SOLUTIONS EXERCISES (i) - Maths for Biology

Computational Methods in Ecology and Evolution
Imperial College London
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Lecturer: Alberto Pascual-García.

Correspondence: alberto.pascual.garcia@gmail.com

Exercises Functions

General questions

Functions domains. 1. The domain of the sin function is $\mathbb{R}$ and, thus, we only need to analyse the domain of the argument $\frac{x+1}{x-1}$. We easily see that the latter is not defined for $x = 1$, and thus the domain of the function will be $\mathbb{R} - \{1\}$.

2. Again, we start analysing the domain of the most "external" function which, in this case, is the logarithm. The domain of the logarithm is $(0, \infty)$, therefore, we should look for the values of the argument that are strictly positive, which means that the following inequality should hold:

$$\frac{x^2 + x + 3}{x^2 + 1} - 1 > 0$$

or, equivalently, that

$$\frac{x^2 + x + 3}{x^2 + 1} > 1$$

solving it we find that

$$x^2 + x + 3 > x^2 + 1 \Longrightarrow x > -2$$

As we see, the argument is positive for values of x lower than zero, and thus the fraction in the argument of the logarithm allow us to "extend" the domain of the logarithm from $(0, \infty)$ to $(-2, \infty)$, which is the domain of $m(x)$.

Functions parity.

$$f(x) = \frac{\sin(x) + x^3}{2x^2 + \cos(x) + 4}$$

SOLUTION: We evaluate the parity making $x = -x$:

$$f(-x) = \frac{\sin(-x) + (-x)^3}{2(-x)^2 + \cos(-x) + 4} = \frac{-\sin(x) - x^3}{2x^2 + \cos(x) + 4} = -f(x)$$

and, thus, the function is odd.

$$g(x) = \frac{3x^4 + x^2}{x^5 + 1}$$

SOLUTION:

$$g(-x) = \frac{3(-x)^4 + (-x)^2}{(-x)^5 + 1} = \frac{3x^4 + x^2}{-x^5 + 1}$$

And this function is not odd nor even.

$$n(x) = \frac{\sin(x)\cos(x)}{x}$$

SOLUTION:

$$n(-x) = \frac{\sin(-x)\cos(-x)}{-x} = \frac{-\sin(x)\cos(x)}{-x} = n(x)$$

therefore, the function is even.

Functions periodicity.

$$f(x) = \sin(x) + \cos(x)$$

SOLUTION: It is not so straightforward to analyse the periodicity of functions as it was the parity, and we should rely on our previous knowledge of the functions analysed and on some intuition. In this case, we know that both functions have period 2π, so it is natural to start checking this value:

$$f(x + 2\pi) = \sin(x + 2\pi) + \cos(x + 2\pi) = \sin(x) + \cos(x)$$

and the function has period 2π as well, and there is no other number α lower than 2π making that $f(x+\alpha) = f(x)$.

$$m(x) = \frac{\sin(x)}{\cos(x)}$$

SOLUTION: Again, we could start considering the period 2π. And, indeed, $m(x) = m(x+2\pi)$. But we should still discard that there it is not periodic for any smaller period (sub-multiple of 2π). If we consider $2\pi/2 = \pi$, the function is periodic because:

$$m(x + \pi) = \frac{\sin(x + \pi)}{\cos(x + \pi)} = \frac{-\sin(x)}{-\cos(x)} = m(x)$$

and, thus, the period is π, as we should expect because $m(x) = \tan(x)$, which we know has period π. The interesting thing is that, even if both functions have period 2π, depending on how we combine them (as a sum in the previous exercise and as a fraction here) we may obtain a function with a different period!

Polynomial functions

Exercise

Polynomials with	Positive leading coefficient	Negative leading coefficient
Odd degree	$x \to \infty$ $y \to \infty$ $x \to -\infty$ $y \to -\infty$	$x \to \infty$ $y \to -\infty$ $x \to -\infty$ $y \to \infty$
Even degree	$x \to \infty$ $y \to \infty$ $x \to -\infty$ $y \to \infty$	$x \to \infty$ $y \to -\infty$ $x \to -\infty$ $y \to -\infty$

Fractional functions

Fractions decomposition

$$f(x) = \frac{3x}{x^2 - 6x + 8}$$

SOLUTION: The denominator can be factored in the form:

$$x^2 - 6x + 8 = (x - 4)(x - 2).$$

A simple way to remind how this factorization is obtained is to observe that $(x-a)(x-b) = x^2 - (a+b)x + ab$, and then we just need to solve the system of equations:

$$\begin{aligned} a + b &= 6 \\ ab &= 8 \end{aligned}$$

to obtain that $a = 4$ and $b = 2$.

Therefore, the fraction becomes

$$f(x) = \frac{3x}{x^2 - 6x + 8} = \frac{A}{(x-4)} + \frac{B}{(x-2)}.$$

Since the denominator in the first fraction is the common denominator of the other two, we get

$$3x = A(x-2) + B(x-4)$$

and, to obtain A and B we just notice that for $x = 2$ and $x = 4$, one of the terms vanishes. For $x = 4$ we obtain that $A = 6$, and for $x = 2$ we get $B = -3$.