.1.3}$$

Consider the first of these,

$$[\varphi^+(x), \varphi^-(y)] = \int đp \int đp' \, [a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] e^{-ip\cdot x + ip'\cdot y} \tag{7.1.4}$$

$$= \int đp \int đp' \, \delta(\boldsymbol{p} - \boldsymbol{p}') e^{-ip\cdot x + ip'\cdot y} \tag{7.1.5}$$

$$= \int đp \, e^{-ip\cdot(x-y)} \tag{7.1.6}$$

$$=: i\Delta^+(x-y), \tag{7.1.7}$$

where the last line defines Δ^+:

$$\Delta^+(x) := i \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{2\omega(\boldsymbol{p})} e^{ip\cdot x} \tag{7.1.8}$$

The factor of i is a convention to make Δ^+ a real function.

The second commutator can now easily be worked out using antisymmetry:

$$[\varphi^-(x), \varphi^+(y)] = -[\varphi^+(y), \varphi^-(x)] = -i\Delta^+(y-x) =: i\Delta^-(x-y), \tag{7.1.9}$$

where the last equality defines Δ^-.

Combining these results we have

$$[\varphi(x), \varphi(y)] = i\Delta^+(x-y) + i\Delta^-(x-y) =: i\Delta(x-y) \tag{7.1.10}$$

where

$$\Delta(x) \coloneqq \Delta^+(x) + \Delta^-(x) \tag{7.1.11}$$
$$= i \int đp(e^{ip\cdot x} - e^{-ip\cdot x}) \tag{7.1.12}$$
$$= -2 \int đp\ \sin(p \cdot x). \tag{7.1.13}$$

Note that Δ is a real ($\Delta(x) = \Delta(x)^*$), odd ($\Delta(-x) = -\Delta(x)$) function. This is what we would expect since the field is real and the commutator is antisymmetric.

Consider the full form of Δ^+:

$$i\Delta^+(x) = \int \frac{d^4p}{(2\pi)^4} 2\pi\delta(p^2 - m^2)\theta(p_0)e^{-ip\cdot x}. \tag{7.1.14}$$

We can also write Δ^- similarly:

$$i\Delta^-(x) = -\int \frac{d^4p}{(2\pi)^4} 2\pi\delta(p^2 - m^2)\theta(p_0)e^{ip\cdot x}. \tag{7.1.15}$$

If we make a change of variables, $p \to -p$, then $d^4p \to d^4p$, and

$$i\Delta^-(x) = -\int \frac{d^4p}{(2\pi)^4} 2\pi\delta(p^2 - m^2)\theta(-p_0)e^{-ip\cdot x}. \tag{7.1.16}$$

This allows us to write Δ as

$$i\Delta(x) = \int \frac{d^4p}{(2\pi)^4} 2\pi\delta(p^2 - m^2)\varepsilon(p_0)e^{-ip\cdot x} \tag{7.1.17}$$

where

$$\varepsilon(p_0) \coloneqq \theta(p_0) - \theta(-p_0) = \begin{cases} +1 & p_0 > 0, \\ 0 & p_0 = 0, \\ -1 & p_0 < 0, \end{cases} \tag{7.1.18}$$

is the sign function.

Note that for proper orthochronous Lorentz transformations Δ is Lorentz invariant, since the sign of p_0 doesn't change and all other terms are manifestly Lorentz invariant.

7.2 Microcausality

Suppose x is space-like, that is $x^2 < 0$. Then there exists some proper orthochronous Lorentz transformation, Λ, taking x to $-x$. Space-like vectors lie outside the light cone, and we can view this transformation as rotating around the light cone. Since $\Delta^\pm(x)$ are individually Lorentz invariant we can apply a Lorentz transformation to one of them and not the other when computing Δ and get the same result, so

$$\Delta(x) = \Delta^+(x) + \Delta^-(x) \tag{7.2.1}$$
$$= \Delta^+(x) + \Delta^-(\Lambda x) \tag{7.2.2}$$
$$= \Delta^+(x) + \Delta^-(-x) \tag{7.2.3}$$
$$= \Delta^+(x) - \Delta^+(x) \tag{7.2.4}$$
$$= 0, \tag{7.2.5}$$

so Δ vanishes for all space-like points. Hence, $[\varphi(x), \varphi(y)]$ vanishes for all space-like separations, $x - y$.

On the other hand no such transformation exists for time-like or light-like points, since it would have to map from one half of the light cone to the other, which is a discontinuous transformation.

We can interpret this as a requirement for causality, specifically for a nonzero commutator between fields at space-like separated points which happen to have the same time coordinate we'd need some form of communication at this time between these points, and that can't happen between space-like points at the same time without violating causality.

This analysis tells us that Δ is not analytic, it has a sudden jump from zero to nonzero at light-like points.

7.3 Equal Time Commutation Relations

The equal time commutation relations can now be viewed as a special case of the covariant commutation relations. Since we have $\pi = \dot{\varphi}$ we want to compute $[\varphi(t, \boldsymbol{x}), \pi(t, \boldsymbol{x}')] = [\varphi(t, \boldsymbol{x}), \dot{\varphi}(t, \boldsymbol{x}')]$. To do this take the time derivative of Δ:

$$\frac{\partial}{\partial x^0}\Delta(x-y) = i\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2\omega(\boldsymbol{p})}\frac{\partial}{\partial x^0}\left[\mathrm{e}^{ip\cdot(x-y)} - \mathrm{e}^{-ip\cdot(x-y)}\right] \tag{7.3.1}$$

$$= i\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2\omega(\boldsymbol{p})} i\omega(\boldsymbol{p})\left[\mathrm{e}^{ip\cdot(x-y)} + \mathrm{e}^{-ip\cdot(x-y)}\right] \tag{7.3.2}$$

$$= -\frac{1}{2}\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\left[\mathrm{e}^{ip\cdot(x-y)} + \mathrm{e}^{-ip\cdot(x-y)}\right]. \tag{7.3.3}$$

For the equal time commutation relations we have $x^0 = y^0$ and so

$$p\cdot(x-y) = \boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y}). \tag{7.3.4}$$

Hence,

$$\frac{\partial}{\partial x^0}\Delta(x-y) = -\frac{1}{2}\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\left[\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} + \mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right] \tag{7.3.5}$$

$$= -\delta^3(\boldsymbol{x}-\boldsymbol{y}), \tag{7.3.6}$$

where we've recognised the integral representation of the Dirac delta:

$$\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\mathrm{e}^{\pm i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} = \delta^3(\boldsymbol{x}-\boldsymbol{y}). \tag{7.3.7}$$

We then have

$$\frac{\partial}{\partial x^0}[i\Delta(x-y)]\Big|_{x^0=y^0=t} = \frac{\partial}{\partial x^0}[\varphi(t,\boldsymbol{x}),\varphi(t,\boldsymbol{y})] \tag{7.3.8}$$

$$= [\dot{\varphi}(t,\boldsymbol{x}),\varphi(t,\boldsymbol{y})] \tag{7.3.9}$$

$$= [\pi(t,\boldsymbol{x}),\varphi(t,\boldsymbol{y})] \tag{7.3.10}$$

$$= -i\delta^3(\boldsymbol{x}-\boldsymbol{y}) \tag{7.3.11}$$

Hence,

$$[\varphi(t,\boldsymbol{x}),\pi(t,\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y}). \tag{7.3.12}$$

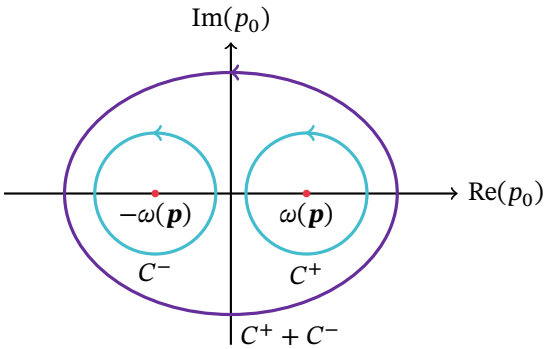


Figure 7.1: The contours used in the contour representation of $\Delta^\pm$ and Δ.

So, we can view the equal time commutation relations as a specific case of the covariant commutation relations. This is why the equal time commutation relations can still produce Lorentz invariant physics.

7.4 Contour Representation

R This section makes use of some complex analysis, in particular contour integrals and the residue theorem. See either *Methods of Theoretical Physics* or *Methods of Mathematical Physics* for details.

The contour representation of $\Delta^\pm$ is

$$\Delta^\pm(x) = -\int_{C^\pm} \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{e}^{-ip\cdot x}}{p^2 - m^2}. \tag{7.4.1}$$

The contours $C^\pm$ are anticlockwise circles containing the poles at $\pm\omega(\boldsymbol{p})$. This is shown in Figure 7.1.

To see why this works start with

$$p^2 - m^2 = (p_0 + \omega(\boldsymbol{p}))(p_0 - \omega(\boldsymbol{p})), \tag{7.4.2}$$

which shows there are poles at $p_0 = \pm\omega(\boldsymbol{p})$. We can then apply the residue theorem, which says that

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_i \mathrm{Res}(f, a_i) \tag{7.4.3}$$

where γ is a closed contour, f is analytic on and inside γ, a_i are the poles of f inside γ, and $\mathrm{Res}(f, a_i)$ is the residue of f at a_i.

We have a particularly easy case of two simple poles. In the case of a simple pole the residue at a_i is given by

$$\mathrm{Res}(f, a_i) = (z - a_i) f(z)|_{z=a_i}, \tag{7.4.4}$$

so

$$\mathrm{Res}\left(\frac{\mathrm{e}^{-ip\cdot x}}{(p_0 + \omega(\boldsymbol{p}))(p_0 - \omega(\boldsymbol{p}))}, \pm\omega(\boldsymbol{p})\right) = \frac{\mathrm{e}^{-ip\cdot x}}{p_0 \pm \omega(\boldsymbol{p})}\bigg|_{p_0 = \mp\omega(\boldsymbol{p})} = \mp\frac{\mathrm{e}^{-ip\cdot x}}{2\omega(\boldsymbol{p})}. \tag{7.4.5}$$

Hence,

$$-\int_{C^+} \frac{\mathrm{d}p_0}{2\pi} \frac{\mathrm{e}^{-ip\cdot x}}{p^2 - m^2} = -i \frac{\mathrm{e}^{-ip\cdot x}}{2\omega(\boldsymbol{p})}\bigg|_{p_0=\omega(\boldsymbol{p})}, \tag{7.4.6}$$

and

$$-\int_{C^+} \frac{\mathrm{d}p_0}{2\pi} \frac{\mathrm{e}^{-ip\cdot x}}{p^2 - m^2} = i \frac{\mathrm{e}^{ip\cdot x}}{2\omega(\boldsymbol{p})}\bigg|_{p_0=-\omega(\boldsymbol{p})}. \tag{7.4.7}$$

That is,

$$-\int_{C^\pm} \frac{\mathrm{d}p_0}{2\pi} \frac{\mathrm{e}^{-ip\cdot x}}{p^2 - m^2} = \mp i \frac{\mathrm{e}^{\mp ip\cdot x}}{2\omega(\boldsymbol{p})}\bigg|_{p_0=\omega(\boldsymbol{p})}. \tag{7.4.8}$$

Having performed the integral over p_0 we are left with an integral over $\boldsymbol{p}$ so

$$-\int_{C^\pm} \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{e}^{-ip\cdot x}}{p^2 - m^2} = \mp i \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \frac{\mathrm{e}^{mpip\cdot x}}{2\omega(\boldsymbol{p})} = \Delta^\pm(x). \tag{7.4.9}$$

The contour representation of Δ is then simply given by

$$\Delta(x) = -\int_C \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{e}^{-ip\cdot x}}{p^2 - m^2} \tag{7.4.10}$$

where $C = C^+ + C^-$ is the combined contour, with the middle parts cancelling out, to give a contour containing both poles as shown in Figure 7.1.

7.5 Δ as Green's Functions

To interpret Δ and $\Delta^\pm$ consider the Klein–Gordon equation,

$$0 = (\partial^2 + m^2)\varphi. \tag{7.5.1}$$

Taking the Fourier transform and using the rule that $\partial_x \to -ip$ we have

$$0 = ((-ip)^2 + m^2)\tilde{\varphi} \implies (p^2 - m^2)\tilde{\varphi} = 0. \tag{7.5.2}$$

Transforming back then gives us the contour representation of Δ or $\Delta^\pm$, depending on the boundary conditions.

This means we can interpret Δ and $\Delta^\pm$ as classical Green's functions for the Klein–Gordon equation, so

$$(\partial^2 + m^2)\Delta(x) = \delta^4(x). \tag{7.5.3}$$

We call Δ and $\Delta^\pm$ propagators, since they tell us how a quantum propagates between states.

Part II

Interactions

Eight

Scattering

8.1 Interactions

Quantum things are generally more difficult than classical things.

Richard Ball

So far, we have only considered Lagrangians which are quadratic in the field, we call these **free field Lagrangians**, since quadratic Lagrangians give linear equations of motion with plane wave solutions, which we interpret as many independent harmonic oscillators, each corresponding to a noninteracting particle. In order to have our particles do anything, such as be measured, we need interactions. This means introducing a nonquadratic term to our Lagrangian.

8.1.1 φ^3 Theory

The simplest nonquadratic term that we can add to our Lagrangian is a cubic term, giving the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}m^2\varphi^2 - \frac{1}{6}g\varphi^3. \tag{8.1.1}$$

The first two terms are just the usual Klein–Gordon Lagrangian, call it $\mathcal{L}$. We can treat this as the Lagrangian for a free particle. The extra term, $-g\varphi^3/6$, is our interaction term, call it $\mathcal{L}_\mathrm{I}$. We call a theory with a cubic term like this a **φ^3 theory** The factor of $1/6$ in this factor is just conventional, its there to make equations simpler later. A φ^n theory would equivalently have a factor of $1/n!$, each time we differentiate this factor becomes closer to 1. The factor of g is called the **coupling constant**. It measures how strong the interaction is.

We can easily apply the Euler–Lagrange equations to this modified Lagrangian, without much changing. We have

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \partial^2\varphi, \qquad \text{and} \qquad \frac{\partial\mathcal{L}}{\partial\varphi} = -m^2\varphi - \frac{1}{3}g\varphi^2. \tag{8.1.2}$$

Hence, we have the equations of motion

$$(\partial^2 + m^2)\varphi = -\frac{1}{2}g\varphi^2. \tag{8.1.3}$$

This is just the Klein–Gordon equation with some sort of source term.

This is now a nonlinear differential equation. These are, in general, hard to solve. There is no general theory and we are usually restricted to specific examples with solutions. Because of this we assume that the interaction isn't very strong, that is that $g \ll 1$, and then we solve by expanding in g and truncating at some appropriate point in our calculations.

8.2 The S Matrix

Most processes in physics are, when it comes down to it, scattering. We see light after it scatters off an object, two objects collide in classical mechanics, two protons collide in the LHC, a whole bunch of particles collide all over the place and a statistical study of this gives statistical mechanics, and so on. Scattering is a very simple process, there are three steps:

- We start in some initial state with free particles of definite momentum, spin, etc.
- The particles undergo some scattering process, which we treat as a black box.
- We finish in some final state with free particles of definite momentum, spin, etc.

In order to allow us to consider free particles we assume that the initial state happens a long long time before the scattering, taking $t \to -\infty$. Similarly we assume the final state is a long long time after the scattering, taking $t \to +\infty$. Call the initial state $|\Psi, -\infty\rangle$ and the final state $|\Psi, +\infty\rangle$. Then an in between state at time t is $|\Psi, t\rangle$.

The initial and final state must somehow be related if we are to pass from one to the other. We take this relation to be of the form

$$|\Psi, +\infty\rangle = S|\Psi, -\infty\rangle \tag{8.2.1}$$

where S, known as the ***S* matrix**, is the operator describing the scattering process.

Now suppose that we start with some fixed, known initial state, $|i\rangle$, so $|\Psi, -\infty\rangle = |i\rangle$. We can choose some complete orthonormal basis of possible final states, $\{|f\rangle\}$, which allows us to expand the final state as

$$|\Psi, +\infty\rangle = \sum_f |f\rangle\langle f|\Psi, +\infty\rangle. \tag{8.2.2}$$

We interpret

$$|\langle f|\Psi, +\infty\rangle|^2 = \mathbb{P}(i \to f) \tag{8.2.3}$$

as the probability that when we start in state $|i\rangle$ our scattering process finishes in state $|f\rangle$.

Using the definition of S we can write the amplitude as

$$\langle f|\Psi, +\infty\rangle = \langle f|S|\Psi, -\infty\rangle = \langle f|S|i\rangle =: S_{fi}, \tag{8.2.4}$$

where $S_{fi} \coloneqq \langle f|S|i\rangle$ is a matrix element of the scattering operator. So the matrix elements of the scattering operator tell us the amplitude for a particular scattering process.

We can expand the final state as

$$|\Psi, +\infty\rangle = \sum_f |f\rangle S_{fi}, \tag{8.2.5}$$

and take the conjugate to get

$$\langle\Psi, +\infty| = \sum_f S^*_{if}\langle f|. \tag{8.2.6}$$

Combining this, and working with normalised states, we get

$$1 = \langle\Psi, +\infty|\Psi, +\infty\rangle = \sum_{f,f'} S_{fi}S^*_{if'}\langle f|f'\rangle = \sum_{f,f'} S_{fi}S^*_{if'}\delta_{ff'} = \sum_f S^*_{if}S_{fi} \tag{8.2.7}$$

or in terms of operators,

$$\mathbb{1} = S^\dagger S, \tag{8.2.8}$$

so the scattering operator is unitary.

We can interpret this as conservation of probability, since $\{|f\rangle\}$ is a complete set of states we must have

$$\sum_f \mathbb{P}(i \to f) = 1, \tag{8.2.9}$$

and hence if we have a definite start state that is also a complete set, so the evolution must be unitary.

Note that the notion of the S matrix as discussed here is more general than in relativistic quantum mechanics since in QFT we can create and destroy particles, so our scattering can include processes like annihilation or pair production.

8.3 Interaction Picture

We now introduce the third, and final, picture of quantum mechanics, called the **interaction picture**, or **Dirac picture**. Suppose we have a Hamiltonian

$$H = H_0 + H_\mathrm{I} \tag{8.3.1}$$

where H_0 describes a free field and H_I an interaction.

In the Dirac picture both operators and states have time dependence. The states evolve as if they were in the Schrödinger picture, but under only the free Hamiltonian, so

$$|\psi, t\rangle_\mathrm{D} \coloneqq \mathrm{e}^{iH_0 t}|\psi, t\rangle_\mathrm{S} = \mathrm{e}^{iH_0 t}\mathrm{e}^{-iHt}|\psi\rangle_\mathrm{H} \tag{8.3.2}$$

where subscripts D, S, and H denote the Dirac, Schrödinger, and interaction pictures respectively. The first equality is the definition of a state in the Dirac picture, and the second comes from inverting the definition of a state in the Heisenberg picture:

$$|\psi\rangle_\mathrm{H} \coloneqq \mathrm{e}^{iHt}|\psi, t\rangle_\mathrm{S}. \tag{8.3.3}$$

Operators in the Dirac picture transform as if they were in the Heisenberg picture, but only under the free Hamiltonian, so

$$A_{\mathrm{D}}(t) = \mathrm{e}^{iH_0 t} A_{\mathrm{S}} \mathrm{e}^{-iH_0 t}. \tag{8.3.4}$$

Take the derivative of the equation defining a state in the Dirac picture, this gives

$$i\frac{\partial}{\partial t}|\psi,t\rangle_{\mathrm{D}} = -H_0 \mathrm{e}^{iH_0 t}|\psi,t\rangle_{\mathrm{S}} + \mathrm{e}^{iH_0 t} i\frac{\partial}{\partial t}|\psi,t\rangle_{\mathrm{S}}. \tag{8.3.5}$$

Now recognise the final term as $\mathrm{e}^{iH_0 t}$ times one side of the Schrödinger equation:

$$i\frac{\partial}{\partial t}|\psi,t\rangle_{\mathrm{S}} = H|\psi,t\rangle_{\mathrm{S}} = H_0|\psi,t\rangle_{\mathrm{S}} + H_{\mathrm{I}}|\psi,t\rangle_{\mathrm{S}}. \tag{8.3.6}$$

Substituting this into the previous equation the first term cancels leaving us with

$$i\frac{\partial}{\partial t}|\psi,t\rangle_{\mathrm{D}} = \mathrm{e}^{iH_0 t} H_{\mathrm{I}}|\psi,t\rangle_{\mathrm{S}} = \mathrm{e}^{iH_0 t} H_{\mathrm{I}} \mathrm{e}^{-iH_0 t}|\psi,t\rangle_{\mathrm{D}} = H_{\mathrm{I}}^{\mathrm{D}}|\psi,t\rangle_{\mathrm{D}}. \tag{8.3.7}$$

This shows that states evolve in time according to the free Hamiltonian, H_0. The time evolution due to the interaction is exhibited by the operators.

We derived the Heisenberg equation by differentiating an arbitrary operator in both pictures. Doing the same here, between the Dirac and Schrödinger picture gives rise to the equation of motion

$$\frac{\partial}{\partial t} A_{\mathrm{D}}(t) = i[H_0, A_{\mathrm{D}}(t)]. \tag{8.3.8}$$

Note that $H_0^{\mathrm{S}} = H_0^{\mathrm{D}} = H_0^{\mathrm{H}}$, but in general $H_{\mathrm{I}}^{\mathrm{D}} \neq H_{\mathrm{I}}^{\mathrm{S}}$, which comes from the general fact that $[H_0, H_{\mathrm{I}}] \neq 0$.

From now on we will work entirely in the Dirac picture. Accordingly we will drop the labels D, S, and H.

8.4 Dyson Series

The **Dyson series** is a formal solution to the evolution equation

$$i\frac{\partial}{\partial t}|\Psi,t\rangle = H_{\mathrm{I}}(t)|\Psi,t\rangle. \tag{8.4.1}$$

Now we make the time dependence of the interaction Hamiltonian explicit. The solution to this is

$$|\Psi,t\rangle = |\Psi,-\infty\rangle - i\int_{-\infty}^{t} \mathrm{d}t_1\, H_{\mathrm{I}}(t_1)|\Psi,t_1\rangle. \tag{8.4.2}$$

To see this just take the derivative:

$$i\frac{\partial}{\partial t}|\Psi,t\rangle = i\frac{\partial}{\partial t}|\Psi,-\infty\rangle + \frac{\partial}{\partial t}\int_{-\infty}^{t} \mathrm{d}t_1\, H_{\mathrm{I}}(t_1)|\Psi,t_1\rangle = H_{\mathrm{I}}(t)|\Psi,t\rangle. \tag{8.4.3}$$

Here the first term vanishes as it is constant and the second term is easily computed by an application of the fundamental theorem of calculus.

The problem is that this solution for $|\Psi, t\rangle$ has $|\Psi, t_1\rangle$ in it, in particular we integrate up to $t_1 = t$. This means that we need to know the state at time t in order to evaluate the state at time t. This seems like a problem. The solution is that we are assuming weak interactions. This means that the operator $H_I(t)$ is, in some sense, "small", and the operator $H_I(t)H_I(t_1)$ is "smaller". Of course, operators don't have sizes in this way, but we can think about specific matrix elements and it all works out. This suggests that we can solve this problem recursively with each level of recursion being a smaller and smaller correction. The first level of recursion is to substitute $|\Psi, t_1\rangle$ with the entire right hand side of Equation (8.4.2), being careful to avoid reusing integration variables, giving

$$|\Psi, t\rangle = |\Psi, -\infty\rangle - i \int_{-\infty}^{t} \mathrm{d}t_1 \, H_I(t_1) \left[|\Psi, -\infty\rangle - i \int_{-\infty}^{t_1} \mathrm{d}t_2 \, H_I(t_2) |\Psi, t_2\rangle \right]. \quad (8.4.4)$$

Note that the second integral is quadratic in the Hamiltonian, so is "smaller" than the rest of the terms. We can repeat this process, writing $|i\rangle = |\Psi, -\infty\rangle$ for compactness, and we get

$$|\Psi, t\rangle = |i\rangle - i \int_{-\infty}^{t} \mathrm{d}t_1 \, H_I(t_1) \left[|i\rangle - i \int_{-\infty}^{t_1} \mathrm{d}t_2 \, H_I(t_2) \left\{ |i\rangle - i \int_{-\infty}^{t_2} \mathrm{d}t_3 \, H_I(t) |\Psi, t_3\rangle \right\} \right].$$

Continuing on like this we have a series

$$\begin{aligned} |\Psi, t\rangle = |i\rangle &+ (-i) \int_{-\infty}^{t} \mathrm{d}t_1 \, H_I(t_1) |i\rangle + (-i)^2 \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 H_I(t_1) H_I(t_2) |i\rangle \\ &+ (-i)^3 \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \int_{-\infty}^{t_2} \mathrm{d}t_3 \, H_I(t_1) H_I(t_2) H_I(t_3) |i\rangle + \cdots . \end{aligned} \quad (8.4.5)$$

Each term in this series is "smaller" than the previous term and so we can truncate at some point once we've achieved the required level of accuracy.

Taking t to infinity we get

$$\begin{aligned} |\Psi, +\infty\rangle = |i\rangle &+ (-i) \int_{-\infty}^{\infty} \mathrm{d}t_1 \, H_I(t_1) |i\rangle + (-i)^2 \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 H_I(t_1) H_I(t_2) |i\rangle \\ &+ (-i)^3 \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \int_{-\infty}^{t_2} \mathrm{d}t_3 \, H_I(t_1) H_I(t_2) H_I(t_3) |i\rangle + \cdots . \end{aligned} \quad (8.4.6)$$

Recognising that this has the form of an operator, albeit an infinite series of integrals, acting on the initial state to give the final state we can identify this operator as the S matrix:

$$\begin{aligned} S = \mathbb{1} &+ (-i) \int_{-\infty}^{\infty} \mathrm{d}t_1 \, H_I(t_1) + (-i)^2 \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 H_I(t_1) H_I(t_2) \\ &+ (-i)^3 \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \int_{-\infty}^{t_2} \mathrm{d}t_3 \, H_I(t_1) H_I(t_2) H_I(t_3) + \cdots . \end{aligned} \quad (8.4.7)$$

Or, more compactly,

$$S = \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \cdots \int_{-\infty}^{t_{n-1}} \mathrm{d}t_n \, H_I(t_1) H_I(t_2) \cdots H_I(t_n). \quad (8.4.8)$$

Notice that the times, t_1, t_2, and so on appear in increasing order, that is in the integral $t_1 < t_2 < \cdots < t_n$, we can neglect the endpoints of the integration ranges where we may have equality since individual points don't contribute to an integral.

To proceed we make use of the following identity:

$$\int_{t_a}^{t_b} \mathrm{d}t_1 \int_{t_a}^{t_1} \mathrm{d}t_2 \int_{t_a}^{t_2} \cdots \int_{t_a}^{t_{n-1}} \mathrm{d}t_n \, A(t_1)A(t_2)\cdots A(t_n)$$
$$= \frac{1}{n!} \int_{t_a}^{t_b} \mathrm{d}t_1 \int_{t_a}^{t_b} \mathrm{d}t_2 \cdots \int_{t_a}^{t_b} \mathrm{d}t_n \, \mathrm{T}[A(t_1)A(t_2)\cdots A(t_n)]. \quad (8.4.9)$$

Here t_a and t_b are arbitrary initial and final times with $t_a < t_b$, and A is an arbitrary operator. The operator T is called the **time ordering**. We take the arguments, which are operators, and order according to evaluation time in order of increasing time from left to right, so that the first operator to act on some state is the operator evaluated at the earliest time. That is, for two operators

$$\mathrm{T}[A(t_1), B(t_2)] \coloneqq \begin{cases} A(t_1)B(t_2) & t_1 > t_2 \\ B(t_2)A(t_1) & t_2 > t_1 \end{cases} \quad (8.4.10)$$
$$= \theta(t_1 - t_2)A(t_1)B(t_2) + \theta(t_2 - t_1)B(t_2)A(t_1). \quad (8.4.11)$$

This can then be extended to n operators in the obvious way, although it becomes harder to write in terms of Heaviside step functions.

We'll show how this works for the $n = 2$ case, and then the general case follows by induction. Start with the right hand side of the identity. For $n = 2$ we have

$$I = \int_{t_a}^{t_b} \mathrm{d}t_1 \int_{t_a}^{t_b} \mathrm{d}t_2 \; \mathrm{T}[A(t_1)A(t_2)]. \quad (8.4.12)$$

We can split the integral over t_2 into two parts, one running from t_a to t_1, and the other from t_1 to t_b:

$$I = \int_{t_a}^{t_b} \mathrm{d}t_1 \left[\int_{t_a}^{t_1} \mathrm{d}t_2 \; \mathrm{T}[A(t_1)A(t_2)] + \int_{t_1}^{t_b} \mathrm{d}t_2 \; \mathrm{T}[A(t_1)A(t_2)] \right]. \quad (8.4.13)$$

Consider the second term,

$$\int_{t_a}^{t_b} \mathrm{d}t_1 \int_{t_2}^{t_b} \mathrm{d}t_2 \; \mathrm{T}[A(t_1)A(t_2)]. \quad (8.4.14)$$

This is a double integral. This corresponds to integrating over the triangle in the plane bounded by $t_1 = t_2$, $t_1 = t_a$, and $t_2 = t_b$. We can integrate over the same region in a different way. Instead integrate along t_2 from t_a to t_1 and then along t_1 from t_a to t_b. This corresponds to the integral

$$\int_{t_a}^{t_b} \mathrm{d}t_2 \int_{t_a}^{t_2} \mathrm{d}t_1 \; \mathrm{T}[A(t_1)A(t_2)]. \quad (8.4.15)$$

Now, t_1 and t_2 are just integration variables, and the integrand is symmetric in t_1 and t_2, as $\mathrm{T}[A(t_1)A(t_2)] = \mathrm{T}[A(t_2)A(t_1)]$, the ordering of operators at different times within a time ordering is not important, since we're going to time order

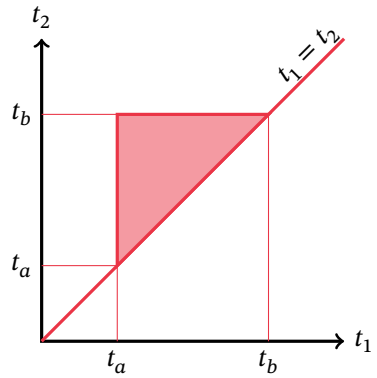


Figure 8.1: The region of integration for the $n = 2$ case of the identity.

them. We are then free to rename t_1 and t_2, and in particular swap them without changing anything else,

$$\int_{t_a}^{t_b} \mathrm{d}t_1 \int_{t_a}^{t_1} \mathrm{d}t_2 \; \mathrm{T}[A(t_1)A(t_2)]. \tag{8.4.16}$$

Combing this back with the first term we get

$$I = 2\int_{t_a}^{t_b} \mathrm{d}t_1 \int_{t_a}^{t_1} \mathrm{d}t_2 \; \mathrm{T}[A(t_1)A(t_2)]. \tag{8.4.17}$$

Noticing with these integration ranges the operators are already time ordered, so we can drop the explicit time ordering, and that $2 = 2!$ and dividing through by 2 we have proven the identity for $n = 2$.

Now, suppose that the identity holds for k integrals, and we have $k+1$ integrals. We can use the induction hypothesis to swap k integrals, and then we pairwise swap integrals as above before the unswapped integral has also been swapped.

We can use this identity to write

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2 \cdots \int_{-\infty}^{\infty} \mathrm{d}t_n \; \mathrm{T}[H_{\mathrm{I}}(t_1)H_{\mathrm{I}}(t_2)\cdots H_{\mathrm{I}}(t_n)]. \tag{8.4.18}$$

We can now replace H_{I} with an integral over the Hamiltonian density giving

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \mathrm{d}x_1 \int \mathrm{d}x_2 \cdots \int \mathrm{d}x_n \; \mathrm{T}[\mathcal{H}_{\mathrm{I}}(x_1)\mathcal{H}_{\mathrm{I}}(x_2)\cdots \mathcal{H}_{\mathrm{I}}(x_n)]. \tag{8.4.19}$$

This is called the **Dyson series**.

Note that the interaction Hamiltonian density is given by $\mathcal{H}_{\mathrm{I}} = -\mathcal{L}_{\mathrm{I}}$. This follows directly from the definition

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\mathrm{I}} = \pi\dot{\varphi} - \mathcal{L} = \pi\dot{\varphi} - \mathcal{L}_0 - \mathcal{L}_{\mathrm{I}} \tag{8.4.20}$$

where the $\pi\dot{\varphi} - \mathcal{L}_0$ term is the unperturbed Hamiltonian density. Using this we can write

$$S = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \mathrm{d}x_1 \int \mathrm{d}x_2 \cdots \int \mathrm{d}x_n \ \mathrm{T}[\mathcal{L}\mathrm{I}(x_1)\mathcal{L}\mathrm{I}(x_2)\cdots\mathcal{L}\mathrm{I}(x_n)] \tag{8.4.21}$$

$$= \exp\left[i \int \mathrm{d}^4 x \mathcal{L}_\mathrm{I}\right]. \tag{8.4.22}$$

So the S matrix can be viewed as the phase factor given by the interaction action. This last step is simply a formal rewriting recognising the exponential series. When expanding make sure to have each integral in each power be over a different variable, and to time order the integrand. From this form we can see that, assuming the action is Lorentz invariant, then so is the S matrix. This works because time ordering is Lorentz invariant, because it can be done using only terms like $\theta(t - t')$, and each such term is invariant under proper orthochronous Lorentz transformations.

8.5 Interpretation

For a given initial state, $|i\rangle$, and final state, $|f\rangle$, we need to find terms in the interaction Lagrangian, $\mathcal{L}_\mathrm{I}$ which give a nonzero transition amplitude for $|i\rangle$ to $|f\rangle$. The entire effect of the interaction is in $\mathcal{L}_\mathrm{I}$, which is an operator given by a polynomial in the fields. This means that really it's just a collection of creation and annihilation operators. Therefore the entire effect of the interaction is to create and destroy particles. We then look for terms with

- annihilation operators destroying the initial state,
- creation operators creating the final state,
- paired creation/annihilation operators creating and then destroying intermediate states.

In terms of particles, we look for terms which destroy the incoming particles, potentially create and then destroy one or more new particles, then create some particles which form the final state. The initial and final particles must be on-shell for us to observe them. The intermediate particles however needn't be on-shell, since we cannot observe them. These are called **virtual particles**. Don't put too much stock in the existence and off-shell nature of these virtual particles, they're really just a mathematical construct with different combinations of intermediate particles corresponding to different terms in the Dyson series.

In the interaction picture the fields satisfy the free field equation, and therefore have the usual creation and annihilation operators. We can therefore use normal ordering, placing all creation operators on the left and all annihilation operators on the right, in order to ensure that incoming particles are destroyed and then new particles are created, rather than the other way round.

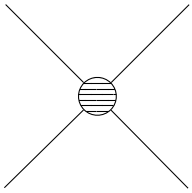
8.6 φ^3 Interactions

In φ^3 theory the interaction Hamiltonian is

$$\mathcal{H}_\mathrm{I} = -\mathcal{L}_\mathrm{I} = \frac{1}{6} g :\varphi^3: , \tag{8.6.1}$$

where we now normal order the φ^3 factor in order to destroy existing particles and then create new particles.

We will consider 2–2 scattering, that is two particles come in, interact in some way, and two particles leave. This can be represented by a diagram like



(8.6.2)

Here the blob represents some unknown black box type interaction. Time flows, in our convention, from left to right, so the left two legs represent the incoming particles, and the right two the outgoing particles. In order to destroy the incoming particles we start with a $\varphi^+\varphi^+$ term. Then to create two new particles we need a $\varphi^-\varphi^-$ term. Recall that we order operators from right to left, as we want the $\varphi^+\varphi^+$ term to act on the initial before the $\varphi^-\varphi^-$ term. We could therefore look for a

$$\varphi^-\varphi^-\varphi^+\varphi^+ \tag{8.6.3}$$

term in the series. However, we won't find such a term since in φ^3 theory we can only have terms with a multiple of 3 factors of $\varphi^\pm$. This means that there are no direct 2–2 scattering events in φ^3 theory, such events must always go via at least one intermediate virtual particle state. So, next we look at the term in the series which is quadratic in the Hamiltonian, and hence has six factors of $\varphi^\pm$. There are two possible orderings of terms with the required starting point of $\varphi^+\varphi^+$ to annihilate the initial state and end point of $\varphi^-\varphi^-$ to create the final state. They are

$$\varphi^-\varphi^-\varphi^-\varphi^+\varphi^+\varphi^+, \qquad \text{and} \qquad \varphi^-\varphi^-\varphi^+\varphi^-\varphi^+\varphi^+. \tag{8.6.4}$$

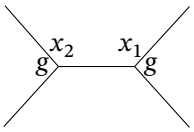
When acting on the initial state the the first vanishes, since after annihilating the initial state we are left with the vacuum state, which we then act on with the annihilator φ^+. So, the only nonzero second term quadratic in the Hamiltonian is

$$\varphi^-\varphi^-\varphi^+\varphi^-\varphi^+\varphi^+. \tag{8.6.5}$$

We interpret this as annihilating the initial state, creating a new particle, annihilating it, and then destroying it again. The amplitude for this particular term is proportional to g^2, since each Hamiltonian introduces a factor of g. Further, by assuming that energy is conserved and everything is local when we annihilate the two initial particles we must immediately create the virtual particle, and all of this must happen at the same position, so this is a single spacetime event, x_2. Then when the virtual particle is annihilated we must immediately, and at the same location, create the new particles, so this is a second spacetime event, x_1. These correspond to the two points at which the interaction Hamiltonians are evaluated so we'll have a nonzero term

$$-\frac{g^2}{36}\int_{-\infty}^{\infty} \mathrm{d}^4x_1 \int_{-\infty}^{\infty} \mathrm{d}^4x_2\, \varphi^-(x_2)\varphi^-(x_2)\varphi^+(x_1)\varphi^-(x_2)\varphi^+(x_2)\varphi^+(x_2). \tag{8.6.6}$$

Pictorially we can represent this as



(8.6.7)

Here the left and right legs represent the incoming and outgoing particles, and the single line in the middle is the virtual particle.

Next, we would consider terms cubic in the Hamiltonian. However, things quickly get out of hand if we proceed as we did here. The cubic term will have nine factors of φ. So, we want a way to do this all more systematically. This is the theory we will develop in the next chapter.

Nine

Wick's Theorem

To proceed we need to be able to deal with time orderings of operators. Our interpretation, of particles being created and destroyed, works best if we normal order things, placing all creation operators on the left and annihilation operators on the right. So, we look for a way to rewrite a time ordering in terms of normal orderings, and other quantities which can be calculated more easily. We proceed with the case of two operators, then we introduce a general theorem central to much of quantum field theory.

9.1 Two Operator Case

Suppose we have two operator valued fields, A and B, which we can write as

$$A = A^+ + A^-, \qquad \text{and} \qquad B = B^+ + B^-, \tag{9.1.1}$$

where A^+ and B^+ are expressions in the annihilation operators and A^- and B^- are expressions in the creation operators. Consider the product $:A(x)B(x'):$. Notice that the normal ordering of a sum is the sum of the normal ordering of the individual terms, that is $:C+D: = :C: + :D:$. Expanding in terms of $A^\pm$ and $B^\pm$ we have

$$\begin{aligned} :A(x)B(x'): &= :[A^+(x)+A^-(x')][B^+(x')+B^-(x')]: \\ &= :A^+(x)B^+(x') + A^+(x)B^-(x') + A^-(x)B^+(x') + A^-(x)B^-(x'): \\ &= A^+(x)B^+(x') + B^-(x')A^+(x) + A^-(x)B^+(x') + A^-(x)B^-(x'). \end{aligned}$$

Now, consider what happens when we subtract this result from $A(x)B(x')$ without the normal ordering, that is

$$A(x)B(x') = A^+(x)B^+(x') + A^+(x)B^-(x') + A^-(x)B^+(x') + A^-(x)B^-(x'). \tag{9.1.2}$$

The only term that doesn't cancel is the second term where we have to swap the ordering, so

$$A(x)B(x') - :A(x)B(x'): = A^+(x)B^-(x') - B^-(x')A^+(x) = [A^+(x), B^-(x')]. \tag{9.1.3}$$

The commutator of creation and annihilation operators is just a number, or more accurately a Dirac delta, and so it doesn't act on states, in particular

$$\langle 0|[A^+(x), B^-(x')]|0\rangle = \langle 0|0\rangle [A^+(x), B^-(x')] = [A^+(x), B^-(x')], \tag{9.1.4}$$

since we choose to normalise the vacuum state. We also have

$$\langle 0|[A^+(x), B^-(x')]|0\rangle = \langle 0|A^+(x)B^-(x')|0\rangle - \langle 0|B^-(x')A^+(x)|0\rangle = \langle 0|A^+(x)B^-(x')|0\rangle, \tag{9.1.5}$$

where the second term vanishes since the annihilation operator $A^+(x)$ acts on the vacuum.

We then have

$$A(x)B(x') = :A(x)B(x'): + \langle 0|A(x)B(x')|0\rangle. \tag{9.1.6}$$

We're nearly there, recall that the definition of time ordering is

$$\mathrm{T}[A(x)B(x')] = \theta(t-t')A(x)B(x') + \theta(t'-t)B(x')A(x), \tag{9.1.7}$$

where t is the time component of x and t' is the time component of x'.

At this point we make the observation that it doesn't matter what order we write operators in if we are normal ordering them, since we're going to change the order anyway, so

$$:A(x)B(x'): = :B(x')A(x):. \tag{9.1.8}$$

Hence, if we normal order the time ordering we have

$$\begin{aligned} :\mathrm{T}[A(x)B(x')]: &= \theta(t-t'):A(x)B(x'): + \theta(t'-t):B(x')A(x): && (9.1.9)\\ &= \theta(t-t'):A(x)B(x'): + \theta(t'-t):A(x)B(x'): && (9.1.10)\\ &= :A(x)B(x'):. && (9.1.11) \end{aligned}$$

So, the normal ordering of a time ordering is just the normal ordering.

Consider Equation (9.1.6). This holds for any product of two operators, in particular if we time order the operators it still works, since we can just separate into the $t < t'$ and $t > t'$ cases and then proceed without the time ordering. Hence,

$$\mathrm{T}[A(x)B(x')] = :A(x)B(x'): + \langle 0|\,\mathrm{T}[A(x)B(x')]|0\rangle. \tag{9.1.12}$$

Here we use $:\mathrm{T}[A(x)B(x')]: = :A(x)B(x'):$ to drop the time ordering in the first term on the right.

This equation, and others like it, are very important in quantum field theory, so important that we introduce a notation to simplify it. We define the **Wick contraction** of two operators, A and B to be

$$\overbrace{A(x)B(x')} := \langle 0|\,\mathrm{T}[A(x)B(x')]|0\rangle. \tag{9.1.13}$$

Note that this is just a number, not an operator. This allows us to state Wick's theorem for two operators.

Lemma 9.1.14 For two operators, A and B, such that $A = A^+ + A^-$ and $B = B^+ + B^-$ where A^+ and B^+ are expressions in the annihilation operator

and A^- and B^- are expressions in the creation operators we have

$$\mathrm{T}[A(x)B(x')] = :A(x)B(x'): + \overbrace{A(x)B(x')}. \tag{9.1.15}$$

Proof. See above work. □

9.2 Three Operator Case

The proof of Wick's theorem, the generalisation of the previous lemma to an arbitrary number of operators, proceeds inductively. It will be simpler to prove if we first prove the three operator case to see how the inductive logic works in detail. As before our goal is to express a time ordered product of operators in terms of their normal ordering and contractions. We now have an extra operator, C, which we again assume can be written as $C = C^+ + C^-$ where C^+ is an expression in the annihilation operators and C^- an expression in the creation operators.

We want to evaluate $\mathrm{T}[A(x_1)B(x_2)C(x_2)]$ where x_i has time component t_i. For simplicity, and without loss of generality, we assume that $t_1 > t_2 > t_3$. If this isn't the case we can reorder freely within the time ordering and then relabel our operators so that it is. We don't consider here the case where the operators are evaluated at the same time, we'll do this later. From now on we suppress the arguments to our operators, but be aware that they are not all evaluated at the same point.

Since, by assumption, A is the first operator evaluated we can move it outside of the time ordering, since it will always appear on the left of any time ordered product of operators:

$$\mathrm{T}[ABC] = A\,\mathrm{T}[BC]. \tag{9.2.1}$$

Now we can apply Lemma 9.1.14 to the time ordering of B and C, giving

$$\mathrm{T}[ABC] = A:BC: + A\overbrace{BC}. \tag{9.2.2}$$

Now express A as $A^+ + A^-$, so

$$\mathrm{T}[ABC] = (A^+ + A^-):BC: + \overbrace{BC} \tag{9.2.3}$$

$$= A^+:BC: + A^+\overbrace{BC} + A^-:BC: + A^-\overbrace{BC}. \tag{9.2.4}$$

Notice that since A^- is a creation operator and is on the left of $:BC:$ we can move it into the normal ordering without changing anything, so

$$\mathrm{T}[ABC] = A^+:BC: + A^+\overbrace{BC} + :A^-BC: + A^-\overbrace{BC}. \tag{9.2.5}$$

Things aren't quite so simple for $A^+:BC:$. In order to bring A^+, which is an annihilation operator, into the normal ordering we need it to be on the right. We can achieve this at the cost of a commutator:

$$A^+:BC: = :BC:A^+ + [A^+, :BC:] = :BCA^+: + [A^+, :BC:]. \tag{9.2.6}$$

Now that A^+ has been brought into the normal ordering we can move it around within the ordering, so

$$A^+ :BC: = :A^+BC: + [A^+, :BC:]. \tag{9.2.7}$$

The commutator can be evaluated with the identity

$$[X, YZ] = XYZ - YZX = XYZ - YXZ + YXZ - YZX = [X, Y]Z + Y[X, Z]. \tag{9.2.8}$$

Since the commutator is just a number the normal ordering doesn't effect it, and so we have

$$A^+ :BC: = :A^+BC: + :[A^+, B^-]C: + :B[A^+, C^-]:. \tag{9.2.9}$$

We've also used here that $[X^+, Y] = [X^+, Y^-]$, since X^+ commutes with Y^+, as both are just formed from annihilation operators.

Exactly as in the two operator case we can replace the commutator with its vacuum expectation value, and then drop the commutator, these are just numbers, so can be brought out of the normal ordering.

$$\begin{aligned} A^+ :BC: &= :A^+BC: + [A^+, B^-]:C: + [A^+, C^-]:B: && (9.2.10)\\ &= :A^+BC: + \langle 0|[A^+, B^-]|0\rangle :C: + \langle 0|[A^+, C^-]|0\rangle :B: && (9.2.11)\\ &= :A^+BC: + \langle 0|A^+B^-|0\rangle :C: + \langle 0|A^+C^-|0\rangle :B:. && (9.2.12) \end{aligned}$$

Notice that since we're normal ordering a single term which decomposes as the sum of creation and annihilation operators we don't really need the normal orderings any more, but we keep them so our result fits the general result.

Going back to our starting point we now have

$$\begin{aligned} \mathrm{T}[ABC] &= A^+ :BC: + A^+\overbrace{BC} + :A^-BC: + A^-\overbrace{BC} && (9.2.13)\\ &= :ABC: + \langle 0|\,\mathrm{T}[A^+B^-]|0\rangle :C: + \langle 0|\,\mathrm{T}[A^+C^-]|0\rangle :B: + A\overbrace{BC}\\ &= :ABC: + \overbrace{AB}:C: + \overbrace{AC}:B:. && (9.2.14) \end{aligned}$$

Here we use

$$A^+\overbrace{BC} + A^-\overbrace{BC} = (A^+ + A^-)\overbrace{BC} = A\overbrace{BC}, \tag{9.2.15}$$

and

$$:A^+BC: + :A^-BC: = :(A^+ + A^-)BC: = :ABC: \tag{9.2.16}$$

The result here is that the time ordering is given by the normal ordering plus all pairs of contractions, normal ordering anything that isn't contracted.

9.3 General Theorem

We can now prove the general theorem relating time orderings of an arbitrary number of operators and their normal orderings and contractions.

Theorem 9.3.1 — Wick's Theorem. Given operators, $A, B, C, D, \ldots$, which can all be written as a sum of an annihilator and creation operators. Suppose A is evaluated at x_1, B at x_2, C at x_3, D at x_4, and so on, and t_i is the time component of x_i. Then if none of the t_i are equal we have

$$\mathrm{T}[ABCD\cdots] = :ABCD\cdots: \tag{9.3.2}$$
$$+ \overbrace{AB}:CD\cdots: + \overbrace{AC}:BD\cdots: + \overbrace{AD}:BC: + \cdots \tag{9.3.3}$$
$$+ \overbrace{BC}:AD\cdots: + \overbrace{BD}:AC\cdots: + \cdots \tag{9.3.4}$$
$$\vdots \tag{9.3.5}$$
$$+ \overbrace{AB\,CD}:\cdots: + \overbrace{AC\,BD}:\cdots: + \overbrace{AD\,BC}:\cdots: + \cdots \tag{9.3.6}$$
$$+ \text{all other possible contractions.} \tag{9.3.7}$$

That is, the time ordering is given by the normal ordering, plus the normal ordering contracting each pair of operators, plus the normal ordering contracting every possible pair of pairs of operators, and so on, doing every single contraction possible and normal ordering anything not contracted.

Proof. We proceed by strong induction on the number of operators. The one operator case is trivial and acts as the basis case, equivalently the two operator case can be the basis case and we then use Lemma 9.1.14 as the proof of our basis case.
Now, suppose that The hypothesis holds for $k-1$ operators for some positive integer k. Take k operators $A, B, C, D\cdots$, evaluated at $x_1, x_2, x_3, x_4, \ldots$ respectively, with t_i the time component of x_i. Without loss of generality we assume $t_1 > t_2 > \cdots > t_k$, if this isn't the case we can reorder within the time ordering and relabel such that it is the case.
We can pull A outside of the time ordering, since it is evaluated last:

$$\mathrm{T}[ABCD\cdots] = A\,\mathrm{T}[BCD\cdots]. \tag{9.3.8}$$

Writing $A = A^+ + A^-$ where A^+ and A^- are expressions in the annihilation and creation operators respectively we have

$$\mathrm{T}[ABCD\cdots] = (A^+ + A^-)\,\mathrm{T}[BCD\cdots]. \tag{9.3.9}$$

The time ordering on the right has $k-1$ operators, so the induction hypothesis allows us to write this as

$$\mathrm{T}[ABCD\cdots] = (A^+ + A^-)(:BCD\cdots: + \overbrace{BC}:B\cdots: \tag{9.3.10}$$
$$+ \text{all contractions not involving } A). \tag{9.3.11}$$

We can trivially put A^- into all of the normal orderings, since it is a creation operator and is on the left. Therefore consider the A^+ terms. We have

$$A^+:BCD\cdots: = :BCD\cdots:A^+ + [A^+, :BCD\cdots:] \tag{9.3.12}$$
$$= :BCD\cdots A^+: + [A^+, :BCD\cdots:] \tag{9.3.13}$$
$$= :A^+BCD\cdots: + [A^+, :BCD\cdots:], \tag{9.3.14}$$

Iteratively applying the identity in Equation (9.2.8) we have

$$A^+ :BCD\cdots: = :A^+BCD\cdots: + [A^+, B^-]:CD\cdots: + [A^+, C^-]:BD\cdots: + [A^+, D^-]:BC\cdots: + \cdots . \quad (9.3.15)$$

As with the two and three operator cases we can replace the commutators with contractions, giving

$$A^+ :BCD\cdots: = :A^+BCD: + \overbrace{AB}:CD\cdots: + \overbrace{AC}:BD\cdots: + \overbrace{AD}:BC\cdots: + \cdots . \quad (9.3.16)$$

So we end up with all contractions containing A and some other operator, normal ordering all other operators.
The exact same logic works for other terms, like

$$A^+\overbrace{BC}:D\cdots: = \overbrace{AD}\overbrace{BC}:\cdots: + \cdots , \quad (9.3.17)$$

where we have all pairs of pairs of contractions with one pair being B and C and the other containing A. Considering all such terms we see that we get all possible contractions involving A. Going back to our result for $\mathrm{T}[ABCD\cdots]$ as containing all contractions not involving A, and recombining A^+ and A^- to get A we prove the theorem. □

Corollary 9.3.18 — Wick's Theorem for Mixed Time Products Suppose that some of the operators in the statement of Theorem 9.3.1 are evaluated at the same time. Then the time ordering is instead given by all contractions not involving contractions of equal time operators and normal ordering everything left over.

Proof. This follows by applying Wick's theorem at unequal times, and then setting some times equal and seeing that contractions of equal time terms vanish, since

$$\mathrm{T}(:A(t,\boldsymbol{x})B(t,\boldsymbol{x}'):) = :A(t,\boldsymbol{x})B(t,\boldsymbol{x}'): \quad (9.3.19)$$

so

$$\overbrace{AB} = \mathrm{T}[:AB:] - :AB: = 0. \quad (9.3.20)$$

□

9.4 Feynman Propagator

Consider the contraction

$$\overbracket{\varphi(x)\varphi(x')} = \langle 0|\,\mathrm{T}[\varphi(x)\varphi(x')]|0\rangle \tag{9.4.1}$$
$$= \theta(t-t')\langle 0|\varphi(x)\varphi(x')|0\rangle + \theta(t'-t)\langle 0|\varphi(x')\varphi(x)|0\rangle \tag{9.4.2}$$

where t is the time component of x and t' is the time component of x'.

We've seen that

$$i\Delta^+(x-x') = [\varphi^+(x), \varphi^-(x')] \tag{9.4.3}$$

is just a number, and so

$$\langle 0|[\varphi^+(x), \varphi^-(x')]|0\rangle = [\varphi^+(x), \varphi^-(x')]\langle 0|0\rangle = [\varphi^+(x), \varphi^-(x')]. \tag{9.4.4}$$

Hence, we have

$$i\Delta^+(x-x') = \langle 0|[\varphi^+(x), \varphi^-(x')]|0\rangle \tag{9.4.5}$$
$$= \langle 0|\varphi^+(x)\varphi^-(x')|0\rangle \tag{9.4.6}$$
$$= \langle 0|\varphi(x)\varphi(x')|0\rangle, \tag{9.4.7}$$

where expanding the commutator the $\varphi^-(x')\varphi^+(x)$ term annihilates the vacuum, and so vanishes, and in the last part expanding φ using $\varphi^\pm$ shows that all terms apart from the $\varphi^+(x)\varphi^-(x')$ term vanish as they involve annihilating the vacuum:

$$\varphi(x)\varphi(x') = (\varphi^+(x) + \varphi^-(x))(\varphi^+(x') + \varphi^-(x')) \tag{9.4.8}$$
$$= \varphi^+(x)\varphi^+(x') + \varphi^+(x)\varphi^-(x') + \varphi^-(x)\varphi^+(x') + \varphi^-(x)\varphi^-(x')$$

Similarly, we have

$$i\Delta^-(x-x') = -\langle 0|[\varphi(x'), \varphi(x)]|0\rangle. \tag{9.4.9}$$

This just follows from $\Delta^+(x) = -\Delta^-(-x)$. We can then define

$$i\Delta_\mathrm{F}(x-x') \coloneqq \overbracket{\varphi(x)\varphi(x')} \tag{9.4.10}$$
$$= \langle 0|\,\mathrm{T}[\varphi(x)\varphi(x')]|0\rangle \tag{9.4.11}$$
$$= \theta(t-t')i\Delta^+(x-x') - \theta(t'-t)i\Delta^-(x-x'). \tag{9.4.12}$$

This is called the **Feynman propagator**. Without the factor of i it's defined as

$$\Delta_\mathrm{F}(x) = \theta(t)\Delta^+(x) - \theta(-t)\Delta^-(x). \tag{9.4.13}$$

The interpretation is as follow:

- For $t > t'$ we have a positive energy, that is Δ^+, mode propagating from x' to x.
- For $t < t'$ we have a negative energy, that is Δ^-, mode propagating from x' to x.

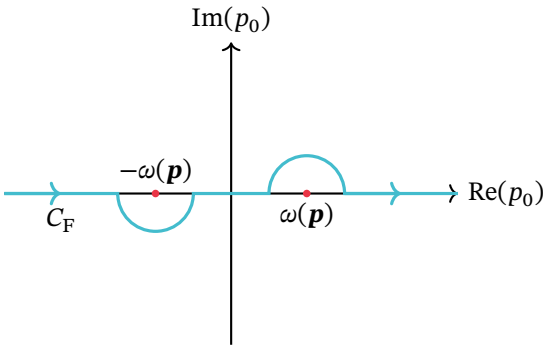


Figure 9.1: The contour used to compute the Feynman propagator.

Notice that we can rewrite the Feynman propagator as

$$\Delta_F(x-x') = \theta(t-t')\Delta^+(x-x') + \theta(t'-t)\Delta^+(x'-x). \tag{9.4.14}$$

This is has the same value but a different interpretation:

- For $t > t'$ we have a positive energy, that is Δ^+, mode propagating from x' to x.
- For $t < t'$ we have a positive energy, that is Δ^+, mode propagating from x to x'.

Comparing these interpretations we see that a negative energy mode (which we'll later associate with antiparticles) is equivalent to a positive energy mode travelling in the opposite direction.

9.4.1 Contour Representation

As well as this interpretation the Feynman propagator has the nice property of automatically encoding the time ordering. This isn't that useful in the current form, since we have explicit Heaviside step functions doing the encoding. However, the Feynman propagator has a contour representation which makes the time ordering less obvious.

The contour representation of Δ_F is

$$\Delta_F(x) = \int_{C_F} \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 - m^2} \tag{9.4.15}$$

where C_F is the contour given by the real axis in the p_0 plane with a slight dip below the pole at $-\omega(\boldsymbol{p})$ and a slight dip above the pole at $\omega(\boldsymbol{p})$. This is shown in Figure 9.1.

To see why this works we consider two cases, first, suppose that $x_0 > 0$. Then close the contour in the lower half plane, so that p_0 has a large, negative, imaginary component and overall $-ip\cdot x$ has a large, negative, real component. Closing with a semicircle Jordan's lemma tells us that the integral along this arc vanishes. Hence, the value of the integral along the C_F contour is equal to the value of the integral around this closed loop. We can apply the residue theorem to the contour about

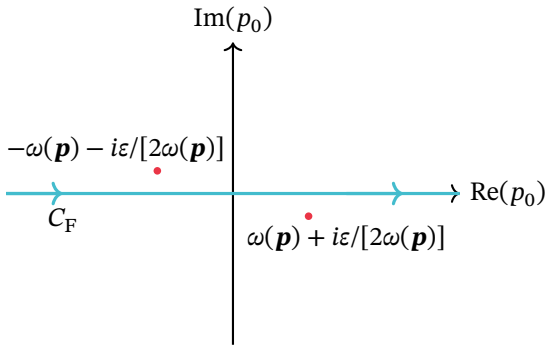


Figure 9.2: The contour for calculating the Feynman propagator using the $i\varepsilon$ prescription.

the loop. There is a single pole enclosed in this loop, the one at $\omega(\boldsymbol{p})$. However, we've already computed the integral of this function around a contour containing the pole $\omega(\boldsymbol{p})$ in Section 7.4, the only difference is a minus sign not in the Feynman propagator contour representation, so we can conclude that the result is $\Delta^+(x)$.

Similarly, if $x_0 < 0$ we close the contour in the upper half plane. Again the integral vanishes along this arc and we can then treat this as a contour containing only the pole at $-\omega(\boldsymbol{p})$, the result of which tells us that this integral gives $-\Delta^-(x)$. So, combining these results we see that we get the expected result of $\pm\Delta^\pm(x)$ depending on the sign of x_0.

9.4.1.1 The $i\varepsilon$ Prescription

This contour integral is not particular nice to evaluate, we have to awkwardly detour around the poles on the real axis. Instead we can slightly modify the integrand to give the representation

$$\Delta_F(x) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 - m^2 + i\varepsilon}. \tag{9.4.16}$$

This moves the two poles slightly off the real axis. Let $\varepsilon > 0$ be a small positive real number, then

$$p^2 - m^2 + i\varepsilon = p_0^2 - \left(\omega(\boldsymbol{p}) - \frac{i\varepsilon}{2\omega(\boldsymbol{p})}\right)^2, \tag{9.4.17}$$

just expand this out to check, and drop terms of order ε^2. The poles are then shifted by $i\varepsilon/[2\omega(\boldsymbol{p})]$, so that the pole which was at $-\omega(\boldsymbol{p})$ is now at $-\omega(\boldsymbol{p}) + i\varepsilon/[2\omega(\boldsymbol{p})]$, and the pole which was at $\omega(\boldsymbol{p})$ is now at $\omega(\boldsymbol{p}) - i\varepsilon/[2\omega(\boldsymbol{p})]$. This is shown in Figure 9.2. We can then integrate along the real axis as normal, and then take $\varepsilon \to 0$ after our computations have finished.

9.4.2 Feynman Propagator as a Green's Function

Like the other propagators the Feynman propagator is a Green's function of the Klein–Gordon equation, that is

$$(\partial^2 + m^2)\Delta_F(x) = \delta^4(x). \tag{9.4.18}$$

The difference is just in the boundary conditions.

The classical retarded and advanced Green's functions, as well as the $\Delta^{\pm}$ propagators are Green's functions for an equation with Neumann boundary conditions, that is, the boundary conditions fix initial conditions on a function and its derivative, in our case we have an initial condition on φ and $\pi = \dot{\varphi}$. This is a classical way of thinking, we have some initial state and we let it evolve according to some equation. The problem is that the assumption is we can know both φ and π at the same time, this is not possible due to the uncertainty principle.

The Feynman propagator on the other hand corresponds to Dirichlet boundary conditions, where we set initial and final conditions on some function, in this case φ. This is much closer to how we think in quantum mechanics, where we consider an initial and final state and transitions between them.

Ten

Feynman Diagrams

This is a great course because there are loads and loads of integrals to do but they're all really easy, so you don't feel any pain but you get to feel proud after.

Richard Ball

In this section we develop the theory of Feynman diagrams. A diagrammatic representation of interactions which have the benefit of greatly simplifying the maths and also seeming quite physical. We stress however, that any one diagram is not a true representation of a physical process, rather it corresponds to a single term in an expansion describing a physical process.

We start from the Dyson expansion,

$$S = \sum_{n=0}^{\infty} S^{(n)}, \tag{10.0.1}$$

where

$$S^{(n)} = \frac{(-i)^n}{n!} \int \mathrm{d}^4 x_1 \cdots \int \mathrm{d}^4 x_n \; \mathrm{T}[\mathcal{H}_{\mathrm{I}}(x_1) \cdots \mathcal{H}_{\mathrm{I}}(x_n)]. \tag{10.0.2}$$

We will use φ^3 theory as an example, but the process is basically the same, but usually with harder integrals, for other theories. This means that

$$\mathcal{H}_{\mathrm{I}}(x) = \frac{g}{3!} :\varphi(x)^3: = \frac{g}{3!} :(\varphi^+(x) + \varphi^-(x))^3:. \tag{10.0.3}$$

We consider low order terms in this series, and develop Feynman diagrams as we go along.

10.1 Evaluating Low Order Terms

10.1.1 $n = 0$

The $n = 0$ term, $S^{(0)}$, is just 1. This represents the case where nothing happens, our initial state just remains and is equal to the final state.

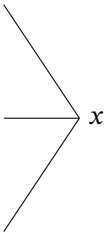
10.1.2 $n = 1$

The $n = 1$ term is

$$S^{(1)} = \frac{-ig}{3!} \int \mathrm{d}^4x \, (\varphi^{+3} + 3\varphi^{-2}\varphi^{+} + 3\varphi^{-}\varphi^{+2} + \varphi^{-3}). \tag{10.1.1}$$

We can consider each term in the integrand separately.

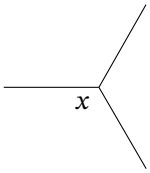
The first term represents three annihilators, therefore this corresponds to three particles in the initial state entering, being destroyed, and then a vacuum final state. As a diagram we represent this as



(10.1.2)

This interaction doesn't conserve momentum, so it must give zero contribution to the overall value of $S^{(1)}$.

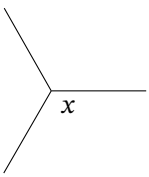
The second term represents an annihilator and two creators, so this corresponds to a single particle entering, being destroyed, and then two new particles being created. Since all three factors of φ are evaluated at the same spacetime point, x, this occurs all at once, which we draw as



(10.1.3)

It's not immediately obvious, but this term cannot actually conserve momentum in φ^3 theory, since all three particles have the same mass, and so it gives zero, which we'll show later.

The third term is two annihilators and one creation operator, it corresponds to two particles being annihilated and then one new particle being created. As a diagram it is

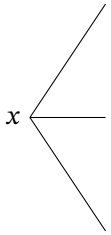


(10.1.4)

Like the previous term this vanishes in φ^3 theory.

The fourth term is three creation operators, so represents three particles being

created from an initial vacuum state. Diagrammatically it is



(10.1.5)

Like the first term this one doesn't conserve momentum and so must have no net contribution to $S^{(1)}$.

10.1.3 $n = 2$

The $n = 2$ term is

$$S^{(2)} = \frac{g^2}{2!(3!)^2} \int \mathrm{d}^4x \int \mathrm{d}^4y \, \mathrm{T}(:\varphi(x)^3: :\varphi(y)^3:). \tag{10.1.6}$$

We can use Wick's theorem to evaluate this normal ordered product. We can then split $S^{(2)}$ further into four terms based on the number of contractions:

$$S^{(2)} = S_0^{(2)} + S_1^{(2)} + S_2^{(2)} + S_3^{(2)}, \tag{10.1.7}$$

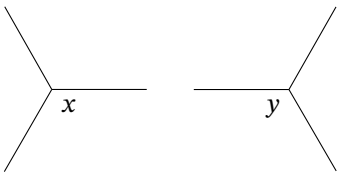
Where $S_i^{(2)}$ is the term in $S^{(2)}$ corresponding to terms with i contractions. Note that since we have six factors of φ we have a maximum of three contractions.

10.1.3.1 No Contractions

The terms with no contractions correspond to the first term in Wick's theorem,

$$:\varphi(x)^3\varphi(y)^3:. \tag{10.1.8}$$

Since there are no contractions between the field evaluated at the two spacetime points, x and y, we get disconnected diagrams, such as



(10.1.9)

This is a single diagram corresponding to two separate processes occurring at x and y. This diagram corresponds to the term

$$\varphi^-(x)\varphi^-(y)\varphi^-(y)\varphi^+(x)\varphi^+(x)\varphi^+(y), \tag{10.1.10}$$

and other terms give similar diagrams.

It can be shown that all disconnected diagrams vanish in φ^3 theory. Further, any disconnected diagram is simply a combination of connected diagrams, so we don't every need to consider disconnected diagrams. In terms of physics disconnected diagrams correspond to multiple unrelated scattering events occurring, and we can simply treat them all separately.

10.1.3.2 One Contraction

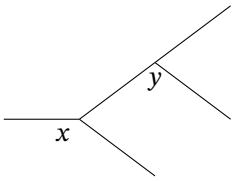
Wick's theorem, generalised for equal times, doesn't have contractions between operators at the same time. Hence, we don't need to consider contractions of $\varphi(x)$ with itself, or $\varphi(y)$ with itself. This means that we have the term

$$3^2 \overbrace{\varphi(x)\varphi(y)} : \varphi(x)^2 \varphi(y)^2 : . \tag{10.1.11}$$

The factor of 3^2 is combinatoric, it comes from there being three ways to select which $\varphi(x)$ field to contract and three ways to select which $\varphi(y)$ field to contract, and then since these are all really identical we can combine them into a single term with weighting 3^2.

The contraction corresponds to a virtual particle propagating from x to y. The four factors of φ in the normal ordering tell us that we'll have four external particles. There are three ways to do this, we can have 1-3, 2-2, or 3-1 interactions, where these are the number of initial particles to final particles.

An example of a 1-3 term is given by the diagram



(10.1.12)

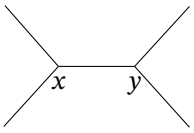
This corresponds to the term

$$\overbrace{\varphi(x)\varphi(y)}\varphi^+(x)\varphi^-(x)\varphi^-(y)\varphi^-(y). \tag{10.1.13}$$

This term, in φ^3 theory, doesn't conserve momentum since all particles have the same mass, so must give no overall contribution to $S^{(2)}$.

Similarly any 3-1 terms must vanish.

More interesting are the 2-2 terms. There are three such terms. The first is



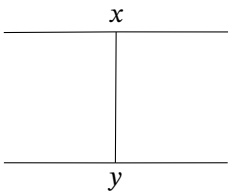
(10.1.14)

This corresponds to the term

$$\overbrace{\varphi(x)\varphi(y)}\varphi^+(x)\varphi^+(x)\varphi^-(y)\varphi^-(y). \tag{10.1.15}$$

We call this the ***s*-channel** diagram.

The second is



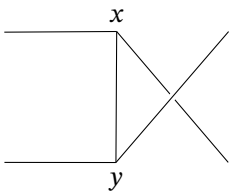
(10.1.16)

This corresponds to the term

$$\overbrace{\varphi(x)\varphi(y)}\varphi^+(x)\varphi^+(y)\varphi^-(x)\varphi^-(y). \tag{10.1.17}$$

We call this the ***t*-channel** diagram.

The third is



(10.1.18)

This corresponds to the term

$$\overbrace{\varphi(x)\varphi(y)}\varphi^+(x)\varphi^+(y)\varphi^-(y)\varphi^-(x) \tag{10.1.19}$$

If we take any of these three diagrams and swap x and y then nothing changes, so this gives us a factor of 2 to include, which nicely cancels the factor of 1/2! in the full expression for $S^{(2)}$.

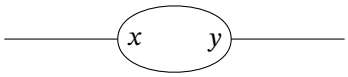
10.1.3.3 Two Contractions

Terms with two contractions correspond to a term proportional to

$$\overbrace{\varphi(x)\varphi(y)}\overbrace{\varphi(x)\varphi(y)}:\varphi(x)\varphi(y):. \tag{10.1.20}$$

The two contractions mean we have two internally propagating virtual particles, and the two normal ordered terms mean we have two external particles.

There is a single such diagram, up to swapping x and y:



(10.1.21)

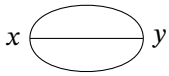
It can be shown that this term actually diverges, giving an infinite result. We call this a one **loop diagram**, since it has a single loop. In contrast the diagrams we have considered so far without loops are **tree diagrams**.

10.1.3.4 Three Contractions

Terms with three contractions correspond to a term proportional to

$$\overbrace{\varphi(x)\varphi(y)}\overbrace{\varphi(x)\varphi(y)}\overbrace{\varphi(x)\varphi(y)}. \tag{10.1.22}$$

This has no external particles and three internal virtual particles propagating from x to y. It corresponds to the diagram



(10.1.23)

This is called a **vacuum bubble** since it is formed entirely of virtual particles, and both the initial and final state are vacuums. Since there are no external particles there are no physically observable contributions from terms like this. It can be shown that bubble diagrams factor out to all levels in all theories, so we don't need to consider them. The correspond to some virtual particles coming into existence and then all annihilating.

10.2 Feynman Diagrams in Momentum Space

As always in QFT you have to get everything right.

Richard Ball

So far with these diagrams we've been working in position space, evaluating the fields at x. External states are momentum eigenstates, this suggests that we should instead work in momentum space instead, which means working with $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$.

We are interested in computing the matrix element

$$S_{fi} = \langle f|S|i\rangle. \tag{10.2.1}$$

We can express $|i\rangle$ and $|f\rangle$ as free particle states created by $a^\dagger(\boldsymbol{p})$ from the vacuum state. These states have definite momentum and all of the particles are on-shell.

Consider the one particle state with momentum $\boldsymbol{p}$. This is created with a single creation operator acting on the vacuum:

$$|\boldsymbol{p}\rangle \coloneqq a^\dagger(\boldsymbol{p})|0\rangle. \tag{10.2.2}$$

Now act on this with $\varphi^+(x)$. We can expand $\varphi^+(x)$ in momentum space using Equation (6.5.7):

$$\varphi^+(x)|\boldsymbol{p}\rangle = \int đp'\, e^{-ip'\cdot x} a(\boldsymbol{p}')a^\dagger(\boldsymbol{p})|0\rangle \tag{10.2.3}$$

$$= \int đp'\, e^{-ip'\cdot x}[a(\boldsymbol{p}'), a^\dagger(\boldsymbol{p}')]|0\rangle \tag{10.2.4}$$

$$= \int đp'\, e^{-ip'\cdot x}\delta(\boldsymbol{p}-\boldsymbol{p}')|0\rangle \tag{10.2.5}$$

$$= e^{-ip\cdot x}|0\rangle. \tag{10.2.6}$$

Here we've used

$$[a(\boldsymbol{p}'), a^\dagger(\boldsymbol{p})]|0\rangle = a(\boldsymbol{p}')a^\dagger(\boldsymbol{p})|0\rangle - a^\dagger(\boldsymbol{p})a(\boldsymbol{p}')|0\rangle = a(\boldsymbol{p}')a^\dagger(\boldsymbol{p})|0\rangle \tag{10.2.7}$$

with the second term vanishing as we annihilate the vacuum.

Taking the conjugate of this result, and using $(\varphi^+)^\dagger = \varphi^-$ we have

$$\langle \boldsymbol{p}|\varphi^-(x) = \langle 0|e^{ip\cdot x}. \tag{10.2.8}$$

Since we're going to be working in momentum space it's useful to have an expression for the contraction in momentum space, we get this through the contour representation of the Feynman propagator:

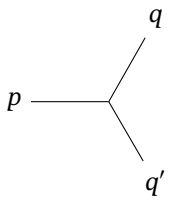
$$\overbrace{\varphi(x)\varphi(y)} = i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\varepsilon} = i\Delta_F(x-y), \tag{10.2.9}$$

where the k^0 integral is performed along the real axis, and we set ε ot zero after our calculations. Note that we've written this here using k, the wavenumber, which is the same as the momentum when $\hbar = 1$. This is mostly to free up the letter p for the momentum of the particles.

We are now in a position to start computing terms in the Dyson series, which we can view as an expansion in g. We won't do all of the ones discussed above, just enough to get the idea.

10.2.1 First Order

Consider the first order term



(10.2.10)

Here we've labelled the particles with their momenta. The initial state contains just a single particle, so its $|i\rangle = |\boldsymbol{p}\rangle$. The final states contain two particles, so the final state is a tensor product of single particle states:

$$|f\rangle = |\boldsymbol{q}, |\boldsymbol{q}'\rangle\rangle = |\boldsymbol{q}\rangle|\boldsymbol{q}'\rangle = |\boldsymbol{q}\rangle \otimes |\boldsymbol{q}'\rangle. \tag{10.2.11}$$

This works because we assume no interaction between the two final particles, at least not at first order in g.

The matrix element we wish to compute is

$$\langle f|S^{(1)}_{1\text{-}2}|i\rangle = \langle \boldsymbol{q}, \boldsymbol{q}'| \left[\frac{-ig}{3!} \int \mathrm{d}^4x 3\varphi^-(x)^2\varphi^+(x) \right] |\boldsymbol{p}\rangle. \tag{10.2.12}$$

Here $S^{(1)}_{1\text{-}2}$ is the term in $S^{(1)}$ corresponding to the term we wish to compute. The factor of 3 in the integral comes from the number of ways we can choose one of the factors of φ to be φ^+. We can show that

$$\langle \boldsymbol{q}, \boldsymbol{q}'|\varphi^-(x)^2 = 2\mathrm{e}^{iq\cdot x}\mathrm{e}^{iq'\cdot x}\langle 0|, \tag{10.2.13}$$

where the factor of two comes from the number of ways we can act on $\langle \boldsymbol{q}, \boldsymbol{q}'|$ with $\varphi^-(x)^2$. We can either act on $\langle \boldsymbol{q}|$ and then $\langle \boldsymbol{q}'|$, or the other way round. We also have

$$\varphi^+(x)|\boldsymbol{p}\rangle = \mathrm{e}^{-ip\cdot x}|0\rangle. \tag{10.2.14}$$

Putting this into our calculation, and noticing that the factors of 2 and 3 cancel with the 3! we have

$$\langle f|S^{(1)}_{1\text{-}2}|i\rangle = -ig \int \mathrm{d}^4x\, \mathrm{e}^{iq\cdot x}\mathrm{e}^{iq'\cdot x}\mathrm{e}^{-ip\cdot x} \tag{10.2.15}$$

$$= -ig(2\pi)^4\delta^4(q + q' - p). \tag{10.2.16}$$

The Dirac delta enforces conservation of momentum, the momentum in, p, must equal the momentum out, $q + q'$, in order to get a nonzero result.

All external particles are on-shell, and in φ^3 theory all particles have the same mass. This means that $p^2 = q^2 = q'^2 = m^2$. Now, consider $(p - q)^2$. Conservation of momentum tells us that $p - q = q'$, and so $(p - q)^2 = q'^2 = m^2$. We can also expand $(p - q)^2$ to get

$$m^2 = (p - q)^2 \tag{10.2.17}$$

$$= p^2 + q^2 - 2p \cdot q \tag{10.2.18}$$

$$= m^2 + m^2 + 2p \cdot q. \tag{10.2.19}$$

Hence, we have $2p \cdot q = m^2$. In the rest frame of the incoming particle we have $p^\mu = (m, \mathbf{0})$, and we can write $q^\mu = (E, \boldsymbol{q})$ without loss of generality. We therefore have

$$2p \cdot q = 2mE = m^2 \implies E = \frac{m}{2}. \tag{10.2.20}$$

This is what we would expect, since the two particles are produced each must get exactly half of the energy available from the first particle, and that energy is just its mass. We also know that $E^2 = m^2 + \boldsymbol{q}^2$, and so $E \geq m$. This means that $E = m/2$ can't satisfy $p = q + q'$. Hence, the contribution from this term vanishes as the condition imposed by the Dirac delta can never be fulfilled.

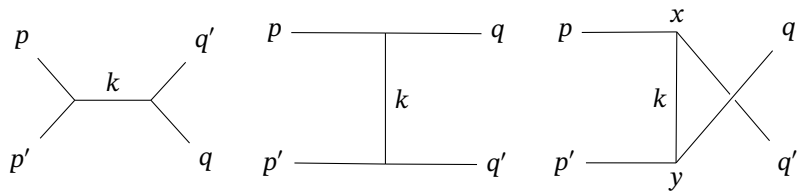
Note that in other theories with multiple particles with different masses it is possible for this term to be nonzero. It's also not true that all vertices like this vanish, the key difference is that if one of the particles is a virtual particle then it needn't satisfy the on-shell condition and so it can have arbitrary momentum, and we can conserve momentum with particles of the same mass.

The key result is that, in φ^3 theory,

$$\langle f|S^{(1)}_{1\text{-}2}|i\rangle = 0. \tag{10.2.21}$$

10.2.2 Second Order

For now we restrict ourselves to tree diagrams, we'll come to loops later. There are three diagrams to consider then:



(10.2.22)

Note that while we assume the same initial and final states the momenta of the virtual particles need not, and in fact aren't, the same, even though we've labelled them all k.

10.2.2.1 s-Channel

Let's start with the s-channel process. We want to compute

$$\langle f|S^{(2)}_s|i\rangle = \langle \boldsymbol{q}, \boldsymbol{q}'| \left[\frac{-g^2}{(3!)^2} \int \mathrm{d}^4x \int \mathrm{d}^4y\, \varphi^-(y)^2 3^2 \overbrace{\varphi(x)\varphi(y)}\varphi^+(x)^2 \right] |\boldsymbol{p}, \boldsymbol{p}'\rangle$$

We can compute the action of $(\varphi^\pm)^2$ on our initial and final states fairly easily:

$$\varphi^+(x)^2|\boldsymbol{p}, \boldsymbol{p}'\rangle = 2\mathrm{e}^{-ip\cdot x}\mathrm{e}^{-ip'\cdot x}|0\rangle, \tag{10.2.23}$$

$$\langle \boldsymbol{q}, \boldsymbol{q}'|\varphi^-(y)^2 = 2\mathrm{e}^{iq\cdot y}\mathrm{e}^{iq'\cdot y}\langle 0|. \tag{10.2.24}$$

Again, the factors of 2 come from there being two orders in which we can act on the paired states.

Noticing that the factor of 3^2 in the integrand, as well as the factors of 2 above, cancel with the $1/(3!)^2$ term, and using $\langle 0|0\rangle = 1$, we have

$$\langle f|S_s^{(2)}|i\rangle = -g^2 \int \mathrm{d}^4x \int \mathrm{d}^4y\, \mathrm{e}^{iq\cdot y}\mathrm{e}^{iq'\cdot y} i \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{e}^{ik\cdot(x-y)}}{k^2 - m^2 + i\varepsilon} \mathrm{e}^{ip\cdot x}\mathrm{e}^{ip'\cdot x}.$$

Performing the x and y integrals we get more Dirac deltas:

$$\langle f|S_s^{(2)}|i\rangle = -ig^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}(2\pi)^4\delta^4(q+q'-k)(2\pi)^4\delta^4(k-p-p')\frac{1}{k^2 - m^2 + i\varepsilon}.$$

The two Dirac deltas ensure that momentum is conserved at each vertex, and hence overall, since we must have $q + q' = k$ and $k = p + p'$ to have a nonzero contribution, and so $p+p' = q+q'$. We can then perform the integral over k using one the second Dirac delta and we get

$$\langle f|S_s^{(2)}|i\rangle = (2\pi)^4\delta^4(q + q' - p - p')\frac{-ig^2}{(p + p')^2 - m^2}. \tag{10.2.25}$$

Note that we can drop the $i\varepsilon$ term since $k^2 = (p+p')^2 \geq 4m^2$, using $p^2 = p'^2 = m^2$, and so the denominator never vanishes. It turns out that this is always the case for tree level diagrams, the $i\varepsilon$ prescription is only needed in loop diagrams. If instead we had used the first Dirac delta to perform the integral we would have gotten a factor of $(q + q')^2$ in the denominator instead, but this is equal to $(p + p')^2$ by momentum conservation.

The Dirac delta in this result is again conserving momentum. The interesting thing is the rest, which we call the **Feynman amplitude** for this process:

$$\mathcal{M}_s = \frac{-ig^2}{(p + p')^2 - m^2}. \tag{10.2.26}$$

10.2.2.2 t-Channel

Now consider the t-channel diagram. Like the s-channel term the t-channel term comes from a term which is sixth order in φ. In particular, it is $:\varphi(x)^3\varphi(y)^3:$, where we call the vertex shared by p and q x, and the vertex shared by p' and q' y. Since we have four external particles we must contract a pair of fields, and since contractions at the same point give zero we must contract $\varphi(x)$ and $\varphi(y)$. There are 3 ways to choose which $\varphi(x)$ and 3 ways to choose which $\varphi(y)$. We then have two factors of $\varphi(x)$ left and two factors of $\varphi(y)$. Looking back at the diagrams we see that a particle is created and destroyed at each vertex, as well as the internal line corresponding to the contraction. This means one of our $\varphi(x)$ values contributes a $\varphi^+(x)$ to this term, destroying an incoming particle, and the other contributes a $\varphi^-(x)$, creating one of the final particles. There are two ways to select which $\varphi(x)$ contributes $\varphi^+(x)$ and which contributes $\varphi^-(x)$. The exact same is true for $\varphi(y)$, so on top of the two factors of 3 from choosing the contraction we have two factors of 2. This gives an overall combinatorial factor of $3 \cdot 3 \cdot 2 \cdot 2 = (3!)^2$, cancelling the $(3!)^2$ which appears in the interaction term. Hence, we have

$$\begin{aligned}&\langle f|S_t^{(2)}|i\rangle \\ &= \langle \boldsymbol{q}, \boldsymbol{q}'|\left[\frac{-g^2}{(3!)^2}\int \mathrm{d}^4x \int \mathrm{d}^4y\,(3!)^2\overbrace{\varphi(x)\varphi(y)}\varphi^-(x)\varphi^-(y)\varphi^+(x)\varphi^+(y)\right]|\boldsymbol{p}, \boldsymbol{p}'\rangle.\end{aligned}$$

Now, as before we act on the initial state with the annihilators and the final state with the creation operators, giving

$$\varphi^+(x)\varphi^+(y)|\boldsymbol{p},\boldsymbol{p}'\rangle = \mathrm{e}^{-ip\cdot x}\mathrm{e}^{-ip'\cdot y}|0\rangle, \tag{10.2.27}$$

$$\langle \boldsymbol{q},\boldsymbol{q}'|\varphi^-(x)\varphi^-(y) = \mathrm{e}^{iq\cdot x}\mathrm{e}^{iq'\cdot y}\langle 0|. \tag{10.2.28}$$

Note that there is no factor of 2 this time, since the particle with momentum p must be destroyed at x in this interaction, and hence we cannot act on this particle with $\varphi^+(y)$, doing so corresponds instead to a u-channel process.

Putting this together with our results we have

$$\langle f|S_t^{(2)}|i\rangle = -g^2\int \mathrm{d}^4x\int \mathrm{d}^4y\,\mathrm{e}^{iq\cdot x}\mathrm{e}^{iq'\cdot y}\int\frac{\mathrm{d}^4x}{(2\pi)^4}\frac{\mathrm{e}^{-ik\cdot(y-x)}}{k^2-m^2+i\varepsilon}\mathrm{e}^{-ip\cdot x}\mathrm{e}^{-ip'\cdot y}.$$

Collecting the exponents into x and y exponentials we have

$$\langle f|S_t^{(2)}|i\rangle = -g^2\int \mathrm{d}^4x\int \mathrm{d}^4y\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{i(q+k-p)\cdot x}\mathrm{e}^{i(q'-k-p')}\frac{1}{k^2-m^2+i\varepsilon}.$$

Performing the x and y integrals gives Dirac deltas:

$$\langle f|S_t^{(2)}|i\rangle = -g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}(2\pi)^4\delta^4(q+k-p)(2\pi)^4\delta^4(q'-k-p')\frac{1}{k^2-m^2+i\varepsilon}.$$

Performing the k integral with the first Dirac delta fixes $k=p-q$, and so we get

$$\langle f|S_t^{(2)}|i\rangle = (2\pi)^4\delta^4(q+q-p-p')\frac{-g^2}{(p-q)^2-m^2}. \tag{10.2.29}$$

As before we have the Dirac delta enforcing energy conservation, and then the amplitude is

$$\mathcal{M}_t = \frac{-ig^2}{(p-q)^2-m^2}. \tag{10.2.30}$$

Similarly we can compute the u-channel process, and we find

$$\langle f|S_u^{(2)}|i\rangle = (2\pi)^4\delta^4(q+q'-p-p')\frac{-ig^2}{(p-q')^2-m^2}. \tag{10.2.31}$$

That is, the amplitudes for the u-channel process is

$$\mathcal{M}_u = \frac{-ig^2}{(p-q')^2-m^2}. \tag{10.2.32}$$

We now introduce the **Mandelstam invariants**, s, t, and u, defined as

$$s = (p+p')^2 = (q+q')^2, \tag{10.2.33}$$

$$t = (p-q)^2 = (p'-q')^2, \tag{10.2.34}$$

$$u = (p-q')^2 = (p'-q)^2. \tag{10.2.35}$$

These are Lorentz invariants characterising each of the three channels.

Consider the sum of these three invariants:

$$\begin{aligned} s+t+u &= (p+p')^2 + (p-q)^2 + (p-q')^2 && (10.2.36)\\ &= p^2 + p'^2 + 2p\cdot p' + p^2 + q^2 - 2p\cdot q + p^2 + q'^2 - 2p\cdot q' \\ &= 6m^2 + 2(p\cdot p' - p\cdot q - p\cdot q') && (10.2.37)\end{aligned}$$

having used the on-shell condition, $p^2 = p'^2 = q^2 = q'^2 = m^2$. Now using conservation of momentum we have $q' = p + p' - q$, and so

$$\begin{aligned} s+t+u &= 6m^2 + 2(p\cdot p' - p\cdot q - p\cdot [p+p'-q]) && (10.2.38)\\ &= 6m^2 + 2(p\cdot p' - p\cdot q - p\cdot p - p\cdot p' + p\cdot q) && (10.2.39)\\ &= 6m^2 + 2m^2 && (10.2.40)\\ &= 4m^2. && (10.2.41)\end{aligned}$$

This shows that, in a theory where all particles have the same mass, the three Mandelstam invariants are not independent.

We are interested in the total matrix element at second order which, ignoring loops, is

$$\langle f|S^{(2)}|i\rangle = \langle \boldsymbol{q}, \boldsymbol{q}'|(S_s^{(2)} + S_t^{(2)} + S_u^{(2)})|\boldsymbol{p}, \boldsymbol{p}'\rangle. \tag{10.2.42}$$

Combing our results we have

$$\langle f|S^{(2)}|i\rangle = (2\pi)^4\delta^4(q+q'-p-p')(-ig^2)\left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right] \tag{10.2.43}$$

and the amplitude is

$$\mathcal{M}^{(2)} = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u \quad = -ig^2\left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right]. \tag{10.2.44}$$

Notice that the amplitude is invariant under swapping p and p' and/or swapping q and q', since this simply swaps t and u if we make one swap, or does nothing if we make two. This implies that our particles are bosons.

Now consider scattering in the centre of mass frame. Here the incoming particles enter with equal and opposite momenta, and therefore also with equal energy since they have the same mass and are on shell. So, $p^\mu = (E, \boldsymbol{p})$ and $p'^\mu = (E, -\boldsymbol{p})$. The energy must split equally between the outgoing particles, so they also have energy E, and the total momentum after the interaction must vanish so they have equal and opposite momenta, $q^\mu = (E, \boldsymbol{q})$ and $q'^\mu = (E, -\boldsymbol{q})$.

Calculating s we have $s = (p+p')^2 = 4E^2 \geq 4m^2$, where the inequality comes from $E^2 = m^2 + \boldsymbol{p}^2 \geq m^2$. This is what justified us dropping the $i\varepsilon$ earlier, since if $s \geq 4m^2$ then $s - m^2$ will never be zero.

Since all particles have the same mass and energy and are on-shell we must have that $\boldsymbol{q}^2 = \boldsymbol{p}^2$, hence

$$t = -(\boldsymbol{p}-\boldsymbol{q})^2 = -2|\boldsymbol{p}|^2(1-\cos\vartheta) \leq 0 \tag{10.2.45}$$

where $\cos\vartheta = \boldsymbol{p}\cdot\boldsymbol{q}/(|\boldsymbol{p}|\boldsymbol{q}||)$. Similarly,

$$u = -(\boldsymbol{p}+\boldsymbol{q})^2 = -2\boldsymbol{p}^2(1+\cos\vartheta) \leq 0. \tag{10.2.46}$$

This means that $t - m^2$ and $u - m^2$ will always be negative, and in particular are never zero allowing us to drop the $i\varepsilon$ term. This also implies that virtual particles are *always* off-shell, as otherwise we would not be able to conserve momentum in 2-1 and 1-2 vertices.

10.3 Feynman Rules

The calculations above all follow roughly the same steps, this suggests that we can formulate a way to go from the diagrams to the final result without having to do these steps every time. This is what Feynman rules do.

For any φ^3 process we have

$$\langle f|S|i\rangle = \delta_{fi} + (2\pi)^4\delta^4\left(\sum p_f - \sum p_i\right)\sum_{n=2}^{\infty}\mathcal{M}^{(n)}. \tag{10.3.1}$$

The Kronecker delta accounts for the case where nothing happens, the Dirac delta enforces energy conservation, where the sums are over all final and then all initial states. The interesting bit is the amplitude, $\mathcal{M}^{(n)}$. Wick's theorem tells us that the total amplitude, $\mathcal{M}$, has contributions from all topologically distinct diagrams[1], and $\mathcal{M}^{(n)}$ has contributions from all topologically distinct n vertex diagrams. Further, we have to consider all diagrams with equal weight due to Wick's theorem, bt the factor of g^n means that we can drop higher order terms, since g is taken to be small.

[1] that is, all diagrams which are not related by rearranging where nodes and edges are

From what we've seen so far we can guess most of the Feynman rules, and the others aren't too bad to demonstrate. The **Feynman rules** for φ^3 theory are

- Every vertex contributes a factor of $-ig$ to the amplitude.
- Every internal line contributes a factor of $i\tilde{\Delta}_{\mathrm{F}}(k) = i/(k^2 - m^2 + i\varepsilon)$ to the amplitude.
- Every external line contributes a factor of 1 to the amplitude.
- Impose conservation of four-momentum at every vertex.
- Every momentum, k, which is not fixed by conservation of four-momentum gives a factor of

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}. \tag{10.3.2}$$

- A symmetry factor (trivial for simple diagrams).

Note that $\tilde{\Delta}_{\mathrm{F}}$ is the Fourier transform of Δ_{F}, which is to say it is the integrand in the contour representation.

Eleven

Getting Measurable Results

Experiments can't measure the S matrix directly. Instead we can measure transition rates between states, including decay rates and cross sections. In this chapter we'll discuss how to compute these from the S matrix for comparison with experimental results.

Our starting point will be

$$S_{fi} = \delta_{fi} + (2\pi)^4 \delta^4 \left(\sum p_f - \sum p_i \right) \mathcal{M}. \tag{11.0.1}$$

The δ_{fi} term corresponds to the case where nothing happens. The Dirac delta, where the sums are over all final and initial particles, enforces conservation of momentum, and $\mathcal{M}$ is the Feynman amplitude, which is what we compute in QFT.

11.1 Transition Rate

Consider the probability, P_{fi}, of transitioning from the initial state $|i\rangle$ to the final state $|f\rangle$. This is simply the square of the S matrix, which is given by

$$P_{fi} = |S_{fi}|^2 = (2\pi)^4 \delta^4(0) (2\pi)^4 \sum_f \delta^4 \left(\sum p_f - \sum p_i \right) |\mathcal{M}|^2. \tag{11.1.1}$$

assuming $i \neq f$, that is that something happens. We have used here $\delta(x)^2 = \delta(0)\delta(x)$ as distributions, which follows since

$$\int \mathrm{d}x \, f(x) \delta(x)^2 = \delta(0) f(0), \tag{11.1.2}$$

having used the sifting property of one of the Dirac deltas. The first sum over f, outside of the Dirac delta, corresponds to a sum, which will turn out to really be an integral, over allowed values of the final momentum.

There's a problem here, in that $\delta^4(0)$ is infinite. Consider

$$(2\pi)^4 \delta^4(p) = \int \mathrm{d}^4 x \, \mathrm{e}^{-ip \cdot x}. \tag{11.1.3}$$

This suggests that

$$(2\pi)^4 \delta^4(0) = \int \mathrm{d}^4 x = VT \tag{11.1.4}$$

where V is the infinite volume of space and T is the infinite length of time. So, VT is the volume of spacetime. To get around the problem we treat the entire system

as if it is in a finite box in spacetime, and then proceed to compute quantities per unit spacetime volume.

We therefore look to compute the transition rate per unit volume,

$$w_{fi} = \frac{P_{fi}}{VT} = \sum_f (2\pi)^4 \delta(\sum p_f - \sum p_i)|\mathcal{M}|^2. \tag{11.1.5}$$

We can proceed using the completeness relation,

$$1 = \int đp\, |\boldsymbol{p}\rangle\langle\boldsymbol{p}|, \tag{11.1.6}$$

which holds if

$$\langle\boldsymbol{p}|\boldsymbol{p}'\rangle = \delta(\boldsymbol{p} - \boldsymbol{p}'). \tag{11.1.7}$$

Writing out the measure in full,

$$1 = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{2\omega(\boldsymbol{p})} |\boldsymbol{p}\rangle\langle\boldsymbol{p}|. \tag{11.1.8}$$

The incident flux is

$$\langle\boldsymbol{p}|\boldsymbol{p}\rangle = 2\omega(\boldsymbol{p})(2\pi)^3\delta^3(\boldsymbol{0}). \tag{11.1.9}$$

Identifying $(2\pi)^3\delta^3(\boldsymbol{0}) = V$ we have

$$\frac{\langle\boldsymbol{p}|\boldsymbol{p}'\rangle}{V} = 2\omega(\boldsymbol{p}). \tag{11.1.10}$$

This implies we have $2\omega(\boldsymbol{p})$ states per unit volume.

Now consider the decay of a single particle, p, into n particles, p_i. The decay rate in the frame of the particle, that is the transition rate per decaying particle, is then

$$\mathrm{d}\Gamma = (2\pi)^4\delta\left(p - \sum p_f\right) \frac{1}{2\omega(\boldsymbol{p})} \prod_f \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3 2E_f} |\mathcal{M}|^2, \tag{11.1.11}$$

where we write $E_f = \omega(\boldsymbol{p_f})$. We can package the measure up into a single thing called the n particle **phase space measure**:

$$(\mathrm{dPS})_n \coloneqq (2\pi)^4\delta^4\left(p - \sum p_f\right) \prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3 2E_f}, \tag{11.1.12}$$

so

$$\mathrm{d}\Gamma = \frac{1}{2E(\boldsymbol{p})} |\mathcal{M}|^2 (\mathrm{dPS})_n, \tag{11.1.13}$$

where $E(\boldsymbol{p}) = \omega(\boldsymbol{p})$.

11.1.1 One to Two Decay

Consider the decay $p \to p_1 + p_2$. When doing an experiment we have detectors covering some solid angle, and so we are interested in not just the decay rate, but the decay rate into some particular solid angle, which is

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega}. \tag{11.1.14}$$

We have two final particles, so we need to compute $(\mathrm{dPS})_2$, which is given by

$$(\mathrm{dPS})_2 = (2\pi)^4\delta^4(p - p_1 - p_2)\frac{\mathrm{d}^3\boldsymbol{p_1}}{(2\pi)^3 2\omega(\boldsymbol{p_1})}\frac{\mathrm{d}^3\boldsymbol{p_2}}{(2\pi)^3 2\omega(\boldsymbol{p_2})}. \tag{11.1.15}$$

We can perform the p_2 integral using one of the Dirac deltas:

$$\int \mathrm{d}^3\boldsymbol{p_2}\delta^4(p_1 + p_2 - p) = \delta(E_1 + E_2 - E), \tag{11.1.16}$$

where we now fix $\boldsymbol{p_2} = \boldsymbol{p} - \boldsymbol{p_1}$ and $E_2 = |\boldsymbol{p_2}|^2 + m_2^2$. We can then express the momentum space volume element for p_1 in terms of polar variables:

$$\mathrm{d}^3 p_1 = |\boldsymbol{p_1}|^2\,\mathrm{d}p_1\,\mathrm{d}\Omega. \tag{11.1.17}$$

Now use

$$E_1^2 = |\boldsymbol{p_1}|^2 + m_1^2 \implies E_1\,\mathrm{d}E_1 = p_1\,\mathrm{d}p_1, \tag{11.1.18}$$

since $\mathrm{d}m_1 = 0$. Then, the partially integrated phase space is

$$\int (\mathrm{dPS})_2 = \frac{1}{4\pi}\mathrm{d}\Omega\frac{p_1\,\mathrm{d}E_1}{E}_2\delta(E_1 + E_2 - E). \tag{11.1.19}$$

We can then write the phase space measure as

$$\int (\mathrm{dPS})_2 = \frac{1}{(4\pi)^2}\frac{|\boldsymbol{p}|}{m}\mathrm{d}\Omega. \tag{11.1.20}$$

Thus in the centre of mass frame the decay rate into some solid angle is

$$\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega}\right)_{\mathrm{CoM}} = \frac{1}{32\pi^2}\frac{|\boldsymbol{p}|}{m^2}|\mathcal{M}|^2. \tag{11.1.21}$$

11.2 Cross Sections

When we consider collisions there is almost no point in considering collisions with three or more particles, since these are so unlikely to occur. So, consider a scattering of two particles, p_1 and p_2. We can work in the lab frame, where, say, the second particle is stationary, so

$$p_1 = (\omega(\boldsymbol{p_1}), \boldsymbol{p_1}), \qquad \text{and} \qquad p_2 = (m_2, \boldsymbol{0}). \tag{11.2.1}$$

We can also write $E_1 = \omega(\boldsymbol{p_1})$.

The incident flux, that is the number of particles crossing unit area per unit time, is given by the volume of particles which would cross through said area in

unit time times the density of particles. The volume is simply $|\boldsymbol{v}|$, where $\boldsymbol{v}$ is the velocity of the particle, since in one unit time the particles will have moved a distance $|\boldsymbol{v}|$ forward. The density of particles is $2E_1$, so the incident flux is $2E_1|\boldsymbol{v}| = 2|\boldsymbol{p_1}|$, where we've used $\boldsymbol{p} = \gamma m\boldsymbol{v}$ and $E = \gamma m$, so $|\boldsymbol{v}| = |\boldsymbol{p}|/E$ as expected.

The number of scattering centres in the target per unit volume is the density of particle states, which is $2E_2 = 2m_2$.

The differential cross section is defined as the transition rate per scattering centre per unit incident flux. The idea being that the number of scattering centres and the incident flux are both dependent on the experiment, and the differential cross section is independent of experimental details. The differential cross section is then

$$\mathrm{d}\sigma = (2\pi)^4\delta^4\left(\sum p_i - \sum p_f\right)\frac{1}{4|\boldsymbol{p_1}|m_2}\left(\prod_f \frac{\mathrm{d}^3\boldsymbol{p_f}}{(2\pi)^3 2E_f}\right)|\mathcal{M}|^2. \tag{11.2.2}$$

Now we can use $(p_1 \cdot p_2)^2 = (E_1 m_2)^2 = (\boldsymbol{p_1} + m_1)^2 m_2^2$ which gives $|\boldsymbol{p_1}|^2 m_2^2 = (p_1 \cdot p_2)^2 - m_1^2 m_2^2$, which is Lorentz invariant. This means that we can write the result in a frame independent way as

$$\mathrm{d}\sigma = \frac{|\mathcal{M}|^2}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}(\mathrm{dPS})_n. \tag{11.2.3}$$

11.2.1 Two to Two Scattering

Consider the case where we have two incoming and two outgoing particles, $p_1 + p_2 \to p_3 + p_4$. In the centre of mass frame we have

$$\int(\mathrm{dPS})_2 = \frac{1}{(4\pi)^2}\frac{p'}{W} \tag{11.2.4}$$

where $p' = |\boldsymbol{p_3}| = |\boldsymbol{p_4}|$, where equality between the magnitude of the momenta of the particles follows by conservation of momentum in the centre of mass frame, meaning $\boldsymbol{p_1} + \boldsymbol{p_2} = \boldsymbol{0}$, so $\boldsymbol{p_3} + \boldsymbol{p_4} = \boldsymbol{0}$ also. As well, $W = E_1 + E_2 = E_3 + E_4$, where the second equality is just conservation of energy.

In the centre of mass frame we have

$$p_1 = (E_1, \boldsymbol{p}), \qquad \text{and} \qquad p_2 = (E_2, -\boldsymbol{p}). \tag{11.2.5}$$

Define $p = |\boldsymbol{p}|$, and so

$$\boldsymbol{p_1} \cdot \boldsymbol{p_2} = E_1 E_2 + \boldsymbol{p}^2 \implies (p_1 \cdot p_2)^2 - m_1^2 m_2^2 = p^2 W^2, \tag{11.2.6}$$

where we've used the on-shell requirement to show this.

Hence, the cross section for scattering per unit solid angle is

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CoM}} = \frac{1}{64\pi^2}\frac{1}{W^2}\left(\frac{p'}{p}\right)^2|\mathcal{M}|^2. \tag{11.2.7}$$

Note that if all particles have the same mass then $p' = p$ and so often we see this equation without the p'/p factor. This is also a valid approximation in the high energy case where the masses provide a negligible contribution to the energy-momentum relation.

11.2.1.1 In φ^3 Theory

In φ^3 theory the two to two scattering gives us the amplitude

$$\mathcal{M} = (-ig^2)\left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right]. \tag{11.2.8}$$

Here $s = W^2$. Hence

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CoM}} = \frac{1}{64\pi^2}\frac{g^4}{s}\left|\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right|^2. \tag{11.2.9}$$

Now consider high energy fixed angle scattering. Then $|\boldsymbol{p}|^2 \gg m^2$, so $|\boldsymbol{p}|^2 \approx E^2$. We also have $s = 4E^2 \approx 4\boldsymbol{p}^2$, $t \approx -2E^2(1-\cos\vartheta)$ and $u \approx -2E^2(1+\cos\vartheta)$. This means that the ratios s/u and s/t are fixed since ϑ is fixed and the factors of E^2 cancel. The angular distribution can be written as

$$\mathrm{d}\Omega = 2\pi\,\mathrm{d}\cos\vartheta, \tag{11.2.10}$$

and so

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\vartheta)}\right)_{\mathrm{CoM}} = \frac{1}{32\pi}\frac{g^4}{s^3}\left(1+\frac{s}{t}+\frac{s}{u}\right)^2, \tag{11.2.11}$$

where we get an extra factor of 2π from the $\int_0^{2\pi}\mathrm{d}\varphi$ integral and the term in brackets follows from neglecting the masses and some trig identities.

Note that dimensionally this all works out since the action, S, has dimensions of $[S] = [\text{length}]^4[g\varphi^3]$, which follows from the action being the spacetime integral of $-g\varphi^3/4!$, and so since action is dimensionless (when $\hbar = 1$) and length and energy have inverse units of each other (when $c = 1$ also) we have $[g\varphi^3] = [\text{energy}]^4$. We know that $[(\partial\varphi)^2] = [\text{energy}]^4$, and so we have $[\varphi] = [\text{energy}]$, since $[\partial] = [\text{length}]^{-1} = [\text{energy}]$. Hence, we must have that $[g] = [\text{energy}]$ and so this entire quantity has units of $[g^4]/[s^3] = [\text{energy}]^4/([\text{momentum}]^2)^3 = [\text{momentum}]^{-2}$, since $[\text{energy}] = [\text{momentum}]$ (when $c = 1$). This tells us that the cross section drops off inversely proportionally to the momentum squared, so at high momentums interactions are much less likely, which is what we would expect, high momentum particles move past each other very quickly without much of a chance for interactions.

Part III

Complex Scalar Fields

Twelve

Charge

Classical fields, such as the electromagnetic field, are always real since they are observables. Wave functions on the other hand are always complex, since their time evolution, $|\psi, t\rangle = e^{iEt}|\psi\rangle$, demands it. In quantum field theory fields can be either real or complex, since they are not observable. However, the action, Lagrangian, and Hamiltonian must be real, this is needed to ensure that the S matrix is unitary, which is required for conservation of probability. Also, the Hamiltonian corresponds to the energy of the system which is observable, and hence real.

12.1 Multiple Real Fields

Consider the common definition of complex numbers which starts something like "a complex number is an ordered pair of real numbers with operations of ...". Much like this definition a complex field can be thought of as a pair of real fields. So, we need to be able to deal with multiple fields. Fortunately our work so far generalises very well to multiple real fields, we just have to sum over fields in certain expressions.

Suppose we have N real scalar fields, φ_r, with $r = 1, \dots, N$. For simplicity, and because it's all we need for the complex case, we assume that all N fields correspond to particles of mass m. The Lagrangian is then

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_r(x))(\partial^\mu \varphi_r(x)) - m^2 \varphi_r(x)^2, \tag{12.1.1}$$

where we have an implied sum over r. We can then define the conjugate fields, and we find

$$\pi_r(x) \coloneqq \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_r(x)} = \dot{\varphi}_r(x). \tag{12.1.2}$$

Any two distinct fields commute, and for two of the same field we have the usual commutation relations. That is, we have the equal time commutation relation

$$[\varphi_r(t, \boldsymbol{x}), \pi_s(t, \boldsymbol{x}')] = i\delta_{rs}\delta(\boldsymbol{x} - \boldsymbol{x}'), \tag{12.1.3}$$

and all other commutators vanish.

The mode expansion is the same as for a single field, but we have independent creation and annihilation operators for each field:

$$\varphi_r(x) = \int \frac{\mathrm{d}^3[p]}{(2\pi)^3} \frac{1}{2\pi(\boldsymbol{p})} [a_r(\boldsymbol{p}) e^{-ip\cdot x} + a_r^\dagger(\boldsymbol{p}) e^{ip\cdot x}]. \tag{12.1.4}$$

These have the covariant commutation relations

$$[a_r(\boldsymbol{p}), a_s^\dagger(\boldsymbol{p}')] = \delta_{rs} 2\omega(\boldsymbol{p})(2\pi)^3 \delta(\boldsymbol{p} - \boldsymbol{p}'), \tag{12.1.5}$$

and all other commutators vanish.

From here we proceed as before, we can define the energy-momentum tensor,

$$T^{\mu\nu}(x) \coloneqq :(\partial^\mu \varphi_r(x))(\partial^\nu \varphi_r(x)) - \eta^{\mu\nu}\mathcal{L}(x):, \tag{12.1.6}$$

with an implied sum over r. We then get the conserved momentum

$$P^\nu = \int \mathrm{d}^3x\, T^{0\nu}(x), \tag{12.1.7}$$

and the conserved current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} \delta\varphi_r, \tag{12.1.8}$$

with an implied sum over r.

The interpretation of $a_r^\dagger(\boldsymbol{p})$ and $a_r(\boldsymbol{p})$ as creation and annihilation operators is the same, each creating/annihilating particles of type r. This follows since we have

$$[H, a_r^\dagger(\boldsymbol{p})] = E a_r^\dagger(\boldsymbol{p}), \qquad \text{and} \qquad [H, a_r(\boldsymbol{p})] = -E a_r(\boldsymbol{p}). \tag{12.1.9}$$

So, $a_r^\dagger(\boldsymbol{p})$ creates a particle of type r and momentum $\boldsymbol{p}$, which we might write as

$$a_r^\dagger(\boldsymbol{p})|0\rangle = |\boldsymbol{p}, r\rangle. \tag{12.1.10}$$

The only time when we *don't* sum over the fields is when defining the number operator, which we want to count particles of type r, not the total number of particles, so we define

$$N_r(\boldsymbol{p}) \coloneqq a_r^\dagger(\boldsymbol{p}) a_r(\boldsymbol{p}) \qquad \text{(no sum on } r\text{)}. \tag{12.1.11}$$

12.2 Complex Scalar Field

Now, consider a complex scalar field, φ. We can write this as a sum of two real fields, φ_1 and φ_2, one multiplied by i:

$$\varphi(x) = \frac{1}{\sqrt{2}}[\varphi_1(x) + i\varphi_2(x)]. \tag{12.2.1}$$

The factor of $1/\sqrt{2}$ here is so that the free Lagrangian comes out as the sum of two free Lagrangians for the real fields φ_1 and φ_2.

Motivated by the need for the Lagrangian to be real we take the usual real scalar field Lagrangian and replace one of every pair of φs with $\varphi^\dagger$. We then treat φ and $\varphi^\dagger$ as two independent fields, since from these we have

$$\varphi_1 = \frac{1}{\sqrt{2}}(\varphi + \varphi^\dagger), \quad \text{and} \quad i\varphi_2 = \frac{1}{\sqrt{2}}(\varphi - \varphi^\dagger), \tag{12.2.2}$$

and we assume φ_1 and φ_2 are independent. Doing so we see that if we also drop the factor of 2 then we get

$$\mathcal{L} = (\partial_\mu \varphi^\dagger)(\partial^\mu \varphi) - m^2 \varphi^\dagger \varphi \tag{12.2.3}$$

$$= \frac{1}{2}(\partial_\mu \varphi_i)(\partial^\mu \varphi_i) - \frac{1}{2} m^2 \varphi_i^2, \tag{12.2.4}$$

with an implied sum over i. Note that we assume both fields have particles of the same mass, since otherwise we cannot guarantee that the Lagrangian will be real.

We can define the conjugate field

$$\pi \coloneqq \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi}^\dagger, \qquad \text{and} \qquad \pi^\dagger = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^\dagger} = \dot{\varphi}. \tag{12.2.5}$$

We then have the equal time commutation relations

$$[\varphi(t, \boldsymbol{x}), \pi(t, \boldsymbol{x}')] = [\varphi^\dagger(t, \boldsymbol{x}), \pi^\dagger(t, \boldsymbol{x}')] = i\delta^3(\boldsymbol{x} - \boldsymbol{x}'), \tag{12.2.6}$$

and all other commutators vanish.

We can define the Hamiltonian as before:

$$H = \int \mathrm{d}^3\boldsymbol{x}\, (\pi\dot{\varphi} + \pi^\dagger \dot{\varphi}^\dagger - \mathcal{L}) \tag{12.2.7}$$

$$= \int \mathrm{d}^3\boldsymbol{x}\, (\pi^\dagger \pi + (\nabla \varphi^\dagger) \cdot (\nabla \varphi) + m^2 \varphi^\dagger \varphi). \tag{12.2.8}$$

This second form makes it clear that this is real.

Computing the Heisenberg equations of motion gives

$$(\partial^2 + m^2)\varphi = 0, \qquad \text{and} \qquad (\partial^2 + m^2)\varphi^\dagger = 0. \tag{12.2.9}$$

12.3 Mode Expansion

The mode expansion is similar to the real case, except, in the real case we needed a and $a^\dagger$ to give a real field. We don't need a real field here, but we do require a second term since this then causes things to cancel when the canonical commutation relations are evaluated outside of the light cone, which enforces causality. As a result we still include two terms in the mode expansion, but we no longer require one to be the Hermitian conjugate of the other, so we introduce some other operator, $b^\dagger$, to replace $a^\dagger$, with the conjugate for consistency with the real case:

$$\varphi(x) = \int đp\, [a(\boldsymbol{p})\mathrm{e}^{-ip\cdot x} + b^\dagger(\boldsymbol{p})\mathrm{e}^{ip\cdot x}]. \tag{12.3.1}$$

We then have

$$\varphi^\dagger(x) = \int đp\, [b(\boldsymbol{p})\mathrm{e}^{-ip\cdot x} + a^\dagger(\boldsymbol{p})\mathrm{e}^{ip\cdot x}]. \tag{12.3.2}$$

We then have the equal time commutation relations

$$[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p})] = [b(\boldsymbol{p}), b^\dagger(\boldsymbol{p})] = đ(\boldsymbol{p} - \boldsymbol{p}'), \tag{12.3.3}$$

and the two independent mode operators commute:

$$[a(\boldsymbol{p}), b(\boldsymbol{p})] = [a^\dagger(\boldsymbol{p}), b^\dagger(\boldsymbol{p})] = 0, \tag{12.3.4}$$

as we as all other commutators vanishing.

We can then show that

$$[H, a^\dagger(\boldsymbol{p})] = Ea^\dagger \qquad [H, a(\boldsymbol{p})] = -Ea(\boldsymbol{p}), \tag{12.3.5}$$

$$[H, b^\dagger(\boldsymbol{p})] = Eb^\dagger \qquad [H, b(\boldsymbol{p})] = -Eb(\boldsymbol{p}). \tag{12.3.6}$$

Recall that a real scalar field can be interpreted as an infinite set of harmonic oscillators, one for each momentum, $\boldsymbol{p}$. We can then interpret the complex scalar field as two independent sets of harmonic oscillators, with two for each momentum, $\boldsymbol{p}$. This leads to the interpretation of $a^\dagger$ and $b^\dagger$ as creating independent particles. These particles are bosons, both with respect to exchanging particles made by the same mode operator, so, for example, both created by $a^\dagger$, and with respect to exchanging particles made by different mode operators, so for example, exchanging a particle created by $a^\dagger$ with a particle created by $b^\dagger$.

The energy-momentum operator is

$$T^{\mu\nu} = :(\partial^\mu\varphi^\dagger)(\partial^\nu\varphi) + (\partial^\nu\varphi^\dagger)(\partial^\mu\varphi) - \eta^{\mu\nu}\mathcal{L}:. \tag{12.3.7}$$

The conserved momentum is

$$P^\nu = \int \mathrm{d}^3\boldsymbol{x}\, T^{0\nu} = \int đp\, p^\nu(a^\dagger(\boldsymbol{p})a(\boldsymbol{p}) + b^\dagger(\boldsymbol{p})b(\boldsymbol{p})) = \int đp\, p^\nu(N_a(\boldsymbol{p}) + N_b(\boldsymbol{p})).$$

This means it is the momentum of the combined two sets of particles which is conserved, not the momentum of each set individually.

12.4 Charge Conservation

There must be something that sets apart particles created by $a^\dagger$ and particles created by $b^\dagger$. In this section we work out what this is. Consider the Lagrangian again,

$$\mathcal{L} = (\partial_\mu\varphi^\dagger)(\partial^\mu\varphi) - m^2\varphi^\dagger\varphi. \tag{12.4.1}$$

Notice that this is invariant under the transformation

$$\varphi \mapsto \mathrm{e}^{i\alpha}\varphi \tag{12.4.2}$$

for some constant α. Notice that α cannot depend on x, since then the derivatives would cause problems. Under this transformation we have $\varphi^\dagger \mapsto \mathrm{e}^{-i\alpha}\varphi^\dagger$ as well. This is a U(1) symmetry[1]. This is an internal symmetry, acting on the fields, rather than on spacetime.

A symmetry implies a conserved quantity, so lets find out what it is[2]. Assume $\alpha \ll 1$, so we have $\exp[i\alpha] \approx 1 + i\alpha$, and hence

$$\varphi \mapsto \mathrm{e}^{i\alpha}\varphi \approx (1 + i\alpha)\varphi = \varphi + i\alpha\varphi, \tag{12.4.3}$$

and hence

$$\delta\varphi = \mathrm{e}^{i\alpha}\varphi - \varphi \approx i\alpha\varphi. \tag{12.4.4}$$

[1] see the *Symmetries of Quantum Mechanics* or *Symmetries of Particles and Fields* course for details

[2] If the title of this section hasn't already given it away!

Similarly,

$$\delta\varphi^\dagger \approx -i\alpha\varphi. \tag{12.4.5}$$

We then have

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi = i\alpha(\partial^\mu\varphi^\dagger)\varphi, \qquad \text{and} \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\dagger)}\delta\varphi^\dagger = -i\alpha\varphi^\dagger(\partial^\mu\varphi). \tag{12.4.6}$$

[3] actually, normal ordering isn't strictly necessary here, but it is in other similar cases so we'll do so here for consistency

The conserved current is then, normal ordering to avoid later issues[3],

$$j^\mu = :\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\dagger)}\delta\varphi^\dagger: \tag{12.4.7}$$

$$= :i\alpha(\partial^\mu\varphi^\dagger)\varphi - i\alpha\varphi^\dagger(\partial^\mu\varphi):. \tag{12.4.8}$$

We can rescale this by a factor of $-\alpha$ to get:

$$j^\mu = i:\varphi^\dagger(\partial^\mu\varphi) - (\partial^\mu\varphi^\dagger)\varphi: \tag{12.4.9}$$

Notice that we can easily show this is conserved, that is that $\partial_\mu j^\mu = 0$:

$$\partial_\mu j^\mu = i:[(\partial_\mu\varphi^\dagger)(\partial^\mu\varphi) + \varphi^\dagger(\partial^2\varphi) - (\partial^2\varphi^\dagger)\varphi - (\partial^\mu\varphi^\dagger)(\partial_\mu\varphi)]: \tag{12.4.10}$$

$$= i:[\varphi^\dagger(\partial^2\varphi) - (\partial^2\varphi^\dagger)\varphi]:. \tag{12.4.11}$$

The equations of motion for the fields give $\partial^2\varphi = -m^2\varphi$ and $\partial^2\varphi^\dagger = -m^2\varphi^\dagger$, so

$$\partial_\mu j^\mu = i:[\varphi^\dagger(-m^2\varphi) - (-m^2\varphi^\dagger)\varphi]: = 0, \tag{12.4.12}$$

where we've used the normal ordering to allow us to swap the order of φ and $\varphi^\dagger$.

Along with this conserved current we have a conserved charge:

$$Q = \int \mathrm{d}^3\boldsymbol{x}\, j^0 = i\int \mathrm{d}^3\boldsymbol{x}\, :(\varphi^\dagger\dot\varphi - \dot\varphi^\dagger\varphi):. \tag{12.4.13}$$

We can write φ and $\varphi^\dagger$ in terms of mode operators, with the time derivatives bringing down a factor of $\pm i\omega(\boldsymbol{p})$, giving

$$Q = i\int \mathrm{d}^3\boldsymbol{x}\int đp\int đp' \tag{12.4.14}$$

$$\Big\{:[a^\dagger(\boldsymbol{p})\mathrm{e}^{ip\cdot x} + b\mathrm{e}^{-ip\cdot x}]i\omega(\boldsymbol{p}')[-a(\boldsymbol{p}')\mathrm{e}^{-ip'\cdot x} + b^\dagger(\boldsymbol{p}')\mathrm{e}^{ip'\cdot x}]:$$

$$- i\omega(\boldsymbol{p}):[a^\dagger(\boldsymbol{p})\mathrm{e}^{ip\cdot x} - b(\boldsymbol{p})\mathrm{e}^{-ip\cdot x}][a(\boldsymbol{p}')\mathrm{e}^{-ip'\cdot x} + b^\dagger(\boldsymbol{p}')\mathrm{e}^{ip'\cdot x}]:\Big\}.$$

After expanding this, integrating over x to get Dirac deltas, and then using these to perform the p' integral we're left with

$$Q = \int đp\, :(a^\dagger(\boldsymbol{p})a(\boldsymbol{p}) - b(\boldsymbol{p})b^\dagger(\boldsymbol{p})): \tag{12.4.15}$$

$$= \int đp(a^\dagger(\boldsymbol{p})a(\boldsymbol{p}) - b^\dagger(\boldsymbol{p})b(\boldsymbol{p})). \tag{12.4.16}$$

From this we can then show that the following commutation relations hold:

$$[Q, a^\dagger(\boldsymbol{p})] = \quad a^\dagger \qquad [Q, a(\boldsymbol{p})] = -a(\boldsymbol{p}), \tag{12.4.17}$$

$$[Q, b^\dagger(\boldsymbol{p})] = -b^\dagger \qquad [Q, b(\boldsymbol{p})] = \quad b(\boldsymbol{p}). \tag{12.4.18}$$

So finally we have a difference between a and b. This leads to the interpretation that

- $a^\dagger(\boldsymbol{p})$ creates a particle with momentum $\boldsymbol{p}$ and "charge" $+1$,
- $b^\dagger(\boldsymbol{p})$ creates a particle with momentum $\boldsymbol{p}$ and "charge" -1,
- $a(\boldsymbol{p})$ annihilates a particle with momentum $\boldsymbol{p}$ and "charge" $+1$,
- $b(\boldsymbol{p})$ annihilates a particle with momentum $\boldsymbol{p}$ and "charge" -1.

Here we write "charge", since this may not be electrical charge, but it can be.

So, a complex scalar field gives a theory with a pair of charged particles. Conservation of Q implies that particles must always be created and destroyed in pairs. This all follows simply from recognising that the Lagrangian must be real and so must depend only on products like $\varphi^\dagger\varphi$.

There is another interpretation of this, which is that $a^\dagger$ creates particles, and $b^\dagger$ creates antiparticles (or vice versa). An antiparticle being a particle with the same mass but opposite charge. Then

- $a^\dagger(\boldsymbol{p})$ creates a particle with momentum $\boldsymbol{p}$ and "charge" $+1$,
- $b^\dagger(\boldsymbol{p})$ creates an antiparticle with momentum $\boldsymbol{p}$ and "charge" -1,
- $a(\boldsymbol{p})$ annihilates a particle with momentum $\boldsymbol{p}$ and "charge" $+1$,
- $b(\boldsymbol{p})$ annihilates an antiparticle with momentum $\boldsymbol{p}$ and "charge" -1.

Notice that destroying a particle of charge $+1$ is the same as creating a particle of charge -1 and vice versa.

For a real field we have $a = b$ and we can interpret this as the statement that the particles in this theory are their own antiparticles.

12.5 Feynman Rules

> *It's useful for you to get used to confusing notation in this course, because it will get a whole lot worse.*
>
> Richard Ball

Consider the mode expansions of the fields:

$$\varphi = \int đp[a(\boldsymbol{p})e^{-ip\cdot x} + b^\dagger(\boldsymbol{p})e^{ip\cdot x}], \tag{12.5.1}$$

$$\varphi^\dagger = \int đp[b(\boldsymbol{p})e^{-ip\cdot x} + a^\dagger(\boldsymbol{p})e^{ip\cdot x}]. \tag{12.5.2}$$

Using these and the commutation relations for the mode operators we can derive the covariant commutator

$$[\varphi(x), \varphi^\dagger(x')] = i\Delta(x - x') \tag{12.5.3}$$

where Δ is the same function as for real scalar fields. The other commutators, $[\varphi(x), \varphi(x')]$ and $[\varphi^\dagger(x), \varphi^\dagger(x)]$, vanish.

A result of this vanishing of of commutators between φs or between $\varphi^\dagger$s is that contractions between two φs or two $\varphi^\dagger$s also vanish, and so the only nonvanishing contractions are

$$\overbrace{\varphi(x)\varphi^\dagger}(x') = \langle 0|\,\mathrm{T}[\varphi(x)\varphi^\dagger(x')]|0\rangle = i\Delta_{\mathrm{F}}(x-x'), \tag{12.5.4}$$

where Δ_{F} is the same as for a real scalar field.

As before our state space is given by direct products of single particle states, which are now labelled with the charge as well as the momentum of the particle:

$$|\boldsymbol{p},+\rangle = a^\dagger(\boldsymbol{p})|0\rangle, \qquad \text{and} \qquad |\boldsymbol{p},-\rangle = b^\dagger(\boldsymbol{p})|0\rangle. \tag{12.5.5}$$

[4] We lower the + and − labels to make room for †.

Writing $\varphi = \varphi_+ + \varphi_-$ as before[4] we have

$$\varphi_+(x)|\boldsymbol{p},+\rangle = \mathrm{e}^{-ip\cdot x}|0\rangle \qquad \varphi_+^\dagger(x)|\boldsymbol{p},-\rangle = \mathrm{e}^{-ip\cdot x}|0\rangle, \tag{12.5.6}$$

$$\langle \boldsymbol{p},+|\varphi_-^\dagger(x) = \mathrm{e}^{ip\cdot x}\langle 0| \qquad \langle \boldsymbol{p},-|\varphi_-(x) = \mathrm{e}^{ip\cdot x}\langle 0|. \tag{12.5.7}$$

So, we conclude that

- φ_+ destroys an incoming particle of charge $+1$,
- $\varphi_+^\dagger$ destroys an incoming particle of charge -1,
- φ_- creates an outgoing particle of charge -1,
- $\varphi_-^\dagger$ creates an outgoing particle of charge $+1$.

From this we can compute all the Feynman rules for any given interaction.

Thirteen

Interactions and Symmetries

13.1 Types of Interaction

We choose to look for interactions which conserve charge. While we can easily construct theories where this isn't the case charge conservation is a common phenomena and we wish our theories to exhibit it. This means we can no longer have a φ^3 interaction, since this would not conserve charge. Even if we consider a $\varphi^3 + (\varphi^\dagger)^3$ interaction, so the Lagrangian is real, we still don't get charge conservation, since the first term transforms as $\varphi^3 \mapsto e^{3i\alpha}\varphi^3$, and the second as $(\varphi^\dagger)^3 \mapsto e^{-3i\alpha}(\varphi^\dagger)^3$. The only way to have a theory conserve charge is if every term in the Lagrangian has as many factors of φ as it does $\varphi^\dagger$. There are a few ways to do this.

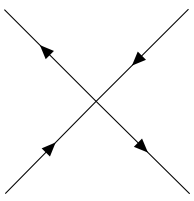
13.1.1 Four Point Interactions

Instead of the three point interaction, φ^3, we can consider a four point interaction, $(\varphi^\dagger\varphi)^2$, where interactions involve four particles at once, this would lead to the interaction Lagrangian

$$\mathcal{L}_I^{(4)} = -\frac{\lambda}{2}(\varphi^\dagger\varphi)^2. \tag{13.1.1}$$

To keep track of charge in our Feynman diagrams it is conventional to put an arrow on our Feynman diagrams pointing in the direction of charge flow. Note that charge flowing backwards in time is the same as negative charge flowing forward in time, that is antiparticles correspond to arrows pointing left and particles to arrows pointing right.

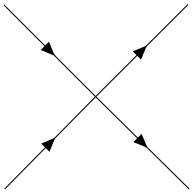
There are three vertices in this theory which are all slightly different, but have the same shape. The first is



(13.1.2)

This corresponds to a particle and antiparticle coming in and a particle and antiparticle going out. The net charge in and out is then 0. So charge is conserved in this vertex.

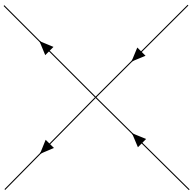
The next vertex is



(13.1.3)

This corresponds to two particles coming in and two particles going out. The net charge in and out is then 2, so charge is conserved in this vertex.

The final vertex is



(13.1.4)

This corresponds to two antiparticles coming in and two antiparticles going out. The net charge in and out is then -2, so charge is conserved in this vertex.

As with the real case the constant factor in this interaction has been chosen such that every vertex gives a factor of $-i\lambda$.

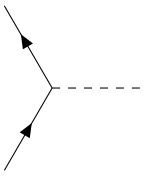
13.1.2 Three Point Interactions

If we introduce a new, neutral, real scalar field, Φ, then we can have a three point interaction of the form

$$\mathcal{L}_{\mathrm{I}}^{(3)} = -y\Phi\varphi^{\dagger}\varphi. \tag{13.1.5}$$

This is known as a **Yukawa interaction**.

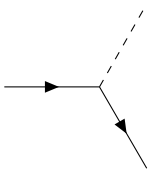
As with φ^3 theory we get three particles in each interaction. There are a variety of vertices we can create in this case. Denoting the neutral particle with a dashed line one possibility is



(13.1.6)

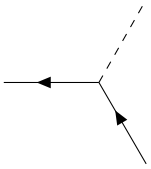
This corresponds to a particle and antiparticle annihilating into a neutral particle.

Another vertex is



(13.1.7)

This corresponds to a particle emitting a neutral particle. Similarly the vertex



(13.1.8)

corresponds to an antiparticle emitting a neutral particle. Notice that in all of these diagrams the total charge is conserved at each vertex.

As before the constant factors are chosen such that each vertex contributes a factor of $-iy$ to the amplitude.

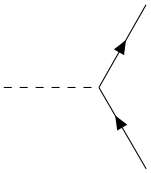
A full Lagrangian in a theory with this interaction may take the form

$$\mathcal{L} = (\partial_\mu \varphi^\dagger)(\partial_\mu \varphi) - m^2 \varphi^\dagger \varphi + \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2} M^2 \Phi^2 - y \Phi \varphi^\dagger \varphi, \qquad (13.1.9)$$

corresponding to a neutral particle, call it s_0, of mass M and charged (anti)particles, call them $s_\pm$, of mass m.

In many ways this theory is not that different from the real φ^3 theory. For example, we still have 2–2 scattering, such as $s_+ s_- \to s_+ s_-$, $s_+ s_+ \to s_+ s_+$, and $s_- s_- \to s_- s_-$. The exchange particle in these cases is a neutral scalar, s_0. Note that in the $s_+ s_- \to s_+ s_-$ case, since the final two particles are distinct, there is no u-channel diagram. Similarly for $s_\pm s_\pm \to s_\pm s_\pm$ there is no s-channel diagram, since the intermediate particle would have to carry a charge of ± 2.

There is one sort of interaction which we don't have in φ^3 theory, and that is decays. If $M > 2m$ then it is possible to have the interaction $s_0 \to s_+ s_-$. This would correspond to an $n = 1$ contribution to the Dyson series, and these terms vanished for φ^3 theory. This would look like



(13.1.10)

The amplitude for this is $\mathcal{M} = -iy$. We can work out the decay rate starting with

$$\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega}\right)_{\mathrm{CoM}} = \frac{1}{32\pi^2} \frac{p}{M^2} |\mathcal{M}|^2, \qquad (13.1.11)$$

where p is the magnitude of the three-momentum of the two charged particles in the centre of mass frame. The amplitude squared will give $|\mathcal{M}|^2 = |-iy|^2 = y^2$. We then have

$$\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega}\right)_{\mathrm{CoM}} = \frac{y^2}{64\pi^2} \frac{\sqrt{M^2 - 4m^2}}{M^2}. \qquad (13.1.12)$$

So we see that we must have $M > 2m$ for this to be nonzero and real, assuming $M, m \geq 0$.

We just said "assuming $M, m \geq 0$", but what if they aren't? In fact, what if they aren't even real? We can analytically extend this decay rate as a function of the

complex variable M. Recall that $\sqrt{f(z)}$ has a branch cut from $f(z) = 0$ to infinity. So the decay rate has a branch cut from $M = 2m$ to infinity. This is part of a general rule, particles correspond to poles, such as the pole at $m = \sqrt{s}$ in $1/(s - m^2)$, and decays correspond to branch cuts, such as the one starting at $M = 2m$ here.

13.2 Symmetries

In this section we'll study the effect of parity, time reversal, and charge conjugation on scalar fields. We'll start from their action on spacetime, and then derive their action on the fields and mode operators. These three symmetries are all $\mathbb{Z}_2$ symmetries, meaning that, apart from doing nothing, we have a single operation and applying this operation twice is the same as doing nothing. The technical word for this is that these symmetries are **idempotent**.

13.2.1 Parity

Parity acts on spacetime by sending $\boldsymbol{x}$ to $-\boldsymbol{x}$. That is

$$\mathcal{P}: \; x^\mu = (t, \boldsymbol{x}) \mapsto \bar{x}^\mu = (t, -\boldsymbol{x}). \tag{13.2.1}$$

Since the direction of space is also changed the momentum is also changed by sending $\boldsymbol{p}$ to $-\boldsymbol{p}$:

$$\mathcal{P}: \; p^\mu = (E, \boldsymbol{p}) \mapsto \bar{p}^\mu = (E, -\boldsymbol{p}). \tag{13.2.2}$$

We will use the notation $\bar{x}$ and $\bar{p}$ with this meaning throughout this section, including when discussing time reversal and charge conjugation.

The definition of parity above is classical. In quantum field theory we have operators and states. Parity is promoted to be a unitary operator, $\mathcal{P}$, on the states, that is $\mathcal{P}^\dagger\mathcal{P} = 1$. Starting with the simplest case, the vacuum state should be invariant under parity, since there's nothing there to change:

$$\mathcal{P}|0\rangle = |0\rangle. \tag{13.2.3}$$

The next simplest state is a single particle state, $|\boldsymbol{p}\rangle$. We expect that the momentum of this particle should be reversed under parity:

$$\mathcal{P}|\boldsymbol{p}\rangle = |-\boldsymbol{p}\rangle. \tag{13.2.4}$$

We can write this momentum state as $|\boldsymbol{p}\rangle = a^\dagger(\boldsymbol{p})|0\rangle$, and so we can identify the action of the parity operator on the state with an equivalent action on $a^\dagger(\boldsymbol{p})$. Since $a^\dagger(\boldsymbol{p})$ is an operator it transforms with two copies of $\mathcal{P}$ as

$$\mathcal{P}a^\dagger(\boldsymbol{p})\mathcal{P}^\dagger = a^\dagger(-\boldsymbol{p}), \tag{13.2.5}$$

so

$$(\mathcal{P}a^\dagger(\boldsymbol{p})\mathcal{P}^\dagger)|0\rangle = a^\dagger(-\boldsymbol{p})|0\rangle = |-\boldsymbol{p}\rangle. \tag{13.2.6}$$

Taking the adjoint of this we have

$$[\mathcal{P}a^\dagger(\boldsymbol{p})\mathcal{P}]^\dagger = (\mathcal{P}^\dagger)^\dagger a(\boldsymbol{p})\mathcal{P}^\dagger = \mathcal{P}a(\boldsymbol{p})\mathcal{P}^\dagger, \tag{13.2.7}$$

and so

$$\mathcal{P}a(\boldsymbol{p})\mathcal{P}^\dagger = [a^\dagger(-\boldsymbol{p})]^\dagger = a(-\boldsymbol{p}). \tag{13.2.8}$$

To derive the effect on the field, which is an operator and so transforms as $\varphi \mapsto \mathcal{P}\varphi\mathcal{P}^\dagger$, we can expand φ in the mode operators:

$$\mathcal{P}\varphi(x)\mathcal{P}^\dagger = \int đp[\mathcal{P}a(\boldsymbol{p})\mathcal{P}^\dagger e^{-ip\cdot x} + \mathcal{P}a^\dagger(\boldsymbol{p})\mathcal{P}^\dagger e^{ip\cdot x}] \tag{13.2.9}$$

$$= -\int đp[a(-\boldsymbol{p})e^{-ip\cdot x} + a^\dagger(-\boldsymbol{p})e^{ip\cdot x}]. \tag{13.2.10}$$

Note that the parity operator doesn't act on the phase factors, even though they depend on position and momentum, they're just scalars and so aren't affected by operators on the Hilbert space. Now take $\boldsymbol{p} \to -\boldsymbol{p}$ in the integral, this doesn't change the measure or region of integration and takes $p \to \bar{p}$, giving

$$\mathcal{P}\varphi(x)\mathcal{P}^\dagger = \int đp[a(\boldsymbol{p})e^{-i\bar{p}\cdot x} + a^\dagger(\boldsymbol{p})e^{i\bar{p}\cdot x}]. \tag{13.2.11}$$

Now notice that $\bar{p}\cdot x = Et + \boldsymbol{p}\cdot\boldsymbol{x} = p\cdot\bar{x}$, and so

$$\mathcal{P}\varphi(x)\mathcal{P}^\dagger = \int đp[a(\boldsymbol{p})e^{-ip\cdot\bar{x}} + a^\dagger(\boldsymbol{p})e^{ip\cdot\bar{x}}] \tag{13.2.12}$$

$$= \varphi(\bar{x}). \tag{13.2.13}$$

So under parity the field transforms as

$$\mathcal{P}:\ \varphi(x) \mapsto \mathcal{P}\varphi(x)\mathcal{P}^\dagger = \varphi(\bar{x}). \tag{13.2.14}$$

This is what we would expect. Note that exactly the same argument can be applied to the conjugate field, and to complex scalar fields, they all transform the same.

One thing that we didn't consider is that quantum states are only defined up to phase. This means that its possible to have an equivalent transformation which introduces a phase, so we could have that parity acts on the single particle state as

$$\mathcal{P}|\boldsymbol{p}\rangle = \eta_p|-\boldsymbol{p}\rangle. \tag{13.2.15}$$

Since $\mathcal{P}$ is idempotent we have $\mathcal{P}\mathcal{P}|\boldsymbol{p}\rangle = \mathcal{P}\eta_p|-\boldsymbol{p}\rangle = \eta_p^2|\boldsymbol{p}\rangle$, and we must have $\mathcal{P}\mathcal{P}|\boldsymbol{p}\rangle = |\boldsymbol{p}\rangle$, so $\eta_p^2 = 1$, meaning $\eta_p = \pm 1$. Above we assumed $\eta_p = 1$, suppose instead that $\eta_p = -1$, then the mode operators must transform as

$$\mathcal{P}:\ a(\boldsymbol{p}) \mapsto \mathcal{P}a(\boldsymbol{p})\mathcal{P}^\dagger = -a(-\boldsymbol{p}), \tag{13.2.16}$$

and hence the field transforms as

$$\mathcal{P}:\ \varphi(x) \mapsto \mathcal{P}\varphi(x)\mathcal{P}^\dagger = -\varphi(\bar{x}). \tag{13.2.17}$$

In this case we say that φ is a **pseudoscalar** field.

13.2.2 Time Reversal

Time reversal acts on spacetime by sending t to $-t$. That is

$$\mathcal{T}: \; x^\mu = (t, \boldsymbol{x}) \mapsto -\bar{x}^\mu = (-t, \boldsymbol{x}). \tag{13.2.18}$$

Since the direction of time reverses the direction of momentum, proportional to the velocity, $\dot{x}$, and hence first order in time derivatives, changes direction. The energy is unaffected. The simplest explanation for why is that the kinetic energy is proportional to $\dot{x}^2$ and so picks up two negatives upon time reversal, cancelling out for no overall change. So, momentum transforms under time reversal as

$$\mathcal{T}: \; p^\mu = (E, \boldsymbol{p}) \mapsto \bar{p}^\mu = (E, -\boldsymbol{p}). \tag{13.2.19}$$

Again when moving to quantum field theory time reversal is promoted to an operator, $\mathcal{T}$ on the states. One effect of time reversal is that the initial and final state are swapped. This means that amplitudes change:

$$\mathcal{T}: \; \langle f|i\rangle \mapsto \langle i|f\rangle = \langle f|i\rangle^*. \tag{13.2.20}$$

So, time reversal means we have to take the complex conjugate of everything. Another way to view this is to consider the time evolution operator, $\exp[-iEt]$. Under time reversal this changes to $\exp[iEt] = \exp[-iEt]^*$. We say that $\mathcal{T}$ is an antiunitary operator.

Definition 13.2.21 — Antiunitary Operator Let $\mathbb{H}$ be a complex Hilbert space with inner product $\langle -, - \rangle$, and let $U: \mathbb{H} \to \mathbb{H}$ be an operator. Then U is an **antiunitary operator** if

$$\langle Ux, Uy\rangle = \langle x, y\rangle^* \tag{13.2.22}$$

for all $x, y \in \mathbb{H}$.

This means that under time reversal the exponential phase factor transforms as

$$\mathcal{T}: \; e^{ip\cdot x} \mapsto e^{(-i)\bar{p}\cdot(-\bar{x})} = e^{i\bar{p}\cdot\bar{x}} = e^{ip\cdot x}. \tag{13.2.23}$$

Consider the transformation of a single particle state:

$$\mathcal{T}|\boldsymbol{p}\rangle = |-\boldsymbol{p}\rangle. \tag{13.2.24}$$

We must therefore have

$$\mathcal{T}a^\dagger(\boldsymbol{p})\mathcal{T}^\dagger = a^\dagger(-\boldsymbol{p}), \qquad \text{and} \qquad \mathcal{T}a(\boldsymbol{p})\mathcal{T}^\dagger = a(-\boldsymbol{p}). \tag{13.2.25}$$

The field then transforms as

$$\begin{aligned}
\mathcal{T}\varphi(x)\mathcal{T}^\dagger &= \int đp[a(-\boldsymbol{p})e^{ip\cdot x} + a^\dagger(-\boldsymbol{p})e^{-ip\cdot x}] && (13.2.26)\\
&= \int đp[a(\boldsymbol{p})e^{i\bar{p}\cdot x} + a(\boldsymbol{p})e^{-i\bar{p}\cdot x}] && (13.2.27)\\
&= \int đp[a(\boldsymbol{p})e^{-ip\cdot(-\bar{x})} + a(\boldsymbol{p})e^{ip\cdot(-\bar{x})}] && (13.2.28)\\
&= \varphi(-\bar{x}). && (13.2.29)
\end{aligned}$$

Here we changed $\boldsymbol{p}$ to $-\boldsymbol{p}$ in the second line. This means the field transforms as

$$\mathcal{T}:\ \varphi(x) \mapsto \mathcal{T}\varphi(x)\mathcal{T}^\dagger = \varphi(-\bar{x}). \tag{13.2.30}$$

Again, we can include a phase factor, η_t, which is restricted to be ± 1. Also all of this logic applies to both the conjugate field and to complex scalar fields.

13.2.3 Charge Conjugation

Classically charge conjugation changes the sign of electric charge. So

$$\mathcal{C}:\ Q \mapsto -Q. \tag{13.2.31}$$

In quantum field theory this corresponds to exchanging particles and antiparticles. Hence, charge conjugation is only interesting for complex scalar fields. Charge conjugation acts on the single particle states according to

$$\mathcal{C}|\boldsymbol{p},+\rangle = |\boldsymbol{p},-\rangle, \qquad \text{and} \qquad \mathcal{C}|\boldsymbol{p},-\rangle = |\boldsymbol{p},+\rangle. \tag{13.2.32}$$

Exchanging particles and antiparticles is the same as exchanging mode operators $a \leftrightarrow b$:

$$\mathcal{C}:\ a(\boldsymbol{p}) \mapsto \mathcal{C}a(\boldsymbol{p})\mathcal{C}^\dagger = b(\boldsymbol{p}), \tag{13.2.33}$$
$$\mathcal{C}:\ a^\dagger(\boldsymbol{p}) \mapsto \mathcal{C}a^\dagger(\boldsymbol{p})\mathcal{C}^\dagger = b^\dagger(\boldsymbol{p}), \tag{13.2.34}$$
$$\mathcal{C}:\ b(\boldsymbol{p}) \mapsto \mathcal{C}b(\boldsymbol{p})\mathcal{C}^\dagger = a(\boldsymbol{p}), \tag{13.2.35}$$
$$\mathcal{C}:\ b^\dagger(\boldsymbol{p}) \mapsto \mathcal{C}b^\dagger(\boldsymbol{p})\mathcal{C}^\dagger = a^\dagger(\boldsymbol{p}). \tag{13.2.36}$$

The action on the field is then

$$\mathcal{C}\varphi(x)\mathcal{C}^\dagger = \int đp[b(\boldsymbol{p})e^{-ip\cdot x} + a^\dagger(\boldsymbol{p})e^{ip\cdot x}] = \varphi^\dagger(x), \tag{13.2.37}$$

so

$$\mathcal{C}:\ \varphi(x) \mapsto \mathcal{C}\varphi(x)\mathcal{C}^\dagger = \varphi^\dagger(x). \tag{13.2.38}$$

Similarly,

$$\mathcal{C}:\ \varphi^\dagger(x) \mapsto \mathcal{C}\varphi^\dagger(x)\mathcal{C}^\dagger = \varphi(x). \tag{13.2.39}$$

For a complex scalar field we also have the current, j^μ. Acting on this with charge conjugation exchanges $\varphi^\dagger$ and φ, giving

$$\mathcal{C}j^\mu\mathcal{C}^\dagger = i\mathcal{C}:\varphi^\dagger\partial^\mu\varphi - \varphi\partial^\mu\varphi^\dagger:\mathcal{C}^\dagger \tag{13.2.40}$$
$$= i:\varphi\partial^\mu\varphi^\dagger - \varphi^\dagger\partial^\mu\varphi: \tag{13.2.41}$$
$$= -j^\mu, \tag{13.2.42}$$

which is what we would expect. Since the total charge is defined as the spatial integral of j^0 we also have

$$\mathcal{C}Q\mathcal{C}^\dagger = -Q, \tag{13.2.43}$$

which is the same as the classical action of charge conjugation.

13.2.4 Acting on the Lagrangian

The free Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}m^2\varphi^2 \tag{13.2.44}$$

for a real scalar field, or

$$\mathcal{L} = (\partial_\mu \varphi^\dagger)(\partial^\mu \varphi) - m^2\varphi^\dagger\varphi \tag{13.2.45}$$

for a complex scalar field. In either case the the result is invariant under $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{C}$.

Now suppose that the interaction Lagrangian is also invariant under $\mathcal{P}$, $\mathcal{T}$ and $\mathcal{C}$. Then, under $\mathcal{P}$ or $\mathcal{C}$ the amplitude transforms as

$$\mathcal{P}: \mathcal{M} \mapsto \mathcal{M}, \tag{13.2.46}$$
$$\mathcal{C}: \mathcal{M} \mapsto \mathcal{M}. \tag{13.2.47}$$

That is, the amplitude is unchanged. Under $\mathcal{T}$ the amplitude transforms as

$$\mathcal{T}: \mathcal{M} \mapsto \mathcal{M}^*. \tag{13.2.48}$$

However, in all three cases $|\mathcal{M}|^2$ is unchanged and so the physics doesn't change.

Invariance of the interaction term under these symmetries is not a given. For example, a φ^3 interaction with a pseudoscalar field gains a minus sign under parity. However, for the purposes of this course we will only consider theories invariant under $\mathcal{P}$, $\mathcal{T}$ and $\mathcal{C}$ separately. In *Gauge Theories in Particle Physics* we will consider theories which aren't invariant under these symmetries individually, but instead are invariant under the combined $\mathcal{CPT}$ symmetry.

Part IV

Dirac Equation

Fourteen

Deriving the Dirac Equation

In this chapter we'll derive the Dirac equation in the way that Dirac did historically. He was motivated by looking for an equation without the negative energies of the Klein–Gordon equation. He failed to find such an equation, but it turns out that the negative energy solutions are actually important, and not just a problem to be eliminated.

14.1 The Dirac Equation

To avoid having to take square roots when computing the energy, which is the source of the negative energy terms in the Klein–Gordon equation, Dirac looked for a wave equation linear in ∂_t. For this to be relativistic Dirac knew that the equation would also have to be linear in ∂_x. Dirac also imposed one further constraint. The equation he was deriving should imply the Klein–Gordon equation since this is how we get wave equations with the relativistic energy momentum relation.

Dirac started with the ansatz

$$\frac{\partial \psi}{\partial t} + \boldsymbol{\alpha} \cdot \nabla \psi + im\beta\psi = 0, \tag{14.1.1}$$

which can be written in a Schrödinger-like form as

$$i\frac{\partial \psi}{\partial t} = H\psi, \qquad \text{where} \qquad H = -i\boldsymbol{\alpha} \cdot \nabla + \beta m = \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m \tag{14.1.2}$$

is the **Dirac Hamiltonian**. It should be noted that this is *not* actually the Hamiltonian for a system satisfying the Dirac equation.

Looking for real energies Dirac posited that H should be Hermitian, and hence $\boldsymbol{\alpha}$ and β should be real. Next Dirac looked for other conditions on $\boldsymbol{\alpha}$ and β which would enforce this equation implying the Klein–Gordon equation. Noticing that the Klein–Gordon equation has second derivatives we need to find a way to extract second derivatives from the Dirac equation. To do this we square the operator, but in such a way that the cross terms cancel, so we compute the product of the operator and its adjoint:

$$\left(\frac{\partial}{\partial t} - \boldsymbol{\alpha} \cdot \nabla - im\beta\right)\left(\frac{\partial}{\partial t} + \boldsymbol{\alpha} \cdot \nabla + im\beta\right)\psi = 0. \tag{14.1.3}$$

Expanding this we have

$$\left(\frac{\partial^2}{\partial t^2} - \alpha_i\alpha_j\frac{\partial^2}{\partial x_i \partial x_j} + m^2\beta^2 - im(\alpha_i\beta + \beta\alpha_i)\frac{\partial}{\partial x_i}\right)\psi = 0. \tag{14.1.4}$$

Looking at this we see that if $\alpha_i\alpha_j = 1$ the cross term vanishes then we get the Klein–Gordon equation. The symmetry of $\partial^2/\partial x_i \partial x_j$ implies that we should look for $\alpha_i\alpha_j$ to be symmetric, so in terms of setting $\alpha_i\alpha_j$ to 1 we impose

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \tag{14.1.5}$$

where $\{A, B\} \coloneqq AB + BA$ is the **anticommutator**. Similarly to get the Klein–Gordon equation we need $\beta^2 = 1$. The cross term will vanish if

$$\{\alpha_i, \beta\} = 0 \tag{14.1.6}$$

Clearly these requirements cannot be satisfied if α_i and β are numbers. We can have relations like these if α_i and β are *matrices* though. Looking for H to be Hermitian means that α_i and β must be *Hermitian* matrices. Suppose α_i and β are $N \times N$ matrices. Then ψ must be an object with N components.

We can go on to derive further properties of α_i and β which will help us to narrow down what matrices they are. First, $\alpha_i^2 = \beta^2 = 1$ the eigenvalues must square to 1, so the eigenvalues of α_i and β are ± 1.

Next consider the trace of α_i. Using the anticommutation relations for α_i and β we have

$$\begin{aligned}
\operatorname{tr}\alpha_i &= \operatorname{tr}(\beta^2\alpha_i) && \beta^2 = 1 && (14.1.7)\\
&= \operatorname{tr}(\beta\alpha_i\beta) && \text{tr is cyclic} && (14.1.8)\\
&= \operatorname{tr}(-\alpha_i\beta^2) && \alpha_i \text{ and } \beta \text{ anticommute} && (14.1.9)\\
&= -\operatorname{tr}(\alpha_i\beta^2) && \text{tr is linear} && (14.1.10)\\
&= -\operatorname{tr}\alpha_i && \beta^2 = 1 && (14.1.11)
\end{aligned}$$

So we can conclude that $\operatorname{tr}\alpha_i = -\operatorname{tr}\alpha_i$ and so $\operatorname{tr}\alpha_i = 0$. We can do exactly the same swapping β and α_i and so $\operatorname{tr}\beta = 0$. This means that α_i and β are *traceless* Hermitian matrices.

In the basis where α_i is diagonal the diagonal is just the eigenvalues, and so the trace is the sum of the eigenvalues. The trace is basis independent so the trace is *always* the sum of eigenvalues. The fact that α_i and β have eigenvalues ± 1 and the sum of the eigenvalues, that is the trace, must vanish means that α_i and β must have an even number of eigenvalues, and hence α_i and β are traceless Hermitian matrices of *even dimension*.

At this point we can just start trying to find a solution for α_i and β. We start with the simplest case, $N = 2$. We know that the Pauli spin matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \text{and} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{14.1.12}$$

form a basis for the 2×2 traceless Hermitian matrices, and so we posit that $\alpha_i = \sigma_i$, or more accurately σ_i is a *representation*[1] of α_i. The problem comes when we look for a 2×2 representation of β. There is no other 2×2 Hermitian traceless matrix, and so we cannot find a value for β in the 2×2 matrices. This may not be a problem. If we're looking at a massless particle then $m = 0$ gets rid of the β term, so we don't need a value for β and we can use 2×2 matrices and 2 component spinors to describe such particles.

[1] see *Symmetries of Quantum Mechanics* or *Symmetries of Particles and Fields* for details on what a representation is.

Move on to the next simplest case, $N = 4$, here there are solutions. One very common representation is the **Dirac representation**, which is chosen such that β is diagonal:

$$\beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \text{and} \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}. \tag{14.1.13}$$

We call these the **Dirac matrices** in this representation.

These are complex, meaning that ψ is an object with four complex components. We call ψ a **spinor**, for reasons we'll see later. As a four component thing it may be more accurate to write ψ with an index, say as ψ_a, but spinor indices are traditionally suppressed unless there is ambiguity. When we do write spinor indices we'll choose values from the start of the alphabet, $a, b, c, \ldots$, as opposed to the Greek letters, $\mu, \nu, \rho, \ldots$ we use for Lorentz indices or $i, j, k, \ldots$ for spatial indices.

14.2 Covariant Form

The Dirac equation as we've written it so far is not *manifestly* covariant. We can fix this by changing up our notation slightly. Start by multiplying through by $i\beta$ to get

$$\left(i\beta\frac{\partial}{\partial t} + i\beta\alpha_j\frac{\partial}{\partial x_j} - m\right)\psi = 0, \tag{14.2.1}$$

where we've used $\beta^2 = 1$, although really we now know that "1" here is an identity matrix, which we leave implicit. Now define $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha^i$. These form a four-vector, $\gamma^\mu = (\gamma^0, \gamma^i) = (\beta, \beta\boldsymbol{\alpha})$. We can then write

$$\left(i\gamma^0\frac{\partial}{\partial x_0} + i\gamma^j\frac{\partial}{\partial x_j} - m\right)\psi = 0, \tag{14.2.2}$$

or more compactly,

$$(i\gamma^\mu\partial_\mu - m)\psi = 0, \tag{14.2.3}$$

or even more compactly,

$$(i\partial\!\!\!/ - m)\psi = 0 \tag{14.2.4}$$

where $a\!\!\!/ \coloneqq \gamma^\mu a_\mu$ is a shorthand called **slash notation**.

Like α_i and β these γ^μ are really $N \times N$ matrices, which we call the **gamma matrices** for obvious reasons. If we reinstate the spinor indices then we might write γ^μ_{ab}, where μ is an index telling us if we are considering γ^0, γ^1, γ^2, or γ^3 and the indices a and b tell us which component of the matrix we are considering.

We can determine the anticommutation relations of γ^μ using those of α_i and β, first, we have $\{\gamma^0, \gamma^0\} = \{\beta, \beta\} = 2\beta^2 = 2$, next $\{\gamma^0, \gamma^i\} = \{\beta, \beta\alpha^i\} = \beta^2\alpha^i + \beta\alpha^i\beta = \beta^2\alpha^i - \alpha^i\beta = 0$. Finally, $\{\gamma^i, \gamma^j\} = \{\beta\alpha^i, \beta\alpha^j\} = \beta\alpha^i\beta\alpha^j + \beta\alpha^j\beta\alpha^i = -\beta\alpha^i\alpha^j\beta - \beta\alpha^j\alpha^i\beta = -\beta\{\alpha^i, \alpha^j\}\beta = -\beta 2\delta^{ij}\beta = 2\delta^{ij}$. Hence,

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{14.2.5}$$

This is called the **Clifford algebra** of the gamma matrices. We take this as the defining property of the gamma matrices, and it is this algebra which is being represented by a particular choice of matrices.

It is useful to define a **fifth gamma matrix**, $\gamma^5 = \gamma_5$. This is *not* part of the Clifford algebra. The 5 is just a label, it's not an index and no index will ever take the value of 5 (while we remain in 4 dimensions, even in more dimensions there are only ever four gamma matrices), we use the label 5 because some sources label the gamma matrices 1 through 4 instead of 0 through 3. This extra matrix is defined as

$$\gamma^5 = \gamma_5 \coloneqq i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{14.2.6}$$

The factor of i here is sometimes left out, in which case γ^5 will be anti-Hermitian.

Using the anticommutator relations we have $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ whenever $\mu \neq \nu$, as well as $(\gamma^0)^2 = 1$ and $(\gamma^i)^2 = -1$, so

$$\begin{aligned}
(\gamma^5)^2 &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 && && (14.2.7)\\
&= \gamma^1\gamma^2\gamma^3(\gamma^0)^2\gamma^1\gamma^2\gamma^3 && \gamma^0 \text{ past 3 } \gamma^\mu && (14.2.8)\\
&= \gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 && (\gamma^0)^2 = 1 && (14.2.9)\\
&= \gamma^2\gamma^3(\gamma^1)^2\gamma^2\gamma^3 && \gamma^1 \text{ past 2 } \gamma^\mu && (14.2.10)\\
&= -\gamma^2\gamma^3\gamma^2\gamma^3 && (\gamma^1)^2 = -1 && (14.2.11)\\
&= \gamma^3(\gamma^2)^2\gamma^3 && \gamma^2 \text{ past 1 } \gamma^\mu && (14.2.12)\\
&= -(\gamma^3)^2 && (\gamma^2)^2 = -1 && (14.2.13)\\
&= 1 && (\gamma^3)^2 = -1. && (14.2.14)
\end{aligned}$$

We can also derive anticommutator relations for γ^5 with the other gamma matrices:

$$\begin{aligned}
\{\gamma^5, \gamma^0\} &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 + i(\gamma^0)^2\gamma^1\gamma^2\gamma^3 && (14.2.15)\\
&= -i(\gamma^0)^2\gamma^1\gamma^2\gamma^3 + i(\gamma^0)^2\gamma^1\gamma^2\gamma^3 && (14.2.16)\\
&= 0, && (14.2.17)\\
\{\gamma^5, \gamma^1\} &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1 + i\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 && (14.2.18)\\
&= i\gamma^0(\gamma^1)^2\gamma^2\gamma^3 - i\gamma^0(\gamma^1)^2\gamma^2\gamma^3 && (14.2.19)\\
&= 0, && (14.2.20)\\
\{\gamma^5, \gamma^2\} &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^2 + i\gamma^2\gamma^0\gamma^1\gamma^2\gamma^3 && (14.2.21)\\
&= -i\gamma^0\gamma^1(\gamma^2)^2\gamma^3 + i\gamma^0\gamma^1(\gamma^2)^2\gamma^3 && (14.2.22)\\
&= 0, && (14.2.23)\\
\{\gamma^5, \gamma^3\} &= i\gamma^0\gamma^1\gamma^2(\gamma^3)^2 + i\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 && (14.2.24)\\
&= i\gamma^0\gamma^1\gamma^2(\gamma^3)^2 - i\gamma^0\gamma^1\gamma^2(\gamma^3)^2 && (14.2.25)\\
&= 0, && (14.2.26)
\end{aligned}$$

so in general

$$\{\gamma^5, \gamma^\mu\} = 0. \tag{14.2.27}$$

The Dirac representation for α_i and β also gives a representation of the gamma matrices, namely

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \text{and} \qquad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}. \tag{14.2.28}$$

As with α_i and β this is just one representation of the gamma matrices, the real definition is just anything satisfying the Clifford algebra. In this representation γ^0 is Hermitian, ${\gamma^0}^\dagger = \gamma^0$, γ^i are anti-Hermitian, ${\gamma^i}^\dagger = -\gamma^i$, and γ^5 is hermitian, ${\gamma^5}^\dagger = \gamma^5$.

We can now show that the Dirac equation implies the Klein–Gordon equation. Start with the Dirac equation,

$$(i\not{\partial} - m)\psi = 0. \tag{14.2.29}$$

Multiply by $i\not{\partial} + m$ to get

$$(i\not{\partial} + m)(i\not{\partial} - m)\psi = 0. \tag{14.2.30}$$

Expanding this the cross terms cancel and we get

$$(-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\psi = 0. \tag{14.2.31}$$

Since $\partial_\mu\partial_\nu$ is symmetric in μ and ν we can replace $\gamma^\mu\gamma^\nu$ with the symmetrised product of γ^μ and γ^ν, namely $\{\gamma^\mu, \gamma^\nu\}/2 = \eta^{\mu\nu}$, since any antisymmetric part of $\gamma^\mu\gamma^\nu$ vanishes in the product with something totally symmetric. Hence we can replace $\gamma^\mu\gamma^\nu$ with $\eta^{\mu\nu}$:

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2)\psi = 0 \implies (\partial^2 + m^2)\psi = 0, \tag{14.2.32}$$

which is exactly the Klein–Gordon equation, so the Dirac equation does imply the Klein–Gordon equation as required.

14.3 Spinor Transformations

We've said that spinors are *not* four-vectors, despite having four components in the Dirac representation. The difference is in their transformation rules. Recall that a four-vector, x, is defined by transforming as

$$x \mapsto x' = \Lambda x \iff x^\mu = x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{14.3.1}$$

for some Lorentz transformation Λ. We now ask how a general spinor transforms under a Lorentz transformation. The Lorentz transformation acts on spacetime, not on the spinor, so we're looking for some spinor, $\psi(x)$, to transform into another spinor, $\psi'(x')$, when x transforms to x'.

Our starting point is the Dirac equation. Before transformation we have

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0, \tag{14.3.2}$$

and after transformation we expect the form of the Dirac equation to remain unchanged, that is

$$(i\gamma'^\mu\partial'_\mu - m)\psi'(x') = 0. \tag{14.3.3}$$

Since m is a scalar it doesn't change under Lorentz transformations. The derivative is just a four-vector, so transforms as $\partial_\mu \mapsto \partial'_\mu = \Lambda^\mu{}_\nu \partial_\nu$. We assume that γ'^μ are still gamma matrices, that is they satisfy the Clifford algebra $\{\gamma'^\mu, \gamma'^\nu\} = 2\eta^{\mu\nu}$. We can then, without loss of generality, take $\gamma'^\mu = \gamma^\mu$, since we know that γ'^μ must be a representation of this Clifford algebra and the physics shouldn't depend on our choice of representation, so we may as well choose the same representation before and after the transformation.

We now assume that ψ transforms according to

$$\psi'(x') = S(\Lambda)\psi(x) \tag{14.3.4}$$

where $S(\Lambda)$ is some 4×4 matrix depending on the Lorentz transformation Λ, but $S(\Lambda)$ is not necessarily a Lorentz transformation. Since the two frames are equivalent descriptions of the same physics it must be possible to go between them in both directions, so $S(\Lambda)$ must be nonsingular. We can then write

$$\psi(x) = S^{-1}(\Lambda)\psi'(x'). \tag{14.3.5}$$

However, note that we can also go from the second frame to the first by the inverse Lorentz transformation, Λ^{-1}, so we must have that

$$S^{-1}(\Lambda) = S(\Lambda^{-1}). \tag{14.3.6}$$

We can now write out the original Dirac equation in terms of the transformed quantities by reversing our transformation, that is $\partial_\mu = \Lambda^\nu{}_\mu \partial'_\nu$ and $\psi(x) = S^{-1}(\Lambda)\psi'(x')$. Then the Dirac equation in the first frame can be written as

$$(i\gamma^\mu \Lambda^\nu{}_\mu \partial'_\nu - m)S^{-1}(\Lambda)\psi'(x') = 0. \tag{14.3.7}$$

Now multiply this on the left by $S(\Lambda)$ and, noting that everything apart from γ^μ commutes with $S(\Lambda)$, we get

$$(iS(\Lambda)\gamma^\mu S^{-1}(\Lambda)\Lambda^\nu{}_\mu \partial'_\nu - m)\psi'(x'). \tag{14.3.8}$$

We see that this takes the form of the Dirac equation if

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda)\Lambda^\nu{}_\mu = \gamma^\nu. \tag{14.3.9}$$

Multiplying on the left by $S^{-1}(\Lambda)$ and the right by $S(\Lambda)$ we get

$$\gamma^\mu \Lambda^\nu{}_\mu = S^{-1}(\Lambda)\gamma^\nu S(\Lambda). \tag{14.3.10}$$

We can think of this as a sort of consistency equation between Lorentz transformations of four-vectors on the left and Lorentz transformations of spinors on the right. Note that γ^μ is a four-vector as far as Lorentz indices are concerned, and so transforms under a single copy of Λ. On the other hand γ^μ_{ab} is a rank 2 tensor for the purposes of spinor indices, and so transforms under two copies of $S(\Lambda)$ and $S^{-1}(\Lambda)$. We can write this out in terms of spinor components as

$$\gamma^\mu_{ab}\Lambda^\nu{}_\mu = S^{-1}(\Lambda)^{ac}\gamma^\nu_{cd}S(\Lambda)^{db}. \tag{14.3.11}$$

The question now is, what is $S(\Lambda)$? To answer this we consider the two basic types of Lorentz transformations.

14.3.1 Boosts

Consider a boost

$$\Lambda = \begin{pmatrix} \cosh\omega & \sinh\omega & 0 & 0 \\ \sinh\omega & \cosh\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{14.3.12}$$

Then we claim that the corresponding spinor transformation matrix, S_B, is given by

$$S_B = \exp\left[\alpha^1 \frac{\omega}{2}\right] = \exp\left[\gamma^0\gamma^1 \frac{\omega}{2}\right]. \tag{14.3.13}$$

Since $(\alpha^1)^2 = 1$ we can expand the exponential into odd and even parts, and since everything is real those parts will be hyperbolic, so

$$\exp\left[\alpha^1 \frac{\omega}{2}\right] = \cosh\frac{\omega}{2} + \alpha^1 \sinh\frac{\omega}{2} = \begin{pmatrix} \mathbb{1}_2 \cosh\frac{\omega}{2} & \sigma^1 \sinh\frac{\omega}{2} \\ \sigma^1 \cosh\frac{\omega}{2} & \mathbb{1}_2 \cosh\frac{\omega}{2} \end{pmatrix}. \tag{14.3.14}$$

In the last step we choose to work in the Dirac representation. Now using $S(\Lambda^{-1}) = S^{-1}(\Lambda)$ and considering the $\mu = 0$ case we can compute the product $S_B^{-1}\gamma^0 S_B$, using

$$S_B^{-1} = \begin{pmatrix} \mathbb{1}_2 \cosh\frac{\omega}{2} & -\sigma^1 \sinh\frac{\omega}{2} \\ -\sigma^1 \sinh\frac{\omega}{2} & \mathbb{1}_2 \cosh\frac{\omega}{2} \end{pmatrix} \tag{14.3.15}$$

and we find

$$S_B^{-1}\gamma^0 S_B = \begin{pmatrix} \mathbb{1}_2 \cosh\omega & \sigma^1 \sinh\omega \\ -\sigma^1 \sinh\omega & -\mathbb{1}_2 \cosh\omega \end{pmatrix} = \gamma^0 \cosh\omega + \gamma^1 \sinh\omega. \tag{14.3.16}$$

This shows that

$$S_B^{-1}\gamma^0 S_B = \Lambda^0{}_\nu \gamma^\nu. \tag{14.3.17}$$

We can make similar arguments for the $\mu = 1, 2, 3$ components by computing $S_B^{-1}\gamma^\mu S_B$. We find that

$$\Lambda^1{}_\nu \gamma^\nu = \gamma^0 \sinh\omega + \gamma^1 \cosh\omega, \quad \Lambda^2{}_\nu \gamma^\nu = \gamma^2, \quad \text{and} \quad \Lambda^3{}_\nu \gamma^\nu = \gamma^3. \tag{14.3.18}$$

This is what we would expect so S_B is the correct transformation matrix.

This generalises to a boost along some arbitrary direction given by the unit vector $\hat{\boldsymbol{n}}$, by

$$S_B = \exp\left[\boldsymbol{\alpha}\cdot\hat{\boldsymbol{n}} \frac{\omega}{2}\right]. \tag{14.3.19}$$

14.3.2 Rotations

Now consider a rotation,

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta & 0 \\ 0 & -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{14.3.20}$$

We claim that the corresponding spinor transformation matrix, S_R, is given by

$$S_R = \exp\left[i\Sigma^3 \frac{\vartheta}{2}\right] \tag{14.3.21}$$

where

$$\Sigma^k = -\frac{i}{4}\varepsilon^{kij}[\gamma^i, \gamma^j] \implies \Sigma^3 = -\frac{1}{2}\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}. \tag{14.3.22}$$

Choosing to work in the Dirac representation in the last step. More generally in the Dirac representation

$$\Sigma^k = \begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}. \tag{14.3.23}$$

We can check that $(\Sigma^3)^2 = 1$, and so we can again expand the exponential in even and odd parts. This time the factor of i means we get normal trig:

$$S_R = \cos\frac{\vartheta}{2} + i\Sigma^3 \sin\frac{\vartheta}{2}. \tag{14.3.24}$$

We can then proceed similarly to the boosts case computing $S_R^{-1}\gamma^\mu S_R$ for each fixed value of μ. We find that

$$\Lambda^0{}_\nu\gamma^\nu = \gamma^0, \qquad \Lambda^1{}_\nu\gamma^\nu = \gamma^1\cos\vartheta + \gamma^2\sin\vartheta, \tag{14.3.25}$$

$$\Lambda^2{}_\nu\gamma^\nu = -\gamma^1\sin\vartheta + \gamma^2\cos\vartheta, \qquad \Lambda^3{}_\nu\gamma^\nu = \gamma^3. \tag{14.3.26}$$

This generalises to a rotation about an axis along $\hat{\boldsymbol{n}}$ as

$$S_R = \exp\left[i\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{n}}\frac{\vartheta}{2}\right]. \tag{14.3.27}$$

14.3.3 General Lorentz Transformation

First note that S_R is unitary, but S_B is not. This corresponds to the fact that the rotations, SO(3), form a compact subgroup of the Lorentz group, whereas the boosts form a noncompact subgroup[2]. However, for both rotations and boosts we have the relation

[2] See *Symmetries of Particles and Fields* for details.

$$S^{-1} = \gamma^0 S^\dagger \gamma^0, \tag{14.3.28}$$

and since any Lorentz transformation can be formed from rotations and boosts this holds in general.

We can combine the two results into one for a general transformation:

$$S(\Lambda) = \exp\left[-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right] \tag{14.3.29}$$

where

$$\sigma^{\mu\nu} \coloneqq \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{14.3.30}$$

and $\omega^{ij} = \varepsilon^{ijk}\vartheta^k$ where $\boldsymbol{\vartheta} = \vartheta\hat{\boldsymbol{n}}$ parametrises a rotation by angle ϑ about the unit vector $\hat{\boldsymbol{n}}$.

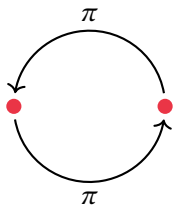


Figure 14.1: Rotating two particles into each other.

Considering just the spatial indices we have

$$-\frac{i}{4}\omega_{ij}\sigma^{ij} = -\frac{i}{4}\varepsilon^{ijk}\vartheta^k\frac{i}{2}[\gamma^i,\gamma^j] = \frac{i}{2}\boldsymbol{\vartheta}\cdot\boldsymbol{\Sigma}, \tag{14.3.31}$$

so we recover the rotation result. Similarly, considering $\mu = 0$ and $\nu = i$ we recover the boost result.

Notice that for a rotation by 2π we have

$$S_R(2\pi) = -S_R(0) = -\mathbb{1}, \tag{14.3.32}$$

so under a 2π rotation a spinor transforms as $\psi(x) \mapsto -\psi(x)$. This means that it takes a rotation of 4π to get back to the start, that is $S_R(4\pi) = \mathbb{1}$. We can interpret this as the Dirac equation describing fermions. The factors of $1/2$ in the exponentials defining S_R and S_B tell us that in particular these fermions are spin $1/2$. One way to convince us that these are fermions is to consider two identical particles described by the Dirac equation. If we rotate about the midpoint between them by π then the particles swap places, as shown in Figure 14.1. However, two rotations by π are equivalent to one rotation by 2π, and so we get a minus sign. This is exactly what we would expect for fermions following Fermi–Dirac statistics, swapping identical fermions gives a negative sign in the wave function.

Fifteen

Dirac Lagrangian

Our goal in this section will be to derive a Lagrangian which produces the Dirac equation as its equation of motion. To do so we first need to look at the sort of quantities that can appear in the Lagrangian. Presumably it includes a spinor, and since the Lagrangian must be real, and spinors are complex, we need to look for a way to take products with spinors in a way that results in a real value, sort of like how we consider products of $\varphi^\dagger$ and φ in a complex scalar Lagrangian. However, since spinors transform in a somewhat more complicated manner than scalars we need to carefully consider which combinations of spinors give us Lorentz scalars.

15.1 Dirac Adjoint

We start by defining the **Dirac adjoint** of a spinor, ψ, to be

$$\bar{\psi} \coloneqq \psi^\dagger \gamma^0. \tag{15.1.1}$$

Recall that under a Lorentz transformation, Λ, a spinor, ψ, transforms as

$$\psi \mapsto \psi' = S(\Lambda)\psi. \tag{15.1.2}$$

The Hermitian conjugate of ψ then transforms as

$$\psi^\dagger \mapsto \psi'^\dagger = \psi^\dagger S(\Lambda)^\dagger. \tag{15.1.3}$$

We want the Lagrangian to be real. For a complex scalar field this was achieved by building it from products of $\varphi^\dagger$ and φ. Consider then $(\gamma^\mu)^\dagger$. Every representation of the gamma matrices is equivalent to a unitary representation. This follows from Maschke's theorem[1] since the gamma matrices generate a finite group, the elements of which are the identity, products of gamma matrices, and their negatives, that this group is finite follows since any product of gamma matrices can be reduced to a product containing each gamma matrix at most once using the anti-commutation relations. Thus we have $(\gamma^\mu)^\dagger = (\gamma^\mu)^{-1}$. The commutation relations tell us that $(\gamma^0)^2 = 1$, so $(\gamma^0)^{-1} = \gamma^0$, and $(\gamma^i)^2 = -1$, so $(\gamma^i)^{-1} = -\gamma^i$. Now using $-\gamma^i = -\gamma^i(\gamma^0)^2 = \gamma^0\gamma^i\gamma^0$, as well as $(\gamma^0)^3 = \gamma^0$, we have the identity

[1] see *Symmetries of Quantum Mechanics* or *Symmetries of Particles and Fields*

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0. \tag{15.1.4}$$

Now consider $S(\Lambda)^\dagger$. If we write $S(\Lambda)$ as

$$S(\Lambda) = \exp\left[-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right] \tag{15.1.5}$$

then, using the fact that the parameters, $\omega_{\mu\nu}$, are real we have

$$S(\Lambda)^\dagger = \exp\left[\frac{i}{4}\omega_{\mu\nu}(\sigma^{\mu\nu})^\dagger\right]. \tag{15.1.6}$$

Now, we need to compute $(\sigma^{\mu\nu})^\dagger$. Start from the definition, $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$. Taking the Hermitian conjugate turns i into $-i$, and reverses the arguments of the commutator and sends them to their Hermitian conjugates, since taking the transpose reverses the order of a product:

$$(\sigma^{\mu\nu})^\dagger = -\frac{i}{2}[(\gamma^\nu)^\dagger, (\gamma^\mu)^\dagger] = \frac{i}{2}[(\gamma^\mu)^\dagger, (\gamma^\nu)^\dagger] = \frac{i}{2}[\gamma^0\gamma^\mu\gamma^0, \gamma^0\gamma^\nu\gamma^0] \tag{15.1.7}$$

where we've used the antisymmetry of the commutator to exchange the order of the arguments. Expanding the commutator whenever we have two neighbouring factors of γ^0 they square to the identity and so we're left only with the outer factors of γ^0, giving

$$(\sigma^{\mu\nu})^\dagger = \frac{i}{2}\gamma^0[\gamma^\mu, \gamma^\nu]\gamma^0 = \gamma^- \sigma^{\mu\nu}\gamma^0. \tag{15.1.8}$$

Hence, we have

$$S(\Lambda)^\dagger = \exp\left[\frac{i}{4}\omega_{\mu\nu}(\sigma^{\mu\nu})^\dagger\right] = \exp\left[\frac{i}{4}\omega_{\mu\nu}\gamma^0\sigma^{\mu\nu}\gamma^0\right]. \tag{15.1.9}$$

Now considering $\exp[\gamma^0 X\gamma^0]$ we have

$$\begin{aligned}\exp\left[\gamma^0 X\gamma^0\right] = \sum_{n=0}^{\infty}\frac{1}{n!}(\gamma^0 X\gamma^0)^n = \sum_{n=0}^{\infty}\frac{1}{n!}\underbrace{\gamma^0 X\gamma^0\gamma^0 X\gamma^0\cdots\gamma^0 X\gamma^0}_{n \text{ factors of } X} \\ = \sum_{n=0}^{\infty}\frac{1}{n!}\gamma^0 X^n\gamma^0 = \gamma^0\exp[X]\gamma^0,\end{aligned} \tag{15.1.10}$$

since the paired factors of γ^0 square to the identity. Hence,

$$S(\Lambda)^\dagger = \exp\left[\frac{i}{4}\omega_{\mu\nu}\gamma^0\sigma^{\mu\nu}\gamma^0\right] = \gamma^0\exp[\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}]\gamma^0 = \gamma^0 S^{-1}(\Lambda)\gamma^0. \tag{15.1.11}$$

Hence the Hermitian conjugate transforms as

$$\psi^\dagger \mapsto \psi'^\dagger = \psi^\dagger S(\Lambda)^\dagger = \psi^\dagger\gamma^0 S^{-1}(\Lambda)\gamma^0 \tag{15.1.12}$$

and so the Dirac adjoint transforms as

$$\bar\psi = \psi^\dagger\gamma^0 \mapsto \bar\psi' = \psi'^\dagger\gamma^0 = \underbrace{\psi^\dagger\gamma^0}_{=\bar\psi} S^{-1}(\Lambda)\underbrace{\gamma^0\gamma^0}_{=1} = \bar\psi S^{-1}(\Lambda). \tag{15.1.13}$$

15.2 Dirac Bilinears

In order to form the Lagrangian we first need to consider what sort of quantities can appear in it. Clearly, the Lagrangian must be a real Lorentz scalar, so we look for ways to form Lorentz scalars from **Dirac bilinears**, which are objects of the form $\bar\psi\Gamma\psi$ where ψ is some spinor satisfying the Dirac equation and Γ is a product

of gamma matrices. It turns out that there are only a finite number of linearly independent possibilities for such objects.

We start with the simplest case, where Γ is a product of zero gamma matrices, $\bar{\psi}\psi$. We can easily see that this is a Lorentz scalar, since it transforms as

$$\bar{\psi}\psi \mapsto \bar{\psi}S^{-1}(\Lambda)S(\Lambda)\psi = \bar{\psi}\psi. \tag{15.2.1}$$

The next simplest case is when Γ is a single gamma matrix, so let $j^\mu = \bar{\psi}\gamma^\mu\psi$. Under a Lorentz transformation we have

$$j^\mu \mapsto j'^\mu = \bar{\psi}'\gamma^\mu\psi' = \bar{\psi}S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\psi. \tag{15.2.2}$$

Now recall that, by definition

$$S^{-1}(\Lambda)\gamma^\nu S(\Lambda) = \gamma^\nu \Lambda^\mu{}_\nu, \tag{15.2.3}$$

so

$$j^\mu \mapsto \bar{\psi}\Lambda^\mu{}_\nu\gamma^\nu\psi = \Lambda^\mu{}_\nu j^\nu. \tag{15.2.4}$$

Hence j^μ is a four-vector.

We can form a scalar from this four vector by contracting with another four-vector. An important case is contraction with the derivative, we have a scalar

$$\bar{\psi}\gamma^\mu\partial_\mu\psi = \bar{\psi}\partial\!\!\!/\psi. \tag{15.2.5}$$

In order to form a Lagrangian it's not enough for the quantities to be scalars, they also need to be real. Starting with $\bar{\psi}\psi$ we have

$$(\bar{\psi}\psi)^\dagger = (\psi^\dagger\gamma^0\psi)^\dagger = \psi^\dagger\gamma^0(\psi^\dagger)^\dagger = \psi^\dagger\gamma^0\psi = \bar{\psi}\psi. \tag{15.2.6}$$

So $\bar{\psi}\psi$ is real. On the other hand we have

$$(\bar{\psi}\partial\!\!\!/\psi)^\dagger = (\psi^\dagger\gamma^0\gamma^\mu\partial_\mu\psi) = (\partial_\mu\psi^\dagger)(\gamma^\mu)^\dagger(\gamma^0)^\dagger\psi = (\partial_\mu\bar{\psi})\gamma^\mu\psi. \tag{15.2.7}$$

Here we've used $(\gamma^\mu)^\dagger(\gamma^0)^\dagger = \gamma^0\gamma^\mu\gamma^0\gamma^0 = \gamma^0\gamma^\mu$. It isn't immediately clear what this means for the realness of $\bar{\psi}\partial\!\!\!/\psi$. When forming the action we integrate over the Lagrangian, so we can consider this quantity under an integral:

$$\int \mathrm{d}^4x\,(\bar{\psi}\partial\!\!\!/\psi)^\dagger = \int \mathrm{d}^4x(\partial_\mu\bar{\psi})\gamma^\mu\psi = \int \mathrm{d}^4x[\partial_\mu(\bar{\psi}\gamma^\mu\psi) - \bar{\psi}\gamma^\mu\partial_\mu\psi]. \tag{15.2.8}$$

Now, the first term is a total derivative, and we are free to add a total derivative to the Lagrangian without changing anything, so we can drop this term without changing the physics. The second term is just $-\bar{\psi}\partial\!\!\!/\psi$. So, for the purpose of forming our Lagrangian $(\bar{\psi}\partial\!\!\!/\psi)$ is equivalent to $-\bar{\psi}\partial\!\!\!/\psi$. Hence, we can think of $\bar{\psi}\partial\!\!\!/\psi$ as a purely imaginary quantity, and so $i\bar{\psi}\partial\!\!\!/\psi$ is a real quantity.

15.3 Dirac Lagrangian

We now have enough to form the Dirac Lagrangian. We know that we can use $\bar{\psi}\psi$ and $i\bar{\psi}\partial\!\!\!/\psi$ in our Lagrangian and after some thought we find that

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi. \tag{15.3.1}$$

The action is then

$$S = \int \mathrm{d}^4x\, \bar{\psi}(i\partial\!\!\!/ - m)\psi. \tag{15.3.2}$$

We can readily check that this is the correct Lagrangian by finding the equations of motion. Since ψ is complex we vary ψ and $\bar{\psi}$ independently. First varying $\bar{\psi}$ we have

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} = (i\not{\partial} - m)\psi, \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = 0 \implies (i\partial\!\!\!/ - m)\psi = 0, \tag{15.3.3}$$

which is exactly the Dirac equation. Now varying ψ we have

$$\frac{\partial\mathcal{L}}{\partial\psi} = -m\bar{\psi}, \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = i\bar{\psi}\gamma^\mu \implies i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 0. \tag{15.3.4}$$

This is simply the Hermitian conjugate of the Dirac equation:

$$\begin{aligned} 0 &= [i\gamma^\mu(\partial_\mu\psi) - m\psi]^\dagger && (15.3.5)\\ &= -i(\partial_\mu\psi)^\dagger(\gamma^\mu)^\dagger - m\psi^\dagger && (15.3.6)\\ &= -i\partial_\mu\psi^\dagger\gamma^0\gamma^\mu\gamma^0 - m\psi^\dagger && (15.3.7)\\ &= -i\partial_\mu\bar{\psi}\gamma^\mu\gamma^0 - m\psi^\dagger. && (15.3.8) \end{aligned}$$

Now multiply by $-\gamma^0$ on the right and we have

$$\begin{aligned} 0 &= -[i\gamma^\mu(\partial_\mu\psi) - m\psi]^\dagger\gamma^0 && (15.3.9)\\ &= i\partial_\mu\bar{\psi}\gamma^\mu(\gamma^0)^2 + m\psi^\dagger\gamma^0 && (15.3.10)\\ &= i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi}, && (15.3.11) \end{aligned}$$

which is the equation derived by varying ψ.

As a complex object the spinor has a $\mathrm{U}(1)$ symmetry given by

$$\psi \mapsto \mathrm{e}^{-i\alpha}\psi \implies \bar{\psi} \mapsto \mathrm{e}^{i\alpha}\bar{\psi}, \tag{15.3.12}$$

and so $\bar{\psi}\psi$ and $\bar{\psi}\partial\!\!\!/\psi$ are invariant under this symmetry. Hence there is a conserved charge. To find it we consider the case of infinitesimal α, where we have

$$\delta\psi = \mathrm{e}^{-i\alpha}\psi - \psi \approx (1 - i\alpha)\psi - \psi = -i\alpha\psi. \tag{15.3.13}$$

Hence the conserved current is, scaling out the factor of α,

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi = \bar{\psi}\gamma^\mu\psi. \tag{15.3.14}$$

The associated conserved charge is

$$Q = \int \mathrm{d}^3\boldsymbol{x}\, j^0 = \int \mathrm{d}^3\boldsymbol{x}\, \bar{\psi}\gamma^0\psi = \int \mathrm{d}^3\boldsymbol{x}\, \psi^\dagger\gamma^0\gamma^0\psi = \int \mathrm{d}^3\boldsymbol{x}\, \psi^\dagger\psi. \tag{15.3.15}$$

We can also derive a Hamiltonian for the Dirac equation. First, we define the conjugate variables

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\psi^\dagger, \qquad \text{and} \qquad \bar{\pi} = \frac{\partial\mathcal{L}}{\partial\dot{\bar{\psi}}} = 0. \tag{15.3.16}$$

Then we form the Hamiltonian density in the usual way:

$$\mathcal{H} = \pi\dot{\psi} + \bar{\pi}\dot{\bar{\psi}} - \mathcal{L} \tag{15.3.17}$$

$$= i\psi^\dagger\dot{\psi} - i\bar{\psi}\partial\!\!\!/\psi + m\bar{\psi}\psi \tag{15.3.18}$$

$$= \bar{\psi}(-i\gamma^i\partial_i + m)\psi. \tag{15.3.19}$$

The Hamiltonian derived by integrating this over spacetime is *not* the "Dirac Hamiltonian", $H = -i\boldsymbol{\alpha}\cdot\nabla + \beta m$. This second quantity was mistakenly identified as the Hamiltonian by Dirac when he first derived his equation and the name has stuck.

It is also possible to derive the energy–momentum tensor for the Dirac Lagrangian, and we find that it is

$$T^{\mu\nu} = \bar{\psi}\gamma^\mu\partial^\nu\psi. \tag{15.3.20}$$

Sixteen

Solutions to the Dirac Equation

16.1 Free Particle Solution

We look for a free particle solution to the Dirac equation,

$$(i\partial\!\!\!/ - m)\psi = 0. \tag{16.1.1}$$

We'll work in the Dirac representation in this chapter. We start with the ansatz of plane wave solutions, multiplied by some spinor so that ψ is a spinor. Assuming that the particle is on-shell, $p^2 = m^2$, we expect this spinor to depend on the three-momentum and mass of the particles, and be independent of position since we're looking for free solutions. So we look for solutions of the form

$$\psi(x) = u(\boldsymbol{p})e^{-ip\cdot x} \tag{16.1.2}$$

for some spinor u. Substituting this into the Dirac equation we have

$$(i\partial\!\!\!/ - m)u(\boldsymbol{p})e^{-ip\cdot x} = iu(\boldsymbol{p})\partial\!\!\!/ e^{-ip\cdot x} - mu(\boldsymbol{p})e^{-ip\cdot x} = 0. \tag{16.1.3}$$

Consider the derivative term:

$$\partial\!\!\!/ e^{-ip\cdot x} = \gamma^\mu \partial_\mu e^{-ip_\nu x^\nu} = -i\gamma^\mu p_\mu e^{-ip\cdot x} = -ip\!\!\!/ e^{-ip\cdot x}. \tag{16.1.4}$$

Hence, substituting this back into the above equation and dividing through by the phase factor we have

$$(p\!\!\!/ - m)u(\boldsymbol{p}) = 0. \tag{16.1.5}$$

In the Dirac representation u is a four-component spinor. We can further split it into two two-component spinors, φ and χ:

$$u = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}. \tag{16.1.6}$$

We then use the block form of the gamma matrices:

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \text{and} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{16.1.7}$$

to write

$$p\!\!\!/ = \gamma^\mu p_\mu = \gamma^0 p_0 - \gamma^i p_i = \begin{pmatrix} p^0 & 0 \\ 0 & -p^0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^i p_i \\ \sigma^i p_i & 0 \end{pmatrix} = \begin{pmatrix} p^0 & -\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & -p^0 \end{pmatrix}. \tag{16.1.8}$$

Then we can write

$$(p\!\!\!/ - m) = \begin{pmatrix} p^0 & -\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & -p^0 \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = m \begin{pmatrix} \varphi \\ \chi \end{pmatrix}. \tag{16.1.9}$$

16.1.1 Positive Energy Solutions

Take $E = p^0$, and then we have

$$\begin{pmatrix} E & -\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & -E \end{pmatrix}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = m\begin{pmatrix}\varphi\\ \chi\end{pmatrix}. \tag{16.1.10}$$

Doing the matrix multiplication on the left we get

$$\begin{pmatrix} E\varphi - \boldsymbol{\sigma}\cdot\boldsymbol{p}\chi \\ \boldsymbol{\sigma}\cdot\boldsymbol{p}\varphi - E\chi \end{pmatrix} = m\begin{pmatrix}\varphi\\ \chi\end{pmatrix}. \tag{16.1.11}$$

So we must have

$$E\varphi - \boldsymbol{\sigma}\cdot\boldsymbol{p}\chi = m\varphi, \tag{16.1.12}$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{p}\varphi - E\chi = m\chi, \tag{16.1.13}$$

rearranging gives

$$(E-m)\varphi = \boldsymbol{\sigma}\cdot\boldsymbol{p}\chi, \tag{16.1.14}$$
$$(E+m)\chi = \boldsymbol{\sigma}\cdot\boldsymbol{p}\varphi. \tag{16.1.15}$$

Solving the second for χ we get

$$\chi = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\varphi. \tag{16.1.16}$$

Substituting this into the first we get

$$(E-m)\varphi = \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2}{E+m}\varphi. \tag{16.1.17}$$

We can calculate $(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2$:

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 = \sigma^i p^i \sigma^j p^j \tag{16.1.18}$$
$$= p^i p^j \sigma^i \sigma^j \tag{16.1.19}$$
$$= p^i p^j \frac{1}{2}\{\sigma^i, \sigma^j\} \tag{16.1.20}$$
$$= p^i p^j \delta^{ij} \tag{16.1.21}$$
$$= p^i p^j \tag{16.1.22}$$
$$= \boldsymbol{p}^2. \tag{16.1.23}$$

Here we've used the fact that $p^i p^j$ is symmetric in i and j to replace $\sigma^i\sigma^j$ with its symmetric part, $\{\sigma^i, \sigma^j\}/2$, since any antisymmetric part vanishes in the product with something symmetric.

Hence, we have

$$(E-m)\varphi = \frac{\boldsymbol{p}^2}{E+m}\varphi \implies \varphi = \frac{\boldsymbol{p}}{E^2-m^2}\varphi. \tag{16.1.24}$$

since $E^2 - m^2 = \boldsymbol{p}^2$ for an on-shell particle. This means that φ is not fixed by the Dirac equation, only its relation to χ is fixed. We can therefore choose φ, and as a two-component spinor we make the choice that

$$\varphi^1 = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \text{and} \qquad \varphi^2 = \begin{pmatrix}0\\1\end{pmatrix}. \tag{16.1.25}$$

This is a basis for the space of two-component spinors allowing us to express arbitrary φ. We can then consider two positive energy solutions, one for each basis spinor:

$$u(\boldsymbol{p}, s) = N\begin{pmatrix} \varphi^s \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\varphi^s \end{pmatrix}. \tag{16.1.26}$$

Here N is some normalisation factor which we'll discuss later.

Note that in the rest frame of the particles we get

$$u(\boldsymbol{0}, s) = N\begin{pmatrix} \varphi_s \\ 0 \end{pmatrix} \tag{16.1.27}$$

which is an eigenvector of the spin operator

$$\frac{1}{2}\Sigma^3 = \frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \tag{16.1.28}$$

with eigenvalues $1/2$ and $-1/2$ for $s = 1$ and $s = 2$ respectively. We therefore interpret φ^1 with spin up particles and φ^2 with spin down particles.

16.1.2 Negative Energy Solutions

If instead we take $p^\mu = (-E, -\boldsymbol{p})$ then we get slightly different solutions. First, we make the alternative ansatz

$$\psi(x) = \upsilon(\boldsymbol{p})\mathrm{e}^{+ip\cdot x}. \tag{16.1.29}$$

Substituting this into the Dirac equation and cancelling the phase factor we get

$$(\not{p} + m)\upsilon(\boldsymbol{p}) = 0. \tag{16.1.30}$$

Following the same steps of writing this as a block matrix and computing the two components and rearranging we get

$$\varphi = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi, \tag{16.1.31}$$

so φ and χ have swapped compared to the positive energy case. Again χ is not fixed after substituting this into the equation from the other component, and we have the solution

$$\upsilon(\boldsymbol{p}, s) = N\begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^s \\ \chi^s \end{pmatrix} \tag{16.1.32}$$

where

$$\chi^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \text{and} \qquad \chi^2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{16.1.33}$$

is a basis for the space of two-component spinors. We choose this basis, rather than the more obvious choice of $\chi^1 = (1, 0)^\mathsf{T}$ and $\chi^2 = (0, 1)^\mathsf{T}$, since we want to interpret the negative energy particles as antiparticles, which have opposite spin to the corresponding particles, and then the negative sign comes from requiring charge conjugation symmetry.

The important thing is that, despite Dirac's best attempts, there are still negative energy solutions. It turns out that these are unavoidable, since they correspond to antiparticles.

16.2 Spin, Angular Momentum and Helicity

We've interpreted

$$S_3 = \frac{1}{2}\Sigma^3 = \frac{1}{2}\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix} \tag{16.2.1}$$

as measuring the spin along the z-axis. We can further define

$$\boldsymbol{S} \coloneqq \frac{1}{2}\boldsymbol{\Sigma} = \frac{1}{2}\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix} \tag{16.2.2}$$

as the spin operator with eigenvalues $\pm 1/2$, and

$$\boldsymbol{S}^2 \coloneqq \frac{3}{4}(\boldsymbol{\Sigma}^3)^2 = \frac{3}{4}\mathbb{1}_4 \tag{16.2.3}$$

is the spin operator with eigenvalues $3/4 = 1/2(1/2+1)$.

The operator $\boldsymbol{\Sigma}$ only commutes with the Dirac Hamiltonian, $\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m$, in the rest frame of the particle. The angular momentum operator, $\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p}$, also only commutes with the Dirac Hamiltonian in the rest frame of the particle. However,

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} \tag{16.2.4}$$

commutes with the Dirac Hamiltonian in any frame, suggesting we interpret it as the total angular momentum operator.

Another choice of operator which commutes in any frame is the **helicity** operator

$$h(\boldsymbol{p}) = \frac{\boldsymbol{\Sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|}. \tag{16.2.5}$$

This measures the spin along the direction of propagation of the particle.

16.3 Normalisation

There are different conventions for how to normalise spinors. We'll make the choice to define

$$N = \sqrt{E+m} \tag{16.3.1}$$

so

$$u(\boldsymbol{p},s) = \sqrt{E+m}\begin{pmatrix}\varphi^s\\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\varphi^s\end{pmatrix}, \qquad \text{and} \qquad v(\boldsymbol{p},s) = \sqrt{E+m}\begin{pmatrix}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^s\\ \chi^s\end{pmatrix}. \tag{16.3.2}$$

We then have

$$u^\dagger u = (E+m)\left[\varphi^\dagger\varphi + \varphi^\dagger\left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\right)^2\varphi\right] \tag{16.3.3}$$

and using $\varphi^\dagger\varphi = 1$ for either spin and $(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 = \boldsymbol{p}^2 = E^2 - m^2$ we have

$$u^\dagger u = (E+m)\left[1 + \frac{E^2-m^2}{(E+m)^2}\right] \tag{16.3.4}$$

$$= (E+m)\left[1 + \frac{(E+m)(E-m)}{(E+m)^2}\right] \tag{16.3.5}$$

$$= E + m + E - m \tag{16.3.6}$$

$$= 2E. \tag{16.3.7}$$

Similarly $v^\dagger v = 2E$.

Defining the Dirac adjoints, $\bar{u} = u^\dagger \gamma^0$ and $\bar{v} = v^\dagger \gamma^0$ we can compute $\bar{u}u$ in the rest frame, since $\bar{u}u$ is a Lorentz invariant scalar, and we find

$$\bar{u}u = (E+m)\left[\begin{pmatrix}\varphi^\dagger & \chi^\dagger\end{pmatrix}\gamma^0\begin{pmatrix}\varphi\\ \chi\end{pmatrix} + \begin{pmatrix}\varphi^\dagger & \chi^\dagger\end{pmatrix}\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})\gamma^0(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{(E+m)^2}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}\right] \tag{16.3.8}$$

$$= E + m \tag{16.3.9}$$

$$= 2m \tag{16.3.10}$$

since the second term vanishes in the rest frame, with $\boldsymbol{p} = \boldsymbol{0}$, and $E = m$ in the rest frame. Similarly we can show that

$$\bar{v}v = -2m. \tag{16.3.11}$$

Other quantities, such as $\bar{u}v$, $\bar{v}u$, and $\bar{u}(\boldsymbol{p}, s)u(\boldsymbol{p}, s')$ with $s \neq s'$ vanish.

We've just computed the inner products of our spinor solutions, we can also compute outer products, $u\bar{u}$. These are most interesting if we sum over spins, where we get the results

$$\sum_s u(\boldsymbol{p}, s)\bar{u}(\boldsymbol{p}, s) = \not{p} + m, \tag{16.3.12}$$

$$\sum_s v(\boldsymbol{p}, s)\bar{v}(\boldsymbol{p}, s) = \not{p} - m. \tag{16.3.13}$$

Note that the sign is the opposite to the sign in $(\not{p} - m)u = (\not{p} + m)v = 0$.

Finally, we can define the projection operators

$$\Lambda_\pm \coloneqq \frac{1}{2m}(\pm\not{p} + m). \tag{16.3.14}$$

These have the effect of projecting on to the positive and negative energy eigenspaces:

$$\Lambda_+ u(\boldsymbol{p}, s) = \frac{1}{2m}(\not{p} + m)u \tag{16.3.15}$$

$$= \frac{1}{2m}\sum_{s'} u(\boldsymbol{p}, s')\bar{u}(\boldsymbol{p}, s')u(\boldsymbol{p}, s) \tag{16.3.16}$$

$$= \frac{1}{2m}\delta_{ss'}u(\boldsymbol{p}, s')2m \tag{16.3.17}$$

$$= u(\boldsymbol{p}, s), \tag{16.3.18}$$

and similarly

$$\Lambda_- v(\boldsymbol{p}, s) = v(\boldsymbol{p}, s). \tag{16.3.19}$$

Since they are projection operators they are idempotent, $\Lambda_\pm^2 = \Lambda_\pm$, and the are also orthogonal projectors, $\Lambda_+\Lambda_- = \Lambda_-\Lambda_+ = 0$. In particular we can show that $\Lambda_+ u = u, \Lambda_+ v = 0, \Lambda_- u = 0$, and $\Lambda_- v = v$, so

$$\Lambda + (\alpha u + \beta v) = \alpha u, \qquad \text{and} \qquad \Lambda_-(\alpha u + \beta v) = \beta v \tag{16.3.20}$$

for $\alpha, \beta \in \mathbb{C}$.

Part V

Quantising the Dirac Equation

Seventeen

Quantisation

17.1 Mode Expansion

The spinor ψ satisfying the Dirac equation is a complex object, so the mode expansion is similar to that of a complex scalar field, as seen in Section 12.3. The difference is that as well as momenta labelling the states we also have spins, and instead of a scalar we want a spinor as a result. So, we sum over spins and include a basis spinor in each term. The mode expansion is then

$$\psi(x) = \sum_s \int đp\, [a_s(\boldsymbol{p})u(\boldsymbol{p}, s)e^{-ip\cdot x} + b_s^\dagger(\boldsymbol{p})v(\boldsymbol{p}, s)e^{ip\cdot x}]. \tag{17.1.1}$$

The adjoint spinor then has the mode expansion

$$\bar{\psi}(x) = \sum_s \int đp\, [b_s(\boldsymbol{p})\bar{v}(\boldsymbol{p}, s)e^{-ip\cdot x} + a_s^\dagger(\boldsymbol{p})\bar{u}(\boldsymbol{p}, s)e^{ip\cdot x}]. \tag{17.1.2}$$

In these expressions u, v, $\bar{u}$, and $\bar{v}$ are just classical spinor-valued functions satisfying the Dirac equation. The objects a_s, b_s, $a_s^\dagger$, and $b_s^\dagger$ are operators on the Hilbert space of states. This means we have five (or eight, depending on how you count) different spaces going on in this expression, and its important not to confuse them:

- Real space, which is the space x lives in.
- Momentum space, which is the space p lives in.
- The spinor space (and its dual) which u, v, $\bar{u}$, and $\bar{v}$ live in.
- The Hilbert space of states (and its dual) which a, b, $a^\dagger$, and $b^\dagger$ act on.
- The space of operators on this Hilbert space (and the space of operators on the dual Hilbert space) which is where a, b, $a^\dagger$, and $b^\dagger$ live.

We'll see that $a_s^\dagger(\boldsymbol{p})$ and $b_s^\dagger(\boldsymbol{p})$ can be interpreted as creating distinct types of particles of charge $\boldsymbol{p}$ and spin s, whereas $a_s(\boldsymbol{p})$ and $b_s(\boldsymbol{p})$ annihilate them. We then interpret particles of type a as particles, and particles of type b as antiparticles.

17.2 Equal Time Anticommutation Relations

Spinors represent spin 1/2 particles, these satisfy Fermi–Dirac statistics, meaning that exchanging two fermions gives a negative sign. The way we encode this is to have the mode operators obey *anti*commutation relations, instead of commutation relations.

We'll start by imposing equal time anticommutation relations, based on the equal time commutation relations for scalar fields, and then we'll explore what this means for the fields. In analogy with the scalar fields we look for the mode operators to anticommute, except possibly when they have the same momentum and spin, that is

$$\{a_r(\boldsymbol{p}), a_s^\dagger(\boldsymbol{p}')\} = \delta_{rs}\delta(\boldsymbol{p}-\boldsymbol{p}'), \tag{17.2.1}$$

$$\{b_r(\boldsymbol{p}), b_s^\dagger(\boldsymbol{p}')\} = \delta_{rs}\delta(\boldsymbol{p}-\boldsymbol{p}'). \tag{17.2.2}$$

All other anticommutation relations are zero, such as $\{a_r(\boldsymbol{p}), a_s(\boldsymbol{p}')\} = 0$ and $\{a_s(\boldsymbol{p}), b_r^\dagger(\boldsymbol{p}')\}$. Note that this implies

$$a_r(\boldsymbol{p})a_r(\boldsymbol{p}) = a_r^\dagger(\boldsymbol{p})a_r^\dagger(\boldsymbol{p}) = b_r(\boldsymbol{p})b_r(\boldsymbol{p}) = b_r^\dagger(\boldsymbol{p})b_r^\dagger(\boldsymbol{p}) = 0. \tag{17.2.3}$$

17.3 Creation and Annihilation

In this section we'll work with and interpret a and $a^\dagger$, however replacing all as with bs everything will still hold. Recall that we can interpret the mode operators for a scalar field theory as creation and annihilation operators because they have the effect of sending an energy eigenstate to an energy eigenstate with a different energy. This in turn follows due to the commutation relation $[a^\dagger(\boldsymbol{p})a(\boldsymbol{p}), a^\dagger(\boldsymbol{p})] = \delta(\boldsymbol{p}-\boldsymbol{p}')$ and $[a^\dagger(\boldsymbol{p})a(\boldsymbol{p}), a_s(\boldsymbol{p})] = -a_r(\boldsymbol{p})\delta(\boldsymbol{p}-\boldsymbol{p}')$, and the product of a and $a^\dagger$ is part of the Hamiltonian.. Well, we can show that the same commutation relation holds in this case too.

Consider the following combination of anticommutators:

$$A\{B,C\}-\{A,C\}B = A(BC+CB)-(AC+CA)B = ABC-CAB = [AB,C]. \tag{17.3.1}$$

We therefore have

$$[a_r^\dagger(\boldsymbol{p})a_r(\boldsymbol{p}), a_s^\dagger(\boldsymbol{p}')] = a_r^\dagger(\boldsymbol{p})\{a_r(\boldsymbol{p}), a_s^\dagger(\boldsymbol{p}')\} - \{a_r^\dagger(\boldsymbol{p}), a_s^\dagger(\boldsymbol{p}')\}a_r(\boldsymbol{p}) \tag{17.3.2}$$

$$= a_r^\dagger(\boldsymbol{p})\delta_{rs}\delta(\boldsymbol{p}-\boldsymbol{p}') \tag{17.3.3}$$

And similarly

$$[a_r^\dagger(\boldsymbol{p})a_r(\boldsymbol{p}), a_s(\boldsymbol{p}')] = -a_r(\boldsymbol{p})\delta_{rs}\delta(\boldsymbol{p}-\boldsymbol{p}'). \tag{17.3.4}$$

This means we can still interpret $a_r^\dagger(\boldsymbol{p})a_r(\boldsymbol{p})$ as a number operator, the eigenvalue being the number of particles of type a with momentum $\boldsymbol{p}$ and spin r. Then $a_r^\dagger(\boldsymbol{p})$ is a creation operator, creating a particle with momentum $\boldsymbol{p}$ and spin r, and $a_r(\boldsymbol{p})$ is an annihilation operator, annihilating a particle with momentum $\boldsymbol{p}$ and spin r. Identical results hold replacing all as with b.

One big difference with the scalar field case is which states can exist, that is what the Fock space is. All states are of the form

$$|\boldsymbol{p_1}, s_1, \boldsymbol{p_2}, s_2, \ldots, \boldsymbol{p_n}, s_n\rangle = a_{s_1}^\dagger(\boldsymbol{p_1})a_{s_2}^\dagger(\boldsymbol{p_2})\cdots a_{s_n}^\dagger(\boldsymbol{p_n})|0\rangle \tag{17.3.5}$$

where for $i \neq j$ either $\boldsymbol{p_i} \neq \boldsymbol{p_j}$ or $s_i \neq s_j$. Suppose that we do try to create two particles with the same momentum and spin, that is, we try to produce the state

$$|\boldsymbol{p}, s, \boldsymbol{p}, s\rangle = a_s^\dagger(\boldsymbol{p}) a_s^\dagger(\boldsymbol{p})|0\rangle. \tag{17.3.6}$$

Then by the anticommutation relation we can swap the two creation operators at the cost of a negative, so we have

$$|\boldsymbol{p}, s, \boldsymbol{p}, s\rangle = -a_s^\dagger(\boldsymbol{p}) a_s^\dagger(\boldsymbol{p})|0\rangle. \tag{17.3.7}$$

This can only hold if $|\boldsymbol{p}, s, \boldsymbol{p}, s\rangle = 0$. This is the genesis of the **Pauli exclusion principle** in quantum field theory. If we try to create a two spin 1/2 particles in the same state we don't get a valid state vector.

Another way of putting this is if we characterise a state by its occupation numbers, that is there are $N_{s_1}(\boldsymbol{p_1})$ particles of type a, momentum $\boldsymbol{p_1}$, and spin s_1, $N_{s_2}(\boldsymbol{p_2})$ particles of type a, momentum $\boldsymbol{p_2}$, and spin s_2, and so on, then $N_s(\boldsymbol{p})$ is either 0 or 1. For bosons on the other hand the occupation number can be any nonnegative integer.

Formally, the **Fock space** (for fermions) is defined to be

$$F_-(\mathbb{H}) = \overline{\bigoplus_{n=0}^{\infty} A\mathbb{H}^{\otimes n}} = \overline{\mathbb{C} \oplus \mathbb{H} \oplus A(\mathbb{H} \otimes \mathbb{H}) \oplus A(\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H})} \tag{17.3.8}$$

where $\mathbb{H}$ is the single particle Hilbert space,

$$\mathbb{H}^{\otimes n} \coloneqq \bigotimes_{i=1}^{n} \mathbb{H} = \underbrace{\mathbb{H} \otimes \cdots \otimes \mathbb{H}}_{n \text{ times}}, \tag{17.3.9}$$

the operator A antisymmetries all tensors, and the line over the top of these statements denotes the Hilbert space completion, adding in the limits of all absolutely convergent series. Note that

$$F_-(\mathbb{H}) = \overline{\bigwedge \mathbb{H}} \tag{17.3.10}$$

where $\bigwedge V$ is the exterior algebra of V, defined by taking the tensor algebra, $T(V)$, and quotienting out by the ideal generated by tensors of the form $x \otimes x$ for $x \in V$.

17.4 Equal Time Anticommutation Relations of the Fields

The canonical momentum is

$$\pi(x) = i\psi^\dagger(x). \tag{17.4.1}$$

We can then compute the equal time anticommutation relations for the field operators, ψ and π, using this. To do this we need to consider spinor indices, which we denote by a and b here, note in the Dirac representation that these run from 0 to 3, as they index into four-component spinors, they aren't Lorentz indices where Latin letters typically run from 1 to 3. The equal time anticommutation relation between ψ and π is

$$\{\psi_a(t, \boldsymbol{x}), \pi_b(t, \boldsymbol{x}')\} = \{\psi_a(t, \boldsymbol{x}), i\psi_b^\dagger(t, \boldsymbol{x}')\} = i\delta_{ab}\delta(\boldsymbol{x} - \boldsymbol{x}'). \tag{17.4.2}$$

Often more useful is the equal time anticommutation relation between ψ and $\bar{\psi}$,

$$\{\psi_a(t,\boldsymbol{x}),\bar{\psi}_b(t,\boldsymbol{x}')\} = \gamma^0_{ab}\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{17.4.3}$$

Also, we have $\{\psi_a(t,\boldsymbol{x}),\psi_b(t,\boldsymbol{x}')\} = \{\bar{\psi}_a(t,\boldsymbol{x}),\bar{\psi}_b(t,\boldsymbol{x}')\} = 0$.

We will prove the equal time anticommutation relation for ψ and $\bar{\psi}$. We have

$$\begin{aligned}\{\psi_a(t,\boldsymbol{x}),\bar{\psi}_b(t,\boldsymbol{x}')\} = \\ \sum_{s,s'}\int đp\int đp'\big[\{a_s(\boldsymbol{p}),a^\dagger_{s'}(\boldsymbol{p}')\}u_a(\boldsymbol{p},s)\bar{u}_b(\boldsymbol{p}',s')\mathrm{e}^{-ip\cdot x+ip'\cdot x'} \\ + \{b^\dagger_s(\boldsymbol{p}),b_{s'}(\boldsymbol{p}')\}v_a(\boldsymbol{p},s)\bar{v}_b(\boldsymbol{p}',s')\mathrm{e}^{ip\cdot x-ip'\cdot x'}\big]\end{aligned} \tag{17.4.4}$$

where the terms with anticommutators mixing the a and b mode operators vanish. Using the anticommutation relation this becomes

$$\begin{aligned}\{\psi_a(t,\boldsymbol{x}),\bar{\psi}_b(t,\boldsymbol{x}')\} = \\ \sum_{s,s'}\int đp\int đp'\big[\delta_{ss'}đ(\boldsymbol{p}-\boldsymbol{p}')u_a(\boldsymbol{p},s)\bar{u}_b(\boldsymbol{p}',s')\mathrm{e}^{-ip\cdot x+ip'\cdot x'} \\ + \delta_{s,s'}đ(\boldsymbol{p}-\boldsymbol{p}')v_a(\boldsymbol{p},s)\bar{v}_b(\boldsymbol{p}',s')\mathrm{e}^{ip\cdot x-ip'\cdot x'}\big].\end{aligned} \tag{17.4.5}$$

We can then use the Kronecker delta to remove one of the sums setting $s = s'$ and the Dirac delta to remove one of the integrals, setting $p' = p$:

$$\begin{aligned}\{\psi_a(t,\boldsymbol{x}),\bar{\psi}_b(t,\boldsymbol{x}')\} = \\ \sum_s\int đp\big[u_a(\boldsymbol{p},s)\bar{u}_b(\boldsymbol{p},s)\mathrm{e}^{-ip\cdot(x-x')}+v_a(\boldsymbol{p},s)\bar{v}_b(\boldsymbol{p},s)\mathrm{e}^{-ip\cdot(x'-x)}\big].\end{aligned} \tag{17.4.6}$$

Now we can use the spin sums

$$\sum_s u(\boldsymbol{p},s)\bar{u}(\boldsymbol{p},s) = \not{p}+m, \qquad \text{and} \qquad \sum_s v(\boldsymbol{p},s)\bar{v}(\boldsymbol{p},s) = \not{p}-m, \tag{17.4.7}$$

giving

$$\{\psi_a(t,\boldsymbol{x}),\bar{\psi}_b(t,\boldsymbol{x}')\} = \int đp\big[(\not{p}+m)_{ab}\mathrm{e}^{-ip\cdot(x-x')}+(\not{p}-m)_{ab}\mathrm{e}^{-ip\cdot(x'-x)}\big].$$

Now we use the fact that we're considering equal time commutation relations, meaning $x^0-x'^0=0$, to rewrite $p\cdot(x-x') = -\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')$, with the negative coming from the metric signature. We also have $\not{p} = \gamma^0 p_0 - \gamma^i p_i$. We can use $p_0 = \omega(\boldsymbol{p})$ to write this as $\not{p} = \gamma^0\omega(\boldsymbol{p}) - \gamma^i p_i$, and the second term doesn't contribute to the integral because it is odd under $p\to -p$, and the measure is even under this transformation. We can also make the transformation $p\to -p$ in the second term, which makes both exponentials the same without changing the value of the integral, and then the mass terms cancel. Hence,

$$\{\psi_a(t,\boldsymbol{x}),\bar{\psi}_b(t,\boldsymbol{x}')\} = \int đp\,2\omega(\boldsymbol{p})\gamma^0_{ab}\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \tag{17.4.8}$$

$$= \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega(\boldsymbol{p})}2\omega(\boldsymbol{p})\gamma^0_{ab}\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \tag{17.4.9}$$

$$= \int\frac{\mathrm{d}^3p}{(2\pi)^3}\gamma^0_{ab}\mathrm{e}^{\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \tag{17.4.10}$$

$$= \gamma^0_{ab}\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{17.4.11}$$

17.5 Hamiltonian

Life is short. Certainly these lectures are.

Richard Ball, skipping lots of algebra

The Hamiltonian is

$$H = \int \mathrm{d}^3\boldsymbol{x}\left(i\psi^\dagger(x)\dot{\psi}(x) - \mathcal{L}(x)\right). \tag{17.5.1}$$

We can insert a factor of $(\gamma^0)^2 = 1$ to get

$$H = \int \mathrm{d}^3\boldsymbol{x}\left(i\psi^\dagger(x)(\gamma^0)^2\partial_0\psi(x) - \mathcal{L}(x)\right) \tag{17.5.2}$$

$$= \int \mathrm{d}^3\boldsymbol{x}\left(i\bar{\psi}(x)\gamma^0\partial_0\psi(x) - \mathcal{L}(x)\right). \tag{17.5.3}$$

Now using $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi = \bar{\psi}(i\gamma^0\partial_0 + i\gamma^i\partial_i - m)\psi$ to cancel the first term and give

$$H = \int \mathrm{d}^3\boldsymbol{x}\left(-i\bar{\psi}(x)\boldsymbol{\gamma}\cdot\nabla\psi(x) + m\bar{\psi}(x)\psi(x)\right) \tag{17.5.4}$$

We can work out the gradient from the mode expansion:

$$\nabla\psi = \sum_s \int đp\,(-i\boldsymbol{p})[a_s(\boldsymbol{p})u(\boldsymbol{p},s)\mathrm{e}^{-ip\cdot x} - b_s^\dagger(\boldsymbol{p})v(\boldsymbol{p},s)\mathrm{e}^{ip\cdot x}]. \tag{17.5.5}$$

Using this the Hamiltonian is given by

$$H = \sum_{s,s'}\int \mathrm{d}^3\boldsymbol{x}\int đp\int đp'\,[(b_s(\boldsymbol{p})\bar{v}(\boldsymbol{p},s)\mathrm{e}^{-ip\cdot x} + a_s^\dagger(\boldsymbol{p})\bar{u}(\boldsymbol{p},s)\mathrm{e}^{ip\cdot x})(-\boldsymbol{p}\cdot\boldsymbol{\gamma})$$

$$(a_{s'}(\boldsymbol{p}')u(\boldsymbol{p}',s')\mathrm{e}^{-ip'\cdot x} - b_{s'}^\dagger(\boldsymbol{p}')v(\boldsymbol{p}',s')\mathrm{e}^{ip'\cdot x}) \tag{17.5.6}$$

$$+ m(b_s(\boldsymbol{p})\bar{v}(\boldsymbol{p},s)\mathrm{e}^{-ip\cdot x} + a_s^\dagger(\boldsymbol{p})\bar{u}(\boldsymbol{p},s)\mathrm{e}^{ip\cdot x}) \tag{17.5.7}$$

$$(a_{s'}(\boldsymbol{p}')u(\boldsymbol{p}',s')\mathrm{e}^{-ip'\cdot x} + b_{s'}^\dagger(\boldsymbol{p}')v(\boldsymbol{p}',s')\mathrm{e}^{ip'\cdot x})] \tag{17.5.8}$$

Expanding everything, then simplifying using spin sums and the anticommutation relations we get the result

$$H = \frac{1}{2}\sum_s \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}[a_s^\dagger(\boldsymbol{p})a_s(\boldsymbol{p}) + b_s^\dagger(\boldsymbol{p})b_s(\boldsymbol{p}) - \delta_{ss}2\omega(\boldsymbol{p})(2\pi)^3\delta(0)]. \tag{17.5.9}$$

We see that this gives us a divergent ground state, which we can deal with by normal ordering. Normal ordering works for fermions almost identically to bosons, but when we exchange two terms to normal order them we include a minus sign. That is, if $\psi = \psi^+ + \psi^-$ then the normal ordering of $\psi_a\psi_b$ is

$$:\psi_a\psi_b: = :(\psi_a^+ + \psi_a^-)(\psi_b^+ + \psi_b^-): \tag{17.5.10}$$

$$= :\psi_a^+\psi_b^+ + \psi_a^+\psi_b^- + \psi_a^-\psi_b^+ + \psi_a^-\psi_b^-: \tag{17.5.11}$$

$$= \psi_a^+\psi_b^+ + \psi_a^+\psi_b^- - \psi_b^+\psi_a^- + \psi_a^-\psi_b^- \tag{17.5.12}$$

This means that

$$:\psi_a\psi_b: = -:\psi_b\psi_a:, \tag{17.5.13}$$

so we can't just reorder terms within a normal ordering in the way we could for bosons.

Suppose we had used commutators instead of anticommutators. As well as a lot of what we've done so far not working the biggest problem would be that the Hamiltonian would have a negative sign in front of the $b_s^\dagger(\boldsymbol{p})b_s(\boldsymbol{p})$ term. This would mean that by adding particles of type b we would be able to make the energy arbitrarily negative, meaning we would have no ground state, and then no way to interpret the theory.

The Hamiltonian is given in terms of the energy–momentum tensor by

$$H = \int \mathrm{d}^3\boldsymbol{x}\, T^{00}, \qquad \text{where} \qquad T^{\mu\nu} = i:\bar{\psi}\gamma^\mu\partial^\nu\psi:. \tag{17.5.14}$$

The conserved three-momentum is

$$\boldsymbol{P} = \sum_s \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{\boldsymbol{p}}{2\omega(\boldsymbol{p})}[a_s^\dagger(\boldsymbol{p})a_s(\boldsymbol{p}) + b_s^\dagger(\boldsymbol{p})b_s(\boldsymbol{p})]. \tag{17.5.15}$$

We also have the expected commutation relations

$$[H, a_s^\dagger(\boldsymbol{p})] = \omega(\boldsymbol{p})a_s^\dagger(\boldsymbol{p}), \qquad [H, a_s(\boldsymbol{p})] = -\omega(\boldsymbol{p})a_s(\boldsymbol{p}), \tag{17.5.16}$$

$$[H, b_s^\dagger(\boldsymbol{p})] = \omega(\boldsymbol{p})b_s^\dagger(\boldsymbol{p}), \qquad [H, b_s(\boldsymbol{p})] = -\omega(\boldsymbol{p})b_s(\boldsymbol{p}). \tag{17.5.17}$$

This means we can interpret $a_s^\dagger(\boldsymbol{p})$ and $b_s^\dagger(\boldsymbol{p})$ as creating particles of energy $\omega(\boldsymbol{p})$ and $a_s(\boldsymbol{p})$ and $b_s(\boldsymbol{p})$ annihilating particles of energy $-\omega(\boldsymbol{p})$.

17.6 Charge Conservation

As the spinors are complex the Dirac Lagrangian is invariant under the U(1) symmetry

$$\psi \mapsto \mathrm{e}^{i\alpha}\psi. \tag{17.6.1}$$

The associated conserved current is

$$j^\mu = :\bar{\psi}\gamma^\mu\psi:, \tag{17.6.2}$$

and the conserved charge is

$$Q = \int \mathrm{d}^3\boldsymbol{x}\, j^0 = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2\omega(\boldsymbol{p})}[a_s^\dagger(\boldsymbol{p})a_s(\boldsymbol{p}) - b_s^\dagger(\boldsymbol{p})b_s(\boldsymbol{p})]. \tag{17.6.3}$$

This means we can interpret particles of type a as having charge $+1$ and particles of charge b as having charge -1, since the charge is given by the number operator for a minus the number operator for b, so the total charge is the number of particles of type a minus the number of particles of type b.

Normally we rescale the charges by a factor $q = -e$, where e is the *magnitude* of charge of the electron. Then the interpretation is that

- particles of type a (particles) have charge $-e$,
- particles of type b (antiparticles) have charge $+e$.

Eighteen

Covariant Anticommutation Relations

18.1 Covariant Anticommutation Relations

The covariant anticommutator, $\{\psi(x), \bar{\psi}(y)\}$, can be calculated by substituting in the mode expansions. To do this we first write ψ as $\psi = \psi^+ + \psi^-$ where

$$\psi^+(x) = \sum_s \int đp\, a_s(\boldsymbol{p}) u(\boldsymbol{p}, s) e^{-ip\cdot x}, \tag{18.1.1}$$

$$\psi^-(x) = \sum_s \int đp\, b_s^\dagger(\boldsymbol{p}) u(\boldsymbol{p}, s) e^{ip\cdot x}, \tag{18.1.2}$$

and similarly for $\bar{\psi}$. Then the only nonzero contributions to the anticommutator will come from $\{\psi^+, \bar{\psi}^-\}$ and $\{\psi^-, \bar{\psi}^-\}$. We'll compute the first here, including explicit spinor indices, a and b:

$$\{\psi_a^+(x), \psi_b^-(y)\} = \sum_{s,s'} \int đp \int đp'\, \{a_s(\boldsymbol{p}), a_{s'}^{\boldsymbol{p}'}\} u_a(\boldsymbol{p}, s) \bar{u}_b(\boldsymbol{p}', s') e^{-ip\cdot x + ip'\cdot y} \tag{18.1.3}$$

$$= \sum_{s,s'} \int đp \int đp'\, \delta_{ss'} đ(\boldsymbol{p} - \boldsymbol{p}') u_a(\boldsymbol{p}, s) \bar{u}_b(\boldsymbol{p}', s') e^{-ip\cdot x + ip'\cdot y} \tag{18.1.4}$$

$$= \sum_s \int đp\, u_a(\boldsymbol{p}, s) \bar{u}_b(\boldsymbol{p}, s) e^{-ip\cdot x + ip\cdot y} \tag{18.1.5}$$

$$= \int đp (\not{p} + m)_{ab} e^{-ip\cdot(x-y)}. \tag{18.1.6}$$

Here we've made use of the spin sum in Equation (16.3.12). Next, notice that

$$\not{\partial} e^{-ip\cdot(x-y)} = \gamma^\mu \partial_\mu e^{-ip\cdot(x-y)} = \gamma^\mu (-ip^\mu) e^{-ip\cdot(x-y)} = -i\not{p} e^{-ip\cdot(x-y)}, \tag{18.1.7}$$

so we can replace $\not{p}$ with $i\not{\partial}$, giving

$$\{\psi_a^+(x), \psi_b^-(y)\} = \int đp (i\not{\partial} + m)_{ab} e^{-ip\cdot(x-y)} \tag{18.1.8}$$

$$= (i\not{\partial} + m)_{ab} \int đp\, e^{-ip\cdot(x-y)} \tag{18.1.9}$$

$$= (i\not{\partial} + m)_{ab} i\Delta^+(x - y), \tag{18.1.10}$$

where Δ^+ is the same propagator as for the scalar case as defined in Section 7.1.

We've seen that squaring the operator in the Dirac equation gives the Klein–Gordon equation, so it shouldn't be too surprising that by acting on results for scalar fields with the Dirac operator we get results for a spinor field. We'll see that this pattern continues through most of the results in this section.

Similarly, we can compute

$$\{\psi_a^-(x), \bar{\psi}_b^+(y)\} = -(i\not{\partial} + m)_{ab} \int e^{ip\cdot(x-y)} đp \tag{18.1.11}$$

$$= (i\not{\partial} + m)_{ab} i\Delta^-(x-y). \tag{18.1.12}$$

We can write these results as

$$\{\psi^\pm(x), \bar{\psi}^\mp(y)\} = iS^\pm(x-y) \tag{18.1.13}$$

where

$$S^\pm(x) \coloneqq (i\not{\partial} + m)\Delta^\pm(x). \tag{18.1.14}$$

We then have

$$\{\psi(x), \bar{\psi}(y)\} = iS(x-y), \tag{18.1.15}$$

where

$$S(x) = S^+(x) + S^-(x) = (i\not{\partial} + m)\Delta(x). \tag{18.1.16}$$

18.1.1 Contour Representation

Recall the contour representation for $\Delta^\pm$:

$$\Delta^\pm(x) = -\int_{C^\pm} \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 - m^2}. \tag{18.1.17}$$

The contours, $C^\pm$ are as given in Figure 7.1. We can use this to give a contour representation for $S^\pm$, going back to writing $\not{p}$ in the place of $i\not{\partial}$, since we can act on the exponential in the contour representation to give

$$S^\pm(x) = -\int_{C^\pm} \frac{d^4p}{(2\pi)^4} \frac{\not{p} + m}{p^2 - m^2} e^{-ip\cdot x}. \tag{18.1.18}$$

Now we can write

$$(\not{p} + m)(\not{p} - m) = p^2 - m^2 \implies \not{p} + m = (p^2 - m^2)(\not{p} - m)^{-1} \tag{18.1.19}$$

and then, in a slight abuse of notation, write the inverse matrix $(\not{p} - m)^{-1}$ as $1/(\not{p} - m)$ and rewrite this equation as

$$S^\pm(x) = -\int_{C^\pm} \frac{d^4p}{(2\pi)^4} \frac{(p^2 - m^2)(\not{p} - m)^{-1}}{p^2 - m^2} e^{-ip\cdot x} = -\int_{C^\pm} \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{\not{p} - m}. \tag{18.1.20}$$

18.2 Fermion Propagator

For fermion operators **time ordering** is defined similarly to bosons, but when we swap two fermion operators we should get a negative sign, so we define the time ordering of the fermion operators $A(x)$ and $B(x')$ to be

$$\mathrm{T}[A(x)B(x')] = \theta(t-t')A(x)B(y) - \theta(t'-t)B(x')A(x). \tag{18.2.1}$$

We can then define the **fermion propagator** as

$$\langle 0|\,\mathrm{T}[\psi(x)\bar{\psi}(x')]|0\rangle. \tag{18.2.2}$$

First, consider the case where $t > t'$, then $\mathrm{T}[\psi(x)\bar{\psi}(x')] = \psi(x)\bar{\psi}(x')$ and we have

$$\begin{aligned}
\langle 0|\psi(x)\bar{\psi}(x')|0\rangle &= \langle 0|[\psi^+(x)+\psi^-(x)][\bar{\psi}^+(x')+\bar{\psi}^-(x')]|0\rangle && (18.2.3)\\
&= \langle 0|[\psi^+(x)\bar{\psi}^+(x')+\psi^+(x)\bar{\psi}^-(x') && (18.2.4)\\
&\qquad +\psi^-(x)\bar{\psi}^+(x')+\psi^-(x)\bar{\psi}^-(x')]|0\rangle && (18.2.5)\\
&= \langle 0|[\psi^+(x)\bar{\psi}^-(x')+\psi^-(x)\bar{\psi}^-(x')]|0\rangle && (18.2.6)\\
&= \langle 0|\{\psi^+(x),\bar{\psi}^-(x')\}|0\rangle && (18.2.7)\\
&= iS^+(x-x'), && (18.2.8)
\end{aligned}$$

where we've used

$$\psi^+(x)|0\rangle = \bar{\psi}^+(x')|0\rangle = 0. \tag{18.2.9}$$

Similarly we find that for $t < t'$

$$\langle 0|\bar{\psi}(x')\psi(x)|0\rangle = iS^-(x-x'). \tag{18.2.10}$$

Hence, we have

$$\langle 0|\,\mathrm{T}[\psi(x)\bar{\psi}(x')]|0\rangle = iS_{\mathrm{F}}(x-x'), \tag{18.2.11}$$

where

$$S_F(x) = \theta(t)S^+(x) - \theta(-t)S^-(x) = (i\partial\!\!\!/ + m)\Delta_{\mathrm{F}}(x) \tag{18.2.12}$$

where Δ_{F} is the Feynman propagator for a scalar field, and $S_F(x)$ is the **Feynman propagator** for a fermion field.

The interpretation is much the same as for a scalar field. If $t' < t$ then we create a fermion at x' and propagate it to x where it is destroyed. If $t < t'$ then we create an antifermion at x and propagate it to x' where it is destroyed. A fermion going in one direction is the same as an antifermion going in the opposite direction.

Note that if we consider an electron then the charge is $-e$, and this corresponds to a particle created by $a^\dagger$ or ψ^+, and the antiparticle, the positron, has charge $+e$, and corresponds to an antiparticle created by $b^\dagger$ or ψ^-.

18.3 Interactions

In this section we'll assume that the fermions are electrons, e^-, and the antifermions are positrons, e^+. The process of deriving the Dyson series doesn't change, since

the interaction Lagrangian is a scalar in any theory. We suppose that the Lagrangian is made of ψ and $\bar{\psi}$ pairs of the form $\bar{\psi}\Gamma\psi$, where Γ is some product of gamma matrices, including γ^5. The only real change to how we consider interactions from a scalar field theory is that in Wick's theorem when we swap the order of two fermions we need to introduce a relative minus sign.

For a fermion we define the **Wick contraction** as

$$\overbrace{\psi(x)\bar{\psi}(x')} \coloneqq \langle 0|\,\mathrm{T}[\psi(x)\bar{\psi}(x')]|0\rangle \tag{18.3.1}$$

$$= iS_{\mathrm{F}}(x-x') \tag{18.3.2}$$

$$= i\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{\not{p}+m}{p^2-m^2+i\varepsilon}\mathrm{e}^{-ip\cdot(x-y)}. \tag{18.3.3}$$

To evaluate the S matrix elements we need to know how $\psi^{\pm}$ and $\bar{\psi}^{\pm}$ act on a single particle state. First consider the single particle state consisting of an electron of momentum $\boldsymbol{p}$ and spin s:

$$|\mathrm{e}^-, \boldsymbol{p}, s\rangle = a_s^\dagger(\boldsymbol{p})|0\rangle. \tag{18.3.4}$$

We can act on this with $\psi^+(x)$:

$$\psi^+(x)|\mathrm{e}^-, \boldsymbol{p}, s\rangle = \sum_{s'}\int đp'\, u(\boldsymbol{p}', s')\mathrm{e}^{-ip'\cdot x}a_{s'}(\boldsymbol{p}')a_s^\dagger(\boldsymbol{p})|0\rangle. \tag{18.3.5}$$

Using the anticommutator relations we have

$$a_{s'}(\boldsymbol{p})a_s^\dagger(\boldsymbol{p}) = \{a_{s'}(\boldsymbol{p}'), a_s^\dagger(\boldsymbol{p})\} - a_s^\dagger(\boldsymbol{p})a_{s'}(\boldsymbol{p}'). \tag{18.3.6}$$

We can then use $a_{s'}(\boldsymbol{p}')|0\rangle = 0$ to rewrite the equation above as

$$\psi^+(x)|\mathrm{e}^-, \boldsymbol{p}, s\rangle = \sum_{s'}\int đp'\, u(\boldsymbol{p}', s')\mathrm{e}^{-ip'\cdot x}\{a_{s'}(\boldsymbol{p}'), a_s^\dagger(\boldsymbol{p})\}|0\rangle \tag{18.3.7}$$

$$= \sum_{s'}\int đp'\, u(\boldsymbol{p}', s')\mathrm{e}^{-ip'\cdot x}\delta_{ss'}đ(\boldsymbol{p}-\boldsymbol{p}')|0\rangle \tag{18.3.8}$$

$$= u(\boldsymbol{p}, s)\mathrm{e}^{-ip\cdot x}|0\rangle. \tag{18.3.9}$$

Similarly,

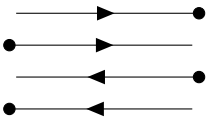
$$\bar{\psi}^+(x)|\mathrm{e}^+, \boldsymbol{p}, s\rangle = \bar{v}(\boldsymbol{p}, s)\mathrm{e}^{-ip\cdot x}|0\rangle, \tag{18.3.10}$$

$$\langle \mathrm{e}^-, \boldsymbol{p}, s|\psi^-(x) = \langle 0|\bar{u}(\boldsymbol{p}, s)\mathrm{e}^{ip\cdot x}, \tag{18.3.11}$$

$$\langle \mathrm{e}^+, \boldsymbol{p}, s|\bar{\psi}^-(x) = \langle 0|v(\boldsymbol{p}, s)\mathrm{e}^{ip\cdot x}. \tag{18.3.12}$$

If we were to proceed with computing various S matrix elements we would see that momentum conservation works the same as for the scalar case, we get a Dirac delta imposing momentum conservation/ The initial and final particles work the same, but with an added spinor, and similarly the internal particles come with an extra spin sum. We can then give the **Feynman rules** as follows, note that a $\bullet$ corresponds to a vertex and a lack of a $\bullet$ corresponds to an external particle.

- For each external line, of momentum $\boldsymbol{p}$ and spin s, one of the following:
  - For each initial electron a factor of $u(\boldsymbol{p}, s)$
  - For each final electron a factor of $\bar{u}(\boldsymbol{p}, s)$
  - For each initial positron a factor of $\bar{v}(\boldsymbol{p}, s)$
  - For each final positron a factor of $v(\boldsymbol{p}, s)$



- For each internal fermion line with four-momentum p a fermion propagator

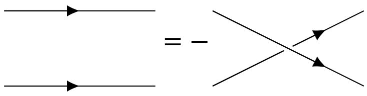
$$i\tilde{S}_{\mathrm{F}}(p) = \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon} = \frac{i}{\not{p} - m + i\varepsilon} \tag{18.3.13}$$

- For each four-momentum, q, not fixed by conserving momentum at each vertex an integral

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4}. \tag{18.3.14}$$

- A phase factor $(-1)^I$, where I is the number of interchanges of neighbouring fermions needed to achieve normal ordering.
- For each vertex some interaction-dependent factor.
- For each closed loop a negative sign and a trace over spinor indices.

These last three points need to be expanded upon. The factor of $(-1)^I$ comes from exchanging fermions to normal order them, which gives a relative minus sign between the amplitudes for the interactions:



(18.3.15)

For the interaction term there are a few interactions we could consider. We could have a purely fermionic interaction Lagrangian, such as

$$\mathcal{L}_{\mathrm{I}}^{(1)} = -G(\bar{\psi}\psi)^2, \tag{18.3.16}$$

where G is some coupling constant. This is the simplest such interaction, since we must have a pair of $\bar{\psi}$ and ψ for the Lagrangian to be a scalar, and we need higher than second order for an interaction term. In this case we get a factor of $-iG$ for each vertex. Another possibility is something like

$$\mathcal{L}_{\mathrm{I}}^{(2)} = -G(\bar{\psi}\gamma^\mu\psi)^2. \tag{18.3.17}$$

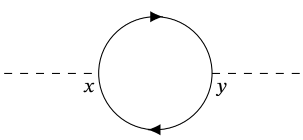
We can also have mixed scalar/fermion interactions, by introducing some scalar field, φ, a possibility is

$$\mathcal{L}_{\mathrm{I}}^{(3)} = -y\varphi\bar{\psi}\psi, \tag{18.3.18}$$

where y is a coupling constant. This is called a **Yukawa interaction**. We get a factor of $-iy$ for each vertex. We can also consider a pseudoscalar interaction,

$$\mathcal{L}_{\mathrm{I}}^{(4)} = -y\varphi\bar{\psi}\gamma^5\psi. \tag{18.3.19}$$

Suppose we go with the scalar interaction $\mathcal{L}_{\mathrm{I}} = -y\varphi\bar{\psi}\psi$. One possible diagram is



(18.3.20)

This corresponds to the term

$$\varphi(x)\overbrace{\bar{\psi}_a(x)\overbrace{\psi_a(x)\varphi(y)\bar{\psi}_b(y)}\psi_b(y)}. \tag{18.3.21}$$

We can compute the contraction with ψ first using Equation (18.3.1), which gives

$$\overbrace{\psi_a(x)\bar{\psi}_b}(y) = iS^{\mathrm{F}}_{ab}(x-y). \tag{18.3.22}$$

Note that we simply move the F label up to make room for the spinor indices, its just the normal propagator, $S^{\mathrm{F}} = S_{\mathrm{F}}$. In order to compute the other contraction, with $\bar{\psi}$ first, we have to swap the two fields, picking up a negative sign, and then we can use Equation (18.3.1) again, giving

$$\begin{aligned}\overbrace{\bar{\psi}_a(x)\psi_b}(y) &= -\overbrace{\psi_b(y)\bar{\psi}_a}(x) &(18.3.23)\\ &= -iS^{\mathrm{F}}_{ba}(y-x). &(18.3.24)\end{aligned}$$

If we then consider the product of these contractions we see that we have

$$\overbrace{\psi_a(x)\bar{\psi}_b}(y)\overbrace{\bar{\psi}_a(x)\psi_b}(y) = [iS^{\mathrm{F}}_{ab}(x-y)][-iS^{\mathrm{F}}_{ba}(y-x)]. \tag{18.3.25}$$

Considering some matrices, A and B, we have $A_{ab}B_{bc} = (AB)_{ac}$, so $A_{ab}B_{ba} = (AB)_{aa} = \mathrm{tr}(AB)$. Hence,

$$\overbrace{\psi_a(x)\bar{\psi}_b}(y)\overbrace{\bar{\psi}_a(x)\psi_b}(y) = [iS^{\mathrm{F}}_{ab}(x-y)][-iS^{\mathrm{F}}_{ba}(y-x)] = -\mathrm{tr}(iS_{\mathrm{F}}(x-y)iS_{\mathrm{F}}(y-x)). \tag{18.3.26}$$

Hence,

$$\varphi(x)\overbrace{\bar{\psi}_a(x)\overbrace{\psi_a(x)\varphi(y)\bar{\psi}_b}(y)\psi_b}(y) = -\varphi(x)\varphi(y)\,\mathrm{tr}(iS_{\mathrm{F}}(x-y)iS_{\mathrm{F}}(y-x)). \tag{18.3.27}$$

Nineteen

Computing Matrix Elements

19.1 Spin Sums

To evaluate decay rates and cross sections we need to evaluate $|\mathcal{M}|^2 = \mathcal{M}^\dagger\mathcal{M}$. Note that $\mathcal{M}$ is just a number, so we could just use $|\mathcal{M}|^2 = \mathcal{M}^*\mathcal{M}$, but $\mathcal{M}$ is typically formed as a product of matrices, and its often easier to compute $\mathcal{M}^\dagger\mathcal{M}$, which is completely equivalent.

A typical experiment may consist of many initial and final particles, all which potentially have spin. Most of the time the initial particles may have any allowed spin, and so we average over all spins for the input particles. In the case of spin 1/2 fermions we treat the initial particles as being in the state

$$|\boldsymbol{p}\rangle = \frac{1}{2}\sum_s |\boldsymbol{p}, s\rangle. \tag{19.1.1}$$

Similarly, the particles at the end of the process may have any spin, although there will be restrictions based on conservation of total angular momentum, however these restrictions are taken care of in the process of computing $\mathcal{M}$ and so we don't need to consider them again. If we are not measuring the final spin then our resulting particles are in a superposition of spin states, and so we treat the final particle state as

$$|\boldsymbol{q}\rangle = \sum_s |\boldsymbol{q}, s\rangle. \tag{19.1.2}$$

For example, in a process with two fermions in both the initial and final state where the initial fermions may have any spin and we don't measure the spin of the final fermions we are actually interested in computing

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\sum_{s,s'}\sum_{r,r'} \mathcal{M}^\dagger\mathcal{M} \tag{19.1.3}$$

where s and s' are the spins of the initial particles and r and r' the spins of the final particles.

The typical amplitude will look like

$$\mathcal{M} = \bar{u}(\boldsymbol{p}', s')\Gamma u(\boldsymbol{p}, s), \tag{19.1.4}$$

possibly with vs in the place of one or more us if we have antiparticles. The $u(\boldsymbol{p}, s)$ corresponds to an incoming fermion, for simplicity we'll assume an electron. The

$\bar{u}(\boldsymbol{p}', s')$ corresponds to an outgoing fermion. The Γ represents the propagator and vertex terms. We then have

$$\begin{aligned}\mathcal{M}^\dagger &= [u^\dagger(\boldsymbol{p}', s')\gamma^0\Gamma u(\boldsymbol{p}, s)]^\dagger && (19.1.5)\\ &= u^\dagger(\boldsymbol{p}, s)\Gamma^\dagger(\gamma^0)^\dagger u(\boldsymbol{p}', s') && (19.1.6)\\ &= \bar{u}(\boldsymbol{p}, s)\bar{\Gamma}u(\boldsymbol{p}', s'). && (19.1.7)\end{aligned}$$

Here we've used $(\gamma^0)^\dagger = \gamma^0$ and defined $\bar{\Gamma} \coloneqq \gamma^0\gamma^\dagger\gamma^0$. If Γ is a product gamma matrices, as is often the case with spin 1/2 particles interacting, then

$$\begin{aligned}\bar{\Gamma} &= \gamma^0\Gamma^\dagger\gamma^0 && (19.1.8)\\ &= \gamma^0(\gamma^\mu\gamma^\nu \ldots \gamma^\rho)^\dagger\gamma^0 && (19.1.9)\\ &= \gamma^0[(\gamma^\rho)^\dagger \cdots (\gamma^\nu)^\dagger(\gamma^\mu)^\dagger]\gamma^0 && (19.1.10)\\ &= \gamma^0[\gamma^0\gamma^\rho\gamma^0 \cdots \gamma^0\gamma^\nu\gamma^0\gamma^0\gamma^\mu\gamma^0]\gamma^0 && (19.1.11)\\ &= \gamma^\rho \cdots \gamma^\nu\gamma^\mu. && (19.1.12)\end{aligned}$$

Here we've used $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ and $(\gamma^0)^2 = 1$. So, the result is $\bar{\Gamma}$ is given by Γ with the order of the gamma matrices reversed. If Γ also contains an odd number of γ^5 matrices then we pick up a negative sign, since $(\gamma^5)^\dagger = \gamma^5 = \gamma^5(\gamma^0)^2 = -\gamma^0\gamma^5\gamma^0$, since γ^5 anticommutes with all gamma matrices.

So, we want to be able to compute

$$\mathcal{M}^\dagger\mathcal{M} = \bar{u}(\boldsymbol{p}, s)\bar{\Gamma}u(\boldsymbol{p}', s')\bar{u}(\boldsymbol{p}', s')\Gamma u(\boldsymbol{p}, s). \tag{19.1.13}$$

Now consider this with explicit spinor indices:

$$\mathcal{M}^\dagger\mathcal{M} = \bar{u}_a(\boldsymbol{p}, s)\bar{\Gamma}_{ab}u_b(\boldsymbol{p}', s')\bar{u}_c(\boldsymbol{p}', s')\Gamma_{cd}u_d(\boldsymbol{p}, s). \tag{19.1.14}$$

After including the spinor indices the terms are just numbers and so commute, so rewrite this as

$$\mathcal{M}^\dagger\mathcal{M} = u_d(\boldsymbol{p}, s)u_d(\boldsymbol{p}, s)\bar{u}_a(\boldsymbol{p}, s)\bar{\Gamma}_{ab}u_b(\boldsymbol{p}', s')\bar{u}_c(\boldsymbol{p}', s')\Gamma_{cd}. \tag{19.1.15}$$

Now consider the averaged matrix element:

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{1}{4}\sum_{s,s'}\mathcal{M}^\dagger\mathcal{M} && (19.1.16)\\ &= \frac{1}{4}\sum_{s,s'}u_d(\boldsymbol{p}, s)\bar{u}_a(\boldsymbol{p}, s)\bar{\Gamma}_{ab}u_b(\boldsymbol{p}', s')\bar{u}_c(\boldsymbol{p}', s')\Gamma_{cd} && (19.1.17)\\ &= \frac{1}{4}\,\mathrm{tr}\left(\sum_{s,s'}u_d(\boldsymbol{p}, s)\bar{u}(\boldsymbol{p}, s)\bar{\Gamma}u_b(\boldsymbol{p}', s')\bar{u}_c(\boldsymbol{p}', s')\Gamma\right). && (19.1.18)\end{aligned}$$

Here we've recognised that

$$u_d(\boldsymbol{p}, s)\bar{u}_a(\boldsymbol{p}, s)\bar{\Gamma}_{ab}u_b(\boldsymbol{p}', s')\bar{u}_c(\boldsymbol{p}', s')\Gamma_{ce} \tag{19.1.19}$$

is the element on row d and column e of a matrix. Setting $e = d$ and using the summation convention as we have above then gives the trace of this matrix.

So, we've reduced the amplitude to a trace of spinors and gamma matrices. We can reduce it further by using the spin sums in Equation (16.3.12),

$$\sum_s u(\boldsymbol{p}, s)\bar{\boldsymbol{p}}, s = \not{p} + m. \tag{19.1.20}$$

We then have

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\,\mathrm{tr}[(\not{p}+m)\bar{\Gamma}(\not{p}+m)\Gamma]. \tag{19.1.21}$$

We can then compute this using various identities for gamma matrices, including, but not limited to,

$$\gamma^\mu\gamma_\mu = 4 \qquad \gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a} \tag{19.1.22}$$

$$\gamma^\mu \not{a}\gamma_\mu = -2\not{a} \qquad \gamma^\mu \not{a}\not{b}\not{c}\not{d}\gamma_\mu = 2(\not{d}\not{a}\not{b}\not{c} + \not{c}\not{b}\not{a}\not{d}) \tag{19.1.23}$$

$$\gamma^\mu \not{a}\not{b}\gamma_\mu = 4a\cdot b \qquad \mathrm{tr}(\not{a}\not{b}) = 4a\cdot b \tag{19.1.24}$$

$$\mathrm{tr}(\not{a}\not{b}\not{c}\not{d}) = 4[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)] \tag{19.1.25}$$

$$\mathrm{tr}(\gamma^5\not{a}\not{b}\not{c}\not{d}) = 4i\varepsilon_{\mu\nu\rho\sigma}a^\mu b^\nu c^\rho d^\sigma \qquad \mathrm{tr}(\not{a}_1\not{a}_2\cdots\not{a}_n) = 0 \text{ for } n \text{ odd} \tag{19.1.26}$$

19.2 Examples

In this section we'll compute some examples arising from a Yukawa interaction,

$$\mathcal{L}_\mathrm{I} = -y\varphi\bar{\psi}\psi. \tag{19.2.1}$$

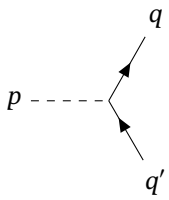
These interactions will be three-point interactions involving a scalar particle, S, of mass M, and a fermion/antifermion pair of mass m, which we assume to be electrons and positrons unless stated otherwise. For this interaction the vertex term is simply $-iy$.

19.2.1 Scalar Decay

So long as $M \geq 2m$ we can have the scalar particle decay into an electron/positron pair,

$$S \to \mathrm{e}^-\mathrm{e}^+. \tag{19.2.2}$$

Suppose the scalar particle has momentum p and the electron and positron have momenta q and q', and spins s and s', respectively. The Feynman diagram for this is



(19.2.3)

Here we have a single vertex contributing $-iy$ to the amplitude. We then have an outgoing electron, contributing $\bar{u}(\boldsymbol{q}, s)$, and an outgoing positron, contributing $v(\boldsymbol{q}', s')$. Hence, the amplitude is

$$\mathcal{M} = (-iy)\bar{u}(\boldsymbol{q}, s)v(\boldsymbol{q}', s'). \tag{19.2.4}$$

The averaged amplitude is then given by summing over the final spins, no factor of 1/4 is needed this time as we don't have any initial spins to average over:

$$\overline{|\mathcal{M}|} = \sum_{s,s'} \mathcal{M}^\dagger \mathcal{M} \tag{19.2.5}$$

$$= \sum_{s,s'} y^2 \bar{v}(\boldsymbol{q}, s') u(\boldsymbol{q}, s) \bar{u}(\boldsymbol{q}, s) v(\boldsymbol{q}', s') \tag{19.2.6}$$

$$= y^2 \sum_{s,s'} \bar{v}_a(\boldsymbol{q}', s') u_a(\boldsymbol{q}, s) \bar{u}_b(\boldsymbol{q}, s) v_b(\boldsymbol{q}', s') \tag{19.2.7}$$

$$= y^2 \sum_{s,s'} u_a(\boldsymbol{q}, s) \bar{u}_b(\boldsymbol{q}, s) v_b(\boldsymbol{q}', s') \bar{v}_a(\boldsymbol{q}', s') \tag{19.2.8}$$

$$= y^2 \sum_{s,s'} \mathrm{tr}[u(\boldsymbol{q}, s) \bar{u}(\boldsymbol{q}, s) v(\boldsymbol{q};, s') \bar{v}(\boldsymbol{q}', s')] \tag{19.2.9}$$

$$= y^2 \,\mathrm{tr}[(\not{q} + m)(\not{q}' - m)] \tag{19.2.10}$$

where we've used the spin sum for the v spinors, $\sum_s v(\boldsymbol{p}, s)\bar{v}(\boldsymbol{p}, s) = \not{p} - m$. We then have

$$\overline{|\mathcal{M}|^2} = y^2 \,\mathrm{tr}[\not{q}\not{q}' - m\not{q} + m\not{q}' - m^2]. \tag{19.2.11}$$

The two cross terms don't contribute, since they each have a single gamma matrix so their trace vanishes. The mass term has an implicit $\mathbb{1}_4$, and so the trace is $4m^2$. The first term has trace $4q \cdot q'$ by one of our identities. Hence,

$$\overline{|\mathcal{M}|^2} = 4y^2(q \cdot q' - m^2). \tag{19.2.12}$$

Conservation of momentum tells us that $p = q + q'$. Squaring this we have $M^2 = 2m^2 + 2q \cdot q'$, and so $q \cdot q' = (M^2 - 2m^2)/2$, which we can use to rewrite the result as

$$\overline{|\mathcal{M}|^2} = 2y^2(M^2 - 4m^2). \tag{19.2.13}$$

The decay rate in the centre of mass frame is then

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega} = \frac{1}{(4\pi)^2} \frac{|\boldsymbol{q}|}{M^2} \overline{|\mathcal{M}|^2} = \frac{y^2}{16\pi^2} M \left(1 - \frac{4m^2}{M^2}\right)^{3/2}. \tag{19.2.14}$$

Here we've used the centre of mass frame so $p^\mu = (M, 0)$, $q^\mu = (E, \boldsymbol{q})$, and $q'^\mu = (E, -\boldsymbol{q})$, and so $p^2 = M^2$ and $p^2 = (q + q')^2 = (2E)^2$ and so $|\boldsymbol{q}|^2 = E^2 - m^2 = M^2/4 - m^2$.

We can integrate this over the sphere to get the total decay rate,

$$\Gamma = 4\pi^2 \frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega} = \frac{y^2}{4\pi} M \left(1 - \frac{4m^2}{M^2}\right)^{3/2}, \tag{19.2.15}$$

where we get the factor of 4π from the integral over the sphere, as there is no angular dependence. There is no angular dependence because the scalar particle has no preferred direction. Preferred directions usually come from spins or polarisations, which naturally select a particular direction.

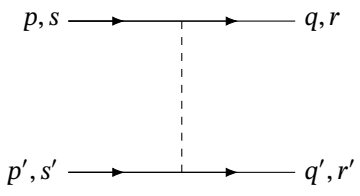
Notice that if we analytically continue the decay rate as a function of M there is a branch point when $M^2 = 4m^2$. We typically get a branch point at the threshold when a process *just* becomes possible.

19.2.2 Elastic Scattering

Consider the elastic scattering process

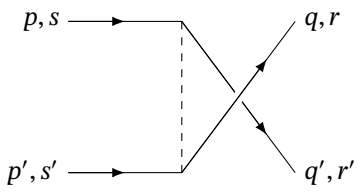
$$e^- e^- \to e^- e^-. \tag{19.2.16}$$

There are two diagrams contributing to this at first order, they are the t-channel diagram,



(19.2.17)

and the u-channel diagram,



(19.2.18)

There is no s-channel diagram as it would require a virtual particle with charge -2, which doesn't exist in this theory.

Suppose the amplitude for the t-channel process is $\mathcal{M}_t$ and the amplitude for the u-channel process is $\mathcal{M}_u$. The total amplitude (to first order) is then

$$\mathcal{M} = \mathcal{M}_t - \mathcal{M}_u. \tag{19.2.19}$$

The negative sign arises due to the crossing of fermion lines swapping the fermions in the u-channel diagram. The amplitude squared is then

$$\mathcal{M}^\dagger \mathcal{M} = \mathcal{M}_t^\dagger \mathcal{M}_t + \mathcal{M}_u^\dagger \mathcal{M}_u - (\mathcal{M}_t^\dagger \mathcal{M}_u + \mathcal{M}_u^\dagger \mathcal{M}_t). \tag{19.2.20}$$

The last term is called the **interference term**.

Call the spins of the momentum p electron s and the spin of the momentum p' particle s'. Note that the spin of the particles cannot change, since the scalar propagator is spinless, and so we don't need to sum over final spins, only average over initial spins.

The amplitude for the t-channel process is

$$\mathcal{M}_t = -y^2 \bar{u}(\boldsymbol{q}, r) u(\boldsymbol{p}, s) \frac{i}{(p-q)^2 - M^2 + i\varepsilon} \bar{u}(\boldsymbol{q}', r') u(\boldsymbol{p}', s'). \tag{19.2.21}$$

The amplitude for the u-channel process is

$$\mathcal{M}_u = -y^2 \bar{u}(\boldsymbol{q}', r') u(\boldsymbol{p}, s) \frac{i}{(p-q')^2 - M^2 + i\varepsilon} \bar{u}(\boldsymbol{q}, r) u(\boldsymbol{p}', s'). \tag{19.2.22}$$

Hence, defining $P_t = i/(t - M^2 + i\varepsilon)$ as the propagator term we have

$$\frac{1}{4}\sum_{s,s',r,r'}\mathcal{M}_t^\dagger\mathcal{M}_t = \frac{y^4}{4}|P_t|^2\sum_{s,s',r,r'}\bar{u}(\boldsymbol{p}',s')u(\boldsymbol{q}',r')\bar{u}(\boldsymbol{p},s)u(\boldsymbol{q},r) \times \bar{u}(\boldsymbol{q},r)u(\boldsymbol{p},s)\bar{u}(\boldsymbol{q}',r')u(\boldsymbol{p}',r') \tag{19.2.23}$$

$$= \frac{y^4}{4}|P_t|^2\sum_{s,s',r,r'}\mathrm{tr}[u(\boldsymbol{p},s)\bar{u}(\boldsymbol{p},s)u(\boldsymbol{q},r)\bar{u}(\boldsymbol{q},r)] \times \mathrm{tr}[u(\boldsymbol{p}',s')\bar{u}(\boldsymbol{p}',s')u(\boldsymbol{q}',r')\bar{u}(\boldsymbol{q}',r')] \tag{19.2.24}$$

$$= \frac{y^4}{4}|P_t|^2\,\mathrm{tr}[(\not{p}' + m)(\not{q}' + m)]\,\mathrm{tr}[(\not{p} + m)(\not{q} + m)] \tag{19.2.25}$$

We can calculate both traces the same way:

$$\mathrm{tr}[(\not{p} + m)(\not{q} + m)] = \mathrm{tr}[\not{p}\not{q} + m\not{p} + m\not{q} + m^2] \tag{19.2.26}$$

$$= 4(p\cdot q + m^2). \tag{19.2.27}$$

So, we have

$$\frac{1}{4}\sum_{s,s',r,r'}\mathcal{M}_t^\dagger\mathcal{M}_t = 4y^4|P_t|^2(p'\cdot q' + 4m^2)(p\cdot q + 4m^2). \tag{19.2.28}$$

Similarly, for the u-channel process, defining $P_u = i/(u - M^2 + i\varepsilon)$, we have

$$\frac{1}{4}\sum_{s,s',r,r'}\mathcal{M}_u^\dagger\mathcal{M}_u = \frac{y^4}{4}|P_u|^2\sum_{s,s',r,r'}\bar{u}(\boldsymbol{p}',s')u(\boldsymbol{q},r)\bar{u}(\boldsymbol{p},s)u(\boldsymbol{q}',r') \times \bar{u}(\boldsymbol{q}',r')u(\boldsymbol{p},s)\bar{u}(\boldsymbol{q},r)u(\boldsymbol{p}',s') \tag{19.2.29}$$

$$= \frac{y^4}{4}|P_u|^2\sum_{s,s',r,r'}\mathrm{tr}[u(\boldsymbol{p},s)\bar{u}(\boldsymbol{p},s)u(\boldsymbol{q}',r')\bar{u}(\boldsymbol{q}',r')] \times \mathrm{tr}[u(\boldsymbol{p}',s')\bar{u}(\boldsymbol{p}',s')u(\boldsymbol{q},r)\bar{u}(\boldsymbol{q},r)] \tag{19.2.30}$$

$$= \frac{y^4}{4}|P_u|^2\,\mathrm{tr}[(\not{p} + m)(\not{q}' + m)]\,\mathrm{tr}[(\not{p}' + m)(\not{q} + m)] \tag{19.2.31}$$

$$= 4y^4|P_u|^2(p\cdot q' + 4m^2)(p'\cdot q + 4m^2). \tag{19.2.32}$$

The first cross term is

$$\sum_{s,s',r,r'}\mathcal{M}_t^\dagger\mathcal{M}_u = \frac{y^4}{4}P_t^*P_u\sum_{s,s',r,r'}\bar{u}(\boldsymbol{p}',s')u(\boldsymbol{q}',r')\bar{u}(\boldsymbol{p},s)u(\boldsymbol{q},r) \times \bar{u}(\boldsymbol{q}',r')u(\boldsymbol{p},s)\bar{u}(\boldsymbol{q},r)u(\boldsymbol{p}',s') \tag{19.2.33}$$

$$= \frac{y^4}{4}P_t^*P_u\sum_{s,s',r,r'}\bar{u}_a(\boldsymbol{p}',s')u_a(\boldsymbol{q}',r')\bar{u}_b(\boldsymbol{p},s)u_b(\boldsymbol{q},r) \times \bar{u}_c(\boldsymbol{q}',r')u_c(\boldsymbol{p},s)\bar{u}_d(\boldsymbol{q},r)u_d(\boldsymbol{p}',s') \tag{19.2.34}$$

$$= \frac{y^4}{4}P_t^*P_u\sum_{s,s',r,r'}u_c(\boldsymbol{p},s)\bar{u}_b(\boldsymbol{p},s)u_b(\boldsymbol{q},r)\bar{u}_d(\boldsymbol{q},r) \times u_d(\boldsymbol{p}',s')\bar{u}_a(\boldsymbol{p}',s')u_a(\boldsymbol{q}',r')\bar{u}_c(\boldsymbol{q}',r') \tag{19.2.35}$$

$$= \frac{y^4}{4} P_t^* P_u \sum_{s,s',r,r'} \mathrm{tr}[u(\boldsymbol{p},s)\bar{u}(\boldsymbol{p},s)u(\boldsymbol{q},r)\bar{u}(\boldsymbol{q},r) \times u(\boldsymbol{p}',s')\bar{u}(\boldsymbol{p}',s')u(\boldsymbol{q}',r')\bar{u}(\boldsymbol{q}',r')] \tag{19.2.36}$$

$$= \frac{y^4}{4} P_t^* P_u \,\mathrm{tr}[(\not{p}+m)(\not{q}+m)(\not{p}'+m)(\not{q}'+m)]. \tag{19.2.37}$$

Expanding this and keeping only terms with an even number of gamma matrices, since the trace of an odd number of gamma matrices vanishes, we get

$$\frac{1}{4}\sum_{s,s',r,r'} \mathcal{M}_t^\dagger \mathcal{M}_u = \frac{y^4}{4} P_t^* P_u \,\mathrm{tr}[\not{p}\not{q}\not{p}'\not{q}' + m^2(\not{p}\not{q}+\not{p}\not{p}'+\not{p}\not{q}'+\not{q}\not{p}'+\not{q}\not{q}'+\not{p}'\not{q}')+m^4].$$

We can then use the identities

$$\mathrm{tr}[\mathbb{1}_4] = 4, \qquad \mathrm{tr}[\not{a}\not{b}] = 4a\cdot b, \tag{19.2.38}$$

$$\mathrm{tr}[\not{a}\not{b}\not{c}\not{d}] = 4[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)], \tag{19.2.39}$$

and we have

$$\frac{1}{4}\sum_{s,s',r,r'} \mathcal{M}_t^\dagger \mathcal{M}_u = y^4 P_t^* P_u \{(p\cdot q)(p'\cdot q') - (p\cdot p')(q\cdot q') + (p\cdot q')(q\cdot p') + m^2[p\cdot q + p\cdot p' + p\cdot q' + q\cdot p' + q\cdot q' + p'\cdot q'] + m^4\}. \tag{19.2.40}$$

Now consider the second cross term:

$$\frac{1}{4}\sum_{s,s',r,r'} \mathcal{M}_u^\dagger \mathcal{M}_t = \frac{y^4}{4} P_u^* P_t \sum_{s,s',r,r'} \bar{u}(\boldsymbol{p}',s')u(\boldsymbol{q},r)\bar{u}(\boldsymbol{p},s)u(\boldsymbol{q}',r') \times \bar{u}(\boldsymbol{q},r)u(\boldsymbol{p},s)\bar{u}(\boldsymbol{q}',r')u(\boldsymbol{p}',s') \tag{19.2.41}$$

$$= \frac{y^4}{4} P_u^* P_t \sum_{s,s',r,r'} \bar{u}_a(\boldsymbol{p}',s')u_a(\boldsymbol{q},r)\bar{u}_b(\boldsymbol{p},s)u_b(\boldsymbol{q}',r') \times \bar{u}_c(\boldsymbol{q},r)u_c(\boldsymbol{p},s)\bar{u}_d(\boldsymbol{q}',r')u_d(\boldsymbol{p}',s') \tag{19.2.42}$$

$$= \frac{y^4}{4} P_u^* P_t \sum_{s,s',r,r'} u_a(\boldsymbol{q},r)\bar{u}_c(\boldsymbol{q},r)u_c(\boldsymbol{p},s)\bar{u}_b(\boldsymbol{p},s) \times u_b(\boldsymbol{q}',r')\bar{u}_d(\boldsymbol{q}',r')u_d(\boldsymbol{p}',s')\bar{u}_a(\boldsymbol{p}',s') \tag{19.2.43}$$

$$= \frac{y^4}{4} P_u^* P_t \sum_{s,s',r,r'} \mathrm{tr}[u(\boldsymbol{q},r)\bar{u}(\boldsymbol{q},r)u(\boldsymbol{p},s)\bar{u}(\boldsymbol{p},s) \times u(\boldsymbol{q}',r')\bar{u}(\boldsymbol{q}',r')u(\boldsymbol{p}',s')\bar{u}(\boldsymbol{p}',s')] \tag{19.2.44}$$

$$= \frac{y^4}{4} P_u^* P_t \sum_{s,s',r,r'} [(\not{q}+m)(\not{p}+m)(\not{q}'+m)(\not{p}'+m)] \tag{19.2.45}$$

$$= \frac{y^4}{4} P_u^* P_t \sum_{s,s',r,r'} [(\not{p}+m)(\not{q}'+m)(\not{p}'+m)(\not{q}+m)] \tag{19.2.46}$$

$$= y^4 P_u^* P_t \{(p\cdot q')(p'\cdot q) - (p\cdot p')(q'\cdot q) + (p\cdot q)(q'\cdot p') + m^2[p\cdot q' + p\cdot p' + p\cdot q + q'\cdot p' + q'\cdot q + p'\cdot q] + m^4\}. \tag{19.2.47}$$

So, since $P_t^* P_u = P_u^* P_t$ after dropping the $i\varepsilon$ terms, we find that $\mathcal{M}_u^\dagger \mathcal{M}_t = \mathcal{M}_t^\dagger \mathcal{M}_u$ in this case and so

$$\frac{1}{4}\sum_{s,s',r,r'}(\mathcal{M}_t^\dagger \mathcal{M}_u + \mathcal{M}_u^\dagger \mathcal{M}^t) = 2y^4 P_t^* P_u\{(p\cdot q)(p'\cdot q') - (p\cdot p')(q\cdot q') + (p\cdot q')(q\cdot p') + m^2[p\cdot q + p\cdot p' + p\cdot q' + q\cdot p' + q\cdot q' + p'\cdot q'] + m^4\}. \quad (19.2.48)$$

Putting this all together the amplitude squared averaged over incoming spins and summed over outgoing spins is

$$\frac{1}{4}\sum_{s,s',r,r'}|\mathcal{M}|^2 = 2y^4\Bigg[\frac{2}{(t-M^2)^2}(p'\cdot q' + 4m^2)(p\cdot q + 4m^2) + \frac{2}{(u-M^2)^2}(p\cdot q' + 4m^2)(p'\cdot q + 4m^2) + \frac{1}{(t-M^2)(u-M^2)}\{(p\cdot q)(p'\cdot q') - (p\cdot p')(q\cdot q') + (p\cdot q')(q\cdot p') + m^2[p\cdot q + p\cdot p' + p\cdot q' + q\cdot p' + q\cdot q' + p'\cdot q'] + m^4\}\Bigg] \quad (19.2.49)$$

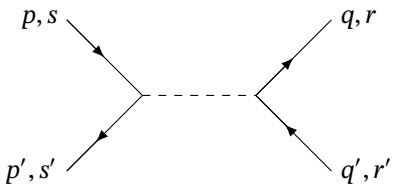
(19.2.50)

19.2.2.1 Annihilation

Consider the annihilation process

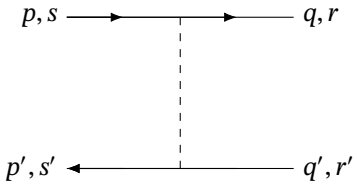
$$e^- e^+ \to e^- e^+. \quad (19.2.51)$$

There are two diagrams contributing to this at first order, they are the t-channel diagram,



(19.2.52)

and the t-channel diagram, where the electron and positron just scatter off each other without annihilating,



(19.2.53)

There is no u-channel diagram as the particles are distinguishable. The total amplitude for this process is

$$\mathcal{M} = \mathcal{M}_s + \mathcal{M}_u, \quad (19.2.54)$$

and so

$$\mathcal{M}^\dagger\mathcal{M} = \mathcal{M}_s^\dagger\mathcal{M}_s + \mathcal{M}_t^\dagger\mathcal{M}_t + (\mathcal{M}_s^\dagger\mathcal{M}_t + \mathcal{M}_t^\dagger\mathcal{M}_s). \tag{19.2.55}$$

The calculation is almost identical to the calculation for electron scattering, except that some particles are replaced with antiparticles, so we need replace some us with $\bar{v}$s and some $\bar{u}$s with vs. We also have an s-channel instead of a u-channel.

Instead of doing this calculation we consider the similar process

$$\mathrm{e}^-\mathrm{e}^+ \to \mu^-\mu^+. \tag{19.2.56}$$

This simplifies matters slightly since there is no t-channel diagram, as the final state particles differ from the initial state particles. For the purposes of quantum field theory we can treat (anti)muons as identical to (anti)electrons but with a different mass, so we just have two fermion fields involved in this interaction.

The s-channel amplitude, which is just the total amplitude to first order as their is only one tree level diagram for this process, is

$$\mathcal{M} = \mathcal{M}_s = (-iy)^2\bar{u}(\boldsymbol{q}, r)v(q', r')\frac{i}{(p+p')^2 - M^2 + i\varepsilon}\bar{v}(\boldsymbol{p}', s')u(\boldsymbol{p}, s). \tag{19.2.57}$$

Averaging over incoming spins and summing over outgoing spins we get

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{1}{4}\sum_{s,s',r,r'}\mathcal{M}^\dagger\mathcal{M} && (19.2.58)\\
&= \frac{y^4}{4}\frac{1}{(s-M^2)^2} && (19.2.59)\\
&\quad\times\sum_{r,r'}\bar{v}(q', r')u(q, r)\bar{u}(q, r)v(q', r')\sum_{s,s'}\bar{u}(p, s)v(p', s')\bar{v}(p', s')u(p, s) && (19.2.60)\\
&= \frac{y^4}{4}\frac{1}{(s-M^2)^2}\operatorname{tr}[(\not{q} + m_\mu)(\not{q}' - m_\mu)]\operatorname{tr}[(\not{p} + m_\mathrm{e})(\not{p}' - m_\mathrm{e})] && (19.2.61)\\
&= \frac{4y^4}{(s-M^2)^2}(q\cdot q' - m_\mu^2)(p\cdot p' - m_\mathrm{e}^2) && (19.2.62)\\
&= \frac{y^4}{(s-M^2)^2}(s - 4m_\mu^2)(s - 4m_\mathrm{e}^2). && (19.2.63)
\end{aligned}$$

Here we assume that s doesn't achieve the value of M^2, so we can drop the $i\varepsilon$ term. In the last step we've used $s = (p+p')^2 = m_\mathrm{e}^2 + m_\mu^2 + 2p\cdot p'$, and a $s = (q+q')^2 = m_\mathrm{e}^2 + m_\mu^2 + 2q\cdot q'$ to rewrite the result in terms of Mandelstam variables.

The differential cross section in the centre of mass frame for this process is then

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\frac{y^4}{(s-M^2)^2}\frac{1}{s}\sqrt{\frac{s-4m_\mu^2}{s-4m_\mathrm{e}^2}}(s-4m_\mathrm{e}^2)(s-4m_\mu^2). \tag{19.2.64}$$

We can easily integrate this since there is no angular dependence to get the total cross section,

$$\frac{1}{16\pi}\frac{y^4}{(s-M^2)^2}\frac{1}{s}\sqrt{\frac{s-4m_\mu^2}{s-4m_\mathrm{e}^2}}(s-4m_\mathrm{e}^2)(s-4m_\mu^2). \tag{19.2.65}$$

Notice that at high energies, when $s \gg m_e, m_\mu, M$, the cross section goes as $1/s$. Essentially meaning that the faster the incoming particles are going the less likely they are to interact and annihilate before passing each other.

Notice that this diverges as $s \to M^2$. We call this a **resonance** for the scalar particle. These can be useful to find new particles, we get a large spike in the cross section when a particle goes on-shell, which tells us the mass of that particle. While the result is divergent here if we included all orders it would turn out that we don't get divergence, just a large peak.

Close to the peak the cross section nearly factors into

$$\mathcal{M}(e^- e^+ \to S)\mathcal{M}(S \to \mu^- \mu^+). \tag{19.2.66}$$

The interpretation is that as the scalar particle becomes *almost* on-shell the process is *almost* a pair of electrons annihilating and then a scalar particle decaying as two separate processes. This is a general feature, that as particles go on-shell amplitudes factor.

Despite using binary on/off-shell or real/virtual particle language there's really much more of a continuum between real and virtual particles. In fact, a lot of what we think of as real particles are just relatively stable virtual particles.

Twenty

CPT For Fermions

In this chapter we'll explore how we can construct parity, time reversal, and charge conjugation operators for Dirac fermions in such a way that the Dirac equation is invariant.

20.1 Parity

Under parity $x^\mu = (t, \boldsymbol{x}) \mapsto \bar{x}^\mu = (t, -\boldsymbol{x}) = x_\mu$, and $\partial_\mu = (\partial_t, \nabla) \mapsto \bar{\partial}_\mu = (\partial_t, -\nabla) = \partial^\mu$. Suppose that under a parity transformation $\psi(x) \mapsto \psi_P(x) = P\psi(\bar{x})$ for some spin matrix, P. We then require the gamma matrices to transform as

$$\gamma^\mu \mapsto P\gamma^\mu P^{-1} = (\gamma^0, -\gamma^i) = \gamma_\mu, \tag{20.1.1}$$

so that starting with the Dirac equation, $(i\partial\!\!\!/ - m)\psi(x)$, under parity we get

$$P(i\gamma^\mu\partial_\mu - m)\psi_(x) = 0 \implies (i\gamma_\mu\partial^\mu - m)\psi_P(x) = 0. \tag{20.1.2}$$

A solution is that $P = P^{-1} = \gamma^0$. As we have $\gamma^0\gamma^i\gamma^0 = -\gamma^i$ and $(\gamma^0)^3 = \gamma^0$, which follows from $(\gamma^0)^2 = 1$ and $\{\gamma^0, \gamma^i\} = 0$. We could more generally take $P = \mathrm{e}^{i\varphi}\gamma^0$, but this doesn't give us anything more interesting.

The adjoint spinor transforms as

$$\bar{\psi} = \psi^\dagger\gamma^0 \mapsto (\gamma^0\psi)^\dagger\gamma^0 = \psi^\dagger\gamma^0\gamma^0 = \bar{\psi}\gamma^0 = \bar{\psi}P^{-1}. \tag{20.1.3}$$

Hence, we have

$$\bar{\psi}\psi \mapsto \bar{\psi}P^{-1}P\psi = \bar{\psi}\psi, \qquad \bar{\psi}\gamma^\mu\psi \mapsto \bar{\psi}P^{-1}\gamma^\mu P\psi = \bar{\psi}\gamma^\mu\psi, \tag{20.1.4}$$

which is what we'd expect for a scalar $\bar{\psi}\psi$ and a vector current, $\bar{\psi}\gamma^\mu\psi$. Importantly, this means that $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ is invariant under parity inversion.

The spinors transform as $u(\boldsymbol{p}, s) \mapsto \gamma^0 u(\boldsymbol{p}, s) = u(-\boldsymbol{p}, s)$, which can be shown easily in the Dirac representation. Spin doesn't change under parity, since $\boldsymbol{r} \times \boldsymbol{p}$ doesn't change, this corresponds to the transformation of the creation operators transforming as

$$\mathcal{P}a_s^\dagger(\boldsymbol{p})\mathcal{P}^\dagger = a_s^\dagger(-\boldsymbol{p}), \qquad \text{and} \qquad \mathcal{P}b_s^\dagger(\boldsymbol{p})\mathcal{P}^\dagger = b_s^\dagger(-\boldsymbol{p}). \tag{20.1.5}$$

This follows by the same argument as for scalar fields.

20.2 Time Reversal

Under time reversal $x^\mu \mapsto -\bar{x}^\mu = -x_\mu$, and $\partial_\mu \mapsto -\bar{\partial}_\mu = -\partial^\mu$. The time reversal operator is antiunitary, we take a complex conjugate when we reverse time. Suppose that $\psi(x) \mapsto \psi_T(x) = T\psi(-\bar{x})$ for some spin matrix T. Then we require the gamma matrices to transform as

$$\gamma^\mu \mapsto T(\gamma^\mu)^* T^{-1} = \gamma_\mu. \tag{20.2.1}$$

As with parity this guarantees the invariance of the Dirac equation, we just replace ψ with ψ_T.

A solution to this is given by noting that in the Dirac representation γ^2 is imaginary and the other gamma matrices are real. The solution then is to take $T = -T^{-1} = \gamma^1\gamma^3$, since $(\gamma^1\gamma^3)^2 = -1$ and we have

$$-\gamma^1\gamma^3\gamma^0\gamma^1\gamma^3 = \gamma^0, \tag{20.2.2}$$
$$-\gamma^1\gamma^3\gamma^1\gamma^1\gamma^3 = -\gamma^1, \tag{20.2.3}$$
$$-\gamma^1\gamma^3\gamma^2\gamma^1\gamma^3 = -\gamma^2, \tag{20.2.4}$$
$$-\gamma^1\gamma^3\gamma^3\gamma^1\gamma^3 = -\gamma^3. \tag{20.2.5}$$

The adjoint spinor transforms as

$$\begin{aligned}\bar{\psi} = \psi^\dagger\gamma^0 \mapsto (\gamma^1\gamma^3\psi)^\dagger\gamma^0 &= \psi^\dagger(\gamma^3)^\dagger(\gamma^1)^\dagger\gamma^0 \\ &= \psi^\dagger\gamma^0(\gamma^3)^\dagger(\gamma^1)^\dagger = \bar{\psi}(\gamma^3)^\dagger(\gamma^1)^\dagger = \bar{\psi}T^{-1}.\end{aligned} \tag{20.2.6}$$

Then, we have

$$\bar{\psi}\psi \mapsto \bar{\psi}T^{-1}T\psi = \bar{\psi}\psi, \qquad \bar{\psi}\gamma^\mu\psi \mapsto \bar{\psi}T^{-1}(\gamma^\mu)^* T\psi = \bar{\psi}\gamma_\mu\psi. \tag{20.2.7}$$

Again, this is what we would expect for a scalar and a vector current. The Lagrangian is unchanged under time reversal.

The spinors transform as $u(\boldsymbol{p}, s) \mapsto Tu(\boldsymbol{p}, s)^* = u(-\boldsymbol{p}, -s)$, which can be shown using the Dirac representation. Alternatively, under time reversal $\boldsymbol{r}$ doesn't change but $\boldsymbol{p} \mapsto -\boldsymbol{p}$, so $\boldsymbol{r} \times \boldsymbol{p} \mapsto -\boldsymbol{r} \times \boldsymbol{p}$.

This corresponds to the creation operators transforming as

$$\mathcal{T}a_s^\dagger(\boldsymbol{p})\mathcal{T}^\dagger = a_{-s}^\dagger(-\boldsymbol{p}), \qquad \text{and} \qquad \mathcal{T}b_s^\dagger(\boldsymbol{p})\mathcal{T}^\dagger = b_{-s}^\dagger(-\boldsymbol{p}). \tag{20.2.8}$$

20.3 Charge Conjugation

Charge conjugation exchanges particles and antiparticles, just as for scalars. Both $\boldsymbol{p}$ and s are left unchanged, corresponding to the mapping of creation operators

$$\mathcal{C}a_s^\dagger(\boldsymbol{p})\mathcal{C}^\dagger = a_s^\dagger(\boldsymbol{p}), \qquad \text{and} \qquad \mathcal{C}b_s^\dagger(\boldsymbol{p})\mathcal{C}^\dagger = b_s^\dagger(\boldsymbol{p}). \tag{20.3.1}$$

The charge conjugated field is then

$$\psi_c = \mathcal{C}\psi\mathcal{C}^\dagger = \sum_s \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2\omega(\boldsymbol{p})}[a_s^\dagger(\boldsymbol{p})v(\boldsymbol{p}, s)\mathrm{e}^{ip\cdot x} + b_s(\boldsymbol{p})u(\boldsymbol{p}, s)\mathrm{e}^{-ip\cdot x}]. \tag{20.3.2}$$

We also have

$$\psi^* = (\psi^\dagger)^\mathsf{T} = \sum_s \int \frac{\mathrm{d}^3[p]}{(2\pi)^3}\frac{1}{2\omega(\boldsymbol{p})}[a_s^\dagger(\boldsymbol{p})u(\boldsymbol{p}, s)^*\mathrm{e}^{ip\cdot x} + b_s(\boldsymbol{p})v(\boldsymbol{p}, s)^*\mathrm{e}^{-ip\cdot x}].$$

(20.3.3)

Here † is the Hermitian adjoint in both the Hilbert space of states and the spinor space. The transpose, T, is only in spinor space. Under charge conjugation we have

$$\psi \mapsto \psi_c = c\psi^* \tag{20.3.4}$$

where c is some spinor matrix such that

$$cu_s^* = v_s, \qquad cv_s^* = u_s, \qquad \text{and} \qquad cc^* = 1. \tag{20.3.5}$$

For the Dirac equation to remain unchanged we need the gamma matrices to transform as

$$c(\gamma^\mu)^* c^{-1} = -\gamma^\mu. \tag{20.3.6}$$

A solution is given by $c = c^{-1} = c^* = i\gamma^2$.

In the Dirac representation

$$c = i\gamma^2 = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} \tag{20.3.7}$$

and then we have

$$cu(\boldsymbol{p}, s)^* = N\begin{pmatrix} \frac{-i\sigma^2\sigma^*\cdot\boldsymbol{p}}{E+m}\varphi^s \\ -i\sigma^2\varphi^s \end{pmatrix} = N\begin{pmatrix} \frac{-\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}(-i\sigma^2\varphi^s) \\ -i\sigma^2\varphi^s \end{pmatrix} = v(\boldsymbol{p}, s), \tag{20.3.8}$$

so long as $\chi_s = -i\sigma^2\varphi_s$, which is the case with our definition of χ_s, and explains the nonobvious choice of $\chi_1 = (0,1)^\mathsf{T}$ and $\chi_2 = (-1,0)^\mathsf{T}$. So, charge conjugation flips the basis spinors.

We can write these equations in terms of the Dirac adjoint:

$$\psi_c = i\gamma^2\psi^* = i\gamma^2\gamma^0\bar{\psi}^\mathsf{T} = C\bar{\psi}^\mathsf{T} \tag{20.3.9}$$

where this equation defines the spinor matrix $C = i\gamma^2\gamma^0$. Under charge conjugation we then have

$$\bar{\psi} \mapsto (i\gamma^2\psi^*)^\dagger\gamma^0 = \psi^\mathsf{T}(-i(\gamma^2)^*\gamma^0) = \psi^\mathsf{T}C = -\psi^\mathsf{T}C^{-1} \tag{20.3.10}$$

and so

$$\bar{\psi}\psi \mapsto -\psi^\mathsf{T}C^{-1}C\bar{\psi}^\mathsf{T} = -\psi^\mathsf{T}\bar{\psi}^\mathsf{T} \tag{20.3.11}$$

and this gives the same physics as $\bar{\psi}\psi$, which is what we would expect for a scalar. Normal ordering, since both fields are evaluated at the same point, $:\bar{\psi}\psi:$ will be invariant under charge conjugation. Similarly,

$$:\bar{\psi}\gamma^\mu\psi: \mapsto -:\psi^\mathsf{T}C^{-1}\gamma^\mu C\bar{\psi}^\mathsf{T}: = \psi^\mathsf{T}(\gamma^{\bar{\mu}})^\mathsf{T}\bar{\psi}^\mathsf{T} = -:\bar{\psi}\gamma^\mu\psi: \tag{20.3.12}$$

where the minus sign comes from commuting fermions. So, as expected, the current is reversed under charge conjugation. The Lagrangian is invariant, up to an overall unobservable total derivative:

$$\mathcal{L} = :\bar{\psi}(i\partial\!\!\!/ - m)\psi: \mapsto :\psi^\mathsf{T}(i(\gamma^\mu)^\mathsf{T}\partial_\mu - m)\bar{\psi}^\mathsf{T}: = :-i\partial_\mu\bar{\psi}\gamma^\mu\psi - m\bar{\psi}\psi:. \tag{20.3.13}$$

To conclude, under parity, time reversal and charge conjugation

$$P: \qquad \psi \mapsto P\psi, \qquad \bar{\psi} \mapsto \bar{\psi}P^{-1}, \qquad P = \gamma^0, \tag{20.3.14}$$

$$T: \qquad \psi \mapsto T\psi, \qquad \bar{\psi} \mapsto \bar{\psi}T^{-1}, \qquad T = \gamma^1\gamma^3, \tag{20.3.15}$$

$$C: \qquad \psi \mapsto C\bar{\psi}^\mathsf{T}, \qquad \bar{\psi} \mapsto -\psi C^{-1}, \qquad P = i\gamma^2\gamma^0. \tag{20.3.16}$$

Part VI

Quantising the Electromagnetic Field

Twenty-One

Quantising the Electromagnetic Field

The hardest of all, except the gravitational field.

Richard Ball

21.1 Classical Electrodynamics

R See *Classical Electrodynamics* for details.

21.1.1 Maxwell's Equations

The electromagnetic field strength tensor is

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{21.1.1}$$

where $A^\mu = (\varphi, \boldsymbol{A})$ is the electromagnetic four-potential. The Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu \tag{21.1.2}$$

gives the classical equations of motion

$$\partial_\mu F^{\mu\nu} = J^\nu, \tag{21.1.3}$$

which is just **Maxwell's equations** in a very compact form.

The four divergence of Maxwell's equation vanishes:

$$0 = \partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu J^\nu, \tag{21.1.4}$$

since $\partial_\nu \partial_\mu$ is symmetric in μ and ν, whereas $F^{\mu\nu}$ is antisymmetric. This equation, $\partial_\nu J^\nu = 0$, is a continuity equation, and tells us that J^ν is a conserved current.

This places a constraint on the field, A^μ. This makes quantising the electromagnetic field significantly more challenging than quantising scalar or fermionic fields. Fundamentally this is due to gauge invariance, a property not present for the scalar or fermion fields we have considered so far.

21.1.2 Gauge Invariance

The Lagrangian is **gauge invariant**, by this we mean that the transformation

$$A^\mu \mapsto A^\mu - \partial^\mu \chi \tag{21.1.5}$$

for some scalar field χ, with everywhere continuous third derivatives, leaves the Lagrangian, and hence the action and the physics, unchanged. Under this transformation we have

$$\begin{aligned} F^{\mu\nu} &\mapsto \partial^\mu(A^\nu - \partial^\nu\chi) - \partial^\nu(A^\mu - \partial^\mu\chi) && (21.1.6)\\ &= \partial^\mu A^\nu - \partial^\mu\partial^\nu - \partial^\nu A^\mu + \partial^\nu\partial^\mu\chi && (21.1.7)\\ &= \partial^\mu A^\nu - \partial^\nu A^\mu && (21.1.8)\\ &= F^{\mu\nu}. && (21.1.9) \end{aligned}$$

We also have

$$A_\mu J^\mu \mapsto A_\mu J^\mu - (\partial_\mu\chi)J^\mu. \tag{21.1.10}$$

So, the action, assuming that we integrate over some volume, V, such that the scalar field χ vanishes on the boundary of V, transforms as

$$\begin{aligned} S[A_\mu] &\mapsto S[A_\mu] - \int_V \mathrm{d}^4x\,(\partial_\mu\chi)J^\mu && (21.1.11)\\ &= S[A_\mu] - \int_V \mathrm{d}^4x\,[\partial_\mu(\chi J^\mu) - \chi\partial_\mu J^\mu] && (21.1.12)\\ &= S[A_\mu] - \int_{\partial V} \mathrm{d}S_\mu\,\chi J^\mu + \int_V \mathrm{d}^4x\chi\partial_\mu J^\mu && (21.1.13)\\ &= S[A_\mu] && (21.1.14) \end{aligned}$$

where the boundary integral over ∂V vanishes by assumption and the volume integral over V vanishes as $\partial_\mu J^\mu = 0$.

We can think of $\partial_\mu\chi$ as a zero mode of the equations of motion, which are

$$\begin{aligned} J^\mu &= \partial_\nu F^{\nu\mu} && (21.1.15)\\ &= \partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) && (21.1.16)\\ &= \partial_\nu\partial^\nu A^\mu - \partial_\nu\partial^\mu A^\nu && (21.1.17)\\ &= \partial^2 A^\mu - \partial^\nu\partial^\mu A_\nu && (21.1.18)\\ &= \eta^{\mu\nu}\partial^2 A_\nu - \partial^\nu\partial^\mu A_\nu && (21.1.19)\\ &= (\eta^{\mu\nu}\partial^2 - \partial^\nu\partial^\mu)A_\nu. && (21.1.20) \end{aligned}$$

However, we have

$$(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)\partial_\nu\chi = \partial^2\partial^\mu\chi - \partial^\mu\partial^2\chi = 0, \tag{21.1.21}$$

and so the operator $\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu$ has a zero eigenvalue, and so has no inverse. This means that the Green's function for the equations of motion, and hence the propagator, doesn't exist.

Gauge invariance, which requires current conservation, implies a constraint, which reduces the degrees of freedom. This means that at most three of the components of A^μ, which are the degrees of freedom, are independent.

21.1.3 Canonical Constraints

Consider the specific case when $\mu = 0$, then we have

$$(\eta^{0\nu}\partial^2 - \partial^\nu\partial^0)A_\nu = J^0. \tag{21.1.22}$$

Separating spatial and temporal components we have

$$\ddot{A}^0 - \nabla^2 A^0 - \ddot{A}^0 + \nabla\cdot\dot{\boldsymbol{A}} = J^0 \implies -\nabla^2 A^0 + \nabla\cdot\dot{\boldsymbol{A}} = J^0. \tag{21.1.23}$$

This means that A^0 is not a genuine dynamical variable as there is no $\dot{A}^0$ term in the equation of motion, and this in turn implies that there is no quadratic $(\dot{A}^0)^2$ term in the Lagrangian.

Now consider the Hamiltonian formalism. We have the canonical momenta

$$\pi^\mu(x) = \frac{\partial\mathcal{L}}{\partial\dot{A}_\mu} = -F^{0\mu}(x). \tag{21.1.24}$$

This means we have $\pi^0 = -F^{00} = 0$, as $F^{\mu\nu}$ is antisymmetric. There is then no momentum conjugate to A^0, further solidifying our argument that it is not a valid dynamical variable. For the spatial components we have

$$\pi^i = -F^{0i} = -\dot{A}^i + \partial^i A^0 = E^i \implies \boldsymbol{\pi} = -\dot{\boldsymbol{A}} + \nabla\varphi = \boldsymbol{E}. \tag{21.1.25}$$

Taking the divergence then gives

$$\nabla\cdot\boldsymbol{\pi} = \nabla^2 A^0 - \nabla\cdot\dot{\boldsymbol{A}} = J^0 \iff \nabla\cdot\boldsymbol{E} = \rho, \tag{21.1.26}$$

where $J^\mu = (\rho, \boldsymbol{J})$. This imposes an extra constraint on the π^i, they must satisfy $\partial_i\pi^i = J^0$, and so only two of the canonical momenta are independent. So, we loose another degree of freedom, meaning we only have two left.

21.1.4 Quantum Theory

In the quantum theory we expect A_μ to correspond to a massless spin 1 particle, the photon. We know that massless particles only have two degrees of freedom, their helicity, which is ± 1. So, of the four fields, A^μ, we only expect two of them to be physical, agreeing with our arguments above.

Note that this is a fundamental fact for massless particles, we can't fix it by choosing a different Lagrangian, or by changing what dynamical variables we consider. Essentially the arguments above don't work if we include a mass term, $-m^2A^2/2$, in the Lagrangian. A full proof of this is beyond this course, but a group theoretical argument in its favour is given in *Symmetries of Particles and Fields*. Essentially the representations of the Poincaré group corresponding to $p^2 = m^2 = 0$ but with nonzero momentum have only two degrees of freedom.

21.1.5 Approaches to Quantisation

There are multiple approaches that are commonly taken to quantise the electromagnetic field and get around these problems, the four most common are:

- develop the Hamiltonian dynamics of the constrained system and then use this to aid in our quantisation,

- ignore the problem and quantise away, introducing spurious degrees of freedom which we fix later by imposing constraints, called the **Gupta–Bleuler method**,
- modify the Lagrangian by adding in a term which breaks gauge invariance, this is particularly suited to the path integral formalism, so will be addressed in the other half of this course,
- adopt a gauge in which there are only two physical degrees of freedom, quantise, and then generalise to restore manifest Lorentz covariance and gauge invariance, this keeps the physics clear, but is somewhat cumbersome.

It is the last of these approaches which we shall adopt here. The motivation is similar to solving a problem in relativity where we compute a value in a specific frame, write it in a manifestly Lorentz invariant way, and then it must hold in all frames.

21.2 Quantisation in the Coulomb Gauge

Consider the gauge transformation $A^\mu \mapsto A^\mu - \partial^\mu \chi$. We've already seen that this doesn't change the physics. The spatial part of this transformation is $\nabla \cdot \boldsymbol{A} \mapsto \nabla \cdot \boldsymbol{A} - \nabla^2 \chi$. We can then choose χ such that $\nabla \cdot \boldsymbol{A} = 0$.

For a free field the equation of motion implies that $\nabla^2 \boldsymbol{A} = \boldsymbol{0}$, and so we can, given suitable boundary conditions, take $A^0 = 0$. Adding in this second condition gives the Coulomb gauge, which is explicitly given by setting

$$A^0 = 0, \qquad \text{and} \qquad \nabla \cdot \boldsymbol{A} = 0. \tag{21.2.1}$$

Note that these conditions are not Lorentz invariant, but they are consistent with the invariant Lorenz gauge condition, $\partial_\mu A^\mu = 0$. However, since all gauge choices are equivalent we can still use these conditions to get Lorentz invariant physics. Having two conditions, such as those above, fixes two degrees of freedom.

The free Maxwell equations in the Coulomb gauge are simply $\partial^2 \boldsymbol{A} = \boldsymbol{0}$. The solutions to these are plane waves, with some vector amplitude,

$$\boldsymbol{A} = \boldsymbol{\varepsilon}^r(\boldsymbol{k}) \mathrm{e}^{-ik\cdot x}. \tag{21.2.2}$$

We include the label $r = 1, 2$ here labelling the two different solutions we expect to find. We call $\boldsymbol{\varepsilon}^r$ the **polarisation**.

Substituting this ansatz into the equation of motion we have

$$0 = \partial^2 [\boldsymbol{\varepsilon}^r(\boldsymbol{k}) \mathrm{e}^{-ik\cdot x}] = -k^2 \boldsymbol{\varepsilon}^r(\boldsymbol{k}) \mathrm{e}^{-ik\cdot x}, \tag{21.2.3}$$

so, we must have $\boldsymbol{k}^2 = 0$. The condition that $\nabla \cdot \boldsymbol{A} = 0$ gives

$$0 = \nabla \cdot (\boldsymbol{\varepsilon}^r(\boldsymbol{k}) \mathrm{e}^{-ik\cdot x}) = i\boldsymbol{k} \cdot \boldsymbol{\varepsilon}^r(\boldsymbol{k}) \mathrm{e}^{-ik\cdot x}, \tag{21.2.4}$$

so we must have $\boldsymbol{k} \cdot \boldsymbol{\varepsilon}^r(\boldsymbol{k}) = 0$. This means that $\boldsymbol{\varepsilon}^r$ are transverse.

We're already not Lorentz invariant, so we may as well pick a frame. Choose one in which $\boldsymbol{k} = |\boldsymbol{k}|(0, 0, 1)$. There are two common choices for $\boldsymbol{\varepsilon}^r$, since the only requirement is that they are linearly independent and orthogonal to $\boldsymbol{\varepsilon}^r$. The first is $\boldsymbol{\varepsilon}^1 = (1, 0, 0)$ and $\boldsymbol{\varepsilon}^2 = (0, 1, 0)$. The second is circular polarisation, where

$\boldsymbol{\varepsilon}^{\pm} = (\boldsymbol{\varepsilon}^1 \pm i\boldsymbol{\varepsilon}^2)/\sqrt{2}$. This second choice is nice because $\boldsymbol{\varepsilon}^{\pm}$ correspond to helicity eigenstates, however the increased complexity of the polarisation vectors is a drawback. It doesn't matter which we choose, its simply a choice of basis, so we'll work generally and allow for either.

While we're making semi-arbitrary choices we may as well make another and define $\boldsymbol{\varepsilon}^3 = \boldsymbol{k}/|\boldsymbol{k}|$, a unit vector in the direction of $\boldsymbol{k}$. So, $\boldsymbol{\varepsilon}^3 = (0, 0, 1)$ in the chosen frame.

Choosing one of these bases one can easily check that

$$\boldsymbol{\varepsilon}^r \cdot \boldsymbol{\varepsilon}^{s*} = \delta^{rs}, \qquad \text{and} \qquad \sum_{r=1,2,3} \varepsilon_i^r \varepsilon_j^{r*} = \delta_{ij}, \tag{21.2.5}$$

where complex conjugation allows for complex polarisations, like the circularly polarised states. Rearranging the second of these we get

$$\sum_{r=1,2} \varepsilon_i^r \varepsilon_j^{r*} = \delta_{ij} - \varepsilon_i^3 \varepsilon_j^{3*} = \delta_{ij} - \frac{k_i k_j}{|\boldsymbol{k}|^2} =: P_{ij}(\boldsymbol{k}) \tag{21.2.6}$$

where this defines the **transverse projector**, $P_{ij}(\boldsymbol{k})$. This has the property

$$\begin{aligned} P_{ij}P_{jk} &= \left(\delta_{ij} - \frac{k_i k_j}{|\boldsymbol{k}|^2}\right)\left(\delta_{jk} - \frac{k_j k_k}{|\boldsymbol{k}|^2}\right) && (21.2.7)\\ &= \delta_{ij}\delta_{jk} - \delta_{ij}\frac{k_j k_k}{|\boldsymbol{k}|^2} - \frac{k_i k_j}{|\boldsymbol{k}|^2}\delta_{jk} + \frac{k_i k_j k_j k_k}{|\boldsymbol{k}|^4} && (21.2.8)\\ &= \delta_{ik} - \frac{k_i k_k}{|\boldsymbol{k}|^2} - \frac{k_i k_k}{|\boldsymbol{k}|^2} + \frac{k_i k_k}{|\boldsymbol{k}|^2} && (21.2.9)\\ &= \delta_{ik} - \frac{k_i k_k}{|\boldsymbol{k}|^2} && (21.2.10)\\ &= P_{ik}. && (21.2.11) \end{aligned}$$

Now expand the free electromagnetic field in terms of the complete set of Coulomb gauge plane wave solutions. Since the field is real we have a term and its Hermitian conjugate:

$$\boldsymbol{A}(x) = \sum_{r=1,2} \int đk[\boldsymbol{\varepsilon}_r(\boldsymbol{k}) a_r(\boldsymbol{k} \mathrm{e}^{-ik\cdot x}) + \boldsymbol{\varepsilon}_r^*(\boldsymbol{k}) a_r^\dagger(\boldsymbol{k}) \mathrm{e}^{ik\cdot x}] \tag{21.2.12}$$

where we have $k^2 = k_0^2 - |\boldsymbol{k}|^2 = 0$ and $k_0 = \omega(\boldsymbol{k})$. Note that whether the labels r are up or down is not important, and so here we move them to make room for the complex conjugate.

To quantise we assume the usual commutation relations,

$$[a_r(\boldsymbol{k}), a_s^\dagger(\boldsymbol{k}')] = \delta_{rs} đ(\boldsymbol{k} - \boldsymbol{k}'), \tag{21.2.13}$$

with all other commutators vanishing. We therefore expect that $a_r^\dagger(\boldsymbol{k})$ creates a photon with momentum $\boldsymbol{k}$ and energy $\omega(\boldsymbol{k}) = \sqrt{\boldsymbol{k}^2} = |\boldsymbol{k}|$, since $m = 0$.

In the Coulomb gauge the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\dot{A}_i^2 - \frac{1}{2}(\partial_i A_j)^2. \tag{21.2.14}$$

The Hamiltonian density is then

$$\mathcal{H} = \frac{1}{2}\dot{A}_i^2 + \frac{1}{2}(\partial_i A_j)^2. \tag{21.2.15}$$

The Hamiltonian is then given by

$$H = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x} : \dot{\boldsymbol{A}}^2 : \tag{21.2.16}$$

$$= \sum_{r=1,2} \int đk\, \omega(\boldsymbol{k}) a_r^\dagger(\boldsymbol{k}) a_r(\boldsymbol{k}). \tag{21.2.17}$$

As usual we can write the field as a positive and negative energy component, $\boldsymbol{A} = \boldsymbol{A}_+ + \boldsymbol{A}_-$, and compute the **covariant commutation relations**:

$$iD_+^{ij}(x-y) \coloneqq [A_+^i(x), A_-^j(y)] \tag{21.2.18}$$

$$= \sum_{r,r'=1,2} \int đk \int đk'\, \varepsilon_r^i(\boldsymbol{k}) \varepsilon_{r'}^j(\boldsymbol{k}')^* [a_r(\boldsymbol{k}), a_{r'}^\dagger(\boldsymbol{k}')] \mathrm{e}^{-ik\cdot x + ik'\cdot y} \tag{21.2.19}$$

$$= \sum_{r,r'=1,2} \int đk \int đk'\, \varepsilon_r^i(\boldsymbol{k}) \varepsilon_{r'}^j(\boldsymbol{k}')^* \delta_{rr'} \delta(\boldsymbol{k}-\boldsymbol{k}') \mathrm{e}^{-ik\cdot x + ik'\cdot y} \tag{21.2.20}$$

$$= \sum_{r=1,2} \int đk\, \varepsilon_r^i(\boldsymbol{k}) \varepsilon_r^j(\boldsymbol{k})^* \mathrm{e}^{-ik(x-y)} \tag{21.2.21}$$

$$= \int đk P^{ij}(\boldsymbol{k}) \mathrm{e}^{-ik\cdot(x-y)}. \tag{21.2.22}$$

Now define

$$iD_-^{ij}(x-y) = [A_-^i(x), A_+^j(y)] = -D_+^{ij}(y-x). \tag{21.2.23}$$

We then define the **Feynman propagator**,

$$iD_{\mathrm{F}}^{ij}(x-y) \coloneqq \langle 0|\,\mathrm{T}[A^i(x)A^j(y)]|0\rangle = i\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{P^{ij}(\boldsymbol{k})}{k^2 + i\varepsilon} \mathrm{e}^{-ik\cdot(x-y)}. \tag{21.2.24}$$

This isn't covariant, but it does contain all the relevant physics.

Since $\nabla \exp[-ik\cdot(x-y)] = \boldsymbol{k}\exp[-ik\cdot(x-y)]$ we can, at least formally, rewrite the projector as

$$P^{ij}(\boldsymbol{k})\Delta_{\mathrm{F}}(x-y) = \left(\delta^{ij} - \frac{\nabla^i\nabla^j}{\nabla^2}\right)\Delta_{\mathrm{F}}(x-y) \tag{21.2.25}$$

where Δ_{F} is the Feynman propagator for a massless scalar field. The Feynman propagator for photons is then given by

$$D_{\mathrm{F}}^{ij}(x-y) = \left(\delta^{ij} - \frac{\nabla^i\nabla^j}{\nabla^2}\right)\Delta_{\mathrm{F}}(x-y). \tag{21.2.26}$$

21.3 Equal Time Commutation Relations

The best thing I can say about this is it's unpleasant.

Richard Ball

We can derive the **equal time commutation relations** from the covariant commutation relations by noting that $\pi^i = -\dot{A}^i$, and so

$$[A^j(t,\boldsymbol{y}),\pi^i(t,\boldsymbol{x})] = [\dot{A}^i(t,\boldsymbol{x}),A^j(t,\boldsymbol{y})] \tag{21.3.1}$$

$$= \frac{\partial}{\partial x_0} iD^{ij}(x-y)\Big|_{x^0=y^0} \tag{21.3.2}$$

$$= \frac{\partial}{\partial x_0}\int \frac{\mathrm{d}^4k}{(2\pi)^k}\frac{1}{2\omega(\boldsymbol{k})}P^{ij}(\boldsymbol{k})\mathrm{e}^{-ik\cdot(x-y)}\Big|_{x^0=y^0} \tag{21.3.3}$$

$$= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2\omega(\boldsymbol{k})}P^{ij}(\boldsymbol{k})(-ik^0)\mathrm{e}^{-ik\cdot(x-y)}\Big|_{x^0=y^0} \tag{21.3.4}$$

$$= -i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2}P^{ij}(\boldsymbol{k})\mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \tag{21.3.5}$$

$$= -\frac{i}{2}\left(\delta^{ij}-\frac{\nabla^i\nabla^j}{\nabla^2}\right)\int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \tag{21.3.6}$$

$$= -\frac{i}{2}\left(\delta^{ij}-\frac{\nabla^i\nabla^j}{\nabla^2}\right)\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{21.3.7}$$

shouldn't have factor of i/2 Here we've used $k^0 = \omega(\boldsymbol{k})$, which follows since $\omega(\boldsymbol{k}) = \sqrt{|\boldsymbol{k}|^2} = |\boldsymbol{k}|$, as $m = 0$, and $k^2 = (k^0)^2 - |\boldsymbol{k}|^2 = 0$, so $k^0 = |\boldsymbol{k}| = \omega(\boldsymbol{k})$.

This result is not very nice, its not even a real function, but a distribution. It should be understood formally as simply a rewriting of Equation (21.3.5).

21.4 Recovering Covariance

Define a timelike unit four-vector, n, such that in the frame in which we define the Coulomb gauge $n^\mu = (1,\boldsymbol{0})$. Then we can write the Coulomb gauge conditions, $A^0 = 0$ and $\nabla\cdot\boldsymbol{A} = 0$, as

$$n^\mu A_\mu = 0, \qquad \text{and} \qquad \partial^\mu A_\mu = 0. \tag{21.4.1}$$

The first is obvious, since the definition of n^μ gives $n^\mu A_\mu = A^0$, the second then follows since $\partial^\mu A_\mu = \partial^0 A_0 - \partial^i A_i = -\partial^i A_i = -\nabla\cdot\boldsymbol{A}$ since A^0 vanishes by the first condition. This rewriting of the conditions, although fairly trivial, allows us to use them in an frame, since they are now manifestly covariant. The results may still be dependent on n in this new frame however, which means we still need to know how the new frame relates to the one used to define the Coulomb gauge in order to know what n is in the new frame.

Since we have $k^\mu = (|\boldsymbol{k}|,\boldsymbol{k})$ in the frame defining the Coulomb gauge we have $k^2 = 0$ and $k\cdot n = k^0 = |\boldsymbol{k}|$. We then have

$$\frac{k^\mu}{k\cdot n} - n^\mu = \left(\frac{|\boldsymbol{k}|}{|\boldsymbol{k}|}-1,\frac{\boldsymbol{k}}{|\boldsymbol{k}^2|}\right) = \left(0,\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\right). \tag{21.4.2}$$

We can extend the polarisation vectors to be four-vectors by defining the four-vector $\varepsilon_r^\mu = (0, \boldsymbol{\varepsilon}_r)$ in the frame defining the Coulomb gauge. We then have $n \cdot \varepsilon = 1 \cdot 0 - \mathbf{0} \cdot \boldsymbol{\varepsilon}_r = 0$, as well as $k \cdot \boldsymbol{\varepsilon}_r = |\boldsymbol{k}| \cdot 0 - \boldsymbol{k} \cdot \boldsymbol{\varepsilon}_r = 0$, since $\boldsymbol{k} \cdot \boldsymbol{\varepsilon}_r = 0$. We also have $\boldsymbol{\varepsilon}_r \cdot \boldsymbol{\varepsilon}_s^* = \delta_{rs}$, and so $\varepsilon_r \cdot \varepsilon_s^* = -\delta_{rs}$.

We can now expand the field in a manifestly covariant mode expansion:

$$A^\mu(x) = \sum_{r=1,2} \int \text{đ}k [\varepsilon_r^\mu(\boldsymbol{k}) a_r(\boldsymbol{k}) \mathrm{e}^{-ik\cdot x} + \varepsilon_r^\mu(\boldsymbol{k})^* a_r^\dagger(\boldsymbol{k}) \mathrm{e}^{ik\cdot x}]. \tag{21.4.3}$$

In the frame defining the Coulomb gauge we have

$$-\eta^{\mu\nu} + n^\mu n^\nu = \begin{cases} 0 & \text{if } \mu = 0 \text{ or } \nu = 0, \\ \delta^{ij} & \text{if } \mu = i = 1,2,3 \text{ and } \nu = j = 1,2,3. \end{cases} \tag{21.4.4}$$

We also have

$$\left(\frac{k^\mu}{k\cdot n} - n^\mu\right)\left(\frac{k^\nu}{k\cdot n} - n^\nu\right) = \begin{cases} 0 & \text{if } \mu = 0 \text{ or } \nu = 0, \\ k^i k^j/|\boldsymbol{k}|^2 & \text{if } \mu = i = 1,2,3 \text{ and } \nu = j = 1,2,3. \end{cases} \tag{21.4.5}$$

Recall that the polarisation vectors have the completeness relation

$$\sum_{r=1,2} \varepsilon_r^i \varepsilon_r^{j*} = \delta^{ij} - \frac{k^i k^j}{|\boldsymbol{k}|^2}. \tag{21.4.6}$$

We can then use the results we've just derived to replace δ^{ij} with $-\eta^{\mu\nu} + n^\mu n^\nu$ and $k^i k^j/|\boldsymbol{k}|$ with $(k^\mu/(k\cdot n) - n^\mu)(k^\nu/(k\cdot n) - n^\nu)$, giving the manifestly covariant

$$\sum_{r=1,2} \varepsilon_r^\mu \varepsilon_r^{\nu*} = -\eta^{\mu\nu} + n^\mu n^\nu - \left(\frac{k^\mu}{k\cdot n} - n^\mu\right)\left(\frac{k^\nu}{k\cdot n} - n^\nu\right) \tag{21.4.7}$$

$$= -\eta^{\mu\nu} + n^\mu n^\nu - \frac{k^\mu k^\nu}{(k\cdot n)^2} + \frac{k^\mu n^\nu + n^\mu k^\nu}{k\cdot n} - n^\mu n^\nu \tag{21.4.8}$$

$$= -\eta^{\mu\nu} - \frac{k^\mu k^\nu}{(k\cdot n)^2} + \frac{k^\mu n^\nu + n^\mu k^\nu}{k\cdot n} \tag{21.4.9}$$

$$=: P^{\mu\nu}(k) \tag{21.4.10}$$

where we now define the manifestly covariant quantity $P^{\mu\nu}(k)$.

We can further simplify this by defining the null four-vector $\bar{k}^\mu = (|\boldsymbol{k}|, -\boldsymbol{k})$, which is such that $\bar{k}^2 = 0$, and $k \cdot \bar{k} = 2|\boldsymbol{k}| = 2(k\cdot n)^2$, so we can write

$$\frac{k^\mu + \bar{k}^\mu}{2k\cdot n} = \frac{(2|\boldsymbol{k}|, \mathbf{0})}{2|\boldsymbol{k}|} = n^\mu \tag{21.4.11}$$

and so

$$P^{\mu\nu}(k) = -\eta^{\mu\nu} - \frac{k^\mu k^\nu}{(k\cdot n)^2} + \frac{n^\mu k^\nu + k^\mu n^\nu}{k\cdot n} \tag{21.4.12}$$

$$= -\eta^{\mu\nu} - \frac{k^\mu k^\nu}{(k\cdot n)^2} + \frac{1}{k\cdot n}\left[\frac{k^\mu + \bar{k}^\mu}{2k\cdot n}k^\nu + k^\mu\frac{k^\nu+\bar{k}^\nu}{2k\cdot n}\right] \tag{21.4.13}$$

$$= -\eta^{\mu\nu} - \frac{k^\mu k^\nu}{(k\cdot n)^2} + \frac{1}{2(k\cdot n)^2}(k^\mu k^\nu + \bar{k}^\mu k^\nu + k^\mu k^\nu + k^\mu\bar{k}^\nu)$$

$$= -\eta^{\mu\nu} + \frac{1}{2(k\cdot n)^2}(\bar{k}^\mu k^\nu + k^\mu\bar{k}^\nu) \tag{21.4.14}$$

$$= -\eta^{\mu\nu} + \frac{k^\mu\bar{k}^\nu + \bar{k}^\mu k^\nu}{k\cdot\bar{k}}. \tag{21.4.15}$$

We can now check that, so long as the photon is on-shell, so $k^2 = \bar{k}^2 = 0$, $P^{\mu\nu}$ is a projection operator, that is $P^{\mu\alpha}P_\alpha{}^\nu \propto P^{\mu\nu}$:

$$P^{\mu\alpha}(k)P_\alpha{}^\nu(k) = \left(-\eta^{\mu\alpha} + \frac{k^\mu\bar{k}^\alpha + \bar{k}^\mu k^\alpha}{k\cdot\bar{k}}\right)\left(-\delta_\alpha{}^\nu + \frac{k_\alpha\bar{k}^\nu + \bar{k}_\alpha k^\nu}{k\cdot\bar{k}}\right) \tag{21.4.16}$$

$$= \eta^{\mu\nu} - \frac{k^\mu\bar{k}^\nu + \bar{k}^\mu k^\nu}{k\cdot\bar{k}} - \frac{k^\mu\bar{k}^\nu + \bar{k}^\mu k^\nu}{k\cdot\bar{k}} \tag{21.4.17}$$

$$+ \frac{1}{(k\cdot\bar{k})^2}(k^\mu\bar{k}^\alpha + \bar{k}^\mu k^\alpha)(k_\alpha\bar{k}^\nu + \bar{k}_\alpha k^\nu) \tag{21.4.18}$$

$$\tag{21.4.19}$$

now using

$$(k^\mu\bar{k}^\alpha + \bar{k}^\mu k^\alpha)(k_\alpha\bar{k}^\nu + \bar{k}_\alpha k^\nu) = k^\mu\bar{k}^\alpha k_\alpha\bar{k}^\nu + k^\mu\bar{k}^\alpha\bar{k}_\alpha k^\nu$$
$$+ \bar{k}^\mu k^\alpha k_\alpha\bar{k}^\nu + \bar{k}^\mu k^\alpha\bar{k}_\alpha k^\nu \tag{21.4.20}$$

$$= k^\mu\bar{k}^\nu k\cdot\bar{k} + \bar{k}^\mu k^\nu k\cdot\bar{k} \tag{21.4.21}$$

we have

$$P^{\mu\alpha}(k)P_\alpha{}^\nu(k) = \eta^{\mu\nu} - 2\frac{k^\mu\bar{k}^\nu + \bar{k}^\mu k^\nu}{k\cdot\bar{k}} + \frac{k^\mu\bar{k}^\nu + \bar{k}^\mu k^\nu}{k\cdot\bar{k}} \tag{21.4.22}$$

$$= -P^{\mu\nu}(k). \tag{21.4.23}$$

The operator $P^{\mu\nu}(k)$ has two independent zero modes, since we have

$$P^{\mu\nu}(k)k_\nu = \left(-\eta^{\mu\nu} + \frac{k^\mu\bar{k}^\nu + \bar{k}^\mu k^\nu}{k\cdot\bar{k}^\nu}\right)k_\nu \tag{21.4.24}$$

$$= -k^\mu + \frac{k^\mu(k\cdot\bar{k}) + \bar{k}^\mu k^2}{k\cdot\bar{k}} \tag{21.4.25}$$

$$= 0, \tag{21.4.26}$$

since $k^2 = 0$. Similarly,

$$P^{\mu\nu}(k)\bar{k}_\nu = 0. \tag{21.4.27}$$

We can then write the propagator in a manifestly covariant way:

$$iD_{\rm F}^{\mu\nu}(x-y) = \langle 0|\,\mathrm{T}[A^\mu(x)A^\nu(y)]|0\rangle = i\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{P^{\mu\nu}(k)}{k^2+i\varepsilon}\mathrm{e}^{-ik\cdot(x-y)}. \tag{21.4.28}$$

Note that $P^{\mu\nu}$ are only projection operators if k is on-shell, that is $k^2 = 0$. While this isn't the case in the integral it is true at the pole at $k^2 = -i\varepsilon$ in the limit $\varepsilon \to 0$, and since this is the only contribution to the integral we can treat them as projection operators inside the integral. This doesn't hold when we consider internal photons which can be off-shell. These off-shell virtual photons will have extra, non-physical, degrees of freedom which real photons don't have.

Note that despite having rewritten everything in a covariant way we are still in the Coulomb gauge. The choice of n^μ or $\bar{k}^\mu$ picks out a particular direction, so we still need to know about the initial frame in order to transform n^μ and $\bar{k}^\mu$ into the new frame. We will look for physical results independent of this specific direction, and hence independent of the choice of gauge.

21.5 General Linear Gauge

> *The only thing that's changed is that $\bar{k}$, which I don't know what it is, is now an α, which I don't know what it is. But it is pretty general.*
>
> Richard Ball

When we have results in a particular gauge we can move to a different gauge by a gauge transformation $A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) - \partial_\mu\chi(x)$. We now consider moving from the Coulomb gauge to a general linear gauge, that is a gauge defined by linear gauge fixing coefficients, i.e., we define the gauge by $L_\mu A^\mu = 0$ where L_μ is some linear operator. For example, the Coulomb gauge is a linear gauge as we can define it by requiring $n_\mu A^\mu = 0$ and $\partial_\mu A^\mu = 0$. In the new gauge we have the gauge condition $L'_\mu A'^\mu = 0$. Substituting in the transformation law we have

$$0 = L'_\mu A'^\mu = L'_\mu(A^\mu - \partial^\mu\chi) \implies L'_\mu\partial^\mu\chi = L'_\mu A^\mu. \tag{21.5.1}$$

This means that χ is always some linear function of A.

With this knowledge we can expand χ in a mode expansion in a similar way to A:

$$\chi(x) = \sum_{r=1,2}\int \text{đ}k\,[c_r(\boldsymbol{k})a_r(\boldsymbol{k})\mathrm{e}^{-ik\cdot x} + c_r^*(\boldsymbol{k})a_r^\dagger(\boldsymbol{k})\mathrm{e}^{ik\cdot x}] \tag{21.5.2}$$

for some, potentially complex, function c_r. Then the mode expansion for the gauge transformed field is

$$\begin{aligned} A'^\mu(x) &= A^\mu(x) - \partial^\mu\chi(x) && (21.5.3)\\ &= \sum_{r=1,2}\int \text{đ}k\,[(\varepsilon_r^\mu(\boldsymbol{k}) + ik^\mu c_r(\boldsymbol{k}))a_r(\boldsymbol{k})\mathrm{e}^{-ik\cdot x} && (21.5.4)\\ &\qquad + (\varepsilon_r^\mu(\boldsymbol{k})^* - ik^\mu c_r^*(\boldsymbol{k}))a_r^\dagger(\boldsymbol{k})\mathrm{e}^{ik\cdot x}] && (21.5.5)\end{aligned}$$

where the factors of $\pm ik^\mu$ come from performing the derivatives.

We can interpret this as the gauge transformation acting on the polarisation vectors according to

$$\varepsilon_r^\mu(\boldsymbol{k}) \mapsto \varepsilon_r^\mu(\boldsymbol{k}) + ik^\mu c_r(\boldsymbol{k}). \tag{21.5.6}$$

This extends to a transformation of the projection operators,

$$P^{\mu\nu}(k) = \sum_r \varepsilon_r^\mu \varepsilon_r^{\nu *} \mapsto P^{\mu\nu}(k) + ik^\mu \sum_r c_r \varepsilon_r^{\nu *} - ik^\nu \sum_r c_r^* \varepsilon_r^\mu + k^\mu k^\nu \sum_r c_r c_r^*$$
$$= -\eta^{\mu\nu} + k^\mu \alpha^\nu + \alpha^{\mu *} k^\nu \tag{21.5.7}$$

where

$$\alpha^\mu = \frac{\bar{k}^\mu}{k \cdot \bar{k}} + i \sum_r c_r \varepsilon_r^{\mu *} + \frac{1}{2} k^\mu \sum_r |c_r|^2. \tag{21.5.8}$$

So, in a general linear gauge $P^{\mu\nu}$ changes from the Coulomb gauge by replacing $\bar{k}^\mu/(k \cdot \bar{k})$ with α^μ.

In general α is complex. For on-shell photons, $k^2 = 0$, we have $\alpha \cdot k = 1$ and $|\alpha|^2 = 2 \sum) r |c_r|^2 \geq 0$, so α is time-like or light-like. We still have $P^{\mu\nu} k_\nu = 0$, but $P^{\mu\nu} \alpha_\nu \neq 0$ in general. In these more general gauges $P^{\mu\nu}$ is actually no longer a projection operator, although we still call it a projection operator.

The physics must still work in these general gauges, yet we have these extra unphysical modes for α. We will soon see how it is that this can all work out.

Twenty-Two

Feynman Rules for Photons

It's the projection operator that isn't a projection operator!

Richard Ball

22.1 The Feynman Rules

We consider an external, and hence on-shell, photon with momentum $\boldsymbol{k}$ and polarisation r. This corresponds to the state

$$|\boldsymbol{k}, r\rangle = a_r^\dagger(\boldsymbol{k})|0\rangle. \tag{22.1.1}$$

We can then compute $A_+^\mu(x)|\boldsymbol{k}, r\rangle$, where

$$A_+^\mu(x) = \sum_{r=1,2} \int \text{đ}k\, \varepsilon_r^\mu(\boldsymbol{k}) a_r(\boldsymbol{k}) \mathrm{e}^{-ik\cdot x} \tag{22.1.2}$$

is the positive energy part of the field A^μ. We then have

$$A_+^\mu(x)|\boldsymbol{k}, r\rangle = \sum_{r'=1,2} \int \text{đ}k'\, \varepsilon_r^\mu(\boldsymbol{k}) a_{r'}(\boldsymbol{k}') \mathrm{e}^{-ik'\cdot x} a_r^\dagger(\boldsymbol{k})|0\rangle \tag{22.1.3}$$

$$= \sum_{r'=1,2} \int \text{đ}k'\, \varepsilon_r^\mu(\boldsymbol{k}) \{[a_{r'}(\boldsymbol{k}), a_r^\dagger(\boldsymbol{k})] + a_r^\dagger(\boldsymbol{k}) a_{r'}(\boldsymbol{k})\} \mathrm{e}^{-ik'\cdot x}|0\rangle \tag{22.1.4}$$

$$= \sum_{r'=1,2} \int \text{đ}k'\, \varepsilon_r^\mu(\boldsymbol{k}) \delta_{rr'} \text{đ}(\boldsymbol{k} - \boldsymbol{k}') \mathrm{e}^{-ik'\cdot x}|0\rangle \tag{22.1.5}$$

$$= \varepsilon_r^\mu(\boldsymbol{k}) \mathrm{e}^{-ik\cdot x}|0\rangle \tag{22.1.6}$$

where we've used $a_{r'}(\boldsymbol{k})|0\rangle = 0$. This is similar to the results for the scalar field and spinor field, but we have a polarisation vector where we had a spinor for the spinor field and nothing for the scalar field.

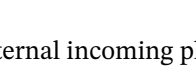
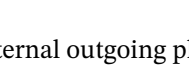
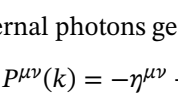
The **Wick contraction** for the photon field is then given by

$$\overbrace{A^\mu(x)A^\nu(y)} = \langle 0|\,\mathrm{T}[A^\mu(x)A^\nu(y)]|0\rangle \tag{22.1.7}$$

$$= iD_{\mathrm{F}}^{\mu\nu}(x - y) \tag{22.1.8}$$

$$= \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{iP^{\mu\nu}(k)}{k^2 + i\varepsilon} \mathrm{e}^{-ik\cdot(x-y)}. \tag{22.1.9}$$

The Feynman rules for photons are then

- External incoming photons get a factor of $\varepsilon_r^\mu(\boldsymbol{k})$: r  μ.
- External outgoing photons get a factor of $\varepsilon_r^\mu(\boldsymbol{k})^*$: μ  r.
- Internal photons get a factor of $iP^{\mu\nu}(k)/(k^2+i\varepsilon)$: μ  ν.

Note that $P^{\mu\nu}(k) = -\eta^{\mu\nu} + k^\mu\alpha^\nu + \alpha^{\mu*}k^\nu$ and $\varepsilon_r^\mu(\boldsymbol{k})$ and α^μ are gauge dependent. However, this gauge dependence must fall out before we compute the final amplitude, since the physics cannot depend on the gauge.

22.2 Ward Identities

The interaction Hamiltonian is $A^\mu J_\mu$. For gauge invariance J_μ must be a conserved current. And gauge invariance is needed to reduce the four degrees of freedom in A^μ to the two physical degrees of freedom. Note that any instance of J_μ must be contracted with A^μ in order to have a gauge invariant action.

In momentum space the conserved current is

$$\tilde{J}_\mu(k) = \int \mathrm{d}^4x\, J_\mu(x)\, \mathrm{e}^{ik\cdot x}. \tag{22.2.1}$$

We then have

$$\begin{aligned} 0 &= \int \mathrm{d}^4x\, [\partial^\mu J_\mu(x)]\mathrm{e}^{ik\cdot x} && (22.2.2)\\ &= \int \mathrm{d}^4x\, \partial^\mu[J_\mu(x)\mathrm{e}^{ik\cdot x}] - \int \mathrm{d}^4x\, J_\mu(x)\partial^\mu \mathrm{e}^{ik\cdot x} && (22.2.3)\\ &= \text{surface term} - \int \mathrm{d}^4x\, ik^\mu J_\mu(x)\mathrm{e}^{ik\cdot x} && (22.2.4)\\ &= -ik^\mu \tilde{J}_\mu(x) && (22.2.5) \end{aligned}$$

since the surface term doesn't contribute. So, the condition for conservation of $J^\mu(x)$ becomes $k^\mu\tilde{J}_\mu = 0$ in momentum space.

Now consider some amplitude we wish to calculate which happens to have a single photon in the initial state. This amplitude is given by

$$\mathcal{M} = \langle\cdots|\,\mathrm{T}\left[\cdots(-i)\textstyle\int \mathrm{d}^4x J_\mu(x)A^\mu(x)\right]|k,r;\cdots\rangle \tag{22.2.6}$$

where $\cdots$ represent all the other particles in the initial and final states, and the terms in the time ordering corresponding to them, including contractions of internal particles. The only term we need explicitly in the time ordered product is the term destroying the incoming photon. The $A^\mu(x)$ in this term will *not* be Wick contracted, since it corresponds to an external state.

We can rewrite this with the Feynman rules for an incoming photon:

$$\begin{aligned} \mathcal{M} &= \langle\cdots|\,\mathrm{T}\left[\cdots(-i)\textstyle\int \mathrm{d}^4x J_\mu(x)\varepsilon_r^\mu(\boldsymbol{k})\mathrm{e}^{-ik\cdot x}\right]|k,r;\cdots\rangle && (22.2.7)\\ &= \langle\cdots|\,\mathrm{T}\left[\cdots(-i)\tilde{J}_\mu(k)\varepsilon_r^\mu(\boldsymbol{k})\right]|k,r;\cdots\rangle. && (22.2.8) \end{aligned}$$

Under a gauge transformation we have $\varepsilon_r^\mu(\boldsymbol{k}) \mapsto \varepsilon_r^\mu(\boldsymbol{k}) + ik^\mu c_r(\boldsymbol{k})$, but we know that $k^\mu \tilde{J}_\mu = 0$ for a conserved current, so this gauge transformation doesn't actually change the amplitude. So the amplitude is gauge independent for an external incoming photon. The same argument also shows that the amplitude is gauge independent for an external outgoing photon.

Often we want to factor out the gauge dependent term, writing

$$\mathcal{M} = \varepsilon_r^\mu(\boldsymbol{k})\mathcal{M}_\mu, \tag{22.2.9}$$

where $\mathcal{M}_\mu$ is the **reduced amplitude**. Gauge independence then can be written in the form

$$k^\mu \mathcal{M}_\mu = 0, \tag{22.2.10}$$

which is called the **Ward identity**.

22.3 Polarisation Sums

Often we don't measure the polarisation of the incoming or outgoing photons. In this case we average over incoming photons and sum over outgoing photons. Suppose we have a single external photon. Then we can write the amplitude as $\mathcal{M} = \varepsilon_r^\mu(\boldsymbol{k})\mathcal{M}_\mu$. Then we have

$$\begin{aligned}
\sum_{r=1,2} |\mathcal{M}|^2 &= \mathcal{M}^\mu \mathcal{M}^{\nu *} \sum_{r=1,2} \varepsilon_\mu^r \varepsilon_\nu^{r*} && (22.3.1)\\
&= \mathcal{M}^\mu \mathcal{M}^{\nu *} P_{\mu\nu}(k) && (22.3.2)\\
&= \mathcal{M}^\mu \mathcal{M}^{\nu *}(-\eta_{\mu\nu} + k_\mu \alpha_\nu + \alpha_\mu^* k_\nu) && (22.3.3)\\
&= -\mathcal{M}^\mu \mathcal{M}_\mu^* && (22.3.4)
\end{aligned}$$

where we've used $\mathcal{M}^\mu k_\mu = 0$ in the last line.

What this means is that in practice we can use the rule

$$\sum_{r=1,2} \varepsilon_\mu^r \varepsilon_\nu^{r*} \to -\eta_{\mu\nu} \tag{22.3.5}$$

to evaluate polarisation sums.

Now consider

$$-|\mathcal{M}|^2 = -\mathcal{M}^\mu \mathcal{M}_\mu^* = -|\mathcal{M}^0|^2 + |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 + |\mathcal{M}^3|^2. \tag{22.3.6}$$

Despite the minus sign this quantity is actually positive definite. We can show this by taking $k^\mu = (k, 0, 0, k)$, and then ew have

$$0 = \mathcal{M}^\mu k_\mu = \mathcal{M}^0 k - \mathcal{M}^3 k \implies \mathcal{M}^0 = \mathcal{M}^3. \tag{22.3.7}$$

This means that the contribution of the nonphysical longitudinal and time-like polarisations cancel out and we are left with just $\mathcal{M}^1$ and $\mathcal{M}^2$ terms, corresponding to the physical, transverse, polarisation states.

22.4 Internal Photons

Consider a process with a single internal photon. This photon must be created and then destroyed, so we expect a contraction in the time ordered product between the fields creating and annihilating the photon. This means the amplitude will be of the form

$$\mathcal{M} = \langle \cdots | \, \mathrm{T}\big[\cdots \int \mathrm{d}^4x \, \overbrace{A^\mu(x) J_\mu(x) \int \mathrm{d}^4y \, A^\nu(y)} J_\nu(y) \big] | \cdots \rangle \tag{22.4.1}$$

where the dots represent the external particles, the fields creating and annihilating them, and any other contractions of fields corresponding to internal particles.

We can replace the contraction with the propagator, giving

$$\begin{aligned}\mathcal{M} &= \langle \cdots | \, \mathrm{T}\big[\cdots \int \mathrm{d}^4x \int \mathrm{d}^4y \, iD_\mathrm{F}^{\mu\nu}(x-y) J_\mu(x) J_\nu(y) \big] | \cdots \rangle &(22.4.2)\\ &= \langle \cdots | \, \mathrm{T}\left[\cdots \int \mathrm{d}^4x \int \mathrm{d}^4y \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{iP^{\mu\nu}(k)}{k^2 + i\varepsilon} \mathrm{e}^{-ik\cdot(x-y)} J_\mu(x) J_\nu(y) \right] | \cdots \rangle \\ &= \langle \cdots | \, \mathrm{T}\left[\cdots \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{iP^{\mu\nu}(k)}{k^2 + i\varepsilon} \tilde{J}_\mu(-k) \tilde{J}_\nu(k) \right] | \cdots \rangle. &(22.4.3)\end{aligned}$$

Notice that $P^{\mu\nu}$ is gauge dependent.

We have that

$$\begin{aligned}P^{\mu\nu}(k) \tilde{J}_\mu(-k) \tilde{J}_\nu(k) &= (-\eta^{\mu\nu} + k^\mu \alpha^\nu + \alpha^{\mu *} k^\nu) \tilde{J}_\mu(k) \tilde{J}_\nu(-k) &(22.4.4)\\ &= -\tilde{J}^\mu(k) \tilde{J}_\mu(-k) &(22.4.5)\end{aligned}$$

where we've used $k^\mu \tilde{J}_\mu(k) = 0$, and so

$$\mathcal{M} = \langle \cdots | \, \mathrm{T}\left[\cdots \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{i}{k^2 + i\varepsilon} \tilde{J}_\mu(-k) \tilde{J}_\nu(k) \right] | \cdots \rangle, \tag{22.4.6}$$

so the gauge dependence drops out, as it must.

However, this "proof" only holds when k is on-shell, which is required for the $k^\mu \tilde{J}_\mu = 0$ step. This isn't too much of a problem as in the integral the only contribution is from the pole at $k^2 = -i\varepsilon$, where the photon is pretty much on-shell. In fact, one can show that the gauge dependence will always drop out. However, this requires us to consider the renormalisation of the divergences and makes use of the Ward–Takahashi identities which are beyond the scope of this course[1].

[1] See *Gauge Theories in Particle Physics*

In practice we don't need to use the full projection operators, and we can just simplify the Feynman rule for internal photons to a factor of

$$\mu \bullet\!\!\sim\!\!\overset{k}{\sim}\!\!\sim\!\!\bullet \, \nu = \frac{-i\eta_{\mu\nu}}{k^2 + i\varepsilon}. \tag{22.4.7}$$

There is a gauge where this holds exactly, i.e., $\alpha = 0$. This is called the **Feynman gauge**.

22.5 Covariant Gauges

It is possible to do the gauge fixing such that $\partial_\mu A^\mu = 0$ in multiple ways. We get a family of gauges. These gauges are more restricted than a general linear gauge. In these gauges the Feynman propagator is

$$iD_\mathrm{F}^{\mu\nu}(x-y) = \frac{iP^{\mu\nu}(k)}{k^2 + i\varepsilon} \tag{22.5.1}$$

with

$$P^{\mu\nu}(k) = -\eta^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2 + i\varepsilon}, \tag{22.5.2}$$

where ξ is the gauge parameter, which parametrises the gauges in this family. This corresponds to choosing to take α^μ to be proportional to $k^\mu/(k^2 + i\varepsilon)$.

Looking for gauge independent results is then as simply as looking for our final results to be independent of ξ. Often people work in this general family of gauges, leaving ξ as a free parameter, and then use the lack of ξ dependence at the end of a calculation as a check that nothing's gone wrong. The two gauges we have mentioned so far by name are part of this gauge family, in particular $\xi = 1$ gives the Feynman gauge and $\xi = 0$ gives the Lorenz gauge.

Part VII

Quantum Electrodynamics

Twenty-Three

Quantum Electrodynamics

We mean that in a precise mathematical sense, not in a physicist sense where it's just kind of mostly true.

Richard Ball

23.1 Electromagnetic Interactions

For photons we require two degrees of freedom. We get this by using gauge invariance to reduce the four degrees of freedom of a vector to two degrees of freedom. Gauge invariance requires a conserved current. We've seen that for electrons we have a conserved current due to the U(1) symmetry, $\psi \mapsto e^{i\alpha}\psi$ for some constant α. The conserved current in this case is $J^\mu = \bar{\psi}\gamma^\mu\psi$. We would expect this to couple to the photon field, A_μ.

The simplest way to include photons and electrons is by defining the following Lagrangian:

$$\mathcal{L}_{\text{CED}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi - q\bar{\psi}A\!\!\!/\psi. \tag{23.1.1}$$

The first term is the kinetic term for photons, giving Maxwell's equations, the second is the kinetic and mass term for electrons, giving the Dirac equation, the final term is the interaction, which is simply $j^\mu A_\mu$. The quantity $q = -e$ is the charge of the electron, and it measures how strongly the photon and electron are coupled, which is to say how strongly the electron "feels" the electromagnetic force. Notice that this is a three point interaction, much like the φ^3 or $\bar{\psi}\psi\varphi$ examples we've seen so far in the course.

We can write this Lagrangian more compactly as

$$\mathcal{L}_{\text{CED}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(iD\!\!\!/ - m)\psi \tag{23.1.2}$$

Here

$$D_\mu = \partial_\mu + iqA_\mu\psi \tag{23.1.3}$$

is the **covariant derivative**. Noting that $iD\!\!\!/ = i\partial\!\!\!/ - qA\!\!\!/$ it is clear that this is an equivalent way of writing the Lagrangian.

23.1.1 Gauge Symmetry

Notice that this Lagrangian maintains the **global symmetry** $\psi \mapsto e^{i\alpha}\psi$. By global here we mean that the symmetry is the same at all positions, that is α is a constant. In fact, we can promote this to a **local symmetry**, $\psi \mapsto e^{i\alpha(x)}\psi$, where we now have α depending on position. This works because we have have constructed the Lagrangian so that the resulting change cancels out, provided that the gauge field, A_μ, transforms as

$$A_\mu \mapsto A_\mu - \frac{1}{q}\partial_\mu \alpha, \tag{23.1.4}$$

which is just the usual gauge transformation with $\chi(x) = \alpha(x)/q$.

We can readily check this is a symmetry of the Lagrangian. We've already shown that such a transformation of A doesn't change $F^{\mu\nu}$, and so we need only consider the electron and interaction terms. First consider the electron term, we have

$$\bar{\psi} i\not{\partial}\psi \mapsto e^{-i\alpha}\bar{\psi} i\not{\partial}(e^{i\alpha}\psi) \tag{23.1.5}$$
$$= e^{-i\alpha}\bar{\psi}e^{i\alpha} i\not{\partial}\psi + e^{-i\alpha}\bar{\psi} i(-i\not{\partial}\alpha)e^{i\alpha}\psi \tag{23.1.6}$$
$$= \bar{\psi} i\not{\partial}\psi + \bar{\psi}(\not{\partial}\alpha)\psi. \tag{23.1.7}$$

Now consider the interaction term:

$$q\bar{\psi}\not{A}\psi \mapsto qe^{-i\alpha}\bar{\psi}\left(\not{A} - \frac{1}{q}\not{\partial}\alpha\right)e^{i\alpha}\psi \tag{23.1.8}$$
$$= q\bar{\psi}\not{A}\psi - \bar{\psi}(\not{\partial}\alpha)\psi \tag{23.1.9}$$

and we see that these two extra terms after transformation exactly cancel, leaving the whole Lagrangian invariant under this transformation.

In terms of the covariant derivative we have the transformation

$$D_\mu = \partial_\mu + iqA_\mu \mapsto \partial_\mu + iqA_\mu + i\partial_\mu\alpha \tag{23.1.10}$$

and we see that this extra term exactly cancels the term we get from the derivative of $\exp[i\alpha]$.

Note that this symmetry, called a U(1) gauge symmetry, is exact, it is required to fix the number of degrees of freedom. It is not like the global symmetry which may be broken.

23.1.2 Equations of Motion

The field equations can be derived fairly easily from the Lagrangian. They are just slight modifications of the free photon and free electron equations of motion, in fact they differ only by the fact that we now have a term with A_μ and an extra $\bar{\psi}$ term. First, we have

$$\frac{\partial\mathcal{L}}{\partial A_\mu} = q\bar{\psi}\gamma^\mu\psi \tag{23.1.11}$$

giving the equation of motion

$$\partial_\mu F^{\mu\nu} = q\bar{\psi}\gamma^\nu\psi. \tag{23.1.12}$$

Second, we have

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = q\not{A}\psi \tag{23.1.13}$$

giving the equation of motion

$$(i\not{\partial} - m)\psi = q\not{A}\psi. \tag{23.1.14}$$

Compare these to the free terms, which we can recover by setting $q = 0$:

$$\partial_\mu F^{\mu\nu} = 0, \qquad \text{and} \qquad (i\not{\partial} - m)\psi = 0. \tag{23.1.15}$$

Notice that we can interpret $\psi\gamma^\nu\psi$ as the fermion current, and so $q\psi\gamma^\nu\psi$ is the charge current, which is gives the expected Maxwell equation with sources,

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{23.1.16}$$

.

These equations are nonlinear and, in general, somewhere between very difficult and impossible to solve, even classically. Fortunately the coupling is weak. The fine structure constant is

$$\alpha = \frac{e^2}{4\pi} = \frac{q^2}{4\pi} \approx \frac{1}{137} \approx 0.0073 \ll 1. \tag{23.1.17}$$

This means that we can do perturbation theory in q.

When we do perturbation theory the next to leading order (NLO) term is a correction on the order of 1 %, since α is on the order of 1 % and every extra vertex contributes a factor of q to the amplitude, and so a factor of $q^2 \sim \alpha$ to the cross section or decay rate, or whatever we're calculating. The next to next to leading order (NNLO) term is then on the order of 0.01 %.

23.1.3 Other Symmetries

Notice that under parity $j^\mu \mapsto j_\mu$ and $A^\mu \mapsto A_\mu$, under time reversal $j^\mu \mapsto -j_\mu$ and $A^\mu \mapsto -A_\mu$, and under charge conjugation $j^\mu \mapsto -j^\mu$ and $A^\mu \mapsto -A^\mu$. Hence the Lagrangian is invariant under charge conjugation, parity, and time reversal separately, and so is also invariant under any combination of these, including CPT, the application of all three.

23.1.4 Quantising

The Lagrangian for quantum electrodynamics is exactly the same as for classical electrodynamics, we just interpret the fields as operators and normal order:

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}{:}F^{\mu\nu}F_{\mu\nu}{:} + {:}\bar{\psi}(i\not{D} - m)\psi{:}. \tag{23.1.18}$$

23.2 QED Perturbation Theory

As discussed previously we must resort to a perturbative approach. We start with the S matrix, and the previously derived perturbation expansion

$$S = \sum_{n=1}^{\infty} S^{(n)} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \mathrm{d}^4x_1 \cdots \int \mathrm{d}^4x_n \; \mathrm{T}[\mathcal{H}_{\text{I}}(x_1) \cdots \mathcal{H}_{\text{I}}(x_n)] \tag{23.2.1}$$

where

$$\mathcal{H}_{\text{I}} = -\mathcal{L}_{\text{I}} = -e{:}\bar{\psi}(x)\not{A}(x)\psi(x){:}. \tag{23.2.2}$$

23.2.1 First Order

We can write the fields in terms of positive and negative energy parts, giving

$$\mathcal{H}_I = -e:(\bar{\psi}^+ + \bar{\psi}^-)(\not{A}^+ + \not{A}^-)(\psi^+ + \psi^-): \tag{23.2.3}$$

$$= -e:[\bar{\psi}^+\not{A}^+\psi^+ + \bar{\psi}^+\not{A}^+\psi^- + \bar{\psi}^-\not{A}^+\psi^+ + \bar{\psi}^-\not{A}^+\psi^- \tag{23.2.4}$$

$$+ \bar{\psi}^+\not{A}^-\psi^+ + \bar{\psi}^+\not{A}^-\psi^- + \bar{\psi}^-\not{A}^-\psi^+ + \bar{\psi}^-\not{A}^-\psi^-]: \tag{23.2.5}$$

Each term here corresponds to a vertex involving a photon and either two electrons, two positrons, or an electron and a positron, conserving charge. None of these processes are valid on their own. They are forbidden by conservation of momentum. It is not possible to have any of these processes if all three particles are on-shell. There are really only two distinct possibilities First, we could have an initial photon and final electron and positron, or initial electron and positron and final photon. Second, we could have an electron (or positron) and photon in the initial (or final) state and an electron (or positron) in the final (or initial) state. If this was the case then denoting the momentum of the initial electron by p, the initial photon by k and the final electron by p' we have $p + k = p'$, so $(p+k)^2 = p^2 + k^2 + 2p\cdot k = m^2 + 2p\cdot k$ and also $(p+k)^2 = p'^2 = m^2$, so we have $p\cdot k = 0$, which is only possible if $k = 0$, since in the rest frame of the initial electron we have $p^\mu = (E, \mathbf{0})$ and $k^\mu = (|\boldsymbol{k}|, \boldsymbol{k})$, so $p\cdot k = E|\boldsymbol{k}| = 0$ and $E \geq m > 0$ so we have $|\boldsymbol{k}| = 0$ meaning $\boldsymbol{k} = \mathbf{0}$ and so $k = 0$, but a photon without momentum isn't really anything.

Hence, we must have

$$\langle f|S^{(1)}|i\rangle = 0. \tag{23.2.6}$$

As with the scalar case we can show that this is the case with an explicit, but lengthy, calculation, but this isn't particularly enlightening and doesn't give us new physics beyond the observation that these processes cannot occur.

23.2.2 Second Order

The second order term is

$$S^{(2)} = -\frac{e^2}{2}\int \mathrm{d}^4x \int \mathrm{d}^4y\ \mathrm{T}[:\bar{\psi}(x)\not{A}(x)\psi(x)::\bar{\psi}(y)\not{A}(y)\psi(y):]. \tag{23.2.7}$$

With Wick's theorem we can further specify the number of contractions contributing to a second order term.

23.2.2.1 Zero Contractions

The zero contraction case is simply the first order case squared, and hence gives zero. In terms of diagrams we have two disconnected single vertex diagrams, and these diagrams are just zero for the reasons discussed above. That is, we have

$$S_A^{(2)} = -\frac{e^2}{2}\int \mathrm{d}^4x \int \mathrm{d}^4y\ :\bar{\psi}(x)\not{A}(x)\psi(x)\bar{\psi}(y)\not{A}(y)\psi(y): \tag{23.2.8}$$

and

$$\langle f|S_A^{(2)}|i\rangle = 0. \tag{23.2.9}$$

23.2.2.2 One Contraction

We can further specify whether the contraction is between electron fields or photon fields. Note that for a nonzero result we must contract either A with another A or ψ with $\bar{\psi}$. The contraction must occur between fields at different positions.

Electron Fields. In this case the we consider the term

$$S_B^{(2)} = -\frac{e^2}{2}\int \mathrm{d}^4x \int \mathrm{d}^4y\, [:\bar{\psi}(x)\not{A}(x)\overbrace{\psi(x)\bar{\psi}}(y)\not{A}(y)\psi(y):$$
$$+ :\overbrace{\bar{\psi}(x)\not{A}(x)\psi(x)\bar{\psi}(y)\not{A}(y)\psi}(y):]. \quad (23.2.10)$$

Actually, the two terms here are the same. It can be shown that

$$:\overbrace{\bar{\psi}(x)\not{A}(x)\psi(x)\bar{\psi}(y)\not{A}(y)ps}(y): = :\bar{\psi}(y)\not{A}\overbrace{\psi(x)\bar{\psi}}(y)\not{A}(y)\psi(y):. \quad (23.2.11)$$

After a change of variables $x \leftrightarrow y$ in the second term this then gives

$$S_B^{(2)} = -e^2\int \mathrm{d}^4x \int \mathrm{d}^4y\, :\bar{\psi}(x)\not{A}(x)\overbrace{\psi(x)\bar{\psi}}(y)\not{A}(y)\psi(y): \quad (23.2.12)$$

This contains a single electron propagator. There are then two photons and two electrons/positrons to place in the initial and final states. To conserve momentum we must have two particles in the initial state and two in the final state. This leads to two processes.

Compton Scattering. This is the process

$$\mathrm{e}^-\gamma \to \mathrm{e}^-\gamma, \quad (23.2.13)$$

or equivalently $\mathrm{e}^+\gamma \to \mathrm{e}^+\gamma$. This corresponds to the term in $S_B^{(2)}$ where we have $\psi^+(y)$ to annihilate the initial electron and $\bar{\psi}^-(x)$ to create the final electron. We can then annihilate the initial photon with either $\not{A}^+(x)$ or $\not{A}(y)$, and then the corresponding $\not{A}^-(y)$ or $\not{A}^-(x)$ creates the final photon.

There are therefore two terms contributing to this process,

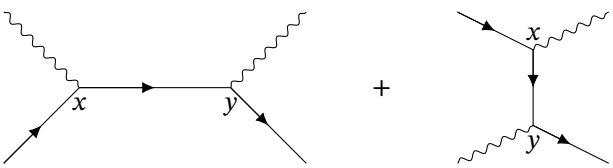
$$S^{(2)}_{\text{Compton}} = S_a^{(2)} + S_b^{(2)}, \quad (23.2.14)$$

with

$$S_a^{(2)} = -e^2\int \mathrm{d}^4x \int \mathrm{d}^4y\, \bar{\psi}^-(x)\gamma^\mu iS_{\mathrm{F}}(x-y)\gamma^\nu A_\mu^-(x)A_\nu^+(y)\psi^+(y), \quad (23.2.15)$$

$$S_b^{(2)} = -e^2\int \mathrm{d}^4x \int \mathrm{d}^4y\, \bar{\psi}^-(x)\gamma^\mu iS_{\mathrm{F}}(x-y)\gamma^\nu A_\nu^-(y)A_\mu^+(x)\psi^+(y). \quad (23.2.16)$$

In terms of diagrams these correspond to

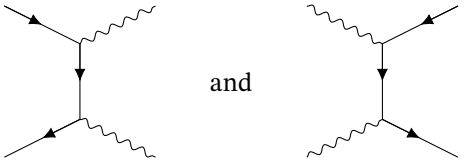


(23.2.17)

Diphoton Process. A diphoton process, as the name suggests, is a process in which there are two photons in either the final or initial state. To second order there are two such processes:

$$e^- e^+ \to \gamma\gamma, \qquad \text{and} \qquad \gamma\gamma \to e^- e^+. \tag{23.2.18}$$

These correspond to the same terms in the Dyson expansion integral as Compton scattering, but with different initial and final states. The Feynman diagrams for these processes are



(23.2.19)

This is an inelastic process, the particles in the final state are different to the particles in the initial state. Notice that these diagrams are both rotated versions of the first diagram in Equation (23.2.17). This sort of observation can save a lot of work when evaluating diagrams, as we'll see later. As well as these two diagrams we also have the same diagrams with the photons crossed.

Photon Fields. In this case we consider the term

$$S_C^{(2)} = -\frac{e^2}{2}\int \mathrm{d}^4x \int \mathrm{d}^4y \, :\bar{\psi}(x)\overbrace{\not{A}(x)\psi(x)\bar{\psi}(y)}\not{A}(y)\psi(y):. \tag{23.2.20}$$

This term corresponds to all processes with a single photon propagator. There are two classes of such a process.

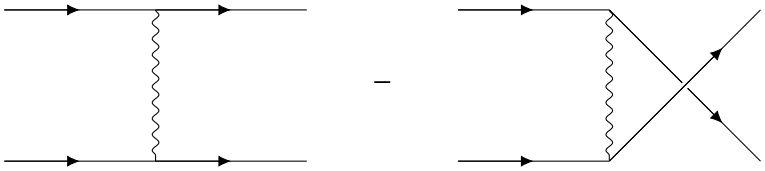
Møller Scattering. The first, **Møller scattering**, is the process

$$e^- e^- \to e^- e^-, \tag{23.2.21}$$

as well as the equivalent process with all positrons. We can select the term in the S matrix which gives this process when sandwiched between two electron states as initial and final states:

$$S^{(2)}_{\text{Møeller}} = -\frac{e^2}{2}\int \mathrm{d}^4x \int \mathrm{d}^4y \, :\bar{\psi}^-(x)\gamma^\mu\psi^+(x)\bar{\psi}^-(y)\gamma^\nu\bar{\psi}^+(y):iD^{\mathrm{F}}_{\mu\nu}(x-y). \tag{23.2.22}$$

This corresponds to the diagrams



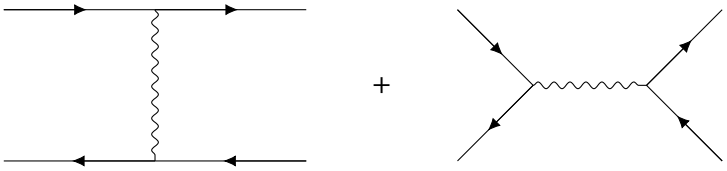
(23.2.23)

The relative minus sign comes from crossing the fermion lines.

Bhabha Scattering. The second, is the very similar process of **Bhabha scattering**,

$$e^+e^- \to e^+e^-. \tag{23.2.24}$$

This corresponds to the diagrams



(23.2.25)

Note that the second diagram here is the first diagram in Møller scattering.

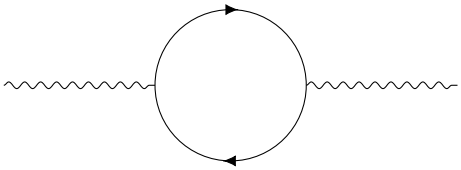
23.2.2.3 Two Contractions

We can also have two contractions. One contraction must always be between electron fields, the second can be either between electron fields or photon fields.

Electron Fields. The term corresponding to two contractions between electron fields is

$$S_E^{(2)} = -\frac{e^2}{2}\int \mathrm{d}^4x \int \mathrm{d}^4y :\bar{\psi}(x)\not{A}(x)\psi(x)\bar{\psi}(y)\not{A}(y)\psi(y):. \tag{23.2.26}$$

This corresponds to the diagram



(23.2.27)

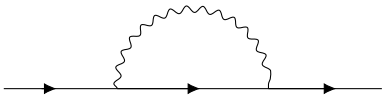
This is called the **vacuum polarisation diagram** or the **photon self energy**. This diagram is important when considering the renormalisation of the A field[1].

[1] See *Gauge Theories in Particle Physics.*

Photon Field. The term corresponding to a contraction between electron fields and a contraction between photon fields is

$$S_D^{(2)} = -\frac{e^2}{2}\int \mathrm{d}^4x \int \mathrm{d}^4y :[\bar{\psi}\not{A}(x)\psi(x)\bar{\psi}(y)\not{A}(y)\psi(x)+\bar{\psi}(x)\not{A}(x)\psi(x)\bar{\psi}(y)\not{A}(y)\psi(y)]:. \tag{23.2.28}$$

This corresponds to the diagram



(23.2.29)

This is called the **electron self energy**. This diagram is important when considering the renormalisation of the ψ field[2].

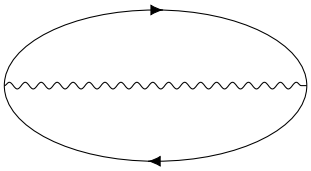
[2] See *Gauge Theories in Particle Physics.*

23.2.2.4 Three Contractions

Finally, there is one way to have three contractions:

$$S_F^{(2)} = -\frac{e^2}{2}\int \mathrm{d}^4x \int \mathrm{d}^4y \, :\bar{\psi}(x)\not{A}(x)\psi(x)\bar{\psi}(y)\not{A}(y)\psi(y): . \tag{23.2.30}$$

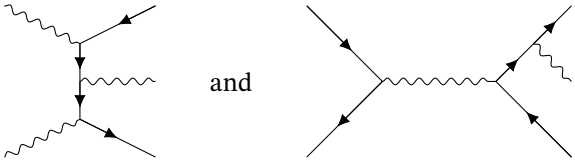
This corresponds to the diagram



(23.2.31)

23.2.3 Third Order

There are a lot of third order terms, too many to cover all of them, so we just give examples here. Essentially there are two ways to increase the order of a term. The first is to add another external particle, which in this case must be a single photon to conserve charge, such as in the following diagrams:

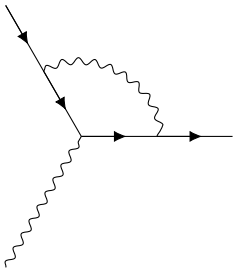


(23.2.32)

The extra photon being emitted corresponds to the previously discussed processes but with **bremsstrahlung**, the process of emitting a radiation as an electron slows down[3].

[3] bremsstrahlung comes from the German *bremsen* meaning to break and *Strahlung* meaning radiation.

The second way to increase the order is to add an extra loop to an existing process. For example, we can consider one of the vertices with an electron and photon in the initial state and a single electron in the final state, which doesn't occur on its own due to momentum conservation. We can a loop to this to get the diagram



(23.2.33)

This process can happen since we've now got a virtual particle, actually three of them, since we actually annihilate the incoming electron, producing a virtual electron and a virtual photon, then annihilate this virtual electron and the incoming photon, and produce a virtual electron, which we then annihilate along with the virtual photon, producing a real electron.

23.3 Feynman Rules for QED

The only new rule for QED, in addition to the rules we've seen already for fermion fields and gauge fields is the vertex term, corresponding to the interaction,

$$\mathcal{L}_{\mathrm{I}} = e\bar{\psi}\not{A}\psi. \tag{23.3.1}$$

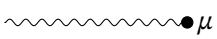
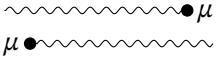
Each vertex term gives a factor of $ie\gamma^{\mu}$. Notice that γ^{μ} has both (implicit) spinor indices and (explicit) Lorentz indices. It is the interaction of these indices that conveys information about the spin of an electron and the polarisation of a photon in a way that allows for the total angular momentum to be conserved.

We now list the Feynman rules for QED. First we have the rules for external spin 1/2 (anti)particles with momentum p and spin s, note that $\bullet$ corresponds to a vertex, whereas a lack of a $\bullet$ corresponds to an external particle:

Incoming particle	$u(\boldsymbol{p}, s)$	
Outgoing particle	$\bar{u}(\boldsymbol{p}, s)$	
Incoming antiparticle	$\bar{v}(\boldsymbol{p}, s)$	
Outgoing antiparticle	$v(\boldsymbol{p}, s)$	

(23.3.2)

Next, we have the rules for spin 1 external particles, which in QED is just photons. For a photon with momentum p and polarisation r the rules are as follows:

Incoming photon	$\varepsilon^{r}_{\mu}(p)$	
Outgoing photon	$\varepsilon^{r}_{\mu}(p)^{*}$	

(23.3.3)

Then there are internal lines, for a photon propagator of momentum k we have

$$\text{Photon propagator} \quad \frac{iP^{\mu\nu}(k)}{k^2 + i\varepsilon} \to -\frac{i\eta^{\mu\nu}}{k^2 + i\varepsilon} \quad \mu \bullet \sim\sim\sim \bullet \nu \tag{23.3.4}$$

For a spin 1/2 fermion propagator of mass m and momentum p we have

$$\text{Photon propagator} \quad -\frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon} \quad \bullet\!\!-\!\!-\!\!-\!\!\bullet \tag{23.3.5}$$

Finally, for each QED vertex we have

$$\text{Vertex} \quad ie\gamma^{\mu} \quad \mu \bullet \tag{23.3.6}$$

In addition we have the usual rules for Feynman diagrams:

- Impose four-momentum conservation at each vertex.
- Integrate over any unfixed four momenta.
- Go along an arrow and write the corresponding terms with spinor indics from right to left as you encounter them.
- For every closed fermion loop take a trace and include a minus sign.
- For every crossed pair of external fermions include a minus sign.

In order to evaluate the resulting amplitudes we need the spin sums:

$$\sum_s u(\boldsymbol{p},s)\bar{u}(\boldsymbol{p},s) = \not{p} + m, \tag{23.3.7}$$

$$\sum_s v(\boldsymbol{p},s)\bar{v}(\boldsymbol{p},s) = \not{p} - m, \tag{23.3.8}$$

and also the polarisation sums

$$\sum_{r=1,2} \varepsilon_r^\mu(k)\varepsilon_r^\nu(k) = P^{\mu\nu} \to -\eta^{\mu\nu}. \tag{23.3.9}$$

Twenty-Four

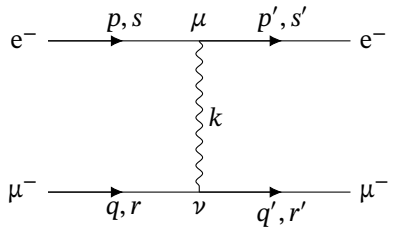
Electron-Muon Scattering

24.1 Amplitude

We want to compute the cross section for elastic electron-muon scattering,

$$e^-\mu^- \to e^-\mu^-. \tag{24.1.1}$$

This is the same as the cross section for $e^+\mu^+ \to e^+\mu^+$. The muon is exactly the same as an electron for the purposes of QED, the only difference is the mass. The mass of the electron is $m_e = 0.511\,\text{MeV}$ and the mass of the muon is $m_\mu = 105\,\text{MeV}$. Importantly this means that the electron and muon are distinguishable, so the only tree level diagram is the t-channel diagram



(24.1.2)

Take the incoming electron as having momentum p and spin s, the outgoing electron as having momentum p' and spin s', the incoming muon as having momentum q and spin r, and the outgoing muon as having momentum q' and spin r'.

If the photon has momentum k then conservation of momentum at each vertex tells us $k = p - p' = q' - q$, note that the overall sign depends on which direction we define the photon as going in, but importantly we have $p - p'$ and $q' - q$, since one is emitting the photon and the other is absorbing it.

We can use the Feynman rules to read off the amplitude from this diagram. Start with the electron line. Following it in the direction of the arrow we have an incoming electron, giving a factor of $u(p,s)$, then a QED vertex, giving a factor of $ie\gamma_\mu$, and then an outgoing electron, giving a factor of $\bar{u}(p',s')$. We then write these from right to left in this order giving $\bar{u}(p',s')ie\gamma_\mu u(p,s)$. The muon term is almost identical: $\bar{u}(q',r')ie\gamma_\mu u(q,r)$. Note that as well as the momentum and spin being different these spinors also refer to a particle of different mass, which is not explicit in the notation used here. Finally, we have the photon propagator, giving a factor of $iP^{\mu\nu}(k)/(k^2 + i\varepsilon)$. Putting this all together the amplitude is

$$\mathcal{M} = (ie)^2\bar{u}(p',s')\gamma_\mu u(p,s)\bar{u}(q',r')\gamma_\nu u(q,r)\frac{iP^{\mu\nu}(k)}{k^2 + i\varepsilon}. \tag{24.1.3}$$

24.2 Currents

This is the truth, up to higher order diagrams.

Richard Ball

Define the electron and muon currents as

$$j_\mu \coloneqq \bar{u}(p',s')\gamma_\mu u(p,s), \qquad \text{and} \qquad J_\nu = \bar{u}(q',r')\gamma_\nu u(q,r) \tag{24.2.1}$$

respectively. Then the amplitude is

$$\mathcal{M} = -ie^2 j_\mu J_\nu \frac{P^{\mu\nu}(k)}{k^2}. \tag{24.2.2}$$

Note that we can drop the $i\varepsilon$ prescription since if the photon goes on-shell, so $k^2 = 0$, then the diagram splits into two QED vertices, both of which have zero amplitude, so for nonzero contributions the photon must be off-shell.

Now consider $k^\mu j_\mu$, we have

$$\begin{aligned} k^\mu j_\mu &= k^\mu \bar{u}(p',s')\gamma_\mu u(p,s) && (24.2.3)\\ &= \bar{u}(p',s')\not{k}u(p,s) && (24.2.4)\\ &= \bar{u}(p',s')(\not{p} - \not{p}')u(p,s) && (24.2.5)\\ &= \bar{u}(p',s')(m-m)u(p,s) && (24.2.6)\\ &= 0. && (24.2.7) \end{aligned}$$

Here we've used the fact that $k = p - p'$ and then the relation $(\not{p} - m)u(p,s) = 0$, and its conjugate $\bar{u}(p',s')(\not{p}' - m) = 0$, to rewrite $\not{p}u(p,s) = mu(p,s)$ and $\bar{u}(p',s')\not{p}' = m\bar{u}(p',s')$. So, j_μ, the electron current, is a conserved current. The exact same argument with $k = q' - q$ proves that J_μ, the muon current, is also a conserved current. Using this we have

$$j_\mu J_\nu P^{\mu\nu}(k) = j_\mu J_\nu(-\eta^{\mu\nu} + k^\mu \alpha^\nu + k^\nu \alpha^{*\mu}) = -j_\mu J^\mu. \tag{24.2.8}$$

This is independent of α and hence gauge invariant. The amplitude is then

$$\mathcal{M} = ie^2 \frac{j_\mu J^\mu}{k^2}. \tag{24.2.9}$$

24.3 Quantum Mechanics Comparison

In QFT we don't have forces and potentials and all that nonsense.

Richard Ball

You can throw away your classical electrodynamics textbook now.

Richard Ball

We can do a nonrelativistic calculation if we assume that the more massive particle, in this case the muon, is stationary and produces a Coulomb potential in which the lighter particle, the electron, scatters. To compare our result to this we take the nonrelativistic limit.

In the nonrelativistic limit we have $|\boldsymbol{j}| \ll j^0$, since there is, relatively, little motion in the nonrelativistic limit. We also have $|\boldsymbol{p}|, |\boldsymbol{p}'| \ll m$. The zeroth component of the electron current is

$$j^0 = \bar{u}(p', s')\gamma^0 u(p, s) \tag{24.3.1}$$
$$= u^\dagger(p', s')(\gamma^0)^2 u(p, s) \tag{24.3.2}$$
$$= u^\dagger(p', s')u(p, s). \tag{24.3.3}$$

Now considering the spinors in the Dirac representation,

$$u(p, s) = \sqrt{E + m_\mathrm{e}}\begin{pmatrix} \varphi^s \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m_\mathrm{e}}\varphi^s \end{pmatrix} \approx \sqrt{2m_\mathrm{e}}\begin{pmatrix} \varphi^s \\ 0 \end{pmatrix}. \tag{24.3.4}$$

We also have $(\varphi^{s'})^\dagger\varphi^s = \delta^{s's}$, so $j^0 = u^\dagger(p', s')u(p, s) \approx 2m_\mathrm{e}\delta_{ss'}$. This means that at low momentum spin is conserved since $\delta_{ss'}$ enforces that the spin of the incoming electron is the same as the spin of the outgoing electron for a nonzero contribution to the amplitude. We can also apply exactly the same logic to the muon getting $J^0 \approx 2m_\mu\delta_{rr'}$.

Now pick some frame in which $p = (E, \boldsymbol{p}) \approx (m_\mathrm{e}, \boldsymbol{p})$, and $p' = (E', \boldsymbol{p}') \approx (m_\mathrm{e}, \boldsymbol{p}')$. Then

$$k^2 = (p - p')^2 \approx -(\boldsymbol{p} - \boldsymbol{p}')^2 = -\boldsymbol{k}^2. \tag{24.3.5}$$

The amplitude is then

$$\mathcal{M} \approx -4ie^2\frac{m_\mathrm{e}m_\mu}{\boldsymbol{k}^2}. \tag{24.3.6}$$

In nonrelativistic quantum mechanics scattering of a particle in a potential $V(\boldsymbol{x})$ the differential cross section is given by[1]

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m^2}{16\pi^2}|\tilde{V}(\boldsymbol{k})|^2 \tag{24.3.7}$$

where $\tilde{V}$ is the Fourier transform of V. Compare this to the result

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2W^2}\frac{p'}{p}|\mathcal{M}|^2 \tag{24.3.8}$$

and noting that in the limit since $m_\mu \gg m_\mathrm{e}$ at low momentum we have $W \approx m_{\mu^-}$, where W is the sum of the energies of the initial particles. Comparing the factors out the front we must have

$$\mathcal{M} = -2im_\mathrm{e}m_\mu\tilde{V}(\boldsymbol{k}). \tag{24.3.9}$$

To get the same result with both approaches we must have $\tilde{V}(\boldsymbol{k}) = 2e^2/\boldsymbol{k}^2$. We can then compute V:

$$V(\boldsymbol{r}) = 2e^2\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\frac{\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{r}}}{\boldsymbol{k}^2} \tag{24.3.10}$$
$$= 2e^2\int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}\int_0^{\pi}\frac{\mathrm{d}\vartheta}{2\pi}\int_0^{\infty}\frac{\mathrm{d}k}{2\pi}\,k^2\sin\vartheta\,\frac{\mathrm{e}^{-ikr\cos\vartheta}}{k^2}, \tag{24.3.11}$$

[1] See *Principles of Quantum Mechanics* for a derivation.

where we've moved to spherical polar coordinates (r, ϑ, φ). The φ integral gives a factor of 2π, cancelling with one of the other factors. We can make the substitution $u = \cos\vartheta$, and so $\mathrm{d}u = -\sin(\vartheta)\mathrm{d}\vartheta$, and the limits of the ϑ integral become -1 and 1, after using the minus from the change of variables sign to flip the order of the limits. We then have

$$V(\boldsymbol{r}) = \frac{e^2}{2\pi^2}\int_{-1}^{1}\mathrm{d}u\int_0^\infty \mathrm{d}k\,\mathrm{e}^{-ikru} \tag{24.3.12}$$

$$= \frac{e^2}{2\pi^2}\int_0^\infty \left[\frac{i}{kr}\mathrm{e}^{-ikru}\right]_{u=-1}^{u=1} \tag{24.3.13}$$

$$= \frac{e^2}{2\pi^2}\int_0^\infty \frac{i}{kr}[\mathrm{e}^{-ikr} - \mathrm{e}^{ikr}] \tag{24.3.14}$$

$$= \frac{e^2}{\pi^2}\int_0^\infty \frac{1}{kr}\sin(kr) \tag{24.3.15}$$

$$= \frac{e^2}{2\pi r} \tag{24.3.16}$$

The result should be $e^2/(4\pi r)$ having used the result

$$\int_0^\infty \mathrm{d}x\frac{\sin x}{x} = \frac{1}{2}\int_{-\infty}^\infty \mathrm{d}x\frac{\sin x}{x} = \frac{\pi}{2}. \tag{24.3.17}$$

It is actually possible to develop all of classical electrodynamics in this way, since we can make this result Lorentz invariant and boost to an arbitrary frame.

24.4 Cross Section

We assume that the spin of the particles is unknown, so we sum over initial and final spins. We'll later divide by 4 to average over initial spins. This requires us to compute

$$\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{k^4}\sum_{\text{spins}} j_\mu^\dagger J^{\mu\dagger} j_\nu J^\nu \tag{24.4.1}$$

$$= \frac{e^4}{k^4}\left(\sum_{s,s'} j_\mu^\dagger j_\nu\right)\left(\sum_{r,r'} J^{\mu\dagger}J^\nu\right). \tag{24.4.2}$$

Consider the first sum:

$$L_{\mu\nu} \coloneqq \frac{1}{2}\sum_{s,s'} j_\mu^\dagger j_\nu \tag{24.4.3}$$

$$= \frac{1}{2}\sum_{s,s'}\bar{u}(p,s)\gamma_\mu u(p',s')\bar{u}(p',s')\gamma_\nu u(p,s) \tag{24.4.4}$$

$$= \frac{1}{2}\sum_{s,s'}\bar{u}_a(p,s)\gamma_{\mu ab}u_b(p',s')\bar{u}_c(p',s')\gamma_{\nu cd}u_d(p,s) \tag{24.4.5}$$

$$= \frac{1}{2}\sum_{s,s'}u_d(p,s)\bar{u}_a(p,s)\gamma_{\mu ab}u_b(p',s')\bar{u}_c(p',s')\gamma_{\nu cd} \tag{24.4.6}$$

$$= \frac{1}{2}\sum_{s,s'}\mathrm{tr}[u(p,s)\bar{u}(p,s)\gamma_\mu u(p',s')\bar{u}(p',s')\gamma_\nu] \tag{24.4.7}$$

$$= \frac{1}{2}\,\mathrm{tr}[(\not{p} + m_\mathrm{e})\gamma_\mu(\not{p}' + m_\mathrm{e})\gamma_\nu] \tag{24.4.8}$$

$$= \frac{1}{2}\,\mathrm{tr}[\not{p}\gamma_\mu\not{p}'\gamma_\nu + m_\mathrm{e}^2\gamma_\mu\gamma_\nu] \tag{24.4.9}$$

having kept only terms with an even number of gamma matrices, since terms with an odd number of gamma matrices vanish in the trace. Now consider the identities

$$\mathrm{tr}[\not{a}\not{b}] = 4a\cdot b \tag{24.4.10}$$

$$\mathrm{tr}[\not{a}\not{b}\not{c}\not{d}] = 4[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)]. \tag{24.4.11}$$

The first of these can be rewritten as $a^\mu b^\nu\,\mathrm{tr}[\gamma_\mu\gamma_\nu] = 4a^\mu b^\nu\eta_{\mu\nu}$, so $\mathrm{tr}[\gamma_\mu\gamma_\nu] = 4\eta_{\mu\nu}$. Similarly, the second can be rewritten as

$$b^\mu d^\nu\,\mathrm{tr}[\not{a}\gamma_\mu\not{c}\gamma_\nu] = 4[a^\rho b^\mu\eta_{\rho\mu}c^\sigma d^\nu\eta_{\sigma\nu} - a^\rho c^\sigma\eta_{\rho\sigma}b^\mu d^\nu\eta_{\mu\nu} + a^\rho d^\nu\eta_{\rho\nu}b^\mu c^\sigma\eta_{\mu\sigma}]$$

and so

$$\mathrm{tr}[\not{a}\gamma_\mu\not{c}\gamma_\nu] = 4[a^\rho\eta_{\rho\mu}c^\sigma\eta_{\sigma\nu} - a^\rho c^\sigma\eta_{\rho\sigma}\eta_{\mu\nu} + a^\rho\eta_{\rho\nu}c^\sigma\eta_{\mu\sigma}] \tag{24.4.12}$$

$$= 4[a_\mu c_\nu - (a\cdot c)\eta_{\mu\nu} + a_\nu c_\mu]. \tag{24.4.13}$$

Then we have

$$L_{\mu\nu} = \frac{1}{2}(\mathrm{tr}[\not{p}\gamma_\mu\not{p}'\gamma_\nu] + m_\mathrm{e}^2\,\mathrm{tr}[\gamma_\mu\gamma_\nu]) \tag{24.4.14}$$

$$= 2(p_\mu p'_\nu + p_\nu p'_\mu - (p\cdot p')\eta_{\mu\nu} + m_\mathrm{e}^2\eta_{\mu\nu}). \tag{24.4.15}$$

Now use

$$k^2 = (p-p')^2 = p^2 + p'^2 - 2p\cdot p' = 2m_\mathrm{e}^2 - 2p\cdot p' \tag{24.4.16}$$

to write this as

$$L_{\mu\nu} = 2\left(p_\mu p'_\nu + p_\nu p'_\mu + \frac{1}{2}k^2\eta_{\mu\nu}\right). \tag{24.4.17}$$

We also get the same result for the term with muon currents. Combing these the spin averaged amplitude squared is

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 \tag{24.4.18}$$

$$= \frac{4e^2}{k^4}\left(p_\mu p'_\nu + p_\nu p'_\mu + \frac{1}{2}k^2\eta_{\mu\nu}\right)\left(q_\mu q'_\nu + q_\nu q'_\mu + \frac{1}{2}k^2\eta^{\mu\nu}\right) \tag{24.4.19}$$

$$= \frac{4e^2}{k^4}\Big[(p\cdot q)(p'\cdot q') + (p\cdot q')(p'\cdot q) + (p\cdot q')(p'\cdot q) \tag{24.4.20}$$

$$+ (p\cdot q)(p'\cdot q') + \frac{1}{2}k^2(p\cdot p' + p\cdot p')$$

$$+ \frac{1}{2}k^2(q\cdot q' + q\cdot q') + \frac{1}{4}k^4\eta_{\mu\nu}\eta^{\mu\nu}\Big]$$

$$= \frac{4e^2}{k^4}\Big[(p\cdot q)^2 + (p\cdot q')^2 + (p\cdot q')^2 \tag{24.4.21}$$

$$+ (p\cdot q)^2 + k^2(p\cdot p') + k^2(q\cdot q') + k^4\Big]$$

$$= 4e^2\left[1 + \frac{p\cdot p' + q\cdot q'}{k^2} + 2\frac{(p\cdot q)^2 + (p\cdot q')^2}{k^4}\right]. \tag{24.4.22}$$

Here we've used $p + q = p' + q'$, and hence

$$\begin{aligned}(p+q)^2 &= p^2 + q^2 + 2p\cdot q = m_e^2 + m_\mu^2 \\ &= (p'+q')^2 = p'^2 + q'^2 + 2p'\cdot q' = m_e^2 + m_\mu^2 + 2p'\cdot q',\end{aligned} \tag{24.4.23}$$

so $p \cdot q = p' \cdot q'$, and similarly,

$$(p-q')^2 = m_e^2 + m_\mu^2 - 2p\cdot q' = (p'-q)^2 = m_e^2 + m_\mu^2 - 2p'\cdot q, \tag{24.4.24}$$

and so $p \cdot q' = p' \cdot q$.

In the high energy limit this simplifies. We can introduce the Mandelstam variables

$$s = (p+q)^2 \approx 2p\cdot q = 2p'\cdot q', \tag{24.4.25}$$

$$t = (p-p')^2 \approx -2p\cdot p' = -2q\cdot q' = k^2, \tag{24.4.26}$$

$$u = (p-q')^2 = -2p\cdot q' = -2p'\cdot q, \tag{24.4.27}$$

where we neglect the m_e^2 and m_μ^2 terms in the high energy limit. Then the spin averaged amplitude squared is

$$\overline{|\mathcal{M}|^2} = 4e^4\left[1 - \frac{t}{t} + 2\frac{s^2/4 + u^2/4}{t^2}\right] \tag{24.4.28}$$

$$= 2e^4\frac{s^2+u^2}{t^2}. \tag{24.4.29}$$

Finally, using $W^2 = s$ and $p/p' \approx 1$, the centre of mass cross section, in this high energy limit, is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{2s}\frac{s^2+u^2}{t^2} \tag{24.4.30}$$

where

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137} \tag{24.4.31}$$

is the fine structure constant.

In the centre of mass frame at high energies we have $p^\mu = (E, \boldsymbol{p})$, $p'^\mu = (E, \boldsymbol{p}')$, $q^\mu = (E, -\boldsymbol{p})$, and $q'^\mu = (E, -\boldsymbol{p}')$. If the scattering angle is ϑ then we have $\cos\vartheta = \boldsymbol{p}\cdot\boldsymbol{p}'/|\boldsymbol{p}||\boldsymbol{p}'|$. Then the Mandelstam variables become

$$s \approx 2p\cdot q = 4E^2, \tag{24.4.32}$$

$$t \approx -2p\cdot p' = -2E^2(1-\cos\vartheta) = -4E^2\sin^2\frac{\vartheta}{2}, \tag{24.4.33}$$

$$u \approx -2p\cdot q' = -2E^2(1+\cos\vartheta) = -4E^2\cos^2\frac{\vartheta}{2}. \tag{24.4.34}$$

Hence, we have

$$\frac{t}{s} = -\sin^2\frac{\vartheta}{2}, \quad \frac{u}{s} = -\cos^2\frac{\vartheta}{2}, \quad \text{and} \quad tu = \tan^2\frac{\vartheta}{2}. \tag{24.4.35}$$

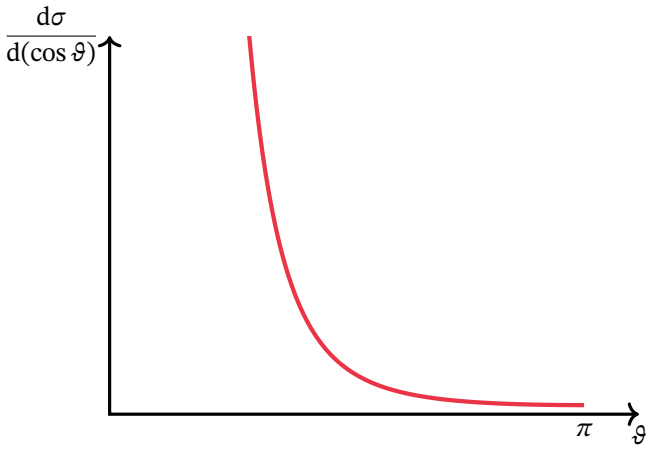


Figure 24.1: The cross section for electron-muon scattering plotted as a function of the scattering angle. Note that the divergence at zero vanishes if we properly account for the masses of the particles.

The cross section is then

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8E^2}\frac{1+\cos^4(\vartheta/2)}{\sin^4(\vartheta/2)}. \tag{24.4.36}$$

The solid angle element is $d\Omega = 2\pi d(\cos\vartheta)$, where the 2π comes from the φ integral, which will always be 2π since there is no φ dependence. We can then write the cross section as

$$\frac{d\sigma}{d(\cos\vartheta)} = \frac{\pi\alpha^2}{4E^2}\frac{1+\cos^4(\vartheta/2)}{\sin^4(\vartheta/2)}. \tag{24.4.37}$$

Notice that this goes as $1/E^2$. This must be true of all high energy cross sections with dimensionless coupling constants since

$$\left[\frac{d\sigma}{d\Omega}\right] = [\text{area}] = [\text{length}]^2 = \frac{1}{[\text{energy}]^2} \tag{24.4.38}$$

and at high energies the only non-negligible energy scale is E, the energy of the incoming particles. The particle masses are all negligible.

The cross section is plotted in Figure 24.1. We see that the cross section is largest at small ϑ, corresponding to not much of an interaction, the particles basically just glance past each other. At $\vartheta = \pi$ the cross section is smallest, which we can think of as corresponding to a head on collision, which is very unlikely. Note that the seeming divergence of the cross section at $\vartheta = 0$ is due to us discarding the masses. If we kept them then the denominator would not vanish and so the cross section would be finite.

Twenty-Five

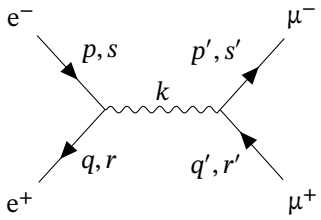
Electron-Positron Annihilation

25.1 Amplitude

We want to compute the cross section for inelastic electron-positron annihilation,

$$e^- e^+ \to \mu^- \mu^+. \tag{25.1.1}$$

We choose the final state to be $\mu^-\mu^+$, rather than e^-e^+, since this reduces the number of diagrams. The only tree level diagram is



(25.1.2)

Conservation of momentum at each vertex tells us that $k = p+q = p'+q'$. Starting with the electron arrow we have a incoming electron, $u(p,s)$, then a QED vertex, $ie\gamma_\mu$, and then an incoming positron, $\bar{v}(q,r)$, we write these in the reverse order to which they were encountered: $ie\bar{v}(q,r)\gamma_\mu u(p,s)$. Similarly for the muon arrow we have an outgoing antimuon, $v(q',r')$, a QED vertex, $ie\gamma_\nu$, and then an outgoing muon, $\bar{u}(p',s')$, and again we write these in the reverse order to which they were encountered: $ie\bar{u}(p',s')\gamma_\nu v(q',r')$. Finally, we have a propagator, $iP^{\mu\nu}(k)/(k^2+i\varepsilon)$. The full amplitude is then

$$\mathcal{M} = (ie)^2 \bar{v}(q,r)\gamma_\mu u(p,s)\bar{u}(p',s')\gamma_\nu v(q',r')\frac{P^{\mu\nu}(k)}{k^2+i\varepsilon}. \tag{25.1.3}$$

To show that this is gauge independent consider

$$\bar{v}(q,r)\not{k}u(p,s) = \bar{v}(q,r)(\not{p}+\not{q})u(p,s) = \bar{v}(q,r)(m_e - m_e)\bar{u}(p,s) = 0, \tag{25.1.4}$$

where we've used $(\not{p}+m_e)u(p,s)=0$ and $\bar{v}(\not{p}-m_e)=0$. Similarly

$$\bar{u}(p',s')\not{k}v(q',r') = \bar{u}(p',s')(\not{p}'+\not{q}')v(q',r') = \bar{u}(p',s')(-m_\mu+m_\mu)v(q',r') = 0.$$

This means that the terms in $P^{\mu\nu}$ proportional to k vanish, and so the α dependence drops out. We can then replace $P^{\mu\nu}$ with $-\eta^{\mu\nu}$, giving

$$\mathcal{M} = (ie)^2 \bar{u}(q,r)\gamma_\mu u(p,s)\bar{u}(p',s')\gamma^\mu v(q',r'). \tag{25.1.5}$$

We can also drop the $i\varepsilon$ term since the photon cannot go on-shell, if it did then this diagram would factorise into two vertices, both of which give zero.

We can then average over the initial spins and sum over final spins to get the spin averaged amplitude squared

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{e^4}{k^4}\frac{1}{4}\left(\sum_{s,r}\bar{u}(p,s)\gamma_\mu v(q,r)\bar{v}(q,r)\gamma_\nu u(p)\right) \\
&\qquad \times\left(\sum_{s',r'}\bar{v}(q',r')\gamma^\mu u(p',s')\bar{u}(p',s')\gamma^\nu v(q',r')\right) && (25.1.6)\\
&= \frac{e^4}{k^4}\frac{1}{4}\operatorname{tr}[u(p,s)\bar{u}(p,s)\gamma_\mu v(q,r)\bar{v}(q,r)\gamma_\nu] \\
&\qquad \times \operatorname{tr}[v(q',r')\bar{v}(q',r')\gamma^\mu u(p',s')\bar{u}(p',s')\gamma^\nu] && (25.1.7)\\
&= \frac{e^4}{k^4}\frac{1}{4}\operatorname{tr}[(\not{p}+m_\mathrm{e})\gamma_\mu(\not{q}-m_\mathrm{e})\gamma_\nu]\operatorname{tr}[(\not{q}'-m_\mu)\gamma^\mu(\not{p}'+m_\mu)\gamma^\nu] && (25.1.8)\\
&= \frac{e^4}{k^4}\frac{1}{4}\operatorname{tr}[\not{p}\not{q}\gamma_\mu\gamma_\nu - m_\mathrm{e}^2\gamma_\mu\gamma_\nu]\operatorname{tr}[\not{q}'\not{p}'\gamma^\mu\gamma^\nu - m_\mu\gamma^\mu\gamma^\nu] && (25.1.9)\\
&= 4\frac{e^4}{k^4}(p_\mu q_\nu + q_\mu p_\nu - (p\cdot q + m_\mathrm{e}^2)\eta_{\mu\nu}) \\
&\qquad \times (p'_\mu q'_\nu + q'_\mu p'_\nu - (p'\cdot q' + m_\mu^2)\eta^{\mu\nu}) && (25.1.10)\\
&= 4\frac{e^4}{k^4}\left(p_\mu q_\nu + q_\mu p_\nu - \frac{k^2}{2}\eta_{\mu\nu}\right)\left(p'_\mu q'_\nu + q'_\mu p'_\nu - \frac{k^2}{2}\eta^{\mu\nu}\right) && (25.1.11)\\
&= 4\frac{e^4}{k^4}\left[1 - \frac{p\cdot q + p'\cdot q'}{k^2} + 2\frac{(p\cdot p')^2 + (p\cdot q')^2}{k^4}\right]. && (25.1.12)
\end{aligned}$$

Here we've used trace identities derived in a very similar way to the previous calculation, as well as $(p+q)^2 = k^2$ which gives $k^2 = 2(m_\mathrm{e}^2 + p\cdot q)$, and other similar identities derived in this way.

In the high energy limit[1] we have

$$s = (p{+}q)^2 = k^2 \approx 2p{\cdot}q, \quad t = (p{-}p')^2 \approx -2p{\cdot}p', \quad \text{and} \quad u = (p{-}q')^2 \approx -2p{\cdot}q'.$$

[1] There is no low energy limit, since we need the excess energy in order to form the heavy muon from the light electrons.

The average squared amplitude is then simply

$$\overline{|\mathcal{M}|^2} \approx 2e^4\frac{t^2+u^2}{s^2}. \tag{25.1.13}$$

The cross section is then

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\vartheta} = \frac{\pi\alpha^2}{4E^2}\left(\cos^4\frac{\vartheta}{2} + \sin^4\frac{\vartheta}{2}\right) \tag{25.1.14}$$

where ϑ is the scattering angle. This uses identities relating the Mandelstam variables and the scattering angle derived in the previous calculation.

Figure 25.1 shows the cross section for an electron and positron to annihilate into a pair of muons. Notice that it is symmetric about $\pi/2$, where it is also minimised. This corresponds to the fact that in the centre of mass frame the muon is equally likely to go in either the direction of the electron or the positron. We can think or the electron as "turning into" a muon and continuing on, for $\vartheta = 0$, or the positron "turning into" a muon and continuing on, for $\vartheta = \pi$, and both are equally likely (even though the latter doesn't conserve charge, since this isn't really what's happening it's fine).

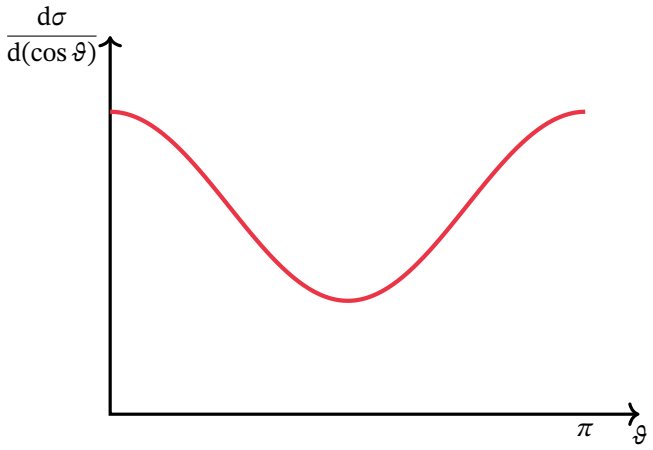


Figure 25.1: The cross section for electron-positron annihilation producing muons as a function of ϑ, the scattering angle.

25.2 Crossing Symmetry

Consider the two QED calculation's we've done so far,

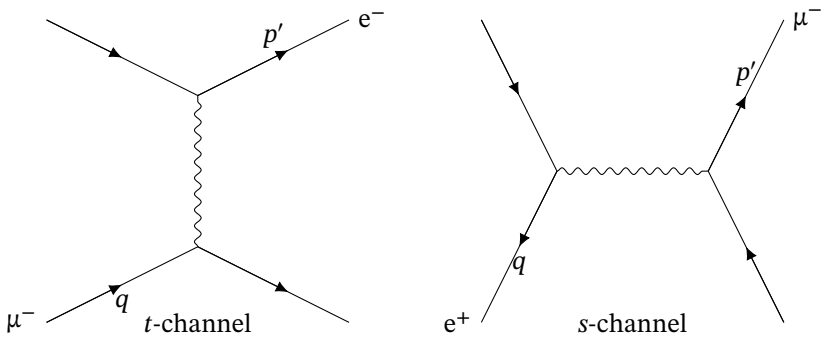
$$e^-\mu^- \to e^-\mu^-, \qquad \text{and} \qquad e^-e^+ \to \mu^-\mu^+. \tag{25.2.1}$$

For these processes we find the following results in the high energy limit:

$$\overline{|\mathcal{M}|^2} = 2e^4\frac{s^2+u^2}{t^2}, \qquad \text{and} \qquad \overline{|\mathcal{M}|^2} = 2e^4\frac{t^2+u^2}{s^2}. \tag{25.2.2}$$

These are very similar. Notice that the first process is an t-channel process, meaning the virtual photon has momentum squared t, and so we would get a pole if the virtual photon were to go on shell, although as previously discussed this is not possible. Similarly the second process is an s-channel process, and so the virtual photon has momentum squared s, and so we would get a pole if the virtual photon were to go on shell.

The similarities between these two results are not a coincidence. They are due to something called **crossing symmetry**. Consider the tree level diagrams for these processes:



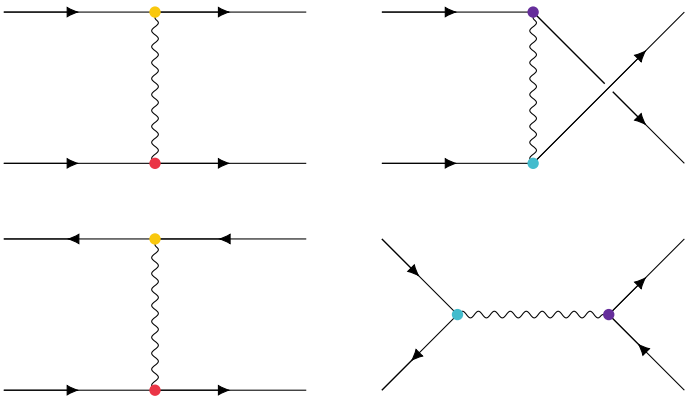
(25.2.3)

Here we limit the number of labels to make the following discussion clearer. Ignore what's happening in the diagram, just focus on the external particles. On the left we have an incoming muon with momentum q, and on the right we have an outgoing muon with momentum p'. On the left we have an outgoing electron with momentum p', on the right we have an incoming *positron* with momentum q.

If we swap these legs of the diagrams, picking up a minus sign to have the electron's direction in spacetime reversed (so it becomes a positron) then these diagrams transform into each other. That is, to go from the amplitude for electron muon scattering to the amplitude for electron-positron annihilation we can just make the swap $p' \leftrightarrow q$. In terms of the Mandelstam variables this corresponds to $t \leftrightarrow s$, which we can see clearly transforms one amplitude into the other. This is the so-called crossing symmetry for this pair of diagrams.

Crossing symmetry is a general property of S matrix elements in any theory, and follows from unitarity and the CPT theorem. The crossing symmetry applies to the elements of the S matrix, and by extension total amplitudes up to some order. It does not apply to individual diagrams. It just so happened that in the preceding example we only had one diagram for each process to tree level.

Consider the following diagrams:



(25.2.4)

The top two contribute to Møeller scattering, $e^-e^- \to e^-e^-$, and the bottom two contribute to Bhabha scattering, $e^-e^+ \to e^-e^+$. Notice the coloured vertices. One can go from Møeller scattering to Bhabha scattering if one takes each coloured vertex in the top line to the corresponding vertex in the bottom line. After reorienting the lines going into the vertex it is then possible to manipulate the Møeller diagrams to give the Bhabha diagrams and vice versa. To do so notice that in the u-channel Møeller diagram the outgoing particle line at the bottom left, the one we usually label as momentum q', becomes an incoming *anti*particle in the s-channel Bhabha diagram, in the position we usually label q. Similarly the incoming and outgoing particles with momenta q and q' in the t-channel Møeller process become outgoing and incoming *anti*particles with momenta q' and q in the t-channel Bhabha process.

So, these two processes are related by $q \leftrightarrow -q'$, with the negative coming from turning a particle into an antiparticle or vice versa. In terms of Mandelstam variables we have $s \leftrightarrow u$. If we were to compute the amplitudes for each diagram we would see that these operations do not map them onto each other. However, in

the high energy limit, it can be shown that the amplitude for Møeller scattering is

$$\overline{|\mathcal{M}|^2} = 2e^4\left(\frac{u-s}{t}+\frac{t-s}{u}\right)^2, \tag{25.2.5}$$

and the amplitude for Bhabha scattering is

$$\overline{|\mathcal{M}|^2} = 2e^4\left(\frac{s-u}{t}+\frac{t-u}{s}\right)^2, \tag{25.2.6}$$

and these do transform into each other under $s \leftrightarrow u$. This is what we mean by the crossing symmetry being a property of the S matrix element rather than individual diagrams.

25.3 Properties of the S matrix

It is possible to get quite a long way just considering properties of the S matrix. We've already come across the three most important:

- Unitarity, this is required for conservation of probability.
- Crossing symmetry, this corresponds to turning some particles into antiparticles and vice versa (i.e., the CPT theorem) and rearranging the kinematics.
- Analyticity in s, t, and u.

This last property, meaning that elements of the S matrix are analytic functions of s, t, and u, has not been mentioned explicitly so far, but we have discussed analytic continuation of certain related functions. For an analytic function to have nontrivial structure, i.e., be more than just a constant, it must have poles, this is Liouville's theorem[2].

[2] See the *Complex Analysis* part of *Methods of Theoretical Physics*

Twenty-Six

Compton Scattering

26.1 Amplitude

They're damn clever these particles, they know what's going on.

Part VIII

Gaussian Integrals

Twenty-Seven

Gaussian Integrals

This part of the course is full of **Gaussian integrals**, so we'll spend some time to get comfortable with them, starting from the simplest one-dimensional case, and then adding in constants, before moving on to the arbitrary dimension case.

27.1 One Dimension

27.1.1 No Constant Coefficient

It is well known that

$$Z_1 \coloneqq \int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left\{-\frac{1}{2}x^2\right\} = \sqrt{2\pi}. \tag{27.1.1}$$

This follows by considering

$$Z_1^2 = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \, \exp\left\{-\frac{1}{2}(x^2+y^2)\right\} \tag{27.1.2}$$

$$= \int_0^{2\pi} \mathrm{d}\vartheta \int_0^{\infty} \mathrm{d}r \, r \exp\left\{-\frac{1}{2}r^2\right\} \tag{27.1.3}$$

$$= 2\pi \left[-\exp\left\{-\frac{1}{2}r^2\right\}\right]_0^{\infty} \tag{27.1.4}$$

$$= 2\pi, \tag{27.1.5}$$

and so $Z_1 = \sqrt{Z_1^2} = 2\pi$.

27.1.2 With Coefficient

If we include a coefficient, a, with $\mathrm{Re}(a) > 0$ then we have

$$Z_a \coloneqq \int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left\{-\frac{1}{2}ax^2\right\} = \sqrt{\frac{2\pi}{a}}, \tag{27.1.6}$$

For $a \in \mathbb{R}$ this follows by a change of variables, $x \mapsto u/\sqrt{a}$, so $\mathrm{d}x = \mathrm{d}u/\sqrt{a}$, and since $a > 0$ and so $\sqrt{a} > 0$ the integration limits are unchanged so we have

$$Z_a = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} \mathrm{d}u \, \exp\left\{-\frac{1}{2}u^2\right\} = \frac{1}{\sqrt{a}} Z_1 = \sqrt{\frac{2\pi}{a}}. \tag{27.1.7}$$

The same thing holds for $a \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$, but we have to consider some contour integral and its not as straight forward to show.

27.1.3 With Linear Term

Now consider

$$Z_a(b) \coloneqq \int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left\{-\frac{1}{2}ax^2 + bx\right\} = \sqrt{\frac{2\pi}{a}} \exp\left\{-\frac{b^2}{2a}\right\}. \tag{27.1.8}$$

This is shown by completing the square:

$$-\frac{1}{2}ax^2 + bx = -\frac{a}{2}\left(x + \frac{b}{a}\right)^2 + \frac{b^2}{2a}. \tag{27.1.9}$$

Hence,

$$Z_a(b) = \int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left\{-\frac{1}{2}a\left(x - \frac{b}{a}\right)\right\} \exp\left\{-\frac{b^2}{2a}\right\}. \tag{27.1.10}$$

Making a change of variables $x \mapsto u + b/a$, so $\mathrm{d}x = \mathrm{d}u$ and the integration region is unchanged we get

$$Z_a(b) = \exp\left\{-\frac{b^2}{2a}\right\} \int_{-\infty}^{\infty} \mathrm{d}u \, \exp\left\{-\frac{1}{2}au^2\right\} = \exp\left\{-\frac{b^2}{2a}\right\} Z_a = \sqrt{\frac{2\pi}{a}} \exp\left\{-\frac{b^2}{2a}\right\}.$$

27.2 Arbitrary Dimension

27.2.1 Diagonal Case

Let $x \in \mathbb{R}^n$, then

$$Z_I \coloneqq \int \mathrm{d}^n x \, \exp\left\{-\frac{1}{2}x^{\mathsf{T}}x\right\} = (2\pi)^{n/2} \tag{27.2.1}$$

where an integral without bounds is understood to be over all of $\mathbb{R}^n$. This follows since $x^{\mathsf{T}}x = x_i x_i$ and we then can factorise this into a product of n one-dimensional Gaussian integrals, each giving a factor of $\sqrt{2\pi}$:

$$Z_I = (Z_1)^n = (2\pi)^{n/2}. \tag{27.2.2}$$

Now let $A \in \mathcal{M}_n(\mathbb{C})$ be diagonal with $\mathrm{Re}(A_{ij}) > 0$ for all $i, j = 1, \dots, n$. Then we have

$$Z_A \coloneqq \int \mathrm{d}^n x \, \exp\left\{-\frac{1}{2}x^{\mathsf{T}}Ax\right\} = (2\pi)^{n/2} \left(\prod_{i=1}^{n} \frac{1}{\sqrt{A_{ij}}}\right). \tag{27.2.3}$$

This follows since $x^{\mathsf{T}}Ax = x_i A_{ij} x_j$, and for A diagonal we have $x^{\mathsf{T}}Ax = x_i^2 A_{ii}$, so Z_A factorises into a product of one-dimensional Gaussian integrals:

$$Z_A = \prod_{i=1}^{n} Z_{A_{ii}} = (2\pi)^{n/2} \frac{1}{\sqrt{A_{ij}}}. \tag{27.2.4}$$

Now notice that for a diagonal matrix

$$\det A = \prod_{i=1}^{n} A_{ii}, \tag{27.2.5}$$

so

$$Z_A = (2\pi)^{n/2} (\det A)^{-1/2}. \tag{27.2.6}$$

27.2.2 Diagonalisable Case

Suppose that instead of being diagonal A is symmetric, and hence diagonalisable. That is, there exists some $D \in \mathrm{SO}(n)$ such that $A' = D^{-1}AD$ is diagonal. Consider the change of variables $x \mapsto Dx$, and $x^\mathsf{T} \mapsto x^\mathsf{T} D^\mathsf{T} = x^\mathsf{T} D^{-1}$. Since $\det D = 1$ we have $\mathrm{d}^n x \mapsto \det(D)\mathrm{d}^n x = \mathrm{d}^n x$. Under this transformation we then have

$$\begin{aligned} Z_A &= \int \mathrm{d}^n x \, \exp\left\{-\frac{1}{2}x^\mathsf{T} A x\right\} && (27.2.7)\\ &= \int \mathrm{d}^n x \, \exp\left\{-\frac{1}{2}x^\mathsf{T} D^{-1} A D x\right\} && (27.2.8)\\ &= (2\pi)^{n/2}(\det(D^{-1}AD))^{-1/2} && (27.2.9)\\ &= (2\pi)^{n/2}(\det A)^{-1/2}, && (27.2.10) \end{aligned}$$

having used

$$\det(D^{-1}AD) = \det(D^{-1})\det(A)\det(D) = \det(D)^{-1}\det(A)\det(D) = \det A.$$

This shows that if $A \in \mathcal{M}_n(\mathbb{C})$ is a symmetric matrix with $\mathrm{Re}(A_{ij}) \geq 0$ for all $i, j = 1, \dots, n$ and the eigenvalues of A, call them a_i, are nonzero, which guarantees that $\det A \neq 0$, then

$$Z_A \coloneqq \int \mathrm{d}^n x \, \exp\left\{-\frac{1}{2}x^\mathsf{T} A x\right\} = (2\pi)^{n/2}(\det A)^{-1/2}. \tag{27.2.11}$$

27.2.3 With Linear Term

Now let $b \in \mathbb{R}^n$ and consider

$$Z_A(b) \coloneqq \int \mathrm{d}^n x \, \exp\left[-\frac{1}{2}x^\mathsf{T} A x + b^\mathsf{T} x\right]. \tag{27.2.12}$$

Consider the change of variables

$$x = u + \Delta b \tag{27.2.13}$$

where $\Delta = A^{-1}$. We then have $\mathrm{d}^n x = \mathrm{d}^n u$, and there is no change to the region of integration. Note that since A is symmetric so is Δ. We then have

$$\begin{aligned} -\frac{1}{2}x^\mathsf{T} A x &= -\frac{1}{2}(u^\mathsf{T} + b^\mathsf{T}\Delta^\mathsf{T})A(u + \Delta b) && (27.2.14)\\ &= -\frac{1}{2}(u^\mathsf{T} A u + u^\mathsf{T} A \Delta b + b^\mathsf{T}\Delta^\mathsf{T} A u + b^\mathsf{T}\Delta^\mathsf{T} A \Delta b). && (27.2.15) \end{aligned}$$

Now we can use $\Delta^\mathsf{T} = \Delta$ and $u^\mathsf{T} b = u_i b_i = b^\mathsf{T} u$ to get

$$\begin{aligned} -\frac{1}{2}x^\mathsf{T} A x &= -\frac{1}{2}(u^\mathsf{T} A u + u^\mathsf{T} b + b^\mathsf{T} u + b^\mathsf{T}\Delta b) && (27.2.16)\\ &= -\frac{1}{2}u^\mathsf{T} A u - b^\mathsf{T} u - b\Delta b. && (27.2.17) \end{aligned}$$

Similarly,

$$b^\mathsf{T} x = b^\mathsf{T} u + b^\mathsf{T}\Delta b, \tag{27.2.18}$$

and so

$$-\frac{1}{2}x^\top Ax + b^\top x = -\frac{1}{2}u^\top Au - b^\top u - \frac{1}{2}b^\top \Delta b + b^\top u + b^\top \Delta b \tag{27.2.19}$$

$$= -\frac{1}{2}u^\top Au + \frac{1}{2}b^\top \Delta b. \tag{27.2.20}$$

Hence, we have

$$\exp\left\{-\frac{1}{2}x^\top Ax + b^\top x\right\} = \exp\left\{-\frac{1}{2}u^\top Au\right\}\exp\left\{\frac{1}{2}b^\top \Delta b\right\}. \tag{27.2.21}$$

Note that these are just exponentials of numbers, so everything commutes and the usual exponential laws apply. Finally, we have

$$Z_A(b) = \int \mathrm{d}^n x\, \exp\left\{-\frac{1}{2}x^\top Ax + b^\top x\right\} \tag{27.2.22}$$

$$= \exp\left\{\frac{1}{2}b^\top \Delta b\right\}\int \mathrm{d}^n u\, \exp\left\{-\frac{1}{2}u^\top Au\right\} \tag{27.2.23}$$

$$= Z_A \exp\left\{\frac{1}{2}b^\top \Delta b\right\} \tag{27.2.24}$$

$$= (2\pi)^{n/2}(\det A)^{-1/2}\exp\left\{-\frac{1}{2}b^\top \Delta b\right\}. \tag{27.2.25}$$

27.3 Generating Function

Let μ be a measure on $\mathbb{R}^n$. For our purposes a measure, μ, is something such that $\mathrm{d}\mu(x) = \mathrm{d}^n x\, \Omega(x)$ where Ω is some integrable function. We define the **expectation value** of $F\colon\ \mathbb{R}^n \to \mathbb{R}$ with respect to μ to be

$$\langle F\rangle_\mu \coloneqq \int \mathrm{d}\mu(x)\, F(x) = \int \mathrm{d}^n x\, \Omega(x) F(x). \tag{27.3.1}$$

We assume that the measure is normalised such that

$$\int \mathrm{d}\mu(x) = 1. \tag{27.3.2}$$

This allows for a probability interpretation of our results.

We then define the **generating function**

$$Z_\mu(b) \coloneqq \langle \mathrm{e}^{b^\top x}\rangle_\mu = \int \mathrm{d}\mu(x)\, \exp\{b^\top x\}. \tag{27.3.3}$$

We can expand the exponential in a series to get

$$\exp\{b^\top x\} = \sum_{\ell=0}^{\infty}\frac{1}{\ell!}(b^\top x)^n = \sum_{\ell=0}^{\infty}\sum_{i_1,\ldots,i_\ell=1}^{n} b_{i_1}\cdots b_{i_\ell}x_{i_1}\cdots x_{i_\ell}. \tag{27.3.4}$$

Applying this to the generating function, and realising that we can use the linearity of the integral to pull out all the sums and factors of b_i, we have

$$Z_\mu(b) = \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\sum_{i_1,\ldots,i_\ell=1}^{n} b_{i_1}\cdots b_{i_\ell}\langle x_{i_1}\cdots x_{i_\ell}\rangle_\mu \tag{27.3.5}$$

We call

$$\langle x_{i_1} \cdots x_{i_\ell} \rangle_\mu = \int \mathrm{d}\mu(x)\, x_{i_1} \cdots x_{i_\ell} \tag{27.3.6}$$

the **correlation function** of x with respect to the measure μ. In particular it's called the ℓ-point correlation function, since it has ℓ points, x_{i_1} through x_{i_ℓ}.

From this work we can see that the generating function is simply a polynomial in b_i, after computing the integrals in the correlation functions they are just numbers. We can then take derivatives to get

$$\frac{\partial}{\partial_k} Z_\mu(b) = \int \mathrm{d}\mu(x)\, \frac{\partial}{\partial b_k} \exp\{b_i x_i\} = \int \mathrm{d}\mu(x)\, x_k \exp\{b_i x_i\}. \tag{27.3.7}$$

This allows us to write the correlation function as

$$\langle x_{i_1} \cdots x_{i_\ell} \rangle_\mu = \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_\ell}} Z_\mu(b)\Big|_{b=0}, \tag{27.3.8}$$

since $\exp[b^\mathsf{T} x]|_{b=0} = 1$.

We can use this same method to find the expectation value of any function with a Taylor series. If

$$F(x) = \sum_{\ell=0}^{\infty} \sum_{i_1,\ldots,i_\ell=1}^{n} F_{i_1,\ldots,i_\ell} x_{i_1} \cdots x_{i_\ell} \tag{27.3.9}$$

for some constants $F_{i_1,\ldots,i_\ell}$, then

$$\langle F \rangle_\mu = \sum_{\ell=0}^{\infty} \sum_{i_1,\ldots,i_\ell=1}^{n} F_{i_1,\ldots,i_\ell} \langle x_{i_1} \cdots x_{i_\ell} \rangle_\mu. \tag{27.3.10}$$

27.4 Gaussian Generating Functions

Consider the **Gaussian measure**

$$\mathrm{d}\mu_0(x) \coloneqq \mathrm{d}^n x\, \Omega_0(x) = \mathrm{d}^n x\, \mathcal{N}_0 \exp\left\{-\frac{1}{2} x^\mathsf{T} A x\right\} \tag{27.4.1}$$

where $A \in \mathcal{M}_n(\mathbb{C})$ is a symmetric matrix with $\mathrm{Re}(A_{ij}) \geq 0$ for all $i, j = 1, \ldots, n$ and with nonzero eigenvalues, and $\mathcal{N}_0$ is the normalisation factor

$$\mathcal{N}_0 = (2\pi)^{-n/2} (\det A)^{1/2} = Z_A^{-1} \tag{27.4.2}$$

which ensures that

$$\int \mathrm{d}\mu_0(x) = 1. \tag{27.4.3}$$

We can then define the **Gaussian generating function**,

$$Z_0(b) = \frac{Z_A(b)}{Z_A} = \exp\left\{\frac{1}{2} b^\mathsf{T} \Delta b\right\}. \tag{27.4.4}$$

Then the **Gaussian correlation function** is

$$\langle x_{i_1} \cdots x_{i_\ell} \rangle_0 := \langle x_{i_1} \cdots x_{i_\ell} \rangle_{\mu_0} = \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_\ell}} \exp\left\{\frac{1}{2} b^\top \Delta b\right\}\Big|_{b=0}. \tag{27.4.5}$$

If F is an arbitrary function with a Taylor series as in Equation (27.3.9) then

$$\langle F \rangle_0 = \sum_{\ell=0}^{\infty} \sum_{i_1,\ldots,i_\ell=1}^{n} F_{i_1,\ldots,i_\ell} \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_\ell}} \exp\left\{\frac{1}{2} b^\top \Delta b\right\}\Big|_{b=0} \tag{27.4.6}$$

$$= F(\partial/\partial b) \exp\left\{\frac{1}{2} b^\top \Delta b\right\}\Big|_{b=0} \tag{27.4.7}$$

where $F(\partial/\partial b)$ is a formal operator expression given by "evaluating" the Taylor series for F at the partial derivatives:

$$F(\partial/\partial b) = \sum_{\ell=0}^{\infty} \sum_{i_1,\ldots,i_\ell=1}^{n} F_{i_1,\ldots,i_\ell} \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_\ell}}. \tag{27.4.8}$$

27.4.1 One Point Gaussian Correlator

Consider the one point Gaussian correlation function,

$$\langle x_k \rangle_0 = \frac{\partial}{\partial b_k} \langle \exp\{b^\top x\} \rangle_0 \Big|_{b=0} = \frac{\partial}{\partial b_k} \exp\left\{\frac{1}{2} b^\top \Delta b\right\}. \tag{27.4.9}$$

We have

$$\frac{\partial}{\partial b_k} \exp\left\{\frac{1}{2} b_i \Delta_{ij} b_j\right\} = \frac{1}{2}[\Delta_{kj} b_j + b_i \Delta_{ik}] \exp\{b^\top \Delta b\} = \Delta_{kj} b_j \exp\{b^\top \Delta b\}, \tag{27.4.10}$$

where we've used the symmetry of Δ and $\partial b_i / \partial b_k = \delta_{ik}$. We then have

$$\langle x_k \rangle_0 = (\Delta_{kj} b_j) \exp\left\{\frac{1}{2} b^\top \Delta b\right\}\Big|_{b=0} = 0. \tag{27.4.11}$$

We could have realised that this must vanish since our integrand is an odd function of x over an even range. We can interpret the one point correlation as the mean of a normally distributed value.

27.4.2 Two Point Gaussian Correlator

Consider the two point Gaussian correlation function,

$$\langle x_k x_l \rangle_0 = \frac{\partial}{\partial b_k} \frac{\partial}{\partial b_l} \langle \exp\{b^\top x\} \rangle_0 \Big|_{b=0}. \tag{27.4.12}$$

We've already done the first derivative, so we just have to compute

$$\langle x_k x_l \rangle_0 = \frac{\partial}{\partial b_l} \left((\Delta_{kjb_j}) \exp\left\{\frac{1}{2} b^\top \Delta b\right\} \right)\Big|_{b=0} \tag{27.4.13}$$

$$= [\delta_{kkj} \Delta_{kj} + (\Delta_{kj} b_j)(\Delta_{li} b_i)] \exp\left\{\frac{1}{2} b^\top b\right\}\Big|_{b=0} \tag{27.4.14}$$

$$= \Delta_{kl}. \tag{27.4.15}$$

We can interpret the two point correlation as the covariance of two normally distributed values.

27.4.3 General Gaussian Correlator

Now consider the ℓ-point Gaussian correlation function

$$\langle x_{i_1} \cdots x_{i_\ell} \rangle_0 = \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_\ell}} \langle \exp\{b^{\mathsf{T}} x\} \rangle_0 \Big|_{b=0} . \tag{27.4.16}$$

In general the derivatives will either act on the exponential term, giving a factor of Δb, or they can act on an existing Δb term giving a Δ term. The only nonzero terms after setting $b = 0$ are those where each Δb has been acted on by a derivative to remove the b dependence. This requires an even number of derivatives to get a nonzer result. In the case when ℓ is even the process for determining the result is as follows:

- Write all pairings (i_p, i_q), which can be formed from the indices $i_1, \ldots, i_\ell$. Treat (i_p, i_q) and (i_q, i_p) as the same and no index can be in a pair with itself.
- Build the set, P, of all sets of pairings, so the elements of P are sets of $\ell/2$ pairs such that every index appears in exactly one pair.
- The correlator is given by

$$\langle x_{i_1} \cdots x_{i_\ell} \rangle_0 = \sum_P \Delta_{i_p i_q} \cdots \Delta_{i_r i_s} . \tag{27.4.17}$$

This is **Wick's theorem**.

To make this clearer lets consider some examples. First, the two point correlator, $\langle x_{i_1} x_{i_2} \rangle_0$. We have two indices, i_1 and i_ℓ, so there is a single pairing, (i_1, i_2), and we have

$$\langle x_{i_1} x_{i_2} \rangle_0 = \Delta_{i_1 i_2} . \tag{27.4.18}$$

We can also write this as

$$\langle x_{i_1} x_{i_2} \rangle_0 = \overbrace{x_{i_1} x_{i_2}} \tag{27.4.19}$$

where

$$\overbrace{x_{i_1} x_{i_2}} := \Delta_{i_1 i_2} . \tag{27.4.20}$$

Now consider the four point correlator, $\langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle_0$. The set of possible pairings is

$$P = \{\{(i_1, i_2), (i_3, i_4)\}, \{(i_1, i_3), (i_2, i_4)\}, \{(i_1, i_4), (i_2, i_3)\}\} . \tag{27.4.21}$$

The correlator is then

$$\langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle_0 = \Delta_{i_1 i_2} \Delta_{i_3 i_4} + \Delta_{i_1 i_3} \Delta_{i_2 i_3} + \Delta_{i_1 i_4} \Delta_{i_2 i_3} \tag{27.4.22}$$

$$= \overbrace{x_{i_1} x_{i_2}} \overbrace{x_{i_3} x_{i_4}} + \overbrace{x_{i_1} x_{i_2} x_{i_3} x_{i_4}} + \overbrace{x_{i_1} x_{i_2} x_{i_3} x_{i_4}} . \tag{27.4.23}$$

Finally, lets do the six point correlation function. We'll jump straight to the contractions:

$$\langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} \rangle = x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} + x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} \tag{27.4.24}$$

$$+ x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} + x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} \tag{27.4.25}$$

$$+ x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} + x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} \tag{27.4.26}$$

$$+ x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} + x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} \tag{27.4.27}$$

$$+ x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} + x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} \tag{27.4.28}$$

$$+ x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} + x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} \tag{27.4.29}$$

$$+ x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} + x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} \tag{27.4.30}$$

$$+ x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} \tag{27.4.31}$$

Code 27.4.32 Here's some Mathematica code to generate all pairs. WARNING: Horribly inefficient code! It generates all permutations of the indices, groups them into pairs, then eliminates repeats, so its $\mathcal{O}(\ell!)$ in memory, I can't even run it for $\ell = 12$.

```
Union[Sort /@
  Union[Sort /@ Partition[#[[1]], 2] & /@
   Table[
     {#[[i]], #[[i + 1]]}, {i, 1, Length[#], 2}] &@
   Permutations[Range[ℓ]]
(* replace ℓ with number of indices (even) *)
```

As you can see the number of indices grows quickly. Suppose we have $2m$ indices for some integer m. Then there are $2m-1$ options for what to pair the first variable with, since it can't be paired with itself. Then take the next unpaired variable, there are $2m-3$ variables to pair it with, since it can't be paired with itself or either of the already paired variables. The next unpaired variable has $2m-5$ potential pairings, and so on, until we have four variables left, one of these has three options for pairing, and then we're left with two variables which we have to pair. That is, the number of pairings is given by

$$\text{number of pairings} = (2m-1)(2m-3)\cdots 3 \cdots 1 = (2m-1)!! \tag{27.4.33}$$

where !! is the double factorial, defined recursively as $n!! = n(n-2)!!$ with $1!! = 0!! = 1$. Here are the number of pairings for the first few nonzero correlators:

2	4	6	8	10	12	14	16	18	20
1	3	15	105	945	10 395	135 135	2 027 025	34 459 425	654 729 075

So its fair to say that evaluating general correlators with more than a couple of points quickly becomes intractable.

Twenty-Eight

Perturbed Correlators

28.1 General Theory

As with the Gaussian case we can write the expected value of any analytic function, F, with respect to some measure, μ, as a sum over correlators with respect to μ:

$$\langle F\rangle_\mu = \sum_{\ell=0}^{\infty}\sum_{i_1,\ldots,i_\ell=1}^{n} F_{i_1\ldots i_\ell}\langle x_{i_1}\cdots x_{i_\ell}\rangle_\mu \tag{28.1.1}$$

$$= \sum_{\ell=0}^{\infty}\sum_{i_1,\ldots,i_\ell=1}^{n} F_{i_1\ldots i_\ell}\frac{\partial}{\partial b_{i_1}}\cdots\frac{\partial}{\partial b_{i_\ell}}Z_\mu(b)\Big|_{b=0} \tag{28.1.2}$$

$$= F(\partial/\partial b)Z_\mu(b)|_{b=0}. \tag{28.1.3}$$

We now consider a slightly more general measure, $\mathrm{d}\mu(x) = \mathrm{d}^n x\,\Omega(x)$ where

$$\Omega(x) = \frac{1}{Z_\lambda}\mathrm{e}^{-S_\lambda(x)} \tag{28.1.4}$$

where

$$S_\lambda(x) = S_0(x) + \lambda V(x) \tag{28.1.5}$$

with $S_0(x) = x^\mathsf{T}Ax/2$, giving the usual Gaussian measure, and V being some "potential" acting as a perturbation. We will assume λ is small and consider expansions in λ. The factor Z_λ is given by

$$Z_\lambda = \int \mathrm{d}^n x\,\exp[-S_\lambda(x)] \tag{28.1.6}$$

$$= \int \mathrm{d}^n x\,\exp[-S_0(x)]\exp[-\lambda V(x)] \tag{28.1.7}$$

$$= Z_0\langle\exp[-\lambda V(x)]\rangle_0. \tag{28.1.8}$$

We will be interested in computing the value of

$$\frac{Z_\lambda}{Z_0} = \langle\exp[-\lambda V(x)]\rangle_0 = \sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}\langle V(x)^k\rangle_0. \tag{28.1.9}$$

We can work out the expectation value once we know V by expanding V.

28.2 x^4 Potential

The first instant of a long series of asking "how do we count?"

Luigi Del Debbio

Consider the example

$$V(x) = \frac{1}{4!}\sum_{i=1}^{n} x_i^4. \tag{28.2.1}$$

We can quickly write down a series for Z_λ/Z_0 in terms of correlators:

$$\frac{Z_\lambda}{Z_0} = 1 - \lambda\frac{1}{4!}\sum_i \langle x_i^4\rangle_0 + \frac{\lambda^2}{2}\frac{1}{(4!)^2}\sum_i\sum_j \langle x_i^4 x_j^4\rangle_0 + \mathcal{O}(\lambda^3). \tag{28.2.2}$$

Now we just need to compute the correlators.

Starting with $\langle x_i^4\rangle_0$. This is just a special case of the four point correlator in Equation (27.4.22) where we set all indices equal. This means that we have

$$\langle x_i x_i x_i x_i\rangle_0 = \Delta_{ii}\Delta_{ii} + \Delta_{ii}\Delta_{ii} + \Delta_{ii}\Delta_{ii} \tag{28.2.3}$$

$$= \overbrace{x_i x_i}\,\overbrace{x_i x_i} + \overbrace{x_i\,\overbrace{x_i x_i}\,x_i} + \overbrace{x_i\,\overbrace{x_i x_i}\,x_i} \tag{28.2.4}$$

$$= 3\Delta_{ii}^2. \tag{28.2.5}$$

Notice that we could have worked this out without needing to do the full four point correlator for arbitrary indices, we just need to find a way to count the number of pairings. In this case there are 3 choices for what to pair the first x_i with, then the two remaining x_is must be paired, so there are 3 terms, and all three are the same.

This is good because as we saw for the six point correlator things quickly get out of hand, and the next term involves an 8 point correlator, which would be even worse at $(8-1)!! = 105$ terms. So, what sort of pairings are there going to be in this correlator? First there are going to be pairings akin to two copies of the four point correlator, with all x_is paired with x_is, and all x_js paired with x_js. There are nine ways to do this, getting a factor of three from pairing up x_is and a factor of three from pairing up x_js. This means we have the term $9\Delta_{ii}^2\Delta_{jj}^2$.

The next logical term to consider is one which pairs one x_i with one x_j, however, if we do so we're left with an odd number of x_is, so another of these must pair with an x_j. So, our next term pairs two x_is with two x_js, and then the remaining x_is are paired with each other and likewise for the remaining x_js. How many ways are there to do this? There are actually (at least) two ways to count this. First, we have $\binom{4}{2}$ ways to choose the two x_i to pair with x_js. Then we have four choices of which x_j to pair with the first and 3 choices of which x_j to pair with the second, for a total of $\binom{4}{2}\cdot 4\cdot 3 = 72$ terms. The second way to count starts with choosing two x_is, again giving a factor of $\binom{4}{2}$, and then choosing two x_js, for another factor of $\binom{4}{2}$. We can then pair up between these pairs in two ways, pairing the first of each pair, or the first of the x_is and the second of the x_js, for a factor of 2, giving $\binom{4}{2}^2\cdot 2 = 72$. Hence, we have the term $72\Delta_{ij}^2\Delta_{ii}\Delta_{jj}$.

There's no way to pair up exactly three x_is with x_js, so the only other term is pairing all four x_is with an x_j. This can be done by holding the x_i in some fixed

order, then considering all permutations of the four x_js, and pairing them up in order for each permutation. So, we have a factor of $4! = 24$. The final term is then $24\Delta_{ij}^4$.

At this point there are some checks we can do. First, we should, including repeated terms, have 8 indices, and 4 Δs in each term. Second, adding up the combinatorial factors should give 105, the total number of ways to pair 8 things, and indeed this does work out. So, we conclude that

$$\langle x_i^4 x_j^4 \rangle_0 = 9\Delta_{ii}^2\Delta_{jj}^2 + 72\Delta_{ij}^2\Delta_{ii}\Delta_{jj} + 24\Delta_{ij}^4. \tag{28.2.6}$$

Hence, to second order, we have

$$\frac{Z_\lambda}{Z_0} = 1 - \frac{\lambda}{4!}\sum_i 3\Delta_{ii}^2 + \frac{\lambda^2}{2(4!)^2}\sum_{i,j}[9\Delta_{ii}^2\Delta_{jj}^2 + 72\Delta_{ij}^2\Delta_{ii}\Delta_{jj} + 24\Delta_{ij}^4] \tag{28.2.7}$$

$$= 1 - \frac{\lambda}{8}\sum_i \Delta_{ii}^2 + \lambda^2\sum_{i,j}\left[\frac{1}{128}\Delta_{ii}^2\Delta_{jj}^2 + \frac{1}{16}\Delta_{ij}^2\Delta_{ii}\Delta_{jj} + \frac{1}{48}\Delta_{ij}^4\right] \tag{28.2.8}$$

28.2.1 Diagrammatic Representation

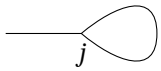
There is a diagrammatic representation of terms like this. For each propagator, Δ_{ij}, we draw a line connecting the nodes i and j:

$\Delta_{ij} =$  (28.2.9)

Note that the position of the nodes doesn't matter, just how they're connected. For terms like Δ_{ii} with a repeated index we have a sum over i when these appear. As a diagram a repeated index means a closed loop, and so any closed loop implies a sum:

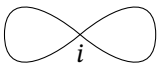
$\sum_i \Delta_{ii} =$  (28.2.10)

Another example might be the term $\sum_j \Delta_{ij}\Delta_{jj}$, which corresponds to the diagram

$\sum_j \Delta_{ij}\Delta_{jj} =$  (28.2.11)

Fun Fact™: Diagrams like this one are called tadpole diagrams!

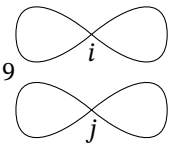
Using this we can write the first order (in λ) Δ_{ii}^2 term as

$\sum_i \Delta_{ii}^2 =$  (28.2.12)

Similarly we can write the following

$$\langle x_i^4 x_j^4 \rangle_0 = 9\Delta_{ii}^2\Delta_{jj}^2 + 72\Delta_{ij}^2\Delta_{ii}\Delta_{jj} + 24\Delta_{ij}^4. \tag{28.2.13}$$

as

$\sum_{i,j}\langle x_i^4 x_j^4 \rangle = 9$  $+72$  $+24$  (28.2.14)

28.2.2 Computing $\log(Z_\lambda/Z_0)$

Suppose we want to compute $\log(Z_\lambda/Z_0)$. To do this we notice that $Z_\lambda/Z_0 = 1 + \mathcal{O}(\lambda)$, and so we can use the Taylor expansion

$$\log(1+\varepsilon) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\varepsilon^n \approx \varepsilon - \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 - \cdots . \tag{28.2.15}$$

Since we already have ε here as a power series in λ we should work this out in powers of λ. The first order term has only a single contribution, from the first order term in λ. That is, to first order

$$\log\left(\frac{Z_\lambda}{Z_0}\right) = -\frac{\lambda}{8}\sum_i \Delta_{ii}^2 + \mathcal{O}(\lambda^2). \tag{28.2.16}$$

For the second order term there are two contributions, we have the first order term squared, which also picks up a factor of $-1/2$ from the expansion of the log:

$$-\frac{1}{2}\frac{\lambda^2}{8^2}\sum_i \Delta_{ii}^2 \sum_j \Delta_{jj}^2 = -\frac{1}{2}\frac{\lambda^2}{128}\sum_{i,j}\Delta_{ii}^2\Delta_{jj}^2. \tag{28.2.17}$$

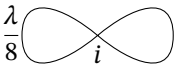
We also have the first order in ε term from the second order in λ term:

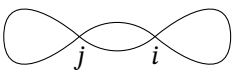
$$\lambda^2 \sum_{i,j}\left[\frac{1}{128}\Delta_{ii}^2\Delta_{jj}^2 + \frac{1}{16}\Delta_{ij}^2\Delta_{ii}\Delta_{jj} + \frac{1}{48}\Delta_{ij}^4\right]. \tag{28.2.18}$$

Notice that the first term here cancels with the other λ^2 term, to give the second order result

$$\log\left(\frac{Z_\lambda}{Z_0}\right) = -\frac{\lambda}{8}\sum_i \Delta_{ii}^2 + \lambda^2\left[\frac{1}{128}\Delta_{ii}^2\Delta_{jj}^2 + \frac{1}{16}\Delta_{ij}^2\Delta_{ii}\Delta_{jj} + \frac{1}{48}\Delta_{ij}^4\right] + \mathcal{O}(\lambda^3). \tag{28.2.19}$$

In terms of diagrams, this corresponds to

$$\log\left(\frac{Z_\lambda}{Z_0}\right) = -\frac{\lambda}{8}$$ 

$$+\frac{\lambda^2}{16}$$ 

$$+\frac{\lambda^2}{48}$$ 

$$+\mathcal{O}(\lambda^3). \tag{28.2.20}$$

Notice that there are no disconnected diagrams, they cancelled out. In fact disconnected diagrams will cancel out here to all orders, and this isn't just the case for this potential. Taking the log of the generating function yields the generating function of connected contributions. We'll prove this later in Section 32.3.

28.3 Correlators

That's it. That's QFT done.

Luigi Del Debbio

Much of the rest of the course will be focused on computing various correlators with respect to some perturbed Gaussian, and of course linking these mathematical expressions to physics. We'll often be interested in calculating

$$\begin{aligned}\langle x_{i_1}\cdots x_{i_\ell}\rangle_\lambda &= \frac{1}{Z_\lambda}\int \mathrm{d}^n x \exp[-S_\lambda(x)] x_{i_1}\cdots x_{i_\ell} && (28.3.1)\\ &= \frac{Z_0}{Z_\lambda}\frac{1}{Z_0}\int \mathrm{d}^n x \exp[-S_\lambda(x)] x_{i_1}\cdots x_{i_\ell} && (28.3.2)\\ &= \frac{Z_0}{Z_\lambda}\frac{1}{Z_0}\int \mathrm{d}^n x \exp[-S_0(x)] \sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!} V(x)^k x_{i_1}\cdots x_{i_\ell} && (28.3.3)\\ &= \frac{Z_0}{Z_\lambda}\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}\langle V(x)^k x_{i_1}\cdots x_{i_\ell}\rangle_0. && (28.3.4)\end{aligned}$$

The good news is that we've just seen how to compute Z_λ/Z_0, so all we need to do is invert this result to get Z_0/Z_λ. This can be done in a similar way to the logarithm by again noticing that $Z_\lambda/Z_0 = 1 - \mathcal{O}(\lambda)$ and then using the result

$$\frac{1}{1-\varepsilon} = \sum_{n=0}^{\infty}\varepsilon^n \approx 1 + \varepsilon + \varepsilon^2 + \cdots . \tag{28.3.5}$$

The results we get will be of the form

$$\langle x_{i_1}\cdots x_{i_\ell}\rangle_0 = \mathcal{O}(\lambda^0) + \mathcal{O}(\lambda) + \mathcal{O}(\lambda^2) + \cdots . \tag{28.3.6}$$

The order of λ^0 (i.e., constant) term will be the result we get with an unperturbed Gaussian. We then have first order corrections, and second order corrections, and so on.

Part IX

Path Integrals in Quantum Mechanics

Twenty-Nine

Path Integrals in QM

29.1 Preliminaries

In this chapter we will develop the theory of path integrals in nonrelativistic quantum mechanics. This should then rapidly generalise to relativistic quantum field theory, but for now we stick to the nonrelativistic case to avoid unnecessary complications.

Working in quantum mechanics we have a position operator, $\hat{Q}$, with eigenstates $|q\rangle$, with eigenvalues q:

$$\hat{Q}|q\rangle = q|q\rangle. \tag{29.1.1}$$

We consider a system evolving under the Hamiltonian $\hat{H}$. We will be interested in computing the transition amplitude between two position eigenstates at two different times. That is, we want to compute

$$\langle q', t'|q, t\rangle \tag{29.1.2}$$

where $t' > t$. This amplitude can be computed in terms of the states $|q\rangle$ and $|q'\rangle$ and the time evolution operator:

$$\langle q', t'|q, t\rangle = \langle q'|e^{-i\hat{H}(t'-t)}|q\rangle. \tag{29.1.3}$$

This is where path integrals come in.

29.2 Constructing the Path Integral

> *We assume this limit exists, and I have no intrest in proving it does.*
>
> Luigi Del Debbio

Let $T = t' - t$, so we're computing

$$\langle q', t'|q, t\rangle = \langle q'|e^{-i\hat{H}T}|q\rangle. \tag{29.2.1}$$

We can split up the time interval between t and t' into $n-1$ steps of length $\varepsilon = T/n$. Define $t_k = t + k\varepsilon$, so $t_0 = t$ and $t_n = t + n\varepsilon = t + T = t'$. We can then take our single factor of $\exp[-i\hat{H}T]$ and split it into n factors of $\exp[-i\hat{H}\varepsilon]$:

$$\langle q', t'|q, t\rangle = \langle q'|\underbrace{e^{-i\hat{H}\varepsilon}\cdots e^{-i\hat{H}\varepsilon}}_{n \text{ factors}}|q\rangle. \tag{29.2.2}$$

Now take the completeness relation,

$$\int \mathrm{d}q\, |q\rangle\langle q| = \mathbb{1}. \tag{29.2.3}$$

We can insert this $n-1$ times between each of the factors of $\exp[-i\hat{H}t]$ giving

$$\int \left(\prod_{k=1}^{n-1} \mathrm{d}q_k\right) \langle q|\mathrm{e}^{-i\hat{H}\varepsilon}|q_{n-1}\rangle\langle q_{n-1}|\mathrm{e}^{-i\hat{H}\varepsilon}|q_{n-2}\rangle\langle q_{n-2}|\cdots|q_1\rangle\langle q_1|\mathrm{e}^{-i\hat{H}\varepsilon}|q\rangle. \tag{29.2.4}$$

Now suppose that ε is small, or equivalently n is large. We can expand the exponential:

$$\mathrm{e}^{-i\hat{H}\varepsilon} = 1 - i\varepsilon\hat{H} + \mathcal{O}(\varepsilon^2). \tag{29.2.5}$$

Suppose we have the Hamiltonian

$$\hat{H} = \frac{\hat{P}^2}{2} + V(\hat{Q}) \tag{29.2.6}$$

where we choose units such that the mass is unity. First consider the terms with the potential, here we simply replace the argument of V with the appropriate position, using $V(\hat{Q})|q\rangle = \sum_{\ell=1}^{\infty} V^{(\ell)}(0)\hat{Q}^\ell|q\rangle = \sum_{\ell=1}^{\infty} V^{(\ell)}(0)q^\ell = V(q)|q\rangle$, and so

$$\begin{aligned}
\langle q_k|V(\hat{Q})|q_{k-1}\rangle &= V(q_{k-1})\langle q_k|q_{k-1}\rangle && (29.2.7)\\
&= V(q_{k-1})\delta(q_k - q_{k-1}) && (29.2.8)\\
&= V\left(\frac{q_k + q_{k-1}}{2}\right)\delta(q_k - q_{k-1}) && (29.2.9)
\end{aligned}$$

where in the last step we note that for nonzero solutions we have $q_k = q_{k-1}$, and in this case $(q_k + q_{k-1})/2 = q_k$. This choice is required to make things converge later. We can then insert the integral representation of the Dirac delta,

$$\delta(q - q') = \int \frac{\mathrm{d}p}{2\pi}\mathrm{e}^{ip(q-q')}, \tag{29.2.10}$$

giving

$$\langle q_k|V(\hat{Q})|q_{k-1}\rangle = \int \frac{\mathrm{d}p_k}{2\pi} V\left(\frac{q_k + q_{k-1}}{2}\right)\mathrm{e}^{ip_k(q_k - q_{k-1})}. \tag{29.2.11}$$

For the momentum term we can insert a complete set of states,

$$\int \mathrm{d}p|p\rangle\langle p| = \mathbb{1}, \tag{29.2.12}$$

giving

$$\begin{aligned}
\frac{1}{2}\langle q_k|\hat{P}^2|q_{k-1}\rangle &= \int \mathrm{d}p_k\, \frac{1}{2}\langle q_k|\hat{P}^2|p_k\rangle\langle p_k|q_{k-1}\rangle && (29.2.13)\\
&= \int \mathrm{d}p_k\, \frac{p_k^2}{2}\langle q_k|p_k\rangle\langle p_k|q_{k-1}\rangle && (29.2.14)\\
&= \int \frac{\mathrm{d}p_k}{2\pi}\, \frac{p_k^2}{2}\mathrm{e}^{ip_k(q_k - q_{k-1})} && (29.2.15)
\end{aligned}$$

where in the last step we've recognised the amplitudes as the wave function of the state momentum eigenstate $|p_k\rangle$ in the position basis,

$$\langle q|p\rangle = \frac{1}{\sqrt{2\pi}} e^{ipq}. \tag{29.2.16}$$

We can take this and go back to writing $1 - i\varepsilon\hat{H}$ as $\exp[-i\hat{H}\varepsilon]$, and the results will agree to first order in ε, that is

$$\langle q_k|e^{-i\hat{H}\varepsilon}|q_{k-1}\rangle = \int \frac{dp_k}{2\pi} \exp\left\{-i\varepsilon\left[\frac{p_k^2}{2} + V\left(\frac{q_k + q_{k-1}}{2}\right)\right]\right\} \exp[ip_k(q_k - q_{k-1})] + \mathcal{O}(\varepsilon^2). \tag{29.2.17}$$

We can recognise the term in the first exponential as the Hamiltonian evaluated at $\tilde{q}_k = (q_k + q_{k-1})/2$ and p_k, allowing us to rewrite this as

$$\langle q_k|e^{-i\hat{H}\varepsilon}|q_{k-1}\rangle = \int \frac{dp_k}{2\pi} \exp[-i\varepsilon H(\tilde{q}_k, p_k)] \exp[ip_k(q_k - q_{k-1})] + \mathcal{O}(\varepsilon^2). \tag{29.2.18}$$

Now combine the two exponentials and factor out $i\varepsilon$:

$$\langle q_k|e^{-i\hat{H}\varepsilon}|q_{k-1}\rangle = \int \frac{dp_k}{2\pi} \exp\left\{i\varepsilon\left[p_k \frac{q_k - q_{k-1}}{\varepsilon} - H(\tilde{q}_k, p_k)\right]\right\} + \mathcal{O}(\varepsilon^2). \tag{29.2.19}$$

We can now put all of these factors together to get the desired result, although we have to take the limit of $n \to \infty$, so $\varepsilon \to 0$, in order to make it exact. We end up with a product of the above terms, with the product of exponentials becoming a sum in the exponential. We should take the limit in such a way that $T = \varepsilon n$ is constant. The result is

$$\langle q', t'|q, t\rangle = \tag{29.2.20}$$
$$\lim_{n\to\infty} \int \left(\prod_{k=1}^{n-1} dq_k\right)\left(\prod_{j=1}^{n} \frac{dp_j}{2\pi}\right) \exp\left\{i\varepsilon \sum_{m=1}^{n}\left[p_m \frac{q_m - q_{m-1}}{\varepsilon} - iH(\tilde{q}_m, p_m)\right]\right\}.$$

We assume that this limit exists. In the limit the sum becomes an integral and $(q_m - q_{m-1})/\varepsilon$ becomes the velocity, the distinction between q_m and $\tilde{q}_m$ is not important in the limit. We introduce the following shorthand for this expression:

$$\langle q', t'|q, t\rangle = \int \mathcal{D}q\, \mathcal{D}p \exp\left\{i\int_t^{t'} d\tau\, [p(\tau)\dot{q}(\tau) - H(q(\tau), p(\tau))]\right\}. \tag{29.2.21}$$

This is the **path integral** form of the amplitude. Note that we absorb the $1/(2\pi)$ factor for each momentum integral into the measure.

29.2.1 Quadratic Momentum

For Hamiltonians, like the one we had above, which are quadratic in the momentum we can partially evaluate this integral. Consider the integral

$$\int \frac{dp_k}{2\pi} \exp\left[ip_k(q_k - q_{k-1}) - i\varepsilon\frac{p_k^2}{2}\right] = (2\pi i\varepsilon)^{-1/2} \exp\left[i\varepsilon\left(\frac{q_k - q_{k-1}}{\varepsilon}\right)^2\right],$$

where we've used Equation (27.1.8) with $a = i\varepsilon$ and $b = q_k - q_{k-1}$. Using this we can rewrite the amplitude as

$$\langle q', t'|q, t\rangle = \lim_{n\to\infty} \int \left(\prod_{k=1}^{n-1} \mathrm{d}q_k\right) (2\pi i\varepsilon)^{-n/2} \exp\left\{i\varepsilon \sum_{m=1}^{n} \left[\frac{1}{2}\left(\frac{q_m - q_{m-1}}{\varepsilon}\right)^2 - V(\tilde{q}_m)\right]\right\} \tag{29.2.22}$$

In the limit $(q_m - q_{m-1})/\varepsilon$ becomes $\dot{q}$, and the sum becomes an integral. We then have

$$\frac{1}{2}\dot{q}^2 - V(q) = L(q, \dot{q}), \tag{29.2.23}$$

so we can identify the exponent as the integral of the Lagrangian, which is just the action, $S[q]$. We can also absorb the $(2\pi i\varepsilon)^{-n/2}$ into the measure and, again assuming the limit exists, we get the shorthand

$$\langle q', t'|q, t\rangle = \int_{q,q'} \mathcal{D}q \, \exp\left[i \int_t^{t'} \mathrm{d}\tau \, L(q, \dot{q})\right] = \int_{q,q'} \mathcal{D}q \, \mathrm{e}^{-iS[q]}. \tag{29.2.24}$$

Here we write q, q' as the limit of the integrals to remind us that q and q' are the fixed endpoints.

The interpretation of this integral is that we split time into very narrow slices. We then integrate over all positions at each time. An alternative way of viewing this, rather than considering one time slice at a time, is that we consider all possible paths between q and q', integrating over them with the weighting $\exp\{iS[q]\}$. This gives the quantum amplitude.

Consider this equation in units where $\hbar \neq 1$. Then since the action has units of $\hbar$ and the argument to the exponential must be dimensionless we must have that

$$\langle q't'|q, t\rangle = \int_{q,q'} \mathcal{D}q \, \mathrm{e}^{-iS[q]/\hbar}. \tag{29.2.25}$$

Now consider the classical limit, where $\hbar$ is much smaller than any other quantity with the same units appearing in our expressions. We can treat this as taking the limit $\hbar \to 0$. In this limit $1/\hbar$ is very large, and so small changes in the rest of the exponent cause large oscillations in the value. Most of these oscillations cancel out, and only when $S[q]$ is stationary do we get a meaningful contribution. This is the root of the principal of least (or stationary) action. This can be made rigorous with the method of stationary phase, see the course *Methods of Mathematical Physics* for more details. Now set $\hbar = 1$ again.

29.3 Correlators

29.3.1 One Point Correlator

Suppose we wish to calculate the matrix element

$$\langle q', t'|\hat{Q}(\bar{t})|q, t\rangle. \tag{29.3.1}$$

We assume that $t < \bar{t} = t + \bar{k}\varepsilon = t_{\bar{k}} < t'$, this is a reasonable assumption since we'll be taking time to be a continuous parameter in the limit. We can write the operator $\hat{Q}(\bar{t})$ in the Schrödinger picture as

$$\hat{Q}(t) = \mathrm{e}^{i\hat{H}\bar{t}}\hat{Q}\mathrm{e}^{-i\hat{H}\bar{t}}. \tag{29.3.2}$$

We can then write the correlator in terms of time independent states:

$$\langle q', t'|\hat{Q}(\bar{t})|q, t\rangle = \langle q'|e^{-i\hat{H}(t'-\bar{t})}\hat{Q}e^{i\hat{H}(\bar{t}-t)}|q\rangle. \tag{29.3.3}$$

We follow the same procedure as before, splitting time into narrow slices. We then split the exponential into many identical factors of $\exp[-i\hat{H}\varepsilon]$:

$$\langle q', t'|\hat{Q}(\bar{t})|q, t\rangle = \langle q'|\underbrace{e^{-i\hat{H}\varepsilon}\cdots e^{-i\hat{H}\varepsilon}}_{\bar{k}\text{ factors}}\hat{Q}\underbrace{e^{-i\hat{H}\varepsilon}\cdots e^{-i\hat{H}\varepsilon}}_{n-\bar{k}\text{ factors}}|q\rangle. \tag{29.3.4}$$

We can then proceed to insert the identity a bunch in terms of the completeness relation. The result will be mostly the same, but we'll have one factor different. Splitting into kinetic and potential terms again we'll have

$$\langle q_{\bar{k}}|\hat{Q}(\bar{t})V(\hat{Q})|q_{\bar{k}-1}\rangle = q_{\bar{k}-1}V(\tilde{q}_{\bar{k}})\delta(q_{\bar{k}} - q_{\bar{k}-1}), \tag{29.3.5}$$

which differs from the other terms by a factor of $q_{\bar{k}}$. The result is that we get an extra factor of $q_{\bar{k}}$ in our integral:

$$\langle q', t'|\hat{Q}(\bar{t})|q, t\rangle = \tag{29.3.6}$$
$$\lim_{n\to\infty}\left(\prod_{k=1}^{n-1}dq_k\right)\left(\prod_{j=1}^{n}\frac{dp_j}{2\pi}\right)q_{\bar{k}-1}\exp\left\{i\varepsilon\sum_{m=1}^{n}\left[p_m\frac{q_m - q_{m-1}}{\varepsilon} - H(\tilde{q}_m, p_m)\right]\right\}.$$

Which, assuming the limit exists, we write as

$$\langle q', t'|\hat{Q}(\bar{t})|q, t\rangle = \int \mathcal{D}q\,\mathcal{D}p\,q(\bar{t})\exp\left\{i\int_t^{t'}d\tau\,[p(\tau)\dot{q}q(\tau) - H(q,p)]\right\}. \tag{29.3.7}$$

If the Hamiltonian has quadratic dependence on momentum we can again partially complete the integral to get

$$\langle q', t'|\hat{Q}(\bar{t})|q, t\rangle = \tag{29.3.8}$$
$$\lim_{n\to\infty}\int\left(\prod_{k=1}^{n-1}dq_k\right)(2\pi i\varepsilon)^{-n/2}q_{\bar{k}}\exp\left\{i\delta\sum_{m=1}^{n}\left[\frac{1}{2}\left(\frac{q_m - q_{m-1}}{\varepsilon}\right)^2 - V(\tilde{q}_m)\right]\right\}.$$

If this limit exists then we write this as

$$\langle q', t'|\hat{Q}(\bar{t})|q, t\rangle = \int_{q,q'}\mathcal{D}q\,q(\bar{t})\exp\left[i\int_t^{t'}d\tau\,L(q,\dot{q})\right]. \tag{29.3.9}$$

29.3.2 Two Point Correlator

Suppose we wish to calculate

$$\langle q', t'|\hat{Q}(\bar{t}_1)\hat{Q}(\bar{t}_2)|q, t\rangle. \tag{29.3.10}$$

We assume that $t < \bar{t}_2 < \bar{t}_1 < t'$. Exactly the same logic as before applies, except we now have two of these altered factors, so get two factors of q in our final path integral. The result is

$$\langle q', t'|\hat{Q}(\bar{t}_1)\hat{Q}(\bar{t}_2)|q, t\rangle = \int_{q,q'}\mathcal{D}q\,q(\bar{t}_1)q(\bar{t}_2)\exp\left[i\int_t^{t'}d\tau\,L(q,\dot{q})\right]. \tag{29.3.11}$$

Here we see something interesting. On the right hand side it doesn't matter if we write $q(\bar{t}_1)q(\bar{t}_2)$ or $q(\bar{t}_2)q(\bar{t}_1)$, since these are just numbers. However, the order of $\hat{Q}(\bar{t}_1)\hat{Q}(\bar{t}_2)$ is important. This ambiguity is due to the assumption that $\bar{t}_2 < \bar{t}_1$. If this isn't the case then the path integral here instead represents

$$\langle q', t' | \hat{Q}(\bar{t}_2)\hat{Q}(\bar{t}_1) | q, t \rangle. \tag{29.3.12}$$

We can compactly summarise this by writing

$$\langle q', t' | \, \mathrm{T}[\hat{Q}(\bar{t}_1)\hat{Q}(\bar{t}_2)] | q, t \rangle = \int_{q,q'} \mathcal{D}q \, q(\bar{t}_1)q(\bar{t}_2) \exp\left[i \int_t^{t'} \mathrm{d}\tau \, L(q, \dot{q}) \right] \tag{29.3.13}$$

where

$$\mathrm{T}[A(t)B(t')] = \theta(t - t')A(t)B(t) + \theta(t' - t)B(t)A(t) \tag{29.3.14}$$

is the time ordered product of the operators $A(t)$ and $B(t')$.

Taking this process to its logical conclusion we have

$$\langle q', t' | \, \mathrm{T}[\hat{Q}(\bar{t}_1) \cdots \hat{Q}(\bar{t}_\ell)] | q, t \rangle = \int_{q,q'} \mathcal{D}q \, q(\bar{t}_1) \cdots q(\bar{q}_\ell) \exp\left[i \int_t^{t'} \mathrm{d}\tau \, L(q, \dot{q}) \right]. \tag{29.3.15}$$

Thirty

Generating Functions

30.1 Functional Derivatives

Consider a **functional**, F, which associates a number, $F[u]$, to a function, $u : \mathbb{R}^D \to \mathbb{R}$. The functional derivative, $\delta F/\delta u(x)$, is defined such that if we vary the function u,

$$u(x) \mapsto u(x) + \delta u(x), \tag{30.1.1}$$

then the variation in F is

$$\delta F = F[u + \delta u] - F[u] = \int \mathrm{d}^D x \, \frac{\delta F}{\delta u(x)} \delta u(x). \tag{30.1.2}$$

Compare this to to the case of a function $f : \mathbb{R}^D \to \mathbb{R}$ when we vary each argument according to $x_k \mapsto x_k + \delta x_k$ and so f varies as

$$\delta f = f(x + \delta x) - f(x) = \sum_k \frac{\partial f}{\partial x_k} \delta x_k. \tag{30.1.3}$$

In this function case we have the identity

$$\frac{\partial}{\partial x_j} x_i = \delta_{ij}, \tag{30.1.4}$$

and in the functional case we have the analogous identity

$$\frac{\delta}{\delta f(x)} f(y) = \delta(x - y). \tag{30.1.5}$$

This, and the fact that the product rule and chain rule also apply to functional derivatives, is pretty much all we'll need.

30.2 Sources

Consider two time dependent functions, f and h. We can add terms proportional to these into our path integral. We call such terms source terms. In particular, we'll be interested in the path integral with sources given below:

$$\langle q', t' | q, t \rangle_{f,h} \tag{30.2.1}$$
$$\coloneqq \int \mathcal{D}p \, \mathcal{D}q \, \exp\left\{ i \int_t^{t'} \mathrm{d}\tau \, [p(\tau)\dot{q}(\tau) - H(q(\tau), p(\tau))] + f(\tau)q(\tau) + h(\tau)p(\tau) \right\}.$$

Consider the following functional derivative:

$$A(\bar{\tau}) = \frac{\delta}{\delta f(\bar{\tau})} \exp\left[i \int_t^{t'} \mathrm{d}\tau\, f(\tau) q(\tau) \right] \tag{30.2.2}$$

$$= \left(\frac{\delta}{\delta f(\bar{\tau})} i \int_t^{t'} \mathrm{d}\tau\, f(\tau) q(\tau) \right) \exp\left[i \int_t^{t'} \mathrm{d}\tau\, f(\tau) q(\tau) \right], \tag{30.2.3}$$

where the last step is an application of the chain rule. We now assume we can move the derivative through the integral to get

$$A(\bar{\tau}) = \left(i \int_t^{t'} \mathrm{d}\tau\, \frac{\delta}{\delta f(\bar{\tau})} f(\tau) q(\tau) \right) \exp\left[i \int_t^{t'} \mathrm{d}\tau\, f(\tau) q(\tau) \right] \tag{30.2.4}$$

$$= \left(i \int_t^{t'} \mathrm{d}\tau\, \delta(\tau - \bar{\tau}) q(\tau) \right) \exp\left[i \int_t^{t'} \mathrm{d}\tau\, f(\tau) q(\tau) \right] \tag{30.2.5}$$

$$= i q(\bar{\tau}) \exp\left[i \int_t^{t'} \mathrm{d}\tau\, f(\tau) q(\tau) \right]. \tag{30.2.6}$$

Hence, we can generate factors of $q(\bar{\tau})$ through

$$q(\bar{\tau}) \exp\left[i \int_t^{t'} \mathrm{d}\tau\, f(\tau) q(\tau) \right] = \frac{1}{i} \frac{\delta}{\delta f(\bar{\tau})} \exp\left[i \int_t^{t'} \mathrm{d}\tau\, f(\tau) q(\tau) \right]. \tag{30.2.7}$$

We can also apply this to $\langle q', t' | q, t \rangle_{f,h}$ to get factors of $q(\bar{\tau})$, since all of the terms without $f(\tau)$ in them don't interact with the functional derivative. That is, we have

$$\frac{1}{i} \frac{\delta}{\delta f(\bar{\tau})} \langle q', t' | q, t \rangle_{f,h} \coloneqq \int \mathcal{D}p\, \mathcal{D}q\, q(\bar{\tau}) \tag{30.2.8}$$

$$\exp\left\{ i \int_t^{t'} \mathrm{d}\tau\, [p(\tau)\dot{q}(\tau) - H(q(\tau), p(\tau))] + f(\tau) q(\tau) + h(\tau) p(\tau) \right\}.$$

We can apply the functional derivative again to get another factor,

$$\left(\frac{1}{i} \frac{\delta}{\delta f(\bar{\tau}_1)} \right) \left(\frac{1}{i} \frac{\delta}{\delta f(\bar{\tau}_2)} \right) \langle q', t' | q, t \rangle_{f,h} \coloneqq \int \mathcal{D}p\, \mathcal{D}q\, q(\bar{\tau}_1) q(\bar{\tau}_2) \tag{30.2.9}$$

$$\exp\left\{ i \int_t^{t'} \mathrm{d}\tau\, [p(\tau)\dot{q}(\tau) - H(q(\tau), p(\tau))] + f(\tau) q(\tau) + h(\tau) p(\tau) \right\}.$$

Repeating this we have

$$\left(\frac{1}{i} \frac{\delta}{\delta f(\bar{\tau}_1)} \right) \cdots \left(\frac{1}{i} \frac{\delta}{\delta f(\bar{\tau}_n)} \right) \langle q', t' | q, t \rangle_{f,h} \coloneqq \int \mathcal{D}p\, \mathcal{D}q\, q(\bar{\tau}_1) \cdots q(\bar{\tau}_n) \tag{30.2.10}$$

$$\exp\left\{ i \int_t^{t'} \mathrm{d}\tau\, [p(\tau)\dot{q}(\tau) - H(q(\tau), p(\tau))] + f(\tau) q(\tau) + h(\tau) p(\tau) \right\}.$$

Comparing this to Equation (29.3.15) we see that

$$\langle q', t' | \mathrm{T}[\hat{Q}(t_1) \cdots \hat{Q}(t_n)] | q, t \rangle = \left(\frac{1}{i} \frac{\delta}{\delta f(t_1)} \right) \cdots \left(\frac{1}{i} \frac{\delta}{\delta f(t_2)} \right) \langle q', t' | q, t \rangle \Big|_{f=h=0}. \tag{30.2.11}$$

30.3 Projection onto the Ground State

Often it is useful to compute the amplitude for a system to evolve from the vacuum state at time t to the vacuum state at time t' under some external sources, f and h. That is, we want to compute $\langle 0, t'|0, t\rangle$, where $|0, t\rangle$ is the vacuum state at time t. Note that this is an eigenstate of the Hamiltonian, *not* the position. We can insert a complete set of states twice to get

$$\langle 0, t'|0, t\rangle_{f,h} = \int \mathrm{d}q\,\mathrm{d}q'\,\langle 0, t'|q', t'\rangle_{f,h}\langle q', t'|q, t\rangle_{f,h}\langle q, t|0, t\rangle_{f,h} \tag{30.3.1}$$

$$= \int \mathrm{d}q\,\mathrm{d}q'\,\varphi_0^*(q')\langle q', t'|q, t\rangle_{f,h}\varphi_0(q). \tag{30.3.2}$$

Here we use the ground state wave function in the position basis:

$$\varphi_0(q) = \langle q, t|0, t\rangle, \implies \varphi_0^*(q) = \langle 0, t|q, t\rangle. \tag{30.3.3}$$

More generally if $\hat{H}$ is the Hamiltonian for our system then we have the energy eigenstate $|n\rangle$, which is such that

$$\hat{H}|n\rangle = E_n, \tag{30.3.4}$$

and the energy eigenfunction

$$\varphi_n(q) = \langle q|n\rangle. \tag{30.3.5}$$

We assume that the energy levels are defined such that the ground state energy vanishes, that is $E_0 = 0$.

Now suppose that f and h have support in $[t, t']$. That is, outside the interval $[t, t']$ f and h vanish. Suppose also that we have times T and T' such that $T < t < t' < T'$. Then between T and t, and t' and T', the sources vanish and we can easily compute $\langle q, t|Q, T\rangle$ by inserting a complete set of energy eigenstates:

$$\langle q, t|Q, T\rangle_{f,h} = \langle q|\mathrm{e}^{-i\hat{H}(t-T)}\langle Q| \tag{30.3.6}$$

$$= \sum_n \langle q|\mathrm{e}^{-i\hat{H}(t-T)}|n\rangle\langle n|Q\rangle \tag{30.3.7}$$

$$= \sum_n \mathrm{e}^{-iE_n(t-T)}\langle q|n\rangle\langle n|Q\rangle \tag{30.3.8}$$

$$= \sum_n \mathrm{e}^{-iE_n(t-T)}\varphi_n(q)\varphi_n^*(Q). \tag{30.3.9}$$

We can analytically extend this result to time with a small imaginary part, $T_I = (1 - i\varepsilon)T$ where $\varepsilon > 0$. We then have

$$\mathrm{e}^{-iE_n(t-T_I)} = \mathrm{e}^{-iE_n(t-T+i\varepsilon T)} = \mathrm{e}^{-iE_n(t-T)}\mathrm{e}^{\varepsilon E_n T}. \tag{30.3.10}$$

If we consider the limit $T \to -\infty$ then all states apart from the ground state, where $E_0 = 0$, will be exponentially suppressed. We then have

$$\lim_{T\to-\infty}\langle q, t|Q, T_I\rangle_{f,h} = \langle q|0\rangle\langle 0|Q\rangle = \varphi_0(q)\varphi_0^*(Q). \tag{30.3.11}$$

We can do the same with $\langle Q', T'|q', t'\rangle$, we get

$$\langle Q', T'|q', t'\rangle_{f,h} = \langle Q|e^{-i\hat{H}(T'-t')}|t\rangle \tag{30.3.12}$$
$$= \sum_n \langle Q'|e^{-i\hat{H}(T'-t')}|n\rangle\langle n|q'\rangle \tag{30.3.13}$$
$$= \sum_n e^{-iE_n(T'-t')}\langle Q'|n\rangle\langle n|q'\rangle \tag{30.3.14}$$
$$= \sum_n e^{-iE_n(T'-t')}\varphi_n(Q')\varphi_n^*(q'). \tag{30.3.15}$$

Analytically extending this result to $T_I' = (1-i\varepsilon)T'$, and so

$$e^{-iE_n(T_I'-t')} = e^{-iE_n(T'-i\varepsilon T'-t')}e^{-iE_n(T'-t')}e^{-\varepsilon E_n T'}, \tag{30.3.16}$$

so this time all states apart from the ground state are exponentially suppressed in the limit $T' \to \infty$. We then have

$$\lim_{T'\to\infty}\langle Q', T_I'|q', t'\rangle_{f,h} = \langle Q'|0\rangle\langle 0|q'\rangle = \varphi_0(Q')\varphi_0^*(q'). \tag{30.3.17}$$

Putting these two results together we find that

$$\lim_{\substack{T\to-\infty\\ T'\to+\infty}} \frac{\langle Q', (1-i\varepsilon)T'|Q, (1-i\varepsilon)T\rangle_{f,h}}{\varphi_0^*(Q)\varphi_0(Q')} = \langle 0, t'|0, t\rangle_{f,h}. \tag{30.3.18}$$

Notice that we can get the same result by the transformation $\hat{H} \mapsto (1-i\varepsilon)\hat{H}$. Note that the only dependence on the boundary is in the derivative on the left hand side, which is simply a normalisation factor which doesn't effect the values of derivatives with respect to the sources.

30.4 Weyl Ordering

For more complicated Hamiltonians we may have to consider terms with products of $\hat{P}$ and $\hat{Q}$. We then need a prescription for ordering terms in the Hamiltonian so that the amplitude matches the path integral. We've used the mid-point prescription, evaluating the potential at $\tilde{q}_k = (q_k + q_{k-1})/2$, to the same effect in this simple case. In the more complex case we use the **Weyl ordering**. Here the order of a product of powers of some operators, $\hat{A}$ and $\hat{B}$, is defined by considering the operator $(\alpha\hat{A} + \beta\hat{B})^n$ and expanding in powers of α and β:

$$(\alpha\hat{A} + \beta\hat{B})^n = \sum_k \frac{n!}{k!\ell!}\alpha^k\beta^\ell[\hat{A}^k\hat{B}^\ell]. \tag{30.4.1}$$

Here $[\hat{A}^k\hat{B}^\ell]$ is the Weyl ordered product. Notice that $\ell = n - k$ and

$$\frac{n!}{k!\ell!} = \binom{n}{k} = \binom{n}{\ell} \tag{30.4.2}$$

Example 30.4.3 Suppose we want to compute $[\hat{A}\hat{B}]$. This is quadratic in the operators so we consider $(\alpha\hat{A}+\beta\hat{B})^2$:

$$(\alpha\hat{A}+\beta\hat{B})^2 = \alpha^2\hat{A} + \alpha\beta(\hat{A}\hat{B}+\hat{B}\hat{A}) + \beta\hat{B}^2. \tag{30.4.4}$$

We then take the terms proportional to $\alpha\beta$ and, after dividing by $n!/(k!\ell!) = 2!/(1!1!) = 2$ we get the Weyl ordering,

$$[\hat{A}\hat{B}] = \frac{1}{2}(\hat{A}\hat{B}+\hat{B}\hat{A}). \tag{30.4.5}$$

Now suppose we want to compute all third order Weyl orderings. We consider

$$(\alpha\hat{A}+\beta\hat{B})^3\alpha^3\hat{A}^3 + \alpha^2\beta(\hat{A}^2\hat{B}+\hat{A}\hat{B}\hat{A}+\hat{B}\hat{A}^2) + \alpha\beta^2(\hat{A}\hat{B}^2+\hat{B}\hat{A}\hat{B}+\hat{B}^2\hat{A}) + \beta^3\hat{B}^3. \tag{30.4.6}$$

We then have

$$[\hat{A}^3] = \frac{3!1!}{3!}\hat{A}^3 = \hat{A}^3, \tag{30.4.7}$$

$$[\hat{A}^2\hat{B}] = \frac{2!1!}{3!}(\hat{A}^2\hat{B}+\hat{A}\hat{B}\hat{A}+\hat{B}\hat{A}^2) = \frac{1}{3}(\hat{A}^2\hat{B}+\hat{A}\hat{B}\hat{A}+\hat{B}\hat{A}^2), \tag{30.4.8}$$

$$[\hat{A}\hat{B}^2] = \frac{1!2!}{3!}(\hat{A}\hat{B}^2+\hat{B}\hat{A}\hat{B}+\hat{B}^2\hat{A}) = \frac{1}{3}(\hat{A}\hat{B}^2+\hat{B}\hat{A}\hat{B}+\hat{B}^2\hat{A}), \tag{30.4.9}$$

$$[\hat{B}^3] = \frac{1!3!}{3!}\hat{B}^3 = \hat{B}^3. \tag{30.4.10}$$

Finally, suppose we want to compute $[\hat{A}^2\hat{B}^2]$. Expanding $(\alpha\hat{A}+\beta\hat{B})^5$ we find the relevant term is

$$\alpha^3\beta^2(\hat{A}^3\hat{B}^2 + \hat{A}^2\hat{B}\hat{A}\hat{B} + \hat{A}^2\hat{B}^2\hat{A} + \hat{A}\hat{B}\hat{A}^2\hat{B} + \hat{A}\hat{B}\hat{A}\hat{B}\hat{A} + \hat{A}\hat{B}^2\hat{A}^2 + \hat{B}\hat{A}^3\hat{B} + \hat{B}\hat{A}^2\hat{B}\hat{A} + \hat{B}\hat{A}\hat{B}\hat{A}^2 + \hat{B}^2\hat{A}^3) \tag{30.4.11}$$

and so the Weyl ordered product is this result divided by $5!/(3!2!) = 10$

$$[\hat{A}^3\hat{B}^2] = \frac{1}{10}(\hat{A}^3\hat{B}^2 + \hat{A}^2\hat{B}\hat{A}\hat{B} + \hat{A}^2\hat{B}^2\hat{A} + \hat{A}\hat{B}\hat{A}^2\hat{B} + \hat{A}\hat{B}\hat{A}\hat{B}\hat{A} + \hat{A}\hat{B}^2\hat{A}^2 + \hat{B}\hat{A}^3\hat{B} + \hat{B}\hat{A}^2\hat{B}\hat{A} + \hat{B}\hat{A}\hat{B}\hat{A}^2 + \hat{B}^2\hat{A}^3) \tag{30.4.12}$$

Part X

Path Integrals in Scalar Field Theory

Thirty-One

Noninteracting Scalar Field Theory

31.1 Scalar Field Theory Recap

The Lagrangian for a free real scalar field, φ, is

$$\mathcal{L}_0(\varphi, \partial_\mu\varphi) = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}m^2\varphi^2. \tag{31.1.1}$$

Applying the Euler–Lagrange equations,

$$\partial_\mu\left(\frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\varphi)}\right) - \frac{\partial\mathcal{L}_0}{\partial\varphi} = 0 \tag{31.1.2}$$

gives the Klein–Gordon equation,

$$(\partial^2 + m^2)\varphi = 0. \tag{31.1.3}$$

The action associated with this Lagrangian is, in D dimensions,

$$S_0[\varphi] = \int \mathrm{d}^D x \left[\frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}m^2\varphi^2\right]. \tag{31.1.4}$$

There is already some physics going on here. The Lagrangian and action tell us what the degrees of freedom of the system are, in this case we just have one, the field, φ. We can also examine the symmetries of the action. Here we have one symmetry, Lorentz invariance. Recall that if x transforms as

$$x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu \iff x \mapsto x' = \Lambda x \tag{31.1.5}$$

for some Lorentz transformation Λ then the field transforms as

$$\varphi(x) \mapsto \varphi'(x') = \varphi(x), \tag{31.1.6}$$

since by definition the field is a scalar. Writing x' as $\Lambda x'$ in the middle of this we have $\varphi(x) \mapsto \varphi'(\Lambda x)$. Redefining our variable so that x is the position in the new frame we have $\varphi(x) \mapsto \varphi'(x) = \varphi(\Lambda^{-1}x)$. If this is confusing think of a one-dimensional function, $f : \mathbb{R} \to \mathbb{R}$. If we want to translate this function to the right, in the positive x direction, by, say a units, then we transform it as $f(x) \mapsto f'(x) = f(x - a)$, so we subtract. The logic is the same here. Our transformation moves the point of evaluation, so we have to move it back.

We define the conjugate momentum as

$$\Pi(x) = \frac{\partial \mathcal{L}_0}{\partial(\partial_0 \varphi)} = \partial_0 \varphi(x) = \dot{\varphi}(x). \tag{31.1.7}$$

The Hamiltonian is then

$$\mathcal{H}(\varphi, \Pi) = \Pi(x)\dot{\varphi}(x) - \mathcal{L}_0(\varphi, \partial_\mu \varphi) \tag{31.1.8}$$

$$= \frac{1}{2}\Pi(x)^2 + \frac{1}{2}\sum_{k=1}^{3}(\partial_k \varphi(x))^2 + \frac{1}{2}m^2\varphi(x)^2. \tag{31.1.9}$$

31.2 The Path Integral

We could derive the path integral in scalar field theory in a very similar way to how we derived it for quantum mechanics, but we won't. Instead we'll assume the form is the same after making the usual substitutions which take us from quantum mechanics to quantum field theory. These substitutions are as follows:

- $t \to x$, we change from a single coordinate to D coordinates;
- $q(t) \to \varphi(x)$, we change from position to the field, φ;
- $\hat{Q}(t) \to \hat{\varphi}(x)$, we change from the position operator to the field operator.

Recall the quantum mechanical path integral:

$$\langle q, t | q', t' \rangle_f = \int \mathcal{D}q \, \exp\{i(S_0[q] + qf)\}. \tag{31.2.1}$$

Moving to scalar field theory we replace this with

$$Z_0[J] \coloneqq \langle 0|0\rangle_J = \int \mathcal{D}\varphi \, \exp\{i(S_0[\varphi] + J \cdot \varphi)\} \tag{31.2.2}$$

where

$$J \cdot \varphi \coloneqq \int \mathrm{d}^D x \, J(x)\varphi(x). \tag{31.2.3}$$

This path integral is just a Gaussian integral, the one complication being that it's an infinite dimensional Gaussian integral. We can make this more obvious with a bit of rewriting. First, consider the action:

$$S_0[\varphi] \coloneqq \int \mathrm{d}^D x \, \mathcal{L}_0 = \frac{1}{2}\mathrm{d}^D x \, [(\partial_\mu \varphi)(\partial^\mu \varphi) - m^2\varphi^2]. \tag{31.2.4}$$

Take the first term and integrate by parts to get

$$\int \mathrm{d}^D x \, (\partial_\mu \varphi)(\partial^\mu \varphi) = \int \mathrm{d}^D x \, [\partial_\mu(\varphi \partial^\mu \varphi) - \varphi \partial^2 \varphi] = -\int \mathrm{d}^D x \, \varphi \partial^2 \varphi \tag{31.2.5}$$

where in the last step we assume that the field vanishes sufficiently quickly at infinity that we can ignore the boundary term. The action is then

$$S_0[\varphi] = \frac{1}{2}\int \mathrm{d}^D x \, \varphi(x)[-\partial^2 - m^2]\varphi(x). \tag{31.2.6}$$

Now, we insert a factor of $-i\varepsilon\varphi^2$ for some $\varepsilon > 0$, which we will later take to zero. This ensures convergence. The modified action is then

$$S_0[\varphi] = \frac{1}{2}\int \mathrm{d}^D x\, \varphi(x)[-\partial^2 - m^2 - i\varepsilon]\varphi(x). \tag{31.2.7}$$

Now define

$$K(x,x') \coloneqq \delta(x-x')[-\partial^2 - m^2]. \tag{31.2.8}$$

Then we can write the action as

$$S_0[\varphi] = \frac{1}{2}\int \mathrm{d}^D x\, \mathrm{d}^D x'\, \varphi(x)K(x,x')\varphi(x'). \tag{31.2.9}$$

Hence,

$$Z_0[0] = \int \mathcal{D}\varphi\, \exp\left[i\frac{1}{2}\int \mathrm{d}^D x\, \mathrm{d}^D x'\, \varphi(x)K(x,x')\varphi(x')\right]. \tag{31.2.10}$$

$$Z_A = \int \mathrm{d}^n x\, \exp\left[-\frac{1}{2}x^\top A x\right] = \int \mathrm{d}^n x\, \exp\left[-\sum_i \sum_j x_i A_{ij} x_j\right]. \tag{31.2.11}$$

We can see that the kernel, $K(x,x')$, plays the role of A and our sums become integrals, but other than that not much changes.

It's easier to compute the action if we move to momentum space, since in momentum space $\partial^2\varphi$ simply becomes $-p^2\tilde{\varphi}$. To do this we write φ as the inverse transform of $\tilde{\varphi}$:

$$\varphi(x) = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \mathrm{e}^{-ip\cdot x}\tilde{\varphi}(p). \tag{31.2.12}$$

We then have

$$S_0[\varphi] = \frac{1}{2}\int_x \int_{p,p'} \mathrm{e}^{-ip\cdot x}\tilde{\varphi}(p)(p'^2 - m^2 + i\varepsilon)\mathrm{e}^{-ip'\cdot x}\tilde{\varphi}(p'). \tag{31.2.13}$$

Here we've introduced the shorthand notation

$$\int_x \coloneqq \int \mathrm{d}^D x, \quad \int_p \coloneqq \int \frac{\mathrm{d}^D p}{(2\pi)^D}, \quad \text{and} \quad \int_{p,p'} = \int \frac{\mathrm{d}^D p}{(2\pi)^D}\int \frac{\mathrm{d}^D p'}{(2\pi)^D} \tag{31.2.14}$$

and so on. We can combine the two exponentials into $\exp[-i(p+p')\cdot x]$, and then performing the x integral gives us a Dirac delta:

$$S_0[\varphi] = \frac{1}{2}\int_{p,p'} \tilde{\varphi}(p)(p'^2 - m^2 + i\varepsilon)\tilde{\varphi}(p')\delta(p+p'). \tag{31.2.15}$$

Using the Dirac delta we can perform the p' integral to get

$$S_0[\varphi] = \frac{1}{2}\int_p \tilde{\varphi}(p)(p^2 - m^2 + i\varepsilon)\tilde{\varphi}(-p). \tag{31.2.16}$$

We can simplify this slightly using a Fourier transform identity. Let $f : \mathbb{R} \to \mathbb{R}$ be a real function whose Fourier transform exists. Then we have

$$\tilde{f}(p) = \int \mathrm{d}x\, \mathrm{e}^{ipx} f(x). \tag{31.2.17}$$

Hence, we have

$$\tilde{f}(-p) = \int \mathrm{d}x\, \mathrm{e}^{-ipx} f(x) = \left[\int \mathrm{d}x \mathrm{e}^{ipx} f(x)\right]^* = \tilde{f}(p)^*. \tag{31.2.18}$$

That is, for a real function, $f(-p) = f(p)^*$. This same result holds in arbitrary dimensions by an identical proof.

Using this we can write the action as

$$S_0[\varphi] = \frac{1}{2}\int_p \tilde{\varphi}(p)(p^2 - m^2 + i\varepsilon)\tilde{\varphi}(p)^*. \tag{31.2.19}$$

31.2.1 With a Source Term

Now we want to compute

$$Z_0[J] = \int \mathcal{D}\varphi\, \exp\left[i\frac{1}{2}\int_p \tilde{\varphi}(p)(p^2 - m^2 + i\varepsilon)\tilde{\varphi}(p)^* + \int \mathrm{d}^D x\, J(x)\varphi(x)\right]. \tag{31.2.20}$$

This is the infinite dimensional analogue of

$$Z_A(b) = \int \mathrm{d}^n x\, \exp\left[-\frac{1}{2}x^\top A x + b^\top x\right] \tag{31.2.21}$$

$$= \int \mathrm{d}^n x\, \exp\left[-\frac{1}{2}\sum_i \sum_j x_i A_{ij} x_j + \sum_i b_i x_i\right]. \tag{31.2.22}$$

The source term simply has the sum turn into an integral. To compute this integral we completed the square by making a substitution, $x = y + \Delta b$, where $\Delta = A^{-1}$. We can do the same here, making the substitution in momentum space:

$$\tilde{\chi}(p) = \tilde{\varphi}(p) + \frac{\tilde{J}(p)}{p^2 - m^2 + i\varepsilon}. \tag{31.2.23}$$

Notice that it's much easier to do this in Fourier space since we can literally divide by $p^2 - m^2 + i\varepsilon$. To do this in real space we'd have to somehow invert the operator $-\partial^2 - m^2 + i\varepsilon$. The finite dimensional analogue of this is that going to Fourier space is a basis change into a basis in which A is diagonal, and so easily inverted. In a sense $\tilde{K}(p, p')$ is "diagonal" in momentum space, in particular we get independent modes, one for each momentum.

Before we do this we should Fourier transform the source term as well. We have

$$J(x) = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \mathrm{e}^{-ip\cdot x} \tilde{J}(p). \tag{31.2.24}$$

Substituting this, and the Fourier transform of the field, into the source term we have

$$\int_x J(x)\varphi(x) = \int_x \int_{p,p'} \tilde{J}(p)\tilde{\varphi}(p')\mathrm{e}^{i(p+p')\cdot x} \tag{31.2.25}$$

$$= \int_{p,p'} \tilde{J}(p)\tilde{\varphi}(p')(2\pi)^D \delta(p+p') \tag{31.2.26}$$

$$= \int_p \tilde{J}(p)\tilde{\varphi}(-p). \tag{31.2.27}$$

Hence,

$$S_0[\varphi] + J\cdot\varphi = \frac{1}{2}\int_p \left[\tilde{\varphi}(p)(p^2 - m^2 + i\varepsilon)\tilde{\varphi}(-p) + \tilde{J}(p)\tilde{\varphi}(-p) + \tilde{J}(-p)\tilde{\varphi}(p)\right]. \tag{31.2.28}$$

The seemingly extra term of $\tilde{J}(-p)\tilde{\varphi}(p)$ comes from factoring out the half, giving two $\tilde{J}(p)\tilde{\varphi}(-p)$ terms. We then recognise that sine the integration range is symmetric all that matters is that the arguments of $\tilde{J}$ and $\tilde{\varphi}$ in this term differ by a sign. So, we can make the substitution $p \to -p$ in one of the factors then giving the final term above.

Making this change of variables we can add the source term to the action:

$$S_0[\varphi] + J\cdot\varphi = \frac{1}{2}\int_p \Bigg[\left(\tilde{\chi}(p) - \frac{\tilde{J}(p)}{p^2 - m^2 + i\varepsilon}\right) \tag{31.2.29}$$

$$\times (p^2 - m^2 + i\varepsilon)\left(\tilde{\chi}(-p) - \frac{\tilde{J}(-p)}{p^2 - m^2 + i\varepsilon}\right) \tag{31.2.30}$$

$$+ \tilde{J}(p)\left(\tilde{\chi}(p) - \frac{\tilde{J}(p)}{p^2 - m^2 + i\varepsilon}\right) \tag{31.2.31}$$

$$+ \tilde{J}(-p)\left(\tilde{\chi}(-p) - \frac{\tilde{J}(-p)}{p^2 - m^2 + i\varepsilon}\right)\Bigg] \tag{31.2.32}$$

Expanding this most terms cancel and we're left with

$$S_0[\varphi] + J\cdot\varphi = \frac{1}{2}\int_p \left[\tilde{\chi}(p)(p^2 - m^2 + i\varepsilon)\tilde{\chi}(-p) - \tilde{J}(p)\frac{1}{p^2 - m^2 + i\varepsilon}\tilde{J}(-p)\right]. \tag{31.2.33}$$

Hence, we have

$$Z_0[J] = \int \mathcal{D}\chi \exp\left\{ i\frac{1}{2}\int_p \left[\tilde{\chi}(p)(p^2 - m^2 + i\varepsilon)\tilde{\chi}(-p)\right.\right. \tag{31.2.34}$$

$$\left.\left. - \tilde{J}(p)\frac{1}{p^2 - m^2 + i\varepsilon}\tilde{J}(-p)\right]\right\} \tag{31.2.35}$$

$$= \exp\left[-\frac{i}{2}\int_p \tilde{J}(p)\frac{1}{p^2 - m^2 + i\varepsilon}\tilde{J}(-p)\right] \tag{31.2.36}$$

$$\times \int \mathcal{D}\chi \exp\left[\frac{i}{2}\int_p \tilde{\chi}(p)(p^2 - m^2 + i\varepsilon)\tilde{\chi}(-p)\right] \tag{31.2.37}$$

$$= \mathcal{N}\exp\left[-\frac{i}{2}\int_p \tilde{J}(p)\frac{1}{p^2 - m^2 + i\varepsilon}\tilde{J}(-p)\right] \tag{31.2.38}$$

where

$$\mathcal{N} = \int \mathcal{D}\chi \exp\left[\frac{i}{2}\int_p \tilde{\chi}(p)(p^2 - m^2 + i\varepsilon)\tilde{\chi}(-p)\right] \tag{31.2.39}$$

is a normalisation factor. This normalisation factor is analgous to the $(\det A)^{-1/2}$ factor in the finite dimensional case. Exactly how we're taking the determinant of an infinite dimensional operator and then inverting this is not important, since we always consider ratios, $Z_0[J]/Z_0[0]$, where this normalisation vanishes.

We can write this result as

$$Z_0[J] = \mathcal{N}\exp\left[-\frac{i}{2}\int \mathrm{d}^D x\, \mathrm{d}^D x'\, J(x)\Delta(x,x')J(x')\right] \tag{31.2.40}$$

where

$$\Delta(x,x') = \Delta(x-x') = \int_p \frac{\mathrm{e}^{-ip\cdot(x-x')}}{p^2 - m^2 + i\varepsilon} \tag{31.2.41}$$

is the **Feynman propagator**. We find this by doing the inverse transform

$$\tilde{J}(p) = \int \mathrm{d}^D x\, \mathrm{e}^{ip\cdot x}J(x), \qquad \text{and} \qquad \tilde{J}(-p) = \int \mathrm{d}^D x\, \mathrm{e}^{-ip\cdot x}J(x), \tag{31.2.42}$$

so

$$\begin{aligned}\int_p \tilde{J}(-p)\frac{1}{p^2 - m^2 + i\varepsilon}\tilde{J}(p) &= \int_p \int_{x,x'} \mathrm{e}^{-ip\cdot x}J(x)\frac{1}{p^2 - m^2 + i\varepsilon}\mathrm{e}^{ip\cdot x'}J(x') \\ &= \int_p \int_{x,x'} J(x)\frac{1}{p^2 - m^2 + i\varepsilon}J(x')\mathrm{e}^{-ip\cdot(x-x')}.\end{aligned}$$

We can show that, in analogy to the finite dimensional integrals, Δ is, in a sense, the inverse of K, more specifically,

$$I = \int \mathrm{d}^D z\, K(x,z)\Delta(z,x') = \delta(x-x'). \tag{31.2.43}$$

This follows by substituting in the definitions:

$$I = \int \mathrm{d}^D z \int_p \delta(x-z)[-\partial^2 - m^2 - i\varepsilon]\frac{\mathrm{e}^{-ip\cdot(z-x')}}{p^2 - m^2 + i\varepsilon} \tag{31.2.44}$$

$$= \int \mathrm{d}^D z \delta(x-z)[-\partial^2 - m^2 - i\varepsilon]\frac{\delta(z-x')}{p^2 - m^2 + i\varepsilon} \tag{31.2.45}$$

The two terms $[-\partial^2 - m^2 - i\varepsilon]$ and $1/(p^2 - m^2 + i\varepsilon)$ cancel as they are inverses once we move the first to Fourier space, and we are left with the product $\delta(x-z)\delta(z-x')$, which reduces to $\delta(x-x')$ when we perform the z integral.

31.2.2 Using the Path Integral

We can use the path integral to compute correlators, much like in Section 30.2. We normalise so that $\mathcal{N} = 1$. Consider, for example, the vacuum expectation of the time ordered product of two fields:

$$\begin{aligned}
\langle 0|\,\mathrm{T}[\varphi(x_1)\varphi(x_2)]|0\rangle &= \left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x_2)}\right) Z_0[J]\Big|_{J=0} \\
&= \left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x_2)}\right) \exp\left[-\frac{i}{2}\int_{x,x'} J(x)\Delta(x,x')J(x')\right] \\
&= \left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\left[-\frac{1}{2}\int_{x'}\Delta(x_2,x')J(x) - \frac{1}{2}\int_x J(x)\Delta(x,x_2)\}\right] \\
&\qquad \times \exp\left[-\frac{i}{2}\int_{x,x'} J(x)\Delta(x,x')J(x')\right] \\
&= -\frac{1}{i}\frac{1}{2}\{\Delta(x_1,x_2)+\Delta(x_2,x_1)\} \\
&= i\Delta(x_1,x_2)
\end{aligned}$$

where we set $J = 0$ in the penultimate term and then used the fact that Δ is symmetric in its arguments. This must be the case since $\mathrm{T}[\varphi(x_1)\varphi(x_2)]$ is symmetric in x_1 and x_2, and we've used this in the past to *define* Δ (see Equation (9.4.11)). We can show its the case using the definition here also, since

$$\Delta(x) = \int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{\mathrm{e}^{-ip\cdot x}}{p^2 - m^2 + i\varepsilon} \overset{p\to -p}{=} \int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{\mathrm{e}^{ip\cdot x}}{p^2 - m^2 + i\varepsilon} = \Delta(-x). \quad (31.2.46)$$

Note that even if D is odd we still don't have to change the overall sign since the integration region, $\mathbb{R}^D$, is even so we're integrating over all possible value of p or $-p$ either way so the change in sign cancels with exchanging the limits.

In terms of physics this symmetry is due to invariance under translations, which means that the propagator can only depend on the *relative* positions of its arguments, not on the actual positions, meaning we can always write it as a function of the difference in positions, and then all that changes is the sign if we swap the two arguments and the argument above tells us the sign is not important.

Thirty-Two

Interacting Scalar Field Theory

32.1 General Theory

In this chapter we consider a scalar field theory with some interaction given by the potential, V, which is a function of the field, φ. This is described by the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + V(\varphi) \tag{32.1.1}$$

$$= \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}m^2\varphi^2 + V(\varphi). \tag{32.1.2}$$

The action for this Lagrangian is then

$$S[\varphi] = S_0[\varphi] + S_{\text{int}}[\varphi] \tag{32.1.3}$$

$$= \int \mathrm{d}^D x \left[\frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}m^2\varphi^2\right] + \int \mathrm{d}^D x V(\varphi). \tag{32.1.4}$$

We will be interested in computing the path integral

$$Z[J] = \langle 0|0\rangle_J = \int \mathcal{D}\varphi \, \exp\left\{i\left[S_0[\varphi] + \int \mathrm{d}^D x \, V(\varphi) + J\cdot\varphi\right]\right\}, \tag{32.1.5}$$

where

$$J\cdot\varphi = \int \mathrm{d}^D x \, J(x)\varphi(x). \tag{32.1.6}$$

Now expand V term in a power series, first write

$$Z[J] = \int \mathcal{D}\varphi \exp\left\{i\int \mathrm{d}^D x \, V(\varphi)\right\} \exp\left\{i(S_0[\varphi] + J\cdot\varphi)\right\}. \tag{32.1.7}$$

Then expand the first exponential:

$$Z[J] = \int \mathcal{D}\varphi \sum_{n=0}^{\infty} \frac{1}{n!}\left[i\int \mathrm{d}^D x \, V(\varphi(x_n))\right]^n \exp\{i(S_0[\varphi] + J\cdot\varphi)\}. \tag{32.1.8}$$

Now expand the potential as well, giving

$$Z[J] = \int \mathcal{D}\varphi \sum_{n=0}^{\infty} \frac{1}{n!}\left[i\int \mathrm{d}^D x \sum_{k=0}^{\infty} \frac{1}{k!}\frac{\partial V}{\partial \varphi}\varphi^k\right]^n \exp\{i(S_0[\varphi] + J\cdot\varphi)\}. \tag{32.1.9}$$

We can obtain each factor of φ by acting on the exponential with $\delta/\delta J(x)$. This also brings down a factor of i, so we include a factor of $1/i$ for each derivative giving

$$Z[J] = \int \mathcal{D}\varphi \sum_{n=0}^{\infty} \frac{1}{n!} \left[i \int \mathrm{d}^D x \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial V}{\partial \varphi} \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right)^k \right]^n \exp\{i(S_0[\varphi] + J \cdot \varphi)\}.$$

We can package the derivatives up into $V(-i\delta/\delta J(x))$, which is just a formal term standing for exactly the expansion above, giving

$$Z[J] = \int \mathcal{D}\varphi \sum_{n=0}^{\infty} \frac{1}{n!} \left[i \int \mathrm{d}^D x \, V\left(\frac{1}{i} \delta/\delta J(x) \right) \right]^n \exp\{i(S_0[\varphi] + J \cdot \varphi)\}. \quad (32.1.10)$$

Finally we can package the exponential back up giving

$$Z[J] = \int \mathcal{D}\varphi \exp\left\{ i \int \mathrm{d}^D x \, V\left(\frac{1}{i} \frac{\mathrm{d}}{\mathrm{d}J(x)} \right) \right\} \exp\{i(S_0[\varphi] + J \cdot \varphi)\}. \quad (32.1.11)$$

Now the first exponential is independent of φ, so we can pull it outside of the integral to get

$$Z[J] = \exp\left\{ i \int \mathrm{d}^D x \, V\left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right) \right\} \int \mathcal{D}\varphi \exp\{i(S_0[\varphi] + J \cdot \varphi)\} \quad (32.1.12)$$

$$= \exp\left\{ i \int \mathrm{d}^D x \, V\left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right) \right\} Z_0[J], \quad (32.1.13)$$

where we recognise

$$Z_0[J] = \int \mathcal{D}\varphi \exp\{i(S_0[\varphi] + J \cdot \varphi)\} \quad (32.1.14)$$

as the amplitude for the noninteracting system.

32.2 φ^3 Interactions

We will consider as an example the potential

$$V(\varphi) = \frac{g}{3!}\varphi^3. \quad (32.2.1)$$

Then we can use

$$V\left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right) = \frac{g}{3!} \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right)^3. \quad (32.2.2)$$

Expand the exponential and $Z_0[J]$ in power series, using Equation (31.2.40) for $Z_0[J]$, this gives

$$Z[J] = \sum_{V=0}^{\infty} \frac{1}{V!} \left[i \frac{g}{3!} \int_x \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right)^3 \right]^V \sum_{P=0}^{\infty} \frac{1}{P!} \left[-\frac{1}{2} \int_{y,z} J(y) i\Delta(y-z) J(z) \right]^P. \quad (32.2.3)$$

This is a perturbative expansion for $Z[J]$, and is a function of J, all spacetime points, x, y, and z, are integrated over.

Consider what happens for some fixed values of V and P. We have $2P$ factors of J, and $3V$ derivatives with respect to J, so we end up with $E = 2P - 3V$ factors of J.

We will again introduce a diagrammatic notation to write the result of integrals like these. First, a propagator is written as a line:

$$i\Delta(y-z) = \quad x \text{ ——— } y \tag{32.2.4}$$

We denote an integral of $J(x)$ as solid dot at the end of a line:

$$i\int \mathrm{d}^D x\, J(x) = \quad x \bullet\text{——— } y \tag{32.2.5}$$

Note that the line also gives a propagator, which we're not worried about here. We denote an integral and a factor of g with a vertex, here with three legs, each of which again represents a propagator which we aren't worried about here:

$$ig\int \mathrm{d}^D x = \quad \underset{x}{\curlyvee} \tag{32.2.6}$$

We have to carefully count how many times each term contributes to the overall amplitude. We need to count the number of ways we can change the term without changing anything. We have the following symmetries to account for:

- Permutations of the derivatives with respect to $J(x)$, evaluated at the same x. That is, if we have the $V = 2$ term

$$\frac{1}{2!}\left(i\frac{g}{3!}\int_{x_1}\left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)^3\right)\left(i\frac{g}{3!}\int_{x_2}\left(\frac{1}{i}\frac{\delta}{\delta J(x_2)}\right)^3\right)\cdots \tag{32.2.7}$$

  then we can exchange two of the derivatives with respect to $J(x_1)$, but we can't exchange a derivative with respect to $J(x_1)$ and a derivative with respect to $J(x_2)$. In total this symmetry gives a factor of $(3!)^V$.

- Permutations of propagators and relabelling dummy integration variables. There are P propagators (which is why we called this index P in the first place) and so we have a factor of $P!$.

- Swapping the ends of propagators, since they're symmetric functions, this gives a factor of 2^P.

- Permuting integrals over x in the first term, which corresponds to permuting vertices. There are V vertices (which is why we called this index V in the first place) and so we have a factor of $V!$.

Notice that Equation (32.2.3) has factors of $1/(3!)^V$, $1/P!$, $1/2^P$, and $1/V!$, and so all of these factors cancel out, indeed this is why we choose to have the factor of 3! and 1/2 in the initial definitions. However, this only works if all of the operations described above are distinct, and this isn't always the case. If two of these operations give the same term then we don't want to double count them.

For example, consider the $V = 1, P = 2$ term:

$$i\frac{g}{3!}\int \mathrm{d}^D x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right) \times \frac{1}{2!}\left(-\frac{1}{2}\right)^2 \int_{y_1,z_1} J(y_1)i\Delta(y_1-z_1)J(z_1)\int_{y_2,z_2} J(y_2)i\Delta(y_2-z_2)J(z_2).$$

Now focus on the contribution from this term when the second derivative acts on the first factor of J and the third derivative acts on the second factor of J:

$$i\frac{g}{3!}\int \mathrm{d}^D x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right) \times \frac{1}{2!}\left(-\frac{1}{2}\right)^2 \int_{y_1,z_1} J(y_1)i\Delta(y_1-z_1)J(z_1)\int_{y_2,z_2} J(y_2)i\Delta(y_2-z_2)J(z_2).$$

Now swap the second and third derivative, which changes nothing, but keep track of which J they act on

$$i\frac{g}{3!}\int \mathrm{d}^D x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right) \times \frac{1}{2!}\left(-\frac{1}{2}\right)^2 \int_{y_1,z_1} J(y_1)i\Delta(y_1-z_1)J(z_1)\int_{y_2,z_2} J(y_2)i\Delta(y_2-z_2)J(z_2).$$

We can now swap the first two factors of J, and make a change of variables $y_1 \leftrightarrow z_1$, giving

$$i\frac{g}{3!}\int \mathrm{d}^D x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right) \times \frac{1}{2!}\left(-\frac{1}{2}\right)^2 \int_{y_1,z_1} J(y_1)i\Delta(y_1-z_1)J(z_1)\int_{y_2,z_2} J(y_2)i\Delta(y_2-z_2)J(z_2).$$

This is just the term we started with but with the colours swapped. This means that these two swaps are inverses, and so swapping the derivatives and reverse swapping the Js (which is just swapping them in this case where there are only 2) are the same operation, and so if we include both the factor of $P! = 2$ and $(3!)^V = 6$ then we'll be overcounting the contributions from this term.

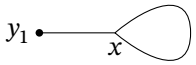
To see this we can perform three derivatives in this term. They will each act on one of four factors of J, specifically one of $J(y_1)$, $J(z_1)$, $J(y_2)$, and $J(z_2)$. The first derivative will have four options for which J to act on, the second derivative will have three options, and the third will have two, so we include a factor of $4 \cdot 3 \cdot 2$. It doesn't matter which J we don't act on, since we can always make a change of variables to choose the argument of J to be any of y_1, z_1, y_2 and z_2. So without loss of generality we assume that nothing acts on $J(y_1)$. Then the three other Js are acted on, and so each gives a Dirac delta, $\delta(x-\xi)$ for each $J(\xi)$. Hence this term becomes

$$i\frac{g}{3!}\frac{1}{2!}\left(-\frac{1}{2}\right)^2 4\cdot 3\cdot 2\int_{x,y_1,z_1,y_2,z_2} J(y_1)\delta(x-z_1)\delta(x-y_2)\delta(x-z_2)i\Delta(y_1-z_1)i\Delta(y_2-z_2). \tag{32.2.8}$$

Cancelling some factors and using the Dirac deltas to perform three of the integrals we get

$$g\frac{1}{2}\int_{x,y_1} J(y_1)\Delta(y_1-x)\Delta(x-x). \tag{32.2.9}$$

Now we can draw the diagram representing this integral, we have a single source, so one solid dot, at y_1 this is then connected by a propagator to x, and then we have another propagator from x to x, which gives a loop:



(32.2.10)

Notice that this diagram has a single mirror symmetry along the propagator line from y_1 to x. This 2-fold symmetry exactly corresponds to the factor of $1/2$ that we are left with in the integral. In general the integral will have a factor of 1 over the order of the symmetry group of the diagram.

32.3 Disconnected Diagrams

A disconnected diagram is any diagram made from two or more pieces with no propagators connecting them. We can consider some arbitrary diagram, D, which may or may not be connected. We can always decompose D as a product of its connected components, which will just be D itself if it's connected. Let C_I be the Ith connected diagram and suppose it appears n_I times in D. Then diagram C_I contributes a factor of $C_I^{n_I}$ to D, up to a symmetry factor. We include in C_I the symmetry factor for this diagram, and so the only symmetry not accounted for is permutations of repeated connected subdiagrams. In particular we get a factor of $n_I!$ for each connected subdiagram. Thus, we can express D as

$$D = \frac{1}{S_D} = \prod_I C_I^{n_I}, \qquad \text{where} \qquad S_D = \prod_I n_I!. \tag{32.3.1}$$

We can then work to all orders in perturbation theory and express $Z[J]$ as a sum over all diagrams, D, which we can regard as a function of $\{n_I\}$, the number of times the connected diagram C_I appears as a subdiagram of D:

$$Z[J] = \sum_{\{n_I\}} D(\{n_I\}) = \sum_{\{n_I\}} \prod_I \frac{1}{n_I!} C_I^{n_I}. \tag{32.3.2}$$

We can swap the order of the product and, then the sum is over fixed I and we take all possible diagrams so n_I runs from 0 to ∞, giving

$$Z[J] = \prod_I \sum_{n_I=0}^{\infty} \frac{1}{n_I!} C_I^{n_I} = \prod \exp\{C_I\} = \exp\left\{\sum_I C_I\right\}. \tag{32.3.3}$$

When normalising $Z[J]$ we choose to have $Z[0] = 1$, which corresponds to replacing $Z[J]$ with $Z[J]/Z[0]$. Since $Z[0]$ consists of all diagrams with no sources we have

$$Z[0] = \exp\left\{\sum_{I|E=0} C_I\right\} \tag{32.3.4}$$

where the sum is taken over diagrams with $E = 0$. We then have

$$\frac{Z[J]}{Z[0]} = \exp\left\{\sum_I C_I\right\}\exp\left\{-\sum_{I|E=0} C_I\right\} = \exp\left\{\sum_{I|E\neq 0} C_I\right\} = \exp\left\{\sum_I{}' C_I\right\} \quad (32.3.5)$$

where the sum in the last two is taken over all connected diagrams with no sources.

Now consider the log of this value:

$$\log\left(\frac{Z[J]}{Z[0]}\right) = \sum_{I|E\neq 0} C_I =: iW[J]. \quad (32.3.6)$$

This is a sum over connected diagrams with no sources. The factor of i is chosen so that $W[J]$ is real for a real scalar field.

That disconnected diagrams don't contribute to this value proves a similar statement about Gaussian correlators made in Section 28.2.2.

Part XI

Scalar Field Correlators

Thirty-Three

Correlators

33.1 Correlators

The generating functional in φ^3 theory is

$$Z[J] = \sum_{V=0}^{\infty} \frac{1}{V!}\left[\frac{ig}{3!}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right]^V \sum_{P=0}^{\infty}\frac{1}{P!}\left[-\frac{1}{2}\int \mathrm{d}^D y \int \mathrm{d}^D z\, J(y) i\Delta(y-z) J(z)\right]^P.$$

We want to calculate the n-point correlator

$$G^{(n)}(x_1,\ldots,x_n) \coloneqq \langle 0|\,\mathrm{T}\{\varphi(x_1)\cdots\varphi(x_n)\}|0\rangle \tag{33.1.1}$$

$$= \left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\cdots\left(\frac{1}{i}\frac{\delta}{\delta J(x_n)}\right) Z[J]\Big|_{J=0}. \tag{33.1.2}$$

Since $Z[J]$ is just a power series in g we can also expand $G^{(n)}$ as a power series in g with coefficients $G^{(n,V)}$ defined by

$$G^{(n)}(x_1,\ldots,x_n) = \sum_{V=0}^{\infty} G^{(n,V)}(x_1,\ldots,x_n) \tag{33.1.3}$$

In order to have a nonzero contribution we need only consider terms with no sources left after performing the derivatives. That is, we want $n = E$, where $E = 2P - 3V$ is the number of sources.

33.2 Two Point Correlator

As an example we'll compute the two point correlator,

$$G^{(2)}(x_1,x_2) = \left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x_2)}\right) Z[J]\Big|_{J=0}. \tag{33.2.1}$$

to order of g^2. Since $n = 2$, and we require $E = n$ for a nonzero contribution, we need to choose P to satisfy $2 = 2P - 3V$.

33.2.1 $V = 0$

For $V = 0$ we must have $P = 1$ for a nonzero contribution. We can easily read off the $V = 0$ and $P = 1$ contribution to the generating functional, it is

$$-\frac{1}{2}\int_{y,z} J(y) i\Delta(y-z) J(z) = \quad \bullet\!\!-\!\!-\!\!\bullet \tag{33.2.2}$$

We could have gone the other way here. If we had first drawn the diagram then we can see from this that there are two sources, the dots, connected by a propagator

So we want to compute

$$G^{(2,0)}(x_1,x_2) = \left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x_2)}\right) Z[J]\Big|_{J=0} \tag{33.2.3}$$

$$= -\frac{1}{2}\left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x_2)}\right)\int_{y,z} J(y) i\Delta(y-z)J(z) \tag{33.2.4}$$

$$= -\frac{1}{2}\left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\Big[\int_{y,z} \delta(x_1-y)\Delta(y-z)J(z) \tag{33.2.5}$$

$$+ \int_{y,z} J(y) i\Delta(y-z)\delta(x_1-z)\Big] \tag{33.2.6}$$

$$= -\frac{1}{2}\left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\Big[\int_{z} \Delta(x_1-z)J(z) \tag{33.2.7}$$

$$+ \int_{y} J(y)\Delta(y-x_1)\Big] \tag{33.2.8}$$

$$= -\left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\int_{y} J(y)\Delta(y-x_1) \tag{33.2.9}$$

where in the last step we perform a change of integration variables $z \to y$ in the first integral and then use the symmetry of Δ, $\Delta(a-b) = \Delta(b-a)$, which is a result of translation invariance. Performing the last integral we get

$$G^{(2,0)}(x_1,x_2) = i\int_y \delta(x_2-y)\Delta(y-x_1) \tag{33.2.10}$$

$$= i\Delta(x_2-x_1). \tag{33.2.11}$$

We can represent this diagrammatically as

$$G^{(2,0)}(x_1,x_2) = i\Delta(x_1-x_2) = \quad x_1 \text{ ——— } x_2 \tag{33.2.12}$$

where we now fix the two endpoints and so no longer have the symmetry factor of 2 from swapping the ends of the propagator.

Recall that we can write the propagator as the inverse Fourier transform of its momentum space equivalent:

$$G^{(2,0)}(x_1,x_2) = i\Delta(x_1-x_2) = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \mathrm{e}^{ip\cdot(x_1-x_2)} \frac{i}{p^2-m^2+i\varepsilon}. \tag{33.2.13}$$

33.2.2 $V = 1$

For $V = 1$ we require the equation $2 = 2P - 3$ to hold, but there is no integer solution for this, and hence there is no nonzero contribution from $V = 1$. This is a peculiarity of φ^3 theory, not a general rule. We saw this through explicit calculations of the $V = 1$ terms in Chapter 10.

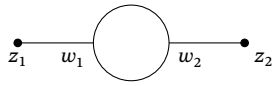
33.2.3 $V = 2$

For a nonzero contribution with $V = 2$ we require the equation $2 = 2P - 6$ to hold, and it does if $P = 4$. We could go through the exercise of writing out the

$V = 2$ and $P = 4$ contribution to $Z[J]$ and then computing the functional derivative with respect to $J(x_1)$ and $J(x_2)$, but this isn't that fun. We'd have six derivatives to compute, all of which can act on any of eight factors of J, its a mess. Instead we can realise that a $V = 2\,P = 4$ contribution consists of a diagram with two vertices and four propagators. Before computing the derivative we must have two sources, so that our final result has no sources. So, we can simply write down all diagrams with two vertices, four propagators, and two sources. It turns out that there are two distinct diagrams satisfying these requirements.

33.2.3.1 The First Diagram

The following diagram has two vertices, four propagators[1], and two sources:



(33.2.14)

[1] Note that the circle is actually two propagators, one making up the top and the other the bottom.

Note that the labels are just to assist in constructing the integral.

Let's start with the symmetry factor. We have two planes of mirror symmetry, vertical and horizontal, and no other symmetries. So the symmetry factor is $2 \cdot 2 = 4$. We then have two source factors, at z_1 and z_2, propagators from z_1 to w_1 and w_2 to z_2, and then two propagators from w_1 to w_2. The integral corresponding to this diagram is

$$I = \frac{1}{4}\int_{z_1,z_2,w_1,w_2} J(z_1)i\Delta(z_1 - w_1)[i\Delta(w_1 - w_2)]^2 i\Delta(w_2 - z_2)J(z_2). \quad (33.2.15)$$

Notice that there are four factors of i giving an overall factor of 1.

We can compute the first derivative, splitting into two terms with the product rule, then getting Dirac deltas, which we use to remove one integral, and then performing a change of variables to show both terms are the same, so we can recombine them and pick up a factor of 2 giving the result

$$\frac{1}{i}\frac{\delta I}{\delta J(x_2)} = \frac{1}{2i}\int_{z_1,w_1,w_2} J(z_1)\Delta(z_1 - w_1)\Delta(w_1 - w_2)^2\Delta(w_2 - x_2). \quad (33.2.16)$$

We can perform the next derivative, now with only one factor of J to act on, and we again get a Dirac delta which we use to remove one of the integrals for the final result

$$G_a^{(2,2)}(x_1, x_2) = \left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\left(\frac{1}{i}\frac{\delta I}{\delta J(x_2)}\right) \quad (33.2.17)$$

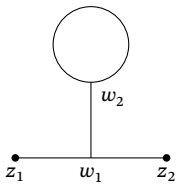
$$= -\frac{1}{2}\int_{w_1,w_2} \Delta(x_1 - w_1)\Delta(w_1 - w_2)^2\Delta(w_2 - x_2). \quad (33.2.18)$$

The subscript a here simply labels that this is the contribution from the first diagram of this type.

33.2.3.2 The Second Diagram

The other diagram with two vertices, four propagators[2], and two sources is

[2] This time the circle is just a single propagator connecting the vertex to itself.



(33.2.19)

Again, the labels are just to help us construct the integral.

Starting with the symmetry factor we have a vertical plane of mirror symmetry, and we can also take the top propagator, forming a circle, swap the ends, and the diagram doesn't change. So the symmetry factor is $2 \cdot 2 = 4$. We then have two source factors at z_1 and z_2, propagators from z_1 to w_1, w_1 to w_2, and w_1 to z_2. We also have a propagator from w_2 back to w_2, which will give a factor of $\Delta(0)$. The integral corresponding to this diagram is

$$I = \frac{1}{4}\int_{z_1,z_2,w_1,w_2} J(z_1)i\Delta(z_1-w_1)i\Delta(w_1-w_2)i\Delta(0)i\Delta(w_2-z_2)J(z_2). \quad (33.2.20)$$

Notice that again the factors of i cancel.

We can compute the first derivative, much like for the previous diagram, and the result is

$$\frac{1}{i}\frac{\delta I}{\delta J(x_2)} = \frac{1}{2i}\int_{z_1,w_1,w_2} J(z_1)\Delta(z_1-w_1)\Delta(w_1-w_2)\Delta(0)\Delta(w_2-x_2). \quad (33.2.21)$$

Computing the second derivative we have

$$G_b^{(2,2)}(x_1,x_2) = \left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\left(\frac{1}{i}\frac{\delta I}{\delta J(x_2)}\right) \quad (33.2.22)$$

$$= -\frac{1}{2}\int_{w_1,w_2} \Delta(x_1-w_1)\Delta(w_1-w_2)\Delta(0)\Delta(w_1-x_2). \quad (33.2.23)$$

Again, the subscript b simply labels that this is the contribution from the second diagram of this type.

33.3 Momentum Space

Its often easier to do computations in Fourier space. One example of this is the correlator, which is given by

$$\Delta(x) = \int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{\mathrm{e}^{-ip\cdot x}}{p^2-m^2+i\varepsilon} \quad (33.3.1)$$

in real space, but in Momentum space is simply

$$\tilde{\Delta}(p) = \frac{1}{p^2-m^2+i\varepsilon}. \quad (33.3.2)$$

We can Fourier transform the correlators we have already calculated to work out the correlator in momentum space, but first, consider the Fourier transform of the two-point correlator, $G^{(n)}$, in a general theory. This is given by

$$\tilde{G}^{(2)}(p_1,p_2) = \int \mathrm{d}^D x_1\,\mathrm{d}^D x_2\,\mathrm{e}^{ip_1\cdot x_1}\mathrm{e}^{ip_2\cdot x_2}G^{(2)}(x_1,x_2). \quad (33.3.3)$$

Now perform a change of variables for the x_1 integral, defining $\xi = x_1 - x_2$. The measure and limits are invariant under translations, as is $G^{(2)}(x_1, x_2)$. This means we can rewrite $G^{(2)}$ as a function of $x_1 - x_2 = \xi$, so $G^{(2)}(x_1 - x_2) = G^{(2)}(\xi)$. Doing so we have

$$\begin{aligned}
\tilde{G}^{(2)}(p_1, p_2) &= \int \mathrm{d}^D\xi \mathrm{d}^D x_2 \, \mathrm{e}^{ip_1\cdot(\xi+x_2)}\mathrm{e}^{ip_2\cdot x_2} G^{(2)}(\xi) && (33.3.4)\\
&= \int \mathrm{d}^D\xi \mathrm{d}^D x_2 \, \mathrm{e}^{ip_1\cdot\xi}\mathrm{e}^{i(p_1+p_2)\cdot x_2} G^{(2)}(\xi) && (33.3.5)\\
&= \int \mathrm{d}^D\xi \, \mathrm{e}^{ip_1\cdot\xi}(2\pi)^D\delta(p_1+p_2) G^{(2)}(\xi) && (33.3.6)\\
&= (2\pi)^D\delta(p_1+p_2)\tilde{G}^{(2)}(p_1), && (33.3.7)
\end{aligned}$$

where in the last step we recognise $\tilde{G}^{(2)}(p_1)$ as the Fourier transform of the single variable function depending on $x_1 - x_2 = \xi$.

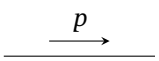
What we see here is that the 2-point correlator depends on only 1 momentum value, the other being fixed by the Dirac delta, so that all momenta sum to zero. This is a general fact. In momentum spae the n-point correlator will depend on $n-1$ momenta values.

33.3.1 $V = 0$

We already know that $G^{(2)}(x_1, x_2) = i\Delta(x_1 - x_2)$, so we trivially have

$$\tilde{G}^{(2,0)}(p) = \frac{i}{p^2 - m^2 + i\varepsilon}. \tag{33.3.8}$$

We represent this diagrammatically as



(33.3.9)

We use arrows to denote the direction of the momentum.

33.3.2 $V = 2$

33.3.2.1 The First Diagram

We can similarly compute the contribution from the first diagram in momentum space by computing the Fourier transform of the known result:

$$G_a^{(2,0)}(x_1, x_2) = -\frac{1}{2}\int_{w_1,w_2} \Delta(x_1 - w_1)\Delta(w_1 - w_2)^2\Delta(w_2 - x_2). \tag{33.3.10}$$

We start by writing the propagators as inverse Fourier transforms of the momentum space propagator, so

$$\begin{aligned}
G_a^{(2,0)}(x_1, x_2) = -\frac{1}{2}\int_{w_1,w_2}\int_{k_1,k_2,k_3,k_4} &\frac{\mathrm{e}^{-ik_1\cdot(x_1-w_1)}}{k_1^2 - m^2 + i\varepsilon}\frac{\mathrm{e}^{-ik_2\cdot(w_1-w_2)}}{k_2^2 - m^2 + i\varepsilon}\\
&\times \frac{\mathrm{e}^{-ik_3\cdot(w_1-w_2)}}{k_3^2 - m^2 + i\varepsilon}\frac{\mathrm{e}^{-ik_4\cdot(w_2-x_2)}}{k_4^2 - m^2 + i\varepsilon}. \qquad (33.3.11)
\end{aligned}$$

Now comes the actual Fourier transform, we have to Fourier transform both x_1 and x_2, giving

$$\tilde{G}_a^{(2,0)}(p_1,p_2) = -\frac{1}{2}\int_{x_1,x_2}\int_{w_1,w_2}\int_{k_1,k_2,k_3,k_4} e^{-ip_1\cdot x_1}e^{-ip_2\cdot x_2}\frac{e^{-ik_1\cdot(x_1-w_1)}}{k_1^2-m^2+i\varepsilon} \times \frac{e^{-ik_2\cdot(w_1-w_2)}}{k_2^2-m^2+i\varepsilon}\frac{e^{-ik_3\cdot(w_1-w_2)}}{k_3^2-m^2+i\varepsilon}\frac{e^{-ik_4\cdot(w_2-x_2)}}{k_4^2-m^2+i\varepsilon} \quad (33.3.12)$$

where the Fourier transform is

$$\tilde{f}(p) = \int_x e^{-ip\cdot x} f(x). \quad (33.3.13)$$

Now we can rewrite the exponentials collecting x_1 and x_2 together:

$$\tilde{G}_a^{(2,0)}(p_1,p_2) = -\frac{1}{2}\int_{x_1,x_2}\int_{w_1,w_2}\int_{k_1,k_2,k_3,k_4} e^{-i(p_1+k_1)\cdot x_1}e^{-i(p_2-k_4)\cdot x_2} \times \frac{e^{ik_1\cdot w_1}}{k_1^2-m^2+i\varepsilon}\frac{e^{-ik_2\cdot(w_1-w_2)}}{k_2^2-m^2+i\varepsilon}\frac{e^{-ik_3\cdot(w_1-w_2)}}{k_3^2-m^2+i\varepsilon}\frac{e^{-ik_4\cdot w_2}}{k_4^2-m^2+i\varepsilon}. \quad (33.3.14)$$

We can now perform the integrals over x_1 and x_2, and we'll get Dirac deltas from these first two exponentials:

$$\tilde{G}_a^{(2,0)}(p_1,p_2) = -\frac{1}{2}\int_{w_1,w_2}\int_{k_1,k_2,k_3,k_4} (2\pi)^D\delta(p_1+k_1)(2\pi)^D(p_2-k_4) \times \frac{e^{ik_1\cdot w_1}}{k_1^2-m^2+i\varepsilon}\frac{e^{-ik_2\cdot(w_1-w_2)}}{k_2^2-m^2+i\varepsilon}\frac{e^{-ik_3\cdot(w_1-w_2)}}{k_3^2-m^2+i\varepsilon}\frac{e^{-ik_4\cdot w_2}}{k_4^2-m^2+i\varepsilon}. \quad (33.3.15)$$

The Dirac deltas allow us to easily perform two of the k integrals, setting $k_1 = -p_1$ and $k_4 = p_2$. Note that the k integrals, being momentum integrals, come with implicit factors of $1/(2\pi)^D$, and so the k_1 and k_4 integrals cancel the factor of $(2\pi)^{2D}$:

$$\tilde{G}_a^{(2,0)}(p_1,p_2) = -\frac{1}{2}\int_{w_1,w_2}\int_{k_2,k_3} \frac{e^{-ip_2\cdot w_1}}{p_2^2-m^2+i\varepsilon}\frac{e^{-ik_2\cdot(w_1-w_2)}}{k_2^2-m^2+i\varepsilon} \times \frac{e^{-ik_3\cdot(w_1-w_2)}}{k_3^2-m^2+i\varepsilon}\frac{e^{-ip_1\cdot w_2}}{p_1^2-m^2+i\varepsilon}. \quad (33.3.16)$$

Next rewrite the exponentials again, this time collecting factors of w_1 and w_2. We get

$$\tilde{G}_a^{(2,0)}(p_1,p_2) = -\frac{1}{2}\int_{w_1,w_2}\int_{k_2,k_3} e^{-i(p_2+k_2+k_3)\cdot w_1}e^{-i(-k_2-k_3+p_1)\cdot w_2} \times \frac{1}{p_2^2-m^2+i\varepsilon}\frac{1}{k_2^2-m^2+i\varepsilon}\frac{1}{k_3^2-m^2+i\varepsilon}\frac{1}{p_1^2-m^2+i\varepsilon}. \quad (33.3.17)$$

We can then perform the w_1 and w_2 integrals, and we get another two Dirac deltas:

$$\tilde{G}_a^{(2,0)}(p_1,p_2) = -\frac{1}{2}\int_{k_2,k_3} (2\pi)^D\delta(p_2+k_2+k_3)(2\pi)^D\delta(-k_2-k_3+p_1) \times \frac{1}{p_2^2-m^2+i\varepsilon}\frac{1}{k_2^2-m^2+i\varepsilon}\frac{1}{k_3^2-m^2+i\varepsilon}\frac{1}{p_1^2-m^2+i\varepsilon}. \quad (33.3.18)$$

We can then perform the k_2 integral using the first Dirac delta setting $k_2 = -p_2 - k_3$, and again cancelling one of the $(2\pi)^D$ factors with one implicit in the k_2 integral, giving

$$\tilde{G}_a^{(2,0)}(p_1,p_2) = -\frac{1}{2}\int_{k_3}(2\pi)^D\delta(p_2+k_3-k_3+p_1)\frac{1}{p_2^2-m^2+i\varepsilon} \times \frac{1}{(p_2+k_3)^2-m^2+i\varepsilon}\frac{1}{k_3^2-m^2+i\varepsilon}\frac{1}{p_1^2-m^2+i\varepsilon}. \quad (33.3.19)$$

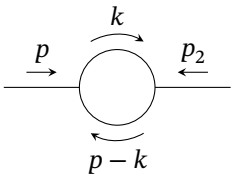
Note that we use the square to get rid of the negatives in the second denominator. We can rename $k_3 \to k$ and rewrite this as

$$\tilde{G}_a^{(2,0)}(p_1,p_2) = -\frac{1}{2}(2\pi)^D\delta(p_1+p_2)\frac{1}{p_2^2-m^2+i\varepsilon} \times\left\{\int_k \frac{1}{(p_2+k)^2-m^2+i\varepsilon}\frac{1}{k^2-m^2+i\varepsilon}\right\}\frac{1}{p_1^2-m^2+i\varepsilon}. \quad (33.3.20)$$

Notice that the Dirac delta means we can set $p_2 = -p_1$ and then we have $(-p_1 + k)^2 = (p_1 - k)^2$ in the first denominator in the integral. Then the result can be written as a function of $p = p_1$ alone, with the only p_2 term being in the Dirac delta:

$$\tilde{G}_a^{(2,0)}(p) = -\frac{1}{2}(2\pi)^D\delta(p+p_2)\frac{1}{p^2-m^2+i\varepsilon} \times\left\{\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{(p-k)^2-m^2+i\varepsilon}\frac{1}{k^2-m^2+i\varepsilon}\right\}\frac{1}{p^2-m^2+i\varepsilon}. \quad (33.3.21)$$

We can also express this in terms of a momentum space diagram:



(33.3.22)

Notice that at each vertex the total momentum is conserved, that is the sum of the momenta on the arrows pointing towards the vertex is equal to the sum of the momenta on the arrows pointing out of the vertex.

33.4 Feynman Rules

Rather than go through the integrals and Fourier transforms every time we can just start with a diagram in momentum space and use the following Feynman rules to turn it into the momentum space propagator:

- An n-point correlator has n lines.
- We leave one end of each external line free, and attach the others to vertices.
- The ith external line has momentum p_i, which we take as incoming momentum, pointing towards the vertex, we denote this with an arrow along the line, labelled with the four-momentum.

- Four-momenta flow along the arrows and the total momentum is conserved at each vertex, with arrows pointing in taken as positive and arrows pointing out as negative. Ensuring this will fix all but L momenta in a diagram with L loops.
- The value of the diagram is given by the product of the factors
  - $i/(p^2 - m^2 + i\varepsilon)$ for each line with momentum p,
  - $1/i$ for each external end of a line (corresponding to the $1/i$ factor that comes with each $\delta/\delta J(x)$ factor),
  - $ig(1/i)^N$ for each vertex connecting N lines,
  - A diagram with L loops will have L internal momenta not fixed by momentum conservation. We integrate over each unfixed momentum with measure $\mathrm{d}^D p/(2\pi)^D$.
  - $1/S$ where S is the symmetry factor of the diagram given by fixing external lines but allowing internal propagators to vary.

Thirty-Four

Pology

34.1 Physical States

The eigenstates of the Hamiltonian form a complete set of states. They can be classified into three types:

- The vacuum state, or ground state, $|0\rangle$, is the lowest energy eigenstate of the Hamiltonian. It corresponds to a state with no particles.
- The single particle states, $|\boldsymbol{p}, \sigma\rangle$, are classified by their spatial momentum, $\boldsymbol{p}$, and σ, which represents some collection of quantum numbers. The energy of these particles is given by the relativistic dispersion relation[1]

$$E_p = \sqrt{\boldsymbol{p}^2 + m_{\text{phys}}^2}. \tag{34.1.1}$$

  Here m_{phys} is the physical mass of the state. Note that this is *not* necessarily the same as the mass appearing in the Lagrangian. We choose normalisation such that[2]

$$\langle \boldsymbol{p}, \sigma | \boldsymbol{p}', \sigma' \rangle = \delta_{\sigma\sigma'} 2E_p (2\pi)^{D-1} \delta^{(D-1)}(\boldsymbol{p} - \boldsymbol{p}'). \tag{34.1.2}$$

- The multiparticle states, $|\boldsymbol{P}, n\rangle$, are classified by the total spatial momentum, $\boldsymbol{P}$, and a collection of other quantum numbers, n. The energy of the multiparticle state is $\sqrt{\boldsymbol{P}^2 + M^2}$ where M^2 is a parameter included in n, and doesn't necessarily correspond to the mass of the particles.

The completeness relation for the eigenstates of the Hamiltonian is

$$\mathbb{1} = |0\rangle\langle 0| + \sum_\sigma \int \mathrm{d}\Omega_p \, |\boldsymbol{p}, \sigma\rangle\langle \boldsymbol{p}, \sigma| + \sum_n \int \mathrm{d}\Omega_P \, |\boldsymbol{P}, n\rangle\langle \boldsymbol{P}, n| \tag{34.1.3}$$

where[3]

$$\mathrm{d}\Omega_p = \frac{\mathrm{d}^{D-1}p}{(2\pi)^{D-1}} \frac{1}{2E_p}. \tag{34.1.4}$$

The sums over n and σ are just a shorthand meaning we should sum over all discrete parameters and integrate over all continuous parameters.

[1] Note that we've been calling this quantity $\omega(\boldsymbol{p})$ in the first half of the course.

[2] Note that we've been calling this quantity $\delta_{\sigma\sigma'}\delta(\boldsymbol{p} - \boldsymbol{p})$ in the first half off the course.

[3] Note that we've been calling this quantity $đp$ in the first half of the course.

34.2 Correlators in Momentum Space

We want to compute the correlator

$$\tilde{G}^{(n)}(p_1,\dots,p_n) = \int \mathrm{d}^D x_1 \cdots \mathrm{d}^D x_n \, \mathrm{e}^{ip_1\cdot x_1}\cdots \mathrm{e}^{ip_n\cdot x_n}\langle 0|\,\mathrm{T}[\varphi(x_1)\cdots\varphi(x_n)]|0\rangle.$$

To do so we consider the sector of the nD-dimensional integration region where, for some r, we have x_1^0 through x_r^0 greater than x_{r+1}^0 through x_n^0. That is,

$$\min\{x_1^0,\dots,x_r^0\} \geq \max\{x_{r+1}^0, x_n^0\}. \tag{34.2.1}$$

We can then split the time ordered product:

$$\mathrm{T}[\varphi(x_1)\cdots\varphi(x_n)] = \mathrm{T}[\varphi(x_1)\cdots\varphi(x_r)]\,\mathrm{T}[\varphi(x_{r+1})\cdots\varphi(x_n)]. \tag{34.2.2}$$

The contribution from this particular time ordering is

$$I = \int_{x_1,\dots,x_n} \exp[i(p_1\cdot x_1+\cdots+p_n\cdot x_n)]\theta(\min\{x_1^0,\dots,x_r^0\}-\max\{x_{r+1}^0,\dots,x_n^0\}) \times \langle 0|\,\mathrm{T}[\varphi(x_1)\cdots\varphi(x_r)]\,\mathrm{T}[\varphi(x_{r+1})\cdots\varphi(x_n)]|0\rangle. \tag{34.2.3}$$

We're going to insert a complete set of states between the two time ordered products, but first, we do some rewriting.

Suppose we have an analytic function, f, depending on some value z. Consider the following expression, where p is the momentum operator in the z direction:

$$\exp[ipa] = \exp\left[a\frac{\partial}{\partial z}\right] f(z) = \sum_{n=0}^{\infty}\frac{1}{n!}a^n\frac{\partial^n}{\partial z^n}f(z) = f(z+a). \tag{34.2.4}$$

This is what we mean when we say that momentum is the generator of translations[4], exponentiating the momentum operator, up to a factor of i, gives a translation. This works in an arbitrary number of dimensions, so we have

[4] For technical details see *Symmetries of Particles and Fields*

$$\mathrm{e}^{ip\cdot a}f(z) = f(z+a). \tag{34.2.5}$$

So, the operator $\exp[ip\cdot a]$ acts on a function to produce a translation. Similarly it acts on an operator to produce a translation. If $F(z)$ is an operator then we have

$$\mathrm{e}^{ip\cdot a}F(z)\mathrm{e}^{-ip\cdot a} = F(z+a), \tag{34.2.6}$$

where as usual we act on operators by sandwiching them between the operator and its inverse.

We can use this to rewrite our time ordered products. Define new variables, y_i, for $i = 1,\dots,r$ such that

$$x_i = x_1 + y_i. \tag{34.2.7}$$

Note that $y_1 = 0$. Then we have

$$\begin{aligned}\varphi(x_1)\cdots\varphi(x_r) &= \varphi(x_1)\varphi(x_1+y_2)\cdots\varphi(x_1+y_r) &(34.2.8)\\ &= \mathrm{e}^{ip\cdot x_1}\varphi(0)\mathrm{e}^{-ip\cdot x_1}\mathrm{e}^{ip\cdot x_1}\varphi(y_2)\mathrm{e}^{-ip\cdot x_1}\cdots\mathrm{e}^{ip\cdot x_1}\varphi(y_r)\mathrm{e}^{-ip\cdot x_1}\\ &= \mathrm{e}^{ip\cdot x_1}\varphi(0)\varphi(y_2)\cdots\varphi(y_r)\mathrm{e}^{-ip\cdot x_1}. &(34.2.9)\end{aligned}$$

We'll focus on the single particle term after inserting the complete set of states, this term has

$$\langle 0|\,\mathrm{T}[\varphi(x_1)\cdots\varphi(x_r)]|\boldsymbol{p},\sigma\rangle\langle\boldsymbol{p},\sigma|\,\mathrm{T}[\varphi(x_{r+1})\cdots\varphi(x_n)]. \tag{34.2.10}$$

So, for the first time ordered product we need to compute

$$\begin{aligned}\langle 0|\,\mathrm{T}[\varphi(x_1)\cdots\varphi(x_r)]|\boldsymbol{p},\sigma\rangle &= \langle 0|\mathrm{e}^{ip\cdot x_1}\,\mathrm{T}[\varphi(0)\varphi(y_2)\cdots\varphi(y_r)]\mathrm{e}^{-ip\cdot x_1}|\boldsymbol{p},\sigma\rangle \\ &= \mathrm{e}^{-ip\cdot x_1}\langle 0|\,\mathrm{T}[\varphi(0)\varphi(y_2)\cdots\varphi(y_r)]|\boldsymbol{p},\sigma\rangle. \end{aligned}\tag{34.2.11}$$

To achieve this last equality we act on the single particle state with $\exp[-ip\cdot x]$, resulting in the same value, but now p is the four-momentum of the single particle state rather than the momentum operator, and so this exponential is a scalar. We similarly act with $\exp[ip\cdot x]$ on the vacuum, giving $\exp[0]=1$.

We can do the same analysis with the second time ordered product, defining y_i for $i=r+1,\ldots,n$ such that

$$x_i = x_{r+1} + y_i. \tag{34.2.12}$$

We then have

$$\varphi(x_{r+1})\cdots\varphi(x_n) = \mathrm{e}^{ip\cdot x_{r+1}}\varphi(0)\varphi(y_{r+2})\cdots\varphi(y_n)\mathrm{e}^{-ip\cdot x_{r+1}}, \tag{34.2.13}$$

and so

$$\langle\boldsymbol{p},\sigma|\,\mathrm{T}[\varphi(x_{r+1})\cdots\varphi(x_n)]|0\rangle = \mathrm{e}^{ip\cdot x_{r+1}}\langle 0|\,\mathrm{T}[\varphi(0)\varphi(y_{r+2})\cdots\varphi(y_n)]|0\rangle. \tag{34.2.14}$$

Now consider the argument to the step function. If we shift all arguments to a min/max by a fixed amount then the minimum/maximum is found at the shift value plus the minimum/maximum size of the shift, and hence we have

$$\begin{aligned}&\min\{x_1^0,\ldots,x_r^0\} - \max\{x_{r+1}^0,\ldots,x_n^0\} \\ &\qquad = x_1^0 - x_{r+1}^0 + \min\{0,y_2,\ldots,y_r\} - \max\{0,y_{r+2},\ldots,y_n\}.\end{aligned}\tag{34.2.15}$$

We now use the integral representation of the step function:

$$\theta(\tau) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{\mathrm{e}^{-i\tau\omega}}{\omega+i\varepsilon}. \tag{34.2.16}$$

For some real value $\varepsilon>0$ which we will take to zero after computing the integral. Suppose $\omega=a+ib$. Then $-i\tau\omega=\tau(-ia+b)$. In order to close the contour and compute the integral we want to choose the contour such that the exponential goes to zero as ω goes to infinity in some way, so that if we close with an arc to complete a semicircle the value of the integral along the arc will vanish by Jordan's lemma[5]. If $\tau>0$ then the integral along the arc vanishes when b goes to $-\infty$, and so we close in the lower half plane, containing the pole at $\omega=-i\varepsilon$. In this case going anticlockwise about the contour means going from ∞ to $-\infty$, flipping the limits of the integral, so we include an extra factor of -1 to account for this. We can then use the residue theorem to compute the value of the integral:

[5] see *Methods of Theoretical Physics* or *Methods of Mathematical Physics* for details.

$$-2\pi i\,\mathrm{Res}\left(\frac{\mathrm{e}^{-i\tau\omega}}{\omega+i\varepsilon},\omega=-i\varepsilon\right) = -2\pi i\mathrm{e}^{-i\tau(-i\varepsilon)} = -2\pi i\mathrm{e}^{-\varepsilon\tau}. \tag{34.2.17}$$

Taking $\varepsilon \to 0$, and so taking the exponential term to 1, we have

$$\theta(\tau) = -\frac{1}{2\pi i}(-2\pi i) = 1. \tag{34.2.18}$$

On the other hand, if $\tau < 0$ then the integral along the arc vanishes when b goes to ∞, and so we close in the upper half plane, containing no poles, and so by Cauchy's theorem the integral vanishes and

$$\theta(\tau) = 0. \tag{34.2.19}$$

So, this integral does indeed represent the Heaviside step function.

Using this, and the shifted minimum and maximum, we can rewrite the Heaviside step function in our integral as

$$\begin{aligned}&\theta(\min\{x_1^-, \dots, x_r^0\} - \max\{x_{r+1}^0, \dots, x_n^0\}) \\ &= -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \exp[-i\omega(x_1^0 - x_{r+1}^0 + \min\{0, y_2, \dots, y_r\} - \max\{0, y_{r+2}, \dots, y_n\})].\end{aligned} \tag{34.2.20}$$

Now consider Equation (34.2.3). We can write out the part of the integrand with dependence on x_1 and x_{r+1} as

$$\exp\left[i\left(\sum_{j=1}^{r} p_j \cdot x_1 - \omega x_1^0 - p \cdot x_1\right)\right] \exp\left[i\left(\sum_{j=r+1}^{n} p_j \cdot x_{r+1} + \omega x_{r+1}^0 + p \cdot x_{r+1}\right)\right].$$

The sums over $p_j \cdot x_1$ and $p_j \cdot x_{r+1}$ come from the exponentials as part of the definition of the Fourier transform after our change of variables, for example we have

$$e^{ip_2 \cdot x_2} = e^{ip_2 \cdot (x_1 + y_2)} = e^{ip_2 \cdot x_1} e^{ip_2 \cdot y_2}, \tag{34.2.21}$$

$$e^{ip_n \cdot x_n} = e^{ip_n \cdot (x_{r+1} + y_n)} = e^{ip_n \cdot x_{r+1}} e^{ip_n \cdot y_n}, \tag{34.2.22}$$

and we aren't considering the $\exp[-ip_2 \cdot y_2]$ or $\exp[-ip_n \cdot y_n]$ factors as we're only interested in x_1 and x_{r+1} dependence here.

The ωx_1^0 and ωx_{r+1}^0 terms come from the x_1^0 and x_{r+1} terms in the integral representation of the Heaviside step function.

Finally, we have the $p \cdot x_1$ and $p \cdot x_{r+1}$ terms from acting on $|\boldsymbol{p}, \sigma\rangle$ with $\exp[-ip \cdot x_1]$ and $\exp[ip \cdot x_{r+1}]$.

Note that in **??** we will be integrating over x_1 and x_{r+1} from the Fourier transform, ω from the integral representation of the Heaviside step function, and p from the integral over the momenta of single particle states in the completeness relation.

Computing the integral over this term with all x_1 and x_{r+1} dependence we get Dirac deltas. Splitting into the space and time components we get

$$\begin{aligned}(2\pi)^D \delta^{(D-1)}\left(\boldsymbol{p} - \sum_{j=1}^{r} \boldsymbol{p}_j\right)\delta\left(E_p + \omega - \sum_{j=1}^{r} p_j^0\right) \\ \times (2\pi)^D \delta^{(D-1)}\left(\boldsymbol{p} + \sum_{j=r+1}^{n} \boldsymbol{p}_j\right)\delta\left(E_p + \omega + \sum_{j=r+1}^{n} p_j^0\right)\end{aligned} \tag{34.2.23}$$

where $E_p = p^0$.

Taking this we can compute the single particle contribution to Equation (34.2.3) is

$$\begin{aligned}
&\int_{y_2,\dots y_r,y_{r+2},\dots y_n} \exp[i(p_2\cdot y_2+\cdots+p_r\cdot y_r+p_{r+2}\cdot y_{r+2}+\cdots+p_n\cdot y_n)]\\
&\times\left(-\frac{1}{2\pi i}\right)\int \mathrm{d}\omega\,\frac{1}{\omega+i\varepsilon}\exp[-i\omega(\min\{0,y_2,\dots,y_r\}-\max\{0,y_{r+2},\dots,y_n\})]\\
&\times\sum_\sigma\int \mathrm{d}\Omega_p\,\langle 0|\,\mathrm{T}[\varphi(0)\varphi(y_2)\cdots\varphi(y_r)]|\boldsymbol{p},\sigma\rangle\langle\boldsymbol{p},\sigma|\,\mathrm{T}[\varphi(0)\varphi(y_{r+2}\cdots\varphi(y_n))]|0\rangle\\
&\times(2\pi)^D\delta^{(D-1)}\Big(\boldsymbol{p}-\sum_{j=1}^r\boldsymbol{p}_j\Big)\delta\Big(E_p+\omega-\sum_{j=1}^r p_j^0\Big)\\
&\times(2\pi)^D\delta^{(D-1)}\Big(\boldsymbol{p}+\sum_{j=r+1}^n\boldsymbol{p}_j\Big)\delta\Big(E_p+\omega+\sum_{j=r+1}^n p_j^0\Big).
\end{aligned}\tag{34.2.24}$$

While this looks daunting we can simplify it greatly using the Dirac deltas. Performing the integral over p, with the measure $\mathrm{d}\Omega_p$ we set

$$\boldsymbol{p}=\sum_{j=1}^r\boldsymbol{p}_j=-\sum_{r+1}^n\boldsymbol{p}_j=:\boldsymbol{q},\tag{34.2.25}$$

and performing the ω integral we set

$$\omega=\sum_{j=1}^r p_j^0-E_p=-\sum_{j=r+1}^n p_j^0-E_p:=q^0-E_p.\tag{34.2.26}$$

That is, we define the four-momentum

$$q=\sum_{j=1}^r p_j=-\sum_{j=r+1}^n p_j.\tag{34.2.27}$$

We can also cancel one of the factors of $(2\pi)^D$ with the factor in the integration measure $\mathrm{d}\Omega_p$. We are then left with a factor of $1/(2E_p)$ from this measure.

The result of these steps is

$$\begin{aligned}
&(2\pi)^D\delta^{(D)}\Big(\sum_{j=1}^n p_j\Big)\Big(-\frac{1}{2\pi i}\Big)\frac{1}{q^0-E_p+i\varepsilon}\frac{1}{2E_p}\\
&\times\int_{y_2,\dots,y_r,y_{r+2},\dots y_n}\exp[i(p_2\cdot y_2+\cdots+p_r\cdot y_r+p_{r+2}\cdot y_{r+2}+\cdots+p_n\cdot y_n)]\\
&\times\exp[-i(q_0-E_p)(\min\{0,y_2,\dots,y_r\}-\max\{0,y_{r+2},\dots,y_n\})]\\
&\times\sum_\sigma\langle 0|\,\mathrm{T}[\varphi(0)\varphi(y_2)\cdots\varphi(y_r)]|\boldsymbol{q},\sigma\rangle\langle\boldsymbol{q},\sigma|\,\mathrm{T}[\varphi(0)\varphi(y_{r+2})\cdots\varphi(y_n)]|0\rangle
\end{aligned}\tag{34.2.28}$$

Notice that we have a pole when $q^0-E_p+i\varepsilon=0$. This means there is a pole whenever some sum of r momenta is on shell. The residue at this pole will be the matrix elements,

$$\langle 0|\,\mathrm{T}[\varphi(0)\varphi(y_2)\cdots\varphi(y_r)]|\boldsymbol{q},\sigma\rangle\langle\boldsymbol{q},\sigma|\,\mathrm{T}[\varphi(0)\varphi(y_{r+2})\cdots\varphi(y_n)]|0\rangle,\tag{34.2.29}$$

times the constant factors.

This result can be written more compactly as

$$(2\pi)^D \delta(p_1 + \cdots + p_n) \frac{1}{q^2 - m^2_{\text{phys}} + i\varepsilon} \times \sum_\sigma M_{0|q\sigma}(p_2, \dots, p_r) M_{q\sigma|0}(p_{r+2}, \dots, p_n) \quad (34.2.30)$$

where

$$M_{0|q\sigma}(p_2, \dots, p_r) = \int \mathrm{d}^D y_2 \cdots \mathrm{d}^D y_r \, \mathrm{e}^{ip_2 \cdot y_2} \cdots \mathrm{e}^{ip_r \cdot y_r} \times \langle 0| \, \mathrm{T}[\varphi(0)\varphi(y_2) \cdots \varphi(y_r)] |q, \sigma\rangle, \quad (34.2.31)$$

$$M_{q\sigma|0}(p_{r+2}, \dots, p_n) = \int \mathrm{d}^D y_{r+1} \cdots \mathrm{d}^D y_n \, \mathrm{e}^{ip_{r+2} \cdot x_{r+2}} \cdots \mathrm{e}^{ip_n \cdot y_n} \times \langle q, \sigma| \, \mathrm{T}[\varphi(0)\varphi(y_{r+2}) \cdots \varphi(y_n)] |0\rangle. \quad (34.2.32)$$

To see how we go from $q^0 - E_p + i\varepsilon$ to $q^2 - m^2_{\text{phys}} + i\varepsilon$ consider the following limit:

$$\lim_{q^0 \to E_p} \frac{1}{q^0 - E_p} \frac{1}{2E_p} = \lim_{q^0 \to E_p} \frac{1}{q^0 - E_p} \frac{1}{E_p + E_p} \quad (34.2.33)$$

$$= \lim_{q^0 \to E_p} \frac{1}{q^0 - E_p} \frac{1}{q^0 + E_p} \quad (34.2.34)$$

$$= \lim_{q^0 \to E_p} \frac{1}{(q^0)^2 - E_p^2} \quad (34.2.35)$$

$$= \lim_{q^0 \to E_p} \frac{1}{(q^0)^2 - \boldsymbol{q}^2 - m^2_{\text{phys}}} \quad (34.2.36)$$

$$= \lim_{q^0 \to E_p} \frac{1}{q^2 - m^2_{\text{phys}}}. \quad (34.2.37)$$

So, when $q^0 \approx E_p$, so near the pole, we can change $q^0 - E_p$ to $q^2 - m^2_{\text{phys}}$. This makes it more obvious that the pole occurs when some sum of momenta goes on-shell.

Thirty-Five

Källén–Lehmann Representation

35.1 Motivation

In the previous chapter we didn't consider the contribution due to multiparticle states, we just looked at single particle states. In this chapter we'll look specifically at the two-point correlation function and we'll consider multiparticle states. We'll also drop the dependence on other quantum numbers, σ, for the single particle state.

The two point correlator is defined as

$$G^{(2)}(x,y) \coloneqq \langle 0|\,\mathrm{T}[\varphi(x)\varphi(y)]|0\rangle = i\Delta_{\mathrm{F}}(x-y). \tag{35.1.1}$$

The Fourier transform of this is

$$\tilde{G}^{(2)}(p_1,p_2) = \int \mathrm{d}^D x\,\mathrm{d}^D y\,\mathrm{e}^{ip_1\cdot x}\mathrm{e}^{ip_2\cdot y} i\Delta(x-y) \tag{35.1.2}$$

$$= \int \mathrm{d}^D\xi\,\mathrm{d}^D y \mathrm{e}^{ip_1\cdot\xi}\mathrm{e}^{i(p_1+p_2)\cdot y} i\Delta(\xi) \tag{35.1.3}$$

$$= (2\pi)^D\delta(p_1+p_2)\int \mathrm{d}^D\xi \mathrm{e}^{ip\cdot\xi} i\Delta(\xi) \tag{35.1.4}$$

where we've made the substitution $\xi = x-y$.

We make the normalisation choice that

$$\langle 0|\varphi(x)|0\rangle = 0, \qquad \text{and} \qquad \langle \boldsymbol{p}|\varphi(0)|0\rangle = 1, \tag{35.1.5}$$

where $|\boldsymbol{p}\rangle$ is the single particle state. With these choices the free theory result is that the Fourier transformed propagator is

$$\tilde{\Delta}_{\mathrm{F}}(p) = \frac{i}{p^2-m^2+i\varepsilon} = \quad \overset{p}{\longrightarrow} \tag{35.1.6}$$

That is, we have $m_{\mathrm{phys}} = m$, where m_{phys} is the mass of the particles and m is the mass parameter appearing in the Lagrangian.

35.2 Interacting Theory

Consider the case when $x^0 > y^0$, so $\mathrm{T}[\varphi(x)\varphi(y)] = \varphi(x)\varphi(y)$. Then we can insert the identity in the form of the complete set of states summing over the vacuum,

single particle states, and multiparticle states, and we get

$$\langle 0|\varphi(x)\varphi(y)|0\rangle = \langle 0|\varphi(x)|0\rangle\langle 0|\varphi(y)|0\rangle + \int \mathrm{d}\Omega_p \langle 0|\varphi(x)|\boldsymbol{p}\rangle\langle \boldsymbol{p}|\varphi(y)|0\rangle + \sum_n \int \mathrm{d}\Omega_P \langle 0|\varphi(x)|\boldsymbol{P}, n\rangle\langle \boldsymbol{P}, n|\varphi(y)|0\rangle. \tag{35.2.1}$$

Notice that the completeness relation is defined by summing over all physical states, that means all states we integrate over are on-shell. In particular, if we have a single particle state with momentum $p^\mu = (p^0, \boldsymbol{p})$ then we have $E_p = p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$ and so we must have $p^2 = m^2_{\text{phys}}$, since $E_p = \sqrt{\boldsymbol{p}^2 + m^2_{\text{phys}}}$ by definition. Similarly, we have $P^0 = \sqrt{\boldsymbol{P}^2 + M^2}$, and so $P^2 = M^2$.

Consider the field $\varphi(x)$. We can write this in terms of $\varphi(0)$ by introducing the translation operator, $\exp[i\hat{P} \cdot x]$, and we then have

$$\varphi(x) = \mathrm{e}^{i\hat{P}\cdot x}\varphi(0)\mathrm{e}^{-i\hat{P}\cdot x}, \tag{35.2.2}$$

and we can do the same for $\varphi(y)$. We then have

$$\langle 0|\varphi(x)|\boldsymbol{p}\rangle = \langle 0|\varphi(0)\mathrm{e}^{-ip\cdot x}|\boldsymbol{p}\rangle = \langle 0|\varphi(0)\mathrm{e}^{-ip\cdot x}|\boldsymbol{p}\rangle \tag{35.2.3}$$

and this exponential is just a numerical factor. Similarly, we have

$$\langle \boldsymbol{p}|\varphi(y)|0\rangle = \langle \boldsymbol{p}|\mathrm{e}^{i\hat{P}\cdot y}\varphi(0)\mathrm{e}^{-i\hat{P}\cdot y}|0\rangle = \langle \boldsymbol{p}|\mathrm{e}^{ip\cdot y}\varphi(0)|0\rangle. \tag{35.2.4}$$

We can also do the same in the multiparticle state, and we just replace p with $\boldsymbol{P}$ and $|\boldsymbol{p}\rangle$ with $|\boldsymbol{P}, n\rangle$. Doing the same in the vacuum term leaves us with factors of $|0\rangle\varphi(0)\langle 0| = 0$, so this term vanishes. Furthermore, for the single particle state we have $\langle 0|\varphi(0)|\boldsymbol{p}\rangle = 1$, and so the single particle state simply becomes an integral over an exponential factor, leaving us with

$$\langle 0|\varphi(x)\varphi(y)|0\rangle = \int \mathrm{d}\Omega_p\, \mathrm{e}^{-ip\cdot(x-y)} + \sum_n \int \mathrm{d}\Omega_P\, \mathrm{e}^{-iP\cdot(x-y)} |\langle 0|\varphi(0)|\boldsymbol{P}, n\rangle|^2. \tag{35.2.5}$$

It would be tempting to compute the integral now, but we have to be careful as the integral is over states which are on-shell, and whether a state is on-shell depends on M^2, which is one of the parameters that we are summing over with $\sum_n$.

We define the **spectral density**, ρ, to be

$$\rho(s) \coloneqq \sum_n |\langle 0|\varphi(0)|\boldsymbol{P}, n\rangle|^2 \delta(s - M^2). \tag{35.2.6}$$

This is a function which is given by a sum over matrix elements when $s = M^2$. This allows us to turn a sum over n into an integral over s, in a density-of-states type argument. We then have

$$\langle 0|\varphi(x)\varphi(y)|0\rangle = \int \mathrm{d}\Omega_p\, \mathrm{e}^{-ip\cdot(x-y)} + \int_{4m^2_{\text{phys}}}^{\infty} \mathrm{d}s\, \rho(s) \int \mathrm{d}\Omega_P \mathrm{e}^{-iP\cdot(x-y)}. \tag{35.2.7}$$

We start the integral at $4m^2_{\text{phys}}$ since we must have $M \geq 2m$ in order for a two-point process to occur, and so $M^2 \geq 4m^2_{\text{phys}}$.

If $x^0 < y^0$ then we simply swap x and y:

$$\langle 0|\,\mathrm{T}[\varphi(x)\varphi(y)]|0\rangle = \langle 0|\varphi(y)\varphi(x)|0\rangle \tag{35.2.8}$$
$$= \int \mathrm{d}\Omega_p\, \mathrm{e}^{-ip\cdot(y-x)} + \int_{4m^2_{\text{phys}}}^{\infty} \mathrm{d}s\,\rho(s) \int \mathrm{d}\Omega_P \mathrm{e}^{-iP\cdot(y-x)}.$$

In general we then have

$$\langle 0|\,\mathrm{T}[\varphi(x)\varphi(y)]|0\rangle = \theta(x^0-y^0)\langle 0|\varphi(x)\varphi(y)|0\rangle + \theta(y^0-x^-)\langle 0|\varphi(y)\varphi(x)|0\rangle. \tag{35.2.9}$$

We can then evaluate this using the contour representation of the propagator:

$$\Delta_{\mathrm{F}}(x-y) = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{\mathrm{e}^{-ip\cdot(x-y)}}{p^2 - m^2_{\text{phys}} + i\varepsilon} \tag{35.2.10}$$
$$= \int \frac{\mathrm{d}^{D-1} p}{(2\pi)^{D-1}} \mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \int \frac{\mathrm{d}p^0}{2\pi} \frac{\mathrm{e}^{-ip^0(x^0-y^0)}}{p^2 - m^2_{\text{phys}} + i\varepsilon}. \tag{35.2.11}$$

We integrate along the real axis in the complex p^0 plane. There are poles just below the real axis at E_p and just above the real axis at $-E_p$. First suppose $x^0 > y^0$, as can be seen in Figure 9.2. Then $\exp[-ip^0(x^0 - y^0)]$ vanishes as $p^0 \to \infty$, so we close the contour with a semicircle in the upper half plane. The integral along this semicircle vanishes by Jordan's lemma and we're left with just the integral along the real axis. Using the residue theorem the value is then given by the residue at the pole positioned just above $-E_p$. Focussing just on the p^0 integral we have

$$I = \int \frac{\mathrm{d}p^0}{2\pi} \frac{\mathrm{e}^{-ip^0(x^0-y^0)}}{p^2 - m^2_{\text{phys}} + i\varepsilon} \tag{35.2.12}$$
$$= \int \frac{\mathrm{d}p^0}{2\pi} \frac{\mathrm{e}^{-ip^0(x^0-y^0)}}{(p^0 - E_p + i\varepsilon)(p^0 + E_p - i\varepsilon)} \tag{35.2.13}$$
$$= 2\pi i\,\mathrm{Res}\left(\frac{\mathrm{e}^{-ip^0(x^0-y^0)}}{(p^0 - E_p + i\varepsilon)(p^0 + E_p - i\varepsilon)}, p^0 = -E_p + i\varepsilon \right) \tag{35.2.14}$$
$$= 2\pi i \frac{\mathrm{e}^{-ip^0(x^0-y^0)}}{p^0 + E_p - i\varepsilon}\bigg|_{p_0 = -E_p + i\varepsilon}. \tag{35.2.15}$$

Note that the $i\varepsilon$ appearing in the second line is *not* the same as the $i\varepsilon$ appearing in the first line. We could compute some function of $i\varepsilon$ and E_p such that this is the case, but all that really matters is that we have a small positive $\varepsilon > 0$ such that the $i\varepsilon$ allows us to avoid the contours, so we simply rename ε as we go. The full integral is then

$$\Delta_{\mathrm{F}}(x-y) = \int \mathrm{d}\Omega_p \mathrm{e}^{-ip\cdot(x-y)} \tag{35.2.16}$$

for $x^0 > y^0$, and we can do something similar if $x^0 < y^0$ closing instead in the lower half plane.

The result of this computation is then

$$\langle 0|\,\mathrm{T}[\varphi(x)\varphi(y)]|0\rangle \tag{35.2.17}$$
$$= \int \frac{\mathrm{d}^D p}{(2\pi)^D} \mathrm{e}^{-ip\cdot(x-y)} \left[\frac{i}{p^2 - m^2_{\text{phys}} + i\varepsilon} + \int_{4m^2_{\text{phys}}}^{\infty} \mathrm{d}s\,\rho(s) \frac{i}{p^2 - s + i\varepsilon} \right].$$

The propagator in momentum space is then given by

$$\tilde{\Delta}_{\mathrm{F}}(p) = \frac{i}{p^2 - m^2_{\mathrm{phys}} + i\varepsilon} + \int_{4m^2_{\mathrm{phys}}}^{\infty} \mathrm{d}s\, \rho(s) \frac{i}{p^2 - s + i\varepsilon}. \tag{35.2.18}$$

This is called the **Källén–Lehmann representation**.

Thirty-Six

LSZ, Optical Theorem, and Ward Identities

36.1 The S Matrix

In a scattering process with an arbitrary number of initial and final particles we need to compute the overlap of a state in the distant past, called the in state, and a state in the distant future, called the out state. The most common example is a $2 \to n$ scattering, where we start with two particles of momenta $\boldsymbol{k_1}$ and $\boldsymbol{k_2}$, and end with n particles of momenta $\boldsymbol{p_1}, \dots, \boldsymbol{p_n}$. We then want to compute

$$\langle \boldsymbol{p_1}, \dots, \boldsymbol{p_n}; \text{out} | \boldsymbol{k_1}, \boldsymbol{k_2}; \text{in} \rangle, \tag{36.1.1}$$

where $|\boldsymbol{k_1}, \boldsymbol{k_2}; \text{in}\rangle$ is considered to be the state $|\boldsymbol{k_1}, \boldsymbol{k_2}, t = -\infty\rangle$ and $|\boldsymbol{p_1}, \dots, \boldsymbol{p_n}; \text{out}\rangle$ is the state $|\boldsymbol{p_1}, \dots, \boldsymbol{p_n}, t = +\infty\rangle$. The S matrix is defined to be the operator such that we can take the states at any time and compute the matrix element of S and get the result above, that is

$$\langle \boldsymbol{p_1}, \dots, \boldsymbol{p_n}; \text{out} | \boldsymbol{k_1}, \boldsymbol{k_2}; \text{in} \rangle = \langle \boldsymbol{p_1}, \dots, \boldsymbol{p_n} | S | \boldsymbol{k_1}, \boldsymbol{k_2} \rangle. \tag{36.1.2}$$

We usually write the S matrix as

$$S = \mathbb{1} + iT, \tag{36.1.3}$$

where T is called the **transition matrix**. This corresponds to the two things that can happen in a scattering process, either nothing happens, corresponding to $\mathbb{1}$, or we have some sort of transition from the input state to another state, so we get end up with the state $iT|\boldsymbol{k_1}, \boldsymbol{k_2}\rangle$. We then compute the overlap of this with the desired final state to work out the probability of a particular outcome.

36.2 LSZ Reduction

LSZ reduction, named for Harry Lehmann, Kurt Symanzik, and Wolfhart Zimmermann, is the result of setting $r = 1$ in Section 34.2. This means there will be a pole in the momentum space correlator whenever a single field is on-shell. The n-point correlation function then has, at least, n poles, each corresponding to one of the momenta $p_i^2 = m_{\text{phys}}^2$. Note that we count higher order poles as multiple poles here. The residue at this multiple pole point gives the S matrix for an n particle scattering process. Suppose we have m particles in the initial state and m' particles

in the final state, with $m+m'=n$. Then we consider the residue at the pole given by taking m of the initial particles, with momenta p_j, to have $p_j^2 = m_{\text{phys}}^2$, and m' of the final particles, with momenta p'_k, to have $p'^2_k = m^2_{\text{phys}}$. This gives

$$\langle p'_1, \dots, p'_{m'}; \text{out} | p_1, \dots, p_m; \text{in}\rangle = \langle p'_1, \dots, p'_m | S | p_1, \dots, p_m\rangle \tag{36.2.1}$$

$$= \lim_{p_j^2, p'^2_k \to m^2_{\text{phys}}} \prod_{k=1}^{m'} (p'^2_k - m^2_{\text{phys}} + i\varepsilon) \prod_{j=1}^{m} (p_j^2 - m^2_{\text{phys}} + i\varepsilon) \times \tilde{G}^{(n)}(p_1, \dots, p_m, -p'_1, \dots, -p'_{m'}). \tag{36.2.2}$$

This holds when

$$\langle \boldsymbol{p} | \varphi(0) | 0\rangle = 1. \tag{36.2.3}$$

The LSZ reduction formula above gives a way to represent quantum amplitudes with Feynman diagrams in momentum space. We can compute the correlator by adapting the Feynman rules in Section 33.4 with the following modifications:

- We associate an outgoing momentum to the external lines that correspond to particles in the final state, giving a relative minus sign compared to the normal Feynman rules when all momenta are taken to be inwards.
- We multiply each external line by a factor of $-i(p^2 - m^2_{\text{phys}} + i\varepsilon)$, this is part of computing the residue. Correlators multiplied by these factors are called **truncated correlators** or **amputated correlators**

36.3 Optical Theorem

Physical constraints are expressed by relations between correlators. One such physical constraint is the unitarity of the S matrix, corresponding to conservation of probability. If we have $S^\dagger S = \mathbb{1}$ and $S = \mathbb{1} + iT$ then

$$(\mathbb{1} + iT)^\dagger (\mathbb{1} + iT) = (\mathbb{1} - iT^\dagger)(\mathbb{1} + iT) = \mathbb{1} - iT^\dagger + iT^\dagger + T^\dagger T \tag{36.3.1}$$

and so we must have

$$-iT^\dagger + iT^\dagger + T^\dagger T = 0 \implies -i(T - T^\dagger) = T^\dagger T. \tag{36.3.2}$$

Consider a matrix element of this equation between some initial state, $|a\rangle$, and some final state, $|b\rangle$. We also factor out a Dirac delta corresponding to conservation of total momentum, so we have

$$\langle b|T|a\rangle = (2\pi)^D \delta(P_a - P_b)\mathcal{M}(a \to b) \tag{36.3.3}$$

where P_a and P_b are the total momenta of the states $|a\rangle$ and $|b\rangle$. This equation defines $\mathcal{M}(a \to b)$, the amplitude for the scattering process where we start in state $|a\rangle$ and end in state $|b\rangle$. The left hand side gives us

$$-i\langle b|T|a\rangle + i\langle b|T^\dagger|a\rangle = -i\langle b|T|a\rangle + i\langle a|T|a\rangle^* \tag{36.3.4}$$

$$= -i(2\pi)^D \delta(P_a - P_b)[\mathcal{M}(a \to b) - \mathcal{M}(b \to a)^*]. \tag{36.3.5}$$

On the right hand side if we insert a complete set of states, $\{|f\rangle\}$, summed over quantum numbers, f and integrated over momenta, which we also denote by f, then we have

$$\begin{aligned}\langle b|T^\dagger T|a\rangle &= \sum_f \int \mathrm{d}\Omega_f \, \langle b|T^\dagger|f\rangle\langle f|T|a\rangle \qquad (36.3.6)\\ &= \sum_f \int \mathrm{d}\Omega_f \, (2\pi)^D \delta(P_b - P_f)\mathcal{M}(b\to f)^*(2\pi)^D\delta(P_f-P_a)\mathcal{M}(a\to f)\\ &= (2\pi)^D\delta(P_a-P_b)\sum_f \int \mathrm{d}\Omega_f\,(2\pi)^D\delta(P_a-P_f)\mathcal{M}(b\to f)^*\mathcal{M}(a\to f)\end{aligned}$$

where we use the product of Dirac deltas identity $\delta(x-y)\delta(y-z) = \delta(x-z)\delta(y-z)$, essentially, setting x to y then y to z is the same as setting x to z and y to z.

Now consider the special case where $a = b$, then the left hand side gives us

$$-i(2\pi)^D\delta(0)[\mathcal{M}(a\to a) - \mathcal{M}(a\to a)^*] = -i(2\pi)^D\delta(0)2i\,\mathrm{Im}[\mathcal{M}(a\to a)], \quad (36.3.7)$$

where we've used $z - z^* = x + iy - (x - iy) = 2iy$. The right hand side gives

$$\begin{aligned}\langle b|T^\dagger T|a\rangle &= (2\pi)^D\delta(0)\sum_f\int \mathrm{d}\Omega_f\,(2\pi)^D\delta(P_a-P_f)\mathcal{M}(a\to f)^*\mathcal{M}(a\to f)\\ &= (2\pi)^D\delta(0)\sum_f\int \mathrm{d}\Omega_f\,(2\pi)^D\delta(P_a-P_f)|\mathcal{M}(a\to f)|^2. \qquad (36.3.8)\end{aligned}$$

Equating these we have

$$2\,\mathrm{Im}[\mathcal{M}(a\to a)] = \sum_f\int \mathrm{d}\Omega_f\,(2\pi)^D\delta(P_a-P_f)|\mathcal{M}(a\to f)|^2. \qquad (36.3.9)$$

This is the **optical theorem**. It relates the imaginary part of the amplitude for $a \to a$ to the total cross section for $a \to f$ summed over all final states.

36.4 Ward Identities

Another example of physical constraints giving relationships between correlators are the Ward identities. These are equalities between field correlators due to symmetries of the system. In classical mechanics symmetries correspond to conserved currents, by Noether's theorem. The Ward identities are the quantum field theory analogue of this.

Start by considering the symmetry transformation of the field

$$\varphi(x) \mapsto \varphi'(x) = \varphi(x) + \varepsilon\delta\varphi(x) \qquad (36.4.1)$$

such that for some constant $\varepsilon > 0$ the action is unchanged. If we then allow ε to depend on position the variation in the action is given by

$$\delta S = \int \mathrm{d}^D x \frac{\delta S}{\delta\varphi(x)}\varepsilon(x)\delta\varphi(x) = -\int \mathrm{d}^D x\,\varepsilon(x)\partial_\mu j^\mu(x) \qquad (36.4.2)$$

where the last step is to integrate by parts. Here j^μ is exactly the conserved current from Noether's theorem. The transformation to perform a change of integration variables in the following path integral:

$$\int \mathcal{D}\varphi\, \mathrm{e}^{iS[\varphi]}O(\varphi) = \int \mathcal{D}\varphi'\, \mathrm{e}^{iS[\varphi']}O(\varphi') \qquad (36.4.3)$$

where O is some generic function of the field. Expanding the right hand side to order ε we have

$$\int \mathcal{D}\varphi \, \mathrm{e}^{iS[\varphi]} O(\varphi) = \int \mathcal{D}\varphi \, e^{iS[\varphi]} (1 + i\delta S[\varphi])(O(\varphi) + \delta O). \tag{36.4.4}$$

Notice that taking the 1 term from $1 + i\delta S[\varphi]$ we just get the left hand side, so we can cancel these and consider just the $i\delta S[\varphi]$ term. We can substitute in the expression for δS from before and a similar expression for the variation in O, that is δO, giving:

$$\int \mathcal{D}\varphi \, \mathrm{e}^{iS[\varphi]} \left[-i \int \mathrm{d}T D x \varepsilon(x) \partial_\mu j^\mu(x) O(\varphi) + \int \mathrm{d}^D x \frac{\mathrm{d}O(\varphi)}{\mathrm{d}\varphi(x)} \varepsilon(x) \delta\varphi(x) \right] = 0. \tag{36.4.5}$$

We can rearrange this and write it as

$$\int \mathrm{d}^D x \, \varepsilon(x) \left[-i \langle \partial_\mu j^\mu(x) O(\varphi) \rangle + \left\langle \frac{\delta O(\varphi)}{\delta\varphi(x)} \delta\varphi(x) \right\rangle \right] = 0. \tag{36.4.6}$$

This is sometimes called the **integrated Ward identity**. It holds for all ε, vanishing on the boundary, and so we get the **Ward identity**

$$-i \langle \partial_\mu j^\mu(x) O(x) \rangle + \left\langle \frac{\delta O(\varphi)}{\delta\varphi(x)} \delta\varphi(x) \right\rangle = 0. \tag{36.4.7}$$

This result encodes two important physical statements:

- Symmetry in QFT becomes relations between correlators. This concept holds beyond perturbation theory and is important when doing renormalisation.
- Current conservation in QFT is realised at the level of the insertion of $\partial_\mu j^\mu$ in field correlators, up to terms from the variation of O. If O is a product of local fields then this variation is localised in spacetime. The contributions are then all proportional to Dirac deltas. These terms are called **contact terms**.

Note that we assumed invariance of the integration measure under this transformation, that is $\mathcal{D}\varphi = \mathcal{D}\varphi'$. If this isn't the case we get extra terms in the Ward identities, which we call the **anomalous Ward identities**.

Part XII

Fermion Fields

Thirty-Seven

Fermion Fields

37.1 Fermion Fields as Grassmann Variables

The operators corresponding to fermionic fields anticommute. However, in a path integral we don't have operators. Instead we treat the fermionic fields, ψ and $\bar{\psi}$, as independent Grassmann variables. See Chapter C for a more detailed explanation, but essentially a Grassmann algebra consists of mutually anticommuting generators, ϑ_i and $\bar{\vartheta}_i$, such that

$$\{\vartheta_i, \vartheta_j\} = \vartheta_i\vartheta_j + \vartheta_j\vartheta_i = 0. \tag{37.1.1}$$

Notice that this implies $\vartheta_i^2 = 0$, since we have $\vartheta_i\vartheta_i = -\vartheta_i\vartheta_i$, where we've swapped ϑ_i and ϑ_i on the right hand side, you just can't tell. All terms in the Grassmann algebra are then linear combinations of products of these generators where each appears at most once in the product.

So, if ψ is a fermionic field, that is a spinor, we have the anticommutation relation

$$\{\psi_\alpha(x), \psi_\beta(y)\} = 0. \tag{37.1.2}$$

Here α and β are spinor indices. In order to compute anything in the path integral formalism we need to be able to take functional derivatives. For this to work consistently with the Grassmann algebra we need the functional derivative with respect to a Grassmann variable to be another Grassmann variable. This means that if we apply these to some functional, F, we have

$$\frac{\delta^2 F}{\delta\psi_\alpha(x)\delta\psi_\beta(y)} = -\frac{\delta F}{\delta\psi_\beta(x)\psi_\alpha(x)}, \tag{37.1.3}$$

and in particular this implies

$$\frac{\delta^2 F}{\delta\psi_\alpha(x)\delta\psi_\alpha(x)} = 0. \tag{37.1.4}$$

We will have terms of the form

$$\int \mathrm{d}^D y\, [\bar{\eta}(y)\psi(y) + \bar{\psi}(y)\eta(y)] \tag{37.1.5}$$

appearing in the (exponent in the) path integral. Here η and $\bar{\eta}$ are two independent source terms/ As with the scalar case we take derivatives with respect to the source

terms to generate factors of the fields. We can then use

$$I_1 = \frac{\delta}{\delta\bar{\eta}(x)} \int \mathrm{d}^D y\, \bar{\eta}(y)\psi(y) \tag{37.1.6}$$

$$= \int \mathrm{d}^D y\, \frac{\delta\bar{\eta}(y)}{\delta\bar{\eta}(x)}\psi(y) \tag{37.1.7}$$

$$= \int \mathrm{d}^D y\, \delta(y-x)\psi(y) \tag{37.1.8}$$

$$= \psi(x). \tag{37.1.9}$$

Similarly, being cautious about commuting Grassmann variables and picking up a negative, we have

$$I_2 = \frac{\delta}{\delta\eta(x)} \int \mathrm{d}^D y\, \bar{\psi}(y)\eta(y) \tag{37.1.10}$$

$$= \int \mathrm{d}^D y\, \frac{\delta}{\delta\eta(x)}\bar{\psi}(y)\eta(y) \tag{37.1.11}$$

$$= -\int \mathrm{d}^D y\, \bar{\psi}(y)\frac{\delta}{\delta\eta(x)}\eta(y) \tag{37.1.12}$$

$$= -\int \mathrm{d}^D y\, \bar{\psi}(y)\delta(x-y) \tag{37.1.13}$$

$$= -\bar{\psi}(x). \tag{37.1.14}$$

37.2 Free Theory

In a free theory for fermions the action is given by

$$S_0[\psi,\bar{\psi}] = \int \mathrm{d}^D x\, \bar{\psi}(x)(i\not{\partial} - m)\psi(x). \tag{37.2.1}$$

Note that the operator $i\not{\partial} - m$ is an operator both on the spacetime, through the derivative, but also on the spinor space, through the γ^μ hidden in the $\not{\partial} = \gamma^\mu\partial_\mu$.

Roughly we can, when $D = 4$, think of $i\not{\partial} - m$ as a 4×4 matrix on spinor space, made up of derivatives. Note that the 4 in $D = 4$ is the spacetime dimension and the 4 in 4×4 is the dimension of the spinor space. These values are related, but not necessarily equal, their equality here is a consequence of the fact that[1] $2 \times 2 = 2 + 2 = 4$.

[1] see *Symmetries of Particles and Fields*

We can rewrite the action with spinor indices as follows:

$$S_0[\psi,\bar{\psi}] = \int \mathrm{d}^D x\, \bar{\psi}_\alpha(x)(i\not{\partial} - m)_{\alpha\beta}\psi_\beta(x). \tag{37.2.2}$$

We can then expand upon what we mean by the operator $(i\not{\partial} - m)_{\alpha\beta}$ by noticing that there is an implicit identity multiplied by m, and so

$$(i\not{\partial} - m)_{\alpha\beta} = (i\gamma^\mu\partial_\mu - mI)_{\alpha\beta} = i\gamma^\mu_{\alpha\beta}\partial_\mu - m\delta_{\alpha\beta}. \tag{37.2.3}$$

Taking the derivative of the action with respect to $\bar{\psi}$ we get the classical equations of motion

$$\frac{\delta}{\delta\bar{\psi}(x)}S_0[\psi,\bar{\psi}] = 0 \implies (i\not{\partial} - m)\psi(x) = 0. \tag{37.2.4}$$

That is, the classical equation of motion is simply Dirac's equation. This means that in the classical limit the path integral will be dominated by the solutions to the Dirac equation.

Like the scalar case we can write a generating functional for the free theory by introducing two independent sources, η and $\bar{\eta}$, and defining

$$Z_0[\eta,\bar{\eta}] \coloneqq \int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\exp\{i(S_0[\psi,\bar{\psi}] + \bar{\eta}\cdot\psi + \bar{\psi}\cdot\eta)\} \tag{37.2.5}$$

where

$$\bar{\eta}\cdot\psi \coloneqq \int \mathrm{d}^D y\,\bar{\eta}(y)\psi(y), \qquad \text{and} \qquad \bar{\psi}\cdot\eta \coloneqq \int \mathrm{d}^D y\,\bar{\psi}(y)\eta(y). \tag{37.2.6}$$

Notice that the exponent is just a quadratic form in ψ and $\bar{\psi}$, and so this generating functional is just a Gaussian, which we can integrate to give

$$Z_0[\eta,\bar{\eta}] = \exp\left\{-\int \mathrm{d}^D x \int \mathrm{d}^D y\,\bar{\eta}(x)S(x-y)\eta(y)\right\} \tag{37.2.7}$$

where S is the **Feynman propagator**, and is given by

$$S_{\alpha\beta}(x-y) = \int_p \mathrm{e}^{-ip\cdot(x-y)}\frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\varepsilon}. \tag{37.2.8}$$

As with the finite dimensional case for Gaussians S is the inverse of the kernel of the quadratic form, which is a fancy way fo saying that $S(x-y)$ is the inverse of $i\not{\partial} - m$.

In the free theory the propagator is given by the two point correlator:

$$\langle 0|\,\mathrm{T}[\psi_\alpha(x)\bar{\psi}_\beta(y)]|0\rangle_0 = S_{\alpha\beta}(x-y). \tag{37.2.9}$$

As with the scalar case we can use functional derivatives acting on the generating functional to calculate this result. However, due to the extra negative sign instead of considering $1/i$ for the derivatives with respect to η we just take i:

$$\begin{aligned}
&\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)}i\frac{\delta}{\delta\eta(y)}Z_0[\eta,\bar{\eta}]\Big|_{\eta=\bar{\eta}=0} \qquad (37.2.10)\\
&\quad= \frac{\delta}{\delta\bar{\eta}(x)}\frac{\delta}{\delta\eta(y)}\exp\left\{-\int_{w,z}\bar{\eta}(w)S(w-z)\eta(z)\right\}\Big|_{\eta=\bar{\eta}=0}\\
&\quad= \frac{\delta}{\delta\bar{\eta}(x)}\left(\int_{w,z}\bar{\eta}(w)S(w-z)\delta(z-y)\right)\\
&\qquad\qquad\times\exp\left\{-\int_{w,z}\bar{\eta}(w)S(w-z)\eta(z)\right\}\Big|_{\eta=\bar{\eta}=0}
\end{aligned}$$

note that we picked up a minus from the exponent, and a minus for commuting the derivative with $\bar{\eta}(w)$

$$\begin{aligned}
&= \frac{\delta}{\delta\bar{\eta}(x)}\left(\int_w \bar{\eta}(w)S(w-y)\right) \\
&\qquad \times \exp\left\{-\int_{w,z}\bar{\eta}(w)S(w-z)\eta(z)\right\}\Big|_{\eta=\bar{\eta}=0} \\
&= \left(\int_w \delta(w-x)S(w-y)\right)\exp\left\{-\int_{w,z}\bar{\eta}(w)S(w-z)\eta(z)\right\} \\
&\qquad \times\left(\int_w \bar{\eta}(w)S(w-y)\right)\left(-\int_{w,z}\delta(w-z)S(w-z)\eta(w-z)\right)\Big|_{\eta=\bar{\eta}=0} \\
&= S(x-y), \qquad (37.2.11)
\end{aligned}$$

setting $\eta = \bar{\eta} = 0$ in the last step.

Notice that the linear term in p means the propagator is no longer symmetric under exchanging x and y, and so in the diagrammatic representation we now add a line to the propagator to show this:

$S(x-y) = x \longrightarrow y$, and $S_{\alpha\beta}(x-y) = x,\alpha \longrightarrow y,\beta$. (37.2.12)

We can compute a general correlator using Wick's theorem modified for Fermions:

$$\langle 0|\,\mathrm{T}[\psi_{\alpha_1}(x_1)\cdots\psi_{\alpha_n}(x_n)\bar{\psi}_{\beta_1}(y_1)\cdots\bar{\psi}_{\beta_n}(y_n)]|0\rangle_0 \qquad (37.2.13)$$

$$= \left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}_{\alpha_1}(x_1)}\right)\cdots\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}_{\alpha_n}(x_n)}\right)\left(i\frac{\delta}{\delta\eta_{\beta_n}(y_n)}\right)\left(i\frac{\delta}{\delta\eta_{\beta_1}(y_n)}\right)Z_0[\eta,\bar{\eta}]\Big|_{\eta=\bar{\eta}=0} \qquad (37.2.14)$$

$$= \sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)S_{\alpha_1\beta_{\sigma(1)}}(x_1 - y_{\sigma(1)}) \qquad (37.2.15)$$

where S_n is the symmetric group on n objects, that is the group of all permutations of $\{1,\ldots,n\}$, and σ is a permutation sending 1 to $\sigma(1)$ with $\mathrm{sgn}(\sigma)$ being the number of swaps required to reorder $\sigma(1),\sigma(2),\ldots,\sigma(n)$ to $1,2,\ldots,n$.

37.3 Interacting Theory

If interactions occur through a potential $V(\psi,\bar{\psi})$ then the action is given by

$$S[\psi,\bar{\psi}] = S_0[\psi,\bar{\psi}] + \int \mathrm{d}^D x\, V(\psi,\bar{\psi}). \qquad (37.3.1)$$

The generating functional for the interacting theory is

$$Z[\eta,\bar{\eta}] = \int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\exp\{i(S[\psi,\bar{\psi}] + \bar{\eta}\cdot\psi + \bar{\psi}\cdot\eta)\} \qquad (37.3.2)$$

$$= \exp\left\{i\int \mathrm{d}^D x\, V\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)}, i\frac{\delta}{\delta\eta(x)}\right)\right\}Z_0[\eta,\bar{\eta}] \qquad (37.3.3)$$

$$= \sum_{V=0}^{\infty}\frac{1}{V!}\left[i\int \mathrm{d}^D x\, V\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)}, i\frac{\delta}{\delta\eta(x)}\right)\right]^V \qquad (37.3.4)$$

$$\times\sum_{P=0}^{\infty}\frac{1}{P!}\left[-\int \mathrm{d}^D y\int \mathrm{d}^D z\,\bar{\eta}(y)S(y-z)\eta(z)\right]^P. \qquad (37.3.5)$$

Here we factor out the interaction from the generating functional leaving the free generating functional. We then do the usual trick of writing a function, here the potential, as a Taylor series in ψ and $\bar{\psi}$, and then replacing the fields with derivatives acting on the free generating functional. In the last step we expand the two exponentials as power series. We choose to normalise such that

$$Z[0,0] = 1. \tag{37.3.6}$$

Now consider a the particular potential

$$V(\psi,\bar{\psi}) = \frac{g}{2}\bar{\psi}(x)\psi(x)\bar{\psi}(x)\psi(x). \tag{37.3.7}$$

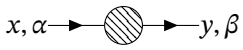
Then, we have

$$Z[\eta,\bar{\eta}] = \sum_{V=0}^{\infty}\frac{1}{V!}\left[i\frac{g}{2}\int \mathrm{d}^D x\,\frac{\delta}{\delta\bar{\eta}(x)}\frac{\delta}{\delta\eta(x)}\frac{\delta}{\delta\bar{\eta}(x)}\frac{\delta}{\delta\eta(x)}\right]^V \tag{37.3.8}$$

$$\times\sum_{P=0}^{\infty}\frac{1}{P!}\left[-\int \mathrm{d}^D y\int \mathrm{d}^D z\,\bar{\eta}(y)S(y-z)\eta(z)\right]^P. \tag{37.3.9}$$

Consider a term with some fixed values of V and P. We will have P factors of η to start with in this term, and we remove $2V$ by differentiating. So, we have $P-2V$ factors of η, and similarly we'll have $P-2V$ factors of $\bar{\eta}$.

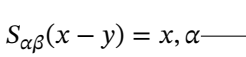
Consider an interaction represented by



(37.3.10)

where the blob in the middle represents some interaction process with one fermion entering and one fermion leaving. We must have one factor of η and one factor of $\bar{\eta}$ for the incoming and outgoing fermions. We'll consider only connected diagrams.

The $V=0$ case must then have $P=1$ in order to have $P-2V=1$. This is just the propagator term though, corresponding to



(37.3.11)

The $V=1$ term must have $P=3$. This means we have three propagators and one vertex. Since this is a quartic interaction we'll have four legs connected at a vertex. As usual we have to compute a constant factor, lets start with the terms coming from the expansion. For $V=1$ we'll have one factor of i and one factor of $g/2$. For $P=3$ we have a factor of $1/3!$. Next we have symmetry factors. Note that most of the operations that left a diagram invariant in the scalar case are no longer valid, as the propagators are no longer symmetric. There are still some choices that we can make when constructing the diagram though. First, we have to choose one end before the arrow of the three propagators to correspond to the incoming particle at x. We then have to choose one end after of the arrow of the two remaining propagators to correspond to the outgoing particle at y. This gives a factor of $3\cdot 2$. There is one final step. We must have two incoming and two outgoing lines at the vertex, that is we must have two propagators start at the vertex and two propagators end at the vertex. We've fixed the start of one propagator as x and the end of another propagator as y. We must then take one of the two unfixed starts and fix it as one of the outgoing particles, this gives a factor of 2. There are two options here, either we picked the propagator with the other end unfixed, in which

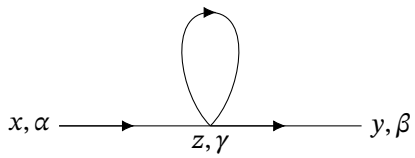
case since it can't be external it must loop back around to the vertex, leaving only one way to connect the two propagators with a single fixed end, or we fixed the start of the propagator with the other end fixed at y, in which case we have to connect the unfixed propagator in a loop and the unfixed end of the other propagator goes into the vertex.

The overall factor is

$$ig\frac{3\cdot 2\cdot 2}{3!\cdot 2} = ig. \tag{37.3.12}$$

This shouldn't be a surprising result at this point.

The diagram corresponding to this term is



(37.3.13)

The symmetry factor of this diagram is 1, since the arrows mean we can't do any of the swapping or mirroring we did for scalar diagrams. The overall value of the diagram is

$$ig\int_{x,y}\bar{\eta}_\alpha(x)S_{\alpha\gamma}(x-z)S_{\gamma\gamma}(z-z)S_{\gamma\beta}(z-y)\eta_\beta(y). \tag{37.3.14}$$

Part XIII

Gauge Fields

Thirty-Eight

Gauge Fields

38.1 What's a Gauge Field

A **gauge field** is a field with a nontrivial local symmetry leaving the action invariant. This symmetry is given by the action of some Lie group[1].

[1] see *Symmetries of Quantum Mechanics* or *Symmetries of Particles and Fields*

We'll focus on the simplest case here, where the Lie group is U(1). This is the symmetry group corresponding to electromagnetism, which is primarily what we consider here, but the standard model also has a U(1) symmetry corresponding to electroweak interactions.

In this case we consider the action

$$S[A] = \int \mathrm{d}^D x \left(-\frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x)\right) \tag{38.1.1}$$

where

$$F^{\mu\nu}(x) \coloneqq \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) \tag{38.1.2}$$

is the electromagnetic field strength tensor and $A^\mu = (\varphi, \boldsymbol{A})$ is the electromagnetic field (also known as the electromagnetic potential).

Applying the Euler Lagrange equations to this one finds the classical equations of motion[2]

[2] see *Classical Electrodynamics* for this calculation

$$\partial_\mu F^{\mu\nu} = 0. \tag{38.1.3}$$

This is simply Maxwell's equations with no sources.

The action is invariant under the transformation

$$A_\mu(x) \mapsto A^\Lambda_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x) \tag{38.1.4}$$

where Λ is some function of spacetime with continuous second derivatives. We call this transformation a **gauge transformation**.

We can readily check that $F_{\mu\nu}$ is invariant under this transformation:

$$\begin{aligned} F_{\mu\nu} &\mapsto \partial_\mu(A_\nu + \partial_\nu \Lambda) - \partial_\nu(A_\mu + \partial_\mu \Lambda) && (38.1.5)\\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda && (38.1.6)\\ &= \partial_\mu A_\nu - \partial_\nu A_\mu && (38.1.7)\\ &= F_{\mu\nu}. && (38.1.8) \end{aligned}$$

Hence, the action is also invariant under this transformation.

38.2 First Attempt at a Path Integral

The gauge potential is quadratic in the fields. We can rewrite the action as

$$S[A] = \frac{1}{2}\int \mathrm{d}^D x\, A_\mu(x)[\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu]A_\nu(x). \tag{38.2.1}$$

This follows by expanding

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= (\partial_\mu A_\nu)(\partial^\mu A^\nu) + (\partial_\nu A_\mu)(\partial^\nu A^\mu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu) - (\partial_\nu A_\mu)(\partial^\mu A^\nu), \end{aligned} \tag{38.2.2}$$

and then noticing that the first term here is

$$(\partial_\mu A_\nu)(\partial^\mu A^\nu) = \partial_\mu(A_\nu\partial^\mu A^\nu) - A_\nu\partial_\mu(\partial^\mu A^\nu) = \partial_\mu(A_\nu\partial^\mu A^\nu) - A_\nu\partial^2 A^\nu \tag{38.2.3}$$

and the first term here is a total derivative, so becomes a surface term with an application of the divergence theorem, and so doesn't contribute to the variation in the action if we choose the variation to vanish on the surface. We can then write $A_\nu\partial^2 A^\nu = A_\mu\eta^{\mu\nu}\partial^2 A_\nu$. The second term in $F_{\mu\nu}F^{\mu\nu}$ also has the same form, the indices are just renamed, so we get a factor of 2. The third term is

$$(\partial_\mu A_\nu)(\partial^\nu A^\mu) = \partial_\mu(A_\nu\partial^\nu A^\mu) - A_\nu\partial_\mu(\partial^\nu A^\mu) = \partial_\mu(A_\nu\partial^\nu A^\mu) - A_\nu\partial^\mu\partial^\nu A_\mu \tag{38.2.4}$$

and the first term here again doesn't contribute to the variation in the action. The fourth term is the same after renaming of indices.

Having written the action in the form

$$S[A] = \frac{1}{2}\int \mathrm{d}^D x\, A_\mu(x)K^{\mu\nu}(x)A_\nu(x) \tag{38.2.5}$$

where

$$K^{\mu\nu}(x) = \eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu \tag{38.2.6}$$

we may be tempted to recognise this as a quadratic form and then define the path integral to be the Gaussian integral

$$Z[A] = \int \mathcal{D}A\, \exp\left\{\frac{i}{2}\int \mathrm{d}^D x\, A_\mu(x)[\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu]A_\nu(x)\right\}. \tag{38.2.7}$$

However, we quickly run into a problem. In order to compute this integral we need to compute the inverse of the kernel, $K^{\mu\nu}$, but we find that $K^{\mu\nu}$ has zero modes and hence is not invertible. This is easiest to see if we move to Fourier space. We can write $A_\mu(x)$ as

$$A_\mu(x) = \int \frac{\mathrm{d}^D k}{(2\pi)^D}\tilde{A}_\mu(k)\mathrm{e}^{-ik\cdot x} \tag{38.2.8}$$

and we then have

$$S[A] = \frac{1}{2}\int \mathrm{d}^D x \int \frac{\mathrm{d}^D k}{(2\pi)^D}\int \frac{\mathrm{d}^D k'}{(2\pi)^D}\tilde{A}_\mu(k)\mathrm{e}^{-ik\cdot x}[\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu]\tilde{A}_\nu(k')\mathrm{e}^{-ik'\cdot x} \tag{38.2.9}$$

$$= \frac{1}{2}\int \mathrm{d}^D x \int_{k,k'} \tilde{A}_\mu(k)\mathrm{e}^{-ik\cdot x}[-\eta^{\mu\nu}k'^2 + k'^\mu k'^\nu]\tilde{A}_\nu(k')\mathrm{e}^{-ik'\cdot x} \tag{38.2.10}$$

$$= \frac{1}{2}\int_{k,k'} \tilde{A}_\mu(k)[-\eta^{\mu\nu}k'^2 + k'^\mu k'^\nu]\tilde{A}_\nu(k')\mathrm{e}^{-i(k'+k)\cdot x} \tag{38.2.11}$$

$$= \frac{1}{2}\int_{k,k'} \tilde{A}_\mu(k)[-\eta^{\mu\nu}k'^2 + k'^\mu k'^\nu]\tilde{A}_\nu(k')(2\pi)^D\delta(k+k') \tag{38.2.12}$$

$$= \frac{1}{2}\int \frac{\mathrm{d}^D k}{(2\pi^D)}\tilde{A}_\mu(k)[-\eta^{\mu\nu}k^2 + k^\mu k^\nu]\tilde{A}_\nu(-k). \tag{38.2.13}$$

The we can see that, defining $\tilde{K}^{\mu\nu}(k) = -k^2\eta^{\mu\nu} + k^\mu k^\nu$, we have

$$\tilde{K}^{\mu\nu}(k)k_\nu = -k^2\eta^{\mu\nu}k + k^\mu k^\nu k_\nu = -k^2 k^\mu + k^\mu k^2 = 0. \tag{38.2.14}$$

So, $\tilde{K}^{\mu\nu}$, and hence $K^{\mu\nu}$, is not invertible.

This generalises to any term $A^{(0)}_\mu(x) = \partial_\mu\Lambda(x)$ in the gauge field, this becomes $\tilde{A}^{(0)}_\mu(k) = -ik_\mu\tilde{\Lambda}$ in Fourier space and so is an eigenfunction of the kernel with zero eigenvalue. This means that any gauge term is a zero mode of the kernel.

Note that, for on-shell particles,

$$\tilde{K}^{\mu\nu}(k) = -k^2\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) = -k^2\pi^{\mu\nu}(k) \tag{38.2.15}$$

where $\pi^{\mu\nu}(k)$ is a projection operator, with $\pi^{\mu\nu}(k)k_\nu = 0$, so we project out the gauge terms from the action.

We interpret this as two field configurations differing only by a gauge transformation having the same weight (action) in the average (path integral) and so giving the same statistical weighted average. Note that in Euclidean space the factor of i in the path integral becomes -1, and this analogy of a weighted average becomes exact with the Boltzmann distribution replacing the complex amplitude.

38.3 Faddeev–Popov Procedure

Being physicists we're going to forget that this is undefined and see what happens. Otherwise we'd be mathematicians.

Luigi Del Debbio

We started with something undefined and ended up with something ... undefined.

Luigi Del Debbio

A **gauge orbit** is a set of fields related by a gauge transformation,

$$\Omega_A \coloneqq \{A^\Lambda_\mu(x) \mid \Lambda \text{ is a gauge transformation}\}. \tag{38.3.1}$$

This is exactly an orbit in the group theoretical sense, with the group action being that of the gauge group, here $\mathrm{U}(1)$.

All of the fields in a gauge orbit describe the same physical state, just in different gauges. In the path integral we want to include one representative for each gauge orbit. We select such a representative by taking it to be the solution to some **gauge fixing** condition,

$$G(A_\mu(x)) = 0, \tag{38.3.2}$$

where G is some function of the fields.

By definition of the Dirac delta we have

$$\int \mathcal{D}G\, \delta(G) = 1. \tag{38.3.3}$$

We can perform a change of variables, considering G to be a function of Λ, and we have

$$1 = \int \mathcal{D}\Lambda\, \delta(G(A^\Lambda_\mu)) \det\left(\frac{\delta G(A^\Lambda_\mu)}{\delta \Lambda}\right). \tag{38.3.4}$$

Now consider what happens when we plug this identity into the path integral, $Z[A]$, now with an added source term, $J \cdot A$ in the exponential:

$$Z[A] = \int \mathcal{D}A\, \mathcal{D}\Lambda \det\left(\frac{\delta G}{\delta \Lambda}\right) \delta(G(A^\Lambda_\mu)) \exp\{iS[A] + iJ \cdot A\}. \tag{38.3.5}$$

Suppose we choose

$$G(A^\Lambda_\mu) = \partial_\mu(A^\mu + \partial^\mu \Lambda), \tag{38.3.6}$$

and we want to work in the Lorentz gauge, where $\partial_\mu A^\mu = 0$. Then we have $G(A^\Lambda_\mu) = \partial^2 \Lambda$, and so

$$\frac{\delta G(A^\Lambda_\mu(x))}{\delta \Lambda(y)} = \delta(x-y)\partial^2. \tag{38.3.7}$$

Notice that this doesn't depend on the gauge field, A_μ, and so we don't need to work out the determinant in full, since it cancels in the normalisation. This is particular to the simple case of $\mathrm{U}(1)$ gauge symmetries. We then have

$$Z[A] = \int \mathcal{D}\Lambda\, \mathcal{D}A\, \delta(G(A^\Lambda_\mu)) \exp\{iS[A] + iJ \cdot A\} \tag{38.3.8}$$

$$= \int \mathcal{D}\Lambda\, \mathcal{D}\Lambda\, \delta(G(A^\Lambda_\mu)) \exp[iS[A^\Lambda] + iJ \cdot A^\Lambda]. \tag{38.3.9}$$

We can replace A with A^Λ in the action because we know the action is invariant under gauge transformations. Replacing A with A^Λ in $J \cdot A$ gives a term $J \cdot \Lambda$, after

integrating by parts, but we set $J = 0$ when computing correlators, so this term doesn't change anything. We then drop the Λ notation, writing

$$Z[A] \propto \int \mathcal{D}\Lambda\, \mathcal{D}A\, \delta(G(A_\mu)) \exp\{iS[A] + iJ \cdot A\}. \tag{38.3.10}$$

What this process has done is move the explicitly undefined part of the path integral into the $\int \mathcal{D}\Lambda$ part, which is exactly the volume of the gauge orbit, and will be normalised out. The rest of the path integral is free from problems when we work with a single representative of Ω_A, which is what the $\delta(G(A_\mu))$ term does.

We can consider a generalised gauge condition by setting

$$G(A_\mu) = \partial_\mu A^\mu - \omega(x) \tag{38.3.11}$$

for some function ω. We then have

$$Z[A] \propto \int \mathcal{D}A\, \delta(\partial_\mu A^\mu - \omega) \exp\{iS[A] + iJ \cdot A\}. \tag{38.3.12}$$

Changing ω corresponds to changing the representative of Ω_A we pick, but doesn't change the physics. This means that all choices of ω are equivalent. We can then choose *all* possible functions ω, and average over them with a Gaussian weight, giving

$$Z[A] \propto \int \mathcal{D}\omega\, \exp\left\{-i \int \mathrm{d}^D x\, \frac{\omega(x)^2}{2\xi}\right\} \int \mathcal{D}A\, \delta(\partial^\mu A_\mu - \omega) \exp[iS[A] + iJ \cdot A].$$

We can use the Dirac delta to compute the ω integral and set $\omega(x) = \partial^\mu A_\mu$, giving

$$\begin{aligned} Z[A] &\propto \int \mathcal{D}A\, \exp\left\{-i \int \mathrm{d}^D x \frac{(\partial^\mu A_\mu)^2}{2\xi}\right\} \exp\{iS[A] + iJ \cdot A\} \\ &= \int \mathcal{D}A \exp\left\{i \int \mathrm{d}^D x \left[\frac{1}{2} A_\mu(x) \left(\eta^{\mu\nu}\partial^2 - \left(1 - \frac{1}{\xi}\right)\partial^\mu \partial^\nu\right) A_\nu\right] + iJ \cdot A\right\}. \end{aligned} \tag{38.3.13}$$

We can then define the kernel

$$K^{\mu\nu}(x) = \eta^{\mu\nu}\partial^2 - \left(1 - \frac{1}{\xi}\right)\partial^\mu \partial^\nu. \tag{38.3.14}$$

In momentum space this becomes

$$\tilde{K}^{\mu\nu}(k) = -\eta^{\mu\nu}k^2 + \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu. \tag{38.3.15}$$

Now notice that k_ν is no longer a zero mode of this operator, and instead has eigenvalue $-1/\xi$.

From this we can compute the inverse, which will be the **Feynman propagator** in momentum space for the photon field:

$$\tilde{D}_{\mathrm{F}}^{\mu\nu}(k) = \frac{-i}{k^2 + i\varepsilon}\left[\eta^{\mu\nu} - (1 - \xi)\frac{k^\mu k^\nu}{k^2}\right]. \tag{38.3.16}$$

We call ξ the **gauge fixing parameter**. If we set it to zero we get what is called the **Landau gauge**, and if we set it to one we get the **Feynman gauge**.

We can then define the generating functional for the fee theory to be the value of the integral above, which we can now compute as a Gaussian integral since the kernel is invertible,

$$Z_0[J] = \exp\left[\frac{1}{2}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\, \tilde{J}_\mu(x)\tilde{D}_{\mathrm{F}}^{\mu\nu}(k)\tilde{J}_\nu(-k)\right], \tag{38.3.17}$$

or in position space,

$$Z_0[J] = \exp\left[-\frac{1}{2}\int \mathrm{d}^D\, \mathrm{d}^D y\, J_\mu(x) D_{\mathrm{F}}^{\mu\nu}(x-y) J_\nu(y)\right] \tag{38.3.18}$$

where

$$D_{\mathrm{F}}^{\mu\nu}(x-y) = \int \frac{\mathrm{d}^D k}{(2\pi)^D} \tilde{D}_{\mathrm{F}}^{\mu\nu}(k) \mathrm{e}^{-ik\cdot(x-y)} \tag{38.3.19}$$

$$= -i\langle 0|\, \mathrm{T}[A^\mu(x)A^\nu(t)]|0\rangle \tag{38.3.20}$$

$$= x \sim\sim\sim\sim\sim\sim\sim y \tag{38.3.21}$$

is the **photon propagator**.

Part XIV

Renormalisation

Thirty-Nine

Dimensional Regularisation

I will probably regret this choice, but that's OK.

Luigi Del Debbio

So far we've encountered many expressions with divergent integrals, usually these are of the form $\int \mathrm{d}^D k$. Our goal now is to find a way of modifying our theories to somehow constrain these infinities. This is called **regularisation**. We will then try to understand the structure of these divergences and see what they tell us about physics.

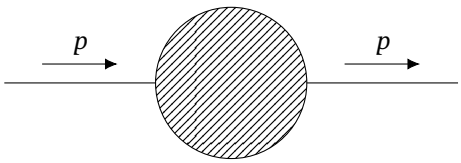
39.1 Setup

We will start with an extended example, for the two-point correlator for a scalar field, φ, with a φ^3 interaction in D dimensions. We work in momentum space, where this correlator is $\tilde{G}^{(2)}(p_1, p_2)$. As usual the correlator is of the form

$$\tilde{G}^{(2)}(p_1, p_2) = (2\pi)^D \delta(p_1 + p_2) i\tilde{\Delta}_{\mathrm{F}}(p^2) \tag{39.1.1}$$

where the Dirac delta conserves overall momentum, the $(2\pi)^D$ is just part of the normalisation, and $\tilde{\Delta}_{\mathrm{F}}$ is the Feynman propagator for the scalar particles in momentum space. We write it as a function of p^2, the momentum of the propagating particle. We've seen that in general n-point correlators depend on $n-1$ momenta, which justifies this choice, and we've seen that momentum only enters scalar correlators as its square.

If we consider one incoming particle and one outgoing particle then this process can be drawn as the Feynman diagram



(39.1.2)

where the blob represents all possible processes with a single initial and final particle.

We assume that $\tilde{G}^{(2)}$ is analytic, and so can be expressed as

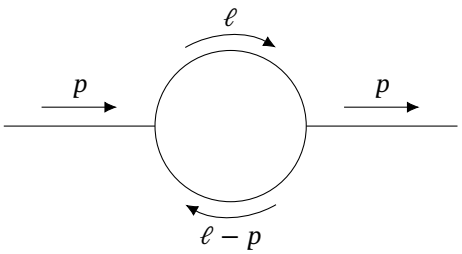
$$\tilde{G}^{(2)}(p_1, p_2) = (2\pi)^D \delta(p_1 + p_2) \sum_k g^k i\tilde{\Delta}_{\mathrm{F}}^{(k)}(p^2) \tag{39.1.3}$$

where g is the coupling constant appearing in the interaction Lagrangian, $\mathcal{L}_{\mathrm{int}} = -g\varphi^3/3!$, and $\tilde{\Delta}_{\mathrm{F}}^{(k)}$ is the contribution of the kth order term to the total propagator.

We will focus on the second order term, in particular we will ignore the factor of $(2\pi)^D \delta(p_1 + p_2)$ and just compute

$$g^2 i\tilde{\Delta}_{\mathrm{F}}^{(2)}(p^2). \tag{39.1.4}$$

As a second order term we have two vertices, each connected to three lines. With two external particles, one incoming and one outgoing, this corresponds to the diagram



(39.1.5)

Using the Feynman rules for φ^3 theory we can write down the term corresponding to this diagram. First, we have a symmetry factor of 2, recalling that in momentum space we fix the two ends of the diagram, so the only symmetry is the horizontal mirror symmetry. We then have a factor of $(ig)^2$, since there are two vertices. As shown in the picture momentum conservation at each vertex leaves one momentum unfixed, ℓ, and so we have an integral over ℓ, this will give us a divergent result. We also have four propagators, with momenta p, ℓ, $\ell - p$, and p again. This gives the result

$$\begin{aligned} g^2 i\tilde{\Delta}_{\mathrm{F}}^{(2)}(p^2) = \frac{1}{2}(ig)^2 \frac{i}{p^2 - m^2 + i\varepsilon} \\ \times \left[\int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \frac{i}{\ell^2 - m^2 + i\varepsilon} \frac{i}{(\ell - p)^2 - m^2 + i\varepsilon} \right] \frac{i}{p^2 - m^2 + i\varepsilon}. \end{aligned} \tag{39.1.6}$$

Write

$$i\tilde{\Delta}(p^2) = \frac{i}{p^2 - m^2 + i\varepsilon} \tag{39.1.7}$$

for the free propagators and

$$i\Pi(p^2) = \frac{g^2}{2} \int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \frac{1}{\ell^2 - m^2 + i\varepsilon} \frac{1}{(\ell - p)^2 - m^2 + i\varepsilon} \tag{39.1.8}$$

for the rest. We then want to compute

$$g^2 i\tilde{\Delta}_{\mathrm{F}}^{(2)}(p^2) = i\tilde{\Delta}(p^2) i\Pi(p^2) i\tilde{\Delta}(p^2). \tag{39.1.9}$$

Specifically, we want to compute $\Pi(p^2)$, which is a divergent integral.

39.2 Feynman Parametrisation

To proceed we use an identity relating the a product of reciprocals to a product of integrals of the reciprocal of a sum, which should hopefully be easier to compute. This identity, known as **Feynman parametrisation** is

$$\frac{1}{A_1^{\alpha_1}\cdots A_n^{\alpha_n}} = \frac{\Gamma(\alpha_1+\cdots+\alpha_n)}{\Gamma(\alpha_1)\cdots\Gamma_{\alpha_n}} \times \int_0^1 \mathrm{d}x_1\, x_1^{\alpha_1-1}\cdots\int_0^1 \mathrm{d}x_n\, x_n^{\alpha_n-1}\delta\left(1-\sum_{j=1}^n x_j\right)\frac{1}{(x_1A_1+\cdots+x_nA_n)^{\alpha_1+\cdots+\alpha_n}}. \tag{39.2.1}$$

This is valid so long as zero is not contained in the convex hull[1] of the complex numbers A_i and $\mathrm{Re}(\alpha_i)\geq 0$.

[1] the convex hull of some points is the smallest convex shape containing alll of the points, essentially imagine shrink wrapping the points.

Here

$$\Gamma(z) = \int_0^\infty \mathrm{d}t\, t^{z-1}\mathrm{e}^{-t} \tag{39.2.2}$$

is the gamma function. Note that $\Gamma(n) = (n-1)!$ for $n\in\mathbb{N}\setminus\{0\}$ and Γ has poles at all nonpositive integers (i.e., negative integers and zero).

In our case we want to use this identity to aid in computing the integral, we therefore have

$$A_1 = \ell^2 - m^2 + i\varepsilon, \quad A_2 = (\ell-p)^2 - m^2 + i\varepsilon, \quad \text{and} \quad \alpha_1=\alpha_2=1. \tag{39.2.3}$$

Since both A_1 and A_2 have the same imaginary part their "convex hull", in this case the line joining them, is parallel to, and above, the real axis, so doesn't contain zero. Hence, we have

$$i\Pi(p^2) = \frac{g^2}{2}\int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{1}{\ell^2-m^2+i\varepsilon}\frac{1}{(\ell-p)^2-m^2+i\varepsilon} \tag{39.2.4}$$

$$= \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}\int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\int_0^1\mathrm{d}x_1\int_0^1\mathrm{d}x_2\,\delta(1-x_1-x_2) \times\frac{1}{[x_1\ell^2+x_2(\ell-p)^2+(x_1+x_2)(-m^2+i\varepsilon)]^2} \tag{39.2.5}$$

now notice that $\Gamma(2)=1!=1$ and $\Gamma(1)=0!=1$, and using the Dirac delta we can perform the x_2 integral, replacing x_2 with $1-x_1$, and renaming $x_1=x$, giving

$$i\Pi(p^2) = \int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\int_0^1\mathrm{d}x\,\frac{1}{[x\ell^2+(1-x)(\ell-p)^2-m^2+i\varepsilon]^2}. \tag{39.2.6}$$

Somehow we then get the following, suggested substitution is $M^2 = m^2 - x(1-x)p^2$ and $q=\ell-xp$, but $q=\ell+xp$ seems to get closer???

$$i\Pi(p^2) = \frac{g^2}{2}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^Dq}{(2\pi)}\frac{1}{(q^2-M^2+i\varepsilon)^2}. \tag{39.2.7}$$

Suppose $D=6$, then we expect this integral to go as $q^6/q^4 = q^2$, so we expect quadratic divergence. This naive dimension counting only works in Euclidean space however.

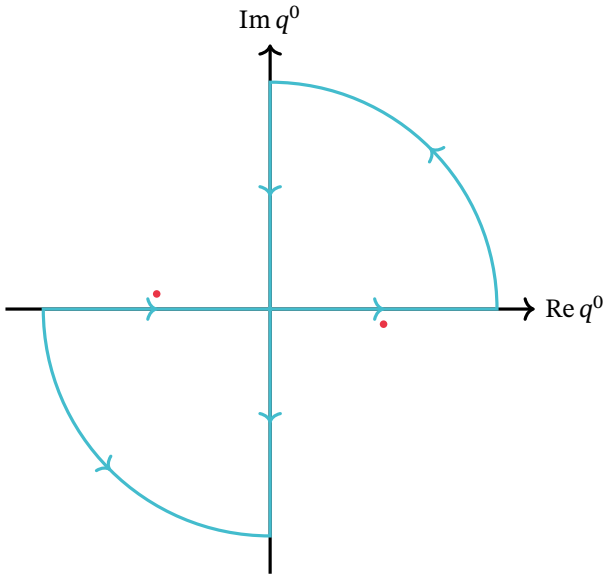


Figure 39.1: The contour used to justify the Wick rotation.

39.3 Wick Rotation

In computing $\Pi(p^2)$ we have the time integral

$$\int \frac{\mathrm{d}q^0}{2\pi} \frac{1}{(q^2 - M^2 + i\varepsilon)^2}. \tag{39.3.1}$$

Consider the contour shown in Figure 39.1. This contains no poles, and so the integral around the contour is zero. The integral along the two quarter circles also vanishes as we take them to infinity, since we can bound it by $1/(q^0)^2$. So the integral along the real axis is the negative of the integral from $+i\infty$ to $-i\infty$.

This means we can make the transformation of variables $q^0 = iq^0_{\mathrm{E}}$ and $\boldsymbol{q} = \boldsymbol{q}_{\mathrm{E}}$. Then with the usual Euclidean metric, the factor of i, and the additional overall minus sign we get the required metric. We also have $q^2 - M^2 = (iq^0_{\mathrm{E}})^2 + \boldsymbol{q}_{\mathrm{E}} \cdot \boldsymbol{q}_{\mathrm{E}} - M^2$ So we can instead compute

$$\Pi(p^2) = \frac{g^2}{2} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D q_{\mathrm{E}}}{(2\pi)^D} \frac{1}{(q^2_{\mathrm{E}} + M^2 + i\varepsilon)^2}. \tag{39.3.2}$$

Now we can make a dimension counting argument. The denominator goes as q^4_{E}, and the measure goes as q^D_{E}, so for $D \geq 4$ we expect the integral to be divergent. Suppose then that $D = 6$. This integral will them be quadratically divergent. We call this a **UV divergence**, since the divergence occurs due to unbounded (and therefore high) energies, as opposed to an IR divergence, which occurs due to very small energies.

We can decrease this divergence by considering the derivative with respect to p^2. We have to be careful computing this since $M^2 = m^2 - x(1-x)p^2$ has p^2 dependence:

$$\frac{\mathrm{d}}{\mathrm{d}p^2}\Pi(p^2) = -g^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D q_{\mathrm{E}}}{(2\pi)^D} \frac{\mathrm{d}M^2}{\mathrm{d}p^2} \frac{1}{(q^2_{\mathrm{E}} + M^2 + i\varepsilon)^3}. \tag{39.3.3}$$

In $D = 6$ the integrand and measure together go as $1/q_{\mathrm{E}}$, so we expect a logarithmic divergence, a great improvement on quadratic, but still divergent. We can take another derivative. Notice that $\mathrm{d}M^2/\mathrm{d}p^2 = -x(1-x)$ does not have p^2 dependence. Thus,

$$\frac{\mathrm{d}^2}{\mathrm{d}((p^2))^2}\Pi(p^2) = 3g^2\int_0^1 \mathrm{d}x\,[x(1-x)]^2\int\frac{\mathrm{d}^D q_{\mathrm{E}}}{(2\pi)^D}\frac{1}{(q_{\mathrm{E}}^2+M^2+i\varepsilon)^4}. \quad (39.3.4)$$

This is now convergent if $D = 6$.

We can start with $\Pi''(p^2)$ and use this finite value to reconstruct $\Pi(p^2)$ by integrating twice, giving

$$\Pi(p^2) = \Pi(\mu_1^2) + \Pi'(\mu_2^2)(p^2-\mu_1^2) + \int_{\mu_1^2}^{p^2}\mathrm{d}s'\int_{\mu_2^2}^{s'}\mathrm{d}s\,\Pi''(s). \quad (39.3.5)$$

Here μ_1 and μ_2 are some energy/momentum scales at which the values of Π and Π' are known and fixed, these arise as constants of integration. The quadratic divergence is still present, in the $\Pi(\mu_1^2)$ factor, as is the logarithmic divergence, in the $\Pi'(\mu_2^2)$ term. The final integral though is finite, so we've managed to isolate the divergences to needing to fix the value of the function and its derivative at some arbitrary values.

39.4 Regularisation

UV divergences appear at high momenta. We therefore modify our theory at high energies, or equivalently at low distances, such that the divergences disappear. There are multiple ways to do this, some of the most common are listed here:

- Sharp cut-off in Euclidean momenta at some value Λ^2, essentially picking a large, finite, number to replace infinity in the integral bounds.
- Pauli–Villars regulator: we modify the propagators to be less divergent at large values of the momentum:

$$\frac{1}{p^2-m^2+i\varepsilon} \to \frac{1}{p^2-m^2+i\varepsilon} - \frac{1}{p^2-M^2+i\varepsilon}. \quad (39.4.1)$$

  Here M is a mass scale.
- Schwinger-time regularisation: rewrite the integral in such a way that it is exponentially suppressed for large values of some parameter, introducing a divergence at 0 in the process, and then introduce a cut-off at low values of the parameter.
- Dimensional regularisation: work in generic dimension, D, and define the integrals by analytically continuation from a region where they converge.

It is this last approach which we shall take here.

39.4.1 Dimensional Regularisation

Dimensional regularisation, or **dim reg**, is the process by which we treat D as an arbitrary parameter which we can vary. We then compute the integrals for convergent values of D, then extend these results by analytic continuation. Consider the quantity

$$I_D = \int \frac{\mathrm{d}^D q_\mathrm{E}}{(2\pi)^D} \frac{1}{(q_\mathrm{E}^2 + M^2)^2}. \tag{39.4.2}$$

We can move to polar coordinates by considering q_E^2 as the radial coordinate squared, and using the angular measure $\mathrm{d}\Omega_D$. We then have

$$I_D = \int \frac{\mathrm{d}\Omega_D}{(2\pi)^D} \frac{1}{2} \int_0^\infty \mathrm{d}q_\mathrm{E}^2 \, (q_\mathrm{E}^2)^{D/2-1} \frac{1}{(q_\mathrm{E}^2 + M^2)^2}. \tag{39.4.3}$$

Now we make this integral dimensionless:

$$I_D = \int \frac{\mathrm{d}\Omega_D}{(2\pi)^D} \frac{1}{2} \left(\frac{1}{M^2}\right)^{2-D/2} \int_0^\infty \mathrm{d}\left(\frac{q_\mathrm{E}^2}{M^2}\right) \left(\frac{q_\mathrm{E}^2}{M^2}\right)^{D/2-1} \frac{1}{(1 + q_\mathrm{E}^2/M^2)^2}. \tag{39.4.4}$$

Notice how this separates the angular dependence, the dimensions, and then a dimensionless integral. Define $\xi = 1/(1 + q_\mathrm{E}^2/M^2)$, we then have

$$I_D = \frac{1}{(2\pi)^D} \frac{2\pi^{D/2}}{\Gamma(D/2)} \frac{1}{2} \left(\frac{1}{M^2}\right)^{2-D/2} \int_0^1 \mathrm{d}\xi \, \xi^{1-D/2} (1-\xi)^{D/2-1}. \tag{39.4.5}$$

Here we've made a change of variables and also used the solid angle in D dimensions,

$$\Omega_D = \int \mathrm{d}\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}, \tag{39.4.6}$$

note that if $D = 3$ we get $2\pi^{3/2}/\Gamma(3/2) = 4\pi$, which is what we would expect. Here

$$\Gamma(z) \coloneqq \int_0^\infty \mathrm{d}t \, t^{z-1} \mathrm{e}^{-t} \tag{39.4.7}$$

is the **gamma function**.

The integral here can be identified as the **beta function**

$$B(w,z) = \frac{\Gamma(w)\Gamma(z)}{\Gamma(w+z)} = \int_0^1 \mathrm{d}\xi \, \xi^{w-1} (1-\xi)^{z-1}. \tag{39.4.8}$$

Hence, we can write the integral as

$$\int_0^1 \mathrm{d}\xi \, \xi^{1-D/2} (1-\xi)^{D/2-1} = B\left(2 - \frac{D}{2}, \frac{D}{2}\right) = \frac{\Gamma(2 - D/2)\Gamma(D/2)}{\Gamma(2)}. \tag{39.4.9}$$

Note that $\Gamma(2) = 1! = 1$. Then we have

$$I_D = \frac{2}{(4\pi)^{D/2}} \frac{1}{\Gamma(D/2)} \frac{1}{2} \left(\frac{1}{M^2}\right)^{2-D/2} B\left(2 - \frac{D}{2}, \frac{D}{2}\right) \tag{39.4.10}$$

$$= \frac{1}{(4\pi)^{D/2}} \Gamma\left(2 - \frac{D}{2}\right) \left(\frac{1}{M^2}\right)^{2-D/2}. \tag{39.4.11}$$

The $1/(4\pi)^{D/2}$ factor is characteristic of one loop integrals. The $(1/M^2)^{2-D/2}$ factor carries all of the dimensions of this value. The divergence is constrained to the $\Gamma(2-D/2)$ term, since Γ has poles at nonpositive integers, that is at 0 and at negative integers. This means we have divergences when $D = 4, 6, 8, \ldots$.

We then have

$$\Pi(p^2) = \frac{g^2}{2}\int_0^1 \mathrm{d}x\, \frac{1}{(4\pi)^{D/2}}\Gamma\left(2-\frac{D}{2}\right)\left(\frac{1}{M^2}\right)^{2-D/2}. \tag{39.4.12}$$

Remember that M^2 has x dependence.

39.5 Generalisation

We can generalise this computation to

$$\int \frac{\mathrm{d}^D_{q_\mathrm{E}}}{(2\pi)^D}\frac{(q_\mathrm{E}^2)^a}{(q_\mathrm{E}^2+M^2)^b} = \frac{\Gamma(b-a-D/2)\Gamma(a+D/2)}{(4\pi)^{D/2}\Gamma(b)\Gamma(D/2)}(M^2)^{-(b-a-D/2)}. \tag{39.5.1}$$

As before we can factor out the angular integral giving

$$\int \frac{\mathrm{d}^D q_\mathrm{E}}{(2\pi)^D}\frac{(q_\mathrm{E})^a}{(q_\mathrm{E}^2+M^2)^b} = \frac{2\pi^{D/2}}{\Gamma(D/2)}\frac{1}{(2\pi)^D}\int_0^\infty \mathrm{d}q\, \frac{q^{2a+D-1}}{(q^2+M^2)^b} \tag{39.5.2}$$

$$= \frac{2}{(4\pi)^{D/2}\Gamma(D/2)}\int_0^\infty \mathrm{d}q\, \frac{q^{2a+D=1}}{(q^2+M^2)^b}. \tag{39.5.3}$$

Comparing this to the generalised result above we see that

$$\int_0^\infty \mathrm{d}q\, \frac{q^{2\alpha}}{(q^2+M^2)^\beta} = \frac{\Gamma(\beta-\alpha-1/2)\Gamma(\alpha+1/2)}{2\Gamma(\beta)}(M^2)^{-(\beta-\alpha-1/2)}. \tag{39.5.4}$$

Forty

Structure of the Divergences

Now that we have manipulated our expression into a more manageable form we can begin to investigate the structure of the divergences. To do so we start with some dimensional analysis.

40.1 Dimensions

If a quantity has dimensions of mass to some power we call this power the mass dimension. For example, momentum has mass dimension 1, and lengths have mass dimension -1. Denote by $[-]$ the mass dimension of some quantity, so if E is an energy then $[E] = 1$. Note that since mass dimension is an exponentiated quantity we have $[ab] = [a] + [b]$.

Consider Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) + \frac{1}{2}m^2\varphi^2 + \frac{1}{3!}g^3\varphi^3. \tag{40.1.1}$$

We know that the action,

$$S = \int \mathrm{d}^D x\, \mathcal{L}, \tag{40.1.2}$$

is dimensionless, and since $[\mathrm{d}^D x] = -D$ we must have $[\mathcal{L}] = D$. Taking the first term of the Lagrangian we know that $[\partial_\mu] = [\partial^\mu] = [1/x] = 1$, and so we must have $[\varphi] = (D-2)/2$, so that $[(\partial_\mu\varphi)(\partial^\mu\varphi)] = D$.

we must also have $[g\varphi^3] = [g] + [\varphi^3] = [\mathcal{L}]$, and so we have

$$[g] = [\mathcal{L}] - [\varphi^3] = [\mathcal{L}] - 3[\varphi] = D - 3\frac{D-2}{2} = \frac{6-D}{2}. \tag{40.1.3}$$

Notice that in $D = 6$ this means the coupling constant is dimensionless. This is the motivation for working in $D = 6$, it's the only dimension in which we have a dimensionless coupling constant allowing us to do perturbation theory.

As previously mentioned in dimensional regularisation we take D to be a variable. In particular we can take $D = 6 - 2\varepsilon$ for some $\varepsilon > 0$. Note that this ε is not the same as the ε appearing in the $i\varepsilon$ prescription. Then we have

$$[\varphi] = \frac{D-2}{2} = \frac{6-2\varepsilon-2}{2} = 2 - \varepsilon, \tag{40.1.4}$$

and

$$[g] = [\mathcal{L}] + 3[\varphi] = 6 - 2\varepsilon - 6 + 3\varepsilon = \varepsilon. \tag{40.1.5}$$

So, as expected, away from $D = 6$ we have a dimensionful coupling. We can fix this by rescaling $g \mapsto g\tilde{\mu}^\varepsilon$, where g is again dimensionless and $\tilde{\mu}$ has dimensions of energy, and so we still have $[g\tilde{\mu}^\varepsilon] = \varepsilon$.

40.2 Computing the Integral

Now consider $\Pi(p^2)$ again. We can now write it as

$$\Pi(p^2) = \frac{g^2}{2}\tilde{\mu}^{2\varepsilon} \int_0^1 \mathrm{d}x \, \frac{1}{(4\pi)^{3-\varepsilon}} \Gamma(\varepsilon - 1) \left(\frac{1}{M^2}\right)^{\varepsilon - 1}. \tag{40.2.1}$$

Collecting together the terms without ε or x dependence into a single parameter, α, this becomes

$$\Pi(p^2) = \frac{\alpha}{2} \int_0^1 \mathrm{d}x \, \Gamma(\varepsilon - 1) M^2 \left(\frac{4\pi\tilde{\mu}^2}{M^2}\right)^\varepsilon \tag{40.2.2}$$

where

$$\alpha = \frac{g^2}{(4\pi)^3}. \tag{40.2.3}$$

Suppose that n is a positive integer, then there exists an expansion for the gamma function for small ε:

$$\Gamma(\varepsilon - n) = \frac{(-1)^n}{n!} \left[\frac{1}{\varepsilon} - \gamma + \sum_{k=1}^{n} \frac{1}{k} + \mathcal{O}(\varepsilon)\right]. \tag{40.2.4}$$

Here $\gamma \approx 0.577216$ is the **Euler–Mascheroni constant**. Using this we have

$$\Gamma(\varepsilon - 1) = -\left[\frac{1}{\varepsilon} - \gamma + 1 + \mathcal{O}(\varepsilon)\right]. \tag{40.2.5}$$

It's fine to work to constant order, $\mathcal{O}(\varepsilon^0)$, since we'll be taking $\varepsilon \to 0$.

Notice that we can rewrite the term with ε as an exponent as the exponential of a log, and then expand the log:

$$\left(\frac{4\pi\tilde{\mu}^2}{M^2}\right)^\varepsilon = \exp\left\{\varepsilon \log\left(\frac{4\pi\tilde{\mu}^2}{M^2}\right)\right\} \tag{40.2.6}$$

$$= 1 + \varepsilon \log\left(\frac{4\pi\tilde{\mu}^2}{M^2}\right) + \mathcal{O}(\varepsilon^2). \tag{40.2.7}$$

Note that we have to go to order ε here since in the product with $\Gamma(\varepsilon - 1)$ we'll have a $1/\varepsilon$ term, so the overall order is ε^0.

Hence, we have

$$\begin{aligned}\Pi(p^2) &= -\frac{\alpha}{2} \int_0^1 \mathrm{d}x \left[\frac{1}{\varepsilon} - \gamma + 1 + \mathcal{O}(\varepsilon)\right] M^2 \left[1 + \varepsilon \log\left(\frac{4\pi\tilde{\mu}^2}{M^2}\right) + \mathcal{O}(\varepsilon^2)\right] \\ &= -\frac{\alpha}{2} \int_0^1 \mathrm{d}x \, M^2 \left[\frac{1}{\varepsilon} - \gamma + 1 + \log\left(\frac{4\pi\tilde{\mu}^2}{M^2}\right) + \mathcal{O}(\varepsilon)\right].\end{aligned} \tag{40.2.8}$$

Note the divergence in the first term when $\varepsilon \to 0$. Now write out M^2 in full, we have

$$\Pi(p^2) = -\frac{\alpha}{2}\int_0^1 \mathrm{d}x\, M^2 \left[\frac{1}{\varepsilon} - \gamma + 1 + \log\left(\frac{4\pi\tilde{\mu}^2}{M^2}\right) + \mathcal{O}(\varepsilon)\right]. \tag{40.2.9}$$

We can compute the integral for the $M^2(1/\varepsilon + 1)$ term since $1/\varepsilon + 1$ has no x dependence and $M^2 = m^2 - x(1-x)p^2$, so using

$$\int_0^1 \mathrm{d}x\, M^2 = \int_0^1 \mathrm{d}x\, [m_\mathrm{R}^2 - x(1-x)p^2] = m_\mathrm{R}^2 - \frac{1}{6}p^2. \tag{40.2.10}$$

we have

$$\Pi(p^2) = \frac{\alpha}{2}\left[\left(\frac{1}{\varepsilon}+1\right)\left(\frac{1}{6}p^2 - m^2\right) + \int_0^1 \mathrm{d}x\, M^2\left(\gamma + \log\left(\frac{M^2}{4\pi\tilde{\mu}^2}\right)\right)\right] \tag{40.2.11}$$

where the argument of the log is flipped as we absorb the minus sign. Now define

$$\mu^2 = \frac{4\pi\tilde{\mu}^2}{e^\gamma}, \tag{40.2.12}$$

and this becomes

$$\Pi(p^2) = \frac{\alpha}{2}\left[\left(\frac{1}{\varepsilon}+1\right)\left(\frac{1}{6}p^2 - m^2\right) + \int_0^1 \mathrm{d}x\, M^2(x,p^2)\log\left(\frac{M^2(x,p^2)}{\mu^2}\right)\right]. \tag{40.2.13}$$

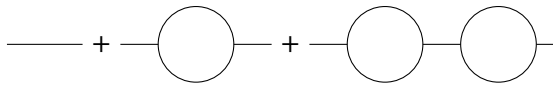
Recall that before we had divergences in the $\Pi(\mu_1^2)$ and $\Pi'(\mu_2^2)$ terms, but not in the $\Pi''(s)$ term, so the divergences appeared in the constant and linear terms. This is what we see here with the constant term $-m^2/\varepsilon$ and the linear (in p^2) term $p^2/(6\varepsilon)$, which are the only places this diverges.

Notice that the results depends on an arbitrary energy scale, μ. This is an unavoidable part of regularisation, however we do it we must introduce some energy scale in such a way that our final expressions depend on the value at this energy scale.

40.3 One Particle Irreducible Diagrams

We've considered only the diagram in Equation (39.1.5) for second order contributions. We also saw that the diagram in Equation (33.2.19) contributes at second order. The reason we don't have to consider this diagram is that it is not **one particle irreducible**, or **1PI|seeone particle irreducible**, meaning that by removing a single propagator we get a disconnected diagram. Such diagrams can always be constructed as a sum of one particle irreducible diagrams connected with additional propagators.

In particular, we can evaluate $\tilde{\Delta}_\mathrm{F}(p^2)$ to all orders by considering the series of diagrams of the form of Equation (39.1.5) connected by propagators:

$i\tilde{\Delta}_\mathrm{F}(p^2) =$  $+ \cdots$ (40.3.1)

$$= i\tilde{\Delta}(p^2) + i\tilde{\Delta}(p^2)(-i\Pi(p^2))i\tilde{\Delta}(p^2) \tag{40.3.2}$$

$$+ i\tilde{\Delta}(p^2)(-i\Pi(p^2))i\tilde{\Delta}(p^2)(-i\Pi(p^2))i\tilde{\Delta}(p^2) + \cdots \tag{40.3.3}$$

$$= i\tilde{\Delta}(p^2) + i\tilde{\Delta}(p^2)[(-i\Pi(p^2))i\tilde{\Delta}(p^2)] \tag{40.3.4}$$
$$+ i\tilde{\Delta}(p^2)[(-i\Pi(p^2))i\tilde{\Delta}(p^2)]^2 + \cdots \tag{40.3.5}$$
$$= i\tilde{\Delta}(p^2)\sum_{k=0}^{\infty}[\Pi(p^2)\tilde{\Delta}(p^2)]^k \tag{40.3.6}$$
$$= i\tilde{\Delta}(p^2)\frac{1}{\Pi(p^2)\tilde{\Delta}(p^2)}, \tag{40.3.7}$$

where we've identified the geometric series in the last step and applied the known result

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \tag{40.3.8}$$

for all $z \in \mathbb{C}$ with $|z| < 1$.

As a result, assuming convergence, we have

$$i\tilde{\Delta}_{\mathrm{F}}(p^2) = \frac{i}{p^2 - m^2 - \Pi(p^2) + i\varepsilon} \tag{40.3.9}$$

where we reintroduce the $i\varepsilon$ prescription, note that this ε is different to the ε of dimensional regularisation.

40.4 Comparison with Källén–Lehmann

Recall the result

$$\tilde{\Delta}_{\mathrm{F}}(p^2) = \frac{1}{p^2 - m_{\mathrm{phys}}^2}\int_{4m_{\mathrm{phys}}^2}^{\infty} \mathrm{d}s\,\rho(s)\frac{1}{p^2 - s} \tag{40.4.1}$$

from Equation (35.2.18), replacing p as a parameter with p^2. This has a pole at the square of the physical mass of the particle, and the residue at this pole is 1. This is because we chose the normalisation

$$\langle \boldsymbol{p}|\varphi(0)|0\rangle = 1. \tag{40.4.2}$$

Compare this to the result we just derived here:

$$\tilde{\Delta}_{\mathrm{F}}(p^2) = \frac{1}{p^2 - m^2 - \Pi(p^2)}. \tag{40.4.3}$$

This has a pole when $p^2 - m^2 - \Pi(p^2) = 0$. So, we must have that

$$m_{\mathrm{phys}}^2 - m^2 = \Pi(m_{\mathrm{phys}}^2). \tag{40.4.4}$$

Notice that $m \neq m_{\mathrm{phys}}$ most of the time, in fact we only have $m = m_{\mathrm{phys}}$ if there is no interaction term.

Consider the denominator $p^2 - m^2 - \Pi(p^2)$, and expand about $p^2 = m_{\mathrm{phys}}^2$:

$$p^2 - m^2 - \Pi(p^2) = (p^2 - m_{\mathrm{phys}}^2)[1 - \Pi'(m_{\mathrm{phys}}^2)] + \mathcal{O}([p^2 - m_{\mathrm{phys}}^2]^2). \tag{40.4.5}$$

Then we can calculate the residue:

$$\operatorname{Res}\left(\tilde{\Delta}_{\mathrm{F}}(p^2), p^2 = m^2_{\mathrm{phys}}\right) \coloneqq \lim_{p^2 \to m^2_{\mathrm{phys}}} (p^2 - m^2_{\mathrm{phys}})\tilde{\Delta}_{\mathrm{F}}(p^2) \tag{40.4.6}$$

$$= \lim_{p^2 \to m^2_{\mathrm{phys}}} \frac{(p^2 - m^2_{\mathrm{phys}})}{(p^2 - m^2_{\mathrm{phys}})[1 - \Pi'(m^2_{\mathrm{phys}})] + \mathcal{O}([p^2 - m^2_{\mathrm{phys}}]^2)} \tag{40.4.7}$$

$$= \frac{1}{1 - \Pi'(m^2_{\mathrm{phys}})}. \tag{40.4.8}$$

So, in general the residue is not 1. This means that the field φ as appearing in this section does not have the required normalisation to agree with the Källén–Lehmann representation. We must therefore *renormalise*. Note that in the free theory $\Pi'(m^2_{\mathrm{phys}}) = 0$ and so the field is properly normalised.

The general idea that we develop from this observation is that for an interacting theory the fields and constants appearing in the Lagrangian, called the **bare fields** and **bare couplings**, are not physical. The physical variables are then defined in terms of these by some well-specified prescription, a process called renormalisation.

Forty-One

Renormalisation

41.1 Renormalising

The Lagrangian for a φ^3 scalar field theory is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}m^2\varphi^2 + \frac{1}{3!}g\varphi^3. \tag{41.1.1}$$

Define the renormalised field, φ_R, according to

$$\varphi = Z^{1/2}\varphi_\mathrm{R} \tag{41.1.2}$$

where Z is some finite constant to be defined later. The Lagrangian in terms of the renormalised field is

$$\mathcal{L} = \frac{1}{2}Z(\partial_\mu\varphi_\mathrm{R})(\partial^\mu\varphi_\mathrm{R}) - \frac{1}{2}m^2Z\varphi_\mathrm{R}^2 + \frac{1}{3!}gZ^{3/2}\varphi_\mathrm{R}^3. \tag{41.1.3}$$

Now define

$$m^2Z = Z_{m^2}m_\mathrm{R}^2, \qquad \text{and} \qquad gZ^{3/2} = Z_g g_\mathrm{R} \tag{41.1.4}$$

where Z_{m^2} and Z_g are finite constants to be defined later. Then we can write the Lagrangian as

$$\mathcal{L} = \frac{1}{2}Z(\partial_\mu\varphi_\mathrm{R})(\partial^\mu\varphi_\mathrm{R}) - \frac{1}{2}Z_{m^2}m_\mathrm{R}^2\varphi_\mathrm{R}^2 + \frac{1}{3!}Z_g g_\mathrm{R}\varphi_\mathrm{R}^3. \tag{41.1.5}$$

We can rewrite the Lagrangian as

$$\begin{aligned}\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi_\mathrm{R})(\partial^\mu\varphi_\mathrm{R}) - \frac{1}{2}m_\mathrm{R}^2\varphi_\mathrm{R}^2 + \frac{1}{3!}g_\mathrm{R}\varphi_\mathrm{R}^3 \\ + \frac{1}{2}\delta_Z(\partial_\mu\varphi_\mathrm{R})(\partial^\mu\varphi_\mathrm{R}) - \frac{1}{2}\delta_{m^2}m_\mathrm{R}^2\varphi_\mathrm{R}^2 + \frac{1}{3!}\delta_g g_\mathrm{R}\varphi_\mathrm{R}^3.\end{aligned} \tag{41.1.6}$$

Here $\delta_Z \coloneqq Z - 1$, $\delta_{m^2} \coloneqq Z_{m^2} - 1$, and $\delta_g \coloneqq Z_g - 1$. We can fix δ_Z, δ_{m^2} and δ_g to fix the values of Z, Z_{m^2} and Z_g. Notice that the Lagrangian now takes the form of the initial Lagrangian, called the **bare Lagrangian**, now with extra subscripts R, plus some extra terms, with factors of δ_Z, δ_{m^2}, and δ_g, which are called the **counter terms**.

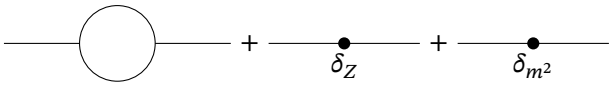
We can interpret these extra terms as more interactions, and therefore they give us more vertices in our Feynman rules. We can then recalculate $\Pi(p^2)$ again

in renormalised perturbation theory. The result is the same as before, plus contributions from the counter terms:

$$\Pi(p^2) = \frac{\alpha}{2}\left[\left(\frac{1}{\varepsilon}+1\right)\left(\frac{1}{6}p^2 - m_{\mathrm{R}}^2\right) + \int_0^1 \mathrm{d}x\, M^2 \log\left(\frac{M^2}{\mu^2}\right)\right] + \delta_Z p^2 - \delta_{m^2} m_{\mathrm{R}}^2 + \mathcal{O}(\alpha^2) \quad (41.1.7)$$

Note that M^2 is now taken as a function of m_{R}^2, $M^2 = m_{\mathrm{R}}^2 - x(1-x)p^2$. Similarly in principle α depends on the renormalised value of g, g_{R}, but it turns out not to matter for this particular calculation. The form of the counter terms here comes from expanding the generating functional, now defined as the exponential of the action with the counter terms in it. Expanding to first order in g_{R} as we did to find $\Pi(p^2)$ initially we get extra factors with $(\partial_\mu \varphi_{\mathrm{R}})(\partial^\mu \varphi_{\mathrm{R}})$ and $m_{\mathrm{R}}^2 \varphi_{\mathrm{R}}^2$, which we can act on with functional derivatives. Moving to Fourier space turns the ∂_μ into p_μ, giving the p^2 factor seen above.

We can write this diagrammatically as

$\Pi(p^2) =$  (41.1.8)

Where we've introduced two new vertices connecting two particles for each of the second order counter terms.

Consider the integral in $\Pi(p^2)$. We can rewrite this as

$$\int_0^1 \mathrm{d}x\, M^2 \log\left(\frac{M^2}{\mu^2}\right) = \int_0^1 \mathrm{d}x\, M^2 \left[\log\left(\frac{M^2}{m_{\mathrm{R}}^2}\right) - \log\left(\frac{\mu^2}{m_{\mathrm{R}}^2}\right)\right] \quad (41.1.9)$$

having used m_{R} to keep the arguments of the logarithms dimensionless. This is a useful thing to do because the second logarithm doesn't have any x dependence, meaning we can easily do the integral as the only x dependence appears through the factor of $M^2 = m_{\mathrm{R}}^2 - x(1-x)p^2$, which is easy to integrate:

$$\int_0^1 \mathrm{d}x\, M^2 = \int_0^1 \mathrm{d}x\, (m_{\mathrm{R}}^2 - x(1-x)p^2) = m_{\mathrm{R}}^2 - \frac{1}{6}p^2. \quad (41.1.10)$$

Hence, we have

$$\int_0^1 \mathrm{d}x\, M^2 \log\frac{M^2}{\mu^2} = \int_0^1 \mathrm{d}x\, M^2 \log\left(\frac{M^2}{m_{\mathrm{R}}^2}\right) + \left(\frac{1}{6}p^2 - m_{\mathrm{R}}^2\right)\log\left(\frac{\mu^2}{m_{\mathrm{R}}^2}\right). \quad (41.1.11)$$

Then we have

$$\Pi(p^2) = \frac{\alpha}{2}\Bigg[\left(\frac{1}{\varepsilon}+1\right)\left(\frac{1}{6}p^2 - m_{\mathrm{R}}^2\right) + \int_0^1 \mathrm{d}x\, M^2 \log\left(\frac{M^2}{m_{\mathrm{R}}^2}\right) + \left(\frac{1}{6}p^2 - m_{\mathrm{R}}^2\right)\log\left(\frac{\mu^2}{m_{\mathrm{R}}^2}\right)\Bigg] + \delta_Z p^2 - \delta_{m^2} m_{\mathrm{R}}^2 + \mathcal{O}(\alpha^2) \quad (41.1.12)$$

$$= \frac{\alpha}{2}\left[\left(\frac{1}{\varepsilon} + \log\left(\frac{\mu^2}{m_{\mathrm{R}}^2}\right) + 1\right)\left(\frac{1}{6}p^2 - m_{\mathrm{R}}^2\right) + \int_0^1 \mathrm{d}x\, M^2 \log\left(\frac{M^2}{m_{\mathrm{R}}^2}\right)\right] + \delta_Z p^2 - \delta_{m^2} m_{\mathrm{R}}^2 + \mathcal{O}(\alpha^2) \tag{41.1.13}$$

$$= \left[\frac{\alpha}{6}\left(\frac{1}{2\varepsilon} + \log\left(\frac{\mu}{m_{\mathrm{R}}}\right) + \frac{1}{2}\right) + \delta_Z\right] p^2 \tag{41.1.14}$$

$$- \left[\alpha\left(\frac{1}{2\varepsilon} + \log\left(\frac{\mu}{m_{\mathrm{R}}}\right) + \frac{1}{2}\right) + \delta_{m^2}\right] m_{\mathrm{R}}^2 \tag{41.1.15}$$

$$+ \frac{\alpha}{2}\int_0^1 \mathrm{d}x\, M^2 \log\left(\frac{M^2}{m_{\mathrm{R}}^2}\right) + \mathcal{O}(\alpha^2) \tag{41.1.16}$$

where we've used $\log z^2 = 2\log z$ to remove the square in the logarithm's argument.

As mentioned earlier since Z and Z_{m^2} are free parameters we can fix them by choosing values for $\delta_Z = Z - 1$ and $\delta_{m^2} = Z_{m^2} - 1$. We do this here, defining

$$\delta_Z = -\frac{\alpha}{6}\left[\frac{1}{2\varepsilon} + \log\left(\frac{\mu}{m_{\mathrm{R}}}\right) + \frac{1}{2}\right] + \kappa_Z \quad + \mathcal{O}(\alpha^2), \tag{41.1.17}$$

$$\delta_{m^2} = -\alpha\left[\frac{1}{2\varepsilon} + \log\left(\frac{\mu}{m_{\mathrm{R}}}\right) + \frac{1}{2}\right] + \kappa_{m^2} + \mathcal{O}(\alpha^2), \tag{41.1.18}$$

where κ_Z and κ_{m^2} are finite constants. This is a sensible choice since it cancels the dependence of $\Pi(p^2)$ on the regulator, ε, and the scale, μ. There is some dependence on ε still through the $\mathcal{O}(\varepsilon)$ terms which we've dropped, but in the limit of $\varepsilon \to 0$ this dependence also vanishes. We are left with

$$\Pi(p^2) = \frac{\alpha}{2}\int_0^1 \mathrm{d}x\, M^2 \log\left(\frac{M^2}{m_{\mathrm{R}}^2}\right) + \mathcal{O}(\alpha^2) \tag{41.1.19}$$

41.2 Renormalisation Scheme

A **renormalisation scheme** is a prescription for defining the finite parts of these divergent integrals. A renormalisation scheme consists of a collection of **renormalisation conditions**, which are values we define with some extra input not coming just from the Lagrangian. We need as many renormalisation conditions as there are renormalised variables, so in our case we need three renormalisation conditions. However, in this section we'll only give two because the two point function we have dealt with so far just happens not to depend on g_{R}.

We will use the **on-shell scheme**, where we fix the renormalised mass parameter appearing in the renormalised Lagrangian to be exactly the physical mass, $m_{\mathrm{R}}^2 = m_{\mathrm{phys}}^2$. This is a physical input, we need the measured mass of the particle. This gives the first renormalisation condition: $\Pi(m_{\mathrm{R}}^2) = 0$, since we've seen that after renormalising we have a pole when $p^2 - m_{\mathrm{R}}^2 - \Pi(p^2) = 0$, so we must have $\Pi(m_{\mathrm{R}}^2) = 0$ in order to have $m_{\mathrm{R}}^2 - m_{\mathrm{R}}^2 - \Pi(m_{\mathrm{R}}^2) = 0$.

The second condition we use is that the residue at the pole must be 1, as in the Källén–Lehmann representation. This can be achieved by noticing that the pole has the form

$$\frac{1}{1 - \Pi'(m_{\mathrm{phys}}^2)} = \frac{1}{1 - \Pi'(m_{\mathrm{R}}^2)} \tag{41.2.1}$$

and this is 1 if $\Pi'(m_{\mathrm{R}}^2) = 1$, giving us the second renormalisation condition.

We can use these renormalisation conditions to rewrite $\Pi(p^2)$ as follows:

$$\Pi(p^2) = \frac{\alpha}{2}\int_0^1 \mathrm{d}x\, M^2 \log\left(\frac{M^2}{M_0^2}\right) + A(p^2 - m_{\mathrm{R}}^2) \tag{41.2.2}$$

where $M_0^2(x) \coloneqq M^2(x, m_{\mathrm{R}}^2) = m_{\mathrm{R}}(1 - x + x^2)$ and A is some constant to be determined. The first term vanishes when $p^2 = m_{\mathrm{R}}^2$ since we get $\log 1 = 0$. The second term also clearly vanishes in this case.

Now consider the derivative of $\Pi'(p^2)$:

$$\Pi'(p^2) = \frac{\alpha}{2}\int_0^2 \mathrm{d}x \left[\frac{\mathrm{d}M^2}{\mathrm{d}p^2}\log\left(\frac{M^2}{M_0^2}\right) + M^2\frac{M_0^2}{M^2}\frac{\mathrm{d}}{\mathrm{d}p^2}\left(\frac{M^2}{M_0^2}\right)\right] + A. \tag{41.2.3}$$

The first term, with the log, vanishes when $p^2 = m_{\mathrm{R}}^2$ since we again have $\log 1 = 0$. The second term simplifies to $\partial M^2/\partial p^2 = -x(1-x)$, and so we have

$$\Pi'(m_{\mathrm{R}}^2) = \frac{\alpha}{2}\int_0^1 \mathrm{d}x\,(x-1)x + A. \tag{41.2.4}$$

Demanding that this is zero we see that we must have

$$A = \frac{\alpha}{12}. \tag{41.2.5}$$

So,

$$\Pi(p^2) = \frac{\alpha}{2}\int_0^1 \mathrm{d}x\, M^2(x,p^2)\log\left(\frac{M^2(x,p^2)}{M_0^2(x)}\right) + \frac{\alpha}{12}(p^2 - m_{\mathrm{R}}^2) + \mathcal{O}(\alpha^2). \tag{41.2.6}$$

Notice that this has no dependence on μ, which is good, and also no dependence on the renormalised mass, since $m_{\mathrm{R}} = m_{\mathrm{phys}}$. Note that any renormalisation of parameters in α has an effect on the order of α^2, so for this one-loop calculation to first order in α it doesn't matter if we use $\alpha = g^2/(4\pi)^3$ or $\alpha = g_{\mathrm{R}}^2/(4\pi)^3$.

Forty-Two

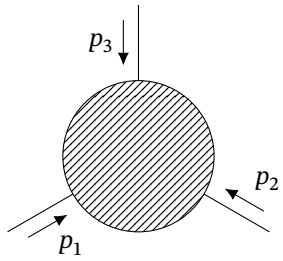
Three Point Functions

42.1 Setup

In this chapter we'll compute the connected three point correlation function in momentum space. This is given by

$$\tilde{G}_c^{(3)}(p_1,p_2,p_3) = \int \mathrm{d}^D x_1\, \mathrm{d}^D x_2 \mathrm{d}^D x_3\, \mathrm{e}^{ip_1\cdot x_1}\mathrm{e}^{ip_2\cdot x_2}\mathrm{e}^{ip_3\cdot x_3} G_c^{(3)}(x_1,x_2,x_3) \tag{42.1.1}$$

$$= \tag{42.1.2}$$



Here $G_c^{(3)}(x_1,x_2,x_3)$ is the connected position space three point correlator, the subscript c reminding us of the "connected" part of that definition. The variables x_i are positions here but note that later we will use the same symbols to refer to the Feynman parameters.

The connected three point correlator in position space is given by

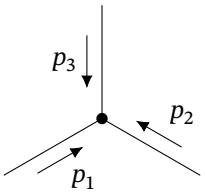
$$G_c^{(3)}(x_1,x_2,x_3) = \left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x_2)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x_3)}\right) W[J]\Big|_{J=0}. \tag{42.1.3}$$

Here we use $W[J] = -i\log(Z[J])$ since this gives us the contribution of the connected diagrams.

There are $E = 2P - 3V$ external legs if there are P propagators and $3V$ vertices. We're looking for three external legs, so $E = 3$.

42.1.1 First Order

The first order term, $V = 1$, is the tree level diagram



(42.1.4)

In momentum space this diagram is given simply by the (renormalised) vertex term, $iZ_g g_R$, and the usual momentum conservation factor:

$$(2\pi)^D \delta(p_1 + p_2 + p_3) i Z_g g_R. \tag{42.1.5}$$

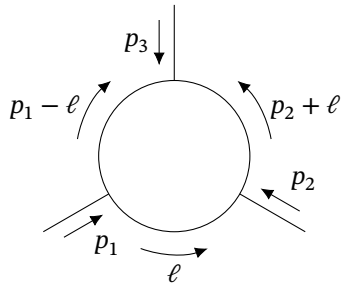
42.1.2 Second Order

There are no integer solutions to $E = 2P - 3V$ with $E = 3$ and $V = 2$.

42.1.3 Third Order

> *I'm going to try to do this in a smart way, which means probably wrong.*
>
> Luigi Del Debbio

We can have $E = 2P - 3V = 3$ if $V = 3$ and $P = 6$. The only connected diagram with 6 propagators, 3 vertices, and 3 external particles is



(42.1.6)

This diagram is given by

$$(2\pi)^D \delta(p_1 + p_2 + p_3)(iZ_g g_R)^3 \frac{i}{p_1^2 - m_R^2} \frac{i}{p_2^2 - m_R^2} \frac{i}{p_3^2 - m_R^2} (iV^{(3,3)}) \tag{42.1.7}$$

where $iV^{(3,3)}$ is the 3 point order 3 divergent integral

$$iV^{(3,3)}(ig_R)^3 \int \frac{d^D \ell}{(2\pi)^D} \frac{i}{\ell^2 - m_R^2} \frac{i}{(p_2 + \ell)^2 - m_R^2} \frac{i}{(p_1 - \ell)^2 - m_R^2}. \tag{42.1.8}$$

We've used the fact that $Z_g = 1 + \mathcal{O}(g_R^2)$ to drop the Z_g from the vertex factor since we are working only to third order and $Z_g^3 = 1 + \mathcal{O}(g_R^6)$.

Using the Feynman parametrisation we can rewrite the integrand:

$$\frac{i}{\ell^2 - m_R^2}\frac{i}{(p_2+\ell)^2 - m_R^2}\frac{i}{(p_1-\ell)^2 - m_R^2} = \int \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \tag{42.1.9}$$
$$\times\delta(1-x_1-x_2-x_3)2[x_1(p_1-\ell)^2 + x_2(p_2+\ell)^2 + x_3\ell^2 - m_R^2]^{-3}.$$

Here we've collected the mass terms to get $(x_1+x_2+x_3)m_R^2$ and then used the Dirac delta to take $x_1+x_2+x_3 = 1$. The factor of 2 comes from $\Gamma(1+1+1)/\Gamma(1)^3 = 2! = 2$. Note that now x_i are Feynman parameters, not positions, and run from 0 to 1. For compactness define the measure

$$\mathrm{d}F_3 = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,2\delta(1-x_1-x_2-x_3) \tag{42.1.10}$$

Expanding the term in the square brackets we get

$$X = x_1(p_1-\ell)^2 + x_2(p_2+\ell)^2 + x_3\ell^2 - m_R^2 \tag{42.1.11}$$
$$= (x_1+x_2+x_3)\ell^2 - 2(x_1p_1 + x_2p_2)\cdot\ell + x_1p_1^2 + x_2p_2^2 - m_R \tag{42.1.12}$$
$$= \ell^2 - 2(x_1p_1 - x_2p_2)\cdot\ell + x_1p_1^2 + x_2p_2^2 - m_R \tag{42.1.13}$$

where we've again used the Dirac delta to set $x_1 + x_2 + x_3 = 1$. Completing the square we get

$$X = [\ell - (x_1p_1 - x_2p_2)]^2 - (x_1p_1 - x_2p_2)^2 + x_1p_1^2 + x_2p_2^2 - m_R^2 \tag{42.1.14}$$
$$= [\ell - (x_1p_1 - x_2p_2)]^2 - x_1^2p_1^2 - x_2^2p_2^2 + 2x_1x_2p_1\cdot p_2$$
$$+ x_1p_1^2 + x_2p_2^2 - m_R^2 \tag{42.1.15}$$
$$= [\ell - (x_1p_1 - x_2p_2)]^2 + x_1(1-x_1)p_1^2 + x_2(1-x_2)p_2^2 \tag{42.1.16}$$
$$+ 2x_1x_2p_1\cdot p_2 - m_R^2. \tag{42.1.17}$$

We want to write this in the form $q^2 - M^2$ as before. To do this we can use the momentum conservation Dirac delta to write $p_1 + p_2 = p_3$, and hence $p_3^2 = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1\cdot p_2$, so $2p_1\cdot p_2 = p_3^2 - p_1^2 - p_2^2$. So we have

$$X = [\ell - (x_1p_1 - x_2p_2)]^2 + x_1(1-x_1)p_1^2 + x_2(1-x_2)p_2^2 \tag{42.1.18}$$
$$+ x_1x_2(p_3^2 - p_1^2 - p_2^2) - m_R^2. \tag{42.1.19}$$

Now using the Feynman parametrisation Dirac delta we can write $1-x_1 = x_2+x_3$. Hence the p_1^2 term after the first term of X is

$$x_1(1-x_1) - 2x_1x_2 = x_1x_2 + x_1x_3 - x_1x_2 = x_1x_3, \tag{42.1.20}$$

similarly $1 - x_2 = x_1 + x_3$ and so the p_2^2 term after the first of X is

$$x_2(1-x_2) - 2x_1x_2 = x_2x_1 + x_2x_3 - 2x_1x_2 = x_2x_3. \tag{42.1.21}$$

Finally, the p_3^2 term is

$$x_1x_2. \tag{42.1.22}$$

Hence,

$$X = [\ell - (x_1p_1 - x_2p_2)]^2 + x_1x_3p_1^2 + x_2x_3p_2^2 + x_1x_2p_3^2 - m_R^2 \tag{42.1.23}$$
$$= q^2 - M^2 \tag{42.1.24}$$

where

$$q^2 = [\ell - (x_1 p_1 - x_2 p_2)]^2, \tag{42.1.25}$$
$$M^2 = m_{\mathrm{R}}^2 - x_1 x_3 p_1^2 - x_2 x_3 p_2^2 - x_1 x_2 p_3^2. \tag{42.1.26}$$

So we are looking to compute

$$iV^{(3,3)} = (ig_{\mathrm{R}})^3 \int \mathrm{d}F_3 \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{i^3}{(q^2 - M^2)^3}. \tag{42.1.27}$$

We now perform a Wick rotation to Euclidean spacetime, setting $q_{\mathrm{E}}^0 = iq_0$ and $\boldsymbol{q}_{\mathrm{E}} = -\boldsymbol{q}$. We pick up an overall minus sign since the exponent, 3, is odd, and we pick up a factor of i from the time component. This factor of $-i$, combined with the factor of i^3 already present from the propagators, cancels to give -1:

$$iV^{(3,3)} = -(ig_{\mathrm{R}})^3 \int \mathrm{d}F_3 \int \frac{\mathrm{d}^D q_{\mathrm{E}}}{(2\pi)^D} \frac{1}{(q_{\mathrm{E}}^2 + M^2)^3}. \tag{42.1.28}$$

In $D = 6$ dimensions the integral goes as $q_{\mathrm{E}}^6/q_{\mathrm{E}}^6$, so we expect at most logarithmic divergence.

42.2 Regularise

We regularise using dimensional regularisation. Set $D = 6 - 2\varepsilon$ and replace the coupling g_{R} with $g_{\mathrm{R}}\tilde{\mu}^{\varepsilon}$, where g_{R} is dimensionless and $\tilde{\mu}$ carries the dimensions of the coupling. We can then use the generalised integral in Equation (39.5.1) to get the result

$$\int \frac{\mathrm{d}^D q_{\mathrm{E}}}{(2\pi)^D} \frac{1}{(q_{\mathrm{E}}^2 + M^2)^3} = \frac{\Gamma(3 - D/2)\Gamma(D/2)}{(4\pi)^{D/2}\Gamma(3)\Gamma(D/2)} (M^2)^{-(3-D/2)}. \tag{42.2.1}$$

So in $D = 6 - 2\varepsilon$ dimensions we have

$$\int \frac{\mathrm{d}^D q_{\mathrm{E}}}{(2\pi)^D} \frac{1}{(q_{\mathrm{E}}^2 + M^2)^3} = \frac{\Gamma(\varepsilon)}{2(4\pi)^{3-\varepsilon}} (M^2)^{-\varepsilon}. \tag{42.2.2}$$

So computing $V^{(3,3)}$ we get the result

$$iV^{(3,3)} = -(ig_{\mathrm{R}}\tilde{\mu}^{\varepsilon})^3 \int \mathrm{d}F_3 \frac{\Gamma(\varepsilon)}{2(4\pi)^{3-\varepsilon}} (M^2)^{-\varepsilon} \tag{42.2.3}$$
$$= -g_{\mathrm{R}}\tilde{\mu}^{\varepsilon} \frac{i^3}{(4\pi)^3} \int \mathrm{d}F_3 \left(\frac{4\pi\tilde{\mu}^2}{M^2}\right)^{\varepsilon} \frac{\Gamma(\varepsilon)}{2}. \tag{42.2.4}$$

Cancelling factors of i we get

$$V^{(3,3)} = g_{\mathrm{R}}\tilde{\mu}^{\varepsilon} \frac{g_{\mathrm{R}}^2}{(4\pi)^3} \int \mathrm{d}F_3 \left(\frac{4\pi\tilde{\mu}^2}{M^2}\right)^{\varepsilon} \frac{\Gamma(\varepsilon)}{2}. \tag{42.2.5}$$

We can expand $\Gamma(\varepsilon)$ as

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon) \tag{42.2.6}$$

and we can expand the other factor in the integrand as

$$\left(\frac{4\pi\tilde{\mu}^2}{M^2}\right)^{\varepsilon} = \exp\left\{\varepsilon\log\left(\frac{4\pi\tilde{\mu}^2}{M^2}\right)\right\} = 1 + \varepsilon\log\left(\frac{4\pi\tilde{\mu}^2}{M^2}\right) + \mathcal{O}(\varepsilon^2). \tag{42.2.7}$$

Combining these the integrand can be expanded as

$$\Gamma(\varepsilon)\left(\frac{4\pi\tilde{\mu}^2}{M^2}\right) = \frac{1}{\varepsilon} + \log\left(\frac{4\pi\tilde{\mu}^2}{\mathrm{e}^{\gamma}M^2}\right) + \mathcal{O}(\varepsilon) \tag{42.2.8}$$

$$= \frac{1}{\varepsilon} + \log\left(\frac{\mu^2}{M^2}\right) + \mathcal{O}(\varepsilon) \tag{42.2.9}$$

where $\mu^2 \coloneqq 4\pi\tilde{\mu}^2/\mathrm{e}^{\gamma}$.

We then have

$$\frac{V^{(3,3)}}{g_{\mathrm{R}}\tilde{\mu}^{\varepsilon}} = \frac{\alpha}{2}\int \mathrm{d}F_3\left[\frac{1}{\varepsilon} - \log\left(\frac{M^2}{\mu^2}\right) + \mathcal{O}(\varepsilon)\right] \tag{42.2.10}$$

where $\alpha = g_{\mathrm{R}}^2/(4\pi)^3$. Here we don't bother to replace the $\tilde{\mu}$ with μ since it's raised to the power of ε and so $\tilde{\mu}^{\varepsilon} \to 1$ as $\varepsilon \to 0$ and we don't need to worry that we now have two (related) scales. Written in this form all of the divergence is in the $1/\varepsilon$ term and all of the μ dependence is in the log.

Now we add in the counter terms. The only three point counter term is $ig_{\mathrm{R}}\delta_g$, and so we have

$$\frac{V^{(3,3)}}{g_{\mathrm{R}}\tilde{\mu}^{\varepsilon}} + \text{counterterms} = \delta_g + \frac{\alpha}{2}\int \mathrm{d}F_3\left(\frac{1}{\varepsilon} - \log\left(\frac{M^2}{\mu^2}\right) + \mathcal{O}(\varepsilon)\right) \tag{42.2.11}$$

$$= \delta_g + \frac{\alpha}{2}\int \mathrm{d}F_3\left(\frac{1}{\varepsilon} - \log\left(\frac{M^2}{m_{\mathrm{R}}^2}\right) - \log\left(\frac{m_{\mathrm{R}}^2}{\mu^2}\right) + \mathcal{O}(\varepsilon)\right). \tag{42.2.12}$$

The $1/\varepsilon$ and $\log(m_{\mathrm{R}}^2/\mu^2)$ terms have no x dependence, so we can easily perform the integral over these terms since

$$\int \mathrm{d}F_3 = \int \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\; 2\delta(1 - x_1 - x_2 - x_3) = 1 \tag{42.2.13}$$

and we have

$$\frac{V^{(3,3)}}{g_{\mathrm{R}}\tilde{\mu}^{\varepsilon}} + \text{counterterms} = \delta_g + \frac{\alpha}{2}\left(\frac{1}{\varepsilon} - \log\left(\frac{m_{\mathrm{R}}^2}{\mu^2}\right)\right) - \frac{\alpha}{2}\int \mathrm{d}F_3\,\log\left(\frac{M^2}{m_{\mathrm{R}}^2}\right). \tag{42.2.14}$$

We can now choose δ_g to get rid of the divergence, $1/\varepsilon$, and scale dependence, μ:

$$\delta_g = -\frac{\alpha}{2}\left[\frac{1}{\varepsilon} - \log\left(\frac{m_{\mathrm{R}}^2}{\mu^2}\right)\right] \tag{42.2.15}$$

To do this we introduce the renormalisation condition that

$$V^{(3,3)}(0,0,0) = 0 \tag{42.2.16}$$

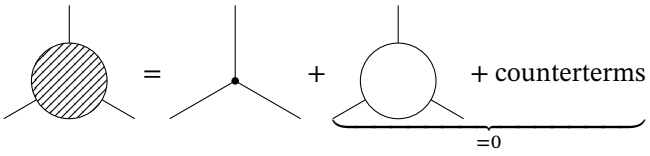
where we're considering $V^{(3,3)}(p_1, p_2, p_3)$ as a function of the external particle's momenta. Combining this with the two other renormalisation conditions of the

on-shell scheme we have three renormalisation conditions, which is enough to fully specify the three renormalised quantities φ_{R}, m_{R}, and g_{R}. We then have

$$\frac{1}{g_{\mathrm{R}}\tilde{\mu}^{\varepsilon}} V^{(3,3)}(0,0,0) = -\frac{\alpha}{2} \int \mathrm{d}F_3 \, \log(1) \tag{42.2.17}$$

which is zero as required.

Diagrammatically we have



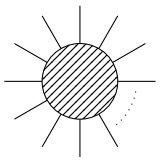
$$+ \text{ counterterms} = ig_{\mathrm{R}}. \tag{42.2.18}$$

Forty-Three

Systematic Approach to Divergences

43.1 Degree of Divergence

Consider the n point correlator:



(43.1.1)

This will always take the form

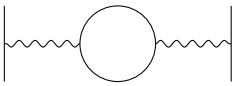
$$\tilde{G}^{(n)}(p_1,\dots,p_n) = (2\pi)^D \delta(p_1+\cdots+p_n) \int \mathrm{d}^D \ell_1 \cdots \mathrm{d}^D \ell_m \frac{\ell_1^{\alpha_1}\cdots\ell_m^{\alpha_m}}{\ell_1^2\cdots\ell_m^2} \quad (43.1.2)$$

where we neglect terms in the numerator and denominator not maximising the power of the integration variables, ℓ_i. The **superficial degree of divergence**, or simply the **degree of divergence**, for a particular diagram, γ, is defined as

$$\delta_\gamma \coloneqq (\text{power of } \ell \text{ in numerator, including } \mathrm{d}^D\ell) - (\text{power of } \ell \text{ in denominator}).$$

If $\delta_\gamma > 0$ then we expect that the diagram is divergent. However, this isn't a hard and fast rule, since there can be other effects that this naive power counting approach doesn't account for, this is what the "superficial" part of superficial degree of divergence is for.

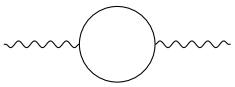
We have already seen that the superficial degree of divergence for $\Pi(p^2)$ predicts a quadratic divergence, and this is indeed the case. It is possible for the superficial degree of divergence to both under and over count the actual degree of divergence. For example, the QED diagram



(43.1.3)

in $D = 4$ dimensions has degree of divergence $4 - 6 = -2$, since the two internal photons give a factor of $1/\ell^2$ each and the two two electrons in the loop give $1/\ell$ each for a total of $1/\ell^6$, and then the $\mathrm{d}^4\ell$ factor gives ℓ^4. So the superficial degree of

divergence predicts it converges. However, it is actually logarithmically divergent. This is due to the presence of the divergent subdiagram



(43.1.4)

which has superficial degree of divergence $4 - 2 = 2$, having only two internal electrons, but is actually only logarithmically divergent as the Ward identities can be used to cancel some of the powers of ℓ.

43.2 Degree of Divergence in φ^3 Theory

Consider a diagram with E external lines, P propagators, L loops, I internal lines, and V vertices. Immediately from this we see that $P = E + I$, and we've already seen that in φ^3 theory $E = 2P - 3V$, which we can rewrite as $E = 3V - 2I$.

The number of loops, L, gives the total number of integrals appearing in the final result. The number of internal lines, I, gives the number of integrals appearing in the expression before we compute the integrals we can do. The number of vertices, V, gives the number of Dirac deltas appearing in the expression, and since we want a final result with one Dirac delta, $\delta(p_1 + \cdots + p_n)$, we can use these to do $V - 1$ integrals, and so we have

$$L = I - V + 1. \tag{43.2.1}$$

The superficial degree of divergence in φ^3 theory in D dimensions is

$$\delta_\gamma = DL - 2I, \tag{43.2.2}$$

with DL being the number of powers coming from L integrals with measure $\mathrm{d}^D\ell$. In φ^3 theory there can be no other terms appearing in the numerator Then I is the number of factors of $1/\ell^2$ coming from each internal propagator, which is the only way for terms to enter the denominator. Using the formulae we've seen so far we can rewrite this as

$$\begin{aligned}
\delta_\gamma &= D(I - V + 1) - 2I && (43.2.3)\\
&= I(D-2) - VD + D && (43.2.4)\\
&= (D-2)\left(\frac{3V}{2} - \frac{E}{2}\right) - VD + D && (43.2.5)\\
&= D - V\left(3\frac{D-2}{2} - D\right) - \frac{D-2}{2}E. && (43.2.6)
\end{aligned}$$

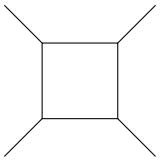
So far this holds in φ^3 theory, and any other theory in which there is no momentum dependence for the vertices, we can now specialise to φ^3 theory. In φ^3 theory the field has mass dimension $[\varphi] = (D-2)/2$ and the coupling constant has mass dimension $[g] = 3(D-2)/2 - D$, which is such that in $D = 6$ we have $[g] = 0$. Hence

$$\delta_\gamma = D - [g]V + [\varphi]E. \tag{43.2.7}$$

Specialising further to $D = 6$ we have

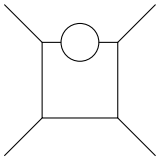
$$\delta_\gamma = 6 - 2E. \tag{43.2.8}$$

Notice that this depends only on the number of external particles. So we can add internal propagators without changing the superficial degree of divergence. For example, consider the diagram



(43.2.9)

which has degree of divergence $\delta_\gamma = 6 - 2 \cdot 4 = -2$, and does indeed converge as this suggests. However, by adding a loop we get the diagram



(43.2.10)

which also has degree of divergence $\delta_\gamma = 6-2\cdot 4 = -2$, however it has a divergent subdiagram,

(43.2.11)

and so this diagram actually diverges. This divergence vanishes after renormalising as then this subdiagram is convergent.

Notice that in general dimension increasing E while keeping V fixed reduces the degree of divergence.

If $[g] > 0$ then there are only a finite number of diagrams with positive degree of divergence. Theories with only a finite number of divergent diagrams are called **super-renormalisable**.

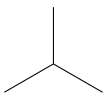
If $[g] = 0$ then at every order in perturbation theory the same correlators are divergent. For example in φ^3 theory with $D = 6$ only the 2 and 3 point correlators are divergent. These divergent diagrams can occur as subdiagrams, meaning that every divergent diagram is either a 2 or 3 point correlator or contains a 2 or 3 point correlator as a subdiagram. In this case all divergences can be reabsorbed by imposing a finite number of of renormalisation conditions, 3 in the case of φ^3 theory renormalising φ, m, and g. Theories in which this is possible are called **renormalisable**.

If $[g] < 0$ then all correlators have divergences for sufficiently large V and we call the theory **non-renormalisable**.

43.3 1PI Vertices

A **1PI vertex** is a 1PI diagram without external propagators or the usual momentum conserving factor $(2\pi)^D \delta(p_1 + \cdots p_n)$. Define $iV^{(n)}(p_1, \dots, p_n)$ to be the sum of all 1PI diagrams with $n = E$ external lines after removing the external propagators and the Dirac delta.

In φ^3 theory the only tree level contribution is $V^{(3)}$, formed from the diagram



(43.3.1)

Any other combination of vertices at tree level will produce a diagram which is not 1PI.

At one loop in φ^3 theory we have $L = 1$, and $L = I - V + 1$, so $I = V$, giving $E = 3V - 2I = V = I$. With two vertices, two external lines, and two internal lines we have

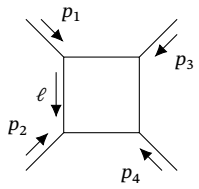
$$iV^{(2)} = \tag{43.3.2}$$

and with three vertices, three external lines, and three internal lines we have

$$iV^{(2)} = \tag{43.3.3}$$

These have degree of divergence $\delta_\gamma = D - 2E$, in $D = 6$ we have $\delta_\gamma \geq 0$ only for $E = 2, 3$, and all other 1PI diagrams are "finite" in the sense that the superficial degree of divergence is less than zero.

For $n = 4$ the first 1PI diagram is



(43.3.4)

Then $iV^{(4)}$, to leading order, is given by the contribution of this diagram, plus the contribution of this diagram with p_2 and p_4 exchanged, plus the contribution of this diagram with p_3 and p_4 exchanged:

$$\begin{aligned} iV^{(4)} = (ig_{\mathrm{R}})^4 \int \frac{\mathrm{d}^D\ell}{(2\pi)^D} \frac{i}{\ell - m_{\mathrm{R}}^2} \frac{i}{(\ell + p_2)^2 - m_{\mathrm{R}}^2} \frac{i}{(\ell + p_2 + p_4)^2 - m_{\mathrm{R}}^2} \\ \times \frac{i}{(\ell - p_1)^2 - m_{\mathrm{R}}^2} + (p_2 \leftrightarrow p_4) + (p_3 \leftrightarrow p_4) + \mathcal{O}(g^6). \end{aligned} \tag{43.3.5}$$

Here we've dropped the Z_g in the $(iZ_g g_{\mathrm{R}})^4$ as it contributes at a higher order. We've also used $p_2 + p_3 + p_4 = -p_1$, which follows from momentum conservation, to simplify the denominator in the final fraction.

Going through the process of doing the Feynman parametrisation and Wick transformation we get terms of the form

$$\int \frac{\mathrm{d}^D q_{\mathrm{E}}}{(2\pi)^D} \frac{1}{(q_{\mathrm{E}}^2 + M^2)^4} = \frac{1}{(2\pi)^D} \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty \mathrm{d}q_{\mathrm{E}} \frac{q_{\mathrm{E}}^5}{(q_{\mathrm{E}}^2 + M^2)^4} \tag{43.3.6}$$

$$\overset{D=6}{=} \frac{1}{6(4\pi)^3} \frac{1}{M^2}, \tag{43.3.7}$$

which is finite.

43.4 Correlator Dimensions

The n point correlator in position space has mass dimension

$$[G^{(n)}(x_1, \ldots, x_n)] = [\langle \varphi(x_1) \cdots \varphi(x_n) \rangle] = n[\varphi]. \tag{43.4.1}$$

In momentum space the mass dimension is then

$$[\tilde{G}^{(n)}(p_1, \dots, p_n)] = \left[\int \mathrm{d}^D x_1 \cdots \mathrm{d}^D x_n \, G^{(n)}\right] = n([\varphi] - D). \tag{43.4.2}$$

The n point 1PI vertex has mass dimension

$$[V^{(n)}(p_1, \dots, p_n)] = n[\varphi] - nD + D + 2n = D - n[\varphi], \tag{43.4.3}$$

this comes from taking the n point correlator, with mass dimension $n[\varphi] - nD$, and cancelling the external propagators, each proportional to $1/p^2$, and the last equality follows using $[\varphi] = (D-2)/2$.

Now consider a theory with n particle interactions with coupling g_n. Then $[g_n] + n[\varphi] = D$ and $[g_n] = [V^{(n)}]$. We also have

$$[V^{(n)}] = DL - 2I + \sum_m V_m [g_m] = \delta_\gamma + \sum_m V_m [g_m] \tag{43.4.4}$$

where V_m is the number of m point vertices. Rearranging this the superficial degree of divergence is

$$\delta_\gamma = [V^{(n)}] - \sum_m V_m [g_m]. \tag{43.4.5}$$

For example, consider $V^{(4)}$, two contributions to this come from

(43.4.6)

and the diagram in Equation (43.3.4). We have $[g_4] = 4 = D = -2$ in $D = 6$, and $[g_3] = 0$ as usual in φ^3 theory. The diagram in Equation (43.3.4) then has degree of divergence $4 - D - 4(3 - D/2) = -2$ in $D = 6$. So this diagram is superficially convergent.

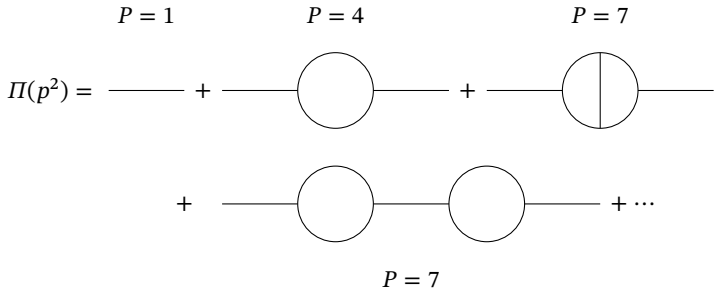
43.5 Higher Orders

We can consider 1PI contributions to the two point correlator, $\Pi(p^2)$, and three point vertex, $V^{(3)}$. To do so we adjust Z, Z_m, and Z_g to satisfy the renormalisation conditions.

For general vertices we can use the **skeleton expansion**. This expansion is defined as

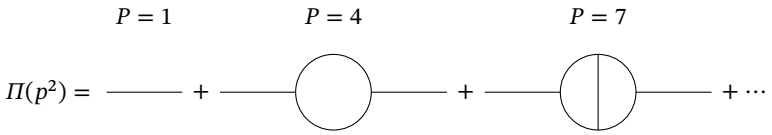
- sum over 1PI diagrams
- omitting diagrams which are just corrections to $\Pi(p^2)$ or $V^{(3)}$,
- replacing all vertices and propagators with $V^{(3)}$ and Δ_{F} respectively.

Starting with the $E = 2$ case, which is just $\Pi(p^2)$, we have $E = 2P - 3V$. A naive sum of diagrams would be



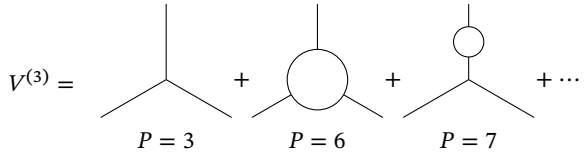
(43.5.1)

However, the last diagram written here is not 1PI, so we discard it to get



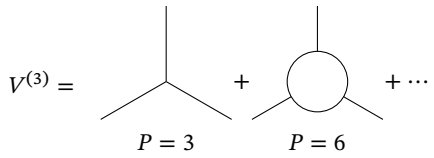
(43.5.2)

The $E = 3$ case is just $V^{(3)}$, and a naive sum of diagrams is



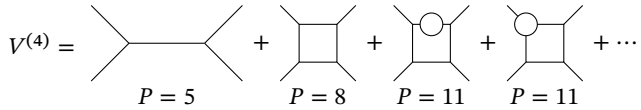
(43.5.3)

The last diagram is not 1PI, so we discard it giving



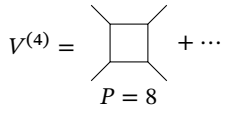
(43.5.4)

Now consider the $E = 4$ case, which is $V^{(4)}$. A naive sum of diagrams is



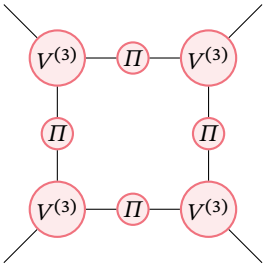
(43.5.5)

The first diagram here is not 1PI, the third and fourth are just one loop corrections to the propagator and the vertex respectively. So we are left with just



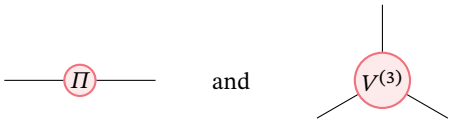
(43.5.6)

The final step is to replace each propagator with $\Pi(p^2)$, which is Δ_{F} to second order, and each vertex with $V^{(3)}$, giving

$V^{(4)} =$ 

(43.5.7)

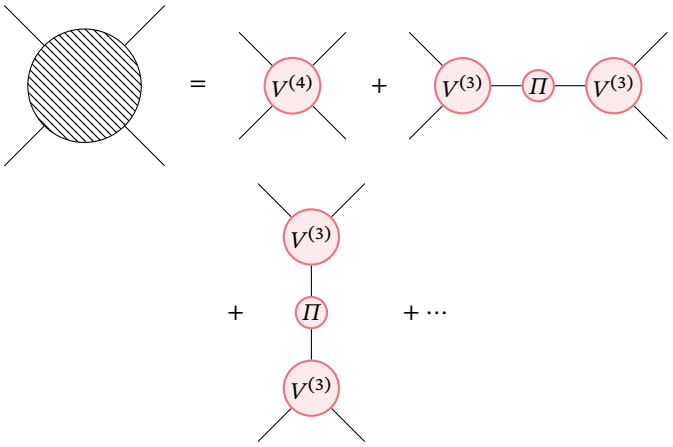
here we interpret



(43.5.8)

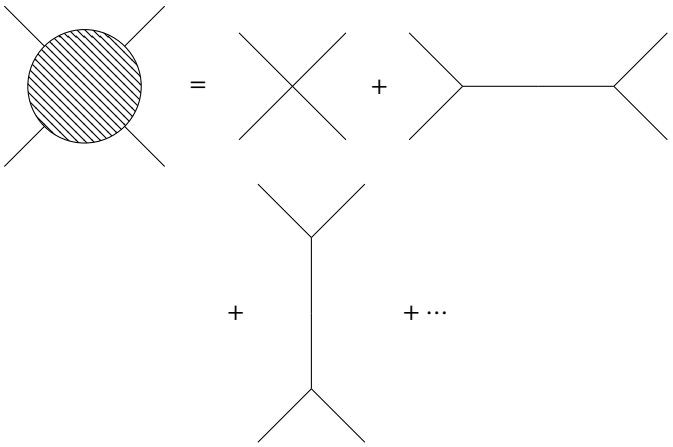
as a propagator $\Pi(p^2)$ and a vertex with vertex term $V^{(3)}$. Note that the propagator is not supposed to be a vertex, it's just a line with a label.

Using this result we can write a general four-point correlator as



(43.5.9)

which is just the usual expansion,



(43.5.10)

with n-point interactions and replacing the propagators with $\Pi(p^2)$ and the n point vertices with $V^{(n)}$.

Forty-Four

More on Renormalisation Schemes

44.1 $\overline{\text{MS}}$ Scheme

Where $\overline{MS}$ stands for minimal subtraction ... bar.

Luigi Del Debbio

Consider the regularised, but not renormalised,

$$\Pi(p^2) = \frac{\alpha_R}{2}\left[\left(\frac{1}{\varepsilon}+1\right)\left(\frac{1}{6}p^2 - m_R^2\right) + \int_0^1 \mathrm{d}x\, M^2 \log\left(\frac{M^2}{\mu^2}\right)\right] + \delta_Z p^2 - \delta_{m^2} m_R^2 + \mathcal{O}(\alpha^2). \quad (44.1.1)$$

In the on-shell scheme the renormalisation conditions are $\Pi(m_R^2) = 0$, and $\Pi'(m_R^2) = 0$. These are enforced by setting $m_R = m_{\text{phys}}$, and requiring that the residue has a value of 1. As a result we get the following term appearing in $\Pi'(p^2)$

$$-\int_0^1 \mathrm{d}x\, x(1-x)\log\left(\frac{M^2}{\mu^2}\right). \quad (44.1.2)$$

This is singular when $m_R^2 = m_{\text{phys}}^2 = 0$.

We can avoid this singularity by choosing a different renormalisation scheme, known as the **$\overline{\text{MS}}$ scheme**. This scheme is not based on physical requirements, like the on-shell scheme, instead we simply choose counter terms to cancel the ε dependence. We will denote renormalised quantities in the $\overline{\text{MS}}$ scheme with bars. We choose the counter terms

$$\delta_Z = -\frac{\bar{\alpha}}{12}\frac{1}{\varepsilon}, \qquad \text{and} \qquad \delta_{m^2} = -\frac{\bar{\alpha}}{2}\frac{1}{\varepsilon}. \quad (44.1.3)$$

Then we have

$$\Pi_{\overline{\text{MS}}}(p^2) = \frac{\bar{\alpha}}{12}(p^2 - 6\bar{m}^2) + \frac{\bar{\alpha}}{2}\int_0^1 \mathrm{d}x\, M^2 \log\left(\frac{M^2}{\mu^2}\right) + \mathcal{O}(\bar{\alpha}^2). \quad (44.1.4)$$

This has the advantage of the integrand being nonsingular for all $x \in [0,1]$, but the disadvantage of not allowing us to remove the μ dependence.

Recall that in the on-shell scheme we had

$$\Pi_{\mathrm{R}}(p^2) = \frac{\alpha}{12}(p^2 - m_{\mathrm{R}}^2) + \frac{\alpha}{2}\int_0^1 \mathrm{d}x\, M^2 \log\left(\frac{M^2}{M_0^2}\right). \tag{44.1.5}$$

This has a pole at $m_{\mathrm{R}}^2 = m_{\mathrm{phys}}^2$ as expected. Note that in general $\Pi_{\overline{\mathrm{MS}}}(p^2) \neq \Pi_{\mathrm{R}}(p^2)$. That is, $\Pi(p^2)$ is scheme dependent.

We need to work out the relation between the $\overline{\mathrm{MS}}$ mass, $\bar{m}$, and the physical mass, m_{phys}. The propagator should have a pole when $p^2 = m_{\mathrm{phys}}^2$. This means that we should have

$$m_{\mathrm{phys}}^2 - \bar{m}^2 - \Pi_{\overline{\mathrm{MS}}}(m_{\mathrm{phys}}^2) = 0, \tag{44.1.6}$$

since this appears in the denominator of the expression for the full propagator as seen in Equation (40.3.9). More precisely, since we are working in perturbation theory we actually require that

$$m_{\mathrm{phys}}^2 - \bar{m}^2 = \Pi_{\overline{\mathrm{MS}}}(m_{\mathrm{phys}}^2) = \mathcal{O}(\bar{\alpha}). \tag{44.1.7}$$

This means that differences between m_{phys}^2 and $\bar{m}^2$ are on the order of $\bar{\alpha}$, i.e. $m_{\mathrm{phys}}^2 = \bar{m}^2 + \mathcal{O}(\bar{\alpha})$, so we can replace m_{phys}^2 with $\bar{m}^2$ up to order $\bar{\alpha}$. Thus, we have that

$$m_{\mathrm{phys}}^2 = \bar{m}^2 + \Pi_{\overline{\mathrm{MS}}}(\bar{m}^2) + \mathcal{O}(\bar{\alpha}) \tag{44.1.8}$$

$$= \bar{m}^2 - \frac{\bar{\alpha}}{2}\left[\frac{1}{6}\bar{m}^2 - \bar{m}^2 + \int_0^1 \mathrm{d}x\, M_0^2 \log\left(\frac{M_0^2}{\mu^2}\right)\right] + \mathcal{O}(\bar{\alpha}^2). \tag{44.1.9}$$

Here we've used $M_0^2 \coloneqq M^2(p^2 = \bar{m}^2, x) = \bar{m}^2(1 - x + x^2)$. We can then split the logarithm as

$$\log\frac{M_0^2}{\mu^2} = \log\frac{\bar{m}^2}{\mu^2} + \log(1 - x + x^2). \tag{44.1.10}$$

The first term is then independent of the integration variable, x, so we can perform the integral over this term. The second term is simply an integral with no physical parameters, so just gives a numerical constant. Hence, this result can be written as

$$m_{\mathrm{phys}}^2 = \bar{m}^2\left[1 + \frac{5}{12}\bar{\alpha}\left(\log\frac{\mu^2}{\bar{m}^2} + C'\right) + \mathcal{O}(\bar{\alpha}^2)\right]. \tag{44.1.11}$$

The left hand side, m_{phys}^2, is a physical quantity, so must be scale independent. This means that the right hand side must also be scale independent. From this we can deduce that $\bar{m}$ and $\bar{\alpha}$ depend on μ in such a way that the right hand side doesn't depend on μ. That the coupling, $\bar{\alpha}$, and mass depend on the scale is what we call the **running of coupling constants**.

44.1.1 Running of Mass

We make use of the log derivative

$$\frac{\mathrm{d}}{\mathrm{d}(\log\mu)} = \mu\frac{\mathrm{d}}{\mathrm{d}\mu} \tag{44.1.12}$$

to evaluate the μ dependence of quantities. This is useful as this is a dimensionless operator, and as such is scale invariant, meaning any scale dependence enters only through the dependence of the original quantity. We also take logs before taking derivatives.

Applying this to the left hand side we have

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\log(m_{\text{phys}}^2) = \frac{1}{2}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\log m_{\text{phys}} = 0. \tag{44.1.13}$$

Taking logs of the right hand side we get

$$\begin{aligned}\frac{1}{2}\log m_{\text{phys}} &= \log(\bar{m}^2) + \log\left[1 + \frac{5}{12}\bar{\alpha}\left(\log\frac{\mu^2}{\bar{m}^2} + C'\right) + \mathcal{O}(\bar{\alpha}^2)\right] && (44.1.14)\\ &\approx \frac{1}{2}\log\bar{m} + \frac{5}{12}\bar{\alpha}\left(\log\frac{\mu^2}{\bar{m}^2} + C'\right) + \mathcal{O}(\bar{\alpha}^2) && (44.1.15)\\ &= \frac{1}{2}\log\bar{m} + \frac{5}{12}\bar{\alpha}\left(\frac{1}{2}\log\frac{\mu}{\bar{m}} + C'\right) + \mathcal{O}(\bar{\alpha}^2) && (44.1.16)\end{aligned}$$

having used $\log(1+x) \approx x$. So, we have

$$\log m_{\text{phys}} = \log\bar{m} + \frac{5}{12}\bar{\alpha}\left(\log\frac{\mu}{\bar{m}} + 2C'\right) + \mathcal{O}(\bar{\alpha}^2). \tag{44.1.17}$$

Taking the log derivative we get

$$\begin{aligned}0 &= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\left[\log\bar{m} + \frac{5}{12}\bar{\alpha}\left(\log\frac{\mu}{\bar{m}} + 2C'\right) + \mathcal{O}(\bar{\alpha}^2)\right] && (44.1.18)\\ &= \frac{1}{\bar{m}}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{m} + \frac{5}{12}\bar{\alpha} - \frac{5}{12}\frac{1}{\bar{m}}\bar{\alpha}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{m} + \frac{5}{12}\log\left(\frac{\mu}{\bar{m}}\right)\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{\alpha} && (44.1.19)\\ &= \frac{1}{\bar{m}}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{m} + \frac{5}{12}\bar{\alpha} + \mathcal{O}(\bar{\alpha}^2). && (44.1.20)\end{aligned}$$

Here we've identified that terms like

$$\bar{\alpha}\frac{1}{\bar{m}}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{m}, \qquad \text{and} \qquad \bar{m}\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{\alpha} \tag{44.1.21}$$

are actually second order in $\bar{\alpha}$.

This tells us that

$$\frac{1}{\bar{m}}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{m} = -\frac{5}{12}\bar{\alpha} + \mathcal{O}(\bar{\alpha}^2). \tag{44.1.22}$$

We call the quantity on the left the **mass anomalous dimension**, $\gamma_m(\bar{\alpha})$. This result is it's value in a one-loop calculation. We'll see more anomalous dimensions and discuss their interpretation later. Note that the definition

$$\gamma_m(\bar{\alpha}) \coloneqq \frac{1}{\bar{m}}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{m} \tag{44.1.23}$$

is non-perturbative and can be extended to any renormalisation scheme. The result above however is only valid perturbatively in the $\overline{\text{MS}}$ scheme.

44.2 Generic Scheme

Consider a generic renormalisation scheme. The bare action, $S[\varphi]$, contains divergences. We introduce a regulator, be it a cut off scale, Λ, or a parameter like ε in dimensional regularisation. We renormalise quantities, defining

$$\varphi = Z^{1/2}\varphi_{\mathrm{R}}, \qquad m^2 = \frac{Z_{m^2}}{Z} m_{\mathrm{R}}^2, \qquad \text{and} \qquad g = \frac{Z_g}{Z^{3/2}} g_{\mathrm{R}}. \tag{44.2.1}$$

In general Z will be a function of the regularised quantities m_{R} and g_{R}, as well as some scale, μ, and some dimensionless parameters, which we'll collectively call ε. We'll assume that we use a cut-off, Λ, which will be a large value with dimensions of energy. Then we can consider Z as a function of dimensionless parameters:

$$Z = Z(g_{\mathrm{R}}, m_{\mathrm{R}}/\mu, \mu/\Lambda, \varepsilon). \tag{44.2.2}$$

We can compute the correlators in the bare regulated theory in the usual way:

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle_{\Lambda, g_{\mathrm{R}}, m_{\mathrm{R}}} = \frac{1}{\mathcal{Z}[0]} \int_{\Lambda} \mathcal{D}\varphi \, \mathrm{e}^{iS[\varphi]} \varphi(x_1) \cdots \varphi(x_n). \tag{44.2.3}$$

Here the subscripts on the correlator remind us of the variables it depends on. The quantity $\mathcal{Z}[0]$ is the generating functional in the absence of sources, normally denoted $Z[0]$, but we're now using Z for renormalisation values. The subscript Λ on the path integral reminds us that we are using a cut-off value, Λ.

We can write the correlator in terms of the regulated values as follows:

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle_{\Lambda, g, m} \tag{44.2.4}$$

$$= Z(g_{\mathrm{R}}, m_{\mathrm{R}}/\mu, \mu/\Lambda)^{n/2} \frac{1}{\mathcal{Z}[0]} \int \mathcal{D}\varphi_{\mathrm{R}} \, \mathrm{e}^{iS_{\mathrm{R}}[\varphi_{\mathrm{R}}]} \varphi_{\mathrm{R}}(x_1) \cdots \varphi_{\mathrm{R}}(x_n) \tag{44.2.5}$$

$$= Z(g_{\mathrm{R}}, m_{\mathrm{R}}/\mu, \mu/\Lambda)^{n/2} \langle \varphi_{\mathrm{R}}(x_1) \cdots \varphi_{\mathrm{R}}(x_n) \rangle_{\mu, g_{\mathrm{R}}, m_{\mathrm{R}}}. \tag{44.2.6}$$

Here $\langle \varphi_{\mathrm{R}}(x_1) \cdots \varphi_{\mathrm{R}}(x_n) \rangle_{\mu, g_{\mathrm{R}}, m_{\mathrm{R}}}$ is the renormalised correlator. It contains the necessary counter terms to make it finite.

To remove the cut-off we take the $\Lambda \to \infty$ limit,

$$\lim_{\Lambda\to\infty} Z(g_{\mathrm{R}}, m_{\mathrm{R}}/\mu, \mu/\Lambda)^{-n/2} \langle \varphi(x_1) \cdots \varphi(x_n) \rangle_{\Lambda, g, m} = \lim_{\Lambda\to\infty} \langle \varphi_{\mathrm{R}}(x_1) \cdots \varphi_{\mathrm{R}}(x_n) \rangle_{\mu, g_{\mathrm{R}}, m_{\mathrm{R}}}. \tag{44.2.7}$$

The correlator on the right is finite and has no Λ dependence, so this limit should also be finite. In practice we don't usually take this limit but just fix Λ to be some value much larger than all other relevant energy scales.

The left hand side above is independent of μ, so

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \langle \varphi(x_1) \cdots \varphi(x_n) \rangle_{\Lambda, g, m} = 0. \tag{44.2.8}$$

The right hand side must also be independent of μ as well. Using this and applying the chain rule we have

$$0 = \left[\mu \frac{\partial}{\partial \mu} + \left(\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g_{\mathrm{R}} \right) \frac{\partial}{\partial g_{\mathrm{R}}} + \frac{1}{m_{\mathrm{R}}} \left(\mu \frac{\mathrm{d}}{\mathrm{d}\mu} m_{\mathrm{R}} \right) m_{\mathrm{R}} \frac{\partial}{\partial m_{\mathrm{R}}} \right] Z(g_{\mathrm{R}}, m_{\mathrm{R}}/\mu, \mu/\Lambda)^{-n/2} \langle \varphi_{\mathrm{R}}(x_1) \cdots \varphi_{\mathrm{R}}(x_n) \rangle_{\mu, m_{\mathrm{R}}, g_{\mathrm{R}}}. \tag{44.2.9}$$

This is a **renormalisation group** equation.

The analysis so far in this section has been completely nonperturbative, so long as we can compute the correlators nonperturbatively. We need perturbation theory to compute g_R and m_R. We introduced the anomalous mass dimension previously,

$$\gamma_m(\bar{\alpha}) \coloneqq \frac{1}{m_R}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}m_R, \tag{44.2.10}$$

and we now introduce

$$\beta(\bar{\alpha}) \coloneqq \mu\frac{\mathrm{d}}{\mathrm{d}\mu}g_R. \tag{44.2.11}$$

We can write the renormalised couplings as functions of the bare couplings, and then we get

$$\langle\varphi_R(x_1)\cdots\varphi_R(x_n)\rangle_{\mu,g_R,m_R} = Z(g, m/\Lambda, \mu/\Lambda)^{-n/2}\langle\varphi(x_1)\cdots\varphi(x_n)\rangle_{\Lambda,m,g}. \tag{44.2.12}$$

Notice that we have now changed to m/Λ instead of m/μ. It doesn't matter which of these we use, with three quantities with dimensions of mass, m, μ, and Λ, we can make two independent dimensionless quantities, exactly which two we use is just a choice of basis in parameter space. Using this we can get the bare couplings as functions of Λ such that the renormalised correlator is constant, that is independent of Λ:

$$\Lambda\frac{\mathrm{d}}{\mathrm{d}\Lambda}\langle\varphi_R(x_1)\cdots\varphi_R(x_n)\rangle_{\mu,g_R,m_R} = 0 + \mathcal{O}(1/\Lambda, \varepsilon). \tag{44.2.13}$$

Any dependence on Λ or ε will be power suppressed.

44.2.1 $\overline{\text{MS}}$ Scheme

> *They got the Nobel prize for this, which I would never get because I would get the sign wrong.*
>
> Luigi Del Debbio

Now we go back to working in the $\overline{\text{MS}}$ scheme and dimensional regularisation. We can expand Z as a power series in $\bar{\alpha}$ and ε:

$$Z = 1 + \sum_{n=1}^{\infty}\frac{a_n(\bar{\alpha})}{\varepsilon^n} \tag{44.2.14}$$

where $a_n(\bar{\alpha})$ are polynomials in $\bar{\alpha}$. We saw that with the choice of the counter terms as in Equation (44.1.3) we have

$$a_1(\bar{\alpha}) = -\frac{\bar{\alpha}}{12} + \mathcal{O}(\bar{\alpha}^2). \tag{44.2.15}$$

These counter terms followed from a one-loop calculation, hence the $\mathcal{O}(\bar{\alpha}^2)$.

Similarly, we can expand Z_{m^2} in a power series:

$$Z_{m^2} = 1 + \sum_{n=1}^{\infty}\frac{b_n(\bar{\alpha})}{\varepsilon^n} \tag{44.2.16}$$

where $b_n(\bar{\alpha})$ are polynomials in $\bar{\alpha}$. Again, our choice of counter terms gives

$$b_1(\bar{\alpha}) = -\frac{\bar{\alpha}}{2} + \mathcal{O}(\bar{\alpha}^2). \tag{44.2.17}$$

We can expand Z_g in a power series:

$$Z_g = 1 + \sum_{n=1}^{\infty} \frac{c_n(\bar{\alpha})}{\varepsilon^n}. \tag{44.2.18}$$

Renormalising the three point vertex in the $\overline{\text{MS}}$ scheme we would find that

$$c_1(\bar{\alpha}) = -\frac{\bar{\alpha}}{2} + \mathcal{O}(\bar{\alpha}^2). \tag{44.2.19}$$

The bare coupling is given by

$$\alpha = \frac{g^2}{(4\pi)^3} = Z_g^2 Z^{-3} \mu^{2\varepsilon} \frac{g_{\text{R}}^2}{(4\pi)^3} = Z_g^2 Z^{-3} \mu^{2\varepsilon} \bar{\alpha}. \tag{44.2.20}$$

The bare coupling should be independent of μ, so

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \log \alpha = 0. \tag{44.2.21}$$

Define

$$\log(Z_g^2 Z^{-3}) =: G(\bar{\alpha}, \varepsilon) = \sum_{n=1}^{\infty} \frac{G_n(\bar{\alpha})}{\varepsilon^n} \tag{44.2.22}$$

where we've used

$$Z_g^2 Z^{-3} = \left(1 + \sum_{k=1}^{\infty} \frac{c_k(\bar{\alpha})}{\varepsilon^k}\right)^2 \left(1 + \sum_{n=1}^{\infty} \frac{a_n(\bar{\alpha})}{\varepsilon^n}\right)^{-3} \tag{44.2.23}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{G_n(\bar{\alpha})}{\varepsilon^n} \tag{44.2.24}$$

where $G_n(\bar{\alpha})$ are polynomials in α formed from products of $a_k(\bar{\alpha})$ and $c_\ell(\bar{\alpha})$. We can then take the log to get

$$\log(Z_g^2 Z^{-3}) = \log\left(1 + \sum_{n=1}^{\infty} \frac{G_n(\bar{\alpha})}{\varepsilon^n}\right) \approx \sum_{n=1}^{\infty} \frac{G_n(\bar{\alpha})}{\varepsilon^n}. \tag{44.2.25}$$

Expanding to first order in $\bar{\alpha}$ we get

$$G_1(\bar{\alpha}) = 2a_1(\bar{\alpha}) - 3c_1(\bar{\alpha}) = -\frac{3}{4}\bar{\alpha} + \mathcal{O}(\bar{\alpha}^2). \tag{44.2.26}$$

Now consider

$$0 = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \log \alpha \tag{44.2.27}$$

$$= \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \log(Z_g^2 Z^{-3} \mu^{2\varepsilon} \bar{\alpha}) \tag{44.2.28}$$

$$= \mu \frac{\mathrm{d}}{\mathrm{d}\mu} [\log \bar{\alpha} + \log(Z_g^2 Z^{-3}) + 2\varepsilon \log(\mu)] \tag{44.2.29}$$

$$\approx \frac{1}{\bar{\alpha}} \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \bar{\alpha} + \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \sum_{n=1}^{\infty} \frac{G_n(\bar{\alpha})}{\varepsilon^n} + 2\varepsilon \tag{44.2.30}$$

$$= \frac{1}{\bar{\alpha}} \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \bar{\alpha} + \left(\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \bar{\alpha}\right) \frac{\partial}{\partial \bar{\alpha}} \sum_{n=1}^{\infty} \frac{G_n(\bar{\alpha})}{\varepsilon^n} + 2\varepsilon. \tag{44.2.31}$$

Rearranging this we have

$$\left(\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{\alpha}\right)\left(1+\frac{\bar{\alpha}G_1'(\bar{\alpha})}{\varepsilon}+\frac{\bar{\alpha}G_2'(\bar{\alpha})}{\varepsilon^2}+\cdots\right)=-2\varepsilon\bar{\alpha}. \tag{44.2.32}$$

The right hand side is $\mathcal{O}(\varepsilon)$. The second bracket on the left hand side has only nonpositive powers of ε, so we make an asatz that the first bracket on the left hand side has an $\mathcal{O}(\varepsilon)$ term, so we can write

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{\alpha}=-2\varepsilon\bar{\alpha}+\beta(\bar{\alpha}) \tag{44.2.33}$$

for some function β. We then have

$$(-2\varepsilon\bar{\alpha}+\beta(\bar{\alpha}))\left(1+\frac{\bar{\alpha}G_1'(\bar{\alpha})}{\varepsilon}+\frac{\bar{\alpha}G_2'(\bar{\alpha})}{\varepsilon^2}+\cdots\right)=-2\varepsilon\bar{\alpha}. \tag{44.2.34}$$

Consider the $\mathcal{O}(\varepsilon^0)$ term, i.e. the constant term with respect to ε, equating coefficients we have

$$-2\bar{\alpha}^2G_1'(\bar{\alpha})+\beta(\bar{\alpha})=0. \tag{44.2.35}$$

Hence,

$$\beta(\bar{\alpha})=-\frac{3}{2}\bar{\alpha}^2+\mathcal{O}(\bar{\alpha}^3). \tag{44.2.36}$$

Now consider the $\mathcal{O}(\varepsilon^{-1})$ term, equating coefficients again we have

$$\bar{\alpha}\beta(\bar{\alpha})G_1'(\bar{\alpha})-2\bar{\alpha}^2G_2'(\bar{\alpha})=0, \tag{44.2.37}$$

so $G_2'(\bar{\alpha})=-3\bar{\alpha}G_1'(\bar{\alpha})/4+\mathcal{O}(\bar{\alpha}^2)$.

Notice that β starts at $\mathcal{O}(\bar{\alpha}^2)$, meaning that previously we have neglected this term when working to $\mathcal{O}(\bar{\alpha})$. For sufficiently small values of $\bar{\alpha}$ as $\bar{\alpha}$ increases $\beta(\bar{\alpha})\sim-\bar{\alpha}^2$ decreases. For larger values of $\bar{\alpha}$ the $\mathcal{O}(\bar{\alpha}^3)$ terms may become important for the behaviour of β. For even larger values of $\bar{\alpha}$ perturbation theory breaks entirely. For large values of μ we have $\bar{\alpha}\to 0$, which is called **asymptotic freedom**, at large enough energy scales there is essentially no coupling, so no interactions and everything behaves as if it were free.

We can follow the same process for the regularised mass, $\bar{m}$, which is related to the bare mass by

$$m=Z_{m^2}^{1/2}Z^{1/2}\bar{m}. \tag{44.2.38}$$

Define

$$M(\bar{\alpha},\varepsilon):=\log(Z_{m^2}^{1/2}Z^{1/2})=\sum_{n=1}^{\infty}\frac{M_n(\bar{\alpha})}{\varepsilon^n}. \tag{44.2.39}$$

A one-loop calculation gives

$$M_1(\bar{\alpha})=\frac{1}{2}b_1(\bar{\alpha})-\frac{1}{2}a_1(\bar{\alpha}). \tag{44.2.40}$$

Since the bare mass is independent of the scale, μ, we have

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\log m = 0. \tag{44.2.41}$$

Hence, we have

$$\begin{aligned}
0 &= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\log(Z_{m^2}^{1/2}Z^{1/2}\bar{m}) && (44.2.42)\\
&= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}[M(\bar{\alpha},\varepsilon)+\log\bar{m}] && (44.2.43)\\
&= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}M + \frac{1}{\bar{m}}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{m} && (44.2.44)\\
&= \frac{\partial M}{\partial\bar{\alpha}}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{\alpha} + \frac{1}{\bar{m}}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{m} && (44.2.45)\\
&= \frac{\partial M}{\partial\bar{\alpha}}(-2\varepsilon\bar{\alpha}-\beta(\bar{\alpha})) + \frac{1}{\bar{m}}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{m}. && (44.2.46)
\end{aligned}$$

Hence, we have

$$\frac{1}{\bar{m}}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{m} = [2\varepsilon\bar{\alpha}-\beta(\bar{\alpha})]\sum_{n=1}^{\infty}\frac{M_n'(\bar{\alpha})}{\varepsilon^n}. \tag{44.2.47}$$

The left hand side has no poles with respect to ε, so we must drop terms on the right hand side which would otherwise give poles, meaning we keep only the terms with positive powers of ε giving

$$\frac{1}{\bar{m}}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{m} = 2\bar{\alpha}M_1'(\bar{\alpha}), \tag{44.2.48}$$

which is $\mathcal{O}(\varepsilon^0)$. The other terms must all cancel out, giving, for example,

$$2\bar{\alpha}M_2'(\bar{\alpha}) - \beta(\bar{\alpha})M_1'(\bar{\alpha}) = 0 \tag{44.2.49}$$

for the $\mathcal{O}(\varepsilon)$ term.

Using

$$M_1(\bar{\alpha}) = \frac{1}{2}b_1(\bar{\alpha}) - \frac{1}{2}a_1(\bar{\alpha}) = -\frac{\bar{\alpha}}{4} + \frac{\bar{\alpha}}{24} + \mathcal{O}(\bar{\alpha}^2) = -\frac{5\bar{\alpha}}{24} + \mathcal{O}(\bar{\alpha}^2) \tag{44.2.50}$$

we get

$$M_1'(\bar{\alpha}) = -\frac{5}{24}. \tag{44.2.51}$$

The anomalous mass dimension is then

$$\gamma_m(\bar{\alpha}) \coloneqq \frac{1}{\bar{m}}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{m} = 2\bar{\alpha}M_1'(\bar{\alpha}) = -\frac{5}{12}\bar{\alpha} + \mathcal{O}(\bar{\alpha}^2). \tag{44.2.52}$$

This is the same result that we got by requiring that the physical mass be independent of the scale.

Now consider the full propagator, $\Delta_{\mathrm{F}}(p^2)$, which can be written in terms of the two point renormalised propagator, $\Delta_{\mathrm{F,R}}(p^2)$ as

$$\Delta_{\mathrm{F}}(p^2) = Z(\bar{\alpha})\Delta_{\mathrm{F,R}}(p^2). \tag{44.2.53}$$

The left hand side is scale invariant,

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\log\Delta_{\mathrm{F}}(p^2)=0, \tag{44.2.54}$$

and so we have

$$\begin{aligned}
0&=\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\log\left(Z(\bar{\alpha})\Delta_{\mathrm{F,R}}(p^2)\right) && (44.2.55)\\
&=\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\log(Z(\bar{\alpha}))+\frac{1}{\Delta_{\mathrm{F,R}}}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\Delta_{\mathrm{F,R}}(p^2;\bar{\alpha},\bar{m},\mu) && (44.2.56)\\
&=\left(\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{\alpha}\right)\frac{\partial}{\partial\bar{\alpha}}\log(Z(\bar{\alpha})) && (44.2.57)\\
&\quad+\frac{1}{\Delta_{\mathrm{F,R}}}\left[\mu\frac{\partial}{\partial\mu}+\left(\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{\alpha}\right)\frac{\partial}{\partial\bar{\alpha}}+\left(\frac{1}{\bar{m}}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{m}\right)\bar{m}\frac{\partial}{\partial\bar{m}}\right]\Delta_{\mathrm{F,R}}(p^2;\bar{\alpha},\bar{m},\mu)
\end{aligned}$$

where we've explicitly written in the dependence of $\Delta_{\mathrm{F,R}}$ on $\bar{\alpha}$, $\bar{m}$, and μ.

Defining the **field anomalous dimension**, γ_φ, as

$$\gamma_\varphi\coloneqq\frac{1}{2}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\log(Z(\bar{\alpha}))=\frac{1}{2}\left(\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{\alpha}\right)\frac{\partial}{\partial\bar{\alpha}}\log(Z(\bar{\alpha})) \tag{44.2.58}$$

we find that in this case we have

$$\begin{aligned}
\gamma_\varphi&=\frac{1}{2}\left(-2\varepsilon\bar{\alpha}+\beta(\bar{\alpha})\right)\left(\frac{a_1'(\bar{\alpha})}{\varepsilon}+\cdots\right) && (44.2.59)\\
&=-\bar{\alpha}a_1'(\bar{\alpha}) && (44.2.60)\\
&=\frac{1}{12}\bar{\alpha}+\mathcal{O}(\bar{\alpha}^2) && (44.2.61)
\end{aligned}$$

We can then rewrite the result above as

$$0=2\gamma_\varphi+\frac{1}{\Delta_{\mathrm{F,R}}}\left[\mu\frac{\partial}{\partial\mu}+\left(\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{\alpha}\right)\frac{\partial}{\partial\bar{\alpha}}+\gamma_m(\bar{m})\bar{m}\frac{\partial}{\partial\bar{m}}\right]\Delta_{\mathrm{F,R}}(p^2;\bar{\alpha},\bar{m},\mu) \tag{44.2.62}$$

Rearranging this gives

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\bar{\alpha})\frac{\partial}{\partial\bar{\alpha}}+\gamma_m(\bar{\alpha})\bar{m}\frac{\partial}{\partial\bar{m}}+2\gamma_\varphi(\bar{\alpha})\right]\Delta_{\mathrm{F,R}}(p^2;\bar{\alpha},\bar{m},\mu)=0. \tag{44.2.63}$$

This is another renormalisation group equation, this time for the two point function.

Notice that the $\overline{\mathrm{MS}}$ scheme is mass independent, all quantities are functions of $\bar{\alpha}$, and not $\bar{m}$. This is not the case for all schemes. Most of the quantities computed here are renormalisation scheme dependent. However, it can be shown that the first two terms of $\beta(\bar{\alpha})$ are scheme independent, as are the first components of the anomalous dimensions γ_m and γ_φ.

For an n point correlator we will have $n\gamma_\varphi$ in place of $2\gamma_\varphi$.

This renormalisation group equation simply says that $Z(\bar{\alpha})\Delta_{\mathrm{F,R}}(p^2)$ is independent of μ, this means that

$$Z(\bar{\alpha}(\mu))\Delta_{\mathrm{F,R}}(p^2;\bar{\alpha}(\mu),\bar{m}(\mu),\mu)=Z(\bar{\alpha}(\lambda\mu))\Delta_{\mathrm{F,R}}(p^2;\bar{\alpha}(\lambda\mu),\bar{m}(\lambda\mu),\lambda\mu), \tag{44.2.64}$$

that is, rescaling μ doesn't change anything. This allows us to take $\lambda=\mathrm{e}^t$ and then take the limit of $t\to 0$ and expand to first order to get an infinitesimal version of the renormalisation group equation.

Forty-Five

Scale Transformations

45.1 Scale Transformations

A **scale transformation** is a transformation of coordinates and fields such that all quantities with the same dimension are scaled by the same amount. For example, the scale transformation defined by $\mu \mapsto \lambda\mu$ rescales all quantities with dimensions of energy by a factor of λ, so we have $p \mapsto \lambda p$ and $\bar{m} \mapsto \lambda\bar{m}$. The coordinates have units of inverse energy, so $1/x$ has units of energy, and so must scale as $1/x \mapsto \lambda/x$, meaning that $x \mapsto x/\lambda$ under this scale transformation. Note that this is different to the transformation $\mu \mapsto \lambda\mu$ at the end of the previous chapter where we were rescaling only μ, and the only change to, for example, $\bar{m}$, was because $\bar{m}$ is a function of μ.

In general the two point function for a scalar field goes as $1/p^2$, and so $\mu^2\Delta_{\mathrm{F,R}}$ is dimensionless, and so invariant under scale transformations. Under a scale transformation $\mu^2 \mapsto \lambda^2\mu^2$, $p^2 \mapsto \lambda^2 p^2$, $\bar{\alpha} \mapsto \bar{\alpha}$, and $\bar{m} \mapsto \lambda\bar{m}$. Invariance of $\mu^2\Delta_{\mathrm{F,R}}$ under a scale transformation is then expressed as

$$\mu^2\Delta_{\mathrm{F,R}}(p^2;\bar{\alpha},\bar{m},\mu) = \lambda^2\mu^2\Delta_{\mathrm{F,R}}(\lambda^2 p^2;\bar{\alpha},\lambda\bar{m},\lambda\mu). \tag{45.1.1}$$

This implies that the two point function transforms as

$$\Delta_{\mathrm{F,R}}(p^2;\bar{\alpha},\bar{m},\mu) = \lambda^2\Delta_{\mathrm{F,R}}(\lambda^2 p^2;\bar{\alpha},\lambda\bar{m},\lambda\mu). \tag{45.1.2}$$

Now take $\lambda = \mathrm{e}^t$ and $t \ll 1$, we then get

$$\Delta_{\mathrm{F,R}}(p^2;\bar{\alpha},\bar{m},\mu) = \mathrm{e}^{2t}\Delta_{\mathrm{F,R}}(\mathrm{e}^{2t}p^2;\bar{\alpha},\mathrm{e}^t\bar{m},\mathrm{e}^t\mu). \tag{45.1.3}$$

Expanding the exponentials to first order we get

$$\Delta_{\mathrm{F,R}}(p^2;\bar{\alpha},\bar{m},\mu) \approx (1+2t)\Delta_{\mathrm{F,R}}((1+2t)p^2;\bar{\alpha},(1+t)\bar{m},(1+t)\mu). \tag{45.1.4}$$

Expanding $\Delta_{\mathrm{F,R}}$ in t we then get

$$\Delta_{\mathrm{F,R}}(p^2;\bar{\alpha},\bar{m},\mu) \approx \tag{45.1.5}$$
$$(1+2t)\left[\Delta_{\mathrm{F,R}}(p^2;\bar{\alpha},\bar{m},\mu) + 2tp^2\frac{\partial}{\partial p^2} + \bar{m}\frac{\partial}{\partial\bar{m}} + \mu\frac{\partial}{\partial\mu}\right]\Delta_{\mathrm{F,R}}(p^2;\bar{\alpha},\bar{m},\mu).$$

Simplifying this gives

$$t\left[\mu\frac{\partial}{\partial\mu} + \bar{m}\frac{\partial}{\partial\bar{m}} + 2p^2\frac{\partial}{\partial p^2} + 2\right]\Delta_{\mathrm{F,R}}(p^2;\bar{\alpha},\bar{m},\mu) = 0. \tag{45.1.6}$$

The numerical coefficients of the derivatives here are just the mass dimensions of the quantities, this is why we use log derivatives, since here we are adding these dimensions, whereas dimensions are usually multiplied.

Compare this result with the renormalisation group equation of the last chapter, setting them equal and cancelling terms to get

$$\left[2p^2\frac{\partial}{\partial p^2} - \beta(\bar{\alpha})\frac{\partial}{\partial\bar{\alpha}} + (1-\gamma_m(\bar{\alpha}))\bar{m}\frac{\partial}{\partial\bar{m}} + 2(1-\gamma_\varphi(\bar{\alpha}))\right]\Delta_{\mathrm{F,R}}(p^2;\bar{\alpha},\bar{m},\mu) = 0.$$

Naively we may have expected that the coefficients of the $\bar{m}$ derivative and the constant term in brackets would be the mass dimension and field dimension, 1 and 2 respectively. However, this is not what we see, instead we see that these terms are modified by a factor of $1-\gamma_m(\bar{\alpha})$ and $1-\gamma_\varphi(\bar{\alpha})$ respectively. This is why we call γ_m and γ_φ **anomalous dimensions**, they modify expressions which would other be just the dimensions, seemingly changing the dimensions of quantities.

We can combine finite scale transformations, $\Delta_{\mathrm{F,R}} = \lambda^2\Delta_{\mathrm{F,R}}$, and finite renormalisation group equations, $Z(\mu)\Delta_{\mathrm{F,R}}(\mu) = Z(\lambda\mu)\Delta_{\mathrm{F,R}}(\lambda\mu)$ to get an integral form.

Start with the renormalisation group equation

$$\Delta_{\mathrm{F,R}}(p^2;\bar{\alpha},\bar{m},\mu) = \frac{Z(\lambda\mu)}{Z(\mu)}\Delta_{\mathrm{F,R}}(p^2;\bar{\alpha}(\lambda\mu),\bar{m}(\lambda\mu),\lambda\mu), \tag{45.1.7}$$

Rescale $\Delta_{\mathrm{F,R}}$ by a factor of $1/\lambda$ giving

$$\Delta_{\mathrm{F,R}}(p^2;\bar{\alpha},\bar{m},\mu) = \frac{1}{\lambda^2}\frac{Z(\lambda\mu)}{Z(\mu)}\Delta_{\mathrm{F,R}}(p^2/\lambda^2;\bar{\alpha}(\lambda\mu),\bar{m}(\lambda\mu)/\lambda,\mu) \tag{45.1.8}$$

This tells us that $\Delta_{\mathrm{F,R}}$ at some particular momentum and mass is simply given by scaling $\Delta_{\mathrm{F,R}}$ at another momentum and mass by a factor of $Z(\lambda\mu)/[Z(\mu)\lambda^2]$.

45.2 Asymptotic Behaviour

Now consider the limit of infinite momentum,

$$\lim_{p^2\to\infty}\Delta_{\mathrm{F,R}}(p^2;\bar{\alpha}(\mu),\bar{m}(\mu),\mu) = \lim_{p^2\to\infty}\left[\frac{1}{\lambda^2}\frac{Z(\lambda\mu)}{Z(\mu)}\right]\Delta_{\mathrm{F,R}}(p_0^2;\bar{\alpha}(\lambda\mu),\bar{m}(\lambda\mu)/\lambda,\mu)$$

with $p_0 = p/\lambda$ held constant as $p^2\to\infty$ and $\lambda\to\infty$.

In the free theory we have

$$\Delta_{\mathrm{F,R}}(p^2) = \frac{1}{p^2-m^2} = \frac{1}{p^2}\frac{1}{1-m^2/p^2} \sim \frac{1}{p^2}, \tag{45.2.1}$$

so in the $p^2\to\infty$ limit we can treat the particles as if they were massless, since $m/p \ll 1$. For an interacting theory we instead have

$$\Delta_{\mathrm{F,R}}(p^2) \sim \frac{1}{p^{2(1-\gamma_\varphi)}}. \tag{45.2.2}$$

For a theory with asymptotic freedom, $\bar{\alpha}(\lambda\mu)\to 0$ as $\lambda\to\infty$, as $p^2\to\infty$ we have $\bar{\alpha}\to 0$ so $\gamma_\varphi\to 0$, and we can treat this theory as a free theory, hence the term asymptotic *freedom*.

45.3 Large Logs

Let's do what you were all told not to in school.
It's nice to break stupid rules.
As long as you do it correctly it's fine.

Luigi Del Debbio

We often consider quantities such as

$$\log \frac{\mu^2}{m_{\mathrm{R}}^2}, \qquad \text{or} \qquad \log \frac{\mu^2}{p^2}. \tag{45.3.1}$$

Generally we need the argument of the logarithm to be of the form

$$\frac{\text{arbitrary scale}}{\text{physical scale}} \tag{45.3.2}$$

possibly raised to some power, to get a dimensionless argument. The problem is that if μ is far from m_{R} or p then these logarithms take on large values. In this case the perturbation scales as α times a large log, which is not what we want for perturbation theory. The solution is to tune the value of μ we use to the scale of the processes we are interested in. The renormalisation group equations then tell us how to tune α, m, and other parameters to get small logs and no μ dependence in any physical quantities.

45.4 Tuning Parameters

*The next thing is, ..., no, I'll tell you more later, we'll run out of time *looks at notes* well actually later is now.*

Luigi Del Debbio

Recall the following nonperturbative definition

$$\beta(\bar{\alpha}) \coloneqq \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \bar{\alpha}. \tag{45.4.1}$$

So long as we don't tell the mathematicians we can multiply through by $\mathrm{d}\mu$ to get

$$\frac{\mathrm{d}\alpha}{\beta(\alpha)} = \frac{\mathrm{d}\mu}{\mu}. \tag{45.4.2}$$

integrating this and choosing integration limits μ_0 and μ for the μ integral and $\bar{\alpha} = \bar{\alpha}(\mu)$ and $\bar{\alpha}_0 = \bar{\alpha}(\mu_0)$ for the α integral we get

$$\int_{\bar{\alpha}_0}^{\bar{\alpha}} \frac{\mathrm{d}\alpha}{\beta(\alpha)} = \int_{\mu_0}^{\mu} \frac{\mathrm{d}\mu}{\mu} = \log \frac{\mu}{\mu_0}. \tag{45.4.3}$$

This gives

$$\frac{\mu}{\mu_0} = \exp\left\{\int_{\bar{\alpha}_0}^{\bar{\alpha}} \frac{\mathrm{d}\alpha}{\beta(\alpha)}\right\}. \tag{45.4.4}$$

Similarly, we can consider

$$\gamma_m \coloneqq \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \log \bar{m} \implies \mathrm{d}(\log \bar{m}) = \frac{\mathrm{d}\mu}{\mu}\gamma_m = \frac{\mathrm{d}\alpha}{\beta(\alpha)}\gamma_m \tag{45.4.5}$$

which gives the integral equation

$$\frac{\bar{m}(\mu)}{\bar{m}(\mu_0)} = \exp\left\{\int_{\bar{\alpha}_0}^{\bar{\alpha}} \mathrm{d}\alpha\, \frac{\gamma_m(\alpha)}{\beta(\alpha)}\right\}. \tag{45.4.6}$$

The step where we replace $\mathrm{d}\mu/\mu$ with $\mathrm{d}\alpha/\beta(\alpha)$ only works in a mass independent scheme.

Again, a similar process gives

$$\frac{Z(\mu)}{Z(\mu_0)} = \exp\left\{2\int_{\bar{\alpha}_0}^{\bar{\alpha}} \mathrm{d}\alpha\, \frac{\gamma_\varphi(\alpha)}{\beta(\alpha)}\right\}. \tag{45.4.7}$$

We can now use the leading order results of the previous sections:

$$\beta(\bar{\alpha}) = -\frac{3}{2}\bar{\alpha}^2, \qquad \gamma_m(\bar{\alpha}) = -\frac{5}{12}\bar{\alpha}, \qquad \text{and} \qquad \gamma_\varphi(\bar{\alpha}) = \frac{1}{12}\bar{\alpha}. \tag{45.4.8}$$

Plugging these into the above expressions we get the following results we get

$$\log\frac{\mu}{\mu_0} = -\frac{2}{3}\int_{\bar{\alpha}_0}^{\bar{\alpha}} \mathrm{d}\alpha\, \frac{1}{\alpha^2} = -\frac{2}{3}\left(\frac{1}{\bar{\alpha}} - \frac{1}{\bar{\alpha}_0}\right). \tag{45.4.9}$$

Now introducing a new parameter, Λ, defined such that

$$\frac{1}{\bar{\alpha}_0} = \frac{3}{2}\log\left(\frac{\mu_0}{\Lambda}\right) \tag{45.4.10}$$

we have

$$\frac{1}{\bar{\alpha}} = \frac{3}{2}\log\left(\frac{\mu}{\Lambda}\right). \tag{45.4.11}$$

Thus,

$$\bar{\alpha}(\mu) = \frac{1}{b_0 \log(\mu/\Lambda)} \tag{45.4.12}$$

for some constant b_0, which to one loop is $b_0 = 3/2$. It turns out that at one loop the relation between $\bar{\alpha}$ and Λ is the same in all schemes.

Consider the limiting behaviour of $\bar{\alpha}(\mu)$. As $\mu \to \infty$ the logarithm becomes infinite and so $\bar{\alpha} \to 0$, this is asymptotic freedom. When μ is on the order of Λ then $\mu/\Lambda \approx 1$, so $\log(\mu/\Lambda) \approx 0$, and $\bar{\alpha}$ becomes very large. This means we cannot do perturbation theory when μ is on the order of Λ.

Note that Λ is a one loop quantity, it is just the boundary conditions of the integrals above, so it makes sense that things would break down on the order of Λ. The value of Λ is scheme dependent and it is not a physical quantity. People often claim that Λ is physical, but it really isn't.

We want to take $\bar{\alpha}_0 \to 0$ in the limit of the integral

$$\int_{\bar{\alpha}_0}^{\bar{\alpha}} \frac{\mathrm{d}\alpha}{\beta(\alpha)}. \tag{45.4.13}$$

However, since $\beta(\alpha) \sim \alpha^2$ this gives a divergent integral. This is because we are only considering the leading order term in $\beta(\alpha)$. Instead, suppose that

$$\beta(\alpha) = -b_0\alpha^2 - b_1\alpha^3 + \cdots . \tag{45.4.14}$$

This works even nonperturbatively, since β will always be analytic. Then we have

$$\frac{1}{\beta(\alpha)} = -\frac{1}{b_0\alpha^2 + b_1\alpha^3 + \cdots} \tag{45.4.15}$$

$$= -\frac{1}{b_0\alpha^2}\frac{1}{1 + \frac{b_1}{b_0}\alpha + \cdots} \tag{45.4.16}$$

$$\approx -\frac{1}{b_0\alpha^2}\left(1 - \frac{b_1}{b_0}\alpha + \cdots\right) \tag{45.4.17}$$

assuming that $b_1\alpha/b_0 + \cdots \ll 1$ for the binomial expansion to apply. We then have that

$$\frac{1}{\beta(\alpha)} + \frac{1}{b_0\alpha^2} - \frac{b_1}{b_0^2\alpha} \neq 0 \tag{45.4.18}$$

for $\alpha > 0$, and so we can take the limit as $\alpha \to 0$ and the integral will converge. Using this we have

$$\frac{\mu}{\mu_0} = \exp\left\{\int_{\bar{\alpha}_0}^{\bar{\alpha}} \mathrm{d}\alpha\, \frac{1}{\beta(\alpha)}\right\} \tag{45.4.19}$$

$$= \exp\left\{\int_{\bar{\alpha}_0}^{\bar{\alpha}} \mathrm{d}\alpha\, \left[\left(\frac{1}{\beta(\alpha)} + \frac{1}{b_0\alpha^2} - \frac{b_1}{b_0^2\alpha}\right) - \frac{1}{b_0\alpha^2} + \frac{b_1}{b_0^2\alpha}\right]\right\}. \tag{45.4.20}$$

Now we can perform the integral over the last two terms, giving

$$\frac{\mu}{\mu_0} = \exp\left\{\frac{1}{b_0\bar{\alpha}} - \frac{1}{b_0\bar{\alpha}_0} + \frac{b_1}{b_0^2}\log\bar{\alpha} - \frac{b_1}{b_0^2}\log\bar{\alpha}_0\right\} \times\left\{\int_{\bar{\alpha}_0}^{\bar{\alpha}} \mathrm{d}\alpha\left(\frac{1}{\beta(\alpha)} + \frac{1}{b_0\alpha^2} - \frac{b_1}{b_0^2\alpha}\right)\right\}. \tag{45.4.21}$$

Now we write the integral over $[\bar{\alpha}_0, \bar{\alpha}]$ as an integral over $[0, \bar{\alpha}]$ minus an integral over $[0, \bar{\alpha}_0]$, giving

$$\frac{\mu}{\mu_0} = \exp\left\{\frac{1}{b_0\bar{\alpha}} - \frac{1}{b_0\bar{\alpha}_0}\right\}\left(\frac{\bar{\alpha}}{\bar{\alpha}_0}\right)^{b_1/b_0^2} \tag{45.4.22}$$

$$\times \exp\left\{\int_0^{\bar{\alpha}} \mathrm{d}\alpha\left(\frac{1}{\beta(\alpha)} + \frac{1}{b_0\alpha^2} - \frac{b_1}{b_0^2\alpha}\right) - \int_0^{\bar{\alpha}_0} \mathrm{d}\alpha\left(\frac{1}{\beta(\alpha)} + \frac{1}{b_0\alpha^2} - \frac{b_1}{b_0^2\alpha}\right)\right\}$$

Rearranging this we get

$$\mu \exp\left\{-\frac{1}{b_0\bar{\alpha}}\right\}\bar{\alpha}^{-b_1/b_0^2}\exp\left\{\int_0^{\bar{\alpha}} \mathrm{d}\alpha\left(\frac{1}{\beta(\alpha)}+\frac{1}{b_0\alpha^2}-\frac{b_1}{b_0^2\alpha}\right)\right\}$$
$$= \mu_0 \exp\left\{-\frac{1}{b_0\bar{\alpha}_0}\right\}\bar{\alpha}_0^{-b_1/b_0^2}\exp\left\{\int_0^{\bar{\alpha}_0} \mathrm{d}\alpha\left(\frac{1}{\beta(\alpha)}+\frac{1}{b_0\alpha^2}-\frac{b_1}{b_0^2\alpha}\right)\right\}. \tag{45.4.23}$$

The right hand side here is independent of μ, and the left hand side is independent of μ_0. Using this we get a nonperturbative definition of Λ. This equation is an integrated form of

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bar{\alpha} = \beta(\bar{\alpha}) \tag{45.4.24}$$

with the boundary condition involving Λ.

45.5 Changing Renormalisation Schemes

Suppose we have two renormalisation schemes, the first involving couplings α and the second couplings α'. We want α and α' to differ only by higher order terms, since the tree level results are scheme independent. This implies that we can write α' as a function of α:

$$\alpha'(\alpha) = \alpha + c_1\alpha^2 + c_2\alpha^3 + \cdots . \tag{45.5.1}$$

We also have

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\alpha = \beta(\alpha) = -(b_0\alpha^2 + b_1\alpha^3 + \cdots). \tag{45.5.2}$$

The same definition in the second scheme gives

$$\beta'(\alpha') = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\alpha' \tag{45.5.3}$$
$$= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}(\alpha + c_1\alpha^2 + \cdots) \tag{45.5.4}$$
$$= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\alpha + c_1\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\alpha^2 + \cdots \tag{45.5.5}$$
$$= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\alpha + 2c_1\mu\alpha\frac{\mathrm{d}}{\mathrm{d}\mu}\alpha + \cdots \tag{45.5.6}$$
$$= \beta(\alpha) + 2c_1\alpha\beta(\alpha) + \cdots \tag{45.5.7}$$
$$= -(b_0\alpha^2 + b_1\alpha^3 + \cdots) - 2c_1\alpha(b_0\alpha^2 + b_1\alpha^3 + \cdots) + \cdots \tag{45.5.8}$$
$$= -b_0\alpha^2 - (b_1 + 2c_1b_0)\alpha^3 + \cdots . \tag{45.5.9}$$

We can compare this to the expanded version of $\beta'(\alpha')$:

$$-b_0\alpha^2 - (b_1 + 2c_1b_0)\alpha^3 + \cdots = -b_0'\alpha'^2 - b_1'\alpha'^3 + \cdots \tag{45.5.10}$$
$$= -b_0(\alpha + c_1\alpha^2 + c_2\alpha^3 + \cdots)^2 \tag{45.5.11}$$
$$- b_1'(\alpha + c_1\alpha^2 + c_2\alpha^3 + \cdots)^3 \tag{45.5.12}$$
$$= -b_0\alpha^2 - b_1'\alpha^3 - 2b_1'c_1\alpha^3 + \cdots \tag{45.5.13}$$

So, we see that

$$b_0 = b_0', \qquad \text{and} \qquad b_1 = b_1'. \tag{45.5.14}$$

This shows that the first two terms of $\beta(\alpha)$ are scheme independent, or **universal**. This does not extend beyond the first two terms.

Similarly one can show that the first terms of both γ_m and γ_φ are universal, and these start at $\mathcal{O}(\alpha)$, so agree on the linear term in any scheme. It can also be shown that $\Lambda = \Lambda'$ at one loop.

In the renormalisation group equations μ is just a dummy variable, so we can take $\bar{m}$ as the same in both schemes. We can take different values of μ in different schemes. This is often desirable in calculations as we can then choose μ to give small logs.

Consider

$$\beta'(\alpha') = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\alpha' = \mu\frac{\mathrm{d}\alpha'}{\mathrm{d}\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}\mu} = \frac{\mathrm{d}\alpha'}{\mathrm{d}\alpha}\beta(\alpha). \tag{45.5.15}$$

This means that the zeros of β are the same as the zeros of β'. In other words, the zeros of β are scheme independent. This is important because in statistical mechanics zeros of β correspond to phase transitions, which cannot be scheme dependent since they are physical.

Performing the other integrals above in a similar way to before we have

$$\bar{m}(\mu) = \bar{m}(\mu_0)\left(\frac{\bar{\alpha}}{\bar{\alpha}_0}\right)^{5/12}, \quad \text{and} \quad Z(\mu) = Z(\mu_0)\left(\frac{\bar{\alpha}}{\bar{\alpha}_0}\right)^{-1/12}. \tag{45.5.16}$$

At one loop $\bar{\alpha}$ evolves logarithmically, so we get a logarithmic correction here between the two schemes. Scaling μ by a factor of λ gives

$$\frac{1}{\lambda}\bar{m}(\lambda\mu) = \frac{1}{\lambda}\bar{m}(\mu)\left(\frac{\bar{\alpha}}{\bar{\alpha}_0}\right)^{5/12}, \tag{45.5.17}$$

so we get a logarithmic addition to the naive scaling.

Every nontrivial ($\alpha \neq 0$) fixed point ($\beta(\alpha) = 0$) gives us $\gamma_\varphi \approx \gamma_\varphi^*$, where γ_φ^* is the field anomalous dimension in the second renormalisation scheme. Then

$$\frac{Z(\mu)}{Z(\mu_0)} = \exp\left\{\int_{\mu_0}^{\mu}\mathrm{d}\tilde{\mu}\,\frac{\gamma_\varphi^*}{\tilde{\mu}}\right\} = \left(\frac{\mu}{\mu_0}\right)^{\gamma_\varphi^*}. \tag{45.5.18}$$

So rather than scaling linearly $Z(\mu)/Z(\mu_0)$ scales as $(\mu/\mu_0)^{\gamma_\varphi^*}$, further justifying it's name as an anomalous dimension.

Part XV

Effective Theories

Forty-Six

Effective Theories

46.1 Effective Theories

For the purposes of this part we will consider a φ^4 theory in $D = 4$ dimensions. This is specified by the Lagrangian

$$\mathcal{L} = \frac{1}{2}Z(\partial_\mu\varphi_\mathrm{R})(\partial^\mu\varphi_\mathrm{R}) - \frac{1}{2}Z_{m^2}m_\mathrm{R}^2\varphi_\mathrm{R}^2 - \frac{1}{4!}Z_\lambda\lambda_\mathrm{R}\varphi_\mathrm{R}^4. \tag{46.1.1}$$

The on shell-scheme gives us φ_R and m_R^2 the same way it works for φ^3. We get λ_R by considering the four-point vertex at zero momentum and setting $\lambda_\mathrm{R} = V^{(4)}(0,0,0,0)$, so at zero-momentum the vertex is just λ_R as one might expect.

The generating functional is

$$\mathcal{Z}[J] = \int \mathcal{D}\varphi_\mathrm{R}\ \exp\{iS[\varphi_\mathrm{R}] + iJ\cdot\varphi_\mathrm{R}\}. \tag{46.1.2}$$

We can perform a Wick rotation to move to a Euclidean path integral. Set $\tau = it$ and $\varphi_\mathrm{R}^\mathrm{E}(\tau,\boldsymbol{x}) = \varphi_\mathrm{R}(it,\boldsymbol{x})$, this makes $\varphi_\mathrm{R}^\mathrm{E}$ and φ_R different fields, but since the fields appear only as dummy variables in the path integral we will continue to call the field φ_R. The generating functional is then

$$\mathcal{Z}[J] = \int \mathcal{D}\varphi_\mathrm{R}\ \exp\{-S_\mathrm{E}[\varphi_\mathrm{R}] + J\cdot\varphi_\mathrm{R}\} \tag{46.1.3}$$

where the Euclidean action is given by

$$S_\mathrm{E}[\varphi_\mathrm{R}] = \int \mathrm{d}^Dx\left[\frac{1}{2}(\partial_\mu\varphi_\mathrm{R})(\partial^\mu\varphi_\mathrm{R}) + \frac{1}{2}Z_{m^2}m_\mathrm{R}^2\varphi_\mathrm{R}^2 + \frac{1}{4!}Z_\lambda\lambda_\mathrm{R}\varphi_\mathrm{R}^4\right]. \tag{46.1.4}$$

We now move to momentum space by writing the field as the inverse Fourier transform:

$$\varphi_\mathrm{R}(x) = \int \frac{\mathrm{d}^Dp}{(2\pi)^D}\mathrm{e}^{ip\cdot x}\tilde{\varphi}_\mathrm{R}(p). \tag{46.1.5}$$

Note that in Euclidean space $p_\mu = -i\partial_\mu$, whereas in Minkowski space $p_\mu = i\partial_\mu$. We choose the Fourier transform such that $\mathrm{e}^{ip\cdot x}$ are eigenfunctions of the momentum operator with the same eigenvalues in both real and Fourier space.

The action as a functional of the Fourier transformed field is

$$S_\mathrm{E}[\tilde{\varphi}_\mathrm{R}] = \frac{1}{2}\int_p \tilde{\varphi}_\mathrm{R}(p)^*[Zp^2 + Z_{m^2}m_\mathrm{R}^2]\tilde{\varphi}_\mathrm{R}(p) + \frac{Z_\lambda\lambda_\mathrm{R}}{4!}\int_{p_1,p_2,p_3,p_4}(2\pi)^D\delta(p_1+p_2+p_3+p_4)\tilde{\varphi}_\mathrm{R}(p_1)\tilde{\varphi}_\mathrm{R}(p_2)\tilde{\varphi}_\mathrm{R}(p_3)\tilde{\varphi}_\mathrm{R}(p_4),$$

(46.1.6)

Note that $p^2 = \boldsymbol{p}^2 + p_4^2 \geq 0$ in Euclidean space and $p_4 = ip_0$.

Now introduce a UV scale, Λ, such that $\Lambda \gg m_{\mathrm{R}}$, meaning that Λ is much larger than the typical energy scales of the theory. We can then separate momentum space into two parts, one with $p^2 > \Lambda^2$ and the other with $p^2 < \Lambda^2$, which we call high and low energies respectively. In Euclidean momentum space the boundary between high and low energies is a 3-sphere. We will restrict ourselves to low energy physics by only having sources for low energy degrees of freedom. That is, we set $\tilde{J}(p) = 0$ for $p^2 > \Lambda^2$.

Now introduce the notation $\varphi_{\mathrm{R}}^+ = \varphi_{\mathrm{R}}(p)$ with $p^2 > \Lambda^2$ and $\varphi_{\mathrm{R}}^- = \varphi_{\mathrm{R}}(p)$ with $p^2 < \Lambda^2$. We can then split the path integral into two integrals, one over each region:

$$\mathcal{Z}[J] = \int \mathcal{D}\varphi_{\mathrm{R}}^- \, \mathcal{D}\varphi_{\mathrm{R}}^+ \, \exp\{-S_{\mathrm{E}}[\varphi_{\mathrm{R}}] + J \cdot \varphi_{\mathrm{R}}^-\}. \tag{46.1.7}$$

We can now do the φ_{R}^+ integral, which is easier since there is no source for φ_{R}^+. We get the result

$$\mathcal{Z}[J] = \int \mathcal{D}\varphi_{\mathrm{R}}^- \, \exp\{-S_{\mathrm{eff}}[\varphi_{\mathrm{R}}^-; \Lambda] + J \cdot \varphi_{\mathrm{R}}^-\}. \tag{46.1.8}$$

Here S_{eff} is the **effective action**, or **Wilsonian effective action**, it only includes the degrees of freedom we are interested in, the low energy ones. It is, in general, a functional of both the low energy modes, φ_{R}^-, and also the cutoff value we choose, Λ.

We now have two ways to do computations, we can do them in the full theory or the effective theory. These must both give the same answer, since we haven't made any approximations yet. We can use this to determine S_{eff} without having to actually do the path integral over φ_{R}^+.

The Ward identities in both the full and effective theory must be the same, which means that the symmetries in both theories are the same. Recall that the action encodes the degrees of freedom and the symmetries of the system. We keep all symmetries in the effective theory, but we only take a subset of the degrees of freedom. In general the effective action is of the form

$$S_{\mathrm{eff}}[\varphi_{\mathrm{R}}; \Lambda] = \tag{46.1.9}$$
$$\int \mathrm{d}^D x \left[\frac{1}{2} Z(\Lambda)(\partial_\mu \varphi_{\mathrm{R}})(\partial^\mu \varphi_{\mathrm{R}}) + \frac{1}{2} m^2(\Lambda) \varphi_{\mathrm{R}}^2 + \frac{1}{4!} \lambda(\Lambda) \varphi_{\mathrm{R}}^4 + \sum_{d=6}^{\infty} c_{di}(\Lambda) O_{di} \right].$$

Notice that all of the constants now depend on Λ. The extra terms in the sum at the end are of the form scalar, $c_{di}(\Lambda)$, called the **low energy coupling**, or LEC, times an operator, O_{di}. The index d labels the mass dimension of the operator, and the index i allows for more than one term at each mass dimension. These must be chosen such that the dimensions work out to give a dimensionless action. Defining O_{di} to have mass dimension $[O_{di}] = d$ this means that c_{di} must have mass dimension $[c_{di}] = 4 - d$, so that $c_{di} O_{di}$ has mass dimension $[c_{di} O_{di}] = 4$, which combines with the integration measure with mass dimension $[\mathrm{d}^4 x] = -4$ to give a dimensionless action. Note that sometimes people instead use the dimensionless $\hat{c}_{di}(\Lambda) \coloneqq \Lambda^{d-4} c_{di}(\Lambda)$ in which case the dimensions appear explicitly through an

extra factor of Λ^{d-4}. This can be useful as it decouples scaling due to powers of Λ from genuine Λ dependence of the LEC.

The original theory is Lorentz invariant, so the effective theory must be too. The full theory also has a $\mathbb{Z}_2$ symmetry $\varphi \mapsto -\varphi$, and so the effective theory must preserve this also.

Note that both the full and effective theory give finite values. The full theory is renormalised, and the finite theory includes a cut-off at energies greater than Λ.

We can now turn the situation on its head if we like. Suppose that we have a theory described by some action which works well at low energies for some energy scale Λ. It is possible, in fact probable, that this is not the fundamental theory, but instead is an effective theory of some more fundamental theory. In this case we call the full theory the UV completion of the theory. For example, many believe that the standard model, described by the symmetry group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, is an effective theory of some larger theory. Various suggestions have been made for this larger theory, usually by extending the symmetry to some larger group[1], such as SO(10) or E_8, both of which contain $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, or by including some new kind of symmetry, such as a fermion-boson symmetry in SUSY[2]. Such a theory is called a grand unified theory, or GUT.

[1] see *Symmetries of Particles and Fields*

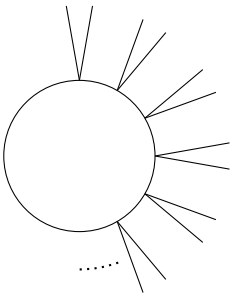
[2] see *Symmetries of Particles and Fields* or *Particle Physics*

46.2 Matching

Matching is the process of comparing computations in the full and effective theory in order to compute the effective action. We use the 1PI vertices for this procedure. In a φ^4 theory every vertex has four legs. If we have V vertices, I internal legs and E external legs then we must have $4V = 2I + E$. If we have L loops then we have $L = I - V + 1$. Combining these gives $2V = 2L - 2 + E$.

46.2.1 φ^{2n} Term

For a given value of E, that is a fixed value of n in $V^{(n)}$, the leading order is the term with the fewest loops. To determine c_{di} we can consider terms O_{di} of the form φ^{2n} for $2n \geq 6$. We take only even powers since we want to keep the $\mathbb{Z}_2$ symmetry $\varphi \mapsto -\varphi$. The term φ^{2n} gives the leading order diagram with $E = 2n$ external legs, $V = n$ vertices, and $L = 1$ loops:

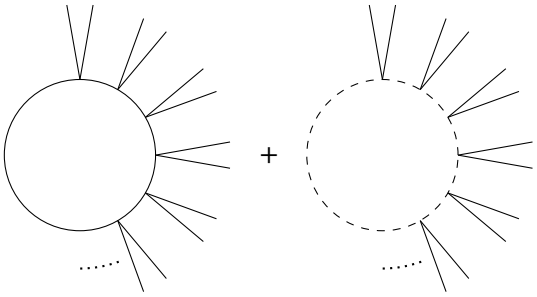


(46.2.1)

Notice that there is no tree level diagram in this case.

In the effective theory we have a very similar diagram but the virtual particle in the loop can have any energy, including energies greater than Λ. The energies of the external particles are constrained to be lower than the cut-off. We can then

split the diagram into two:



(46.2.2)

Here the dashed line is a high energy particle and the normal line is a low energy particle.

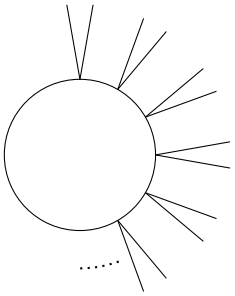
A computation in the full theory gives the expected vertex term of λ_R. This is a leading order calculation, so actually what we have is

$$Z_\lambda \lambda_R = \lambda_R + \mathcal{O}(\lambda_R^2). \tag{46.2.3}$$

A computation in the effective theory is more complicated. First, there is a tree level term in the effective theory, since $O_{2n,i}$ gives us a $2n$-vertex. For simplicity we'll assume only one term of each mass dimension, and drop the i index. This $2n$-vertex gives the vertex term

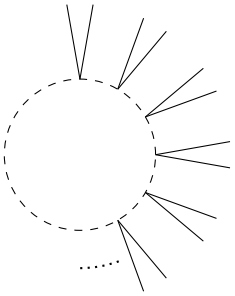
$$-(2n)!c_{2n}(\Lambda). \tag{46.2.4}$$

Note that we need to include the $(2n)!$ combinatoric term, which usually cancels with the factor we include in the Lagrangian, such as the $1/4!$ in the φ^4 term, but we haven't included this factor for the additional terms in the effective action so it instead appears in the vertex term. We also have the one loop term with a low energy loop. Thus in the effective field theory we get the term



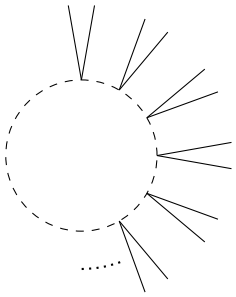
$$-(2n)!c_{2n}(\Lambda). \tag{46.2.5}$$

Matching the previous expression we have



$$= -(2n)!c_{2n}(\Lambda). \tag{46.2.6}$$

We can also use the Feynman rules to compute this diagram, for simplicity take momentum of all external particles to be zero. We then get



$$= \frac{(-\lambda_R)^n (2n)!}{2^n n \cdot 2} \int_\Lambda^\infty \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2 + m_{\text{phys}}^2} \right)^n + \mathcal{O}(\lambda_R^{n+1}).$$

Notice that the integral starts at Λ, since we are considering a high energy loop. We are considering $2n \geq 6$ so this integral goes as $k^4/k^{2n} = k^{4-2n} \leq k^{-2}$ and so the integral converges. If we perform this integral then we get

$$\frac{(\lambda_R/2)^n (2n)!}{32\pi^2 n(n-2)} \frac{1}{\Lambda^{2n-4}} + \mathcal{O}(\lambda_R^{n+1}, m_{\text{phys}}^2/\Lambda^2). \tag{46.2.7}$$

The $m_{\text{phys}}^2/\Lambda^2$ comes from considering the propagator

$$\frac{1}{k^2 + m_{\text{phys}}^2} = \frac{1}{k^2} \frac{1}{1 + m_{\text{phys}}^2/k^2} \approx \frac{1}{k^2} \left(1 - \frac{m_{\text{phys}}^2}{k^2} \right) \approx \frac{1}{k^2}. \tag{46.2.8}$$

Since $k^2 \in [\Lambda^2, \infty)$ and $m_{\text{phys}} \ll \Lambda$ this approximation is valid. Setting the result above to be equal to $-(2n)! c_{2n}(\Lambda)$ we see that the only Λ dependence comes from the dimensions, $1/\Lambda^{2n-4}$. This holds for all c_d, we have power suppression according to dimension. However, other loop diagrams do have Λ dependence.

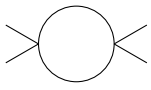
46.2.2 φ^4 Term

Now consider the φ^4 term. To one loop we have

$$-\lambda(\Lambda) = -Z_\lambda \lambda_R +$$

(46.2.9)

The left hand side is the result of a computation in the effective field theory and the right hand side the same computation in the full theory. The left hand side is simply the φ^4 four-point vertex, the right hand side also contains a single high energy loop. In both computations there is also a term given by the following diagram, with the loop being low energy in both cases (as opposed to before where the loop in the full theory was over all energies) which cancels:



(46.2.10)

Computing the remaining diagram we have

$$\lambda(\Lambda) = -Z_\lambda \lambda_R + \frac{3}{2}(-\lambda_R)^2 \int_\Lambda^\infty \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2 + m_{\text{phys}}^2} \right)^2. \tag{46.2.11}$$

Using $-\lambda_R = V^{(4)}(0,0,0,0)$ as our renormalisation condition another computation gives

$$-\lambda_R = -Z_\lambda \lambda_R + \frac{3}{2}(-\lambda_R)^2 \int_0^\infty \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2 + m^2_{\text{phys}}} \right)^2 . \tag{46.2.12}$$

The only difference between these two results is the regions of integration, $[\Lambda, \infty)$ and $[0, \infty)$. The difference is then the finite integral

$$\frac{3}{2}(-\lambda_R)^2 \int_0^\Lambda \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2 + m^2_{\text{phys}}} \right)^2 . \tag{46.2.13}$$

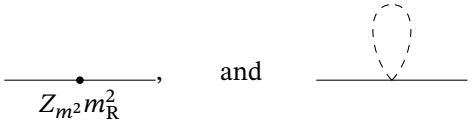
Computing this integral we have

$$\lambda(\Lambda) - \lambda_R = \frac{3}{16\pi^2} \lambda_R^2 \left(\log \frac{\Lambda}{m_R} - \frac{1}{2} \right) + \mathcal{O}(\lambda_R^3). \tag{46.2.14}$$

The effect in the effective field theory is a shift proportional to $\lambda_R^2 \log(\Lambda^2/m_R^2)$. The log dependence on Λ should not be surprising since λ_R is dimensionless. Note that this is a large log since $\Lambda^2 \gg m_R^2$ is our starting assumption.

46.2.3 φ^2 Term

Now consider the φ^2 term. In the full theory we have two leading order contributions:



(46.2.15)

The effective theory only has the two point vertex contributing $m^2(\Lambda)$. These diagrams have no p dependence, meaning $Z(\Lambda) = 1$, A full computation gives

$$m^2(\Lambda) = Z_{m^2} m_R^2 + \frac{1}{2}\lambda_R \int_\Lambda^\infty \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2_{\text{phys}}} + \mathcal{O}(\lambda_R^2), \tag{46.2.16}$$

and so

$$m^2_{\text{phys}} = Z_{m^2} m_R^2 + \frac{1}{2}\lambda_R \int_0^\infty \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2_{\text{phys}}}. \tag{46.2.17}$$

Combing these results we have

$$m^2(\Lambda) = m^2_{\text{phys}} - \frac{1}{2}\lambda_R \int_0^\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + + m^2_{\text{phys}}} \tag{46.2.18}$$

$$= m^2_{\text{phys}} - \frac{\lambda_R}{16\pi^2} \left[\Lambda^2 - m^2_{\text{phys}} \log \frac{\Lambda^2}{m^2_{\text{phys}}} \right]. \tag{46.2.19}$$

So we get a correction scaling with Λ^2 which is what we expect since the dimensions of $m^2(\Lambda)$ are energy squared.

46.2.4 New Cut-off

We can now forget the full theory, even though we haven't computed $Z(\Lambda)$, this term is just an overall rescaling of the fields. Now choose a new scale $\Lambda_0 > \Lambda$ and define $m^2(\Lambda_0)$, $\lambda(\Lambda_0)$ and set $c_{2n,i}(\Lambda_0) = 0$. We can integrate out the modes between Λ and Λ_0 giving

$$\exp\{-S[\varphi;\Lambda]\} = \int_{\Lambda\leq|p|<\Lambda_0} \mathcal{D}\varphi \, \exp\{-S[\varphi,\Lambda_0]\}. \tag{46.2.20}$$

We then have

$$m^2(\Lambda) = m^2(\Lambda_0) + \frac{1}{2}\lambda(\Lambda_0)\int_\Lambda^{\Lambda_0} \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2+m^2(\Lambda_0)^2} \tag{46.2.21}$$

$$= m^2(\Lambda_0) + \frac{1}{16\pi^2}\lambda(\Lambda_0)(\Lambda_0^2-\Lambda^2) + \cdots . \tag{46.2.22}$$

The second term here comes from the second diagram in Equation (46.2.15). Similarly, we have

$$\lambda(\Lambda) = \lambda(\Lambda_0) - \frac{3}{2}\int_\Lambda^{\Lambda_0} \frac{\mathrm{d}^4k}{(2\pi)^4}\left(\frac{1}{k^2+m^2(\Lambda_0)}\right)^2 \tag{46.2.23}$$

$$= \lambda(\Lambda_0) - \frac{3}{16\pi^2}\lambda(\Lambda_0)^2\log\frac{\Lambda_0}{\Lambda}. \tag{46.2.24}$$

Finally,

$$c_{2n,i}(\Lambda) = \text{number}\cdot\lambda(\Lambda_0)^n\left(\frac{1}{\Lambda^{2n-4}} - \frac{1}{\Lambda_0^{2n-4}}\right) \tag{46.2.25}$$

for $2n \geq 6$, so $2n-4>0$.

What's important here is that the dominant correction to $m^2(\Lambda)$ is $\sim \Lambda_0^2$, the dominant correction to $\lambda(\Lambda)$ is $\sim \log(\Lambda_0/\Lambda)$, and the dominant correction to $c_{2n,i}(\Lambda)$ is $\sim \Lambda_0^{4-2n}$. So, all the corrections scale with the dimensions.

We call $c_{d,i}$ irrelevant couplings since physics at the Λ scale doesn't depend on $c_{d,i}$ at the higher Λ_0 scale.

We can do this process again, defining some $\tilde{\Lambda}_0 > \Lambda_0$. In this way we can build up a hierarchy of scales. The precision of any result compared between two different scales is given by the ratio of the scales.

! All remarks made here are to leading or next to leading order, and may not hold at higher orders.

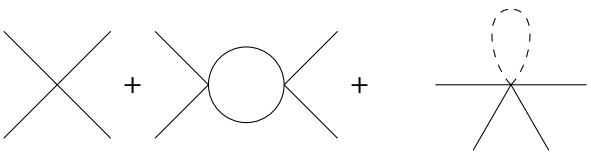
This analysis breaks down on the order of Λ.

Generally we want

$$\frac{m^2(\Lambda_0)}{\Lambda_0^2} \ll 1, \qquad \text{and} \qquad \frac{c_{2n,i}(\Lambda_0)}{\Lambda_0^{4-2n}} \ll 1. \tag{46.2.26}$$

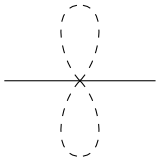
This means that c_{di} are now power suppressed corrections to loops.

The φ^6 vertex, the most dominant correction, accounts for the six point vertex correction in the last diagram in the following expression for $\lambda(\Lambda)$:



(46.2.27)

This correction goes as $\sim c_6\Lambda_0^2 \ll 1$, so it's power suppressed. Similarly we get a φ^6 correction for $m^2(\Lambda)$. This takes the form



$$\sim (c_6\Lambda_0^2)\Lambda_0^2. \tag{46.2.28}$$

The $c_6\Lambda_0^2$ term is small, but Λ_0^2 is large, so this term is not power suppressed. This correction *is* suppressed by being a two-loop diagram, but that is a different form of suppression.

We now have a problem. We need a fine tuning of the coefficients c_{di} in order to have a large difference between $m^2(\Lambda)$ and Λ_0^2. For example, we probe the standard model with the LHC on the $\Lambda_0 \sim 10\,\text{TeV} = 10^{12}\,\text{eV}$ scale. The GUT scale, the scale at which many believe the effective theory of the standard model fails and we need a deeper theory, is $\sim 10^{15}\,\text{eV}$. The mass scale in question here is that of the Higgs boson, $m(\Lambda) \approx m_\text{H} = 125\,\text{GeV} \ll \Lambda_0^2$. The need for this fine tuning in the standard model is called the **naturalness problem**.

46.3 Non-Examinable Remarks

We can generalise the definition of $\beta(\bar{\alpha})$ for a generic coupling, c:

$$\beta(c) \coloneqq \Lambda\frac{\mathrm{d}}{\mathrm{d}\Lambda}c(\Lambda). \tag{46.3.1}$$

We want $\beta(c)$ to evolve in such a way that the low energies are unaffected. We've seen that for the case of λ in φ^4 theory we have

$$\beta(\lambda) = \frac{3}{16\pi^2}\lambda^2 \geq 0. \tag{46.3.2}$$

Thus

$$\int_{\lambda(m_\text{phys}^2)}^{\infty}\frac{\mathrm{d}\lambda}{\beta(\lambda)} = \log\frac{\Lambda_\text{max}}{m_\text{phys}} \tag{46.3.3}$$

with

$$\Lambda_\text{max} = m_\text{phys}\exp\left\{\frac{16\pi^2}{3}\frac{1}{\lambda_{m_\text{phys}}}\right\}. \tag{46.3.4}$$

We can only take Λ_max to infinity if $\lambda(m_\text{phys}^2)$ vanishes sufficiently quickly, in which case there is no interaction. This means that φ^4 theory can only be renormalised at all scales if there are no interactions. This is called the **triviality problem**. The same problem occurs in QED as well, fortunately however the coupling in this case, α, is very small and Λ_max is very large, so Λ_max, the point at which things start to break, is much larger than all energies which we see.

Appendices

A

Pre-Course Revision

The content of this appendix is based on the notes released before the course began to recap key ideas from previous courses. In particular, it covers material from Quantum Theory and Classical Electrodynamics.

A.1 Special Relativity

A.1.1 Four Vectors

Consider some event occurring at the spatial position $\boldsymbol{x}$ and at time t. The coordinates of this event, in four-dimensional Minkowski space, form a **contravariant** four-vector, which we write with upper indices:

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, \boldsymbol{x}). \tag{A.1.1}$$

The **covariant** four-vector has lower indices:

$$x_\mu = (x_0, x_1, x_2, x_3) = (ct, -\boldsymbol{x}). \tag{A.1.2}$$

Similarly a general four-vector has a contravariant and covariant form:

$$a^\mu = (a^0, a^1, a^2, a^3) = (a^0, \boldsymbol{a}), \quad \text{and} \quad a_\mu = (a_0, a_1, a_2, a_3) = (a^0, -\boldsymbol{a}). \tag{A.1.3}$$

In particular, $a^0 = a_0$ and $a^i = -a_i$, where we follow the convention that Greek indices run from 0 to 3 and Latin indices run from 1 to 3.

We move between upper and lower indices using the **metric tensor**, η, also denoted g:

$$a^\mu = \eta^{\mu\nu} a_\nu, \qquad \text{and} \qquad a_\mu = \eta_{\mu\nu} a^\nu, \tag{A.1.4}$$

where we apply the summation convention, summing over repeated indices with one up and one down. The covariant and contravariant components of the Minkowski metric are given by

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{A.1.5}$$

and the mixed components are given by

$$\eta_\mu{}^\nu = \eta^\nu{}_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta_\mu{}^\nu = \delta^\nu{}_\mu. \tag{A.1.6}$$

Here $\delta^\mu{}_\nu$ is the Kronecker delta, defined to be 1 if $\mu = \nu$ and 0 otherwise. Note that we have made a choice of signs here.

Sometimes the Minkowski metric is instead chosen to be

$$\eta^{\mu\nu} \overset{!}{=} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A.1.7}$$

These two conventions are known as $(+---)$ and $(-+++)$ for short, where the signs refer to the signs along the diagonal.

The **scalar product** of two four-vectors, a and b, in Minkowski space is defined to be

$$a \cdot b \coloneqq a^\mu b_\mu = a_\mu b^\mu = a_\mu b_\nu \eta^{\mu\nu} = a^\mu b^\nu \eta_{\mu\nu} = a^0 b^0 - \boldsymbol{a} \cdot \boldsymbol{b} \tag{A.1.8}$$

where $\boldsymbol{a} \cdot \boldsymbol{b} = a^1 b^1 + a^2 b^2 + a^3 b^3$ is the normal three-dimensional scalar product.

A.1.2 Lorentz Transformations

Lorentz transformations are linear transformations on four-vectors which leave the scalar product invariant. If Λ is a Lorentz transformation then the four-vector a transforms as

$$a^\mu \to a'^\mu = \Lambda^\mu{}_\nu a^\nu. \tag{A.1.9}$$

These are the **homogeneous Lorentz transformations**, and they form a group called the **homogeneous Lorentz group**, or just the Lorentz group, denoted $\mathrm{O}(1,3)$, where $(1,3)$ is the signature of the metric, one time component and three spatial components. Including translations as well we get the Poincaré group, $\mathbb{R}^{1,3} \rtimes \mathrm{O}(1,3)$, where $\mathbb{R}^{1,3}$ denotes translations in Minkowski space[1].

[1] for more details on groups see Symmetries of Quantum Mechanics or Symmetries of Particles and Fields

The "standard" Lorentz transformation is a **boost** along the x-axis, where the boosted frame moves at a constant speed, v, along the x-axis:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\omega & -\sinh\omega & 0 & 0 \\ -\sinh\omega & \cosh\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.1.10}$$

where

$$\tanh\omega = \beta \coloneqq \frac{v}{c}, \tag{A.1.11}$$

$$\cosh\omega = \gamma \coloneqq \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-v^2/c^2}}, \tag{A.1.12}$$

$$\sinh\omega = \gamma\beta. \tag{A.1.13}$$

In terms of coordinates we have

$$ct' = \gamma\left(ct - \frac{v}{c}x\right), \tag{A.1.14}$$

$$x' = \gamma(x - vt), \tag{A.1.15}$$

$$y' = y, \tag{A.1.16}$$

$$z' = z. \tag{A.1.17}$$

Note that the Lorentz group contains ordinary three-dimensional rotations as a subgroup, $\mathrm{O}(3)$, for example, the following Lorentz transformation is a clockwise rotation by φ about the z-axis, leaving time unaffected:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A.1.18}$$

By definition the scalar product is invariant under the action of the Lorentz group. This means that for some Lorentz transformation Λ and four-vectors a and b we have

$$b'^{\mu}b'^{\nu}\eta_{\mu\nu} = \Lambda^{\mu}{}_{\rho}a^{\rho}\Lambda^{\nu}{}_{\sigma}b^{\sigma}\eta_{\mu\nu} = a^{\rho}b^{\sigma}\eta_{\rho\sigma}. \tag{A.1.19}$$

The last equality is given by computing the scalar product after the transformation. Since a and b are arbitrary this means we must have

$$\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}\eta_{\mu\nu} = \eta_{\rho\sigma}. \tag{A.1.20}$$

We can rewrite this as

$$(\Lambda^{\mathsf{T}})_{\rho}{}^{\mu}\eta_{\mu\nu}\Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}. \tag{A.1.21}$$

In matrix notation we then have $\Lambda^{\mathsf{T}}\eta\Lambda = \eta$. Taking the determinant we have

$$\det(\Lambda^{\mathsf{T}}\eta\Lambda) = \det(\Lambda^{\mathsf{T}})\det(\eta)\det(\Lambda) = \det(\eta) \implies (\det\Lambda)^2 = 1, \tag{A.1.22}$$

where we've used $\det(AB) = \det(A)\det(B)$, and $\det(A^{\mathsf{T}}) = \det(A)$. Hence, we have

$$\det\Lambda = \pm 1 \tag{A.1.23}$$

for all $\Lambda \in \mathrm{O}(1,3)$. Transformations with $\det\Lambda = +1$ are called **proper Lorentz transformations**, and form a subgroup, $\mathrm{SO}(1,3)$. Lorentz transformations with $\det\Lambda = -1$ are called **improper Lorentz transformations**.

Now set the free indices, ρ and σ, to 0 in Equation (A.1.20) and we get

$$\Lambda^{\mu}{}_{0}\Lambda^{\nu}{}_{0}\eta_{\mu\nu} = \eta_{00} = 1. \tag{A.1.24}$$

Writing out the components this becomes

$$\Lambda^{0}{}_{0}\Lambda^{0}{}_{0} - \Lambda^{1}{}_{0}\Lambda^{1}{}_{0} - \Lambda^{2}{}_{0}\Lambda^{2}{}_{0} - \Lambda^{3}{}_{0}\Lambda^{3}{}_{0} = 1 \tag{A.1.25}$$

which gives

$$(\Lambda^{0}{}_{0})^2 = 1 + \sum_{i=1}^{3}(\Lambda^{i}{}_{0})^2 \geq 1, \tag{A.1.26}$$

where the inequality follows from the fact that $\Lambda^i{}_0$ is real, so its square is nonnegative. We can conclude from this that either

$$\Lambda^0{}_0 \geq 1 \quad \text{or} \quad \Lambda^0{}_0 \leq -1. \tag{A.1.27}$$

We call transformations with $\Lambda^0{}_0 \geq 1$ **orthochronous Lorentz transformations**, these form a subgroup, $\mathrm{O}^+(1,3)$. There is also a subgroup of proper orthochronous Lorentz transformations, $\mathrm{SO}^+(1,3)$.

We can divide homogeneous Lorentz transformations into four classes, or in more technical language, the Lie group $\mathrm{O}(1,3)$ has four connected components:

$$\begin{array}{lcc} & \det\Lambda = +1 & \det\Lambda = -1 \\ \Lambda^0{}_0 \geq +1 & \text{I} & \text{II} \\ \Lambda^0{}_0 \leq -1 & \text{III} & \text{IV} \end{array} \tag{A.1.28}$$

It is component I, the proper orthochronous Lorentz transformations, which contains the identity and so is the most important. In particular, it is the Lie group generated by the Lie algebra $\mathfrak{so}(1,3)$.

Any transformation of type II, III, or IV can be generated by combing a proper orthochronous transformation of type I with one of the following:

- **Spatial inversion**: The parity operation $x^0 \to x^0$ and $\boldsymbol{x} \to -\boldsymbol{x}$:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.1.29}$$

- **Time reversal**: the operation $x^0 \to -x^0$ and $\boldsymbol{x} \to \boldsymbol{x}$:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A.1.30}$$

- **Spacetime inversion**: The operation $x^\mu \to -x^\mu$:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.1.31}$$

If x and y are four vectors then the **spacetime interval** $s^2 \coloneqq (x-y)^\mu (x-y)_\mu$ is invariant under Lorentz transformations, since its just a scalar product of some four-vector $x - y$.

A.1.3 Classification of Four-Vectors

We classify four-vectors, a, by whether their scalar product with themselves is positive, negative, or zero:

- if $a^2 > 0$ we call a **time-like**,
- if $a^2 = 0$ we call a **light-like**,
- if $a^2 < 0$ we call a **space-like**.

A.1.4 Differential Operators

The gradient operator in normal $\mathbb{R}^n$ consists of spatial derivatives at each component. The generalisation to Minkowski space includes a time derivative, and a factor of c for dimensional consistency:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right). \tag{A.1.32}$$

Note that the derivative with respect to a contravariant vector gives a covariant operator. As with normal four-vectors we can raise the index to get the operator

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right). \tag{A.1.33}$$

The Laplacian, $\nabla^2 = \nabla \cdot \nabla$, can also be generalised to Minkowski space as the scalar product of ∂ with itself, we call the result the **d'Alembertian**:

$$\partial^2 \coloneqq \partial_\mu \partial^\mu = \partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}. \tag{A.1.34}$$

This is also denoted $\Box^2$ or $\Box$. Note that the wave equation for some field, φ, can then be written as $\partial^2 \varphi = 0$.

A.1.5 Momentum and Energy

The four-momentum, p, is a four-vector given by

$$p^\mu = \left(\frac{E}{c}, \boldsymbol{p}\right) \tag{A.1.35}$$

where E is the energy of the particle and $\boldsymbol{p}$ is the relativistic three-momentum of the particle.

For a free particle of mass m we have that

$$p^2 = \frac{E^2}{c^2} - \boldsymbol{p} \cdot \boldsymbol{p} = m^2 c^2. \tag{A.1.36}$$

This can be rewritten as

$$E^2 = |\boldsymbol{p}|^2 c^2 + m^2 c^4, \tag{A.1.37}$$

which is known as the **energy-momentum relation**, or the **mass-shell condition**.

Particles for which $p^\mu p_\mu = m^2 c^2$ are said to be on their **mass-shell**.

A.2 Electromagnetic Theory

A.2.1 Maxwell's Equations

We use **Heaviside–Lorentz units**, where $\varepsilon_0 = 1$. In these units **Maxwell's equations** are

$$\nabla \cdot \boldsymbol{B} = 0, \tag{ME1}$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \tag{ME2}$$

$$\nabla \times \boldsymbol{B} = \boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}, \tag{ME3}$$

$$\nabla \cdot \boldsymbol{E} = \rho. \tag{ME4}$$

Here $\boldsymbol{E}$ is the electric field, $\boldsymbol{B}$ is the magnetic field, $\boldsymbol{j}$ is the current density, and ρ is the charge density.

Any vector field with vanishing divergence, including $\boldsymbol{B}$, can be written as the curl of another field, say $\boldsymbol{A}$, then $\nabla\cdot\boldsymbol{B} = 0$ follows from the identity that $\nabla\cdot(\nabla\times\boldsymbol{A}) = 0$ for any vector field $\boldsymbol{A}$. Hence, we can rewrite the first equation as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{ME1'}$$

Similarly we can write $\boldsymbol{E}$ as

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\Phi \tag{ME2'}$$

since $\nabla \times (\nabla\Phi) = \boldsymbol{0}$ for all scalar fields Φ and the curl of the first term just gets us back to $\boldsymbol{B}$, so this equation can replace the second of Maxwell's equations.

Combing the third and fourth equation it is possible to derive the continuity equation

$$\nabla \cdot \boldsymbol{j} + \frac{1}{c}\frac{\partial \rho}{\partial t} = 0. \tag{A.2.1}$$

Defining the electromagnetic **four-current**, $j^{\mu}_{\text{em}} \coloneqq (\rho, \boldsymbol{j})$ then we can write this as

$$\partial_{\mu} j^{\mu}_{\text{em}} = 0. \tag{A.2.2}$$

This is a statement of charge conservation.

A.2.2 Four-Vector Potential

We can define the **four-potential** as

$$A^{\mu} \coloneqq (\Phi, \boldsymbol{A}). \tag{A.2.3}$$

Recall the identity

$$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}. \tag{A.2.4}$$

Substitute the modified Maxwell equations into the third Maxwell equation, giving

$$\nabla \times (\nabla \times \boldsymbol{A}) = \boldsymbol{j} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \frac{1}{c}\nabla\frac{\partial \Phi}{\partial t}. \tag{A.2.5}$$

On the left we can use the identity to get

$$\nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \boldsymbol{j} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \frac{1}{c}\nabla\frac{\partial \Phi}{\partial t}. \tag{A.2.6}$$

Rearranging this gives

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} + \nabla\left[\nabla \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\right] = \partial^2 \boldsymbol{A} + \nabla\left[\nabla \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\right] = \boldsymbol{j}. \tag{A.2.7}$$

Doing something similar with the fourth Maxwell equation we get

$$-\frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{A}) - \nabla^2 \Phi = \rho. \tag{A.2.8}$$

We can combine these two equations into a covariant form:

$$\partial^2 A^{\mu} - \partial^{\mu}\partial_{\nu} A^{\nu} = j^{\mu}_{\text{em}}. \tag{A.2.9}$$

A.2.3 Gauge Transformations

We can define two new potentials, $\tilde{\boldsymbol{A}}$ and $\tilde{\Phi}$, via the **gauge transformation**

$$\tilde{\boldsymbol{A}} = \boldsymbol{A} - \nabla\chi \quad \text{and} \quad \tilde{\Phi} = \Phi + \frac{1}{c}\frac{\partial\chi}{\partial t}, \tag{A.2.10}$$

or in covariant form,

$$\tilde{A}^\mu = A^\mu + \partial^\mu\chi. \tag{A.2.11}$$

Here χ is any scalar field (with second derivatives). Importantly the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ don't change if we make this transformation. We can use this to choose potentials satisfying the **Lorenz gauge** condition

$$\nabla\cdot\boldsymbol{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0, \tag{A.2.12}$$

or in covariant form,

$$\partial_\nu A^\nu = 0. \tag{A.2.13}$$

In this case Maxwell's equations reduce to two decoupled equations,

$$\partial^2\boldsymbol{A} = \boldsymbol{j}, \quad \text{and} \quad \partial^2\Phi = \rho, \tag{A.2.14}$$

or in covariant form,

$$\partial^2 A^\mu = j^\mu_{\text{em}}. \tag{A.2.15}$$

Note that the Lorenz gauge doesn't fix the potentials uniquely, since

$$\nabla\cdot\tilde{\boldsymbol{A}} + \frac{1}{c}\frac{\partial\tilde{\Phi}}{\partial t} = \nabla\cdot\boldsymbol{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} - \nabla^2\chi + \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2}. \tag{A.2.16}$$

So if the original potential, A^μ, satisfies $\partial_\mu A^\mu = 0$ then we can choose to have $\partial_\mu\tilde{A}^\mu = 0$ so long as χ is such that $\partial^2\chi = 0$.

A.2.4 Covariant Form of Maxwell's Equations

The **Maxwell tensor**, or **electromagnetic field strength tensor** is defined to be

$$F^{\mu\nu} \coloneqq \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{A.2.17}$$

Note that there is a different sign convention used, for example, by Mandl and Shaw, where

$$F^{\mu\nu} \overset{!}{\coloneqq} \partial^\nu A^\mu - \partial^\mu A^\nu. \tag{A.2.18}$$

We can think of the electromagnetic field strength tensor as the four-dimensional "curl" of the four-potential[2]

The electromagnetic field strength tensor is manifestly gauge invariant, since any gauge transformation cancels between the two terms.

[2] this becomes precise in the language of geometric algebra, where we replace curl with the wedge product, and the wedge product of two vectors gives a bivector, which for our purposes is a two index tensor.

Considering the space-time and space-space components of F we can fairly simply show that they correspond to the normal electric and magnetic fields:

$$F^{i0} = \partial^i A^0 - \partial^0 A^i = \frac{\partial \Phi}{\partial x^i} - \frac{1}{c}\frac{\partial A^i}{\partial t} = E^i, \tag{A.2.19}$$

$$F^{ij} = \partial^i A^j - \partial^j A^i = -\frac{\partial A^i}{\partial x^i} + \frac{\partial A^i}{\partial x^j} = -\varepsilon^{ijk} B^k. \tag{A.2.20}$$

Here ε^{ijk} is the Levi-Civita symbol, which is defined to be 1 if ijk is an even permutation of 123, -1 if it's an odd permutation, and 0 otherwise.

We can write F as a 4×4 matrix:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_z \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{A.2.21}$$

We can then write Equation (A.2.9) as

$$\partial_\mu F^{\mu\nu} = j^\nu_{\text{em}}. \tag{A.2.22}$$

Conservation of the electromagnetic four-current is then very obvious, since

$$\partial_\nu j^\nu_{\text{em}} = \partial_\nu \partial_\mu F^{\mu\nu} = 0, \tag{A.2.23}$$

since $F^{\mu\nu}$ is antisymmetric in μ and ν and $\partial_\nu \partial_\mu$ is symmetric in μ and ν.

A.2.5 Interaction of Matter with EM Fields

The classical Hamiltonian for a free, non-relativistic particle is

$$H = \frac{|\boldsymbol{p}|^2}{2m}. \tag{A.2.24}$$

For a particle with charge q in an electromagnetic field we modify the Hamiltonian through the **minimal coupling prescription**,

$$H \to H - q\Phi, \quad \text{and} \quad \boldsymbol{p} \to \boldsymbol{p} - \frac{q}{c}\boldsymbol{A}, \tag{A.2.25}$$

giving the classical Hamiltonian for a non-relativistic particle in an electromagnetic field:

$$H = \frac{|\boldsymbol{p} - q\boldsymbol{A}/c|^2}{2m} + q\Phi. \tag{A.2.26}$$

Note that this equation is in Heaviside–Lorentz units, in SI units there is no factor of c.

This Hamiltonian is chosen such that the force on a particle due to the electromagnetic field is

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\right), \tag{A.2.27}$$

which is the Lorentz force in Heaviside–Lorentz units, again the factor of c is absent in SI units.

A.3 Relativistic Quantum Mechanics

A.3.1 Klein–Gordon Equation

The **Schrödinger equation** for a free particle of mass m is

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\boldsymbol{r},t) = i\hbar\frac{\partial}{\partial t}\Psi(\boldsymbol{r},t). \tag{A.3.1}$$

One way to obtain this is from the non-relativistic energy-momentum relation,

$$E = \frac{|\boldsymbol{p}|^2}{2m} = H, \tag{A.3.2}$$

by substituting in the relevant operators:

$$E \to i\hbar\frac{\partial}{\partial t}, \quad \text{and} \quad \boldsymbol{p} \to -i\hbar\nabla. \tag{A.3.3}$$

To obtain a relativistic equation then we should use the relativistic energy-momentum relation,

$$E^2 = |\boldsymbol{p}|^2c^2 + m^2c^4. \tag{A.3.4}$$

Then after the operator substitution we have

$$-\hbar^2\frac{\partial^2}{\partial t^2}\varphi(\boldsymbol{r},t) = -\hbar^2c^2\nabla^2\varphi(\boldsymbol{r},t) + m^2c^2\varphi(\boldsymbol{r},t). \tag{A.3.5}$$

This is the **Klein–Gordon equation**.

We can write the Klein–Gordon equation in a manifestly covariant form as

$$\left(\partial^2 + \frac{m^2c^2}{\hbar^2}\right)\varphi(x) = 0 \qquad \text{or} \qquad (\partial^2 + \mu^2)\varphi(x) = 0 \tag{A.3.6}$$

where $\mu = mc/\hbar$. We can jump straight to this covariant form by instead using the operator prescription

$$p^\mu \to i\hbar\frac{\partial}{\partial x^\mu} = i\hbar\left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right). \tag{A.3.7}$$

Note that if $m = 0$ then the Klein–Gordon equation reduces to a wave equation, $\partial^2\varphi(x) = 0$.

The Klein–Gordon equation is valid for any spin 0 particle.

A.3.1.1 Free Particle Solutions

The Klein–Gordon equation has plane-wave solutions,

$$\varphi(\boldsymbol{r},t) = \exp[i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t]. \tag{A.3.8}$$

Substituting this into the Klein–Gordon equation we see it is a solution as long as ω, $\boldsymbol{k}$, and m are related by

$$\hbar^2\omega^2 = \hbar^2c^2|\boldsymbol{k}|^2 + m^2c^4. \tag{A.3.9}$$

Taking the square root we have

$$\hbar\omega = \pm\sqrt{\hbar^2c^2|\boldsymbol{k}|^2 + m^2c^4}. \tag{A.3.10}$$

These solutions are eigenfunctions of the momentum operator and the energy operator, with eigenvalues $\boldsymbol{p} = \hbar\boldsymbol{k}$ and $E = \hbar\omega$ respectively. Interpreting $\hbar\omega$ as the allowed energy of the free particle there is ambiguity in the sign of the total energy. We have both positive and negative energy solutions. Initially this was viewed as a problem, but later people realised that both sorts of solutions were important, with negative energy solutions corresponding to antiparticles.

Define the four-vector $k^\mu = (\omega/c, \boldsymbol{k})$. Then the solutions to the Klein–Gordon equation can be written in a covariant form as

$$\varphi(x) = \exp[-ik\cdot x] = \exp[-ik^\mu x_\mu] = \exp[-ip^\mu x_\mu/\hbar]. \tag{A.3.11}$$

This allows us to interpret the four-momentum as $p^\mu = \hbar k^\mu$.

If φ is a solution of the Klein–Gordon equation then so is its complex conjugate, φ^*. The general solution is a superposition of plane-wave solutions:

$$\varphi(x) = \int \frac{\mathrm{d}^3x}{2E_p(2\pi)^3}[f(p)\exp[-ip\cdot x/\hbar] + g(p)\exp[ip\cdot x/\hbar]]. \tag{A.3.12}$$

Here the factors in the denominator of the measure are included for future convenience, and

$$E_p \coloneqq +\sqrt{|\boldsymbol{p}|^2c^2 + m^2c^4} \tag{A.3.13}$$

is defined to be positive.

A.3.2 Continuity Equation and Probability Interpretation

Denote the Schrödinger equation by (SE) and its conjugate by (SE)*, that is

$$(\mathrm{SE}) = -\frac{\hbar^2}{2m}\nabla^2\Psi(\boldsymbol{r},t) - i\hbar\frac{\partial}{\partial t}\Psi(\boldsymbol{r},t) = 0. \tag{A.3.14}$$

Then by considering

$$\Psi^*(\mathrm{SE}) - \Psi(\mathrm{SE}) = 0 \tag{A.3.15}$$

we can derive a continuity equation

$$\nabla\cdot\boldsymbol{j} + \frac{\partial\rho}{\partial t} = 0, \tag{A.3.16}$$

where

$$\rho \coloneqq \Psi^*\Psi, \quad \text{and} \quad \boldsymbol{j} \coloneqq -\frac{i\hbar}{2m}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*). \tag{A.3.17}$$

These are the **probability density** and **probability current** respectively.

We can do the same for the Klein–Gordon equation and obtain the same continuity equation, but now with

$$\rho \coloneqq \frac{i\hbar}{2mc^2}\left(\varphi^*\frac{\partial\varphi}{\partial t} - \varphi\frac{\partial\varphi^*}{\partial t}\right), \quad \text{and} \quad \boldsymbol{j} \coloneqq -\frac{i\hbar}{2m}(\varphi^*\nabla\varphi - \varphi\nabla\varphi^*). \tag{A.3.18}$$

Here we chose to make $\boldsymbol{j}$ identical to the non-relativistic Schrödinger current. We can show that in the non-relativistic limit ρ reduces to $\varphi^*\varphi$. There is a problem however, the candidate for the probability density, $\rho(x)$, is no longer positive definite, the negative energy solutions will have $\rho(x) < 0$. This means that there is no simple probability density interpretation for ρ from the Klein–Gordon equation.

A.3.3 Dirac Equation

Dirac, aiming to avoid the problems with negative energy/probability solutions, looked for a relativistic equation linear in $\partial/\partial t$. He then proceeded to argue that the equation must be linear in spatial derivatives, since in relativity we have to treat space and time on an equal footing. We therefore start with an equation of the form

$$i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{r},t) = \hat{H}\psi(\boldsymbol{r},t) \tag{A.3.19}$$

where $\hat{H}$ is linear in space and time derivatives. These requirements mean that we can write out a candidate Hamiltonian with some unknown coefficients:

$$i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{r},t) = -i\hbar c\left(\alpha^1\frac{\partial}{\partial x^1} + \alpha^2\frac{\partial}{\partial x^2} + \alpha^3\frac{\partial}{\partial x^3}\right)\psi(\boldsymbol{r},t) + \beta mc^2\psi(\boldsymbol{r},t). \tag{A.3.20}$$

We choose to have α^i and β be dimensionless by including the required factors of c, $\hbar$, and m, we also include a conventional factor of $-i$. Using the operator prescription $\hat{\boldsymbol{p}} = -i\hbar\nabla$ we can write this more compactly as

$$i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{r},t) = (c\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} + \beta mc^2)\psi(\boldsymbol{r},t). \tag{A.3.21}$$

This is the **Dirac equation** for a free particle. We assume that the Hamiltonian is independent of position and time, which is what we would want for free particle solutions. Hence α^i and β must be independent of position and time, so commute with the derivatives, such as $\hat{\boldsymbol{p}}$, but not necessarily each other.

Next, we impose the relativistic energy-momentum equation:

$$\hat{H}^2\psi(\boldsymbol{r},t) = (c^2|\hat{\boldsymbol{p}}|^2 + m^2c^4)\psi(\boldsymbol{r},t). \tag{A.3.22}$$

We know that

$$\hat{H}^2\psi(\boldsymbol{r},t) = (c\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} + \beta mc^2)(c\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} + \beta mc^2)\psi(\boldsymbol{r},t). \tag{A.3.23}$$

Expanding the right hand side, while being careful about the ordering of terms, we get

$$\hat{H}^2\psi(\boldsymbol{r},t) = \left\{c^2[(\alpha^1)^2(\hat{p}^1)^2 + (\alpha^2)^2(\hat{p}^2)^2 + (\alpha^3)^2(\hat{p}^3)^2] + m^2c^4\beta^2\right\}\psi(\boldsymbol{r},t) \tag{A.3.24}$$

$$+\left\{(\alpha^1\alpha^2 + \alpha^2\alpha^1)\hat{p}^1\hat{p}^2 + (\alpha^2\alpha^3 + \alpha^3\alpha^2)\hat{p}^2\hat{p}^3\right\}\psi(\boldsymbol{r},t) \tag{A.3.25}$$

$$+ mc^3\left\{(\alpha^1\beta + \beta\alpha^1)\hat{p}^1 + (\alpha^2\beta + \beta\alpha^2)\hat{p}^2 + (\alpha^3\beta + \beta\alpha^3)\hat{p}^3\right\}\psi(\boldsymbol{r},t). \tag{A.3.26}$$

The relativistic energy-momentum relation is satisfied if

$$(\alpha^1)^2 = (\alpha^2)^2 = (\alpha^3)^2 = \beta^2 = 1, \tag{A.3.27}$$

$$\alpha^i\alpha^j + \alpha^j\alpha^i = 0, \tag{A.3.28}$$

$$\alpha^i\beta + \beta\alpha^i = 0, \tag{A.3.29}$$

which can be written more compactly sa

$$\{\alpha^i,\alpha^j\} = 2\delta^{ij}, \quad \{\alpha^i,\beta\} = 0, \quad \text{and} \quad \beta^2 = 1, \tag{A.3.30}$$

where $\{A, B\} = AB + BA$ is the **anticommutator**.

Clearly α^i and β cannot just be numbers for these requirements to hold. Instead, we assume they are matrices. Since $\hat{H}$ is Hermitian they must be Hermitian, and hence are square matrices. This means they must have real eigenvalues. Since these matrices square to the identity their squares have eigenvalues whose product is 1, since the determinant is the product of the eigenvalues and the determinant of the unit matrix is 1. Since the eigenvalues are real the eigenvalues of α^i and β must be ± 1.

Lemma A.3.31 We have that $\operatorname{tr} \alpha^i = \operatorname{tr} \beta = 0$.

Proof. Consider $\operatorname{tr} \alpha^i$. Since $\beta^2 = 1$ we can insert β^2 into the argument of the trace and then use the cyclic property to get

$$\operatorname{tr} \alpha^i = \operatorname{tr}(\beta^2 \alpha^i) = \operatorname{tr}(\beta \alpha^i \beta). \tag{A.3.32}$$

Now using the anticommutation relations we can replace $\alpha^i \beta$ with $-\beta \alpha^i$ and then take the negative outside the trace to get

$$\operatorname{tr} \alpha^i = \operatorname{tr}(\beta \alpha^i \beta) = \operatorname{tr}(\beta(-\beta \alpha^i)) = -\operatorname{tr}(\beta^2 \alpha^i) = -\operatorname{tr}(\alpha^i) \tag{A.3.33}$$

since $\beta^2 = 1$, and so $\operatorname{tr} \alpha^i = -\operatorname{tr} \alpha^i$, which must mean $\operatorname{tr} \alpha^i = 0$.
Similarly, consider $\operatorname{tr} \beta$, we have

$$\operatorname{tr} \beta = \operatorname{tr}((\alpha^i)^2 \beta) = \operatorname{tr}(\alpha^i \beta \alpha^i) = -\operatorname{tr}((\alpha^i)^2 \beta) = -\operatorname{tr} \beta, \tag{A.3.34}$$

and so $\operatorname{tr} \beta = 0$. □

So, if α^i and β are $n \times n$ matrices with eigenvalues ± 1 since the trace is the sum of the eigenvalues and the trace vanishes n must be even so that the eigenvalues can cancel pairwise. There is no 2×2 set of Hermitian traceless 2×2 matrices satisfying the anticommutation relations, although the Pauli matrices come close, satisfying

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}, \tag{A.3.35}$$

but there is no viable candidate for β.

The simplest representation is 4×4. The **standard representation** is

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \alpha^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{A.3.36}$$

$$\alpha^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad \alpha^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{A.3.37}$$

These can be written in a 2×2 block form as

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{and} \qquad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \tag{A.3.38}$$

where $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices. We can also write these as $\alpha^i = \sigma^1 \otimes \sigma^i$. It's not too difficult to check that these matrices have the desired properties.

Since the Hamiltonian is a 4×4 matrix the wave function, $\psi(\boldsymbol{r}, t)$, must be a four component column vector:

$$\psi(\boldsymbol{r}, t) = \begin{pmatrix} \psi_1(\boldsymbol{r}, t) \\ \psi_2(\boldsymbol{r}, t) \\ \psi_3(\boldsymbol{r}, t) \\ \psi_4(\boldsymbol{r}, t) \end{pmatrix}. \tag{A.3.39}$$

A.3.4 Probability Density

The Dirac equation for a free particle is

$$i\hbar \frac{\partial}{\partial t}\psi(\boldsymbol{r}, t) = (-i\hbar c\boldsymbol{\alpha} \cdot \vec{\nabla} + \beta mc^2)\psi(\boldsymbol{r}, t). \tag{A.3.40}$$

Here $\vec{\nabla}$ is the normal gradient operator acting to the right. From this we can construct a probability density. Take the Hermitian conjugate of this equation:

$$-i\hbar \frac{\partial}{\partial t}\psi^\dagger(\boldsymbol{r}, t) = \psi^\dagger(\boldsymbol{r}, t)\left(i\hbar c\boldsymbol{\alpha} \cdot \overleftarrow{\nabla} + \beta mc^2\right). \tag{A.3.41}$$

Here $\overleftarrow{\nabla}$ denotes a gradient operator acting to the left. Note that $\psi^\dagger$ is a row vector. Multiplying the first equation by $\psi^\dagger$ on the left, and the second by ψ on the right we obtain

$$\begin{aligned} i\hbar \frac{\partial}{\partial t}(\psi^\dagger\psi) &= -i\hbar c(\psi^\dagger\boldsymbol{\alpha} \cdot \vec{\nabla}\psi + \psi^\dagger\boldsymbol{\alpha} \cdot \overleftarrow{\nabla}\psi) && \text{(A.3.42)} \\ &= -i\hbar c(\psi^\dagger\boldsymbol{\alpha} \cdot \nabla\psi + (\nabla\psi^\dagger) \cdot \boldsymbol{\alpha}\psi) && \text{(A.3.43)} \\ &= -i\hbar c\nabla \cdot (\psi^\dagger\boldsymbol{\alpha}\psi). && \text{(A.3.44)} \end{aligned}$$

We can write this as a continuity equation,

$$\nabla \cdot \boldsymbol{j} + \frac{1}{c}\frac{\partial \rho}{\partial t} = 0, \tag{A.3.45}$$

where

$$\rho \coloneqq \psi^\dagger\psi, \qquad \text{and} \qquad \boldsymbol{j} = \psi^\dagger\boldsymbol{\alpha}\psi. \tag{A.3.46}$$

We can write this in a covariant form:

$$\partial_\mu j^\mu = 0, \tag{A.3.47}$$

where $j = (\rho, \boldsymbol{j})$ is the **four-probability current**.

This tells us that $\psi^\dagger\psi$ transforms as the time component of a four-vector, and $\psi^\dagger\boldsymbol{\alpha}\psi$ transforms as the corresponding space part.

Note that $\rho = |\psi|^2$ is positive definite, solving one of our earlier problems with the Klein–Gordon equation.

A.3.5 Free Particle Solutoins

The Dirac equation has plane-wave solutions:

$$\psi(\boldsymbol{r},t)=\psi(x)=\exp[-ik_\mu x^\mu]u(\boldsymbol{p})=\exp\left[-\frac{i}{\hbar}(cp_0t-\boldsymbol{p}\cdot r)\right]u(\boldsymbol{p}). \quad \text{(A.3.48)}$$

Here $u(\boldsymbol{p})$ is a four component column vector.

Substituting this into the Dirac equation we get

$$p^0u=(\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta mc)u. \quad \text{(A.3.49)}$$

We interpret this solution as representing a particle of energy cp^0 and momentum $\boldsymbol{p}$. Putting in the matrices α^i and β this becomes

$$\begin{pmatrix} -p^0+mc & 0 & p^3 & p^1-ip^2 \\ 0 & -p^0+mc & p^1+ip^2 & -p^3 \\ p^3 & p^1-ip^2 & -(p^0+mc) & 0 \\ p^1+ip^2 & -p^3 & 0 & -(p^0+mc) \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}=0. \quad \text{(A.3.50)}$$

The condition for a nontrivial solution for u_i to exist is that the determinant of the matrix vanishes. Computing the determinant we find that the determinant vanishes exactly when

$$[m^2c^2+|\boldsymbol{p}|^2-(p^0)^2]^2=0, \quad \text{(A.3.51)}$$

which is exactly the energy-momentum relation. Taking the square root we have

$$p^0=\pm\sqrt{m^2c^2+|\boldsymbol{p}|^2}, \quad \text{(A.3.52)}$$

so we still have negative energy solutions.

A.3.5.1 Positive Energy Solutions

First suppose $p^0>0$, that is

$$p^0=+\sqrt{m^2c^2+|\boldsymbol{p}|^2}=:p^0_+=\frac{E}{c}, \quad \text{(A.3.53)}$$

where we choose to take $E>0$ as the magnitude of the energy. Substituting into the first two rows of Equation (A.3.50) we get

$$0=(-p^0_++mc)u_1+p_3u_3+(p^1-ip^2)u_4, \quad \text{(A.3.54)}$$
$$0=(-p^0_++mc)u_2+(p^1+ip^2)u_3-p^3u_4. \quad \text{(A.3.55)}$$

Only two of the four u_i are linearly independent, so we are free to fix two values. The conventional choice is either $u_1=1$ and $u_2=0$, or $u_1=0$ and $u_2=1$. The first choice gives

$$u_3=\frac{p^3}{p^0_++mc}, \quad \text{and} \quad u_4=\frac{p^1+ip^2}{p^0_++mc}, \quad \text{(A.3.56)}$$

whereas the second gives

$$u_3=\frac{p^1-ip^2}{p^0_++mc}, \quad \text{and} \quad u_4=\frac{-p^3}{p^0_++mc}. \quad \text{(A.3.57)}$$

A.3.5.2 Negative Energy Solutions

Now suppose $p^0 < 0$, that is

$$p^0 = -\sqrt{m^2c^2 + |\boldsymbol{p}|^2} =: p^0_- = -\frac{E}{c}, \tag{A.3.58}$$

where again, we choose to have $E > 0$. We can follow a similar process to the positive energy solutions, but for the lower two rows of Equation (A.3.50). Again, we make an arbitrary choice for two components, the conventions being either $u_3 = 1$ and $u_4 = 0$, or $u_3 = 0$ and $u_4 = 1$. The first choice leads to

$$u_1 = \frac{p^3}{p^0_- - mc}, \quad \text{and} \quad u_2 = \frac{p^1 + ip^2}{p^0_- - mc}, \tag{A.3.59}$$

and the second gives

$$u_1 = \frac{p^1 - ip^2}{p^0_- - mc}, \quad \text{and} \quad u_2 = \frac{-p^3}{p^0_- - mc}. \tag{A.3.60}$$

A.3.5.3 Summary

The positive energy solutions with four-momentum $p^\mu_+ = (p^0_+, \boldsymbol{p})$ are

$$\omega^1(\boldsymbol{p}) = \begin{pmatrix} 1 \\ 0 \\ \frac{p^3}{p^0_+ + mc} \\ \frac{p^1 + ip^2}{p^0_+ + mc} \end{pmatrix}, \quad \text{and} \quad \omega^2(\boldsymbol{p}) = \omega^1(\boldsymbol{p}) = \begin{pmatrix} 0 \\ 1 \\ \frac{p^1 - ip^2}{p^0_+ + mc} \\ \frac{-p^3}{p^0_+ + mc} \end{pmatrix}. \tag{A.3.61}$$

The negative energy solutions with four-momentum $p^\mu_- = (p^0_-, -\boldsymbol{p})$ are

$$\omega^1(\boldsymbol{p}) = \begin{pmatrix} -\frac{p^1 - ip^2}{p^0_+ + mc} \\ \frac{p^3}{p^0_+ + mc} \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \omega^2(\boldsymbol{p}) = \omega^1(\boldsymbol{p}) = \begin{pmatrix} -\frac{p^3}{p^0_+ + mc} \\ -\frac{p^1 + ip^2}{p^0_+ + mc} \\ 1 \\ 0 \end{pmatrix}. \tag{A.3.62}$$

Note that there are varying conventions on how the ω^μ are labelled.

A.3.5.4 Rest Frame Solutions

When the particle is at rest, that is $\boldsymbol{p} = \boldsymbol{0}$, we have

$$\omega^1(\boldsymbol{0}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \omega^2(\boldsymbol{0}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \omega^3(\boldsymbol{0}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \omega^4(\boldsymbol{0}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{A.3.63}$$

These are such that the solutions

$$\psi^{(1)} = \exp\left[-\frac{imc^2 t}{\hbar}\right]\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \psi^{(2)} = \exp\left[-\frac{imc^2 t}{\hbar}\right]\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{A.3.64}$$

are degenerate in energy. There should therefore be another operator which commutes with the Hamiltonian and whose eigenvalues label the states. One such operator is

$$\Sigma^3 \coloneqq \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.3.65}$$

The rest frame four-component spinors, $\omega^\mu(\mathbf{0})$, are eigenvectors of Σ^3 with eigenvalues ± 1. This suggests that we can interpret the Dirac equation as describing a spin 1/2 particle, and the eigenvalues of Σ^3 are interpreted as the 3rd component of the spin in the rest frame.

The notation Σ^3 suggests we should define

$$\boldsymbol{\Sigma} \coloneqq \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \quad \text{or} \quad \Sigma^i \coloneqq \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}. \tag{A.3.66}$$

We then have

$$\left(\frac{1}{2}\hbar\boldsymbol{\Sigma}\right) \cdot \left(\frac{1}{2}\hbar\boldsymbol{\Sigma}\right) = \frac{3}{4}\hbar^2\hat{1} \tag{A.3.67}$$

and $\hbar\Sigma^3/2$ has eigenvalues of $\pm\hbar/2$ suggesting that the spin operator is

$$\hat{\boldsymbol{s}} = \frac{1}{2}\hbar\boldsymbol{\Sigma}. \tag{A.3.68}$$

Note that $\boldsymbol{\Sigma}$ does not commute with the Hamiltonian in frames other than the rest frame, when $\boldsymbol{p} = \mathbf{0}$. The operator $\hat{\boldsymbol{L}} = \hat{\boldsymbol{r}} \times \hat{\boldsymbol{p}}$ also doesn't commute with the Hamiltonian. However, the operator

$$\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \hat{\boldsymbol{s}} = \hat{\boldsymbol{r}} \times \hat{\boldsymbol{p}} + \frac{1}{2}\hbar\boldsymbol{\Sigma} \tag{A.3.69}$$

does commute with the Hamiltonian. This suggests we can interpret $\hat{\boldsymbol{J}}$ as the total angular momentum.

When $\boldsymbol{p} \neq \mathbf{0}$ there are two independent states for any fixed four-momentum. Therefore there exists some operator which commutes with $c\boldsymbol{\alpha} \cdot \hat{\boldsymbol{p}} + \beta mc^2$ and labels the states. One option is the **helicity operator**

$$\hat{h}(\boldsymbol{p}) = \begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} & 0 \\ 0 & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} \end{pmatrix}. \tag{A.3.70}$$

This commutes with $c\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} + \beta mc^2$ and has eigenvalues ± 1. We can therefore chose general plane-wave states with $\boldsymbol{p} \neq \mathbf{0}$ to be helicity states.

B

Hamiltonian Mode Expansion

The Hamiltonian for the free Klein–Gordon field is

$$H = \frac{1}{2}\int \mathrm{d}^3x \left\{(\pi(x))^2 + (\nabla\varphi(x))^2 + m^2(\varphi(x))^2\right\} \equiv H_1 + H_2 + H_3. \tag{B.0.1}$$

In Section 6.3 we claimed that the time independent contribution from each term is

$$H_1 \to \frac{1}{8}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left\{a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + a^\dagger(\boldsymbol{p})a(\boldsymbol{p})\right\}, \tag{B.0.2}$$

$$H_2 \to \frac{1}{8}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\boldsymbol{p}^2}{\omega(\boldsymbol{p})^2}\left\{a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + a^\dagger(\boldsymbol{p})a(\boldsymbol{p})\right\}, \tag{B.0.3}$$

$$H_3 \to \frac{1}{8}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{m^2}{\omega(\boldsymbol{p})^2}\left\{a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + a^\dagger(\boldsymbol{p})a(\boldsymbol{p})\right\}. \tag{B.0.4}$$

We also claimed that the time dependent parts cancel in $H_1 + H_2 + H_3$. In this section we complete the full calculation showing this to be true.

Start with the mode expansion of $\varphi(x)$:

$$\varphi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega(\boldsymbol{p})}[a(\boldsymbol{p})\mathrm{e}^{-ip\cdot x} + a^\dagger(\boldsymbol{p})\mathrm{e}^{ip\cdot x}]. \tag{B.0.5}$$

Taking the time derivative we get

$$\pi(x) = \dot{\varphi}(x) = -\frac{i}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}[a(\boldsymbol{p})\mathrm{e}^{-ip\cdot x} - a^\dagger(\boldsymbol{p})\mathrm{e}^{ip\cdot x}]. \tag{B.0.6}$$

We can then compute π^2:

$$\pi(x)^2 = -\frac{1}{4}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\int \frac{\mathrm{d}^3p'}{(2\pi)^3}[a(\boldsymbol{p})\mathrm{e}^{-ip\cdot x} - a^\dagger(\boldsymbol{p})\mathrm{e}^{ip\cdot x}] \\ [a(\boldsymbol{p}')\mathrm{e}^{-ip'\cdot x} - a^\dagger(\boldsymbol{p}')\mathrm{e}^{ip'\cdot x}] \tag{B.0.7}$$

Expand the integrand to get

$$a(\boldsymbol{p})a(\boldsymbol{p}')\mathrm{e}^{-i(p+p')\cdot x} - a(\boldsymbol{p})a^\dagger(\boldsymbol{p}')\mathrm{e}^{-i(p-p')\cdot x} \\ - a^\dagger(\boldsymbol{p})a(\boldsymbol{p}')\mathrm{e}^{i(p-p')\cdot x} + a^\dagger(\boldsymbol{p})a^\dagger(\boldsymbol{p}')\mathrm{e}^{i(p+p')\cdot x} \tag{B.0.8}$$

Now consider the identities

$$\int \mathrm{d}^3x\, \mathrm{e}^{\pm i(p+p')\cdot x} = (2\pi)^3\delta(\boldsymbol{p}+\boldsymbol{p}')\mathrm{e}^{\pm 2i\omega(\boldsymbol{p})t} \tag{B.0.9}$$

$$\int \mathrm{d}^3x\, \mathrm{e}^{\pm i(p-p')\cdot x} = (2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{p}'). \tag{B.0.10}$$

With these we can perform the x integral to get

$$\begin{aligned} H_1 = -\frac{1}{8}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\int \frac{\mathrm{d}^3p'}{(2\pi)^3}[a(\boldsymbol{p})a(\boldsymbol{p}')(2\pi)^3\delta(\boldsymbol{p}+\boldsymbol{p}')\mathrm{e}^{-2i\omega(\boldsymbol{p})t} \\ - a(\boldsymbol{p})a^\dagger(\boldsymbol{p}')(2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{p}') - a^\dagger(\boldsymbol{p})a(\boldsymbol{p}')(2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{p}') \\ + a^\dagger(\boldsymbol{p})a^\dagger(\boldsymbol{p}')(2\pi)^3\delta(\boldsymbol{p}+\boldsymbol{p}')\mathrm{e}^{2i\omega(\boldsymbol{p})t}]. \end{aligned} \tag{B.0.11}$$

Now performing the p' integral we get

$$\begin{aligned} H_1 = -\frac{1}{8}\int \frac{\mathrm{d}^3p}{(2\pi)^3}[a(\boldsymbol{p})a(-\boldsymbol{p})\mathrm{e}^{-2i\omega(\boldsymbol{p})} - a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) \\ + a^\dagger(\boldsymbol{p})a(\boldsymbol{p}) + a^\dagger(\boldsymbol{p})a^\dagger(-\boldsymbol{p})\mathrm{e}^{2i\omega(\boldsymbol{p})t}]. \end{aligned} \tag{B.0.12}$$

The time independent part of this is

$$H_1 \to \frac{1}{8}\int \frac{\mathrm{d}^3p}{(2\pi)^3}[a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + a^\dagger(\boldsymbol{p})a(\boldsymbol{p})]. \tag{B.0.13}$$

The time dependent part is

$$H_1^t = -\frac{1}{8}\int \frac{\mathrm{d}^3p}{(2\pi)^3}[a(\boldsymbol{p})a(-\boldsymbol{p})\mathrm{e}^{-2i\omega(\boldsymbol{p})} + a^\dagger(\boldsymbol{p})a^\dagger(-\boldsymbol{p})\mathrm{e}^{2i\omega(\boldsymbol{p})t}]. \tag{B.0.14}$$

This should (hopefully) cancel out with the other two terms.

Now consider the second term. We start by computing $\nabla\varphi$ in terms of the mode operators, which gives

$$\nabla\varphi(x) = \frac{i}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\boldsymbol{p}}{\omega(\boldsymbol{p})}[a(\boldsymbol{p})\mathrm{e}^{-ip\cdot x} - a^\dagger(\boldsymbol{p})\mathrm{e}^{ip\cdot x}]. \tag{B.0.15}$$

We can then compute $(\nabla\varphi)^2$:

$$\begin{aligned} (\nabla\varphi(x))^2 = -\frac{1}{4}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\boldsymbol{p}\cdot\boldsymbol{p}'}{\omega(\boldsymbol{p})\omega(\boldsymbol{p}')}[a(\boldsymbol{p})\mathrm{e}^{-ip\cdot x} - a^\dagger(\boldsymbol{p})\mathrm{e}^{ip\cdot x}] \\ [a(\boldsymbol{p}')\mathrm{e}^{-ip'\cdot x} - a^\dagger(\boldsymbol{p}')\mathrm{e}^{ip'\cdot x}]. \end{aligned} \tag{B.0.16}$$

Expanding the integrand we get

$$\begin{aligned} a(\boldsymbol{p})a(\boldsymbol{p}')\mathrm{e}^{-i(p+p')\cdot x} - a(\boldsymbol{p})a^\dagger(\boldsymbol{p}')\mathrm{e}^{-i(p-p')\cdot x} \\ - a^\dagger(\boldsymbol{p})a(\boldsymbol{p}')\mathrm{e}^{i(p-p')\cdot x} + a^\dagger(\boldsymbol{p})a^\dagger(\boldsymbol{p}')\mathrm{e}^{i(p+p')\cdot x}. \end{aligned} \tag{B.0.17}$$

We can then perform the integral over x and we'll get Dirac deltas:

$$\begin{aligned} H_2 = -\frac{1}{8}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\int \frac{\mathrm{d}^3p'}{(2\pi)^3}\frac{\boldsymbol{p}\cdot\boldsymbol{p}'}{\omega(\boldsymbol{p})\omega(\boldsymbol{p}')}[a(\boldsymbol{p})a(\boldsymbol{p}')(2\pi)^3\delta(\boldsymbol{p}+\boldsymbol{p}')\mathrm{e}^{-2i\omega(\boldsymbol{p})t} \\ - a(\boldsymbol{p})a^\dagger(\boldsymbol{p}')(2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{p}') - a^\dagger(\boldsymbol{p})a(\boldsymbol{p}')(2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{p}') \\ + a^\dagger(\boldsymbol{p})a^\dagger(\boldsymbol{p}')(2\pi)^3\delta(\boldsymbol{p}+\boldsymbol{p}')\mathrm{e}^{2i\omega(\boldsymbol{p})t}] \end{aligned} \tag{B.0.18}$$

Performing the p' integral we get

$$H_2 = -\frac{1}{8}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{|\boldsymbol{p}|^2}{\omega(\boldsymbol{p})^2}[-a(\boldsymbol{p})a(-\boldsymbol{p})\mathrm{e}^{-2i\omega(\boldsymbol{p})t} - a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) - a^\dagger(\boldsymbol{p})a(\boldsymbol{p}) - a^\dagger(\boldsymbol{p})a^\dagger(-\boldsymbol{p})\mathrm{e}^{2i\omega(\boldsymbol{p})t}]. \quad \text{(B.0.19)}$$

Note the extra negative sign from setting $\boldsymbol{p} = -\boldsymbol{p}$ in the factor out the front for the time dependent parts. The time independent part of this is

$$H_2 \to \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{|\boldsymbol{p}|^2}{\omega(\boldsymbol{p})^2}[a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + a^\dagger(\boldsymbol{p})a(\boldsymbol{p})]. \quad \text{(B.0.20)}$$

The time dependent part is

$$H_2^t = \frac{1}{8}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{|\boldsymbol{p}|^2}{\omega(\boldsymbol{p})^2}[a(\boldsymbol{p})a(-\boldsymbol{p})\mathrm{e}^{-2i\omega(\boldsymbol{p})t} + a^\dagger(\boldsymbol{p})a^\dagger(-\boldsymbol{p})\mathrm{e}^{2i\omega(\boldsymbol{p})t}]. \quad \text{(B.0.21)}$$

Finally, consider the φ^2 term:

$$\varphi(x)^2 = \frac{1}{4}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\int \frac{\mathrm{d}^3 p'}{(2\pi)^3}\frac{1}{\omega(\boldsymbol{p})\omega(\boldsymbol{p}')}[a(\boldsymbol{p})\mathrm{e}^{-ip\cdot x} + a^\dagger(\boldsymbol{p})\mathrm{e}^{ip\cdot x}] [a(\boldsymbol{p}')\mathrm{e}^{-ip'\cdot x} + a^\dagger(\boldsymbol{p}')\mathrm{e}^{ip'\cdot x}]. \quad \text{(B.0.22)}$$

Expand the integrand to get

$$a(\boldsymbol{p})a^\dagger(\boldsymbol{p}')\mathrm{e}^{-i(p+p')\cdot x} + a(\boldsymbol{p})a^\dagger(\boldsymbol{p}')\mathrm{e}^{-i(p-p')\cdot x} + a^\dagger(\boldsymbol{p})a(\boldsymbol{p}')\mathrm{e}^{i(p-p')\cdot x} + a^\dagger(\boldsymbol{p})a^\dagger(\boldsymbol{p}')\mathrm{e}^{i(p+p')\cdot x}. \quad \text{(B.0.23)}$$

We can then perform the x integral to get Dirac deltas:

$$H_3 = \frac{1}{8}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\int \frac{\mathrm{d}^3 p'}{(2\pi)^3}\frac{m^2}{\omega(\boldsymbol{p})\omega(\boldsymbol{p}')}[a(\boldsymbol{p})a(\boldsymbol{p}')(2\pi)^3\delta(\boldsymbol{p}+\boldsymbol{p}')\mathrm{e}^{-2i\omega(\boldsymbol{p})t} + a(\boldsymbol{p})a^\dagger(\boldsymbol{p}')(2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{p}') + a^\dagger(\boldsymbol{p})a(\boldsymbol{p}')(2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{p}') + a^\dagger(\boldsymbol{p})a^\dagger(\boldsymbol{p}')(2\pi)^3\delta(\boldsymbol{p}+\boldsymbol{p}')\mathrm{e}^{2i\omega(\boldsymbol{p})t}] \quad \text{(B.0.24)}$$

Performing the p' integral we get

$$H_3 = \frac{1}{8}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{m^2}{\omega(\boldsymbol{p})^2}[a(\boldsymbol{p})a(-\boldsymbol{p})\mathrm{e}^{-2i\omega(\boldsymbol{p})} + a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + a^\dagger(\boldsymbol{p})a(\boldsymbol{p}) + a^\dagger(\boldsymbol{p})a^\dagger(-\boldsymbol{p})\mathrm{e}^{2i\omega(\boldsymbol{p})t}] \quad \text{(B.0.25)}$$

The time independent part of this is

$$H_3 \to \frac{1}{8}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{m^2}{\omega(\boldsymbol{p})^2}[a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + a^\dagger(\boldsymbol{p})a(\boldsymbol{p})]. \quad \text{(B.0.26)}$$

The time dependent part is

$$H_3^t = \frac{1}{8}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{m^2}{\omega(\boldsymbol{p})^2}[a(\boldsymbol{p})a(-\boldsymbol{p})\mathrm{e}^{-2i\omega(\boldsymbol{p})t} + a^\dagger(\boldsymbol{p})a^\dagger(-\boldsymbol{p})\mathrm{e}^{2i\omega(\boldsymbol{p})t}]. \quad \text{(B.0.27)}$$

Now we consider the time dependent parts. First notice that each has a factor of

$$\frac{1}{8}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}[a(\boldsymbol{p})a(-\boldsymbol{p})\mathrm{e}^{-2i\omega(\boldsymbol{p})t} + a^\dagger(\boldsymbol{p})a^\dagger(-\boldsymbol{p})\mathrm{e}^{2i\omega(\boldsymbol{p})}], \tag{B.0.28}$$

so we focus on the terms not present here:

$$-1 + \frac{|\boldsymbol{p}|^2}{\omega(\boldsymbol{p})^2} + \frac{m^2}{\omega(\boldsymbol{p})^2} = \frac{|\boldsymbol{p}|^2 + m^2}{\omega(\boldsymbol{p})^2} - 1 = 1 - 1 = 0 \tag{B.0.29}$$

since by definition $\omega(\boldsymbol{p})^2 = |\boldsymbol{p}|^2 + m^2$.

C

Grassmann Variables

C.1 What Are They

Definition C.1.1 — Grassmann Algebra Let V be an n-dimensional vector space over a field, $\mathbb{F}$, with a basis $\{\vartheta_i\}$ for $i = 1, \dots, n$. The **Grassmann algebra** is defined to be the exterior algebra of V:

$$\mathcal{A} = \bigwedge V \coloneqq \mathbb{C} \oplus V \oplus (V \wedge V) \oplus (V \wedge V \wedge V) \oplus \cdots \tag{C.1.2}$$

where $\wedge$ is the totally antisymmetric exterior product. The elements of this algebra are called **Grassmann numbers** or **Grassmann variables**.

Less abstractly, a Grassmann algebra, $\mathcal{A}$, over $\mathbb{R}$ or $\mathbb{C}$, is constructed from a set, $\{\vartheta_i\}$, which can be added and scaled like vectors, and we define an antisymmetric product on these elements, so $\vartheta_i\vartheta_j = -\vartheta_j\vartheta_i$, or

$$\{\vartheta_i, \vartheta_j\} = 0. \tag{C.1.3}$$

A general element of this algebra is of the form

$$z = z_0 + \sum_{k=1}^{n} z_{i_1,i_2,\dots,i_k} \vartheta_{i_1} \vartheta_{i_2} \cdots \vartheta_{i_k}, \tag{C.1.4}$$

for $z_0, z_{i_1,\dots,i_k} \in \mathbb{F}$ and ϑ_{i_j} the basis vectors of V. That is, a general element is a polynomial in the basis vectors such that each basis vector appears at most once in each product, since antisymmetry requires that any term with a repeated basis vector vanishes.

If we have n generators then the Grassmann algebra is a vector space of dimension 2^n. We can always pick some canonical ordering for basis vectors, say ϑ_1, ϑ_2, and so on, and reorder any term to be in this ordering, up to a sign change. The basis for the Grassmann algebra is formed of products $\vartheta_{i_1}\vartheta_{i_2}\cdots\vartheta_{i_k}$ with $1 \le k \le n$, such that $i_j < i_{j+1}$ for all $j = 1, \dots, n$. There are 2^n such objects, since there are $\binom{n}{k}$ terms with k generators, and

$$\sum_{k=1}^{n} \binom{n}{k} = 2^n \tag{C.1.5}$$

is a well known identity.

C.2 Parity

Definition C.2.1 — Parity Let $\mathcal{A}$ be a Grassmann algebra. **parity** is defined as an automorphism, $P : \mathcal{A} \to \mathcal{A}$ defined on a single generator by

$$P(\vartheta_i) = -\vartheta_i \tag{C.2.2}$$

and as an algebra automorphism we have $P(\vartheta_i\vartheta_j) = P(\vartheta_i)P(\vartheta_j)$, and so on, and we then extend P linearly to be defined on all of $\mathcal{A}$.
Parity acting on a monomial gives

$$P(\vartheta_{i_1} \cdots \vartheta_{i_k}) = (-1)^k \vartheta_{i_1} \cdots \vartheta_{i_k}. \tag{C.2.3}$$

Parity can be thought of as a generalisation of reflection through the origin. As such it defines two eigenspaces, $\mathcal{A}^{\pm}$. If $a \in \mathcal{A}^+$ then $P(a) = a$, and if $a \in \mathcal{A}^-$ we have $P(a) = -a$. Note that a general sum of Grassmann variables may not be in either of these eigenspaces, for example, $\vartheta_1 + \vartheta_2\vartheta_3$ maps to $-\vartheta_1 + \vartheta_2\vartheta_3$ under parity.

C.3 Grassmann Differentiation

Definition C.3.1 — Grassmann Differentiation For a Grassmann algebra $\mathcal{A}$ we can define **Grassmann differentiation** to be a linear mapping, $D : \mathcal{A} \to \mathcal{A}$ satisfying the modified product rule

$$D(a_1a_2) = P(a_1)D(a_2) + D(a_1)a_2. \tag{C.3.2}$$

This product rule is such that

$$DP + PD = 0, \tag{C.3.3}$$

that is differentiation and parity anticommute. We can think of the parity operation as arising because we have to commute D past a_1 in order to take the derivative of a_2.

Note that if $a \in \mathcal{A}^{\pm}$ then $D(a) \in \mathcal{A}^{\mp}$, that is, the derivative changes the parity of a product of Grassmann variables.

We can introduce differential operators, $\partial/\partial\vartheta_i$, with respect ot a Grassmann variable, ϑ_i, by requiring that the identity

$$\frac{\partial\vartheta_j}{\partial\vartheta_i} = \delta_{ij} \tag{C.3.4}$$

holds. These then satisfy the anticommutation relations

$$\frac{\partial}{\partial\vartheta_i}\frac{\partial}{\partial\vartheta_j} + \frac{\partial}{\partial\vartheta_j}\frac{\partial}{\partial\vartheta_i} = 0, \qquad \text{and} \qquad \vartheta_i\frac{\partial}{\partial\vartheta_j} + \frac{\partial}{\partial\vartheta_j}\vartheta_i = 0, \tag{C.3.5}$$

in other words we can define the derivatives and generators as operators on $\mathcal{A}$ satisfying the above anticommutation relations.

Consider some functions $\sigma : \mathcal{A} \to \mathcal{A}^-$ and $x : \mathcal{A} \to \mathcal{A}^+$, and another function $f : \mathcal{A}^- \times \mathcal{A}^+ \to \mathcal{A}$. Then we have the chain rule

$$\frac{\partial}{\partial\vartheta} f(\sigma, x) = \frac{\partial\sigma}{\partial\vartheta}\frac{\partial f}{\partial\sigma} + \frac{\partial x}{\partial\vartheta}\frac{\partial f}{\partial x}. \tag{C.3.6}$$

Suppose now that $F : \mathbb{C} \to \mathbb{C}$ is an analytic function, at least in the region of interest. Then this function has a Taylor series at 0 given by

$$F(x) = F(0) + x\frac{\partial F}{\partial x}\bigg|_{x=0} + \mathcal{O}(x^2). \tag{C.3.7}$$

We can extend F to a function of Grassmann variables through its Taylor series, and this is particularly simple, since we have

$$F(\vartheta) = F(0) + \vartheta\frac{\partial F}{\partial x}\bigg|_{x=0} \tag{C.3.8}$$

exactly, since higher powers vanish by antisymmetry.

C.4 Grassmann Integration

Definition C.4.1 — Grassmann Integration Let $\mathcal{A}$ be a Grassmann algebra over a field, $\mathbb{F}$, and $D : \mathcal{A} \to \mathcal{A}$ a differential operator on $\mathcal{A}$. We can define another linear operator $I : \mathcal{A} \to \mathcal{A}$ called **Grassmann integration**. It is defined by requiring the following properties:

- I is linear:

$$I(\lambda_1 a_1 + \lambda_2 a_2) = \lambda_1 I(a_1) + \lambda_2 I(a_2) \tag{C.4.2}$$

  for all $\lambda_1, \lambda_2 \in \mathbb{F}$ and $a_1, a_2 \in \mathcal{A}$.

- The derivative anticommutes with parity:

$$PI + IP = 0. \tag{C.4.3}$$

- $DI = ID = 0$.
- If $D(a) = 0$ for some $a \in \mathcal{A}$ then $I(ba) = I(b)a$ for all $b \in \mathcal{A}$.

Note that all of these properties are satisfied by D, so we can define integration and differentiation to be the same:

$$\int \mathrm{d}\vartheta\, a = \frac{\partial}{\partial\vartheta} a \tag{C.4.4}$$

for all $\vartheta, a \in \mathcal{A}$.

The requirement that $ID = 0$ is the requirement that the integral of a total derivative vanishes, which we often require for integration by parts, and that an derivative of an integral vanishes is then just a statement that we want to have $DI + ID = 0$.

This definition gives

$$\int \mathrm{d}\vartheta\, f(\vartheta) = \frac{1}{\lambda}\int \mathrm{d}\vartheta'\, f(\lambda\vartheta+\mu) \tag{C.4.5}$$

for $\lambda,\mu\in\mathbb{F}$. This follows by writing ϑ as a function of ϑ', namely $\vartheta(\vartheta')=\lambda\vartheta'+\mu$. We then have

$$\begin{aligned}
\int \mathrm{d}\vartheta\, f(\vartheta) &= \frac{\partial}{\partial\vartheta} f(\vartheta) && \text{(C.4.6)}\\
&= \left(\frac{\partial}{\partial\vartheta'} f(\vartheta(\vartheta'))\right)\frac{\partial\vartheta'}{\partial\vartheta} && \text{(C.4.7)}\\
&= \left(\frac{\partial}{\partial\vartheta'} f(\vartheta(\vartheta'))\right)\left(\frac{\partial\vartheta}{\partial\vartheta'}\right)^{-1} && \text{(C.4.8)}\\
&= \left(\frac{\partial}{\partial\vartheta'} f(\vartheta(\vartheta'))\right)\frac{1}{\lambda} && \text{(C.4.9)}\\
&= \frac{1}{\lambda}\int \mathrm{d}\vartheta'\, f(\vartheta(\vartheta')) && \text{(C.4.10)}\\
&= \frac{1}{\lambda}\int \mathrm{d}\vartheta'\, f(\lambda\vartheta'+\mu). && \text{(C.4.11)}
\end{aligned}$$

This generalises further to

$$\int \mathrm{d}\vartheta_1\cdots\mathrm{d}\vartheta_k = \int \mathrm{d}\vartheta'_1\cdots\mathrm{d}\vartheta'_k\, J(\vartheta') \tag{C.4.12}$$

where J is the *inverse* of the Jacobian for normal integration, meaning

$$J^{-1} = \det\frac{\partial\vartheta_i}{\partial\vartheta'_j}. \tag{C.4.13}$$

Definition C.4.14 — Complex Conjugation Let $\mathcal{A}$ be a Grassmann algebra generated by $\{\vartheta_i\}$ with $i=1,\dots,n$. We can define another Grassmann algebra with $2n$ generators, $\{\vartheta_i,\bar{\vartheta}_i\}$. We think of $\bar{\vartheta}_i$ as the **complex conjugate** of ϑ_i.

This process of taking complex conjugates acts much like the Hermitian conjugate, in particular, $\bar{\bar{\vartheta}}_i=\vartheta_i$, $\overline{(\lambda a+\mu b)}=\lambda^*\bar{a}+\mu^*\bar{b}$ for $\lambda,\mu\in\mathbb{F}$ and $a,b\in\mathcal{A}$, and $\overline{ab}=\bar{b}\bar{a}$ for $a,b\in\mathcal{A}$.

This can be used to define a Grassmann Gaussian exponential:

$$\exp\left[\sum_{i,j=1}^{n}\bar{\vartheta}_i M_{ij}\vartheta_j\right]. \tag{C.4.15}$$

This is defined through the Taylor series, which vanishes beyond the first order term, so we can first pull out one of the sums giving

$$\prod_{i=1}^{n}\exp\left[\bar{\vartheta}_i\sum_{j=1}^{n}M_{ij}\vartheta_j\right], \tag{C.4.16}$$

and then expand the exponential giving

$$\prod_{i=1}^{n}\left(1+\bar{\vartheta}_i\sum_{j=1}^{n}M_{ij}\vartheta_j\right). \tag{C.4.17}$$

Acronyms

Symbols
1PI: One Particle Irreducible 283

C
CCR: Canonical Commutation Relation 24

E
ETCR: Equal Time Commutation Relation 29

G
GUT: Grand Unified Theory 323

L
LEC: Low Energy Coupling 322

N
NLO: next to leading order 175
NNLO: next to next to leading order 175

Q
QCD: Quantum Chromodynamics 1
QED: Quantum Electrodynamics 1
QFT: Quantum Field Theory 1

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