Problem #1

Max Root of the Tree:

$$V_{1,1} = \bigvee_{i=1}^{3} V_{1,i,1}$$
$$V_{1,2} = \bigvee_{i=1}^{3} \left(V_{1,1} \Leftrightarrow\ V_{1,i,1}\right) \wedge V_{1,i,2}$$ $$V_{1,2} = \bigvee_{i=1}^{3} \left(\left(V_{1,1} \Rightarrow V_{1,i,1}\right) \wedge \left(V_{1,i,1} \Rightarrow V_{1,1}\right)\right)\ \wedge V_{1,i,2}$$ $$V_{1,2} = \bigvee_{i=1}^{3} \left(\left(\bar{V}_{1,1} \vee V_{1,i,1}\right) \wedge \left(V_{1,1} \vee\ \bar{V}_{1,i,1}\right)\right)\ \wedge V_{1,i,2}$$ $$V_{1,2} = \bigvee_{i=1}^{3} \left(\bar{V}_{1,1} \wedge \bar{V}_{1,i,1} \wedge V_{1,i,2}\right) \vee \left(V_{1,1} \wedge\ V_{1,i,1} \wedge V_{1,i,2}\right)$$
$$V_{1,3} = \bigvee_{i=1}^{3} \left(V_{1,1} \Leftrightarrow\ V_{1,i,1}\right) \wedge \left(V_{1,2} \Leftrightarrow\ V_{1,i,2}\right) \wedge V_{1,i,3}$$ $$V_{1,3} = \bigvee_{i=1}^{3} \left(\left(V_{1,1} \wedge\ V_{1,i,1}\right) \vee \left(\bar{V}_{1,1} \wedge\ \bar{V}_{1,i,1}\right)\right) \wedge \left(\left(V_{1,2} \wedge\ V_{1,i,2}\right) \vee \left(\bar{V}_{1,2} \wedge\ \bar{V}_{1,i,2}\right)\right)\ \wedge V_{1,i,3}$$ $$V_{1,3} = \bigvee_{i=1}^{3} \left(V_{1,1} \wedge\ V_{1,i,1} \wedge V_{1,2} \wedge\ V_{1,i,2} \wedge V_{1,i,3}\right) \vee \left(V_{1,1} \wedge\ V_{1,i,1} \wedge \bar{V}_{1,2} \wedge\ \bar{V}_{1,i,2} \wedge V_{1,i,3}\right) \vee \left(\bar{V}_{1,1} \wedge\ \bar{V}_{1,i,1} \wedge V_{1,2} \wedge\ V_{1,i,2} \wedge V_{1,i,3}\right) \vee \left(\bar{V}_{1,1} \wedge\ \bar{V}_{1,i,1} \wedge \bar{V}_{1,2} \wedge\ \bar{V}_{1,i,2} \wedge V_{1,i,3}\right)$$

Min Level one of the Tree Root of the Tree:

$\displaystyle V_{1,i,1} = \bigwedge_{j=i}^{3} V_{1,i,j,1}$
$\displaystyle V_{1,i,2} = \bigwedge_{j=1}^{3} \left(V_{1,i,1} \Leftrightarrow V_{1,i,j,1}\right) \wedge V_{1,i,j,2}$ $\displaystyle V_{1,i,2} = \bigwedge_{j=1}^{3} \left(\left(V_{1,i,1} \wedge V_{1,i,j,1}\right) \vee \left(\bar{V}_{1,i,1} \wedge \bar{V}_{1,i,j,1}\right)\right) \wedge V_{1,i,j,2}$ $\displaystyle V_{1,i,2} = \bigwedge_{j=1}^{3} \left(\left(V_{1,i,1} \wedge V_{1,i,j,1} \wedge V_{1,i,j,2}\right) \vee \left(\bar{V}_{1,i,1} \wedge \bar{V}_{1,i,j,1} \wedge V_{1,i,j,2}\right)\right)$
$\displaystyle V_{1,i,3} = \bigwedge_{j=1}^{3} \left(V_{1,i,1} \Leftrightarrow V_{1,i,j,1}\right) \wedge \left(V_{1,i,2} \Leftrightarrow V_{1,i,j,2}\right) \wedge V_{1,i,j,3}$ $\displaystyle V_{1,i,3} = \bigwedge_{j=1}^{3} \left(\left(V_{1,i,1} \wedge V_{1,i,j,1}\right) \vee \left(\bar{V}_{1,i,1} \wedge \bar{V}_{1,i,j,1}\right)\right) \wedge \left(\left(V_{1,i,2} \wedge V_{1,i,j,2}\right) \vee \left(\bar{V}_{1,i,2} \wedge \bar{V}_{1,i,j,2}\right)\right) \wedge V_{1,i,j,3}$ $\displaystyle V_{1,i,3} = \bigwedge_{j=1}^{3} \left(V_{1,i,1} \wedge V_{1,i,j,1} \wedge V_{1,i,2} \wedge V_{1,i,j,2} \wedge V_{1,i,j,3}\right) \vee \left(V_{1,i,1} \wedge V_{1,i,j,1} \wedge \bar{V}_{1,i,2} \wedge \bar{V}_{1,i,j,2} \wedge V_{1,i,j,3}\right)$ $\displaystyle \vee \left(\bar{V}_{1,i,1} \wedge \bar{V}_{1,i,j,1} \wedge V_{1,i,2} \wedge V_{1,i,j,2} \wedge V_{1,i,j,3}\right) \vee \left(\bar{V}_{1,i,1} \wedge \bar{V}_{1,i,j,1} \wedge \bar{V}_{1,i,2} \wedge \bar{V}_{1,i,j,2} \wedge V_{1,i,j,3}\right)$

Problem #2

Question: The pigeonhole principle PHP_n^{n+1} says any function from $n+1$ pigeons into n holes must result in two pigeons in the same hole. Let $P_{i,j}$ be a variable expressing that pigeon i gets mapped to hole j. Consider the $n=3$ case.

Express the following as propositional formulas:

a. **Every i gets mapped to some j.**
b. **Some j is mapped to by i and i' where $i \neq i'$.**

The conjunction of these two statements is a propositional formula for PHP_3^4. Convert $\neg$ PHP to clausal form and give a resolution refutation for this statement. Finally, trace the execution of DPLL on this formula.

Part A:

$$\bigwedge_{i=1}^{4}\left(P_{i,1} \wedge \overline{P_{i,2}} \wedge \overline{P_{i,3}}\right) \vee \left(\overline{P_{i1}} \wedge P_{i,2} \wedge \overline{P_{i,3}}\right) \vee \left(\overline{P_{i,1}} \wedge \overline{P_{i,2}} \wedge P_{i,3}\right)$$

Part B:

$$\bigvee_{j=1}^{3}\left(P_{1,j} \wedge P_{2,j}\right) \vee \left(P_{1,j} \wedge P_{3,j}\right) \vee \left(P_{1,j} \wedge P_{4,j}\right) \vee \left(P_{2,j} \wedge P_{3,j}\right) \vee \left(P_{2,j} \wedge P_{4,j}\right) \vee \left(P_{3,j} \wedge P_{4,j}\right)$$