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Theoretical Physics

Symmetries of Quantum Mechanics

Course Notes

Symmetries of Quantum Mechanics

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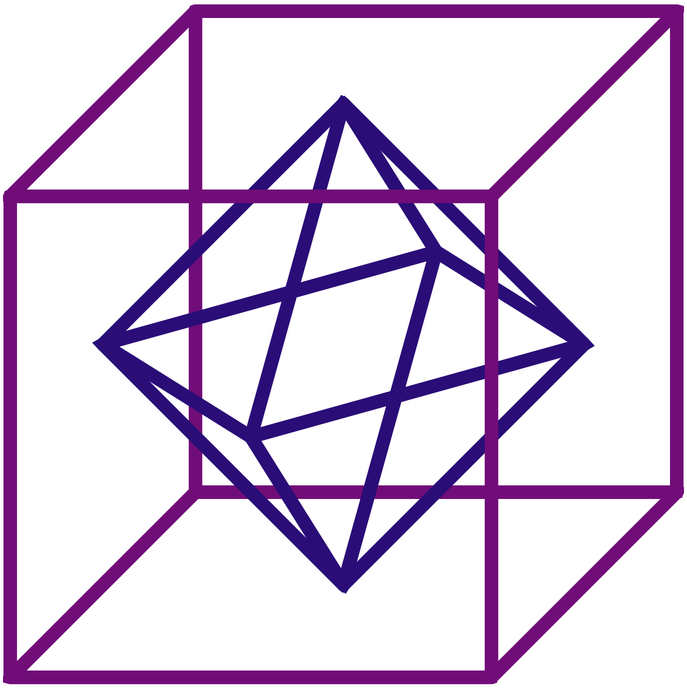
Abstract

These are my notes from the course symmetries of quantum mechanics. I took this course as a part of the theoretical physics degree at the University of Edinburgh.

These notes were last updated at 11:52 on February 16, 2022. For notes on other topics see `https://github.com/WilloughbySeago/Uni-Notes`.



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Introduction

1.1 Binary Operations

Definition 1.1.1 — Binary Operation A **binary operation** on a set X is a map, $f\colon X \times X \to X$.

R We say that X is **closed** under the binary operation since combining two elements of X gives another element of X.

Notation 1.1.2 Binary operations are usually written with infix notation, for example, the binary operation $\cdot\colon X \times X \to X$ maps $(x, x') \mapsto x \cdot x'$ whereas for a normal function, say, $f\colon X \times X \to X$, we usually use prefix notation: $(x, x') \mapsto f(x, x')$.
The other common notation when there is only one (obvious) choice of binary operation is juxtaposing the two elements, for example $(x, x') \mapsto xx'$. This is exactly what we do with multiplication most of the time rather than writing $x \cdot x$, or, $x \times x$. We will use this notation most of the time, particularly when the binary operation is denoted $\cdot$, and we will not comment on it further.

The notion of a binary operation is very general. We typically restrict ourselves to various classes of binary operations which are easier to work with due to possessing various properties.

1.1.1 Associativity

Definition 1.1.3 — Associativity We say that the binary operation $\cdot\colon X \times X \to X$ is **associative** if

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z. \tag{1.1.4}$$

From this it follows that an associative binary operation and any number of elements in a product the answer will be the same no matter how we write the brackets and so we usually don't write any brackets at all. For example, with four elements two possible ways to write the product of four elements are

$$(x_1x_2)(x_3x_4) = x_1(x_2(x_3x_4)) = x_1x_2x_3x_4. \tag{1.1.5}$$

Writing $x_3x_4 = x$ it follows that $(x_1x_2)(x_3x_4) = (x_1x_2)x = x_1(x_2x) = x_1(x_2(x_3x_4))$ where the second equality is where we apply the associativity axiom.

■ **Example 1.1.6 — Function Composition** Denote by $\mathrm{Hom}(A, B)$ the set of functions from A to B. We define **function composition** to be the binary operation $\circ\colon \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$ for sets A, B, and C, such that for $f \in \mathrm{Hom}(B, C)$ and $g \in \mathrm{Hom}(A, B)$ we have

$$(f \circ g)(a) = f(g(a)) \tag{1.1.7}$$

for all $a \in A$. An alternative way of saying this is that the following diagram commutes, meaning that the result is independent of the path taken:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[dr, "f\circ g"'] & B \arrow[d, "f"] \\
& C.
\end{tikzcd} \tag{1.1.8}$$

Function composition is associative. That is if $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$, and $h \in \mathrm{Hom}(C, D)$ then

$$(f\circ(g\circ h))(d) = f((g\circ h)(d)) = f(g(h(d))) = (f\circ g)(h(d)) = ((f\circ g)\circ h)(d),$$

or in other words

$$f \circ (g \circ h) = (f \circ g) \circ h \tag{1.1.9}$$

and so $\circ$ is associative.
The commutative diagram expressing this fact is

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] \arrow[rrr, bend right, "h\circ g\circ f"'] & B \arrow[r, "g"] \arrow[rr, bend left, "h\circ g"] & C \arrow[r, "h"] & D.
\end{tikzcd} \tag{1.1.10}$$

One important corollary is that matrix multiplication is just composition of linear maps and so matrix multiplication is associative.

Many of the binary operations that we are familiar with are associative, such as addition, and multiplication, but not all, for example, subtraction isn't associative, consider $5 - (2 - 3) = 6$ and $(5 - 2) - 3 = 0$.

■ **Example 1.1.11 — Nonassociativity** An example of a binary operation that *isn't* associative is the vector cross product, $\times\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ (note that the first $\times$ is the cross product of vectors and the second one is the Cartesian product of sets). This can be shown by an example. Take $\boldsymbol{a} = (1, 1, 0)^\top$,

$\boldsymbol{b} = (0,1,0)^\top$, and $\boldsymbol{c} = (0,0,1)^\top$. Then

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \implies (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \tag{1.1.12}$$

whereas

$$\boldsymbol{b} \times \boldsymbol{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \implies \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1.1.13}$$

1.1.2 Identity

Definition 1.1.14 — Identity Given a set, X, and a binary operation on X, $\cdot\colon X \times X \to X$, we say that $e \in X$ is the **identity** if

$$x \cdot e = e \cdot x = x \tag{1.1.15}$$

for all $x \in X$.

Notation 1.1.16 — Identities There are many notations for identities since it is an idea that emerged in many different areas before being unified by group theory and other algebraic concepts. The notation we will choose typically depends on both what the elements of X are and the nature of $\cdot$. For example,

- if the elements of X are matrices then the identity may be denoted I, or $\mathbb{1}$,
- if the elements of X are functions then the identity may be denoted id, or ι,
- if $\cdot$ can be thought of as multiplication then the identity is often denoted 1, and
- if $\cdot$ can be thought of as addition (in which case we are more likely to denote the operation $+$) then the identity is often denoted 0.

■ **Example 1.1.17 — Identities**

- The identity for multiplication in $\mathbb{R}$ is 1.
- The identity for addition in $\mathbb{Q}$ is 0.
- The identity for matrix multiplication is I, which has δ_{ij} as elements.
- The identity function is $\mathrm{id}_X\colon X \times X \to X$ defined by $\mathrm{id}_X(x) = x$ for all $x \in X$.

Not all binary operations have an identity, for example, there is no identity for the cross product. It is also important that the identity must be an element of X. For example if we set $X = \mathbb{Z}_{>0}$[1] and take our operation to be addition then there is no identity since $0 \notin \mathbb{Z}_{>0}$.

1.1.3 Inverse

Definition 1.1.18 — Inverse Given a set, X, and a binary operation on X, $\cdot\colon X \times X \to X$, such that $e \in X$ acts as an identity element then we say that $x \in X$ has an **inverse** in X if there exists some $x^{-1} \in X$ such that

$$x \cdot x^{-1} = x^{-1} \cdot x = e. \tag{1.1.19}$$

Notation 1.1.20 — Inverses If we think of the binary operation, $\cdot$, as multiplicative then we write the inverse of x as x^{-1}, taking inspiration from division being the inverse of multiplication.
If we think of the binary operation, $+$, as additive then we write the inverse of x as $-x$, and we write $y - x$ as shorthand for $y + (-x)$, taking inspiration from subtraction being the inverse of multiplication.

As with the identity it is important that the inverse is an element of X. For example, taking $X = \mathbb{N}$[2] and our operation to be addition we have an identity, $0 \in \mathbb{N}$, but no inverses (apart from 0, which is its own inverse), since, for example, -3 is the inverse of 3, but $-3 \notin \mathbb{N}$.

1.2 Groups

Definition 1.2.1 — Group Formally a **group** is an ordered pair, $(G, \cdot)$, where G is a set and $\cdot$ is a binary operation on G satisfying the following **group axioms**:

1. **Associativity**: For all $g_1, g_2, g_3 \in G$
$$g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3. \tag{1.2.2}$$
2. **Identity**: There exists some $e \in G$ such that $e \cdot g = g \cdot e = g$.
3. **Inverse**: For all $g \in G$ there exists some g^{-1} such that
$$g \cdot g^{-1} = g^{-1} \cdot g = e \tag{1.2.3}$$
where e is the identity of the group.

R In practice we don't really think of groups as an ordered pair, $(G, \cdot)$, but as a set and an operation on the set and rather than saying "the group $(G, \cdot)$" most of the time we will say "the group G under $\cdot$", or simply "the group G" when it is clear what the group operation.

Some sources include a fourth axiom:

4. **Closure**: The product of two elements of a group is another element of the group.

This is implicit however in the definition of a binary operation as a function $X \times X \to X$ and so we leave it out. It will be important to consider when we think about subgroups by restricting the binary operation to a subset.

[1] $\mathbb{Z}_{>0} \coloneqq \{1, \dots\} = \mathbb{N} \setminus \{0\}$

[2] $\mathbb{N} \coloneqq \{0, 1, \dots\} = \mathbb{Z}_{>0} \cup \{0\}$.

Notation 1.2.4 — Multiple Groups Say G and H are two groups of interest. Then we will use G and H as subscripts to differentiate between the two groups. For example the product of two elements in G may be written as $g \cdot_G g'$, as opposed to $g \cdot_H g'$, which is the product of H applied to elements of G, as we may sometimes have reason to do if, say, $H \subseteq G$. The identity in H may be denoted e_H, and so $e_H \cdot_H h = h$, but we may not have $h \cdot_G e_H = h$, since a different operation means we can have a different identity.

Lemma 1.2.5 The identity of a group is unique.

Proof. Suppose that G is a group and $e, e' \in G$ both act as identities. That is $e'g = ge = g$ for all $g \in G$. Then $e = e'e = e'$ where the first equality holds by the identity property of e' and the second by the identity property of e. This means $e = e'$ and so the identity is unique. □

Lemma 1.2.6 The inverse of a group element is unique.

Proof. Suppose that G is a group and $g \in G$ is such that $h, h' \in G$ act as inverses to G. That is $hg = gh' = e$ where $e \in G$ is the identity of G. Right multiplying $hg = gh' = e$ by h' we have $hgh' = gh'h'$. Using the inverse property of h' on both sides we have $he = eh'$ which implies $h = h'$ and so the inverse is unique. □

Lemma 1.2.7 Let G be a group and $g, h \in G$. Then $(gh)^{-1} = h^{-1}g^{-1}$.

Proof. The defining property of the inverse is that $gg^{-1} = e$ so we simply need to show that $(gh)(h^{-1}g^{-1}) = e$ and we can then identify that $h^{-1}g^{-1} = (gh)^{-1}$. Using associativity we can rewrite the brackets in the expression however we like and so we have $(gh)(h^{-1}g^{-1}) = g(hh^{-1})g^{-1} = geg^{-1} = gg^{-1} = e$. □

Lemma 1.2.8 Let G be a group, and $g \in G$. Then $(g^{-1})^{-1} = g$.

Proof. If $(g^{-1})^{-1} = g$ then we expect that $(g^{-1})^{-1}g^{-1} = e$. We can identify that $(g^{-1})^{-1}g^{-1} = (gg^{-1})^{-1}$ using Lemma 1.2.7. We then have $(g^{-1})^{-1}g^{-1} = (gg^{-1})^{-1} = e^{-1} = e$, since $ee = e$, so clearly $e^{-1} = e$. □

1.2.1 Examples of Groups

▪ **Example 1.2.9 — Additive Groups** $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ all form groups under addition. In particular the identity is 0, and the inverse of x is $-x$.

These same sets don't form groups under multiplication. The identity of multiplication is 1, and there is no number which acts as an inverse for multiplication by zero, that is there are no solutions to $0x = 1$ in $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. Noticing that zero causes issues with division it's sensible to consider these sets with zero removed. Denote by $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$ the set of integers with zero removed. This is not a group since, for example, the multiplicative inverse of 2 is $1/2$, and $1/2 \notin \mathbb{Z}^*$.

■ **Example 1.2.10 — Multiplicative Groups** $\mathbb{Q}^*$, $\mathbb{R}^*$, and $\mathbb{C}^*$ all form a group under multiplication, where we use the notation $\mathbb{Q}^* := \mathbb{Q}\setminus\{0\}$, $\mathbb{R}^* := \mathbb{R}\setminus\{0\}$, and $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. The identity is 1 and the inverse of x is $1/x$.

Definition 1.2.11 — Permutation Formally a **permutation** on n objects is a bijection

$$\sigma\colon \{1, \dots, n\} \to \{1, \dots, n\}. \tag{1.2.12}$$

This idea can be extended to any set of size n, not just $\{1, \dots, n\}$. Informally we can think of a permutation as a way of ordering the n objects such that the mth object is in position $\sigma(m)$.

■ **Example 1.2.13 — Permutation Group** The **permutation group**, S_n, defined as the set of all permutations on n objects, is a group under function composition. What this means is that if we permute the objects then permute them again we will have a permutation of the objects (closure), we can always leave the objects in the order they are (identity), and we can always undo a permutation (inverse).

■ **Example 1.2.14 — Cyclic Group** Take some $n \in \mathbb{Z}_{>0}$. We define $\mathbb{Z}_n := \{e^{2i\pi m/n} \mid m = 0, \dots, n-1\}$. This is a group under multiplication, called the **cyclic group** of order n.
Instead define $\mathbb{Z}_n := \{0, \dots, n-1\}$. This is a group under addition modulo n. This is actually the same group as the previous definition of $\mathbb{Z}_n$ (they are isomorphic, a term defined later in Definition 2.1.1).

■ **Example 1.2.15 — Rotation Group** Define $\mathrm{O}(3) := \{O \in \mathcal{M}_3(\mathbb{R}) \mid O^\top O = \mathbb{1}\}$. This is a group under matrix multiplication. The identity is the identity matrix and the inverse is the normal matrix inverse, which is guaranteed to exist since for $O \in \mathrm{O}(3)$ we have $\det O = \pm 1 \neq 0$. This is called the **rotation group**.

R Strictly this is the fundamental representation of $\mathrm{O}(3)$.

1.2.2 Basic Definitions

Definition 1.2.16 — Group Size Given some group G we classify it as **finite**, **discrete**, or **continuous**, depending on whether G has a finite number of elements, the same number of elements as $\mathbb{Z}$, or more elements than $\mathbb{Z}$.

Recall that two sets have the same cardinality if there is a bijection between them. For example, $\mathbb{Z}$, $\mathbb{Z}_{>0}$, $\mathbb{N}$, and $\mathbb{Q}$ all have the same cardinality. A set is larger than a second set if there is an injective function from the first set to the second, but not vice versa. For example, there are more real numbers than integers.

■ **Example 1.2.17** Of the groups mentioned so far S_n, and $\mathbb{Z}_n$ are finite. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}^*$, and $\mathbb{Q}^*$ are discrete. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^*$, $\mathbb{C}^*$, and $\mathrm{O}(3)$ are continuous.

Definition 1.2.18 — Order The **order** of a finite group, G, is the number of elements in G, denoted $|G|$.
Given some group, G, the **order** of $g \in G$ is the smallest $n \in \mathbb{Z}_{>0}$ such that $g^n = e$, where e is the group identity and g^n has the expected meaning of the product of g with itself n times.
Note that the order of the identity is always 1.

■ **Example 1.2.19 — Order** The order of S_n is $|S_n| = n!$. The order of $\mathbb{Z}_n$ is $|\mathbb{Z}_n| = n$.
The order of $\mathrm{e}^{2i\pi 3/9} = \mathrm{e}^{2i\pi/3} \in \mathbb{Z}_9$ is 3 since $(\mathrm{e}^{2i\pi/3})^1 = \mathrm{e}^{2i\pi/3}$, $(\mathrm{e}^{2i\pi/3})^2 = \mathrm{e}^{4i\pi/3}$, and $(\mathrm{e}^{2i\pi/3})^3 = \mathrm{e}^{6i\pi/3} = \mathrm{e}^{2i\pi} = 1$, which is the identity of $\mathbb{Z}_n$.

Definition 1.2.20 — Abelian A group, G, is **Abelian** if all of its elements commute. That is $gg' = g'g$ for all $g \in G$. If this is not the case we say that G is non-Abelian.

■ **Example 1.2.21 — Abelian** Of the groups mentioned so far $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}^*$, $\mathbb{R}^*$, $\mathbb{C}^*$, and $\mathbb{Z}^*$ are Abelian. S_n and $\mathrm{O}(3)$ are non-Abelian.

Definition 1.2.22 — Subgroup Let G be a group. We say that H is a **subgroup** of G, denoted $H \subseteq G$, if

- H is a subset of G, and
- H is a group under the group operation of G restricted to elements of H.

A subgroup is said to be a **proper subgroup** if the subgroup is not equal to the full group, or the **trivial group**, $\{e\}$.

■ **Example 1.2.23 — Subgroup** $\mathbb{Z}_3$ is a subgroup of $\mathbb{Z}_9$ since both are groups and $\mathbb{Z}_3 = \{1, \mathrm{e}^{2i\pi/3}, \mathrm{e}^{2i\pi 2/3}\} \subset \mathbb{Z}_9$.

Definition 1.2.24 — Conjugate Given a group, G, we say that $g_1, g_2 \in G$ are **conjugate** if there exists $g \in G$ such that $g_1 = g g_2 g^{-1}$.

Lemma 1.2.25 Let G be a group. Then the relation $\sim$ defined by $g \sim h$ if g and h are conjugate in G is an equivalence relation.

Proof. Let $a \in G$. Then $a = eae^{-1}$ where e is the identity of G. This shows that $a \sim a$ and so $\sim$ is reflexive.
Let $a, b \in G$ be such that $a \sim b$. Then $a = gbg^{-1}$ for some $g \in G$. Right multiplying by g and left multiplying by g^{-1} this becomes $g^{-1}ag = g^{-1}gbg^{-1}g = b$. Noticing that $g^{-1} = g' \in G$ and $g = g'^{-1}$ this becomes $b = g'ag'^{-1}$ which shows that $b \sim a$ and so $\sim$ is symmetric.
Let $a, b, c \in G$ be such that $a \sim b$ and $b \sim c$. Then there exists $g, g' \in G$ such that $a = gbg^{-1}$ and $b = g'cg'^{-1}$. Inserting the second equation into the first we see that $a = gg'cg'^{-1}g^{-1}$. Now we write $g'' = gg' \in G$ and notice from Lemma 1.2.7 that $g'^{-1}g^{-1} = (gg')^{-1} = g''^{-1}$ we can write $a = g''cg''^{-1}$ and so $a \sim c$, meaning that $\sim$ is transitive. Hence $\sim$ is an equivalence relation. □

Definition 1.2.26 — Generators Given a set $\{g_i\} \subseteq G$ we say that $\{g_i\}$ **generate** G if all elements of G can be written as a product of g_i. We call g_i **generators**.
The **rank** of a group is the size of the smallest set of generators.
If the rank of a group is 1 then there is one generator, g, and all elements are of the form g^n. We call such a group **cyclic**.

Definition 1.2.27 — Centre The **centre** of the group G is the set

$$Z(G) := \{z \in G \mid gz = zg \text{ for all } g \in G\}. \tag{1.2.28}$$

That is the centre is the set of all elements that commute with all other elements.

R Notice that the identity is always in the centre.

R The Z comes from the German *Zentrum* for centre.

Lemma 1.2.29 Given a group G the centre, $Z(G)$, is a subgroup of G.

Proof. Clearly $e \in Z(G)$ since $eg = ge$ for all $g \in G$. Let $z, z' \in Z(G)$, then $zz' \in Z(G)$ since

$$(zz')g = z(z'g) = z(gz') = (zg)z' = (gz)z' = g(zz') \tag{1.2.30}$$

for all $g \in G$. Finally let $z \in Z(G)$, then $z^{-1} \in Z(G)$ since if $gz = zg$ for all $g \in G$ then left and right multiplying by z^{-1} we get $z^{-1}gzz^{-1} = z^{-1}g = z^{-1}zgz^{-1} = gz^{-1}$, and so $z^{-1} \in Z(G)$. We have shown that $Z(G)$ is a group and by construction it is a subset of G so $Z(G)$ is a subgroup of G. □

■ **Example 1.2.31 — Centre** If G is Abelian then $Z(G) = G$ The centre of S_3 is the trivial group. The centre of $\mathrm{O}(3)$ is $Z(\mathrm{O}(3)) = \{\mathbf{1}, -\mathbf{1}\}$.

Theorem 1.2.32 — Subgroup Criteria. Let G be a group and let H be a nonempty subset of G. Then H is a subgroup of G if and only if $g_1 g_2^{-1} \in H$ for all $g_1, g_2 \in H$.

Proof. Suppose H is a subgroup of G and $g_1, g_2 \in H$. The group axioms require that $g_2^{-1} \in H$. In order for H to be closed we must have $g_1 g_2^{-1} \in H$. Hence if H is a subgroup of G then $g_1 g_2^{-1} \in H$ for all $g_1, g_2 \in H$.
Now suppose that $g_1 g_2^{-1} \in H$ for all $g_1, g_2 \in H$. Take some $g \in H$ and the condition gives $gg^{-1} = e \in H$, so the identity is in H. Using this we have $eg^{-1} = g^{-1} \in H$, so all elements of H have inverses in H. Take some $g_1, g_2 \in H$. We now know that $g_2^{-1} \in H$ and so using Lemma 1.2.8 we get $g_1(g_2^{-1})^{-1} = g_1 g_2 \in H$, thus H is closed under the operation. Hence H is a group, and by definition it is a subset of G so H is a subgroup of G. □

1.2.3 Cayley Tables

Definition 1.2.33 — Cayley Table Given a finite group, G, we can list all possible products of pairs of group elements in a table, called a **Cayley table**, or **multiplication table**. This is done by listing the elements along the edge in some chosen order, usually starting with the identity, and taking the value in the ith row and jth column as the product of the ith and jth element of the group in the chosen order. That is

$$\begin{array}{c|ccccc}
G & e & a & b & c & \cdots \\ \hline
e & e & a & b & c & \cdots \\
a & a & a^2 & ab & ac & \cdots \\
b & b & ba & b^2 & bc & \cdots \\
c & c & ca & cb & c^2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array} \tag{1.2.34}$$

The Cayley table for $\mathbb{Z}_2$ is

$$\begin{array}{c|cc}
\mathbb{Z}_2 & 1 & -1 \\ \hline
1 & 1 & -1 \\
-1 & -1 & 1
\end{array} \tag{1.2.35}$$

The Cayley table for $\mathbb{Z}_3$ is

$$\begin{array}{c|ccc}
\mathbb{Z}_3 & 1 & \mathrm{e}^{2i\pi/3} & \mathrm{e}^{2i\pi 2/3} \\ \hline
1 & 1 & \mathrm{e}^{2i\pi/3} & \mathrm{e}^{2i\pi 2/3} \\
\mathrm{e}^{2i\pi/3} & \mathrm{e}^{2i\pi/3} & \mathrm{e}^{2i\pi 2/3} & 1 \\
\mathrm{e}^{2i\pi 2/3} & \mathrm{e}^{2i\pi 2/3} & 1 & \mathrm{e}^{2i\pi/3}
\end{array} \tag{1.2.36}$$

This isn't that easy to read. There is a perhaps simpler way to think of $\mathbb{Z}_n$, as the set $\{0, \dots, n-1\}$, with addition modulo n as an operation. Using this we get the Cayley table

$\mathbb{Z}_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

(1.2.37)

Notice that the structure of these two tables is the same if we make the identification $1 \leftrightarrow 0$, $e^{2i\pi/3} \leftrightarrow 2$, and $e^{2i\pi 2/3} \leftrightarrow 2$. This is made clearer by colouring the entries in to the tables matching the colours based on this correspondence:

$\mathbb{Z}_3$	1	$e^{2i\pi/3}$	$e^{2i\pi 2/3}$
1	1	$e^{2i\pi/3}$	$e^{2i\pi 2/3}$
$e^{2i\pi/3}$	$e^{2i\pi/3}$	$e^{2i\pi 2/3}$	1
$e^{2i\pi 2/3}$	$e^{2i\pi 2/3}$	1	$e^{2i\pi/3}$

$\mathbb{Z}_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

(1.2.38)

We will see in a bit that what we really are saying here is that the multiplicative group $\{1, e^{2i\pi/3}, e^{2i\pi 2/3}\}$ and the group $\{0, 1, 2\}$ under addition modulo 3 are isomorphic, and so have the same structure and all of their group theoretical properties are the same. For this reason we often simply think of them as being the same and just consider a single group $\mathbb{Z}_3$ using whichever of these groups is most useful at the moment.

We can make a similar identification between the group generated by multiplication of $e^{2i\pi/n}$ and the group of $\{0, \dots, n-1\}$ under addition modulo n, both of which can be thought of as $\mathbb{Z}_n$ by identifying $1 \leftrightarrow 0$ and $e^{2i\pi m/n} \leftrightarrow m$. For example we can thin, of $\mathbb{Z}_2$ as addition modulo 2 on $\{0, 1\}$.

Notation 1.2.39 — Cycle Notation A **k-cycle** is a way of writing a permutation down. For $a_i \in \{0, 1, \dots, n\}$ we write $(a_1\, a_2\, \dots\, a_m)$ to denote the permutation in S_n that sends a_1 to a_2, a_2 to a_3, and so on, sending a_{m-1} to a_m, and finally a_m to a_1.
Using this notation the identity permutation is denoted $()$.

For example, consider the 2-cycle $(1\,2)$ acting on the objects tuple (a, b, c). This sends 1, which here is the first object, a, to 2, which here is the second object b, and sends the second object to the first object. We can write this as

$$(1\,2)(a, b, c) = (b, a, c). \tag{1.2.40}$$

Applying this 2-cycle a second time we get

$$(1\,2)^2(a, b, c) = (1\,2)(1\,2)(a, b, c) = (1\,2)(b, a, c) = (a, b, c) \tag{1.2.41}$$

and so we see that $(1\,2)^2 = ()$.

Now consider the 3-cycle $(1\,2\,3)$ acting on (a, b, c). We see that

$$(1\,2\,3)(a, b, c) = (c, a, b), \tag{1.2.42}$$

$$(1\,2\,3)^2 = (1\,2\,3)(c, a, b) = (b, c, a), \tag{1.2.43}$$

$$(1\,2\,3)^3 = (1\,2\,3)(b, c, a) = (a, b, c), \tag{1.2.44}$$

so $(1\,2\,3)^3 = ()$.

Carrying on like this we can build up the Cayley table for S_3, notice that all 2-cycles are self-inverses (i.e. they are order 2, so square to give the identity):

S_3	$()$	$(1\,2)$	$(2\,3)$	$(1\,3)$	$(1\,2\,3)$	$(3\,2\,1)$
$()$	$()$	$(1\,2)$	$(2\,3)$	$(1\,3)$	$(1\,2\,3)$	$(3\,2\,1)$
$(1\,2)$	$(1\,2)$	$()$	$(1\,2\,3)$	$(3\,2\,1)$	$(2\,3)$	$(1, 3)$
$(2\,3)$	$(1\,3)$	$(3\,2\,1)$	$()$	$(1\,2\,3)$	$(1\,3)$	$(1\,2)$
$(1\,3)$	$(1\,3)$	$(1\,2\,3)$	$(3\,2\,1)$	$()$	$(1\,2)$	$(2\,3)$
$(1\,2\,3)$	$(1\,2\,3)$	$(1\,3)$	$(1\,2)$	$(2\,3)$	$(3\,2\,1)$	$()$
$(3\,2\,1)$	$(3\,2\,1)$	$(2\,3)$	$(1\,3)$	$(1\,2)$	$()$	$(3\,2\,1)$

(1.2.45)

Cayley tables can be a useful way to visualise group operations for small groups. For example, we can see that S_3 has as a subgroup $\{(), (1\,2)\}$, which is the upper left hand corner of the table, and that this is equivalent to $\mathbb{Z}_2$ after making the correspondence $1 \leftrightarrow ()$ and $-1 \leftrightarrow (1\,2)$, again this can be seen more easily by colouring in the relevant entries:

$\mathbb{Z}_2$	1	-1
1	1	-1
-1	-1	1

S_3	$()$	$(1\,2)$	$(2\,3)$	$\cdots$
$()$	$()$	$(1\,2)$	$(2\,3)$	$\cdots$
$(1\,2)$	$(1\,2)$	$()$	$(1\,2\,3)$	$\cdots$
$(2\,3)$	$(1\,3)$	$(3\,2\,1)$	$()$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

(1.2.46)

Theorem 1.2.47 — Rearrangement Theorem. The rows and columns of a multiplication table are permutations of the group. That is they contain each element of the group exactly once.

Proof. Suppose that there is a row of the Cayley table for G such that $g \in G$ appears more than once, say this is the row associated with $g' \in G$. That means that there exist two elements $g_1, g_2 \in G$ such that $g'g_1 = g'g_2 = g$. Applying the left inverse to g' we get $g_1 = g_2$, and so g cannot appear more than once.
Since all columns of the table must be filled and there are $|G|$ columns and $|G|$ elements in order to have no repeats each element must appear once. □

Identifying the permutations giving the rows with the element of the group that is associated with that row we get the next theorem. The statement of the theorem is in terms of isomorphisms which we will define shortly but for now think of "is isomorphic to" as meaning "is equivalent to in the sense of the Cayley tables above having the same structure after renaming elements". Skip the proof until we've covered isomorphisms and then come back and look at it.

Theorem 1.2.48 — Cayley's Theorem. Any finite group is isomorphic to a subgroup of the symmetric group.

Proof. Let G be a finite group and let $S_{|G|}$ be the permutation group of order $|G|$. The for each $g \in G$ we can define $\sigma_g \colon G \to G$ to by $\sigma_g(g') = gg'$. This function is invertible since $\sigma_{g^{-1}}$ is its inverse, as can be seen by considering $\sigma_{g^{-1}}(\sigma_g(g')) = \sigma_{g^{-1}}(gg') = g^{-1}gg' = g'$. This means that σ_g is bijective and hence is a permutation on the set G.

Now define $\varphi\colon G \to S_{|G|}$ by $\varphi(g) = \sigma_g$. Then φ is a homomorphism since

$$\begin{aligned}(\varphi(gg'))(g'') = \sigma_{gg'}(g'') = gg'g'' = \sigma_g(g'g'')\\ = \sigma_g(\sigma_{g'}(g'')) = (\sigma_g \circ \sigma_{g'})(g'') = (\varphi(g)\varphi(g'))(g''). \qquad (1.2.49)\end{aligned}$$

Now suppose $\varphi(g) = \varphi(g')$, then $\sigma_g = \sigma_{g'}$, meaning $gg'' = g'g''$ for all $g'' \in G$, which means that $g = g'$ since we can apply g''^{-1} to the right of this equation. This shows that φ is injective.

The function $\tilde{\varphi}\colon G \to \operatorname{Im}(\varphi)$ given by $\tilde{\varphi}(g) = \varphi(g)$ is a surjective. It remains only to show that $\operatorname{Im}\varphi$ is a subgroup of $S_{|G|}$. This will be proven in Lemma 2.1.21 and so we have proven the theorem. □

Cayley's theorem is similar in nature to the Whitney embedding theorem which states that any manifold can be embedded into Euclidean space, $\mathbb{R}^n$, for suitable n. We just swap "manifold" with "group", "embedding" with "isomorphism", and "Euclidean space, $\mathbb{R}^n$" with "a subgroup of the permutation group, S_n".

1.3 Almost Groups

In this course we are interested in the study of groups. But why? Well, it turns out that groups are the natural way to discuss symmetries, and symmetries occur all over the place in physics. Broadly if we have some sort of symmetry we can always do nothing (identity) and undo the symmetry (inverses). In the act of chaining symmetries one after another it also shouldn't matter how we combine them (associativity). This is why we use groups to define symmetries. It is worth wandering briefly what happens if we relax some of the group axioms. We aren't the first to think this and many of the resulting structures have already got names. Throughout the next few chapters we will prove many theorems and lemmas. Many of theses hold for some relaxed version of a group, just check which of the group axioms are used in the proof.

The most relaxed form of a group is just a set. This isn't that interesting on its own so we move on to the next most relaxed form of a group.

Definition 1.3.1 — Magma A **magma**, $(X, \cdot)$, is a set, X, and a binary operation, $\cdot\colon X \times X \to X$.

■ Example 1.3.2 — Magma

- $(\mathbb{R}^3, \times)$ is a magma where $\mathbb{R}^3$ is the space of three-tuples of real numbers, (x, y, z) and $\times$ is the usual vector cross product such that

$$(x, y, z) \times (a, b, c) \coloneqq (yc - zb, za - xc, xb - ya). \qquad (1.3.3)$$

- $(\mathbb{R}, f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}; f(x, y) = x^3y + 2x)$ is a magma.

Definition 1.3.4 — Semigroup A **semigroup**, $(X, \cdot)$, is a magma such that $\cdot$ is associative.

■ **Example 1.3.5 — Semigroup**

- $(\mathbb{Z}_{>0}, +)$ is a semigroup.
- $(\{f^{n*}\}, *)$ is a semigroup where f is some (sufficiently smooth) function, $*$ is convolution, and $f^{n*} = f * f * \cdots * f$ for n copies of f with $n \in \mathbb{Z}_{>0}$.
- $(X, \cdot)$ is a semigroup called the null semigroup if there exists $0 \in X$ such that $xy = 0$ for all $x, y \in X$.

Definition 1.3.6 — Monoid A **monoid**, $(X, \cdot, 1)$, is a semigroup such that $1 \in X$ acts as an identity for $\cdot$.

■ **Example 1.3.7 — Monoid**

- $(\mathbb{N}, +, 0)$ is a monoid.
- $(\mathbb{Z}, \cdot, 1)$ is a monoid.
- More generally if $(R, \cdot, +, 1, 0)$ is a ring (with unity) then $(R, \cdot, 1)$ is a monoid (and $(R, +, 0)$ is an Abelian group).
- $(\mathcal{M}_n(\mathbb{F}), \cdot)$ is a monoid where $\mathcal{M}_n(\mathbb{F})$ is the set of $n \times n$ square matrices over the field $\mathbb{F}$ and $\cdot$ is matrix multiplication.
- If $(X, \cdot)$ is a semigroup then $(X \cup \{e\}, \cdot, e)$ is a monoid where we extend the definition of $\cdot$ to $X \cup \{e\}$ by defining $xe = ex = x$ for all $x \in X$ and $e^2 = e$.
- $(\{\mathrm{True}, \mathrm{False}\}, \mathrm{XOR}, \mathrm{False})$ is a monoid.
- $(\mathcal{P}(A), \cup, \emptyset)$ is a monoid where $\mathcal{P}(A)$ is the power set of some set A, $\cup$ is the union of sets, and $\emptyset$ is the empty set.
- $(\mathcal{P}(A), \cap, A)$ is a monoid where $\cap$ is the intersection of sets.
- $(\mathrm{Hom}_{\mathbf{Set}}(X, X), \circ, \mathrm{id}_X)$ is a monoid where $\mathrm{Hom}_{\mathbf{Set}}(X, X)$ is the set of all functions $X \to X$ for some set X, more generally $\mathrm{Hom}_{\mathbf{C}}(A, B)$ is the class of all morphisms from A to B in some category $\mathbf{C}$. Also $\circ$ is composition of functions and id_X is the identity function, defined by $\mathrm{id}_X(x) = x$ for all $x \in X$. It turns out that all monoids are isomorphic to a submonoid of a monoid of this type, this is similar to all groups being isomorphic to a subgroup of a permutation group, in particular by requiring invertible functions for a group we get bijections instead of just normal functions, and for finite groups bijections can be interpreted as permutations.

- $(\mathrm{End}_{\mathbf{Grp}}(G), \circ, \mathrm{id}_G)$ is a monoid where G is a group, $\mathrm{End}_{\mathbf{Grp}}(G)$ is the set of all **endomorphisms** of G, which is to say homomorphisms (to be defined in the next section) of G with itself, $\mathrm{End}_{\mathbf{Grp}}(G) := \mathrm{Hom}_{\mathbf{Grp}}(G, G)$.

- More generally $(\mathrm{End}_{\mathbf{C}}(X), \circ, \mathrm{id}_X)$ is a monoid where $\mathbf{C}$ is a category, X is an object of $\mathbf{C}$, and $\mathrm{End}_{\mathbf{C}}(X)$ is the set of morphisms from X to X in $\mathbf{C}$, that is $\mathrm{End}_{\mathbf{C}}(X) = \mathrm{Hom}_{\mathbf{C}}(X, X)$. The previous example is subsumed by this one taking $\mathbf{C} = \mathbf{Grp}$, the category of groups with group homomorphisms as morphisms.

- $(\mathrm{Hom}_{\mathbf{C}}(\bullet, \bullet), \circ, \mathrm{id}_\bullet)$ is a monoid where $\mathbf{C}$ is a category with a single object, $\bullet \in \mathrm{Obj}(\mathbf{C})$. The elements of this monoid are the morphisms from this object to itself with morphism composition as a binary operation and the identity morphism, $\mathrm{id}_\bullet$, as the identity. This operation is associative and the identity exists by the definition of a category. If all morphisms are isomorphisms then they are invertible and so this is a group.

We can now define a group having worked our way up to it one property at a time.

Definition 1.3.8 — Group A **group**, $(X, \cdot, 1, -^{-1})$, is a monoid, $(X, \cdot, 1)$, equipped with a function $-^{-1} \colon X \longrightarrow X$ such that x^{-1} acts as the inverse of x for all $x \in X$.

Note that we previously just wrote $(G, \cdot)$, leaving 1 and $-^{-1}$ out of the notation. In fact we usually don't even write the operations, we just say

- Let M be a magma with operation $\cdot$, for $(M, \cdot)$,

- Let M be a monoid with operation $\cdot$ and identity 1 for $(M, \cdot, 1)$, and

- Let G be a group with operation $\cdot$, identity 1, and inverses g^{-1} for $(G, \cdot, 1, -^{-1})$.

Two

Morphisms and Cosets

2.1 Morphisms

Definition 2.1.1 — Morphism A **homomorphism** between groups G and H is a map $\varphi\colon G \to H$ which preserves the group product. That is for all $g, g' \in G$ we have

$$\varphi(gg') = \varphi(g)\varphi(g'). \tag{2.1.2}$$

R The product gg' on the left is the group product of G whereas the product $\varphi(g)\varphi(g')$ on the right is the group product of H. We can emphasise this by writing $\varphi(g \cdot_G g') = \varphi(g) \cdot_H \varphi(g')$.

An **isomorphism** between groups G and H is a bijective homomorphism. If there exists an isomorphism between G and H then we say that G and H are **isomorphic** and denote this $G \cong H$.

An isomorphism preserves all group structure. That means we can think of isomorphic groups as being the same group, just with the labels of the elements and the group operation renamed. We've already seen one example of this, $\{\pm 1, \pm i\}$ with the group operation of multiplication is isomorphic to, and hence considered the same as, $\{0, 1, 2, 3\}$ with the group operation of addition modulo 4.

In group theory we are almost always only interested in properties holding "up to isomorphism". For example, we may say "there is one group up to isomorphism with some property", by which we actually mean that all groups with this property are isomorphic. Often the "up to isomorphism" is left implicit and we just say "there is one group with some property".

Homomorphism comes from ὁμός (*homos*) meaning same, and μορφή (*morphe*) meaning shape or form. Isomorphism comes from ἴσος (*isos*) meaning equivalent or equal, and μορφή (*morphe*) meaning shape or form.

Note that the relation $\cong$ on the set of all groups defined by $G \cong H$ if G and H are isomorphic is an equivalence relation (see Example A.1.25). This is what justifies us saying that two isomorphic groups are the same. Isomorphism is exactly what we mean when we say two groups are equivalent, rather than the stricter meaning of being exactly equal.

■ **Example 2.1.3 — Trivial Examples** Consider the trivial group, $\{e\}$, consisting of a single element, which must act as an identity. Then $\varphi\colon G \to \{e\}$ for some group G, defined by $\varphi(g) = e$ for all $g \in G$ is a homomorphism since $\varphi(gg') = e = ee = \varphi(g)\varphi(g')$. This is not an isomorphism unless $G = \{e\}$.
Similarly there exists a homomorphism between any two groups, G and H, by sending all elements of G to the identity of H.
All groups are isomorphic to themselves since the identity function, $\mathrm{id}_G\colon G \to G$, defined by $\mathrm{id}_G(g) = g$ for all $g \in G$ is an isomorphism since $\mathrm{id}_G(gg') = gg' = \mathrm{id}_G(g)\mathrm{id}_G(g')$ and id_G is a self inverse, so bijective.

■ **Example 2.1.4 — Groups of Order 2 and 3** $\mathbb{Z}_2$ and S_2 are isomorphic. First notice that there are two permutations on 2 objects, we either leave them as is, $()$, or swap them, $(1\,2)$. Then $() \mapsto 1$ and $(1\,2) \mapsto -1$ is an isomorphism. To see this note that $(1\,2)(1\,2) = ()$, that is swapping and swapping back has no net effect, and so

$$\varphi(1 \cdot 1) = \varphi(1) = () = ()() = \varphi(1)\varphi(1), \tag{2.1.5}$$
$$\varphi((-1) \cdot (-1)) = \varphi(1) = () = (1\,2)(1\,2) = \varphi(-1)\varphi(-1), \tag{2.1.6}$$
$$\varphi(1 \cdot (-1)) = \varphi(-1) = (1\,2) = ()(1\,2) = \varphi(1)\varphi(-1). \tag{2.1.7}$$

The final $\varphi((-1)\cdot 1)$ case is covered by the $\varphi(1\cdot(-1))$ case since both groups are Abelian.
In fact all groups of order two are isomorphic to $\mathbb{Z}_2$ under the isomorphism of sending the identity to 1 and the non-identity to the -1. Similarly all groups of order three are isomorphic to $\mathbb{Z}_3$.

■ **Example 2.1.8 — Discrete Isomorphisms** The group $\mathbb{Z}$ under addition is isomorphic to the group $2\mathbb{Z}$ under addition. Here $n\mathbb{Z}$ is understood to be the set of integer multiples of n, so $2\mathbb{Z}$ is the set of even integers.
One isomorphism, $\varphi\colon \mathbb{Z} \to 2\mathbb{Z}$, is the obvious choice of $\varphi(n) = 2n$. First we check that this is a homomorphism, given some $m, n \in \mathbb{Z}$ we have

$$\varphi(n + m) = 2(n + m) = 2n + 2m = \varphi(n) + \varphi(m), \tag{2.1.9}$$

so this is indeed a homomorphism.
Next we check that this is injective. Suppose $\varphi(n) = \varphi(m)$ for two elements $n, m \in \mathbb{Z}$. Then $2n = 2m$, which readily implies $n = m$, and so φ is injective.
Finally we check that this is surjective. Consider some $n \in 2\mathbb{Z}$, since this is even[a] $2|n$ and so $n/2$ is an integer. It follows that for each $n \in 2\mathbb{Z}$ we have $n/2 \in \mathbb{Z}$ as the element such that $\varphi(n/2) = n$, and so φ is surjective.
Note that we could also have identified that $\varphi^{-1}(n) = n/2$ is the inverse of φ. Either way φ is a bijective homomorphism and hence an isomorphism.

[a] Recall that $m|n$ means m divides n, meaning that n is an integer multiple of m and n/m is an integer.

■ **Example 2.1.10 — Continuous Isomorphisms** The groups $(\mathbb{R},+)$ and $(\mathbb{R}_{>0},\cdot)$ of positive real numbers under multiplication are isomorphic.
One isomorphism between these is $x \mapsto \mathrm{e}^x$, which is a homomorphism since

$$\mathrm{e}^{x+y} = \mathrm{e}^x\mathrm{e}^y \tag{2.1.11}$$

for all $x, y \in \mathbb{R}$ and is bijective since $x \mapsto \ln x$ is the inverse.

■ **Example 2.1.12 — Complex Numbers as Matrices** The multiplicative group of complex numbers is isomorphic to the subset of 2×2 real matrices

$$\left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| a, b \in \mathbb{R}\right\}. \tag{2.1.13}$$

An isomorphism between these two groups is given by

$$\varphi(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \tag{2.1.14}$$

We first check that this is a homomorphism:

$$\varphi((a+bi)(c+di)) = \varphi((ac-bd)+(ad+bc)i) = \begin{pmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{pmatrix} \tag{2.1.15}$$

and

$$\varphi(a+bi)\varphi(c+di) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac-bc & ad+bc \\ -bc-ad & bd+ac \end{pmatrix} \tag{2.1.16}$$

So this is indeed a homomorphism.
We can see that this is bijective by noticing that the inverse is simply

$$\varphi^{-1}\left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\right) = a+bi. \tag{2.1.17}$$

What we have done here is find a two-dimensional real representation of $\mathbb{C}^*$. Notice that if we restrict ourselves to complex numbers with unit modulus then we can write $a = \cos\vartheta$ and $b = \sin\vartheta$ which allows us to make an identification between complex numbers of unit modulus and two-dimensional rotations. Denoting the multiplicative group of complex numbers with unit modulus by $\mathbb{T}$, the group of 1×1 unitary matrices by $\mathrm{U}(1)$ and the group of two-dimensional rotations by $\mathrm{SO}(2)$ what we see here is that $\mathrm{U}(1) \cong \mathbb{T} \cong \mathrm{SO}(2)$, where the isomorphism between $\mathbb{T}$ and $\mathrm{U}(1)$ is the obvious one mapping $z \in \mathbb{T}$ to $(z) \in \mathrm{U}(1)$ and the isomorphism between $\mathbb{T}$ and $\mathrm{SO}(2)$ is the restriction of φ to $\mathbb{T}$.

It isn't until we get to groups of order 4 that we get two groups which *aren't* isomorphic. The two groups of order 4 are $\mathbb{Z}_2$ and the **Klein *Vierergruppe***, $\mathbb{Z}_2\times\mathbb{Z}_2$[1], which has the unique property for a group of this order that all non-trivial (i.e. not

[1] this notation will make sense when we talk about direct products of groups in Definition 5.1.1

the identity) elements are of order 2. These two groups have the Cayley tables

$$\begin{array}{r|rrrr} \mathbb{Z}_4 & 1 & -1 & i & -i \\ \hline 1 & 1 & -1 & i & -i \\ -1 & -1 & 1 & -i & i \\ i & i & -i & -1 & 1 \\ -i & -i & i & 1 & -1 \end{array} \qquad \begin{array}{c|cccc} \mathbb{Z}_2 \times \mathbb{Z}_2 & e & a & b & c \\ \hline e & e & a & b & c \\ a & a & e & c & b \\ b & b & c & e & a \\ c & c & b & a & e \end{array} \tag{2.1.18}$$

It is possible to fill the second one of these in by starting with the first row and column, which are simple taking e as the identity, and the leading diagonal, which must be e in every row since we have declared all nontrivial elements to be of order 2. This leaves the last two slots on the a line open, since each element must appear exactly once in each row and column these slots must be c and b. Continuing on we can fill out the rest of the table. Notice that we don't need to ever discuss what the elements e, a, b, and c are. It is enough to know that they form a group with this property of squaring to the identity. We will see later one possible set of elements that naturally form a group of this structure in Example 5.1.12.

There are some immediate consequences of these definitions that are worth considering.

Lemma 2.1.19 — Homomorphisms Map Identities to Identities Let G and H be groups. Then if $\varphi\colon G \longrightarrow H$ is a homomorphism $\varphi(e_G) = e_H$ where e_G and e_H are the identities of G and H respectively.

Proof. By definition $\varphi(gg') = \varphi(g)\varphi(g')$ for all $g, g' \in G$. In particular we have $\varphi(e_G g) = \varphi(e_G)\varphi(g)$, and $\varphi(e_G g) = \varphi(g) = e_H\varphi(g)$. Right multiplying these two results by $\varphi(g)^{-1}$ we have $\varphi(e_G) = e_H$. □

Lemma 2.1.20 — Homomorphisms Map Inverses to Inverses Let G and H be groups and $\varphi\colon G \longrightarrow H$ a homomorphism. Then $\varphi(g^{-1}) = \varphi(g)^{-1}$.

Proof. By definition $\varphi(gg') = \varphi(g)\varphi(g')$ for all $g, g' \in G$. By Lemma 2.1.19 we have $\varphi(e_G) = e_H$ where e_G and e_H are the identities of G and H. We then have $\varphi(gg^{-1}) = \varphi(e_G) = e_H$, and also $\varphi(gg^{-1}) = \varphi(g)\varphi(g^{-1})$. From this we see that $e_H = \varphi(g)\varphi(g^{-1})$, which is to say that $\varphi(g^{-1}) = \varphi(g)^{-1}$ since $\varphi(g)\varphi(g^{-1})$ gives the identity, which defines the inverse. □

Lemma 2.1.21 — The Image of a Homomorphism is a Subgroup Let G and H be groups and $\varphi\colon G \longrightarrow H$ a homomorphism. Then $\operatorname{Im}\varphi = \varphi(G)$ is a subgroup of H.

Proof. Consider $h, h' \in \operatorname{Im}\varphi$. Then there exists $g, g' \in G$ such that $h = \varphi(g)$ and $h' = \varphi(g')$. Since G is a group $gg'^{-1} \in G$. We then have $\varphi(gg'^{-1}) = \varphi(g)\varphi(g'^{-1}) = \varphi(g)\varphi(g')^{-1} = hh'^{-1}$ by Lemma 2.1.20 and the defining property of a homomorphism. This means that $hh'^{-1} \in \operatorname{Im}\varphi$ and hence $\operatorname{Im}\varphi$ is a subgroup of H by the subgroup criterion of Theorem 1.2.32. □

Lemma 2.1.22 Every group of rank 1 is isomorphic to some cyclic group.

Proof. First suppose that G is a finite group of order n. Then elements of G are of the form g^i for some $i = 0, \dots, n-1$. In particular $g^0 = e$. Notice that $f^i = g^j$ if $i \equiv j \bmod n$.

The map $\varphi\colon G \longrightarrow \mathbb{Z}_n$ defined by $\varphi(g^i) = i$ is then an isomorphism, using $\mathbb{Z}_n$ as the group of integers under addition modulo n. Clearly $\varphi(g^i g^j) = \varphi(g^{i+j}) = i + j = \varphi(g^i) + \varphi(g^j)$ where addition outside of the argument of φ occurs modulo n.

The inverse of this map is simply $\varphi^{-1}(i) = g^i$ and so this is a bijection. Meaning that φ is an isomorphism. □

Lemma 2.1.23 A homomorphism is injective if and only if its kernel is trivial.

Proof. Let G and H be groups and $\varphi\colon G \longrightarrow H$ a homomorphism. Suppose $\ker\varphi = \{e_G\}$, where e_G is the identity of G. Suppose $\varphi(g) = \varphi(h)$ for some $g, h \in G$. Then $\varphi(g)^{-1} = \varphi(h)^{-1}$ so $\varphi(g)\varphi(h)^{-1} = e_H$, where e_H is the identity of H. Now using Lemma 2.1.20 we can write this as $\varphi(g)\varphi(h^{-1}) = e_H$, and using the defining property of the homomorphism this gives us $\varphi(gh^{-1}) = e_H$. Finally using Lemma 2.1.19 we have $gh^{-1} = e_G$. Right multiplying by h we have $g = h$, and so φ is injective.

Suppose instead that φ is injective. Let $g \in \ker\varphi$. Then $\varphi(g) = e_H$. However, we know from Lemma 2.1.19 that $e_G \in \ker\varphi$, and so $\varphi(e_G) = e_H = \varphi(g)$, then the injective nature of φ means $e_G = g$, and so $\ker\varphi = \{e_G\}$. □

2.2 Group Presentations

Definition 2.2.1 — Group Presentations A **group presentation** is a way of defining a specific group. A generic group presentation is of the form

$$G = \langle S \mid C \rangle \tag{2.2.2}$$

which we read as "G is the group generated by the elements of S subject to the constraints C".

▪ **Example 2.2.3 — Cyclic Groups** The cyclic group, $\mathbb{Z}_n$, has the group presentation

$$\mathbb{Z}_n = \langle a \mid a^n = e \rangle. \tag{2.2.4}$$

We can identify $a = \mathrm{e}^{2i\pi/n}$ as one possible generator with the group operation of multiplication but it need not be the only one. For example if $n = 3$ then $a = \mathrm{e}^{2i\pi 2/3}$ also works.

If instead we take the group operation to be addition modulo n then $a = 1$ is a generator. For the $n = 3$ case we can also choose $a = 2$ as a generator.

■ **Example 2.2.5 — Permutation Group** The following is group presentations of S_3:

$$S_3 = \langle a, b, c \mid a^2 = b^2 = c^3 = abc = e \rangle. \tag{2.2.6}$$

We have a fair amount of choice here about exactly which elements a, b, and c are, clearly $e = ()$ is the identity. We can then choose a and b to be any of the 2-cycles, $(1\,2)$, $(1\,3)$, and $(2\,3)$, and c as one of the 3-cycles, $(1\,2\,3)$ or $(3\,2\,1)$.

Group presentations aren't unique. For example, the following is another valid presentation of S_3:

$$S_3 = \langle A, B \mid A^2 = B^2 = (AB)^3 = e \rangle. \tag{2.2.7}$$

We can see from this that A and B must be 2-cycles and AB is a three cycle.

■ **Example 2.2.8 — Quaternion Group** The **quaternion group** is the group with the presentation

$$Q := \langle -e, i, j, k \mid (-e)^2 = e, i^2 = j^2 = k^2 = ijk = -e \rangle. \tag{2.2.9}$$

This group is of order $|Q| = 8$, with $Q = \{\pm e, \pm i, \pm j, \pm k\}$, where by $-i$ we mean $(-e)i$.

Making the identification of $e = 1$ and taking i, j, and k, as the quaternions $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ respectively it is clear that this is a subset of the quaternions, $\mathbb{H}$. We can think Q being to $\mathbb{H}$ as $\mathbb{Z}_4 = \{\pm 1, \pm i\}$ is to $\mathbb{C}$, or $\mathbb{Z}_2 = \{\pm 1\}$ is to $\mathbb{R}$.

Another identification we can make is $-e = -\mathbf{1}$, $i = \sigma_1$, $j = \sigma_2$, and $k = \sigma_3$, where σ_i are the Pauli matrices. In fact the Pauli matrices are a representation of the quaternion group.

2.3 Cosets

Definition 2.3.1 — Coset Given some group G and subgroup H we define for each $g \in G$ the left (right) **coset** to be the set

$$gH := \{gh \mid h \in H\} \tag{2.3.2}$$

$$(Hg := \{hg \mid h \in H\}). \tag{2.3.3}$$

Typically we will state and prove things for left cosets and then the equivalent statement about right cosets will hold and be proven in exactly the same way. We will often refer simply to cosets when we mean left cosets.

Definition 2.3.4 — Partition Given a nonempty set, X, we say that the collection of sets $\{P_i \subseteq X\}$ is a **partition** or **decomposition** of X if

- $\bigcup_i P_i = X$, and
- $P_i \cap P_j = \emptyset$ if $i \neq j$.

That is every element of X is in exactly one of P_i. We can assume that P_i are non-empty.

Lemma 2.3.5 Let G be a group with subgroup H. Then the set of all cosets, gH, partitions G. Further, all cosets are of the same size, meaning $|gH| = |H|$ for all $g \in G$.

Proof. In order for the cosets to be a partition we must show that two cosets are either equal or disjoint. Consider some element $g_1 \in G$ which is not in some coset g_2H. Clearly this means that the two cosets g_1H and g_2H are equal since $e \in H$ so $g_1e = g_1$ is in g_1H. Suppose then that $g_1H \cap g_2H \neq \emptyset$. Then it follows that there exist some $h_1, h_2 \in H$ such that $g_1h_1 = g_2h_2$. This then means that $g_1 = g_2h_2h_1^{-1}$, however, since H is a group $h_2h_1^{-1} \in H$, and so $g_1 = g_2h$ for some $h = h_2h_1^{-1} \in H$ meaning that $g_1 \in g_2H$, which contradicts our earlier assumption. Hence $g_1H \cap g_2H = \emptyset$. Combining this with noticing that for $g \in G$ we have $g \in gH$ since $e \in H$ and so $g = ge$ means that $g \in gH$ proves the first part of the statement, that gH partition G.
Consider the map $g_1H \longrightarrow g_2H$ defined by $g_1h \longmapsto g_2h$ for $h \in H$. This is invertible since inverses are unique and hence $|g_1H| = |g_2H|$. In particular taking $g_1 = g$ and $g_2 = e$ we have $|gH| = |H|$. □

The fact that gH partition G into sets of equal size allows us to prove the next famous theorem. But first, a definition.

Definition 2.3.6 — Index Given a finite group G with subgroup H we define the **index** of H in G to be

$$[G : H] := \frac{|G|}{|H|}. \tag{2.3.7}$$

Theorem 2.3.8 — Lagrange's Theorem. Given a finite group G with subgroup H the index $[G : H]$ is an integer.

Proof. The cosets partition G into sets of size $|H|$. Suppose that there are n distinct cosets. Then $|G| = n|H|$, meaning that $|G|/|H| = n$. □

Lagrange's theorem says that a subset can be a subgroup only if the cardinality of the subset divides the order of the group. Notice that just because this holds does not mean that the subset is a subgroup. There is also no requirement that just because a number divides the order of the group that there is a subgroup of that order. Lagrange's theorem is much better for ruling out possible subgroups than it is for actually finding them.

▪ **Application 2.3.9** In particle physics and statistical mechanics if a continuous global symmetry given by the group G is broken to some subgroup H then it is possible to formulate an effective field theory in terms of cosets. For example in the theory of strong interactions, quantum chromodynamics

> (QCD), the breaking of left-right symmetry, known as chiral symmetry, gives rise to the effective theory of pions, known as chiral perturbation theory.

Three

Group Action

3.1 Group Action

Definition 3.1.1 Let G be a group and X a set. A **group action** is a map, $\varphi\colon G \times X \to X$, where we use the notation $\varphi(g, x) = g \mathbin{.} x$. This map must be compatible with the group structure, by which we mean

- $e \mathbin{.} x = x$ ($\varphi(e, x) = x$) for all $x \in X$, and
- $(gg') \mathbin{.} x = g \mathbin{.} (g' \mathbin{.} x)$ ($\varphi(gg', x) = \varphi(g, \varphi(g', x))$) for all $g, g' \in G$ and $x \in X$.

! Be careful to distinguish "." used to the group action and "$\cdot$" used to denote a group product, in general $g \mathbin{.} g' \neq g \cdot g'$. This is another good reason *not* to use a dot to denote the group product.

R Technically what we have defined here is a *left* group action. We can also define a right group action in a similar way. Let G be a group and X a set. The right group action is a map, $\varphi\colon X \times G \to X$, where we use the notation $\varphi(x, g) = x \mathbin{.} g$. This map must be compatible with the group structure, by which we mean

- $x \mathbin{.} e = x$ ($\varphi(x, e) = x$) for all $x \in X$, and
- $x \mathbin{.} (gg') = (x \mathbin{.} g) \mathbin{.} g'$ ($\varphi(x, gg') = \varphi(\varphi(x, g), g')$) for all $g, g' \in G$ and $x \in X$.

The difference between left and right group actions is subtle. For a left group action if we act on x with the product gg' then g' acts first, for a right group action g acts first.

An alternative definition for finite groups is that a group action is a group homomorphism from G into $S_{|X|}$. That is we can think of the action as taking an element of G and then determining how to reorder the elements of X based on this choice.

Identifying $S_{|X|}$ with bijections from X to X we can further define a group action to be a homomorphism $\varphi\colon G \to \mathrm{Aut}(X)$, where $\mathrm{Aut}(X)$ is the group of automorphisms on X, which is to say exactly the set of bijections $X \to X$, with the group product being function composition.

■ **Example 3.1.2 — General Examples** For any group G and set X the trivial group action is

$$g \,.\, x = x \tag{3.1.3}$$

for all $x \in X$.
For the special case of $X = G$ we have a variety of choices for the group action of a group on itself:

- Left multiplication: $g \,.\, x = gx$ for all $g, x \in G$,
- Right multiplication: $g \,.\, x = xg$ for all $g, x \in G$ (strictly this is a *right* group action), and
- Conjugation: $g \,.\, x = gxg^{-1}$.

■ **Example 3.1.4 — Specific Examples** Let S_n be the permutation group on n objects and X the set of all tuples $(x_1, \dots, x_n)$ such that x_i are unique. Then S_n acts on $x = (1, \dots, n) \in X$ by permuting the elements. Notice that by acting on x with S_n we can get any element of X and that $|X| = n!$.
The group $(\mathbb{Z}, +)$ acts on the set $\mathbb{R}$ as $m \,.\, r = (-1)^m r$ for $m \in \mathbb{Z}$ and $r \in \mathbb{R}$.

■ **Example 3.1.5 — Representation** The group action of a group on a linear space is called a representation and will be the subject of study of much of the rest of this course.
The group action of a group on a nonlinear space is called a nonlinear representation, and is beyond the scope of this course.

■ **Application 3.1.6** Hilbert spaces describing wave functions are linear spaces. Particles correspond to representations of the Lorentz group, $\mathrm{O}(1,3)$, which is the set of all Lorentz transformations with matrix multiplication as a group action, as well as various internal symmetry groups. That is group actions of the Lorentz group, on the Hilbert space of wave functions gives particles.

■ **Application 3.1.7 — Gauge Theory** In a gauge theory the gauge group acts on the gauge potential.
One family of groups that appears in this context are the **unitary groups**

$$\mathrm{U}(n) := \{U \in \mathcal{M}_n(\mathbb{C}) \mid U^\dagger U = UU^\dagger = \mathbf{1}\}. \tag{3.1.8}$$

In particular we often deal with

$$\mathrm{U}(1) = \{z \in \mathbb{C} \mid z^* z = zz^* = |z|^2 = 1\} = \{\mathrm{e}^{i\varphi} \mid \varphi \in \mathbb{R}\} \tag{3.1.9}$$

which is the group of complex numbers with unit modulus[a]. This particular group is the gauge group of (quantum) electrodynamics and it's action on the gauge potential is

$$\mathrm{e}^{i\varphi(x)} \,.\, A_\mu = A_\mu + \partial_\mu \varphi(x). \tag{3.1.10}$$

Recall that adding the derivative of a function to the electromagnetic potential doesn't change the electric or magnetic fields, $-\nabla A^0$ and $\nabla \times \boldsymbol{A}$.

R See the notes for quantum theory for more discussion of gauge invariance, and the notes from the particle physics part of relativity, nuclear, and particle physics for a discussion of QED.

[a] Strictly the group of complex numbers with unit modulus is the circle group, $\mathbb{T}$, and $\mathrm{U}(1)$ is the group of 1×1 unitary matrices over the complex numbers, however, under the obvious correspondence that $(z) \in \mathrm{U}(1)$ should correspond to $z \in \mathbb{T}$ these two groups are isomorphic and therefore we don't distinguish between them.

3.2 Orbits and Stabilisers

Definition 3.2.1 — Orbit Given a group G which acts on the set X for each $x \in X$ we define the **orbit** of x to be the set, $g(x)$, containing all elements of X which are reached by acting on x with an element of G. That is

$$g(x) \coloneqq \{g \,.\, x \mid g \in G\} \subseteq X. \tag{3.2.2}$$

Notation 3.2.3 Sometimes the orbit is denoted $G(x)$ or using a notation similar to cosets $G \,.\, x$. Yet another notation is $\mathrm{Orb}_G(x)$, or $\mathrm{Orb}(x)$ when the group is clear.

■ **Example 3.2.4** Consider the group of rotations in the plane about some point $\boldsymbol{a}$. The orbit of some other point $\boldsymbol{b}$ is the circle of radius $|\boldsymbol{a} - \boldsymbol{b}|$ around the point $\boldsymbol{a}$.

Consider the group of integers, $\mathbb{Z}$, acting on $\mathbb{R}$ by $n \,.\, r = r + n$ for $n \in \mathbb{Z}$ and $r \in \mathbb{R}$. The orbit of $1 \in \mathbb{R}$ is the set $\mathbb{Z} \subset \mathbb{R}$.

Consider the group $\mathbb{R}^*$ acting on some vector space, V, by $r \,.\, \boldsymbol{v} = r\boldsymbol{v}$ for $r \in \mathbb{R}$ and $\boldsymbol{v} \in V$. Then the orbit of $\boldsymbol{v}$ is all vectors parallel to $\boldsymbol{v}$.

Definition 3.2.5 — Stabiliser Given a group G which acts on the set X for each $x \in X$ we define the **stabiliser** of x to be the subset, G_x, of G which leaves x invariant under the group action. That is

$$G_x \coloneqq \{g \in G \mid g \,.\, x = x\}. \tag{3.2.6}$$

Notation 3.2.7 Sometimes the stabiliser is denoted $\mathrm{Stab}_G(x)$, or $\mathrm{Stab}(x)$ where the group is clear.

■ **Example 3.2.8** Consider the group S_4, which acts on the set of strings of four objects by permutation. The stabiliser of (a, b, c, c) is $\mathrm{Stab}_{S_4}((a, b, c, c)) = \{(), (3\,4)\}$.

■ **Example 3.2.9** Consider the group $\mathrm{GL}(2, \mathbb{R})$ acting on $\mathbb{R}^2$ by matrix multiplication. Since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \tag{3.2.10}$$

the stabiliser of $(1, 0)^\top$ is

$$\mathrm{Stab}_{\mathrm{GL}(2,\mathbb{R})}((1, 0)^\top) = \left\{ \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \middle| b, d \in \mathbb{R} \text{ and } d \neq 0 \right\}. \tag{3.2.11}$$

Note that we require $d \neq 0$ so that $\det A = d \neq 0$ since $A \in \mathrm{GL}(2, \mathbb{R})$ must be invertible.

■ **Example 3.2.12** Let G be a group which acts on itself by conjugation. Then the stabiliser of $a \in G$ is

$$\mathrm{Stab}_G(a) = \{g \in G \mid gag^{-1} = a\}, \tag{3.2.13}$$

which we can think of as the set of all elements, g, which commute with a, which follows by right multiplying the condition by g to get $gag^{-1}g = ga = ag$. This is sometimes called the **centraliser** of a, denoted $C_G(a)$. Compare this to the centre of G, $Z(G)$, which is the set of all commuting elements in G.

Lemma 3.2.14 If G is a group and acts on the set X then the stabiliser is a subgroup of G for any element of X.

Proof. Let $x \in X$. Suppose that $g, h \in \mathrm{Stab}(x)$, that is $g\,.\,x = h\,.\,x = x$. Then $(gh^{-1})\,.\,x = g\,.\,(h^{-1}\,.\,x)$. Since $h^{-1}h = e$ we have that $(h^{-1}h)\,.\,x = e\,.\,x = x$, we also have that $(h^{-1}h)\,.\,x = h^{-1}\,.\,(h\,.\,x) = h^{-1}\,.\,x$ Hence $h^{-1}\,.\,x = x$. Using this we have $(gh^{-1})\,.\,x = g\,.\,x = x$, and so $gh^{-1} \in \mathrm{Stab}(x)$. Thus by the subgroup criterion (Theorem 1.2.32) $\mathrm{Stab}(x)$ is a subgroup of G. □

Theorem 3.2.15 — Orbit-Stabiliser Theorem. Let G be a group and $g \in G$. Define an action of G on some set X with $x \in X$. The map $\varphi g(x) \to G/\mathrm{Stab}(x)$, where $G/\mathrm{Stab}(x)$ is the set of all cosets of $\mathrm{Stab}(x)$, defined by $g\,.\,x \mapsto g\,\mathrm{Stab}(x)$ defines a bijection.

Proof. Since $\mathrm{Stab}(x)$ is a subgroup of G by Lemma 3.2.14 we can define cosets, gG_x, which partition G by Lemma 2.3.5. Hence the map is surjective.
To demonstrate that this map is injective we need to prove that if $g_1\,\mathrm{Stab}(x) =$

$g_2 \operatorname{Stab}(x)$ then $g_1 . x = g_2 . x$. Notice that if $g_1 \operatorname{Stab}(x) = g_2 \operatorname{Stab}(x)$ then there exists some $g \in \operatorname{Stab}(x)$ such that $g_1 = g_2 g$ by the definition of a coset. We therefore have

$$g_1 . x = (g_2 g) . x = g_2 . g . x = g_2 . x \tag{3.2.16}$$

where in the last step we have used $g \in \operatorname{Stab}(x)$ and so by definition $g . x = x$. We have therefore demonstrated that this map is injective. Hence the map is a bijection. □

Corollary 3.2.17 Given a finite group, G, which acts on a set X for all $x \in X$ we have

$$|G| = |\operatorname{Stab}(x)||\operatorname{Orb}(x)|. \tag{3.2.18}$$

Proof. By Lagrange's theorem (Theorem 2.3.8) $|G|/|G_x|$ is an integer. Further we can identify this as the number of sets in the partition of G by $\operatorname{Stab}(x)$, which has size $|G/\operatorname{Stab}(x)|$, which is exactly $|\operatorname{Orb}(x)|$, since by the orbit-stabiliser theorem (Theorem 3.2.15) the set of sets partitioning G by $\operatorname{Stab}(x)$, $G/\operatorname{Stab}(x)$ is in bijection with the set of orbits, $\operatorname{Orb}(x)$. □

Consider the limiting cases of this theorem. If $\operatorname{Stab}(x) = G$ then $|\operatorname{Orb}(x)| = 1$, since all elements of G leave x fixed. Clearly $|G| = |\operatorname{Stab}(x)|$, and so $|G| = |\operatorname{Stab}(x)||\operatorname{Orb}(x)|$ in accordance with the theorem.

If $\operatorname{Stab}(x) = \{e\}$ then $|\operatorname{Orb}(x)| = |G|$, since all elements of $G \setminus \{e\}$ must change x. Then $|G| = |\operatorname{Stab}(x)||\operatorname{Orb}(x)|$ in accordance with the theorem.

Theorem 3.2.19 — Cauchy's Theorem. Let G be a group and let p be prime. If $|G|/p$ is an integer then there exists an element of order p in G. That is there exists $g \in G$ with $g \neq e$ such that $g^p = e$ where e is the identity of G.

Proof. Let G be a group with $|G|/p$ an integer for some prime p. Define G^p to be the set of p-length strings of elements of G. That is $x \in G^p$ is of the form $(x_1, \dots, x_p)$ for $x_i \in G$. Define $X \subset G^p$ to be the set of elements $(x_1, \dots, x_p)$ such that $x_1 \cdots x_p = e$.
The size of X is $|X| = |G|^{p-1}$. This follows since for x_1 we can pick any element of G. For x_2 we can pick any element of G. So on until x_{p-1} for which we can pick any element of G. We then have to chose x_p such that $x_p = (x_1 \cdots x_{p-1})^{-1}$. We then have

$$x_1 \cdots x_p = (x_1 \cdots x_{p-1})x_p = (x_1 \cdots x_{p-1})(x_1 \cdots x_{p-1})^{-1} = e. \tag{3.2.20}$$

Hence we make a choice from a set of size $|G|$ $p - 1$ times and so we have $|G|^{p-1}$ possible elements in X.
Since $|X|$ is a multiple of $|G|$ and $|G|$ is divisible by p $|X|$ is also divisible by p.
Define the group action of $\mathbb{Z}_p$ on X by cyclic permutation. That is given

$m \in \mathbb{Z}_p$ and $x = (x_1, \dots, x_p) \in X$ define

$$m \, . \, x = (x_{1+m}, \dots, x_{p+m}) = (x_{m+1}, \dots, x_p, x_1, \dots, x_m) \tag{3.2.21}$$

where addition in the indices is done modulo m to so the indices remain in $\{1, \dots, m\}$.
Clearly $|\mathbb{Z}_p| = p$. Then by Corollary 3.2.17 it follows that

$$p = |\mathbb{Z}_p| = |\mathrm{Stab}_{\mathbb{Z}_p}(x)||\mathrm{Orb}_{\mathbb{Z}_p}(x)|. \tag{3.2.22}$$

Since p is prime either $|\mathrm{Stab}_{\mathbb{Z}_p}(x)| = p$ and $|\mathrm{Orb}_{\mathbb{Z}_p}(x)| = 1$ or $|\mathrm{Stab}_{\mathbb{Z}_p}(x)| = 1$ and $|\mathrm{Orb}_{\mathbb{Z}_p}(x)| = p$.
If $|\mathrm{Orb}_{\mathbb{Z}_p}(x)| = 1$ then $x \in X$ must be of the form $(g, \dots, g)$ for some $g \in G$ since if this wasn't the case then a cyclic permutation would not leave x invariant. An example of this case is $x = (e, \dots, e)$. This cannot be the only example since then $|X|$ is not divisible by p, since we can write $|X|$ as $|X| = np + m1$, where n is the number of orbits of length p and m is the number of orbits of length 1. Clearly in order for $|X|$ to be divisible by p we need m to be divisible by p. Therefore there must be some $g \in G$ such that $g \neq e$ but $(g, \dots g) \in X$. By the definition of X this means that $g \cdots g = g^p = e$, and so g is some element of order p. □

■ **Example 3.2.23** S_3 is order 6 and hence has elements of order 2 and 3, for example $(1\,2)$ is of order 2 and $(1\,2\,3)$ is of order 3.
$\mathbb{Z}_6$ is of order 6 and hence has elements of order 2 and 3. Using addition modulo 6 as the group operation 3 is of order 2 and 2 is of order 3 in this group.

Corollary 3.2.24 If G is a group and $|G| = p$ for prime p then G is isomorphic to $\mathbb{Z}_p$.

Proof. By Cauchy's theorem (Theorem 3.2.19) G has an element of order p. That is there exists $g \in G$ such that $g^p = e$, further $g^m \neq e$ for $m < p$ since m doesn't divide p. Since there are p elements of G and each g^m must be distinct for $m < p$ it follows that all elements of G are of the form g^m for some $m \in [0, p) \cap \mathbb{Z}$. Therefore G is cyclic and of order p and therefore G is isomorphic to $\mathbb{Z}_p$. □

Four

Normal Subgroups

4.1 Normal Subgroups

Definition 4.1.1 — Normal Subgroup Let G be a group and N be a subgroup of G. Then we say that N is a **normal subgroup** of G if N is invariant under the group action, $G \times N \to N$, of conjugation. That is for all $n \in N$ and all $g \in G$ we have that $gng^{-1} \in N$. A normal subgroup is also referred to as an **invariant subgroup**.

> **Notation 4.1.2** If N is a normal subgroup of G we denote this $N \trianglelefteq G$. If N is a normal subgroup of G, and $N \neq G$, then we denote this $N \triangleleft G$.

We can think of normal subgroups as groups which are invariant under relabelling of the elements, this relabelling is done by conjugation. Compare this to matrix transformation. Suppose that M is a matrix in some basis, $\{\boldsymbol{e}_i\}$, and that $\{\boldsymbol{e}'_i\}$ is some other basis related to the first by $\boldsymbol{e}'_i = T_{ij}\boldsymbol{e}_j$. Then in this new basis M becomes TMT^{-1}. M describes the same transform in both bases but with different components.

■ **Example 4.1.3** For any group, G, both G and the trivial group, $\{e\}$, are normal subgroups.
If G is an Abelian group and H is a subgroup of G then H is a normal subgroup since $ghg^{-1} = gg^{-1}h = h \in H$ for all $g \in G$ and $h \in H$.

■ **Example 4.1.4** Recall that the quaternion group has the presentation

$$Q = \langle -e, i, j, k \mid (-e)^2 = e, i^2 = j^2 = k^2 = ijk = -e \rangle. \tag{4.1.5}$$

We can define $\mathbb{Z}_2$ to have the presentation $\mathbb{Z}_2 = \langle -e \mid (-e)^2 = e \rangle$, which has elements $\{e, -e\}$. This is a normal subgroup of Q. To see this note that

$i^{-1} = -i$, $j^{-1} = -j$, $k^{-1} = -k$, and $-e^{-1} = -e$ so

$$\begin{aligned}
eee^{-1} = ee(-e) &= -e^3 = -e, && (4.1.6)\\
(-e)e(-e)^{-1} = (-e)ee &= -e^3 = -e, && (4.1.7)\\
e(-e)e^{-1} = e(-e)(-e) &= e^3 = e, && (4.1.8)\\
(-e)(-e)(-e)^{-1} = (-e)(-e)e &= e^3 = e, && (4.1.9)\\
iei^{-1} = ie(-i) &= -i^2 = e, && (4.1.10)\\
i(-e)i^{-1} = i(-e)(-i) &= i^2 = -e, && (4.1.11)\\
(-i)e(-i)^{-1} = (-i)ei &= -i^2 = e, && (4.1.12)\\
(-i)(-e)(-i)^{-1} = (-i)(-e)i &= i^2 = -e. && (4.1.13)\\
& && (4.1.14)
\end{aligned}$$

We also get the same results if we replace all is with j or k and so $\mathbb{Z}_2$ is invariant under conjugation and hence a normal subgroup of Q, $\mathbb{Z}_2 \lhd Q$.

In the previous example all elements of $\mathbb{Z}_2$ map to themselves under conjugation by elements of Q not in $\mathbb{Z}_2$. However, this need not be the case for a normal subgroup, as the next example shall show.

■ **Example 4.1.15** The permutations $A_3 = \{(), (1\,2\,3), (3\,2\,1)\}$ form a normal subgroup of S_3, this subgroup is called the alternating group and we will study it more in the future.
First we have to show that A_3 is a subgroup, which we can do via the subgroup criterion by considering $(1\,2\,3)(3\,2\,1)^{-1} = (1\,2\,3)(1\,2\,3) = (3\,2\,1) \in A_3$ and $(3\,2\,1)(1\,2\,3)^{-1} = (3\,2\,1)(3\,2\,1) = (1\,2\,3) \in A_3$, and clearly if we have eg^{-1} or ge^{-1} with $e = ()$ and $g = (1\,2\,3)$ or $g = (3\,2\,1)$ we will simply get g or g^{-1}, both of which are in A_3 since $(1\,2\,3)^{-1} = (3\,2\,1)$.

■ **Example 4.1.16** Invertible transformations of some vector space form a group, since the composition of two transformations is again a transformation, by definition for an *invertible* transformation the inverse exists, and the identity transformation is in this set. Given a basis for this vector space we can write these transformations as matrices with nonzero determinants, and in doing so define the **general linear group**:

$$\mathrm{GL}(n, \mathbb{F}) := \{A \in \mathcal{M}_n(\mathbb{F}) \mid \det A \neq 0\}. \tag{4.1.17}$$

Here $\mathbb{F}$ is a field, n is the dimension of the vector space, and $\mathcal{M}_n(\mathbb{F})$ are all square $n \times n$ matrices with entries from $\mathbb{F}$. Recall that $A \in \mathcal{M}_n(\mathbb{F})$ is invertible if and only if $\det A \neq 0$.
Another group we can define is the **special linear group**:

$$\mathrm{SL}(n, \mathbb{F}) := \{A \in \mathcal{M}_n(\mathbb{F}) \mid \det A = 1\} \subseteq \mathrm{GL}(n, \mathbb{F}). \tag{4.1.18}$$

$\mathrm{SL}(n, \mathbb{F})$ is a subgroup of $\mathrm{GL}(n, \mathbb{F})$ since if $A, B \in \mathrm{SL}(n, \mathbb{F})$ then $\det(AB^{-1}) = \det(A)\det(B^{-1}) = \det(A)/\det(B) = 1/1 = 1$, so $AB^{-1} \in \mathrm{SL}(n, \mathbb{F})$.

Further, $\mathrm{SL}(n, \mathbb{F})$ is a normal subgroup of $\mathrm{GL}(n, \mathbb{F})$ since if $A \in \mathrm{SL}(n, \mathbb{F})$ and $B \in \mathrm{GL}(n, \mathbb{F})$ we have

$$\det(BAB^{-1}) = \det(B)\det(A)\det(B^{-1}) = \det(B)\det(A)\frac{1}{\det(B)} = \det(A) = 1$$

so BAB^{-1} is again in $\mathrm{SL}(n, \mathbb{F})$.

There is an easier way to test if a subgroup is normal, and it is to consider the cosets.

Theorem 4.1.19. Let H be a subgroup of some group, G. Then H is a normal subgroup of G if and only if the left and right cosets are equal. That is $gH = Hg$ for all $g \in G$.

Proof. First, suppose that H is a normal subgroup. Consider some $g \in G$ and $h \in H$. Then $ghg^{-1} \in H$, call $h' = ghg^{-1}$. Similarly since $g^{-1} \in G$ we have $g^{-1}hg = g^{-1}h(g^{-1})^{-1} \in H$, call $h'' = g^{-1}hg$. Then $gh = ghg^{-1}g = h'g \in Hg$ and $hg = gg^{-1}hg = gh'' \in gH$. Since $gh \in Hg$ for all $h \in H$ it follows that $gH \subseteq Hg$. Since $hg \in gH$ for all $h \in H$ it follows that $Hg \subseteq gH$. Hence $gH = Hg$.
Now, suppose that $gH = Hg$ for all $g \in G$. Then for some $g \in G$ and $h \in H$ we have $gh \in gH = Hg$, which means there exists some $h' \in H$ such that $gh = h'g$. Left multiplying by g^{-1} we have $ghg^{-1} = h' \in H$, and so H is invariant under conjugation and hence normal in G. □

4.2 Coset Groups

Definition 4.2.1 — Coset Product Let G be a group and N be a normal subgroup of G. Then we can define G/N to be the set of cosets of N, that is

$$G/N \coloneqq \{gN \mid g \in N\} = \{Ng \mid g \in G\}. \tag{4.2.2}$$

We can define a binary operation on G/N according to

$$(gN)(g'N) \coloneqq (gg')N. \tag{4.2.3}$$

With this operation G/N forms a group which we call the **quotient group** or **factor group**.

Theorem 4.2.4. If G is a group and N is a normal subgroup of G then G/N with the operation defined above is a group.

Proof. We must first show that the operation is well defined. That is that no matter which element of gN we choose to represent gN we get the same

result. Suppose $gN = g'N$ and $\tilde{g}N = \tilde{g}'N$, the operation is well defined only if $(g\tilde{g})N = (g'\tilde{g}')N$. For this to be the case it is sufficient that $g'\tilde{g}' \in (g\tilde{g})N$. Since $g'N = gN$ we know that $g' = gn$ for some $n \in N$ and since $\tilde{g}'N = \tilde{g}N$ we know that $\tilde{g}' = \tilde{g}\tilde{n}$ for some $\tilde{n} \in N$. We then have $g'\tilde{g}' = gn\tilde{g}\tilde{n}$. We need to show that this is of the form $g\tilde{g}n''$ for some $n'' \in N$.
Now, by definition since N is normal we have $\tilde{g}n'\tilde{g}^{-1} = n$ for some $n' \in N$. From this it follows that $\tilde{g}n' = n\tilde{g}$ We then have

$$\begin{aligned} g'\tilde{g}' &= (gn)(\tilde{g}\tilde{n}) && (4.2.5)\\ &= g(n\tilde{g})\tilde{n} && (4.2.6)\\ &= g(\tilde{g}n')\tilde{n} && (4.2.7)\\ &= (g\tilde{g})(n'\tilde{n}) && (4.2.8)\\ &= (g\tilde{g})n'' && (4.2.9) \end{aligned}$$

where $n'' = n'\tilde{n} \in N$. hence this operation is well defined.
It remains to show that G/N is a group. To do so notice that $eN = N$ acts as an identity: $(eN)(gN) = (eg)N = gN$ for all $g \in G$, and that $g^{-1}N$ is the inverse of gN, since $(gN)(g^{-1}N) = (gg^{-1})N = eN = N$. Associativity follows from associativity of the group product, that is if $g_1, g_2, g_3 \in G$ then

$$\begin{aligned} [(g_1N)(g_2N)](g_3N) &= [(g_1g_2)N](g_3N) && (4.2.10)\\ &= ((g_1g_2)g_3)N && (4.2.11)\\ &= (g_1(g_2g_3))N && (4.2.12)\\ &= (g_1N)[(g_2g_3)N] && (4.2.13)\\ &= (g_1N)[(g_2N)(g_3N)]. && (4.2.14) \end{aligned}$$

Hence G/N is a group. □

■ **Example 4.2.15** Recall from Example 4.1.4 that the quaternion group, Q, has as a normal subgroup $\mathbb{Z}_2$.
The quotient group in this case is $Q/\mathbb{Z}_2$. To see this we first write out all cosets:

$$\mathbb{Z}_2 = \{e, -e\}, \quad i\mathbb{Z}_2 = \{i, -i\}, \quad j\mathbb{Z}_2 = \{j, -j\}, \quad \text{and} \quad k\mathbb{Z}_2 = \{k, -k\}.$$

We can then construct the group table:

$Q/\mathbb{Z}_2$	$\mathbb{Z}_2$	$i\mathbb{Z}_2$	$j\mathbb{Z}_2$	$k\mathbb{Z}_2$
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$i\mathbb{Z}_2$	$j\mathbb{Z}_2$	$k\mathbb{Z}_2$
$i\mathbb{Z}_2$	$i\mathbb{Z}_2$	$\mathbb{Z}_2$	$k\mathbb{Z}_2$	$j\mathbb{Z}_2$
$j\mathbb{Z}_2$	$j\mathbb{Z}_2$	$k\mathbb{Z}_2$	$\mathbb{Z}_2$	$i\mathbb{Z}_2$
$k\mathbb{Z}_2$	$k\mathbb{Z}_2$	$j\mathbb{Z}_2$	$i\mathbb{Z}_2$	$\mathbb{Z}_2$

(4.2.16)

Here we have used $ij = k$, $jk = i$, and $ki = j$, $i^2 = j^2 = k^2 = -e$, and that permuting any two of i, j, and k adds a negative sign.
Comparing this Cayley table with those in Equation (2.1.18) we see that $Q/\mathbb{Z}_2 \cong \mathbb{Z}_2{\times}\mathbb{Z}_2$. In particular we have the correspondence $\mathbb{Z}_2 \leftrightarrow e$, $i\mathbb{Z}_2 \leftrightarrow a$, $j\mathbb{Z}_2 \leftrightarrow b$, and $k\mathbb{Z}_2 \leftrightarrow c$.

▪ **Example 4.2.17** Let $n\mathbb{Z}$ denote the integer multiples of n, viewed as a subgroup of $\mathbb{Z}$. Then $n\mathbb{Z}$ is a normal subgroup of $\mathbb{Z}$, since $\mathbb{Z}$ is Abelian and so left and right cosets are equal. The cosets are of the form $m + n\mathbb{Z} = \{m+nk \mid k \in \mathbb{Z}\}$. There is then a natural isomorphism $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}_n$, namely $m + n\mathbb{Z} \mapsto \mathrm{e}^{2i\pi m/n}$. For this reason a lot of people denote the group we have been calling $\mathbb{Z}_n$ by $\mathbb{Z}/n\mathbb{Z}$.

Definition 4.2.18 — Simple Group If a group has no proper normal subgroups then we say that it is a **simple group**.

Thinking of the quotient groups, G/N, as division of groups, as the notation suggests, it makes sense to view simple groups as the "primes" of groups, in that they can't be further divided.

Finite simple groups are classified up to automorphism. That is given any finite simple group it will be isomorphic to a group from a known set of groups. This classification of finite simple groups is not simple. The proof consists of tens of thousands of pages across several hundred journal articles written by approximately 100 authors over a period of more than 50 years, finally being completed in 2008.

Theorem 4.2.19. If G is a group and H is a subgroup of G and $[G : H] = 2$ then H is a normal subgroup of G.

Proof. Since $[G : H] = |G|/|H| = 2$ there are two cosets, namely H and gH for $g \in G \setminus H$. Considering right cosets instead we have H and Hg, again for $g \in G \setminus H$. By Lemma 2.3.5 we also know that the cosets partition G, so that $G = H \cup gH = H \cup Hg$. Since H is the same in both of these and cosets are disjoint we must have that $gH = Hg$ and so H is a normal subgroup of G by Theorem 4.1.19. □

Note that in this case $|G/H| = |G|/|H| = [G : H] = 2$ and so $G/H \cong \mathbb{Z}_2$.

As the name of the next theorem suggests there are multiple "isomorphism theorems" but we will discuss only on the first.

Theorem 4.2.20 — First Isomorphism Theorem. Let G and H be groups and $\varphi\colon G \to H$ a homomorphism. Then

1. The image of φ, $\operatorname{Im}\varphi$, is a subgroup of H.
2. The kernel of φ, $\ker\varphi$, is a normal subgroup of G.
3. $G/\ker\varphi \cong \operatorname{Im}\varphi$.

Proof. The first statement is simply Lemma 2.1.21.
Recall that $\ker\varphi = \{k \in G \mid \varphi(k) = e\}$ where e is the identity of H. For the proof of the second statement consider $k_1, k_2 \in \ker\varphi$. Then

$$\varphi(k_1k_2^{-1}) = \varphi(k_1)\varphi(k_2^{-1}) = \varphi(k_1)\varphi(k_2)^{-1} = ee = e \tag{4.2.21}$$

so $k_1k_2^{-1} \in \ker\varphi$ and so by the subgroup criterion (Theorem 1.2.32) $\ker\varphi$ is a subgroup of G. Here we used Lemma 2.1.20 to allow us to identify $\varphi(k_2^{-1}) = \varphi(k_2)^{-1}$.

It remains to show that $\ker\varphi$ is a *normal* subgroup of G. Take $k \in \ker\varphi$ and $g \in G$. Then

$$\varphi(gkg^{-1}) = \varphi(g)\varphi(k)\varphi(g^{-1}) = \varphi(g)e\varphi(g^{-1}) = \varphi(g)\varphi(g^{-1}) = \varphi(gg^{-1}) = \varphi(e) = e \tag{4.2.22}$$

where we have used Lemma 2.1.19 to identify $\varphi(e) = e$. This shows that $gkg^{-1} \in \ker\varphi$ and so $\ker\varphi$ is a normal subgroup of G. This proves the second point.

For the third point we need to show that there is an isomorphism $\psi\colon G/\ker\varphi \longrightarrow \operatorname{Im}\varphi$. To do so we consider the obvious choice that $\psi(gK) = \varphi(g)$, where $K = \ker\varphi$. Since ψ is defined on representatives of the cosets we need to show that it is well defined. Let $g_1, g_2 \in G$ be in the same coset, that is $g_1K = g_2K$. It follows that $(g_1^{-1}K)(g_2K) = (g_1g_2)K$ but also $(g_1^{-1}K)(g_2K) = (g_1^{-1}K)(g_1K) = (g_1^{-1}g_1)K = K$ and so $g_1^{-1}g_2 \in K$. Thus, $\varphi(g_1^{-1}g_2) = e$, since $K = \ker\varphi$. We then have $\varphi(g_1^{-1}g_2) = \varphi(g_1^{-1})\varphi(g_2) = \varphi(g_1)^{-1}\varphi(g_2)$ having used the definition of the homomorphism and Lemma 2.1.20. We then have $\varphi(g_1)^{-1}\varphi(g_2) = e$, and so $\varphi(g_1) = \varphi(g_2)$. This shows that ψ is well defined.

Next, we verify that ψ is a homomorphism. To do so let $g_1, g_2 \in G$ and so $g_1K, g_2K \in G/K$. We then have

$$\begin{aligned}
\psi((g_1K)(g_2K)) &= \psi((g_1g_2)K) && (4.2.23)\\
&= \varphi(g_1g_2) && (4.2.24)\\
&= \varphi(g_1)\varphi(g_2) && (4.2.25)\\
&= \psi(g_1K)\psi(g_2K), && (4.2.26)
\end{aligned}$$

so ψ is a homomorphism.

Finally we show that ψ is an isomorphism, that is that it is bijective. The kernel of ψ consists of all cosets $gK \in G/K$ such that $\varphi(g) = e$, but these are exactly the elements $g \in G$ such that $g \in K = \ker\varphi$. Hence the kernel of ψ is the trivial group $\{K\} \subset G/K$. This proves that ψ is injective by Lemma 2.1.23.

Finally let $h \in \operatorname{Im}\varphi$. Then there exists $g \in G$ such that $\varphi(g) = h$. We then have $\psi(gK) = \varphi(g) = h$, and so ψ is surjective.

So ψ is a bijective homomorphism and hence an isomorphism. □

An alternative statement of the first isomorphism theorem is that the following diagram commutes:

$$\begin{tikzcd}
G \arrow[r, "\varphi"] \arrow[d, "\pi"] & \operatorname{Im}\varphi \subseteq H \\
G/\ker\varphi . \arrow[ur, "\sim", "\psi"'] &
\end{tikzcd} \tag{4.2.27}$$

Here $\pi\colon G \longrightarrow G/\ker\varphi$ is the **natural projection** defined by $\pi(g) = g\ker\varphi$. We use the notation $X \xrightarrow{\sim} Y$ to denote an isomorphism from X to Y.

One of the main uses of the first isomorphism theorem is to quickly check if a subgroup is normal. This works since if H is a subgroup of some group G and there exists some homomorphism $\varphi\colon G \to \tilde{G}$ for some group $\tilde{G}$ such that $H = \ker\varphi$ then by the first isomorphism theorem $\ker\varphi$, and hence H, is normal in G.

Five

Products of Groups

5.1 Direct Products

Definition 5.1.1 — Direct Product Let H and J be groups. Then we define a new group, $H \times J$, called the **direct product** of H and J such that

$$H \times J := \{(h, j) \mid h \in H \text{ and } j \in J\}. \tag{5.1.2}$$

We extend the products of the two groups to define a product

$$(h, j)(h', j') = (hh', jj') \tag{5.1.3}$$

for $h, h' \in H$ and $j, j' \in J$.

We can think of the direct product as extending the Cartesian product to groups. Notice that for finite groups $|H \times J| = |H||J|$.

Theorem 5.1.4. The direct product of two groups is a group.

Proof. Let H and J be groups. Consider $h_1, h_2, h_3 \in H$ and $j_1, j_2, j_3 \in J$. Then

$$\begin{aligned}
(h_1, j_1)[(h_2, j_2)(h_3, j_3)] &= (h_1, j_1)(h_2h_3, j_2j_3) && (5.1.5)\\
&= (h_1(h_2h_3), j_1(j_2j_3)) && (5.1.6)\\
&= ((h_1h_2)h_3, (j_1j_2)j_3) && (5.1.7)\\
&= (h_1h_2, j_1j_2)(h_3, j_3) && (5.1.8)\\
&= [(h_1, j_1)(h_2, j_2)](h_3, j_3) && (5.1.9)
\end{aligned}$$

so associativity in $H \times J$ follows from associativity in H and J. let e_H be the identity of H and e_J the identity of J. Then

$$(h, j)(e_H, e_J) = (he_H, je_J) = (h, j) \tag{5.1.10}$$

for all $h \in H$ and $j \in J$ and so $e_{H\times J} = (e_H, e_J)$ is the identity of $H \times J$. Finally notice that for all $h \in H$ and $j \in J$ we have

$$(h, j)(h^{-1}, j^{-1}) = (hh^{-1}, jj^{-1}) = (e_H, e_J) = e_{H\times J} \tag{5.1.11}$$

and since $h^{-1} \in H$ and $j^{-1} \in J$ we have that $(h^{-1}, j^{-1}) \in H \times J$ acts as the inverse for $(h, j) \in H \times J$.
Hence $H \times J$ is a group. □

■ **Example 5.1.12 — Klein *Vierergruppe*** The **Klein *Vierergruppe*** is most simply defined as the direct product group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Note that

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\} \tag{5.1.13}$$

and all elements square to $(1, 1)$, which is the identity, for example, $(-1, 1)^2 = ((-1)^2, 1^2) = (1, 1)$. Hence $\mathbb{Z}_2 \times \mathbb{Z}_2$ matches our earlier definition of the Klein *Vierergruppe*, namely the unique group of order 4 such that all elements square to the identity.

■ **Example 5.1.14** Let $\mathbb{R}$ be the group of real numbers under addition. Then $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the group of two-component vectors, (x, y), with addition defined by

$$(x, y) + (x', y') = (x + x', y + y'). \tag{5.1.15}$$

The direct product is both commutative and associative up to isomorphism. That is $G \times H \cong H \times G$ and $G \times (H \times J) \cong (G \times H) \times J$, with the obvious isomorphisms $(g, h) \mapsto (h, g)$ and $(g, (h, j)) \mapsto ((g, h), j)$, respectively. The later means that we can define the direct product of multiple groups in a sensible way and so we typically write $G \times H \times J$ and (g, h, j) and so on.

The order of $(h, j) \in H \times J$ is the lowest common multiple of the orders of h and j. In particular if the orders of h and j are relatively prime then the order of (h, j) is the product of the orders of h and j. This means that if H and J are cyclic groups of relatively prime orders m and n their direct product is again cyclic:

$$\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_{mn}. \tag{5.1.16}$$

Given two groups, G and H, it is common to view G and H as subgroups of $G \times H$, which can be done since we can identify $G \cong \{e\} \times H$ and $H \cong G \times \{e\}$ under the obvious isomorphisms $g \mapsto (g, e)$ and $h \mapsto (e, h)$, respectively. Making this identification both G and H are normal subgroups of $G \times H$. We can show this easily for G and it can be shown similarly for H:

$$(g, h)(g', e)(g, h)^{-1} = (g, h)(g', e)(g^{-1}, h^{-1}) = (gg'g^{-1}, heh^{-1}) = (gg'^{-1}, e) \in G \times \{e\} \tag{5.1.17}$$

since $gg'g^{-1} \in G$ as G is a group.

5.2 Semidirect Product

Definition 5.2.1 — Semidirect Product Let H and J be groups and $\varphi\colon H \times J \to J$ a group action of H on J. Then we can define a new group, $H \ltimes J$, as the set

$$\{(h, j) \mid h \in H \text{ and } j \in J\} \tag{5.2.2}$$

and the group product

$$(h, j)(h', j') = (hh', j\varphi(h, j')) = (hh', j(h \,.\, j')) \tag{5.2.3}$$

for $h, h' \in H$ and $j, j' \in J$.

> R The vertical line in the symbol $\ltimes$ goes with the group which acts on the other group.

For each group action we technically have two semidirect product groups, $H \ltimes J$ and $J \rtimes H$, but these differ only in the order of elements, (h, j) vs. (j, h), and are isomorphic. The order of $H \ltimes J$ is $|H \ltimes J| = |H||J|$.

■ **Example 5.2.4 — Isometries of Euclidean Space** The **isometries of Euclidean space** is the group which preserves Euclidean distance. This group is typically denoted $\mathrm{ISO}(n)$ or $\mathrm{E}(n)$, where n is the dimension of the space on which the group acts. This group consists of rotations, reflections and translations. Transformations in the first two categories, rotations and reflections, form $\mathrm{O}(n)$, that is the group of transformations which preserve Euclidean distance and leave the origin invariant. Translations can be viewed as $\mathbb{R}^n$. A point in Euclidean space can be viewed as $\boldsymbol{x} \in \mathbb{R}^n$, which is somewhat confusing since $\mathbb{R}^n$ appears twice, as both the translations and the Euclidean space upon which they act.

We can write $\mathrm{ISO}(n)$ as a semidirect product, $\mathrm{ISO}(n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$. We just need to work out the correct group action. First we need to state how $\mathrm{ISO}(n)$ acts on $\mathbb{R}^n$ (as Euclidean space). This is simple, given a rotation and/or reflection $R \in \mathrm{O}(n)$ and translation $\boldsymbol{a} \in \mathbb{R}^n$ this acts on $\boldsymbol{x} \in \mathbb{R}^n$ as $(R, \boldsymbol{a}).\boldsymbol{x} = R\boldsymbol{x}+\boldsymbol{a}$, that is we rotate $\boldsymbol{x}$ with R and then translate by $\boldsymbol{a}$, notice that the order is important, which we will see means that $\mathrm{ISO}(n)$ is *not* a direct product of $\mathrm{O}(n)$ and $\mathbb{R}^n$.

Consider what happens when we act on some $\boldsymbol{x} \in \mathbb{R}^n$ (as a point in Euclidean space) by two isometries, $(R, \boldsymbol{a})$ and $(R', \boldsymbol{a}')$, where $R, R' \in \mathrm{O}(n)$ and $\boldsymbol{a}, \boldsymbol{a}' \in \mathbb{R}^n$ (as the group of translations). We then have

$$\begin{aligned}
(R', \boldsymbol{a}')(R, \boldsymbol{a}) \,.\, \boldsymbol{x} &= (R', \boldsymbol{a}') \,.\, (R, \boldsymbol{a}) \,.\, \boldsymbol{x} && (5.2.5)\\
&= (R', \boldsymbol{a}) \,.\, (R\boldsymbol{x} + \boldsymbol{a}) && (5.2.6)\\
&= R'(R\boldsymbol{x} + \boldsymbol{a}) + \boldsymbol{a}' && (5.2.7)\\
&= R'R\boldsymbol{x} + R'\boldsymbol{a} + \boldsymbol{a}' && (5.2.8)\\
&= (R'R, R'\boldsymbol{a} + \boldsymbol{a}') \,.\, \boldsymbol{x}. && (5.2.9)
\end{aligned}$$

So we identify the group action associated with the semidirect product as

$$R' \,.\, \boldsymbol{a} = R'\boldsymbol{a}, \qquad \text{or} \qquad \varphi(R', \boldsymbol{a}) = R\boldsymbol{a}, \tag{5.2.10}$$

which is probably what most people would expect, the rotation (or reflection) acts by rotating (or reflecting).

We can further generalise the isometries of Euclidean space by dropping the requirement that lengths be preserved and allowing uniform scaling. In this case the group of symmetries is the affine group, $\mathrm{Aff}(V)$, which is the semidirect product $\mathrm{GL}(V) \ltimes V$ where $\mathrm{GL}(V)$ acts on the vector space V with the expected action, $M \,.\, \boldsymbol{v} = M\boldsymbol{v}$ for $M \in \mathrm{GL}(V)$ and $\boldsymbol{v} \in V$. This contains $\mathrm{ISO}(V)$ as a subgroup.

▪ Example 5.2.11 — Dihedral Group The **dihedral group** of order $2n$ can be defined abstractly as

$$D_n := \langle r, s \mid r^n = s^2 = e, s^{-1}rs = r^{-1} \rangle. \tag{5.2.12}$$

This can be identified as the group of symmetries of the regular n-gon. Here r is identified as a rotation by $2\pi/n$ and s as a mirror symmetry or inversion in a perpendicular bisector of one of the sides.

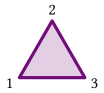
! Some sources denote the dihedral group of order $2n$ by D_{2n}, since it has $2n$ elements, whereas we denote it D_n, as it is the group of symmetries of the regular n-gon.

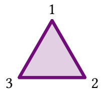
We can identify a subgroup generated by s as $\mathbb{Z}_2 = \{e, s\}$. We can identify a subgroup generated by r as $\mathbb{Z}_n = \{e, r, \dots, r^{n-1}\}$. It turns out that D_n can then be written as the semidirect product $D_n \cong \mathbb{Z}_2 \ltimes \mathbb{Z}_n$.
To see this it is best to just consider a few examples. First we identify $\mathbb{Z}_2 = \{\pm 1\}$, and $\mathbb{Z}_n = \{\mathrm{e}^{2i\pi m/n}\}$. A few examples of products in D_n are then

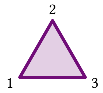
$$(+1, \mathrm{e}^{2i\pi m/n})(+1, \mathrm{e}^{2i\pi m'/n}) = (+1, \mathrm{e}^{2i\pi(m+m')/n}), \tag{5.2.13}$$
$$(-1, \mathrm{e}^{2i\pi m/m})(+1, \mathrm{e}^{2i\pi m'/n}) = (-1, \mathrm{e}^{2i\pi(m-m')/n}), \tag{5.2.14}$$
$$(+1, \mathrm{e}^{2i\pi m/n})(+1, \mathrm{e}^{2i\pi m'/n}) = (-1, \mathrm{e}^{2i\pi(m+m')/n}). \tag{5.2.15}$$

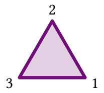
That is the group action associated with the semidirect product is $\pm 1 \,.\, \mathrm{e}^{2i\pi m/n} = \mathrm{e}^{\pm 2i\pi m/n}$. We can further identify this as $1 \,.\, z = z$ and $-1 \,.\, z = z^*$.

It is worth examining the dihedral group more, particularly as it comes up in geometry and chemistry. Starting with geometry we claimed that D_3 is the group of symmetries of an equilateral triangle. By this we mean that D_3 acts on an equilateral triangle such that there is no noticeable change. However, in order to keep track of what is happening we label the corners of the triangle, but these labels have no meaning besides keeping track of how D_3 is acting. Taking r to be a clockwise rotation by 120° and s to be a reflection in the vertical. Graphically the action of r and s on the triangle is

$r \,.$ 

$=$ 

, and $s \,.$ 

$=$ 

(5.2.16)

We should check that these symmetries really correspond to D_3 as defined by the presentation above. First, notice that three rotations correspond to a rotation by $3 \cdot 120° = 360°$, which is the same as no rotation at all. Second, notice that repeating

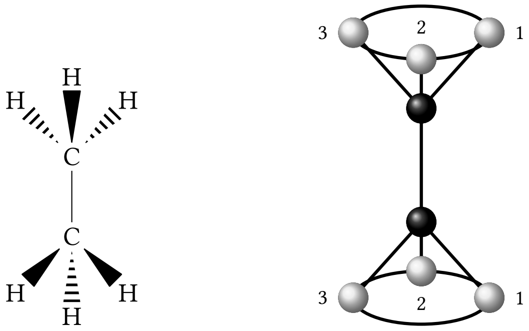
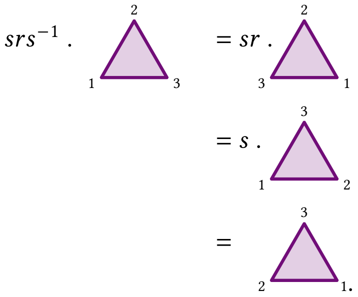


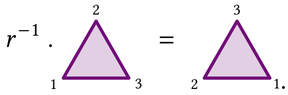
Figure 5.1: Ethane molecule.

a mirror symmetry undoes it so $s^2 = e$. Finally consider the action of srs^{-1} on the triangle, noticing that $s^{-1} = s$, we have



(5.2.17)

Now compare this to the action of r^{-1}, which is an anticlockwise rotation by 120°:



(5.2.18)

Noticing that these are the same we see that D_3 does truly describe the symmetries of the equilateral triangle.

As well as the equilateral triangle D_3 is also the symmetry group of an ideal C_2H_6 molecule. This molecule is shown in Figure 5.1. Here we identify r as a rotation around the carbon-carbon bond and s as a rotation by 180° about the perpendicular bisector to this bond in the page, such that the numbers on the upper and lower hydrogens match. Shown here is staggered ethane, which is such that viewed end on the hydrogens don't line up. D_3 is also the symmetry group of eclipsed ethane, where, when viewed end on, the hydrogens line up, we just change s to be inversion about the centre of the carbon-carbon bond.

Six

Permutation Groups

6.1 Symmetric Group

Definition 6.1.1 — Permutation a **permutation**, σ, on n objects is a bijection $\sigma \colon X \to X$ where $|X| = n$.

Typically we identify X as $\{1, \dots, n\}$.

Definition 6.1.2 — Symmetric Permutation Group The **symmetric group** on n objects is the set of all permutations on n objects with function composition as a group operation. This group is denoted S_n.

The order of S_n is $n!$, since we can choose to permute the first element to any of n possible options, the second to any of $n-1$ options, and so on giving $n(n-1)\cdots 1 = n!$ choices.

Lemma 6.1.3 The symmetric group on n objects is a group.

Proof. Let $\sigma, \rho \in S_n$. Then both of these are bijections on some set X with $|X| = n$. Their composition is defined by $(\sigma \circ \rho)(x) = \sigma(\rho(x))$ for all $x \in X$. Straight away we see that this is indeed a permutation, since $\sigma \circ \rho \colon X \to X$ and the inverse is $\rho^{-1} \circ \sigma^{-1}$, the existence of said inverses in S_n is guaranteed as $\sigma \in S_n$ is a bijection and so its inverse exists and is also a bijection. This shows that S_n is closed.
The identity function, $\mathrm{id}_X \colon X \to X$ defined by $\mathrm{id}_X(x) = x$ for all $x \in X$ is a permutation and $\mathrm{id}_X \circ \sigma = \sigma$ for all $\sigma \in S_n$. As previously discussed σ has the inverse σ^{-1}, which is such that $\sigma \circ \sigma^{-1} = \mathrm{id}_X$. Hence S_n is a group. □

S_n acts on the set of all tuples $(x_1, \dots, x_n)$ where $x_i \in X$ are distinct in the obvious way, namely by permuting the elements:

$$\sigma \,.\, (x_1, \dots, x_n) = (\sigma(x_1), \dots, \sigma(x_n)). \tag{6.1.4}$$

Definition 6.1.5 — Cycle A k-**cycle** is a way of writing a certain permutation. Namely $(a_1 \; \dots \; a_k)$ with $a_i \in X$ is the permutation that sends a_1 to a_2, a_2 to a_3, and so on until a_{k-1} to a_k and a_k to a_1. All $x \in X$ such that $x \neq a_i$

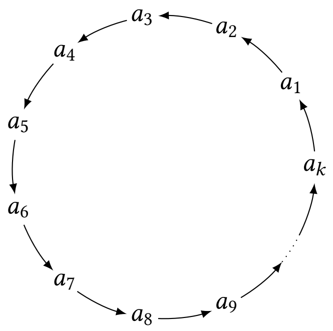


Figure 6.1: The k-cycle $(a_1\ \dots\ a_k)$.

are left unchanged.
A 2-cycle is also called a **transposition**.
The identity is usually written as () when using cycle notation, although we could also write it as a 1-cycle, (a) for any $a \in X$.
Given a k-cycle $(a_1\ \dots\ a_k)$ we can start on any element of this cycle, so this is equivalent to $(a_m\ \dots\ a_k\, a_1\ \dots\ a_{m-1})$.

■ **Example 6.1.6** Consider S_4, this contains the 3-cycles $(1\,4\,2)$ and $(1\,2\,3)$. We can work out their product by considering their action on some 4-tuple (a, b, c, d):

$$\begin{aligned}(1\,4\,2)(1\,2\,3)\,.\,(a\,b\,c\,d) &= (1\,4\,2)\,.\,(c, a, b, d) && (6.1.7)\\ &= (a, d, b, c) && (6.1.8)\\ &= (2\,3\,4)\,.\,(a, b, c, d), && (6.1.9)\end{aligned}$$

hence, we have $(1\,4\,2)(1\,2\,3) = (2\,3\,4)$.

Definition 6.1.10 — Disjoint Cycles Two cycles are disjoint if no element of X appears in both cycles.

Lemma 6.1.11 All permutations can be written as a product of disjoint cycles.

Proof. We proceed by induction on the size of $n = |X|$. Clearly if $n = 1$ then the only permutation is the identity, ().
Let $\sigma \in S_n$ and suppose that all cycles in S_{n-1} can be written as disjoint cycles. For simplicity we will take $X = \{1, \dots, n\}$. If $\sigma(n) = n$ then we can consider σ as a permutation on $\{1, \dots, n-1\}$ leaving n fixed and we are done since this can be written as a product of disjoint cycles. If $\sigma(n) = k \neq n$ then consider the permutation $\rho = (n\,k)\sigma$. We have that $\rho(n) = (n\,k)\sigma(n) = (n\,k)(k) = n$, here we are treating $(n\,k)$ as a function, $(n\,k)\colon \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$, defined by $(n\,k)(n) = k$, $(n\,k)(k) = n$ and $(n\,k)(a) = a$ for $a \neq n, k$. So

we can think of ρ as being a permutation on $\{1, \dots, n-1\}$, and hence can be written as a product of disjoint cycles, $\rho = \tau_1 \cdots \tau_r$. The cycles τ_i only contain the numbers $1, \dots, n-1$, and each appears in at most one of these cycles.

Clearly $(n\,k)(n\,k) = ()$, and so it follows that $\sigma = (n\,k)(n\,k)\sigma = (n\,k)\tau_1 \cdots \tau_r$. If k doesn't appear in any of the cycles τ_i then we are done as this is a product of disjoint cycles. Disjoint cycles commute, since by being disjoint they act on different elements of the tuple $(x_1, \dots, x_n)$, and so don't interact, meaning the order doesn't matter. Using this we are free to assume that the cycle in which k appears, if it appears, is τ_1.

We are free to start on any element of the cycle so we write $\tau_1 = (k\,a_1\ \dots\ a_m)$. We then have

$$(n\,k)\tau_1 = (n\,k)(k\,a_1\ \dots\ a_m) = (n\,k\,a_1\ \dots\ a_m). \tag{6.1.12}$$

This follows by considering $(n\,k)\tau_1(k) = (n\,k)(a_1) = a_1$, $(n\,k)\tau_1(n) = (n\,k)(n) = k$, $(n\,k)\tau_1(a_m) = (n\,k)(k) = n$, and $(n\,k)\tau_1(a_i) = (n\,k)a_{i+1} = a_{i+1}$ for $i \neq m$. It follows then that we can write

$$\sigma = (n\,k\,a_1\ \dots\ a_m)\tau_2 \cdots \tau_r \tag{6.1.13}$$

which is a product of disjoint cycles.

Hence by induction we can write any permutation in S_n as a product of disjoint cycles for all $n \in \mathbb{N}$. □

One question that we may reasonably ask is how many m-cycles are there in S_n for some fixed $m \in \{1, \dots, n\}$. If the order of a cycle didn't matter then there would be $\binom{n}{m}$ m-cycles in S_n. However, the order does matter. Suppose we have chosen our m terms to appear in the cycle. We can start with any of them, reducing the number of choices that give distinct cycles by a factor of $1/m$. There are then $(m-1)!$ choices for ordering the $m-1$ elements remaining, giving the number of m-cycles to be

$$\binom{n}{m}\frac{1}{m}(m-1)! = \frac{n!}{(n-m)!m!}\frac{1}{m}(m-1)! = \frac{n!}{(n-m)!} \tag{6.1.14}$$

where we have used $m(m-1)! = m!$.

Theorem 6.1.15. The transpositions generate S_n. That is, all permutations can be written as a product of 2-cycles.

Proof. All elements of S_n are simply k-cycles for some $k \in \{1, \dots, n\}$. An arbitrary k-cycle can be written as

$$(a_1\,a_2\ \dots\ a_k) = (a_1\,a_k)(a_1\,a_{k-1}) \cdots (a_1\,a_2). \tag{6.1.16}$$

To see why this works we consider three cases. First, if this acts on some a_i with $i \neq 1$, on the left hand side we clearly see that a_i maps to a_{i+1}. On the right hand side a_i commutes with cycles until a cycle with a_i occurs, this cycle will be $(a_1\,a_i)$, and so a_i will be sent to a_1. The next cycle is then $(a_1\,a_{i+1})$, and hence a_1 maps to a_{i+1} which then commutes with all of the remaining cycles. Hence a_i will map to a_{i+1}.

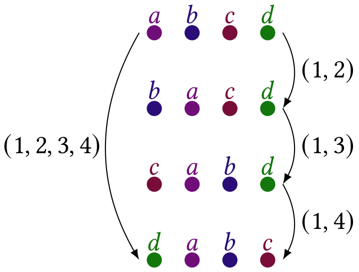


Figure 6.2: The permutation $(1\,2\,3\,4)$ acts on (a, b, c, d), which can be done in steps where each step is a transposition.

The second case is when this acts on a_1, in which case the first cycle sends a_1 to a_2, which then commutes with all remaining cycles and so a_1 maps to a_2, which is what we want.
The final case is trivial, its where this acts on some $a \neq a_i$ for any i, in which case on both the left and right this element is not changed and we are finished. □

The above theorem is fairly obvious. It states that we can do any permutation just by swapping two items at a time. This is demonstrated in Figure 6.2.

Notice that this theorem implies that the rank of S_n is at most

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{1}{2}n(n-1). \tag{6.1.17}$$

although we will see it is less than this.

Lemma 6.1.18 The transpositions, $\{(1\,2), (1\,3), \dots, (1\,n)\}$, generate S_n.

Proof. Notice that

$$(a_1\,a_2) = (1\,a_1)(1\,a_2)(1\,a_1). \tag{6.1.19}$$

To see this we can consider this acting on $(1\ \dots\ a_1\ \dots\ a_2\ \dots)$:

$$\begin{aligned}(1\,a_1)(1\,a_2)(1\,a_1)(1\ \dots\ a_1\ \dots\ a_2\ \dots) & \qquad (6.1.20)\\ = (1\,a_1)(1\,a_2)(a_1\ \dots\ 1\ \dots\ a_2\ \dots) & \qquad (6.1.21)\\ = (1\,a_1)(a_1\ \dots\ a_2\ \dots\ 1\ \dots) & \qquad (6.1.22)\\ = (1\ \dots\ a_2\ \dots\ a_1\ \dots) & \qquad (6.1.23)\end{aligned}$$

so the action of $(1\,a_1)(1\,a_2)(1\,a_1)$ is to swap a_1 and a_2, which is exactly the action of $(a_1\,a_2)$. Using this we can generate any transposition from transpositions of the form $(1\,a)$. Therefore by Theorem 6.1.15 transpositions of this form generate S_n. □

Notice that this theorem implies that the rank of S_n is at most n, for $n > 3$ this is an improvement on our previous bound.

Lemma 6.1.24 The transpositions, $\{(1\,2), (2\,3), \dots, (n-1\,n)\}$, generate S_n.

Proof. Notice that

$$(1\,k) = g(1\,2)g^{-1} \tag{6.1.25}$$

where $g = (k-1\,k)\cdots(3\,4)(2\,3)$. To see this first notice that $g^{-1} = (2\,3)(3\,4)\cdots(k-1\ \ k)$. Then notice that the action of g^{-1} is to exchange $k-1$ and k, then swap k and $k-2$, and then k and $k-3$, and so on until k and 2 have been swapped. Then $(1\,2)$ swaps k and 1. We then use g to swap 2 and 3 back, then 3 and 4, and so on until we swap $k-2$ and $k-1$ back to their original positions. The result is that 1 and k swap, which is exactly what $(1\,k)$ does.

Using this we can generate any transposition of the form $(1\,a)$ from transpositions of the form $(k-1\ \ k)$, and so by Lemma 6.1.18 these transpositions generate S_n. □

We can think of this proof as consisting of a basis change from the $(1\,a)$ transpositions case, and similarly for the next proof.

Lemma 6.1.26 The cycles, $\{(1\,2), (1\,2\,\dots\,n)\}$, generate S_n.

Proof. Notice that

$$(k\ \ k+1) = g(1\,2)g^{-1} \tag{6.1.27}$$

where $g = (1\,2\,\dots\,n)^{k-1}$. To see this first notice that $g^{-1} = (n\,\dots\,2\,1)^{k-1}$. The action of g^{-1} is then to cycle backwards through the elements $k-1$ times. The result is that k and $k+1$ end up in the first two positions. These are then swapped by $1, 2$. Then g cycles forwards through the elements $k-1$ times and we end up with the same tuple but with k and $k+1$ swapped, which is exactly what $(k\ \ k+1)$ does.

Using this we can generate any transposition of the form $(k\,k+1)$, and so by Lemma 6.1.24 these two cycles generate S_n. □

Corollary 6.1.28 The rank of S_n is 2.

Proof. S_n is generated by $\{(1\,2), (1\,\dots\,n)\}$ by Lemma 6.1.26, so S_n is of rank 2. □

6.2 Alternating Group

We have seen that any permutation can be written as a product of transpositions by Theorem 6.1.15. It turns out that the number of transpositions it takes to write any given permutation is unambiguously even or odd. We then term the permutation as even or odd based on the parity of the number of transpositions it takes to write it. This is the obvious definition of even and odd permutations but it isn't that easy to work with so we use an equivalent, but easier to work with, definition.

Definition 6.2.1 — Sign of a Permutation Let S_n be the symmetric group on n letters. Define the **Vandermonde polynomial** to be

$$\begin{aligned} P(x_1, \dots, x_n) &= P(\boldsymbol{x}) && (6.2.2)\\ &:= (x_1 - x_2)(x_1 - x_2) \cdots (x_1 - x_n) && (6.2.3)\\ &\qquad \cdot (x_2 - x_3) \cdots (x_2 - x_n) \cdots (x_{n-1} - x_n) && (6.2.4)\\ &= \prod_{\substack{i,j \in \{1,\dots,n\} \\ i<j}} (x_i - x_j). && (6.2.5) \end{aligned}$$

This polynomial is such that exchanging any two variables, x_i and x_j, results in the sign of the polynomial changing.

Define X to be the set of all permutations of $(x_1, \dots, x_n)$. We can define a group action, $\varphi \colon S_n \times X \longrightarrow X$, in the usual way as S_n acting on $\boldsymbol{x} \in X$ by permutation.

Now define $\psi \colon S_n \longrightarrow \{\pm 1\}$ by $\psi(\sigma) = P(\sigma \,.\, \boldsymbol{x})/|P(\boldsymbol{x})|$. Then if $\psi(\sigma) = 1$ we say σ is an **even permutation** and if $\psi(\sigma) = -1$ we say that σ is an **odd permutation**. This agrees with the "parity of the number of transpositions" definition since each transposition swaps two variables and so an even number of transpositions corresponds to an even number of swaps and hence no overall sign change, similarly an odd number of transpositions will result in a sign change.

Definition 6.2.6 — Alternating Group The **alternating group**, A_n, is the group of all even permutations on n letters.

Theorem 6.2.7. The alternating group, A_n, is a normal subgroup of the symmetric group, S_n.

Proof. We claim that $\psi \colon S_n \longrightarrow \mathbb{Z}_2$ as defined in Definition 6.2.1 is a group homomorphism. First notice that $\psi(\sigma\rho) = P(\sigma\rho \,.\, \boldsymbol{x})/|P(\boldsymbol{x})|$. Now consider $\psi(\sigma)\psi(\rho) = P(\sigma.\boldsymbol{x})P(\rho.\boldsymbol{x})/|P(\boldsymbol{x})|^2$. Hence $\psi(\sigma)\psi(\rho) = 1$ if σ and ρ have the same parity and $\psi(\sigma)\psi(\rho) = -1$ if σ and ρ have opposite parities. Suppose $\sigma = s_1 \cdots s_k$ and $\rho = r_1 \cdots r_m$ where s_i and r_i are transpositions. Then $\sigma\rho = s_1 \cdots s_k r_1 \cdots r_m$ is a product of $k+m$ transpositions. This is even if k and m are both even, or both odd, and odd if k and m have opposite parities. Therefore $\psi(\sigma\rho) = P(\sigma\rho \,.\, \boldsymbol{x})/|P(\boldsymbol{x})|$ is 1 if σ and ρ are the same parity and -1 if they are opposite parities. Hence ψ is a homomorphism.

The kernel of ψ is A_n, since even permutations map to 1 by definition. Hence by Theorem 4.2.20 A_n is a normal subgroup of S_n. □

Lemma 6.2.8 The three-cycles generate A_n.

Proof. Notice that $(1\, a_1)(1\, a_2) = (1\, a_1\, a_2)$. Since transpositions of the form $(1\, a)$ generate S_n and A_n consists of all permutations which can be written as a product of an even number of transpositions we can always pair up transpositions like this to write elements of A_n as a product of three cycles. Hence the three-cycles generate A_n. □

Notice that the order of A_n is $|A_n| = |S_n|/|\mathbb{Z}_2| = n!/2$.

Seven

Applications

7.1 Platonic Solids

Definition 7.1.1 — Platonic Solid A **platonic solid** is a regular convex polyhedron.

There are five platonic solids, the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. These have 4, 6, 8, 12, and 20 faces, respectively, and are formed from equilateral triangles, squares, equilateral triangles, regula pentagons, and equilateral triangles, respectively.

Each platonic solid has an associated symmetry group, which acts on the solid leaving it invariant. These symmetry groups are all permutation groups, both symmetric and alternating. We can think of them as acting by permuting the vertices of the solid. The reason why the full symmetry group is not necessarily all of S_n is because certain permutations aren't allowed, for example, if two vertices are connected by an edge then this must be the case after permuting vertices.

The dual of a platonic solid is the platonic solid you get if you swap the vertices and faces, that is if you join the centre of two faces if they have a common edge. This is demonstrated in Figure 7.1 for the cube and octahedron. The tetrahedron is its own dual, the cube and octahedron are dual and the dodecahedron and icosahedron are dual. Duals have the same symmetry group.

Consider the tetrahedron, with the symmetry group A_4. This group is of order $4!/2 = 12$. A_4 is generated by two symmetries shown in Figure 7.2.

Table 7.1: The platonic solids, along with the number of faces, F, vertices, V, and edges, E, which are related by Euler's formula, $V - E + F = 2$, the regular polygons that make up their faces, and their symmetry groups. Notice that duals have the same symmetry groups, the same number of edges and that the number of faces and vertices are swapped.

Sold	F	V	E	Polygon	Symmetry Group
Tetrahedron	4	4	6	Triangle	A_4
Cube	6	8	12	Square	S_4
Octahedron	8	6	12	Triangle	S_4
Dodecahedron	12	20	30	Pentagon	A_5
Icosahedron	20	12	30	Triangle	A_5

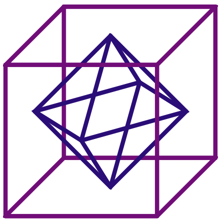


Figure 7.1: The octahedron is the dual of the cube.

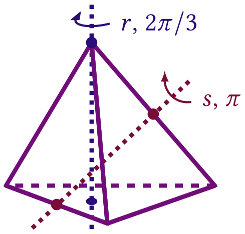


Figure 7.2: Two possible symmetries of the tetrahedron

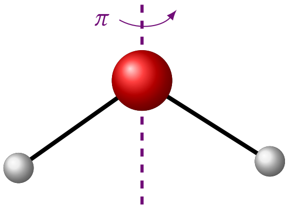


Figure 7.3: Water, with symmetry group $\mathbb{Z}_2$, has a permanent dipole.

For the case of the cube we can view S_4 as acting by permuting the diagonals of the cube.

7.2 Molecules

For a molecule to have a permanent electric dipole it must necessarily have some level of asymmetry, since a spherically symmetric molecule cannot have a preferred direction for a dipole to lie along. For example, water, has a permanent dipole, and it has as a symmetry group $\mathbb{Z}_2$, as shown in Figure 7.3.

Any molecule with symmetry group $\mathbb{Z}_n$ with $n \in \{2, 3, \dots\}$ cannot have a permanent electric dipole perpendicular to the symmetry axis. For example, water's electric dipole is aligned with its symmetry axis.

Consider again the case of ethene, C_2H_6, shown in Figure 5.1. This has symmetry group $D_3 \cong \mathbb{Z}_2 \ltimes \mathbb{Z}_3$. With $\mathbb{Z}_3$ corresponding to rotations by $2\pi/3$ about the carbon-carbon bond and $\mathbb{Z}_2$ corresponding to either inversion about the middle of the carbon-carbon bond or rotations about the perpendicular bisector to the carbon-carbon bond, depending on whether the molecule is eclipsed or staggered. Either way C_2H_6 cannot have a permanent dipole since the two axis of symmetry are orthogonal. For example, suppose there was an electric dipole and it aligned with the $\mathbb{Z}_3$ symmetry axis. Then the action of $\mathbb{Z}_2$ would be to reverse the dipole, and hence this isn't a valid permanent dipole.

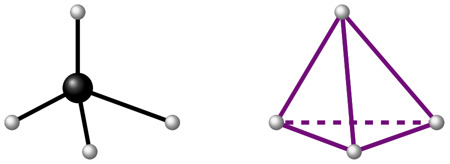
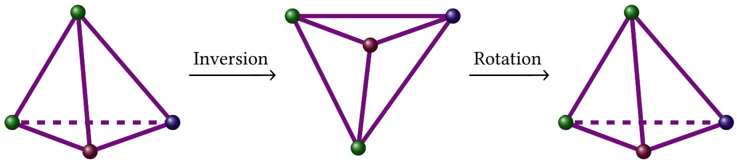


Figure 7.4: Methane is a tetragonal molecule, meaning that it has a central atom, here carbon, and four atoms arranged around it making the points of a tetrahedron.

It can be shown that only molecules with a cyclic symmetry group, $\mathbb{Z}_n$, can have permanent dipoles.

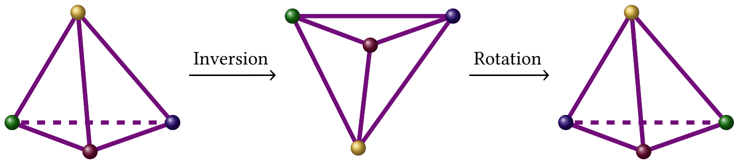
Another example of symmetry applications to molecules is the chirality of molecules. A molecule is chiral if it is different from its mirror image. Another way of saying this is that a molecule is chiral if it does not admit an improper rotation axis, an improper rotation being a rotation followed by an inversion.

A tetragonal molecule is one where there is a centre atom with four atoms around it at the points of a tetrahedron, such as methane, CH_4, shown in Figure 7.4. An example of a tetragonal molecule which is achiral is CCl_2BrI, this is shown here:



(7.2.1)

On the other hand CFClBrI is chiral, this is shown here:



(7.2.2)

Notice that the green and blue are swapped after the inversion and rotation.

Part II

Representation Theory

Eight

Basics of Representation Theory

R Material in this section applies both to finite groups and compact groups. We won't worry too much about what it means for a group to be compact, we just note that the Lie groups $\mathrm{U}(1) \cong \mathrm{SO}(2)$ and $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2$ are compact. More generally $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{Spin}(n)$, $\mathrm{U}(n)$, and $\mathrm{SU}(n)$ are compact.

8.1 Representation Definition

There are two essentially equivalent definitions of a representation:

Definition 8.1.1 — Representation A **representation** of a group, G, is a group action of G on a linear space, V. That is $\varphi\colon G \times V \to V$ is a representation if it is a group action.
A **representation** of a group, G, is a homomorphism with the automorphism group of a linear space. That is $\rho\colon G \to \mathrm{GL}(V)$ is a representation if it is a homomorphism.

The equivalence of these two definitions is simple, if $\varphi(g, \boldsymbol{v}) = g \,.\, \boldsymbol{v} = \rho(g)\boldsymbol{v}$ for all $g \in G$ and $\boldsymbol{v} \in V$ then φ and ρ are the same representation in the two slightly different definitions.

When the homomorphism is clear it is common to refer to V as the representation, rather than ρ.

Representation theory gives us a way to do concrete calculations in a group. We have implicitly been using representation theory already, for example we have used $\mathbb{Z}_n = \{\mathrm{e}^{2i\pi m/n} \mid m = 0, \dots, n-1\}$ as *the* cyclic group of order n. Strictly this is actually a representation of a more general cyclic group, defined by $\langle g \mid g^n = e\rangle$. The linear space in question is $\mathbb{C}$, and $\mathrm{GL}(\mathbb{C}) = \mathbb{C} \setminus \{0\}$. The group action is rotation by $2\pi m/n$.

Representation theory is particularly useful because we have developed a lot of tools for dealing with computations in linear spaces since they are common in other areas of maths and physics. Representation theory allows us to use these to work with groups.

■ **Application 8.1.2** When we solve the quantum harmonic oscillator we can do so in different linear spaces, such position space, $|x\rangle$, momentum space, $|p\rangle$, or number-of-particles space, $|n\rangle$. Each one of these corresponds to studying the same underlying physics on a different linear space, which

> we can think of as being different representations.
> If a system has a certain symmetry described by a group, G, then this is expressed in the mathematics by G acting on the Hilbert space of states, which is to say as a representation of G on the Hilbert space of states.

If the relevant linear space is finite dimensional, say $\dim V = n$ then we can associate $\mathrm{GL}(V)$ with $\mathrm{GL}(n, \mathbb{F})$, the group of $n \times n$ matrices with entries in $\mathbb{F}$, and so we can associate $\rho(g)$ with a matrix.

8.2 Pedestrian Approach

R In this section we will get an idea of how representation can be used without being too worried about precise definitions.

Consider S_3. By Lemma 6.1.18 $\{(1\,2), (1\,3)\}$ generates S_n, so we can define a representation by how it maps these two elements to the linear space. Inspired by S_3 as a permutation group we look for a representation that conserves this permuting ability. In particular the obvious choice of 3 things to act on in a linear space are the 3 basis vectors of a 3-dimensional space, such as $\mathbb{R}^3$. We use the standard basis, $\boldsymbol{e}_1 = (1, 0, 0)^\top$, $\boldsymbol{e}_2 = (0, 1, 0)^\top$, and $\boldsymbol{e}_3 = (0, 0, 1)^\top$. We can easily construct matrices that permute these, for example we want $\rho((1\,2))\boldsymbol{e}_1 = \boldsymbol{e}_2$, $\rho((1\,2))\boldsymbol{e}_2 = \boldsymbol{e}_2$ and $\rho((1\,2))\boldsymbol{e}_3 = \boldsymbol{e}_3$. This fully determines the matrix $\rho((1\,2))$:

$$\rho((1\,2)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8.2.1}$$

Similarly, we have

$$\rho((1\,3)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{8.2.2}$$

This is called the **permutation representation** of S_3, and can easily be generalised to S_n.

At this point there are a few questions we should consider. First, are there any other representations? The answer to this is yes, and we'll see some later. The second is can we find a simpler representation, for some sense of simpler. The answer is again yes. In particular notice that $\boldsymbol{v} = (1, 1, 1)^\top$ is an common eigenvector for both of these matrices, and hence there is an invariant subspace, $\operatorname{span}\{\boldsymbol{v}\}$, which is unchanged by this representation. It can be shown that this allows us to perform a basis change and write these matrices in block diagonal form. We can then define a new representation, ρ', in this block diagonal basis such that

$$\rho'((1\,2)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix}, \qquad \text{and} \qquad \rho'((1\,3)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b_{11} & b_{12} \\ 0 & b_{21} & b_{22} \end{pmatrix}. \tag{8.2.3}$$

This is simpler because it is fully determined by the 2×2 block matrices on the diagonal. We say that the permutation representation is reducible.

8.3 Basic Definitions

Definition 8.3.1 — Faithful If $\rho\colon G \to \mathrm{GL}(V)$ is a representation of G then we say ρ is **faithful** if it is injective, that is $\rho(g) = \rho(g')$ implies $g = g'$. If this is not the case then we say ρ is **unfaithful**.

The permutation representation of S_3 is faithful.

Definition 8.3.2 — Trivial Representation *A* **trivial representation** is $\rho\colon G \to \mathrm{GL}(V)$ defined by $\rho(g) = \rho(e) = \mathbf{1}_V$, where $\mathbf{1}_V$ is the identity in $\mathrm{GL}(V)$ and V is an *arbitrary vector space*.
The **trivial representation** is $\rho\colon G \to \mathrm{GL}(V)$ defined by $\rho(g) = \rho(e) = \mathbf{1}_V$, where $\mathbf{1}_V$ is the identity in $\mathrm{GL}(V)$ and V is a *one-dimensional vector space*.

A trivial representation is maximally unfaithful since all elements map to the same operator.

Definition 8.3.3 — Unitary Representation If $\rho\colon G \to \mathrm{U}(V)$ is a homomorphism then we say that ρ is a **unitary representation**. That is a unitary representation is one in which $\rho(g)$ is a unitary operator for all $g \in G$.

We have been working with a unitary representation of $\mathbb{Z}_n$ as $\{\mathrm{e}^{2i\pi m/n} \mid m = 0, \dots, n-1\}$.

Definition 8.3.4 — Equivalence of Representations If ρ and ρ' are representations of G on V the we say that ρ and ρ' are **equivalent** if they are represented by a **similarity transform**, that is $\rho(g) = S\rho'(g)S^{-1}$ for some $S \in \mathrm{GL}(V)$. Notice that S must be the same for all $g \in G$.

As the name suggests the equivalence of representations, $\sim$, is an equivalence relation. Clearly $\rho \sim \rho$ since $\rho(g) = \mathbf{1}_V\rho(g)\mathbf{1}_V^{-1}$. Also, if $\rho \sim \rho'$ then $\rho(g) = S\rho'(g)S^{-1}$ for some $S \in \mathrm{GL}(V)$, and so $\rho'(g) = S^{-1}\rho(g)S$, identifying $S = (S^{-1})^{-1}$ and knowing that if $S \in \mathrm{GL}(V)$ we also have $S^{-1} \in \mathrm{GL}(V)$ we see that $\rho' \sim \rho$. Finally, if $\rho \sim \rho'$ and $\rho' \sim \rho''$ there exists $S, T \in \mathrm{GL}(V)$ such that $\rho(g) = S\rho'(g)S^{-1}$ and $\rho'(g) = T\rho''(g)T^{-1}$, and hence $\rho(g) = ST\rho''(g)T^{-1}S^{-1} = (ST)\rho(g)(ST)^{-1}$ and since $S, T \in \mathrm{GL}(V)$ it follows that $ST \in \mathrm{GL}(V)$, and hence $\rho \sim \rho''$.

Definition 8.3.5 — Invariant Subspace Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of the group G on the linear space V. Let W be a subspace of V. Then we call W a **invariant subspace** if $\rho(g)\boldsymbol{w} \in W$ for all $g \in G$ and $\boldsymbol{w} \in W$.
Using the group-action definition of a representation W is an invariant subspace if $\mathrm{Orb}(\boldsymbol{w}) \subseteq W$ for all $\boldsymbol{w} \in W$.

A trivial representation, $\rho(g) = \mathbf{1}_V$, has V as an invariant subspace. Our earlier example of S_3 in the permutation representation has $\mathrm{span}\{(1,1,1)^{\top}\}$ as an invariant subspace. All representations leave the trivial subspace, $\{\mathbf{0}\}$, invariant.

Definition 8.3.6 — Irreducible We call a representation **irreducible** if it has no invariant subspaces, apart from the trivial zero-dimensional subspace, $\{\mathbf{0}\}$. If a representation can be written in block diagonal form then it is **reducible**

It is common to shorten "irreducible representation" to **irrep**.

For finite groups and continuous compact groups the reducible representations can be written as a direct sum of irreducible representations. For this reason we often care only about irreducible representations.

The permutation representation of S_3 is reducible.

Lemma 8.3.7 If a representation has invariant subspaces then we can write it in block diagonal form.

Definition 8.3.8 — Real and Complex Representations A representation, $\rho\colon G \to \mathrm{GL}(V)$, is **real** if either V is a real vector space or the representation is equivalent to a representation which can be thought of as acting on a real vector space by reducing the field of scalars to $\mathbb{R}$. Alternatively ρ is equivalent to ρ^*.

A **complex representation** is a representation, $\rho\colon G \to \mathrm{GL}(V)$, where ρ is not equivalent to ρ^*.

We will refine this notion later to include pseudo-real representations.

The permutation representation of S_3 is real, and hence so is ρ', even though it is possible that a_{ij} and/or b_{ij} are not real, since ρ and ρ' are equivalent.

■ **Application 8.3.9** Unitary representations are important in quantum physics. They are the natural language to describe symmetries on the Hilbert space of states since they preserve the inner product, and hence the probability of being in a given state.

Further there is a certain view from which particles *are* irreducible representations of the Poincaré group, $\mathbb{R}^{1,3} \ltimes \mathrm{O}(1,3)$. We then associate complex representations with charged particles and real representations with neutral particles.

8.4 Some Theorems

Theorem 8.4.1 — Maschke's Theorem. Any representation of a finite group is equivalent to a unitary representation.

Proof. Let G be a finite group, V a vector space, and $\rho\colon G \to \mathrm{GL}(V)$ a representation. Let $\langle -, - \rangle$ be an inner product on G. The statement of the theorem is equivalent to stating that we can define a new inner product on V such that this inner product is invariant under the action of this representation. This works since the two inner products will be related by a change of basis, and hence this new inner product can be viewed as the old inner product after a similarity transform.

The inner product that we define is $\langle -, - \rangle_G$ and it is defined for $x, y \in V$ as

$$\langle x, y \rangle_G := \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)x, \rho(g)y \rangle. \tag{8.4.2}$$

We can think of this inner product being defined by acting on the old inner product with the representation and then averaging over G.
We need to show that $\langle -, - \rangle_G$ is an inner product and that it is invariant with respect to the representation. By definition $\rho(g)x, \rho(g)y \in V$ and so $\langle \rho(g)x, \rho(g)y \rangle$ is positive definite. This carries through the sum and so $\langle x, y \rangle_G$ is positive definite. Linearity and conjugate symmetry of $\langle -, - \rangle_G$ similarly follows from these same properties for $\langle -, - \rangle$ without any complications since G is finite.
Now consider what happens when we first act on V with $\rho(g')$ for some $g' \in G$. Using the fact that ρ is a homomorphism we then have

$$\begin{aligned} \langle \rho(g')x, \rho(g')y \rangle_G &= \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)\rho(g')x, \rho(g)\rho(g')y \rangle & (8.4.3) \\ &= \sum_{g \in G} \langle \rho(gg')x, \rho(gg')y \rangle & (8.4.4) \\ &= \sum_{g'' \in G} \langle \rho(g'')x, \rho(g'')y \rangle & (8.4.5) \\ &= \langle x, y \rangle_G. & (8.4.6) \end{aligned}$$

In the penultimate step we have used the fact that as g takes on all values in G gg' necessarily also takes on all values in G. This is due to the fact that in a Cayley table each column must contain every element of G exactly once. Hence summing over g with factors of gg' is the same as summing over $g'' = gg'$, only the order of the terms changes and since the inner product gives an element of the base field this sum is commutative.
Finally we remark that

$$\langle x, y \rangle_G = \langle \rho(g')x, \rho(g')y \rangle_G = \langle \rho(g')^\dagger \rho(g')x, y \rangle_G \qquad (8.4.7)$$

using the property of inner products that $\langle x, Ay \rangle = \langle A^\dagger x, y \rangle$ for any inner product, $\langle -, - \rangle$, and operator A. Hence we can identify that $\rho(g')^\dagger \rho(g') = \mathbf{1}$, and so ρ is a unitary representation. □

It turns out that being irreducible is an incredibly strong requirement, so much so that it doesn't really leave much wiggle room, as the next theorem shows. Schur's lemma states that there is no room for non-trivial homomorphisms between irreducible representations.

Theorem 8.4.8 — Schur's Lemma. Let $\rho \colon G \to \mathrm{GL}(V)$ and $\rho \colon G \to \mathrm{GL}(V')$ be irreducible representations of the group G on some finite dimensional vector spaces V and V'. Let $T \colon V \to V'$ be a linear map satisfying $\rho'(g) \circ T = T \circ \rho(g)$ for all $g \in G$ where $\circ$ is composition of functions. Then

1. either T is an isomorphism or T is trivial.
2. If $V = V'$ then $T = \lambda \mathbf{1}$ for $\lambda \in \mathbb{C}$ and $\mathbf{1}$ being the identity map on V.

Proof. The first step is to notice that $\ker T$ and $\operatorname{Im} T$ are invariant subspaces. Recall that $\ker T := \{v \in V \mid Tv = 0\}$. To show that $\ker T$ is an invariant subspace we need to show that $\rho(g)v \in \ker T$ for all $v \in \ker T$. To do this we notice that for $v \in \ker T$ we have

$$\begin{aligned} T\rho(g)v &= (T \circ \rho(g))v = (\rho'(g) \circ T)v \\ &= \rho'(g)Tv = \rho'(g)(Tv) = \rho'(g)0 = 0 \end{aligned} \tag{8.4.9}$$

where the final equality follows since linear maps map 0 to 0. We have therefore shown that $T\rho(g)v = 0$ for all $v \in \ker T$ and hence $\rho(g)v \in \ker T$ so $\ker T$ is an invariant subspace of V under ρ.

Now recall that $\operatorname{Im} T = \{v' \in V' \mid v' = T(v) \text{ for some } v \in V\}$. This is an invariant subspace if $\rho'(g)v \in \operatorname{Im} T$ for all $v' \in \operatorname{Im} T$. To show this we notice that for $v' \in \operatorname{Im} T$ we have $v \in V$ such that $v' = Tv$ and so

$$\rho'(g)v' = \rho'(g)Tv = (\rho'(g) \circ T)v = (T \circ \rho(g))v = T\rho(g)v \tag{8.4.10}$$

and so $\rho'(g)v'$ is of the form $T\rho(g)v$ and $\rho(g)v \in V$ meaning $\rho'(g)v' \in \operatorname{Im} T$. Hence $\operatorname{Im} T$ is an invariant subspace of V' under ρ'.

By definition ρ and ρ' are irreducible representations and therefore have no nontrivial invariant subspaces. This means that the invariant subspace $\ker T$ must be $\{0\}$ or V, and similarly $\operatorname{Im} T$ must be $\{0\}$ or V'. We now treat this by cases.

- Suppose $\ker T = \{0\}$. Then $\operatorname{Im} T \neq \{0\}$ since all $v \in V$ with $v \neq 0$ map to something other than 0, meaning that $\operatorname{Im} T$ must contain nonzero elements. Hence $\operatorname{Im} T = V'$. It follows that T is an isomorphism since a trivial kernel implies that $T \colon V \longrightarrow V'$ is injective since $V' = \operatorname{Im} T$ this map is also surjective.

- Suppose $\ker T = V$. Then $\operatorname{Im} T = \{0\}$, since by definition all $v \in V$ map to 0. Hence T is the trivial zero function, $T(v) = 0$ for all $v \in V$.

This finishes the proof of the first statement.

For the second statement suppose T is nontrivial, that is $T \neq 0$. Then T has at least one nonzero eigenvalue, $\lambda \in \mathbb{C}$, since if all eigenvalues are zero then T is trivial. Now define a second linear map $U := T - \lambda\mathbf{1}$. Then by construction at least one eigenvalue of U is 0 and so U is not an isomorphism, since all vectors parallel to the eigenvector with eigenvalue 0 are mapped to 0. More formally this means that $\dim(\ker U) \geq 1$, in fact the dimension is the multiplicity of the eigenvalue 0.

Since $U = T - \lambda\mathbf{1}$ it is clear that $\rho'(g) \circ U = U \circ \rho(g)$ and so by the first part of this theorem $U = 0$, since U is not an isomorphism. Hence $U = 0 = T - \lambda\mathbf{1}$ which we can rearrange to get $T = \lambda\mathbf{1}$. □

An equivalent statement to the first part of Schur's lemma is that the following diagram commuting for all $g \in G$ only if T is trivial or an isomorphism:

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\rho(g)"'] & V' \arrow[d, "\rho'(g)"] \\
V \arrow[r, "T"] & V'
\end{tikzcd} \tag{8.4.11}$$

An alternative statement of the second part of Schur's lemma, which is often state as the full version of Schur's lemma in the physics literature, is the following corollary.

Corollary 8.4.12 Let ρ be an irreducible representation of a group G on some finite dimensional vector space, V, and T be some linear map on this same vector space such that $[T, \rho(g)] = 0$ for all $g \in G$. Then $T = \lambda \mathbf{1}$.

R Here $[A, B] \coloneqq AB - BA$ is the usual **commutator**.

The final theorem for this section relates writing representations as direct sums of irreducible representations. See Definition A.2.54 for the definition of direct sums.

Theorem 8.4.13 — Decomposability Theorem. Let $\rho\colon G \longrightarrow \mathrm{GL}(V)$ be a reducible representation for some compact group G. Then we can write

$$\rho(g) = m_1\rho_1(g) \oplus m_2\rho_2(g) \oplus \cdots \oplus m_k\rho_k(g) = \bigoplus_{i=1}^{k} m_i\rho_i(g). \tag{8.4.14}$$

where $\rho_i\colon G \longrightarrow \mathrm{GL}(V)$ are irreducible representations and $m_i \in \mathbb{Z}_{>0}$. By $m_i\rho_i(g)$ we mean

$$m_i\rho_i(g) \coloneqq \underbrace{\rho_i(g) \oplus \cdots \oplus \rho_i(g)}_{m_i \text{ times}} = \bigoplus_{j=1}^{m_i} \rho_i(g). \tag{8.4.15}$$

Proof. □

Nine

Character Theory of Finite Groups

9.1 Basics of Character Theory

Definition 9.1.1 — Character Let G be a finite group and ρ a representation of G. Then we define the **character**, $\chi(g)$, of some $g \in G$ as

$$\chi(g) \coloneqq \operatorname{tr}[\rho(g)]. \tag{9.1.2}$$

Notation 9.1.3 When discussing multiple representations of G we will denote the representation being considered by a subscript, so $\chi_\rho(g) = \operatorname{tr}[\rho(g)]$ and $\chi_{\rho'}(g) = \operatorname{tr}[\rho'(g)]$.

Lemma 9.1.4 The characters of two equivalent representations are equal.

Proof. Recall that the representations ρ and ρ' of G on V are equivalent if there exists $S \in \mathrm{GL}(V)$ such that $\rho'(g) = S\rho(g)S^{-1}$ for all $g \in G$. Therefore

$$\begin{aligned}
\chi_{\rho'}(g) &\coloneqq \operatorname{tr}[\rho'(g)] && (9.1.5)\\
&= \operatorname{tr}[S\rho(g)S^{-1}] && (9.1.6)\\
&= \operatorname{tr}[S^{-1}S\rho(g)] && (9.1.7)\\
&= \operatorname{tr}[\rho(g)] && (9.1.8)\\
&\eqqcolon \chi_\rho(g). && (9.1.9)
\end{aligned}$$

Here we have used the cyclic property of the trace, that $\operatorname{tr}(ABC) = \operatorname{tr}(CAB)$. □

Recall that an equivalence relation is a relation which is symmetric, reflexive, and transitive. The equivalence class of $a \in A$ under some equivalence relation, $\sim$, is the set $[a] \coloneqq \{b \in A \mid a \sim b\}$. The set of all equivalence classes is denoted $A/\sim$.

Definition 9.1.10 A **class function** is a function, $f \colon A \to B$, which takes on the same value for all $b \in [a]$, where $[a]$ is an equivalence class of A under some equivalence relation.

Recall that $x \sim y$ if $x = gyg^{-1}$ is an equivalence relation, in particular it is called conjugacy and the equivalence classes of this equivalence relation are called conjugation classes.

Lemma 9.1.11 The character is a class function on the conjugacy classes.

Proof. Let $x, y \in G$ be in the same conjugacy class. Then $x = gyg^{-1}$ for some $g \in G$. Let ρ be a representation of G. Then

$$\begin{aligned} \chi_\rho(x) &\coloneqq \mathrm{tr}[\rho(x)] && (9.1.12)\\ &= \mathrm{tr}[\rho(gxg^{-1})] && (9.1.13)\\ &= \mathrm{tr}[\rho(g)\rho(y)\rho(g^{-1})] && (9.1.14)\\ &= \mathrm{tr}[\rho(g)\rho(y)\rho(g)^{-1}] && (9.1.15)\\ &= \mathrm{tr}[\rho(g)^{-1}\rho(g)\rho(y)] && (9.1.16)\\ &= \mathrm{tr}[\rho(y)] && (9.1.17)\\ &\eqqcolon \chi_\rho(y). && (9.1.18) \end{aligned}$$

Here we have used the fact that ρ is a homomorphism so $\rho(ab) = \rho(a)\rho(b)$ and $\rho(a^{-1}) = \rho(a)^{-1}$. We have also used the cyclic property of the trace, $\mathrm{tr}(ABC) = \mathrm{tr}(CAB)$. Hence $\chi_\rho(x) = \chi_\rho(y)$ for all $x, y \in [x] \in G/\sim$ where $\sim$ is conjugacy. □

Lemma 9.1.19 Let ρ and ρ' be representations of G. Then $\chi_{\rho\oplus\rho'}(g) = \chi_\rho(g) + \chi_{\rho'}(g)$ for all $g \in G$ and $\chi_{\rho\otimes\rho'}(g) = \chi_\rho(g)\chi_{\rho'}(g)$.

Proof. In this proof we use braket notation. Let $\{|i\rangle\}$ be an orthonormal basis. Then in braket notation with the Einstein summation convention

$$\mathrm{tr}(A) \coloneqq \langle i|A|i\rangle. \tag{9.1.20}$$

Therefore

$$\begin{aligned} \chi_{\rho\oplus\rho'}(g) &\coloneqq \mathrm{tr}([\rho\oplus\rho')(g)] && (9.1.21)\\ &= \mathrm{tr}[\rho(g)\oplus\rho'(g)] && (9.1.22)\\ &= \langle i|(\rho(g)\oplus\rho'(g))|i\rangle && (9.1.23)\\ &= \langle i|\rho(g)|i\rangle + \langle i|\rho'(g)|i\rangle && (9.1.24)\\ &= \mathrm{tr}[\rho(g)] + \mathrm{tr}[\rho'(g)] && (9.1.25)\\ &\eqqcolon \chi_\rho(g) + \chi_{\rho'}(g). && (9.1.26) \end{aligned}$$

Here we have used the fact that

$$(A\oplus B)(|v\rangle\oplus|w\rangle) = A|v\rangle\oplus B|w\rangle \tag{9.1.27}$$

and so

$$(\langle v'|\oplus\langle w'|)(A\oplus B)(|v\rangle\oplus|w\rangle) = (\langle v'|\oplus\langle w'|)(A|v\rangle\oplus B|w\rangle) = \langle v'|A|v\rangle+\langle w'|B|w\rangle.$$

$$\tag{9.1.28}$$

Similarly

$$\begin{align}\chi_{\rho\otimes\rho'}(g) &:= \mathrm{tr}[(\rho\otimes\rho')(g)] \tag{9.1.29}\\ &= \mathrm{tr}[\rho(g)\otimes\rho'(g)] \tag{9.1.30}\\ &= \langle i|(\rho(g)\otimes\rho'(g))|i\rangle \tag{9.1.31}\\ &= \langle i|\rho(g)|i\rangle\langle i|\rho'(g)|i\rangle \tag{9.1.32}\\ &= \mathrm{tr}[\rho(g)]\,\mathrm{tr}[\rho'(g)] \tag{9.1.33}\\ &=: \chi_\rho(g)\chi_{\rho'}(g). \tag{9.1.34}\end{align}$$

Here we have used the fact that

$$(A\otimes B)(|v\rangle\otimes|w\rangle) = A|v\rangle\otimes B|w\rangle. \tag{9.1.35}$$

□

Lemma 9.1.36 Let G be a finite group and ρ a representation of G. Then $\chi_\rho(g^{-1}) = \chi_\rho(g)^*$ for all $g\in G$.

Proof. By Theorem 8.4.1 ρ is equivalent to a unitary representation, ρ'. By Lemma 9.1.4 $\chi_\rho(g) = \chi_{\rho'}(g)$ for all $g\in G$. Since ρ' is a homomorphism $\rho'(g^{-1}) = \rho'(g)^{-1}$ by Lemma 2.1.20. Since ρ' is unitary $\rho'(g)^{-1} = \rho'(g)^\dagger$. Since $\mathrm{tr}(A) = \mathrm{tr}(A^\top)$ it follows that $\mathrm{tr}(A^\dagger) = \mathrm{tr}(A^*) = \mathrm{tr}(A)^*$. Hence

$$\begin{aligned}\chi_\rho(g^{-1}) = \chi_{\rho'}(g^{-1}) = \mathrm{tr}[\rho'(g^{-1})] = \mathrm{tr}[\rho'(g)^{-1}]\\ = \mathrm{tr}[\rho'(g)^\dagger] = \mathrm{tr}[\rho'(g)]^* =: \chi_{\rho'}(g)^* = \chi_\rho(g)^*.\end{aligned} \tag{9.1.37}$$

□

Lemma 9.1.38 Let G be a finite group and ρ a one-dimensional representation of G. Then $\chi_\rho(g) = \rho(g)$, where we make the natural identification of $\rho(g)$ as a 1×1 matrix with the matrix element $\rho(g)_{11}$.

Proof. This is trivially true since the trace of (z) is $\mathrm{tr}[(z)] = z$, so $\rho(g) = (z)$ for some $z\in\mathbb{C}$ and $\chi_\rho(g) = \mathrm{tr}[\rho(g)] = \mathrm{tr}[(z)] = z$. □

9.2 Space of Characters

We can view the characters of a given representation as forming vectors. Given a finite group, G, and a representation ρ we define a vector

$$\chi(\rho) = (\chi_\rho(e), \chi_\rho(g_1), \dots, \chi_\rho(g_{N_c})) \in \mathbb{C}^{N_c} \tag{9.2.1}$$

where g_i is in the kth conjugacy class of G and N_c is the total number of conjugacy classes. It is simply a matter of convention to assign the first conjugacy class to be the one containing the identity.

Definition 9.2.2 — Inner product on the space of characters We can define an inner product on the space of characters as follows:

$$\langle \chi(\rho_a), \chi(\rho_b) \rangle := \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_a}(g)^* \chi_{\rho_b}(g) = \frac{1}{|G|} \sum_{k=1}^{N_c} c_k \chi_{ak}^* \chi_{bk} \tag{9.2.3}$$

where c_k is the number of elements in the kth conjugacy class and $\chi_{ak} := \chi_{\rho_a}(g_k)$ with g_k being a representative member of the kth conjugacy class.

The following theorem gives us a useful way to test if a representation is irreducible.

Theorem 9.2.4 — Orthogonality Theorem. The irreducible representations form an orthonormal set in the space of characters.

Proof. R This proof is beyond the scope of this course.

Suppose V_a and V_b are irreducible representations[a] of some finite group, G. Denote these representations by ρ_{V_a} and ρ_{V_b} respectively. The space of all homomorphisms $V_a \longrightarrow V_b$ is denoted Hom$_($ $V_a, V_b)$, and is equal to $V_a^* \otimes V_b$ where V_a^* is the dual vector space of V_a, we can think of this informally as going from kets to bras, or more formally as the space of linear functions, $V_a \longrightarrow \mathbb{F}$, where $\mathbb{F}$ is the base field of V_a.
Let V_0 denote the vector space of trivial representations, that is the vector space with the associated representation $\rho_{V_0}(g) = \mathbf{1}_{V_0}$ for all $g \in G$.
Schur's lemma (Theorem 8.4.8) states that the only nontrivial homomorphisms compatible with the group structure are isomorphisms. Hence we there are only nontrivial homomorphisms if $V_a \cong V_b$, in which case $\dim(\mathrm{Hom}(V_a, V_b)_0) = 1$, where the subscript 0 denotes projection onto the trivial representation subspace. On the other hand if $V_a \not\cong V_b$ then $\dim(\mathrm{Hom}(V_a, V_b)) = 0$. We can sum this up as $\dim(\mathrm{Hom}(V_a, V_b)_0) = \delta_{ab}$.
The character of the representation $V_a^* \otimes V_b$ is given by

$$\chi(\rho_{V_a^* \otimes V_b}) = \chi(\rho_{V_a})\chi(\rho_{V_b}), \tag{9.2.5}$$

which follows from Lemma 9.1.19 applied to each component of the vectors $\chi(\rho)$. Further

$$\chi(\rho_{V_a^*}) = \chi(\rho_{V_a})^* \tag{9.2.6}$$

since

$$\chi_{\rho_{V_a^*}}(g) = \mathrm{tr}[\rho_{V_a^*}(g)] = \mathrm{tr}[\rho_{V_a}(g)^\dagger] = \mathrm{tr}[\rho_{V_a}(g)]^* = \chi_{\rho_{V_a}}(g)^*, \tag{9.2.7}$$

which follows from the fact that the adjoint (Hermitian conjugate) of a vector is an element of the dual space and vice versa. We therefore have

$$\chi(\rho_{V_a^* \otimes V_b}) = \chi(\rho_{V_a})^* \chi(\rho_{V_b}). \tag{9.2.8}$$

Now define the operator φ according to

$$\varphi = \frac{1}{|G|}\sum_{g\in G}\rho_V(g). \tag{9.2.9}$$

This is a projection operator, meaning $\varphi^2 = \varphi$, in particular it projects onto V_0. This means $\varphi v_0 = v_0$ for all $v_0 \in V_0$ and $\varphi v_0^\perp = 0$ where $v_0^\perp$ is an element of the orthogonal space to V_0, which we denote $V_0^\perp$ and is the space such that $V_0^\perp \oplus V_0 = V$, note that all vectors in $V_0^\perp$ are by construction orthogonal to all vectors in V_0.
To see that φ has these properties notice that applying φ to a vector gives an invariant vector and the only invariant vectors correspond to vectors in the trivial representation subspace. Further, applying φ a second time just rearranges the order of the vectors in the sum, which has no effect since we are averaging over the whole group. The dimension of the space projected onto by a projector is simply the trace of said projector, since we can write a projector in a diagonal form with 1 on the diagonal for basis vectors spanning the subspace and 0 for the other diagonal components. Using the linearity of the trace we then have

$$\dim(V_0) = \operatorname{tr}\varphi = \frac{1}{|G|}\operatorname{tr}[\rho_V(g)] = \frac{1}{|G|}\sum_{g\in G}\chi_{\rho_V}(g). \tag{9.2.10}$$

Combining all of the above we have

$$\begin{aligned}
\delta_{ab} &= \dim(\mathrm{Hom}(V_a, V_b)) && (9.2.11)\\
&= \dim((V_a^* \otimes V_b)_0) && (9.2.12)\\
&= \frac{1}{|G|}\sum_{g\in G}\chi_{\rho_{V_a^*\otimes V_b}}(g) && (9.2.13)\\
&= \frac{1}{|G|}\sum_{g\in G}\chi_{\rho_{V_a}}(g)^*\chi_{\rho_{V_b}}(g) && (9.2.14)\\
&=: \langle\chi(\rho_{V_a}), \chi(\rho_{V_b})\rangle. && (9.2.15)
\end{aligned}$$

This completes the proof. □

[a] of course we're being a bit sloppy with the language here, but this is standard, what we really mean is there exist irreducible representations, $G \to \mathrm{GL}(V_a)$ and $G \to \mathrm{GL}(V_b)$.

The important thing about the orthogonality theorem is all of the corollaries we can prove from it. In the following let ρ be a representation of some finite group G. We can write ρ as

$$\rho = \bigoplus_{i=1}^{k} m_i\rho_i = \bigoplus_{i=1}^{k}\rho_i^{\oplus m_i} = m_1\rho_i \oplus \cdots \oplus m_k\rho_k \tag{9.2.16}$$

where ρ_i are irreducible representations and $m_i \in \mathbb{Z}_{>0}$.

Corollary 9.2.17 A representation is fully characterised by its character since the distinct ρ_i correspond to linearly independent, in fact orthonormal, vectors in the character space. We have

$$\chi_\rho(g) = \sum_{i=1}^{k} m_i \chi_{\rho_i}(g). \tag{9.2.18}$$

Corollary 9.2.19 The multiplicity of some particular irreducible representation, ρ_a, in the decomposition of ρ is

$$m_a = \langle \chi(\rho_a), \chi(\rho) \rangle \tag{9.2.20}$$

Proof. By the linearity of the character

$$\chi(\rho) = \sum_{i=1}^{k} m_i \chi(\rho_i) \tag{9.2.21}$$

Then by the linearity of the inner product we have

$$\begin{align}
\langle \chi(\rho_a), \chi(\rho) \rangle &= \left\langle \chi(\rho_a), \sum_{i=1}^{k} m_i \chi(\rho_i) \right\rangle \tag{9.2.22}\\
&= \sum_{i=1}^{k} m_i \langle \chi(\rho_a), \chi(\rho_i) \rangle \tag{9.2.23}\\
&= \sum_{i=1}^{k} m_i \delta_{ai} \tag{9.2.24}\\
&= m_a \tag{9.2.25}
\end{align}$$

which completes the proof. □

Compare this to the standard way of finding the components of a vector:

$$v^i = \boldsymbol{e_i} \cdot \boldsymbol{v}. \tag{9.2.26}$$

Corollary 9.2.27 We can define a norm on the space of characters by

$$\|\chi(\rho)\|^2 := \langle \chi(\rho), \chi(\rho) \rangle = \sum_i m_i^2 \tag{9.2.28}$$

and ρ is irreducible if and only if $\|\chi(\rho)\| = 1$.

Proof. Suppose ρ is irreducible. Then we can think of ρ as being decomposed as $\rho = 1\rho$, that is $k = 1$ and $m_1 = 1$. Hence we trivially have $\|\chi(\rho)\| = 1$. Suppose instead that $\|\chi(\rho)\| = 1$. Then it follows that $m_1^2 + \cdots + m_k^2 = 1$. Since $m_i \in \mathbb{Z}_{>0}$ it follows that $m_i = 0$ for all but one value of i and for that value of i $m_i = 1$, which means $\rho = 1\rho_i = \rho_i$, so ρ is irreducible. □

Lemma 9.2.29 The character of the identity element corresponds to the dimension of the representation. That is

$$\chi_{\rho_i}(e) = \dim V_i. \tag{9.2.30}$$

Proof. For any representation, $\rho\colon G \to V$ we have $\rho(e) = \mathbf{1}_V$ by Lemma 2.1.19 and for any vector space, V, we have $\operatorname{tr} \mathbf{1}_V = \dim V$, since $\mathbf{1}_V$ is just a matrix with ones on the diagonal. □

Lemma 9.2.31 Let ρ_0 be the trivial representation of some finite group, G. Then $\chi(\rho_0) = (1, \dots, 1) \in \mathbb{C}^{N_c}$.

Proof. This follows since the characters of the trivial representation are all one since $\chi_{\rho_0}(g) = \operatorname{tr}[\rho_0(g)] = \operatorname{tr} \mathbf{1}_{V_0}$ and the vector space of *the* trivial representation is one dimensional so $\operatorname{tr} \mathbf{1}_{V_0} = 1$. □

9.2.1 Dimensionality Theorem

Definition 9.2.32 — Regular Representation Let V be a vector space of dimension $|G|$. Let $\{\boldsymbol{e}_g\}$ be a set of $|G|$ linearly independent vectors in V which we label with group elements, $g \in G$. This set is guaranteed to exist since V is $|G|$-dimensional. We can think of V as the space spanned by these vectors. Define a left action of G on $\{\boldsymbol{e}_g\}$ in the obvious way by

$$g \,.\, \boldsymbol{e}_{g'} := \boldsymbol{e}_{gg'}. \tag{9.2.33}$$

The **regular representation**, $\rho_{\mathrm{R}}\colon G \to \mathrm{GL}(V)$ is defined as the representation associated with this action, that is

$$\rho_{\mathrm{R}}(g)\boldsymbol{e}_{g'} = \boldsymbol{e}_{gg'}. \tag{9.2.34}$$

The regular representation is, in general, reducible, however it does have the following useful property:

$$\chi_{\rho_{\mathrm{R}}}(g) = \begin{cases} 0 & \text{if } g \neq e, \\ |G| & \text{if } g = e. \end{cases} \tag{9.2.35}$$

This follows since for $g \neq e$ the diagonal of $\rho_{\mathrm{R}}(g)$ must be zero, since if they weren't it would mean that $\boldsymbol{e}_{gg} = \boldsymbol{e}_g$, which means that $g^2 = g$, which can only happen for $g = e$. Additionally $\rho_{\mathrm{R}}(e) = \mathbf{1}_V$ and so $\chi_{\mathrm{R}}(e) = \operatorname{tr} \mathbf{1}_V = \dim V = |G|$.

The regular representation can also be viewed as the induced representation of the trivial representation. Induced representations will be defined in .

The main use of the regular representation for us is to prove the following theorem which aids in classifying the irreducible representations.

Theorem 9.2.36 — Dimensionality Theorem. Let G be a group and $\{V_i\}$ be the irreducible representations of G, then

$$|G| = \sum_{i=1}^{N_c} \dim(V_i)^2. \tag{9.2.37}$$

Proof. Let ρ_{R} be the regular representation. This can be written as

$$\rho_R = \bigoplus_{i=1}^{k} \rho_i^{\oplus m_i}. \tag{9.2.38}$$

Notice that we then have

$$\begin{aligned} m_i &= \langle \chi(\rho_i), \chi(\rho_R) \rangle & (9.2.39)\\ &=: \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_i}(g)^* \chi_{\rho_{\mathrm{R}}}(g) & (9.2.40)\\ &= \chi_{\rho_i}(e)^* & (9.2.41)\\ &= \dim V_i. & (9.2.42) \end{aligned}$$

Here we have used the fact that $\chi_{\rho_{\mathrm{R}}}(g) = 0$ for $g \neq e$ and so all terms in the sum vanish apart from the $g = e$ term. The regular representation contributes a factor of $|G|$ to this term, which cancels with the existing normalisation factor to leave just $\chi_{\rho_i}(e)^*$, since $g = e$ in this term. We then apply Lemma 9.2.29 to get $\chi_{\rho_i}(e) = \dim V_i$, and since this is real the complex conjugate does nothing.

Using $m_i = \dim V_i$ we then have

$$\chi_{\rho_{\mathrm{R}}}(g) = \sum_{i=1}^{k} m_i \chi_{\rho_i}(g) = \sum_{i=1}^{k} \dim(V_i) \chi_{\rho_i}(g). \tag{9.2.43}$$

Considering the specific case of $g = e$ this then becomes

$$|G| = \chi_{\rho_{\mathrm{R}}}(e) = \sum_{i=1}^{k} \dim(V_i) \chi_{\rho_i}(e) = \sum_{i=1}^{k} \dim(V_i)^2 \tag{9.2.44}$$

where we have used Lemma 9.2.29 again in the last equality. □

The dimensionality theorem quickly allows us to limit the possible irreducible representations. For example, S_3 ($|S_3| = 3! = 6$) could have either 6 one-dimensional irreducible representations ($6 \cdot 1^2 = 6$) or 2 one-dimensional irreducible representations and 1 two-dimensional irreducible representation ($2 \cdot 1^2 + 1 \cdot 2^2 = 6$). It turns out that the latter is correct.

9.3 Character Tables

The common way to give the information on the characters of the irreducible representations is as a **character table**. Since the character is a class function, that is it is the same for all group elements sharing a conjugacy class, we only write out the conjugacy classes for the character table, rather than the full group. We then have an optional line for the order of the classes. The rows below then list the characters of the irreducible representations on the conjugacy classes. Typically we list the conjugacy class containing the identity first and the trivial representation (one dimensional representation given by the trivial action) first which means that the first row contains all ones.

As an example we now give the character table of S_3 but there are some details here we have yet to discuss.

S_3 classes	$[()]$	$[(1\,2)]$	$[(1\,2\,3)]$
Order	1	3	2
Trivial	1	1	1
Alternating	1	-1	1
Standard	2	0	-1

(9.3.1)

The labels down the side label the irreducible representations, we haven't yet defined the alternating or standard representations so don't worry too much about them.

In this section we will write statements like "the character table is square". What we mean by the character table here is the numbers that we fill in for each conjugacy class and representation ignoring the labels that we give along the edges.

Theorem 9.3.2. The character table is square.

Proof. R This proof is beyond the scope of this course.

The total number of irreducible representations must be at most the number of conjugacy classes since the irreducible representations form a basis in the space of class functions, which is a space of dimension N_c.
Suppose there is a class function, $f\colon G \longrightarrow \mathbb{C}$, such that f is orthogonal to all of the characters of the irreducible representations, that is

$$\langle f, \chi(\rho_i)\rangle = 0 \tag{9.3.3}$$

for all irreducible representations, ρ_i. If we can show that this necessarily means $f = (0, \dots, 0) \in \mathbb{C}^{N_c}$ then this shows that the characters form a complete set of vectors on the class function space, and hence the number of characters is equal to the dimension of the space.
Consider the map

$$\varphi\colon V_i \longrightarrow V_i \qquad \text{where} \qquad \varphi := \sum_{g\in G} f(g)^* \rho_i(\gamma). \tag{9.3.4}$$

Here V_i is the vector space associated with the irreducible representation ρ_i.

It can easily be seen that

$$\begin{aligned}\varphi\rho_i(g') &= \sum_{g\in G} f(g)^*\rho_i(g)\rho_i(g') && (9.3.5)\\ &= \sum_{g\in G} f(g)^*\rho_i(gg') && (9.3.6)\\ &= \sum_{g''\in G} f(g'')^*\rho_i(g'') && (9.3.7)\\ &= \sum_{g'''\in G} \rho_i(g')f(g''')^*\rho_i(g''') && (9.3.8)\\ &= \rho_i(g') \sum_{g'''\in G} f(g''')^* && (9.3.9)\end{aligned}$$

where we use the fact that we are averaging over the group so averaging gg' over g is the same as averaging over $g'' = gg'$. We then have by Schur's lemma (Theorem 8.4.8) that $\varphi = \lambda\mathbf{1}$. Hence,

$$\operatorname{tr}\varphi = \lambda \dim V_i \qquad (9.3.10)$$

and

$$\operatorname{tr}\varphi = \sum_{g\in G} f(g)^*\chi_{\rho_i}(g) = |G|\langle f, \chi(\rho_i)\rangle = 0. \qquad (9.3.11)$$

The last equality being by our assumption that f is orthogonal to all of the characters. We therefore must have that either $\lambda = 0$, in which case $f(g) = 0$ for all $g \in G$, or

$$\sum_{g\in G} f^*(g)\rho_i(g) \qquad (9.3.12)$$

holds for all irreducible representations, and hence for all representations since they can be written as a sum of irreducible representations.

In particular this must hold for the regular representation, ρ_{R}. However, since $\rho_R(g)$ are all linearly independent by definition this means $f(g) = 0$ for all $g \in G$. Either way the result is that $f = (0, \dots, 0)$ and so the characters form a complete basis and hence the number of irreducible representations is equal to the number of conjugacy classes. □

The following theorem is useful but the proof requires somewhat complicated linear algebra so we omit it.

Theorem 9.3.13. This dimension of any irreducible representation divides the order of the group. That is $|G|/\dim V_i \in \mathbb{Z}_{>0}$ where $\rho_i \colon G \to \mathrm{GL}(V_i)$ is an irreducible representation.

Further $\dim V_i$ divides $G/Z(G)$, where $Z(G)$ is the centre of the group, which is the normal subgroup of all commuting elements.

Corollary 9.3.14 All irreducible representations of an Abelian group are one dimensional.

Proof. Let G be an Abelian group. Since the group is Abelian $gg'g^{-1} = g'gg^{-1} = g'$ for all $g, g' \in G$, and so every conjugacy class contains exactly one element. Hence $N_c = |G|$. By the dimensionality theorem (Theorem 9.2.36)

$$|G| = \sum_{i=1}^{N_c} \dim(V_i)^2 = \sum_{i=1}^{|G|} \dim(V_i)^2. \tag{9.3.15}$$

Since $\dim V_i$ divides $|G|$ by Theorem 9.3.13 it follows that $\dim V_i \neq 0$ and so $\dim V_i = 1$ is the only way to have this equation hold. □

The past few theorems combined shows that we know quite a lot about the dimensions of the irreducible representations before we even start to work out what the representations are. This mostly stems from Schur's lemma. We can sum it up in a set of diophantine equations (equations with integer solutions) which must be satisfied.

- $|G| = 1 + \dim(V_2)^2 + \dots + \dim(V_k)^2$,
- $k = N_c$, and
- $|G|/\dim V_i \in \mathbb{Z}_{>0}$.

The 1 in the first equation corresponds to the dimension of the trivial representation, which is always present. These equations are often enough to allow us to work out the dimensions of the irreducible representations without having to explicitly compute them.

Appendices

A

Mathematical Preliminaries

A.1 Basic Mathematics

A.1.1 Notation

Notation A.1.1 — Number Sets The set of natural numbers is

$$\mathbb{N} := \{0, 1, 2, \dots\}. \tag{A.1.2}$$

Note that the inclusion of zero in $\mathbb{N}$ is subject to debate. The set of integers is denoted

$$\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}. \tag{A.1.3}$$

The set of positive integers is denoted

$$\mathbb{Z}_{>0} := \{1, 2, \dots\}. \tag{A.1.4}$$

The set of rational numbers is denoted

$$\mathbb{Q} := \{p/q \mid p, q \in \mathbb{Z} \text{ and } q \neq 0\}. \tag{A.1.5}$$

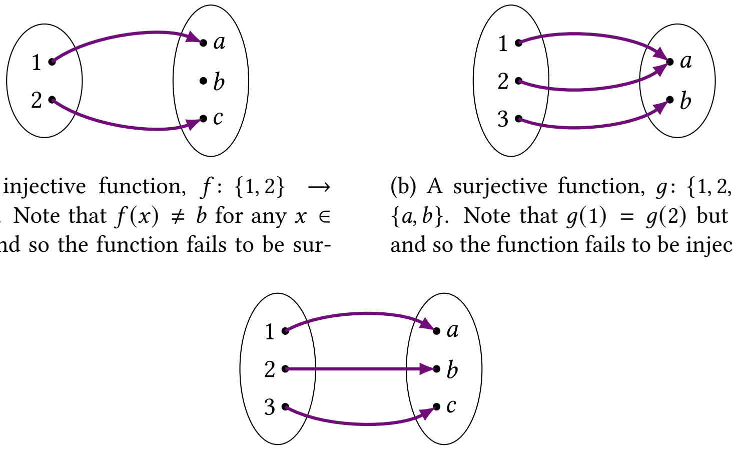
The set of real numbers is denoted $\mathbb{R}$, and the set of complex numbers $\mathbb{C}$. The set of *all* quaternions (as opposed to the quaternion group of order 8) is denoted $\mathbb{H}$.

Notation A.1.6 — Sphere The unit sphere in $n+1$ dimensions is

$$S^n := \{\boldsymbol{x} \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots x_{n+1}^2 = 1\}. \tag{A.1.7}$$

Note that S^n is an n-dimensional manifold, which we view as embedded in $(n+1)$-dimensional Euclidean space, $\mathbb{R}^{n+1}$.
What we normally call the circle is S^1 and what we normally call the sphere is S^2.

Notation A.1.8 — Sets of Matrices We denote the set of $m \times n$ matrices with entries in $\mathbb{F}$ (which is usually a field and usually $\mathbb{R}$ or $\mathbb{C}$) by $\mathcal{M}_{m\times n}(\mathbb{F})$.
We denote the set of square $n \times n$ matrices with entries in $\mathbb{F}$ by $\mathcal{M}_n(\mathbb{F})$.



(a) An injective function, $f\colon \{1,2\} \longrightarrow \{a,b,c\}$. Note that $f(x) \neq b$ for any $x \in \{1,2\}$ and so the function fails to be surjective.

(b) A surjective function, $g\colon \{1,2,3\} \longrightarrow \{a,b\}$. Note that $g(1) = g(2)$ but $1 \neq 2$ and so the function fails to be injective.

(c) A bijective function, $h\colon \{1,2,3\} \longrightarrow \{a,b,c\}$.

Figure A.1: Injective, surjective, and bijective functions.

We denote the set of invertible $n \times n$ square matrices matrices over $\mathbb{F}$, called the general linear group, by

$$\mathrm{GL}(n,\mathbb{F}) = \{A \in \mathcal{M}_n(\mathbb{F}) \mid \det A \neq 0\}. \tag{A.1.9}$$

If $\mathbb{F}$ is evident from context we may simply write $\mathrm{GL}(n)$. If V is an n-dimensional vector space over $\mathbb{F}$ then we may also write this set as $\mathrm{GL}(V)$.

Notation A.1.10 — Einstein Summation Convention When two identical indices appear in the same term then they are summed over, for example,

$$x_i y_i = \sum_i x_i y_i. \tag{A.1.11}$$

A.1.2 Definitions

Definition A.1.12 — Function Types Let $\varphi\colon A \to B$. Then φ is

- **injective** if for all $a, a' \in A$ $\varphi(a) = \varphi(a')$ implies $a = a'$,
- **surjective** if for all $b \in B$ there exists $a \in A$ such that $\varphi(a) = b$, and
- **bijective** if φ is both injective and surjective.

A function is invertible if and only if it is bijective.

Definition A.1.13 — Kernel Given a map $\varphi\colon A \to B$ the **kernel** is defined as the set of elements of A which map to the trivial element of B, which is the zero vector, $\mathbf{0}$, if B is a vector space, or the identity if B is a group:

$$\ker\varphi := \{a \in A \mid \varphi(a) \text{ is the trivial element of } B\} \subseteq A. \tag{A.1.14}$$

Definition A.1.15 — Image Given a map $\varphi\colon A \to B$ the **image** is set of $b \in B$ for which there exists some $a \in A$ such that $\varphi(a) = b$:

$$\operatorname{Im}\varphi = \varphi(A) := \{b \in B \mid \exists\, a \in A \text{ such that } \varphi(a) = b\} \subseteq B. \tag{A.1.16}$$

Definition A.1.17 — Empty Set The **empty set**, $\emptyset$, is the set containing no elements.

Definition A.1.18 — Kronecker Delta The **Kronecker delta**, δ_{ij}, is defined as

$$\delta_{ij} := \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{A.1.19}$$

Note that δ_{ij} are the elements of the identity matrix.

Definition A.1.20 — Levi-Civita Symbol The **Levi-Civita Symbol** in n-indices is the completely asymmetric (pseudo)tensor which is defined so that $\varepsilon_{123\dots n} := 1$. Antisymmetry then means that $\varepsilon_{1\cdots i\cdots j\cdots n} = -\varepsilon_{1\cdots j\cdots i\cdots n}$, for example $\varepsilon_{213\cdots n} = -1$. Antisymmetry also means that Levi–Civita symbol vanishes if it has repeated indices.
Most commonly $n = 3$ and

$$\varepsilon_{ijk} := \begin{cases} 1, & \text{if } (i,j,k) = (1,2,3), (2,3,1), (3,1,2), \\ -1, & \text{if } (i,j,k) = (1,3,2), (2,1,3), (3,2,1), \\ 0, & \text{if any index is repeated.} \end{cases} \tag{A.1.21}$$

Definition A.1.22 — Equivalence Relations Given two sets, A and B, a **relation**, R, is a subset of $A \times B := \{(a,b) \mid a \in A \text{ and } b \in B\} \supseteq R$. We say that $a \in A$ is related to $b \in B$, which we denote with infix notation, $a\,R\,b$, if $(a,b) \in R$.
If $A = B$ in the above definition then we call R a **binary!relation** on A.
A relation, $\sim$, on a set A is a binary relation on A such that the following axioms hold for all $a, b, c \in A$

- $\sim$ is **reflexive**, so $a \sim a$.
- $\sim$ is **symmetric**, so if $a \sim b$ then $b \sim a$.
- $\sim$ is **transitive**, so if $a \sim b$ and $b \sim c$ then $a \sim c$.

A **equivalence class** of an element $a \in A$ under some equivalence relation, $\sim$, is the set

$$[a] := \{x \mid a \sim x\}. \tag{A.1.23}$$

We call elements of $[a]$ representatives of the equivalence class. We denote the set of all equivalence classes by $A/\sim$.

■ **Example A.1.24 — Equivalence Relations** $=$ is the prototypical equivalence relation.
Congruence modulo $m \in \mathbb{Z}_{>0}$ is an equivalence relation on $\mathbb{R}$.
$\sim$ defined by $z \sim w$ if $|z| = |w|$ is an equivalence relation on $\mathbb{C}$.
$\sim$ defined by $\boldsymbol{v} \sim \boldsymbol{u}$ if $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel is an equivalence relation on $\mathbb{R}^n$.

■ **Example A.1.25 — Isomorphism** Isomorphisms, as defined in the text, are equivalence relations:

- Let A be a group, then the identity function, $\mathrm{id}_A \colon A \longrightarrow A$ defined by $\mathrm{id}_A(a) = a$ for all $a \in A$ is an isomorphism since $\mathrm{id}_A(aa') = aa' = \mathrm{id}_A(a)\mathrm{id}_A(a')$ and clearly id_A is invertible, and is its own inverse.

- Let A and B be isomorphic groups. Then there exists some bijection $\varphi \colon A \longrightarrow B$ such that $\varphi(aa') = \varphi(a)\varphi(a')$. Since φ is a bijection $\varphi^{-1} \colon B \longrightarrow A$ exists and is also a bijection. Applying the inverse to both sides of the defining relation we have $\varphi^{-1}(\varphi(aa')) = \varphi^{-1}(\varphi(a)\varphi(a'))$. Since φ is surjective any element of B can be written in the form $b = \varphi(a)$ for some $a \in A$ and so it follows that $\varphi^{-1}(\varphi(aa')) = \varphi^{-1}(b)\varphi(b')$ where $b, b' \in B$ are arbitrary and we choose $a, a' \in A$ to be such that $b = \varphi(a)$ and $b' = \varphi(a')$. From the defining relation for φ we know that $\varphi(aa') = \varphi(a)\varphi(a') = bb'$. It follows that $\varphi^{-1}(\varphi(aa')) = \varphi^{-1}(bb') = \varphi^{-1}(b)\varphi^{-1}(b')$, which means that $B \cong A$.

- Let A, B, and C be groups such that $A \cong B$ and $B \cong C$. Then there exists isomorphisms $\varphi \colon A \longrightarrow B$ and $\psi \colon B \longrightarrow C$. We claim that $\psi \circ \varphi \colon A \longrightarrow C$ is an isomorphism. Clearly $\psi \circ \varphi$ is bijective, since $\varphi^{-1} \circ \psi^{-1}$ is its inverse, as can be seen by considering $(\varphi^{-1} \circ \psi^{-1})((\psi \circ \varphi)(a)) = \varphi^{-1}(\psi^{-1}(\psi(\varphi(a)))) = \varphi^{-1}(\varphi(a)) = a$ for all $a \in A$.

  It remains to show that $\psi \circ \varphi$ is a homomorphism. To do so consider $(\psi \circ \varphi)(aa') = \psi(\varphi(aa')) = \psi(\varphi(a)\varphi(a'))$, which follows since φ is an isomorphism. Now write $\varphi(a) = b$ and $\varphi(a') = b'$, where $b, b' \in B$. We then have $(\psi \circ \varphi)(aa') = \psi(bb') = \psi(b)\psi(b')$, which follows since ψ is an isomorphism. We then have $(\psi \circ \varphi)(aa') = \psi(b)\psi(b') = \psi(\varphi(b))\psi\varphi(b') = (\psi \circ \varphi)(b)(\psi \circ \varphi)(b')$, and so $\psi \circ \varphi$ is a bijective homomorphism and hence an isomorphism, meaning $A \cong C$.

A.2 Linear Algebra

A.2.1 Vectors

Definition A.2.1 — Vector Space A vector space, V, over a field, $\mathbb{F}$, is a set of vectors, V, with two operations, $\cdot : \mathbb{F} \times V \longrightarrow V$, known as scalar multiplication, and $+ : V \times V \longrightarrow V$, known as vector addition, which are defined such that the following hold for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ and $k, k' \in \mathbb{F}$:

1. **Associativity**: $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$,
2. There exists a **zero vector**, $\mathbf{0} \in V$, such that $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$.
3. There exists $-\boldsymbol{u} \in V$ such that $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$. We write this as $\boldsymbol{u} - \boldsymbol{u}$ for short.
4. **Commutativity**: $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$.
5. Distributivity of scalar multiplication over vector addition $k(\boldsymbol{u} + \boldsymbol{v}) = k\boldsymbol{u} + k\boldsymbol{v}$.
6. Distributivity of scalar multiplication over field addition $(k + k')\boldsymbol{u} = k\boldsymbol{u} + k'\boldsymbol{u}$.
7. Compatibility of field and scalar multiplication $(kk')\boldsymbol{u} = k(k'\boldsymbol{u})$.
8. $1\boldsymbol{u} = \boldsymbol{u}$ where 1 is the multiplicative identity of $\mathbb{F}$.

Note that the first three axioms make $(V, +)$ a group and the fourth makes it Abelian.

Definition A.2.2 — Hilbert Space A **Hilbert space**, $\mathcal{H}$, is a vector space over either $\mathbb{R}$ or $\mathbb{C}$, equipped with an inner product that induces a complete metric. We shall assume a complex Hilbert space, for a real Hilbert space simply ignore any complex conjugates and replace $\mathbb{C}$ with $\mathbb{R}$.
An **inner product** is a function $\langle -, - \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$ such that for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{H}$ and $k, k' \in \mathbb{C}$

1. $\langle -, - \rangle$ is linear in it's second argument, that is
$$\langle \boldsymbol{u}, k\boldsymbol{v} + k'\boldsymbol{w} \rangle = k\langle \boldsymbol{u}, \boldsymbol{v} \rangle + k'\langle \boldsymbol{u}, \boldsymbol{w} \rangle. \tag{A.2.3}$$
2. $\langle -, - \rangle$ is conjugate symmetric:
$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle^*. \tag{A.2.4}$$
3. $\langle -, - \rangle$ is positive definite, that is $\langle \boldsymbol{u}, \boldsymbol{u} \rangle \geq 0$ with equality only if $\boldsymbol{u} = \mathbf{0}$.

! Mathematicians often define an inner product to be linear in it's first argument, so
$$\langle k\boldsymbol{u} + k'\boldsymbol{v}, \boldsymbol{w} \rangle = k\langle \boldsymbol{u}, \boldsymbol{w} \rangle + k'\langle \boldsymbol{v}, \boldsymbol{w} \rangle. \tag{A.2.5}$$

The first two axioms are sometimes combined to give an extra axiom that $\langle -,-\rangle$ is conjugate linear in its first argument (or second if we follow the other convention). That is

$$\langle k\boldsymbol{u} + k'\boldsymbol{v}, \boldsymbol{w}\rangle = k^*\langle \boldsymbol{u}, \boldsymbol{w}\rangle + k'^*\langle \boldsymbol{v}, \boldsymbol{w}\rangle. \tag{A.2.6}$$

We can then define a **norm** on $\mathcal{H}$ by $\|\boldsymbol{u}\| := \sqrt{\langle \boldsymbol{u}, \boldsymbol{u}\rangle}$.
The final condition for $\mathcal{H}$ to be a Hilbert space is completeness. Namely that if the series $\sum_{n=0}^{\infty} \boldsymbol{u_n}$ converges absolutely, so that $\sum_{n=0}^{\infty}\|\boldsymbol{u_n}\|$ converges to a finite value then the original series, $\sum_{n=0}^{\infty} \boldsymbol{u_n}$, converges to some vector in $\mathcal{H}$.

■ **Example A.2.7** The set of n-tuples of complex numbers, $\mathbb{C}^n$, is a Hilbert space over $\mathbb{C}$ with the inner product

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \langle (u_1, \dots, u_n), (v_1, \dots, v_n)\rangle := \sum_{i=1}^{n} u_i^* v_i = u_i^* v_i \tag{A.2.8}$$

where in the last term we use the Einstein summation convention.

■ **Example A.2.9 — Functions** The space of square integrable functions on $X \subseteq \mathbb{R}^n$ forms a Hilbert space, denoted $L^2(X)$. A function, $f\colon X \to \mathbb{C}$, is square integrable if[a]

$$\int_X |f(x)|^2 \, \mathrm{d}x \tag{A.2.10}$$

exists and is finite. For example, the function defined by $f(x) = e^{-x^2}$ is an element of $L^2(\mathbb{R})$.
Given $f, g \in L^2(X)$ we define the inner product in this space to be

$$\langle f, g\rangle := \int_X f^*(x) g(x) \, \mathrm{d}x. \tag{A.2.11}$$

R Another subtly that arises here is that we actually need to consider elements of $L^2(X)$ to be equivalence classes of functions which are equal almost everywhere (meaning that the measure of the set of points where they are not equal is zero). Otherwise we may have some functions such that $\langle f, g\rangle = 0$ but $f \neq g$ since f and g disagree on some set of points with a vanishing measure. We say that we are considering the functions mod the equivalence relation of being equal almost everywhere.

This is an important example since we can often identify "square integrable functions" with "possible wave functions", since square-integrability is a requirement for us to be able to normalise a wave function, which we do so by the procedure

$$\psi \to \frac{\psi}{\|\psi\|} = \frac{\psi}{\sqrt{\langle \psi, \psi\rangle}} = \left(\int |\psi(x)|^2 \, \mathrm{d}x\right)^{-1/2} \psi. \tag{A.2.12}$$

This only makes sense if $\int |\psi(x)|^2 \, \mathrm{d}x$ is finite (and nonzero).

[a] for this space to be complete (a requirement for Hilbert spaces) this must be a Lebesgue integral but in physics functions are usually nice enough that we can use the standard Riemann integral, which agrees with the Lebesgue integral when both exists.

Notation A.2.13 — Bra-Ket Notation In physics, particularly in quantum mechanics, we often use **bra-ket notation**, developed by Dirac. We identify vectors, $\boldsymbol{u}$, with **kets**, $|u\rangle$, and dual vectors, $\boldsymbol{v}$, with **bras**, $\langle v|$. The inner product $\langle \boldsymbol{u}, \boldsymbol{v}\rangle$ is then written $\langle v|u\rangle$. This is the notation we will use for most of this course.

Definition A.2.14 — Linear Operator Given two vector spaces, V and W, over some field, $\mathbb{F}$, a function, $f \colon V \longrightarrow W$, is said to be a linear operator if for $\boldsymbol{u}, \boldsymbol{v} \in V$ and $k \in \mathbb{F}$ we have

$$f(\boldsymbol{u}+\boldsymbol{v}) = f(\boldsymbol{u}) + f(\boldsymbol{v}), \qquad \text{and} \qquad f(k\boldsymbol{u}) = kf(\boldsymbol{u}). \tag{A.2.15}$$

Instead of the function notation $f(\boldsymbol{u})$ we typically use a multiplicative notation, $A\boldsymbol{u}$, which is due to the fact that if V is finite dimensional then we can choose a basis and represent a linear map by a matrix. Using bra-ket notation also we have

$$A(|u\rangle + |v\rangle) = A|u\rangle + A|v\rangle, \qquad \text{and} \qquad A(k|u\rangle) = kA|u\rangle. \tag{A.2.16}$$

An operator is **antilinear** if

$$A(|u\rangle + |v\rangle) = A|u\rangle + A|v\rangle, \qquad \text{and} \qquad A(k|u\rangle) = k^*A|u\rangle. \tag{A.2.17}$$

An example of such an operator is the time reversal operator, T, which takes $t \longrightarrow -t$.

For simplicity from now on we will consider the complex vector space $V = \mathbb{C}^N$. This is N-dimensional ($\dim V = N$), which we take to be finite, although many of these ideas work, possibly with slight modification, for infinite dimensional vector spaces. Most of the time we will also consider linear operators from V to V, since this is far more common in practice that linear operators from V to some different vector space, W.

Definition A.2.18 — Basis Given a vector space, V, we say that $\{|e_i\rangle\}$ is a **linearly independent** set if the only solution to

$$\lambda_i|e_i\rangle = |0\rangle, \tag{A.2.19}$$

where $|0\rangle$ is the zero vector, is $\lambda_i = 0$ for all i.
We say that $\{|e_i\rangle\}$ is a **basis** for V if $\{|e_i\rangle\}$ is a linearly independent set and spans V. That is given some $|u\rangle \in V$ we can write

$$|u\rangle = u_i|e_i\rangle \tag{A.2.20}$$

for some $u_i \in \mathbb{C}$.

The number of vectors in a basis is the **dimension** of the vector space, denoted $\dim V$.
We say that two vectors, $|u\rangle, |v\rangle \in V$, are **orthogonal** (with respect to some inner product) if $\langle u|v\rangle = 0$.
We say that a vector, $|u\rangle \in V$, is normalised if $\|u\| = \sqrt{\langle u|u\rangle} = 1$.
We say that $\{|e_i\rangle\}$ is an orthonormal basis for V if it is a basis for V, $|e_i\rangle$ is normalised for all i and all of the basis vectors are mutually orthogonal. These last two conditions are summarised by requiring that

$$\langle e_i|e_j\rangle = \delta_{ij}. \tag{A.2.21}$$

Definition A.2.22 — Completeness Relation Given a vector space, V, and an orthonormal basis, $\{|e_i\rangle\}$, we can write the identity operator, $\mathbf{1}$, as

$$\mathbf{1} = \sum_{i=1}^{N} |e_i\rangle\langle e_i|. \tag{A.2.23}$$

Recall that the identity operator is defined such that

$$\mathbf{1}|u\rangle = |u\rangle \tag{A.2.24}$$

for all $|u\rangle \in V$.

Definition A.2.25 — Partition of the Identity A **projection operator**, P_i, is an operator satisfying

$$P_iP_j = \delta_{ij}P_i, \qquad \text{and} \qquad P_i^\dagger = P_i. \tag{A.2.26}$$

A **partition of the identity** is a collection of projection operators, $\{P_j\}$, such that

$$\mathbf{1} = \sum_{j=1}^{N_P} P_j \tag{A.2.27}$$

where $N_P = |\{P_j\}|$ is the number of projection operators in the partition. We can write each partition operator as

$$P_j = \sum_{i=1}^{N_j} |e_i\rangle\langle e_i| \tag{A.2.28}$$

where N_j are such that

$$\dim V = N = \sum_{j=1}^{N_P} N_j. \tag{A.2.29}$$

Definition A.2.30 — Matrix Element Given a linear operator, $A\colon V \to V$, and an orthonormal basis, $\{|e_i\rangle\}$, we define the **matrix elements** to be

$$A_{ij} := \langle e_i|Ae_j\rangle = \langle e_i|A|e_j\rangle \tag{A.2.31}$$

where A is understood to act on the right and $|Ae_j\rangle := A|e_j\rangle$.
If we know the matrix elements of A we can reconstruct A using

$$A = \sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij}|e_i\rangle\langle e_j|. \tag{A.2.32}$$

Definition A.2.33 — Eigenvalues and Eigenvectors Given a linear operator A, we call $|v_i\rangle$ an **eigenvector** and $\lambda_i \in \mathbb{C}$ an **eigenvalue** if

$$A|v_i\rangle = \lambda_i|v_i\rangle. \tag{A.2.34}$$

There are N solutions to this, which follows from the **characteristic polynomial**, $\det(A - \lambda 1) = 0$, having N solutions, which in turn follows from the fundamental theorem of algebra.

A.2.2 Matrices

Definition A.2.35 — Transpose and Hermitian Conjugate Given a matrix, A, with matrix elements A_{ij}, the **transpose** matrix, $A^\top$, has matrix elements $A^\top_{ij} = A_{ji}$.
A matrix is **symmetric** if $A^\top = A$, or **antisymmetric** if $A^\top = -A$.
Given a matrix, A, with matrix elements A_{ij}, the **Hermitian conjugate**, $A^\dagger$, has matrix elements $A^\dagger_{ij} = A^*_{ji}$. Here * denotes the **complex conjugate**, so $(x+iy)^* = x - iy$ and $(re^{i\vartheta})^* = re^{-\vartheta}$ for $x, y, r, \vartheta \in \mathbb{R}$. That is the Hermitian conjugate is the complex conjugate of the transpose.
A matrix is **Hermitian** if $A^\dagger = A$, or **anti-Hermitian** if $A^\dagger = -A$.

Lemma A.2.36 The eigenvalues of a Hermitian matrix are real.

Proof. Let A be a Hermitian matrix and $\boldsymbol{v}$ an eigenvector with nonzero eigenvalue λ. Note that this means $\boldsymbol{v}$ is nonzero. If 0 is an eigenvalue of A then this is real so we need not consider this case further. By definition $A\boldsymbol{v} = \lambda\boldsymbol{v}$. Taking the Hermitian conjugate of both sides we get $\boldsymbol{v}^\dagger A^\dagger = \lambda^*\boldsymbol{v}^\dagger$, where we have used $(XY)^\dagger = Y^\dagger X^\dagger$. Multiplying both sides on the right by $\boldsymbol{v}$ we get $\boldsymbol{v}^\dagger A\boldsymbol{v} = \lambda^*\boldsymbol{v}^\dagger\boldsymbol{v}$. Identifying $A\boldsymbol{v} = \lambda\boldsymbol{v}$ on the left hand side this becomes $\boldsymbol{v}^\dagger\lambda\boldsymbol{v} = \lambda\boldsymbol{v}^\dagger\boldsymbol{v} = \lambda^*\boldsymbol{v}^\dagger\boldsymbol{v}$. It follows that we must have $\lambda = \lambda^*$, which means we must have $\lambda \in \mathbb{R}$. □

We can choose the eigenvalues of a Hermitian matrix to be orthonormal, and hence they form a basis for the vector space. In this basis the matrix will be diagonal and the values on the diagonal are simply the eigenvalues.

Given a Hermitian matrix, A, with eigenvalues λ_i and corresponding eigenvectors $|v_i\rangle$ we can write this matrix as

$$A = \sum_{i=1}^{N} \lambda_i |v_i\rangle\langle v_i|. \tag{A.2.37}$$

This is diagonalised by the transformation $V^\dagger A V$ where

$$V = \sum_{i=1}^{N} |v_i\rangle\langle e_i| \tag{A.2.38}$$

where $|e_i\rangle$ are the basis vectors in the original basis. It is easy to see that this transform gives the desired result:

$$\begin{aligned} V^\dagger A V &= \underbrace{|e_i\rangle\langle v_i|}_{=V^\dagger} \underbrace{(\lambda_j |v_j\rangle\langle v_j|)}_{=A} \underbrace{|v_k\rangle\langle e_k|}_{=V} && \text{(A.2.39)} \\ &= \lambda_j |e_i\rangle\langle v_i|v_j\rangle\langle v_j|v_k\rangle\langle e_k| && \text{(A.2.40)} \\ &= \lambda_j \delta_{ij}\delta_{jk} |e_i\rangle\langle e_k| && \text{(A.2.41)} \\ &= \lambda_i |e_i\rangle\langle e_i| && \text{(A.2.42)} \end{aligned}$$

This last term is just a diagonal matrix with the eigenvalues, λ_i, as the diagonal elements, which is exactly what we wanted.

For non-Hermitian matrices it is possible that the eigenvalues aren't linearly independent. In this case the best we can do is Jordan normal form where the eigenvalues are on the diagonal and all other entries are either zero or one for elements in the subspace of degenerate eigenvalues.

Definition A.2.43 — Inverse The **inverse** of a matrix, A, is the matrix A^{-1} such that $A^{-1}A = AA^{-1} = \mathbf{1}$. Such a matrix exists only if the determinant is non-zero.

An equivalent requirement for A^{-1} to exist is for A to have no zero eigenvalues. For a Hermitian matrix the inverse in the eigenbasis is simply $A^{-1} = \operatorname{diag}(1/\lambda_1, \dots, 1/\lambda_N)$.

Definition A.2.44 — Orthogonal and Unitary A matrix, O, is **orthogonal** if $O^\top O = \mathbf{1}$, that is $O^\top = O^{-1}$.
A matrix, U, is **unitary** if $U^\dagger U = \mathbf{1}$, that is $U^\dagger = U^{-1}$.

The following holds:

$$\langle u|Av\rangle = \langle u|A|v\rangle = \langle A^\dagger u|v\rangle. \tag{A.2.45}$$

For a unitary matrix, U, this implies

$$\langle Uu|Uv\rangle = \langle u|U^\dagger U|v\rangle = \langle u|\mathbf{1}|v\rangle = \langle u|v\rangle. \tag{A.2.46}$$

We say that unitary transforms preserve the inner product, or that the inner product is invariant under unitary transforms.

Definition A.2.47 — Trace The **trace** of a matrix, A is

$$\operatorname{tr} A := \sum_i \langle e_i | A_i | i \rangle = A_{ii} \tag{A.2.48}$$

where in the last term we are using the Einstein summation convention to sum over i.

The trace of a matrix is simply the sum of its eigenvalues, this doesn't just hold for Hermitian matrices.

The trace is cyclic, meaning $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, $\operatorname{tr}(ABC) = \operatorname{tr}(BCA) = \operatorname{tr}(CAB)$, etc.

The trace is linear, meaning $\operatorname{tr}(kA) = k \operatorname{tr}(A)$ for scalar k.

$\langle A, B \rangle := \operatorname{tr}(A^\dagger B)$ is an inner product on the vector space of matrices. This is called the **Gram–Schmidt inner product**.

Definition A.2.49 — Determinant The **determinant** of a matrix, A, is

$$\det A = |A| =:= \varepsilon_{i_1 \dots i_N} A_{1 i_1} \cdots A_{N i_N} \tag{A.2.50}$$

with summation over indices implied.

The determinant of a matrix is the product of its eigenvalues.

The determinant of a product is the product of the determinants:

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA). \tag{A.2.51}$$

Definition A.2.52 — Diagonal A matrix, A, is **diagonal** if $A_{ij} = 0$ for $i \neq j$. A matrix, A, is **block diagonal** if it can be written in the form

$$A = \begin{pmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & A_n \end{pmatrix} \tag{A.2.53}$$

where A_i are square matrices and the 0s represent matrices where all elements are zero.

A.2.3 Combining Vector Spaces

Definition A.2.54 — Direct Sum Given vector spaces V and W we call $V \oplus W$ the **direct sum**. It is defined by associating with each pair of vectors, $|v_i\rangle \in V$ and $|w_a\rangle \in W$, a vector $|v_i\rangle \oplus |w_i\rangle = |v_i \oplus w_a\rangle \in V \oplus W$ and extending the inner product to

$$\langle v_i \oplus w_a | v_j \oplus w_b \rangle_{V \oplus W} := \langle v_i | v_j \rangle_V + \langle w_a | w_b \rangle_W \tag{A.2.55}$$

where the subscripts denote which vector space the inner product is in. Note that the notation $|v \oplus w\rangle$ is non-standard.

The dimension of $V \oplus W$ is

$$\dim(V \oplus W) = \dim V + \dim W. \tag{A.2.56}$$

Given $A \in \mathrm{GL}(V)$ and $B \in \mathrm{GL}(W)$ the direct sum, $A \oplus B$, acts on $|v\rangle \oplus |w\rangle \in V \oplus W$ as

$$(A \oplus B)(|v\rangle \oplus |w\rangle) := (A|v\rangle) \oplus (B|w\rangle). \tag{A.2.57}$$

This shows we can think of $V \oplus W$ as a $(\dim V + \dim W)$-dimensional vector space with operators represented by $(v+w) \times (v+w)$ block diagonal matrices:

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}. \tag{A.2.58}$$

We then think of $|v\rangle \oplus |w\rangle \in V \oplus W$ as $(v_1, \dots, v_{\dim V}, w_1, \dots, w_{\dim W})$.

An important question is can a given vector space be written as a direct sum of vector spaces, this occurs when considering the irreducibility of representations.

Definition A.2.59 — Direct Product Given vector spaces V and W we call $V \otimes W$ the **direct product**. It is defined by associating with each pair of vectors, $|v_i\rangle \in V$ and $|w_a\rangle \in W$, a vector $|v_i\rangle \otimes |w_a\rangle = |v_i \otimes w_a\rangle \in V \otimes W$ and extending the inner product to

$$\langle v_i \otimes w_a | v_j \otimes w_b \rangle_{V \otimes W} := \langle v_i | v_j \rangle_V \langle w_a | w_b \rangle \tag{A.2.60}$$

where the subscripts denote which vector space the inner product is in. Note that the notation $|v \otimes w\rangle$ is non-standard.
The dimension of $V \otimes W$ is

$$\dim(V \otimes W) = \dim(V)\dim(W). \tag{A.2.61}$$

Given $A \in \mathrm{GL}(V)$ and $B \in \mathrm{GL}(W)$ the direct product, $A \otimes B$, acts on $|v\rangle \otimes |w\rangle \in V \otimes W$ as

$$(A \otimes B)(|v\rangle \otimes |w\rangle) = (A|v\rangle) \otimes (B|w\rangle). \tag{A.2.62}$$

■ **Application A.2.63** In quantum mechanics we can combine states from different Hilbert spaces representing different properties with direct products. For example, given an electron wave function with a spatial component and a spin component the direct product of these gives the state of the electron.

The direct product plays a role in representation theory in terms of what we will call Kronecker products. These can be used to obtain all representations from the fundamental representations.

B

Groups

B.1 Finite Groups

Definition B.1.1 The **trivial group** is the group containing only the identity, $\{e\}$. It is the only group of order 1.

- Order 1.
- Rank 1.
- Cyclic.
- Abelian.

The trivial group is isomorphic to $\mathbb{Z}_1$, S_1, and $\mathrm{SO}(1)$.

Definition B.1.2 $\mathbb{Z}_2$ is the cyclic group of order 2, see Definition B.1.9. It is the only group of order 2.

- Order 2.
- Rank 1.
- Cyclic.
- Abelian.

$\mathbb{Z}_2$ is isomorphic to S_2.

Definition B.1.3 $\mathbb{Z}_3$ is the cyclic group of order 3, see Definition B.1.9. It is the only group of order 3.

- Order 3.
- Rank 1.
- Cyclic.
- Abelian.

Definition B.1.4 $\mathbb{Z}_4$ is the cyclic group of order 4, see Definition B.1.9. It is one of two groups of order 4.

- Order 4.
- Rank 1.
- Cyclic.
- Abelian.

Definition B.1.5 $\mathbb{Z}_2 \times \mathbb{Z}_2$ is the **Klein *Vierergruppe***. It is one of two groups of order 4. It is a direct product of two copies of $\mathbb{Z}_2$.

- Order 4.
- Rank 2.
- Abelian.

Definition B.1.6 S_3 is the permutation group on 3 elements, see Definition B.1.11.

- Order 6.
- Rank 2.
- Non-Abelian.

Definition B.1.7 The **quaternion group**, Q, has the group presentation

$$Q = \langle -e, i, j, k \mid (-e)^2 = e, i^2 = j^2 = k^2 = ijk = e \rangle. \tag{B.1.8}$$

- Order 8.
- Rank 2.
- Non-Abelian.

The Pauli matrices provide a two-dimensional complex representation by the correspondence $(-e, i, j, k) \longrightarrow (-\mathbf{1}, \sigma_1, \sigma_2, \sigma_3)$.

B.1.1 Other Finite Groups

Definition B.1.9 The **cyclic group** of order n, denoted $\mathbb{Z}_n$, is given by the presentation

$$\mathbb{Z}_n = \langle a \mid a^n = e \rangle. \tag{B.1.10}$$

Identifying $a = \mathrm{e}^{2i\pi/n}$ and the operation as multiplication we get a group formed from the nth roots of unity. Identifying $a = 1$ and the operation as addition modulo n we get a group formed from $\{0, \dots, n-1\}$.

- Order n.
- Rank 1.
- Cyclic.
- Abelian.

All finite cyclic groups are isomorphic to $\mathbb{Z}_n$ for some n.

Definition B.1.11 The **permutation group** on n objects is the group of all permutations (bijections) of $\{1, \dots, n\}$, with function composition as the group operation.

- Order $n!$.
- Rank 2.
- Non-Abelian ($n > 2$).

S_1 and S_0 are isomorphic to the trivial group.
S_2 is isomorphic to $\mathbb{Z}_2$.

B.2 Discrete Groups

Definition B.2.1 The integers, $\mathbb{Z}$, under addition.

- Rank 1.
- Cyclic.
- Abelian.

Definition B.2.2 The rational numbers, $\mathbb{Q}$, under addition.

- Abelian.

Definition B.2.3 The nonzero rational numbers, $\mathbb{Q}^*$, under multiplication.

- Abelian.

B.3 Continuous Groups

B.3.1 Scalars

Definition B.3.1 The real numbers, $\mathbb{R}$, under addition.

- Abelian.

$(\mathbb{R}, +)$ is isomorphic to $(\mathbb{R}_{>0}, \cdot)$.

Definition B.3.2 The nonzero real numbers, $\mathbb{R}^*$, under multiplication.

- Abelian.

Definition B.3.3 The complex numbers, $\mathbb{C}$, under addition.

- Abelian.

Definition B.3.4 The nonzero complex numbers, $\mathbb{C}^*$, under multiplication.

- Abelian.

B.3.2 Matrices

Definition B.3.5 The **general linear group**

$$\mathrm{GL}(n,\mathbb{F}) = \{M \in \mathcal{M}_n(\mathbb{F}) \mid \det M \neq 0\}\,. \tag{B.3.6}$$

If V is a vector space of dimension n over $\mathbb{F}$ then this group is also denoted $\mathrm{GL}(V)$. If $\mathbb{F}$ is obvious from context then this group is denoted $\mathrm{GL}(n)$.

- Non-Abelian ($n > 1$).

Definition B.3.7 The **special linear group**

$$\mathrm{SL}(n,\mathbb{F}) = \{M \in \mathcal{M}_n(\mathbb{F}) \mid \det M = 1\}. \tag{B.3.8}$$

If V is a vector space of dimension n over $\mathbb{F}$ then this group is also denoted $\mathrm{SL}(V)$. If $\mathbb{F}$ is obvious from context then this group is denoted $\mathrm{SL}(n)$.

- Non-Abelian ($n > 1$).

$\mathrm{SL}(n,\mathbb{F})$ is a subgroup of $\mathrm{GL}(n,\mathbb{F})$.

Definition B.3.9 The **orthogonal group**

$$\mathrm{O}(n) = \{O \in \mathcal{M}_n(\mathbb{R}) \mid O^\top O = OO^\top = \mathbf{1}\}. \tag{B.3.10}$$

- Non-Abelian ($n > 1$).

$\mathrm{O}(n)$ is a subgroup of $\mathrm{GL}(n,\mathbb{R})$.
$\mathrm{O}(n)$ is the group of distance preserving transformations of Euclidean space which leave the origin invariant.
$\mathrm{O}(n)$ is the group of rotations and inversions of $\mathbb{R}^n$.

Definition B.3.11 The **special orthogonal group**

$$\mathrm{SO}(n) = \{O \in \mathcal{M}_n(\mathbb{R}) \mid O^\top O = OO^\top = \mathbf{1} \text{ and } \det O = 1\}. \tag{B.3.12}$$

- Non-Abelian ($n > 1$).

$\mathrm{SO}(n)$ is a subgroup of $\mathrm{O}(n)$ and $\mathrm{SL}(n,\mathbb{R})$.
$\mathrm{SO}(n)$ is the group of rotations of $\mathbb{R}^n$.
$\mathrm{SO}(2)$ is isomorphic to $\mathrm{U}(1)$ and the circle group, $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ under multiplication.

Definition B.3.13 The **unitary group**

$$\mathrm{U}(n) = \{U \in \mathcal{M}_n(\mathbb{C}) \mid U^\dagger U = UU^\dagger = \mathbf{1}\}. \tag{B.3.14}$$

- Non-Abelian ($n > 1$).

$\mathrm{U}(n)$ is a subgroup of $\mathrm{GL}(n, \mathbb{C})$.
$\mathrm{U}(n)$ is the group which preserves the standard inner product on $\mathbb{C}^n$.
$\mathrm{U}(1)$ is isomorphic to $\mathrm{SO}(2)$ and the circle group, $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ under multiplication.

Definition B.3.15 The **special unitary group**

$$\mathrm{SU}(n) = \{U \in \mathcal{M}_n(\mathbb{C}) \mid U^\dagger U = UU^\dagger = \mathbf{1} \text{ and } \det U = \mathbf{1}\}. \tag{B.3.16}$$

- Non-Abelian ($n > 1$).

$\mathrm{SU}(n)$ is a subgroup of $\mathrm{U}(n)$ and $\mathrm{SL}(n, \mathbb{C})$.

Definition B.3.17 The **isometries of Euclidean space**

$$\mathrm{ISO}(n) = \mathrm{O}(n) \ltimes \mathbb{R}^n \tag{B.3.18}$$

where $(R, \boldsymbol{a})(R', \boldsymbol{a}') \coloneqq (RR', \boldsymbol{a} + R\boldsymbol{a}')$.

- Non-Abelian

$\mathrm{ISO}(n)$ is the group of distance preserving transformations of Euclidean space.
$\mathrm{ISO}(n)$ is the group of rotations, reflections, and translations of $\mathbb{R}^n$.
$\mathrm{ISO}(n)$ has both $\mathrm{O}(n)$ and $\mathbb{R}^n$ as normal subgroups.

Definition B.3.19 The **Lorentz group**, $\mathrm{O}(1, 3)$, is the group of all Lorentz transformations of Minkowski space.

- Non-Abelian.

$\mathrm{O}(1, 3)$ is the group that preserves the quadratic form $(t, x, y, z) \mapsto t^2 - x^2 - y^2 - z^2$.
$\mathrm{SO}^+(1, 3)$ is the subgroup of $\mathrm{O}(1, 3)$ which preserves the orientation of space (S for special, that is unit determinant) and direction of time (that's what the + represents).

Definition B.3.20 The **Poincaré group** is the group of all isometries of Minkowski space, sometimes denoted $\mathrm{ISO}(1, 3)$. That is it is the group of all Lorentz transformations and translations.

- Non-Abelian.

The Poincaré group can be identified as the semidirect product $\mathrm{ISO}(1, 3) =$

> $\mathbb{R}^{1,3} \rtimes \mathrm{O}(1,3)$ where $\mathbb{R}^{1,3}$ is the group of spacetime translations of Minkowski space and $\mathrm{O}(1,3)$ is the Lorentz group.

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