EXERCISES (iv) - Maths for Biology

Computational Methods in Ecology and Evolution
Imperial College London
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Exercises Ordinary Differential Equations (ODEs)

Separated variables Integrate the following ODE:

$$a)\ y' = \frac{x^2}{1-y^2}$$

Plot in your computer several particular solutions.

Homogeneous a) Integrate the following ODE, and look for the particular solution $y(1) = 0$. Can you try to simplify further the expression once you get that particular solution?

$$2(x + 2y)dx + (y - x)dy = 0$$

b) Solve the following ODEs

$$y' = \frac{y^2+2xy}{x^2}$$

Exercise 3. Linear a) Integrate the following ODE:

$$y' + 2y = e^{-x}$$

Plot several particular particular solutions and look for the equation of $y(0) = 0.75$.

b) The Bernouilli equations belong to a kind of differential equations that can be converted into linear ones, and have the form:

$$y' = a(x)y + b(x)y^n.$$

There are still other kind of ODEs due to Ricatti, that can be converted as well into linear, having the form:

$$y' = a(x) + b(x)y + c(x)y^2.$$

We will focus only on Bernouilli's equations. To solve them, we start dividing every member by y^n, what yields:

$$\frac{y'}{y^n} = a(x)y^{(1-n)} + b(x).$$

This looks like kind of linear but there is a term multiplying the derivative that we do not like. Then we do the change $u = y^{(1-n)}$. We need to know as well which is the expression for y' after this change, namely $u' = d\left(y^{(1-n)}\right)/dx = (1-n)y^{-n}y'$, which we rearrange as:

$$\frac{y'}{y^n} = \frac{u'}{1-n}$$

and then

$$\frac{u'}{1-n} = a(x)u + b(x).$$

We still have an annoying factor $(1-n)$, but if we can simply say $\hat{a}(x) = (1-n)a(x)$ and $\hat{b}(x) = (1-n)b(x)$, we will deal with a canonical linear equation:

$$u' = \hat{a}(x)u + \hat{b}(x).$$

You just need to solve the equation as linear, and then undo the changes you made. Now apply this procedure to solve the Bernouilli equation:

$$y' = 5x^2y^5 + \frac{y}{2x}.$$

c) Solve the differential equation:

$$2y' - y = 4\sin(3t)$$

and explain if the general solution is finite when $t \to \infty$. Hint1: Be aware that this behaviour depends on the possible values of the integration constant. Hint2: At some point you will need to integrate by parts twice.

Exact ODEs a) Integrate the following ODE:

$$2x + y^2 + 2xyy' = 0.$$

b) The following ODE is NOT exact.

$$(3xy + y^2) + (x^2 + xy)y' = 0.$$

You should demonstrate that it is not exact and then apply the method for solving exact equations to show that, indeed, it is not possible to find a solution $\psi = c$ fulfilling the conditions of an exact ODE.

ODEs (guess the type) Consider the following first order differential equation (ODE):

$$a)\ y' = \frac{y^2 - x^2}{xy}.$$

1. Obtain the general solution $y = f(x, C)$, being C a constant.
2. Obtain the value of C for the particular solution $y(e^2) = \sqrt{4}e^2$, where e is the Euler number (i.e. the base of the natural logarithm).
3. Explain which is the domain of the particular solution $y = f(x)$ you obtained in the previous step.

Optional exercises

Newton's law of cooling According with the Newton's law of cooling, the rate of heat loss of a body is proportional to the difference between the temperature T of the body and the temperature T_{env} of the environment. The average temperature in Mars is -50 degrees Celsius and it is estimated that a given light clothing equipment that is being tested, makes that human temperature will be reduced from 36.5 degrees to 36 in ten minutes. If we consider a safety limit of 35 degrees, estimate the total time that a human can stay in Mars surface with this equipment. The Newton's law is:

$$\frac{dT}{dt} = k(T - T_{env}).$$

Logistic or Verhulst equation The logistic equation models the growth of the abundance N of a single population species as

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N,$$

where r is the intrinsic growth rate, namely the growth of the species when there is no limiting factor (this would lead to the well-known exponential growth consequence of the Malthusian equation), and K is a saturating limit known as carrying capacity of the environment for the species. It is interpreted as an intrinsic limitation for the species growth due to the fact than, in a given environment, there are limited resources for every species. With your current knowledge of ODEs you are probably thinking in solving this equation immediately. But, as we are dealing with a biological problem, we should try to understand what is this equation telling us before going into the solution.

First of all, you should explore which values of the parameters may allow you to have biologically meaningful values. In this case, it is obvious that any values of the parameters leading to non-positive abundances $N < 0$ would make non sense. This exercise is what is called the exploration of the *feasibility* of the system, and it is made exploring the stationary state of the system, namely looking at $dN/dt = 0$. You should explore the feasibility condition of this system. In this case, it is not very interesting because we have just one species and two parameters, but for multiple species systems performing this first step is critical to avoid working with non-sense models, and because you can already get very interesting biological insights (for instance, in the folder of Theoretical Biology, there is a paper of Soyer that is a good example of this).

Second, as $dN/dt = f(N)$ we can analyse this function. You should plot $f(N)$ versus N, find analytically its maxima and minima and the roots, and explain what is this function telling you from a biological point of view. To interpret this analysis, remember the meaning of the function, if $f(N) < 0$ means that the population is decreasing and the other way around.

Third, look for the solutions. It is a separated variables ODE (N and t) but, to solve the side of N you will need to apply a factorization as in exercise 2b. The solution was given already in the block of exercises of functions, so you may consider to merge that exercise here. Once you got the solution, perform the $\lim_{N\to\infty} N(t)$. The solution you will obtain will be a constant, which is its meaning? Finally, as you have already fighted against the definition of limit, you should not have problems going to the Wikipedia page on "Lyapunov stability" and trying to understand the definitions of Lyapunov stability and Asymptotic stability given in the section "Definition for continuous-time systems". And just enjoy that you feel more powerful than one week ago.