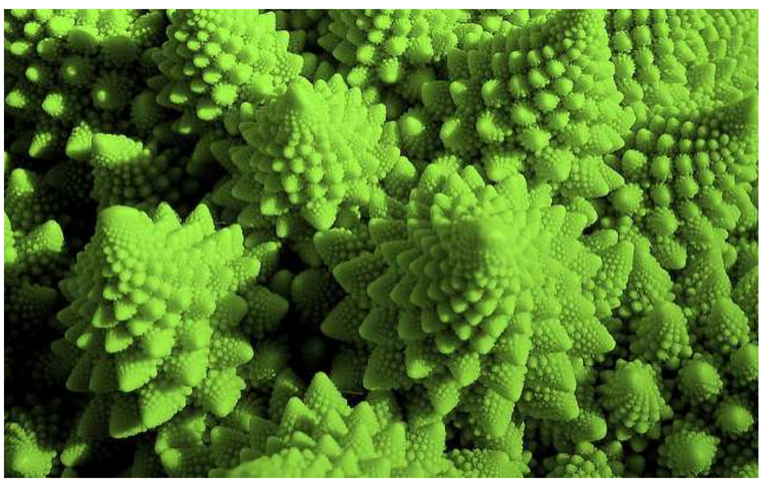
Maths 4Biology

Alberto Pascual-García

This project contains supporting notes and scripts for the course "Maths 4 Biology" of the Master for Computational Methods in Ecology and Evolution at Imperial College London. It is a fork of the the project "Mathematics Boot Camp" proposed by Viktor Jirsa for graduate courses at the Center for Complex Systems & Brain Sciences at Florida Atlantic University. Similar in spirit, by no means it is intended to compete with the traditional introductory courses in mathematics taught at universities. Rather it should be viewed as a synopsis of the mathematical tools most probably encountered in scientific applications. With respect to Jistra's notes, the present project expands on some of the topics and includes first-Order Differential Equations, of oustanding importance in Ecology. On the other hand, Fourier analysis is not considered in the course, although it is maintained in these notes for completeness.



Zürich, December 2018
Institute of Integrative Biology
ETH

Contents

List of Figures

1 Function Theory

1.1 Foundations

Definition: A *function* f is a rule which unambiguously assigns $y \in B$ to each $x \in A$.

"$x \in A$" means that "x is an element of A".

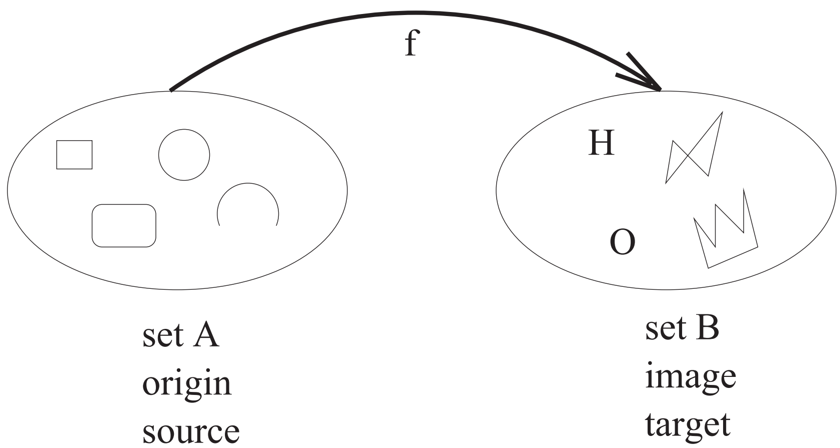


Figure 1: Illustration of a function

Notation:

$$
\begin{array}{rll}
f: & A \to B & \\
f: & A \to B & x \in A,\ y \in B \\
 & y = f(x) & x \in A,\ y \in B \\
 & y = y(x) & x \in A,\ y \in B
\end{array}
$$

Examples:

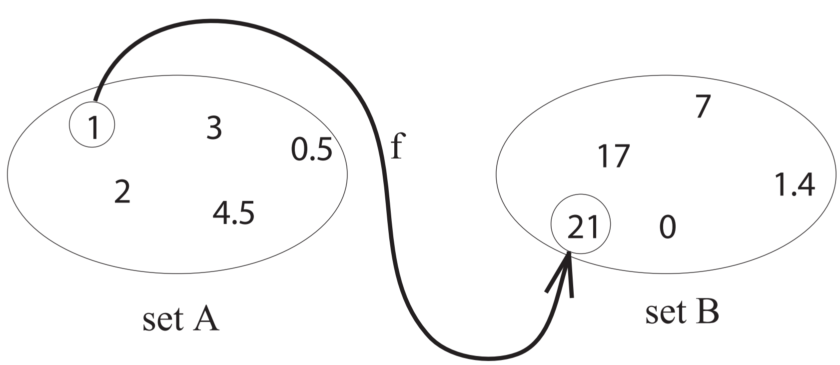
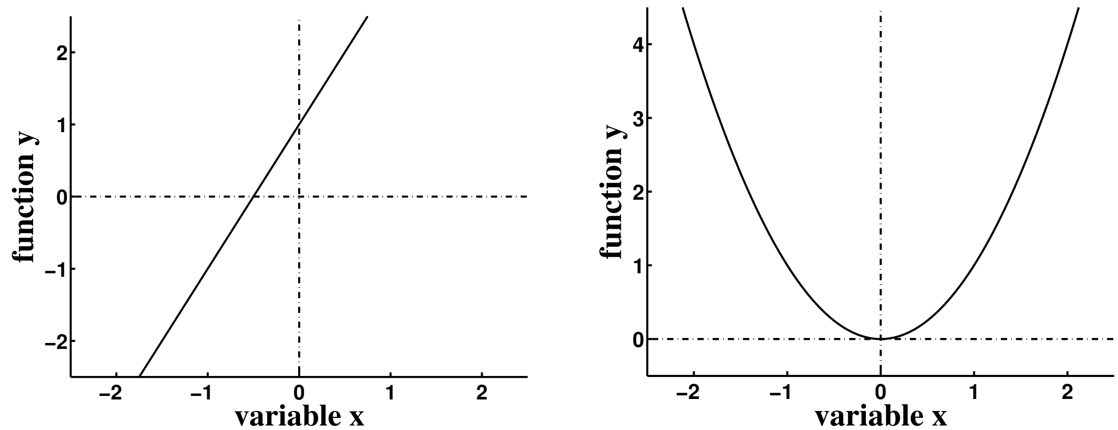


Figure 2: Assigning an element $\{21\} \in B$ to $\{1\} \in A$

Most commonly, functions are defined by equations: $y = f(x) = 2x + 1$ $\quad y = f(x) = x^2$

Graphical representation:



(a) Function $y = 2x + 1$ (b) Function $y = x^2$

Figure 3: A linear and a quadratic function

1.2 Inverse Functions

f^{-1} denotes the inverse of the function f.

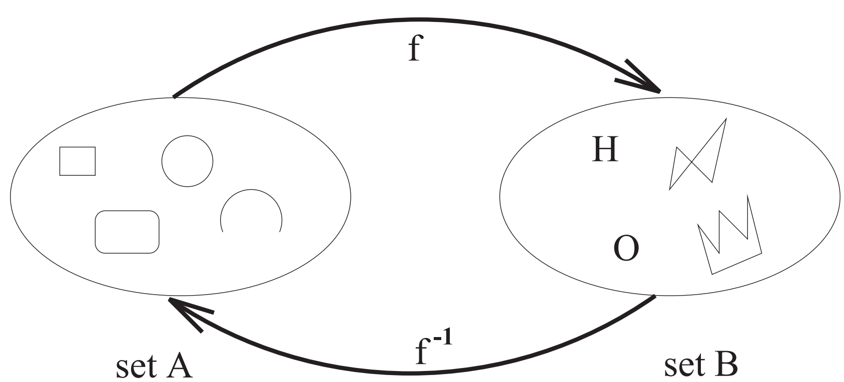


Figure 4: Inverse function

Notation:

$$f^{-1} : \ B \to A$$

$$x = f^{-1}(y) \qquad \text{where} \qquad y = f(x)$$

Graphically the inverse can be constructed as the mirror image of the function at the first bisector. This method always works, but caution is asked for, because the inverse may not be unique and require more detailed discussion.

Example:

$$y = f(x) = 2x + 1 \qquad x = f^{-1}(y) = \frac{1}{2}(y - 1)$$

Note: There is not always an inverse function!

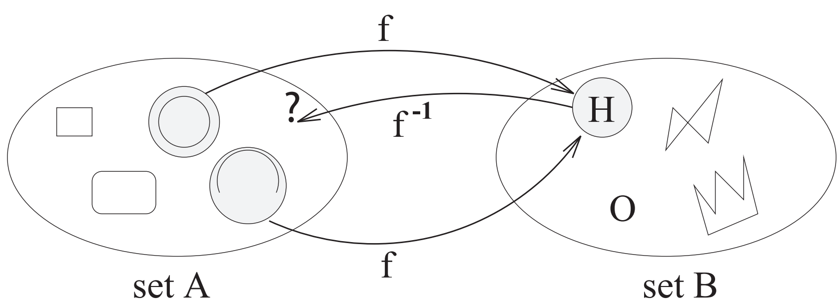
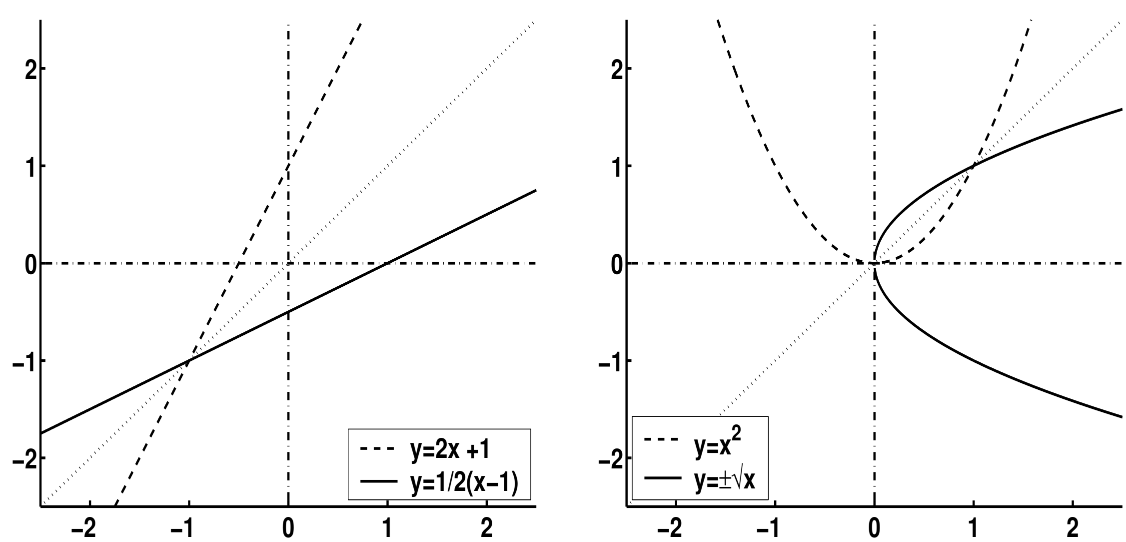


Figure 5: The inverse f^{-1} is not unique, thus not a function.

Example:

$$y = f(x) = x^2$$

$$x = \sqrt{y} \qquad \text{or} \qquad x = -\sqrt{y}$$



(a) $f(x) = 2\,x + 1 \;\rightarrow\; f^{-1}(x) = \frac{1}{2}(x - 1)$ (b) $f(x) = x^2 \;\rightarrow\; f^{-1}(x) = \pm\sqrt{x}$

Figure 6: Graphical construction of the inverse function

1.3 Implicit Functions

A function is not given explicitly as in $y = f(x)$, but implicitly by $F(x, y) = 0$.

Example: Unit circle: $F(x, y) = x^2 + y^2 - 1 = 0$.

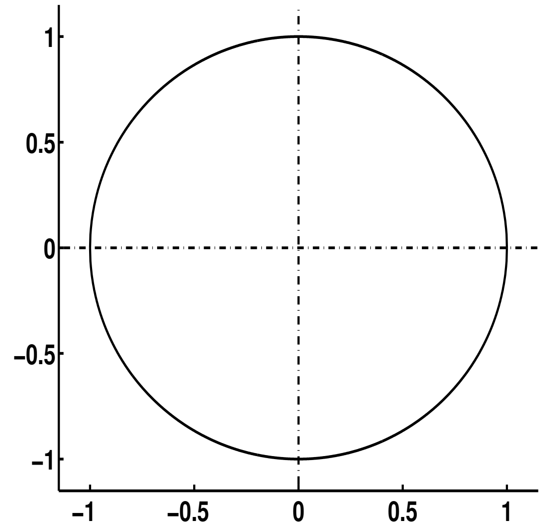


Figure 7: Unit circle

The implicit representation of the unit circle also needs additional conditions to become unique and thus a function: a local neighborhood has to be defined, e.g. $y = \sqrt{1 - x^2}$ for $x \in (-1; 1)$, $y > 0$ and $y = -\sqrt{1 - x^2}$ for $x \in (-1; 1)$, $y < 0$.

1.4 Polynomials

Polynomials are defined as a class of functions of the form

$$y = a_0 + a_1\, x + a_2\, x^2 + \cdots + a_N\, x^N = \sum_{n=0}^{N} a_n\, x^n$$

where the function y is said to be a polynomial of order N.

Example:

$$y = \underbrace{2}_{a_2} x^2 + \underbrace{8}_{a_1} x + \underbrace{4}_{a_0}$$

Goal: To achieve a qualitative understanding of a given function without computing each value.

Approach:

$$y = \sum_{n=0}^{N} a_n\, x^n \qquad \text{where we assume } \; a_N > 0$$

Step 1: If N is even, then $x \to \pm\infty : y \to \pm\infty$; if N is odd, then $x \to \pm\infty : y \to \mp\infty$. Note: If $a_N < 0$, the behavior is the opposite.

Step 2: A polynomial of order N has N roots which are the solutions of $f(x) = 0$.

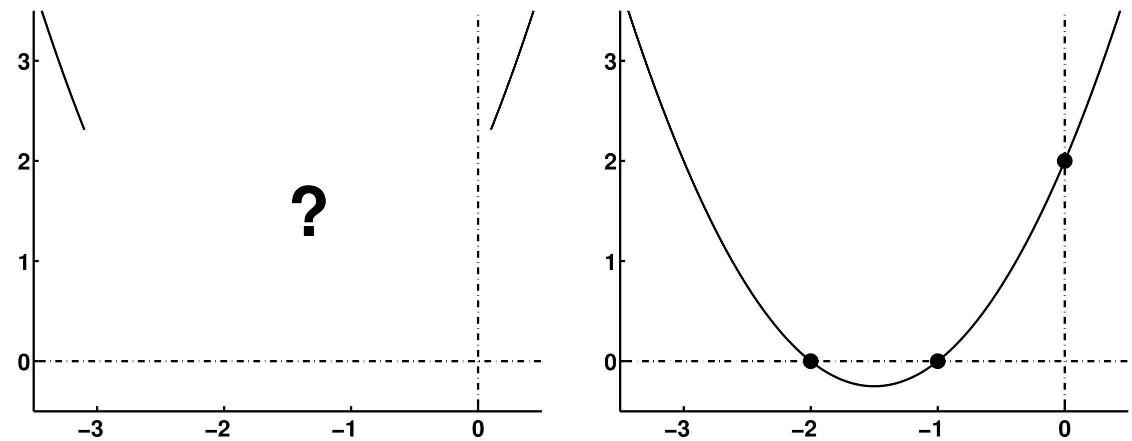
Set $y = 0 : f(x) = x^N + a_{N-1}\, x^{N-1} + \cdots + a_1\, x + a_0 = 0$

$$y = 17x + 4 \qquad y = 0 : x = -\tfrac{4}{17} \qquad \rightarrow \quad 1 \text{ root}$$

$$y = 2x^2 + 8x + 4 \qquad y = 0 : x = \tfrac{1}{4}(-8 \pm \sqrt{8^2 - 4 \cdot 2 \cdot 4}) = \tfrac{1}{4}(-8 \pm \sqrt{32}) = -2 \pm \sqrt{2} \qquad \rightarrow \quad 2 \text{ roots}$$

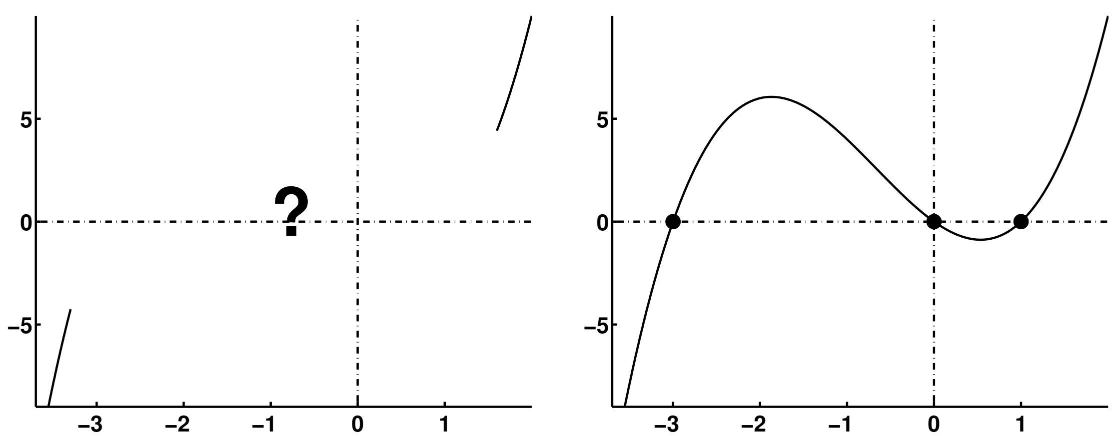
$\Rightarrow$ the roots are the locations where $f(x)$ crosses the x-axis.

Examples: Construct graph of $y = f(x)$



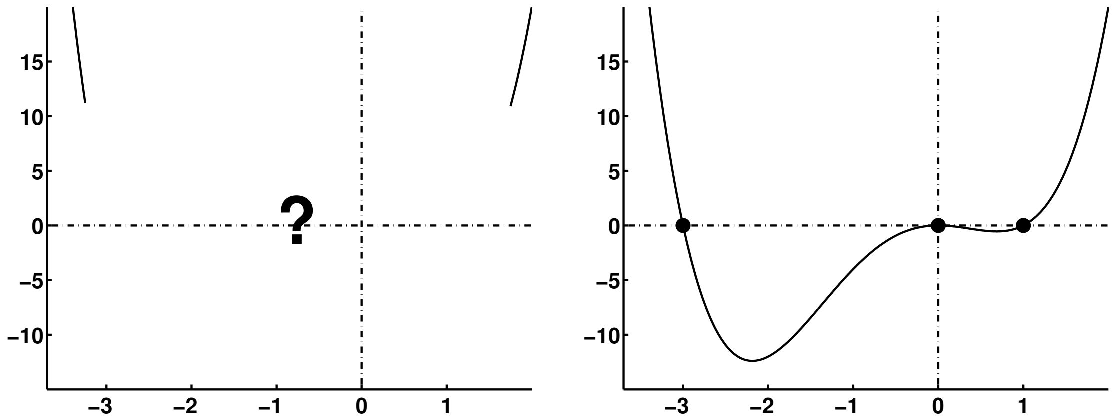
(a) Step 1: $y = x^2 + 3x + 2$ (b) Step 2: $y = x^2 + 3x + 2 = 0$

Figure 8: Graphical construction (the roots are $x = -1$ and $x = -2$)



(a) Step 1: $y = x^3 + 2x^2 - 3x$ (b) Step 2: $y = x^3 + 2x^2 - 3x = 0$

Figure 9: Graphical construction (the roots are $x = -3, x = 0$ and $x = 1$)



(a) Step 1: $y = x^4 + 2x^3 - 3x^2$ (b) Step 2: $y = x^4 + 2x^3 - 3x^2 = 0$

Figure 10: Graphical construction (the roots are $x = -3, x = 0$ and $x = 1$)

Horizontal translation is a horizontal shift of a function by x_0

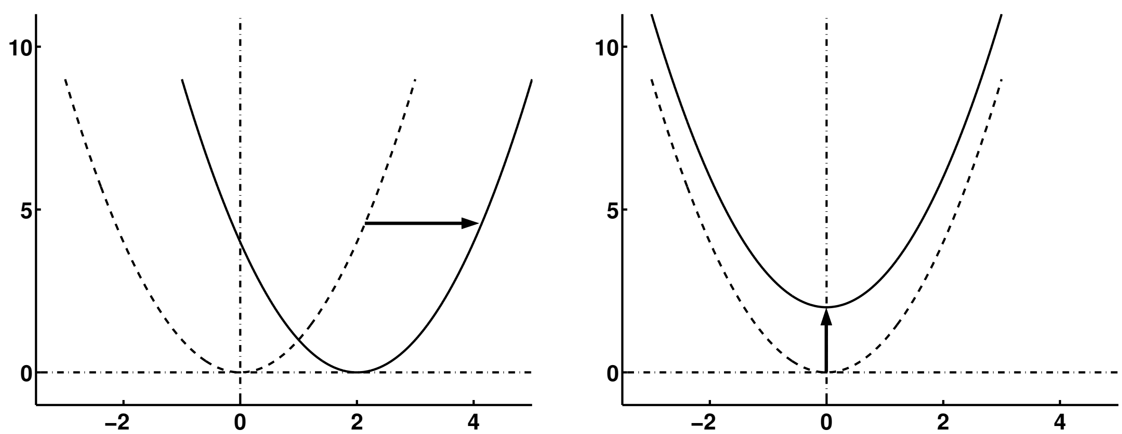
$$y = f(x) \;\rightarrow\; y = f(x - x_0)$$

Example: $y = x^2$ shift by $x_0 = 2$: $y = (x-2)^2 = x^2 - 4x + 4$

Vertical translation is a vertical shift of a function by y_0

$$y = f(x) \;\rightarrow\; y = f(x) + y_0$$

Example: $y = x^2$ shift by $y_0 = 2$: $y = x^2 + 2$



(a) Horizontal shift by $x_0 = 2 : y = (x-2)^2$ (b) Vertical shift by $y_0 = 2 : y = x^2 + 2$

Figure 11: Vertical and horizontal shift of $y = x^2$

1.5 Trigonometric Functions

Trigonometric functions are a class of periodic functions such as

$$y = f(x) = A\sin(kx + \phi) \quad \text{and} \quad y = f(x) = A\cos(kx + \phi)$$

The constant parameters are the amplitude A, the frequency k and the phase (angle) ϕ. The period is defined as $\lambda = 2\pi/k$ and the trigonometric functions fulfill the relation $f(x+\lambda) = f(x)$.

Special values of trigonometric functions (midnight stuff)								
x	0	$\pi/6 = 30°$	$\pi/4 = 45°$	$\pi/3 = 60°$	$\pi/2 = 90°$	$\pi = 180°$	$3\pi/2 = 270°$	$2\pi = 360°$
$\sin x$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	0	-1	0
$\cos x$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	-1	0	1

Horizontal translation (shift) by means of ϕ:

$$\sin(x + \frac{\pi}{2}) = \cos x \qquad \cos(x + \frac{\pi}{2}) = -\sin x \qquad \sin(x - \frac{\pi}{2}) = -\cos x \qquad \cos(x - \frac{\pi}{2}) = \sin x$$

Useful relations between $\cos x$ and $\sin x$:

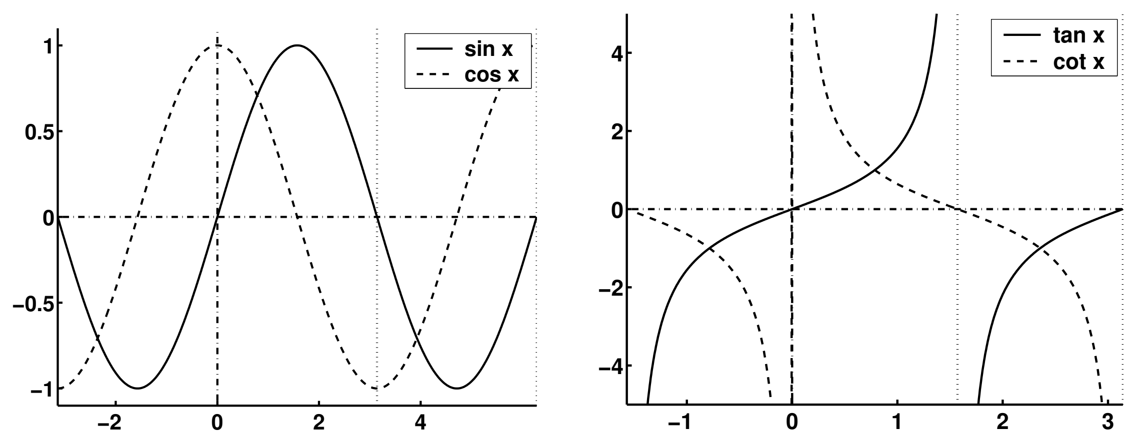
$$\cos^2 x + \sin^2 x = 1$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \qquad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin 2x = 2 \sin x \cos x \qquad \cos 2x = \cos^2 x - \sin^2 x$$

Other trigonometric functions:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$



(a) The functions $y = \sin x$ and $y = \cos x$

(b) The functions $y = \tan x$ and $y = \cot x$. Note that these are π-periodic

Figure 12: The most common trigonometric functions

1.6 Exponential Functions

Exponential functions are functions most commonly used in the form

$$y = Ae^{kx} = A \exp kx$$

with the constant parameters: A amplitude, k growth rate if $k > 0$ and the damping or fall off, if $k < 0$, and e Euler number: 2.714....

Note that $e^0 = 1$ and $e^{-x} = \frac{1}{e^x}$.

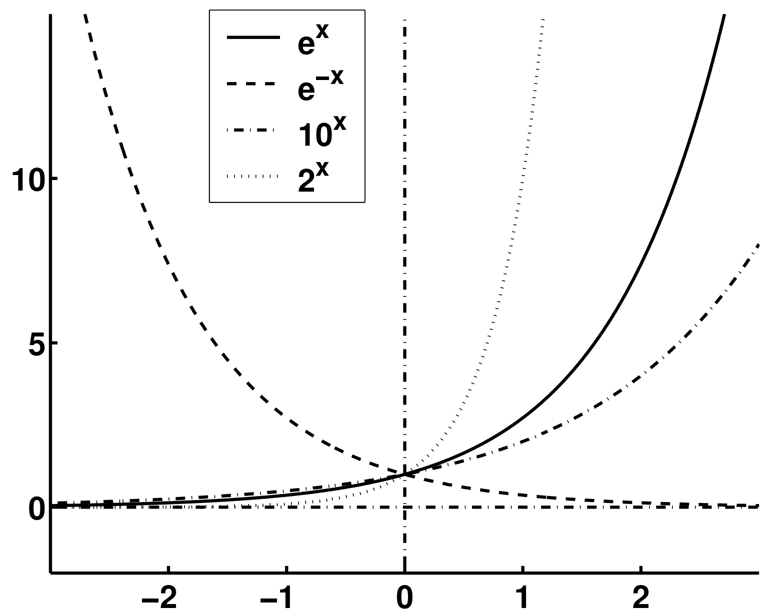


Figure 13: Exponential functions

1.7 Hyperbolic Functions

Hyperbolic functions are of the form

$$y = f(x) = \cosh x = \frac{1}{2}(e^x + e^{-x}) \qquad \text{hyperbolic cosine}$$

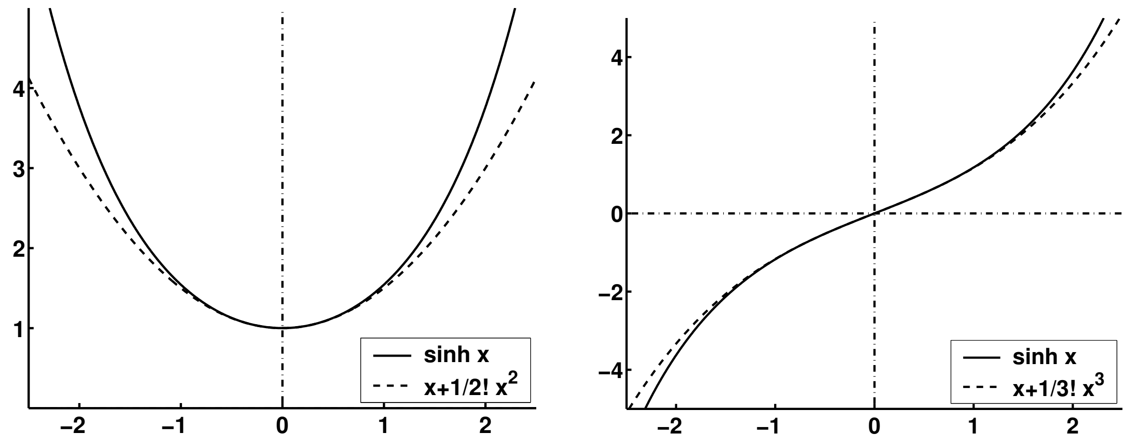
and

$$y = f(x) = \sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \text{hyperbolic sine}$$

They have similar properties as the trigonometric functions such as a representation by exponentials (as we shall see later), and their derivatives convert into each other. But the hyperbolic functions are *not* periodic.

Other hyperbolic functions:

$$y = \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad y = \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



(a) $y = \sinh x$ and $y = x + \frac{1}{3!}\,x^3$ (b) $y = \cosh x$ and $y = 1 + \frac{1}{2!}\,x^2$

Figure 14: Hyperbolic functions

1.8 Basic Inverse Functions

1.8.1 Logarithms

The logarithms are the inverse of the exponential functions:

$$y = a^x \quad \leftrightarrow \quad x = \log_a y \qquad \text{where} \quad 0 < y < \infty$$

Special cases:

$$\begin{array}{llllll} a = e: & y & = & \log_e x & = & \ln x & \text{natural logarithm} \\ a = 10: & y & = & \log_{10} x & = & \mathrm{lg}x & \text{decimal logarithm} \\ a = 2: & y & = & \log_2 x & = & \mathrm{ld}x & \text{dual logarithm} \end{array}$$

Remark: The most commonly used logarithm is $\ln x$, but there are certain applications for other logarithms as well. For instance, the decimal logarithm can be used to find the number of digits in a decimal number ($\mathrm{lg}4821 = 3.683 \rightarrow$ taking the whole number in front of the decimal point and adding 1 gives the number of digits, 4). Similarly, the dual logarithm can be used to find the number of bits or binary digits that are necessary to represent a number n in binary format, i.e. as zeros and ones.

Rules and tricks for dealing with logarithms:

$$\ln x^n = n \ln x$$

$$\ln x_1\, x_2 = \ln x_1 + \ln x_2 \qquad \ln \frac{x_1}{x_2} = \ln x_1 - \ln x_2$$

$$\log_a x = \frac{\ln x}{\ln a}$$

Note: Every logarithm can be expressed in terms of the natural logarithm, and every exponential function can be written in terms of the basis e

$$a^x = e^{x\,\ln a} \quad \text{with} \quad a > 0 \qquad \textbf{Very useful relation!}$$

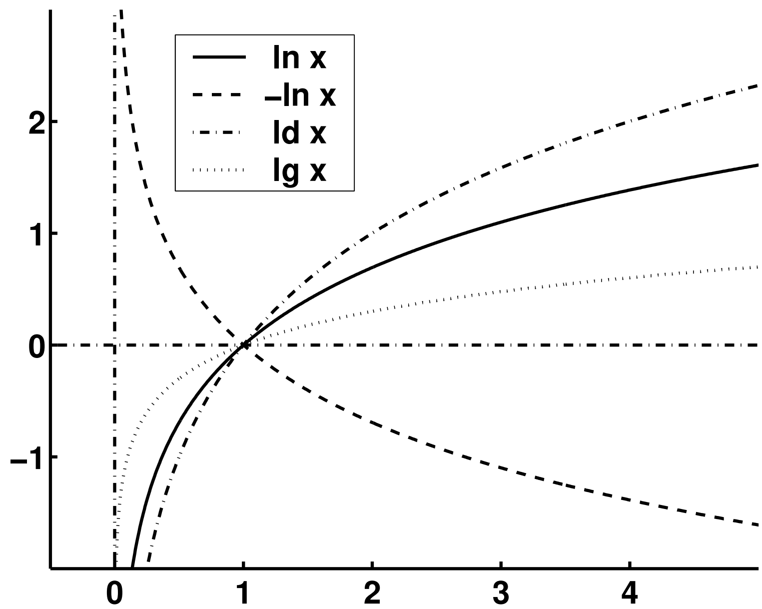
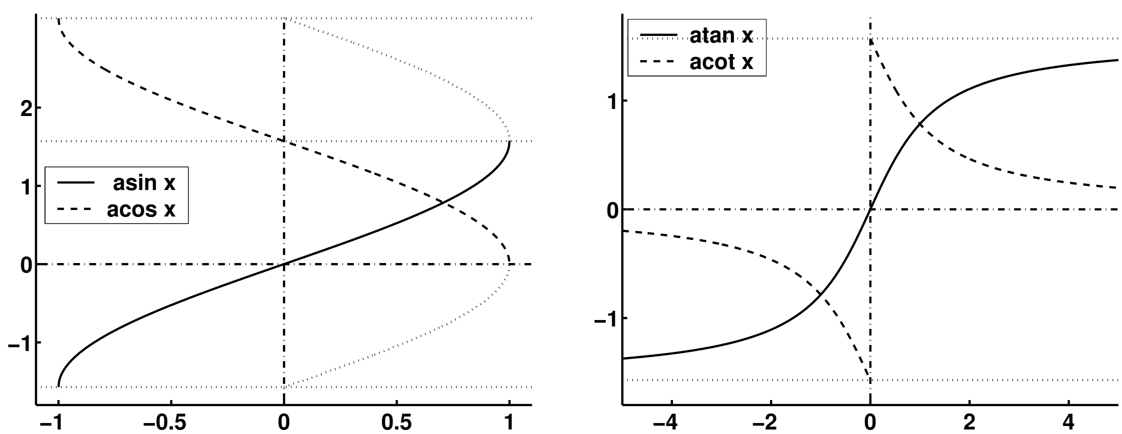


Figure 15: The logarithmic functions $y = \ln(x)$, $y = \lg(x)$, $y = \mathrm{ld}(x)$, and $y = -\ln(x)$.

1.8.2 Other inverse functions

$$
\begin{array}{lcll}
y = \sin x & \rightarrow & x = \arcsin y & \text{arc sine} \\
y = \cos x & \rightarrow & x = \arccos y & \text{arc cosine} \\
y = \tan x & \rightarrow & x = \arctan y & \text{arc tangent} \\
y = \cot x & \rightarrow & x = \mathrm{arccot}\ y & \text{arc cotangent} \\
y = \sinh x & \rightarrow & x = \mathrm{arcsinh}\ y & \\
 & & x = \ln(y + \sqrt{y^2 + 1}) & \text{area sine hyperbolic} \\
y = \cosh x & \rightarrow & x = \mathrm{arccosh}\ y & \\
 & & x = \ln(y + \sqrt{y^2 - 1}) & \text{area cosine hyperbolic}
\end{array}
$$



(a) Arc sine, and arc cosine

(b) Arc tangent, and arc cotangent

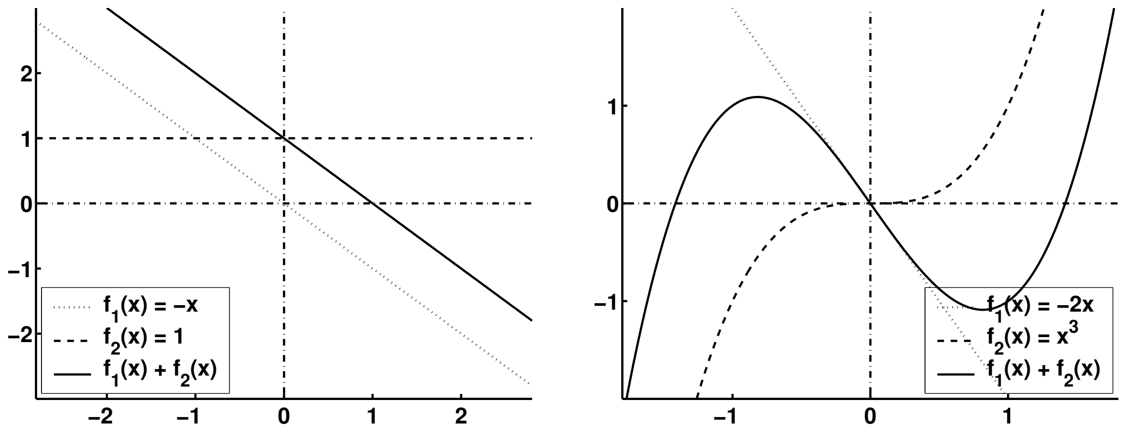
Figure 16: The inverse of the trigonometric functions

1.9 Elementary Combinations of Functions

1.9.1 Superposition

Two functions are superimposed on each other by adding their values for the same x.

$$y = f_1(x) + f_2(x)$$



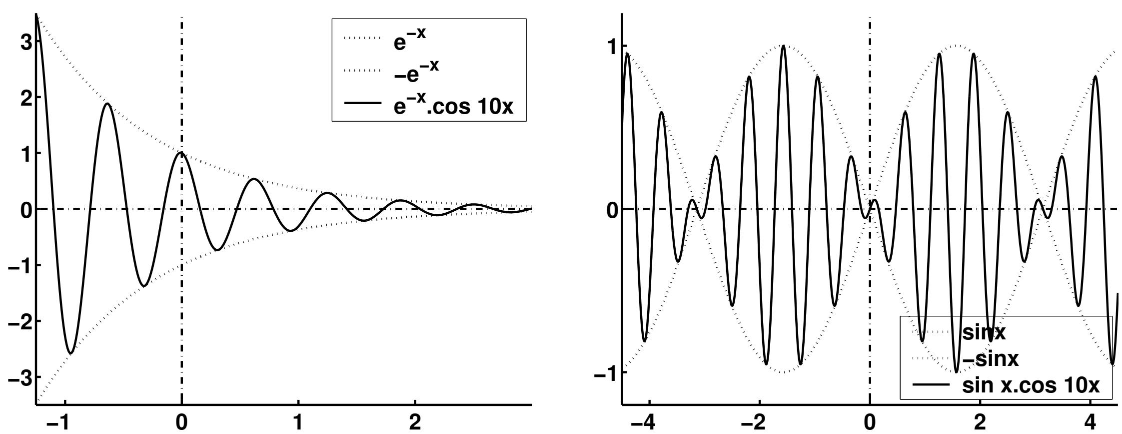
(a) Superposition of $f_1(x) = -x$ and $f_2(x) = 1$ (b) Superposition of $f_1(x) = -2x$ and $f_2(x) = x^3$

Figure 17: Superposition of lines and functions

1.9.2 Modulation

A function is modulated by another function by multiplying their values for the same x.

$$y = f_1(x)\, f_2(x)$$



(a) e^{-x} is the envelope function of $\cos 10x$ (b) $\cos 10x$ is modulated with $\sin x$

Figure 18: Modulation of functions

2 Differential and Integral Calculus

First derivatives of simple functions were studied by Galileo Galilei (1564-1642) and Johannes Kepler (1571-1630). A systematic theory of differential calculus was developed by Isaac Newton (1643-1727) and Gottfried Wilhelm Leibniz (1646-1710).

2.1 Difference Quotient

The difference quotient becomes the differential in the limit $h \to 0$ and describes the slope of a function $y = f(x)$ at a given point x.

$$y'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}$$

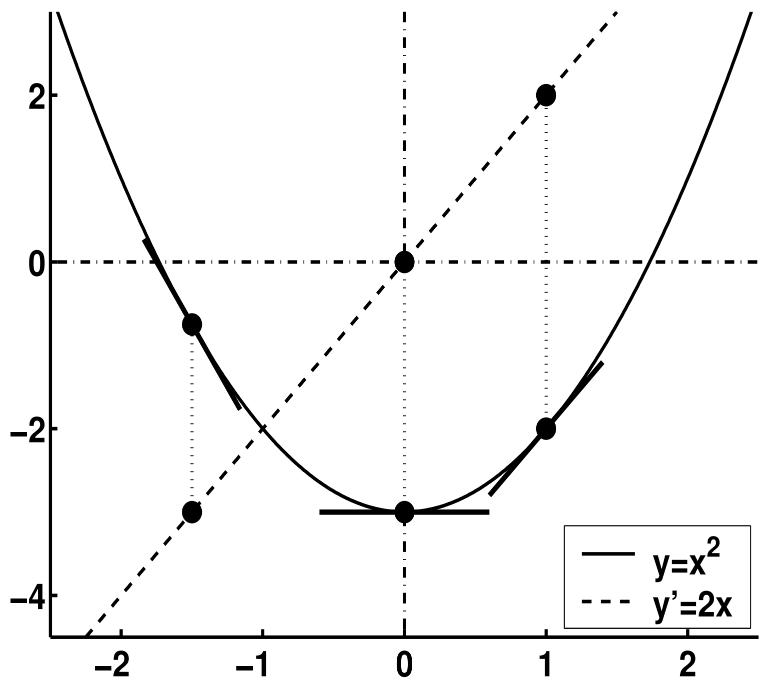


Figure 19: The slope of a curve is found from its derivative.

Notation: The limit value of the difference quotient is called the derivative of a function $f(x)$. Derivatives are denoted by

$$y'(x)\ ,\ \frac{dy}{dx}\ ,\ \frac{df}{dx}\ ,\ \frac{d}{dx}f(x) \qquad \text{or sometimes in physics: } \dot{y}(t)$$

Note: Here we consider first-order derivatives only.

Example: $\quad y = f(x) = x^2$

$$y' = \frac{dy}{dx} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2hx + h^2}{h} = 2x$$

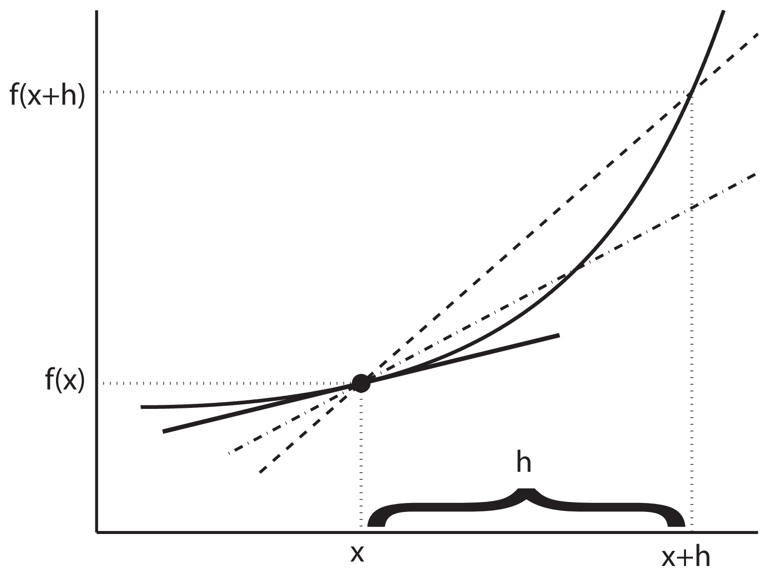


Figure 20: Slope as h$\rightarrow$ 0.

2.2 Derivatives of Elementary Functions

2.2.1 Polynomials

$$y = x^2 \quad \rightarrow \quad \frac{dy}{dx} = 2x \qquad\qquad \text{more general:} \quad y = x^n \quad \rightarrow \quad \frac{dy}{dx} = nx^{n-1}$$

2.2.2 Trigonometric functions

$$y = \sin x \quad \rightarrow \quad \frac{dy}{dx} = \cos x \qquad\qquad y = \cos x \quad \rightarrow \quad \frac{dy}{dx} = -\sin x$$

2.2.3 Exponential functions

$$y = e^x \quad \rightarrow \quad \frac{dy}{dx} = e^x$$

2.2.4 Hyperbolic functions

$$y = \sinh x \quad \rightarrow \quad \frac{dy}{dx} = \cosh x \qquad\qquad y = \cosh x \quad \rightarrow \quad \frac{dy}{dx} = \sinh x$$

2.2.5 Logarithms

$$y = \ln x \quad \rightarrow \quad \frac{dy}{dx} = \frac{1}{x}$$

2.3 The Basic Rules for Calculating Derivatives

If the derivatives of two functions $u(x)$ and $v(x)$ exist on an interval $a < x < b$, then the derivatives of their combinations exist as well, i.e.

$$u+v, \quad \alpha\, u \text{ with } \alpha \in \mathbb{R}, \quad u\, v, \quad \frac{u}{v} \text{ if } v(x) \neq 0 \text{ for } a < x < b$$

Rules:

$(u+v)' = \frac{d}{dx}\{u+v\} = u' + v'$	**derivatives are additive**
$(\alpha\, u)' = \frac{d}{dx}\{\alpha\, u\} = \alpha\, u'$	**multiplication with a scalar**
$(u\, v)' = \frac{d}{dx}\{u\, v\} = u'\, v + u\, v'$	**product rule**
$(\frac{u}{v})' = \frac{d}{dx}\{\frac{u}{v}\} = \frac{u'\, v - u\, v'}{v^2}$	**quotient rule**

Examples:

$$\frac{d}{dx}\{x^{17} + \cos x\} = 17x^{16} - \sin x$$

$$\frac{d}{dx}\{35 \cosh x\} = 35 \frac{d}{dx} \cosh x = 35 \sinh x$$

$$\frac{d}{dx}\{\cos x e^x\} = -\sin x e^x + \cos x e^x = e^x(\cos x - \sin x)$$

$$\frac{d}{dx}\{\frac{\cos x}{e^x}\} = \frac{-\sin x\, e^x - \cos x\, e^x}{e^{2x}} = \frac{-e^x\,(\sin x + \cos x)}{e^{2x}} = -\frac{\sin x + \cos x}{e^x}$$

2.4 The Chain Rule

If $u(x)$ and $v(x)$ have derivatives and the image of $v(x)$ is part of the source set of $u(x)$, then $u(v(x))$ has a derivative.

To understand what this complicated sentence means, consider $\ln(\cos x)$. Here $u(x) = \ln x$ and $v(x) = \cos x$. The source set of $\cos x$ are all real numbers $[-\infty, \infty]$, the image set of the cosine are the numbers in the interval [-1, 1], and the source set of the logarithm are all positive real numbers $]0, \infty]$. Therefore the image set of the cosine and the source set of the logarithm overlap in the interval $]0, 1]$. The source set of $\cos x$ that corresponds to the image set $]0, 1]$ is given by all numbers where $\cos x$ is positive, i.e. $]-\frac{\pi}{2}, -\frac{\pi}{2}[$, $]\frac{3\pi}{2}, -\frac{5\pi}{2}[$, etc., and the function $\ln(\cos x)$ exists and has a derivative for these values of x.

$$[u(v(x))]' = \frac{d}{dx}\{u(v(x))\} = \frac{d\, u(v)}{dv}\, \frac{d\, v(x)}{dx} \qquad \textbf{chain rule}$$

Examples:

$$f(x) = \cos(\alpha x) \quad \rightarrow \quad u(v) = \cos v \quad \text{and} \quad v(x) = \alpha x$$

$$\frac{d}{dx} \cos \alpha x = \frac{d \cos \alpha x}{d\, \alpha x}\, \frac{d\, \alpha x}{dx} = (-\sin \alpha x)\, \alpha = -\alpha \sin \alpha x$$

$$f(x) = (2x+5)^3 \quad \rightarrow \quad u(v) = v^3 \quad \text{and} \quad v(x) = 2x+5$$

$$\frac{d}{dx}\,(2x+5)^3 = \frac{d\,(2x+5)^3}{d\,(2x+5)}\,\frac{d\,(2x+5)}{dx} = 3\,(2x+5)^2\;2 = 6\,(2x+5)^2$$

2.5 Selected problems (the page from hell):

Important note: Now we can take the derivative of ANY analytic function !!!

$$f(x) = e^{\ln x} \quad \rightarrow \quad u(v) = e^v \quad \text{and} \quad v(x) = \ln x$$

$$f'(x) = e^{\ln x}\,\frac{1}{x} = x\,\frac{1}{x} = 1 \qquad \text{of course we started with} \quad f(x) = x \;\;\rightarrow\;\; f'(x) = 1$$

$$f(x) = \sqrt{\sin(3\,\alpha^2\,x^5)} = [\sin(3\,\alpha^2\,x^5)]^{\frac{1}{2}} = u(v(w(x)))$$

$$\rightarrow \quad u(v) = v^{\frac{1}{2}} \quad v(w) = \sin(3\,\alpha^2\,x^5) \quad w(x) = 3\,\alpha^2\,x^5$$

$$f'(x) = \frac{d\,u(v(w(x)))}{dv}\,\frac{d\,v(w(x))}{dw}\,\frac{d\,w(x)}{dx} = \frac{1}{2}\,[\sin(3\,\alpha^2\,x^5)]^{\frac{1}{2}-1}\;\cos(3\,\alpha^2\,x^5)\;3\,\alpha\,5\,x^{5-1}$$

$$= \frac{15\,\alpha\,x^4\;\cos(3\alpha^2 x^5)}{2\sqrt{\sin(3\alpha^2 x^5)}} \qquad \text{who guessed this result ???}$$

$$f(x) = \frac{3x^2 + \cos kx}{\cosh x} \quad \rightarrow \quad f'(x) = \frac{(6x - k\sin kx)\cosh x + (3x^2 + \cos kx)\sinh x}{\cosh^2 x}$$

Also quite ugly, but technically correct !!!

$$f(x) = \cos^2 kx = \cos kx\;\cos kx \quad \rightarrow \quad f'(x) = 2\;\cos kx(-\sin kx)\,k = -2k\;\cos kx\sin kx$$

$$\text{or} \quad \rightarrow \quad (-\sin kx)\,k\;\cos kx + \cos kx(-\sin kx)\,k = -2k\;\cos kx\sin kx$$

$$f(x) = y = (x^5 + e^{\cos kx})^{1/2} \quad \rightarrow \quad y' = \frac{1}{2}(x^5 + e^{\cos kx})^{-1/2}\,(5\,x^4 + e^{\cos kx}(-k\;\sin kx))$$

$$= \frac{5\,x^4 - k\;\sin 2kx\,e^{\cos kx}}{2\,(x^5 + e^{\cos kx})^{1/2}}$$

$$y = x^x = e^{x\ln x} \quad (\text{remember: } a^x = e^{x\ln a}) \quad \rightarrow \quad y' = e^{x\ln x}\,(\ln x + 1) = x^x(\ln x + 1)$$

2.6 Integral Calculus: Definite Integrals

How do you determine the area A enclosed by a function $f(x)$ and the horizontal axis. It is simple if $f(x) = f_0 = const.$

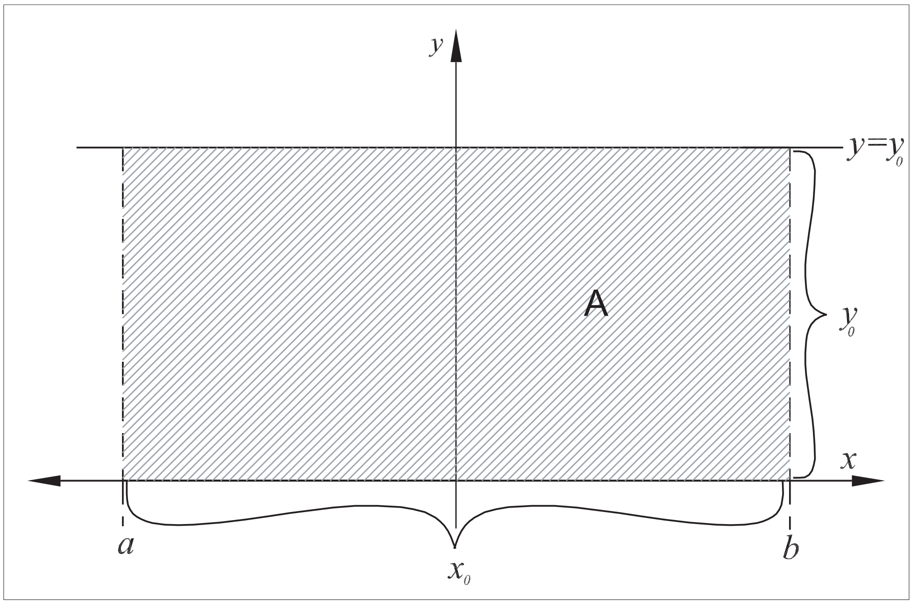


Figure 21: Area A enclosed by the horizontal axis and a horizontal line.

For the general case, divide the area A into subareas A_ν between $x_{\nu-1}$ and x_ν.

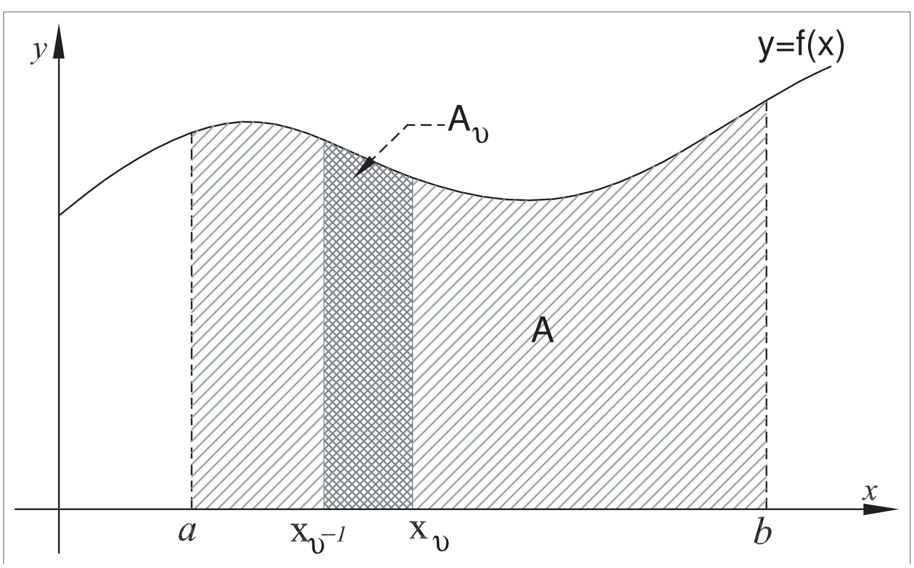


Figure 22: Area enclosed by the function $f(x)$ and the horizontal axis.

Then the subarea A_ν may be approximated by $A_\nu = f(\xi_\nu)(x_\nu - x_{\nu-1})$ for $x_{\nu-1} < \xi_\nu < x_\nu$. There exists always a ξ_ν such that this is true.

Reconstruct the area A as follows:

$$A = \sum_{\nu=1}^{N} = A_\nu = \sum_{\nu=1}^{N} f(\xi_\nu) \underbrace{(x_\nu - x_{\nu-1})}_{\Delta x}$$

This sum is called the *Riemann sum.* The limit of the Riemann sum is the area A and defines the integral.

$$A = \lim_{N\to\infty} \sum_{\nu=1}^{N} f(\xi_\nu)\Delta x = \int_{x=a}^{x=b} f(x)\,dx$$

The area enclosed by $f(x)$ and the horizontal x-axis over an interval $x \in [a, b]$ is given by definite integral

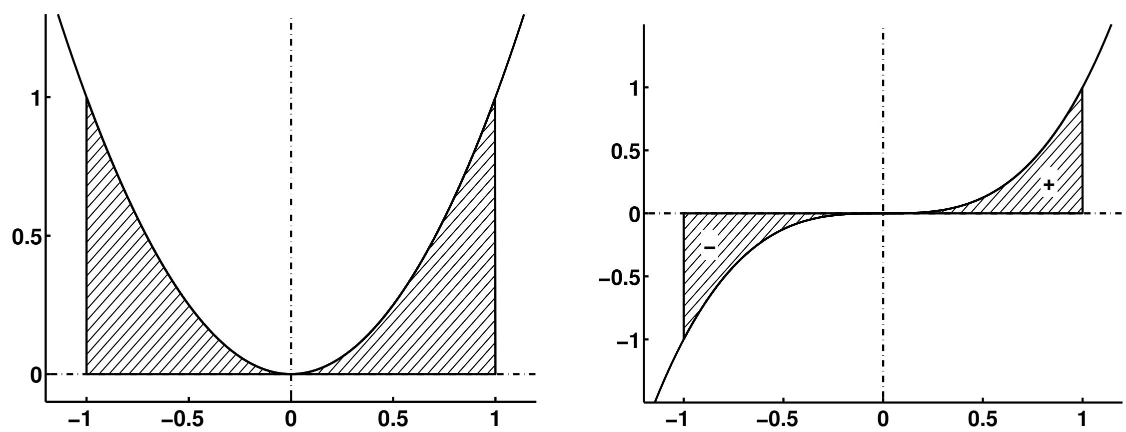
$$\int_a^b f(x)\,dx = F(x)\,\Big|_{x=a}^{x=b} = F(x)\,\Big|_a^b = F(b) - F(a)$$

where $F(x)$ is called the *anti-derivative* of $f(x)$ and

$$f(x) = \frac{dF(x)}{dx} = F'(x) \qquad \text{or, which is equivalent,} \qquad F(x) = \int f(x)\,dx + \text{const}$$

Integration is to some extend the inverse operation of differentiation.

Examples:



(a) Definite integral of $y = x^2$ (b) Definite integral of $y = x^3$

Figure 23: Definite integrals as areas under curves

a) The shaded area is given by

$$A = \int_{-1}^{1} f(x)\,dx = F(1) - F(-1)$$

We know $f(x) = x^2 = F'(x)$. Now we guess $F(x) = 1/3x^3 + c$.

$$A = F(1) - F(-1) = \frac{1}{3}1^3 + c - \{\frac{1}{3}(-1)^3 + c\} = \frac{2}{3}$$

b) Again, the shaded area is given by

$$A = \int_{-1}^{1} f(x)\, dx = F(1) - F(-1)$$

We guess $F(x) = \frac{1}{4}x^4 + c$ and find

$$A = \int_{-1}^{1} f(x)\, dx = F(1) - F(-1) = \frac{1}{4}1^4 + c - \{\frac{1}{4}(-1)^4 + c\} = 0$$

Why does the area A vanish? It actually consists of two areas, A_1 and A_2, which both have the same size, but opposite sign $A_1 = -A_2$.

$$A_1 = F(0) - F(-1) = \frac{1}{4}0^4 + c - \frac{1}{4}(-1)^4 - c = -\frac{1}{4} = -A_2$$

Note: In an integral the area *below* the x-axis is counted negative. In order to calculate the shaded area we have to evaluate all pieces between intersections of the curve with the horizontal axis separately and add up their magnitudes. Here: $A = \mid A_1 \mid + \mid A_2 \mid = \mid -\frac{1}{4} \mid + \mid \frac{1}{4} \mid = \frac{1}{2}$.

2.7 Methods of Integration

Properties of integrals:

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

$$\int_a^b f(x)\, dx = F(b) - F(a) = -(F(a) - F(b)) = -\int_b^a f(x)\, dx$$

$$\int_a^b (f_1(x) + f_2(x))\, dx = \int_a^b f_1(x)\, dx + \int_a^b f(x)_2\, dx$$

$$\int_a^b c\, f(x)\, dx = c \int_a^b f(x)\, dx \qquad \text{where } c \text{ is a constant}$$

Approach (in order of preference):

- **Guess:** Find $F(x)$ such that $\frac{dF(x)}{dx} = f(x)$. For polynomials: $F(a\,x^n) = \frac{a}{n+1}\,x^{n+1}$.

  Note: Here n can be negative or any rational number except -1.

- **Tables:** $F(x)$ can be looked up in mathematical tables of anti-derivatives and/or definite integrals which can be found in e.g. Bronshteyn, Semendjajew or Gradsteyn.

- **Partial integration:** Corresponds to the product rule but only works for special cases.

$$\int f(x)\,g(x)dx = F(x)\,g(x) - \int F(x)\,g'(x)\,dx$$

$$\int_a^b x\,\cosh x\,dx = \underbrace{x}_{g}\underbrace{\sinh x}_{F}\,|_a^b - \int_a^b \underbrace{\sinh x}_{F}\underbrace{1}_{g'}\,dx = b\sinh b - a\sinh a - (\cosh b - \cosh a)$$

- **Substitution:** Corresponds to the chain rule but again only works for special cases.

$$\int_{x=a}^{x=b} f(\phi(x))\,\phi'(x)\,dx = \int_{u=\phi(a)}^{u=\phi(b)} f(u)\,du \qquad \text{where} \qquad u = \phi(x)$$

$$\int_0^\pi \cos^2 x\ \sin x\,dx \quad \text{substitute:} \quad u = \cos x = \phi(x)$$

$$u' = \frac{du}{dx} = -\sin x = \phi'(x) \quad \rightarrow \quad du = -\sin x dx = \phi'(x)dx \quad \rightarrow \quad dx = -\frac{du}{\sin x}$$

Substitute the integral:

$$\int_{x=0}^{x=\pi} \cos^2 x\ \sin x\,\frac{-du}{\sin x} = -\int_{x=0}^{x=\pi} \cos^2 x\,du = -\int_{x=0}^{x=\pi} u^2\,du$$

Express the boundaries in terms of u:

$$x = 0 \quad \rightarrow \quad u = \cos 0 = 1 \qquad\qquad x = \pi \quad \rightarrow \quad u = \cos\pi = -1$$

Insert them and perform the integration:

$$\int_0^\pi \cos^2 x\ \sin x\,dx = -\int_{u=1}^{u=-1} u^2\,du = -\frac{1}{3}u^3\,\Big|_1^{-1} = -\frac{1}{3}(-1)^3 + \frac{1}{3}1^3 = \frac{2}{3}$$

2.8 Symmetries

A function $f(x)$ is called an *even function* if $f(-x) = f(x)$; a function $g(x)$ is called an *odd function* if $g(-x) = -g(x)$. The product of two even functions or the product of two odd functions is an even function; the product of an odd and an even function is an odd function.

The integral over a symmetric interval around $x = 0$ of an odd function vanishes.

$$\int_{-a}^{b=a} g(x)\, dx = \int_{-a}^{b=a} f(x)\, g(x)\, dx = 0 \qquad \text{if } f(-x) = f(x) \quad \text{and} \quad g(-x) = -g(x)$$

Example:

$$\int_{-1}^{1} \underbrace{x^2}_{f(x)} \underbrace{\sin 3x}_{g(x)}\, dx = 0$$

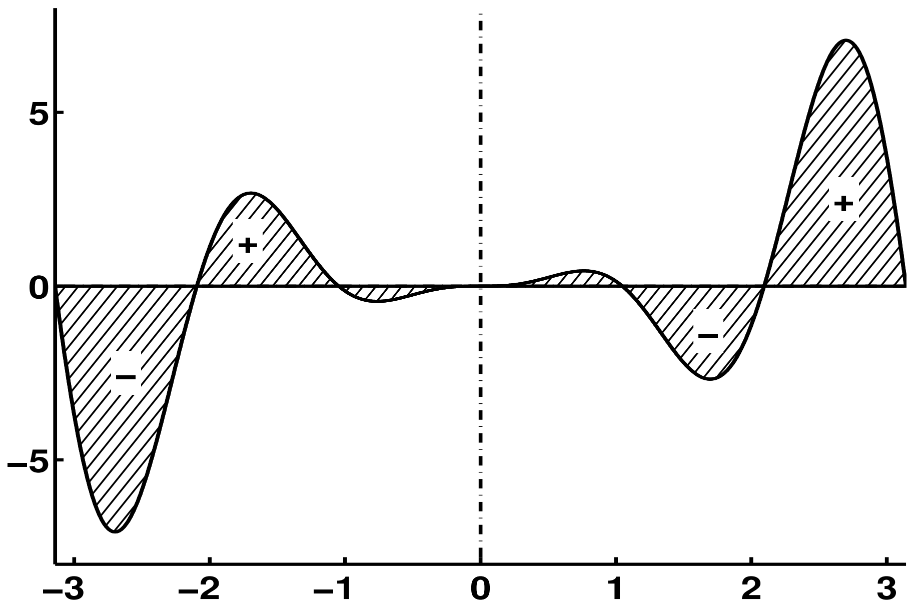


Figure 24: Due to symmetry the integral $\int_{-1}^{1} x^2 \sin 3x\, dx$ vanishes

2.9 Orthogonality of trigonometric functions

The cosine is an even function $\cos(-x) = \cos x$, and the sine is an odd function $\sin(-x) = -\sin x$. Moreover, these trigonometric functions are 2π-periodic, hence it is sufficient to consider integration over windows of 2π only.

$$\int_0^{2\pi} \underbrace{\sin x\, \cos x}_{\frac{1}{2}\sin 2x}\, dx = \frac{1}{2}\int_0^{2\pi} \sin 2x\, dx = -1/4 \cos 2x\, \Big|_0^{2\pi} = 0 \quad \text{or equivalent:} \quad \int_{-\pi}^{\pi} \sin x\, \cos x\, dx = 0$$

$$\int_{-\pi}^{\pi} \sin 2x \, \sin x \, dx = 2 \int_{-\pi}^{\pi} \underbrace{\sin^2 x}_{u^2} \, \underbrace{\cos x \, dx}_{du} = 2 \int_{u=0}^{u=0} u^2 \, du = 0$$

Here we used the substitution $u = \sin x$ and $du = \cos x \, dx$ with the boundaries $x = \pi \to u = 0$ and $x = -\pi \to u = 0$.

More general cases:

$$\int_{-\pi}^{\pi} \cos mx \, \cos nx \, dx = \int_{-\pi}^{\pi} \sin mx \, \sin nx \, dx = \pi \, \delta_{mn}$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0 \;\; \forall \;\; m, \, n \qquad \text{where "}\forall\text{" means "for all"}$$

δ_{mn} is called the *Kronecker delta* which is defined as $\delta_{mn} = 1$ if $m = n$ and $\delta_{mn} = 0$ else.

2.10 Integrals to Infinity

If one or both boundaries of an integral are infinite this does not mean that the area under this curve cannot be finite. A trivial example is given by the integral from $-\infty$ to $+\infty$ over an odd function. This integral vanishes, as seen above, independent of the function as long as it is odd.

A nontrivial example is

$$\int_1^{\infty} \frac{1}{x^2} \, dx = - \left. \frac{-1}{x} \right|_1^{\infty} = -(\underbrace{\frac{1}{\infty}}_{=0} - \frac{1}{1}) = 1$$

however

$$\int_1^{\infty} \frac{1}{x} \, dx = \ln x \, \Big|_1^{\infty} = \ln \infty - \underbrace{\ln 1}_{=0} = \infty$$

In the same way even if a function has a singularity like $\frac{1}{\sqrt{x}}$ for $x \to 0$, the area can still be finite

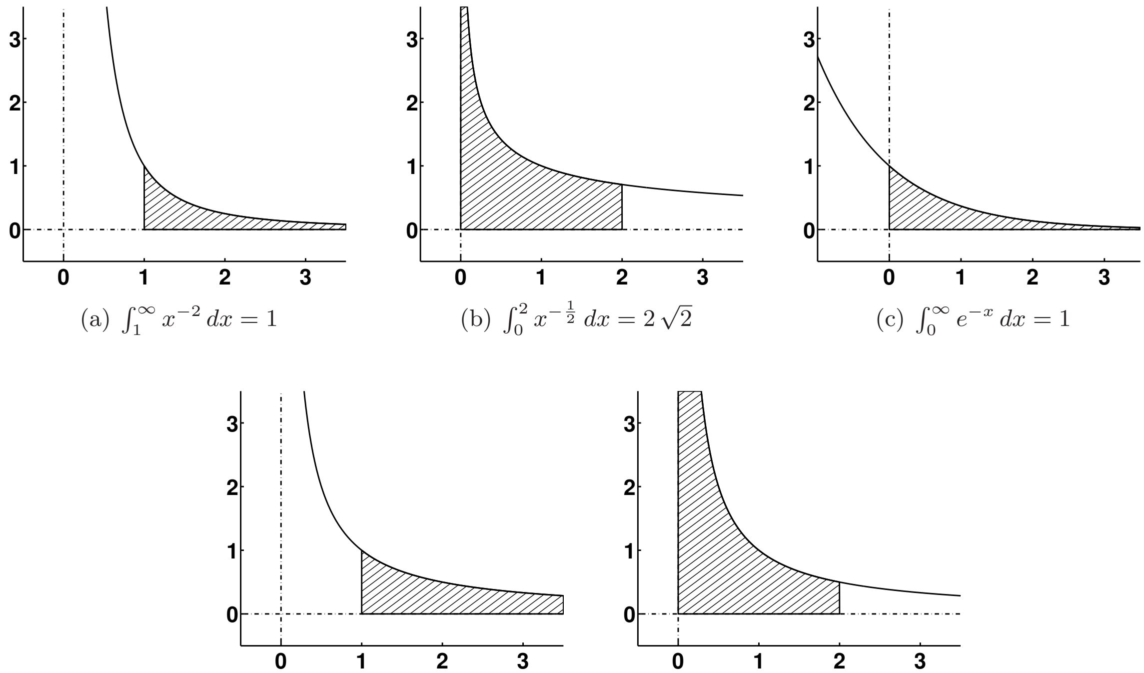
$$\int_0^2 x^{-\frac{1}{2}} \, dx = 2 \, x^{\frac{1}{2}} \, \Big|_0^2 = 2 \sqrt{x} \, \Big|_0^2 = 2 \, (\sqrt{2} - \sqrt{0}) = 2 \sqrt{2}$$

but again

$$\int_0^2 x^{-1} \, dx = \int_0^2 \frac{1}{x} \, dx = \ln x \, \Big|_0^2 = \ln 2 - \underbrace{\ln 0}_{=-\infty} = \infty$$

And finally an exponential function

$$\int_0^{\infty} e^{-x} \, dx = -e^{-x} \, \Big|_0^{\infty} = -(\underbrace{e^{-\infty}}_{=0} - \underbrace{e^0}_{=1}) = 1$$



(a) $\int_1^\infty x^{-2}\,dx = 1$ (b) $\int_0^2 x^{-\frac{1}{2}}\,dx = 2\sqrt{2}$ (c) $\int_0^\infty e^{-x}\,dx = 1$

(d) $\int_1^\infty x^{-1}\,dx = \infty$ (e) $\int_0^2 x^{-1}\,dx = \infty$

Figure 25: Definite integrals that involve infinities

2.11 Functions with no Antiderivative

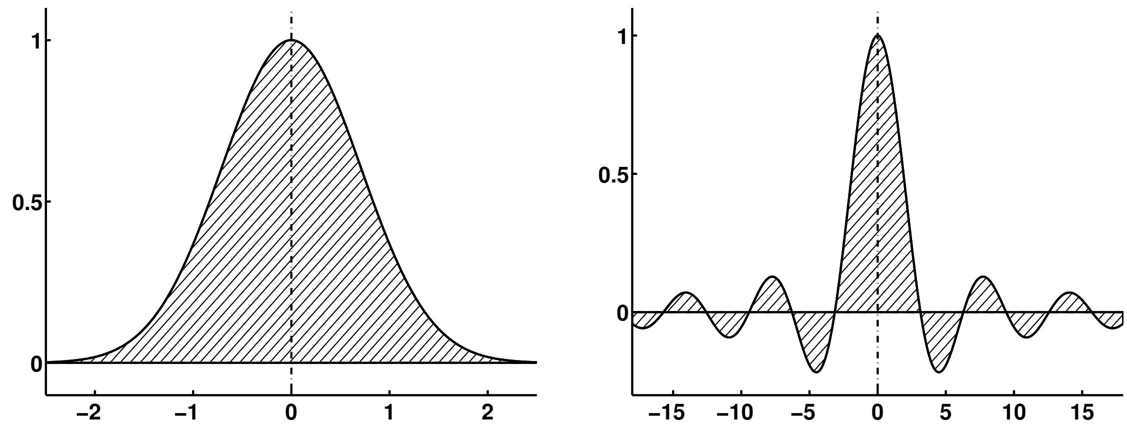
As we have seen, it quite straightforward to calculate the derivatives of quite complicated "monsters" of functions. On the other hand, it is much more difficult to find antiderivatives. To make things worse there are certain functions with very important applications for which an antiderivative does not exist, i.e. it cannot be expressed in terms of elementary functions.

One of these simple functions which do not have an antiderivative is $f(x) = e^{-x^2}$. This is very inconvenient because this function is the famous bell-shaped Gaussian which rules the entire field of statistics, because the probability that an event occurs within a certain interval of a parameter is given by area under this curve. This area unfortunately cannot be calculated using a pocket calculator that has only elementary functions. The definite integral can be found numerically or looked up in tables, and it also has a name: the "error function" erf(x)

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\,du \qquad \text{note:} \qquad \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

A second example of such a function with no antiderivative is the so-called "integral sine" Si(x)

$$\operatorname{Si}(x) = \int_0^x \frac{\sin u}{u}\,du \qquad \text{note:} \qquad \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$$



(a) $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$ (b) $\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi$

Figure 26: Definite integrals over functions with no antiderivatives

3 Vector Algebra

3.1 Vectors

Until now we have dealt only with *scalars* which are one-dimensional entities consisting of a magnitude and a sign. Higher-dimensional entities are composed of several scalars each of which is related to a particular direction. These objects are called vectors and are represented in print by either using bold symbols like $\boldsymbol{x}$ or with an arrow on top as in $\overrightarrow{x}$. An n-dimensional vector has n components x_i with $i = 1, ..., n$. Its magnitude is given by $|\overrightarrow{x}| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$.

Notation: $\overrightarrow{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$ is a column vector and $\overrightarrow{x} = (x_1, x_2, ..., x_n)$ is row vector.

Sometimes a row vector is specifically denoted as $\overrightarrow{x}^T$ (T for transposed).

A vector is graphically represented by an arrow. The vector's magnitude $|\overrightarrow{x}|$ is denoted by the arrow's length. If the starting point of the vector coincides with the origin of the coordinate system, then its end point corresponds to the coordinates of the vector components. Such a vector is called a coordinate vector.

3.2 Elementary Vector Operations

3.2.1 Addition and Subtraction

The sum two vectors can be obtained graphically by either shifting the tail of the second arrow to the head of the first, or by constructing the parallelogram that is defined by the two arrows. The difference between two vectors can be found by adding the vector that has the same length but points into the opposite direction.

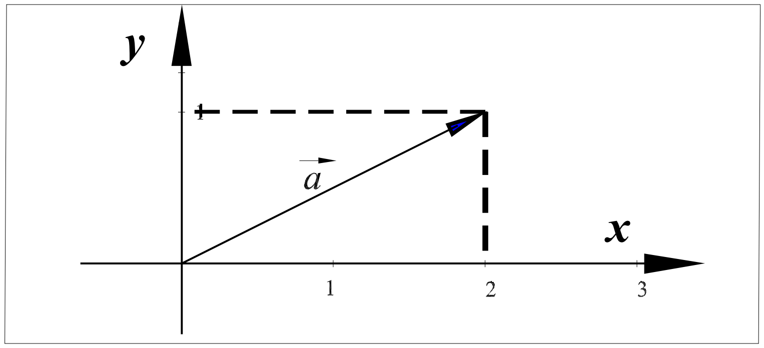
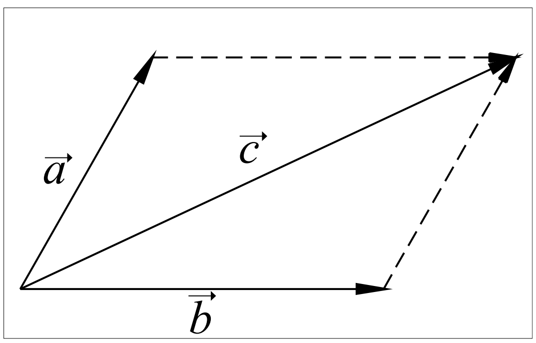
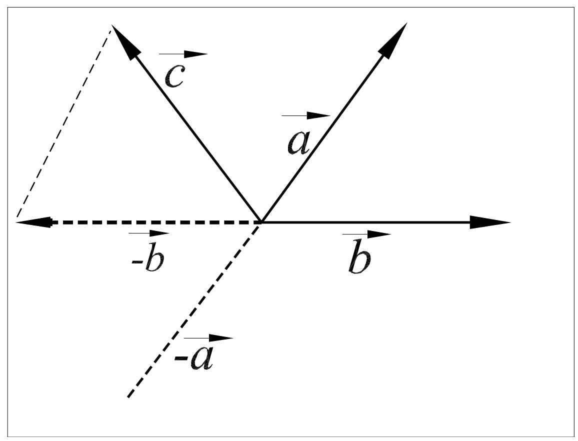


Figure 27: The vector $(1, 2)$ is an arrow from the origin to the point $x = 1$ and $y = 2$.



(a) The sum of two vectors $\vec{c} = \vec{a} + \vec{b}$



(b) The difference between two vectors $\vec{c} = \vec{a} - \vec{b}$

Figure 28: Addition and subtraction of vectors

In components: $\quad \vec{a} + \vec{b} = (a_1 + b_1, ..., a_n + b_n) = (c_1, ..., c_n) = \vec{c}$

Properties:

$$\begin{array}{rcll} \vec{a} + \vec{b} & = & \vec{b} + \vec{a} & \textbf{commutative} \\ (\vec{a} + \vec{b}) + \vec{c} & = & \vec{a} + (\vec{b} + \vec{c}) & \textbf{associative} \end{array}$$

A closed polygon corresponds to the vector sum equal $\vec{0}$.

Important note: Make sure you understand that $\vec{0} \neq 0$!!!!

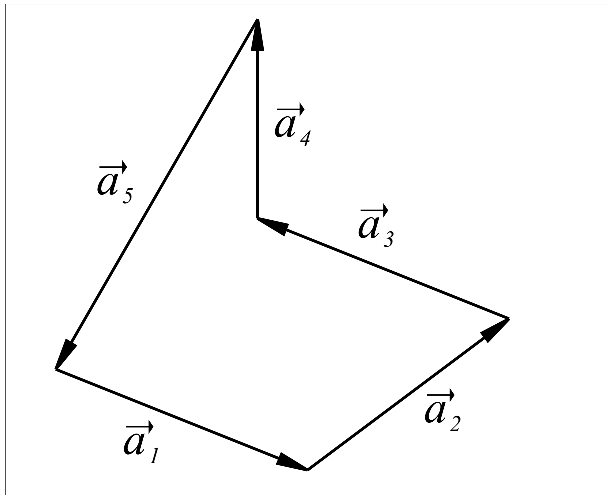


Figure 29: $\overrightarrow{a_1} + \overrightarrow{a_2} + \ldots + \overrightarrow{a_5} = \overrightarrow{0}$

3.2.2 Multiplication of a Vector with a Scalar

A vector can be multiplied with a scalar by multiplying each of the components which results in either stretching or squeezing of the vector and may change its orientation.

$$\overrightarrow{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \overrightarrow{b} = -2\,\overrightarrow{a} = -2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

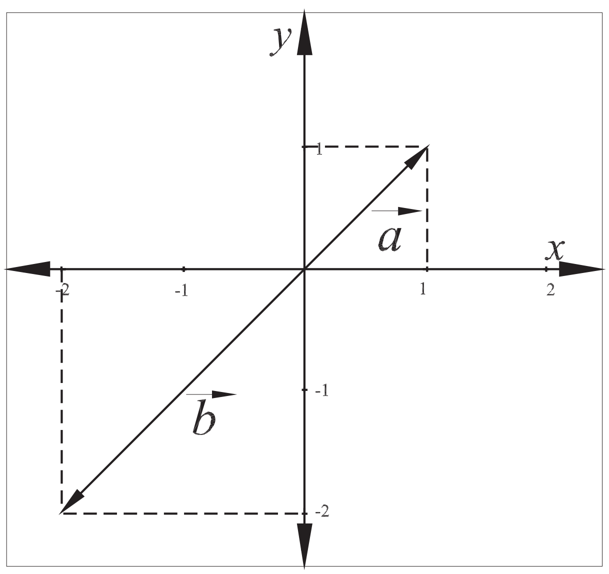


Figure 30: The multiplication of a vector with a scalar $\overrightarrow{b} = -2\,\overrightarrow{a}$

Linear dependence of vectors:

n vectors $\overrightarrow{a_1}, \cdots, \overrightarrow{a_n}$ are called *linearly independent*, if the only way to fulfill

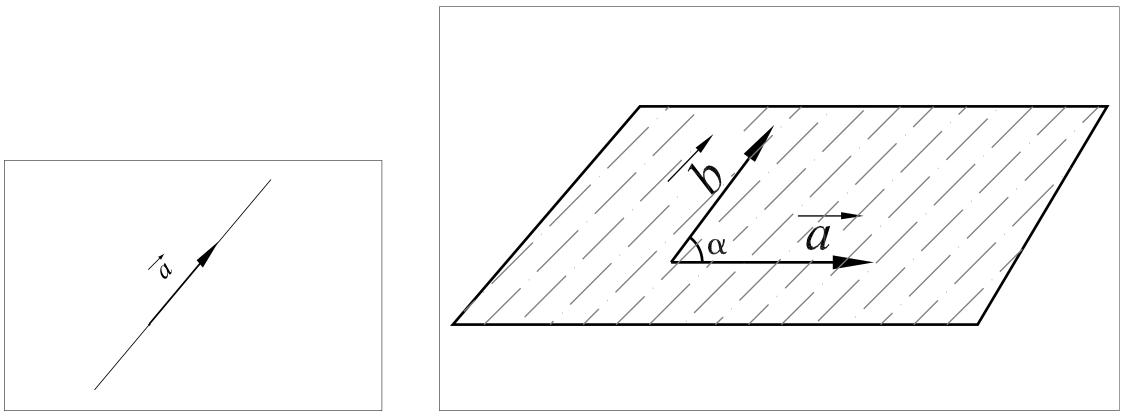
$$\alpha_1\,\overrightarrow{a_1} + \alpha_2\,\overrightarrow{a_2} + \cdots + \alpha_n\,\overrightarrow{a_n} = 0 \quad \text{is} \quad \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

If this relation can be fulfilled with certain $\alpha_i \neq 0$, then the vectors are said to be *linearly dependent*. For instance, imagine $\alpha_1 \neq 0$, all others are free. Then $\vec{a_1}$ may be expressed by the other vectors and is redundant.

$$\vec{a_1} = -\frac{1}{\alpha_1} \sum_{i=2}^{n} \alpha_i \, \vec{a_i} \tag{1}$$

One-dimensional: $\alpha \, \vec{a}$ represents all vectors on a straight line. Such vectors are called collinear.

Two-dimensional: $\alpha_1 \, \vec{a_1} + \alpha_2 \, \vec{a_2}$ represents all vectors in the plane. These vectors are coplanar.



(a) Collinear vectors define a line (b) Two non-collinear vectors span a plane

Figure 31: Collinear and coplanar vectors

3.2.3 Scalar Product

Two vectors $\vec{a}$ and $\vec{b}$ can be multiplied such that the result is a scalar c. This operation is called the *scalar, dot* or *inner* product.

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\alpha \qquad \text{where } \alpha \text{ is the angle between } \vec{a} \text{ and } \vec{b}$$

$$\vec{a} \cdot \vec{b} = a_1\, b_1 + a_2\, b_2 + ... + a_n\, b_n \qquad \text{scalar product in components}$$

The scalar product measures the contribution of vector $\vec{a}$ to vector $\vec{b}$ If the angle between $\vec{a}$ and $\vec{b}$ is 90° the two vectors are orthogonal, there are no contributions at all.

Properties:

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \qquad \textbf{commutative}$$

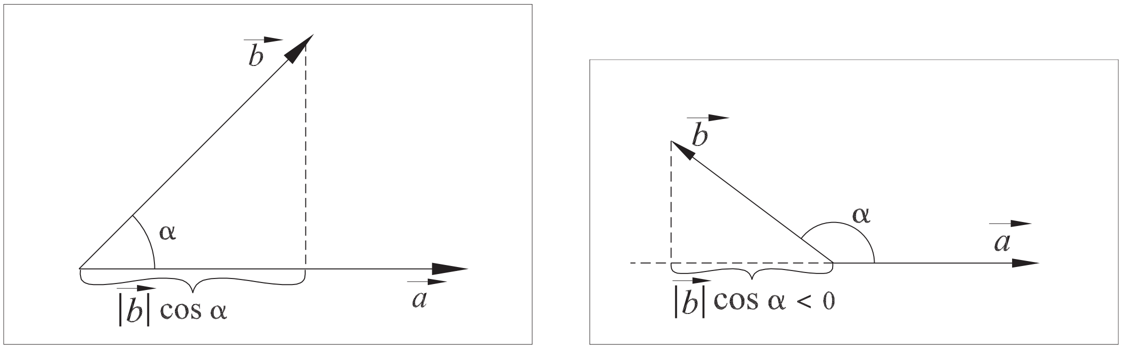
$$(c\,\vec{a}) \cdot \vec{b} = c\,(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\,\vec{b}) \qquad \textbf{associative}$$

$$(\vec{a_1} + \vec{a_2}) \cdot \vec{b} = \vec{a_1} \cdot \vec{b} + \vec{a_2} \cdot \vec{b} \qquad \textbf{distributive}$$

Examples:

$$\vec{a} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \vec{a} \cdot \vec{b} = 2 \cdot 1 + 0 \cdot (-2) = 2$$

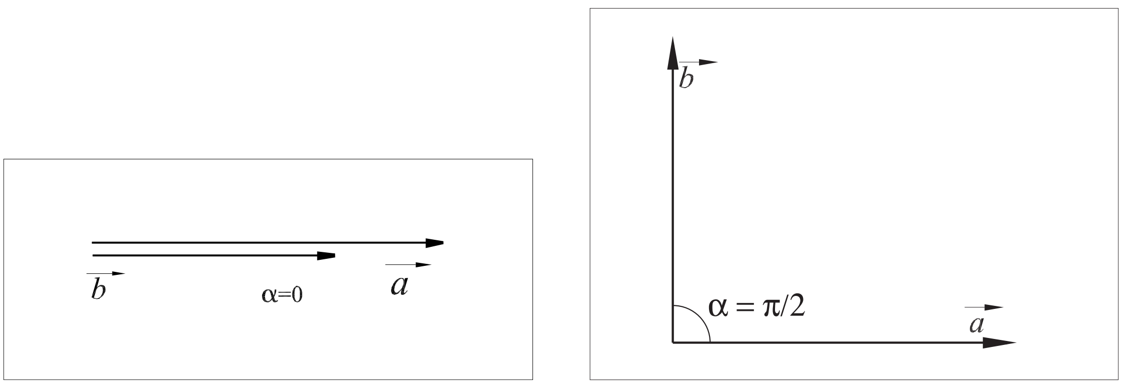
$$|\overrightarrow{a}| = 2 \qquad |\overrightarrow{b}| = \sqrt{5} \quad \rightarrow \quad \cos\alpha = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}||\overrightarrow{b}|} = \frac{2}{2\sqrt{5}} \quad \rightarrow \quad \alpha = \arccos\frac{1}{\sqrt{5}} = 1.107 \approx 63^\circ$$



(a) Projection of $\overrightarrow{b}$ on $\overrightarrow{a}$

(b) If $\alpha > 90^0$ then $\cos\alpha < 0$ and the scalar product is negative

Figure 32: Scalar Product



(a) The dot product has is maximal value in the case $\alpha = 0 \ \rightarrow \cos\alpha = 1$

(b) For $\alpha = 90^\circ$ the scalar product vanishes because $\cos\alpha = 0$

Figure 33: Scalar product for parallel and orthogonal vectors

3.2.4 Vector Product

Two vectors $\vec{a}$ and $\vec{b}$ can be multiplied such that the result is a vector $\vec{c}$. This operation is called the *vector, cross* or *outer* product.

The vector product exists only in three dimensions !!!

$$\vec{a} \times \vec{b} = \vec{c} \qquad\qquad |\vec{c}| = |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\,\sin\alpha$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \qquad \text{vector product in components}$$

The result of a vector product between two non-collinear vectors $\vec{a}$ and $\vec{b}$ is a vector $\vec{c}$ which has a magnitude of $|\vec{a}|\,|\vec{b}|\,\sin\alpha$ and points into the direction perpendicular to the plane defined by $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\vec{c}$ form a right-handed system. To find this direction have your right thumb point into the direction of $\vec{a}$, the right index into the direction of $\vec{b}$, and the right middle finger perpendicular to the plane defined by $\vec{a}$ and $\vec{b}$. There is only one way to do that without hurting yourself seriously. Now the middle finger points into the direction of $\vec{c}$.

Hint: It is imperative that you use the **right** hand for this.

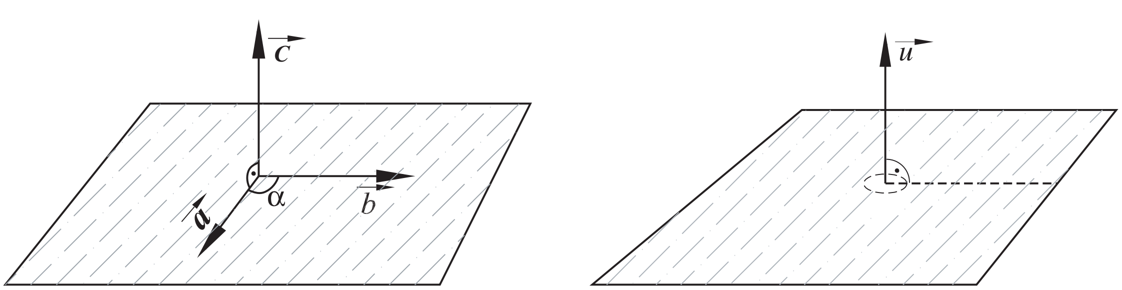
Properties:

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \qquad \text{anti-commutative}$$

$$(c\,\vec{a}) \times \vec{b} = c\,(\vec{a} \times \vec{b}) = \vec{a} \times (c\,\vec{b}) \qquad \text{associative}$$

$$(\vec{a_1} + \vec{a_2}) \times \vec{b} = \vec{a_1} \times \vec{b} + \vec{a_2} \times \vec{b} \qquad \text{distributive}$$

Note: In 3 dimensions a plane can be defined by a point in the plane and its *normal vector* $\vec{n}$.



(a) $\vec{a}$ and $\vec{b}$ span a plane

(b) A plane defined by a point and its normal vector

Figure 34: Vectors in 3-dimensional space

3.3 Matrices

A matrix $\mathbf{A}$ operates on a vector $\overrightarrow{x}$ and transforms, i.e. stretches, squeezes or rotates it.

$$\overrightarrow{y} = \boldsymbol{A}\,\overrightarrow{x} \qquad \text{where} \quad \boldsymbol{A} = \begin{pmatrix} A_{11} & A_{12} & A_{1n} \\ A_{21} & \ddots & \dots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} = A_{ij} \quad \text{is a } \; n \times n \text{ matrix}$$

$$\overrightarrow{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} A_{11}\,x_1 + A_{12}\,x_2 + \dots A_{1n}\,x_n \\ A_{21}\,x_1 + A_{22}\,x_2 + \dots A_{2n}\,x_n \\ \vdots \\ A_{n1}\,x_1 + A_{n2}\,x_2 + \dots A_{nn}\,x_n \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}}_{\boldsymbol{A}} \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\overrightarrow{x}} = \underbrace{\begin{pmatrix} 2 \\ -1 \end{pmatrix}}_{\overrightarrow{y}}$$

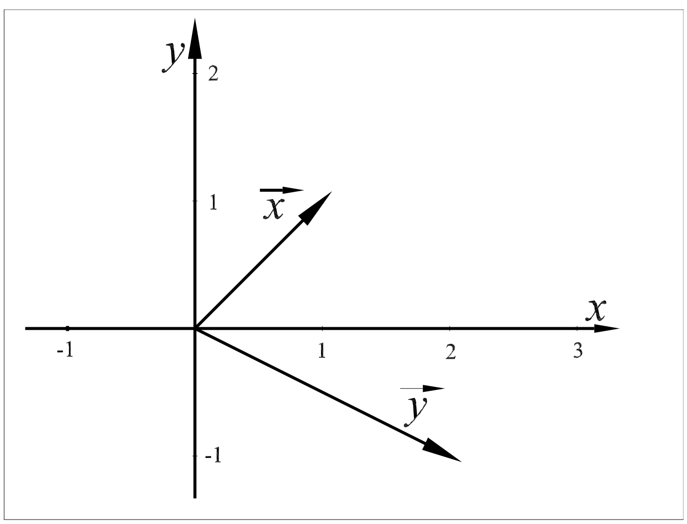


Figure 35: Rotation and scaling of a vector

Properties:

$$\boldsymbol{A} = \boldsymbol{B} \quad \rightarrow \quad a_{ij} = b_{ij}$$

$$\boldsymbol{A} + \boldsymbol{B} \quad \rightarrow \quad a_{ij} + b_{ij}$$

$$c\,\boldsymbol{A} \quad \rightarrow \quad c\,a_{ij}$$

$$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$$

3.4 Multiplication of Matrices

The product of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is found by calculating the scalar products between the *rows* of matrix $\boldsymbol{A}$ and the *columns* of matrix $\boldsymbol{B}$. This implies that the number of columns of matrix $\boldsymbol{A}$ must be the same as the number of rows of matrix $\boldsymbol{B}$.

Examples:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}\, b_{11} + a_{12}\, b_{21} & a_{11}\, b_{12} + a_{12}\, b_{22} \\ a_{21}\, b_{11} + a_{22}\, b_{21} & a_{21}\, b_{12} + a_{22}\, b_{22} \end{pmatrix}$$

$$\boldsymbol{A}\,\boldsymbol{B} = \begin{pmatrix} 5 & 3 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 5 \cdot 2 + 3 \cdot (-1) & 5 \cdot (-3) + 3 \cdot 4 \\ 2 \cdot 2 + 7 \cdot (-1) & 2 \cdot (-3) + 7 \cdot 4 \end{pmatrix} = \begin{pmatrix} 7 & -3 \\ -3 & 22 \end{pmatrix}$$

$$\boldsymbol{B}\,\boldsymbol{A} = \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 2 & 7 \end{pmatrix} = \begin{pmatrix} 4 & -15 \\ 3 & -25 \end{pmatrix} \neq \boldsymbol{A}\,\boldsymbol{B}$$

The multiplication of matrices is NOT commutative!!!!!

In general a $n \times m$ matrix can be multiplied with a $m \times n$ matrix and the result is a $n \times n$ matrix.

$$\boldsymbol{C} = \boldsymbol{A}\,\boldsymbol{B} = \begin{pmatrix} \Sigma_i\, a_{1i}\, b_{i1} & \Sigma_i\, a_{1i}\, b_{i2} & \dots & \Sigma_i\, a_{1i}\, b_{in} \\ \Sigma_i\, a_{2i}\, b_{i1} & \Sigma_i\, a_{2i}\, b_{i2} & \dots & \Sigma_i\, a_{2i}\, b_{in} \\ \vdots & \vdots & & \vdots \\ \Sigma_i\, a_{ni}\, b_{i1} & \Sigma_i\, a_{ni}\, b_{i2} & \dots & \Sigma_i\, a_{ni}\, b_{in} \end{pmatrix} \qquad \text{with } \Sigma_i = \sum_{i=1}^{m}$$

3.5 Transposed Matrix

The transposed of a matrix is found by exchanging the row and column vectors.

$$\boldsymbol{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \dots \\ A_{21} & A_{22} & \cdot & \dots \\ \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \dots & A_{mn} \end{pmatrix} \quad \rightarrow \quad \boldsymbol{A}^{\boldsymbol{T}} = \begin{pmatrix} A_{11} & A_{21} & A_{31} & \dots \\ A_{12} & A_{22} & \cdot & \dots \\ \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \dots & A_{nm} \end{pmatrix}$$

Examples:

$$\boldsymbol{A} = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 4 & 0 \end{pmatrix} \quad \rightarrow \quad \boldsymbol{A}^{\boldsymbol{T}} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \\ 5 & 0 \end{pmatrix}$$

$$\boldsymbol{A}\,\boldsymbol{A}^{\boldsymbol{T}} = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 4 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 4 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 38 & 14 \\ 14 & 17 \end{pmatrix}$$

$$\boldsymbol{A^T}\,\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 5 \\ 1 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 10 & 10 \\ 10 & 25 & 15 \\ 10 & 15 & 25 \end{pmatrix}$$

This should eliminate any remaining doubts that matrix multiplication could be commutative. These matrices are not even the same size.

3.6 Basis vectors

Basis vectors span a coordinate system and can be represented in various ways

$$\vec{i}\,, \vec{j}\,, \vec{k} \qquad \vec{e_1}, \vec{e_2}, \vec{e_3} \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{s} = \begin{pmatrix} x \\ y \end{pmatrix} = x \overbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}^{\vec{e_1}} + y \overbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}^{\vec{e_2}} = x\,\vec{e_1} + y\,\vec{e_2}$$

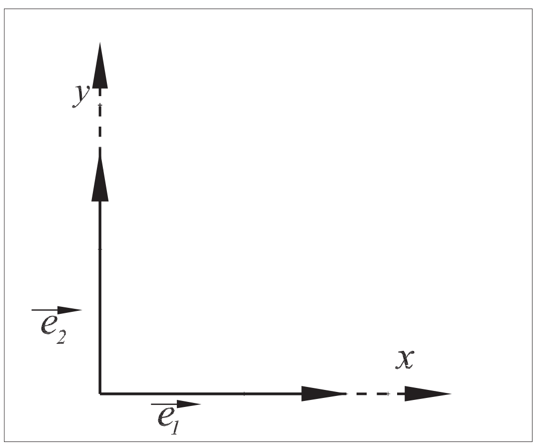


Figure 36: Basis vectors.

3.7 Transformation of coordinate systems

In general the component of a vector depend on the coordinate system used. Coordinate systems can be transformed, which changes the vector components in a certain way and a vector $\vec{s}$ in the old coordinate system becomes the vector $\widetilde{\vec{s}}$ in the new coordinates. The two easiest transformations of a coordinate system are a translation or shift and a rotation around the origin.

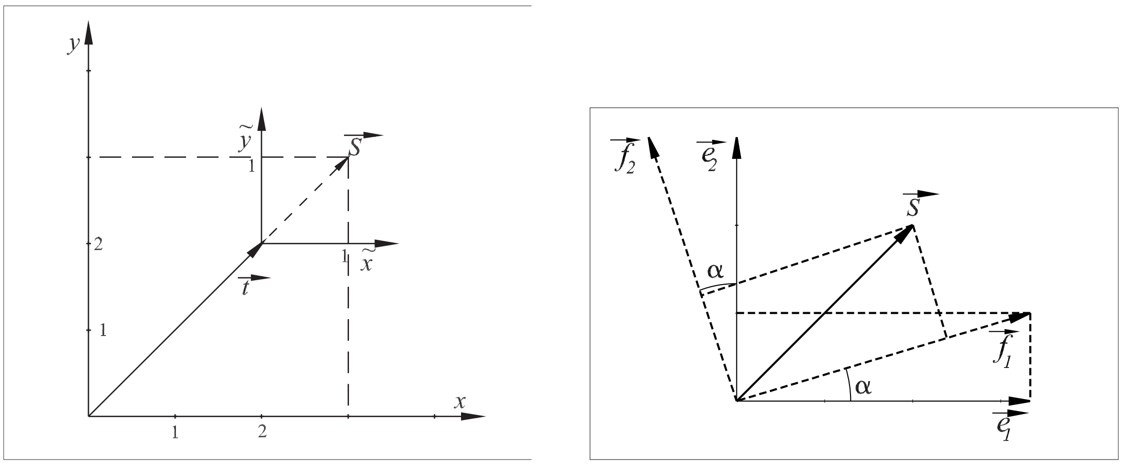
3.7.1 Translation

A translation of the coordinate system is performed by adding a constant vector

$$\vec{s} \to \widetilde{\vec{s}} = \vec{s} + \vec{t} \qquad \text{shifts the coordinate system by } \vec{t} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

Components of $\vec{s}$ in old system: $\vec{s} = \begin{pmatrix} x + t_1 \\ y + t_2 \end{pmatrix} = \begin{pmatrix} 1 + t_1 \\ 2 + t_2 \end{pmatrix}$

Components of $\widetilde{\vec{s}}$ in the new system: $\widetilde{\vec{s}} = \begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



(a) Translation of the coordinate system

(b) Rotation of the coordinate system

Figure 37: Coordinate transformations: Translation and Rotation

3.7.2 Rotation

A rotation of the coordinate system by an angle α around the origin is performed by applying the rotation matrix $\boldsymbol{R}$ to the vector $\widetilde{\vec{s}}$

$$\boldsymbol{R} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \qquad \widetilde{\vec{s}} = \boldsymbol{R}\,\vec{s} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\,\cos\alpha - y\,\sin\alpha \\ x\,\sin\alpha + y\,\cos\alpha \end{pmatrix}$$

The rotation matrix $\boldsymbol{R}$ can be found by calculating the basis vectors $\widetilde{\vec{e_1}}$ and $\widetilde{\vec{e_2}}$ for the new coordinate system

$$\widetilde{\vec{e_1}} = \cos\alpha\,\vec{e_1} + \sin\alpha\,\vec{e_2} \qquad \text{and} \qquad \widetilde{\vec{e_2}} = -\sin\alpha\,\vec{e_1} + \cos\alpha\,\vec{e_2}$$

Representation of a point $\vec{s}$:

$$\vec{s} = \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{x\,\vec{e_1} + y\,\vec{e_2}}_{\text{old system}} = \underbrace{\widetilde{x}\,\widetilde{\vec{e_1}} + \widetilde{y}\,\widetilde{\vec{e_2}}}_{\text{new system}} = \begin{pmatrix} \widetilde{x}\,\cos\alpha + \widetilde{y}\,\sin\alpha \\ -\widetilde{x}\,\sin\alpha + \widetilde{y}\,\cos\alpha \end{pmatrix}$$

Relation between old and new coordinates:

$$\vec{x} = \boldsymbol{A}\,\vec{\widetilde{x}} \qquad \text{where} \qquad \boldsymbol{A} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$\vec{\widetilde{x}} = \boldsymbol{A^T}\,\vec{x} \qquad \text{with} \qquad \boldsymbol{A^T} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \qquad \text{and} \quad \boldsymbol{A}\,\boldsymbol{A^T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} \vec{s} = x\,\vec{e_1} + y\,\vec{e_2} &= \widetilde{x}\,\underbrace{(\cos\alpha\,\vec{e_1} + \sin\alpha\,\vec{e_2})}_{\vec{\widetilde{e_1}}} + \widetilde{y}\,\underbrace{(-\sin\alpha\,\vec{e_1} + \cos\alpha\,\vec{e_2})}_{\vec{\widetilde{e_2}}} \\ &= \underbrace{(\widetilde{x}\,\cos\alpha - \widetilde{y}\,\sin\alpha)}_{x}\,\vec{e_1} + \underbrace{(\widetilde{x}\,\sin\alpha + \widetilde{y}\,\cos\alpha)}_{y}\,\vec{e_2} \end{aligned}$$

3.7.3 Polar coordinates

Polar coordinates are often used if the problem under consideration has a certain symmetry. They are represented by a vector $\vec{e_r}$ from the origin to a point in the plane and a vector $\vec{e_\varphi}$ from that point with the direction tangentially to the unit circle.

$$\vec{S} = \begin{pmatrix} x \\ y \end{pmatrix} = x\,\vec{e_1} + y\,\vec{e_2} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix} = r\,\vec{e_r} + \varphi\,\vec{e_\varphi} \quad \rightarrow \quad \begin{pmatrix} r \\ \varphi \end{pmatrix}_{pol}$$

$$\text{with} \quad r = \sqrt{x^2 + y^2} \quad \varphi = \arctan\frac{y}{x}$$

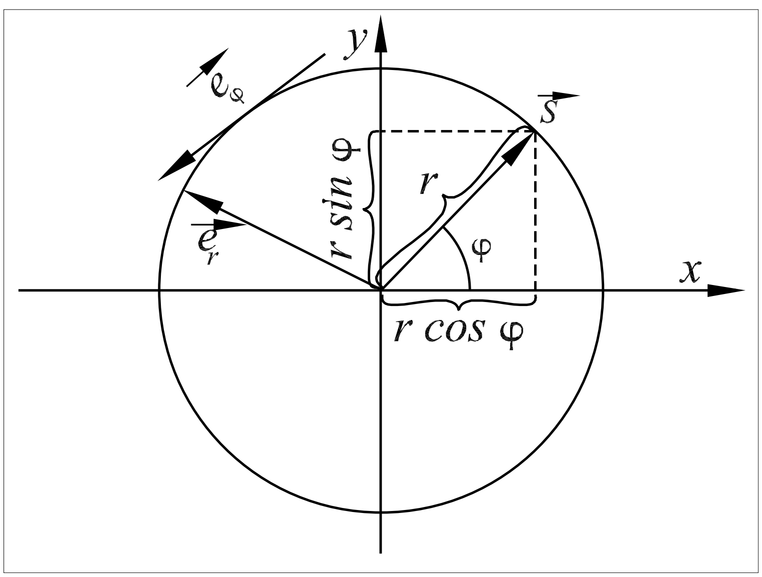


Figure 38: Polar coordinates

Example:

$$\vec{S} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{cart} = x\,\vec{e_1} + y\,\vec{e_2} \qquad \rightarrow \qquad r = \sqrt{1^2 + 1^2} = \sqrt{2} \qquad \varphi = \frac{\pi}{4} = 45^0$$

$$\vec{S} = r\,\vec{e_r} + \frac{\pi}{4}\,\vec{e_\varphi} \quad \rightarrow \quad \begin{pmatrix} \sqrt{2} \\ \frac{\pi}{4} \end{pmatrix}_{pol}$$

Note: The quantity $\begin{pmatrix} \sqrt{2} \\ \frac{\pi}{4} \end{pmatrix}_{pol}$ is *not* a vector!!!!

3.7.4 Non-orthogonal Coordinate Systems

Using a system of basis vectors that is orthogonal and normalized, i.e. $\vec{e_1} \cdot \vec{e_1} = \vec{e_2} \cdot \vec{e_2} = 0$ and $\vec{e_1} \cdot \vec{e_2} = 0$, or more general $\vec{e_i} \cdot \vec{e_j} = \delta_{ij}$ is very convenient because it is straight forward to find a certain component of a vector simply by multiplying with the corresponding basis vector

$$\vec{x} = x_1\,\vec{e_1} + x_2\,\vec{e_2} \;=\; \begin{cases} \vec{x} \cdot \vec{e_1} = x_1\,\overbrace{\vec{e_1} \cdot \vec{e_1}}^{=1} + x_2\,\overbrace{\vec{e_2} \cdot \vec{e_1}}^{=0} = x_1 \\ \vec{x} \cdot \vec{e_2} = x_1\,\underbrace{\vec{e_1} \cdot \vec{e_2}}_{=0} + x_2\,\underbrace{\vec{e_2} \cdot \vec{e_2}}_{=1} = x_2 \end{cases}$$

Sometimes, however it is necessary to represent vectors in a basis system $\vec{u}$, $\vec{v}$ that is not orthogonal. The easiest way to deal with this situation is to introduce a second set of basis vectors, the so-called *adjoint* vectors or *dual basis* $\vec{u}^\dagger$, $\vec{v}^\dagger$ such the relations

$$\vec{u}^\dagger \cdot \vec{u} = \vec{v}^\dagger \cdot \vec{v} = 1 \qquad \text{and} \qquad \vec{u}^\dagger \cdot \vec{v} = \vec{v}^\dagger \cdot \vec{u} = 0$$

are fulfilled. In components these equations read

$$u_1^\dagger\,u_1 + u_2^\dagger\,u_2 = 1 \qquad v_1^\dagger\,v_1 + v_2^\dagger\,v_2 = 1 \qquad u_1^\dagger\,v_1 + u_2^\dagger\,v_2 = 0 \qquad v_1^\dagger\,u_1 + v_2^\dagger\,u_2 = 0$$

which are four equations for the four unknowns $u_i^\dagger$, $v_i^\dagger$ and allows us to determine the adjoint vectors $\vec{u}^\dagger$ and $\vec{v}^\dagger$ if the original basis vectors $\vec{u}$ and $\vec{v}$ are linearly independent, i.e. not collinear.

Now we can express vectors in the basis $\vec{u}$ and $\vec{v}$, and determine their components by multiplying with the corresponding vectors from the adjoint basis

$$\vec{x} = a\,\vec{u} + b\,\vec{v} \;=\; \begin{cases} \vec{x} \cdot \vec{u}^\dagger = a\,\overbrace{\vec{u} \cdot \vec{u}^\dagger}^{=1} + b\,\overbrace{\vec{v} \cdot \vec{u}^\dagger}^{=0} = a \\ \vec{x} \cdot \vec{v} = a\,\underbrace{\vec{u} \cdot \vec{v}^\dagger}_{=0} + b\,\underbrace{\vec{v} \cdot \vec{v}^\dagger}_{=1} = b \end{cases}$$

Note: An ortho-normal basis system is simply the special case where the original and adjoint basis vectors are the same.

3.8 Determinants

The *determinant* is a is a descriptor of a matrix. The determinant of a 2×2 matrix is given by

$$\det \boldsymbol{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{21}\,a_{12}$$

The determinant of a 3×3 matrix it is defined as

$$\det \boldsymbol{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\,a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{21}\,a_{32} - a_{13}\,a_{22}\,a_{31} - a_{11}\,a_{23}\,a_{32} - a_{33}\,a_{12}\,a_{21}$$

Hint: The question arises, of course: ”Who can remember something like this?” Well, it is actually not that difficult using the following construction. First copy the left and the middle column to the right. Then go through this scheme as indicated below: the left to right or southeast diagonals are counted positive, the right to left or southwest diagonals are counted negative, resulting in the formula for the determinant of a 3×3 matrix (unfortunately such a procedure does not exist for higher dimensional matrices and how to find their determinants is beyond the scope of this course).

$$\left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right| \quad \Rightarrow \quad \left\{ \begin{array}{l} \begin{array}{ccccccccc} a_{11} & & a_{12} & & a_{13} & & a_{11} & & a_{12} \\ & \searrow & & \searrow & & \searrow & & & \\ a_{21} & & a_{22} & & a_{23} & & a_{21} & & a_{22} \\ & & & \searrow & & \searrow & & \searrow & \\ a_{31} & & a_{32} & & a_{33} & & a_{31} & & a_{32} \end{array} \qquad \textbf{positive: } + \\ \\ \begin{array}{ccccccccc} a_{11} & & a_{12} & & a_{13} & & a_{11} & & a_{12} \\ & & & \swarrow & & \swarrow & & \swarrow & \\ a_{21} & & a_{22} & & a_{23} & & a_{21} & & a_{22} \\ & \swarrow & & \swarrow & & \swarrow & & & \\ a_{31} & & a_{32} & & a_{33} & & a_{31} & & a_{32} \end{array} \qquad \textbf{negative: } - \end{array} \right.$$

Properties:

If at least two of the column vectors are *linearly dependent* the determinant $\det \boldsymbol{A} = 0$.

$$\det(\boldsymbol{A}\,\boldsymbol{B}) = \det \boldsymbol{A}\,\det \boldsymbol{B}$$

Examples:

$$\begin{array}{rcl} x - 2y & = & \alpha_1 \\ 5\,(x - 2y) & = & \alpha_2 \end{array} \qquad \boldsymbol{A} = \begin{pmatrix} 1 & -2 \\ 5 & -10 \end{pmatrix} \qquad \vec{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$\det \boldsymbol{A} = (-10) \cdot 1 - (-2) \cdot 5 = 0$$

$$\left|\begin{array}{rrr|rr} 3 & 1 & 2 & 3 & 1 \\ 0 & -2 & 2 & 0 & -2 \\ 1 & 3 & 0 & 1 & 3 \end{array}\right| = 3 \cdot (-2) \cdot 0 + 1 \cdot 2 \cdot 1 + 2 \cdot 0 \cdot 3 - 2 \cdot (-2) \cdot 1 - 3 \cdot 2 \cdot 3 - 1 \cdot 0 \cdot 0 = -12$$

Note: The components of the vector product can be found from a determinant

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i}\,(a_2\, b_3 - a_3\, b_2) + \vec{j}\,(a_3\, b_1 - a_1\, b_3) + \vec{k}\,(a_1\, b_2 - a_2\, b_1)$$

3.9 The Inverse of a Matrix $\boldsymbol{A}^{-1}$

The matrix $\boldsymbol{A}$ has an inverse with the property $\boldsymbol{A}\,\boldsymbol{A}^{-1} = \boldsymbol{I} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$ if $\det \boldsymbol{A} \neq 0$

Note:

$$\vec{y} = \boldsymbol{A}\,\vec{x} \qquad \rightarrow \qquad \vec{x} = \boldsymbol{A}^{-1}\,\vec{y}$$

$$\det(\boldsymbol{A}\,\boldsymbol{A}^{-1}) = \det \boldsymbol{I} = 1 = \det \boldsymbol{A}\,\det \boldsymbol{A}^{-1} \qquad \rightarrow \qquad \det \boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}}$$

Inverse of a 2×2 matrix:

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \boldsymbol{A}^{-1} = \frac{1}{\det\ \boldsymbol{A}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{A}\,\boldsymbol{A}^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \frac{1}{\det\ \boldsymbol{A}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$$= \frac{1}{a_{11}\,a_{22} - a_{12}\,a_{21}} \begin{pmatrix} a_{11}\,a_{22} - a_{12}\,a_{21} & 0 \\ 0 & a_{11}\,a_{22} - a_{12}\,a_{21} \end{pmatrix} = \boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3.10 Linear Systems of Equations

A system of the form

$$\begin{array}{lcl} y_1 & = & a_{11}\,x_1 + a_{12}\,x_2 + \dots + a_{1n}\,x_n \\ y_2 & = & a_{21}\,x_1 + a_{22}\,x_2 + \dots + a_{2n}\,x_n \\ \vdots & & \vdots \\ y_n & = & a_{n1}\,x_1 + a_{n2}\,x_2 + \dots + a_{nn}\,x_n \end{array} \qquad \text{or} \qquad \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

is called a *linear system of equations* and can be conveniently written in terms of vectors and matrices $\vec{y} = \boldsymbol{A}\,\vec{x}$. In most cases the coefficients a_{ij} and the values on the left hand side y_i are known, and one is interested in finding a solution, i.e. values for the x_i such that all equations are fulfilled. What are the conditions that such a system has solutions and what are their properties?

We distinguish two cases:

1. $\vec{y} \neq \vec{0}$, i.e. at least one of the $y_i \neq 0$. In this case the system is called *inhomogeneous* and it has a unique solution if $\det \boldsymbol{A} \neq 0$. Then the matrix $\boldsymbol{A}$ has an inverse and the solution is given by $\vec{x} = \boldsymbol{A}^{-1}\,\vec{y}$. For $\det \boldsymbol{A} = 0$ the system has either no solution or infinitely many depending on $\vec{y}$;
2. $\vec{y} = \vec{0}$, i.e. all of the $y_i = 0$. In this case the system is called *homogeneous* and it has always the solution $\vec{x} = \vec{0}$, which is called the *trivial* solution. Non-trivial solutions exist only if $\det \boldsymbol{A} = 0$ and then there are infinitely many.

Examples:

$$\begin{array}{c} 3\,x_1 + x_2 = 6 \\ 3\,x_1 - x_2 = 12 \end{array} \quad \text{inhom.,} \quad \det \boldsymbol{A} = \begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix} = -6 \neq 0 \quad \rightarrow \quad \begin{array}{c} \text{unique} \\ \text{solution} \end{array} \quad \rightarrow \quad \begin{array}{c} x_1 = 3 \\ x_2 = -3 \end{array}$$

$$\begin{array}{c} 3\,x_1 + x_2 = 6 \\ 6\,x_1 + 2\,x_2 = 10 \end{array} \quad \text{inhom.,} \quad \det \boldsymbol{A} = \begin{vmatrix} 3 & 1 \\ 6 & 2 \end{vmatrix} = 0 \quad \rightarrow \quad 12 = 10 \text{ fi} \quad \rightarrow \quad \begin{array}{c} \text{no} \\ \text{solution} \end{array}$$

$$\begin{array}{c} 3\,x_1 + x_2 = 6 \\ 6\,x_1 + 2\,x_2 = 12 \end{array} \quad \text{inhom.,} \quad \det \boldsymbol{A} = \begin{vmatrix} 3 & 1 \\ 6 & 2 \end{vmatrix} = 0 \quad \rightarrow \quad x_2 = -3\,x_1 + 6 \quad \rightarrow \quad \begin{array}{c} \text{infinitely many} \\ \text{solutions} \end{array}$$

$$\begin{array}{c} 3\,x_1 + x_2 = 0 \\ 3\,x_1 - x_2 = 0 \end{array} \quad \text{hom.,} \quad \det \boldsymbol{A} = \begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix} = -6 \quad \rightarrow \quad \begin{array}{c} x_2 = -3\,x_1 \\ x_2 = 3\,x_1 \end{array} \quad \rightarrow \quad \begin{array}{c} \text{trivial} \\ \text{solution} \end{array} \quad \rightarrow \quad \vec{x} = \vec{0}$$

$$\begin{array}{c} 6\,x_1 + 2\,x_2 = 0 \\ 3\,x_1 + x_2 = 0 \end{array} \quad \text{hom.,} \quad \det \boldsymbol{A} = \begin{vmatrix} 6 & 2 \\ 3 & 1 \end{vmatrix} = 0 \quad \rightarrow \quad x_2 = -3\,x_1 \quad \rightarrow \quad \begin{array}{c} \text{infinitely many} \\ \text{solution} \end{array}$$

3.11 Eigenvalues and Eigenvectors

A matrix performs a stretch, squeeze and/or rotation of a vector.The vector and matrix elements depend on the choice of the coordinate system. Since this choice is arbitrary, the question arises whether there is a special or canonical representation which is independent of the coordinate system.

There are distinguished directions [eigendirections (eigen $\sim$ self)] along which a matrix operates. Vectors pointing into these directions are only scaled but not rotated.

$$\boldsymbol{A}\vec{v} = \lambda\vec{v} \qquad \lambda \sim \text{eigenvalue} \quad \vec{v} \sim \text{eigenvector}$$

$$\text{or:} \quad \underbrace{(\boldsymbol{A} - \lambda\,\boldsymbol{I})}_{\boldsymbol{B}}\,\vec{v} = 0 \qquad \text{where} \quad \boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ is the identity matrix.}$$

The linear system of equations given by $\boldsymbol{b}\vec{v} = 0$ is homogeneous and has nontrivial solutions $\vec{v} \neq \vec{0}$ only if $\det\boldsymbol{b} = 0$. The condition for non-vanishing eigenvectors is therefore given by

$$\det(\boldsymbol{A} - \lambda\,\boldsymbol{I}) = 0$$

from which the eigenvalues can be readily found. The eigenvector are then determined by solving

$$(\boldsymbol{A} - \lambda\,\boldsymbol{I})\,\vec{v} = 0$$

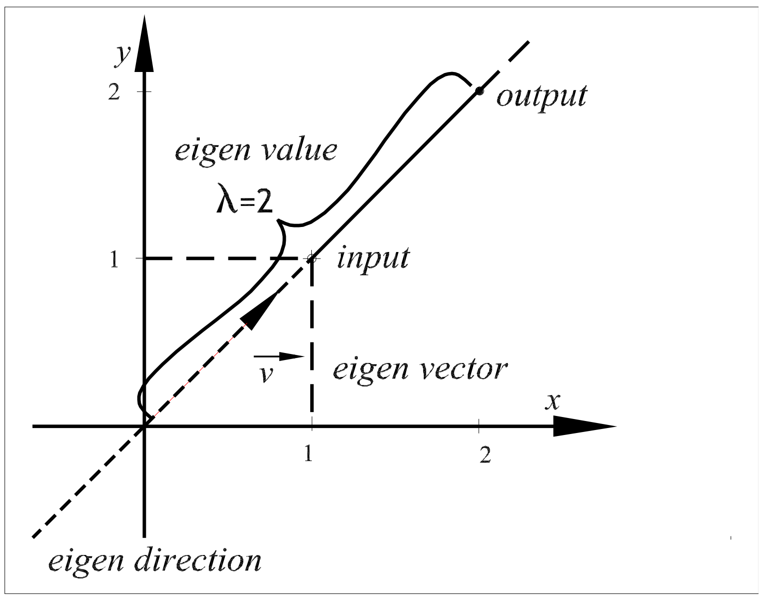


Figure 39: Eigenvalues and eigenvectors

Examples:

$$\boldsymbol{A} = \begin{pmatrix} 13 & 4 \\ 4 & 7 \end{pmatrix} \quad \rightarrow \quad \text{eigenvalues:} \quad \det(\boldsymbol{A} - \lambda\,\boldsymbol{I}) = \begin{vmatrix} 13-\lambda & 4 \\ 4 & 7-\lambda \end{vmatrix} = 0$$

$$\rightarrow \quad \lambda^2 - 20\lambda + 75 = 0 \quad \text{(characteristic polynomial)} \quad \rightarrow \quad \lambda_1 = 15, \quad \lambda_2 = 5$$

$$\lambda_1 = 15: \qquad \begin{array}{rcrcl} 13\,v_1 & + & 4\,v_2 & = & 15\,v_1 \\ 4\,v_1 & + & 7\,v_2 & = & 15\,v_2 \end{array} \qquad \rightarrow \quad v_1 = 2\,v_2$$

$$\rightarrow \quad \text{choose:} \quad v_2 = 1 \quad \rightarrow \quad v_1 = 2 \quad \rightarrow \quad |\vec{v_1}| = \sqrt{5} \quad \Rightarrow \quad \vec{v_1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 5: \quad \boldsymbol{A}\,\vec{v_2} = 5\,\vec{v_2} \quad \rightarrow \quad v_2 = -2\,v_1 \quad \rightarrow \quad \text{choose:} \; v_1 = 1 \quad \rightarrow \quad v_2 = -2 \quad \Rightarrow \quad \vec{v_2} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Note: A matrix is called *symmetric* if $a_{ij} = a_{ji}$. Symmetric Matrices have real eigenvalues and orthogonal eigenvectors $\vec{v_1} \cdot \vec{v_2} = 2 \cdot 1 + 1 \cdot (-2) = 0$.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \qquad \rightarrow \quad \text{eigenvalues:} \quad \det(\boldsymbol{A} - \lambda\,\boldsymbol{I}) = \begin{vmatrix} 1-\lambda & 0 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$\rightarrow \quad (1-\lambda)(2-\lambda) = 0 \quad \text{(characteristic polynomial)} \qquad \rightarrow \quad \lambda_1 = 1, \quad \lambda_2 = 2$$

$$\rightarrow \quad \text{eigenvectors:} \quad \boldsymbol{A}\,\vec{v} = \lambda\,\vec{v} \qquad \rightarrow \quad \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\lambda_1 = 1: \qquad \begin{array}{rcl} v_1 & = & v_1 \\ 2\,v_1 + 2\,v_2 & = & v_2 \end{array} \qquad \rightarrow \quad v_1 = -\frac{1}{2} v_2$$

$$\rightarrow \quad \text{choose:} \quad v_2 = 2 \quad \rightarrow \quad v_1 = -1 \quad \rightarrow \quad |\vec{v_1}| = \sqrt{5} \quad \Rightarrow \quad \vec{v_1} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\lambda_2 = 2: \quad \boldsymbol{A}\,\vec{v_2} = 2\,\vec{v_2} \quad \rightarrow \quad v_1 = 2\,v_1 \quad \rightarrow \quad v_1 = 0 \quad \rightarrow \quad \text{choose:} \; v_2 = 1 \quad \Rightarrow \quad \vec{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

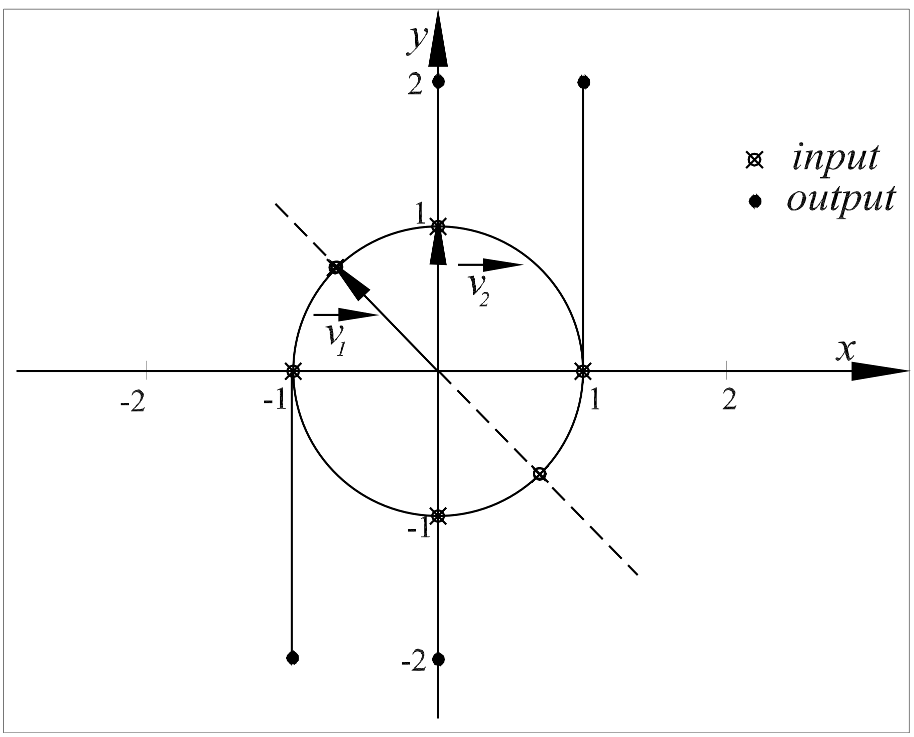


Figure 40: Determining eigenvalues and eigenvectors

4 Complex Number Theory

There are polynomials, such as $f(x) = x^2 + 1$, which do not have a root, $f(x) = 0$ for $x \in \mathbb{R}$. This is one of the reasons to extend the numbers from $x \in \mathbb{R}$ to $z \in \mathbb{C}$ where $\mathbb{C}$ is the set of complex numbers. Beyond this algebraic motivation there are many applications of complex numbers.

4.1 Representations and Basic Properties

A complex number z consists of a pair of real numbers called the real and imaginary part, respectively, and a "new" number 'i' which has the property $i^2 = -1$. While the real numbers can be represented as points on a line, a complex number is given as a point in a plane (called the complex plane) where the coordinates are its real (horizontal axis) and imaginary (vertical axis) part.

$$z = \underbrace{\overbrace{a}^{\text{real}} + i \overbrace{b}^{\text{imaginary}}}_{\text{complex number}} \qquad \text{with} \quad a, b \in \mathbb{R}$$

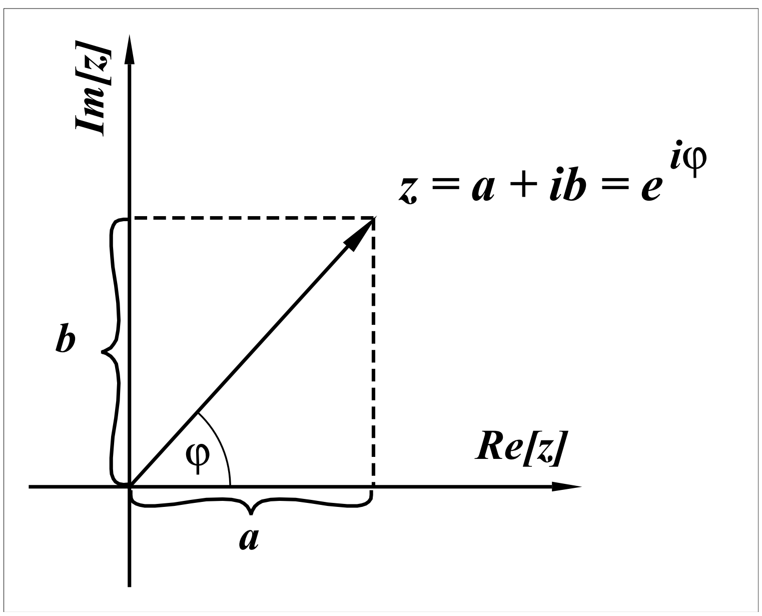


Figure 41: Representation of a complex number

The real and imaginary part, a and b, can be expressed by a distance from the origin r and an angle φ (remember polar coordinates) in terms of $a = r\cos\varphi$ and $b = r\sin\varphi$ which leads to

$$z = a + i\,b = r\,\cos\varphi + i\,r\,\sin\varphi = r\,e^{i\,\varphi} \qquad \text{with} \quad r = \sqrt{a^2 + b^2} \quad \varphi = \arctan\frac{b}{a}$$

Remarks:

- We will proof later by Taylor expansion that the relation $\cos\varphi + i\,\sin\varphi = e^{i\,\varphi}$ is true.
- All basic computations such as addition, subtraction, multiplication or division are defined for $z \in \mathbb{C}$.
- The inverse relation $\varphi = \arctan\frac{b}{a}$ is not unique because $\frac{-b}{-a} = \frac{b}{a}$. However, the first number is the third quadrant whereas the second number is in the first quadrant. Most computer languages, therefore have a second function to calculate the inverse tangent (usually called atan2 or so) which accepts two arguments, i.e. b and a and not only their ratio $\frac{b}{a}$ and returns the correct angle in the $[0, 2\pi]$ or $[-\pi, \pi]$ range.
- The number i is called the imaginary unit and is defined as $i^2 = -1$. It represents a very powerful tool to simplify calculations, in particular when trigonometric functions are involved. From its definition we find readily $i = \pm\sqrt{-1}$, $i^3 = -i$, and $i^4 = 1$.

Rules for dealing with complex numbers: $z = a + i\,b = r\,e^{i\,\varphi}$

$$\begin{aligned}
\text{Addition:} \quad & (a_1 + i\,b_1) + (a_2 + i\,b_2) &= a_1 + a_2 + i\,(b_1 + b_2) \\
\text{Multiplication:} \quad & (a_1 + i\,b_1)\,(a_2 + i\,b_2) &= a_1\,a_2 - b_1\,b_2 + i\,(a_1\,b_2 + a_2\,b_1) \\
& r_1\,e^{i\,\varphi_1}\,r_2\,e^{i\,\varphi_2} &= r_1\,r_2\,e^{i\,(\varphi_1+\varphi_2)}
\end{aligned}$$

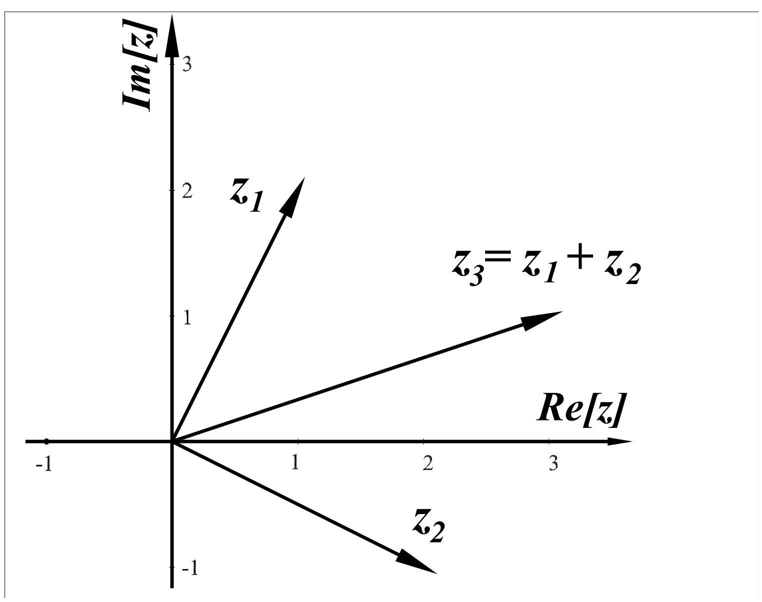


Figure 42: Adding two complex numbers

$\Rightarrow$ Addition of two complex numbers is done by adding their corresponding vectors

$\Rightarrow$ Multiplication of two complex numbers results in the product of the individual amplitudes and the sum of the phases

All the properties of real numbers are still preserved!

Examples: $z_1 = 1 + 2\,i \qquad z_2 = 2 - \,i$

$$z_1 + z_2 = 1 + 2 + i\,(2 - 1) = 3 + i \qquad z_1\,z_2 = 1 \cdot 2 - 2 \cdot (-1) + i\,(1 \cdot (-1) + 2 \cdot 2) = 4 + 3\,i$$

4.2 Complex conjugate

The complex conjugate of $z = a + i\,b$ is defined as $z^* = a - i\,b$

$$z^* = a + i\,b = r\,e^{-i\,\varphi} \qquad z\,z^* = (a + i\,b)\,(a - i\,b) = a^2 + b^2 = |\,z\,|^2 \neq z^2$$

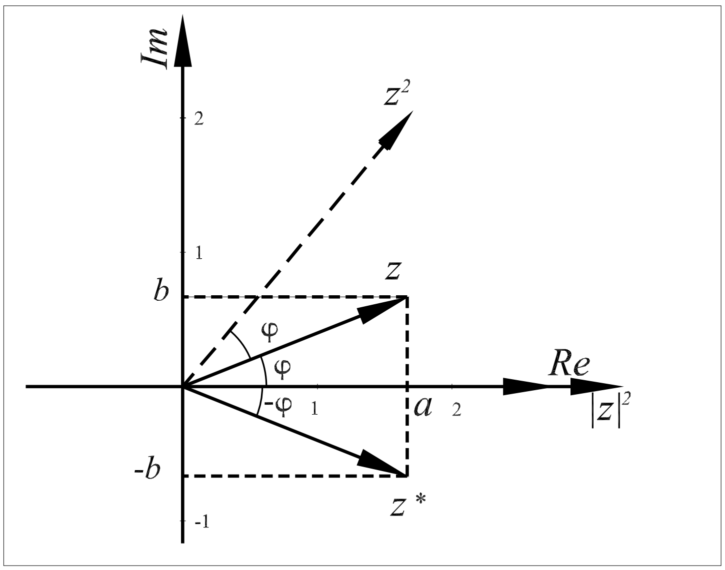


Figure 43: The complex number z and its complex conjugate z^*

Compare the following:

$$\begin{array}{ccccccccccc} z^2 & = & z\,z & = & (a + i\,b)^2 & = & a^2 - b^2 + 2\,i\,a\,b & = & (r\,e^{i\,\varphi})^2 & = & r^2\,e^{2\,i\,\varphi} \\ |\,z\,|^2 & = & z\,z^* & = & (a + i\,b)\,(a - i\,b) & = & a^2 + b^2 & = & r\,e^{i\,\varphi}\,r\,e^{-i\,\varphi} & = & r^2 \end{array}$$

Some more rules:

$$\text{Complex division:} \quad \frac{z_1}{z_2} = \frac{a_1 + i\,b_1}{a_2 + i\,b_2} = \frac{z_1\,z_2^*}{z_2\,z_2^*} = \frac{z_1\,z_2^*}{|\,z_2\,|^2} = \frac{a_1\,a_2 + b_1\,b_2 + i\,(a_2\,b_1 - a_1\,b_2)}{a_2^2 + b_2^2}$$

$$\mathcal{R}(z) = \frac{1}{2}(z + z^*) = \frac{1}{2}(a + i\,b + a - i\,b) = a \qquad \mathcal{I}(z) = \frac{1}{2}(z - z^*) = \frac{1}{2}(a + i\,b - (a - i\,b)) = b$$

$$e^{i\,\varphi} = \cos\varphi + i\,\sin\varphi \qquad \Rightarrow \qquad \cos\varphi = \frac{1}{2}(e^{i\,\varphi} + e^{-i\,\varphi}) \qquad \sin\varphi = \frac{1}{2\,i}(e^{i\,\varphi} - e^{-i\,\varphi})$$

5 Fourier Series

Fourier series go back to a publication by the French mathematician Jean Baptiste Joseph Fourier (1768-1830)

in 1804. A *Fourier series* is the decomposition of a function $f(x)$ into a sum of trigonometric

functions. If not stated otherwise we will assume in the following $f(x) = f(x+T)$, i.e. $f(x)$ it periodic

with a period of T.

A first simple step:

Find a representation of the function $f(x) = A\,\sin(kx - \phi)$ in terms of $\sin kx$ and $\cos kx$.

Here A is called the amplitude, k is the frequency and ϕ is the phase.

It can be shown that $f(x)$ can be written in the form

$$f(x) = A\,\sin(kx-\phi) = a\,\cos kx + b\,\sin kx \qquad \text{with} \qquad a = -A\,\sin\phi \qquad b = A\,\cos\phi$$

$$\text{and amplitude and phase given by:} \qquad A = \sqrt{a^2+b^2} \qquad \phi = -\arctan\frac{a}{b}$$

Example:

$$f(x) = 2\,\sin(x - \frac{\pi}{3}) \quad \to \quad a = -2\,\sin\frac{\pi}{3} = -\sqrt{3} \quad b = 2\,\cos\frac{\pi}{3} = 1 \quad \to \quad f(x) = -\sqrt{3}\cos x + \sin x$$

Superposition of two curves with different frequencies (harmonics):

$$\begin{aligned} f(x) &= A_1\,\sin(kx-\phi_1) + A_2\,\sin(2kx-\phi_2) \\ &= a_1\cos kx + b_1\,\sin kx + a_2\,\cos 2kx + b_2\,\sin 2kx \end{aligned}$$

$\Rightarrow$ By these means arbitrarily complicated functions may be constructed or decomposed.

Most functions can be approximated by a sum over trigonometric functions and some higher harmonics.

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{N} a_n\,\cos nkx + b_n\,\sin nkx \qquad \text{where } a_n \text{ and } b_n \text{ are real constants}$$

In the limit $N \to \infty$ a large class of functions can be represented as a Fourier series

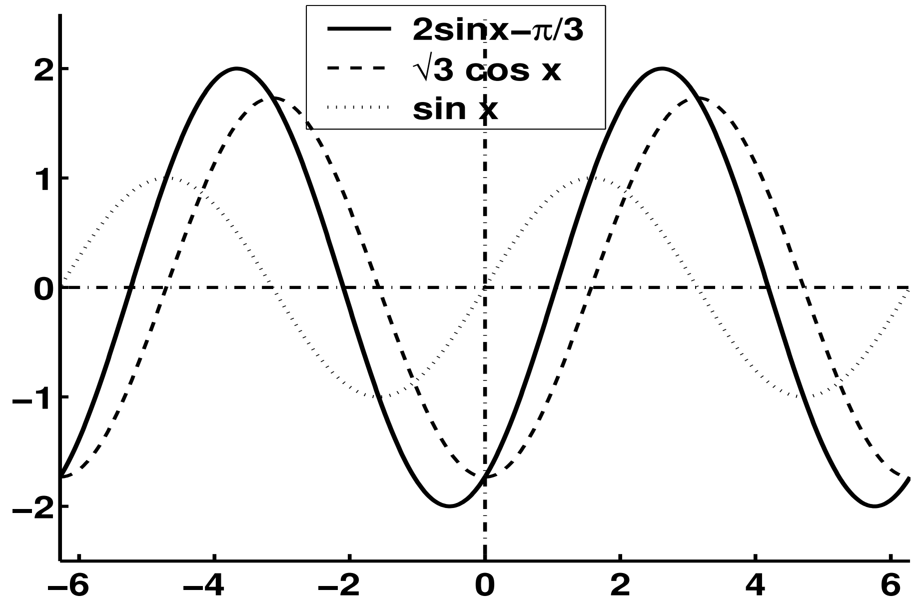


Figure 44: $f(x) = 2\,\sin(x - \frac{\pi}{3})$ expressed as $f(x) = -\sqrt{3}\cos x + \sin x$

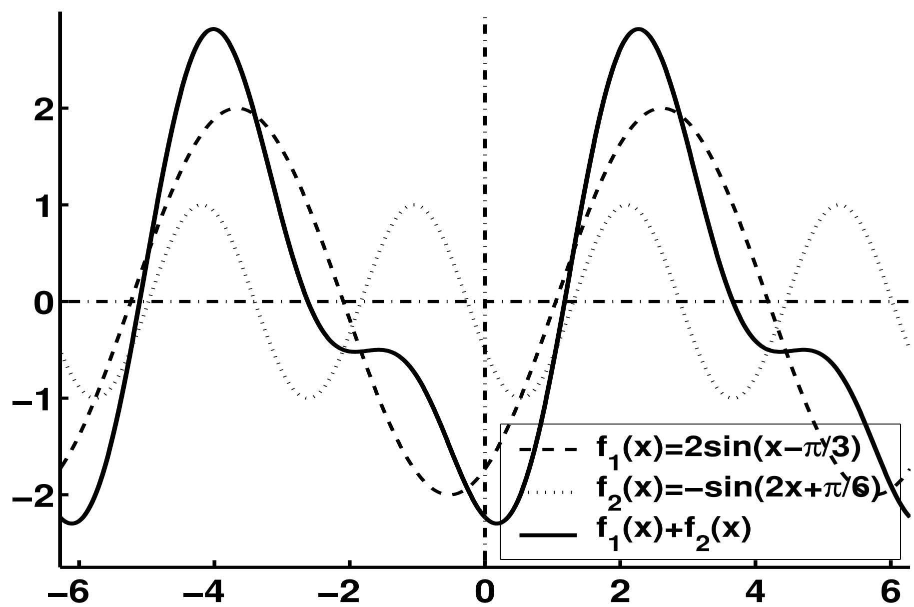


Figure 45: Superposition of two functions with different frequencies and phases

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\,\cos nkx + b_n\,\sin nkx$$

or by making use of $\cos nkx = \frac{1}{2}(e^{inkx} + e^{-inkx})$ and $\sin nkx = \frac{1}{2\,i}(e^{inkx} - e^{-inkx})$

$$f(x) = \underbrace{\frac{a_0}{2}}_{c_0} + \sum_{n=1}^{\infty} \left(\underbrace{\frac{a_n - i\,b_n}{2}}_{c_n} \, e^{inkx} + \underbrace{\frac{a_n + i\,b_n}{2}}_{c_{-n}} \, e^{-inkx} \right) = \sum_{n=-\infty}^{\infty} c_n \, e^{inkx}$$

How can the Fourier coefficients a_n, b_n and c_n be found?

If $f(x)$ can be represented by a Fourier series then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nkx + b_n \sin nkx \qquad \text{or} \qquad f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{inkx}$$

Now remember the orthogonality relations for the trigonometric functions:

$$\int_{-\pi}^{\pi} \sin mkx \, \sin nkx \; dx = \int_{-\pi}^{\pi} \cos mkx \, \cos nkx \; dx = \pi \, \delta_{mn} \qquad \int_{-\pi}^{\pi} \sin nkx \, \cos mkx \; dx = 0$$

and for the exponentials: $\displaystyle\int_{-\pi}^{\pi} e^{imx} \, e^{-inx} \; dx = 2\,\pi\,\delta_{mn}$

So we multiply both sides of the right equation above with e^{-imkx} and integrate from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x) \, e^{-imkx} \; dx = \int_{-\pi}^{\pi} e^{-imkx} \; dx \sum_{n=-\infty}^{\infty} c_n \, e^{inkx} = \sum_{n=-\infty}^{\infty} c_n \underbrace{\int_{-\pi}^{\pi} e^{inkx} e^{-imkx} \; dx}_{2\,\pi\,\delta_{mn}}$$

Because of the orthogonality of the exponential functions the sum disappears and we find for c_n

$$c_n = \frac{1}{2\,\pi} \int_{-\pi}^{\pi} f(x) \, e^{-inkx} \; dx \qquad n = 0, \pm 1, \ldots$$

In a similar way the left equation can be multiplied with $\sin mkx$ and in a second step with $\cos mkx$

and again integrated over x from $-\pi$ to π. Using the orthogonality relations for sine and cosine

the coefficients a_n and b_n can be readily calculated

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, \cos nkx \; dx \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, \sin nkx \; dx \qquad n = 0, 1, \ldots$$

Example: $\quad f(x) = x^2 \quad -\pi \leq x \leq \pi$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin nkx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x^2}_{even}\,\underbrace{\sin nkx}_{odd}\,dx = 0 \qquad \text{symmetries!!}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nkx\;dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nkx\;dx = \frac{2}{\pi}\left[\frac{2x}{(nk)^2}\cos nkx + \left(\frac{x^2}{nk} - \frac{2}{(nk)^3}\right)\sin nkx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\frac{2\pi}{(nk)^2}\cos nk\pi = (-1)^n\frac{4}{(nk)^2}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\;dx = \frac{2}{\pi}\left[\frac{1}{3}x^3\right]_0^{\pi} = \frac{2}{3}\pi^2$$

$$\Rightarrow f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{1}{(nk)^2}\;\cos nkx$$

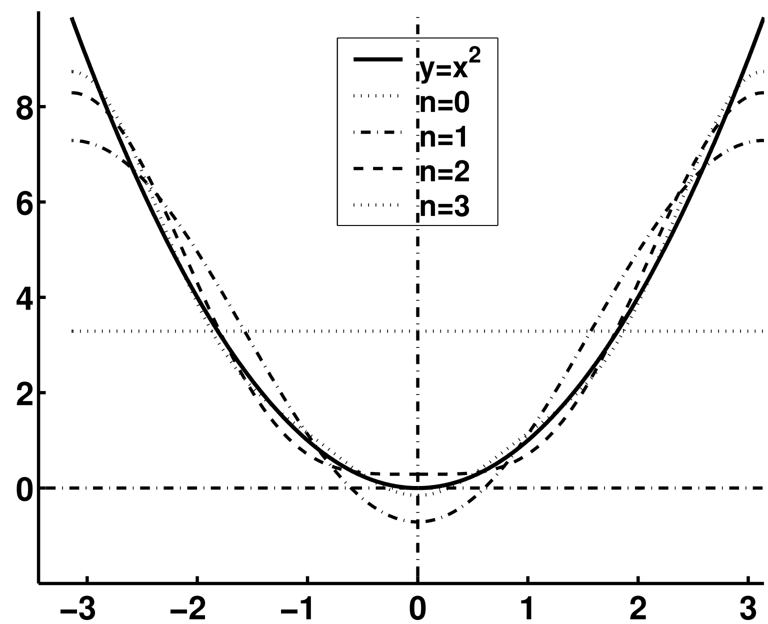


Figure 46: Decomposition of a non periodic function

5.1 Power Spectrum

Power spectrum is the power in the n-th frequency component

plotted over all frequencies.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\underbrace{a_n\,\cos nkx + b_n\,\sin nkx}_{\text{n-th frequency component}}$$

$$\Rightarrow \quad \text{Power:} \quad P = a_n^2 + b_n^2 = A_n^2\,\cos^2\phi_n + A_n^2\,\sin^2\phi_n = A_n^2$$

$$\Rightarrow \quad \text{For the above example:} \quad P = a_n^2 + b_n^2 = 0^2 + \frac{16}{(nk)^4} \propto \frac{1}{n^4}$$

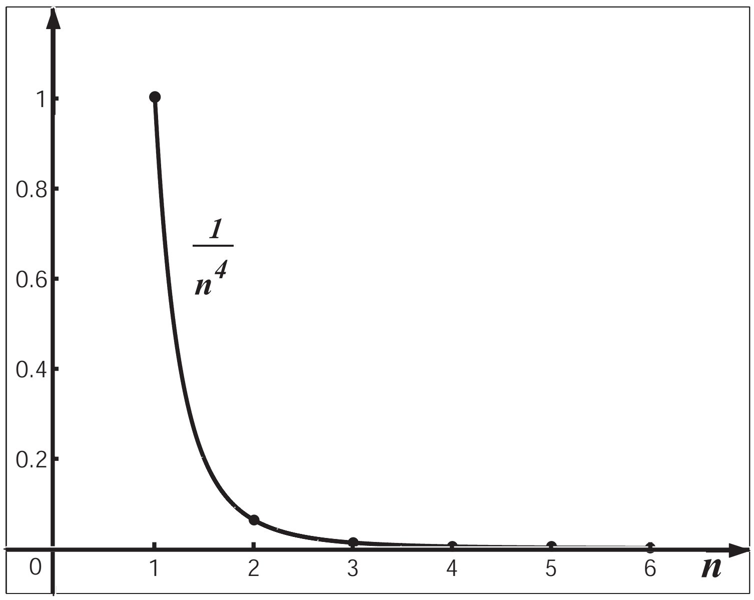


Figure 47: The power spectrum for $f(x) = x^2$

Typical situation: When a time series is not periodic it may be assumed to be periodically contained. If T it the length of the time series, then the smallest frequency in the Fourier series is given by $k = \frac{2\pi}{T}$.

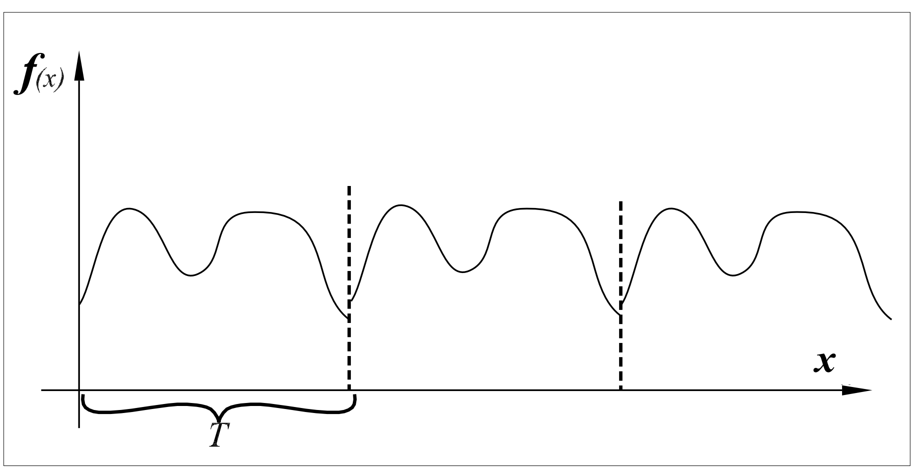


Figure 48: A finite non periodic time series is periodically contained

5.2 Gibbs' Phenomenon

It is an obvious question, whether a Fourier series always gets closer and closer to the value of a given function at all points x. The answer is 'yes', if the function is continuous, and 'no' if the function is discontinuous, i.e. it has jumps. Unless we are very lucky, this is the case, for instance if we assume a non-periodic time series to be periodically contained as above. To see what happens around the discontinuities we look at the function

$$f(x) = \begin{cases} 1 & \text{for} \quad 0 \le x < \pi \\ -1 & \text{for} \quad \pi \le x < 2\pi \end{cases} \qquad \text{which represents a step function}$$

Periodically contained this function can be written as a Fourier series of the form

$$f(x) = \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + + \frac{\sin 7x}{7} + \ ...\right] = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\,x}{2n+1}$$

The figure shows functions obtained from the first (dashed) and the first two terms from the Fourier

series together with the function for $n = 20$. It is obvious, that in the vicinity of the discontinuities the Fourier series does not converge to the values ± 1 but overshoots and then exhibits a fast damped

oscillation which is called 'ringing'. These two effects together, the overshoot and the ringing at discontinuities are known as Gibbs' phenomenon.

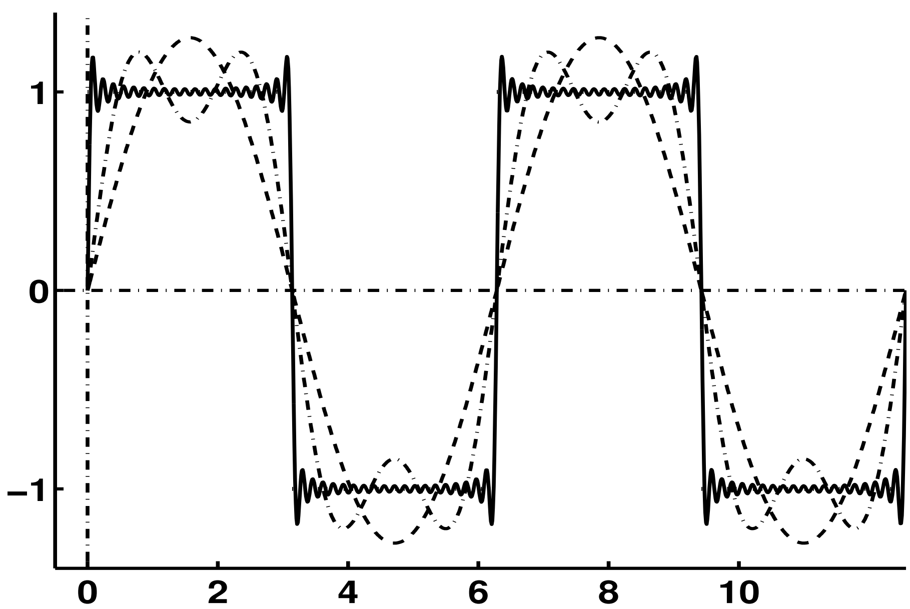


Figure 49: Gibbs' phenomenon

5.3 Important Time-Frequency relations

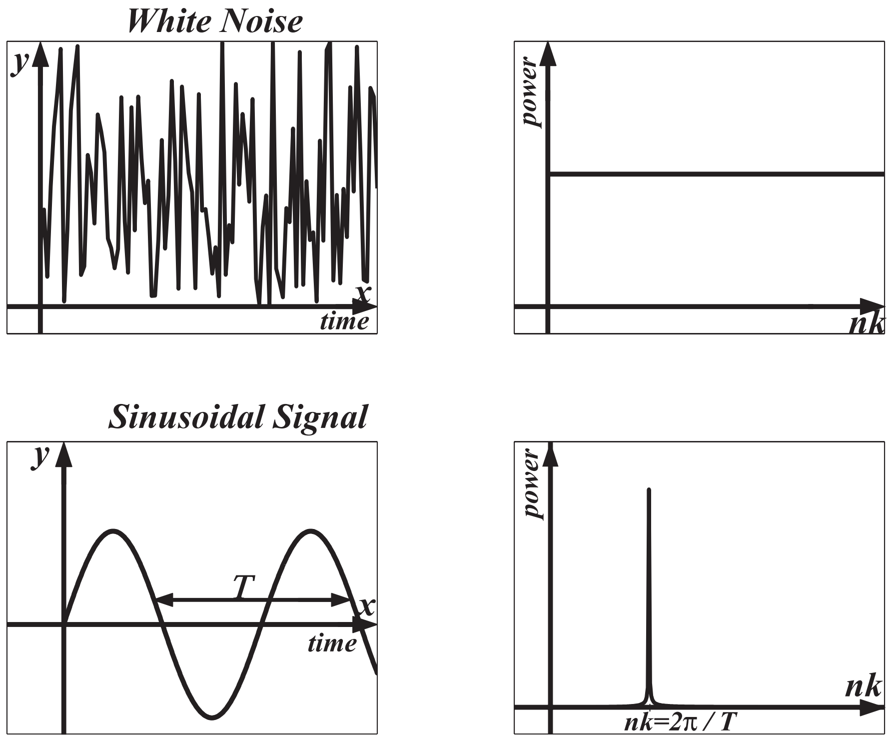


Figure 50: Power spectra of a random signal and a sine wave

5.4 Taylor Series

A Taylor series is an approximation of a function $f(x)$ in the neighborhood of a given point in terms of its derivatives. This technique has been developed by Brook Taylor$(1685 - 1731)$ and first published in 1715.

In a first step we can approximate the function $f(x)$ in the neighborhood around x_0 by the tangent through x_0

$$f(x) = f(x_0) + f'(x_0)\,(x - x_0) + \text{error}$$

Can we do better than this? Yes if higher order derivatives are considered!

$$f(x) = f(x_0) + f'(x_0)\,(x - x_0) + \frac{1}{2!}\,F''(x_0)(x - x_0)^2 + \frac{1}{3!}\,F'''(x_0)(x - x_0)^3 + ...$$

Taylor series: $\quad f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\,\frac{d^n f(x)}{dx^n}\bigg|_{x=x_0} (x - x_0)^n \qquad \text{with} \quad n! = 1 \cdot 2 \cdot 3 \cdot ... \quad \text{n-factorial}$

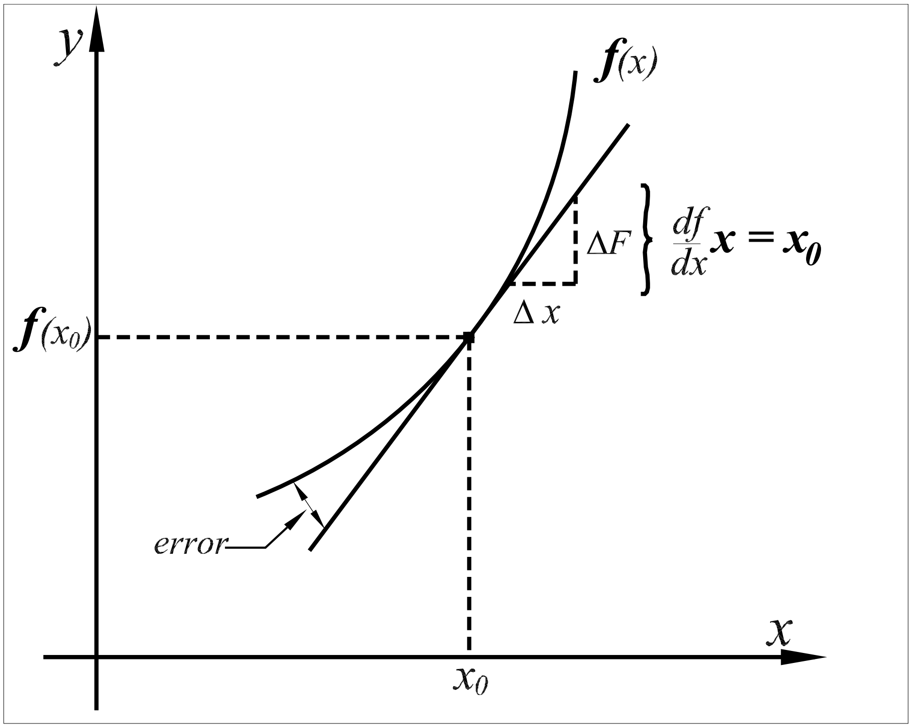


Figure 51: Approximation of a curve at a point x_0 with the tangent at x_0

A function $f(x)$ may be approximated by truncating a Taylor Expansion

around x_0 at the m-th order.

$\Rightarrow$ Polynomial representations of function work well if

the error approaches 0 as the order n increases.

Error estimate:

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} \left. \frac{d^n f(x)}{dx^n} \right|_{x=x_0} (x - x_0)^k + \underbrace{R_n(x)}_{\text{error}}$$

Lagrange formulation of the error R_n in a Taylor expansion that is truncated at order n

$$R_n(x) = \frac{(x - x_0)^{n+1}}{(n+1)!} \left. \frac{d^n f(x)}{dx^n} \right|_{x=\xi} \qquad \xi = x_0 + \delta\,(x - x_0) \quad 0 < \delta < 1$$

Examples:

Approximate the function $\sin x$ up to the 7-th order for x around $x_0 = 0$ (Maclaurin series)

using a Taylor expansion

$$\sin x \approx \underbrace{\sin 0}_{0} + \underbrace{\cos 0 \cdot x}_{x} + \underbrace{-\sin 0 \frac{1}{2!}\, x^2}_{0} - \frac{1}{3!}\, x^3 + \frac{1}{5!}\, x^5 - \frac{1}{7!}\, x^7 \qquad \text{symmetries!!}$$

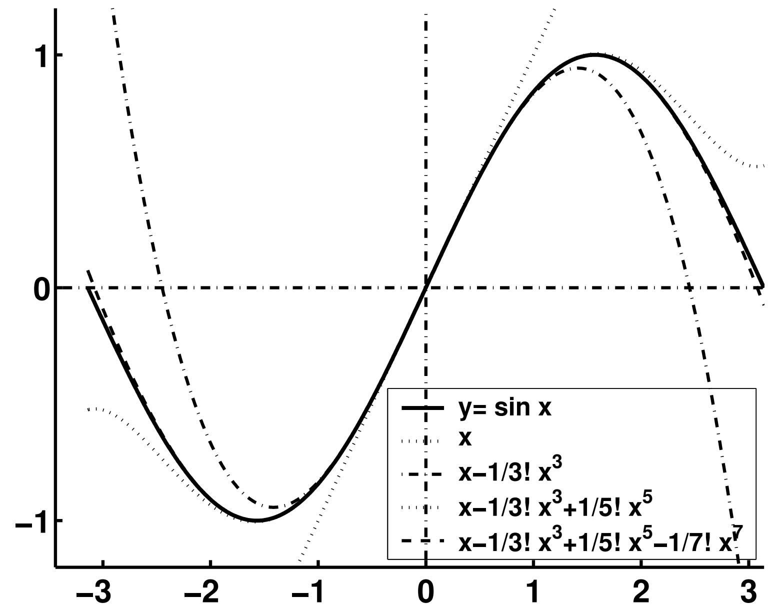


Figure 52: Steps of a Taylor expansion of $\sin x$ around $x_0 = 0$

The further away you move from the expansion point, the more significant the higher order terms!!

Specific expansion of important functions:

$$\sin x = x - \frac{1}{3!}\, x^3 + \frac{1}{5!}\, x^5 - \ .... \ \frac{(-1)^{n+1}}{(2\,n+1)!}\, x^{2\,n+1} \qquad \text{only odd terms}$$

$$\cos x = 1 - \frac{1}{2!}\, x^2 + \frac{1}{4!}\, x^4 - \ .... \ \frac{(-1)^n}{2\,n!}\, x^{2\,n} \qquad \text{only even terms}$$

$$e^x = 1 + \frac{1}{2!}\, x^2 + \frac{1}{3!}\, x^3 + \ .... \ \frac{1}{n!}\, x^n$$

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} + \ .... \ \frac{(-1)^{n+1}}{n!}\, x^{n+1}$$

Euler's Formula: $e^{i\theta} = \cos\theta + i\,\sin\theta$

$$e^{i\,\theta} = 1 + i\theta + \frac{1}{2!}\underbrace{(i\,\theta)^2}_{-\theta^2} + \frac{1}{3!}\underbrace{(i\,\theta)^3}_{-i\,\theta^3} + \frac{1}{4!}\underbrace{(i\,\theta)^4}_{\theta^4} + \frac{1}{5!}\underbrace{(i\theta)^5}_{i\,\theta^5} + ... \qquad \text{Taylor expansion around } \theta = 0$$

$$\Rightarrow \qquad e^{i\,\theta} = \underbrace{1 - \frac{1}{2!}\,\theta^2 + \frac{1}{4!}\,\theta^4 + ...}_{\cos\theta} \quad + i\,(\underbrace{\theta - \frac{1}{3!}\,\theta^3 + \frac{1}{5!}\,\theta^5 + ...}_{\sin\theta}) \qquad \text{q.e.d.}$$

5.5 Finally some fun with Math

Now, let's have a little more fun with complex numbers. From Euler's formula we find

$$e^{i\,\pi} = \cos\pi + i\,\sin\pi = -1$$

Now we take the natural log of both sides

$$\ln\{e^{i\pi}\} = i\pi = \ln(-1) \qquad \text{WHAT'S THAT ??}$$

Didn't your math teachers always tell you that there is not such a thing as the logarithm of a negative number? Well, obviously they lied. But probably for good reasons: They didn't want you to get confused. Because as a smart kid you may have come up with

$$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i \quad \text{or take the log} \quad \ln i = i\frac{\pi}{2}$$

even worse, the logarithm of an imaginary number ? Or multiply the last equation by i

$$i\ln i = i * i\frac{\pi}{2} \qquad \ln(i^i) = -\frac{\pi}{2}$$

and raise into the exponent of e

$$i^i = e^{-\frac{\pi}{2}} \approx 0.2078796 \quad \text{A } \textit{real} \text{ number !!!}$$

Or you can rewrite the monster number π simply as $\pi = -2\,i\ln i$. Confused? Don't worry, you are in excellent company. The American mathematician Benjamin Pierce in the last century called Euler's formula "mysterious". The story goes that after having shown these derivations to his class at Harvard he turned around and declared: "Gentlemen," (there might not have been many women in math classes at this time) ”this is surely true, it is absolutely paradoxical; we cannot understand what it means. But we have proved it, and therefore we know it must be the truth.”

Talking about proofs: Check this one out

$$e^{3i\pi} = \cos 3\pi + i \sin 3\pi = -1 \quad \text{and take the log} \quad 3i\pi = \ln(-1)$$

Uuuups, didn’t we see above that $\ln(-1) = i\pi$? So we have $3i\pi = i\pi$ or $1 = 3$, right ? See, that’s the trouble your teacher wanted to keep you away from.

6 Acknowledgements

The Mathematics Boot Camp would not have been possible without the help of my students, both running the boot camp, as well as writing this script. Here I wish to thank in particular Felix Almonte, Collins Assisi, Frederick Carver and Gautam Vallabha who tutored several afternoon sessions over the years and developed the exercises. Their contribution cannot be overstated. Jaime Morris helped in the typing and formatting of the script and Ajay Pillai did the lion’s share in generating the figures and formatting the script. Without these two, the script would have never evolved beyond a few hand-scribbled notes. Armin Fuchs spent considerable time on debugging the manuscript and took over my class towards the end of my tenure at FAU. Thank you so much!

Epilogue:

There was a young fellow from Trinity
Who took $\sqrt{\infty}$
But the number of digits
Gave him the fidgets
He dropped Math and took up Divinity.

George Gamow