

Smooth transitions from the Schwarzschild vacuum to de Sitter space

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We provide an infinity of spacetimes which contain part of both the Schwarzschild vacuum and de Sitter space. The transition, which occurs below the Schwarzschild event horizon, involves only boundary surfaces (no surface layers). An explicit example is given in which the weak and strong energy conditions are satisfied everywhere (except in the de Sitter section) and the dominant energy condition is violated only in the vicinity of the boundary to the Schwarzschild section. The singularity is avoided by way of a change in topology in accord with a theorem due to Borde.

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Without a quantum theory of gravity the endpoint of complete gravitational collapse remains one of the central issues in physics. The idea of regular black holes (those without internal singularities) goes back at least 40 years [1] and the idea of a transition to de Sitter space across a thin shell in the Schwarzschild vacuum has seen extensive discussion in the literature [2]. However, the introduction of a thin shell is an approximation that allows properties of such a transition to be absorbed into properties of the shell itself and so it is certainly of interest to explore regular black holes which do not contain thin shells. Relatively little work has been done in this area [3] and it is this type of transition that is the subject of this work. Whereas our considerations are classical (and formal), regular black holes are of considerable interest from a nonclassical viewpoint [4].

We consider the nonsingular transition from the Schwarzschild vacuum (\mathcal{V}) to de Sitter space (\mathcal{V}') by way of spacelike boundary surfaces (not surface layers) Σ_1 (below the horizon in Schwarzschild) and Σ_2 (above the cosmological horizon in de Sitter) through a nonvacuum region \mathcal{M} . Part of the resultant maximally extended spacetime is shown in Fig. 1.

In this paper the singularity is avoided by way of a change in topology in accord with a theorem due to Borde [5]. We summarize the theorem here and refer the reader to Borde's papers for details. Suppose that a future causally simple spacetime obeys the null convergence condition (the Ricci tensor R_a^b obeys $R_a^b N^a N_b \geq 0$ for all null vectors N^a), is null geodesically complete to the future and contains an eventually future-trapped surface \mathcal{T} (that is, the divergence along each null geodesic that emanates orthogonally from \mathcal{T} becomes negative somewhere to the future of \mathcal{T}). Then there is a compact slice to the causal future of \mathcal{T} . As explained by Borde, it is the development of this compact slice that allows singularity avoidance [6].

In \mathcal{M} write [7,8]

$$ds_{\mathcal{M}}^2 = \frac{dr^2}{1 - \frac{2M(r)}{r}} + r^2 d\Omega^2 + e^{2\Phi(r)} dT^2, \quad r < 2M(r), \quad (1)$$

where $d\Omega^2$ is the metric of a unit 2-sphere ($d\theta^2 + \sin^2(\theta)d\phi^2$). Note that the metric (1) is not static. (Whereas $\xi^\alpha = \delta_T^\alpha$ is a hypersurface-orthogonal Killing vector, it is spacelike.) For notational convenience, in both \mathcal{V} and \mathcal{V}' write

$$ds_{\mathcal{V}}^2 = \frac{dr^2}{f(r)} + r^2 d\Omega^2 - f(r) dt^2, \quad r < 2m(r), \quad (2)$$

where $f(r) \equiv 1 - \frac{2m(r)}{r}$. Again for notational convenience, on Σ_1 and Σ_2 write

$$ds_{\Sigma}^2 = R^2 d\Omega^2 + d\lambda^2. \quad (3)$$

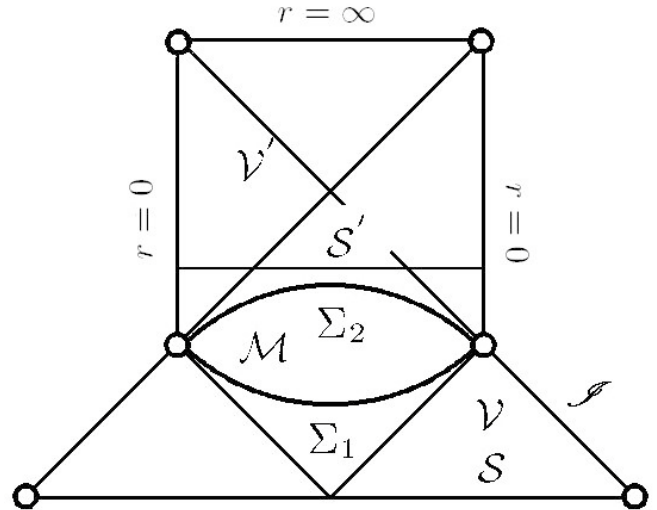


FIG. 1. Global structure of part of the transition from Schwarzschild (\mathcal{V}) to de Sitter space (\mathcal{V}') by way of boundary surfaces Σ_1 and Σ_2 . \mathcal{M} is not vacuum. There is no singularity because of the change in topology. The spacelike hypersurface S' is compact (S^3) but the spacelike hypersurface S is not ($S^2 \times R$). The diagram can be rotated about S to provide a geodesically complete extension.

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Without loss in generality, we have taken θ and ϕ continuous but note that the coordinates in \mathcal{V} and \mathcal{V}' and on Σ_1 and Σ_2 are otherwise distinct. The fact that the coordinates used in (2) are valid only in a neighborhood of Σ_1 and Σ_2 in no way limits our analysis.

The continuity of the first and second fundamental forms at Σ_1 and Σ_2 (the Darmois-Israel junction conditions [9]) result in the conditions

$$R_{\Sigma} = r_{\Sigma} = r_{\Sigma}, \quad (4)$$

$$M_{\Sigma} = m_{\Sigma}, \quad (5)$$

and

$$\Phi' = \frac{m^* r - m}{r^2(2\frac{m}{r} - 1)} \quad (6)$$

at Σ , where $' \equiv \frac{d}{dr}$ and $* \equiv \frac{d}{dr}$. These conditions must hold at both Σ_1 and Σ_2 . We note that since $m^* = 0$ in \mathcal{V} then $\Phi' < 0$ at Σ_1 . Further, since $m = \frac{\Lambda r^3}{6}$ in \mathcal{V}' , where Λ is the cosmological constant, $\Phi' > 0$ at Σ_2 .

In the usual way, the energy density ρ and principal pressures p_1 and p_2 for \mathcal{M} evaluate to

$$\tilde{\rho} = \frac{2}{r^3}(M + \Phi' r(2M - r)), \quad (7)$$

$$\tilde{p}_1 = -\frac{2M'}{r^2}, \quad (8)$$

and

$$\begin{aligned} \tilde{p}_2 = & -\frac{1}{r^3}((\Phi'' + (\Phi')^2)r^2(2M - r) \\ & + \Phi' r(rM' + M - r) + rM' - M), \end{aligned} \quad (9)$$

where \sim indicates division by 8π .

From Eqs. (4), (6), (7), and (8) it follows that

$$\tilde{\rho}_{\Sigma} = \frac{2m_{\Sigma}^*}{r_{\Sigma}^2} \quad (10)$$

and

$$\tilde{p}_{1\Sigma} = -\frac{2M'_{\Sigma}}{r_{\Sigma}^2}. \quad (11)$$

From (10) we then have

$$\tilde{\rho}_{\Sigma_1} = 0 \quad (12)$$

and

$$\tilde{\rho}_{\Sigma_2} = \Lambda. \quad (13)$$

Eqs. (11) and (12) make it clear that the dominant energy condition [10] is necessarily violated in \mathcal{M} at least in the neighborhood of Σ_1 [11]. Equations (11) and (13) make it clear that the dominant energy condition need not be violated throughout \mathcal{M} or at least in the neighborhood of Σ_2 . Equations (11) and (12) also show us that $M' < 0$ in \mathcal{M} in the neighborhood of Σ_1 for the weak (and null) energy condition to be satisfied.

The degree to which energy conditions can be satisfied in \mathcal{M} , subject to the boundary conditions specified, depends on $\Phi(r)$ and $M(r)$. (In the de Sitter section the strong

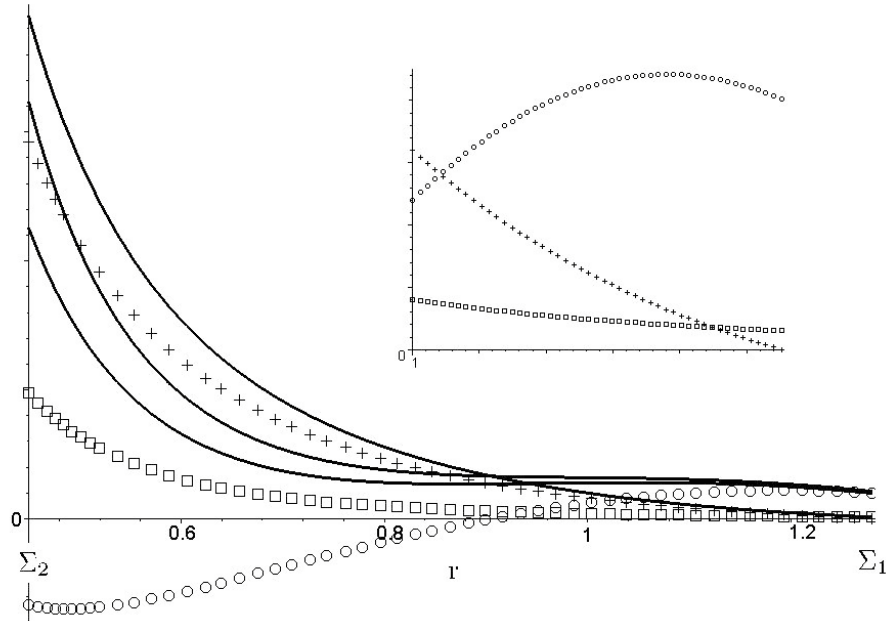


FIG. 2. The energy conditions in \mathcal{M} . The energy density $\tilde{\rho}$ is indicated by a cross, \tilde{p}_1 by a box, and \tilde{p}_2 (which becomes a tension) by a circle. The solid curves give (top down at $r = 0.6$) $\tilde{\rho} + \tilde{p}_1$, $\tilde{\rho} + \tilde{p}_1 + \tilde{p}_2$, and $\tilde{\rho} + \tilde{p}_2$. Detail near Σ_1 where the dominant energy condition fails is shown in the insert.

energy condition is of course violated.) Whereas one could demand isotropy of the pressure (and thereby solve for $M(r)$) we see no reason to do this here since, as explained above, the dominant energy condition is necessarily violated in at least a neighborhood of Σ_1 thus removing at least part of \mathcal{M} from normal classical physics. Rather, we are interested in the degree to which the energy conditions must be violated and we content ourselves here with a simple example so as to explore this.

Take

$$\Phi(r) = (1 - r^2)(r^2 - 2) \quad (14)$$

and

$$M(r) = \frac{2}{r}. \quad (15)$$

It follows that r_{Σ_1} is uniquely determined (~ 1.276) as is r_{Σ_2} (~ 0.451) and therefore $\Lambda (= 12/r_{\Sigma_2}^4)$. The energy conditions are shown in Fig. 2 where it can be seen that the weak, null, and strong energy conditions hold for $r_{\Sigma_2} < r <$

r_{Σ_1} and even the dominant energy condition holds for $r_{\Sigma_2} < r < \sim 0.8r_{\Sigma_1}$.

Clearly there is an infinity of spacetimes which contain part of both the Schwarzschild vacuum and de Sitter space with the transition (below the Schwarzschild event horizon) involving only boundary surfaces (no surface layers). The simple example given above shows that it is not difficult to find examples in which the weak, null, and strong energy conditions hold throughout (excepting of course in the de Sitter section) and the violation of the dominant energy condition is isolated. The singularity is avoided by way of a change in topology in accord with a theorem due to Borde [5].

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- [1] See A. D. Sakharov, Sov. Phys. JETP **22**, 241 (1966); E. B. Gliner, Sov. Phys. JETP **22**, 378 (1966) who both considered de Sitter space as the final state of gravitational collapse. J. M. Bardeen, in Proceedings of GR5, Tiflis, USSR, 1968 (unpublished) presented a modification of the Reissner-Nordström solution with an event horizon but no singularities. It is important to note that Bardeen was well aware of the importance of topology change in his example. This has been discussed in detail by A. Borde, Phys. Rev. D **50**, 3692 (1994).
 - [2] See M. A. Markov, JETP Lett. **36**, 265 (1982); Ann. Phys. (N.Y.) **155**, 333 (1984); M. R. Bernstein, Bull. Am. Phys. Soc. **16**, 1016 (1984); V. P. Frolov, M. A. Markov, and V. F. Mukhanov, Phys. Lett. B **216**, 272 (1989); K. Lake and T. Zannias, Phys. Lett. A **140**, 291 (1989); V. P. Frolov, M. A. Markov, and V. F. Mukhanov, Phys. Rev. D **41**, 383 (1990); R. Balbinot and E. Poisson, Phys. Rev. D **41**, 395 (1990); D. Morgan, Phys. Rev. D **43**, 3144 (1991); M. Visser and D. Wiltshire, Classical Quantum Gravity **21**, 1135 (2004). For a related discussion see E. Poisson and W. Israel, Classical Quantum Gravity **5**, L201 (1988). For related discussions of lightlike boundaries see W. Shen and S. Zhu, Phys. Lett. A **126**, 229 (1988); C. Barrabès and V. P. Frolov, Phys. Rev. D **53**, 3215 (1996); Helv. Phys. Acta **69**, 253 (1996).
 - [3] M. Mars, M. M. Martín-Prats, and J. M. M. Senovilla, Classical Quantum Gravity **13**, L51 (1996) have considered regular Schwarzschild black holes (without transition to de Sitter space). See J. M. M. Senovilla, Gen. Relativ. Gravit. **30**, 701 (1998) for further discussion. I. Dymnikova, Gen. Relativ. Gravit. **24**, 235 (1992); see also I. Dymnikova and E. Galaktionov, gr-qc/0409049 has considered a form of (2) which is de Sitter for $r \rightarrow 0$ and Schwarzschild for $r \rightarrow \infty$. See also E. Elizalde and S. R. Hildebrandt, Phys. Rev. D **65**, 124024 (2002) who consider regular interiors with a prescribed equation of state.
 - [4] The limiting curvature hypothesis (see [2]) has been given a nonclassical implementation; V. Mukhanov and R. Brandenberger, Phys. Rev. Lett. **68**, 1969 (1992); R. Brandenberger, V. Mukhanov, and A. Sornborger, Phys. Rev. D **48**, 1629 (1993) leading to nonclassical regular black hole constructions in 1 + 1 dimensions; M. Trodden, V. Mukhanov, and R. Brandenberger, Phys. Lett. B **316**, 483 (1993). There are many current discussions of nonclassical regular black holes in various contexts. See, for example, L. Modesto, Phys. Rev. D **70**, 124009 (2004); V. Husain and O. Winkler, gr-qc/0410125; T. Hirayama and B. Holdom, Phys. Rev. D **68**, 044003 (2003); D. A. Easson, J. High Energy Phys. **02** (2003) 037; D. A. Easson and R. H. Brandenberger, J. High Energy Phys. **06** (2001) 024.
 - [5] A. Borde, Phys. Rev. D **55**, 7615 (1997); see also A. Borde [1].
 - [6] If the initial spacetime is not asymptotically flat then more elaborate constructions are possible. See K. Lake (in preparation).
 - [7] We use geometrical units and a signature of +2. For convenience, explicit functional dependence is usually shown only on the first appearance of a function.
 - [8] This construction, though unusual, is by no means new. In the older Russian literature this is known as a “T-sphere.” The work on these configurations by I. D. Novikov and V. A. Ruban has recently been reprinted. See the Editor’s notes in Gen. Relativ. Gravit. **33**, 363

(2001); Gen. Relativ. Gravit. **33**, 2255 (2001) and the articles accompanying these notes. Some further discussion (due to K. S. Thorne) is given in section 3.2 of Ya. B. Zel'dovich and I. D. Novikov, *Relativistic Astrophysics Vol. 1 Stars and Relativity* (University of Chicago, Chicago, 1971). See also G. C. McVittie and R. J. Wiltshire, Int. J. Theor. Phys. **14**, 145 (1975). The concept of “T” and “R” regions can be distinguished invariantly as follows: Define the vector field k by $k_a \equiv \nabla_a(C_{bcde}C^{bcde}) = -\nabla_a(\bar{C}_{bcde}\bar{C}^{bcde})$ where C_{bcde} is the Weyl tensor and \bar{C}_{bcde} its dual tensor. In what follows we assume that $C_{bcde}C^{bcde} \neq 0$ except perhaps locally. A T region of spacetime, defined by k , corresponds to time-like k and an R region of spacetime corresponds to space-like k . These are invariant algorithmic generalizations of the classical notions of R and T regions and reduce to them in the spherically symmetric case.

- [9] See, for example, P. Musgrave and K. Lake, Classical Quantum Gravity **13**, 1885 (1996).
- [10] For a discussion of energy conditions see S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973); M. Visser, *Lorentzian Wormholes* (Springer-Verlag, New York, 1996); E. Poisson, *A Relativist's Toolkit: The Mathematics of Black-hole Mechanics* (Cambridge University Press, Cambridge, England, 2004).
- [11] The necessary violation of the dominant energy condition is a very general result. See, for example, Mars *et al.*, [3].
- [12] This is a package which runs within Maple. It is entirely distinct from packages distributed with Maple and must be obtained independently. The GRTensorII software and documentation is distributed freely on the World-Wide-Web from the address <http://grtensor.org>.