ex ① :

1) $\overline{X_1} = \frac{4+6+8}{3} = \frac{18}{3} = 6$

$\overline{X_2} = \frac{5+7+0}{3} = \frac{12}{3} = 4$

donc $Y = \begin{pmatrix} 4-6 & 5-4 \\ 6-6 & 7-4 \\ 8-6 & 0-4 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & 3 \\ 2 & -4 \end{pmatrix}$

$\sigma_{X_1} = \sqrt{\frac{1}{3}\left((-2)^2+0^2+2^2\right)} = \sqrt{\frac{8}{3}} = \frac{\sqrt{8}}{\sqrt{3}} = \frac{2\sqrt{2}}{\sqrt{3}}$

$\sigma_{X_2} = \sqrt{\frac{1}{3}\left(1^2+3^2+(-4)^2\right)} = \frac{\sqrt{26}}{\sqrt{3}}$

donc $Z = \begin{pmatrix} \frac{-2}{\frac{2\sqrt{2}}{\sqrt{3}}} & \frac{1}{\frac{\sqrt{26}}{\sqrt{3}}} \\ \frac{0}{\frac{2\sqrt{2}}{\sqrt{3}}} & \frac{3}{\frac{\sqrt{26}}{\sqrt{3}}} \\ \frac{2}{\frac{2\sqrt{2}}{\sqrt{3}}} & \frac{-4}{\frac{\sqrt{26}}{\sqrt{3}}} \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{3}{2}} & \sqrt{\frac{3}{26}} \\ 0 & \frac{3\sqrt{3}}{\sqrt{26}} \\ \sqrt{\frac{3}{2}} & \frac{-4\sqrt{3}}{\sqrt{26}} \end{pmatrix}$

2) on a $R = \frac{1}{3} R^T R$

$R = \frac{1}{3}\begin{pmatrix} -\sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}} \\ \sqrt{\frac{3}{26}} & \frac{3\sqrt{3}}{\sqrt{26}} & \frac{-4\sqrt{3}}{\sqrt{26}} \end{pmatrix} \times \begin{pmatrix} -\sqrt{\frac{3}{2}} & \sqrt{\frac{3}{26}} \\ 0 & \frac{3\sqrt{3}}{\sqrt{26}} \\ \sqrt{\frac{3}{2}} & \frac{-4\sqrt{3}}{\sqrt{26}} \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} 3 & -2{,}08 \\ -2{,}08 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -0{,}69 \\ -0{,}69 & 1 \end{pmatrix}$

donc est une matrice carré symetrique.

3) Les valeurs propres :

$$\det(R - \lambda I) = \det R = \begin{pmatrix} 1-\lambda & -0,69 \\ -0,69 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)^2 - (-0,69)^2 = (1-\lambda+0,69)(1-\lambda-0,69)$$

$$= (1,69-\lambda)(0,31-\lambda).$$

on a $\boxed{1,69 = \lambda_1}$ ou $\boxed{0,31 = \lambda_2}$

La matrice diagonale.

donc $D = \begin{pmatrix} 1,69 & 0 \\ 0 & 0,31 \end{pmatrix}$

4) Les ~~valeurs~~ vecteurs propres :

on a $\begin{pmatrix} 1 & -0,69 \\ -0,69 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$

* pour $\lambda_1 = 1,69$.

$$\begin{pmatrix} 1 & -0,69 \\ -0,69 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} 1,69x \\ 1,69y \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} x - 0,69y = 1,69x \\ -0,69x + y = 1,69y \end{cases} \Leftrightarrow \begin{cases} x - 1,69x = 0,69y \\ y - 1,69y = 0,69x \end{cases}$$

$$\Leftrightarrow \begin{cases} -0{,}69x = 0{,}69y \\ -0{,}69y = 0{,}69x \end{cases}$$

donc $\boxed{-x = y}$

donc $U_1 = \begin{pmatrix} x \\ -x \end{pmatrix} = x\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow$ pour $\lambda_2 = 0{,}31$.

ona $\begin{pmatrix} 1 & -0{,}69 \\ -0{,}69 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_2\begin{pmatrix} 0{,}31x \\ 0{,}31y \end{pmatrix}$

$$\begin{cases} x - 0{,}69y = 0{,}31x \\ -0{,}69x + y = 0{,}31y \end{cases} \Leftrightarrow \begin{cases} x - 0{,}31x = 0{,}69y \\ y - 0{,}31y = 0{,}69x \end{cases}$$

$$\Leftrightarrow \begin{cases} 0{,}69x = 0{,}69y \\ 0{,}69y = 0{,}69x \end{cases}$$

d'où $\boxed{x = y}$

donc $U_2 = \begin{pmatrix} x \\ x \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$\text{dors } P = \begin{array}{cc} U_1 & U_2 \\ \left(\begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix}\right. & \left.\right) \end{array}$$

donc ellaest perpondeculaire.

5) on a $Tr(R) = \sum_{i=1} r_{ii} = r_{11} + r_{22} = 1 + 1$

donc $Tr(D) \underset{\Leftrightarrow}{=} 2.$

$\boxed{\underline{\text{La qualité d'analyse}} : \dfrac{\lambda_1}{\lambda_1 + \lambda_2}}$ (la somme des vateurs propres)

ex ② :

1) $\overline{X_1} = \dfrac{2+4+6}{3} = \dfrac{12}{3} = 4$

$\overline{X_2} = \dfrac{3+5+1}{3} = \dfrac{9}{3} = 3$

on a $X = \begin{pmatrix} 2-4 & 3-3 \\ 4-4 & 5-3 \\ 6-4 & 1-3 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \\ 2 & -2 \end{pmatrix}$

$\sigma_{X_1} = \sqrt{\frac{1}{3}\left((-2)^2 + 0^2 + (2^2)\right)} = \dfrac{\sqrt{8}}{\sqrt{3}} = \dfrac{2\sqrt{2}}{\sqrt{3}}$

$\sigma_{X_2} = \sqrt{\frac{1}{3}\left(0^2 + 2^2 + (-2)^2\right)} = \dfrac{\sqrt{8}}{\sqrt{3}} = \dfrac{2\sqrt{2}}{\sqrt{3}}$

donc $Z = \begin{pmatrix} \frac{-\cancel{2}}{\frac{\cancel{2}\sqrt{2}}{\sqrt{3}}} & \frac{0}{\frac{2\sqrt{2}}{\sqrt{3}}} \\ \frac{0}{\frac{2\sqrt{2}}{\sqrt{3}}} & \frac{\cancel{2}}{\frac{\cancel{2}\sqrt{2}}{\sqrt{3}}} \\ \frac{\cancel{2}}{\frac{\cancel{2}\sqrt{2}}{\sqrt{3}}} & \frac{-\cancel{2}}{\frac{\cancel{2}\sqrt{2}}{\sqrt{3}}} \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{3}{2}} & 0 \\ 0 & \sqrt{\frac{3}{2}} \\ \sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} \end{pmatrix}$

2)

$$R = \frac{1}{3}\begin{pmatrix} -\sqrt{\frac{3}{2}} & 0 \\ 0 & \sqrt{\frac{3}{2}} \\ \sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} \end{pmatrix} \times \begin{pmatrix} -\sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}} \\ 0 & \sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 3 & -\frac{3}{2} \\ -\frac{3}{2} & 3 \end{pmatrix} = \begin{pmatrix} 1 & -0{,}22 \\ -0{,}22 & 1 \end{pmatrix}$$

donc est une matrice carré symetrique.

3) $\det(R - 2I) = \det R = \begin{pmatrix} 1-\lambda & -0{,}22 \\ -0{,}22 & 1-\lambda \end{pmatrix}$

$= (1-\lambda)^2 - (-0{,}22)^2 = (1-\lambda-0{,}22)(1-\lambda+0{,}22)$

$= (0{,}78-\lambda)(1{,}22-\lambda).$

donc $\boxed{\lambda_2 = 0{,}78}$ ou $\boxed{\lambda_1 = 1{,}22.}$

donc $D = \begin{pmatrix} 1{,}22 & 0 \\ 0 & 0{,}78 \end{pmatrix}$.

$\Rightarrow$ Les vecteurs propres :

$$\begin{pmatrix} 1 & -0{,}22 \\ -0{,}22 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow$ pour $\lambda_2 = 0,78$.

$$\begin{pmatrix} 1 & -0,22 \\ -0,22 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} 0,78x \\ 0,78y \end{pmatrix}$$

$$\Rightarrow \begin{cases} x - 0,22y = 0,78x \\ -0,22x + y = 0,78y \end{cases} \Leftrightarrow \begin{cases} x - 0,78x = 0,22y \\ y - 0,78y = 0,22x \end{cases}$$

$$\Leftrightarrow \begin{cases} 0,22x = 0,22y \\ 0,22y = 0,22x \end{cases} \text{ d'où } \boxed{x = y}$$

$$\text{d'où } v_2 = \begin{pmatrix} x \\ x \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$\Rightarrow$ pour $\lambda_1 = 1,22$

$$\begin{pmatrix} 1 & -0,22 \\ 0,22 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda_2 \begin{pmatrix} 1,22x \\ 1,22y \end{pmatrix}$$

$$\Rightarrow \begin{cases} x - 0,22y = 1,22x \\ -0,22x + y = 1,22y \end{cases} \Leftrightarrow \begin{cases} x - 1,22x = 0,22y \\ y - 1,22y = 0,22x \end{cases}$$

$$\Leftrightarrow \begin{cases} -0,22x = 0,22y \\ -0,22y = 0,22x \end{cases} \text{ d'où } \boxed{-x = y}$$

$$\text{donc } v_1 = \begin{pmatrix} x \\ -x \end{pmatrix} = x \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\text{d'où } P = \begin{array}{cc} v_1 & v_2 \\ \multicolumn{2}{c}{\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}} \end{array}$$

4) la qualité d'analyse :

on a $\lambda_1 = 1,22$ $\lambda_2 = 0,78$

$$100 \times \frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{1,22}{2} = 61\%$$

$$100 \times \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{0,78}{2} = 39\%$$

Exercice ③ : serie ①

1)

$$X = \left(\begin{array}{cc|c} 10 & 10 & 20 \\ 5 & 15 & 20 \\ 15 & 5 & 20 \\ \hline 30 & 30 & 60 \end{array}\right)$$

- le tableau des fréquences relatives et marginal :

$$\boxed{f_{ij} = \frac{n_{ij}}{n}}$$

$$F = \left(\begin{array}{cc|c} \frac{10}{60} & \frac{10}{60} & \frac{20}{60} \\ \frac{5}{60} & \frac{15}{60} & \frac{20}{60} \\ \frac{15}{60} & \frac{5}{60} & \frac{20}{60} \\ \hline \frac{30}{60} & \frac{30}{60} & 1 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} 1/6 & 1/6 & 1/3 \\ \frac{1}{12} & 1/4 & 1/3 \\ 1/4 & 1/12 & 1/3 \\ \hline \frac{1}{2} & 1/2 & 1 \end{array}\right)$$

2) ⓐ

On a $\lambda_{11} = \dfrac{F_{11}}{F_{1.}\sqrt{F_{.1}}} = \dfrac{1/6}{1/3\sqrt{1/2}} = \dfrac{\sqrt{2}}{2}$

$\lambda_{12} = \dfrac{F_{12}}{F_{1.}\sqrt{F_{.2}}} = \dfrac{1/6}{1/3\sqrt{1/2}} = \dfrac{\sqrt{2}}{2}$

donc $\boxed{P_1 = \left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)}$.

on a $\lambda_{21} = \frac{f_{21}}{F_{2.}\sqrt{f_{.1}}} = \frac{1/12}{1/3\sqrt{\frac{1}{2}}} = \frac{\sqrt{2}}{4}$

$\lambda_{22} = \frac{F_{22}}{f_{2.}\sqrt{f_{.2}}} = \frac{1/4}{1/3\sqrt{\frac{1}{2}}} = \frac{3\sqrt{2}}{4}$

$$\boxed{P_2 = \left(\frac{\sqrt{2}}{4}, \frac{3\sqrt{2}}{4}\right)}.$$

on a $\lambda_{31} = \frac{F_{31}}{F_{3.}\sqrt{f_{.1}}} = \frac{1/4}{\frac{1}{3}\sqrt{\frac{1}{2}}} = \frac{3\sqrt{2}}{4}$

$\lambda_{32} = \frac{f_{32}}{F_{3.}\sqrt{f_{.2}}} = \frac{1/12}{\frac{1}{3}\sqrt{\frac{1}{2}}} = \frac{\sqrt{2}}{4}$

$$\boxed{P_3 = \left(\frac{3\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right)}$$

b) la distance X^2 entre point P_i et $P_{i'}$

$$\boxed{d^2(P_i, P_{i'}) = \sum_{i=1}^{m2} \left(\lambda_{ij} - \lambda_{i'j}\right)^2}$$

$d^2(P_2, P_3) = \left(\frac{\sqrt{2}}{4} - \frac{3\sqrt{2}}{4}\right)^2 + \left(\frac{3\sqrt{2}}{4} - \frac{\sqrt{2}}{4}\right)^2 = 1$

$d^2_X(P_3, P_1) = \left(\frac{3\sqrt{2}}{4} - \frac{\sqrt{2}}{4}\right)^2 + \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = \frac{1}{4}$

$d^2_X(P_1, P_2) = \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{4} - \frac{3\sqrt{2}}{4}\right)^2 = \frac{1}{4}$

3) a)

$$\sigma_{12} = \sum_{i=1}^{3} f_{i.} \left(\lambda_{i1} - \sqrt{f_{.1}}\right)\left(\lambda_{i2} - \sqrt{f_{.2}}\right)$$
$$= f_{1.}\left(\lambda_{11} - \sqrt{f_{.1}}\right)\left(\lambda_{12} - \sqrt{f_{.2}}\right) + f_{2.}\left(\lambda_{21} - \sqrt{f_{.1}}\right)\left(\lambda_{22} - \sqrt{f_{.2}}\right)$$
$$+ f_{3.}\left(\lambda_{31} - \sqrt{f_{.1}}\right)\left(\lambda_{32} - \sqrt{f_{.2}}\right)$$

σ_1^2 , σ_2^2 , σ_{12} , σ_{21}

$$V = \begin{pmatrix} \sigma_1^2 , & \sigma_{12} \\ \sigma_2 , & \sigma_2^2 \end{pmatrix}$$

$$\sigma_{12} = \frac{1}{3}\left(\frac{\sqrt{2}}{2} - \sqrt{\frac{1}{2}}\right)\left(\frac{\sqrt{2}}{2} - \sqrt{\frac{1}{2}}\right) + \frac{1}{3}\left(\frac{\sqrt{2}}{4} - \sqrt{\frac{1}{2}}\right)\left(\frac{3\sqrt{2}}{4} - \sqrt{\frac{1}{2}}\right)$$
$$+ \frac{1}{3}\left(\frac{2\sqrt{3}}{4} - \sqrt{\frac{1}{2}}\right)\left(\frac{\sqrt{2}}{4} - \sqrt{\frac{1}{2}}\right) = 0 - \frac{1}{24} - \frac{1}{24} = \frac{-1}{12}$$

$$V = \begin{pmatrix} \sigma_1^2 & \frac{-1}{12} \\ \frac{-1}{12} & \sigma_2^2 \end{pmatrix}$$

$$\sigma_1^2 = \frac{1}{3}\left(\frac{\sqrt{2}}{2} - \sqrt{\frac{1}{2}}\right)^2 + \frac{1}{3}\left(\frac{\sqrt{2}}{4} - \sqrt{\frac{1}{2}}\right)^2 + \frac{1}{3}\left(\frac{3\sqrt{2}}{4} - \sqrt{\frac{1}{2}}\right)^2 = \frac{1}{12}$$

$$\sigma_2^2 = \frac{1}{12}$$

$$\sigma_{2}^{2} = \frac{-1}{12}$$

donc $V \begin{pmatrix} \frac{1}{12} & \frac{-1}{12} \\ \frac{-1}{12} & \frac{1}{12} \end{pmatrix}$.

3) b) Les voteurs propres :

$$\det(V-\lambda I) = \det\begin{pmatrix} 1/12-\lambda & -1/12 \\ -1/12 & 1/12-\lambda \end{pmatrix}$$

$$= \left(\frac{1}{12}-\lambda\right)^2 - \left(\frac{-1}{12}\right)^2$$

$$= \left(\frac{1}{12}-\lambda-\frac{1}{12}\right)\left(\frac{1}{12}-\lambda+\frac{1}{12}\right)$$

$$= -\lambda\left(\frac{2}{12}-\lambda\right)$$

$$\lambda = 0 \quad \text{ou} \quad \lambda = \frac{1}{6}.$$

$$\rightarrow V_B(I) = tr(V) = \frac{1}{6}+0 = \frac{1}{6}.$$

3) c) Variabilité totale de nuage $B(I)$:

$$\boxed{\frac{\lambda\ max}{\sum \lambda} = \frac{1}{6}.}$$