SOLUTIONS EXERCISES (iii) - Maths for Biology

Computational Methods in Ecology and Evolution
Imperial College London
Silwood Park

Course 2017-18

Lecturer: Alberto Pascual-García.

Correspondence: alberto.pascual.garcia@gmail.com (informal questions);

Exercises

Immediate integrals

$$a) \int \sqrt[m]{x^n} dx$$

SOLUTION: We rewrite the integral first as:

$$\int \sqrt[m]{x^n} dx = \int x^{n/m} dx$$

And then we integrate

$$\int x^{n/m} dx = \frac{x^{(n/m)+1}}{n/m+1} + C$$

But, you should realize that, for $n/m = -1$, this solution diverges. Therefore, we should present another solution for that particular case:

$$\int \frac{1}{x} dx = \log |x| + C.$$

In summary, for integrals in which the parameters may take different values, we should be careful and consider the different possibilities.

$$b) \int \tan^2 x dx$$

SOLUTION: In general, when we want to solve an integral, the first thing we should do is to see if any simple transformation leads to an immediate solution. In this case, if we simply add and subtract 1 to the integral we obtain two immediate integrals:

$$\int \tan^2 x dx = \int (\tan^2 x + 1 - 1) dx = \int (\tan^2 x + 1) dx - \int dx = \tan x - x + C.$$

Integral methods (by substitution)

$$a)\ I = \int \frac{x^3}{2+x^8}dx$$

SOLUTION: The first thing we are aware of is that x^3 is the derivative of x^4 (except by a constant). Thus, doing the change $t = x^4$, we get that $dt = 4x^3dx$ and then $x^3dx = (1/4)dt$ and $t^2 = x^8$. If we substitute:

$$I = \frac{1}{4}\int \frac{dt}{2+t^2}.$$

At this point, we should be ready to look for an immediate integral and, indeed, the denominator sounds like familiar because it looks very similar to the derivative of the arctan, but there is a 2 in the denominator that prevent us to obtain this solution immediately. If we divide the denominator by two we will make it vanish, but we need then to divide as well the numerator by two, what yields:

$$I = \frac{1}{4}\int \frac{\frac{1}{2}}{1+\frac{t^2}{2}}dt = \frac{1}{8}\int \frac{1}{1+\left(\frac{t}{\sqrt{2}}\right)^2}$$

Again, we have almost the integral of arctan, but the problem now is that, if we say that $x = t/\sqrt{2}$, to be immediate there should be a factor $1/\sqrt{2}$ in the numerator. We can then just multiply and divide by $1/\sqrt{2}$, and we will have the integral we are looking for. Solve it and undo the change of variables:

$$I = \frac{\sqrt{2}}{8}\int \frac{1/\sqrt{2}}{1+\left(\frac{t}{\sqrt{2}}\right)^2} = \frac{\sqrt{2}}{8}\arctan\left(\frac{t}{\sqrt{2}}\right) + \mathrm{C} = \frac{\sqrt{2}}{8}\arctan\left(\frac{x^4}{\sqrt{2}}\right) + \mathrm{C}.$$

$$b)\ I = \int \frac{1}{x\log x}dx$$

SOLUTION: This integral is easy, because we know that $1/x$ is the derivative of $\log x$, so this integral can be seen as $\int \frac{f'(x)}{f(x)}dx$, which is ideal for a substitution method. We just make $u = \log x$, and we get $du = \frac{1}{x}dx$, being the integral immediate:

$$\int \frac{du}{u} = \log|u| + C = \log|\log(x)| + C.$$

Integral methods (by parts)

$$a)\ \int x^3 \log x dx$$

SOLUTION: We make $\log x = u$ and $x^3dx = dv$, what yields $du = \frac{1}{x}dx$ and $v = \frac{x^4}{4}$. Taking into account the formula to integrate by parts:

$$\int udv = uv - \int vdu$$

we substitute and solve:

$$\int x^3 \log x dx = \frac{x^4}{4}\log x - \int \frac{x^3}{4}dx = \frac{x^4}{4}\log x - \frac{x^4}{16} + C = \frac{x^4}{4}(4\log x - 1) + C.$$

$$b)\ \int e^x \cos(x) dx$$

SOLUTION: This is a (tricky) classical integral. In this case, we make $u = e^x$ and $\cos(x)dx = dv$, yielding $du = e^x dx$ and $v = \sin(x)$, and then

$$I = \int e^x \cos(x) dx = e^x \sin(x) - \int e^x \sin(x) dx = e^x \sin(x) - I_2.$$

But... we did solve anything? The second integral is very similar to the previous one! Maybe these are not bad news, What would happen if we perform the same kind of change to I_2? Let's make $u = e^x$ and $\sin(x)dx = dv$, from which we get $du = e^x dx$ and $v = -\cos(x)$, and then substitute:

$$I = e^x \sin(x) - \left(-e^x \cos x + \int e^x \cos(x) dx \right),$$

and... surprise! you can see that the second integral is precisely the one we are analysing! Therefore, this is equivalent to say:

$$I = e^x \sin(x) - (-e^x \cos x + I)$$

and we can solve the equation for I to obtain the solution:

$$I = \frac{e^x}{2}(\sin(x) + \cos(x)).$$

Integral methods (rational functions)

$$a)\ I = \int \frac{7x-6}{x^2+x-6} dx$$

SOLUTION: When we find rational functions we should simplify them first with the methods we learned. In this case, the degree of the denominator is larger than that of the numerator. Decomposing it in simple fractions yields

$$\frac{1}{x^2+x-6} = \frac{1}{(x-2)(x+3)} = \frac{A}{(x-2)} + \frac{B}{(x+3)} = \frac{A(x+3)+B(x-2)}{(x-2)(x+3)}$$

Equating the numerator to the one we have in the integral yields the system of equations

$$A + B = 7$$

$$3A - 2B = -6$$

whose solution is $A = 8/5$ and $B = -27/5$. We can rewrite the original integral in terms of immediate integrals now:

$$I = \int \frac{7x-6}{(x-2)(x+3)} dx = \frac{8}{5} \int \frac{1}{x-2} dx - \frac{27}{5} \int \frac{1}{x+3} dx$$

$$= \frac{8}{5} \log|x-2| - \frac{27}{5} \log|x+3| + C.$$

$$b)\ \int \frac{x^2+1}{x^4-x^2} dx$$

SOLUTION: As the degree of the polynomial in the denominator is larger than the one in the numerator, we perform the decomposition we learned:

$$\frac{x^2+1}{x^4-x^2} = \frac{x^2+1}{x^2(x+1)(x-1)} = \frac{A}{x^2} + \frac{B}{(x+1)} + \frac{C}{(x-1)}$$

And we obtain the coefficients:

$$A = \left(\frac{x^2+1}{(x+1)(x-1)} \right)_{x=0} = -1$$

$$B = \left(\frac{x^2+1}{x^2(x-1)} \right)_{x=-1} = 1$$

$$C = \left(\frac{x^2+1}{x^2(x+1)} \right)_{x=1} = -1$$

$$I = \int \frac{x^2+1}{x^4-x^2} dx = -\int \frac{dx}{x^2} + \int \frac{dx}{(x+1)} - \int \frac{dx}{(x-1)}$$

$$I = \frac{1}{x} + \log|x-1| - \log|x+1| + C.$$

Applications. **Molecular motor work** SOLUTION: To obtain the work we should integrate the force function over the coordinate in which the movement is operating $W = \int F(x)dx$. The integral is easy, we just need to be careful with the signs and also to be aware that the problem is asking for the net work of the process (and not for the work made by the tweezers on the molecule nor of the molecule on the tweezers). If we do not take into account the sign criteria, the integral we want to solve is:

$$W = \int_{-3}^{2} |(x-1)(x+2)| dx$$

To take into account the sign of the function we should note which are the roots, take some values to see which are the signs, and take different signs between the intervals determined by the roots. Then, we will obtain:

$$W = \int_{-3}^{-2} (x-1)(x+2)dx - \int_{-2}^{1} (x-1)(x+2)dx + \int_{1}^{2} (x-1)(x+2)dx =$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2} - 2x \right]_{-3}^{-2} - \left[\frac{x^3}{3} + \frac{x^2}{2} - 2x \right]_{-2}^{1} + \left[\frac{x^3}{3} + \frac{x^2}{2} - 2x \right]_{1}^{2} = \frac{49}{6} pN \times nm > 0.$$

Therefore, the tweezers exert more force than the one performed by the molecule.