Problem #1

Given a set of *n*, 1-bit numbers, the *max* of those *n* numbers is equal to 0b1 if and only if **any** of those numbers are 0b1. In contrast, the *min* of the *n* 1-bit numbers is 0b1 if and only if **all** *n* numbers are 0b1. Hence, for *n* 1-bit numbers, the *max* and *min* can be written as:

$$max_1 = \bigvee_{i=1}^{n} V_i \qquad\qquad min_1 = \bigwedge_{i=1}^{n} V_i \tag{1}$$

where V_i is the *i*th 1-bit number in the set of *n* numbers.

As an extension of the equation (1) above, consider the case of a set of *k*-bit numbers (where $k > 1$). Since any integer is composed of a series of bits, determining the minimum and maximum of the *n* numbers can be done at the bit level. The most significant bit (MSB) of the max/min is calculated using equation (1); this is because MSB of the maximum/minimum only depends on the MSBs of the set of *n* numbers. For all subsequent bits, *j* (where *j* is a bit index defined as $1 < j \leq k$), extra caution must be shown because it is not enough to only look at the value of the *j*th bit of a number since bit value V_i is only relevant to a min/max operation if and only if all of its previous, more significant bits (i.e. $< j$ and greater than 0) are equivalent to the corresponding bits in the min/max number.

Example: Determine the maximum of a set of three, two-digit numbers { 0b00, 0b10, 0b01 }[1]

To determine the MSB of the maximum, you consider the most significant bit of all three words in the set, which are {0b0, 0b1, 0b0} respectively. Hence, the most significant bit (MSB) of max is equal to the OR of the MSBs of the three numbers in the set. When determining the value of the second bit (i.e. least significant in this case) for the maximum, one can only consider those numbers who preceding bits are equal to (i.e. same as) the previous, more significant bits in *max*. To check for equivalence, use the operator, ⇔, which is defined as:

$$(A \Leftrightarrow B) \equiv (A \wedge B) \vee (\bar{A} \wedge \bar{B}) \tag{2}$$

If one did not check equivalence and used the equation (1) as is, the *max* would erroneously be: 0b11 (with the MSB coming from word 0b10 and the LSB coming from word 0b01).

When one combines the requirement of checking preceding bit equivalence with equation (1), the complete Boolean expression to find the *j*th bit of the minimum and maximum of a set of *n* numbers is:

$$max_j = \bigvee_{i=1}^{n}\left(V_{i,j} \wedge \left(\bigwedge_{k=1}^{j-1}(max_k \Leftrightarrow V_{i,k})\right)\right), \text{for } j > 1 \qquad min_j = \bigwedge_{i=1}^{n}\left(V_{i,j} \vee \left(\bigvee_{k=1}^{j-1}\overline{(min_k \Leftrightarrow V_{i,k})}\right)\right), \text{for } j > 1 \tag{3}$$

where $V_{i,j}$ is the value of the *j*th bit of the *i*th number in the set of *n* numbers.

Minimax Tree Values

Minimax is a logical extension of equation (3). In this problem, each node in the tree has three successors. Depending on whether the node is in a *max* or *min* level, it applies the corresponding equation from (3) on its children to determine the respective value of each of its *k*-bits. For this problem, *k* (i.e. bits per word) is 3 since the numbers are 3-bits in length, and the size of each set of numbers, *n*, is 3 (i.e. the number of successors). The following are the bit

[1] The case of *min* is logical extension of the *max* example and is not shown. However, its equation is provided in (3).

equations for each of the values in the 4 ply minimax as defined in equation (3). For level 1, the bit equations are expressed in conjunctive normal form (CNF). However, since these CNF equations are not in terms of only intermediary values and leaf nodes (e.g. $V_{i,j,k,m,b}$), additional manipulation of the equations is required to complete the question as stated.

Max – Level 0 (Root of the Tree)

$$V_1 = \bigvee_{i=1}^{3} V_{i,1} \tag{4}$$

$$V_2 = \bigvee_{i=1}^{3} (V_1 \Leftrightarrow V_{i,1}) \wedge V_{i,2} \tag{5}$$

$$V_3 = \bigvee_{i=1}^{3} (V_1 \Leftrightarrow V_{i,1}) \wedge (V_2 \Leftrightarrow V_{i,2}) \wedge V_{i,3} \tag{6}$$

Min – Level 1 – Specifically for move m_1.

$$V_{1,1} = \bigwedge_{j=1}^{3} V_{1,j,1} \tag{7}$$

This is already in CNF.

$$V_{i,2} = \bigwedge_{j=1}^{3} \overline{(V_{1,1} \Leftrightarrow V_{1,j,1})} \vee V_{1,j,2}$$

This is converted to CNF via:

$$V_{i,2} = \bigwedge_{j=1}^{3} \left(\left(\overline{(V_{i,1} \wedge V_{i,j,1}) \vee (\overline{V_{i,1}} \wedge \overline{V_{i,j,1}})} \right) \vee V_{i,j,2} \right), \text{Biconditional Elimination (BE)}$$

$$V_{i,2} = \bigwedge_{j=1}^{3} \left(\left(\overline{(V_{i,1} \wedge V_{i,j,1})} \wedge \overline{(\overline{V_{i,1}} \wedge \overline{V_{i,j,1}})} \right) \vee V_{i,j,2} \right), \text{De Morgan's Law (DM)}$$

$$V_{i,2} = \bigwedge_{j=1}^{3} \left(\left((\overline{V_{i,1}} \vee \overline{V_{i,j,1}}) \wedge (V_{i,1} \vee V_{i,j,1}) \right) \vee V_{i,j,2} \right), \text{DM}$$

$$V_{1,2} = \bigwedge_{j=1}^{3} \left((\overline{V_{1,1}} \vee \overline{V_{1,j,1}} \vee V_{1,j,2}) \wedge (V_{1,1} \vee V_{1,j,1} \vee V_{1,j,2}) \right), \text{Distributivity of} \vee \text{over} \wedge \text{(CNF)} \tag{8}$$

$$V_{i,3} = \bigwedge_{j=1}^{3} \overline{(V_{i,1} \Leftrightarrow V_{i,j,1})} \vee \overline{(V_{i,2} \Leftrightarrow V_{i,j,2})} \vee V_{i,j,3}$$

This is converted to CNF via:

$$V_{i,3} = \bigwedge_{j=1}^{3} \left(\left(\overline{(V_{i,1} \wedge V_{i,j,1}) \vee (\overline{V_{i,1}} \wedge \overline{V_{i,j,1}})} \right) \vee \left(\overline{(V_{i,2} \wedge V_{i,j,2}) \vee (\overline{V_{i,2}} \wedge \overline{V_{i,j,2}})} \right) \vee V_{i,j,3} \right), \text{BE}$$

$$V_{i,2} = \bigwedge_{j=1}^{3} \left(\left(\overline{(V_{i,1} \wedge V_{i,j,1})} \wedge \overline{(\overline{V_{i,1}} \wedge \overline{V_{i,j,1}})} \right) \vee \left(\overline{(V_{i,2} \wedge V_{i,j,2})} \wedge \overline{(\overline{V_{i,1}} \wedge \overline{V_{i,j,2}})} \right) \vee V_{i,j,3} \right), \text{DM}$$

$$V_{i,3} = \bigwedge_{j=1}^{3} \left(\left((\overline{V_{i,1}} \vee \overline{V_{i,j,1}}) \wedge (V_{i,1} \vee V_{i,j,1}) \right) \vee \left((\overline{V_{i,1}} \vee \overline{V_{i,j,2}}) \wedge (V_{i,2} \vee V_{i,j,2}) \right) \vee V_{i,j,3} \right), \text{DM}$$

$$V_{1,3} = \bigwedge_{j=1}^{3} \left((V_{i,1} \wedge \overline{V_{i,j,1}}) \vee (\overline{V_{i,1}} \vee V_{i,j,1}) \vee (V_{i,2} \wedge \overline{V_{i,j,2}}) \vee (\overline{V_{i,1}} \vee V_{i,j,2}) \vee V_{i,j,3} \right) \text{ Distributivity} \tag{9}$$

$$V_{1,3} = \bigwedge_{j=1}^{3}\left(\neg\neg\left(\left(V_{i,1} \wedge \overline{V_{i,j,1}}\right) \vee \left(\overline{V_{i,1}} \wedge V_{i,j,1}\right) \vee \left(V_{i,2} \wedge \overline{V_{i,j,2}}\right) \vee \left(\overline{V_{i,1}} \wedge V_{i,j,2}\right) \vee V_{i,j,3}\right)\right) \text{ Double Negation}$$

$$V_{1,3} = \bigwedge_{j=1}^{3}\left(\neg\left(\left(\overline{V_{i,1}} \vee V_{i,j,1}\right) \wedge \left(V_{i,1} \vee \overline{V_{i,j,1}}\right) \wedge \left(\overline{V_{i,2}} \vee V_{i,j,2}\right) \wedge \left(V_{i,1} \vee \overline{V_{i,j,2}}\right) \wedge \overline{V_{i,j,3}}\right)\right), \text{DM}$$

$$\begin{aligned}V_{1,3} = \bigwedge_{j=1}^{3}\Bigg(\neg\Big(&\left(V_{i,1} \wedge V_{i,j,1} \wedge V_{i,2} \wedge V_{i,j,2} \wedge \overline{V_{i,j,3}}\right) \vee \left(V_{i,1} \wedge V_{i,j,1} \wedge \overline{V_{i,2}} \wedge \overline{V_{i,j,2}} \wedge \overline{V_{i,j,3}}\right) \\ &\vee \left(\overline{V_{i,1}} \wedge \overline{V_{i,j,1}} \wedge V_{i,2} \wedge V_{i,j,2} \wedge \overline{V_{i,j,3}}\right) \vee \left(\overline{V_{i,1}} \wedge \overline{V_{i,j,1}} \wedge \overline{V_{i,2}} \wedge \overline{V_{i,j,2}} \wedge \overline{V_{i,j,3}}\right)\Big)\Bigg), \text{Distributivity}\end{aligned}$$

$$\begin{aligned}V_{1,3} = \bigwedge_{j=1}^{3}\Bigg(\Big(&\left(\overline{V_{1,1}} \vee \overline{V_{1,j,1}} \vee \overline{V_{1,2}} \vee \overline{V_{1,j,2}} \vee V_{1,j,3}\right) \wedge \left(\overline{V_{1,1}} \vee \overline{V_{1,j,1}} \vee V_{1,2} \vee V_{1,j,2} \vee V_{1,j,3}\right) \\ &\wedge \left(V_{1,1} \vee V_{1,j,1} \vee \overline{V_{1,2}} \vee \overline{V_{1,j,2}} \vee V_{1,j,3}\right) \wedge \left(V_{1,1} \vee V_{1,j,1} \vee V_{1,2} \vee V_{1,j,2} \vee V_{1,j,3}\right)\Big)\Bigg), \text{DM } \mathbf{(CNF)}\end{aligned}$$

Max – Level 2

$V_{i,j,1} = \bigvee_{k=1}^{3} V_{i,j,k,1}$	(10)
$V_{i,j,2} = \bigvee_{k=1}^{3} \left(V_{i,j,1} \Leftrightarrow V_{i,j,k,1}\right) \wedge V_{i,j,k,2}$	(11)
$V_{i,j,3} = \bigvee_{k=1}^{3} \left(V_{i,j,1} \Leftrightarrow V_{i,j,k,1}\right) \wedge \left(V_{i,j,2} \Leftrightarrow V_{i,j,k,2}\right) \wedge V_{i,j,k,3}$	(12)

Min – Level 3

$V_{i,j,k,1} = \bigwedge_{m=1}^{3} V_{i,j,k,m,1}$	(13)
$V_{i,j,k,2} = \bigwedge_{m=1}^{3} \overline{\left(V_{i,j,k,1} \Leftrightarrow V_{i,j,k,m,1}\right)} \vee V_{i,j,k,m,2}$	(14)
$V_{i,j,k,3} = \bigwedge_{m=1}^{3} \overline{\left(V_{i,j,k,1} \Leftrightarrow V_{i,j,k,m,1}\right)} \vee \overline{\left(V_{i,j,k,2} \Leftrightarrow V_{i,j,k,m,2}\right)} \vee V_{i,j,k,m,3}$	(15)

Non-Tree Intermediary Variables

To simplify the solutions for $V_{1,2,}$ and $V_{1,3}$, three intermediary relations that are not part of the tree will be used. They are shown in equations (16), (17), and (18). Non-tree intermediary relations are used to simplify Boolean expressions and reduce growth in the resulting Boolean expressions.

$\alpha \Leftrightarrow (\beta \Leftrightarrow \gamma)$ $\left(\bar{\alpha} \vee (\beta \Leftrightarrow \gamma)\right) \wedge \left(\alpha \vee \overline{(\beta \Leftrightarrow \gamma)}\right)$, Biconditional Elimination (BE) $\left(\bar{\alpha} \vee \left((\bar{\beta} \vee \gamma) \wedge (\beta \vee \bar{\gamma})\right)\right) \wedge \left(\alpha \vee \overline{\left((\beta \wedge \gamma) \vee (\bar{\beta} \wedge \bar{\gamma})\right)}\right)$, BE	(16)

$$\left(\bar{\alpha} \lor \left((\bar{\beta} \lor \gamma) \land (\beta \lor \bar{\gamma})\right)\right) \land \left(\alpha \lor \left((\beta \lor \gamma) \lor (\bar{\beta} \lor \bar{\gamma})\right)\right), \text{De Morgan's (DM)}$$

$$(\bar{\alpha} \lor \bar{\beta} \lor \gamma) \land (\bar{\alpha} \lor \beta \lor \bar{\gamma}) \land (\alpha \lor \beta \lor \gamma) \lor (\alpha \lor \bar{\beta} \lor \bar{\gamma}), \text{Distribution}$$

$$\alpha \Leftrightarrow (\beta \Leftrightarrow \gamma) \land (\delta \Leftrightarrow \epsilon)$$

$$\left(\bar{\alpha} \lor ((\beta \Leftrightarrow \gamma) \land (\delta \Leftrightarrow \epsilon))\right) \land \left(\alpha \lor \overline{((\beta \Leftrightarrow \gamma) \land (\delta \Leftrightarrow \epsilon))}\right), \text{BE}$$

$$\left(\bar{\alpha} \lor \left(\left((\bar{\beta} \lor \gamma) \land (\beta \lor \bar{\gamma})\right) \land \left((\bar{\delta} \lor \epsilon) \land (\delta \lor \bar{\epsilon})\right)\right)\right) \land \left(\alpha \lor \overline{\left(\left((\bar{\beta} \lor \gamma) \land (\beta \lor \bar{\gamma})\right) \land \left((\bar{\delta} \lor \epsilon) \land (\delta \lor \bar{\epsilon})\right)\right)}\right), \text{BE} \qquad (17)$$

$$(\bar{\alpha} \lor \bar{\beta} \lor \gamma) \land (\bar{\alpha} \lor \beta \lor \bar{\gamma}) \land (\bar{\alpha} \lor \bar{\delta} \lor \epsilon) \land (\bar{\alpha} \lor \delta \lor \bar{\epsilon}) \land (\alpha \lor \bar{\beta} \lor \bar{\gamma} \lor \bar{\delta} \lor \bar{\epsilon}) \land (\alpha \lor \bar{\beta} \lor \bar{\gamma} \lor \delta \lor \epsilon)$$
$$\land (\alpha \lor \beta \lor \gamma \lor \bar{\delta} \lor \bar{\epsilon}) \land (\alpha \lor \beta \lor \gamma \lor \delta \lor \epsilon), \text{Distributivity and DM}$$

$$\alpha \Leftrightarrow \overline{(\beta \Leftrightarrow \gamma)} \lor \overline{(\delta \Leftrightarrow \epsilon)}$$

$$\left(\bar{\alpha} \lor (\overline{(\beta \Leftrightarrow \gamma)} \lor \overline{(\delta \Leftrightarrow \epsilon)})\right) \land \left(\alpha \lor \overline{(\overline{(\beta \Leftrightarrow \gamma)} \lor \overline{(\delta \Leftrightarrow \epsilon)})}\right), \text{BE}$$

$$\left(\bar{\alpha} \lor \left(\overline{\left((\bar{\beta} \lor \gamma) \land (\beta \lor \bar{\gamma})\right)} \lor \overline{\left((\bar{\delta} \lor \epsilon) \land (\delta \lor \bar{\epsilon})\right)}\right)\right) \land \left(\alpha \lor \overline{\left(\overline{\left((\bar{\beta} \lor \gamma) \land (\beta \lor \bar{\gamma})\right)} \lor \overline{\left((\bar{\delta} \lor \epsilon) \land (\delta \lor \bar{\epsilon})\right)}\right)}\right), \text{BE} \qquad (18)$$

$$(\alpha \lor \bar{\beta} \lor \gamma) \land (\alpha \lor \beta \lor \bar{\gamma}) \land (\alpha \lor \bar{\delta} \lor \epsilon) \land (\alpha \lor \delta \lor \bar{\epsilon}) \land (\bar{\alpha} \lor \beta \lor \gamma \lor \delta \lor \epsilon) \land (\bar{\alpha} \lor \bar{\beta} \lor \bar{\gamma} \lor \delta \lor \epsilon)$$
$$\land (\bar{\alpha} \lor \beta \lor \gamma \lor \bar{\delta} \lor \bar{\epsilon}) \land (\bar{\alpha} \lor \bar{\beta} \lor \bar{\gamma} \lor \bar{\delta} \lor \bar{\epsilon}), \text{Distributivity and DM}$$

Intermediary Variables in the Minimax Tree

The previously expressed CNF expressions for $V_{i,b}$ were not in terms of the leaf nodes as required by the problem. To successively substitute successor expressions back into the equations for $V_{i,b}$ would cause the length of the expressions to go exponentially and become even more unmanageable. In this section, a set of intermediary CNF expressions are derived that will be used to express $V_{i,b}$ in terms of the tree's leaf nodes. This final expression of $V_{i,b}$ in terms of leaf nodes is done in the next section.

Max – Level 2

$$V_{i,j,1} \Leftrightarrow \bigvee_{k=1}^{3} V_{i,j,k,1}$$

To convert this to CNF:

$$\left(V_{i,j,1} \land \bigvee_{k=1}^{3} V_{i,j,k,1}\right) \lor \left(\overline{V_{i,j,1}} \land \overline{\bigvee_{k=1}^{3} V_{i,j,k,1}}\right), \text{Biconditional Elimination (BE)}$$

$$\left(\overline{V_{i,j,1}} \lor \bigvee_{k=1}^{3} V_{i,j,k,1}\right) \land \left(V_{i,j,1} \lor \overline{\bigvee_{k=1}^{3} V_{i,j,k,1}}\right), \text{Distributivity} \qquad (19)$$

$$\left(\overline{V_{i,j,1}} \lor \bigvee_{k=1}^{3} V_{i,j,k,1}\right) \land \left(V_{i,j,1} \lor \bigwedge_{k=1}^{3} \overline{V_{i,j,k,1}}\right), \text{De Morgan's Law (DM)}$$

$$\left(\overline{V_{i,j,1}} \lor \bigvee_{k=1}^{3} V_{i,j,k,1}\right) \land \bigwedge_{k=1}^{3} \left(V_{i,j,1} \lor \overline{V_{i,j,k,1}}\right), \text{Distributivity}$$

$$V_{i,j,2} \Leftrightarrow \bigvee_{k=1}^{3} \left(V_{i,j,1} \Leftrightarrow V_{i,j,k,1}\right) \wedge V_{i,j,k,2}$$

Use an Intermediary Variable α_k:

$$\alpha_k \Leftrightarrow \left(V_{i,j,1} \Leftrightarrow V_{i,j,k,1}\right)$$

Using this simplified form, the equation can easily be converted to CNF.

$$V_{i,j,2} \Leftrightarrow \bigvee_{k=1}^{3} \alpha_k \wedge V_{i,j,k,2}\text{, Substitution}$$

$$\left(V_{i,j,2} \vee \bigvee_{k=1}^{3} \alpha_k \wedge V_{i,j,k,2}\right) \wedge \left(\overline{V_{i,j,2}} \vee \overline{\left(\bigvee_{k=1}^{3} \alpha_k \wedge V_{i,j,k,2}\right)}\right)\text{, BE}$$

$$\left(V_{i,j,2} \vee \bigvee_{k=1}^{3} \alpha_k \wedge V_{i,j,k,2}\right) \wedge \left(\overline{V_{i,j,2}} \vee \bigwedge_{k=1}^{3} (\overline{\alpha_k} \vee \overline{V_{i,j,k,2}})\right)\text{, DM}$$

$$\left(V_{i,j,2} \vee \bigvee_{k=1}^{3} \alpha_k \wedge V_{i,j,k,2}\right) \wedge \bigwedge_{k=1}^{3} (\overline{V_{i,j,2}} \vee \overline{\alpha_k} \vee \overline{V_{i,j,k,2}})\text{, DM} \qquad (20)$$

$$\begin{aligned}&\left(V_{i,j,2} \vee \alpha_1 \vee \alpha_2 \vee \alpha_3\right) \wedge \left(V_{i,j,2} \vee \alpha_1 \vee \alpha_2 \vee V_{i,j,3,2}\right) \wedge \left(V_{i,j,2} \vee \alpha_1 \vee V_{i,j,2,2} \vee \alpha_3\right) \wedge \left(V_{i,j,2} \vee \alpha_1 \vee V_{i,j,2,2} \vee V_{i,j,3,2}\right)\\ &\wedge \left(V_{i,j,2} \vee V_{i,j,1,2} \vee \alpha_2 \vee \alpha_3\right) \wedge \left(V_{i,j,2} \vee V_{i,j,1,2} \vee \alpha_2 \vee V_{i,j,3,2}\right) \wedge \left(V_{i,j,2} \vee V_{i,j,1,2} \vee V_{i,j,2,2} \vee \alpha_3\right)\\ &\wedge \left(V_{i,j,2} \vee V_{i,j,1,2} \vee V_{i,j,2,2} \vee V_{i,j,3,2}\right) \wedge \left(\bigwedge_{k=1}^{3} (\overline{V_{i,j,2}} \vee \overline{\alpha_k} \vee \overline{V_{i,j,k,2}})\right)\text{, Distributivity}\end{aligned}$$

This is then combined with the solution of (16) to get a sufficiently expressed expression for use in the CNF:

$$\begin{aligned}&\left(V_{i,j,2} \vee \alpha_1 \vee \alpha_2 \vee \alpha_3\right) \wedge \left(V_{i,j,2} \vee \alpha_1 \vee \alpha_2 \vee V_{i,j,3,2}\right) \wedge \left(V_{i,j,2} \vee \alpha_1 \vee V_{i,j,2,2} \vee \alpha_3\right) \wedge \left(V_{i,j,2} \vee \alpha_1 \vee V_{i,j,2,2} \vee V_{i,j,3,2}\right)\\ &\wedge \left(V_{i,j,2} \vee V_{i,j,1,2} \vee \alpha_2 \vee \alpha_3\right) \wedge \left(V_{i,j,2} \vee V_{i,j,1,2} \vee \alpha_2 \vee V_{i,j,3,2}\right) \wedge \left(V_{i,j,2} \vee V_{i,j,1,2} \vee V_{i,j,2,2} \vee \alpha_3\right)\\ &\wedge \left(V_{i,j,2} \vee V_{i,j,1,2} \vee V_{i,j,2,2} \vee V_{i,j,3,2}\right) \wedge \left(\bigwedge_{k=1}^{3} (\overline{V_{i,j,2}} \vee \overline{\alpha_k} \vee \overline{V_{i,j,k,2}})\right)\\ &\wedge \bigwedge_{k=1}^{3} \Big((\overline{\alpha_k} \vee \overline{V_{i,j,1}} \vee V_{i,j,k,1}) \wedge (\overline{\alpha_k} \vee V_{i,j,1} \vee \overline{V_{i,j,k,1}}) \wedge (\alpha_k \vee V_{i,j,1} \vee V_{i,j,k,1})\\ &\vee (\alpha_k \vee \overline{V_{i,j,1}} \vee \overline{V_{i,j,k,1}})\Big)\end{aligned}$$

$$V_{i,j,3} \Leftrightarrow \bigvee_{k=1}^{3} \left(V_{i,j,1} \Leftrightarrow V_{i,j,k,1}\right) \wedge \left(V_{i,j,2} \Leftrightarrow V_{i,j,k,2}\right) \wedge V_{i,j,k,3}$$

Use an Intermediary Variable γ_k:

$$\gamma_k = \left(V_{i,j,1} \Leftrightarrow V_{i,j,k,1}\right) \wedge \left(V_{i,j,2} \Leftrightarrow V_{i,j,k,2}\right) \qquad (21)$$

This reduces the expression to:

$$V_{i,j,3} \Leftrightarrow \bigvee_{k=1}^{3} \gamma_k \wedge V_{i,j,k,3}$$

The equation above is now in the same form as the revised equation in (20). Using that equation's solution, substitute this statement's symbols.

$$\begin{aligned}&(V_{i,j,3} \lor \gamma_1 \lor \gamma_2 \lor \gamma_3) \land (V_{i,j,3} \lor \gamma_1 \lor \gamma_2 \lor V_{i,j,3,3}) \land (V_{i,j,3} \lor \gamma_1 \lor V_{i,j,2,3} \lor \gamma_3) \land (V_{i,j,3} \lor \gamma_1 \lor V_{i,j,2,3} \lor V_{i,j,3,3}) \\ &\quad \land (V_{i,j,3} \lor V_{i,j,1,3} \lor \gamma_2 \lor \gamma_3) \land (V_{i,j,3} \lor V_{i,j,1,3} \lor \gamma_2 \lor V_{i,j,3,3}) \land (V_{i,j,3} \lor V_{i,j,1,3} \lor V_{i,j,2,3} \lor \gamma_3) \\ &\quad \land (V_{i,j,3} \lor V_{i,j,1,3} \lor V_{i,j,2,3} \lor V_{i,j,3,3}) \land \left(\bigwedge_{k=1}^{3} (V_{i,j,3} \lor \overline{\gamma_k} \lor \overline{V_{i,j,k,3}})\right)\end{aligned}$$

Since γ was used as a simplification, it must be plugged back into the equations using the solution from (17).

$$\begin{aligned}&(V_{1,j,3} \lor \gamma_1 \lor \gamma_2 \lor \gamma_3) \land (V_{1,j,3} \lor \gamma_1 \lor \gamma_2 \lor V_{1,j,3,3}) \land (V_{1,j,3} \lor \gamma_1 \lor V_{1,j,2,3} \lor \gamma_3) \land (V_{1,j,3} \lor \gamma_1 \lor V_{1,j,2,3} \lor V_{1,j,3,3}) \\ &\quad \land (V_{1,j,3} \lor V_{1,j,1,3} \lor \gamma_2 \lor \gamma_3) \land (V_{1,j,3} \lor V_{1,j,1,3} \lor \gamma_2 \lor V_{1,j,3,3}) \land (V_{1,j,3} \lor V_{1,j,1,3} \lor V_{1,j,2,3} \lor \gamma_3) \\ &\quad \land (V_{1,j,3} \lor V_{1,j,1,3} \lor V_{1,j,2,3} \lor V_{1,j,3,3}) \land \left(\bigwedge_{k=1}^{3} (V_{1,j,3} \lor \overline{\gamma_k} \lor \overline{V_{1,j,k,3}})\right) \\ &\quad \land \bigwedge_{k=1} \Big((\overline{\gamma_k} \lor \overline{V_{1,j,1}} \lor V_{1,j,k,1}) \land (\overline{\gamma_k} \lor V_{1,j,1} \lor \overline{V_{1,j,k,1}}) \land (\overline{\gamma_k} \lor \overline{V_{1,j,2}} \lor V_{1,j,k,2}) \\ &\quad \land (\overline{\gamma_k} \lor V_{1,j,2} \lor \overline{V_{1,j,k,2}}) \land (\gamma_k \lor \overline{V_{1,j,1}} \lor \overline{V_{1,j,k,1}} \lor \overline{V_{1,j,2}} \lor \overline{V_{1,j,k,2}}) \\ &\quad \land (\gamma_k \lor \overline{V_{1,j,1}} \lor \overline{V_{1,j,k,1}} \lor V_{1,j,2} \lor V_{1,j,k,2}) \land (\gamma_k \lor V_{1,j,1} \lor V_{1,j,k,1} \lor \overline{V_{1,j,2}} \lor \overline{V_{1,j,k,2}}) \\ &\quad \land (\gamma_k \lor V_{1,j,1} \lor V_{1,j,k,1} \lor V_{1,j,2} \lor V_{1,j,k,2})\Big)\end{aligned}$$

Min – Level 3

$$V_{i,j,k,1} \Leftrightarrow \bigwedge_{m=1}^{3} V_{i,j,k,m,1}$$

$$\left(V_{i,j,1} \land \bigwedge_{m=i}^{3} V_{i,j,k,m,1}\right) \lor \left(\overline{V_{i,j,1}} \land \overline{\bigwedge_{m=i}^{3} V_{i,j,k,m,1}}\right), \text{Biconditional Elimination (BE)}$$

$$\left(\overline{V_{i,j,1}} \lor \bigwedge_{m=i}^{3} V_{i,j,k,m,1}\right) \land \left(V_{i,j,1} \lor \overline{\bigwedge_{m=i}^{3} V_{i,j,k,m,1}}\right), \text{Distributivity}$$

$$\left(\overline{V_{i,j,1}} \lor \bigwedge_{m=i}^{3} V_{i,j,k,m,1}\right) \land \left(V_{i,j,1} \lor \bigvee_{m=1}^{3} \overline{V_{i,j,k,m,1}}\right), \text{DM}$$

$$\bigwedge_{m=1}^{3} (\overline{V_{i,j,1}} \lor V_{i,j,k,m,1}) \land \left(V_{i,j,1} \lor \bigvee_{m=1}^{3} \overline{V_{i,j,k,m,1}}\right), \text{Distributivity}$$

(22)

$$V_{i,j,k,2} \Leftrightarrow \bigwedge_{m=1}^{3} \overline{(V_{i,j,k,1} \Leftrightarrow V_{i,j,k,m,1})} \lor V_{i,j,k,m,2}$$

Use an Intermediary Variable β_m:

$$\beta_m \Leftrightarrow (V_{i,j,k,1} \Leftrightarrow V_{i,j,k,m,1})$$

This reduces the expression to:

(23)

$$\left(V_{i,j,k,2} \Leftrightarrow \bigwedge_{m=1}^{3} \overline{\beta_m} \vee V_{i,j,k,m,2}\right), \text{Substitution}$$

Using this simplified form, the equation can easily be converted to CNF.

$$\left(\overline{V_{i,j,k,2}} \vee \bigwedge_{m=1}^{3} \overline{\beta_m} \vee V_{i,j,k,m,2}\right) \wedge \left(V_{i,j,k,2} \vee \overline{\left(\bigwedge_{m=1}^{3} \overline{\beta_m} \vee V_{i,j,k,m,2}\right)}\right), \text{BE}$$

$$\left(\bigwedge_{m=1}^{3} \left(\overline{V_{i,j,k,2}} \vee \overline{\beta_m} \vee V_{i,j,k,m,2}\right)\right) \wedge \left(V_{i,j,k,2} \vee \bigvee_{m=1}^{3} \left(\beta_m \wedge \overline{V_{i,j,k,m,2}}\right)\right), \text{Distributivity and DM}$$

$$\begin{aligned}
&\left(\bigwedge_{m=1}^{3} \left(\overline{V_{i,j,k,2}} \vee \overline{\beta_m} \vee V_{i,j,k,m,2}\right)\right) \wedge \left(V_{i,j,k,2} \vee \beta_1 \vee \beta_2 \vee \beta_3\right) \wedge \left(V_{i,j,k,2} \vee \beta_1 \vee \beta_2 \vee \overline{V_{i,j,k,3,2}}\right) \\
&\quad \wedge \left(V_{i,j,k,2} \vee \beta_1 \vee \overline{V_{i,j,k,2,2}} \vee \beta_3\right) \wedge \left(V_{i,j,k,2} \vee \beta_1 \vee \overline{V_{i,j,k,2,2}} \vee \overline{V_{i,j,k,3,2}}\right) \\
&\quad \wedge \left(V_{i,j,k,2} \vee \overline{V_{i,j,k,1,2}} \vee \beta_2 \vee \beta_3\right) \wedge \left(V_{i,j,k,2} \vee \overline{V_{i,j,k,1,2}} \vee \beta_2 \vee \overline{V_{i,j,k,3,2}}\right) \\
&\quad \wedge \left(V_{i,j,k,2} \vee \overline{V_{i,j,k,1,2}} \vee \overline{V_{i,j,k,2,2}} \vee \beta_3\right) \wedge \left(V_{i,j,k,2} \vee \overline{V_{i,j,k,1,2}} \vee \overline{V_{i,j,k,2,2}} \vee \overline{V_{i,j,k,3,2}}\right), \text{Distributivity}
\end{aligned}$$

This solution is then combined with the solution of (17):

$$\begin{aligned}
&\left(\bigwedge_{m=1}^{3} \left(\overline{V_{i,j,k,2}} \vee \overline{\beta_m} \vee V_{i,j,k,m,2}\right)\right) \wedge \left(V_{i,j,k,2} \vee \beta_1 \vee \beta_2 \vee \beta_3\right) \wedge \left(V_{i,j,k,2} \vee \beta_1 \vee \beta_2 \vee \overline{V_{i,j,k,3,2}}\right) \\
&\quad \wedge \left(V_{i,j,k,2} \vee \beta_1 \vee \overline{V_{i,j,k,2,2}} \vee \beta_3\right) \wedge \left(V_{i,j,k,2} \vee \beta_1 \vee \overline{V_{i,j,k,2,2}} \vee \overline{V_{i,j,k,3,2}}\right) \\
&\quad \wedge \left(V_{i,j,k,2} \vee \overline{V_{i,j,k,1,2}} \vee \beta_2 \vee \beta_3\right) \wedge \left(V_{i,j,k,2} \vee \overline{V_{i,j,k,1,2}} \vee \beta_2 \vee \overline{V_{i,j,k,3,2}}\right) \\
&\quad \wedge \left(V_{i,j,k,2} \vee \overline{V_{i,j,k,1,2}} \vee \overline{V_{i,j,k,2,2}} \vee \beta_3\right) \wedge \left(V_{i,j,k,2} \vee \overline{V_{i,j,k,1,2}} \vee \overline{V_{i,j,k,2,2}} \vee \overline{V_{i,j,k,3,2}}\right) \\
&\quad \wedge \bigwedge_{m=1}^{3} \Big(\left(\overline{\beta_m} \vee \overline{V_{i,j,k,1}} \vee V_{i,j,k,m,1}\right) \wedge \left(\overline{\beta_m} \vee V_{i,j,k,1} \vee \overline{V_{i,j,k,m,1}}\right) \wedge \left(\beta_m \vee V_{i,j,k,1} \vee V_{i,j,k,m,1}\right) \\
&\quad \vee \left(\beta_m \vee \overline{V_{i,j,k,1}} \vee \overline{V_{i,j,k,m,1}}\right)\Big)
\end{aligned}$$

$$V_{i,j,k,3} \Leftrightarrow \bigwedge_{m=1}^{3} \overline{\left(V_{i,j,k,1} \Leftrightarrow V_{i,j,k,m,1}\right)} \vee \overline{\left(V_{i,j,k,2} \Leftrightarrow V_{i,j,k,m,2}\right)} \vee V_{i,j,k,m,3} \tag{24}$$

Define:

$$\delta_m \Leftrightarrow \overline{\left(V_{i,j,k,1} \Leftrightarrow V_{i,j,k,m,1}\right)} \vee \overline{\left(V_{i,j,k,2} \Leftrightarrow V_{i,j,k,m,2}\right)}$$

The equation above is now in the same form as the revised equation in (23). Using that equation's solution, substitute these symbols.

$$\begin{aligned}
&\left(\bigwedge_{m=1}^{3} \left(\overline{V_{i,j,k,3}} \vee \overline{\delta_m} \vee V_{i,j,k,m,3}\right)\right) \wedge \left(V_{i,j,k,3} \vee \delta_1 \vee \delta_2 \vee \delta_3\right) \wedge \left(V_{i,j,k,3} \vee \delta_1 \vee \delta_2 \vee \overline{V_{i,j,k,3,3}}\right) \\
&\quad \wedge \left(V_{i,j,k,3} \vee \delta_1 \vee \overline{V_{i,j,k,2,3}} \vee \delta_3\right) \wedge \left(V_{i,j,k,3} \vee \delta_1 \vee \overline{V_{i,j,k,2,3}} \vee \overline{V_{i,j,k,3,3}}\right) \wedge \left(V_{i,j,k,3} \vee \overline{V_{i,j,k,1,3}} \vee \delta_2 \vee \delta_3\right) \\
&\quad \wedge \left(V_{i,j,k,3} \vee \overline{V_{i,j,k,1,3}} \vee \delta_2 \vee \overline{V_{i,j,k,3,3}}\right) \wedge \left(V_{i,j,k,3} \vee \overline{V_{i,j,k,1,3}} \vee \overline{V_{i,j,k,2,3}} \vee \delta_3\right) \\
&\quad \wedge \left(V_{i,j,k,3} \vee \overline{V_{i,j,k,1,3}} \vee \overline{V_{i,j,k,2,3}} \vee \overline{V_{i,j,k,3,3}}\right)
\end{aligned}$$

This solution is then combined with the solution of (18):

$$
\begin{aligned}
&\left(\bigwedge_{m=1}^{3}\left(\overline{V_{i,j,k,3}} \vee \overline{\delta_m} \vee V_{i,j,k,m,3}\right)\right) \wedge \left(V_{i,j,k,3} \vee \delta_1 \vee \delta_2 \vee \delta_3\right) \wedge \left(V_{i,j,k,3} \vee \delta_1 \vee \delta_2 \vee \overline{V_{i,j,k,3,3}}\right) \\
&\quad \wedge \left(V_{i,j,k,3} \vee \delta_1 \vee \overline{V_{i,j,k,2,3}} \vee \delta_3\right) \wedge \left(V_{i,j,k,3} \vee \delta_1 \vee \overline{V_{i,j,k,2,3}} \vee \overline{V_{i,j,k,3,3}}\right) \wedge \left(V_{i,j,k,3} \vee \overline{V_{i,j,k,1,3}} \vee \delta_2 \vee \delta_3\right) \\
&\quad \wedge \left(V_{i,j,k,3} \vee \overline{V_{i,j,k,1,3}} \vee \delta_2 \vee \overline{V_{i,j,k,3,3}}\right) \wedge \left(V_{i,j,k,3} \vee \overline{V_{i,j,k,1,3}} \vee \overline{V_{i,j,k,2,3}} \vee \delta_3\right) \\
&\quad \wedge \left(V_{i,j,k,3} \vee \overline{V_{i,j,k,1,3}} \vee \overline{V_{i,j,k,2,3}} \vee \overline{V_{i,j,k,3,3}}\right) \\
&\quad \wedge \bigwedge_{m=1}^{3}\Big(\left(\delta_m \vee \overline{V_{i,j,k,1}} \vee V_{i,j,k,m,1}\right) \wedge \left(\delta_m \vee V_{i,j,k,1} \vee \overline{V_{i,j,k,m,1}}\right) \wedge \left(\delta_m \vee \overline{V_{i,j,k,2}} \vee V_{i,j,k,m,2}\right) \\
&\quad \wedge \left(\delta_m \vee V_{i,j,k,2} \vee \overline{V_{i,j,k,m,2}}\right) \wedge \left(\overline{\delta_m} \vee V_{i,j,k,1} \vee V_{i,j,k,m,1} \vee V_{i,j,k,2} \vee V_{i,j,k,m,2}\right) \\
&\quad \wedge \left(\overline{\delta_m} \vee \overline{V_{i,j,k,1}} \vee \overline{V_{i,j,k,m,1}} \vee V_{i,j,k,2} \vee V_{i,j,k,m,2}\right) \wedge \left(\overline{\delta_m} \vee V_{i,j,k,1} \vee V_{i,j,k,m,1} \vee \overline{V_{i,j,k,2}} \vee \overline{V_{i,j,k,m,2}}\right) \\
&\quad \wedge \left(\overline{\delta_m} \vee \overline{V_{i,j,k,1}} \vee \overline{V_{i,j,k,m,1}} \vee \overline{V_{i,j,k,2}} \vee \overline{V_{i,j,k,m,2}}\right)\Big)
\end{aligned}
$$

Final CNFs

The CNF for $V_{1,1}$ is in equation (25); note it is the conjunction for the final derived equations in (7), (19), and (22). Equation is the full CNF for

$$
V_{1,1} = \bigwedge_{j=i}^{3}\left(V_{1,j,1}\right) \wedge \bigwedge_{j=1}^{3}\left(\left(V_{1,j,1} \Leftrightarrow \bigvee_{k=1}^{3} V_{1,j,k,1}\right) \wedge \bigwedge_{k=1}^{3}\left(V_{1,j,k,1} \Leftrightarrow \bigwedge_{m=i}^{3} V_{1,j,k,m,1}\right)\right)
$$

Final CNF:

$$
V_{1,1} = \left(\bigwedge_{j=1}^{3} V_{1,j,1}\right) \wedge \bigwedge_{j=1}^{3}\left(\left(\left(\overline{V_{i,j,1}} \vee \bigvee_{k=1}^{3} V_{i,j,k,1}\right) \wedge \bigwedge_{k=1}^{3}\left(V_{i,j,1} \vee \overline{V_{i,j,k,1}}\right)\right) \wedge \bigwedge_{k=1}^{3}\left(\bigwedge_{m=1}^{3}\left(\overline{V_{1,j,1}} \vee V_{1,j,k,m,1}\right) \wedge \left(V_{1,j,1} \vee \bigvee_{m=1}^{3} \overline{V_{1,j,k,m,1}}\right)\right)\right)
$$

(25)

The CNF for $V_{1,2}$ is in equation (26). It is the conjunction final derived equations in (8), (20), (23), and (25).

$$
\begin{aligned}
V_{1,2} = &\left(\bigwedge_{j=1}^{3} \overline{\left(V_{1,1} \Leftrightarrow V_{1,j,1}\right)} \vee V_{1,j,2}\right) \\
&\wedge \bigwedge_{j=1}^{3}\left(\left(V_{1,j,2} \Leftrightarrow \left(\bigvee_{k=1}^{3}\left(V_{1,j,1} \Leftrightarrow V_{1,j,k,1}\right) \wedge V_{1,j,k,2}\right)\right) \wedge \bigwedge_{k=1}^{3}\left(V_{1,j,k,2} \Leftrightarrow \bigwedge_{m=1}^{3} \overline{\left(V_{1,j,k,1} \Leftrightarrow V_{1,j,k,m,1}\right)} \vee V_{1,j,k,m,2}\right)\right) \\
&\wedge \left(V_{1,1}\right)
\end{aligned}
$$

(26)

Final CNF:

$$
\begin{aligned}
V_{1,2} = \Bigg(\bigwedge_{j=1}^{3}\left(\left(\overline{V_{1,1}} \vee \overline{V_{1,j,1}} \vee V_{1,j,2}\right) \wedge \left(V_{1,1} \vee V_{1,j,1} \vee V_{1,j,2}\right)\right)\Bigg) \\
\wedge \bigwedge_{j=1}^{3}\Bigg(\Bigg(\left(V_{1,j,2} \vee \alpha_1 \vee \alpha_2 \vee \alpha_3\right) \wedge \left(V_{1,j,2} \vee \alpha_1 \vee \alpha_2 \vee V_{1,j,3,2}\right) \wedge \left(V_{1,j,2} \vee \alpha_1 \vee V_{1,j,2,2} \vee \alpha_3\right) \wedge \left(V_{1,j,2} \vee \alpha_1 \vee V_{1,j,2,2} \vee V_{i,j,3,2}\right) \\
\wedge \left(V_{1,j,2} \vee V_{1,j,1,2} \vee \alpha_2 \vee \alpha_3\right) \wedge \left(V_{1,j,2} \vee V_{1,j,1,2} \vee \alpha_2 \vee V_{1,j,3,2}\right) \wedge \left(V_{1,j,2} \vee V_{1,j,1,2} \vee V_{1,j,2,2} \vee \alpha_3\right) \wedge \left(V_{1,j,2} \vee V_{1,j,1,2} \vee V_{1,j,2,2} \vee V_{1,j,3,2}\right) \\
\wedge \left(\bigwedge_{k=1}^{3}\left(\overline{V_{1,j,2}} \vee \overline{\alpha_k} \vee \overline{V_{1,j,k,2}}\right)\right) \wedge \bigwedge_{k=1}^{3}\left(\left(\overline{\alpha_k} \vee \overline{V_{1,j,1}} \vee V_{i,j,k,1}\right) \wedge \left(\overline{\alpha_k} \vee V_{1,j,1} \vee \overline{V_{1,j,k,1}}\right) \wedge \left(\alpha_k \vee V_{1,j,1} \vee V_{1,j,k,1}\right) \vee \left(\alpha_k \vee \overline{V_{1,j,1}} \vee \overline{V_{1,j,k,1}}\right)\right)\Bigg) \\
\wedge \bigwedge_{k=1}^{3}\Bigg(\Bigg(\bigwedge_{m=1}^{3}\left(\overline{V_{1,j,k,2}} \vee \overline{\beta_m} \vee V_{1,j,k,m,2}\right)\Bigg) \wedge \left(V_{1,j,k,2} \vee \beta_1 \vee \beta_2 \vee \beta_3\right) \wedge \left(V_{1,j,k,2} \vee \beta_1 \vee \beta_2 \vee \overline{V_{1,j,k,3,2}}\right) \wedge \left(V_{1,j,k,2} \vee \beta_1 \vee \overline{V_{1,j,k,2,2}} \vee \beta_3\right) \\
\wedge \left(V_{1,j,k,2} \vee \beta_1 \vee \overline{V_{1,j,k,2,2}} \vee \overline{V_{1,j,k,3,2}}\right) \wedge \left(V_{1,j,k,2} \vee \overline{V_{1,j,k,1,2}} \vee \beta_2 \vee \beta_3\right) \wedge \left(V_{1,j,k,2} \vee \overline{V_{1,j,k,1,2}} \vee \beta_2 \vee \overline{V_{1,j,k,3,2}}\right) \\
\wedge \left(V_{1,j,k,2} \vee \overline{V_{1,j,k,1,2}} \vee \overline{V_{1,j,k,2,2}} \vee \beta_3\right) \wedge \left(V_{1,j,k,2} \vee \overline{V_{1,j,k,1,2}} \vee \overline{V_{1,j,k,2,2}} \vee \overline{V_{1,j,k,3,2}}\right) \\
\wedge \bigwedge_{m=1}^{3}\left(\left(\overline{\beta_m} \vee \overline{V_{1,j,k,1}} \vee V_{1,j,k,m,1}\right) \wedge \left(\overline{\beta_m} \vee V_{1,j,k,1} \vee \overline{V_{1,j,k,m,1}}\right) \wedge \left(\beta_m \vee V_{1,j,k,1} \vee V_{1,j,k,m,1}\right) \vee \left(\beta_m \vee \overline{V_{1,j,k,1}} \vee \overline{V_{1,j,k,m,1}}\right)\right)\Bigg)\Bigg) \\
\wedge \Bigg(\Bigg(\bigwedge_{j=1}^{3} V_{1,j,1}\Bigg) \wedge \bigwedge_{j=1}^{3}\Bigg(\Bigg(\Bigg(\overline{V_{i,j,1}} \vee \bigvee_{k=1}^{3} V_{i,j,k,1}\Bigg) \wedge \bigwedge_{k=1}^{3}\left(V_{i,j,1} \vee \overline{V_{i,j,k,1}}\right)\Bigg) \wedge \bigwedge_{k=1}^{3}\Bigg(\bigwedge_{m=1}^{3}\left(\overline{V_{1,j,1}} \vee V_{1,j,k,m,1}\right) \wedge \Bigg(V_{1,j,1} \vee \bigvee_{m=1}^{3} \overline{V_{1,j,k,m,1}}\Bigg)\Bigg)\Bigg)\Bigg)
\end{aligned}
$$

The CNF for $V_{1,3}$ is in equation (27). It is the conjunction final derived equations in (9), (21), (24), and (26).

$$
\begin{aligned}
V_{1,3} = \left(\bigwedge_{j=1}^{3}\overline{\left(V_{1,1} \Leftrightarrow V_{1,j,1}\right)} \vee \overline{\left(V_{1,2} \Leftrightarrow V_{1,j,2}\right)} \vee V_{1,j,3}\right) \\
\wedge \bigwedge_{j=1}^{3}\Bigg(\Bigg(V_{1,j,k,2} \Leftrightarrow \bigwedge_{m=1}^{3}\overline{\left(V_{1,j,k,1} \Leftrightarrow V_{1,j,k,m,1}\right)} \vee V_{1,j,k,m,2}\Bigg) \\
\wedge \bigwedge_{k=1}^{3}\Bigg(V_{1,j,k,3} \Leftrightarrow \bigwedge_{m=1}^{3}\overline{\left(V_{1,j,k,1} \Leftrightarrow V_{1,j,k,m,1}\right)} \vee \overline{\left(V_{1,j,k,2} \Leftrightarrow V_{1,j,k,m,2}\right)} \vee V_{1,j,k,m,3}\Bigg)\Bigg) \wedge \left(V_{1,2}\right)
\end{aligned} \qquad (27)
$$

Final CNF:

$$
\begin{aligned}
V_{1,3} = \Bigg(& \bigwedge_{j=1}^{3} \Big(\big((\overline{V_{1,1}} \vee \overline{V_{1,j,1}} \vee \overline{V_{1,2}} \vee \overline{V_{1,j,2}} \vee V_{1,j,3}) \wedge (\overline{V_{1,1}} \vee \overline{V_{1,j,1}} \vee V_{1,2} \vee V_{1,j,2} \vee V_{1,j,3}) \wedge (V_{1,1} \vee V_{1,j,1} \vee \overline{V_{1,2}} \vee \overline{V_{1,j,2}} \vee V_{1,j,3}) \\
& \wedge (V_{1,1} \vee V_{1,j,1} \vee V_{1,2} \vee V_{1,j,2} \vee V_{1,j,3}) \big) \Big) \Bigg) \\
& \wedge \bigwedge_{j=1}^{3} \Bigg(\Bigg((V_{1,j,3} \vee \gamma_1 \vee \gamma_2 \vee \gamma_3) \wedge (V_{1,j,3} \vee \gamma_1 \vee \gamma_2 \vee V_{1,j,3,3}) \wedge (V_{1,j,3} \vee \gamma_1 \vee V_{1,j,2,3} \vee \gamma_3) \wedge (V_{1,j,3} \vee \gamma_1 \vee V_{1,j,2,3} \vee V_{1,j,3,3}) \\
& \wedge (V_{1,j,3} \vee V_{1,j,1,3} \vee \gamma_2 \vee \gamma_3) \wedge (V_{1,j,3} \vee V_{1,j,1,3} \vee \gamma_2 \vee V_{1,j,3,3}) \wedge (V_{1,j,3} \vee V_{1,j,1,3} \vee V_{1,j,2,3} \vee \gamma_3) \wedge (V_{1,j,3} \vee V_{1,j,1,3} \vee V_{1,j,2,3} \vee V_{1,j,3,3}) \\
& \wedge \Bigg(\bigwedge_{k=1}^{3} (V_{1,j,3} \vee \overline{\gamma_k} \vee \overline{V_{1,j,k,3}}) \Bigg) \\
& \wedge \bigwedge_{k=1} \big((\overline{\gamma_k} \vee \overline{V_{1,j,1}} \vee V_{1,j,k,1}) \wedge (\overline{\gamma_k} \vee V_{1,j,1} \vee \overline{V_{1,j,k,1}}) \wedge (\overline{\gamma_k} \vee \overline{V_{1,j,2}} \vee V_{1,j,k,2}) \wedge (\overline{\gamma_k} \vee V_{1,j,2} \vee \overline{V_{1,j,k,2}}) \\
& \wedge (\gamma_k \vee \overline{V_{1,j,1}} \vee \overline{V_{1,j,k,1}} \vee \overline{V_{1,j,2}} \vee \overline{V_{1,j,k,2}}) \wedge (\gamma_k \vee \overline{V_{1,j,1}} \vee \overline{V_{1,j,k,1}} \vee V_{1,j,2} \vee V_{1,j,k,2}) \wedge (\gamma_k \vee V_{1,j,1} \vee V_{1,j,k,1} \vee \overline{V_{1,j,2}} \vee \overline{V_{1,j,k,2}}) \\
& \wedge (\gamma_k \vee V_{1,j,1} \vee V_{1,j,k,1} \vee V_{1,j,2} \vee V_{1,j,k,2}) \big) \Bigg) \\
& \wedge \bigwedge_{k=1}^{3} \Bigg(\Bigg(\bigwedge_{m=1}^{3} (\overline{V_{1,j,k,3}} \vee \overline{\delta_m} \vee V_{1,j,k,m,3}) \Bigg) \wedge (V_{1,j,k,3} \vee \delta_1 \vee \delta_2 \vee \delta_3) \wedge (V_{1,j,k,3} \vee \delta_1 \vee \delta_2 \vee \overline{V_{1,j,k,3,3}}) \wedge (V_{1,j,k,3} \vee \delta_1 \vee \overline{V_{1,j,k,2,3}} \vee \delta_3) \\
& \wedge (V_{1,j,k,3} \vee \delta_1 \vee \overline{V_{1,j,k,2,3}} \vee \overline{V_{1,j,k,3,3}}) \wedge (V_{1,j,k,3} \vee \overline{V_{1,j,k,1,3}} \vee \delta_2 \vee \delta_3) \wedge (V_{1,j,k,3} \vee \overline{V_{1,j,k,1,3}} \vee \delta_2 \vee \overline{V_{1,j,k,3,3}}) \\
& \wedge (V_{1,j,k,3} \vee \overline{V_{1,j,k,1,3}} \vee \overline{V_{1,j,k,2,3}} \vee \delta_3) \wedge (V_{1,j,k,3} \vee \overline{V_{1,j,k,1,3}} \vee \overline{V_{1,j,k,2,3}} \vee \overline{V_{1,j,k,3,3}}) \\
& \wedge \bigwedge_{m=1}^{3} \big((\delta_m \vee \overline{V_{1,j,k,1}} \vee V_{1,j,k,m,1}) \wedge (\delta_m \vee V_{1,j,k,1} \vee \overline{V_{1,j,k,m,1}}) \wedge (\delta_m \vee \overline{V_{1,j,k,2}} \vee V_{1,j,k,m,2}) \wedge (\delta_m \vee V_{1,j,k,2} \vee \overline{V_{1,j,k,m,2}}) \\
& \wedge (\overline{\delta_m} \vee V_{1,j,k,1} \vee V_{1,j,k,m,1} \vee V_{1,j,k,2} \vee V_{1,j,k,m,2}) \wedge (\overline{\delta_m} \vee \overline{V_{1,j,k,1}} \vee \overline{V_{1,j,k,m,1}} \vee V_{1,j,k,2} \vee V_{1,j,k,m,2}) \\
& \wedge (\overline{\delta_m} \vee V_{1,j,k,1} \vee V_{1,j,k,m,1} \vee \overline{V_{1,j,k,2}} \vee \overline{V_{1,j,k,m,2}}) \wedge (\overline{\delta_m} \vee \overline{V_{1,j,k,1}} \vee \overline{V_{1,j,k,m,1}} \vee \overline{V_{1,j,k,2}} \vee \overline{V_{1,j,k,m,2}}) \big) \Bigg) \\
& \wedge \Bigg(\bigwedge_{j=1}^{3} \big((\overline{V_{1,1}} \vee \overline{V_{1,j,1}} \vee V_{1,j,2}) \wedge (V_{1,1} \vee V_{1,j,1} \vee V_{1,j,2}) \big) \Bigg) \\
& \wedge \bigwedge_{j=1}^{3} \Bigg(\Bigg((V_{1,j,2} \vee \alpha_1 \vee \alpha_2 \vee \alpha_3) \wedge (V_{1,j,2} \vee \alpha_1 \vee \alpha_2 \vee V_{1,j,3,2}) \wedge (V_{1,j,2} \vee \alpha_1 \vee V_{1,j,2,2} \vee \alpha_3) \wedge (V_{1,j,2} \vee \alpha_1 \vee V_{1,j,2,2} \vee V_{i,j,3,2}) \\
& \wedge (V_{1,j,2} \vee V_{1,j,1,2} \vee \alpha_2 \vee \alpha_3) \wedge (V_{1,j,2} \vee V_{1,j,1,2} \vee \alpha_2 \vee V_{1,j,3,2}) \wedge (V_{1,j,2} \vee V_{1,j,1,2} \vee V_{1,j,2,2} \vee \alpha_3) \wedge (V_{1,j,2} \vee V_{1,j,1,2} \vee V_{1,j,2,2} \vee V_{1,j,3,2}) \\
& \wedge \Bigg(\bigwedge_{k=1}^{3} (\overline{V_{1,j,2}} \vee \overline{\alpha_k} \vee \overline{V_{1,j,k,2}}) \Bigg) \wedge \bigwedge_{k=1}^{3} \big((\overline{\alpha_k} \vee \overline{V_{1,j,1}} \vee V_{i,j,k,1}) \wedge (\overline{\alpha_k} \vee V_{1,j,1} \vee \overline{V_{1,j,k,1}}) \wedge (\alpha_k \vee V_{1,j,1} \vee V_{1,j,k,1}) \vee (\alpha_k \vee \overline{V_{1,j,1}} \vee \overline{V_{1,j,k,1}}) \big) \Bigg) \\
& \wedge \bigwedge_{k=1}^{3} \Bigg(\Bigg(\bigwedge_{m=1}^{3} (\overline{V_{1,j,k,2}} \vee \overline{\beta_m} \vee V_{1,j,k,m,2}) \Bigg) \wedge (V_{1,j,k,2} \vee \beta_1 \vee \beta_2 \vee \beta_3) \wedge (V_{1,j,k,2} \vee \beta_1 \vee \beta_2 \vee \overline{V_{1,j,k,3,2}}) \wedge (V_{1,j,k,2} \vee \beta_1 \vee \overline{V_{1,j,k,2,2}} \vee \beta_3) \\
& \wedge (V_{1,j,k,2} \vee \beta_1 \vee \overline{V_{1,j,k,2,2}} \vee \overline{V_{1,j,k,3,2}}) \wedge (V_{1,j,k,2} \vee \overline{V_{1,j,k,1,2}} \vee \beta_2 \vee \beta_3) \wedge (V_{1,j,k,2} \vee \overline{V_{1,j,k,1,2}} \vee \beta_2 \vee \overline{V_{1,j,k,3,2}}) \\
& \wedge (V_{1,j,k,2} \vee \overline{V_{1,j,k,1,2}} \vee \overline{V_{1,j,k,2,2}} \vee \beta_3) \wedge (V_{1,j,k,2} \vee \overline{V_{1,j,k,1,2}} \vee \overline{V_{1,j,k,2,2}} \vee \overline{V_{1,j,k,3,2}}) \\
& \wedge \bigwedge_{m=1}^{3} \big((\overline{\beta_m} \vee \overline{V_{1,j,k,1}} \vee V_{1,j,k,m,1}) \wedge (\overline{\beta_m} \vee V_{1,j,k,1} \vee \overline{V_{1,j,k,m,1}}) \wedge (\beta_m \vee V_{1,j,k,1} \vee V_{1,j,k,m,1}) \vee (\beta_m \vee \overline{V_{1,j,k,1}} \vee \overline{V_{1,j,k,m,1}}) \big) \Bigg) \Bigg) \\
& \wedge \Bigg(\Bigg(\bigwedge_{j=1}^{3} V_{1,j,1} \Bigg) \wedge \bigwedge_{j=1}^{3} \Bigg(\Bigg(\Bigg(\overline{V_{i,j,1}} \vee \bigvee_{k=1}^{3} V_{i,j,k,1} \Bigg) \wedge \bigwedge_{k=1}^{3} (V_{i,j,1} \vee \overline{V_{i,j,k,1}}) \Bigg) \wedge \bigwedge_{k=1}^{3} \Bigg(\bigwedge_{m=1}^{3} (\overline{V_{1,j,1}} \vee V_{1,j,k,m,1}) \wedge \Bigg(V_{1,j,1} \vee \bigvee_{m=1}^{3} \overline{V_{1,j,k,m,1}} \Bigg) \Bigg) \Bigg) \Bigg)
\end{aligned}
$$

Problem #2

Question: The pigeonhole principle PHP_n^{n+1} says any function from $n + 1$ pigeons into n holes must result in two pigeons in the same hole. Let $P_{i,j}$ be a variable expressing that pigeon i gets mapped to hole j. Consider the $n = 3$ case.

Express the following as propositional formulas:

a. **Every i gets mapped to some j.**
b. **Some j is mapped to by i and i' where $i \neq i'$.**

The conjunction of these two statements is a propositional formula for PHP_3^4. Convert $\neg$ PHP to clausal form and give a resolution refutation for this statement. Finally, trace the execution of DPLL on this formula.

Part A:

Each pigeon can be in only one hole. As such, for a given pigeon, there is a disjunction of conjunctions (i.e. OR of ANDs). Each conjunction explicitly limits the pigeon to a single hole. The subsequent derivation in equation (28) converts the equation to CNF.

$$
\begin{aligned}
&R_1: \bigwedge_{i=1}^{4}\left(\left(P_{i,1} \wedge \overline{P_{i,2}} \wedge \overline{P_{i,3}}\right) \vee \left(\overline{P_{i,1}} \wedge P_{i,2} \wedge \overline{P_{i,3}}\right) \vee \left(\overline{P_{i,1}} \wedge \overline{P_{i,2}} \wedge P_{i,3}\right)\right) \\
&R_1: \bigwedge_{i=1}^{4}\left(\neg\neg\left(\left(P_{i,1} \wedge \overline{P_{i,2}} \wedge \overline{P_{i,3}}\right) \vee \left(\overline{P_{i,1}} \wedge P_{i,2} \wedge \overline{P_{i,3}}\right) \vee \left(\overline{P_{i,1}} \wedge \overline{P_{i,2}} \wedge P_{i,3}\right)\right)\right) \\
&R_1: \bigwedge_{i=1}^{4}\left(\neg\overline{\left(\left(P_{i,1} \wedge \overline{P_{i,2}} \wedge \overline{P_{i,3}}\right) \vee \left(\overline{P_{i,1}} \wedge P_{i,2} \wedge \overline{P_{i,3}}\right) \vee \left(\overline{P_{i,1}} \wedge \overline{P_{i,2}} \wedge P_{i,3}\right)\right)}\right) \\
&R_1: \bigwedge_{i=1}^{4}\left(\neg\left(\overline{\left(P_{i,1} \wedge \overline{P_{i,2}} \wedge \overline{P_{i,3}}\right)} \wedge \overline{\left(\overline{P_{i,1}} \wedge P_{i,2} \wedge \overline{P_{i,3}}\right)} \wedge \overline{\left(\overline{P_{i,1}} \wedge \overline{P_{i,2}} \wedge P_{i,3}\right)}\right)\right) \\
&R_1: \bigwedge_{i=1}^{4}\left(\neg\left(\left(\overline{P_{i,1}} \vee P_{i,2} \vee P_{i,3}\right) \wedge \left(P_{i,1} \vee \overline{P_{i,2}} \vee P_{i,3}\right) \vee \left(P_{i,1} \vee P_{i,2} \vee \overline{P_{i,3}}\right)\right)\right) \\
&R_1: \bigwedge_{i=1}^{4}\left(\neg\left(\left(P_{i,1} \wedge P_{i,2}\right) \vee \left(P_{i,1} \wedge P_{i,3}\right) \vee \left(P_{i,2} \wedge P_{i,3}\right) \vee \left(\overline{P_{i,1}} \wedge \overline{P_{i,2}} \wedge \overline{P_{i,3}}\right)\right)\right) \\
&R_1: \bigwedge_{i=1}^{4}\left(\left(\overline{P_{i,1}} \vee \overline{P_{i,2}}\right) \wedge \left(\overline{P_{i,1}} \vee \overline{P_{i,3}}\right) \wedge \left(\overline{P_{i,2}} \vee \overline{P_{i,3}}\right) \wedge \left(P_{i,1} \vee P_{i,2} \vee P_{i,3}\right)\right) \quad \textbf{(CNF)}
\end{aligned}
\tag{28}
$$

Part B:

The pigeonhole principle is satisfied whenever any two pigeons are in any one hole. This translates to a large disjunction of conjunctions (i.e. disjunctive normal form – DNF) where each conjunction represents pigeons i and k being simultaneously in hole j. The inverse of a DNF is a CNF; the DNF equation will become important when it comes to performing the resolution refutation.

$$
\begin{aligned}
R_2&: \bigvee_{j=1}^{3} (P_{1,j} \wedge P_{2,j}) \vee (P_{1,j} \wedge P_{3,j}) \vee (P_{1,j} \wedge P_{4,j}) \vee (P_{2,j} \wedge P_{3,j}) \vee (P_{2,j} \wedge P_{4,j}) \vee (P_{3,j} \wedge P_{4,j}) \\
\overline{R_2}&: \bigwedge_{j=1}^{3} \left((\overline{P_{1,j}} \vee \overline{P_{2,j}}) \wedge (\overline{P_{1,j}} \vee \overline{P_{3,j}}) \wedge (\overline{P_{1,j}} \vee \overline{P_{4,j}}) \wedge (\overline{P_{2,j}} \vee \overline{P_{3,j}}) \wedge (\overline{P_{2,j}} \vee \overline{P_{4,j}}) \wedge (\overline{P_{3,j}} \vee \overline{P_{4,j}}) \right)
\end{aligned}
\tag{29}
$$

$\boldsymbol{PHP_3^4}$:

The pigeonhole principles states that part R_1 implies R_2. Hence:

$$
(R_1 \Rightarrow R_2) \tag{30}
$$

$\neg\, \boldsymbol{PHP_3^4}$:

To show the pigeonhole principle is valid, take its negation and show that the negation is unsatisfiable. The negation is done in equation (31). Note that the implication is removed via implication elimination.

$$
\begin{gathered}
\neg\,(R_1 \Rightarrow R_2) \\
\neg\,(\overline{R_1} \vee R_2) \\
R_1 \wedge \overline{R_2}
\end{gathered}
\tag{31}
$$

When equation (31) is populated with the resolved equations in (28) and (29), the negation of the pigeonhole principle with three holes and four pigeons, $\neg\, PHP_3^4$, is in CNF. This is shown in equation (32).

$$
\begin{aligned}
&\bigwedge_{i=1}^{4} \left((\overline{P_{i,1}} \vee \overline{P_{i,2}}) \wedge (\overline{P_{i,1}} \vee \overline{P_{i,3}}) \wedge (\overline{P_{i,2}} \vee \overline{P_{i,3}}) \wedge (P_{i,1} \vee P_{i,2} \vee P_{i,3}) \right) \\
&\qquad \wedge \quad \bigwedge_{j=1}^{3} \left((\overline{P_{1,j}} \vee \overline{P_{2,j}}) \wedge (\overline{P_{1,j}} \vee \overline{P_{3,j}}) \wedge (\overline{P_{1,j}} \vee \overline{P_{4,j}}) \wedge (\overline{P_{2,j}} \vee \overline{P_{3,j}}) \wedge (\overline{P_{2,j}} \vee \overline{P_{4,j}}) \wedge (\overline{P_{3,j}} \vee \overline{P_{4,j}}) \right)
\end{aligned}
\tag{32}
$$

Resolution Refutation:

Resolution refutation necessitates combining clauses which have literal(s) whose sign(s) (i.e. positive or negative) is/are complementary. When combining these clauses, the goal for resolution refutation is to find an empty clause. The four step proof of resolution refutation is shown in equations (33) to (36).

From R_1: $$R_3: \frac{(\overline{P_{1,2}} \vee \overline{P_{1,3}}),\ \ (P_{1,1} \vee P_{1,2} \vee P_{1,3})}{P_{1,1}}$$	(33)
From R_1: $$R_4: \frac{(\overline{P_{2,2}} \vee \overline{P_{2,3}}),\ \ (P_{2,1} \vee P_{2,2} \vee P_{2,3})}{P_{2,1}}$$	(34)
From $\overline{R_2}$ and R_3: $$R_5: \frac{(\overline{P_{1,1}} \vee \overline{P_{2,1}}),\quad P_{1,1}}{\overline{P_{2,1}}}$$	(35)
From R_4 and R_5:	(36)

$$R_6: \frac{P_{2,1},\ \overline{P_{2,1}}}{\{\}}$$ Completing the Resolution Refutation since the empty clause was found.	

DPLL Algorithm

In the DPLL algorithm, there are four distinct steps per iteration; the steps and their sequential ordering are:

1. If a clause has been assigned to false, terminate. Similarly, if the assignment satisfies the set of clauses, return the assignment as the expression has been satisfied.
2. Check for any pure symbols (i.e. symbols that have the same sign in all clauses).
3. Check for any unit clauses (i.e. any clause with only one symbol)
4. Choose the first symbol from the list of unassigned symbols. Test assigning both true and false to that symbol and see if either assignment satisfies the expression.

I wrote a program (**HW3_Q2_DPLL.py**) to execute the DPLL algorithm on this CNF. Below is the output from my program. Each step the algorithm took before reaching an empty clause is listed as well as the initial conditions (e.g. clauses and model). This step by step description of the algorithm's operations serves as the trace.

```
The clauses are below.  A plus sign ("+") before a symbol name indicates a
positive literal.
A minus sign ("-") before a symbol name indicates a negated literal.
[['-P1,1', '-P1,2'], ['-P1,1', '-P1,3'], ['-P1,2', '-P1,3'], ['+P1,1', '+P1,2',
'+P1,3'], ['-P2,1', '-P2,2'], ['-P2,1', '-P2,3'], ['-P2,2', '-P2,3'], ['+P2,1',
'+P2,2', '+P2,3'], ['-P3,1', '-P3,2'], ['-P3,1', '-P3,3'], ['-P3,2', '-P3,3'],
['+P3,1', '+P3,2', '+P3,3'], ['-P4,1', '-P4,2'], ['-P4,1', '-P4,3'], ['-P4,2', '-
P4,3'], ['+P4,1', '+P4,2', '+P4,3'], ['-P1,1', '-P2,1'], ['-P1,1', '-P3,1'], ['-
P1,1', '-P4,1'], ['-P2,1', '-P3,1'], ['-P2,1', '-P4,1'], ['-P3,1', '-P4,1'], ['-
P1,2', '-P2,2'], ['-P1,2', '-P3,2'], ['-P1,2', '-P4,2'], ['-P2,2', '-P3,2'], ['-
P2,2', '-P4,2'], ['-P3,2', '-P4,2'], ['-P1,3', '-P2,3'], ['-P1,3', '-P3,3'], ['-
P1,3', '-P4,3'], ['-P2,3', '-P3,3'], ['-P2,3', '-P4,3'], ['-P3,3', '-P4,3']]

The model is:  ['P1,1', 'P1,2', 'P1,3', 'P2,1', 'P2,2', 'P2,3', 'P3,1', 'P3,2',
'P3,3', 'P4,1', 'P4,2', 'P4,3']

Step #1: Try assigning symbol "P1,1" to "True".
Step #2: Symbol "P1,2" is a pure symbol. It was assigned to "False".
Step #3: Symbol "P1,3" is a pure symbol. It was assigned to "False".
Step #4: Unit clause found for symbol "P2,1". It was assigned to "False".
Step #5: Unit clause found for symbol "P3,1". It was assigned to "False".
Step #6: Unit clause found for symbol "P4,1". It was assigned to "False".
Step #7: Try assigning symbol "P2,2" to "True".
Step #8: Symbol "P2,3" is a pure symbol. It was assigned to "False".
Step #9: Unit clause found for symbol "P3,2". It was assigned to "False".
Step #10: Unit clause found for symbol "P3,3". It was assigned to "True".
Step #11: Unit clause found for symbol "P4,2". It was assigned to "False".
Step #12: Unit clause found for symbol "P4,3". It was assigned to "True".
Step #13: Empty clause found. Recursing...

Assigning symbol "P2,2" to "True" failed.
Step #14: Try assigning symbol "P2,2" to "False".
Step #15: Unit clause found for symbol "P2,3". It was assigned to "True".
Step #16: Unit clause found for symbol "P3,3". It was assigned to "False".
```

```
Step #17: Unit clause found for symbol "P3,2". It was assigned to "True".
Step #18: Unit clause found for symbol "P4,2". It was assigned to "False".
Step #19: Unit clause found for symbol "P4,3". It was assigned to "True".
Step #20: Empty clause found. Recursing...

Assigning symbol "P1,1" to "True" failed.
Step #21: Try assigning symbol "P1,1" to "False".
Step #22: Try assigning symbol "P1,2" to "True".
Step #23: Symbol "P1,3" is a pure symbol. It was assigned to "False".
Step #24: Unit clause found for symbol "P2,2". It was assigned to "False".
Step #25: Unit clause found for symbol "P3,2". It was assigned to "False".
Step #26: Unit clause found for symbol "P4,2". It was assigned to "False".
Step #27: Try assigning symbol "P2,1" to "True".
Step #28: Symbol "P2,3" is a pure symbol. It was assigned to "False".
Step #29: Unit clause found for symbol "P3,1". It was assigned to "False".
Step #30: Unit clause found for symbol "P3,3". It was assigned to "True".
Step #31: Unit clause found for symbol "P4,1". It was assigned to "False".
Step #32: Unit clause found for symbol "P4,3". It was assigned to "True".
Step #33: Empty clause found. Recursing...

Assigning symbol "P2,1" to "True" failed.
Step #34: Try assigning symbol "P2,1" to "False".
Step #35: Unit clause found for symbol "P2,3". It was assigned to "True".
Step #36: Unit clause found for symbol "P3,3". It was assigned to "False".
Step #37: Unit clause found for symbol "P3,1". It was assigned to "True".
Step #38: Unit clause found for symbol "P4,1". It was assigned to "False".
Step #39: Unit clause found for symbol "P4,3". It was assigned to "True".
Step #40: Empty clause found. Recursing...

Assigning symbol "P1,2" to "True" failed.
Step #41: Try assigning symbol "P1,2" to "False".
Step #42: Unit clause found for symbol "P1,3". It was assigned to "True".
Step #43: Unit clause found for symbol "P2,3". It was assigned to "False".
Step #44: Unit clause found for symbol "P3,3". It was assigned to "False".
Step #45: Unit clause found for symbol "P4,3". It was assigned to "False".
Step #46: Try assigning symbol "P2,1" to "True".
Step #47: Symbol "P2,2" is a pure symbol. It was assigned to "False".
Step #48: Unit clause found for symbol "P3,1". It was assigned to "False".
Step #49: Unit clause found for symbol "P3,2". It was assigned to "True".
Step #50: Unit clause found for symbol "P4,1". It was assigned to "False".
Step #51: Unit clause found for symbol "P4,2". It was assigned to "True".
Step #52: Empty clause found. Recursing...

Assigning symbol "P2,1" to "True" failed.
Step #53: Try assigning symbol "P2,1" to "False".
Step #54: Unit clause found for symbol "P2,2". It was assigned to "True".
Step #55: Unit clause found for symbol "P3,2". It was assigned to "False".
Step #56: Unit clause found for symbol "P3,1". It was assigned to "True".
Step #57: Unit clause found for symbol "P4,1". It was assigned to "False".
Step #58: Unit clause found for symbol "P4,2". It was assigned to "True".
Step #59: Empty clause found. Recursing...

These clauses are unsatisfiable.
```