SOLUTIONS EXERCISES (ii) - Maths for Biology

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Exercises Limits

Definition of limits 1. SOLUTION: Applying the definition of limit is difficult, but it is a good exercise to get used to mathematical thinking. Applying strictly the definition we would say that:

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; / \; 0 < |x-2| < \delta \Rightarrow |(x^2+x+2)-8| < \varepsilon$$

The strategy is typically to simplify a little bit the expression for the function approximating to the limit, to see if we can get any guess for the choice of δ, which is our objective. We can factorize the polynomial easily:

$$|(x^2+x+2)-8| = |x^2+x-6| = |(x-2)(x+3)| = |x-2||x+3|$$

Then, we have that $|x-2||x+3| < \varepsilon$. If you realize, there is another condition in the definition $0 < |x-2| < \delta$ (the variable approaches the limit we look for), containing the term $|x-2|$. Indeed, if we forget about $0 < |x-2|$, that is essentially telling us that $\delta > 0$, both conditions are very similar:

$$\begin{aligned} |x-2| &< \delta \\ |x-2||x+3| &< \varepsilon, \end{aligned}$$

being the main difference the term $|x+3|$. And, if we find a number c such that $|x+3| < c$ things look even better:

$$\begin{aligned} i)\; |x-2| &< \delta \\ ii)\; |x+3| &< c \\ iii)\; |x-2| &< \varepsilon/c. \end{aligned}$$

We have now one more condition but two of them are very similar, and we just need to focus on finding a value of c that will help us to link conditions i) and iii). Note that we are not interested in any x value but only in those close to 2. We can then go step by step and see what are telling us these conditions. Starting from condition i), imagine that we are at a distance $\delta = 1$ (I choose this value for simplicity only), and you will see that x should be constrained to the interval $(1, 3)$. I obtained this solving the following inequation

$$|x-2| < 1$$

Inequations with absolute values are solved transforming them into two inequations:

$$-1 < x-2 < 1$$

From the right side we obtain that $x < 3$ and, from the left side, that $x > 1$. In summary, from the first condition we know that, when we are approaching to the limit ($x \to 2$), the value of x must be somewhere in the neighborhood of 2, taking the interval the values 1 and 3 for the arbitrary value of δ=1. More generally, the interval is $(2-\delta, \delta+2)$, becoming smaller and smaller this interval around 2 for smaller δ.

Next, let's check which values would take the constant c in condition ii) if we explore x values within this interval. Taking for example $x = 3$ which is the maximum of the interval $(1,3)$, we obtain that $c > 6$. If we now substitute this value in condition iii) we get that $|x-2| < \epsilon/6$, and we end up with

$$\begin{aligned} |x-2| &< \delta \\ |x-2| &< \varepsilon/6. \end{aligned}$$

The definition of limit says that, "for all ε there exist a δ that" so, once we fix ϵ, if we solve for δ this equation $\delta \leq \varepsilon/6$ we know that, independently of the ε you choose, you get a positive δ fulfilling the definition. Independently of ϵ? Not really true, because to get this result we started with an arbitrary value of $\delta = 1$, so δ should fulfill both $\delta < 1$ and $\delta \leq \varepsilon/6$, that can be written as

$$\delta = \min(1, \epsilon/6)\,.$$

To convince yourself, you can select $\varepsilon = 1$ which leads to $\delta = 1/6$. From the first condition, you should be positioned in that distance at $x = 11/6$. If you substitute this value of x in the function and you take the difference to the limit, the value is around 0.81, which is smaller than ε.

2. SOLUTION: Remember that the ceiling function is $\lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\}$. Therefore, in the intervals of interest what happens is that $\lceil x \rceil = 3$ for $(2,3]$, and $\lceil x \rceil = 2$ for $(1,2]$. From this, we can deduce that

$$\lim_{x\to 2^-} \lceil x \rceil = 2 \text{ and } \lim_{x\to 2^+} \lceil x \rceil = 3.$$

Given that the lateral limits are different, the requested limit does not exist.

Indeterminate limits of type 1^∞. Applying the suggested clue to our function, we get

$$\lim_{V\to 0} f(V) = e^{\lim_{V\to 0}(1/\sin(V))(\cos(V)-1)}$$

Now, we can look in the survival manual for the infinitesimal table, and we see that, when $x \to 0$, $1-\cos x \approx \frac{x^2}{2}$ and $\sin x \approx x$. And then

$$\lim_{V\to 0} f(V) = e^{\lim_{V\to 0} \frac{-V^2}{2}\frac{1}{V}} = e^{\lim_{V\to 0} \frac{-V}{2}} = e^0 = 1$$

Exercises Derivatives

Mean value theorem. We consider that the distance made by the herd at a given time is given by a function $f(t)$, and we will make the reasonable assumption that this is a continuous and derivable function. If you realize, we are monitoring a groups of animals, and we expect them to follow more or less smooth trajectories (they should run approximately following a single direction if they run away from a threat!). The interval of time between the first measurement, t_0, and the second one, t_1, can be expressed in hours as $t \in (t_0, t_0 + 1/6)$. And it is within this interval where we expect them to reach the claimed speed v, which is at least at a given time t^* equal to the average speed. Therefore, the mean value theorem states that:

$$v^* = f'(t^*) = \frac{\Delta x}{\Delta t} = \frac{2.5km}{1/6h} = 15.0km/h$$

You may say, well, it is obvious that if their average speed was higher than the highest speed ever recorded, of course they were running faster. Ok, fair enough, but which *exact* value would you determine as the new record? Because moving on average at that speed means that they may be moving above and below that value, and you cannot be completely sure that they actually run exactly at any other value except to this one right? What I argue is that, presenting the result in this way (I demonstrate that, at a given instant of time, I can guarantee that they were running at this speed) instead of saying (on average they were running at this speed) makes a huge difference from an epistemological perspective.

Taylor expansion a) Obtain the order three Taylor's expansion of the following function:

$$f(x) = \log\sqrt{\frac{1+x}{1-x}}$$

at $x_0 = 0$.

SOLUTION: First of all, note that the function is continuous and derivable at $x_0 = 0$, we start rewriting it to simplify the operations:

$$f(x) = \frac{1}{2}\log\frac{1+x}{1-x} = \frac{1}{2}\left(\log(1+x) - \log(1-x)\right).$$

Now, we compute the derivatives:

$$\begin{aligned}
f'(x) &= \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) \\
f''(x) &= \frac{1}{2}\left(\frac{-1}{(1+x)^2} + \frac{1}{(1-x)^2}\right) \\
f'''(x) &= \frac{1}{2}\left(\frac{2}{(1+x)^3} + \frac{2}{(1-x)^3}\right)
\end{aligned}$$

which, evaluated at $x_0 = 0$ yield $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$ and $f'''(0) = 2$. Therefore, the polynomial required is

$$T_3(x) = x + \frac{2}{3!}x^3 = x + \frac{1}{3}x^3.$$

b) Obtain the order three expansion of the function

$$f(x) = \frac{\ln x}{x}$$

at $x_0 = 1$.

SOLUTION: We start computing derivatives:

$$\begin{aligned}
f'(x) &= \frac{1 - \log x}{x^2} \\
f''(x) &= \frac{2\log x - 3}{x^3} \\
f'''(x) &= \frac{11 - 6\mathrm{logx}}{x^4}
\end{aligned}$$

Now we evaluate the function and the derivatives at the point requested: $f(1) = 0$, $f'(1) = 1$, $f''(1) = -3$ and $f'''(1) = 11$. And finally we build the Taylor expansion:

$$T_2(1) = (x-1) - \frac{3}{2}(x-1)^2 + \frac{11}{6}(x-1)^3.$$

Simple derivatives

$$f(x) = \arctan\left(\frac{\sin x}{1+\cos x}\right)$$

SOLUTION: In some of the simplifications we make in the following, take into account that we are using trigonometric relationships that you will find in the survival toolbox.

$$f'(x) = \frac{1}{1 + \frac{\sin^2 x}{(1+\cos x)^2}} \frac{\cos x(1+\cos x) + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{\cos x(1+\cos x)+\sin^2 x}{(1+\cos x)^2+\sin^2 x} = \frac{\cos x+1}{2\cos x+2} = \frac{1}{2}.$$

$$g(x) = \log\left(\frac{2\tan x+1}{\tan x+2}\right)$$

SOLUTION: In this case, we start rewriting the function using logarithmic relationships, and then we use trigonometric ones to simplify the derivative.

$$g(x) = \log\left(2\tan x+1\right) - \log(\tan x+2).$$

$$g'(x) = (1+\tan^2 x)\left(\frac{2}{2\tan x+1} - \frac{1}{\tan x+2}\right) =$$

$$= (\cos x)\left(\frac{2}{2\tan x+1} - \frac{1}{\sin x+2\cos x}\right) =$$

$$= \frac{3}{2\sin^2 x+5\sin x\cos x+2\cos^2 x} = \frac{3}{5\sin x\cos x+2}.$$

Functions of the type $f(x)^{g(x)}$

$$g(x) = (a^2+x^2)^{\arctan(x/a)}.$$

SOLUTION: We follow the example and we apply logarithms:

$$\log(g(x)) = \arctan(x/a)\log(a^2+x^2)$$

and then

$$\frac{g'(x)}{g(x)} = \frac{1/a}{1+(x/a)^2}\log(a^2+x^2) + (\arctan(x/a))\,\frac{2x}{a^2+x^2}$$

if you rewrite the first quotient a little bit, you will see that there is a common factor with the second one which is actually the base of $g(x)$. This helps us to find out a beautiful expression solving for $g'(x)$ (yes, beauty is subjective, but compactness in maths is cool):

$$g'(x) = \left(a\log(a^2+x^2) + 2x\arctan(x/a)\right)(a^2+x^2)^{\left(\arctan\frac{x}{a}\right)-1}.$$

Functions behaviour.

$$f(x) = xe^x$$

SOLUTION: The first derivative of this function is: $f'(x) = e^x(x+1)$. Therefore, it is only equal to zero when $x=-1$ (because $e^x > 0\ \forall x \in \mathbb{R}$). If we explore the neighbourhood of this point we observe that:

- If $x < -1$ then $f'(x) < 0$, and then $f(x)$ decreases in the interval $(-\infty, 1)$.
- If $x > -1$ then $f'(x) > 0$, and then $f(x)$ grows in the interval $(1, \infty)$.
- $x = -1$ and given that it is a continuous function and that it changes its behaviour, there should be a local minimum.

An alternative analysis can be made computing the second derivative, $f''(x) = e^x(x+2)$ which, evaluated at $x = -1$ yields $f''(-1) = e^{-1} > 0$, what confirms that we are dealing with a minimum.

$$g(x) = \log \sqrt{2x^3 + 3x^2}$$

SOLUTION: First, we need to determine the domain of the function. We know that if the root is negative it would only be defined in the complex plane, hence we start considering the values in which the polynomial is positive: $\{x \in \mathbb{R} \ / \ 2x^3 + 3x^2 > 0\} = \{x \in \mathbb{R} \ / \ x > -3/2; \ x \neq 0\}$. Since we know that both $\log x$ and $\sqrt{x}$ grow for positive real values, the calculus of the extremes of $g(x)$ is equivalent to the calculus of the extremes of $h(x) = 2x^3 + 3x^2$.

The derivative of this function is $h'(x) = 6x^2 + 6x$, which vanishes for $x = 0$ and $x = -1$. We should discard the analysis at $x = 0$ because it does not belong to the domain of the function. To analyse what happens for $x = -1$ we compute the second derivative $h''(x) = 12x + 6$ which we evaluate at the critical point, finding $h''(-1) = -6 < 0$, and we conclude that $x = -1$ is a maximum of $h(x)$ and, in turn, of $g(x)$.