Tree

Data Structures
C++ for C Coders

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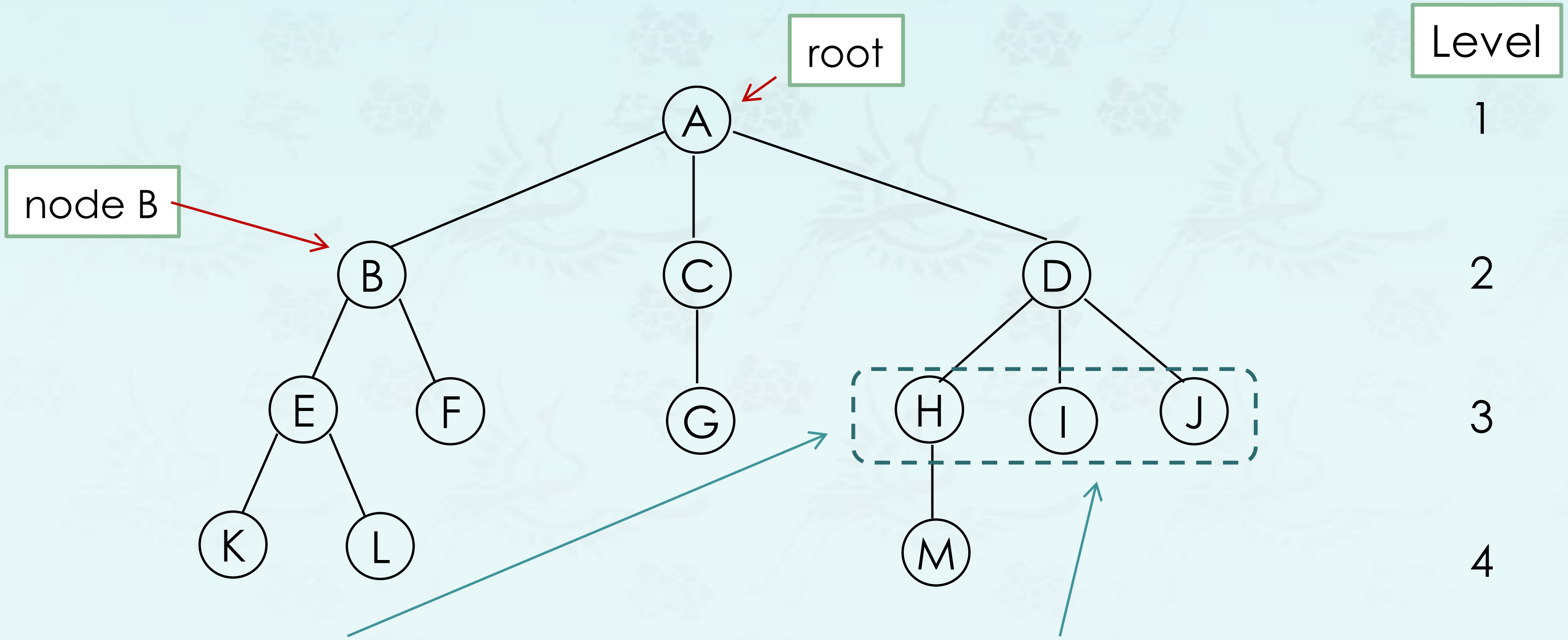
Tree

- ***introduction***
- *binary tree*
- *priority queues & heaps*
- *binary search tree*

Introduction - Terminology

A tree data structure: it is like a linked list that has a **first** node, this node is called as the **root** of the tree.

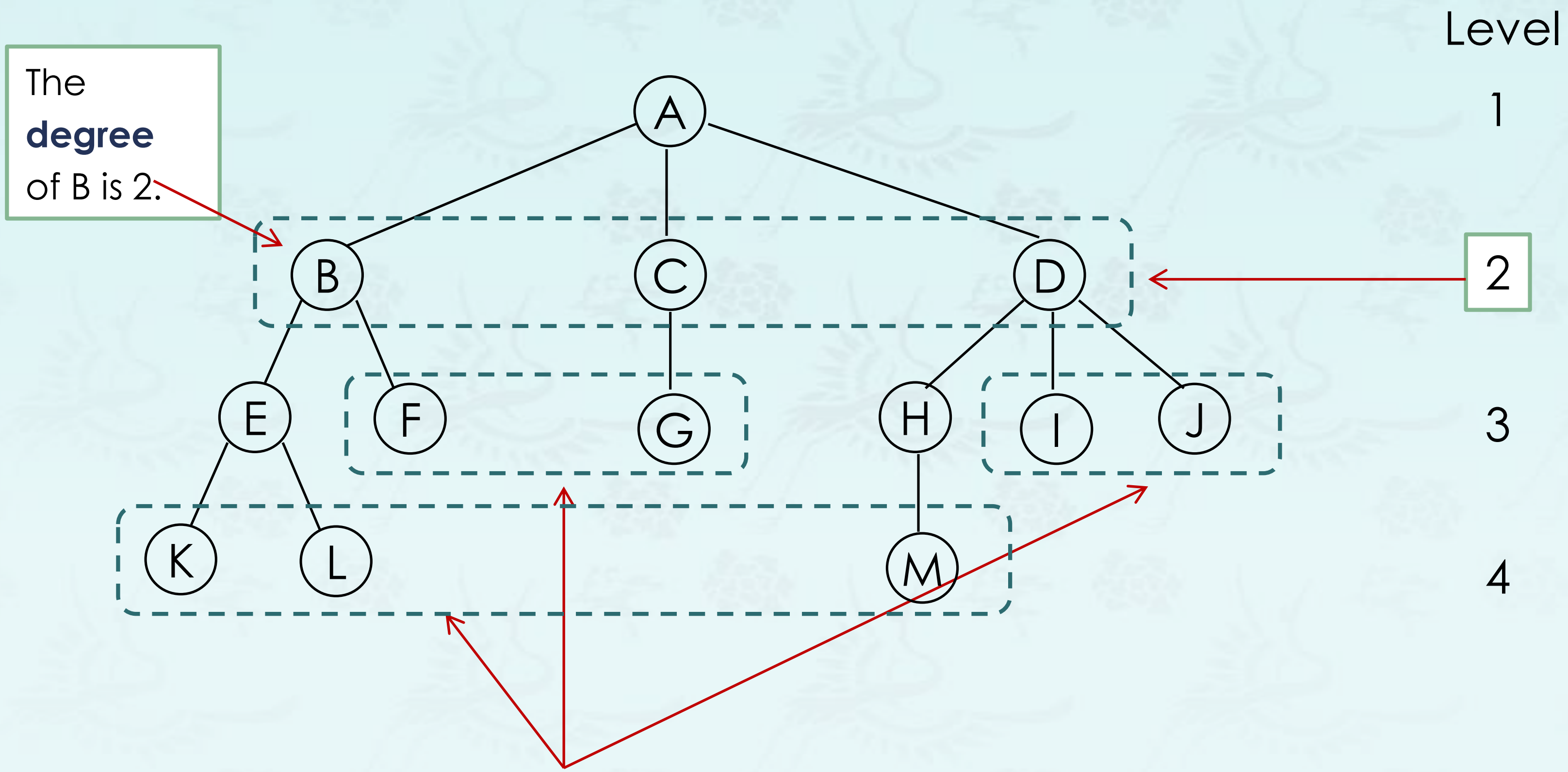
Example. A **tree** with a root storing the value ‘A’



- The **children** of **D** are **H, I, and J**; **H, I, and J** are **siblings**.
- The **parent** of **D** is **A**.

Introduction - Terminology

Definition. child, parent, sibling, degree, leaf nodes, level, height, internal node



- Zero degree nodes are **leaf nodes**, all others are **internal nodes.**
- The **degree** of a node is the number of children.
- The **degree of a tree** is the **maximum of the degree of the nodes** in the tree.
- The **height** or **depth** of a tree is the max level of any nodes in the tree.

Introduction – Representation of trees

Exercise. The tree representing the HTML document below?

```
<html>
   <head>
      <title>Hello World!</title>
   </head>
   <body>
      <h1>This is a <u>Heading</u></h1>
      <p>This is a paragraph with some <u>underlined</u>
text.</p>
   </body>
</html>
```

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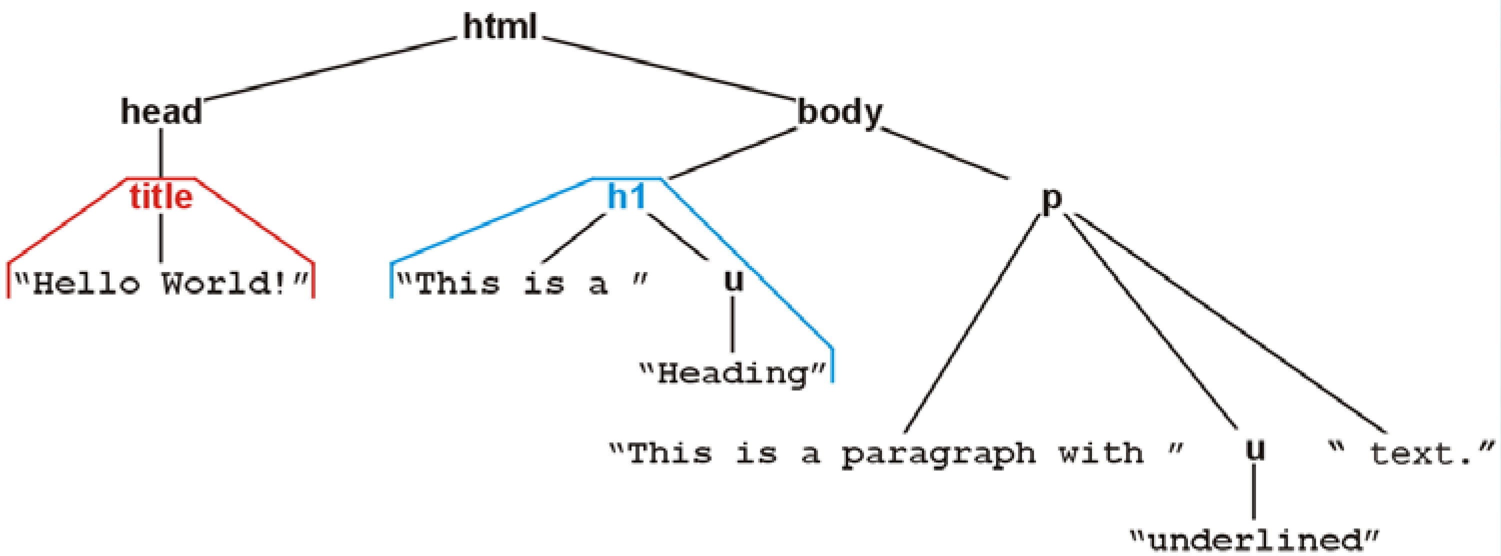


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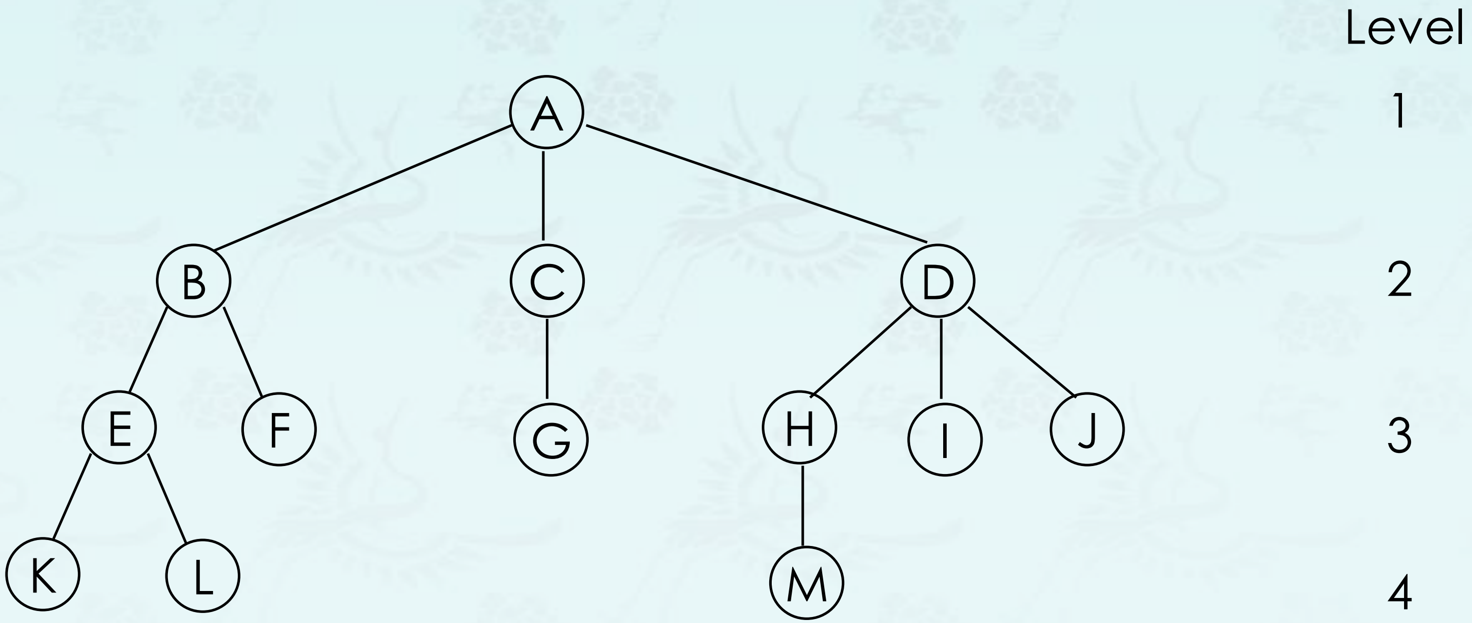
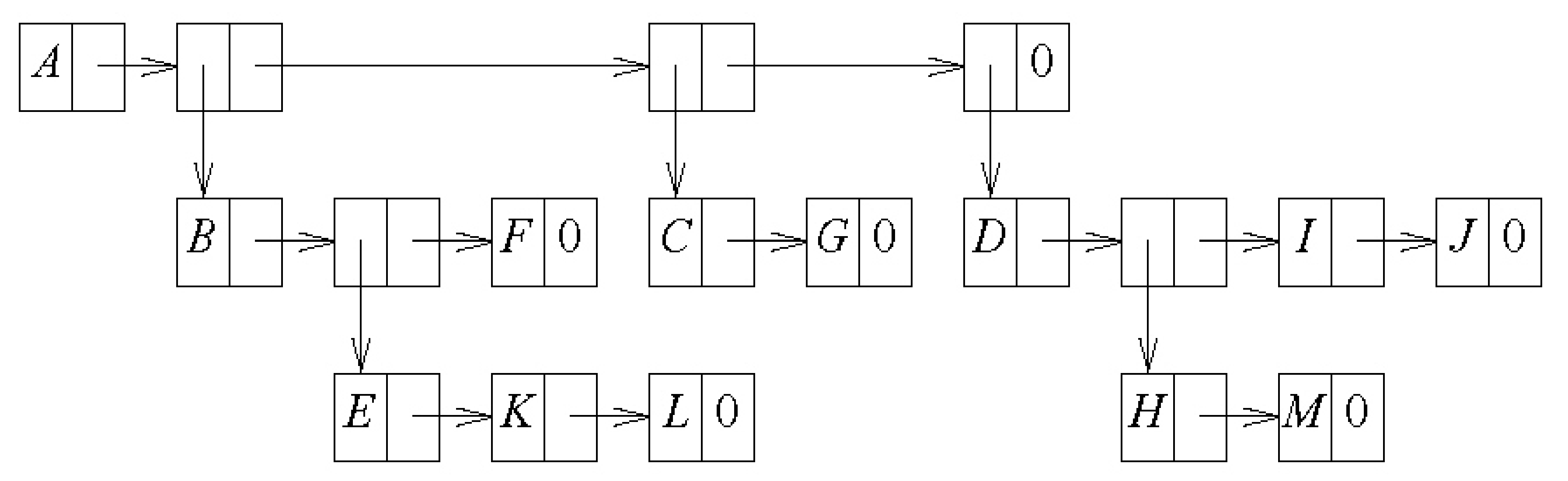
Introduction – Representation of trees

❖ **List representation:** **(A (B (E (K, L), F), C (G), D (H (M), I, J)))**



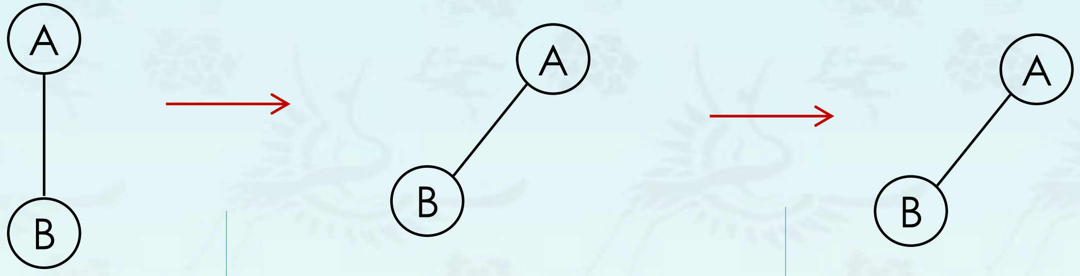
Introduction – Representation of trees

- ❖ List representation: (A (B (E (K, L), F), C (G), D (H (M), I, J)))
- ❖ Memory representation:

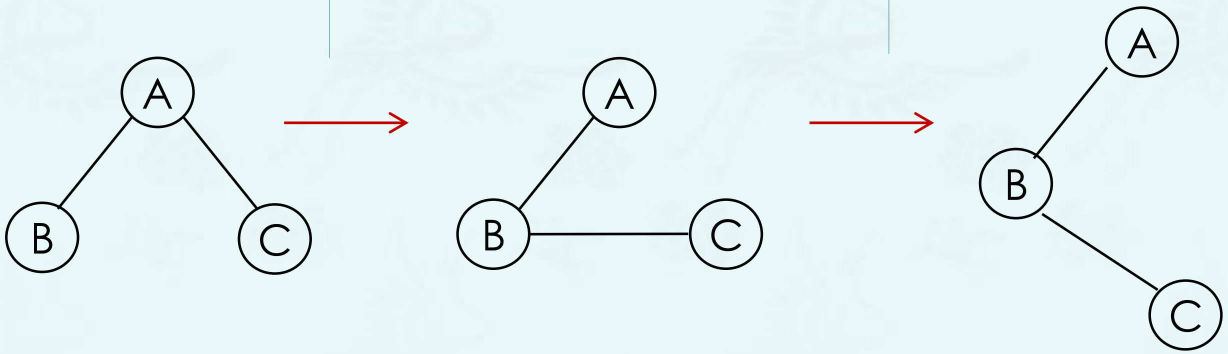


❖ Left child-right child tree representation:

- Rotate the tree **clockwise by 45 degree.** Why?
- To obtain the degree-two tree.
- Note: The root of the tree can never have a sibling.



tree | Left child-right sibling tree | **binary tree**



Tree

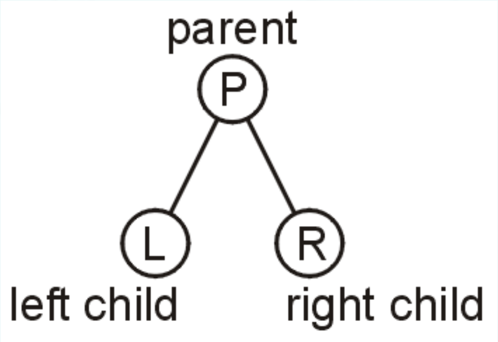
Chapter 5

- *introduction*
- ***binary tree***
- *priority queues & heaps*
- *binary search tree*

Binary trees

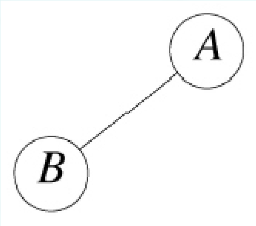
Definition: A tree such that each node has *exactly* two children.

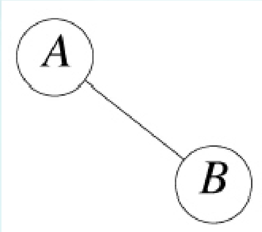
- Notice, exactly two children - not up to two children! (because *exactly* two children means a left child **and/or** right child, no middle child.)
- Each child is either empty or another binary tree.
- Given this constraint, we can label the two children as left and right nodes or subtrees.



Binary trees

Example: two binary trees with two nodes

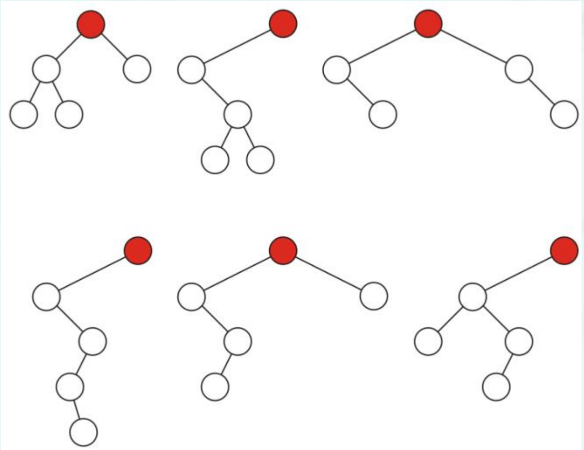




Q: are they two different **binary** trees?

A: Yes!

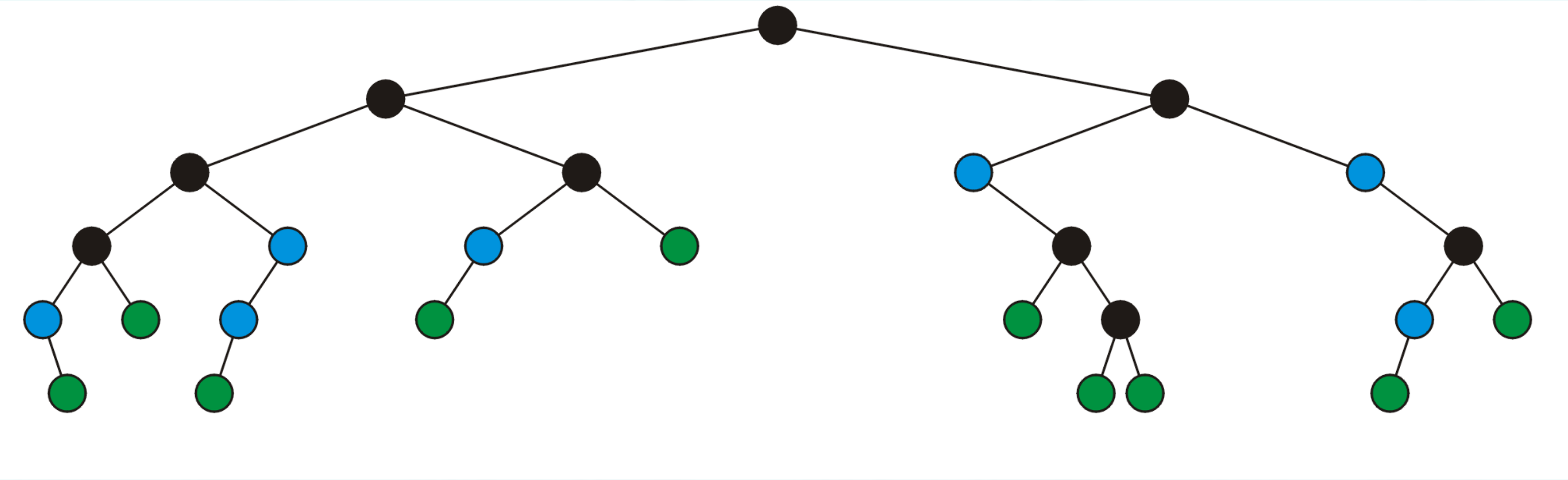
Example: five binary trees with five nodes.



Definition: A **full node** is a node where both left **and** right sub-trees are non-empty trees:

Q: how many full nodes are there?

Q: how many leaf nodes are there?

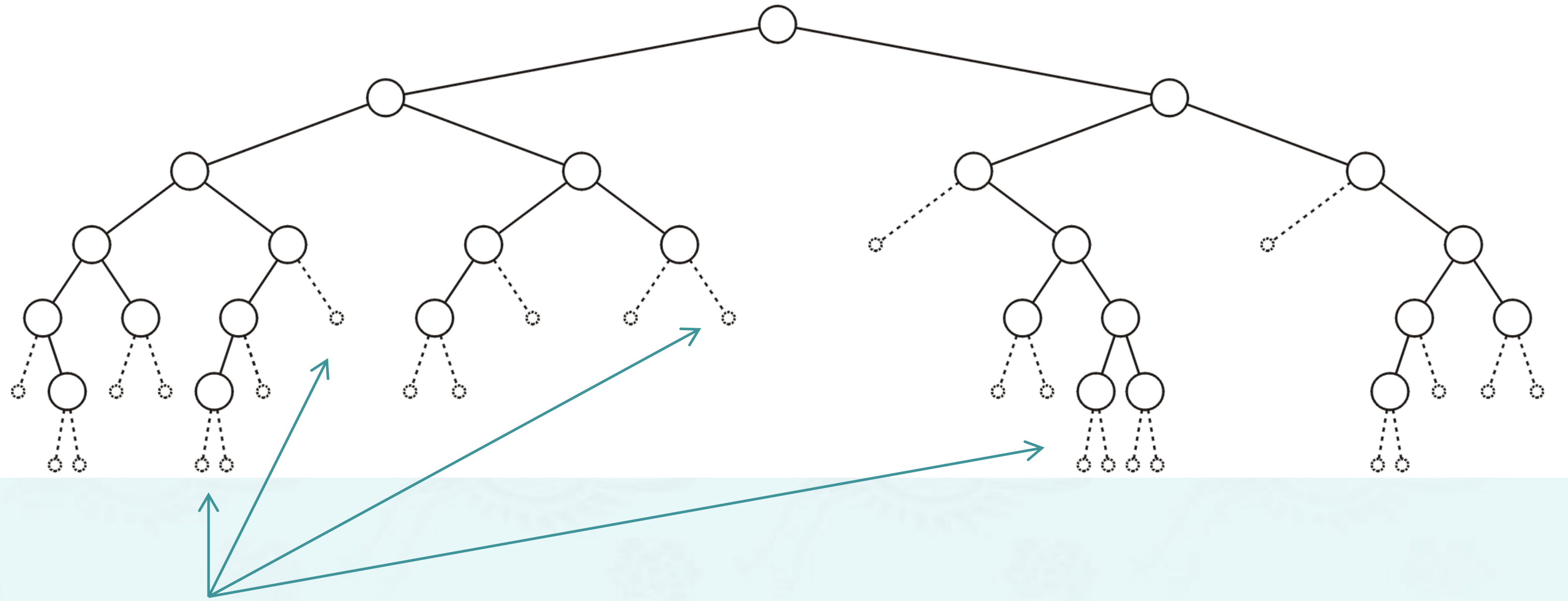


● full nodes ● leaf nodes ● neither

Q: What is the height of the tree?

Q: What is the degree of the tree?

Definition: An ***empty node*** or ***null sub-tree*** is a location where a new leaf node (or a sub-tree) could be inserted.



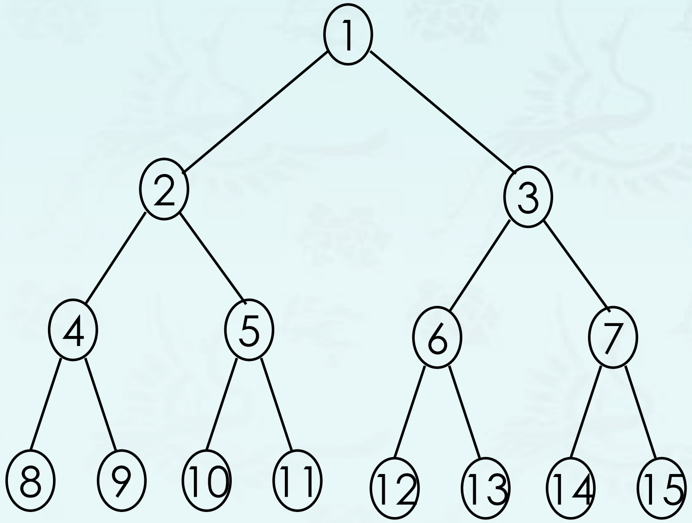
Graphically, the missing branches.

Binary trees

ADT BinaryTree
objects: a finite set of nodes either empty or consisting of a root node, leftBinaryTree, and rightBinaryTree.

functions:
`boolean empty(bt)`
`binaryTree new Node{key, left, right}`
`binaryTree left(bt)`
`element getKey(bt)`
`binaryTree right(bt)`

Observation:
Maximum number of nodes in binary trees in each level and all levels?

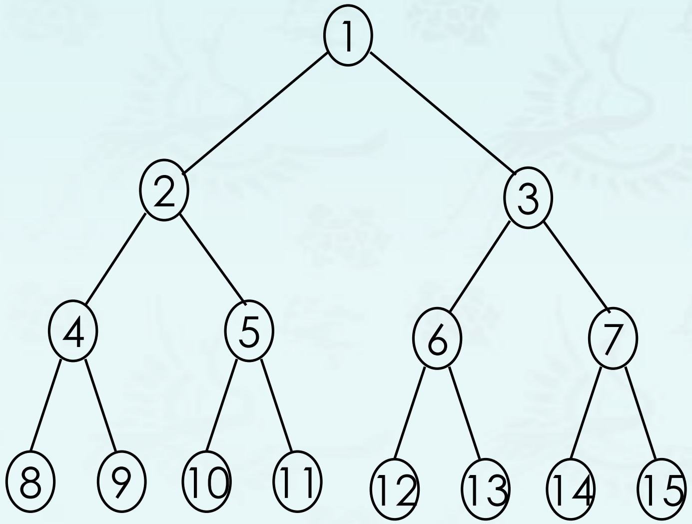


Height	Nodes at one level	Nodes at all levels
1	$2^0 = 1$	$1 = 2^1 - 1$
2	$2^1 = 2$	$3 = 2^2 - 1$
3	$2^2 = 4$	$7 = 2^3 - 1$
4	$2^3 = 8$	$15 = 2^4 - 1$
.	.	.
11	$2^{10} = 1024$	$2047 = 2^{11} - 1$
.	.	.
h		

Binary trees - Properties

Observation:

Maximum number of nodes in binary trees in each level and all levels?



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.	.	.
11	$2^{10} = 1024$	$2047 = 2^{11} - 1$
.	.	.
h	$\mathbf{2^{h-1}}$	$\mathbf{2^h - 1}$

Binary trees - Properties

(1) The maximum number of **nodes on level i** of a binary tree is

[redacted] $i \geq 1$

(2) The maximum number of **nodes in a binary tree of depth k** is

$2^k - 1$, [redacted]

(3) The depth(height) of a **complete binary tree** with **n** nodes is

$\lceil$ [redacted] $\rceil$, $\lceil x \rceil$ is the smallest integer $\geq x$.

$$
\begin{aligned}
n &= 2^h - 1 \\
n + 1 &= 2^h \\
\log(n+1) &= log2^h \\
\log(n+1) &= h \\
h &= \lceil \log(n+1) \rceil \\
h &= \lfloor \log(n) \rfloor + 1
\end{aligned}
$$

Binary trees - Properties

(1) The maximum number of **nodes on** ***level i*** of a binary tree is

$$2^{i-1}, \qquad i \geq 1$$

(2) The maximum number of **nodes in a binary tree of** $\boldsymbol{depth\ k}$ is

$$2^k - 1, \quad k \geq 1$$

Proof (1) by induction on i.

Induction base:

On $\boldsymbol{level\ i\ =\ 1}$, the root is the only node. Hence, $2^{i-1} = 2^{1-1} = 2^0 = 1$.
which is the maximum number of nodes on $level\ i = 1$ → 1
On $\boldsymbol{level\ i\ = 2}$, → $2^{2-1} = 2$

Induction hypothesis:

Assume that the maximum number of nodes on $\boldsymbol{level\ i - 1}$ is $\boldsymbol{2^{i-2}}$.

Induction step:

Since on $\boldsymbol{level\ i\ - 1}$ → 2^{i-2} by hypothesis and
each node has a maximum $degree\ of\ 2$,
the maximum number of nodes on $\boldsymbol{level\ i}$ is $2 * 2^{i-2}$, or $\boldsymbol{2^{i-1}}$

Binary trees - Properties

(1) The maximum number of **nodes on *level i*** of a binary tree is

$$2^{i-1}, \qquad i \geq 1$$

(2) The maximum number of **nodes in a binary tree of *depth k*** is

$$2^k - 1, \quad k \geq 1$$

Proof (2) Using geometric summation:
The maximum number of nodes in a binary tree of ***depth k*** is "the summation of the maximum number of nodes on every level".

$$\sum_{i=1}^{k} (maximum\ numer\ of\ nodes\ on\ level\ i) = \underbrace{2^0}_{\textbf{level 1}} + 2^1 + \ldots + \underbrace{2^{i-1}}_{\textbf{level k}}$$

Binary trees - Properties

(1) The maximum number of **nodes on *level i*** of a binary tree is

$$2^{i-1}, \qquad i \geq 1$$

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Proof (2) Using geometric summation:
The maximum number of nodes in a binary tree of ***depth k*** is "the summation of the maximum number of nodes on every level".

$$\sum_{i=1}^{k}(maximum\ numer\ of\ nodes\ on\ level\ i) = 2^0 + 2^1 + \ldots + 2^{i-1} = \sum_{i=1}^{k} 2^{i-1} = \mathbf{2^k - 1}$$

$$\sum_{i=0}^{n} a^i = 1 + a + a^2 + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1}$$

$$1 + 2 + 2^2 + \cdots + 2^{n-1} + 2^n = \frac{2^{n+1} - 1}{2 - 1}$$

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^{n-1} &= 2^{n+1} - 1 - 2^n \\ &= 2^n(2 - 1) - 1 \\ &= \mathbf{2^n - 1} \end{aligned}$$

(1) The maximum number of **nodes on *level i*** of a binary tree is

$$2^{i-1}, \qquad i \geq 1$$

(2) The maximum number of **nodes in a binary tree of *depth k*** is

$$2^k - 1, \quad k \geq 1$$

Something significant? The depth of a full binary tree of *n* nodes is $\Theta(\log n)$:

Many operations with trees have a run time that goes with the **depth** of some path within the tree; if we have a full binary tree (or something *close* to it), we know that those operations run in $\boldsymbol{O}(\log \boldsymbol{n})$.

Proof:

$n = 2^k - 1$

$n + 1 = 2^k$

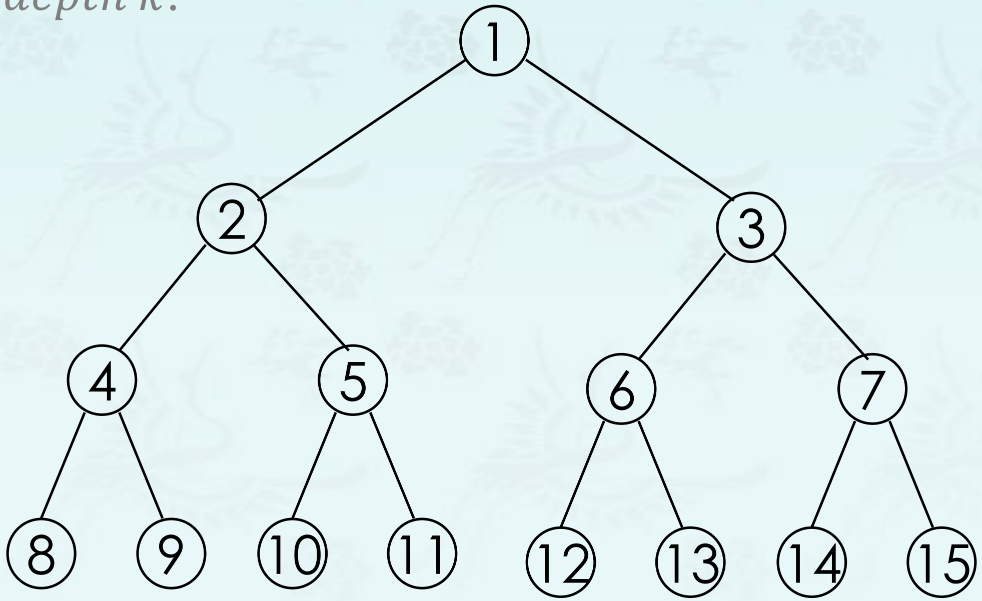
$log_2(n+1) = log_2(2^k)$

$log_2(n+1) = k$

$\Theta(\log n) = k$

Definition: *A **full** binary tree* of $\boldsymbol{depth\ k}$ is a binary tree having **2^k – 1 nodes**, $k \geq 0$.

Definition: A binary tree with n nodes and $depth\ k$ is **complete** iff its nodes correspond to the nodes numbered from 1 to n in the full binary tree of $depth\ k$.

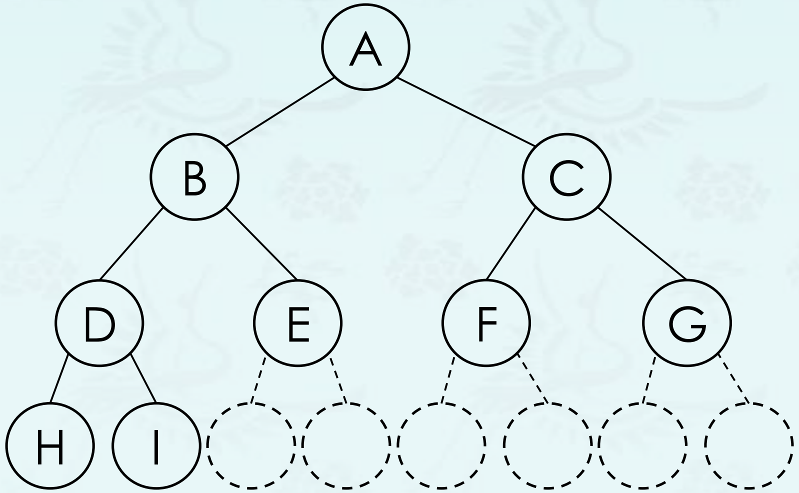


*A **full** binary tree*

Binary trees - Properties

Definition: A ***full*** *binary tree* of depth k is a binary tree of depth k having $\mathbf{2^k - 1}$ nodes, $k \geq 0$.

Definition: A binary tree with n nodes and $depth\ k$ is **complete** **iff** its nodes correspond to the nodes numbered from 1 to n in the full binary tree of $depth\ k$.



*A **complete** binary tree*

(3) The height of a **complete binary tree** with n nodes is

$\lceil log_2\,(n+1) \rceil$, $\lceil x \rceil$ is the smallest integer $\geq\; x$.

Proof (3): The maximum number of **nodes n** of a binary tree with its $height\; k$ or $depth\; k$ is $2^k - 1$, $k \geq 1$.
In a binary tree, it has the maximum number of nodes n of a $n = 2^k - 1$.

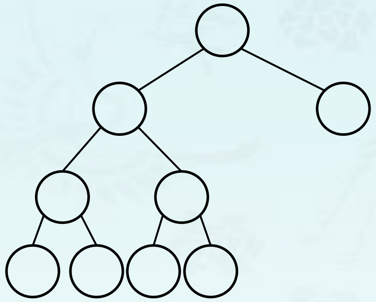
$$n = 2^k - 1,\;\; for\; k \geq 1,$$
$$2^k = n + 1$$
$$log_2\,(2^k) = log_2(n+1)$$
$$k = log_2\,(n+1)$$

$k = \lceil log_2\,(n+1) \rceil$ since k is an integer, to include incomplete trees.

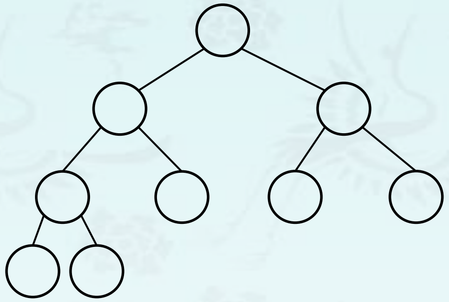
Binary trees - Properties

Definition: A binary tree with n nodes and depth k is **complete iff** its nodes correspond to the nodes numbered from 1 to n in the full binary tree of depth k.

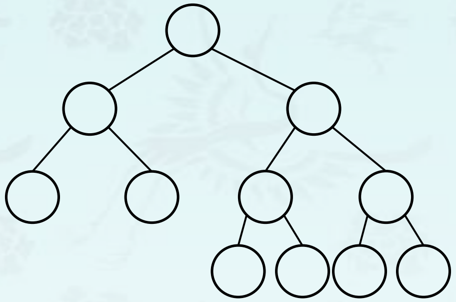
Exercise: identify a **complete** binary tree.



(1)



(2)



(3)

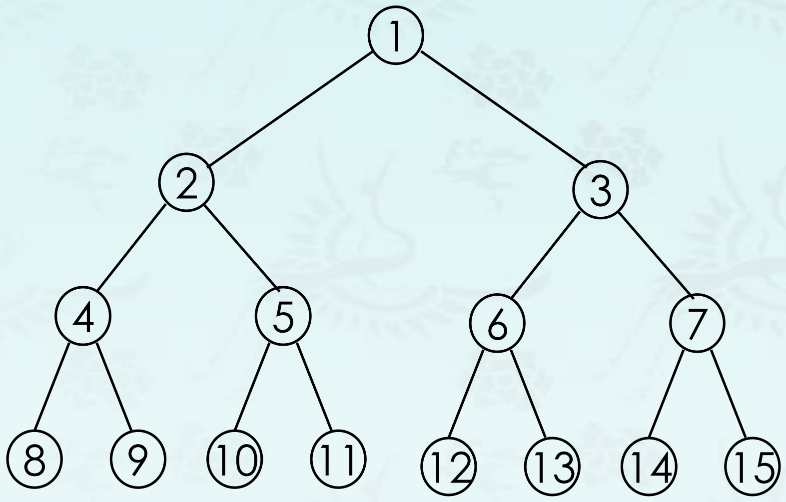
Q. Meanings of a complete tree in terms of ADT?

A. Removals of a node are only allowed from the "last" position.
There is one position available to insert a node every time!

Prof. Youngsup Kim, idebtor@gmail.com, Data Structures, CSEE Dept., Handong Global University

Binary trees - Properties

Problem: representing a binary tree in memory
Hint: remembering a full binary tree with sequential node numbers

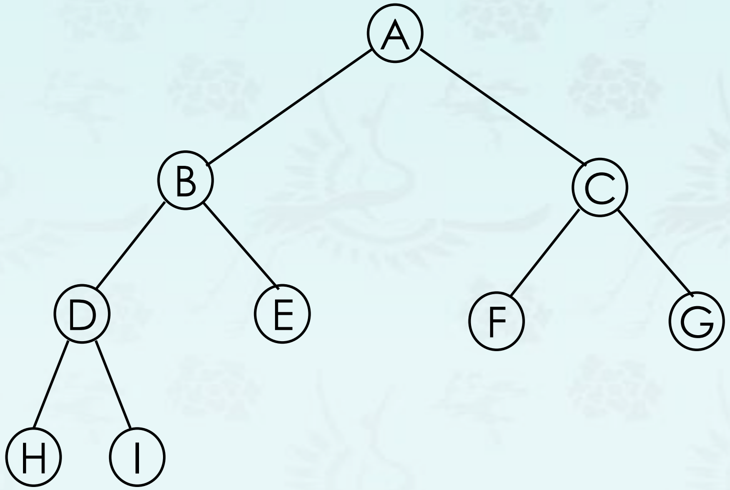


Solution: use one dimensional array to store nodes sequentially.
Any potential problems?

Problems remain: good for a full binary tree, but not good memory usage for a skewed or complete binary tree.

Binary trees – Array representation

Problem: Let's suppose that you have a **complete binary tree** in an array, how can we locate node i's parent or child?
Example: Find its parent, left child and right child at node D.

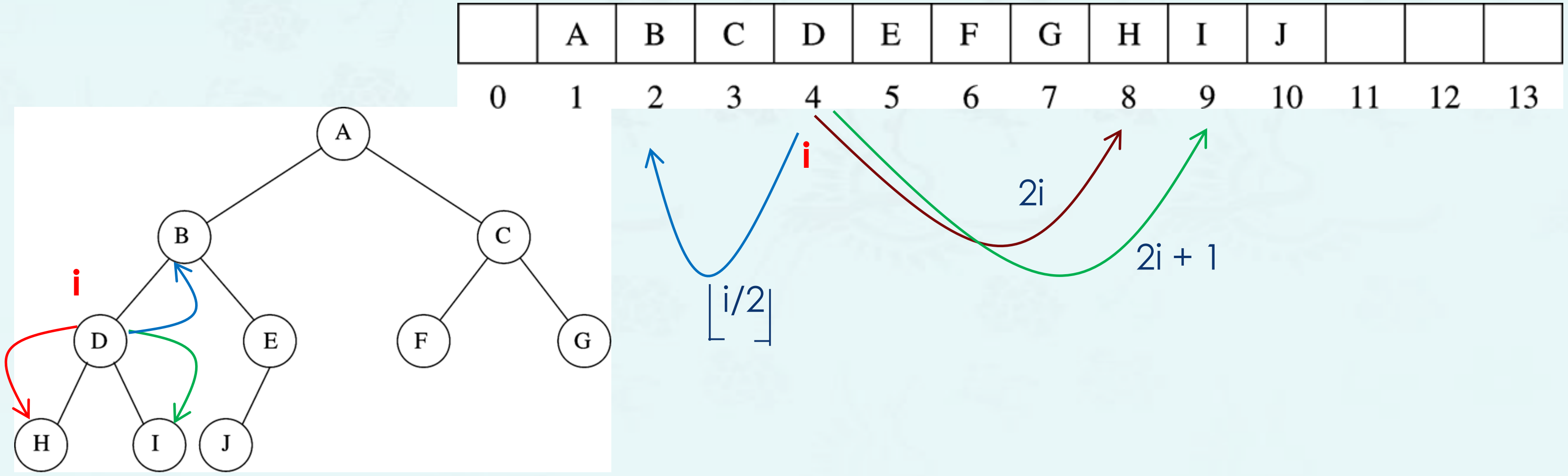


[0]	-
[1]	A
[2]	B
[3]	C
[4]	D
[5]	E
[6]	F
[7]	G
[8]	H
[9]	I

Example: Find its parent, left child and right child at node D.

Lemma 5.4 a **complete** binary tree with n nodes, any node index i, $1 \leq i \leq n$, we have

- Given element at position i in the array
  - Left $child(i)$ = at position $2i$
  - Right $child(i)$ = at position $2i + 1$
  - $Parent(i)$ = at position $\lfloor i/2 \rfloor$



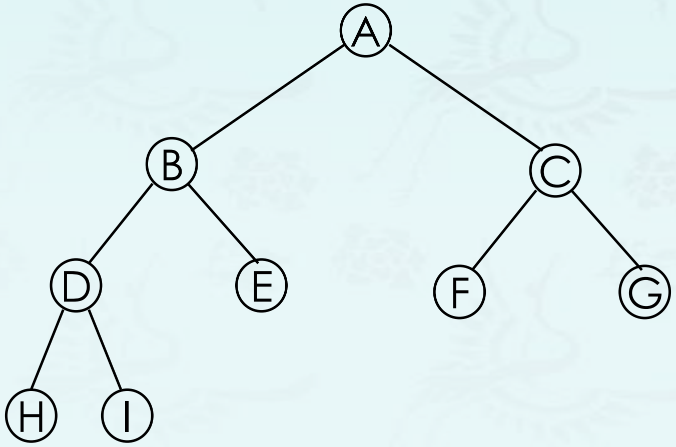
Example: Find its parent, left child and right child at node D.

Lemma 5.4 a **complete** binary tree with n nodes, any node index i, $1 \le i \le n$, we have

(1) $parent(i)$ is at $\lfloor i/2 \rfloor$ $if\ i \ne 1.$ $If\ i = 1$, i is at the root and has no parent.

(2) $leftChild(i)$ is at $2i$ $if\ 2i <= n.$ $If\ 2i > n$, then i has no left child.

(3) $rightChild(i)$ is at $2i+1$ $if\ 2i+1 <= n.$ $If\ 2i+1 > n$, then i has no right child.



[0]	-
[1]	A
[2]	B
[3]	C
[4]	D
[5]	E
[6]	F
[7]	G
[8]	H
[9]	I

Solution:

$parent(i = 4)$ is at 4/2 = 2

$leftChild(4)$ is at 2x4 = 8

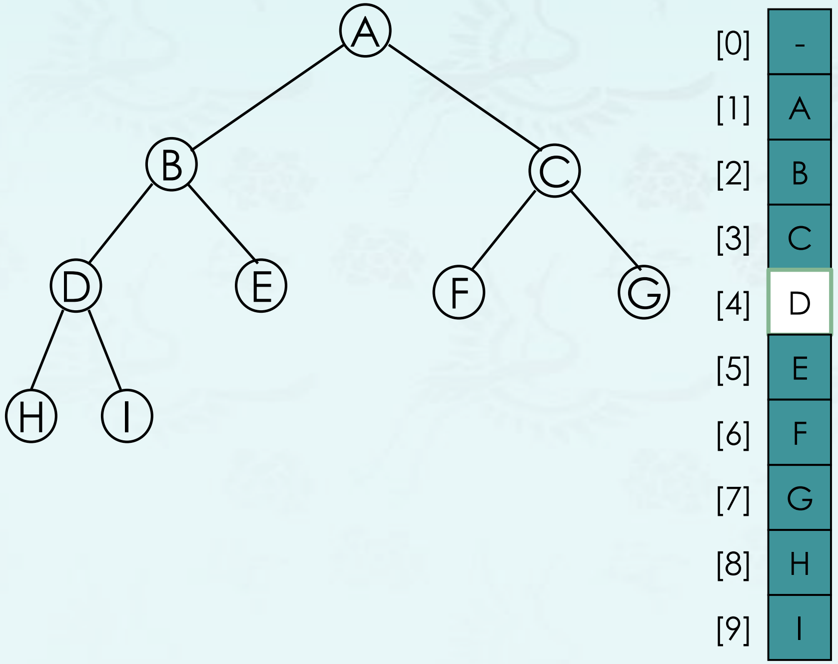
$rightChild(4)$ is at 2x4 + 1 = 9

Wow!
Can we use this to all binary trees?
Why not?

Example: Find its parent, left child and right child at node D.

Lemma 5.4 a **complete** binary tree with n nodes, any node index i, $1 \leq i \leq n$, we have

Wow!
Can we use this to all binary trees?
Why not?



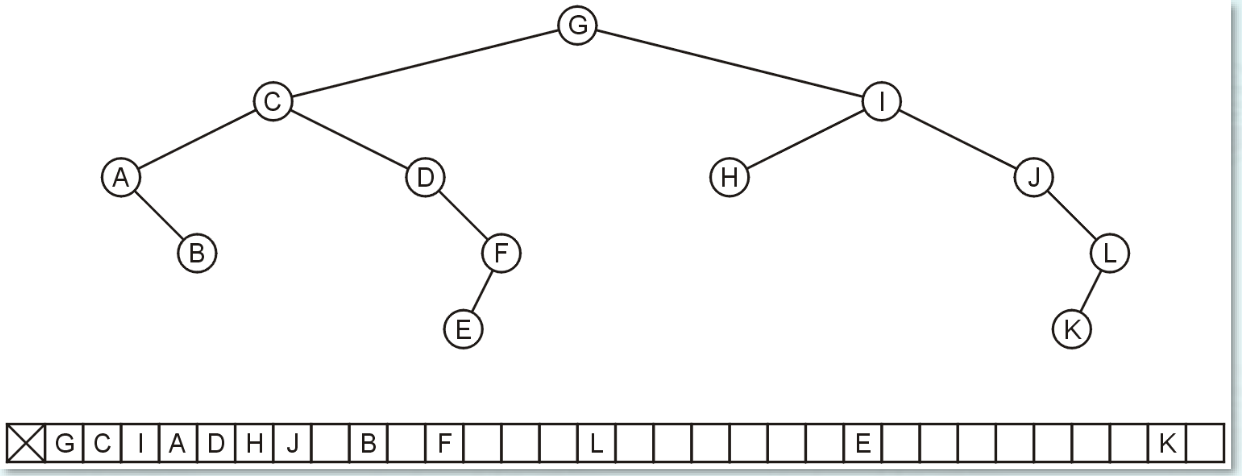
Problem remains:
The problem with storing an arbitrary binary tree using an array is the inefficiency *in memory usage*.

Binary trees – Array representation

Q. Can we use this array rep. to store all binary trees? Why not?

Example: This tree has 12 nodes, and requires an array of 32 elements.

A. Adding one extra node, as a child of node K or E **doubles** the required memory for the array!



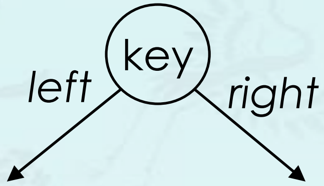
A. In the worst case a skewed tree of depth k will require **$2^k - 1$** space which is O(**2^k**). Of these, **only k** will be used.

Q. What happens when k = n? (Is there such a tree?)

Binary trees – Linked representation

Node representations:

left	key	right



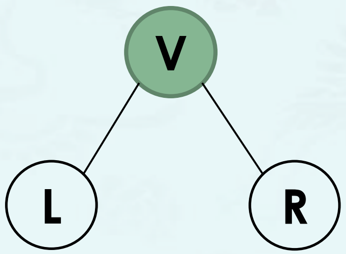

```
struct TreeNode{
    int       key;
    TreeNode* left;
    TreeNode* right;
};
using tree = TreeNode*;
```

Q. Is this node structure good enough?

A. Not easy to find its parent node.
Parent field could be added if necessary

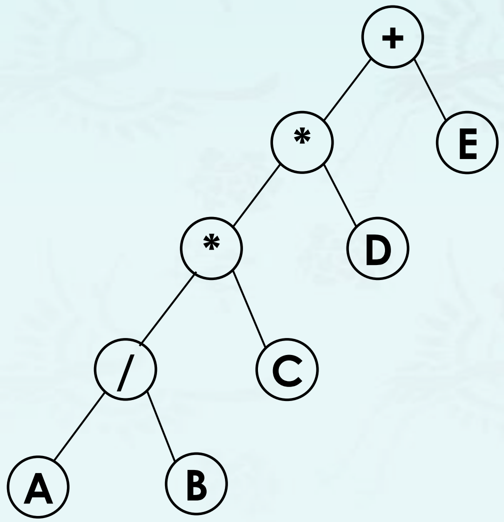
Tree traversal (known as **tree search**) refers to the process of visiting each node in a tree, exactly once, in a systematic way.

- There are three possible moves if we traverse left before right:
  LVR, LRV, VLR.
- These are named **inorder, postorder,** and **preorder** because of the position of the **V** (visiting node) with respect to the L and R.
- There are three types of **depth-first traversal.**



Example: inorder traversal(LVR)

- Moving down the tree toward the left until you can go no farther. Then you "visit" the node, move one node to the right and continue. If you cannot move to the right, go back one more node.




```
void inorder(tree root) {
   if (root == nullptr) return;

   inorder(root->left);
   cout << root->key;
   inorder(root->right);
}
```

Output:

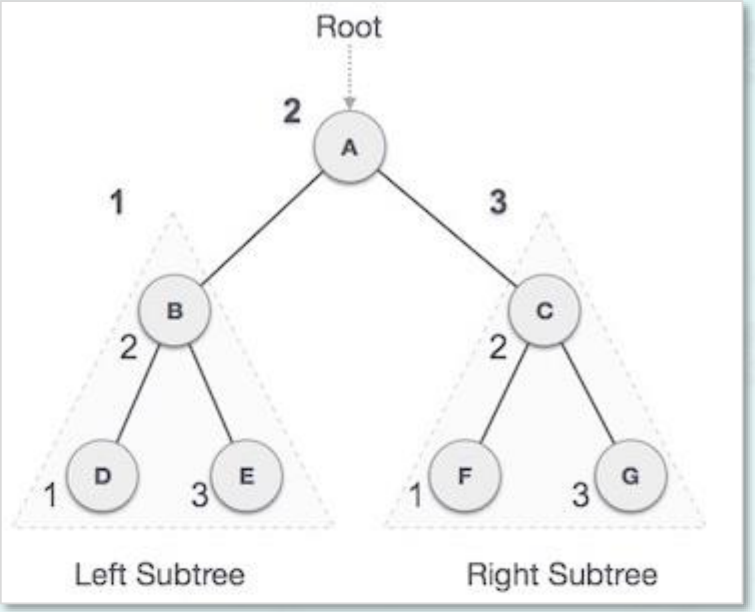
Output(LVR) : A / B * C * D + E

Binary tree traversals

inorder traversal(LVR)

Until all nodes are traversed −

- Step 1 − Recursively traverse left subtree.
- Step 2 − Visit root node.
- Step 3 − Recursively traverse right subtree.

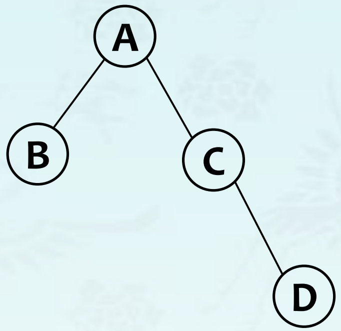


Output:

Output(LVR) : D B E A F C G

Q1: Output(LVR):

Q2: How many times is inorder( ) invoked for the complete traversal?




```
struct TreeNode{
    int      key;
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```

```
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   inorder(root->left);
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```
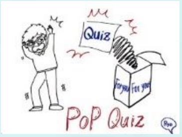
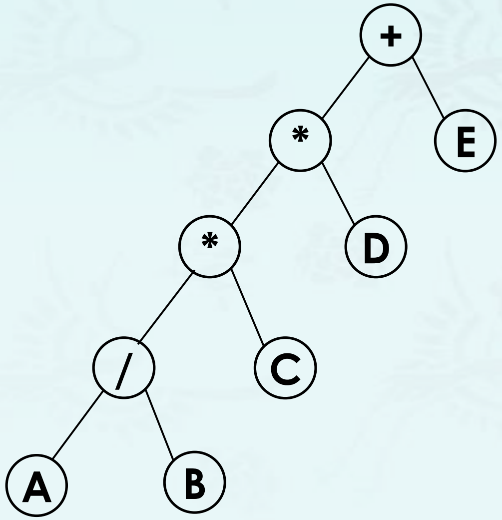

Binary tree traversals



Example: inorder traversal(LVR)

Q: How many times is inorder( ) invoked for the complete traversal?



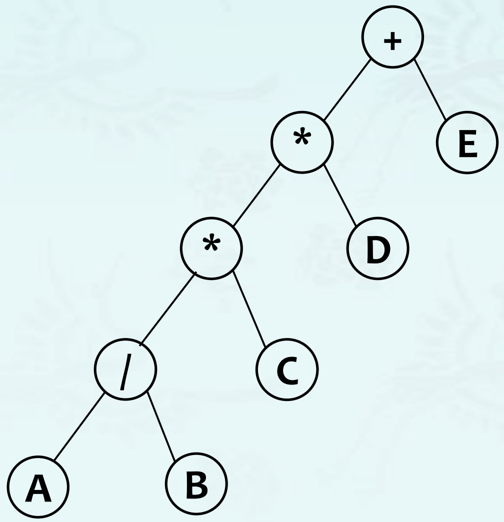

```
void inorder(tree root) {
   if (root == nullptr) return;

   inorder(root->left);
   cout << root->key;
   inorder(root->right);
}
```

Output:

Output(LVR) : A / B * C * D + E

Example: inorder traversal(LVR)

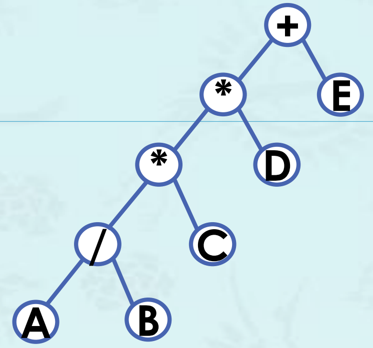
Note: "Since there are 9 nodes in the tree, inorder is invoked 19 times for the complete traversal." This is not a typo.




```
void inorder(tree root) {
   if (root == nullptr) return;

   inorder(root->left);
   cout << root->key;
   inorder(root->right);
}
```

A. **Every leaf node** node must visit (call the function) its left child and right child to make sure they don't have the child. 9 + 5 * 2 = 19

Binary tree traversals

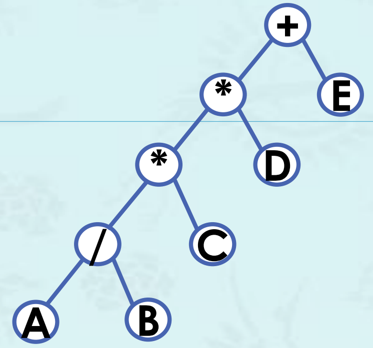



```
void inorder(tree root){
  if (root) {
    inorder(root->left);
    cout << root->key;
    inorder(root->right);
  }
}
```

Example: inorder traversal(LVR)

Call of inorder	root or root→key	Action	inorder	root or root→key	Value Action
1	+		11	C	
2	*		12	NULL	
3	*		11	C	**cout**
4	/		13	NULL	
5	A		2	*	**cout**
6	NULL		14	D	
5	A	**cout**	15	NULL	
7	NULL		14	D	**cout**
4	/	**cout**	16	NULL	
8	B		1	+	**cout**
9	NULL		17	E	
8	B	**cout**	18	NULL	
10	NULL		17	E	**cout**
3	*	**cout**	**19**	NULL	

Binary tree traversals




```
void inorder(tree root){
  if (root) {
    inorder(root->left);
    cout << root->key;
    inorder(root->right);
  }
}
```

Example: inorder traversal(LVR)

Call of inorder		root or root→key	Action	
1		+	1.push	
2		*	2.push	
3		*	3.push	
4		/	4.push	
5		A	5.push	
6		NULL	return	
5	1.pop	A		**cout**
7		NULL	return	
4	2.pop	/		**cout**
8		B	6.push	
9		NULL	return	
8	3.pop	B		**cout**
10		NULL	return	
3	4.pop	*		**cout**

inorder	root or root→key	Value Action

System Stack

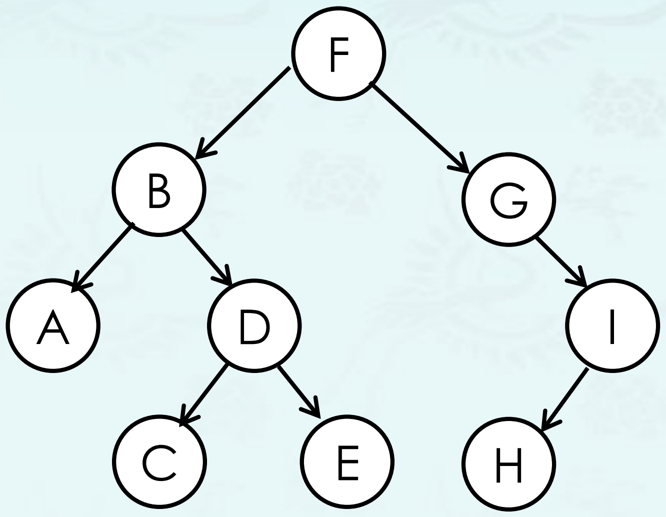
5.push(A) 1.pop
4.push(/) 2.pop 6.push(B) 3.pop
3.push(*) 4.pop
2.push(*)
1.push(+)

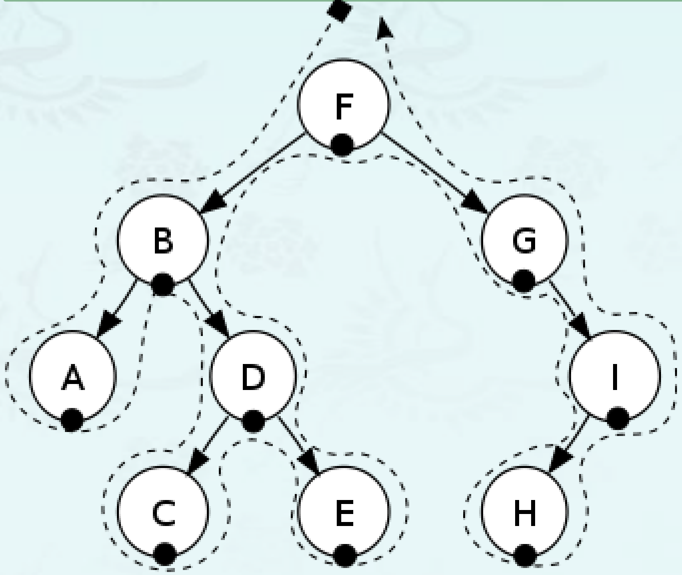
Example: inorder traversal(LVR)

1. Traverse the left subtree.
2. Visit the root.
3. Traverse the right subtree.

Exercise: Output?

A: A, B, C, D, E, F, G, H, I

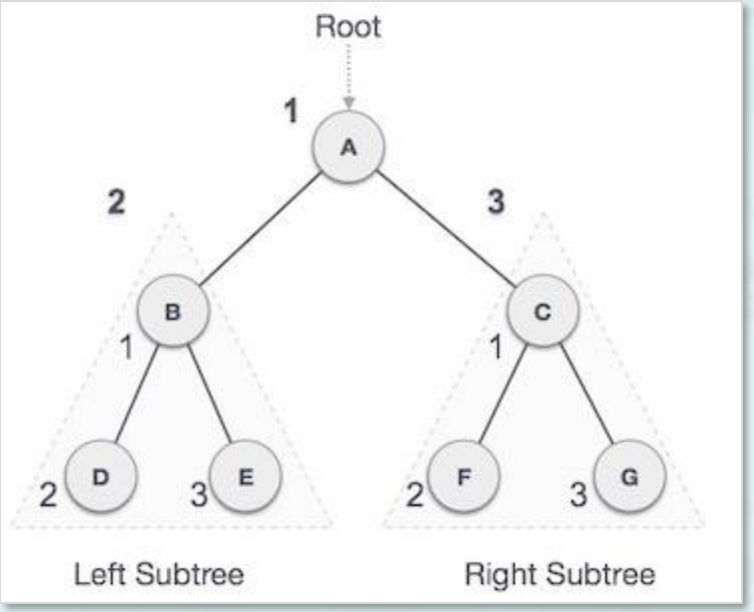




Binary tree traversals

preorder traversal(VLR) Until all nodes are traversed −

- Step 1 − Visit root node.
- Step 2 − Recursively traverse left subtree.
- Step 3 − Recursively traverse right subtree.

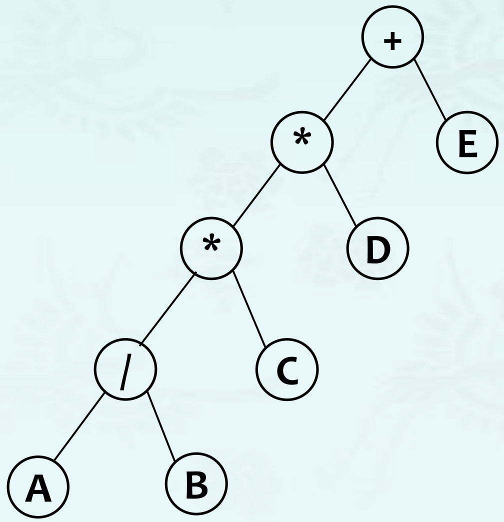


Output:

Output(VLR) : A B D E C F G

Example: preorder traversal(VLR)

- Visit a node, traverse left, and continue. When you cannot continue, move right and begin again or move back until you can move right and resume.




```
void preorder(tree root) {
   if (root == nullptr) return;

   cout << root->key;
   preorder(root->left);
   preorder(root->right);
}
```

Output:

Output(LVR) : + * * / A B C D E

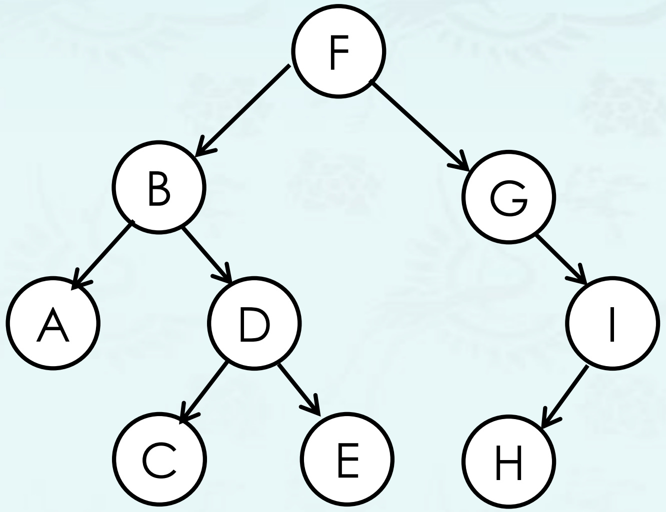
Binary tree traversals

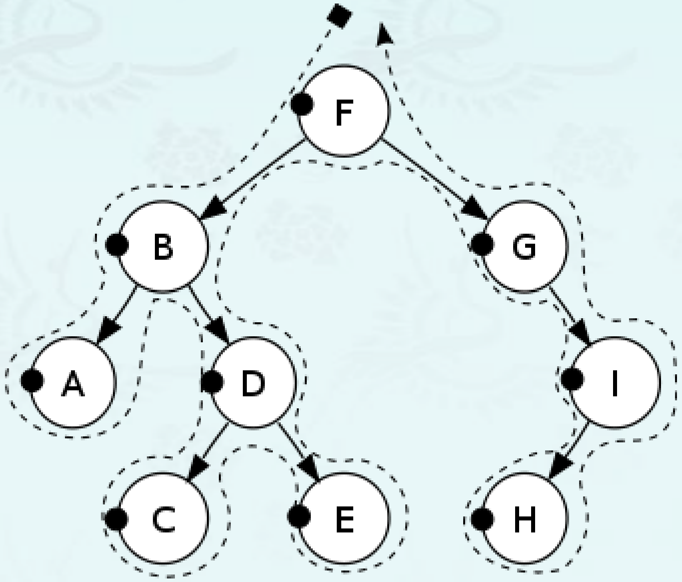
Example: preorder Traversal(VLR)

1. Visit the root.
2. Traverse the left subtree.
3. Traverse the right subtree

Exercise: Output?

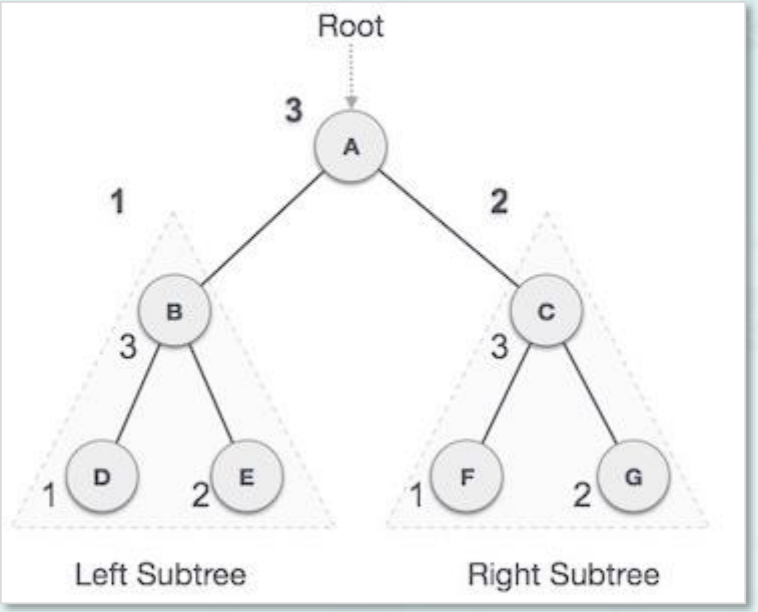
A: F, B, A, D, C, E, G, I, H





postorder traversal(LRV) Until all nodes are traversed −

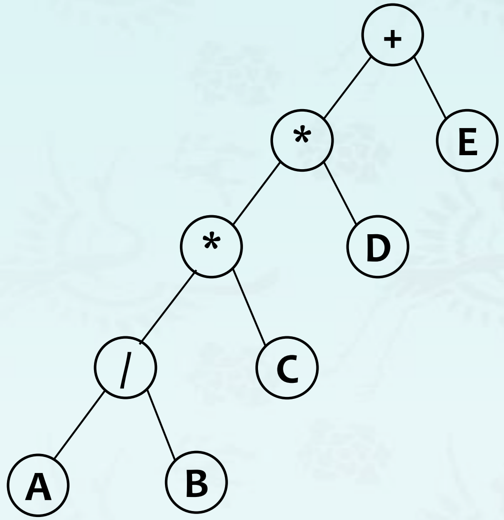
- Step 1 − Recursively traverse left subtree.
- Step 2 − Recursively traverse right subtree.
- Step 3 − Visit root node.



Output:

Output(LRV) : D E B F G C A

Example: postorder traversal(LRV)




```
void postorder(tree root) {
   if (root == nullptr) return;

   postorder(root->left);
   postorder(root->right);
   cout << root->key;
}
```

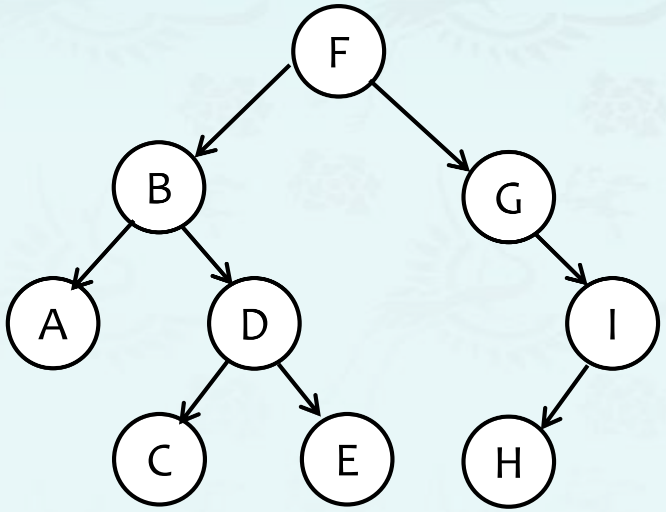
Output:

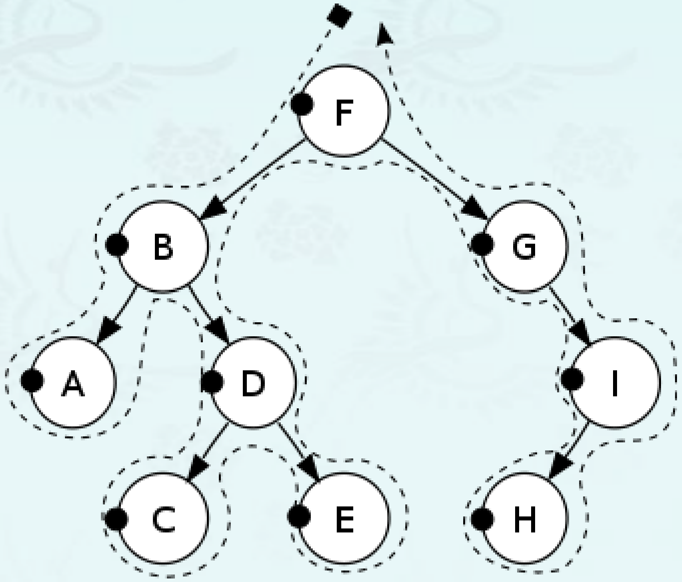
Output(LVR) : A B / C * D * E +

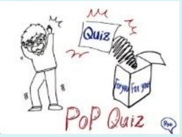
Example: postorder traversal(LR V)
1. Traverse the left subtree.
2. Traverse the right subtree.
3. Visit the root.

Exercise: Output?

A: A C E D B H I G F







Binary tree traversals

Summary

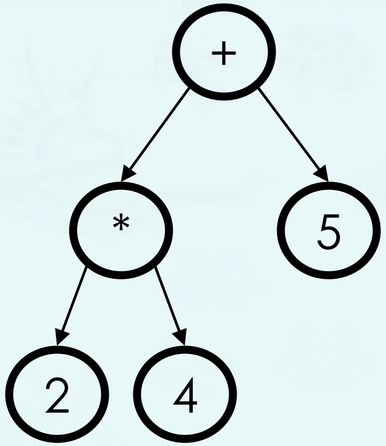
A *traversal* is an order for visiting all the nodes of a tree

- *Preorder*:
- *Inorder*:
- *Postorder*:
- Levelorder:

Preorder: root, left subtree, right subtree

Inorder: left subtree, root, right subtree

Postorder: left subtree, right subtree, root



Binary tree traversals

Summary

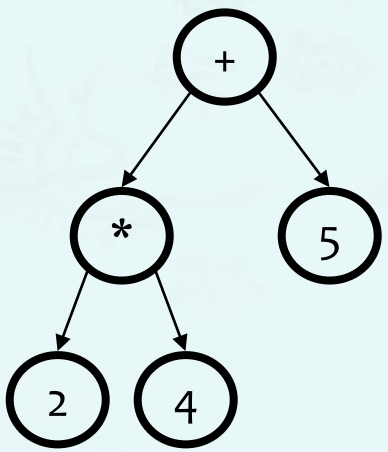
A *traversal* is an order for visiting all the nodes of a tree

- *Preorder*: + * 2 4 5
- *Inorder*: 2 * 4 + 5
- *Postorder*: 2 4 * 5 +
- Levelorder: + * 5 2 4

Preorder: root, left subtree, right subtree

Inorder: left subtree, root, right subtree

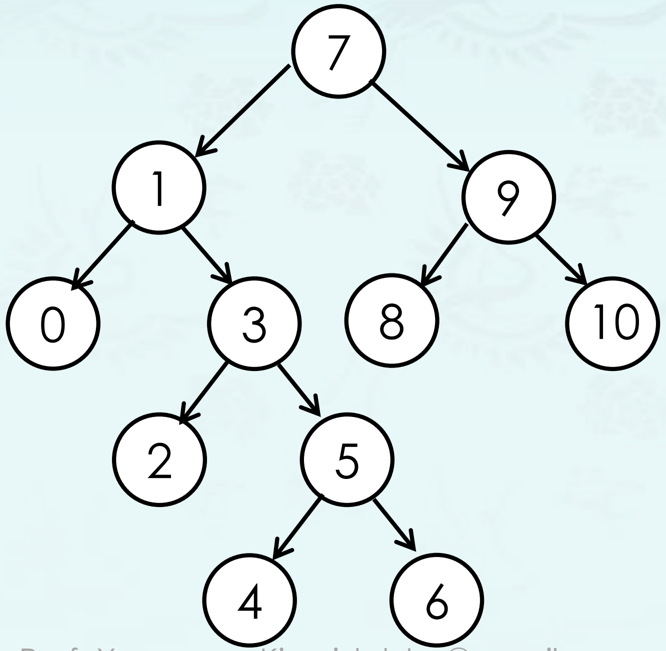
Postorder: left subtree, right subtree, root



Binary tree traversals

Example:

preorder Traversal(VLR)	7, 1, 0, 3, 2, 5, 4, 6, 9, 8, 10
inorder traversal(LVR)	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
postorder traversal(LRV)	0, 2, 4, 6, 5, 3, 1, 8, 10, 9, 7



Binary tree traversals

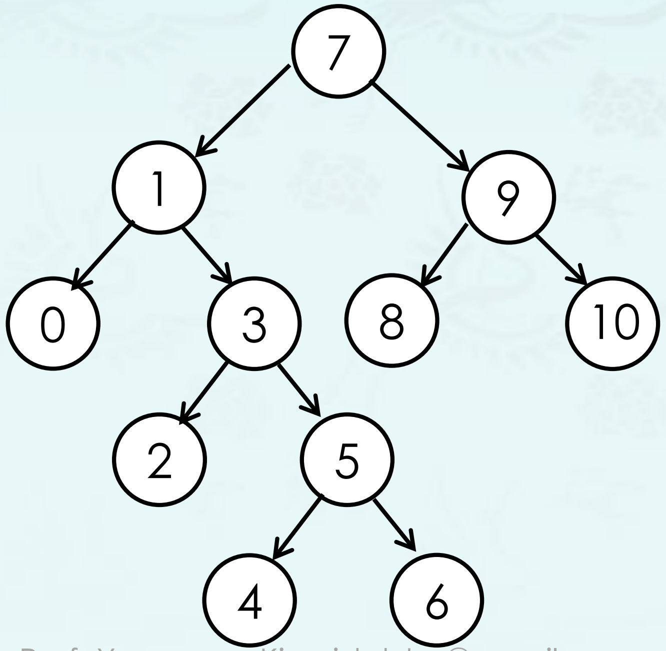
Example:

preorder Traversal(VLR) 7, 1, 0, 3, 2, 5, 4, 6, 9, 8, 10

inorder traversal(LVR) 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

postorder traversal(LRV) 0, 2, 4, 6, 5, 3, 1, 8, 10, 9, 7

Observations:



Binary tree traversals

Observations:

1. If you know you need to explore the roots before inspecting any leaves, you pick **preorder** because you will encounter all the roots before all of the leaves.

2. If you know you need to explore all the leaves before any nodes, you select **postorder** because you don't waste any time inspecting roots in search for leaves.

3. If you know that the tree has an inherent sequence in the nodes, and you want to flatten the tree back into its original sequence, than an **inorder** traversal should be used. The tree would be flattened in the same way it was created. A pre-order or post-order traversal might not unwind the tree back into the sequence which was used to create it.

Tree

- *introduction*
- *binary tree*
- ***priority queues & heaps***
- *binary search tree*