

# Linear and Quadratic Programming (with CGAL)

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*Based on slides by Bernd Gärtner*

# Today

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- Quick introduction to Linear Programming (formulation and representation).
- Quick recap on techniques to solve it (no extensive knowledge needed).
- Linear Programming in CGAL.
- Examples.
- Integer Programming (not required for this class, but important).
- Quadratic Convex Programming (CGAL).

# Linear Programming (LP)

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- Linear programming is a central topic in optimization .
- It provides a powerful tool in modeling many applications .
- LP has attracted most of its attention in optimization during the last six decades for two main reasons :
  - Applicability : There are many real-world applications that can be modeled as linear programming ;
  - Solvability : There are theoretically and practically efficient techniques for solving large-scale problems.

# Linear Programming (LP)

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**We face an optimization problem subject to some constraints.**

- \* **Variables:** describe our choices, the parameters that we are allowed to change;
- \* **Objective function:** Describes the criterion that we wish to minimize (e.g. cost) or maximize (e.g. profit);
- \* **Constraints:** Describe the limitations that we have for the choice of the values of the variables.

# Linear Programming

## Mathematical formulation

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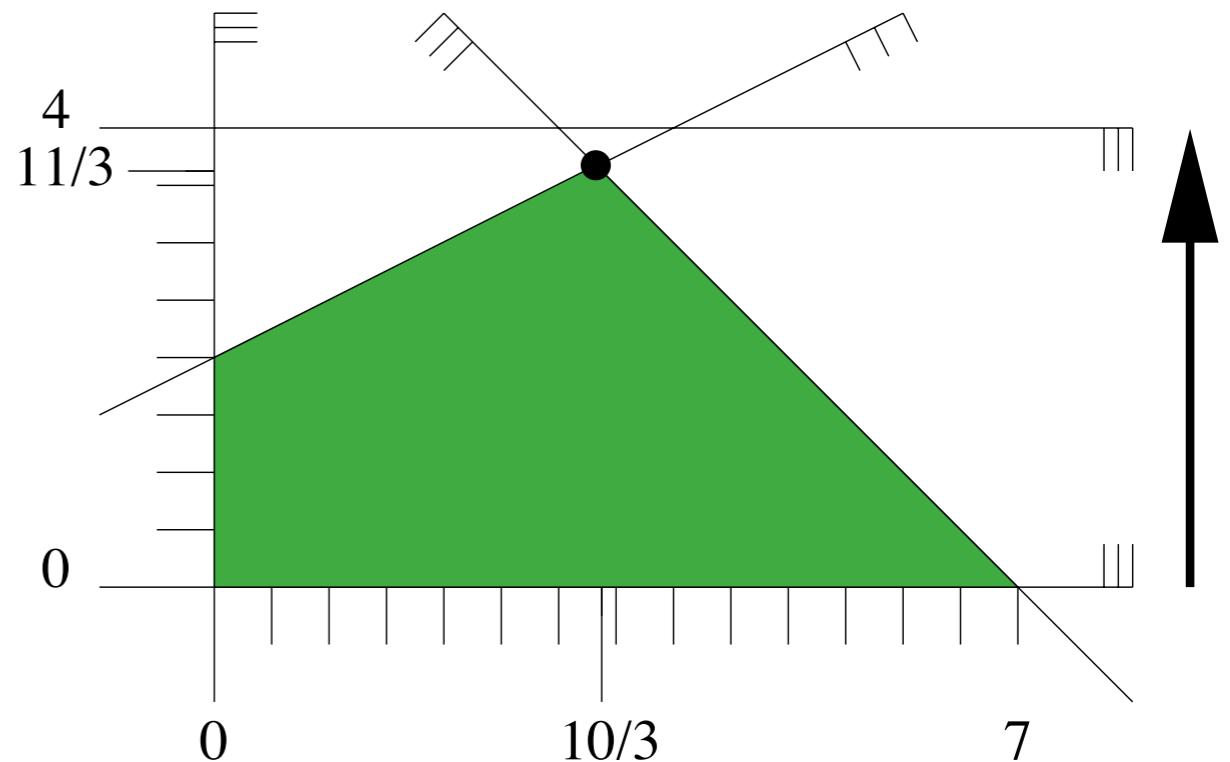
$$\begin{array}{ll} \text{minimize} & -32y + 64 \\ \text{subject to} & x + y \leq 7 \\ & -x + 2y \leq 4 \\ & x \geq 0 \\ & y \geq 0 \\ & y \leq 4 \end{array}$$

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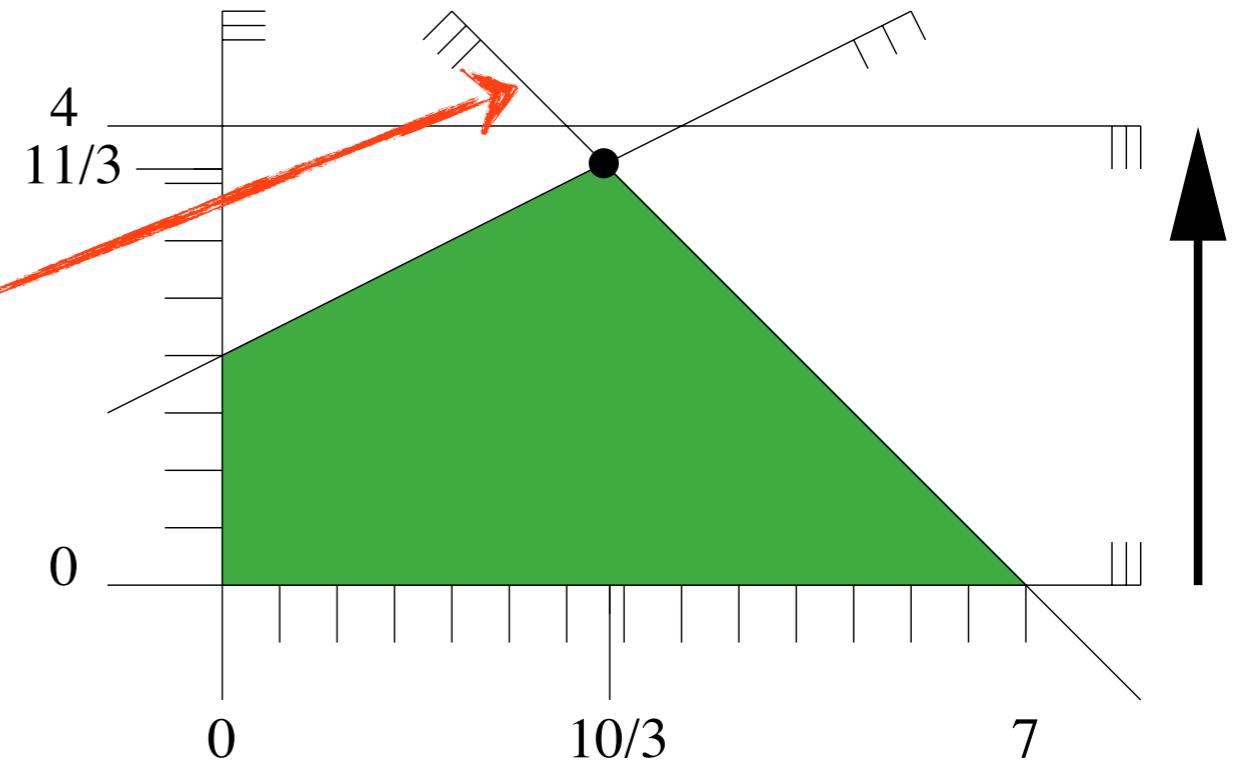


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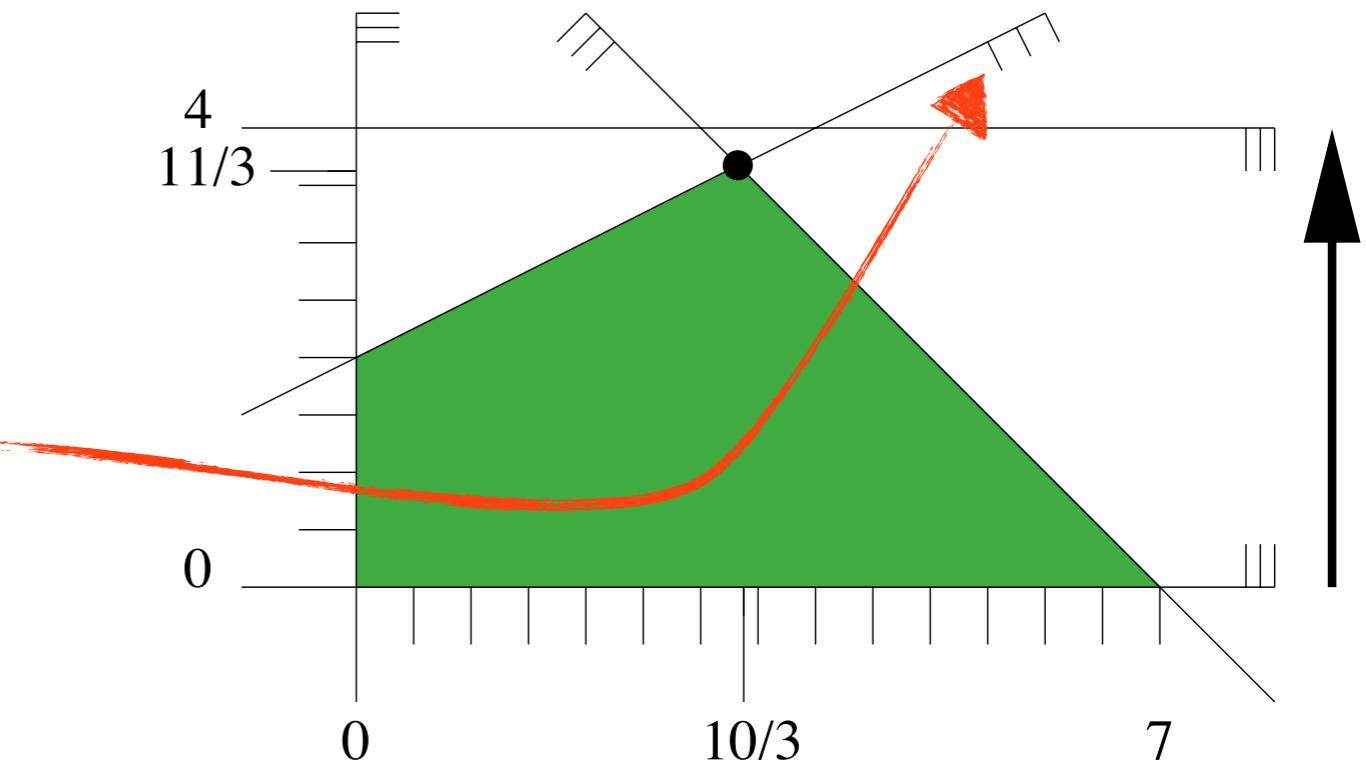
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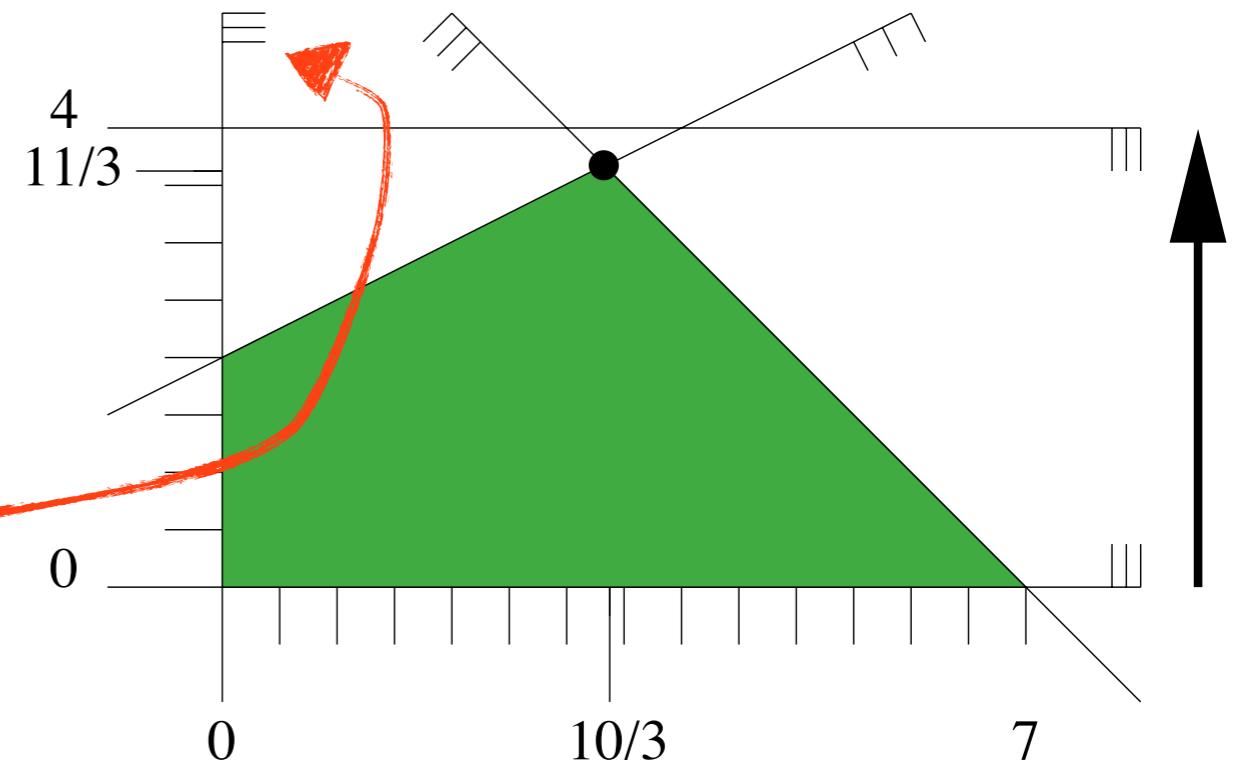
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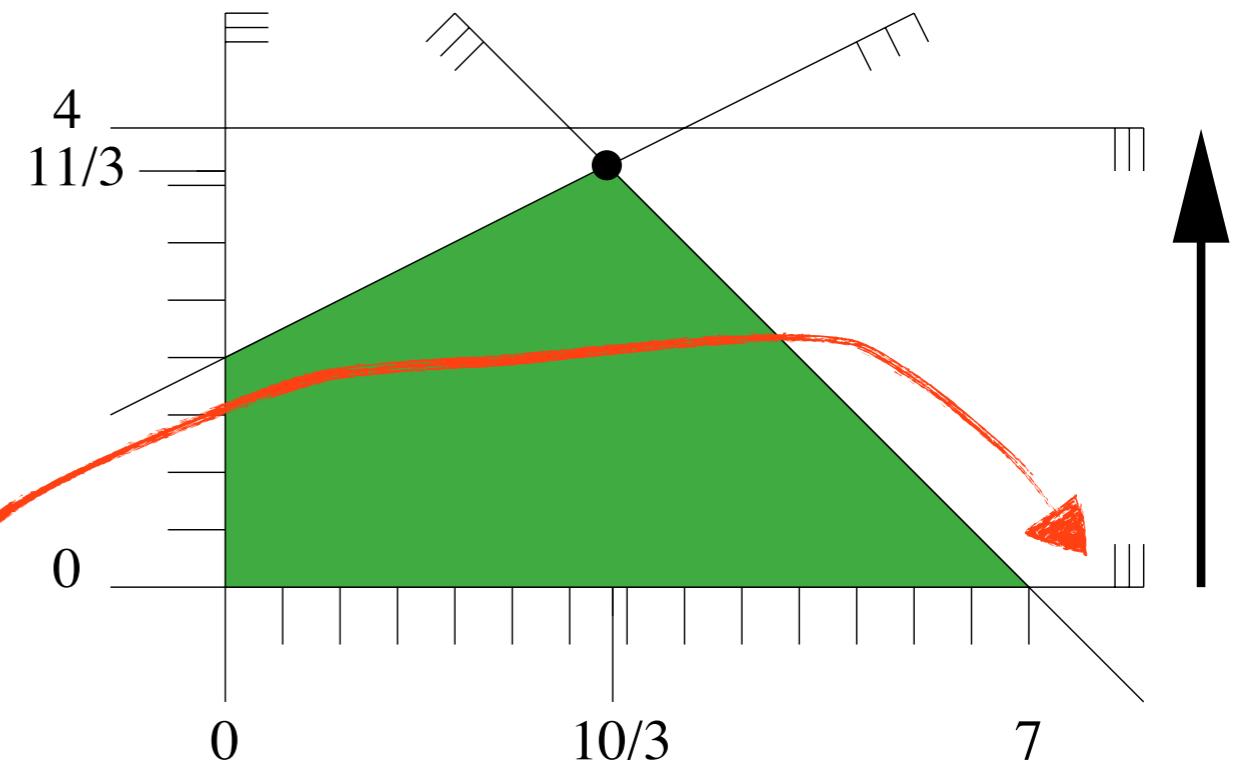
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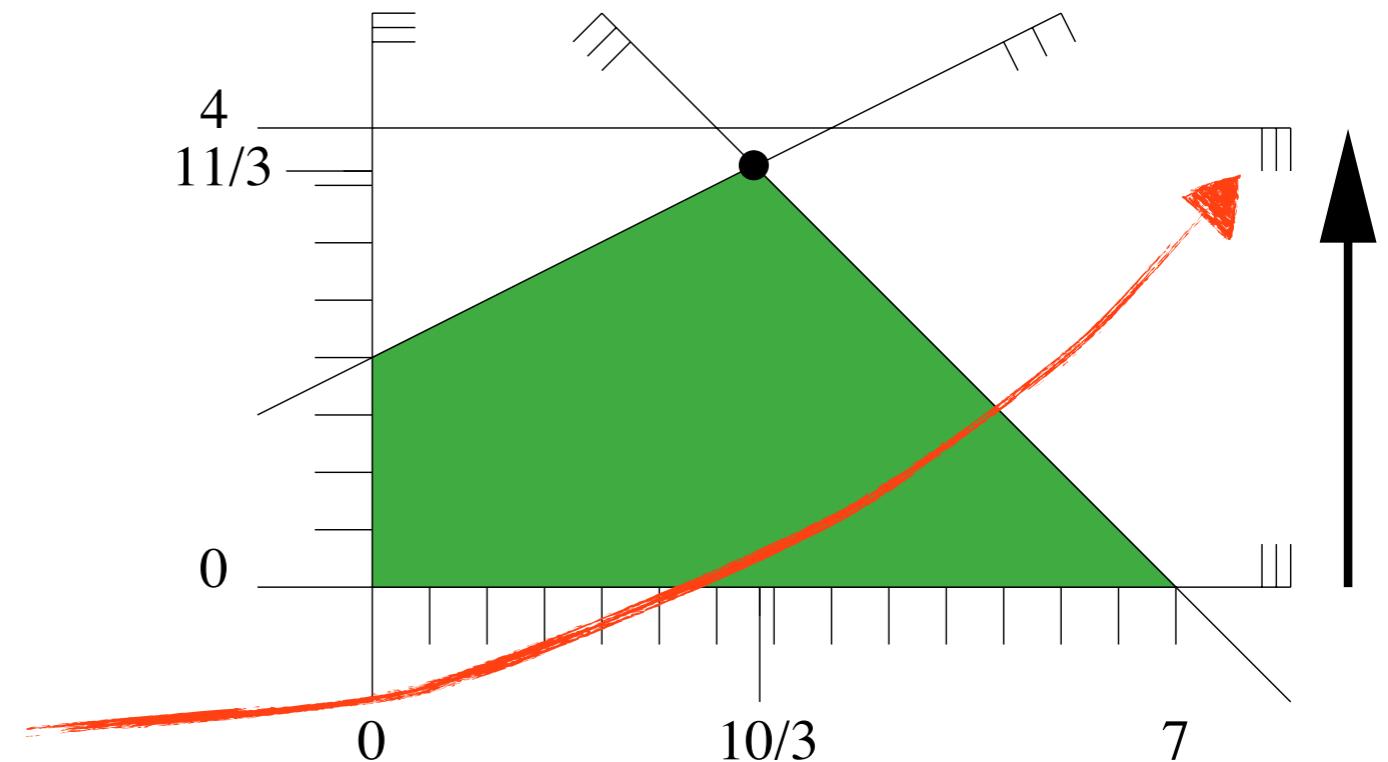


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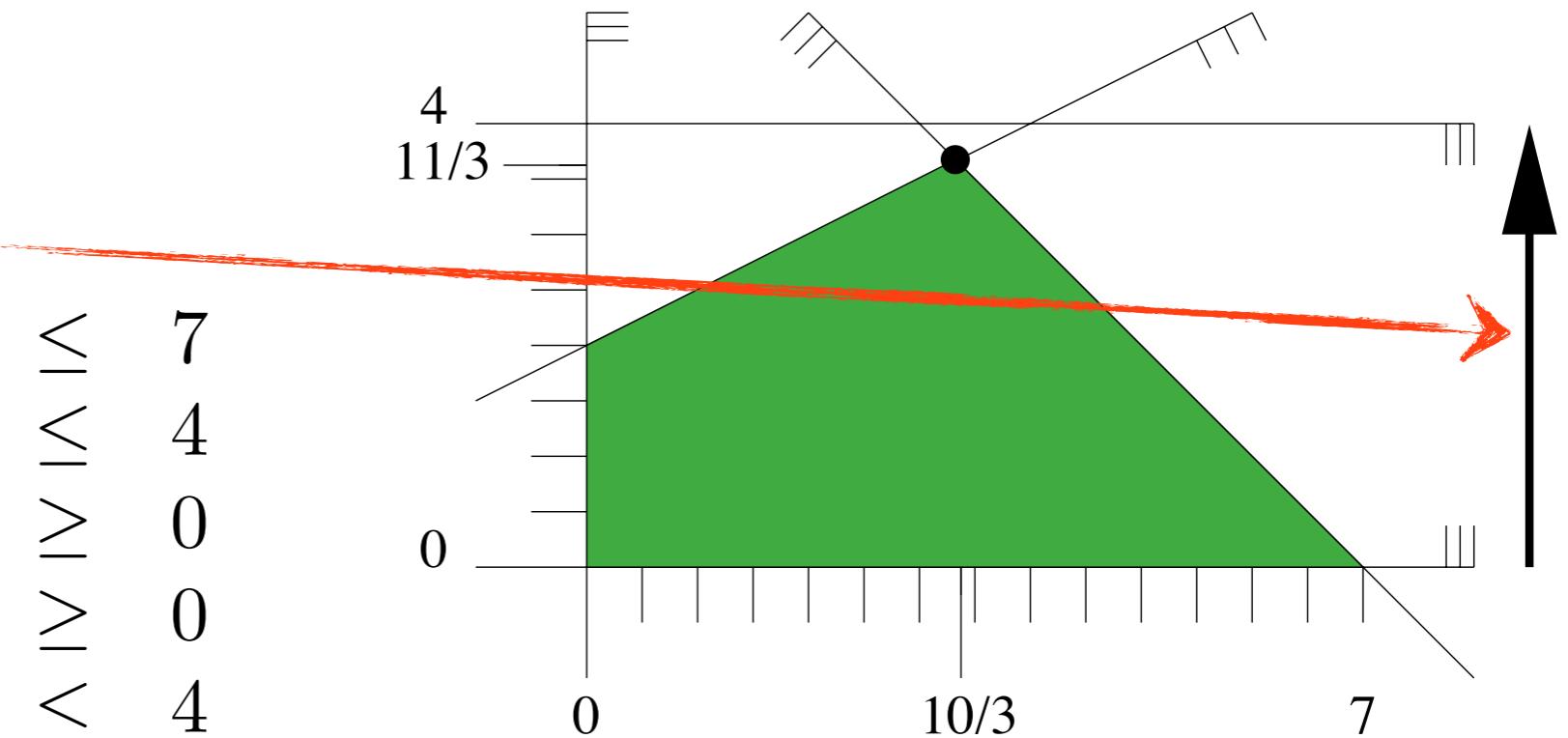


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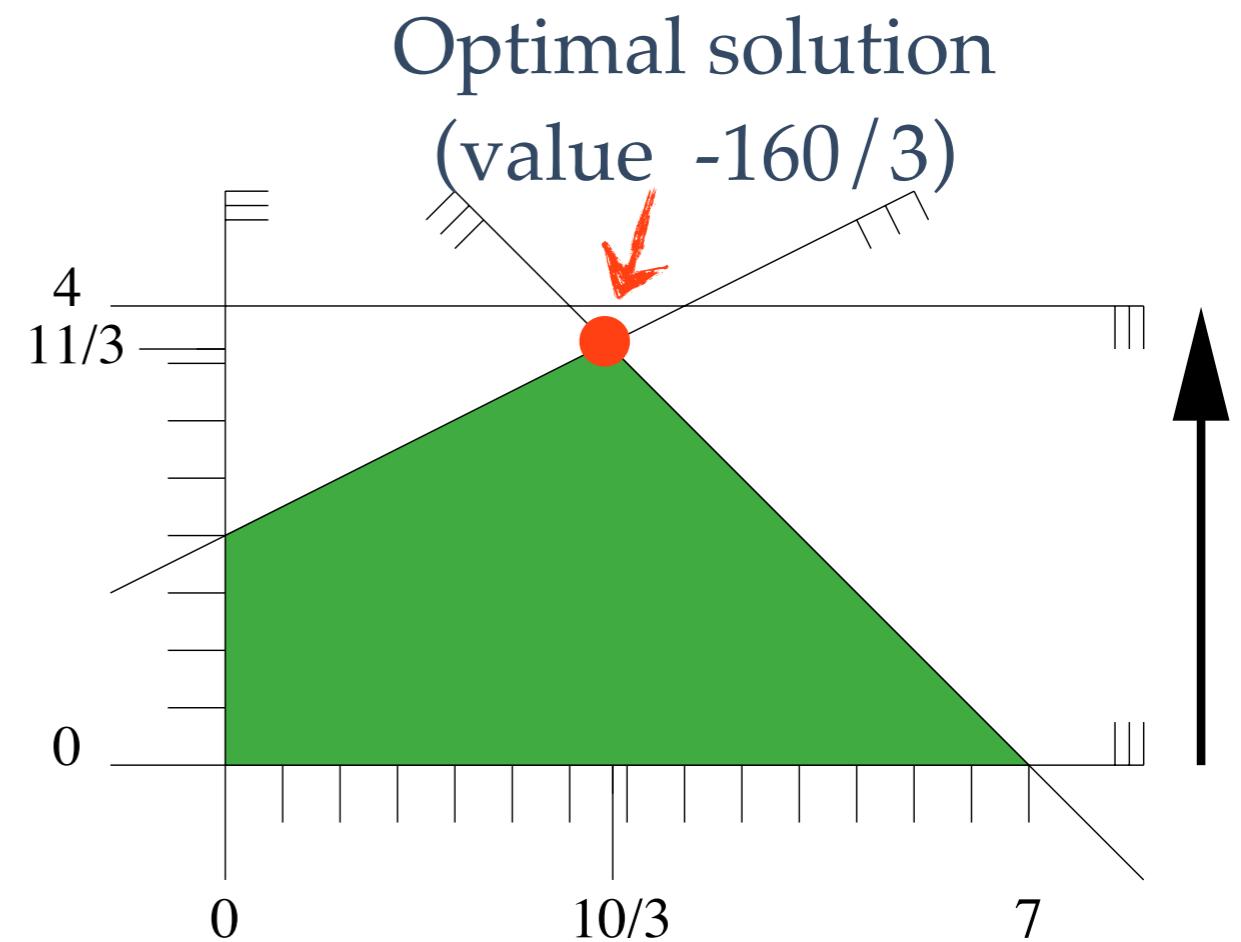


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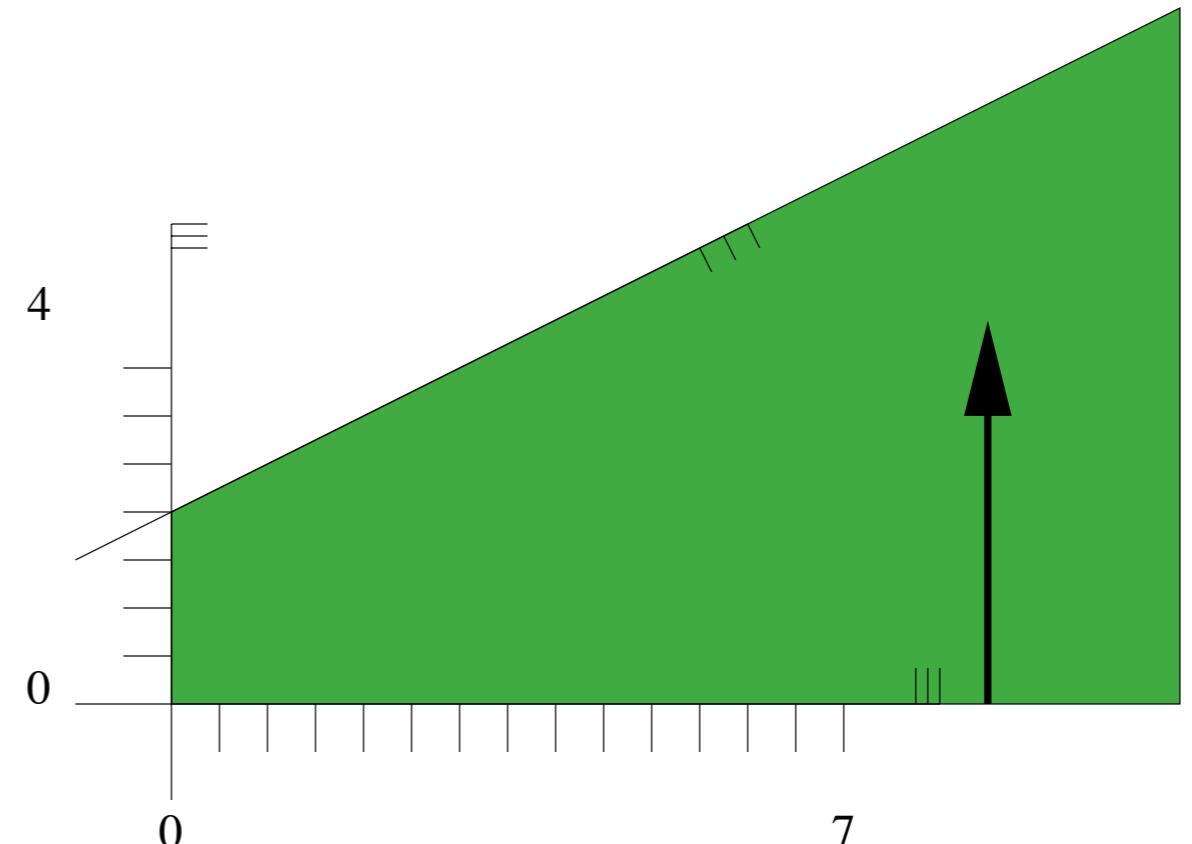
# Linear Programming

## Mathematical formulation

- ❖ **Problem:** Minimize a linear function in  $n$  variables subject to  $m$  linear (in)equation constraints!
- ❖ **Unbounded linear programs:**

minimize  
subject to

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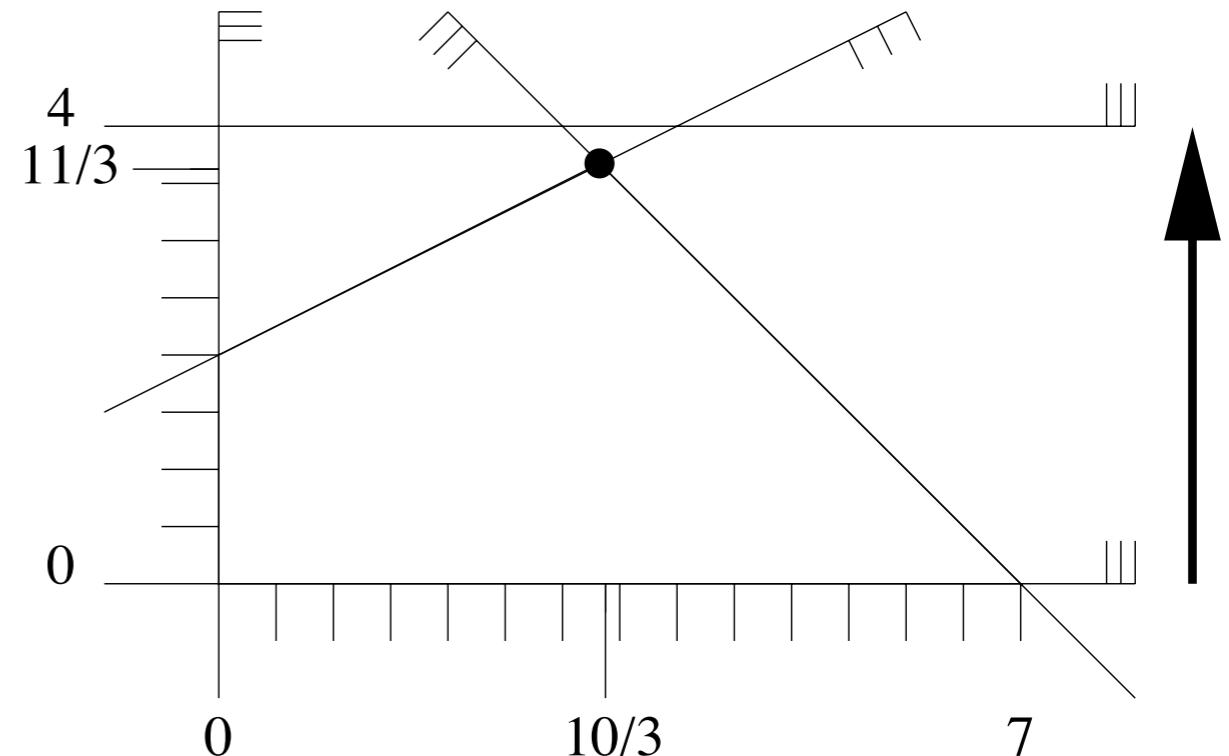


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$$\begin{array}{ccccc} & 1 & 1 & & 7 \\ & -1 & 2 & x & 4 \\ & -1 & 0 & & 0 \\ & 0 & -1 & y & 0 \\ & 0 & 1 & & 4 \end{array} \leq \begin{array}{c} 7 \\ 4 \\ 0 \\ 0 \\ 4 \end{array}$$

A

b

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$$\begin{array}{c} \mathbf{c}^T \\ \hline \begin{matrix} 0 \\ -32 \end{matrix} \end{array} \begin{matrix} x & y \end{matrix} + \begin{matrix} \mathbf{c}_0 \\ 64 \end{matrix} \quad \begin{array}{c} \mathbf{A} \\ \hline \begin{matrix} 1 & 1 \\ -1 & 2 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{matrix} \end{array} \begin{matrix} x \\ y \end{matrix} \leq \begin{matrix} \mathbf{b} \\ \hline \begin{matrix} 7 \\ 4 \\ 0 \\ 0 \\ 4 \end{matrix} \end{array}$$

# Linear Programming

## Mathematical formulation

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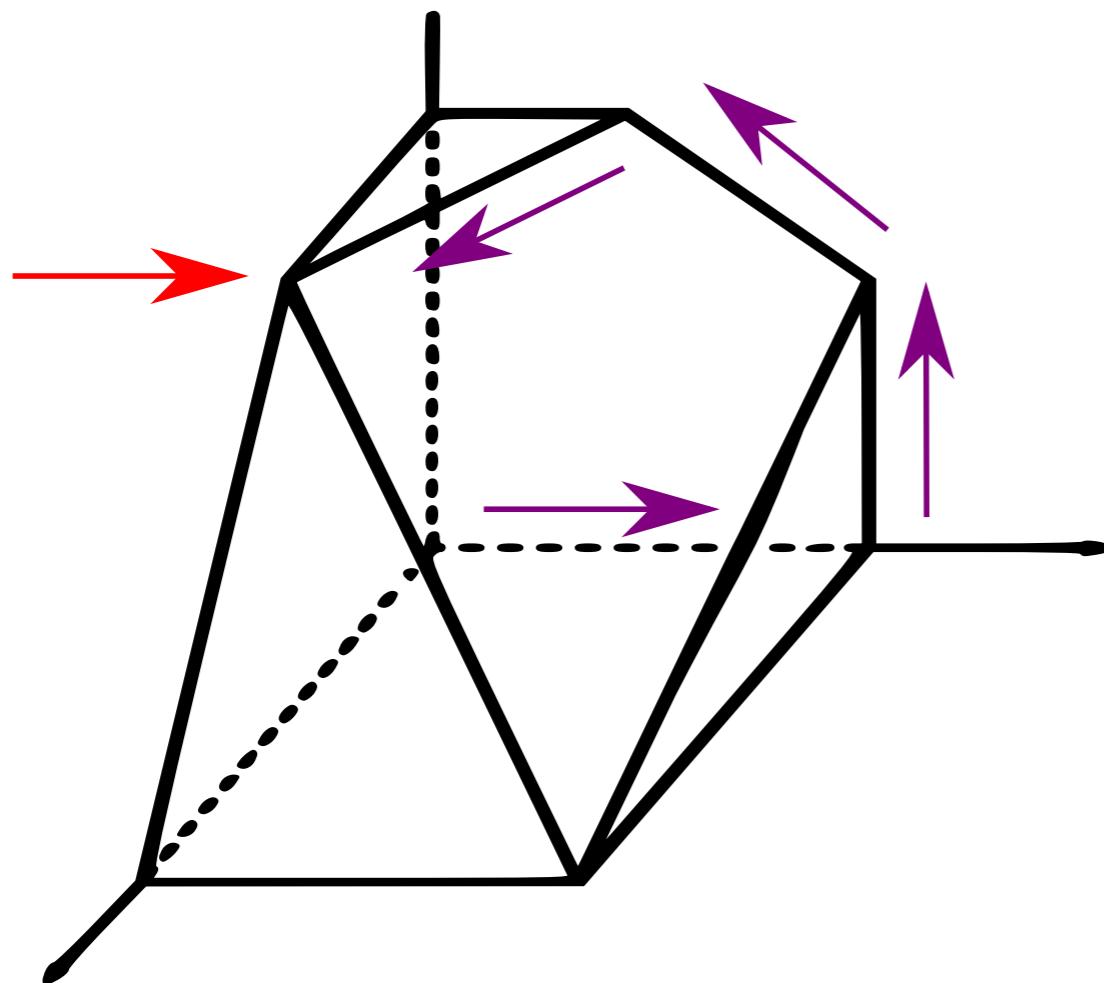
- \* **General form of LP:**

$$\begin{array}{ll}\text{minimize} & c^T x + c_0 \\ \text{subject to} & Ax \geq b \\ & l \leq x \leq u\end{array}$$

$$(x, c, l, u \in \mathbb{R}^n, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \quad c_0 \in \mathbb{R})$$

# Linear Programming

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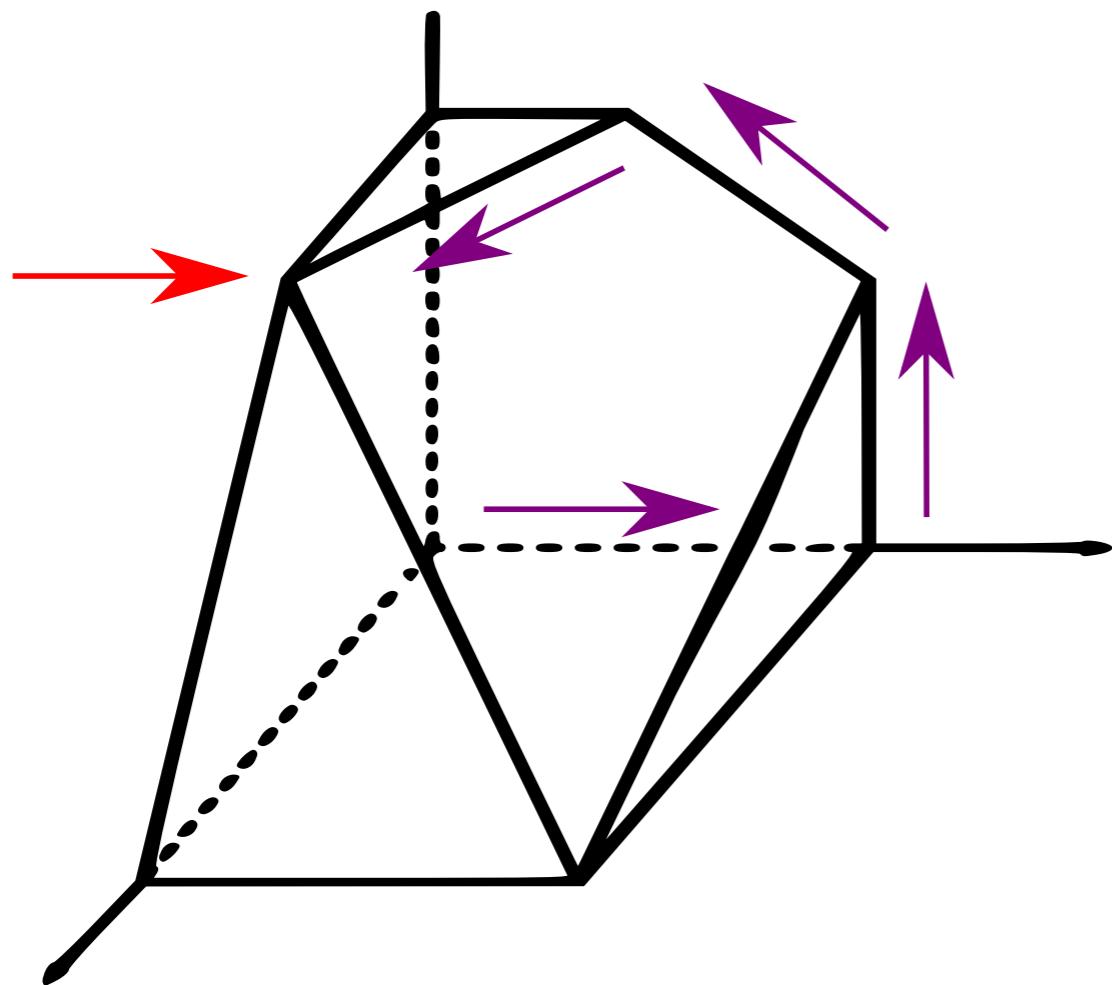


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- The constraints define a convex polyhedron  
 $P = \{x | Ax \leq b\}$

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$$\begin{array}{ll}\text{minimize} & c^T x + c_0 \\ \text{subject to} & Ax \geq b \\ & l \leq x \leq u\end{array}$$

- The constraints define a convex polyhedron  
$$P = \{x | Ax \leq b\}$$
- Simplex-type algorithms walk along the edges of  $P$  to vertices with better objective value.
- To decide on the next edge we use a *pivot rule*.

# Linear Programming

Complexity of Simplex-type algorithms  
with  $n$  variables and  $m$  constraints.

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- $O(\max\{m, n\})$  if  $\min\{m,n\}$  is  $O(1)$   
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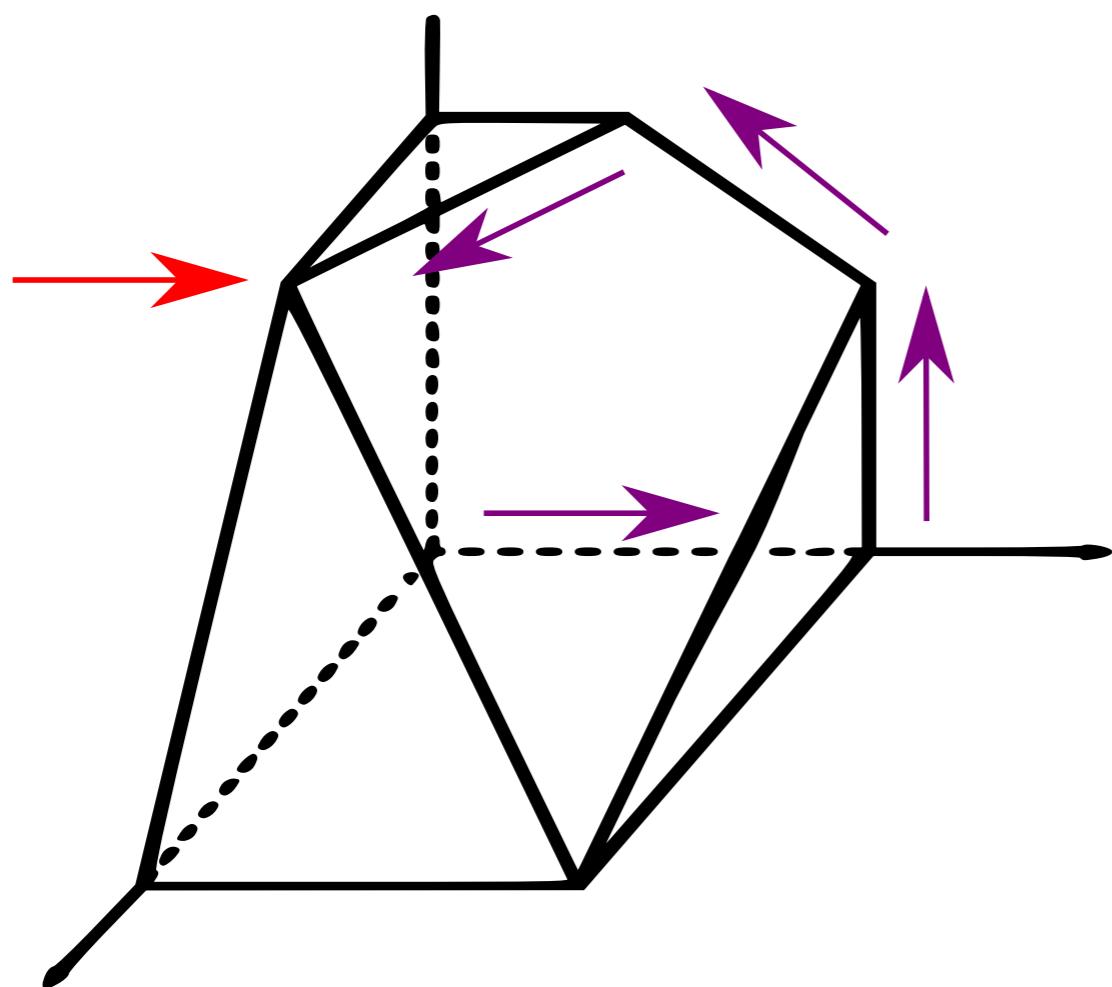
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(hidden constant exponential in  $\min\{m,n\}$ )
- CGAL: Gives exact solutions!  
Works well as long as  $\min\{m,n\} \leq 50$   
(or up to 100)

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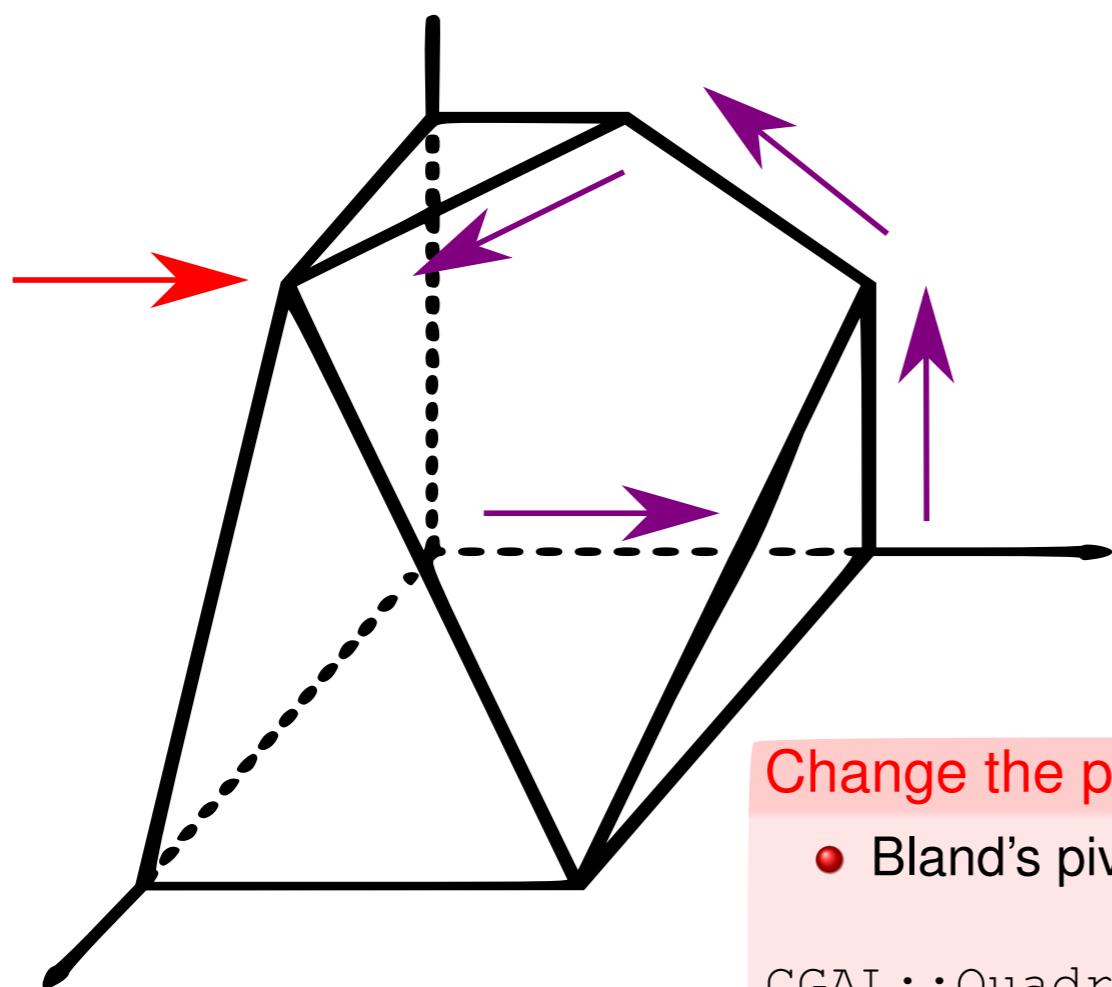
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The default pivot rule strategy might fail:

- it is deterministic, so it picks the same improving edge every time.
- If the algorithm reaches a vertex for the second time, then cycles forever.

# Linear Programming



Change the pivot rule

- Bland's pivot rule avoids cycling (but it is slower...)

```
CGAL::Quadratic_program_options options;  
options.set_pricing_strategy(CGAL::QP_BLAND);  
Solution s = CGAL::SOLVER(program, ET(), options);
```

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- \* General form of LP in CGAL:

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$\leq, =, \text{ or } \geq$  (individually  
for each constraint)

variables →  $(x, c, l, u \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c_0 \in \mathbb{R})$

objective function ↑ ↑ ↑ ↑

lower and upper bounds ↑ ↑

constraint matrix ↗

right-hand side ↗ ↗ ↗ ↗

shift ↗ ↗ ↗ ↗

# Linear Programming ... in CGAL

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- ❖ **Preamble:** Choice of input type and exact internal number type

Gnu  
Multi-  
precision  
Library  
(GMP)

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#include <cassert>
#include <CGAL/basic.h>
#include <CGAL/QP_models.h>
#include <CGAL/QP_functions.h>
#include <CGAL/Gmpz.h>

// choose exact integral type
typedef CGAL::Gmpz ET;

// program and solution types
typedef CGAL::Quadratic_program<int> Program;
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input type

exact internal type

for linear *and* quadratic programs

GMP used internally

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Inside, the LP solver uses quotients of  
the data type specified here.

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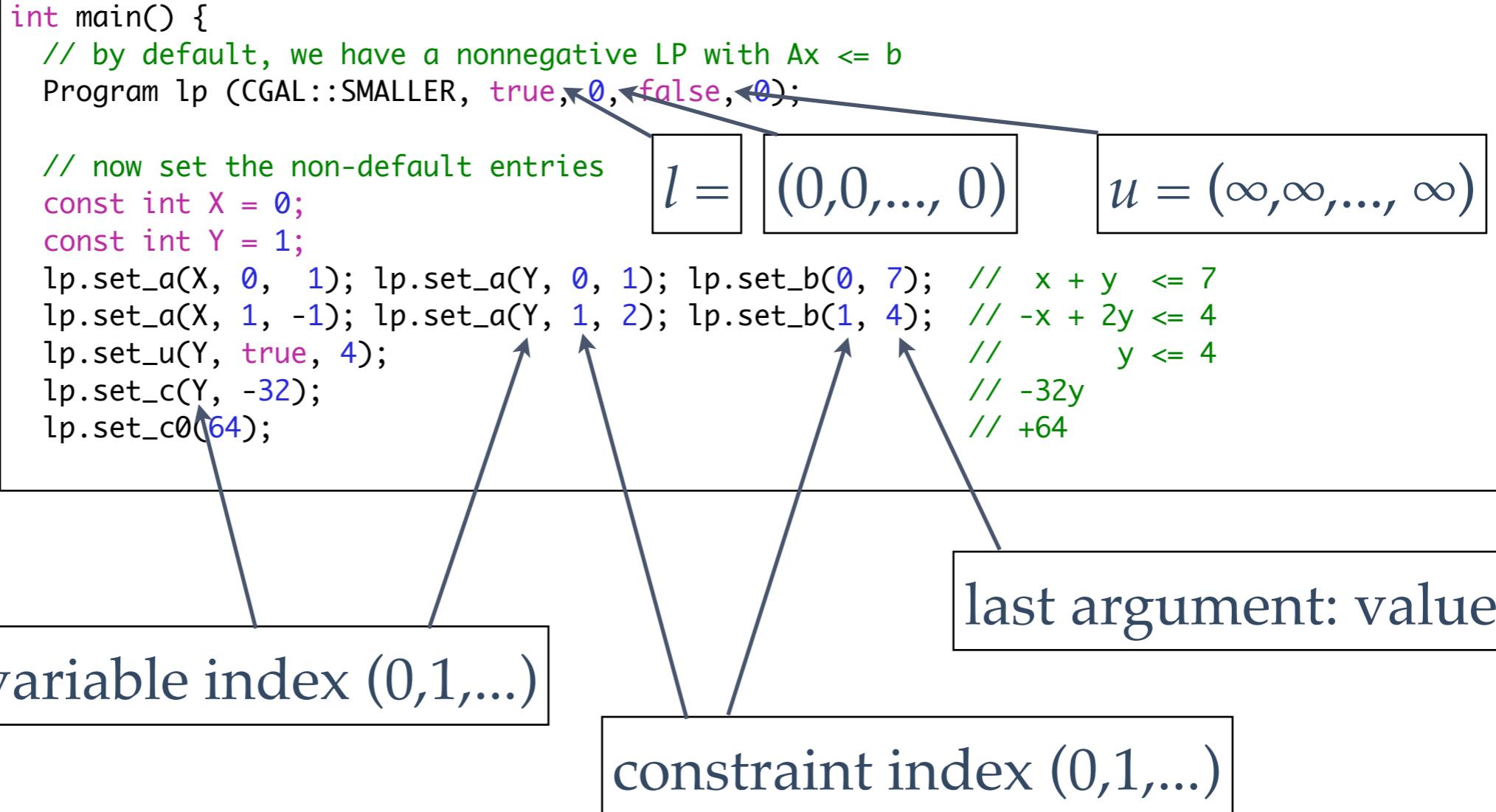
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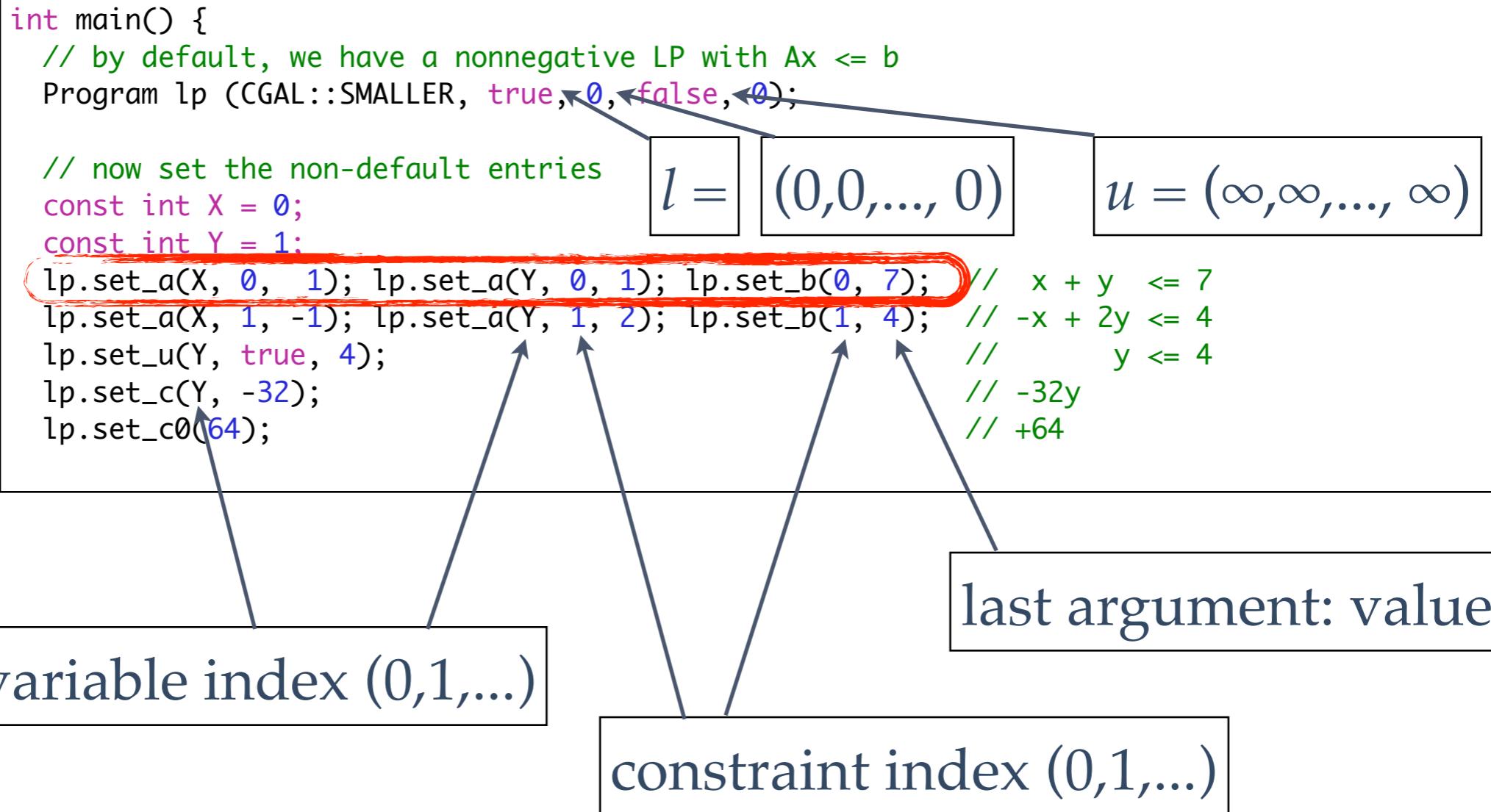
- ❖ **Setup:** Enter the program data



# Linear Programming ... in CGAL

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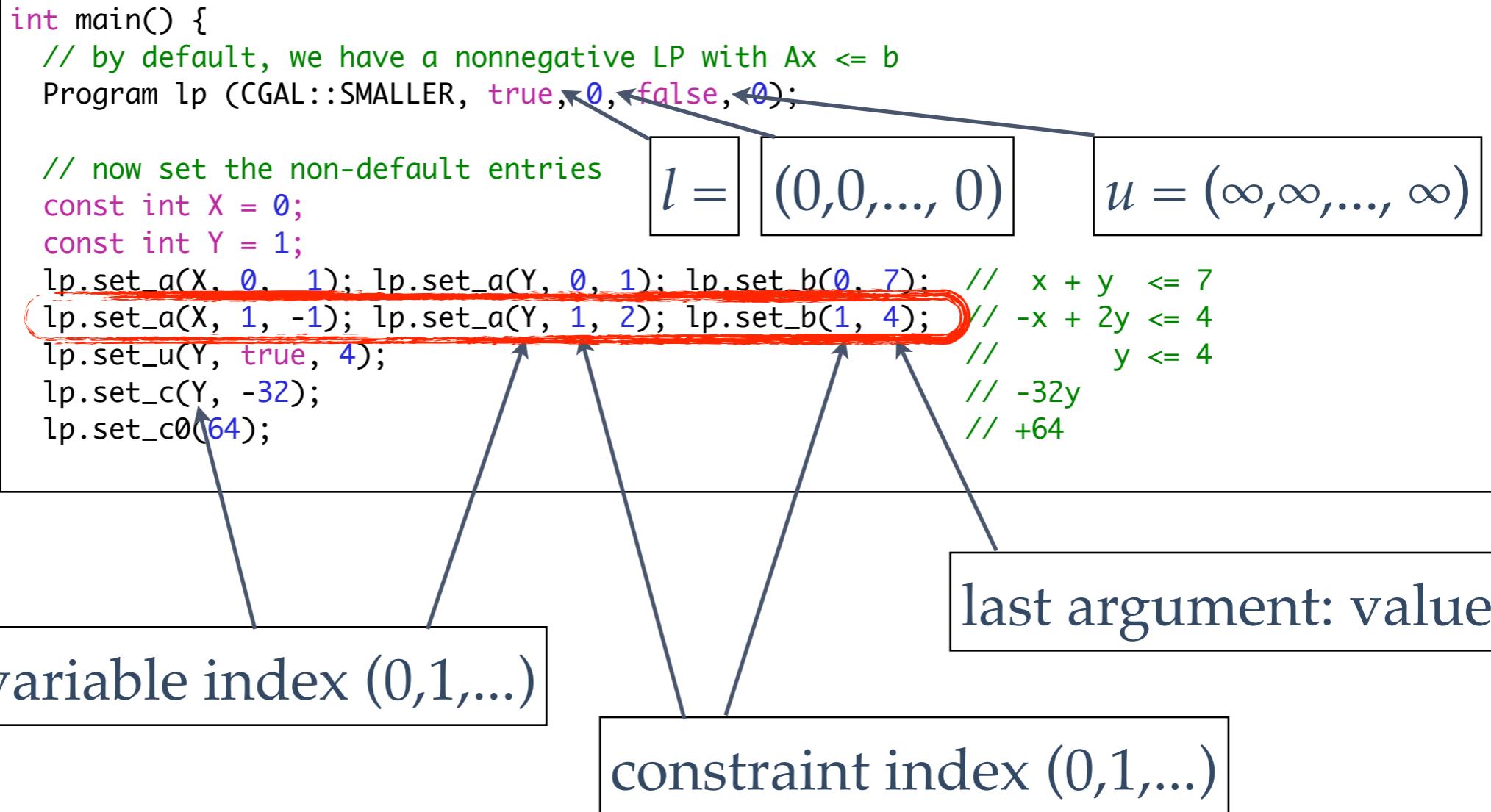
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- Setup: Enter the program data

```
int main() {
    // by default, we have a nonnegative LP with Ax <= b
    Program lp (CGAL::SMALLER, true, 0, false, 0);

    // now set the non-default entries
    const int X = 0;
    const int Y = 1;
    lp.set_a(X, 0, 1); lp.set_a(Y, 0, 1); lp.set_b(0, 7);    // x + y <= 7
    lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4);    // -x + 2y <= 4
    lp.set_u(Y, true, 4);                                     // y <= 4
    lp.set_c(Y, -32);                                         // -32y
    lp.set_c0(64);                                           // +64
```

$l =$

$(0, 0, \dots, 0)$

$u = (\infty, \infty, \dots, \infty)$

variable index (0,1,...)

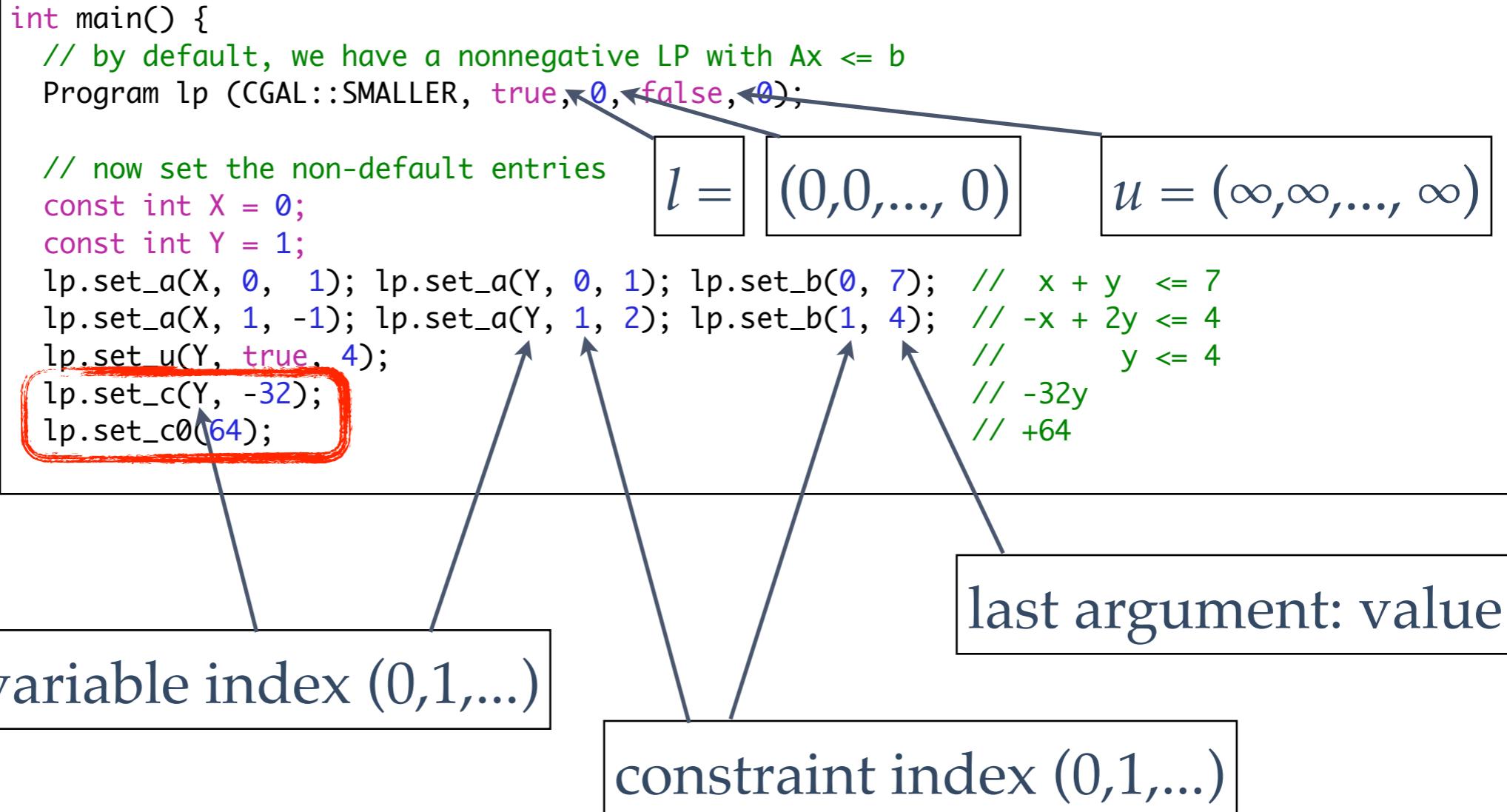
constraint index (0,1,...)

last argument: value

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- Setup: Enter the program data



# Linear Programming ... in CGAL

---

- \* **Solve:** Call the linear programming solver and output solution

```
// solve the program, using ET as the exact type
Solution s = CGAL::solve_linear_program(lp, ET());
assert (s.solves_linear_program(lp));

// output solution
std::cout << s;
return 0;
}
```

independent verification

# Linear Programming ... in CGAL

- \* **Solve:** Call the linear programming solver and output solution

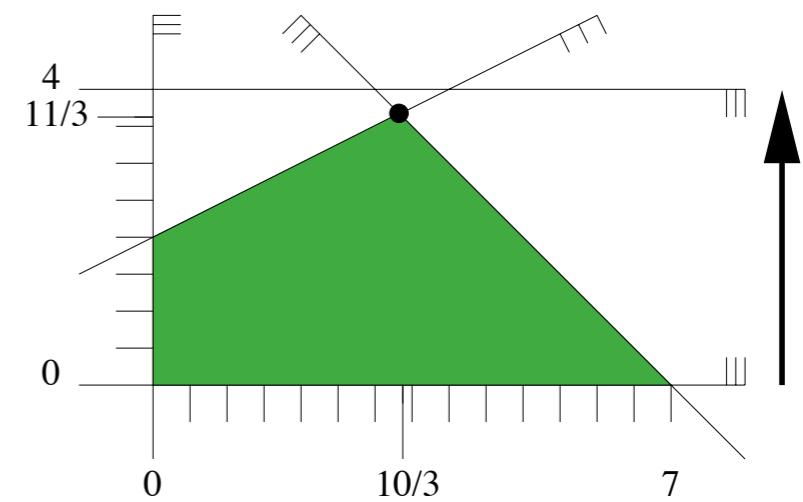
```
// solve the program, using ET as the exact type
Solution s = CGAL::solve_linear_program(lp, ET());
assert (s.solves_linear_program(lp));

// output solution
std::cout << s;
return 0;
}
```

independent verification

- \* **Output:**

```
status: OPTIMAL
objective value: -160/3
variable values:
 0: 10/3
 1: 11/3
```

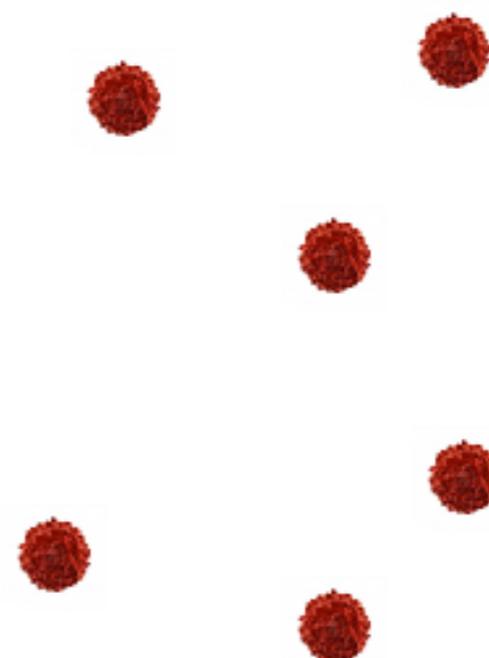


# Linear Programming

## Application I: Cancer Therapy

---

- \* **Given:** locations of cancer cells (red)

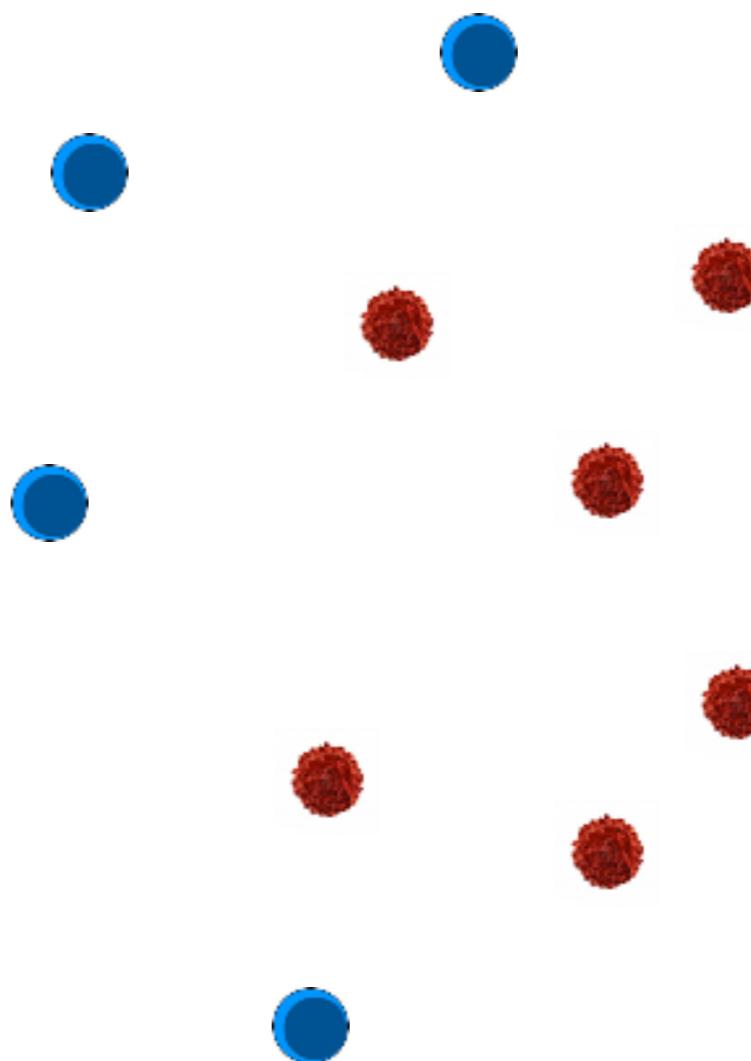


# Linear Programming

## Application I: Cancer Therapy

---

- Given: locations of cancer cells (red) and healthy cells (blue)

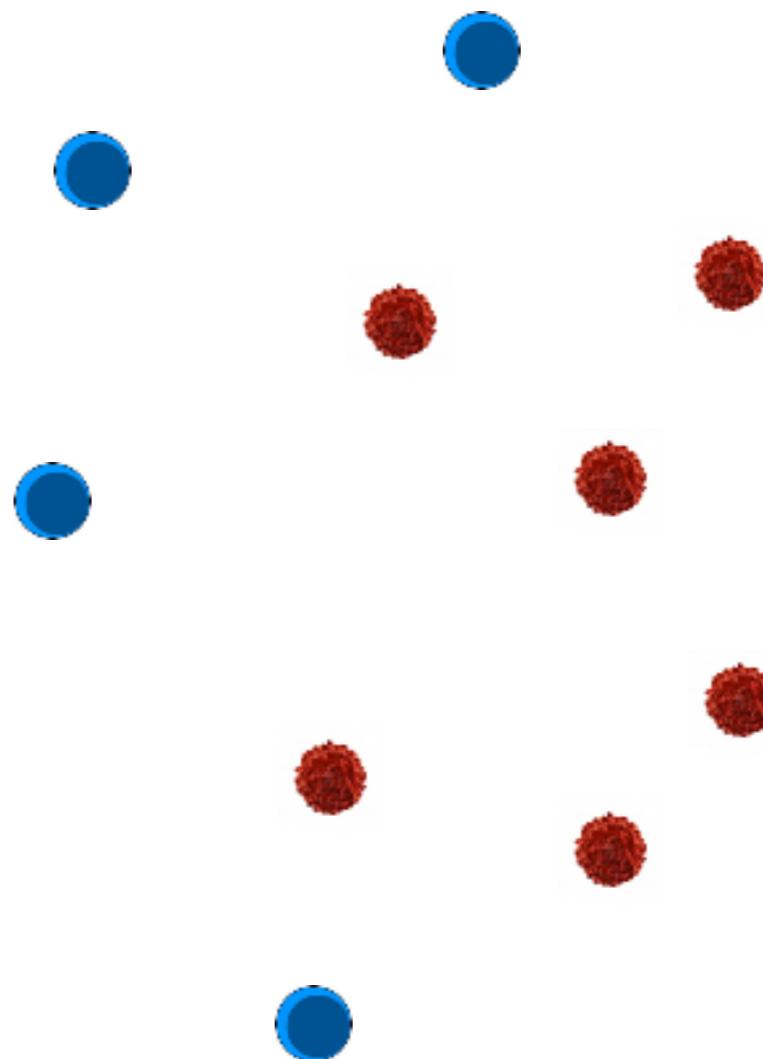


# Linear Programming

## Application I: Cancer Therapy

---

- ❖ **Given:** locations of cancer cells (red) and healthy cells (blue)
- ❖ **Wanted:** center and radius of exposure so that all cancer cells are killed and all healthy cells are unaffected.

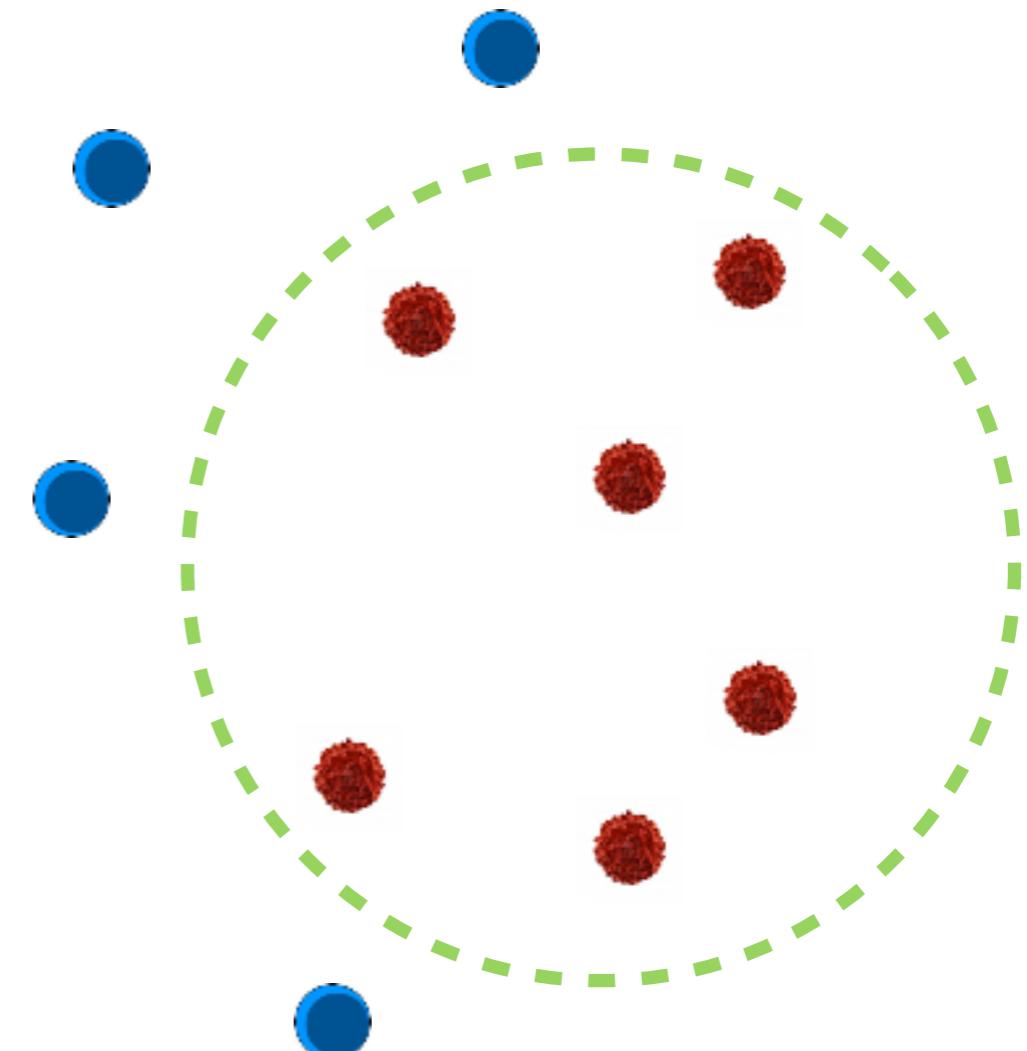


# Linear Programming

## Application I: Cancer Therapy

---

- ❖ **Given:** locations of cancer cells (red) and healthy cells (blue)
- ❖ **Wanted:** center and radius of exposure so that all cancer cells are killed and all healthy cells are unaffected.
- ❖ This may be possible...

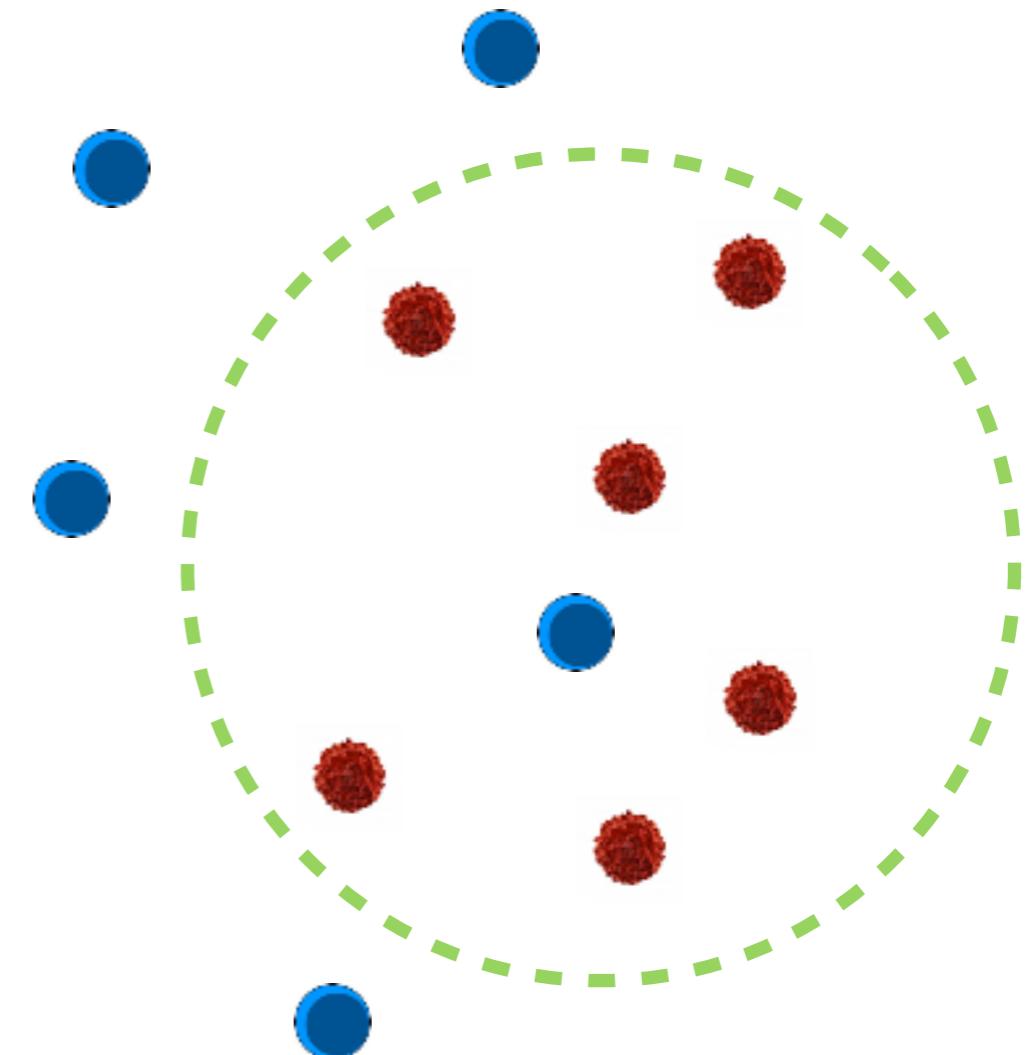


# Linear Programming

## Application I: Cancer Therapy

---

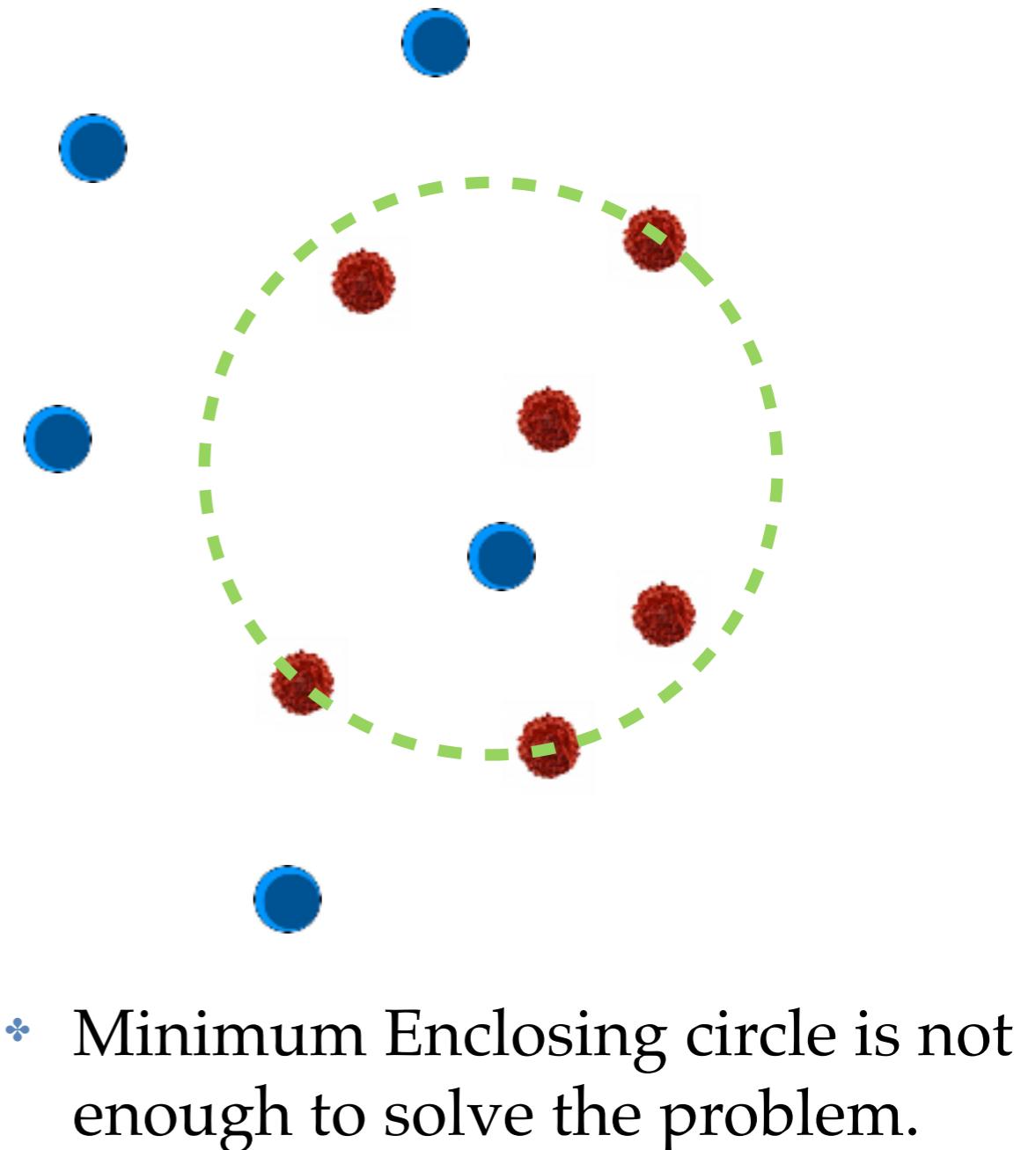
- ❖ **Given:** locations of cancer cells (red) and healthy cells (blue)
- ❖ **Wanted:** center and radius of exposure so that all cancer cells are killed and all healthy cells are unaffected.
- ❖ This may be possible... or not.



# Linear Programming

## Application I: Cancer Therapy

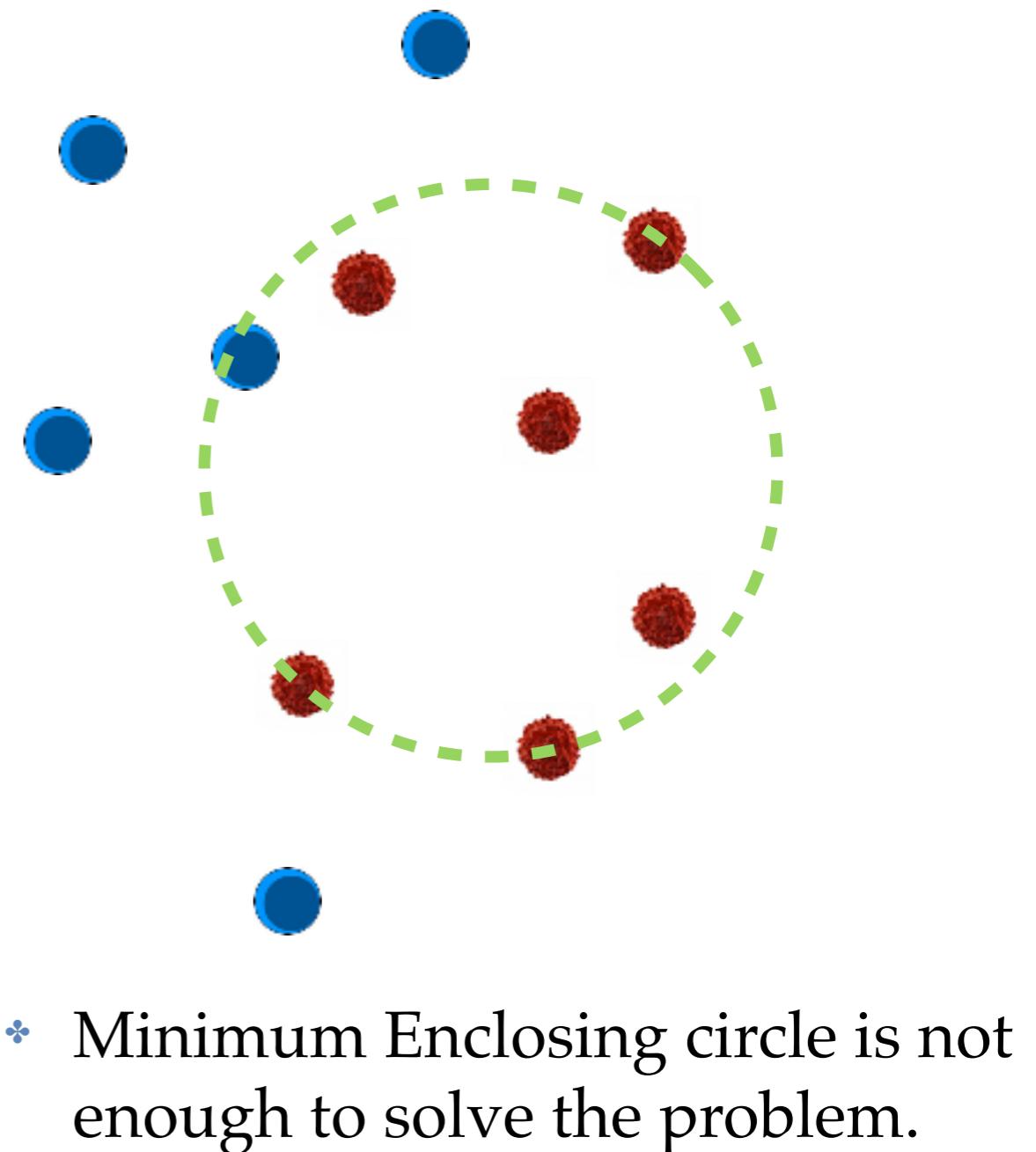
- Given: locations of cancer cells (red) and healthy cells (blue)
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# Linear Programming

## Application I: Cancer Therapy

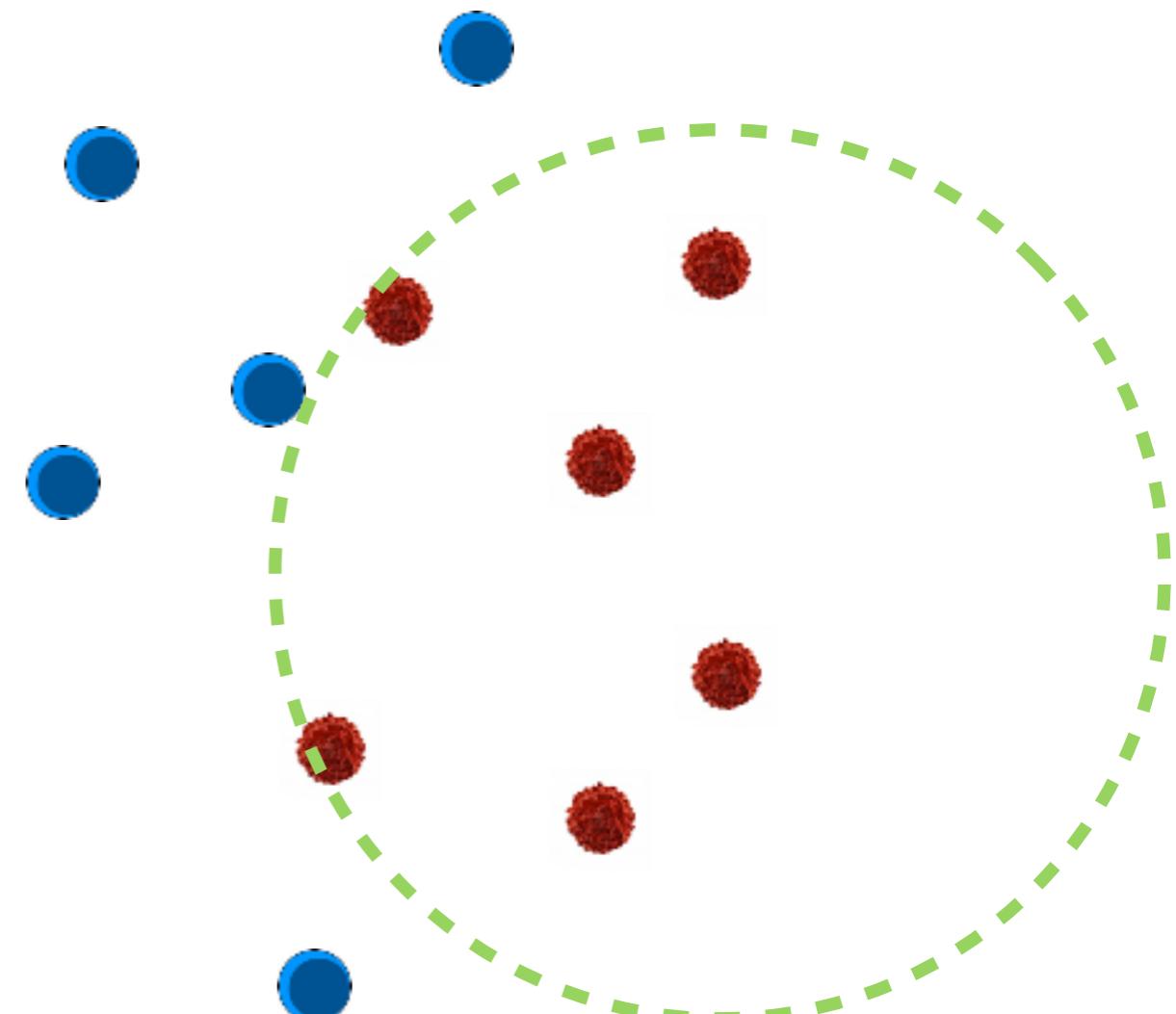
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# Linear Programming

## Application I: Cancer Therapy

- Given: locations of cancer cells (red) and healthy cells (blue)
- Wanted: center and radius of exposure so that all cancer cells are killed and all healthy cells are unaffected.
- This may be possible... or not.



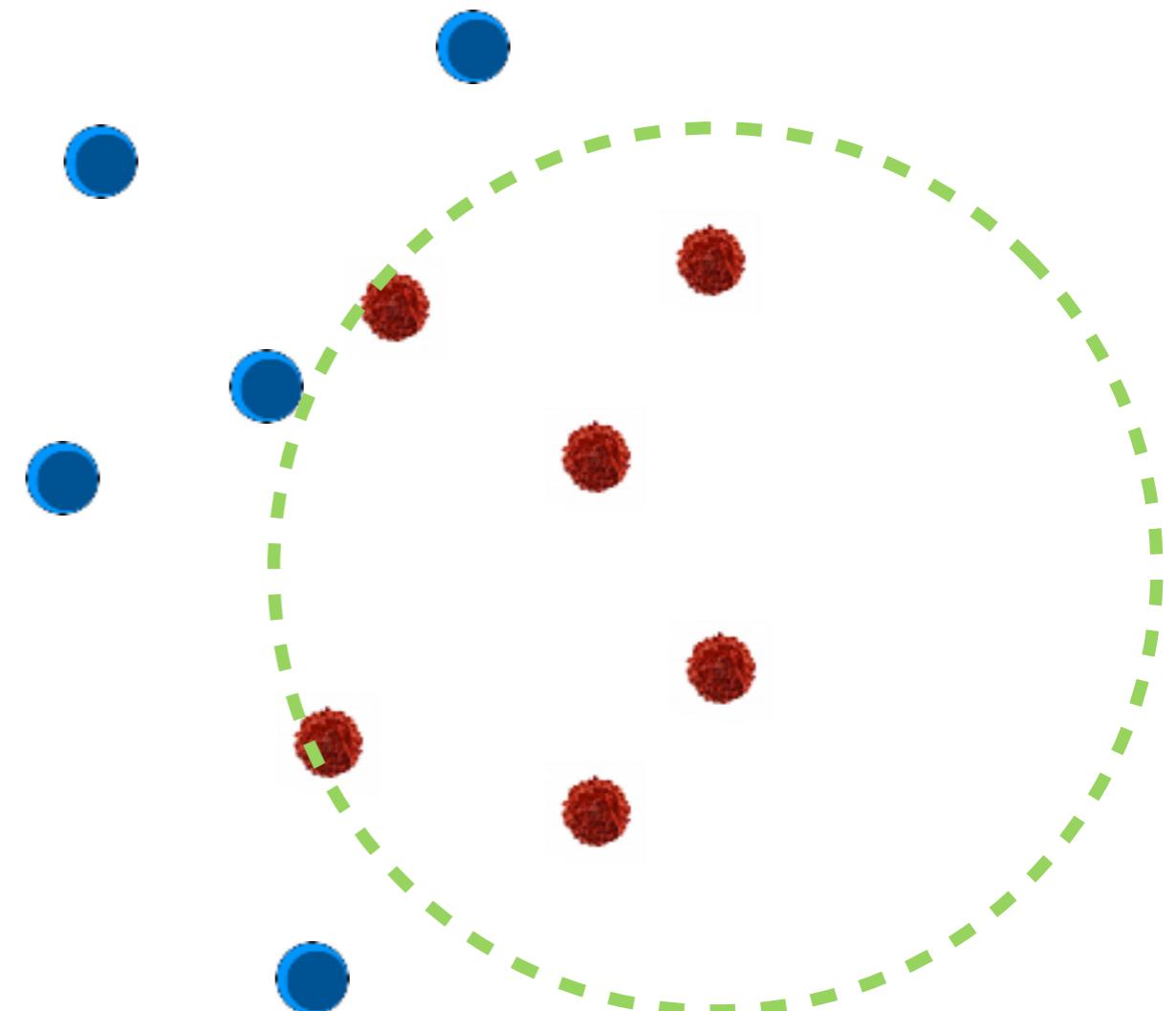
- Minimum Enclosing circle is not enough to solve the problem.

# Linear Programming

## Application I: Cancer Therapy

---

- ❖ **The geometric problem:** Given two finite sets  $R$  and  $B$  in the plane, does there exist a disk that contains  $R$  and is disjoint from  $B$ ?

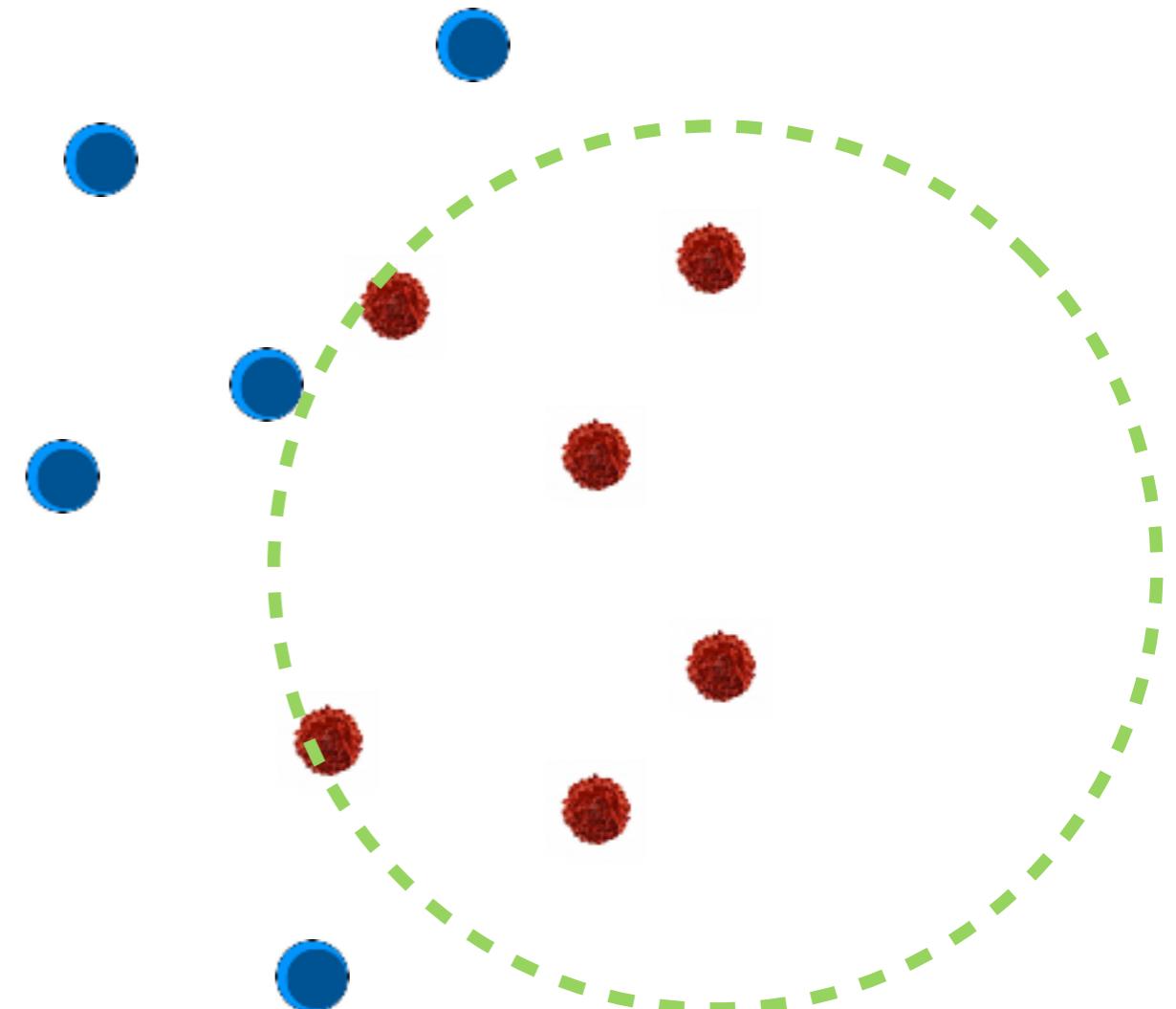


# Linear Programming

## Application I: Cancer Therapy

---

- ❖ **The geometric problem:** Given two finite sets  $R$  and  $B$  in the plane, does there exist a disk that contains  $R$  and is disjoint from  $B$ ?
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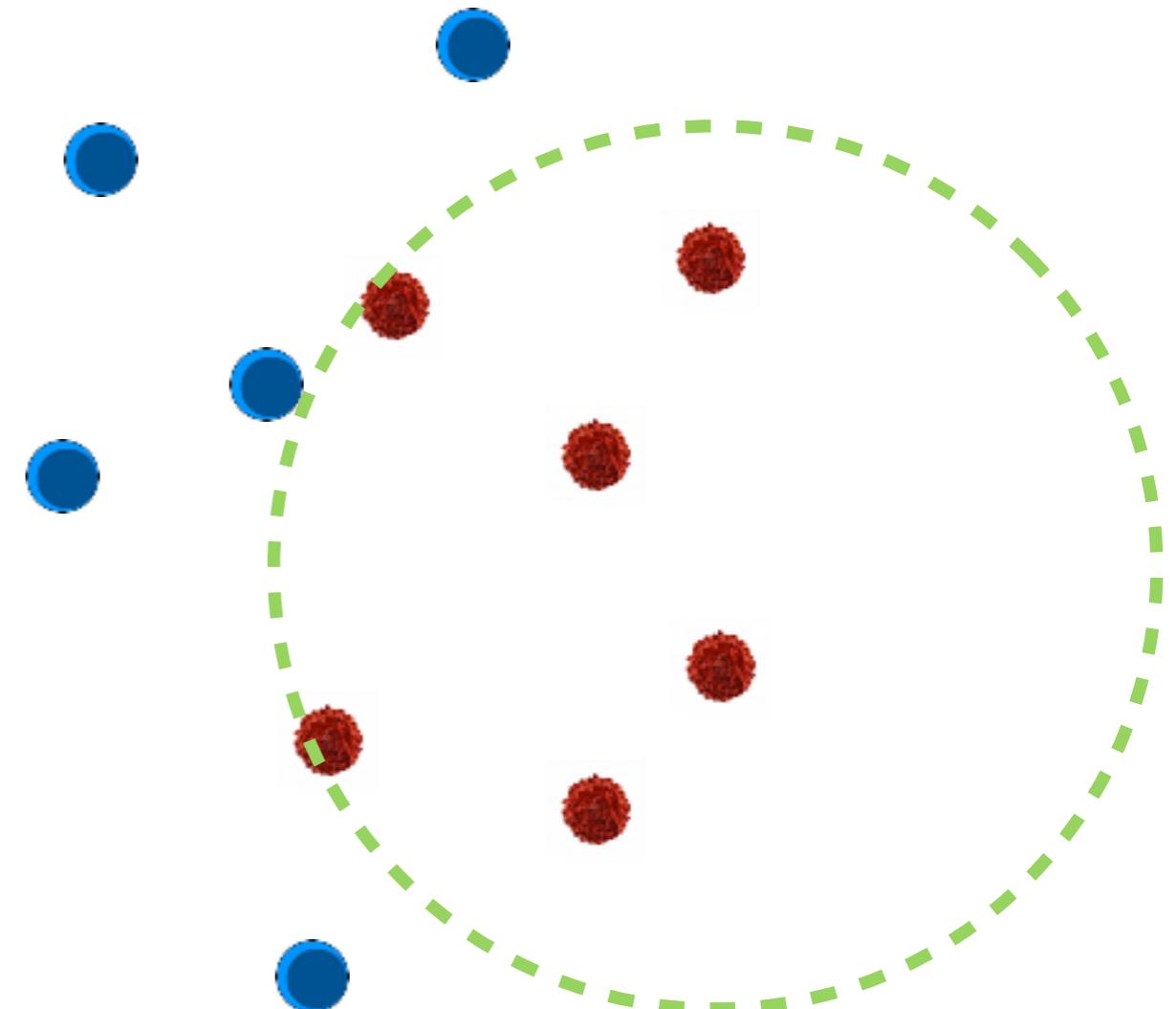


# Linear Programming

## Application I: Cancer Therapy

---

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- ❖ **Constraints:** Blue points should be outside, red points inside.

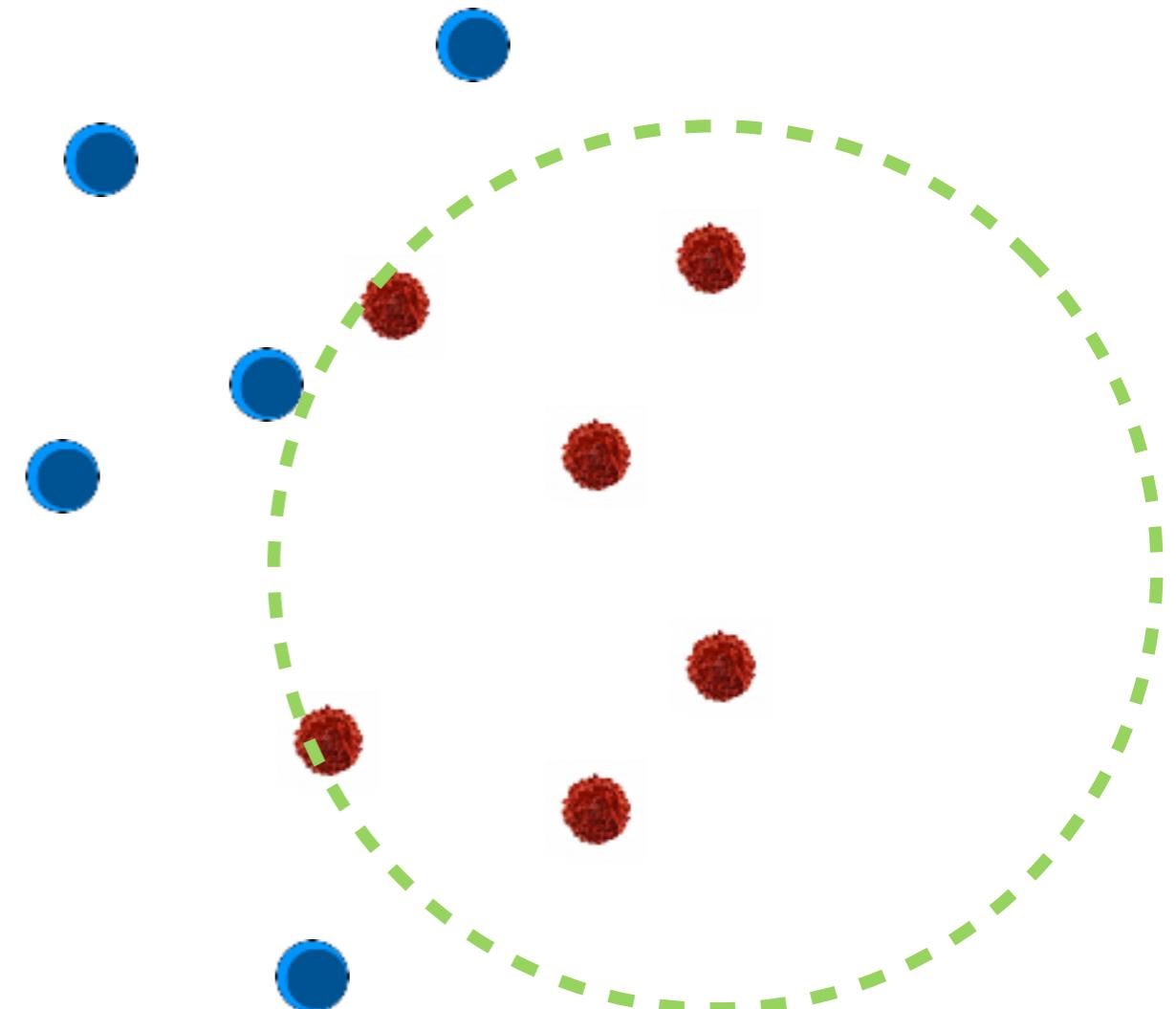


# Linear Programming

## Application I: Cancer Therapy

---

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- ❖ **Problem:** Constraints are not linear...

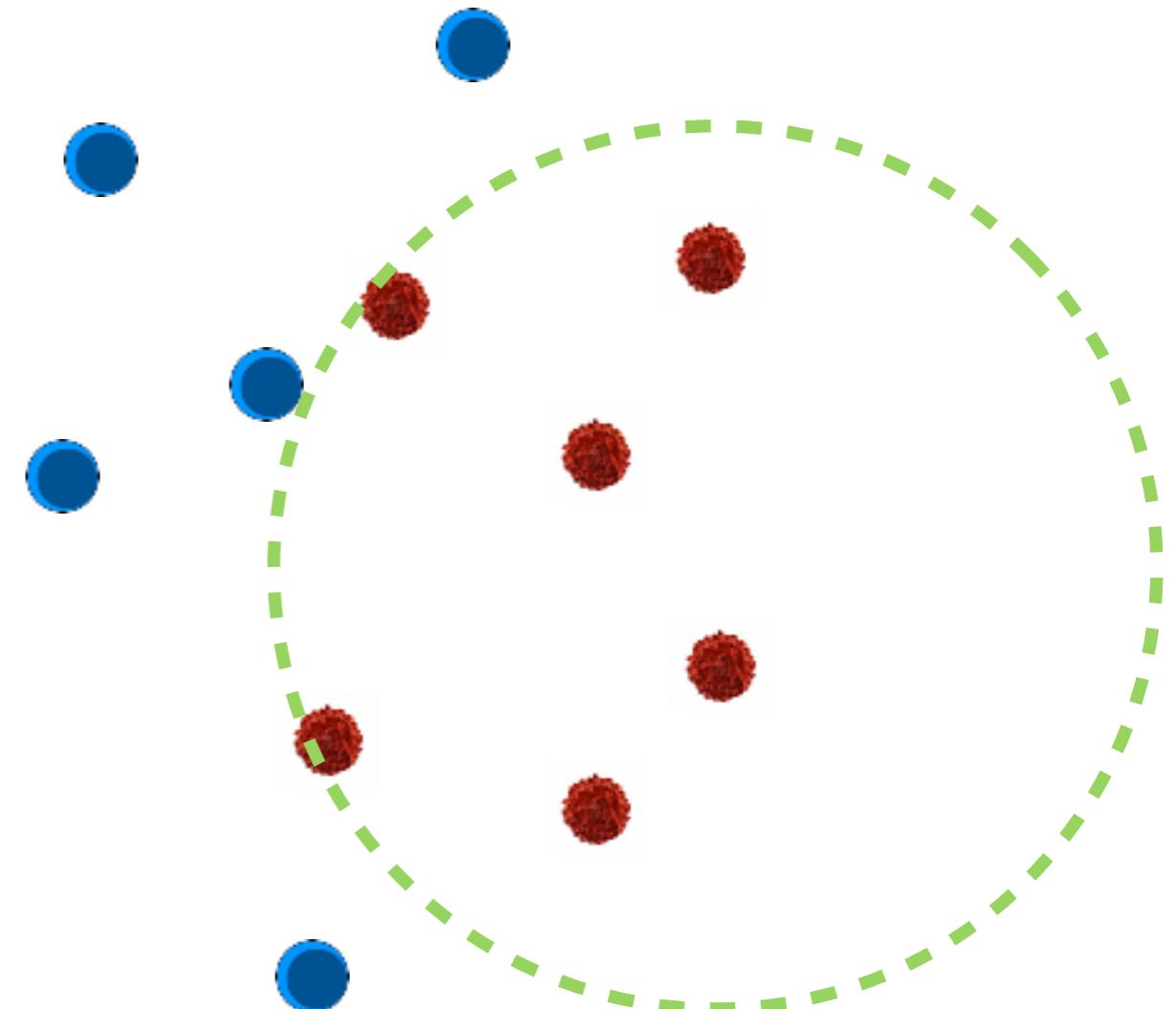


# Linear Programming

## Application I: Cancer Therapy

---

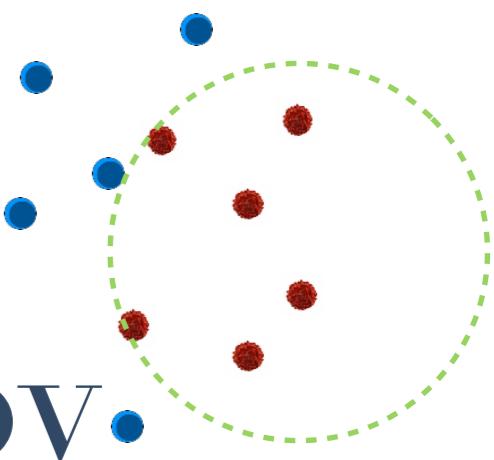
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- ❖ **Constraints:** Blue points should be outside, red points inside.
- ❖ **Problem:** Constraints are not linear... or are they?



# Linear Programming

## Application I: Cancer Therapy

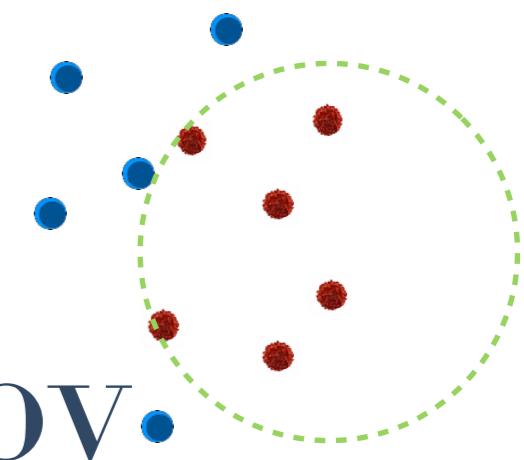
---



- ❖ **Problem:** We want to represent the property of being inside of a circle as a linear constraint.

# Linear Programming

## Application I: Cancer Therapy



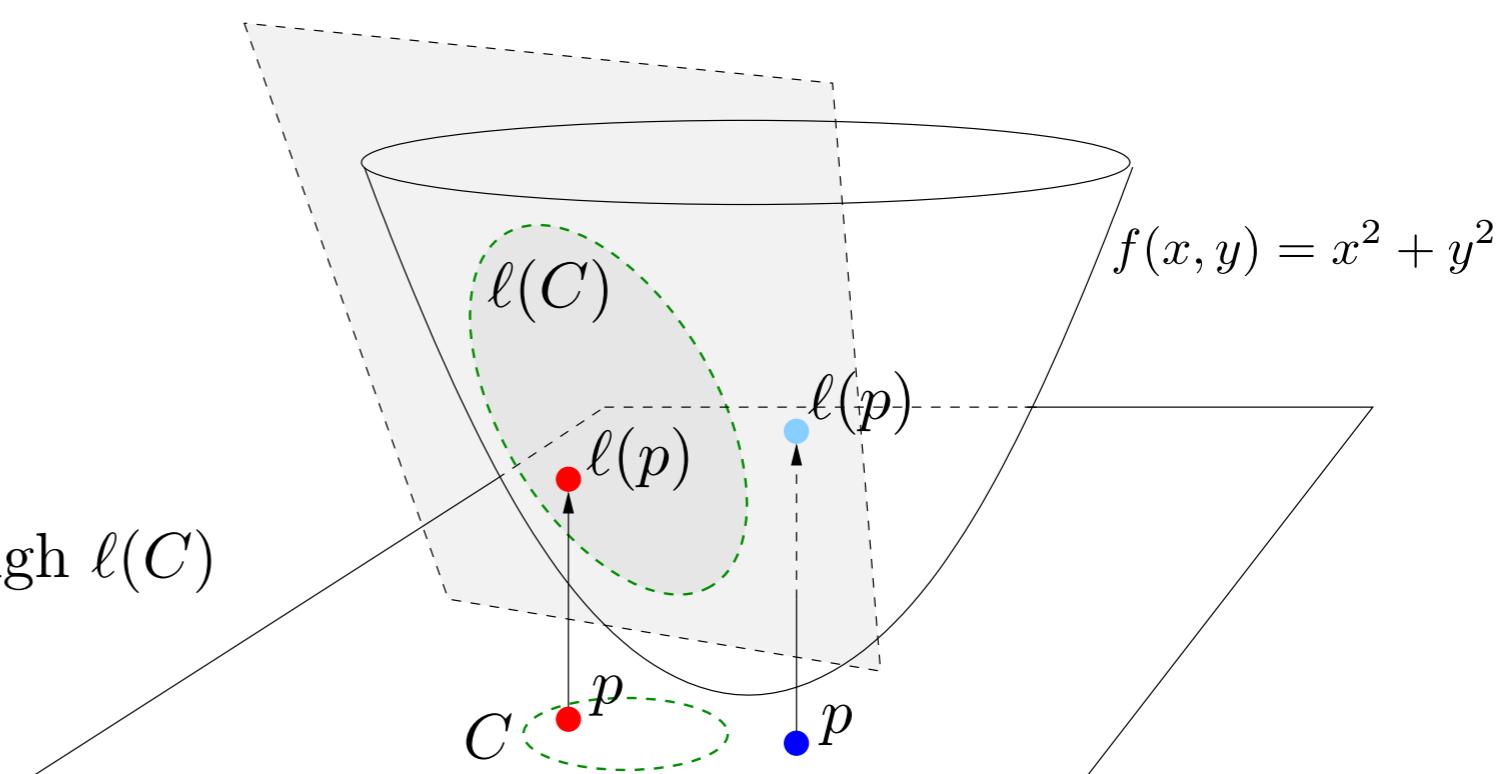
- ❖ **Problem:** We want to represent the property of being inside of a circle as a linear constraint.
- ❖ Apply *lifting map*  $\ell : (x, y) \mapsto (x, y, x^2 + y^2)$

$$p \left\{ \begin{array}{l} \text{inside} \\ \text{on} \\ \text{outside} \end{array} \right\} C$$

$\Updownarrow$

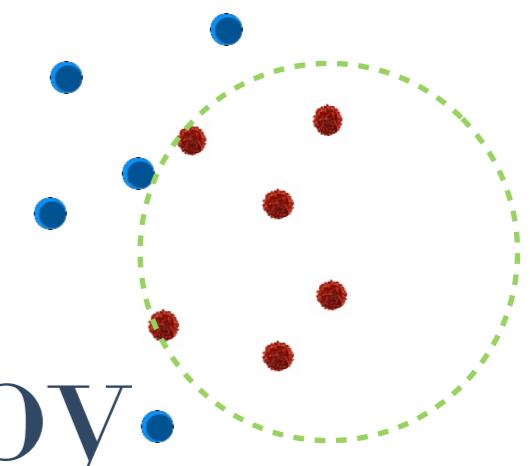
$$\ell(p) \left\{ \begin{array}{l} \text{below} \\ \text{on} \\ \text{above} \end{array} \right\}$$

the plane through  $\ell(C)$



# Linear Programming

## Application I: Cancer Therapy

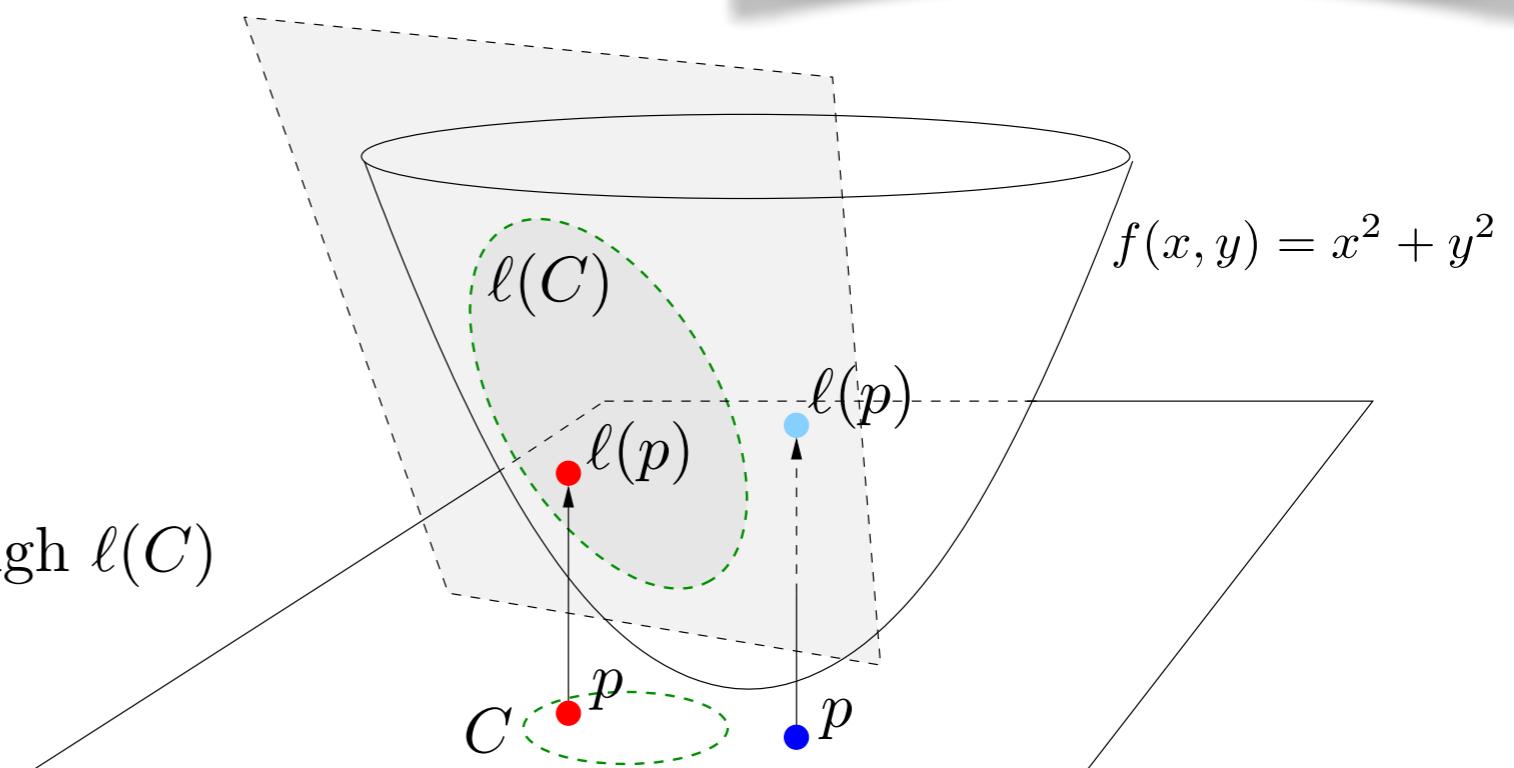


- ❖ **Problem:** We want to represent the property of being inside of a circle as a linear constraint.
- ❖ Apply *lifting map*  $\ell : (x, y) \mapsto (x, y, x^2 + y^2)$

Every circle has a corresponding plane

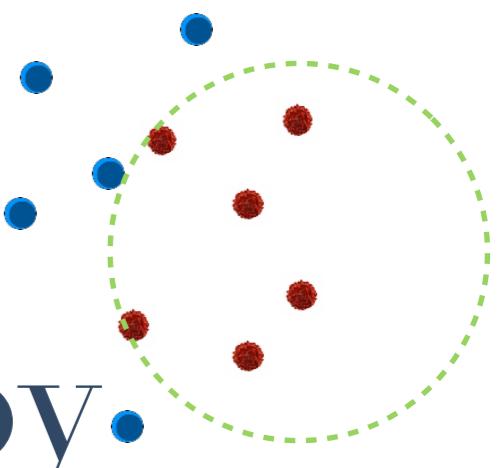
$$p \left\{ \begin{array}{l} \text{inside} \\ \text{on} \\ \text{outside} \end{array} \right\} C$$
$$\Updownarrow$$
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the plane through  $\ell(C)$

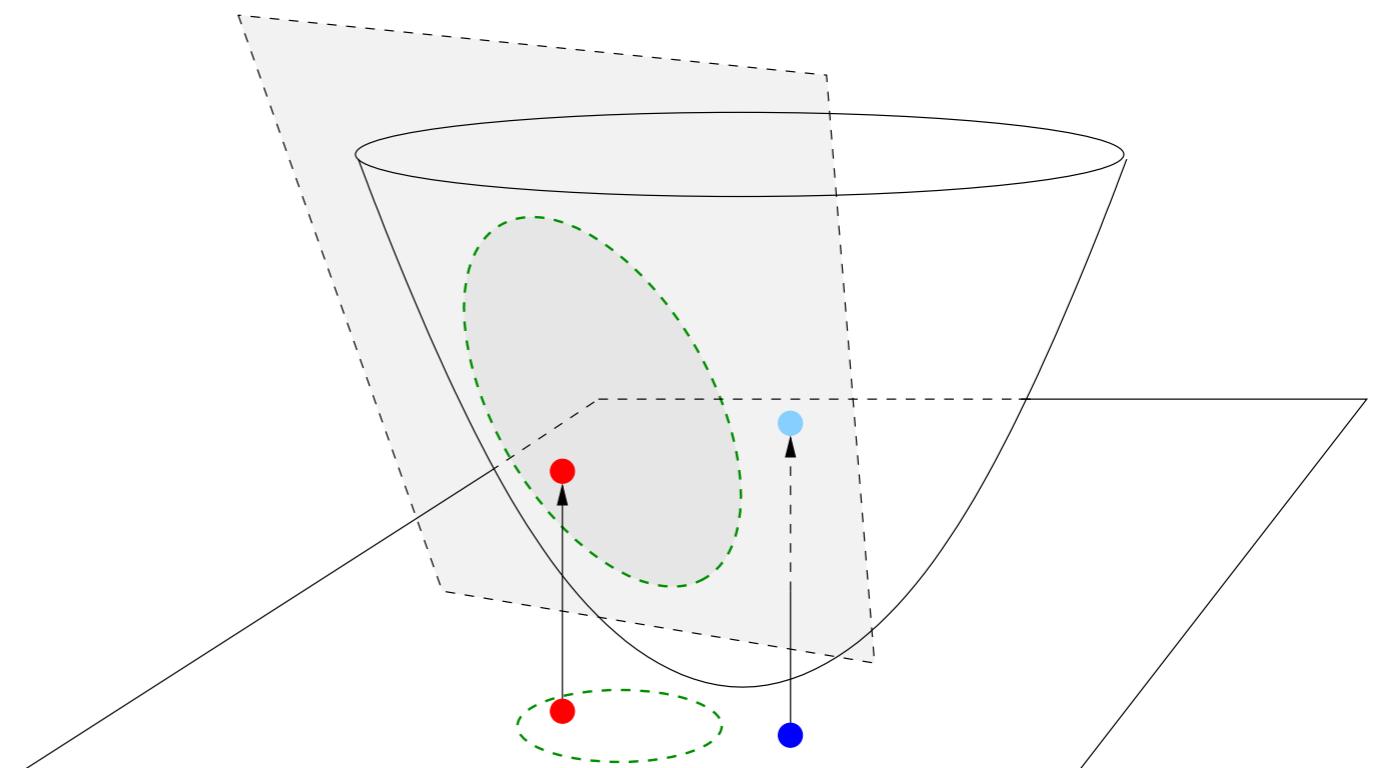


# Linear Programming

## Application I: Cancer Therapy



- ❖ **The geometric problem (lifted space):** Given the lifted sets  $R'$  and  $B'$  in space, is there a plane that has  $R'$  below/on it and  $B'$  above?



# Linear Programming

## Application I: Cancer Therapy

---

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- ❖ This can be solved with linear programming!

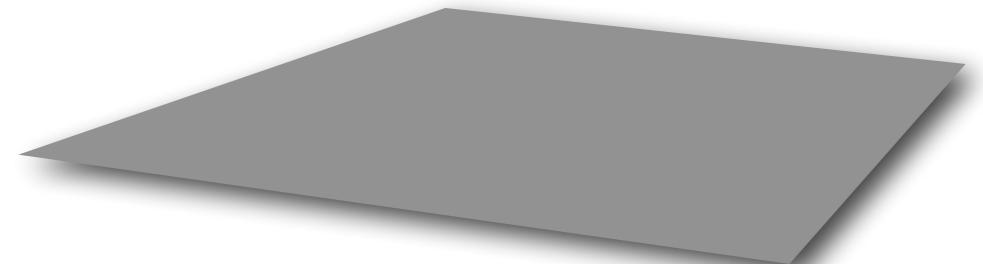
# Linear Programming

## Application I: Cancer Therapy

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$$\text{plane: } z = \alpha x + \beta y + \gamma$$

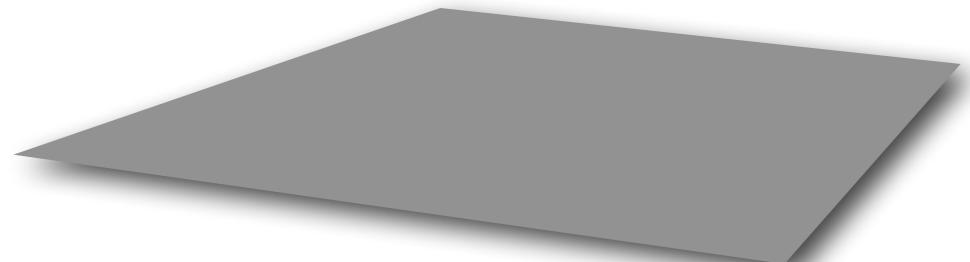


# Linear Programming

## Application I: Cancer Therapy

---

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- ⊕ This can be solved with linear programming!
- ⊕ Find  $\alpha, \beta, \gamma, \delta$  ( $\delta > 0$ ) such that... plane:  $z = \alpha x + \beta y + \gamma$

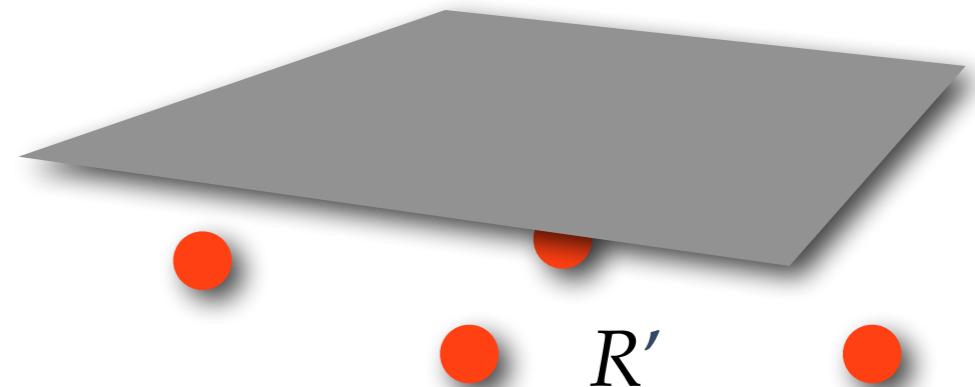


# Linear Programming

## Application I: Cancer Therapy

---

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$$x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R$$

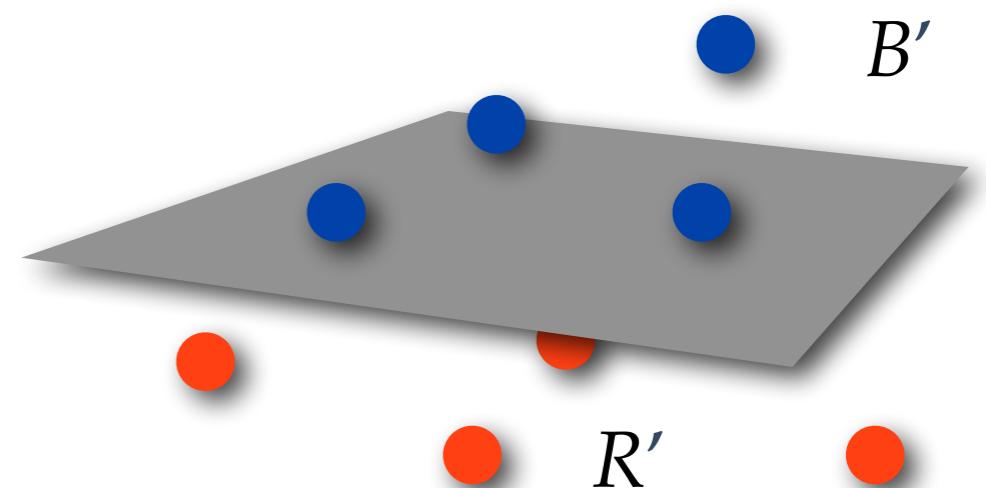
# Linear Programming

## Application I: Cancer Therapy

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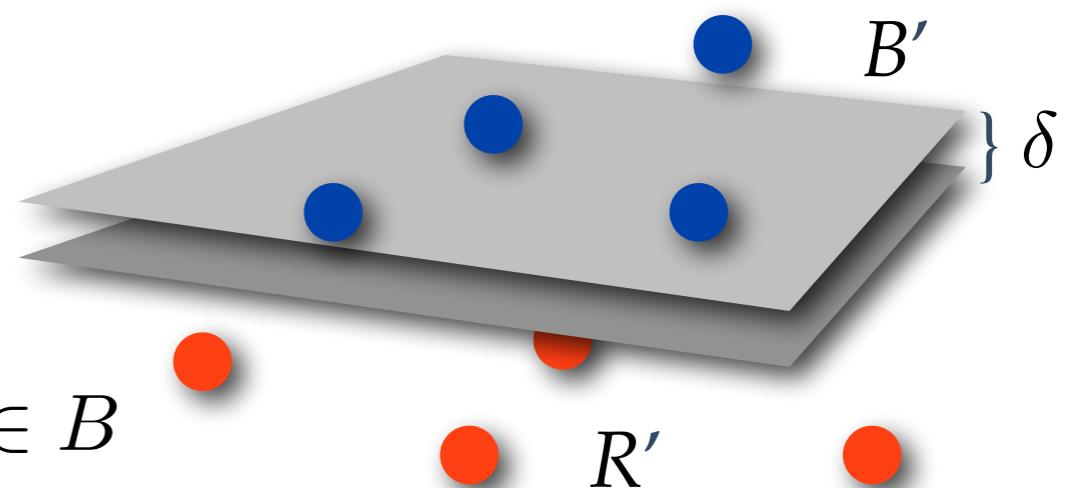
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# Linear Programming

## Application I: Cancer Therapy

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$$x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, \quad (\textcolor{red}{x}, \textcolor{red}{y}) \in B$$

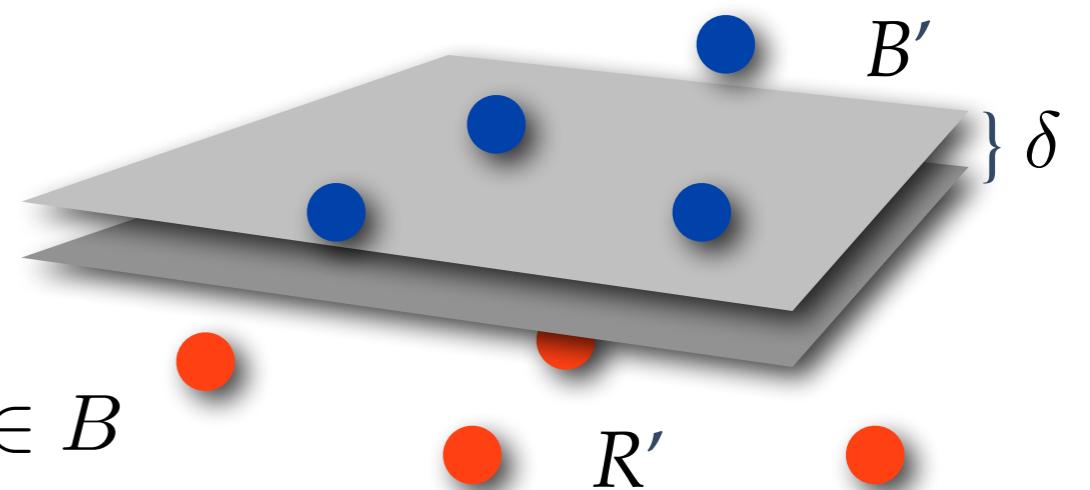
$$x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (\textcolor{red}{x}, \textcolor{red}{y}) \in R$$

# Linear Programming

## Application I: Cancer Therapy

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$$x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (\textcolor{red}{x}, \textcolor{red}{y}) \in R$$

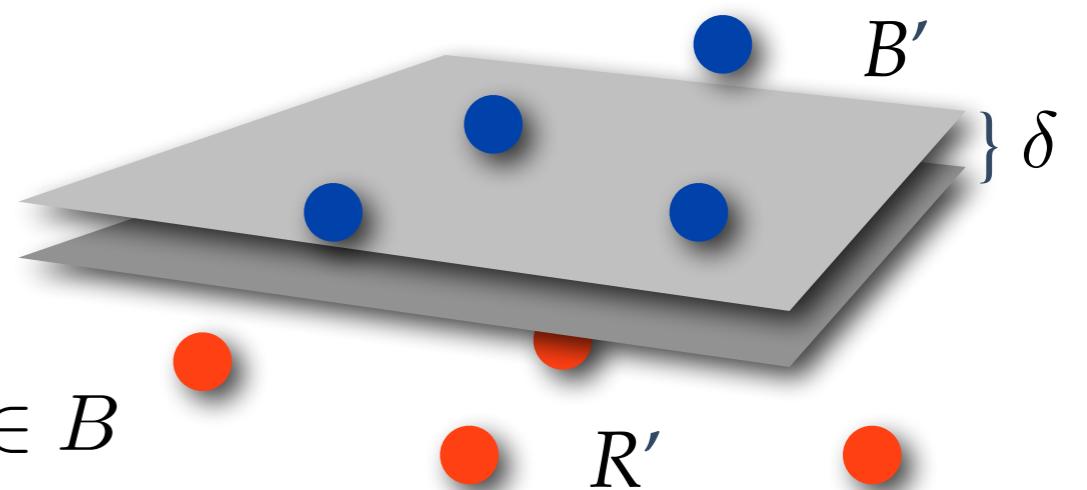
These are linear constraints!

# Linear Programming

## Application I: Cancer Therapy

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$$x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R$$

$$(3, 6) \in R$$

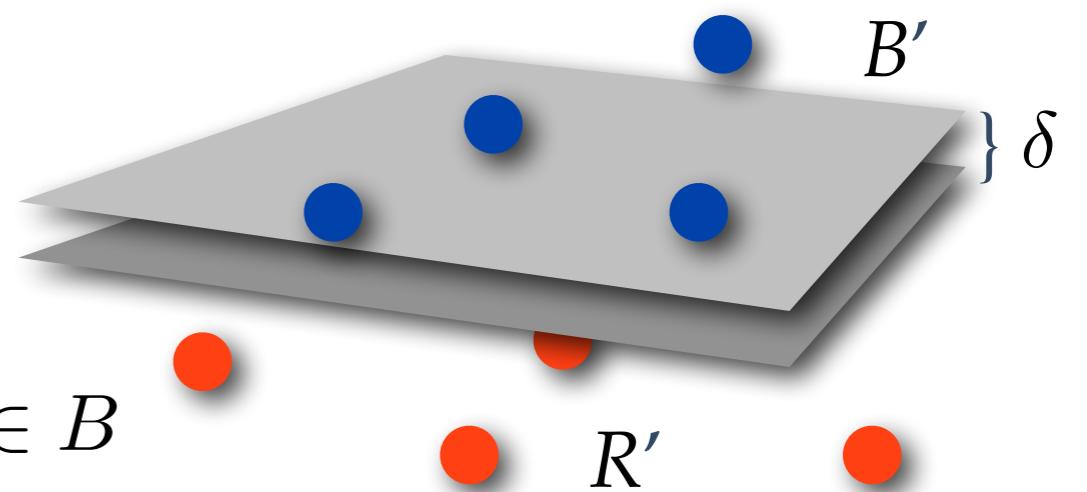
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# Linear Programming

## Application I: Cancer Therapy

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- Find  $\alpha, \beta, \gamma, \delta > 0$  such that...

$$\text{plane: } z = \alpha x + \beta y + \gamma$$



$$x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, \quad (x, y) \in B$$

$$x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R$$

$$(3, 6) \in R \longrightarrow 3^2 + 6^2 \leq 3\alpha + 6\beta + \gamma$$

These are linear constraints!

# Linear Programming

## Application I: Cancer Therapy

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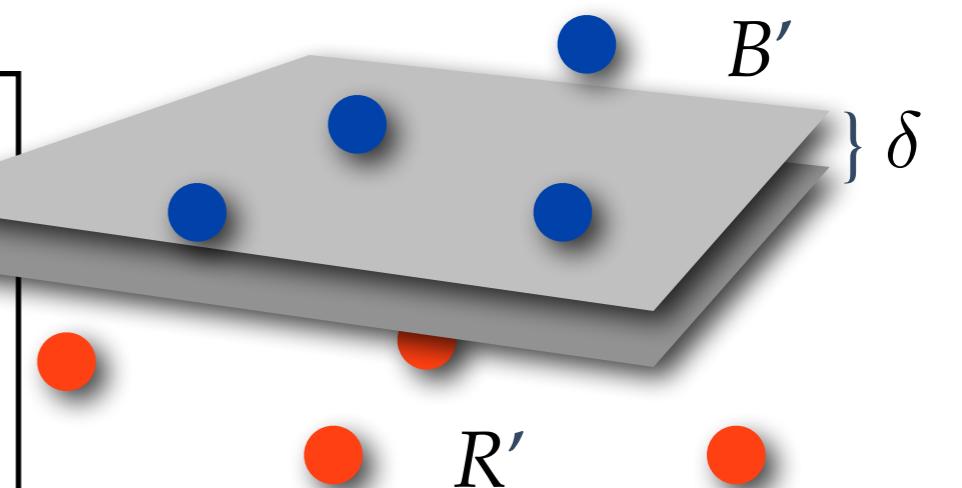
- This can be solved with linear programming!

- Find  $\alpha, \beta, \gamma, \delta > 0$  such that...

$$\begin{aligned} & \text{maximize } \delta \\ & \text{subject to} \\ & \quad x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, \quad (x, y) \in B \\ & \quad x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R \end{aligned}$$

4 variables!!  
 $|B| + |R|$  constraints

$$\text{plane: } z = \alpha x + \beta y + \gamma$$



# Linear Programming

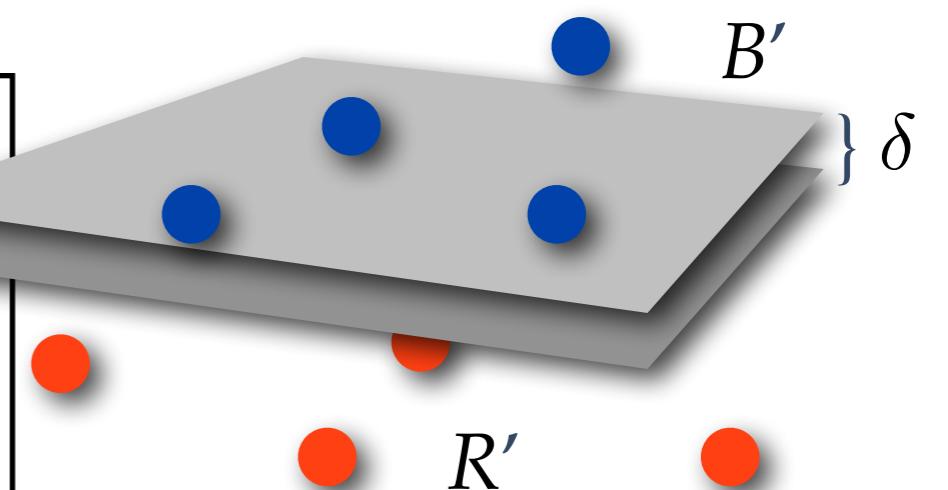
## Application I: Cancer Therapy

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- Find  $\alpha, \beta, \gamma, \delta > 0$  such that...

$$\begin{aligned} & \text{maximize } \delta \\ & \text{subject to} \\ & \quad x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, \quad (x, y) \in B' \\ & \quad x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R' \end{aligned}$$

**4 variables!!**  
 $|B'| + |R'|$  constraints

$$\text{plane: } z = \alpha x + \beta y + \gamma$$



# Linear Programming

## Application I: Cancer Therapy

---

- ❖ **Fact:** Exposure is possible if and only if the following linear program has positive value.

maximize  $\delta$

subject to

$$x^2 + y^2 \geq \alpha \textcolor{red}{x} + \beta \textcolor{red}{y} + \gamma + \delta, \quad (\textcolor{red}{x}, \textcolor{red}{y}) \in B$$

$$x^2 + y^2 \leq \alpha \textcolor{red}{x} + \beta \textcolor{red}{y} + \gamma, \quad (\textcolor{red}{x}, \textcolor{red}{y}) \in R$$

# Linear Programming

## Application I: Cancer Therapy

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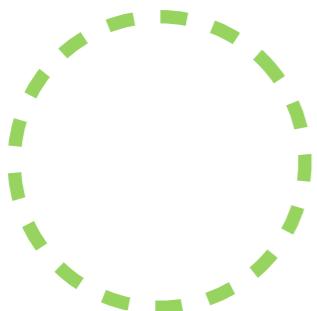
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- \* **Reconstructing the exposure from an optimal solution  $(\alpha, \beta, \gamma, \delta)$ :**



$$= \{(x, y) : x^2 + y^2 = \alpha x + \beta y + \gamma\}$$

# Linear Programming

## Application I: Cancer Therapy

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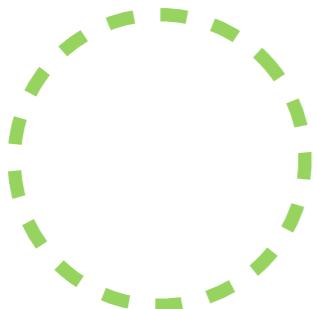
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subject to

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- \* **Reconstructing the exposure from an optimal solution  $(\alpha, \beta, \gamma, \delta)$ :**



$$= \{(x, y) : x^2 + y^2 = \alpha x + \beta y + \gamma\}$$

$$= \{(x, y) : (x - \frac{\alpha}{2})^2 + (y - \frac{\beta}{2})^2 = \gamma + \frac{\alpha^2}{4} + \frac{\beta^2}{4}\}$$

# Linear Programming

## Application I: Cancer Therapy

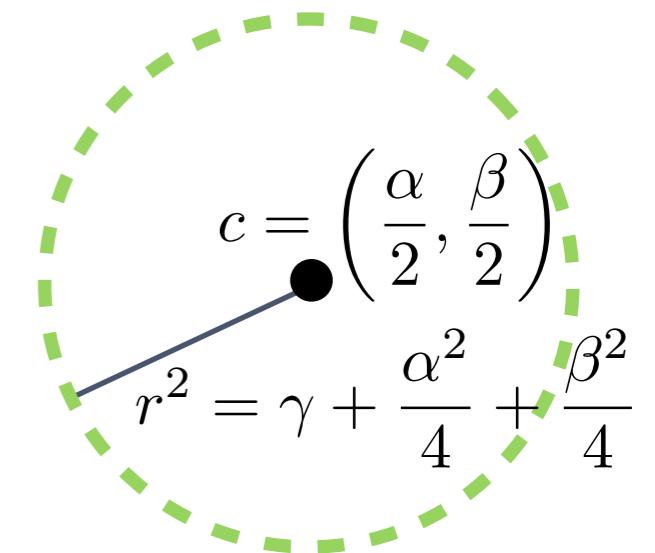
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subject to

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$$= \{(x, y) : (x - \frac{\alpha}{2})^2 + (y - \frac{\beta}{2})^2 = \gamma + \frac{\alpha^2}{4} + \frac{\beta^2}{4}\}$$

# Linear Programming

## Application I: Cancer Therapy

- \* **Implementation in CGAL:**

$$\begin{array}{lll} \text{minimize} & -\delta \\ \text{subject to} & x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, & (x, y) \in B \\ & x^2 + y^2 \leq \alpha x + \beta y + \gamma, & (x, y) \in R \\ & \delta \leq 1 \end{array}$$

Avoids unbounded program

maximize  $c^T x \rightarrow$  minimize  $-c^T x$  and negate resulting value

# Linear Programming

## Application I: Cancer Therapy

---

- Implementation in CGAL: Setup and Solve (Preamble as before)

```

int main() {
    // by default, we have an LP with Ax <= b and no bounds for
    // the four variables alpha, beta, gamma, delta
    Program lp (CGAL::SMALLER, false, 0, false, 0);
    const int alpha = 0;
    const int beta = 1;
    const int gamma = 2;
    const int delta = 3;

    // number of red and blue points
    int m; std::cin >> m;
    int n; std::cin >> n;

    // read the red points (cancer cells)
    for (int i=0; i<m; ++i) {
        int x; std::cin >> x;
        int y; std::cin >> y;
        // set up <= constraint for point inside/on circle:
        // -alpha x - beta y - gamma <= -x^2 - y^2
        lp.set_a (alpha, i, -x);
        lp.set_a (beta, i, -y);
        lp.set_a (gamma, i, -1);
        lp.set_b (i, -x*x - y*y);
    }
}

```

```

// read the blue points (healthy cells)
for (int j=0; j<n; ++j) {
    int x; std::cin >> x;
    int y; std::cin >> y;
    // set up <= constraint for point outside circle:
    // alpha x + beta y + gamma + delta <= x^2 + y^2
    lp.set_a (alpha, m+j, x);
    lp.set_a (beta, m+j, y);
    lp.set_a (gamma, m+j, 1);
    lp.set_a (delta, m+j, 1);
    lp.set_b (m+j, x*x + y*y);
}

// objective function: -delta (the solver minimizes)
lp.set_c(delta, -1);

// enforce a bounded problem:
lp.set_u (delta, true, 1);

// solve the program, using ET as the exact type
Solution s = CGAL::solve_linear_program(lp, ET());
assert (s.solves_linear_program(lp));

```

# Linear Programming

## Application I: Cancer Therapy

$$\begin{aligned}
 & \text{minimize} && -\delta \\
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 & && x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R \\
 & && \delta \leq 1
 \end{aligned}$$

- Implementation in CGAL: Setup and Solve (Preamble as before)

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int main() {
    // by default, we have an LP with Ax <= b and no bounds for
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    Program lp (CGAL::SMALLER, false, 0, false, 0);
    const int alpha = 0;
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    const int gamma = 2;
    const int delta = 3;

    // number of red and blue points
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    for (int i=0; i<m; ++i) {
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```

// read the blue points (healthy cells)
for (int j=0; j<n; ++j) {
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# Linear Programming

## Application I: Cancer Therapy

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# Linear Programming

## Application I: Cancer Therapy

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# Linear Programming

## Application I: Cancer Therapy

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## Application I: Cancer Therapy

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### \* Implementation in CGAL: Output

negate resulting value!

```
// output exposure center and radius, if they exist
if (s.is_optimal() && (s.objective_value() < 0)) {
    // *opt := alpha, *(opt+1) := beta, *(opt+2) := gamma
    CGAL::Quadratic_program_solution<ET>::Variable_value_iterator
        opt = s.variable_values_begin();
    CGAL::Quotient<ET> alpha_opt = *opt +alpha;// +0
    CGAL::Quotient<ET> beta_opt = *(opt+beta); // +1
    CGAL::Quotient<ET> gamma_opt = *(opt+gamma); // +2
    std::cout << "There is a valid exposure:\n";
    std::cout << " Center = (" // (alpha/2, beta/2)
        << alpha_opt/2 << ", " << beta_opt/2
        << ")\n";
    std::cout << " Squared Radius = " // gamma + alpha^2/4 + beta^2/4
        << gamma_opt + alpha_opt*alpha_opt/4 + beta_opt*beta_opt/4 << "\n";
} else
    std::cout << "There is no valid exposure.";
std::cout << "\n";
return 0;
}
```

# Linear Programming

## Application I: Cancer Therapy

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    CGAL::Quadratic_program_solution<ET>::Variable_value_iterator
        opt = s.variable_values_begin(); ← "Pointer" to first
    CGAL::Quotient<ET> alpha_opt = *opt +alpha;// +0 variable of optimal
    CGAL::Quotient<ET> beta_opt = *(opt+beta); // +1 solution
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    std::cout << "There is a valid exposure:\n";
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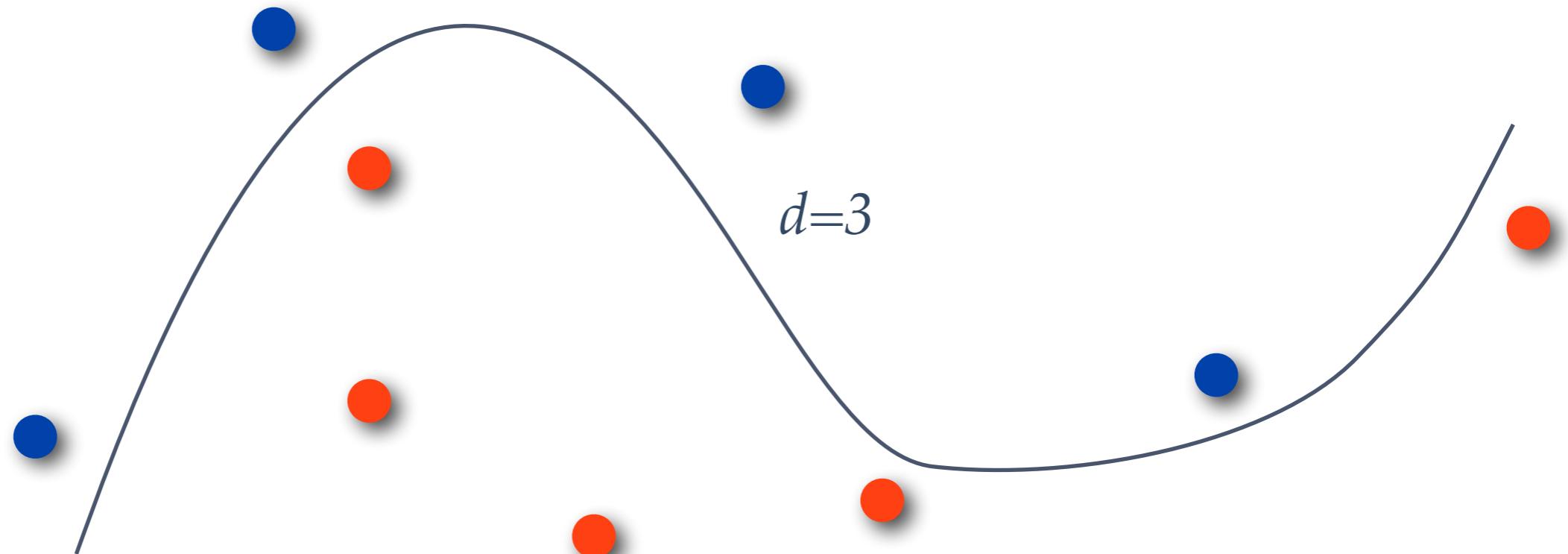
"Pointer" to first  
variable of optimal  
solution

The quotient  
**\* (opt+i)** is  
the value of the  
variable  $x_i$  in the  
optimal solution

# Linear Programming Beyond Cancer Therapy

---

- \* Given a set of R of red and a set B of blue points, can they be separated by the zero set of a polynomial of degree  $d$ ?



# Linear Programming Beyond Cancer Therapy

---

- Given a set of  $R$  of red and a set  $B$  of blue points, can they be separated by the zero set of a polynomial of degree  $d$ ?
- Polynomial of degree 3:

$$ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j$$

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- Linear programming formulation: find  $a,b,c,d,e,f,g,h,i,j$  such that
$$ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j \leq 0, \quad (x, y) \in B$$
$$ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j \geq 0, \quad (x, y) \in R$$

# Linear Programming Beyond Cancer Therapy

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$$ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j \geq 0, \quad (x, y) \in R$$
- This is linear separability in 9-dimensional space, under the generalized lifting map  $(x, y) \rightarrow (x^3, x^2y, xy^2, y^3, x^2, xy, y^2, x, y)$

# Linear Programming

## Further Applications

---

Not necessary for the class, but good to know

- ❖ Linear programming relaxations for hard combinatorial problems

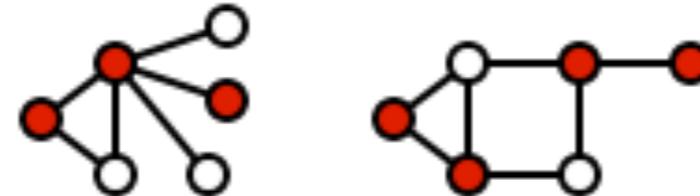
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- \* Linear programming relaxations for hard combinatorial problems
- \* **Vertex Cover:** Given a graph  $G=(V,E)$ , find a smallest subset of vertices (a vertex cover) such that every edge is incident to one vertex of the cover.



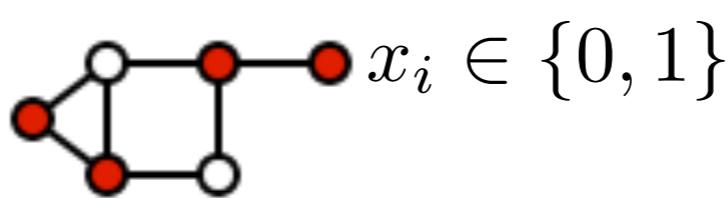
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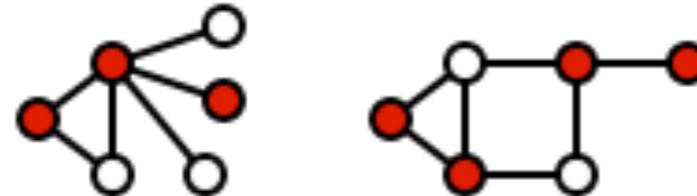
$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n x_i \\ \text{subject to} & x_i + x_j \geq 1 \quad \forall \{i, j\} \in E \\ & 0 \leq x_i \leq 1 \quad \forall i \in V \end{array}$$

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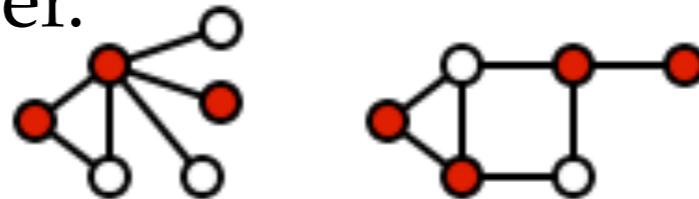
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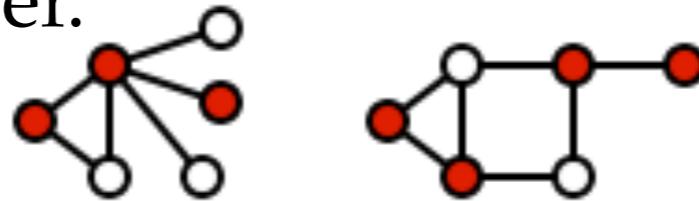
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- **Theorem:**  $C = \{i : x_i^* \geq 1/2\}$  is a vertex cover of size at most 2 opt.

[Ref]

# Linear vs. Integer Programming

---

- ❖ Often, applications lead to linear programs with the additional requirement of *integral solutions* (e.g. vertex cover)

# Linear vs. Integer Programming

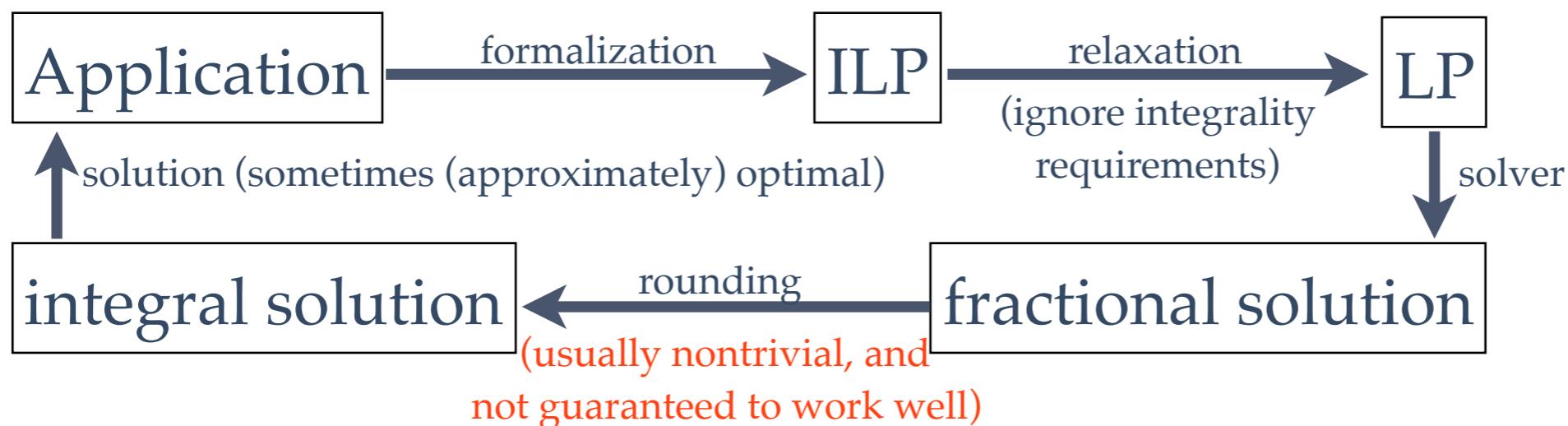
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# Linear vs. Integer Programming

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- Often, applications lead to linear programs with the additional requirement of *integral solutions* (e.g. vertex cover)
- Such programs are called *integer linear programs* (ILP) and are in general much harder to solve than linear programs (NP-hard)
- Typical approach (e.g. vertex cover):



# Quadratic Programming (QP)

---

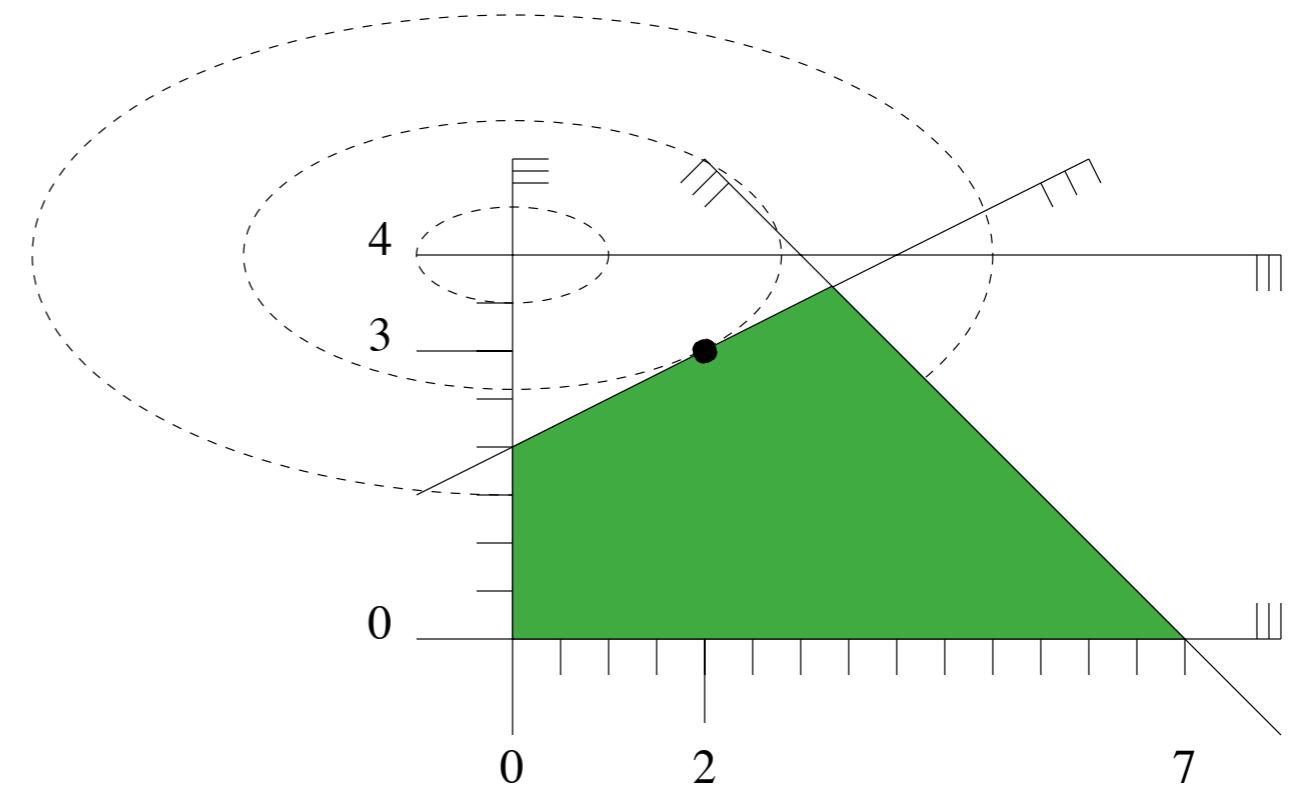
- ❖ **Problem:** Minimize a **convex** quadratic function in  $n$  variables subject to  $m$  linear (in)equality constraints!

# Quadratic Programming

---

- ❖ **Problem:** Minimize a **convex** quadratic function in  $n$  variables subject to  $m$  linear (in)equality constraints!
- ❖ **Example** ( $n=2$ ,  $m=5$ ):

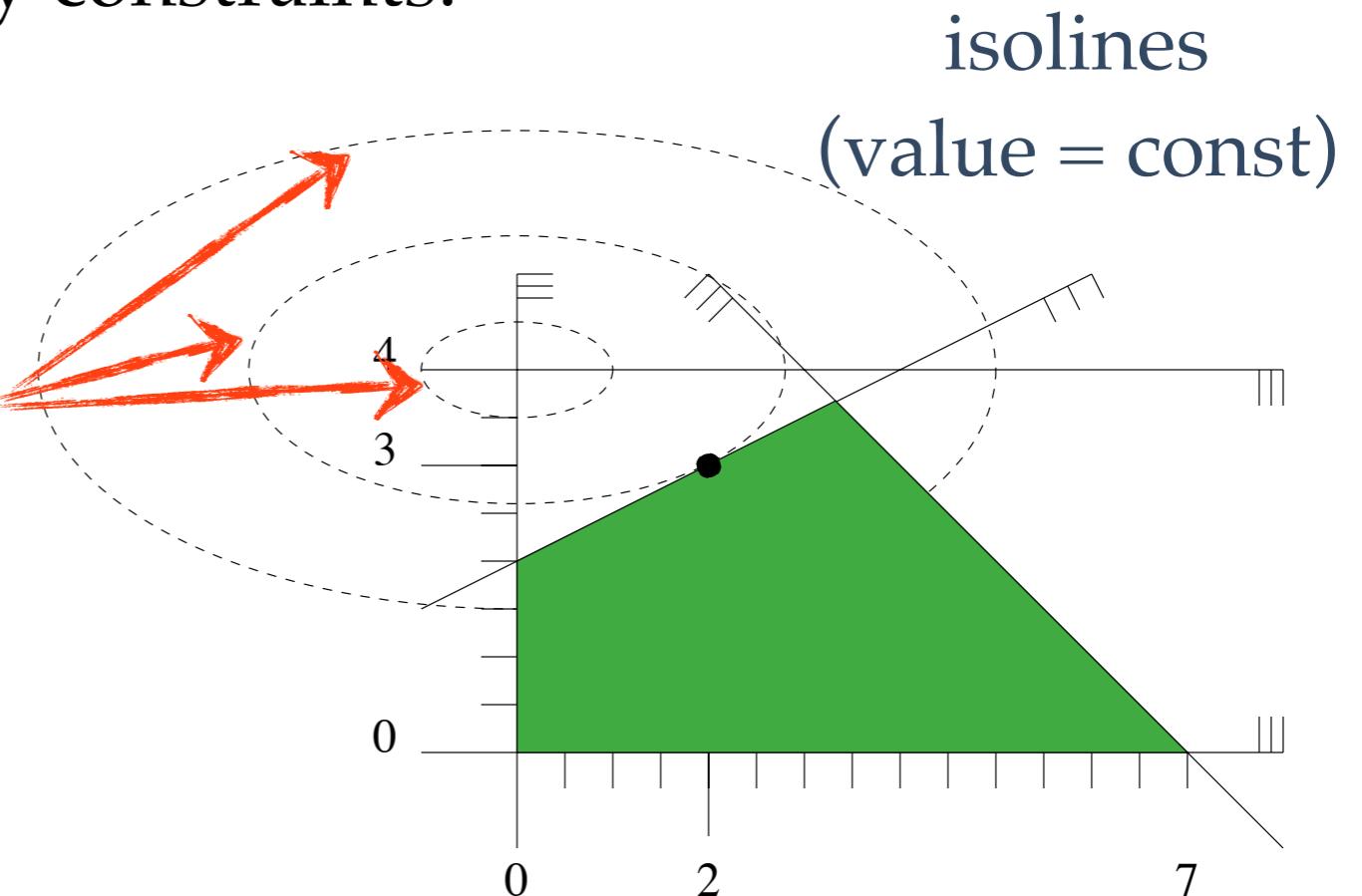
$$\begin{array}{ll} \text{minimize} & x^2 + 4y^2 - 32y + 64 \\ \text{subject to} & \begin{array}{lcl} x + y & \leq & 7 \\ -x + 2y & \leq & 4 \\ x & \geq & 0 \\ y & \geq & 0 \\ y & \leq & 4 \end{array} \end{array}$$



# Quadratic Programming

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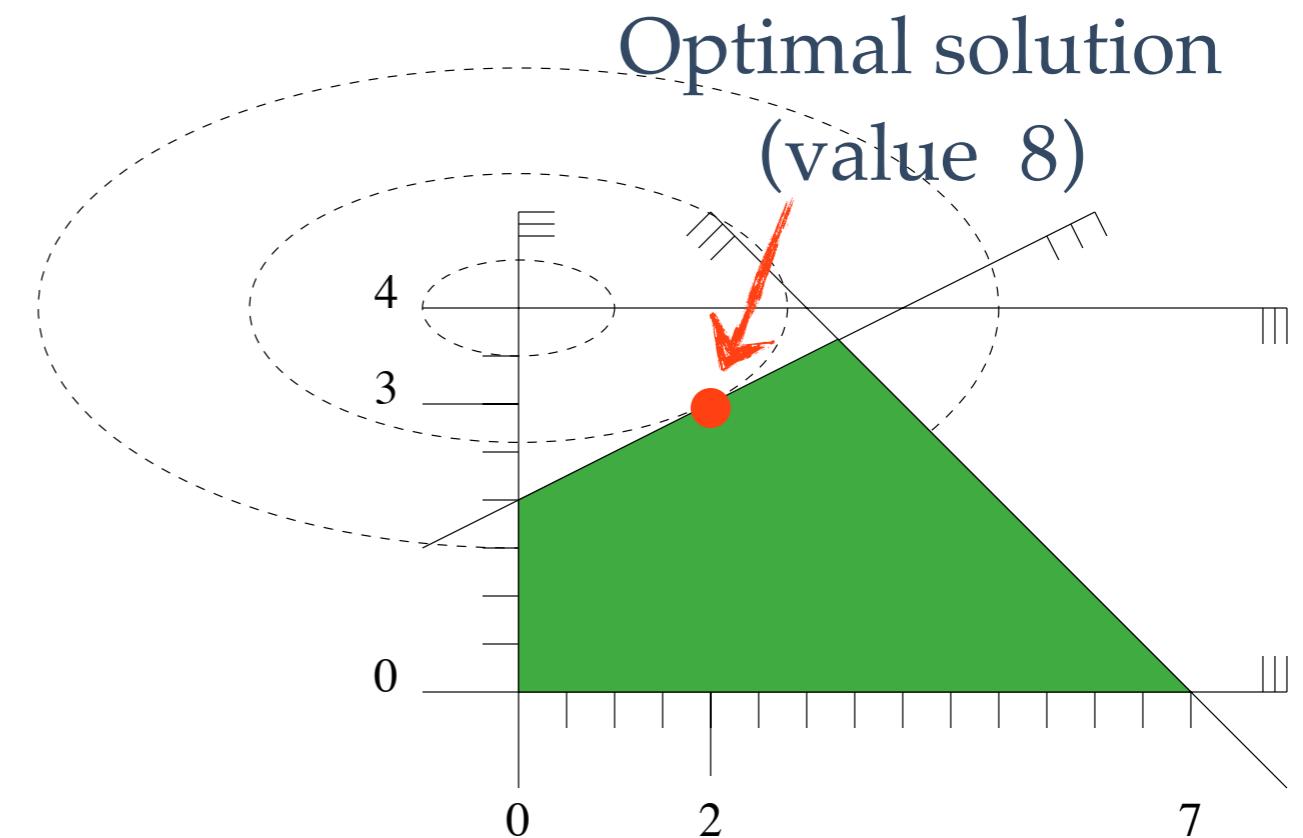


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# Quadratic Programming ... in CGAL

---

- \* **General form of QP in CGAL:**

$$\begin{array}{ll}\text{minimize} & x^T D x + c^T x + c_0 \\ \text{subject to} & Ax \gtrless b \\ & l \leq x \leq u\end{array}$$

$(D \in \mathbb{R}^{n \times n}$  symmetric positive semidefinite)

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The CGAL solver might in this case return solutions that are not optimal, or it might crash.
- \* **Relax:** In the applications, we know from theory that  $D$  is “good”

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( $D \in \mathbb{R}^{n \times n}$  symmetric positive semidefinite)

- \* Example:

$$\begin{array}{ll}\text{minimize} & x^2 + 4y^2 - 32y + 64 \\ \text{subject to} & \begin{array}{lcl}x + y & \leq & 7 \\ -x + 2y & \leq & 4 \\ x & \geq & 0 \\ y & \geq & 0 \\ y & \leq & 4\end{array} \\ & \Rightarrow & D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \checkmark\end{array}$$

# Quadratic Programming Application: Low-Risk Investment

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- ❖ **Risk-tolerant strategy:** Minimize the risk under a given lower bound for the expected return!

# Quadratic Programming Application: Low-Risk Investment

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  - \*  $1, 2, \dots, n$  (e.g. 1 = Swatch shares, 2 = Credit Suisse shares,...)

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# Quadratic Programming Application: Low-Risk Investment

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- ❖ **Example:**  $n=2$

	$r_i$
Swatch shares	10% (0.1)
Credit Suisse shares	51% (0.51)

$v_{ij}$	Swatch shares	Credit Suisse shares
Swatch shares	0.09	-0.05
Credit Suisse shares	-0.05	0.25

# Quadratic Programming Application: Low-Risk Investment

- Example:  $n=2$

	$r_i$		
Swatch shares	10% (0.1)	Swatch shares	Credit Suisse shares
Credit Suisse shares	51% (0.51)	Credit Suisse shares	0.25

Negative correlation: if CS does worse than expected,  
Swatch will probably do better, and vice versa

# Quadratic Programming Application: Low-Risk Investment

- Example:  $n=2$

	$r_i$		
Swatch shares	10% (0.1)	$v_{ij}$	Swatch shares
Credit Suisse shares	51% (0.51)	Credit Suisse shares	-0.05



Read as: standard deviation of return rate is  $\sqrt{0.25} = 0.5$   
(actual return rate could easily be off by 0.5)

# Quadratic Programming Application: Low-Risk Investment

---

- ❖ **Investment strategy:**

$$(x_1, x_2, \dots, x_n), \quad \sum_{i=1}^n x_i = 1, \quad x_i \geq 0 \forall i$$

Meaning: An  $x_i$  fraction of your money goes into investment  $i$

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- \* **Example:** half the money in Swatch shares, half in Credit Suisse shares; expected return rate is

$$\frac{1}{2} \cdot 0.1 + \frac{1}{2} \cdot 0.51 = 0.305 = 30.5\%$$

# Quadratic Programming Application: Low-Risk Investment

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$$\text{Var}\left[\sum_{i=1}^n x_i R_i\right] = \sum_{i=1}^n \sum_{j=1}^n v_{ij} x_i x_j = x^T D x$$

Straightforward calculations

$D = (v_{ij})_{1 \leq i, j \leq n}$  is the *covariance matrix*

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- \* **Example:** half-half Swatch/CS has risk  $\frac{0.09 - 2 \cdot 0.05 + 0.25}{4} = 0.06$

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Straightforward calculations

less than each individual risk!

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# Quadratic Programming Application: Low-Risk Investment

---

- \* **The risk-tolerant case:** Find the investment strategy with lowest risk that guarantees expected return rate  $\rho$  at least!

$$\begin{aligned} & \text{minimize} && \sum_{i=1}^n \sum_{j=1}^n v_{ij} x_i x_j \leftarrow \boxed{\text{risk}} \\ & \text{subject to} && \sum_{i=1}^n r_i x_i \geq \rho \\ & && \sum_{i=1}^n x_i = 1 \quad \boxed{\text{expected return rate}} \\ & && x_i \geq 0, \quad i = 1, \dots, n \\ & && \boxed{\text{strategy}} \end{aligned}$$

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**Fact:** A covariance matrix is positive semidefinite, so this is indeed a convex QP.

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**Fact:** A covariance matrix is positive semidefinite, so this is indeed a convex QP.

- \* **Example:**  $\rho = 0.4$ : 26.8% Swatch, 73.2% Credit Suisse; risk = 0.121

# Low-Risk Investment Example... in CGAL

$$\begin{aligned} & \text{minimize} && \sum_{i=1}^n \sum_{j=1}^n v_{ij} x_i x_j = x^T D x \\ & \text{subject to} && \sum_{i=1}^n r_i x_i \geq \rho \\ & && \sum_{i=1}^n x_i = 1 \\ & && x_i \geq 0, \quad i = 1, \dots, n \end{aligned}$$

- ❖ **Preamble:** This time, the input is rational...

Gnu  
Multi-  
precision  
Library  
(GMP)

```
#include <iostream>
#include <cassert>
#include <CGAL/basic.h>
#include <CGAL/QP_models.h>
#include <CGAL/QP_functions.h>
#include <CGAL/Gmpq.h>

// choose exact Rational type
typedef CGAL::Gmpq ET;

// program and solution types
typedef CGAL::Quadratic_program<ET> Program;
typedef CGAL::Quadratic_program_solution<ET> Solution;
```

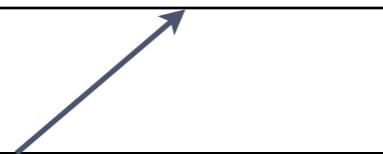
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- \* **Input:** Desired expected return

```
int main() {
    // read minimum expected return rate
    std::cout << "What is your desired expected return rate? ";
    double rho; std::cin >> rho;
```

for example,  $0.4 \approx 40\%$



# Low-Risk Investment Example... in CGAL

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- ❖ **Setup:** Make sure to enter matrix 2D (customary in QP solvers)!

```
// by default, we have a nonnegative QP with Ax >= b
Program qp(CGAL::LARGER, true, 0, false, 0);

// now set the non-default entries:
const int sw = 0;
const int cs = 1;

// constraint on expected return: 0.1 sw + 0.51 cs >= rho
qp.set_a(sw, 0, ET(1)/10);
qp.set_a(cs, 0, ET(51)/100);
qp.set_b( 0, rho);

// strategy constraint: sw + cs = 1
qp.set_a(sw, 1, 1);
qp.set_a(cs, 1, 1);
qp.set_b( 1, 1);
qp.set_r( 1, CGAL::EQUAL); // override default >=

// objective function: 0.09 sw^2 - 0.05 sw cs - 0.05 cs sw + 0.25 cs^2
// we need to specify the entries of the symmetric matrix 2D, on and below the diagonal
qp.set_d(sw, sw, ET(18)/100); // 0.09 sw^2
qp.set_d(cs, sw, -ET(1)/10); // -0.05 cs sw
qp.set_d(cs, cs, ET(1)/2); // 0.25 cs^2
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```

!

$j \leq i$  in `set_d(i, j)`

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- ❖ **Solve:** ...as nonnegative quadratic program (a little faster)

```
// solve the program, using ET as the exact type
assert(qp.is_nonnegative());
Solution s = CGAL::solve_nonnegative_quadratic_program(qp, ET());
assert (s.solves_quadratic_program(qp));
```

Independent verifications.  
Only do something in Debug mode.

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- \* **Output:** query solution status; if feasible, output strategy / risk

```
// output
if (s.status() == CGAL::QP_INFEASIBLE) {
    std::cout << "Expected return rate " << rho << " cannot be achieved.\n";
} else {
    assert (s.status() == CGAL::QP_OPTIMAL);
    Solution::Variable_value_iterator opt = s.variable_values_begin();
    ET sw_ratio = opt->numerator() / opt->denominator();
    ET cs_ratio = (opt+1)->numerator() / (opt+1)->denominator();
    ET risk = s.objective_value().numerator() / s.objective_value().denominator();
    double sw_percent = ceil_to_double(100 * *opt);
    std::cout << "Minimum risk investment strategy:\n"
        << sw_ratio << " ~ " << sw_percent << "%" << " Swatch\n"
        << cs_ratio << " ~ " << 100-sw_percent << "%" << " CS\n"
        << "Risk = " << risk
        << " ~ ." << ceil_to_double(100 * s.objective_value())
        << "\n";
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if (s.status() == CGAL::QP_INFEASIBLE) {
    std::cout << "Expected return rate " << rho << " cannot be achieved.\n";
} else {
    assert (s.status() == CGAL::QP_OPTIMAL);
    Solution::Variable_value_iterator opt = s.variable_values_begin();
    ET sw_ratio = opt->numerator() / opt->denominator();
    ET cs_ratio = (opt+1)->numerator() / (opt+1)->denominator();
    ET risk = s.objective_value().numerator() / s.objective_value().denominator();
    double sw_percent = ceil_to_double(100 * *opt);
    std::cout << "Minimum risk investment strategy:\n"
        << sw_ratio << " ~ " << sw_percent << "%" << " Swatch\n"
        << cs_ratio << " ~ " << 100-sw_percent << "%" << " CS\n"
        << "Risk = " << risk
        << " ~ ." << ceil_to_double(100 * s.objective_value())
        << "\n";
}
```

# Low-Risk Investment Example... in CGAL

$$\begin{aligned}
 & \text{minimize} && \sum_{i=1}^n \sum_{j=1}^n v_{ij} x_i x_j = x^T D x \\
 & \text{subject to} && \sum_{i=1}^n r_i x_i \geq \rho \\
 & && \sum_{i=1}^n x_i = 1 \\
 & && x_i \geq 0, \quad i = 1, \dots, n
 \end{aligned}$$

- \* **Output:** query solution status; if feasible, output strategy / risk

```

// output
if (s.status() == CGAL::QP_INFEASIBLE) {
    std::cout << "Expected return rate " << rho << " cannot be achieved.\n";
} else {
    assert (s.status() == CGAL::QP_OPTIMAL);
    Solution::Variable_value_iterator opt = s.variable_values_begin();
    ET sw_ratio = opt->numerator() / opt->denominator();
    ET cs_ratio = (opt+1)->numerator() / (opt+1)->denominator();
    ET risk = s.objective_value().numerator() / s.objective_value().denominator();
    double sw_percent = ceil_to_double(100 * *opt);
    std::cout << "Minimum risk investment strategy:\n"
        << sw_ratio << " ~ " << sw_percent << "%" << " Swatch\n"
        << cs_ratio << " ~ " << 100-sw_percent << "%" << " CS\n"
        << "Risk = " << risk
        << " ~ ." << ceil_to_double(100 * s.objective_value())
        << "\n";
}

```

# Known Bug :=(

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- ❖ You can't reliably copy or assign instances of the class  
**CGAL::Quadratic\_program\_solution<ET>**
- ❖ **Workaround 1:** If you want to pass or return such instances to / from a function, pass a pointer to the instance instead!
- ❖ **Workaround 2:** If you want to assign a new solution to an existing instance... don't do it!

# Sources and Further Reading

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- ❖ **LP/QP Solver:** Online manual at [www.cgal.org](http://www.cgal.org): Online Manual  
→ Combinatorial Algorithms → Linear and Quadratic Programming Solver
- ❖ **Cancer Therapy:** J. O'Rourke, S. Kosaraju, and N. Megiddo:  
Computing Circular Separability, *Discrete & Computational Geometry* 1:105-113 (1986)
- ❖ **Low-Risk Investment:** H. Markowitz: Portfolio Selection, *Journal of Finance* 7(1): 77-91 (1952)