SOLUTIONS (v) - Maths for Biology

Computational Methods in Ecology and Evolution
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Exercises linear algebra

Basic Properties

Trace Proof that: $\mathrm{tr}(kA) = k\mathrm{tr}(A)$

SOLUTION: Remember that for operators like the trace we use in Latex the typography mathrm. The second property states that $\mathrm{tr}(kA) = k\mathrm{tr}(A)$, therefore:

$$\mathrm{tr}(kA) = \sum_{k=1}^{n} ka_{kk} = k\sum_{k=1}^{n} a_{kk} = k\mathrm{tr}(A)$$

Proof that: $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

SOLUTION: Again, we write explicitly the operations. Note that, as they ask us to compute the trace of the product, both matrices should be not only conformable, but they should lead to a square matrix. This means that, if for a generic product of matrices we would write $AB = \sum_j a_{ij}b_{jk} = c_{ik} = C$, in this case we know that the first dimension of the matrix A (running over the index i) must be equal to the second dimension of the matrix B (running over the index k):

$$\mathrm{tr}(AB) = \mathrm{tr}(\sum_{j}^{n} a_{ij}b_{jk}) = \sum_{i}^{m}\sum_{j}^{n} a_{ij}b_{ji} = \sum_{j}^{n}\sum_{i}^{m} b_{ji}a_{ij} = \mathrm{tr}(\sum_{i}^{m} b_{ji}a_{ik}) = \mathrm{tr}(BA).$$

More complex expressions with matrices SOLUTION: We compute first the square of A:

$$A^2 = \begin{pmatrix} 2 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 10 & 2 \\ 3 & 7 \end{pmatrix}$$

and then

$$g(A) = A^2 - A - 8I = \begin{pmatrix} 10 & 2 \\ 3 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 3 & -1 \end{pmatrix} - \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, the matrix is a root of the polynomial.

Basis and diagonalization

Cayley-Hamilton. In a previous exercise, we observed that the matrix $A = \begin{pmatrix} 1 & 3 \\ 4 & -3 \end{pmatrix}$ was a root of $g(x)$. Now, the Cayley-Hamilton THEOREM states that every matrix is a root of its own characteristic polynomial. Demonstrate that this is true with this matrix as well.

SOLUTION: First of all, we compute the characteristic polynomial making $\det(A - \lambda I) = 0$:

$$\det \begin{pmatrix} 1-\lambda & 3 \\ 4 & -3-\lambda \end{pmatrix} = 0 \Rightarrow -(1-\lambda)(3+\lambda) - 12 = 0 \Rightarrow -(3+\lambda-3\lambda-\lambda^2) - 12 = \lambda^2 + 2\lambda - 15 = 0.$$

We proof now that the matrix is a root of the polynomial, again we compute the square of the matrix first:

$$A^2 = \begin{pmatrix} 1 & 3 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 13 & -6 \\ -8 & 21 \end{pmatrix}$$

and substitute in the polynomial expression

$$g(A) = \begin{pmatrix} 13 & -6 \\ -8 & 21 \end{pmatrix} + 2 \begin{pmatrix} 1 & 3 \\ 4 & -3 \end{pmatrix} - 15 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

q. e. d.

Basis. Determine if the following vectors constitute a basis in $\mathbb{R}^3$:

$$a = (1,1,1);\ b = (1,2,3);\ c = (2,-1,1)$$

SOLUTION: The three vectors constitute a basis if they are linearly independent. We will write the vectors in matrix form and we will see if we can show, using elementary operations, that the matrix can be written in a row echelon form:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{pmatrix}$$

As there are not null rows the three vectors are linearly independent, and then they constitute a basis.

Subspaces. First of all, we should know what is meant with subspaces. When we find a basis of a space, we know that the vectors that constitute the basis are linearly independent. Some times, one of the vectors is linearly dependent, but all the other vectors are independent, and thus they are a basis not of the whole space but of some subspace with less dimensions than the original one. For example, if we get the vectors in the standard basis $e_1 = (1,0,0)$ and $e_2 = (0,1,0)$ and then we consider a vector which is linear combination of these two like $u = e_1 + e_2 = (1,1,0)$, the three vectors $\{e_1, e_2, u\}$ are not a basis of $\mathbb{R}^3$ but $\{e_1, e_2\}$ are a subasis of $\mathbb{R}^3$ (indeed, if we consider the first two components only they are a basis of $\mathbb{R}^2$). In the exercise, they tell us that two Subspaces of $\mathbb{R}^3$ that we will call U and W are described by the following vectors:

$$u_1 = (1,1,-1);\ u_2 = (2,3,-1);\ u_3 = (3,1,-5);$$

$$w_1 = (1,-1,-3);\ w_2 = (3,-2,-8);\ w_3 = (2,1,-3).$$

You should represent each subspace with a matrix and, reducing each matrix to its row canonical form, show that $U = W$.

SOLUTION: We start representing the matrix U and reducing it to row echelon form

$$U = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -1 \\ 3 & 1 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

but the problem is asking us for the canonical form. The canonical form of a matrix can be obtained from the row echelon form considering two additional properties:

- The main value of each row (the first non vanishing element of the row, highlighted above in red) is equal to one.
- The main value of each row is the only non-vanishing element in its column.

Therefore, in the last equivalent matrix we should still perform one more operation to obtain the canonical form. Making $L1 - L2 \rightarrow L1$

$$U \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

we obtain a matrix that fulfills the above conditions. Proceeding similarly with W:

$$W = \begin{pmatrix} 1 & -1 & -3 \\ 3 & -2 & -8 \\ 2 & 1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Changing the basis. Consider the following basis in $\mathbb{R}^3$:

$$S = \{s_1 = (1,2,0), s_2 = (2,3,1), s_3 = (0,2,3)\}$$

You should look for:

- The matrix P that changes the basis from the standard basis $E = \{e_1, e_2, e_3\}$ to the basis of S.

SOLUTION: Since E is the generic base, the matrix we are looking for is given by the S vectors written in columns

$$P = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 1 & 3 \end{pmatrix}.$$

Be careful not to get confused because the first row and column are equal, but this was just a coincidence. How is this result explained? Well, we know that the vectors in the new base can be written as linear combinations of the vectors in the standard base:

$$\begin{aligned} s_1 &= c_{11}e_1 + c_{12}e_2 + c_{13}e_3 \\ s_2 &= c_{21}e_1 + c_{22}e_2 + c_{23}e_3 \\ s_3 &= c_{31}e_1 + c_{32}e_2 + c_{33}e_3 \end{aligned}$$

where the matrix $C = c_{ij}$ contains the coefficients we need to change the base. Let's try to obtain the vector s_2 using the matrix C. If I make $s_2 = Ce_2$ let's see what happens:

$$s_2 = Ce_2 = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (c_{12}, c_{22}, c_{32})$$

which can be written as $s_2 = c_{12}e_1 + c_{22}e_2 + c_{32}e_3$. But... if you see how we wrote s_2 above, this is not correct! Nevertheless, if instead of using the matrix C we get its transpose matrix $C^t = P$, you can try the same exercise ($s_1 = Pe_1$) and you will obtain $s_2 = c_{21}e_1 + c_{22}e_2 + c_{23}e_3$. Therefore, the matrix of change of basis is the transpose of the matrix of the coefficients, the matrix P that we wrote explicitly.

- Take a generic vector $x = (a, b, c)$ and write it in the basis of S.

SOLUTION: Since we don't know in which base it is written the vector x, we assume it is in the standard basis. We want to write x in a new basis S which we know should be written as a linear combination of the vectors of the new basis S as

$$x = us_1 + vs_2 + ws_3$$

being u, v and w three coefficients that we should determine. During the lectures, we said that if the basis is not orthogonal, this problem is painful. A quick check will show us that this is the case because, for example, the scalar product $s_1 s_2 = 5$ so it is not an orthogonal basis. In this case, we learned that we should generate a system of equations projecting x with respect to s_1, s_2 and s_3, respectively, which yields:

$$
\begin{aligned}
x s_1 &= u s_1 s_1 + v s_2 s_1 + w s_3 s_1 \\
x s_2 &= u s_1 s_2 + v s_2 s_2 + w s_3 s_2 \\
x s_3 &= u s_1 s_3 + v s_2 s_3 + w s_3 s_3
\end{aligned}
$$

and, in this system, you have all the scalar products $u s_i$ and $s_i s_j$, therefore you can solve for u, v and w. If you tried to do that, you probably found that it is indeed painful to solve it. So let's look for another strategy. If we have already the system in the basis S and we should find its coordinates in the standard basis, the problem is much easier. We start with the same equation:

$$x = u s_1 + v s_2 + w s_3$$

and we substitute the expression of the s_i vectors in the standard basis:

$$x = u(c_{11}e_1 + c_{12}e_2 + c_{13}e_3) + v(c_{21}e_1 + c_{22}e_2 + c_{23}e_3) + w(c_{31}e_1 + c_{32}e_2 + c_{33}e_3)$$

which, rearranging, leads to

$$x = (uc_{11} + c_{21}v + c_{31}w)e_1 + (uc_{12} + c_{22}v + c_{32}w)e_2 + (uc_{13} + c_{23}v + c_{33}w)e_3$$

and this means that

$$
\begin{aligned}
a &= uc_{11} + c_{21}v + c_{31}w \\
b &= uc_{12} + c_{22}v + c_{32}w \\
c &= uc_{13} + c_{23}v + c_{33}w.
\end{aligned}
$$

But, wait a minute, this means that the coeficients in the standard basis $x = (a, b, c)$ can be obtained by multiplying those in the new basis $y = (u, v, w)$ by the transformation matrix P, i.e. $x = Py$. Then, given that the matrix P is built with three vectors that are linearly independent, it is invertible, and we can then say that $P^{-1}x = y$.

Using the formula for the inverse, I get:

$$P^{-1} = \frac{1}{|P|}\mathrm{adj(P)} = -\frac{1}{5}\begin{pmatrix} 7 & -6 & 4 \\ -6 & 3 & -2 \\ 2 & -1 & -1 \end{pmatrix}.$$

Then, a generic vector $x = (a, b, c)$ will be written in the new coordinates as ($y = P^{-1}x$):

$$y = (u, v, w) = \frac{-1}{5}\left(7a - 6b + 4c,\ -6a + 3b - 2c,\ 2a - b - c\right).$$

At this point, you may be thinking that there is something weird happening. Because, in the previous section, to obtain the coordinates of e_1 in the new basis we made $s_1 = Pe_1$. So now, by symmetry, we may be tempted to say as well that $y = Px$, because x is in the standard basis and y in the basis of S, but there is a subtle question here.

Imagine that you are working with a basis in which you measure in, let's say, meters. And now make a change of basis which consists simply in reducing the scale from meters to centimeters, what means that you divide the scale by 100. Now, consider a vector in the first basis of length one meter, how would you obtain that distance in the new scale? You may say, well, just divide by 100, but not, because otherwise you will say that the vector is of length 1 cm. and we know that it is 100 cm., Therefore, to transform the vector we do the opposite than what we did with the transformation of the basis, in this case multiplying by 100 to keep the vector invariant under the change of scale (namely, to measure the same distance with other units). Therefore, in order to keep vectors invariant under a change of basis, which is equivalent to a change of units, we should make them "contravary" with respect to the change. This is the reason why we say that vectors are "contravariants" (of course there exist as well "covectors", which "covary" with the change, but we are not interested on these here). Consequently, to obtain the new coordinates, we should do instead $y = P^{-1}x$.

- Which is the matrix Q that allows you to change back the coordinates from the basis S to E?

SOLUTION: $Q = P$.

- Bring the coordinates of generic vector you obtained in the basis S back to the basis E.

SOLUTION: Just make the product Py and you will get back the vector $x = (a, b, c)$.

Diagonalization (i). Diagonalize the following matrix A, and find the matrix P such that $D = P^{-1}AP$, being D the diagonal matrix. Once you obtain D and P, demonstrate that $A = PDP^{-1}$. Finally, find the value of A^5.

$$A = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}$$

SOLUTION: We start finding the roots of the characteristic polynomial $\det(A - \lambda I) = 0$:

$$\begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 10 = 0.$$

We get the roots $\lambda_1 = 5$ and $\lambda_2 = -2$, which are the eigenvalues of A. To find the first eigenvector we subtract λ_1 from the diagonal of A:

$$M = \begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix}$$

and we solve the homogeneous system of equations $Mx = 0$:

$$\begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Longrightarrow \begin{array}{rcrcl} -x & + & 2y & = & 0 \\ 3x & - & 6y & = & 0 \end{array}$$

which we observe has only one independent solution, because both equations are proportional. Therefore, we have infinite solutions $v_1 = (2a,\ a)$ and we can get a simple one making $a = 1$, which yields $v_1 = (2,\ 1)$. Proceeding similarly for $\lambda_2 = -2$ you should obtain the vector $v_2 = (-a,\ 3a)$, and we pick up as particular solution $v_2 = (-1,\ 3)$. Therefore, the matrix of the eigenvectors will be the matrix of change of basis

$$P = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}.$$

If we now compute the inverse of P

$$P^{-1} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$$

You can demonstrate that the diagonal matrix D, that we already know after computing the eigenvalues of A:

$$D = \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}$$

can be obtained with the matrix P and its inverse with the formula $D = P^{-1}AP$, and the other way around, we can obtain A from D making $A = PDP^{-1}$.

An interesting use of the diagonal matrices is that we can perform some operations that would be difficult otherwise. Making A^5 would be annoying, it is much easier to compute $A^5 = P^{-1}D^5P$

$$A^5 = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2674 & -1353 \\ -902 & 419 \end{pmatrix}$$

Important Note! we will do the same if we want to compute complex operations on A, like $\sqrt{A}$.

Diagonalization (ii). Consider the following matrix:

$$A = \begin{pmatrix} 1 & -1 \\ 0 & -2 \end{pmatrix}$$

1. Diagonalize the matrix A, and find the matrix P such that $D = P^{-1}AP$, being D diagonal.
2. Find the matrix P^{-1}.
3. Verify the Cayley-Hamilton theorem, which states that any square matrix is a root of its characteristic polynomial.

SOLUTION: 1) We start finding the roots of the characteristic polynomial, $\det(A - \lambda I) = 0$:

$$\begin{vmatrix} 1-\lambda & -1 \\ 0 & -2-\lambda \end{vmatrix} = \lambda^2 + \lambda - 2 = 0.$$

We get the roots $\lambda_1 = 1$ and $\lambda_2 = -2$, which are the eigenvalues of A. To find the first eigenvector we subtract λ_1 from the diagonal of A:

$$M = \begin{pmatrix} 0 & -1 \\ 0 & -3 \end{pmatrix}$$

and we solve the homogeneous system of equations $Mx = 0$:

$$\begin{pmatrix} 0 & -1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Longrightarrow \qquad \begin{array}{rcl} -y & = & 0 \\ -3y & = & 0 \end{array}$$

from which we observe that $y = 0$ and x may have any value, so we can write the first eigenvector as $v_1 = (a,\ 0)$. Getting a simple solution, $a = 1$, it yields $v_1 = (1,\ 0)$. Proceeding similarly for $\lambda_2 = -2$ you should obtain the vector $v_2 = (a,\ 3a)$, and we pick up as particular solution $v_2 = (1,\ 3)$. Therefore, the matrix of the eigenvectors will be the matrix of change of basis

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}.$$

2) The inverse of P can be computed as $P^{-1} = \frac{1}{\det P}\mathrm{adj}(A)$, yielding

$$P^{-1} = \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix}.$$

3) We substitute λ by A in the polynomial and we get

$$A^2 + A - 2I = \begin{pmatrix} 1 & -1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & -2 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & -2 \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = \ldots$$

$$\ldots = \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & -2 \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Diagonalization (iii). Look for the eigenvalues of the following matrix B, and then look for a set of eigenvectors linearly independent. Is it B diagonalizable?

$$B = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$$

SOLUTION: We proceed similarly, looking for the roots of the characteristic polynomial $\det(B - \lambda I) = 0$:

$$\begin{vmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{vmatrix} = \lambda^3 - 12\lambda - 16 = (\lambda+2)^2(\lambda-4) = 0.$$

The roots are $\lambda_1 = -2$ (twice. i.e. it has multiplicity two) and $\lambda_2 = 4$. If we substitute λ_1 we get the following homogeneous system:

$$\begin{array}{rcrrr} x & - & y & +z = & 0 \\ 7x & - & 7y & +z = & 0 \\ 6x & - & 6y & = & 0 \end{array}$$

where it is immediate to see that $x = y$ and $z = 0$, therefore $v_1 = (a,\ a,\ 0)$. It is interesting to note that, even if λ_1 is a "double" root, we only get one linearly independent vector. As the matrix is of dimension 3, we need three linearly independent vectors for the matrix to be diagonalizable, so we should expect that the root with multiplicity two provides two linearly independent vectors. As it is not the case, the matrix will not be diagonalizable. If we substitute now λ_2:

$$\begin{array}{rcrrr} 7x & - & y & +z = & 0 \\ 7x & - & y & +z = & 0 \\ 6x & - & 6y & +6z = & 0 \end{array}$$

we observe that the third equation is not compatible with the first two equations unless $x = 0$, in which case we get $y = z$. Therefore, the second eigenvalue provides only one eigenvector of the kind $v_2 = (0,\ a,\ a)$ and we conclude that, given that the diagonalization only generates two linearly independent vectors, we cannot find a matrix to change the basis, and the matrix B is not diagonalizable.

Diagonalization (iv) Diagonalize the following matrix A, and find the matrix P such that $D = P^{-1}AP$, being D the diagonal matrix. Once you obtain D and P, demonstrate that $A = PDP^{-1}$. Finally, find the value of A^5.

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}.$$

SOLUTION: We start finding the roots of the characteristic polynomial $\det(A - \lambda I) = 0$:

$$\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 = 0.$$

We get the roots $\lambda_1 = 1$ and $\lambda_2 = 4$, which are the eigenvalues of A. To find the first eigenvector we subtract λ_1 from the diagonal of A:

$$M = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

and we solve the homogeneous system of equations $Mx = 0$, which obviously leads to $-x = 2y$ and thus $v_1 = (2a,\ -a)$. Making $a = 1$, it yields $v_1 = (2,\ -1)$. Proceeding similarly for $\lambda_2 = 4$ you should obtain the vector $v_2 = (a,\ a)$, and we pick up as particular solution $v_2 = (1,\ 1)$. Therefore, the matrix of the eigenvectors will be the matrix of change of basis

$$P = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}.$$

If we now compute the inverse of P using the formula $P^{-1} = \frac{1}{\det P}\mathrm{adj}(A)$ we get

$$P^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} -$$

You can demonstrate that the diagonal matrix D, that we already know after computing the eigenvalues of A

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix},$$

can be obtained with the matrix P and its inverse with the formula $D = P^{-1}AP$. And the other way around, we can obtain A from D making $A = PDP^{-1}$. We finally compute $A^5 = PD^5P^{-1}$

$$A^5 = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1024 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 342 & 682 \\ 341 & 683 \end{pmatrix}.$$