# Chapter 12: Linear models with two explanatory factor variables (Two-way analysis of variance)

STATS 201/8

University of Auckland

# Learning Outcomes

In this chapter you will learn about:

- Fitting a model with two explanatory factors, a.k.a., two-way ANOVA
- Interaction plots
- Interpretting the fitted model
- Pairwise comparisons using `emmeans`
- Using Occam's razor to simplify a model
- Relevant `R`-code

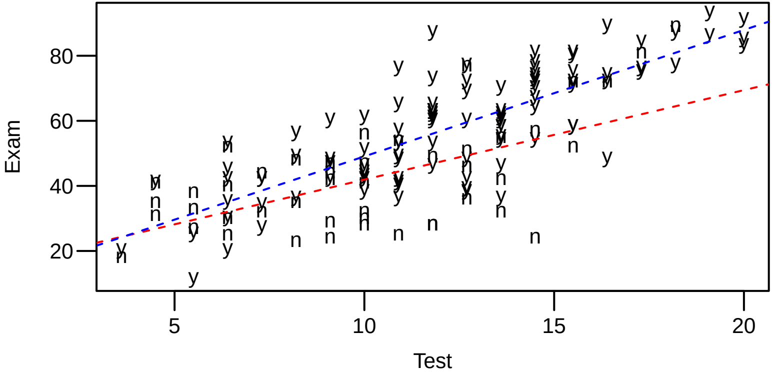
## Section 12.1
## Example: Using test success and attendance to explain exam score

## Two-way ANOVA with interaction

# Exam score vs test success and attendance

In Chapter 8 we investigated whether the effect of a student's test mark on exam score changed depending on whether they regularly attend ~~or not~~.

We saw that those who ~~attended~~ regularly (blue line and "y" for "yes") got more 'return' for each additional test mark.



# Exam score vs test success and attendance. . .

Here we ~~are using~~ the same two explanatory variables as in Chapter 8 but we ~~are going to~~ change the explanatory test score variable so that it only has two states – ~~did they~~ pass the test ~~or not?~~

~~That is, we are going to use the dichotomous factor variable "test success", rather than the raw test score value.~~

We shall also be using attendance as a second explanatory factor.

~~The~~ example we ~~are using here would be called~~ a two-way ANOVA, as there are two explanatory factors.

First, read in the data and change the class of `Attend` to factor:

```
> ## Importing data into R
> Stats20x.df = read.table("Data/STATS20x.txt", header=T)
> Stats20x.df$Attend=factor(Stats20x.df$Attend)
```

# Exam score vs test success and attendance...

We need to transform the numeric `Test` variable into a factor with two levels, pass or not pass.

We can then ask whether passing the test results in better exam marks and vice-versa, on average. We ~~will~~ also ask the same question of regular attendance.

Let us create the new factor variable `Pass.test`:

```
> Stats20x.df$Pass.test=with(Stats20x.df,
+                             factor(ifelse(Test>=10,"pass","nopass")))
> ## Check to see if the call above does what we expect
> min(Stats20x.df$Test[Stats20x.df$Pass.test=="pass"])
[1] 10
> max(Stats20x.df$Test[Stats20x.df$Pass.test=="nopass"])
[1] 9.1
```

# Section 12.2
# Interaction plots

# Exam score vs test success and attendance...

`interactionPlots()`

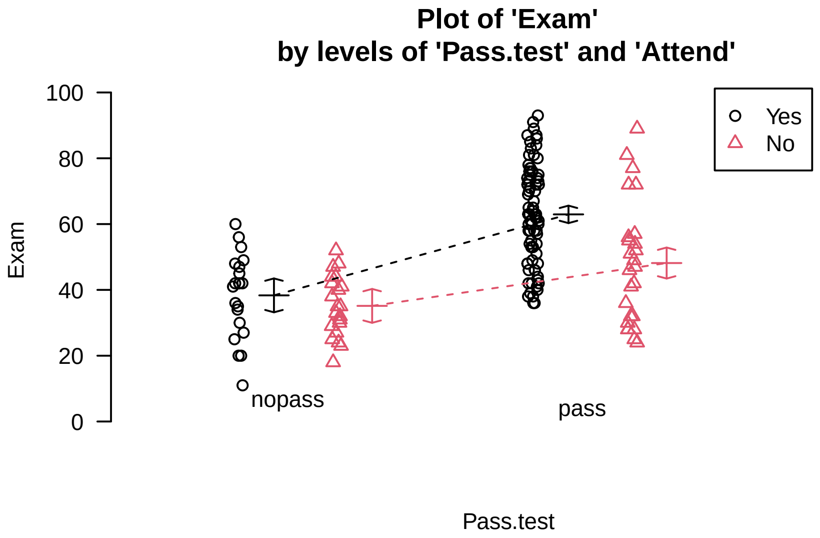
Let us see how these data explain `Exam` by using an `s20x` function `interactionPlots()`.

This is designed specifically for plotting a continuous $Y$ (in our case `Exam`) against two factor variables (here they are `Attend` and the newly created `Pass.test`).

# Exam vs test success and attendance...

`interactionPlots()`...

```
> interactionPlots(Exam ~ Pass.test + Attend, data = Stats20x.df)
```



# Exam vs test success and attendance...

Here we see that ‘attenders’ who pass the test seem to be doing markedly better than most other students. Note that we do not have parallel lines, thereby indicating that there could be an interaction between the two factors.

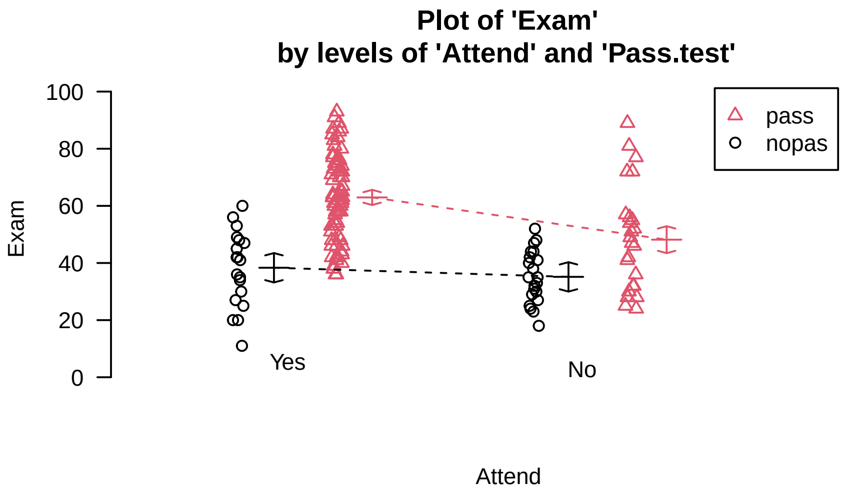
In other words, the effect on exam score of passing the test may depend on whether a student regularly attended or not.

As shown below, we can rearrange the layout of the interaction plot by reversing the order in which the explanatory variables are given in the right-hand side of the model formula argument.

# Exam vs test success and attendance. . .

We still conclude the same insights as above.

```
> interactionPlots(Exam ~ Attend + Pass.test, data = Stats20x.df)
```



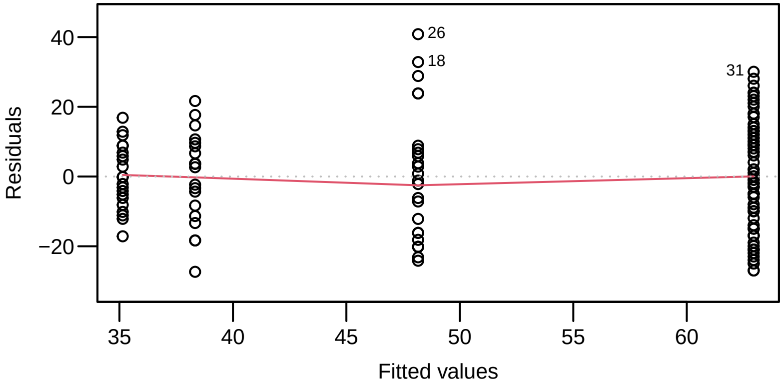
# Section 12.3
# Fitting the interaction model

# Exam vs test success and attendance. . .

## Assumption checks

Let us fit the model with interaction, and check the assumptions.

```
> Exam.fit = lm(Exam ~ Attend*Pass.test, data = Stats20x.df)
> plot(Exam.fit, which=1)
```

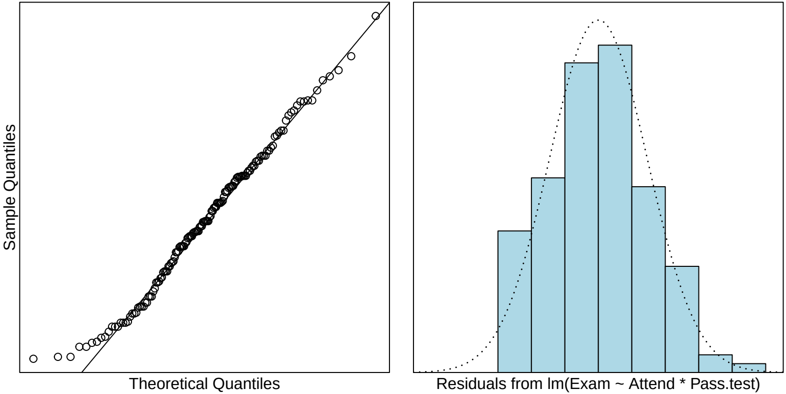



The **EOV** assumption seems to be okay.

# Exam vs test success and attendance...

## Assumption checks...

```
> normcheck(Exam.fit)
```

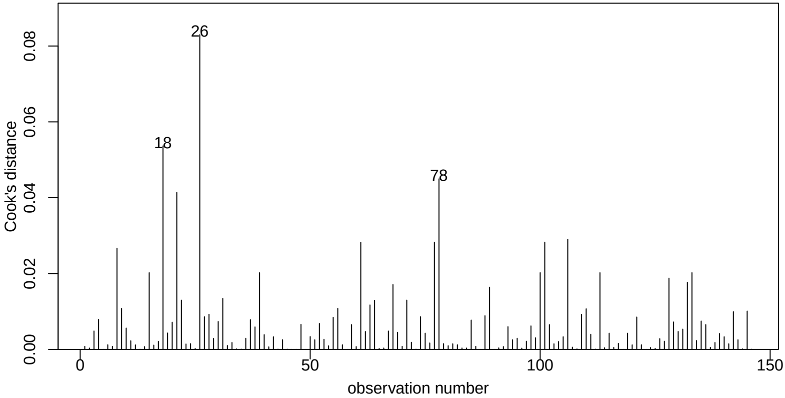



The normality assumption seems to be reasonably good, other than a lack of large negative residuals.

# Exam vs test success and attendance. . .

## Assumption checks. . .

```
> cooks20x(Exam.fit)
```




No unduly influential data points.

# Exam vs test success and attendance...

We conclude that we can trust the output. Let us see what it is telling us.

```
> anova(Exam.fit)
Analysis of Variance Table

Response: Exam
                 Df  Sum Sq Mean Sq F value    Pr(>F)
Attend            1  7630.8  7630.8  34.990 2.364e-08 ***
Pass.test         1 11076.9 11076.9  50.791 4.763e-11 ***
Attend:Pass.test  1   909.7   909.7   4.171   0.04297 *
Residuals       142 30968.4   218.1
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The *P*-value of 0.043 is just under 0.05 and so establishes that there is a significant interaction: The effect of passing the test depends on whether the student has attended ~~class~~ or not.

So, we cannot simply state the effect of passing the test, because the size of this effect depends on whether the student attended or not.

One way to think of this is that we have to consider all 4 ($2 \times 2$) different test success/attendance possibilities separately.

# Exam vs test success and attendance...

Let us investigate what our model tells us in terms of the estimated parameters:

```
> summary(Exam.fit)
```

```
Coefficients:
                        Estimate Std. Error t value Pr(>|t|)
(Intercept)               35.143      3.223  10.905  < 2e-16 ***
AttendYes                  3.190      4.557   0.700  0.48504
Pass.testpass             13.017      4.371   2.978  0.00341 **
AttendYes:Pass.testpass   11.599      5.679   2.042  0.04297 *
---
Residual standard error: 14.77 on 142 degrees of freedom
Multiple R-squared:  0.3878,Adjusted R-squared:  0.3749
F-statistic: 29.98 on 3 and 142 DF,  p-value: 4.452e-15
```

The *P*-value for interaction is the same as before.

Note also that the $R^2 = 39\%$ can be obtained from the ANOVA table above as follows: $R^2 = 100 \times \left(1 - \frac{30968}{30968+910+11077+7631}\right)$ is the proportion of variability that is explained by our model terms.

# Exam vs test success and attendance...

The formula for the above two-way ANOVA can be written as:

$$\texttt{Exam} = \beta_0 + \beta_1 \times \texttt{Attend}_{\texttt{Yes}} + \beta_2 \times \texttt{Pass.test}_{\texttt{pass}} + \beta_3 \times \texttt{Attend}_{\texttt{Yes}} \times \texttt{Pass.test}_{\texttt{pass}} + \varepsilon$$

where $\varepsilon \overset{iid}{\sim} N(0, \sigma^2)$.

This model is relative to the baseline levels of `Attend` and `Pass.test`. These baselines are "No" and "nopass", respectively, since they are the levels with the lowest alphanumeric value.

# Exam vs test success and attendance. . .

## Alternative parameterizations: the group means model

As seen in the previous chapter, another option is to remove the baseline with the addition of -1 in the model formula. One other complication is that we have to use : rather than * when specifying the interaction term.

```
> Exam.fitNoBaseline=lm(Exam~Attend:Pass.test-1,data=Stats20x.df)
> coef(summary(Exam.fitNoBaseline))
                          Estimate Std. Error  t value     Pr(>|t|)
AttendNo:Pass.testnopass  35.14286   3.222594 10.90515 1.707299e-20
AttendYes:Pass.testnopass 38.33333   3.222594 11.89518 4.515124e-23
AttendNo:Pass.testpass    48.16000   2.953556 16.30577 2.405316e-34
AttendYes:Pass.testpass   62.94937   1.661505 37.88696 4.012344e-76
```

This is the group means model. It is simply giving the estimated group mean for each of the four combinations of attendance and test success.

# Exam vs test success and attendance. . .

## Alternative parameterizations: the means and effects model

A further alternative is to use the means and effects model formula for the two-way ANOVA:

$$\texttt{Exam}_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}, \text{ where } \epsilon_{ijk} \overset{iid}{\sim} N(0, \sigma^2)$$

where:

- $\mu$ is the overall mean,
- $\alpha_i$s are the `Attend` effects relative to the overall mean,
- $\beta_j$s are the `Pass.test` effects relative to the overall mean,
- $\gamma_{ij}$s are the interaction effects between levels of `Attend` and `Pass.test` relative to the overall mean.

This is simply a different parameterization of the two-way ANOVA model. That is, there is no change to the ~~model, but just in the way we choose to write it~~.

In this course for Executive Summaries we will typically be interested in estimating relevant pairwise group differences, for which we will once again use `emmeans`.

# Section 12.4
# Interpretting the output using pairwise differences

# Exam vs test success and attendance

## Interpreting the output

In studies in which *all* of the explanatory variables are factors, our interest typically lies in making statistical inferences about the sizes of pairwise differences between the means of different treatment combinations.

We could calculate these pairwise differences from the above output, but what we really need is Tukey-adjusted confidence intervals for the differences. These are needed because of the multi-comparisons problem. Once again, we will use the `emmeans()` function to perform these calculations for us.

In the field of statistics, pairwise comparisons are often called *contrasts*[1] because they are comparing (i.e., contrasting) two different means.

[1] The term *contrasts* includes other more complicated forms of comparisons.

# Exam vs test success and attendance

## Pairwise comparisons

The `emmeans()` function creates a list with two components. We'll store the result in `exam.emm` so that we can print each object separately.

```
> library(emmeans)
> exam.emm <- emmeans(Exam.fit, specs = pairwise ~ Attend:Pass.test)
```

Here, `pairwise` $\sim$ `Attend:Pass.test` is an `R` formula that is specifying an interaction model between `Attend` and `Pass.test`. This is telling `emmeans` that we want pairwise comparisons for every treatment combination.

# Exam vs test success and attendance

## Pairwise comparisons...

The first component of `exam.emm`, named `emmeans`, contains four rows: one per treatment[2] combination of the levels of `Attend` and `Pass.test`.

```
> exam.emm$emmeans
 Attend Pass.test emmean   SE  df lower.CL upper.CL
 No     nopass      35.1 3.22 142     28.8     41.5
 Yes    nopass      38.3 3.22 142     32.0     44.7
 No     pass        48.2 2.95 142     42.3     54.0
 Yes    pass        62.9 1.66 142     59.7     66.2

Confidence level used: 0.95
```

This is equivalent to the output from the coefficient summary table for the `Exam.fitNoBaseline` model, with the addition that ~~(~~`lower.CL`~~)~~ and ~~(~~`upper.CL`~~)~~ give the 95% confidence interval for the treatment means.

[2] The word *treatment* is often used on its own to refer to a combination of the levels of two or more factor variables.

# Exam vs test success and attendance

## Pairwise comparisons...

The second component of `exam.emm` contains information about the six[3] possible pairwise comparisons between the four treatment combinations.

```
> exam.emm$contrasts
 contrast                  estimate   SE  df t.ratio p.value
 No nopass - Yes nopass       -3.19 4.56 142  -0.700  0.8969
 No nopass - No pass         -13.02 4.37 142  -2.978  0.0178
 No nopass - Yes pass        -27.81 3.63 142  -7.669  <.0001
 Yes nopass - No pass         -9.83 4.37 142  -2.248  0.1155
 Yes nopass - Yes pass       -24.62 3.63 142  -6.789  <.0001
 No pass - Yes pass          -14.79 3.39 142  -4.364  0.0001

P value adjustment: tukey method for comparing a family of 4 estimates
```

For example, the second row of the above `contrasts` table says that the *effect* (difference between the means) of the two levels of `Pass.test` conditional on the level of `Attend` = `No` is `-13.02`. So, for students who do not regularly attend lectures, those who pass the test will score an average of 13 points higher in the exam than those who fail.

[3] `choose(4,2)=6`

# Exam vs test success and attendance

## Pairwise comparisons...

For our Executive Summary we require *simultaneous* confidence intervals only for the simple contrasts[4] between means.

These are easily obtained by passing the `contrasts` object to the `confint()` function, i.e.

```
> confint(exam.emm$contrasts)[-c(3,4),]
                 contrast   estimate       SE  df  lower.CL   upper.CL
1 No nopass - Yes nopass  -3.190476 4.557436 142 -15.03851   8.657554
2    No nopass - No pass -13.017143 4.371339 142 -24.38137  -1.652912
5  Yes nopass - Yes pass -24.616034 3.625701 142 -34.04182 -15.190247
6     No pass - Yes pass -14.789367 3.388818 142 -23.59933  -5.979408
```

Note that we have excluded rows 3 and 4 of the `confint` table since they are not simple contrasts.

[4] Only those pairwise comparisons which involve conditioning on the level of one the two factors. That is, they share a common level of one of the factors.

# Exam vs test success and attendance...

## Statements for the Executive Summary

We interpret this output as follows (noting that the effect is always conditional on the level of the other factor):

- We estimate that for students who attend regularly, those who pass the test can expect to get 15 to 34 more marks in the exam than those who do not pass the test.
- For students who do not attend regularly, those who pass the test can expect to get 2 to 24 more marks in the exam than those who do not pass the test.
- For students who pass the test, those who regularly attend can expect to get between 6 and 24 more marks in the exam than those who do not attend regularly.
- And, for those who do not pass the test, those who regularly attend can expect to get between 9 marks less and 15 more marks than those who do attend regularly.

# Exam vs test success and attendance. . .

## Closing remarks

Recall that the data used in this example are from a single STATS 20x summer school class. The above statements are only relevant to the population of students who could have taken the course that summer.

Moreover, the data were collected pre-covid, so it is certainly the case that attendance at lectures would now not have anywhere near as much effect.

# Section 12.5
# Example 2: Using gender and attendance to explain exam score

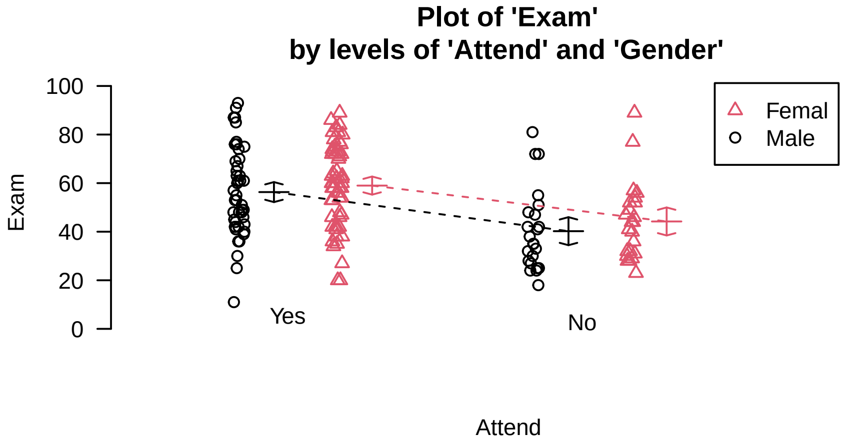
## Two-way ANOVA without interaction

# Exam score vs gender and attendance

Let us do another analysis where we will ask whether the effect of gender (on exam score) changes depending on whether the student attends regularly or not.

# Exam vs gender and attendance

```
> interactionPlots(Exam ~ Attend + Gender, data = Stats20x.df)
```



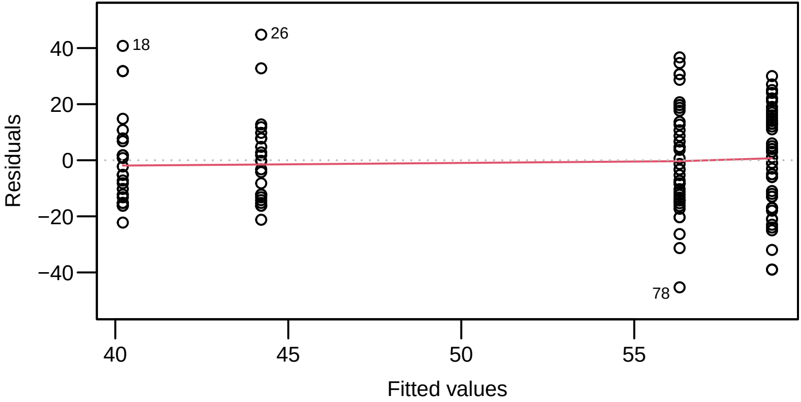

Not so much going on here as our plots are parallel. Looks like there is an effect of attendance (no surprise there), and the parallel lines suggests this effect is the same for both genders. There is little difference between genders.

# Exam vs gender and attendance...

## Assumption checks

Let us fit an interaction model and check the assumptions.

```
> Exam.fit2 = lm(Exam ~ Attend*Gender, data = Stats20x.df)
> plot(Exam.fit2, which=1)
```

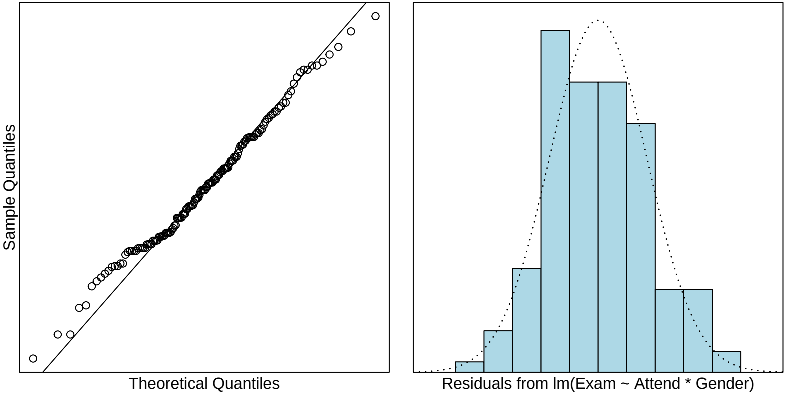



The EOV assumption seems to be okay.

# Exam vs gender and attendance...

## Assumption checks...

```
> normcheck(Exam.fit2)
```

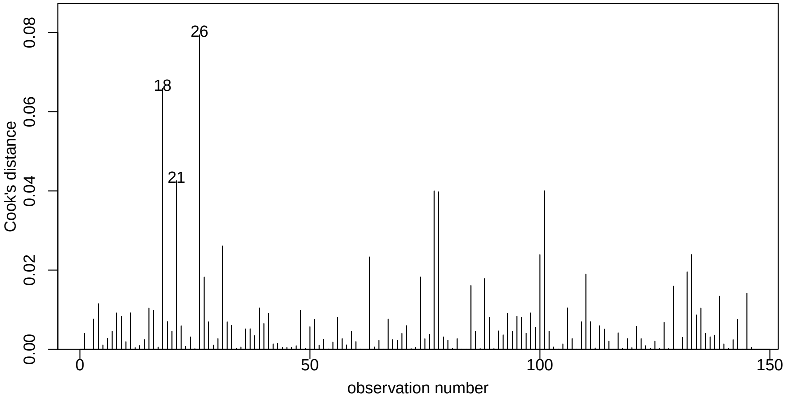



The normality assumption seems to be okay.

# Exam vs gender and attendance. . .

## Assumption checks. . .

```
> cooks20x(Exam.fit2)
```




No unduly influential data points.

# Exam vs gender and attendance...

We can trust the model. Lets see what it is telling us.

```
> anova(Exam.fit2)
Analysis of Variance Table

Response: Exam
              Df Sum Sq Mean Sq F value    Pr(>F)
Attend         1   7631  7630.8 25.4393 1.372e-06 ***
Gender         1    347   346.7  1.1557    0.2842
Attend:Gender  1     14    13.9  0.0463    0.8300
Residuals    142  42594   300.0
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

There is definitely no evidence of an interaction, so we'll apply Occam's razor[5] and fit a simpler additive model (i.e., no interaction term).

[5] "With all thing being equal, the simplest explanation tends to be the right one", William of Ockham, 1287-1347

# Exam vs gender and attendance...

## The additive model

Recall that we use + rather that ∗ in the model formula to fit the additive model.

```
> Exam.fit3 = lm(Exam ~ Attend + Gender, data = Stats20x.df)
> anova(Exam.fit3)
Analysis of Variance Table

Response: Exam
           Df Sum Sq Mean Sq F value    Pr(>F)
Attend      1   7631  7630.8 25.6101 1.264e-06 ***
Gender      1    347   346.7  1.1634    0.2826
Residuals 143  42608   298.0
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

We see that the gender is also not significant here[6], so we again apply Occam's razor and remove this term.

[6] If it were significant then we could look at the output from `emmeans(Exam.fit3, specs = pairwise ~ Attend)` and `emmeans(Exam.fit3, specs = pairwise ~ Gender)` – but this is pointless here since the factors only have two levels and there is only one pairwise comparison and so no multi-comparison issue.

# Exam vs gender and attendance. . .

Removal of the gender term reduces the model to one with just a factor with two levels. This is the two sample t-test scenario of Chapter 5.

```
> Exam.fit4=lm(Exam ~ Attend, data = Stats20x.df)
> summary(Exam.fit4)
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   42.217      2.547  16.578  < 2e-16 ***
AttendYes     15.563      3.077   5.058 1.27e-06 ***
---
Residual standard error: 17.27 on 144 degrees of freedom
Multiple R-squared:  0.1508,Adjusted R-squared:  0.145
F-statistic: 25.58 on 1 and 144 DF,  p-value: 1.271e-06
```

# Section 12.6
# Relevant R-code

# Most of the R-code you need for this chapter

You do not need to create indicator variables - R does that for you. The baseline can be changed if you wish rather than having R choose it for you – see relevant R-code from chapter 9.

Use interaction-plots to inspect the data. Non-parallel lines indicate that interaction may exist.

```
> interactionPlots(Exam ~ Pass.test + Attend, data = Stats20x.df)
```

Fit the interaction model (use the * in the model formula)

```
> Exam.fit = lm(Exam ~ Attend*Pass.test, data = Stats20x.df)
```

and use the ANOVA table to see if there is evidence of interaction.

```
> anova(ExamTestAttend.fit)
```

In the first example we had evidence of interaction (small *P*-value associated with `':'` part of the ANOVA output) and we inspect the pairwise interactions using `emmeans` to correct CIs for multi-comparisons:

```
> exam.emm=emmeans(Exam.fit, specs = pairwise ~ Attend:Gender)
> confint(exam.emm$contrasts)
```

# Most of the R-code you need for this chapter

If you don't have any evidence of interaction then simplify your model to an additive model and then see if the individual terms are significant.

The additive model replaces `*` with a `+` in the model formula.
(e.g. `Exam.fit3 = lm(Exam   Attend + Gender, data = Stats20x.df)`)

If both variables are significant in the additive model then use this model for inference (this model is referred to here as `additive.fit`)

```
> exam.emm=emmeans(additive.fit, specs = variable1)
> confint(exam.emm$contrasts)
```

```
> exam.emm=emmeans(additive.fit, specs = variable2)
> confint(exam.emm$contrasts)
```

Otherwise, delete non-significant variables until you have the simplest model possible.

# Section 12.7
# Alternative parameterizations of the two-way ANOVA model
# (Optional section)

# Alternative parameterizations of two-way ANOVA

## The reference cell model

Recall the reference cell model we used to represent `Exam` score:

$$\texttt{Exam} = \beta_0 + \beta_1 \times \texttt{Attend}_{\texttt{Yes}} + \beta_2 \times \texttt{Pass.test}_{\texttt{pass}} + \beta_3 \times \texttt{Attend}_{\texttt{Yes}} \times \texttt{Pass.test}_{\texttt{pass}} + \varepsilon,$$

where $\varepsilon \overset{iid}{\sim} N(0, \sigma^2)$.

The parameter $\beta_0$ denotes the overall true baseline mean exam score. Notice that neither the `no` level of `Attend` nor the `nopass` level of `Pass.test` appear as subscripts in the above model. This tells us that these are the baseline levels, i.e. $\beta_0$ denotes the mean over `Exam` scores from students who neither regularly attended lectures nor passed the test.

So, what do the parameters $\beta_1, \beta_2,$ and $\beta_3$ represent? To help us answer this question we consider the means model[7] formulation for `Exam` score.

[7] We first encountered the means model for the single factor male fruitflies study in Chapter 11.

# Alternative parameterizations of two-way ANOVA

## The means model

The means model parameterization for exam score is

$$\texttt{Exam}_{ijk} = \mu_{ij} + \varepsilon_{ijk},$$

where $\mu_{ij}$ denotes the true mean exam score of 20x students who are in the $i$th level of `Attend` and $j$th level of `Pass.test` ($i = \texttt{no}$ or `yes`; $j = \texttt{nopass}$ or `pass`). The error term $\varepsilon_{ijk} \overset{iid}{\sim} N(0, \sigma^2)$ denotes the deviation of the $k$th student's exam score from the mean exam score, $\mu_{ij}$.

But, how can we use $\mu_{ij}$ to assess whether one or both of `Attend` and `Pass.test` have an effect on `Exam` score?

# Alternative parameterizations of two-way ANOVA

## Relating the means and reference cell models

We decompose each mean response, $\mu_{ij}$, into four terms:

1. $\mu_{11}$, the baseline or reference-level mean response;
2. $\mu_{i1} - \mu_{11}$, the *main effect*[8] of the $i$th level of the first factor, where $i$ does not equal the baseline level;
3. $\mu_{1j} - \mu_{11}$, the *main effect* of the $j$th level of the second factor, where $j$ does not equal the baseline level;
4. Interaction, the part of $\mu_{ij}$ that is left over after eliminating the contributing components defined by terms 1–3 above, i.e.

$$\begin{aligned} \text{Interaction} &= \mu_{ij} - \mu_{11} - (\mu_{i1} - \mu_{11}) - (\mu_{1j} - \mu_{11}) \\ &= \mu_{ij} - \mu_{i1} - \mu_{1j} + \mu_{11} \end{aligned}$$

[8] A main effect is defined as the difference between the mean response when all factors, except the one of interest, are at the baseline level and the reference-level mean.

# Alternative parameterizations of two-way ANOVA

## Relating the means and reference cell models

We now have the tools to re-express the mean `Exam` score in terms of the main effects of the $i$th level of `Attend` and the $j$th level of `Pass.test`, and their interaction, i.e.

$$\mu_{ij} = \mu_{11} + (\mu_{1j} - \mu_{11}) + (\mu_{i1} - \mu_{11}) + (\mu_{ij} - \mu_{i1} - \mu_{1j} + \mu_{11}).$$

The following two-way table[9] illustrates how each term in the above decomposition relates to each combination of the levels of `Attend` and `Pass.test`:

| Attend | Pass.test | |
|---|---|---|
| | `nopass` | `pass` |
| `no` | $\mu_{11}$ | $\mu_{i1} - \mu_{11}$ |
| `yes` | $\mu_{1j} - \mu_{11}$ | $\mu_{ij} - \mu_{11} - (\mu_{i1} - \mu_{11}) - (\mu_{1j} - \mu_{11})$ |

[9] More generally, differences in the first row of a two-way reference model decomposition table correspond to the main effects of the column factor. The differences in the first column correspond to the row factor main effects. The terms in each of the of the remaining cells, except the reference cell, correspond to interaction effects.

# Alternative parameterizations of two-way ANOVA

## Relating the means and reference cell models

| Factors | | Parameterization | | | |
|---|---|---|---|---|---|
| `Attend` | `Pass.test` | Means | Estimate[10] | Reference cell | Estimate[11] |
| no | nopass | $\mu_{11}$ | 35.1 | $\beta_0 = \mu_{11}$ | 35.1 |
| yes | nopass | $\mu_{21}$ | 38.3 | $\beta_1 = \mu_{21} - \mu_{11}$ | 3.2 |
| no | pass | $\mu_{12}$ | 48.2 | $\beta_2 = \mu_{12} - \mu_{11}$ | 13.1 |
| yes | pass | $\mu_{22}$ | 62.9 | $\beta_3 = \mu_{22} - \mu_{21} - \mu_{12} + \mu_{11}$ | 11.5 |

From the above table we see that:

- $\beta_1$ represents the effect of `Attend = yes` at the reference level of `Pass.test = nopass`
- $\beta_2$ represents the effect of `Pass.test = pass` at the reference level of `Attend = no`
- $\beta_3$ represents the `Attend` *x* `Pass.test` interaction effect when `Attend = yes` and `Pass.test = pass`

[10] See estimates of `Attend`×`Pass.test` treatment means on slide 24.

[11] See regression coefficients table on slide 17.

# Alternative parameterizations of two-way ANOVA

## The reference cell model

The values in the `Estimate` column of the regression summary table[12] result in the following equation for predicted longevity:

$$\begin{aligned}\widehat{\texttt{Exam}} = 35.14 + 3.19 \times \texttt{Attend}_{\texttt{Yes}} + 13.02 \times \texttt{Pass.test}_{\texttt{pass}} \\ + 11.60 \times \texttt{Attend}_{\texttt{Yes}} \times \texttt{Pass.test}_{\texttt{pass}}\end{aligned}$$

[12] See slide 17; Coefficients rounded to 2 decimal places.

# Alternative parameterizations of two-way ANOVA

## Relating the means and effects models

We saw in Chapter 11 that the effects model offers an alternative to the reference model parameterization.[13] To relate the means and effects models we use an alternative decomposition of each mean response, $\mu_{ij}$, into:

1. $\mu = \mu_{..}$, the reference overall mean response;
2. $\alpha_i = \mu_{i.} - \mu$, the *main effect* of the $i$th level of the first factor[14];
3. $\pi_j = \mu_{.j} - \mu$, the *main effect* of the $j$th level of the second factor[14];
4. $(\alpha\tau)_{ij}$, the interaction effect reflecting the component of $\mu_{ij}$ left over after eliminating the components corresponding to the terms defined in 1–3 above, i.e. $(\alpha\tau)_{ij} = \mu_{ij} - \mu - \alpha_i - \pi_j$.

[13] The effects model is the default parameterization used in the analysis of data from designed experiments and, therefore, in STATS 240 (Design and Structured Data).

[14] The distance between the mean response of the first (second) factor at the $i$th ($j$th) level and the overall mean.

# Alternative parameterizations of two-way ANOVA

## The effects model

It directly follows from point 4 above that we can re-express the mean `Exam` score, $\mu_{ij}$, in terms of $\alpha_i$, the effect of the $i$th level of `Attend`, $\pi_j$, the effect of the $j$th level of `Pass.test`, and their interaction, $(\alpha\pi)_{ij}$, i.e.

$$\mu_{ij} = \mu + \alpha_i + \pi_j + (\alpha\pi)_{ij}.$$

This means that the effects model formula for the two-way ANOVA is

$$\texttt{Exam}_{ijk} = \mu + \alpha_i + \pi_j + (\alpha\pi)_{ij} + \epsilon_{ijk},$$

where $\epsilon_{ijk} \overset{iid}{\sim} N(0, \sigma^2)$.

# Alternative parameterizations of two-way ANOVA

## Relating the means and effects models

The following two-way layout illustrates how the above decomposition of $\mu_{ij}$ relates to each combination of the levels of `Attend` and `Pass.test`:

| Attend | Pass.test: `nopass` | Pass.test: `pass` | Row mean | $\alpha_i$, Row effect |
|---|---|---|---|---|
| `no` | $\mu_{11} = 35.1$ | $\mu_{12} = 48.2$ | $\mu_{1\cdot} = 41.7$ | |
| `yes` | $\mu_{21} = 38.3$ | $\mu_{22} = 62.9$ | $\mu_{2\cdot} = 50.6$ | |
| Column mean | $\mu_{\cdot 1} = 36.7$ | $\mu_{\cdot 2} = 55.6$ | $\mu = 52.9$ | |
| $\pi_i$, Column effect | | | | |

```
> emmeans(Exam.fit, specs = pairwise ~ Attend)$emmeans
NOTE: Results may be misleading due to involvement in interactions
 Attend emmean   SE  df lower.CL upper.CL
 No       41.7 2.19 142     37.3     46.0
 Yes      50.6 1.81 142     47.1     54.2

Results are averaged over the levels of: Pass.test
Confidence level used: 0.95
```

# Alternative parameterizations of two-way ANOVA

## Relating the means and effects models

| Factors | | Parameterization | | | |
|---|---|---|---|---|---|
| `Attend` | `Pass.test` | Means | Estimate[15] | Reference cell | Estimate[16] |
| no | nopass | $\mu_{11}$ | 35.1 | $\beta_0 = \mu_{11}$ | 35.1 |
| yes | nopass | $\mu_{21}$ | 38.3 | $\beta_1 = \mu_{21} - \mu_{11}$ | 3.2 |
| no | pass | $\mu_{12}$ | 48.2 | $\beta_2 = \mu_{12} - \mu_{11}$ | 13.1 |
| yes | pass | $\mu_{22}$ | 62.9 | $\beta_3 = \mu_{22} - \mu_{21} - \mu_{12} + \mu_{11}$ | 11.5 |

From the above table we see that:

- $\beta_1$ represents the effect of `Attend = yes` at the reference level of `Pass.test = nopass`
- $\beta_2$ represents the effect of `Pass.test = pass` at the reference level of `Attend = no`
- $\beta_3$ represents the `Attend` *x* `Pass.test` interaction effect when `Attend = yes` and `Pass.test = pass`

[15] See estimates of `Attend`×`Pass.test` treatment means on slide 24.

[16] See regression coefficients table on slide 17.