

# Exercise 1: Rotational Spectra

Sample Solution

Effective: 19.10.2016

1. Calculate the absorption cross sections in the microwave spectral range for the following molecules (Figure 1):

- $HCl$ ,  $ClO$ ,  $CO$ ,  $N_2O$ ,  $O_3$

Questions and answers:

- Estimate the rotational constant  $B$  for  $HCl$  and for  $CO$ .
  - $B_{HCl} \approx 300 \text{ GHz}$
  - $B_{CO} \approx 60 \text{ GHz}$
- Why is  $B$  larger for  $HCl$  than for  $CO$ ?
  - The reduced mass  $\mu$  is larger for  $HCl$ . This is caused by a larger molecule in general and a larger mass difference of the atoms inside the molecule.
- Do you have any idea why  $N_2O$  behaves like a diatomic molecule — and  $O_3$  not?
  - The angles between the atoms inside the molecule differ.  $N_2O$  has flat angles and therefore moments of inertia like a linear molecule ( $I_A = 0, I_B = I_C$ ).  $O_3$  has a more complex structure with differing moments of inertia ( $I_A \neq I_B \neq I_C$ ).
- Calculate the reduced mass of the different molecules from the masses of the individual atoms. (For the diatomic molecules this is trivial. For  $N_2O$ , I think the appropriate mass can be found by careful thinking. Ignore  $O_3$ .)
  - The reduced mass  $\mu$  for a diatomic molecule is given by  $\mu = \frac{m_1 m_2}{m_1 + m_2}$ . The atomic masses of the relevant atoms are  $H (1u)$ ,  $Cl (35u)$ ,  $C (12u)$ ,  $O (16u)$  and  $N (14u)$ . For the regarded molecules the reduced masses are:

$$\mu(HCl) = 0.972u$$

$$\mu(ClO) = 10.98u$$

$$\mu(CO) = 6.857u$$

$$\mu(N_2O) \approx 7.5u$$

- Calculate the bond length of the various molecules (except  $O_3$ ) from the reduced mass and the rotational constant. Verify your result with Google. Again  $N_2O$  needs some extra thinking.
  - For each molecule, the rotational constant  $B$  can be found using Figure 1. The rotational constant (for frequencies) is defined as

$$B = \frac{h}{8\pi^2 I} \quad \text{with} \quad I = \mu r_0^2$$

The equations can be combined and solved for  $r_0$  which is the bond length of the regarded molecule.

$$r_0(HCl) = 132 \text{ pm}$$

$$r_0(ClO) = 158 \text{ pm}$$

$$r_0(CO) = 111 \text{ pm}$$

$$r_0(N_2O) \approx 233 \text{ pm}$$

- Play with different temperatures. How does the rotational spectrum change? Can you explain the changes?
    - Different temperatures lead to different distributions of energy states. With higher temperatures the amount of molecules in high energy states increases and absorption lines at higher frequencies get stronger. When observing very cold temperatures all molecules are in the lowest energy state ( $i = 0$ ) and only one absorption line at the lowest frequency is left.
- Investigate other molecules!
  - Show for a diatomic molecule that the moment of inertia is given by  $I = \mu r_0^2$ .

$$I = \sum_i m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2$$

The center of gravity is defined as  $m_1 r_1 = m_2 r_2$ . Insert this and you get

$$\begin{aligned} I &= m_2 r_2 r_1 + m_1 r_1 r_2 \\ &= r_1 r_2 (m_1 + m_2) \end{aligned} \tag{1}$$

We can do more with the center of gravity equation:

$$\begin{aligned} m_1 r_1 &= m_2 r_2 = m_2 \overbrace{(r_0 - r_1)}^{\text{from def.}} = m_2 r_0 - m_2 r_1 \\ (m_1 + m_2) r_1 &= m_2 r_0 \\ r_1 &= \frac{m_2 r_0}{m_1 + m_2} \end{aligned} \tag{2}$$

$$r_2 = \frac{m_1 r_0}{m_1 + m_2} \tag{3}$$

Insert (2) and (3) into (1):

$$\begin{aligned} I &= \frac{m_2 r_0 \ m_1 r_0 \ (m_1 + m_2)}{(m_1 + m_2)(m_1 + m_2)} \\ &= \frac{m_1 m_2}{m_1 + m_2} r_0^2 = \mu r_0^2 \quad \square \end{aligned}$$

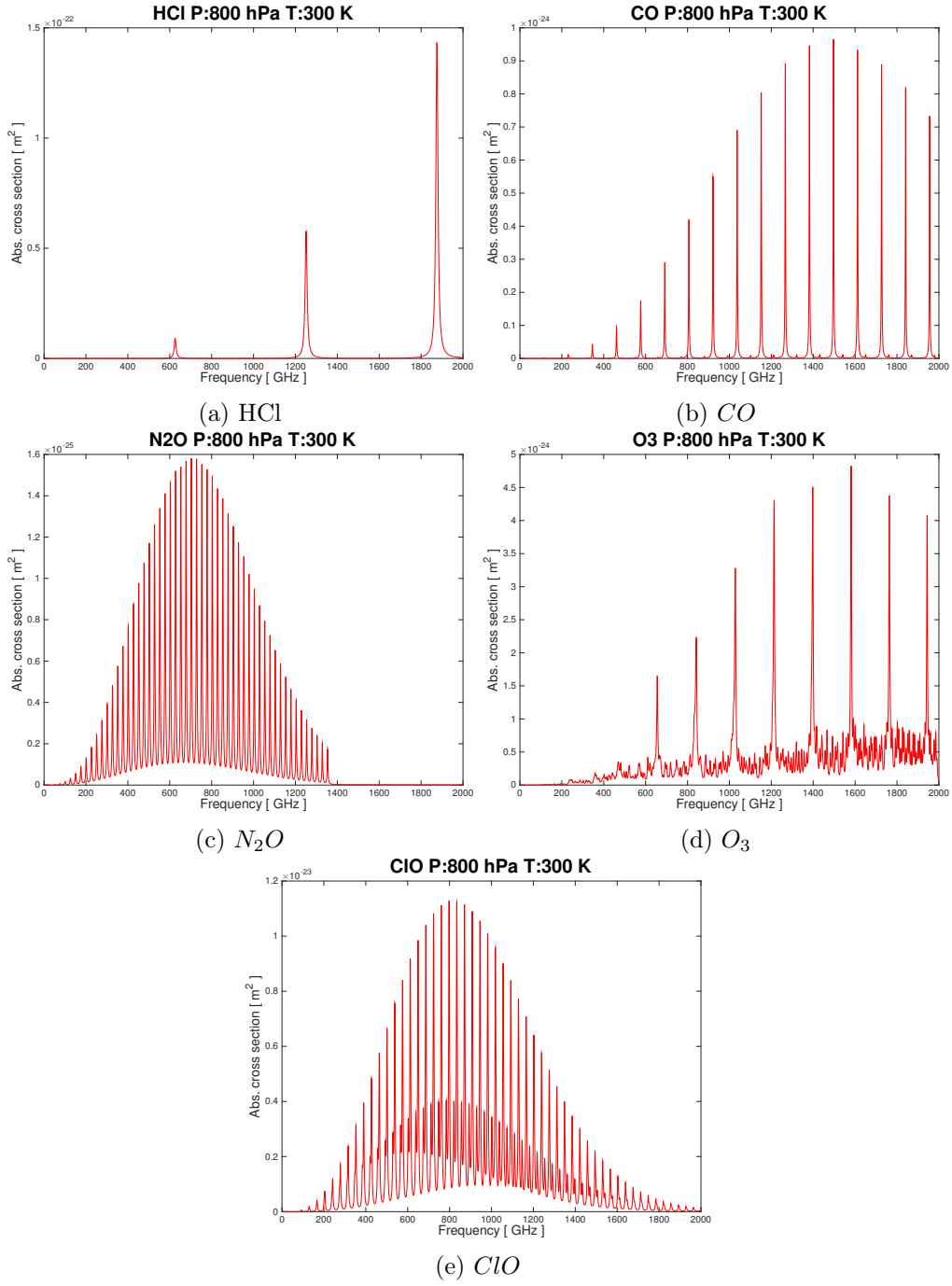


Figure 1: Absorption cross sections of the molecules  $HCl$ ,  $ClO$ ,  $CO$ ,  $N_2O$  and  $O_3$ .