Diagonaliser une matrice A : chercher P et D diagonale / $A = PDP^{-1}$.

$$A^p = P D^p P^{-1} \qquad D^n = \begin{pmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_r^n \end{pmatrix}$$

vp
$\vec{v}$?

- λ valeur propre de A
- X vecteur propre de A associé à λ $(X \neq 0)$

$\Rightarrow AX = \lambda X \iff AX - \lambda X = 0 \iff (A - \lambda I)X = 0$
$\iff X \in \ker(A - \lambda I)$

- sous espace propre de λ : $E_\lambda = \{X \mid AX = \lambda X\} = \ker(A - \lambda I)$

Proposition : Si A est diagonalisable, alors
- les valeurs propres constituent les coefficients de D.
- les vecteurs propres " les colonnes de P disposés dans le même ordre que les valeurs propres

racines $\iff P(X) = 0$ // $\underbrace{X^2 - 1}_{=0} = (X-1)(X+1)$ racines

$X^2 + 1 = (X - i)(X + i)$

$= (X^2+1)^3 (X^2 + X + 1)^2$
$= (X-i)^3 (X+i)^3 (X-j)^2 (X+\bar{j})^2$

Rappelle $P(X) = a_n X^n + \dots + a_1 X + a_0$

$\mathbb{R}$: $\prod (X^2 + aX + b)^m (X - \alpha)^p$, $\Delta < 0$

$\mathbb{C}$: $\prod (X - \alpha_i)^{m_i}$

P scindé si $P(X) = \lambda \prod (X - \alpha_i)^{m_i}$
si $\deg P = n$ $\quad \sum_{i=1}^n m_i = n \quad 1 \leq m_i \leq n$

Proposition $A \in M_n(\mathbb{K})$. On suppose que A possède $\lambda_1, \dots, \lambda_k$ comme ~~racines de A~~ vp

1) Si on a

$$\sum_{i=1}^{k} \dim E_{\lambda_i} = n.$$

Alors A est diagonalisable.

$\deg \chi_A = n$

$P(X) = (X-1)^2(X+1)$

$1 \quad -1 \quad / \quad 2$

$\deg 3$

$= (X-1)(X+1)(X-1)$

$\left(1 \leq \dim E_{\lambda_i} \leq m_i\right)$

2) Si on a

$\dim E_{\lambda_i} = m_i \quad \forall i$ où m_i est l'ordre de multiplicité de λ_i.

3) Si A possède n vp 2 à 2 distinctes alors A est diagonalisable

$= (X - \alpha_1) \cdots (X - \alpha_n)$

4) Si A est triangulaire, alors A est diagonalisable.

Exemple 1) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$.

1) $\chi_A(X) = \det(A - XI) = \begin{vmatrix} 1-X & 0 & 0 \\ 0 & 1-X & 0 \\ 1 & -1 & 2-X \end{vmatrix} = (1-X)(1-X)(2-X)$

$= -(X-2)(X-1)^2$

Donc 2 est une valeur propre simple
et 1 " " " " double

$$AX = \lambda X$$

• **Pour $\lambda = 2$**

$$AX = 2X \iff (A - 2I)X = 0$$

$\mathcal{E}_\lambda$

$$\begin{cases} -1\,x_1 = 0 \\ -1\,x_2 = 0 \\ x_1 - x_2 = 0 \end{cases} \implies \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$E_2 = \operatorname{Ker}(A - 2I) = \operatorname{Vect}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Donc $\dim E_2 = 1$

Pour $\lambda = 1$ $\quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $\qquad E_\lambda = \{ X \mid AX = \lambda X \}$

$$AX = X \iff (A - I)X = 0$$

$$\iff \begin{cases} 0 = 0 \\ 0 = 0 \\ x_1 - x_2 + x_3 = 0 \end{cases} \iff x_1 - x_2 + x_3 = 0$$

$$\iff x_3 = -x_1 + x_2$$

Mais

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ -x_1 + x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ -x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ x_2 \end{pmatrix}$$

$$= x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$M_{3,1}(\mathbb{R})$.

Donc $E_1 = \text{Vect} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

Soient α, β $\mid$ $\alpha \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$E = \{ (x, y, z) \in \mathbb{R}^3 / x - y + z = 0 \}$

$$\Rightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \\ -\alpha + \beta = 0 \end{cases} \Rightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \end{cases}$$

Donc la famille $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ est libre et génératrice donc base. Par conséquent $\dim E_1 = 2$

Puisque $\dim E_1 + \dim E_2 = 3$ alors

A est diagonalisable. Donc $A = PDP^{-1}$

On a $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$; $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \qquad P^{-1} = \frac{1}{\det P} \operatorname{com}(P)^T$$

$$\det P = 1$$

$$\operatorname{com} P = \begin{pmatrix} +1 & 0 & +1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Donc $P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

On a $A^n = P D^n P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$

$$A^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 2^n \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1+2^n & 1-2^n & 2^n \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 3 & -2 & 0 \\ -2 & 2 & 1 \end{pmatrix} \quad //$$

$$2 - (-2 \quad)$$

$$4 + \lambda$$

$$\chi_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 2 & -1 \\ 3 & -2-\lambda & 0 \\ -2 & 2 & 1-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda & 2 & -1 \\ 3 & -2-\lambda & 0 \\ \lambda-2 & 0 & 2-\lambda \end{vmatrix} = (\lambda - 2) \begin{vmatrix} -\lambda & 2 & -1 \\ 3 & -2-\lambda & 0 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= (\lambda - 2) \begin{vmatrix} 1-\lambda & 2 \\ 1-\lambda & -2-\lambda \end{vmatrix} = + (\lambda - 2)(\lambda - 1) \begin{vmatrix} 1 & 2 \\ 1 & -2-\lambda \end{vmatrix}$$

$$= (\lambda - 2)(\lambda - 1) \begin{vmatrix} 0 & 4+\lambda \\ 1 & -2-\lambda \end{vmatrix} = -(\lambda - 2)(\lambda - 1)(\lambda + 4)$$

$$= (\lambda - 2) \begin{vmatrix} -\lambda-1 & 2 & -1 \\ 3 & -2-\lambda & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

$$= -(\lambda - 2) \begin{vmatrix} -\lambda-1 & 2 \\ 3 & -2-\lambda \end{vmatrix} =$$

$$\chi_A(\lambda) = -(\lambda-1)(\lambda-2)(\lambda+4)$$

Les valeurs propres de A sont 1, 2 et -4. Elles sont toutes simples. Donc A est diagonalisable

Pour $\lambda = 1$: Soit $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in E_1$:

$$AX = X \iff (A - I)X = 0$$

$$\iff \begin{cases} -x_1 + 2x_2 - x_3 = 0 \\ 3x_1 - 3x_2 = 0 \\ -2x_1 + 2x_2 = 0 \end{cases} \iff \begin{cases} -x_1 + 2x_2 - x_3 = 0 \\ x_1 - x_2 = 0 \\ -x_1 + x_2 = 0 \end{cases}$$

$$\iff \begin{cases} -x_1 + 2x_1 - x_3 = 0 \\ x_2 = x_1 \end{cases} \iff \begin{cases} x_3 = x_1 \\ x_2 = x_1 \end{cases}$$

Donc $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \\ x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Donc $E_1 = \mathrm{Vect}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ et $\dim E_1 = 1$

$-2 + \frac{3}{2} = \frac{-4+3}{2}$

<u>Pour $\lambda = 2$</u> : Soit $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in E_2$.

$$Ax = 2x \iff \begin{cases} -2x_1 + 2x_2 - x_3 = 0 \\ 3x_1 - 4x_2 = 0 \\ -2x_1 + 2x_2 - x_3 = 0 \end{cases} \iff \begin{cases} -2x_1 + \frac{3}{2}x_1 - x_3 = 0 \\ x_2 = \frac{3}{4}x_1 \end{cases}$$

$$\iff \begin{cases} x_3 = -\frac{1}{2}x_1 \\ x_2 = \frac{3}{4}x_1 \end{cases}$$

Donc $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ \frac{3}{4}x_1 \\ -\frac{1}{2}x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ \frac{3}{4} \\ -\frac{1}{2} \end{pmatrix}$

Donc $E_2 = \text{Vect}\left\{\begin{pmatrix}1\\3/4\\-1/2\end{pmatrix}\right\} = \text{Vect}\left\{\begin{pmatrix}4\\3\\-2\end{pmatrix}\right\}$

Pour $\lambda = -4$: Soit $X = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \in E_{-4}$

$$AX = -4X \iff (A + 4I)X = 0 \iff \begin{cases} 4x_1 + 2x_2 - x_3 = 0 \\ 3x_1 + 2x_2 = 0 \\ -2x_1 + 2x_2 + 5x_3 = 0 \end{cases}$$

$$\iff \begin{cases} 4x_1 - 3x_1 - x_3 = 0 \\ -2x_1 - 3x_1 + 5x_3 = 0 \\ x_2 = -\frac{3}{2}x_1 \end{cases} \iff \begin{cases} x_3 = x_1 \\ x_3 = x_1 \\ x_2 = -3/2\, x_1 \end{cases}$$

$$\iff \begin{cases} x_3 = x_1 \\ x_2 = -3/2\, x_1 \end{cases}$$; Donc $X = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1\\-3/2\,x_1\\x_1\end{pmatrix} = x_1\begin{pmatrix}1\\-3/2\\1\end{pmatrix}$

Donc $E_{-4} = \text{Vect}\left\{\begin{pmatrix}1\\-3/2\\1\end{pmatrix}\right\} = \text{Vect}\left\{\begin{pmatrix}2\\-3\\2\end{pmatrix}\right\}$

Donc $\dim E_{-4} = 1$

Done $\quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 4 & 2 \\ 1 & 3 & -3 \\ 1 & -2 & 2 \end{pmatrix}$

$A = PDP^{-1}$

$$\begin{vmatrix} 1 & 4 & 2 \\ 1 & 3 & -3 \\ 1 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 2 \\ 0 & -1 & -5 \\ 0 & -6 & 0 \end{vmatrix} = \begin{vmatrix} -1 & -5 \\ -6 & 0 \end{vmatrix} = -30$$

$$P^{-1} = \frac{1}{\det P} (\mathrm{Com}\, P)^T$$

$$\mathrm{Com}(P) = \begin{pmatrix} \begin{vmatrix} 3 & -3 \\ -2 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & -3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 4 & 2 \\ -2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 1 & -2 \end{vmatrix} \\ \begin{vmatrix} 4 & 2 \\ 3 & -3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 0 & -5 & -5 \\ -12 & 0 & 6 \\ -18 & 5 & -1 \end{pmatrix}$$

Done $\quad P^{-1} = \frac{-1}{30} \begin{pmatrix} 0 & -12 & -18 \\ -5 & 0 & 5 \\ -5 & 6 & 1 \end{pmatrix}$

$$A^n = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & (-4)^n \end{pmatrix} P^{-1}$$

<u>Exemple d'application</u>

Soit $A \in M_n(\mathbb{R})$. 1/ <u>Exponentielle d'une matrice</u>

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

On <u>définit</u> e^A par :

$$e^A = I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \dots + \frac{1}{n!} A^n + \dots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

1) Si A est diagonale : $A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

Done

$$e^{A} = I + \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} \lambda_1^3 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^3 \end{pmatrix}$$

$$+ \dots + \frac{1}{k!}\begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix} + \dots$$

$$= \begin{pmatrix} 1+\lambda_1+\frac{\lambda_1^2}{2!}+\dots+\frac{\lambda_1^k}{k!}+\dots & 0 & 0 \\ 0 & 1+\lambda_2+\dots+\frac{\lambda_2^k}{k!}+\dots & 0 \\ & & \ddots \\ & & 1+\lambda_n+\frac{\lambda_n^2}{2!}+\dots+\frac{\lambda_n^k}{k!}+\dots \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_1} & & & 0 \\ & e^{\lambda_2} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n} \end{pmatrix}$$

Si A est diagonalisable, alors $\exists P$ inversible et D diagonale telle que $\underline{A = PDP^{-1}}$

$$e^{PDP^{-1}} = e^{A} = I + A + \frac{1}{2!}A^2 + \dots + \frac{1}{k!}A^k + \dots$$

$$= \underset{PP^{-1}}{I} + PDP^{-1} + \frac{1}{2!}PD^2P^{-1} + \dots + \frac{1}{k!}PD^kP^{-1} + \dots$$

$$= P\left(I + D + \frac{1}{2!}D^2 + \dots + \frac{1}{k!}D^k + \dots\right)P^{-1}$$

$$= Pe^{D}P^{-1} = P\begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}P^{-1}.$$

1| ~~Suyfie~~ <u>Suite linéaire</u>

$$(S)\begin{cases} U_{n+1} = 2v_n - w_n \\ V_{n+1} = 3U_n - 2W_n \\ W_{n+1} = -2u_n + 2V_n + w_n \end{cases}$$

On note $X_n = \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix}$

$$(S) \Leftrightarrow \begin{pmatrix} U_{n+1} \\ V_{n+1} \\ W_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 3 & -2 & 0 \\ -2 & 2 & 1 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix}$$

$$\Leftrightarrow X_{n+1} = A X_n$$

$$\Leftrightarrow X_n = A^n X_0$$

$X_{n+1} = AX_n$

$X_n = AX_{n-1}$

$X_1 = AX_0$

$X_2 = AX_1$

$= AAX_0$

$= A^2 X_0$

$X_3 = AX^2$

$= AA^2X_n = A^3 X_0$

$$\boxed{X_n = A^n X_0}$$

$A = \begin{pmatrix} 0 & 2 & -1 \\ 3 & -2 & 0 \\ -2 & 2 & 1 \end{pmatrix}$

$U_{n+1} = \alpha U_n$

$= \alpha(\alpha U_{n-1})$

$= \alpha^2 U_{n-1}$

$= \alpha^2(\alpha U_{n-2})$

$= \alpha^3 U_{n-2}$

$$\boxed{U_{n+1} = \alpha^n U_0}$$

~~si tc q geo tm~~

a^n

$$\boxed{L\, u_n = 0 \text{ si } |q| < 1}$$

Calculs de A^n → diagonalisation

$\int \frac{f'}{f} = \ln f$

2) Systèmes différentielles

$$\begin{cases} x'(t) = y(t) - z(t) \\ y'(t) = 3x(t) - 2y(t) \\ z'(t) = -2x(t) + 2y(t) + z(t) \end{cases}$$

$$\begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 3 & -2 & 0 \\ -2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

$x(0) = x_0,\ y(0) = y_0,\ z(0) = z_0$

Soit $U(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$. Donc

$$\boxed{\begin{cases} U'(t) = A\,U(t) \\ U(0) = U_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \end{cases}}$$

$$\boxed{\begin{cases} x'(t) = \alpha x(t) \\ x(0) = x_0 \end{cases} \iff \boxed{x(t) = x_0 e^{\alpha t}}}$$

$\int \frac{x'(t)}{x(t)} dt = \int \alpha\, dt$

$\ln x(t) = \alpha t + c$

$x(t) = e^c e^{\alpha t}$

$x(t) = M e^{\alpha t}$

$x(0) = x_0 = M$

La solution $\boxed{u(t) = e^{tA} \cdot u_0}$?

$$\boxed{A = P D P^{-1}}$$

$$e^{A} = P e^{D} P^{-1} = P \begin{pmatrix} e^{d_1} & & 0 \\ & \ddots & \\ 0 & & e^{d_n} \end{pmatrix} P^{-1}$$

Devoir

Resoudre le système suivant.

1)
$$\begin{cases} u_{n+1} = 3v_n + 2w_n \\ v_{n+1} = -2u_n + 5v_n + 2w_n \\ w_{n+1} = 2u_n - 3v_n \end{cases}$$

$X_n = A^n X_0$

$e^{tA} = P(\ ) P^{-1}$

2)
$$\begin{cases} x'(t) = 3x(t) - z(t) \\ y'(t) = 2x(t) + 4y(t) + 2z(t) \\ z'(t) = -x(t) + 3z(t) \end{cases}$$

$x(0) = x_0,\ y(0) = y_0$ et $z(0) = z_0$

$U(t) = e^{tA} U_0.$

$$U(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad A = (a_{ij})$$

$$U'(t) = A\,U(t) \iff \begin{cases} x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) + \dots + a_{1n}x_n(t) \\ \vdots \\ x_n(t) = a_{n1}x_1(t) + \dots + a_{nn}x_n(t) \end{cases}$$

si $\underline{A \text{ diagonale}} \rightarrow$

$$\begin{cases} x_1'(t) = a_{11}x_1(t) \\ x_2(t) = a_{22}x_2(t) \\ \vdots \\ x_n'(t) = a_{nn}x_n(t) \end{cases} \iff \begin{cases} x_1(t) = e^{a_{11}t}x_1(0) \\ \vdots \\ x_n(t) = e^{a_{nn}t}x_n(0) \end{cases}$$

$$\iff U(t) = \begin{pmatrix} e^{a_{11}t} & & \\ & \ddots & \\ & & e^{a_{nn}t} \end{pmatrix} U(0)$$

$$\boxed{u(t) = e^{tA}\, u(0)}$$

Si A diagonalisable $A = PDP^{-1}$

$$P^{-1}\Big(u'(t) = A\, u(t)$$

$$= PDP^{-1}u(t) \Big).$$

$$(\star) \iff P^{-1}u'(t) = D P^{-1} u(t)$$

$$Y(t) = P^{-1}u(t)$$

$$(\star) \iff Y'(t) = D\,Y(t) \iff Y(t) = e^{tD}\,Y(0)$$

$$P^{-1}u(t) = e^{tD}P^{-1}u(0)$$

$$\boxed{u(t) = Pe^{tD}P^{-1}u(0)}$$

$$= e^{tPDP^{-1}}\, u(0)$$

$$= e^{tA}\, u(0)$$

$$=$$