Test theory test on Game Theory

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Task description

- Positions of the game are integers in $[1\,..\,(day + month + year)]$, players move in turns
  Eloise-Abelard-etc.
- Moves (for both players) are $(+n)$ where n is the maximal integer in $[1\,..\,(year)]$ such that
  the next position is in the admissible range $[1\,..\,(day + month + year)]$; for example, if
  $day.month.year$ is 24.04.1961, then a player can move from position 12 to the next
  position 1973 (i.e., $n = 1961$), but from position 1234 to 1989 (i.e., $n = 755$).
- A player wins and the play stops as soon as the player moves to the final position $day + month + year$.

1. Game as a finite position game (FPG). Answer

Our game could be defined as $G = (P_E, P_A, M_E, M_A, F_E, F_A)$, where

- P_E and P_A are disjoint finite sets of positions for Eloise and Abelard,
- $M_E \subseteq P_E \times (P_E \cup P_A)$ and $M_A \subseteq P_A \times (P_E \cup P_A)$ are admissible moves of Eloise and Abelard,
- $F_E \subseteq (P_E \cup P_A)$ and $F_E \subseteq (P_E \cup P_A)$ are disjoint final positions (where Eloise and Abelard won already respectively)

Given parameters $day = 9$, $month = 3$, $year = 2000$:

- $P_E = \{(p, E) : p \in \mathbb{Z} \wedge p \in [1; 2012]\}$
- $P_A = \{(p, A) : p \in \mathbb{Z} \wedge p \in [1; 2012]\}$
- $M_E = \{(p_e, p_a) : p_e \in P_E \wedge p_a \in P_A \wedge \exists max(n) : n \in [1; 2000] \wedge n + p \leq 2012, p \in p_e\}$

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- $F_E = \{(2012, A)\}$
- $F_A = \{(2012, E)\}$

Sets of the final positions simply states in what position should be current player to win the game. A, E says who should turn next move from this position.

2. General definition of winning strategy for a player in FPG. Answer

~~A winning strategy for a player~~ $X \in \{E, A\}$ ~~is any subset of~~ $S_X \subseteq M_X$ ~~such that every move leads the player to the winning position from the set of winning positions~~ W_X ~~for a player. In other words, every move from~~ M_X ~~leads to the position~~ $p \notin W_{\overline{X}}$.

$\cancel{S_X = \{(p,q) \in M_X : q \in F_X\} \cup \{(p,q) \in M_X : \nexists x \in P_X : (q,x) \in S_X\}}$

3. Description in the set-theoretic terms backward induction for Eloise and computing all initial game positions where Eloise has a winning strategy. Answer

Eloise has a winning strategy in a position p if and only if:

- $p \in F_E$ or
- $\exists (p,q) \in M_E : p \notin F_E \wedge \forall (q,x) \in M_A : x \in W_E$

Now, compute the all initial positions where Eloise has a winning strategy:

$P_i = \{(p, E) : p \in \mathbb{Z}, p \in [1, 2012)\}$ - set of initial positions. Next step is to apply backward induction:

- $W_E^0 = \emptyset$
- $W_E^1 = \{(p, E) : p \in \mathbb{Z}, p \in [12, 2012)\}$

So, resulting set according to the task is following: $W_E = \{(p, E) : p \in \mathbb{Z}, p \in [12, 2012)\}$, because any move of Eloise $((p, E), (2012, A))$ will lead to $p' \in F_E$ since she turns first. All other positions do not satisfy Eloise's winning strategy constraints, hence we need only one backward step.

4. List all initial game positions where Abelard has a winning strategy. Answer

*~~Let Abelard choose the initial position.~~

- $\cancel{W_A^0 = \emptyset}$
- $\cancel{W_A^1 = \{(p, A) : p \in \mathbb{Z}, p \in [12, 2012)\}}$

~~So, winning positions for Abelard:~~ $\cancel{W_A = \{(p, A) : p \in \mathbb{Z}, p \in [12, 2012)\}}$~~. Since Eloise turns first it takes one move to get into position from~~ $\cancel{W_A^1}$. ~~***So, due to Eloise's first move appropriate set of initial positions for winning strategy of Abelard will be:***~~ $\cancel{W_A^I = \{(p, E) : p \in \mathbb{Z} \wedge p \in [1; 12)\}}$.

5. Definition and sketch for extensive form of a game. Answer

A game (with complete information) in extensive form is a directed labeled tree (infinite maybe, but in our case finite) where

- the root is the initial position, nodes are positions, and the leaves are final positions;
- positions specify turns, all edges are moves by participating players according to their turns;
- all final positions are marked (by a vector of) individual payoffs for each player.

Remark: payoffs presented as Abelard:Eloise

6. Definition and list of all plays of the game in the extensive form that are Pareto optimal. Answer

A play S is *Pareto optimal* if there is no any play S' such that $\pi(S') \geq \pi(S)$ holds componentwise and $\pi_X(S') > \pi_X(S)$ for some player X of the game. In Pareto optimal play the deviant player may increase its payoff.

There are 3 game from left to right as mentioned on the picture in the 5th question. See if it's Pareto optimal:

$S_1 \to \pi_A(S_2) < \pi_A(S_1)$, $\pi(S_3) = \pi(S_1)$, so it's Pareto optimal.

$S_2 \to \pi_E(S_1) < \pi_E(S_2)$, $\pi_E(S_3) < \pi_E(S_2)$, so it's Pareto optimal.

$S_3 \to \pi_A(S_2) < \pi_A(S_3)$, $\pi(S_1) = \pi(S_3)$

Finally, all the games are Pareto optimal.

7. Definition and list of all plays of the game in the extensive form that are Nash equilibrium. Answer

Play $S = (S_A, S_B, ...)$ is acceptable for a player X in A, B, etc. if $\pi_X(S) \geq \pi_X(S_{X:S'_X})$ for any legal strategy's S'_X of the player X. **Nash equilibrium** is any play that is acceptable for all players.

All the plays acceptable by Eloise, because Abelard gives only left only one strategy to her. Play 2 is not acceptable by Abelard, he could increase the payoff simply changing strategy. But contrary to play 1 and 3. **Hence plays 1 and 3 are Nash equilibrium**.

8. Definition and representational of the game in normal form. Answer

Two players' game in the normal form is 2-dimensional finite table (matrix) where

- rows correspond to the strategies of the first player,
- columns correspond to the strategies of the second player,
- each row and each column produce a play and their intersection cell contains vector of individual payoffs.

Let rows will be for the Abelard's strategies, and columns for the Eloises' ones. Hence

	+n	+0
+11-p	1:-1	not a play
+[12..2011]	-1:1	not a play
+2012	not a play	1:-1

Where n in admissible range, p is in range $[0, 10]$.