SOLUTIONS (iv) - Maths for Biology

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Solutions Ordinary Differential Equations

Separated variables

$$a)\ y' = \frac{x^2}{1-y^2}$$

SOLUTION: We can easily separate this equation as:

$$\int (1-y^2)dy = \int x^2 dx$$

which, integrating both members yields

$$3y - y^3 - x^3 = c.$$

Homogeneous Before we start solving this kind of equations, it worths to explain a little bit more why we call to these equations homogeneous and why our beloved change of variables $u = y/x$ works so well. Typically we say that a *function* is homogeneous if the following condition holds:

$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

where λ is a real number and n is an integer that will be called the *degree* of the homogeneous function. Basically, what we say is that if we multiply each variable by λ it will be possible to factorize out lambda, perhaps to the square, cube or any other degree. For instance, the following function is homogeneous of degree two:

$$f(x, y) = 4x^2 + y^2$$

because, if I make $x \to \lambda x$ and $y \to \lambda y$, I obtain:

$$f(\lambda x, \lambda y) = 4(\lambda x)^2 + (\lambda y)^2 = \lambda^2 \left(4x^2 + y^2\right) = \lambda^2 f(x, y).$$

The following function is homogeneous of degree one:

$$f(x, y) = x \cos\left(\frac{x}{y}\right),$$

again, multiplying by λ we get

$$f(\lambda x, \lambda y) = \lambda x \cos\left(\frac{\lambda x}{\lambda y}\right) = \lambda x \cos\left(\frac{x}{y}\right) = \lambda f(x, y).$$

If we deal with differential equations, given the general expression

$$M(x, y)dx + N(x, y)dy = 0,$$

we will say that it is homogeneous if both $M(x, y)$ and $N(x, y)$ are homogeneous of the same degree. This fact is what allow us to simplify the expressions when we make the change, and give us a way to test in advance if the equation is homogeneous. Now try to solve the exercises.

$$a)\ 2(x + 2y)dx + (y - x)dy = 0$$

SOLUTION: The functions $2(x+2y)$ and $(y-x)$ are both homogeneous of degree one. Then, we can simplify first a little bit the expression, because we can multiply both functions for λ and the equation remains the same. Then, what if we say that $\lambda = 1/x$? Multipliying both functions we will get:

$$2(1 + 2y/x)dx + (y/x - 1)dy = 0,$$

which leads to functions containing the familiar y/x expression. Now we can apply the change $y = ux$, which transform the equation into

$$2(1 + 2u)dx + (u - 1)dy = 0.$$

Note that it is not necessary to start simplifying the expression as we did, you can apply directly the usual change. We need still to substitute $\frac{dy}{dx} = \frac{du}{dx}x + u$ (we applied the derivative with respect to x of the product, i.e. $(ux)'$) and, separating variables you should get

$$\frac{-dx}{x} = \frac{(u-1)du}{u^2 + 3u + 2}.$$

If we now integrate, the left-hand side equation is immediate, so let's have a look to the right-hand side. This is a rational function, and the degree of the denominator is larger than the one in the numerator, so we can factorize it as:

$$\frac{(u-1)}{u^2 + 3u + 2} = \frac{A}{(u+1)} + \frac{B}{(u+2)}$$

and you will find that $A = -2$ and $B = 3$. Therefore, to integrate the differential equation we need to consider:

$$\int \frac{-dx}{x} = \int \frac{-2du}{(u+1)} + \int \frac{3du}{(u+2)} \tag{1}$$

and we finally get

$$-\ln|x| + c = -2\ln|u+1| + 3\ln|u+2| = \ln|(u+1)^{-2}(u+2)^3| \tag{2}$$

which, getting back the original variables and simplifying yields the general solution of the differential equation (note that since the constant is arbitrary, we can write $\ln|c|$ instead of c, and hence:

$$\left(\frac{y}{x} + 1\right)^{-2}\left(\frac{y}{x} + 2\right)^3 = cx^{-1}. \tag{3}$$

The problem also asks to evaluate the particular solution $y(1) = 0$, which means that $c = 8$, and thus we obtain

$$\left(\frac{y}{x} + 1\right)^{-2}\left(\frac{y}{x} + 2\right)^3 = 8x^{-1},$$

that allows us to get a nice simplification

$$(2x+y)^3 = 8(x+y)^2.$$

Finally, let's talk about a subtle question about integrals that lead to a logarithm function as primitive, when we have done a change of variables. We know that when we perform the integral $\int dx/x$ we obtain $\ln|x|$ and we take absolute values because the domain of $1/x$ do not contain 0. Above, in the step I labelled as Eq. 1 we reached the expressions shown in Eq. 2, involving the logarithm of a variable, that is wrapping other variables. In particular, taking the absolute value, we "lost" the solutions $u = -1$ and $u = -2$ which, if we consider the original variables, means that we lost $y = -x$ and $y = -2x$. If you substitute into the general solution, Eq. 3, you will see that it is a solution as soon as $c = 0$. This is not the case for the particular solution we were asked to find and this is most of the times the case and the reason why we are not very careful with that, but we should be aware of this fact.

$$b)\ y' = \frac{y^2 + 2xy}{x^2}$$

SOLUTION: We can rewrite the equation as

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + 2\frac{y}{x}$$

and, in this way, we clearly see that it is homogeneous. We apply the change of variables $v = \frac{y}{x}$:

$$x\frac{dv}{dx} + v = v^2 + 2v.$$

We can separate now variables:

$$\frac{dx}{x} = \left(\frac{dv}{v(v+1)}\right)$$

and, if you integrate now, you will face difficulties with the right side of the equation. But you know how to solve that already, just expanding into factors:

$$\frac{1}{v(v+1)} = \frac{A}{v} + \frac{B}{v+1}$$

that leads to $A = 1$ and $B = -1$. You will have the same problem with the logistic equation. We rewrite now the equation:

$$\frac{dx}{x} = \left(\frac{1}{v} - \frac{1}{v+1}\right) dv$$

and both sides are now easy to integrate, what yields

$$\ln|x| + \ln|c| = \ln|v| - \ln|v+1|.$$

Working with the properties of logarithms and taking exponentials in both sides we get

$$cx = \frac{v}{v+1}$$

and we just need to undo the change and rearrange a little bit to obtain

$$y = \frac{cx^2}{\text{1-cx}}.$$

Linear ODEs

$$a)\ y' + 2y = e^{-x}$$

SOLUTION: First of all, we look for the integrating factor:

$$\mu(x) = \exp \int 2dx = e^{2x}.$$

Multiplying in both members by the integrating factor we obtain:

$$y'e^{2x} + 2ye^{2x} = e^{x}$$

and the left side of the equation is clearly $(ye^{2x})'$ and, thus, we can write

$$(ye^{2x})' = e^{x} \Rightarrow (ye^{2x})'dx = e^{x}dx$$

and integrating we obtain

$$y = e^{-x} + ce^{-2x}.$$

$$b)\ y' = 5x^2y^5 + \frac{y}{2x}.$$

SOLUTION. As we anticipated, this is a Bernouilli equation, so first we divide by y^5

$$\frac{y'}{y^5} = 5x^2 + \frac{1}{2xy^4}.$$

Following the proposed method, we should consider that $u = y^{-4}$, and then $u' = (y^{-4})' = -4y^{-5}y'$. Applying these changes we get

$$u' + \frac{u}{x} = -20x^2.$$

Which is already a linear equation. The integating factor is

$$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = x^2$$

what yields

$$(ux^2)' = -20x^4$$

and we integrate obtaining

$$u = \frac{1}{x^2}\left[-24x^5 + c\right]$$

recovering the original variable y leads to the final expression

$$y = \sqrt[4]{\frac{x^2}{-4x^5 + c}}.$$

$$c)\ 2y' - y = 4\sin(3t)$$

SOLUTION: The equation is linear, to look for the integrating factor it is convenient to divide both sides by 2 to apply the formula we do know, in which the term y' is multiplied just by one:

$$y' - y/2 = 2\sin(3t)$$

then the integrating factor will be

$$\mu(t) = e^{-\int \frac{1}{2} dt} = e^{-\frac{t}{2}}$$

that multiplies both sides of the equation, which yields

$$e^{-\frac{t}{2}} y' - e^{-\frac{t}{2}} y/2 = e^{-\frac{t}{2}} 2\sin(3t).$$

As usual, we double check that the left hand side corresponds with the differential we are looking for

$$d(e^{-\frac{t}{2}} y) = e^{-\frac{t}{2}} y' - e^{-\frac{t}{2}} y/2,$$

thus

$$d(e^{-\frac{t}{2}} y) = e^{-\frac{t}{2}} 2\sin(3t)$$

that we can integrate. The right hand side should be integrated by parts twice, which yields

$$y(t) = -\frac{24}{37}\cos(3t) - \frac{4}{37}\sin(3t) + ce^{\frac{t}{2}}.$$

An interesting observation of this solution is that its behaviour when $t \to \infty$ depends on the sign of c. If $c = 0$ the solution is finite but, as soon as it is different from zero, it will tend to ∞ if $c > 0$ and to $-\infty$ if $c < 0$.

Exact ODEs

$$a)\ 2x + y^2 + 2xyy' = 0$$

SOLUTION: It is easy to see that, if we call $M(x, y) = 2x + y^2$ and $N(x, y) = 2xy$, clearly $\partial M/\partial y = M_y = 2y = N_x = \partial N/\partial x$, and we demonstrate that it is exact. Then, we know that the function ψ we should look for arises from the operation $\psi = \int M dx + h(y)$:

$$\psi = \int (2x + y^2) dx + h(y) = x^2 + xy^2 + h(y).$$

We now need to find the function $h(y)$, but we know that $\psi_y = N$, and then $2xy + h'(y) = 2xy$, equation from which we obtain that $h'(y) = 0$. Therefore, $h(y) = 0$, that we substitute in ψ and we present the final solution as $\psi = c$, namely

$$x^2 + xy^2 = c.$$

$$b)\ (3xy + y^2) + (x^2 + xy)y' = 0.$$

SOLUTION. It is straightforward to demonstrate that it is not exact, because $M_y = 2y + 3x \neq 2x + y = N_y$. If we still aim to find the function ψ we find that:

$$\psi = \frac{3}{2}x^2 y + xy^2 + h(y).$$

Next, we make $\psi_y = N$

$$\frac{3}{2}x^2 + 2xy + h'(y) = x^2 + xy$$

obtaining

$$h'(y) = -\frac{1}{2}x^2 - xy.$$

If we integrate this function we get

$$h(y) = -\frac{1}{2}(x^2 y + xy^2),$$

and the final function would be:

$$\psi = x^2 y + \frac{1}{2}xy^2 + c.$$

It is clear that taking the partial derivatives of ψ with respect to x or y will not lead to the functions M and N, which is what allows us to say that ψ is a solution in the exact case.

ODEs (guess the type) Consider the following first order differential equation (ODE):

$$a)\ y' = \frac{y^2 - x^2}{xy}.$$

1. Obtain the general solution $y = f(x, C)$, being C a constant.
2. Obtain the value of C for the particular solution $y(e^2) = \sqrt{4}e^2$, where e is the Euler number (i.e. the base of the natural logarithm).
3. Explain which is the domain of the particular solution $y = f(x)$ you obtained in the previous step.

1) SOLUTION: We have no clue in this exercise about which kind of equation it is, but we can show that it is homogeneous. Both numerator and denominator are order two, so we can start rewritting a little bit the equation dividing both numerator and denominator by x^2:

$$y' = \frac{\frac{y^2}{x^2} - 1}{\frac{y}{x}}.$$

From this expression, it is immediate to see the convenience of the canonical change of variables applied to homogeneous ODEs, which is $u = y/x$. With this change we observe that $y = ux$ and, therefore, we can obtain the substitution for y' applying the derivative of the product

$$y' = \frac{dy}{dx} = \frac{d(ux)}{dx} = \frac{du}{dx}x + u\frac{dx}{dx} = u'x + u.$$

We substitute the change in the ODE and it yields

$$u'x + u = \frac{u^2 - 1}{u} \Rightarrow u'ux + u^2 = u^2 - 1 \Rightarrow u'ux = -1.$$

This results in a separable ODE

$$u'u = \frac{du}{dx}u = \frac{-1}{x} \Rightarrow udu = \frac{-dx}{x}$$

Integrating both sides we obtain

$$\frac{u^2}{2} = \ln|x| + C$$

We now substitute back the original variables to obtain the general solution:

$$y^2 = 2x^2 \ln|x| + C.$$

2) SOLUTION: To obtain the particular solution, we substitute $x = e^2$ and $y = \sqrt{4}e^2$ in the general solution

$$(\sqrt{4}e^2)^2 = 2e^4 \ln|e^2| + C.$$

Using either the definition of logarithm or the properties of logarithms, we observe that $\ln|e^2| = \ln e^2 = 2\ln e = 2$, and then we easily find that $C = 0$.

3) SOLUTION: To obtain the domain of the function $y = f(x)$, we look for possible singularities. The particular solution we have found is:

$$y = \sqrt{2x^2 \ln|x|}$$

so we should be careful with the domains of the logarithm first, and then of the root square. Since in the argument of the logarithm there is the absolute value of x, there are no problems with this function except for $x = 0$, where it diverges to infinity. In addition, we observe that it is multiplied by x^2 and, thus, we will have no problems with negative values. Therefore, the domain of the function would be $\mathbb{R} - \{0\}$.