Problem #9.10

A popular children's riddle is "Brothers and sisters I have none, but that man's father is my father's son." Use the rules of family domain (Section 8.3.2 on page 301) to show who that man is. You may apply any of the inference methods described in this chapter. Why do you think that this riddle is difficult?

This problem in prenex normal form is:

$$\forall x \forall y \exists z \left(\neg\, Brother(me, x) \land \neg\, Sister(me, y) \land \left(Father(z) \Leftrightarrow Son\big(Father(me)\big)\right)\right)$$

In prenex normal form, all quantifiers are on the outside of the expression as above. This can be made into CNF by biconditional elimination. Hence:

$$\forall x \forall y \exists z \left(\neg\, Brother(me, x) \land \neg\, Sister(me, y) \land \left(\overline{Father(z)} \lor Son\big(Father(me)\big)\right) \land \left(Father(z) \lor \overline{Son\big(Father(me)\big)}\right)\right)$$

The term $Son\big(Father(me)\big)$ can be

$$Son\big(Father(me)\big) \Leftrightarrow me \lor Brother(me)$$

This simplifies the original equation to:

$$\forall x \forall y \exists z \left(\neg\, Brother(me, x) \land \neg\, Sister(me, y) \land \left(Father(z) \Leftrightarrow me \lor Brother(me)\right)\right)$$

$Brother(me, x)$ is false for all x so this simplifies to:

$$\forall x \forall y \exists z \left(\neg\, Brother(me, x) \land \neg\, Sister(me, y) \land (Father(z) \Leftrightarrow me)\right)$$

By And Elimination, this simplifies to:

$$\exists z\, Father(z) \Leftrightarrow me$$

The only value assignment to z that will make this statement true is if z is my son.

This riddle is not terribly difficult. However, it obfuscates $Father(z) \Leftrightarrow me$ by wrapping the me object in what are compliementary operations since me has no brothers.

Problem #9.23

From "Horses are animals," it follows that "The head of a horse is the head of an animal." Demonstrate that this inference is valid by carrying out the following steps:

a. Translate the premise and the conclusion into the language of first order logic. Use three predicates: $\boldsymbol{HeadOf(h, x)}$ (meaning "$\boldsymbol{h}$ is the head of $\boldsymbol{x}$"), $\boldsymbol{Horses(x)}$, and $\boldsymbol{Animal(x)}$.

The premise of this statement is "Horses are animals". Rewritten in first-order logic with the defined predicates, this statement is:

$$\forall x\big(Horse(x) \Rightarrow Animal(x)\big)$$

The conclusion of this statement is:

$$\forall y \exists z\big(HeadOf(h, y) \land Horse(y) \Rightarrow HeadOf(h, z) \land Animal(z)\big)$$

b. Negate the conclusion, and convert the premise and the negated conclusion into conjunctive normal form.

By definition:

$$Premise \Rightarrow Conclusion$$

To perform refutation, negate the conclusion and show that:

$$Premise \land \neg\ Conclusion \Leftrightarrow \{\}$$

The premise is already in prenex normal form so the quantifiers can be dropped resulting in:

$$Horse(x) \Rightarrow Animal(x)$$

This can be made into CNF through implication elimination.

$$\big(Horse(x) \lor \overline{Animal(x)}\big) \land \big(\overline{Horse(x)} \lor Animal(x)\big)$$

In the conclusion clause, the existential quantifier can be replaced by making z a function of y (i.e. $f(y)$). Hence, the conclusion becomes:

$$\forall y \left(HeadOf(h, y) \land Horse(y) \Rightarrow HeadOf\big(h, f(y)\big) \land Animal\big(f(y)\big)\right)$$

Again, since all variables are bounded by a universal quantifier, the universal quantifier(s) can be dropped making the statement:

$$HeadOf(h, y) \land Horse(y) \Rightarrow HeadOf\big(h, f(y)\big) \land Animal\big(f(y)\big)$$

When implication elimination is applied to this equation, the result is:

$$\neg\ \big(HeadOf(h, y) \land Horse(y)\big) \lor \left(HeadOf\big(h, f(y)\big) \land Animal\big(f(y)\big)\right)$$