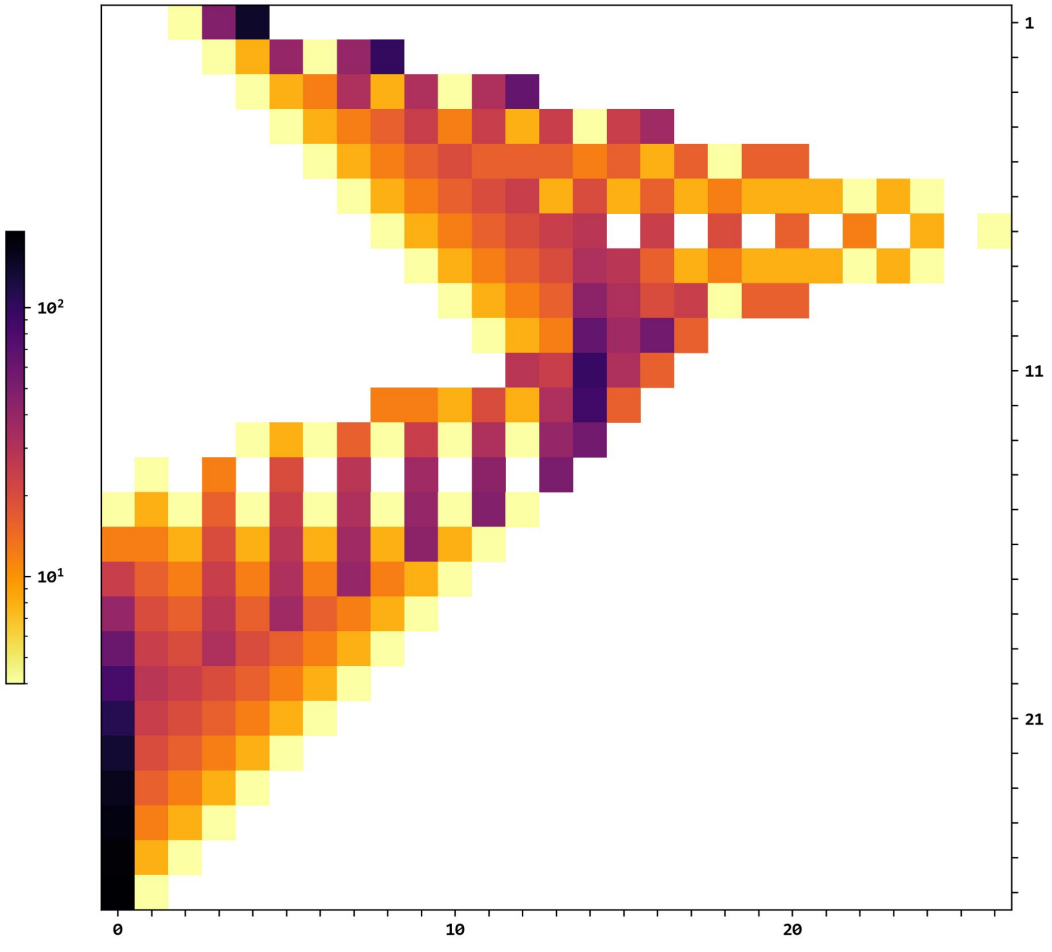
What is a B-Matrix?

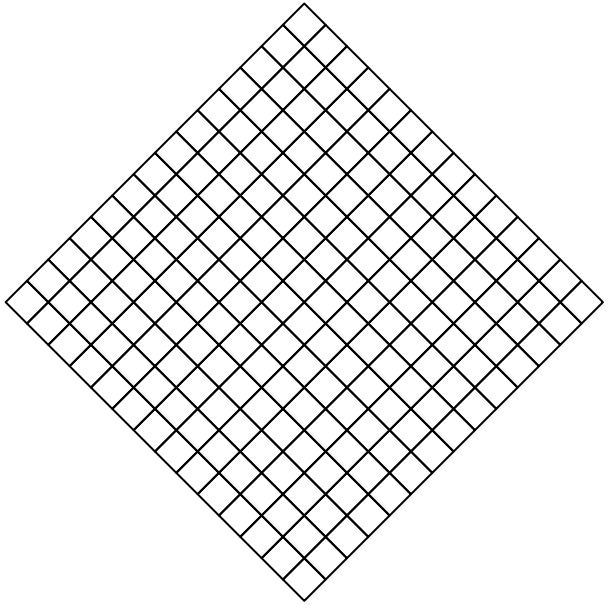
Also known as a network portrait, it is a graph invariant data abstraction about the structural properties of a graph.

It is used in portrait divergence as a metric of similarity between networks.



Bagrow, J. P., Bollt, E. M., Skufca, J. D., & Ben-Avraham, D. (2008). Portraits of complex networks. EPL (Europhysics Letters).

Graph	Square Graph (14x14)
Nodes	196
Edges	364



This grid has the B-Matrix on the right

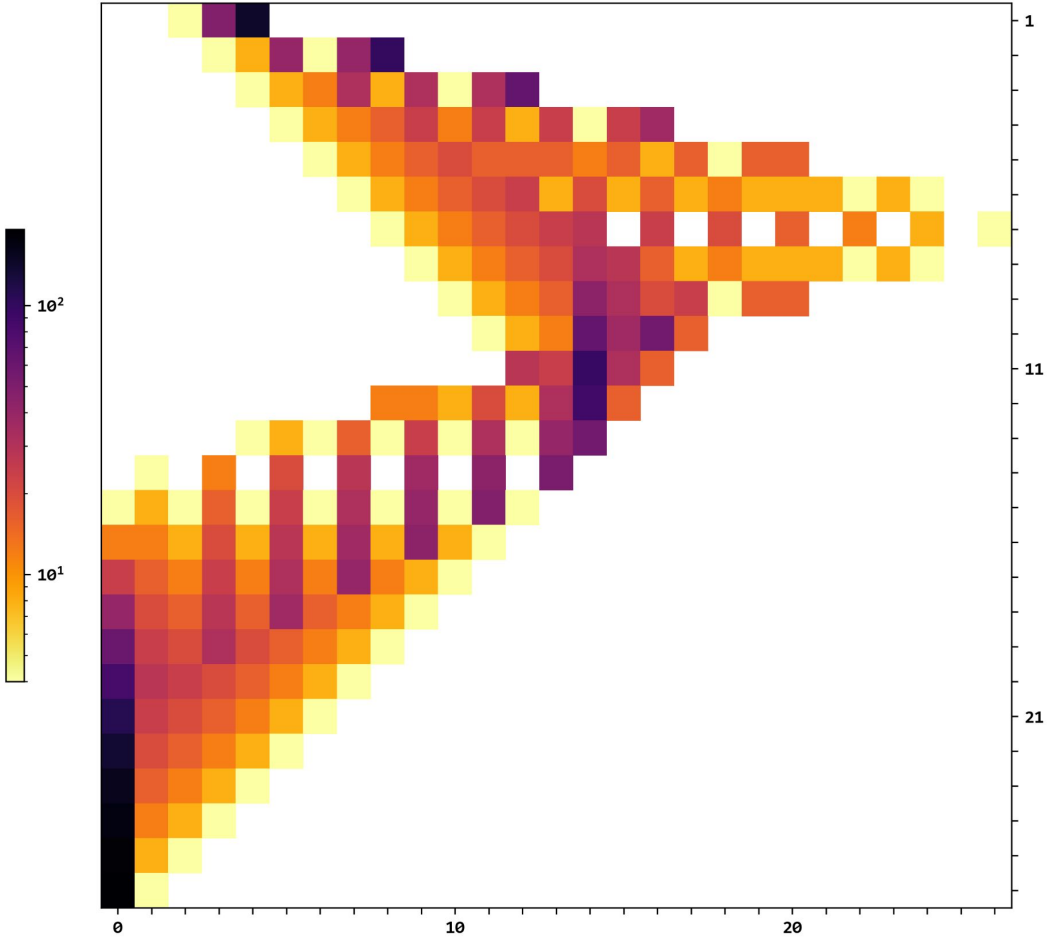
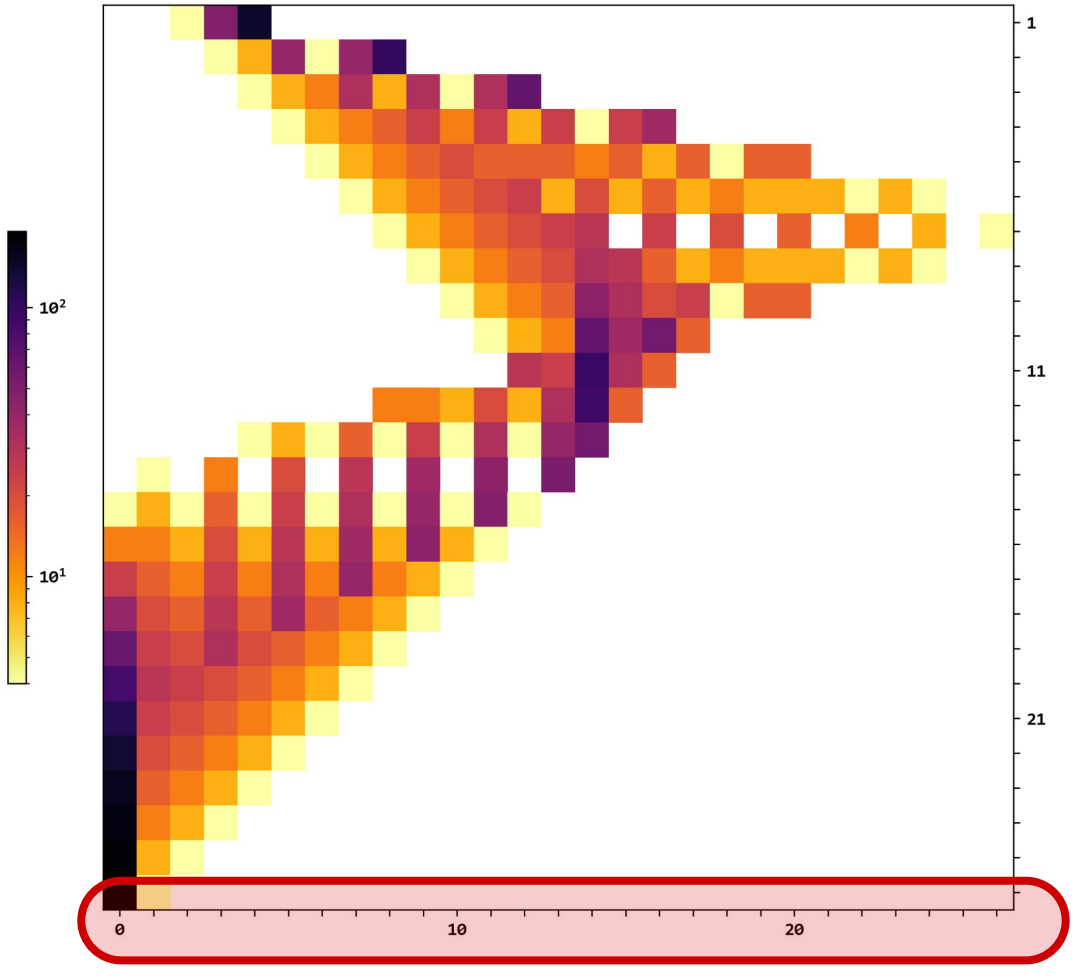


Figure 1 in Bagrow, J. P., & Bollt, E. M. (2019). An information-theoretic, all-scales approach to comparing networks. Applied Network Science.

The horizontal axis is for number of nodes reached by the algorithm at a given step.

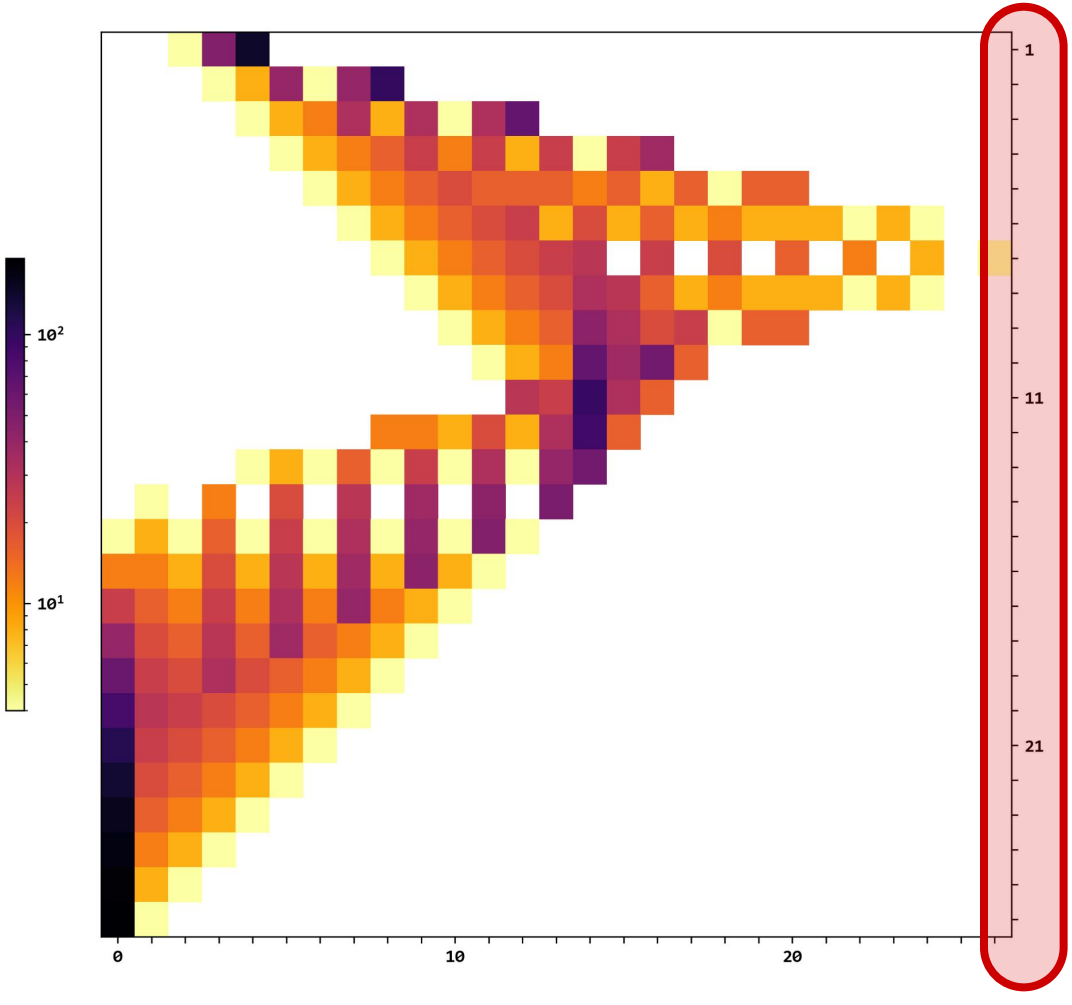
Has index-zero and starts counting at zeroth column, with +1 integer steps.



The vertical axis is the number of hops the algorithm run through.

Has index-zero and starts counting at zeroth row, with +1 integer steps.

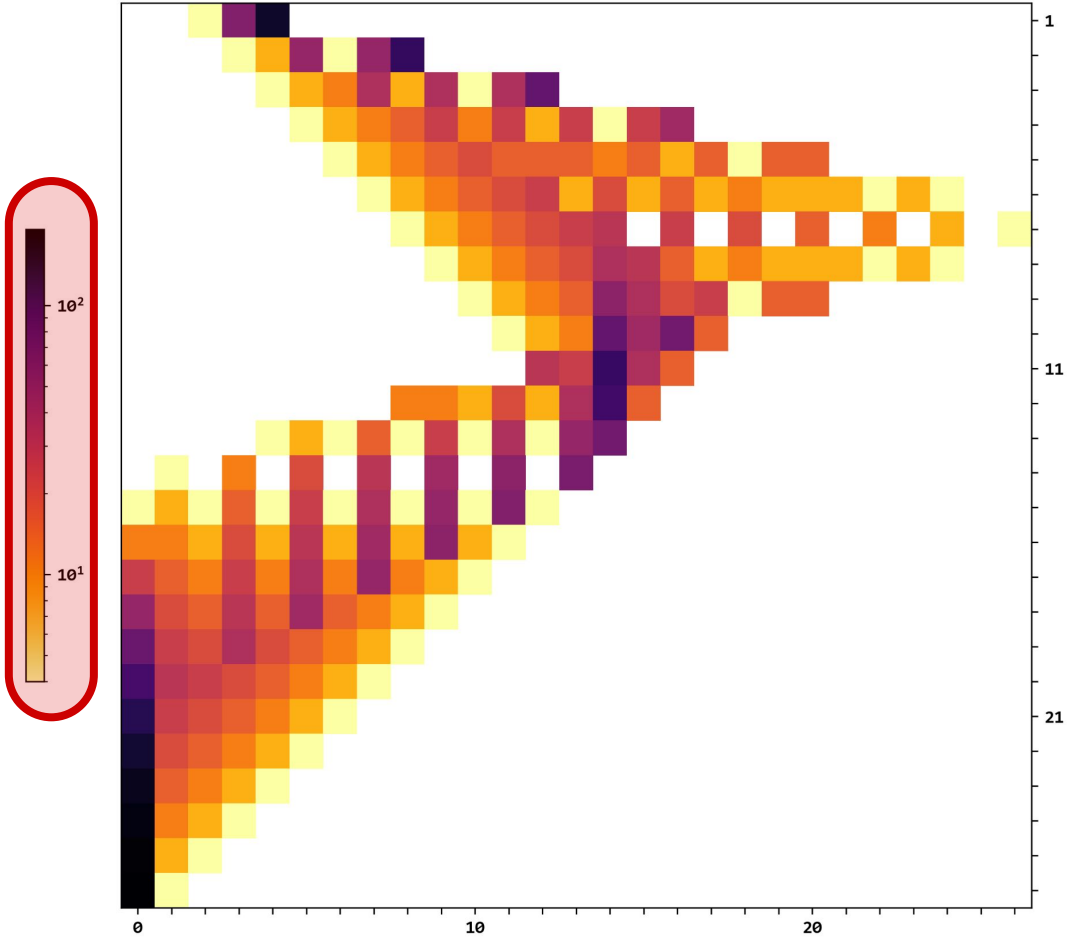
The zeroth row can be omitted in visualizations. The reason why is covered later in this explainer.



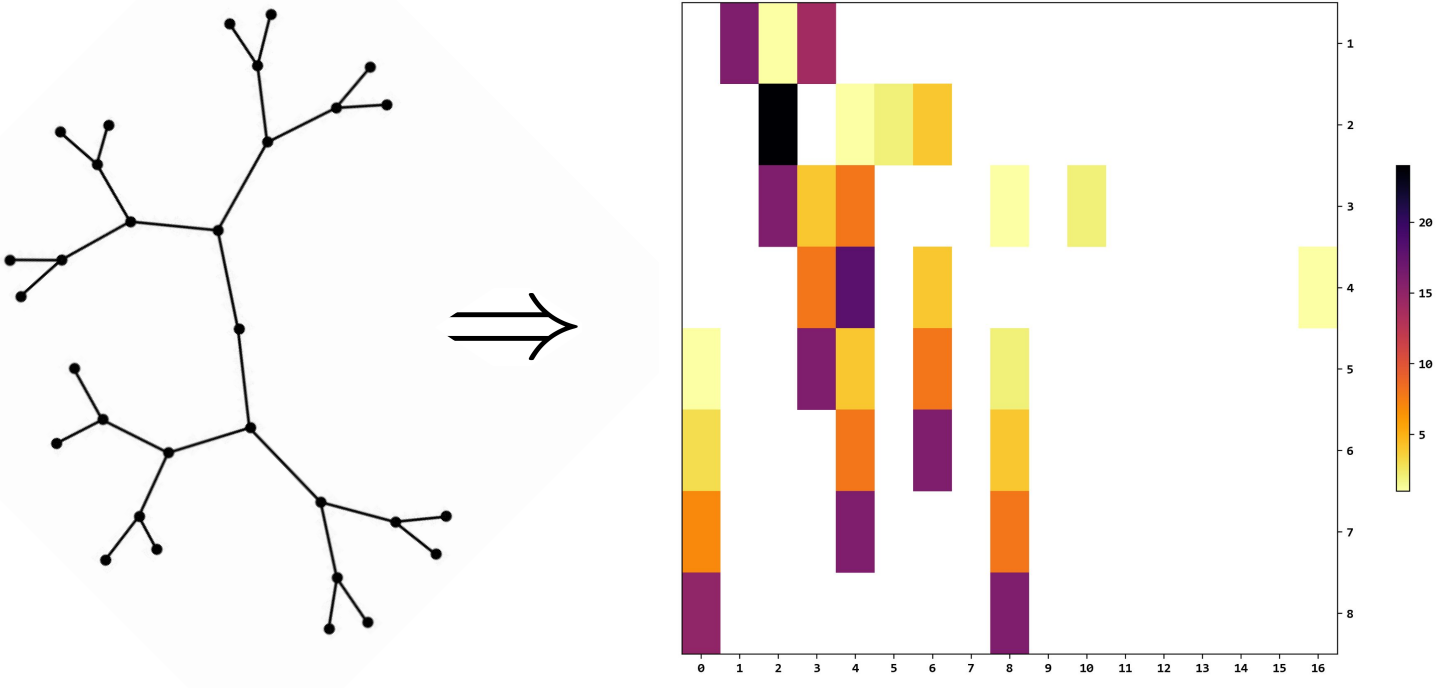
In this color-based B-Matrix portrait the colormap encodes total node count per cell.

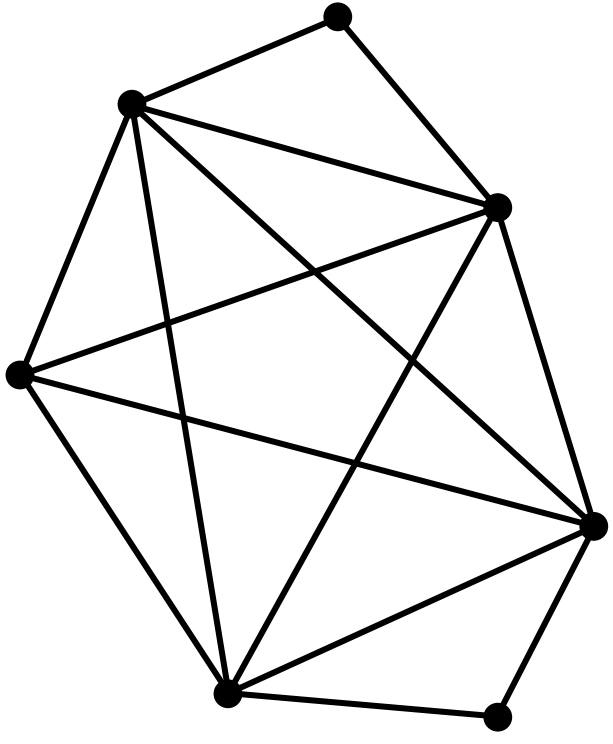
Starts counting at 1, with +1 integer steps, from a default of 0. At the end of the algorithm, cells at 0 become to NaN.

Because the difference between the smallest and largest values in this matrix is larger than 100, this colormap is log-transformed.



But how exactly is a B-Matrix created?





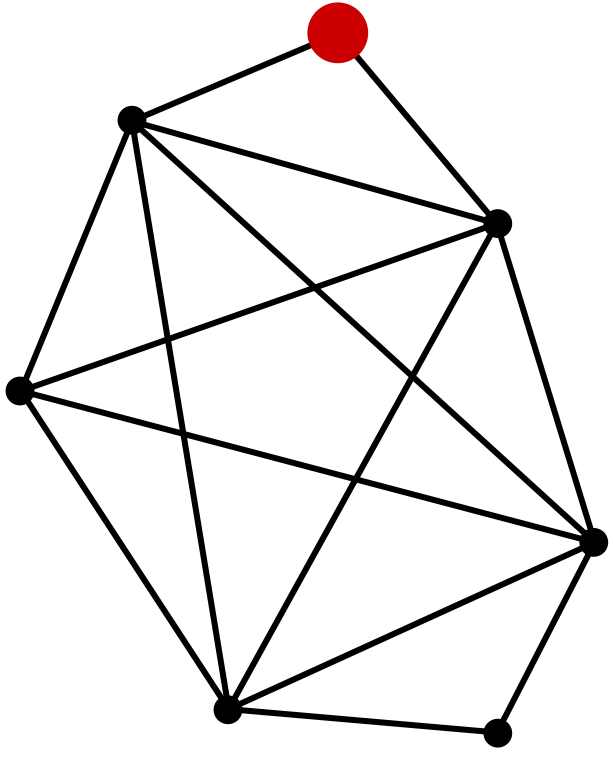
Consider this small graph as an explainer example of how `algorithm_BMatrix.py` works.

- It has 7 nodes and 14 edges
  - Edgelist:

```
1-2
1-3
2-4
2-5
2-6
2-3
3-4
3-5
3-6
4-5
4-6
5-7
5-6
6-7
```

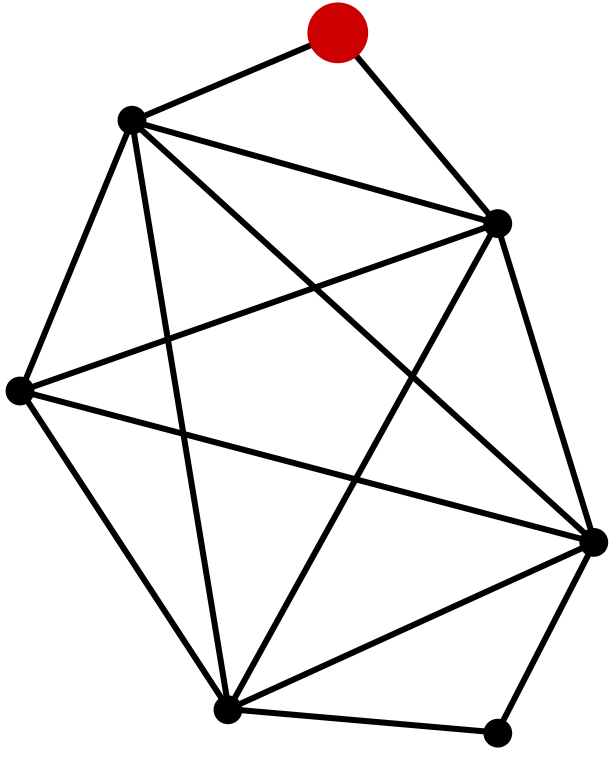
- Snippet:

```
G = nx.graph_atlas(1115)
B_matrix = algorithm_BMatrix(G)
```



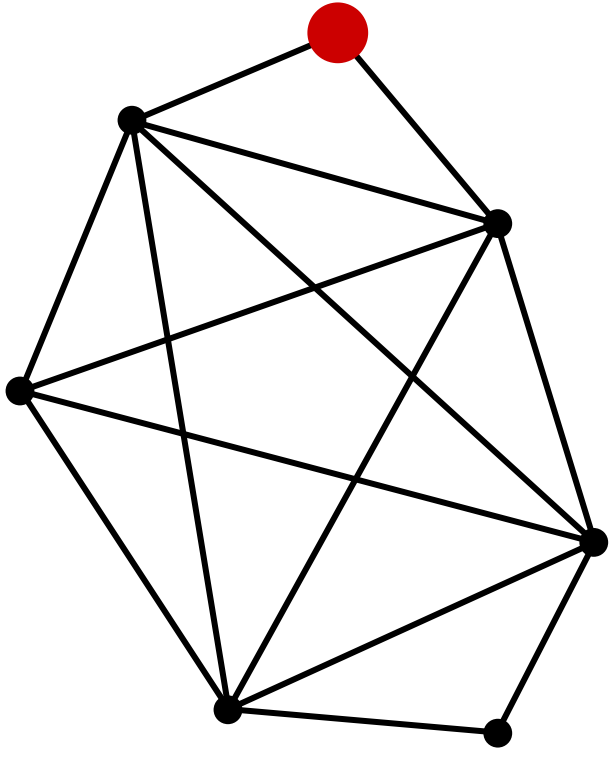
We start by picking a node, and initialize an empty matrix.

0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0



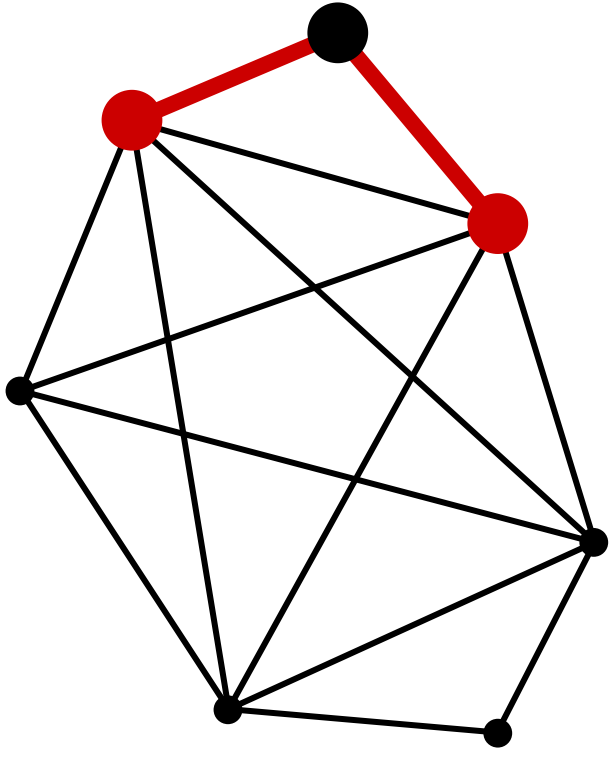
Rows are for number of hops away from this starting node, and columns are for node counts.

0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0



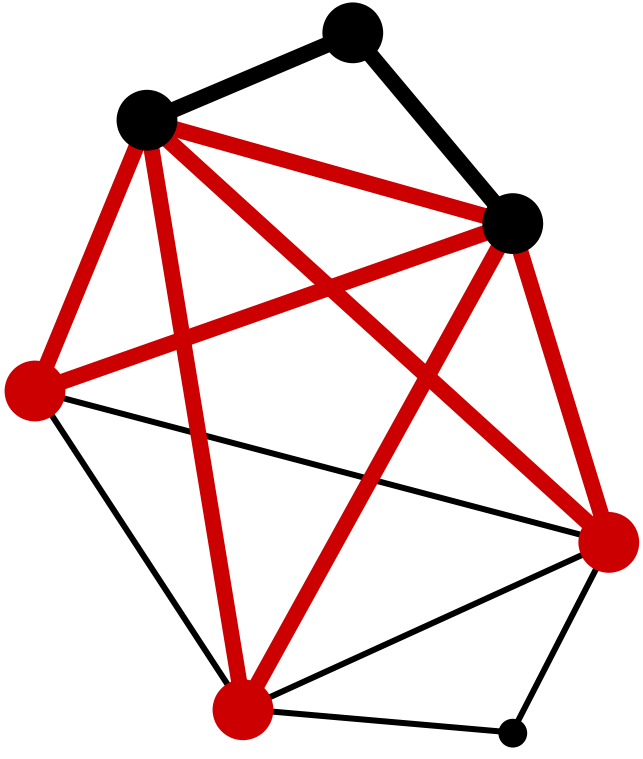
We only have one node at hand (no hops yet),
so we **flip the bit** at zeroth row and first column.

0	1	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0



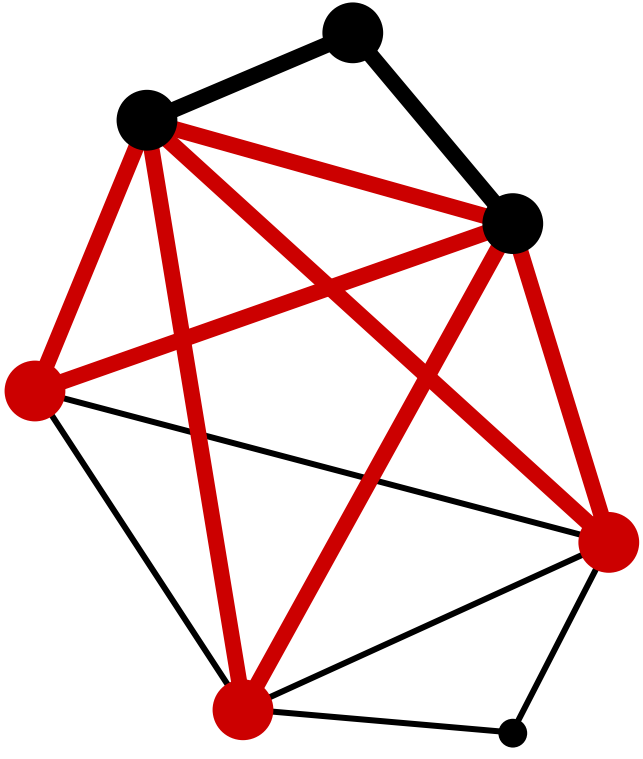
Then at first hop, there are two nodes - so we flip the bit at the first row and second column.

0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0



At second hop there are three nodes, so we flip the bit at the second row and third column.

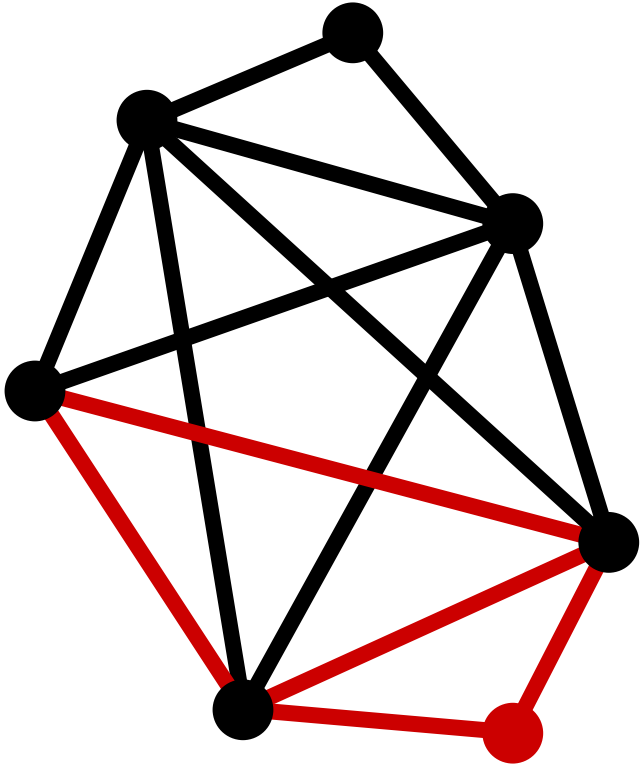
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	0	0



(in literature these hops are known as an *L-shell*, where *L* is the hop number).

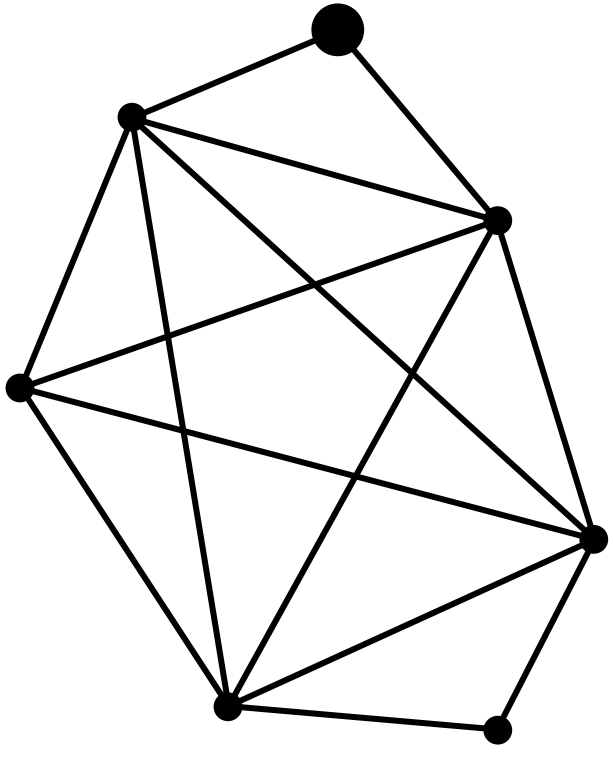
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	0	0

Bagrow, J. P., Bollt, E. M., Skufca, J. D., & Ben-Avraham, D. (2008).
Portraits of complex networks. EPL (Europhysics Letters), 81(6), 68004.



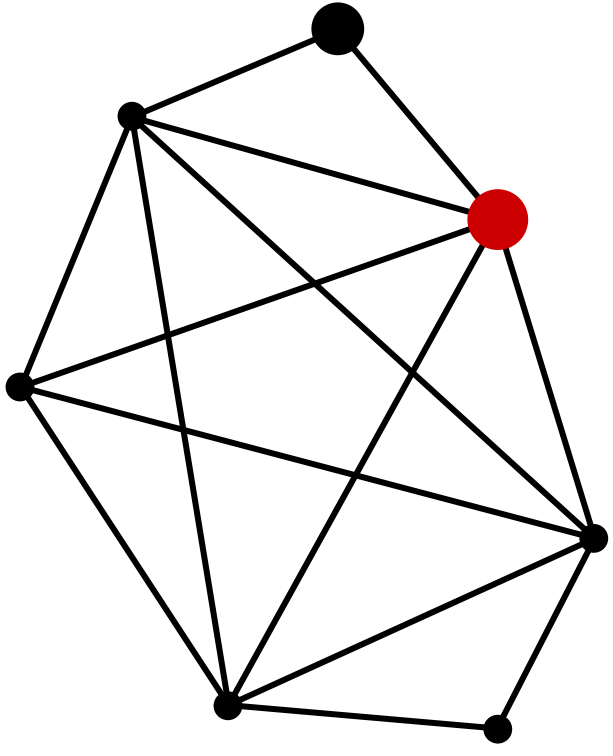
At third hop there is one node (last one), so we flip the bit at the third row and first column.

0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	1	0	0	0	0



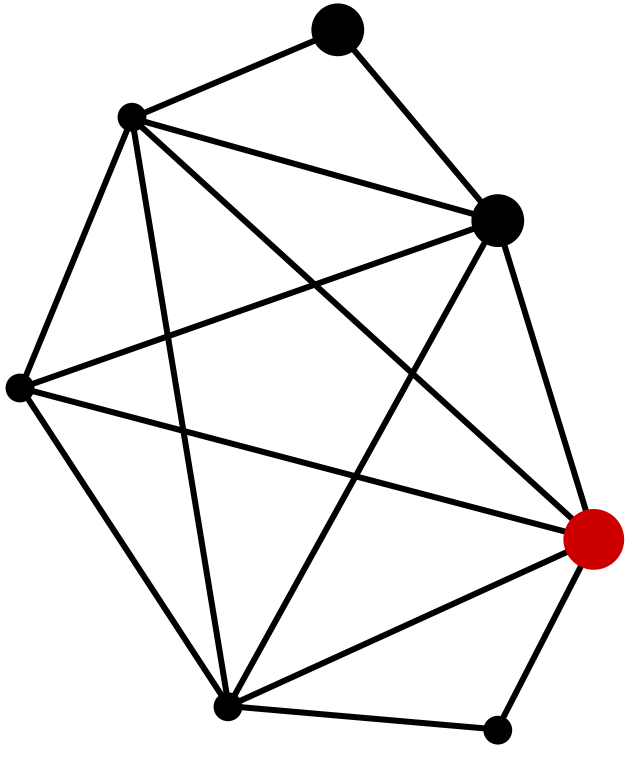
This completes the bit matrix of node 1 (of 7).

0	**1**	0	0	0	0
0	0	**1**	0	0	0
0	0	0	**1**	0	0
0	**1**	0	0	0	0



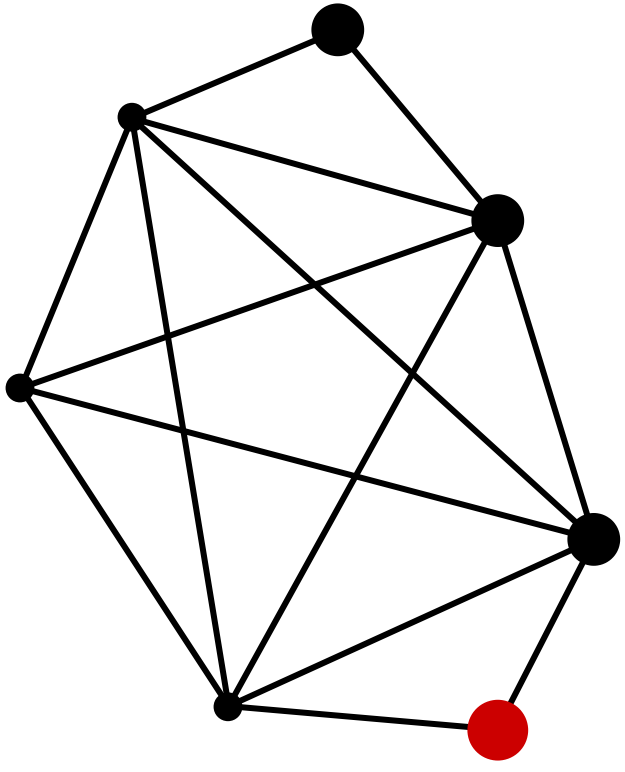
We repeat and get a bit matrix for node 2 (of 7), and we store the already completed matrix.

0	1	0	0	0	0
0	0	0	0	0	1
0	1	0	0	0	0
1	0	0	0	0	0



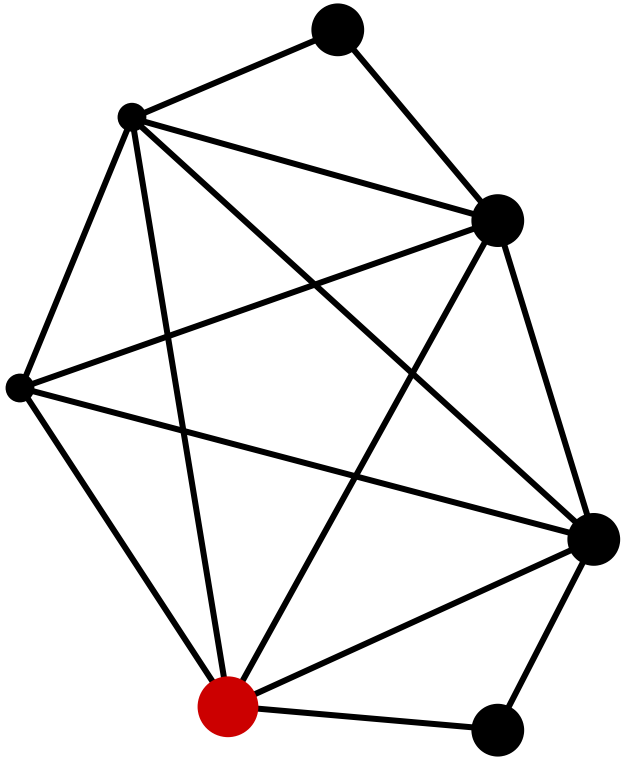
This is the bit matrix of node 3 (of 7), and we save the two already completed matrices.

0	1	0	0	0	0
0	0	0	0	0	1
0	1	0	0	0	0
1	0	0	0	0	0



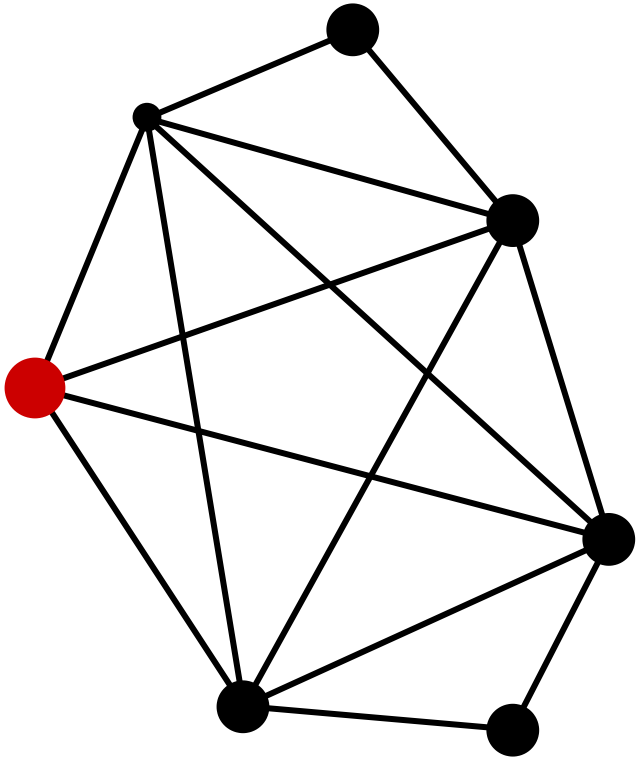
The bit matrix of node 4 (of 7), and we save the three already completed matrices.

0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	1	0	0	0	0



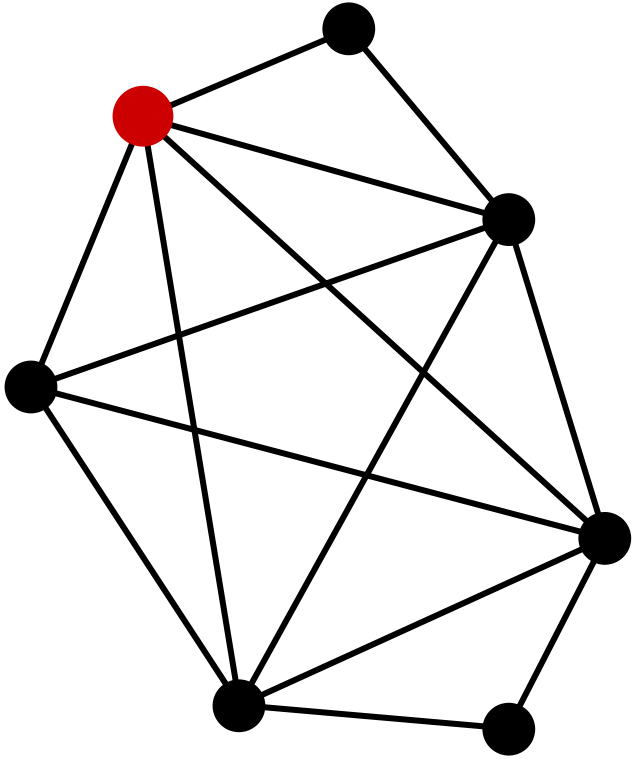
The bit matrix of node 5 (of 7), and we save the four already completed matrices.

0	1	0	0	0	0
0	0	0	0	0	1
0	1	0	0	0	0
1	0	0	0	0	0



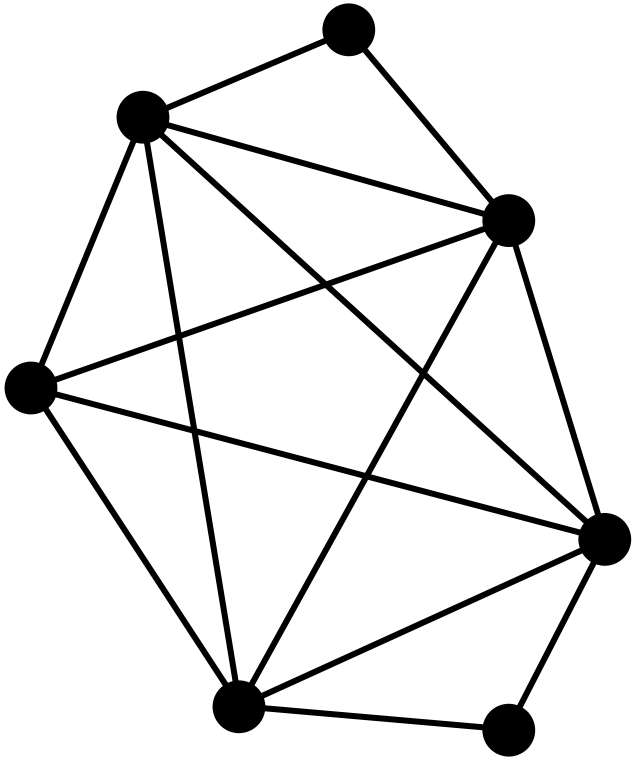
The bit matrix of node 6 (of 7), and we save the five already completed matrices.

0	1	0	0	0	0
0	0	0	0	1	0
0	0	1	0	0	0
1	0	0	0	0	0



Finally, the bit matrix of node 7 (of 7), and we save the six already completed matrices.

0	1	0	0	0	0
0	0	0	0	0	1
0	1	0	0	0	0
1	0	0	0	0	0



We add these seven bit matrices element-wise into a single matrix. This ends the algorithm.

0	7	0	0	0	0
0	0	2	0	1	4
0	4	1	2	0	0
5	2	0	0	0	0

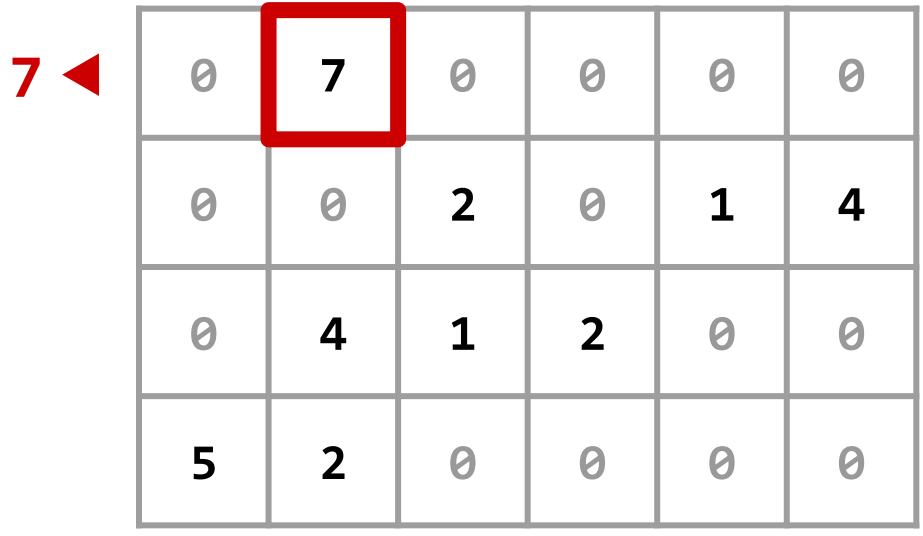
We're done!

The B-Matrix of the graph.

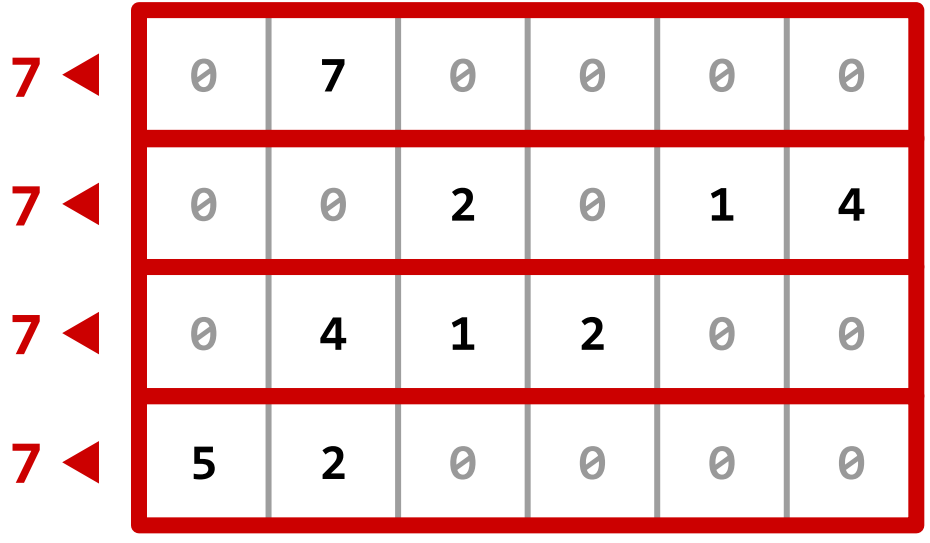
And it has a bunch of properties.

0	7	0	0	0	0
0	0	2	0	1	4
0	4	1	2	0	0
5	2	0	0	0	0

For example, the zeroth row always has total node count in the first column position.

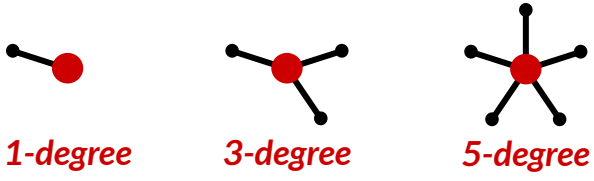


Also each row adds up to the **total number of nodes**.



The first row marks the number of times different node counts happened at exactly one hop away from the starting node.

The node degree is how many edges a node has, so this row is effectively a record of frequency of node degrees.

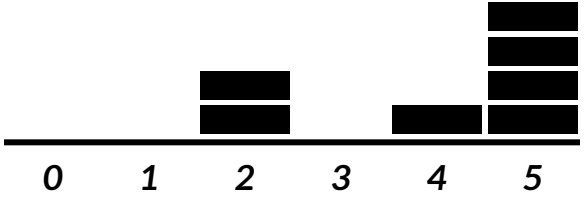


1-degree *3-degree* *5-degree*

2-degree *4-degree*

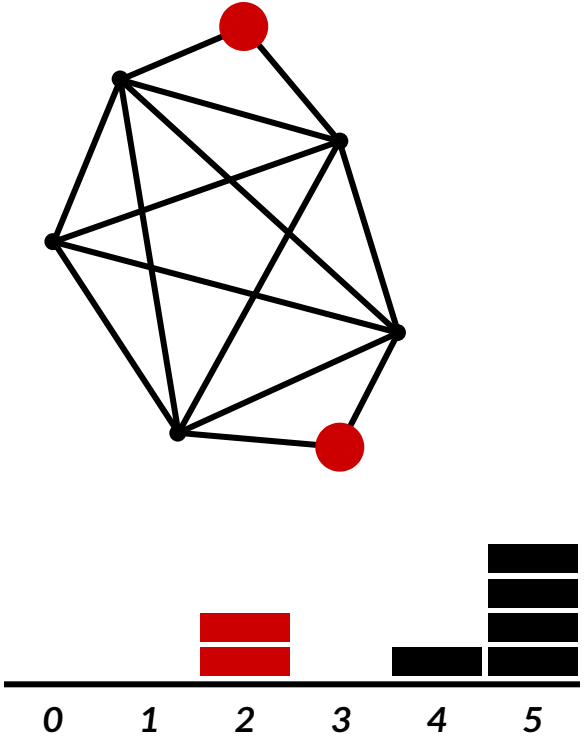
0	7	0	0	0	0
0	0	2	0	1	4
0	4	1	2	0	0
5	2	0	0	0	0
	1	2	3	4	5

We can visualize the distribution of node degrees with a bar chart of counts, or *histogram*.



Frequency of node degrees

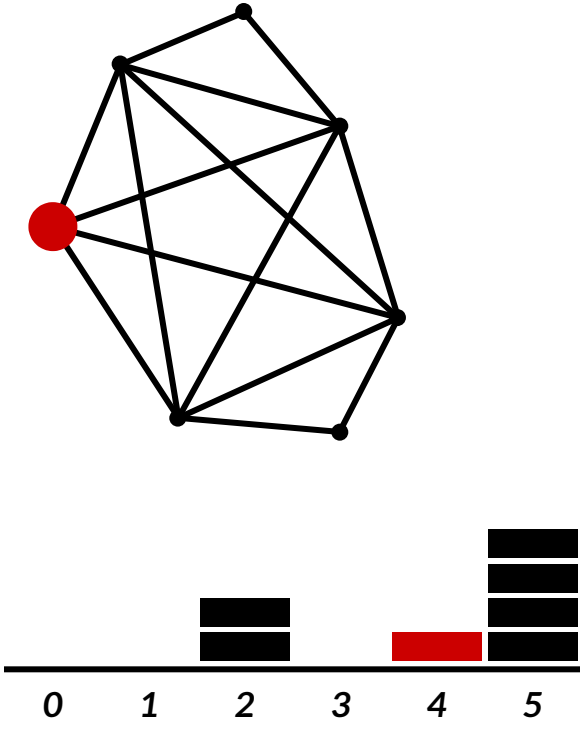
0	7	0	0	0	0
0	0	2	0	1	4
0	4	1	2	0	0
5	2	0	0	0	0



Frequency of node degrees

[Row 1, Column 2] Two nodes of degree 2

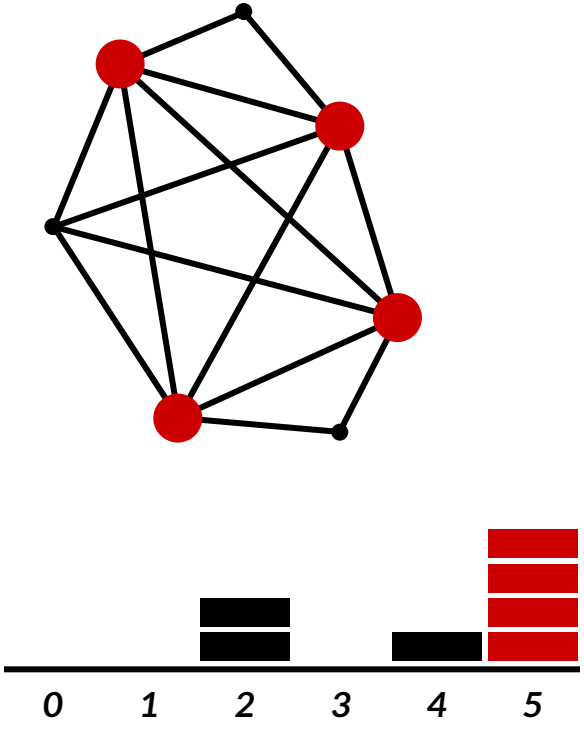
0	7	0	0	0	0
0	0	2	0	1	4
0	4	1	2	0	0
5	2	0	0	0	0



Frequency of node degrees

[Row 1, Column 4] One node of degree 4

0	7	0	0	0	0
0	0	2	0	1	4
0	4	1	2	0	0
5	2	0	0	0	0

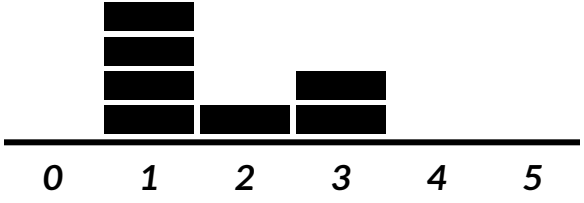


Frequency of node degrees

[Row 1, Column 5] Four nodes of degree 5

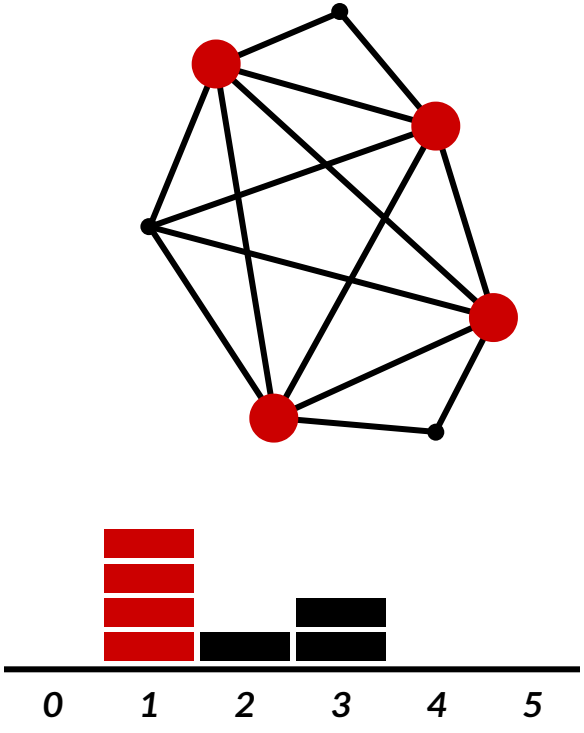
0	7	0	0	0	0
0	0	2	0	1	4
0	4	1	2	0	0
5	2	0	0	0	0

The next row is the distribution of nodes with *2-shell* degrees.



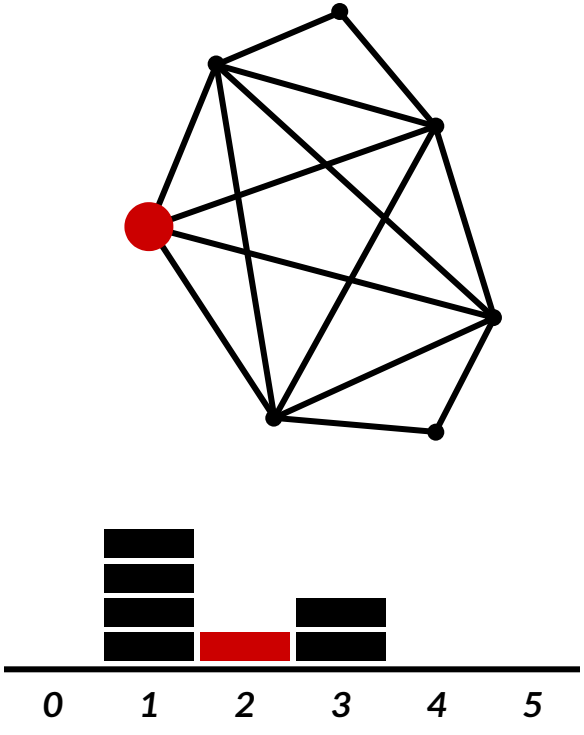
Frequency of node 2-shell degrees

0	7	0	0	0	0
0	0	2	0	1	4
0	4	1	2	0	0
5	2	0	0	0	0



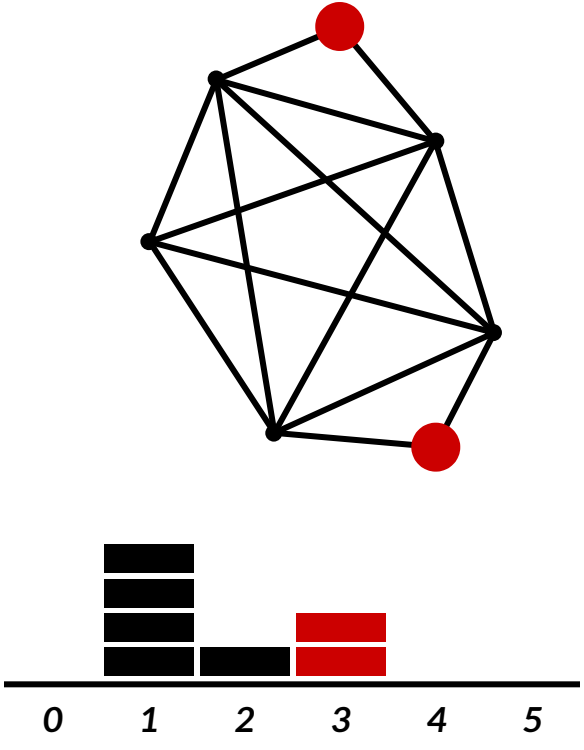
Frequency of node 2-shell degrees

0	7	0	0	0	0
0	0	2	0	1	4
0	4	1	2	0	0
5	2	0	0	0	0



Frequency of node 2-shell degrees

0	7	0	0	0	0
0	0	2	0	1	4
0	4	1	2	0	0
5	2	0	0	0	0

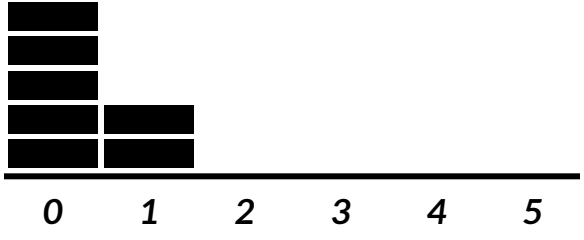


Frequency of node 2-shell degrees

0	7	0	0	0	0
0	0	2	0	1	4
0	4	1	2	0	0
5	2	0	0	0	0

Finally, the last row is the maximum number of hops the algorithm got through before running out of nodes.

In other words, the final hop number is equivalent to the diameter of the graph, *L-shell* of 3 in this case.

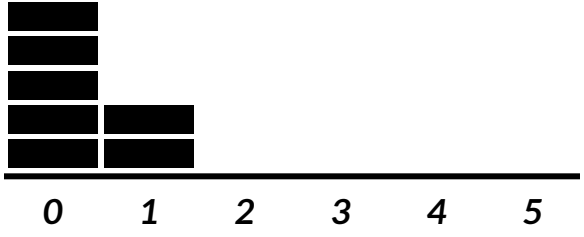


Frequency of node 3-shell degrees

0	7	0	0	0	0
0	0	2	0	1	4
0	4	1	2	0	0
5	2	0	0	0	0

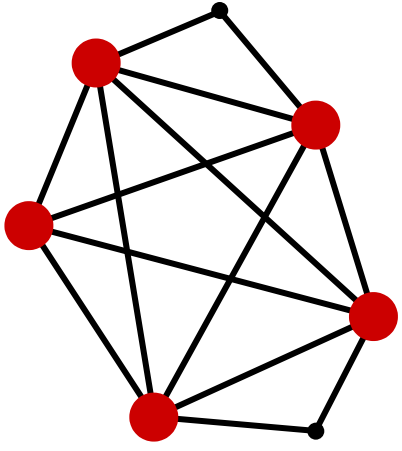
This means that the height of the B-Matrix depends on the graph diameter.

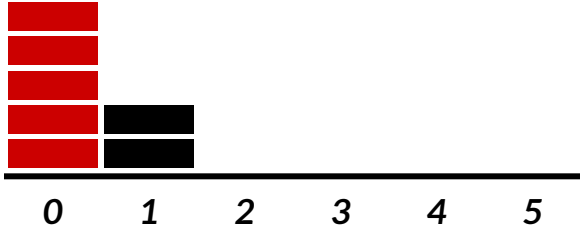
In this example, by the third hop there are only five instances where three hops completely covered all nodes, with two leftover nodes at the ends of the diameter.



Frequency of node 3-shell degrees

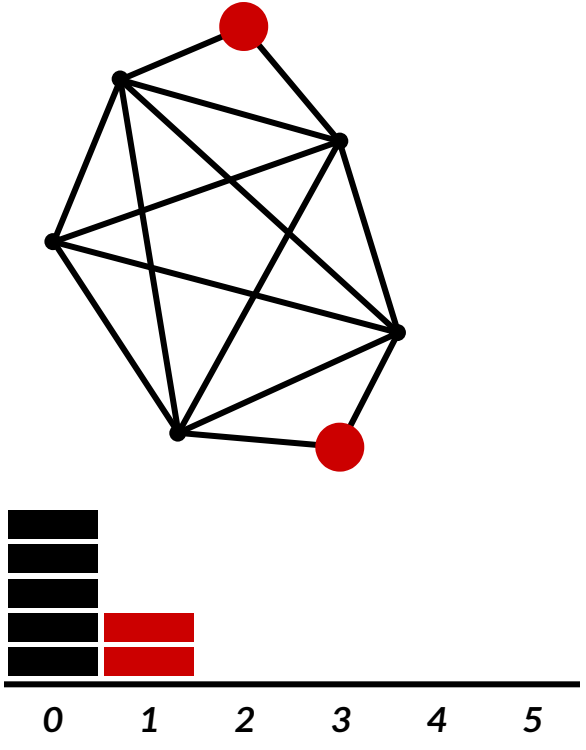
0	7	0	0	0	0
0	0	2	0	1	4
0	4	1	2	0	0
5	2	0	0	0	0





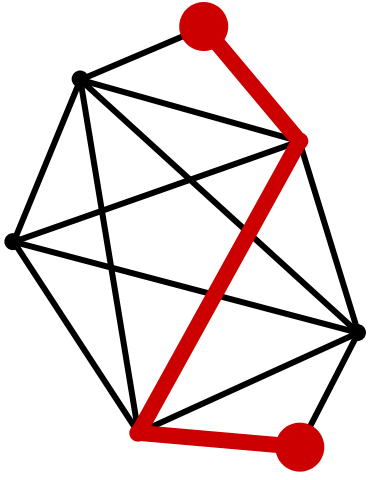
Frequency of node 3-shell degrees

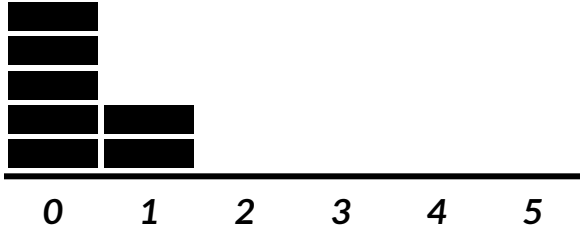
0	7	0	0	0	0
0	0	2	0	1	4
0	4	1	2	0	0
5	2	0	0	0	0



Frequency of node 3-shell degrees

0	7	0	0	0	0
0	0	2	0	1	4
0	4	1	2	0	0
5	2	0	0	0	0





Frequency of node 3-shell degrees

The graph diameter is the length of the shortest path between the two most distanced nodes.

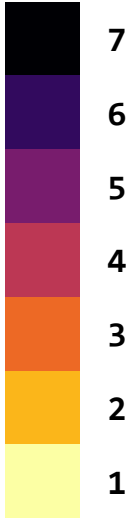
0	7	0	0	0	0
0	0	2	0	1	4
0	4	1	2	0	0
5	2	0	0	0	0

In contrast to this explainer example, the B-Matrix of a real-world graph can be very large.

It is not practical to show this data abstraction directly with numbers, we need a visual encoding.

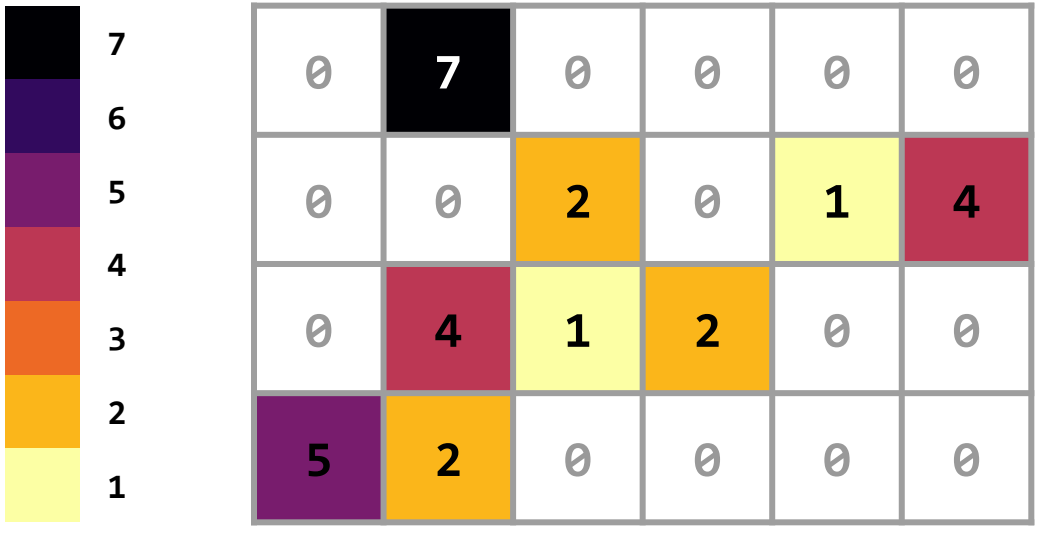
0	7	0	0	0	0
0	0	2	0	1	4
0	4	1	2	0	0
5	2	0	0	0	0

In literature this is solved by mapping the B-Matrix node count to a colormap range.



0	7	0	0	0	0
0	0	2	0	1	4
0	4	1	2	0	0
5	2	0	0	0	0

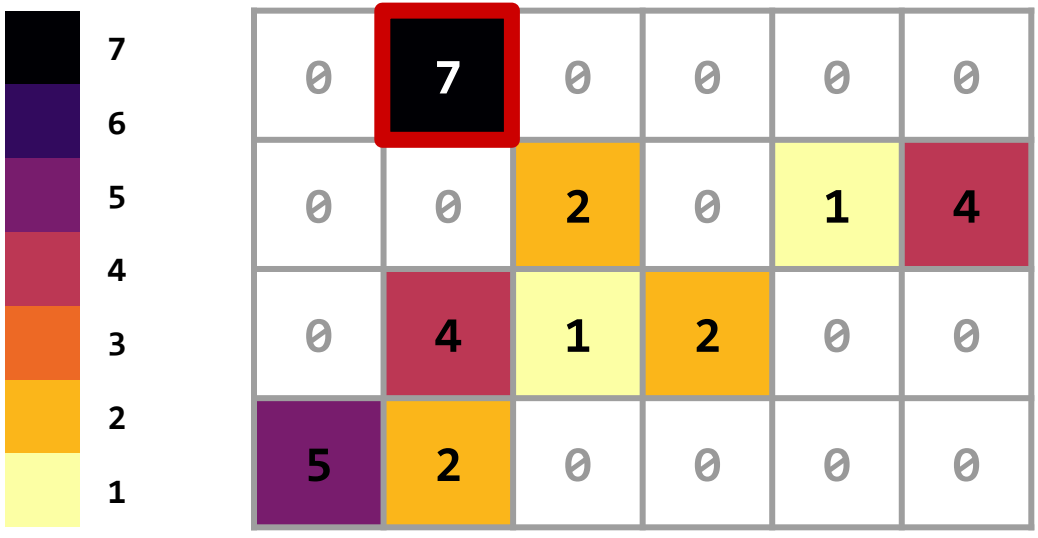
And the result
is a heatmap.



However, we can do one more edit to increase information resolution in this heatmap idiom.

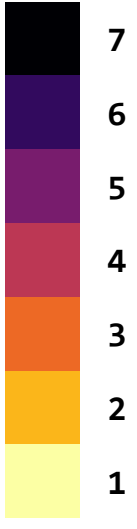
Notice that the zeroth row always has the highest value (which sets the colormap extreme).

In large networks this value can be so high that the colormap must be log-transformed.



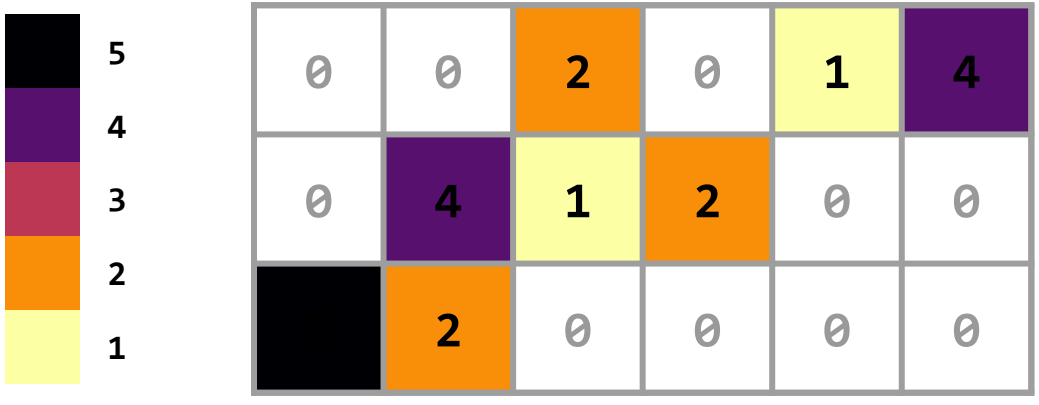
We can safely remove the zeroth row.

This is fine because this row only contains redundant data (total node count).

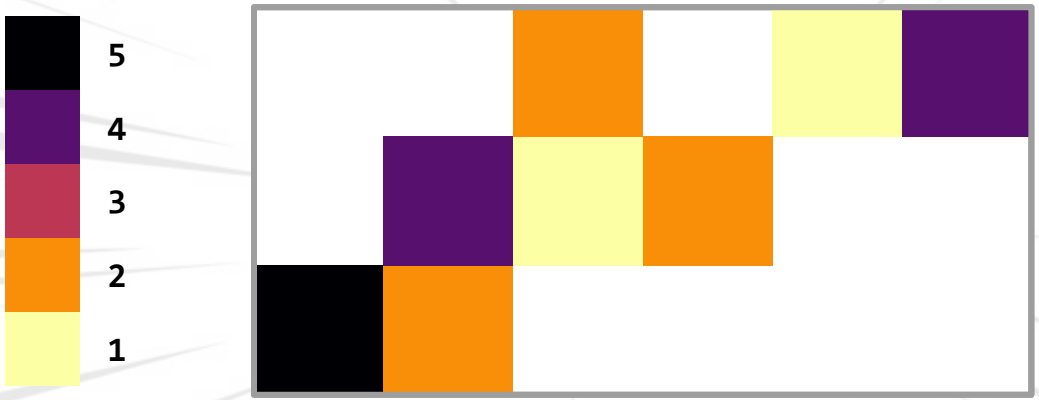


0	0	2	0	1	4
0	4	1	2	0	0
5	2	0	0	0	0

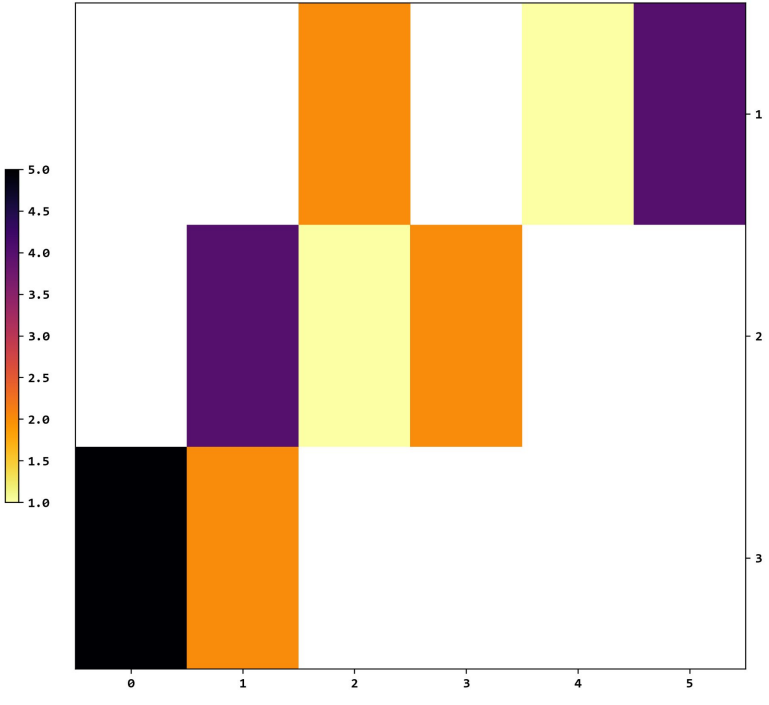
Then we can get higher fidelity by rescaling the colormap.



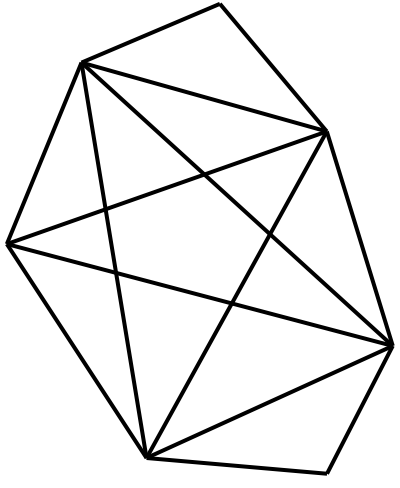
That's it!
This figure is the
network portrait
of the graph.



The script `plot_BMatrix_colormap.py` generates a network portrait from a B-Matrix (example data and figure below).

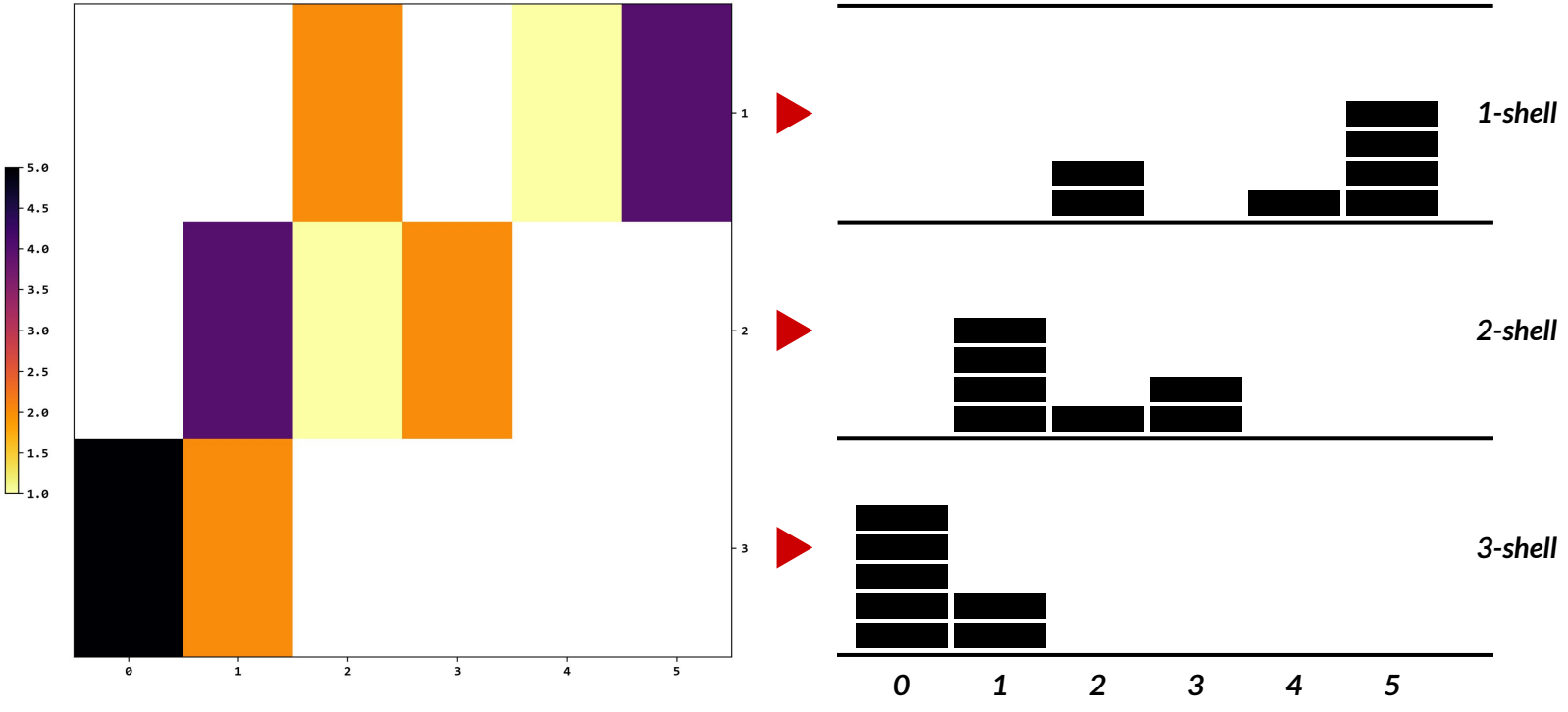


Graph	networkx.graph_atlas(1115)
Nodes	7
Edges	14

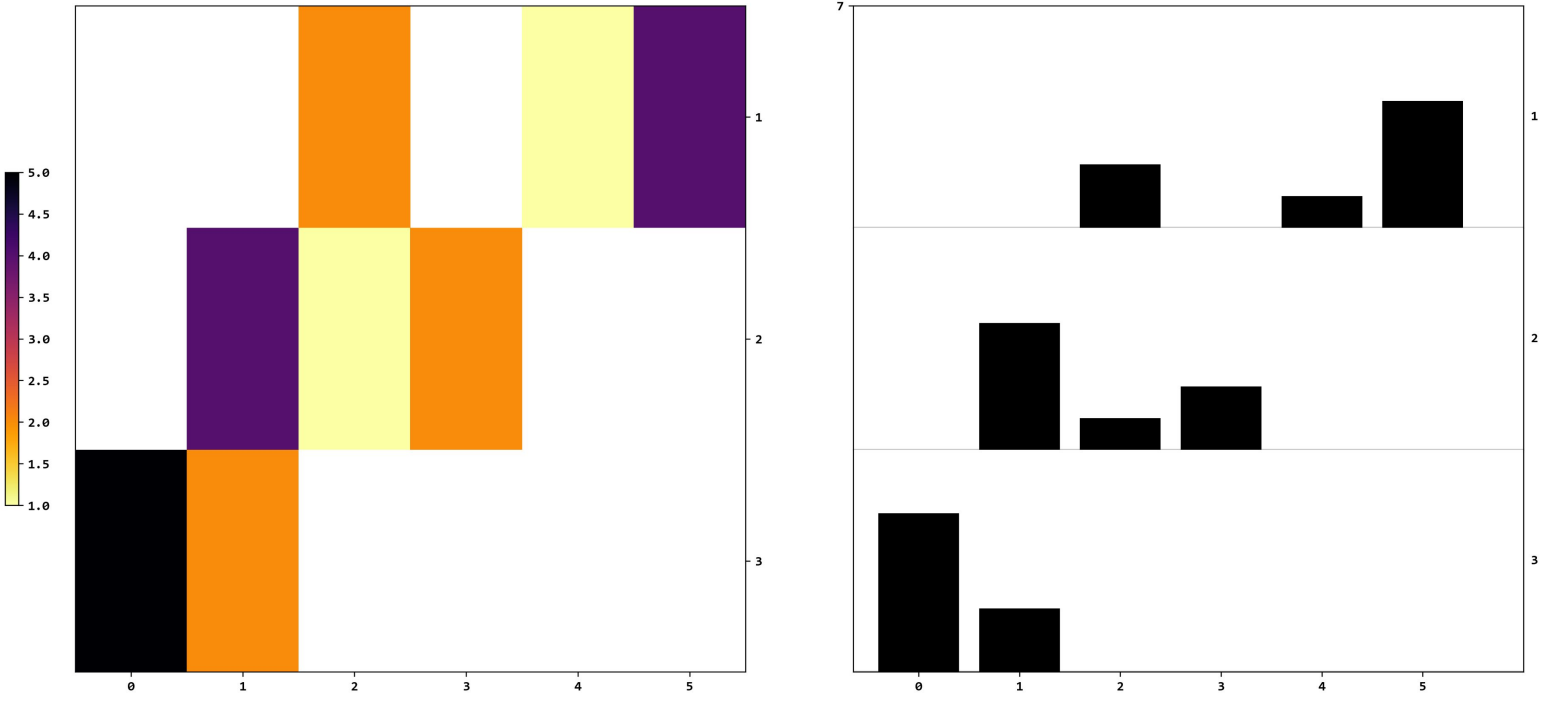


1-2
1-3
2-4
2-5
2-6
2-3
3-4
3-5
3-6
4-5
4-6
5-7
5-6
6-7

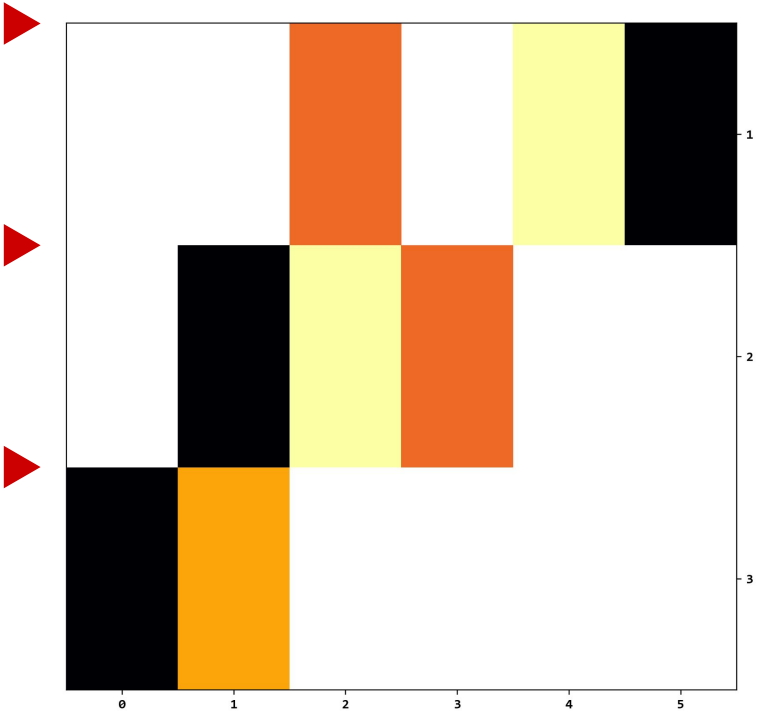
Recall that each row of the B-Matrix can be visually encoded as stacked bars for count data.

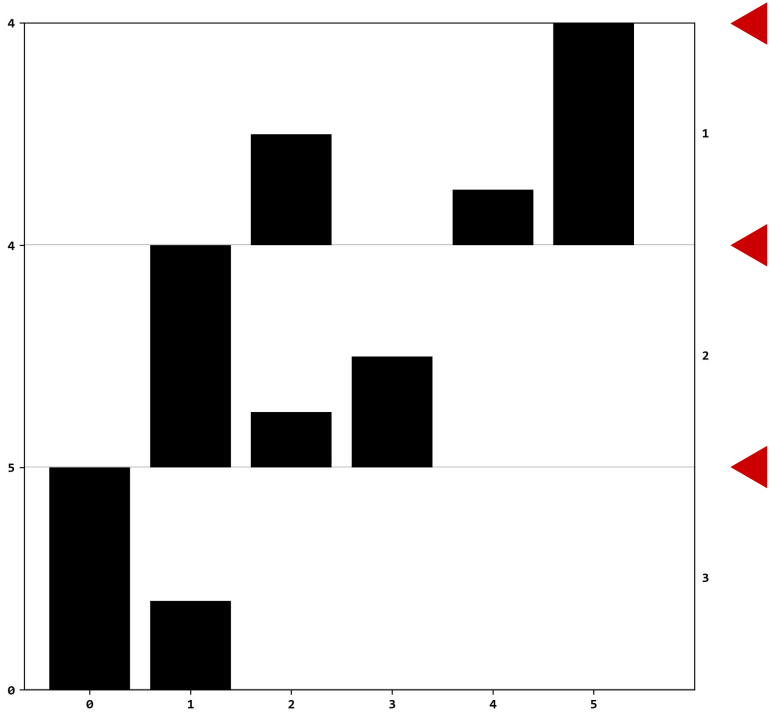


The script `plot_BMatrix_histogram.py` creates this visualization with bars instead of colors for encoding node counts.

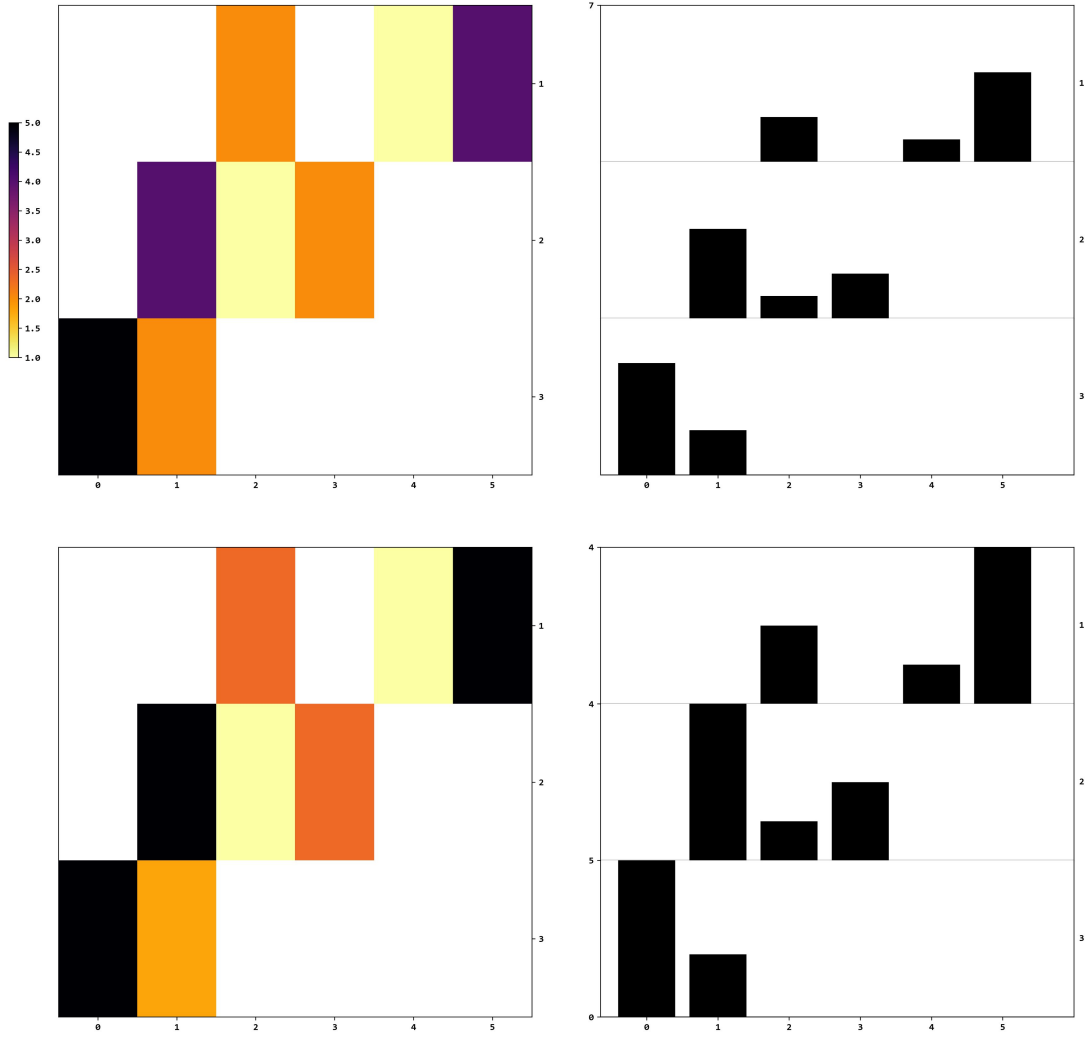


To further support B-Matrix interpretation tasks, both visualization scripts have a `row_normalized=True` option to enhance information discovery within hop level.

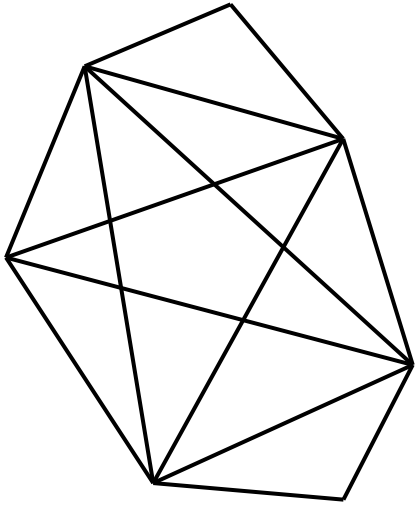


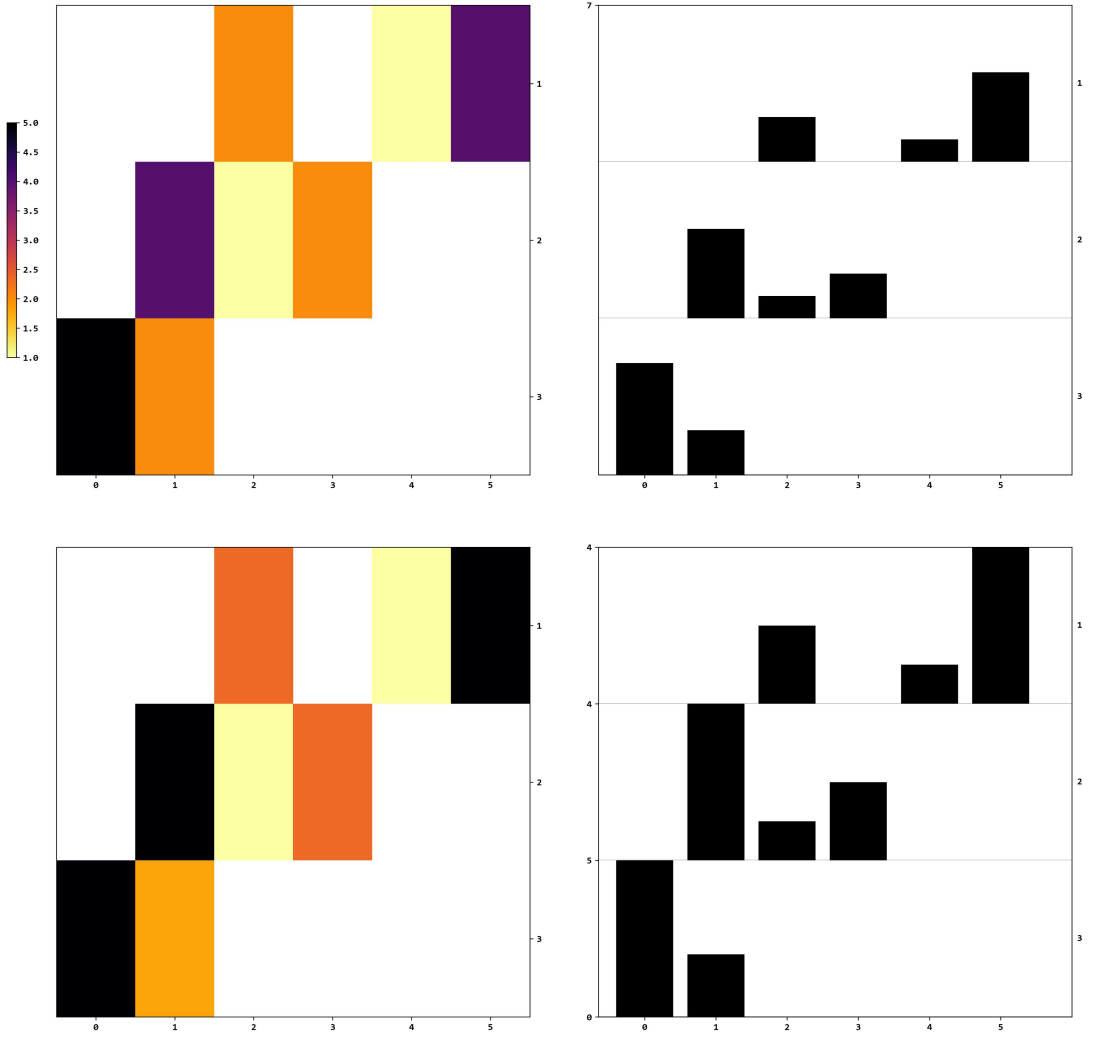


Both visualization scripts render and save these figures at `300dpi` and in `PNG format` with the non-interactive back-end `matplotlib.use("Agg")`.



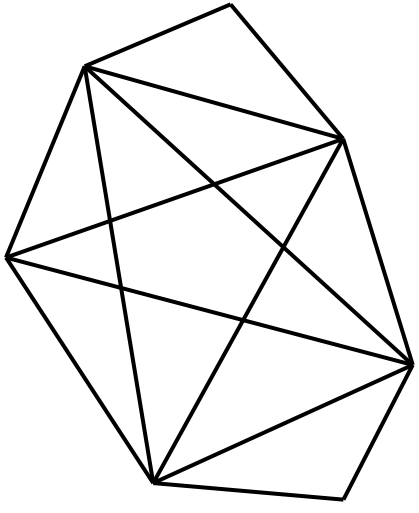
Graph	nx.graph_atlas(1115)
Nodes	7
Edges	14



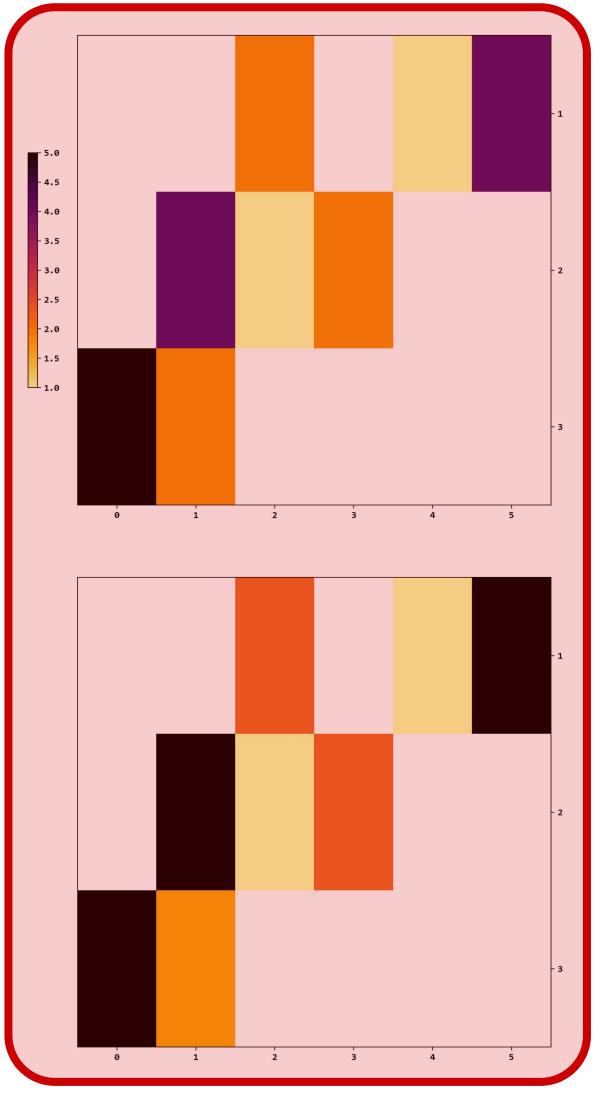


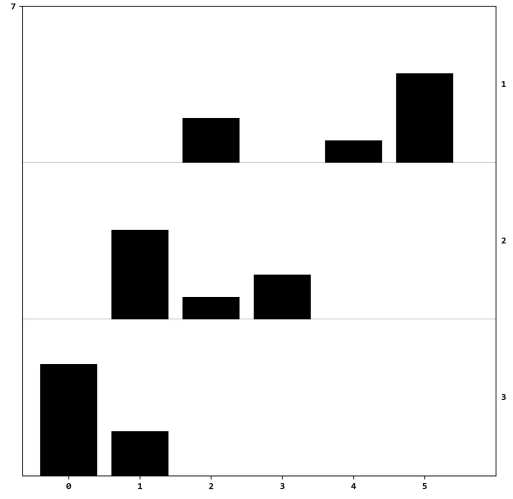
Example explainer graph and its quartet of B-Matrix visualizations.

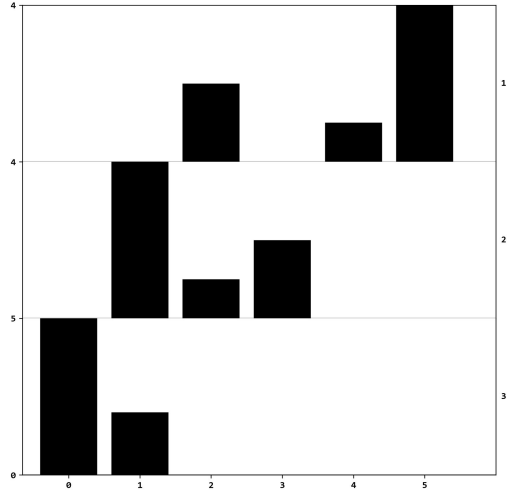
Graph	nx.graph_atlas(1115)
Nodes	7
Edges	14



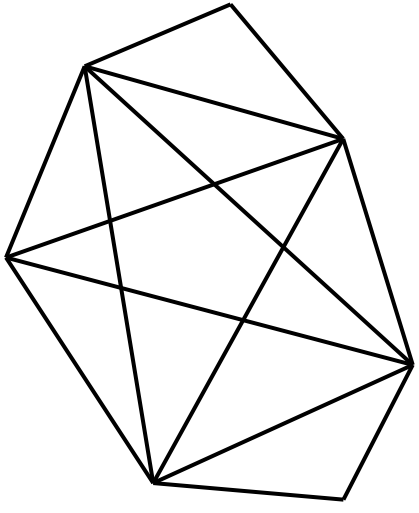
On this left side node counts are encoded with through a colormap.

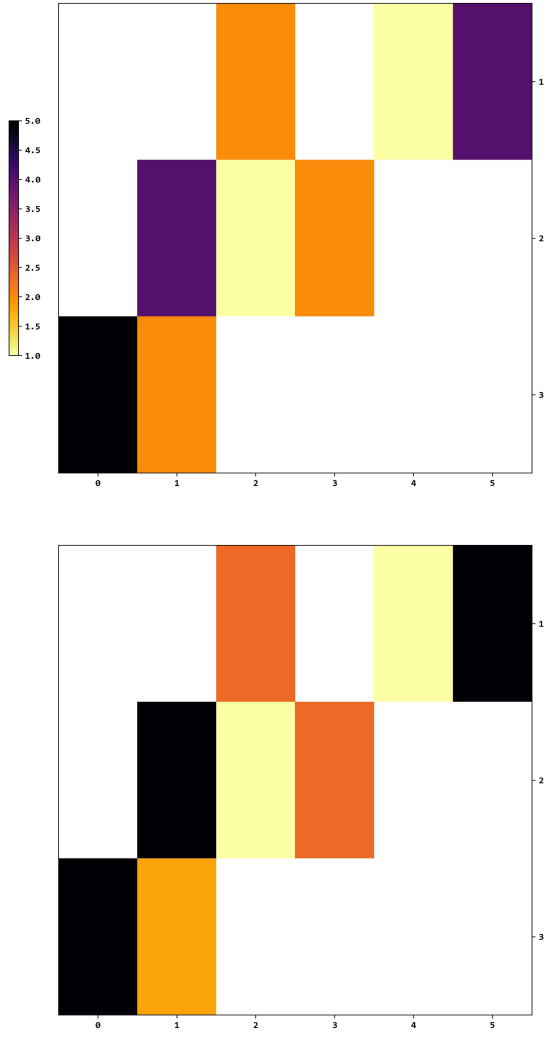


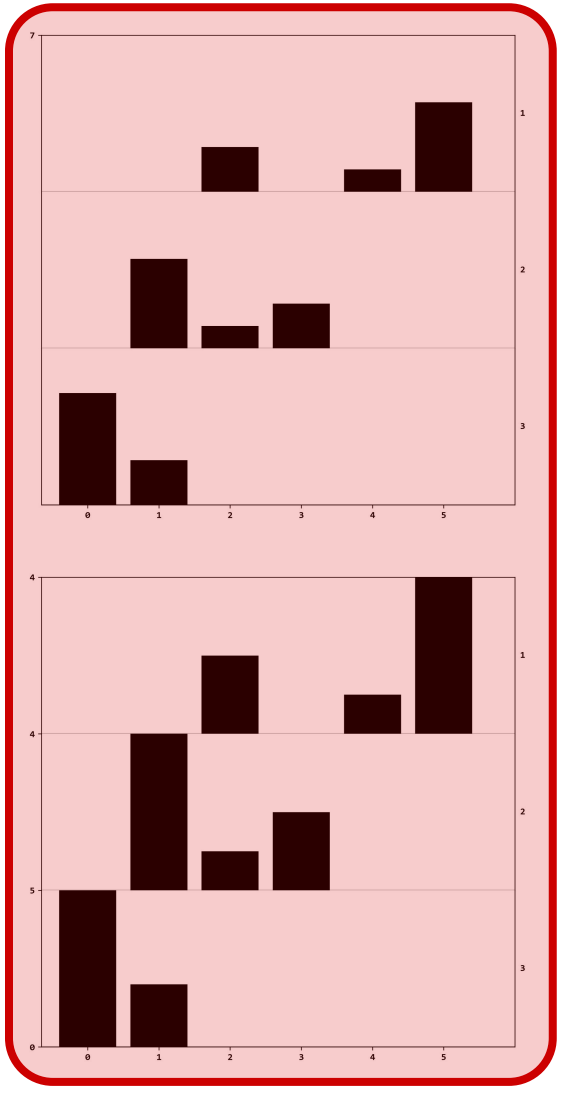




Graph	nx.graph_atlas(1115)
Nodes	7
Edges	14

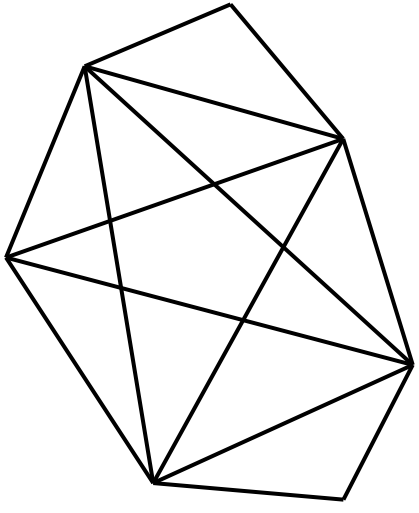


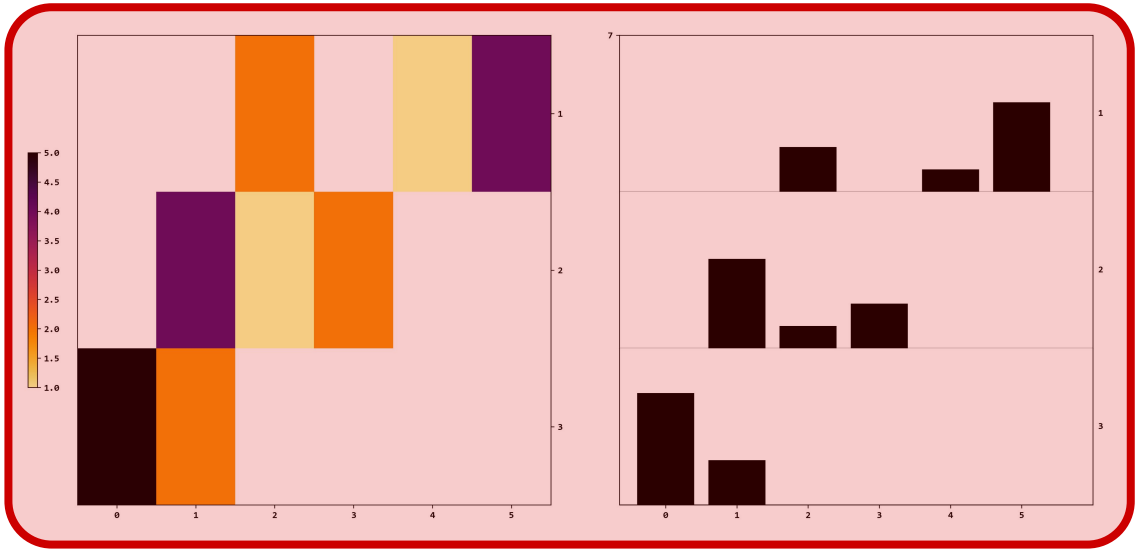


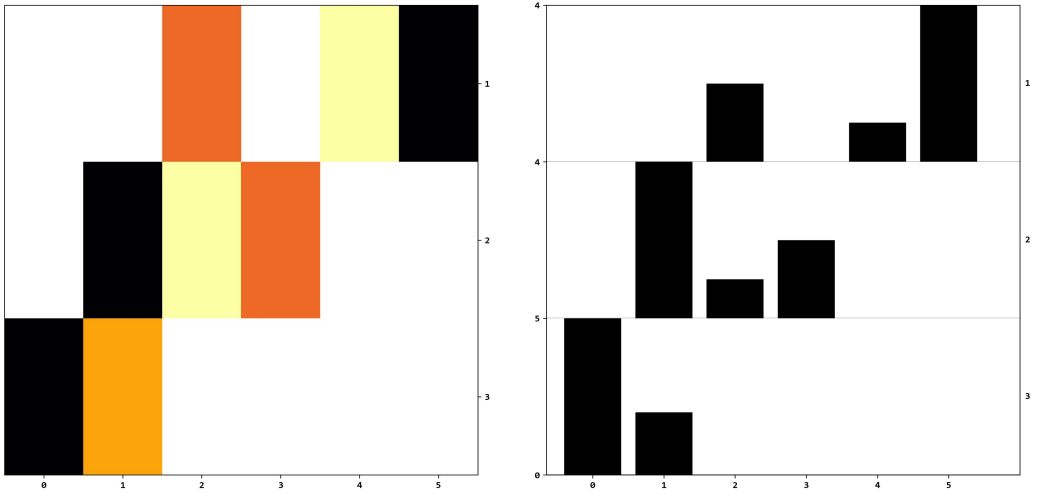


On this right side node counts are encoded with a bar charts (histograms).

Graph	nx.graph_atlas(1115)
Nodes	7
Edges	14

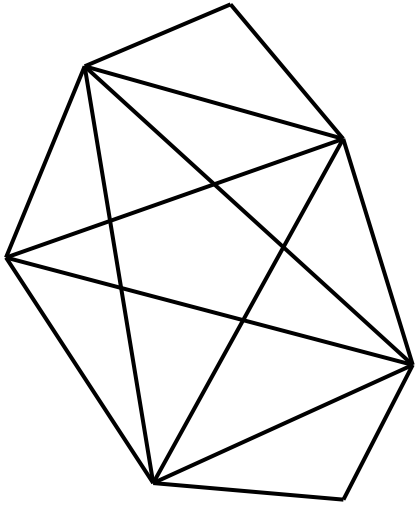


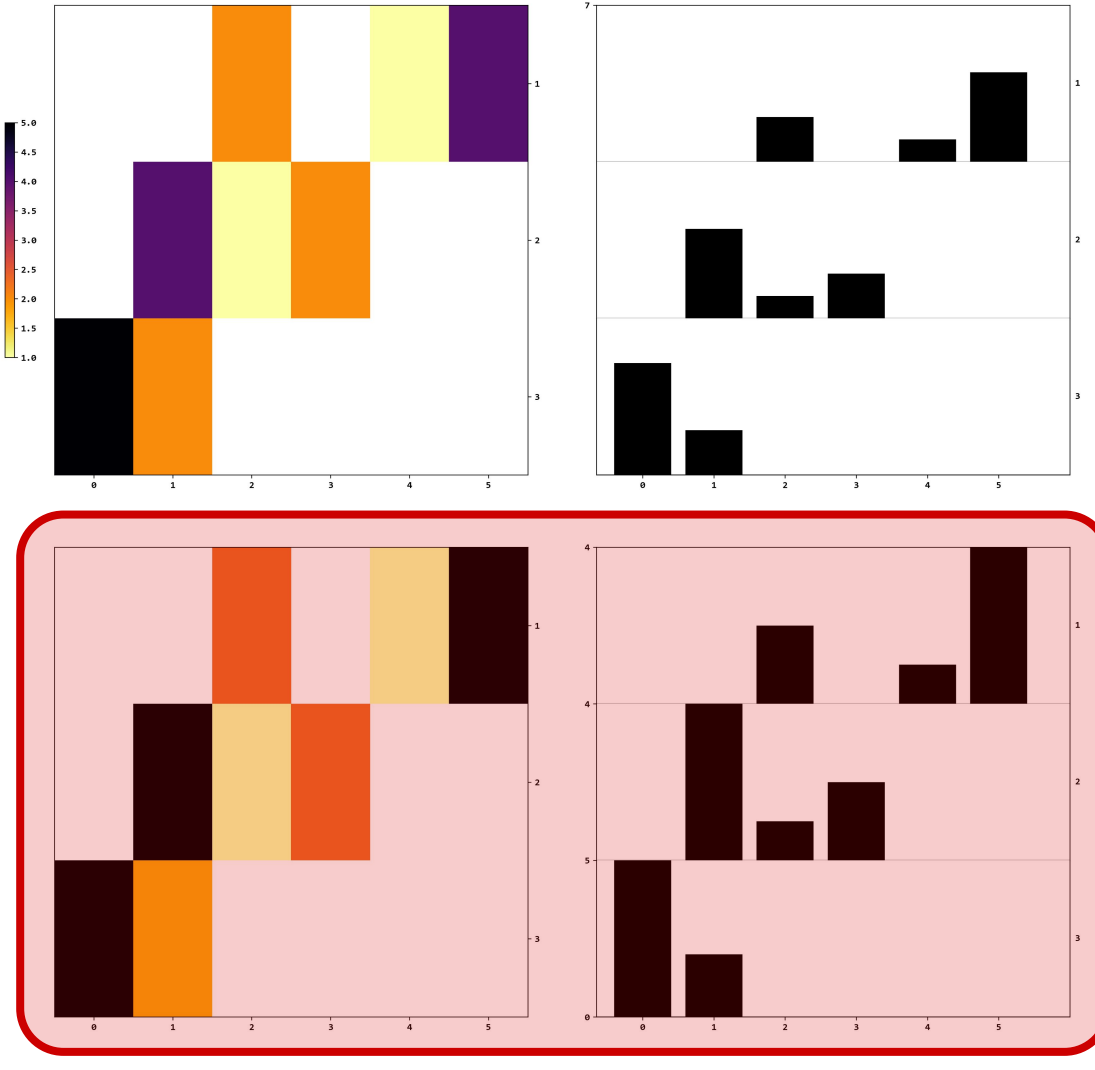




This top section has global normalization (shared range for colormap and y-axis).

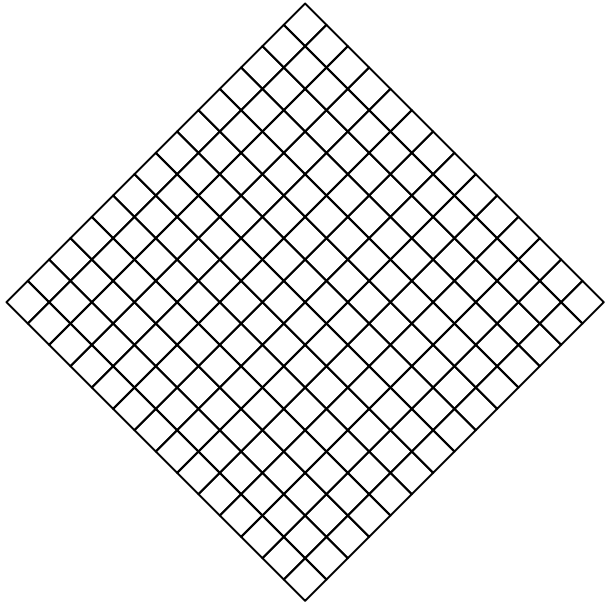
Graph	nx.graph_atlas(1115)
Nodes	7
Edges	14

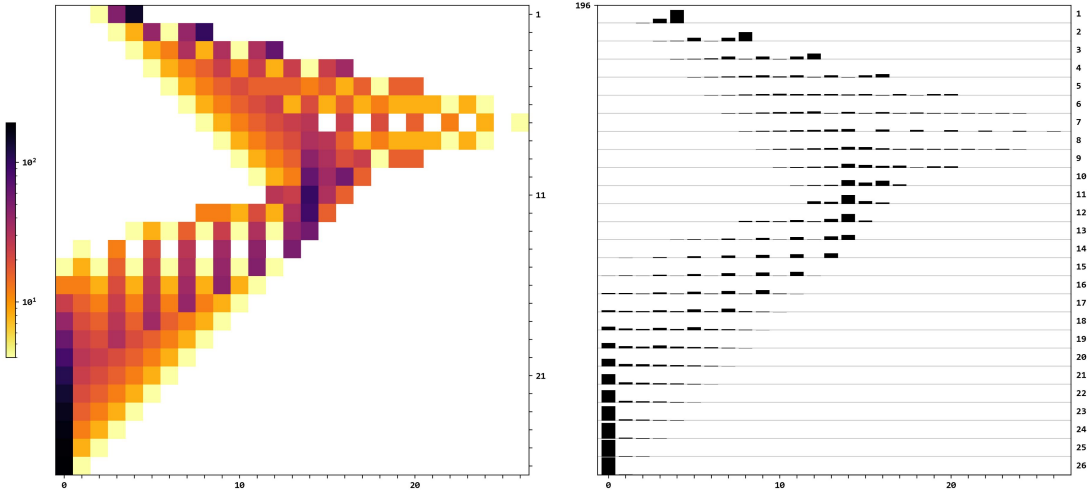


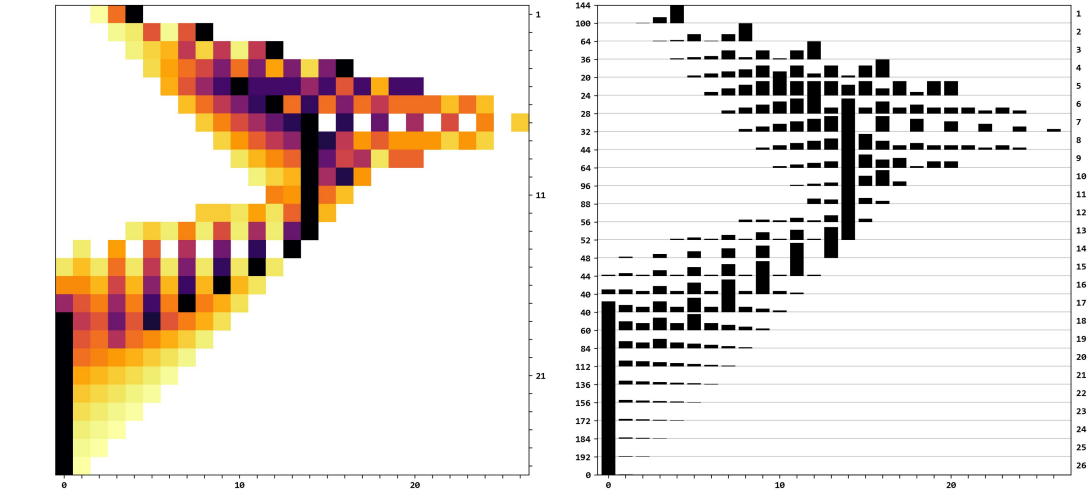


This bottom section has row-normalization (each row has its own colormap and y-axis range).

Graph	Square Grid (14x14)
Nodes	196
Edges	364

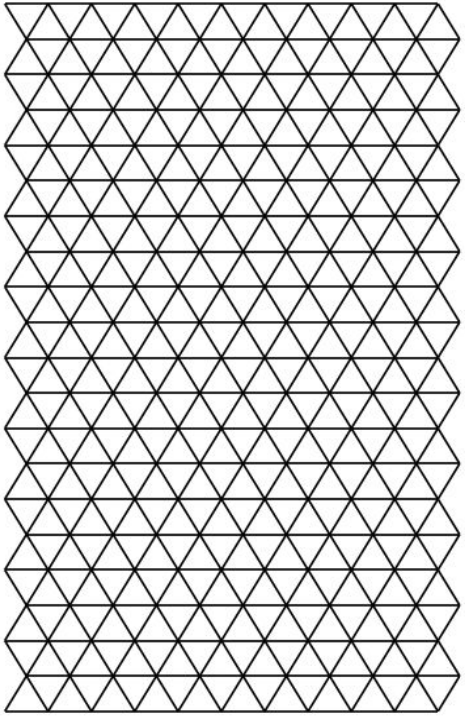


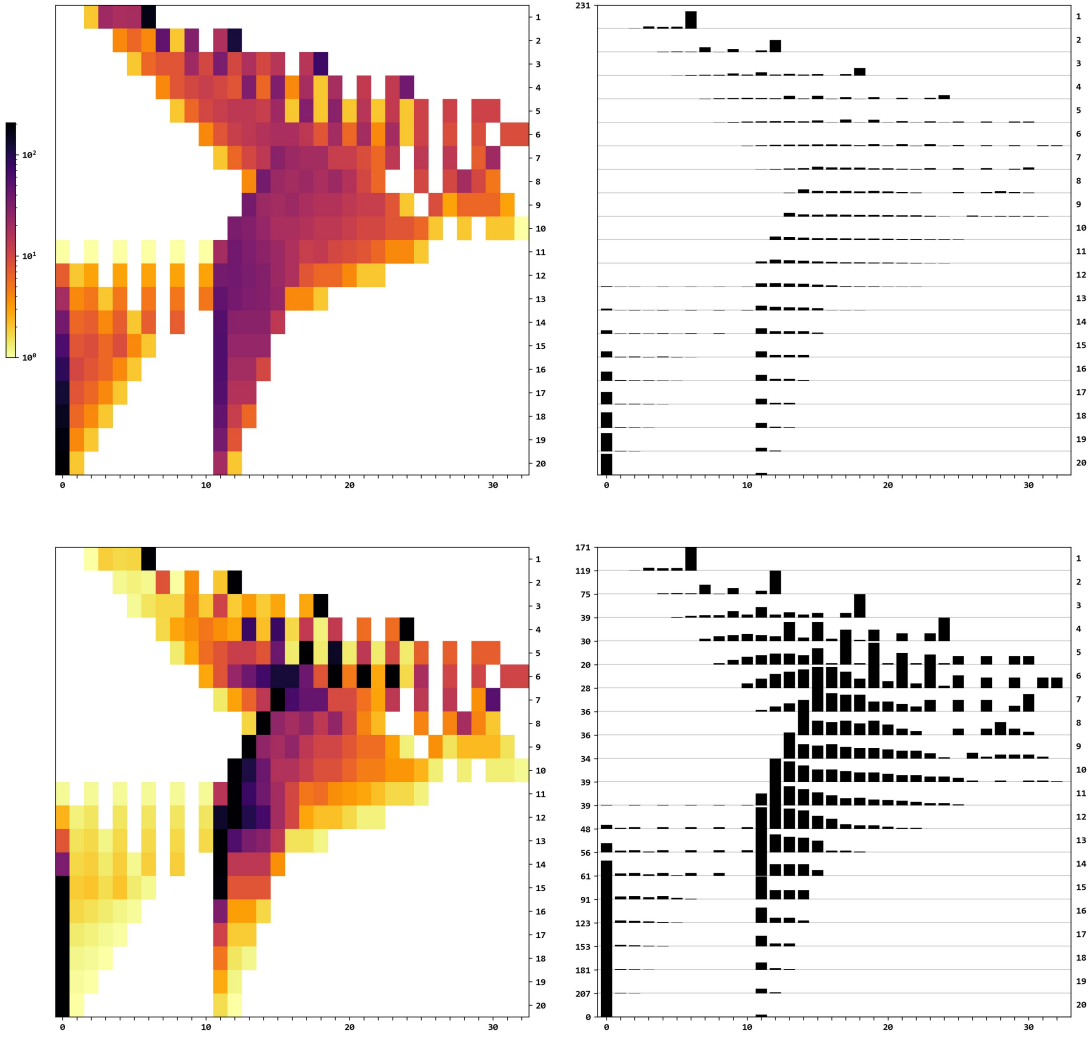




Returning to the example at the beginning, now with a quartet of visualizations.

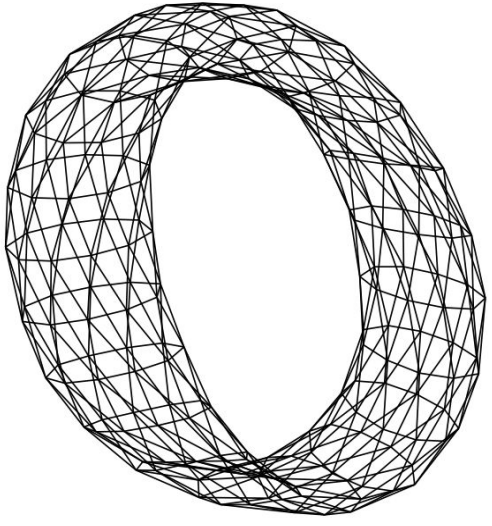
Graph	Planar triangle mesh (20x20)
Nodes	231
Edges	630

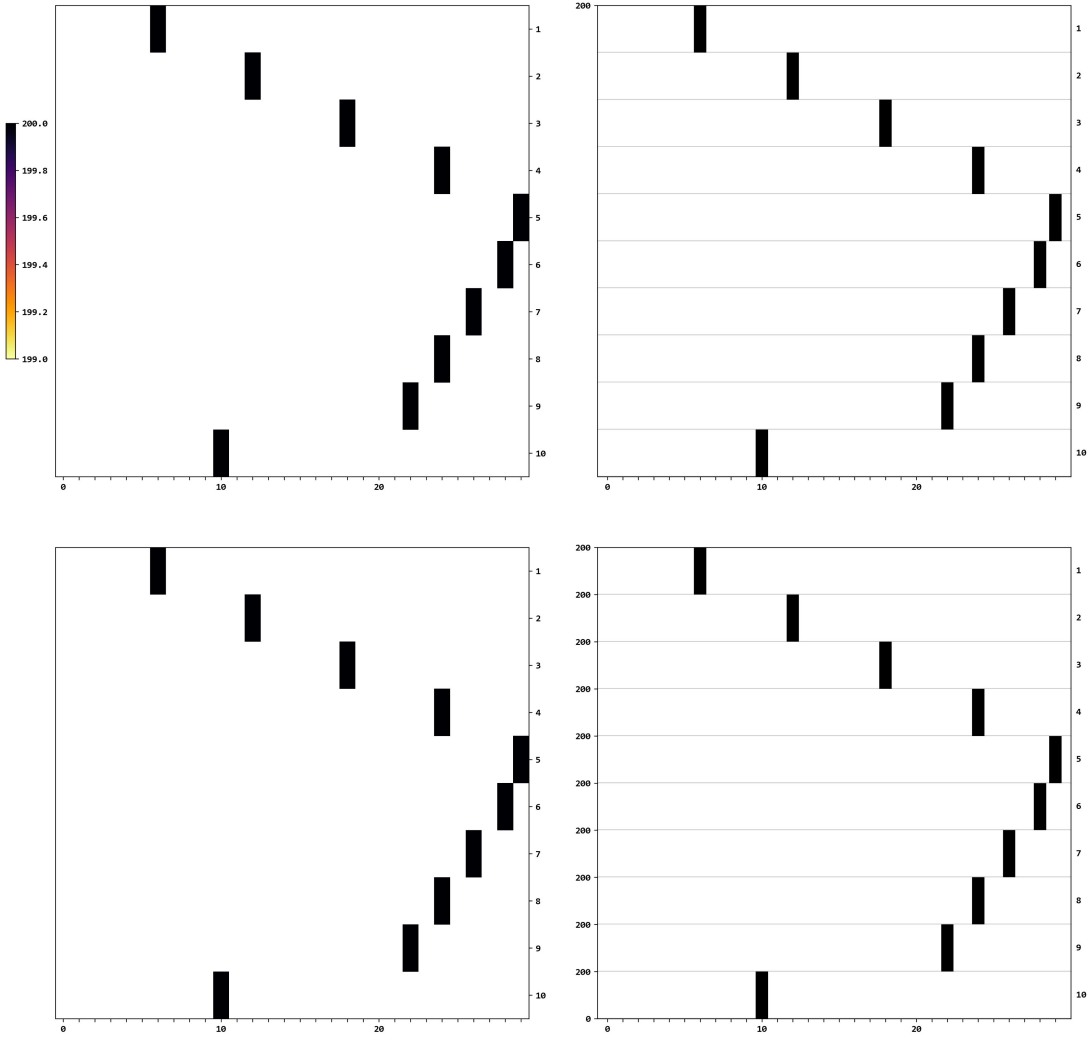




Another regular grid graph.

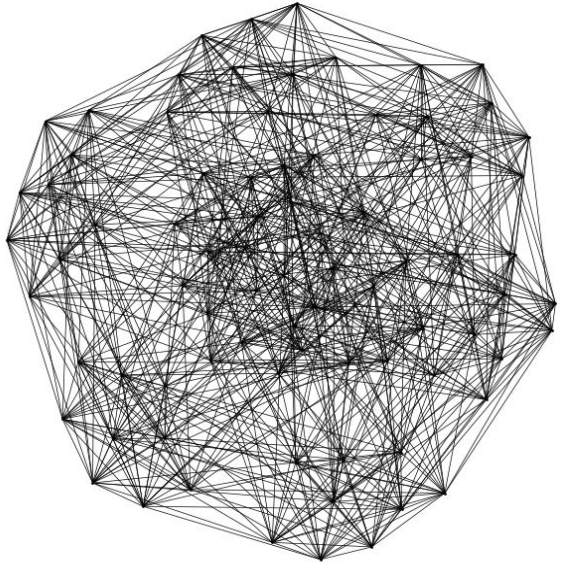
Graph	Closed triangle mesh (20x20)
Nodes	200
Edges	600

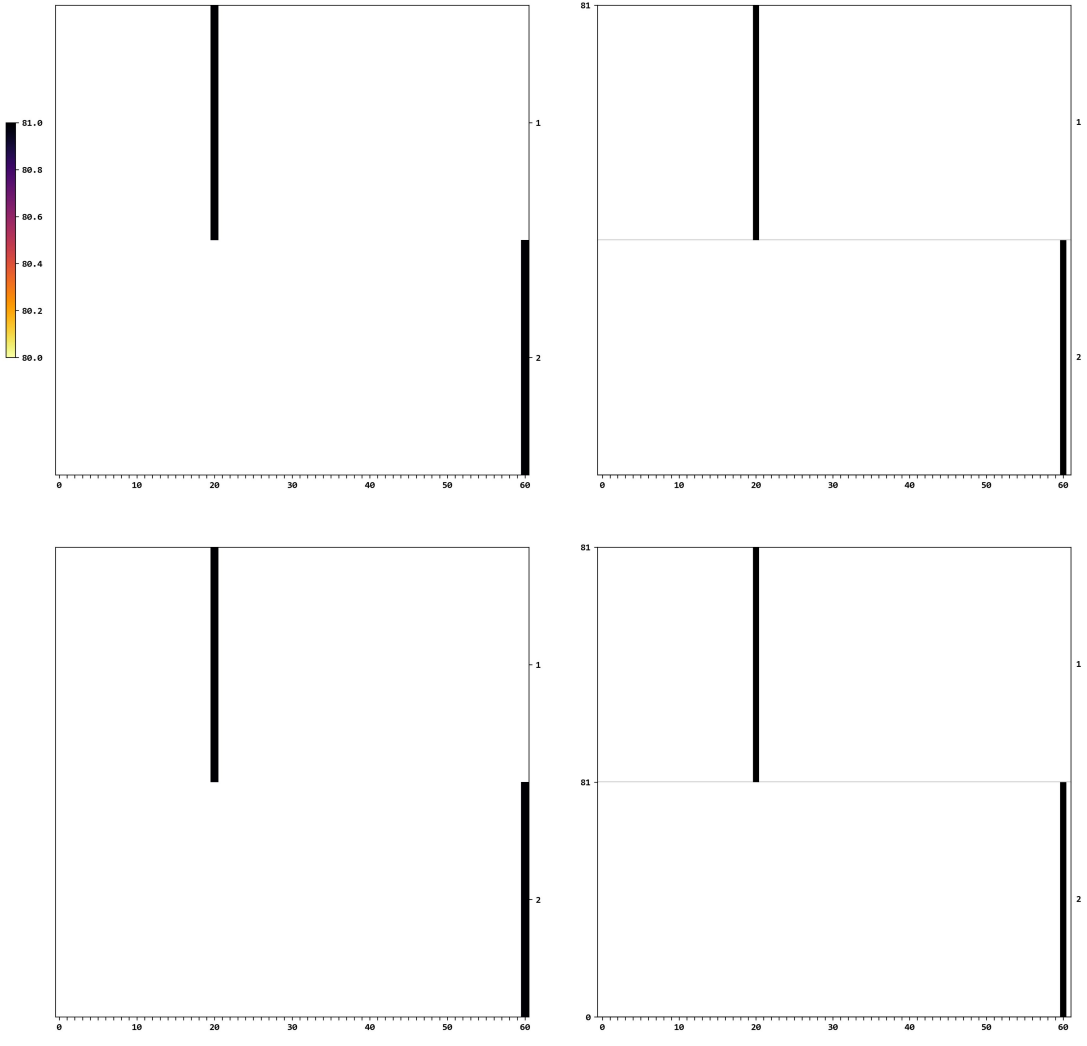




A periodic manifold has equal connections in all directions, so there is no variance in node degrees across hops.

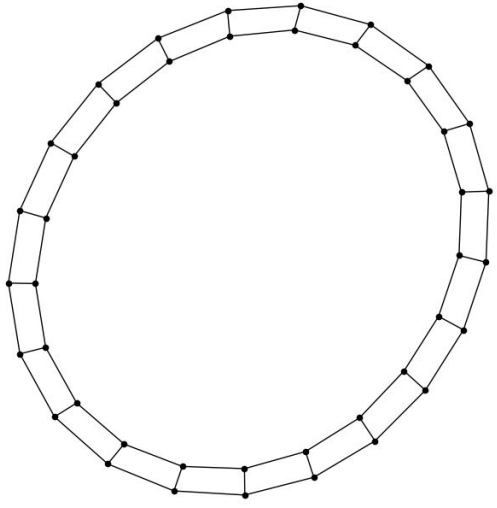
Graph	Classic Sudoku nx.sudoku_graph(n=3)
Nodes	81
Edges	810

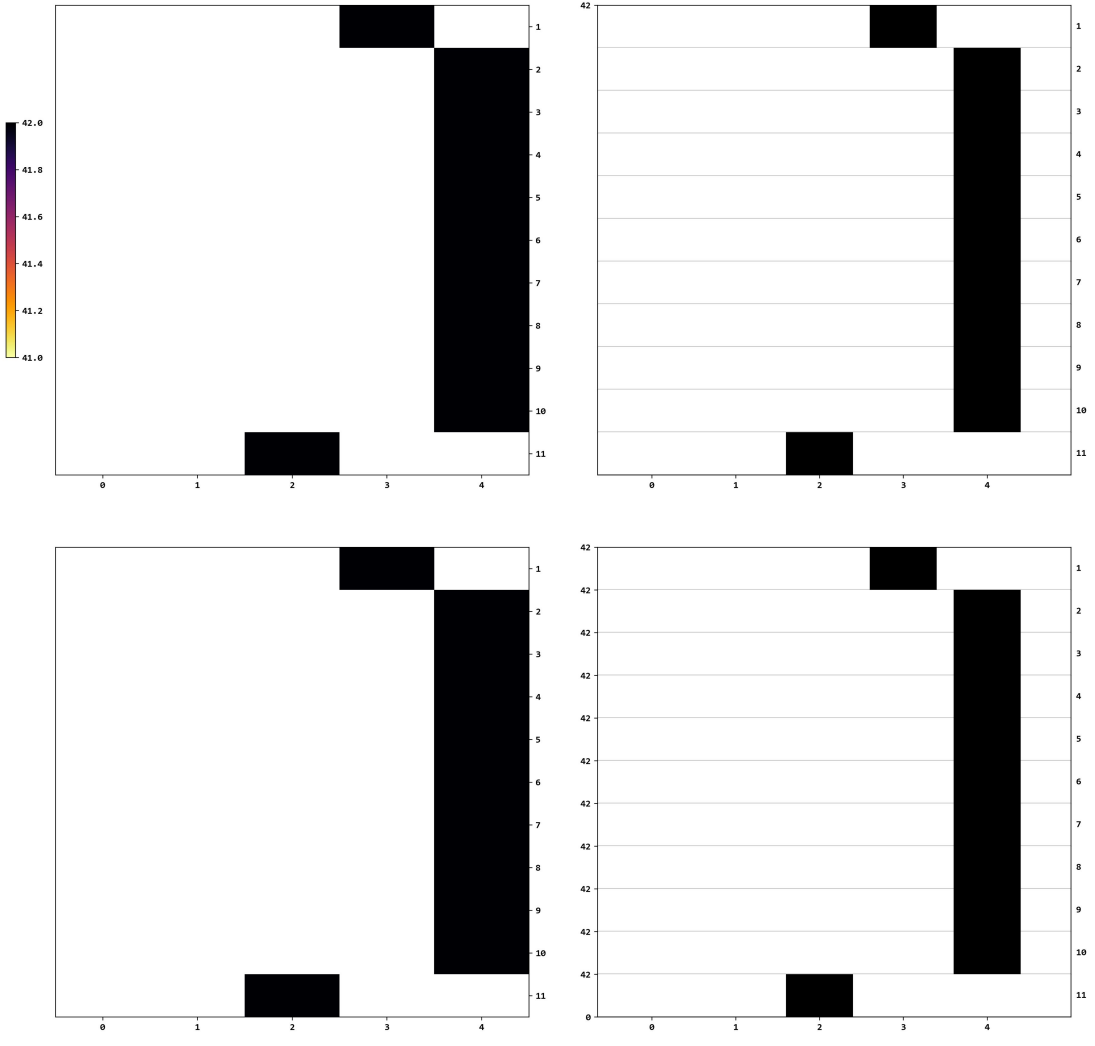




Sudoku rows, columns, and blocks form cliques.

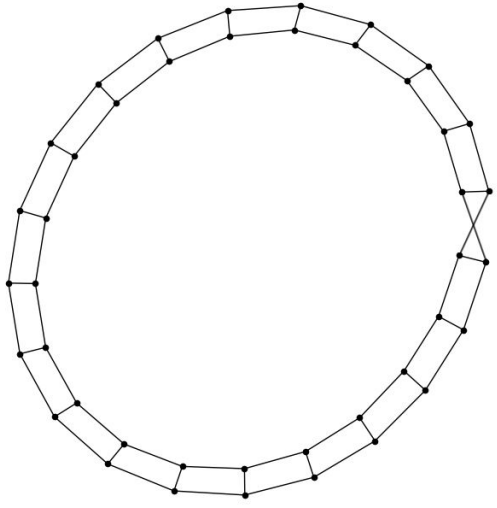
Graph	Planar Strip (21 segments)
Nodes	42
Edges	63

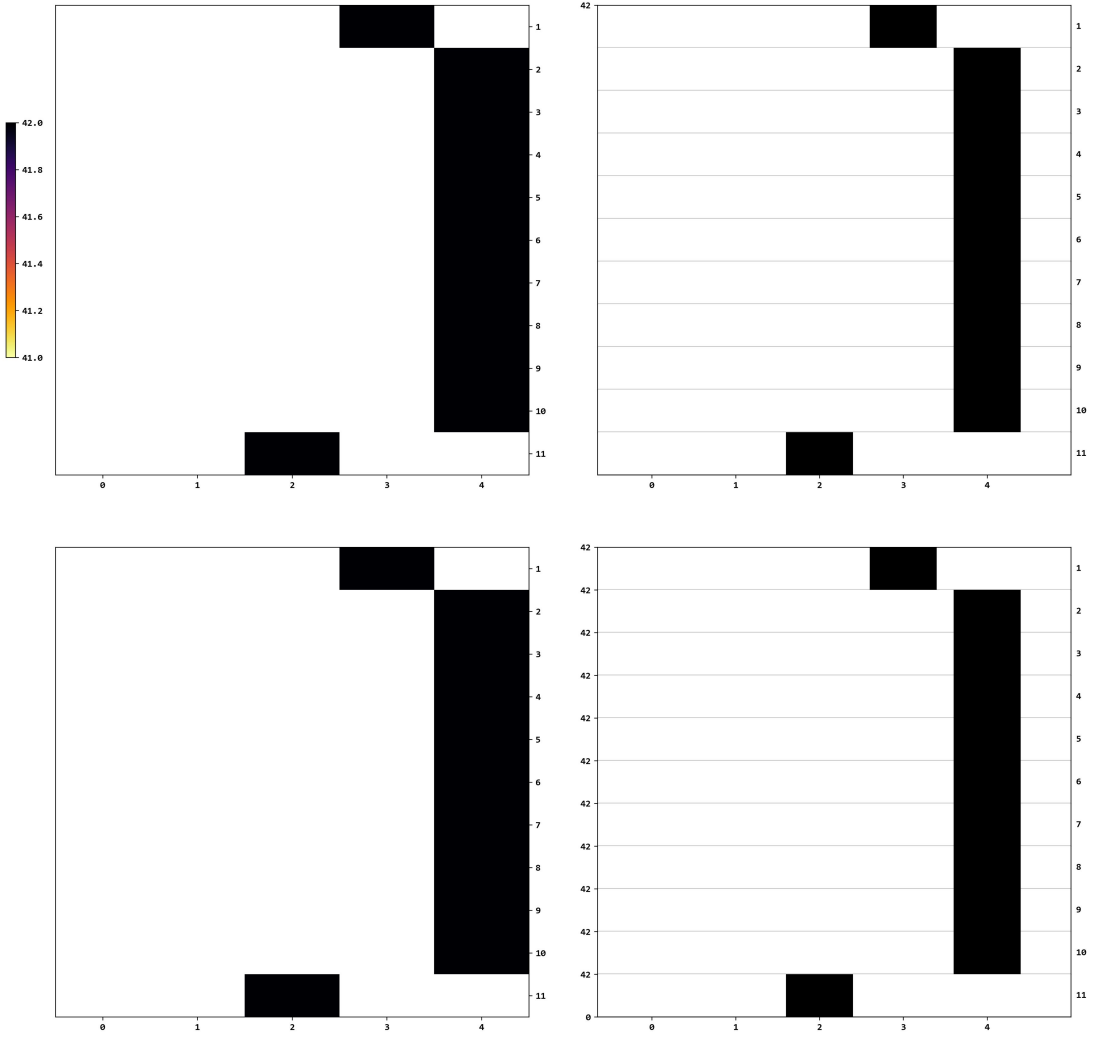




Non-variance of node counts also happens with a planar strip.

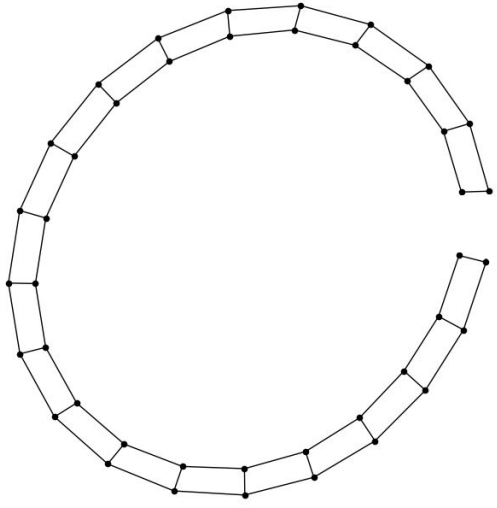
Graph	Möbius Strip (21 segments)
Nodes	42
Edges	63

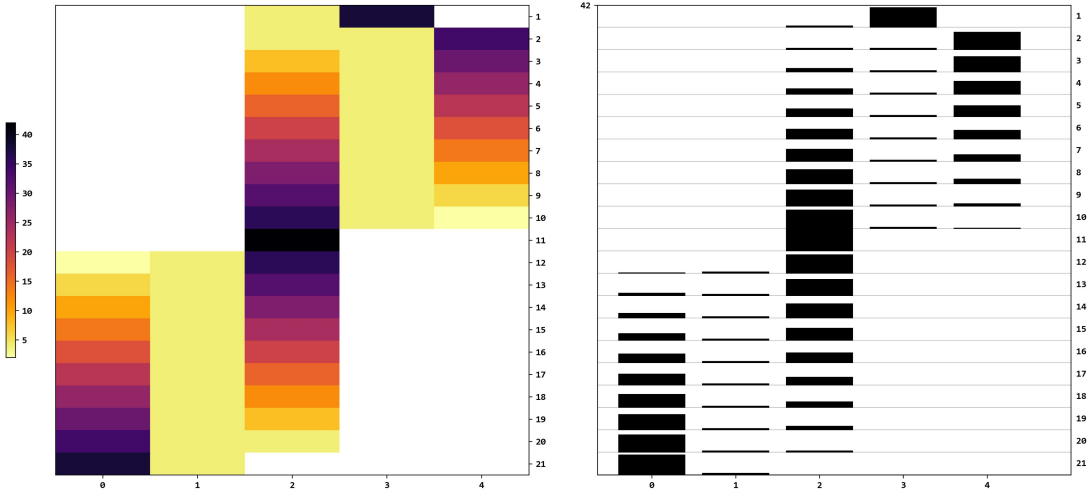


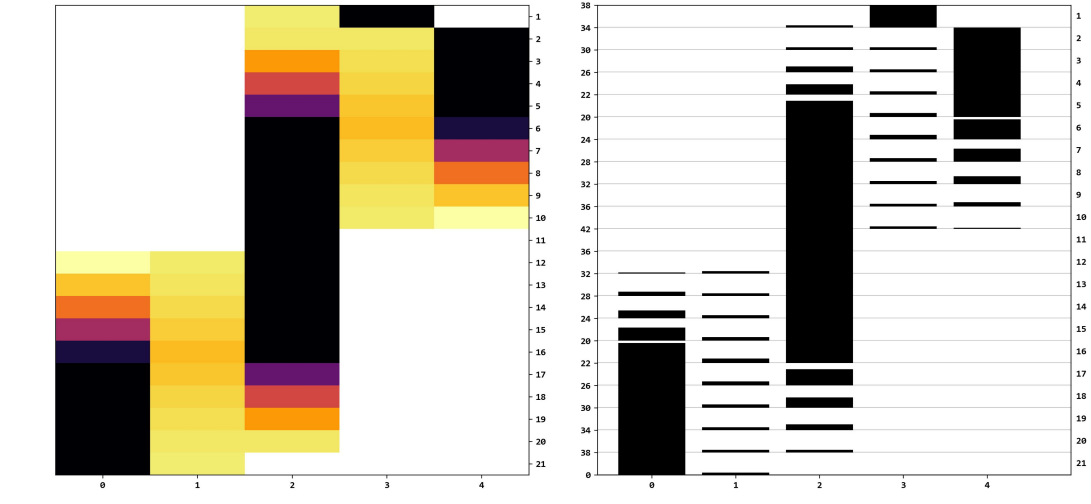


Cutting and twisting into a Möbius strip has no variance effect.

Graph	Linear Strip (20 segments)
Nodes	42
Edges	61

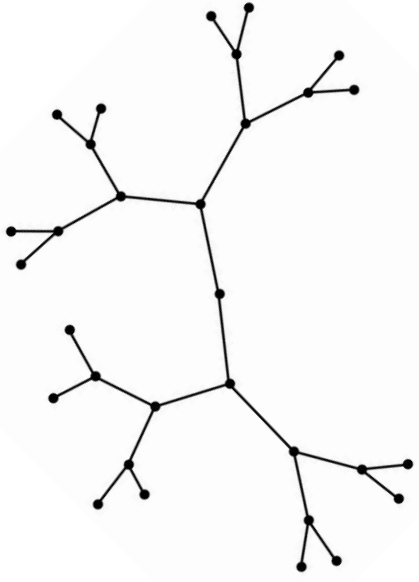


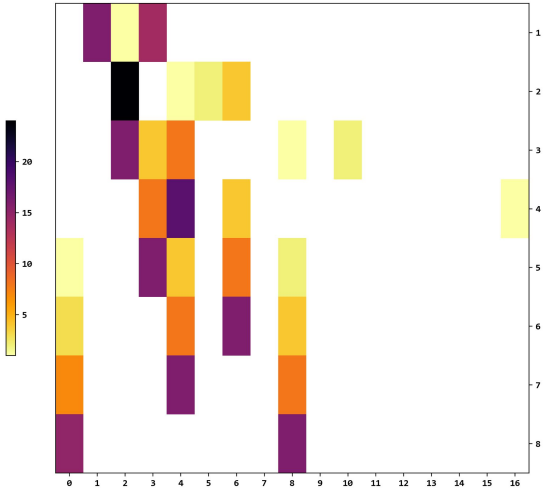


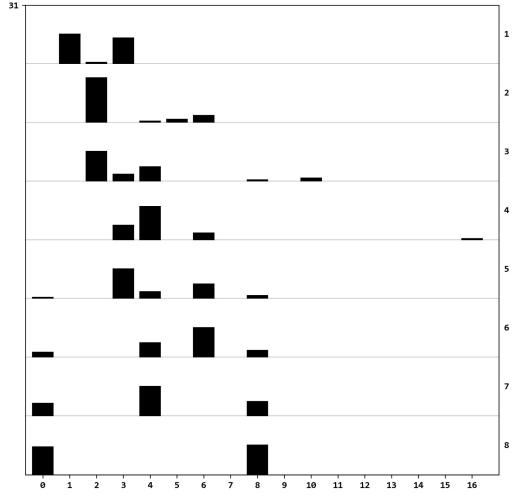


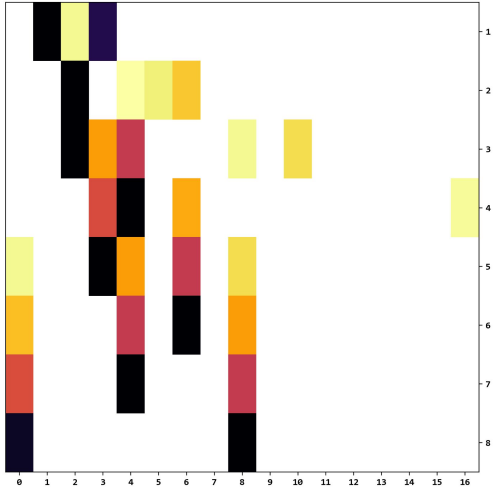
But graphs with dead ends have high column variance and row count.

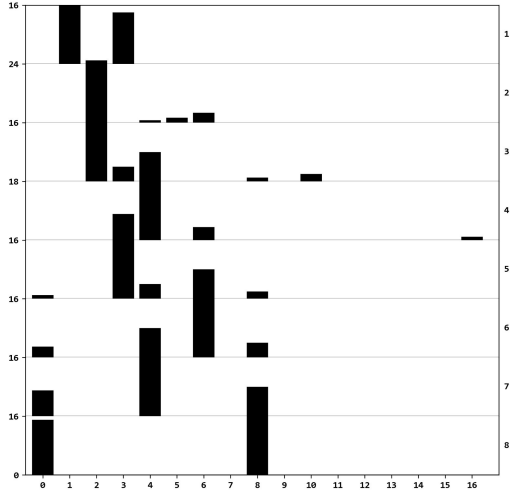
Graph	Fractal Tree nx.balanced_tree(r=2,n=4)
Nodes	31
Edges	30







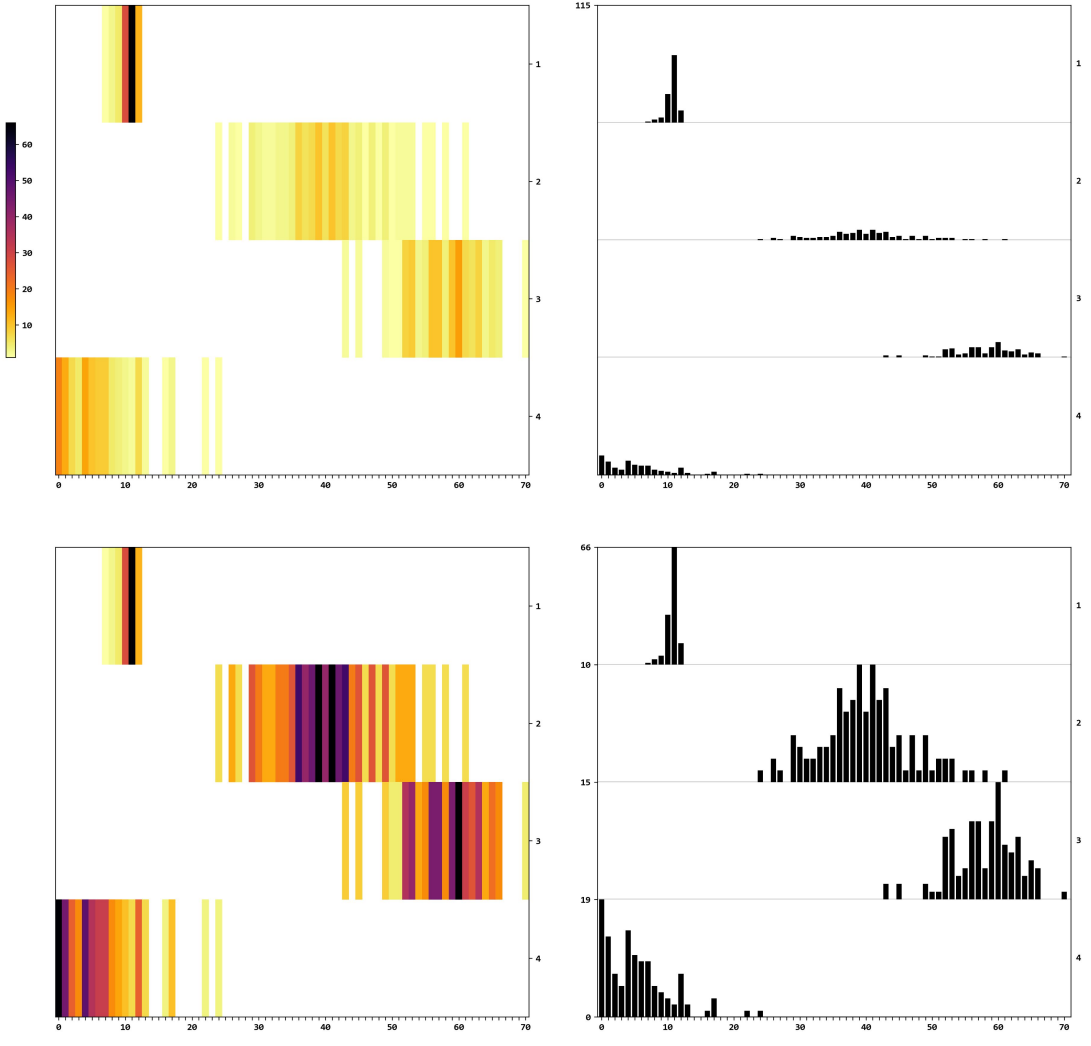




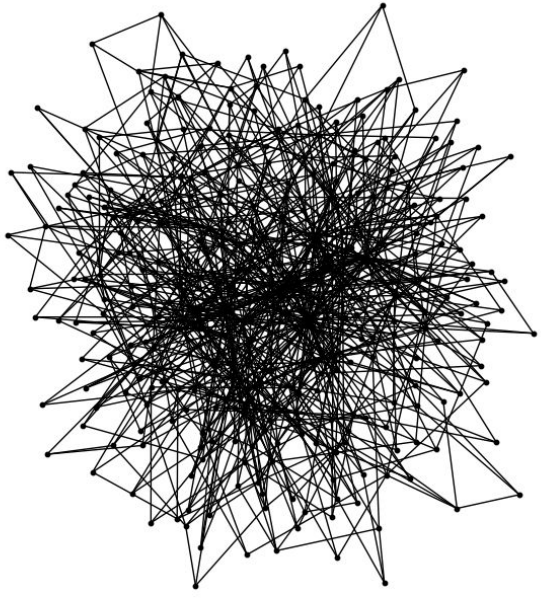
Trees have node variance and vertical stripe patterns.

Graph	Fall 2000 College Football Games personal.umich.edu/~mejn/netdata
Nodes	115
Edges	613

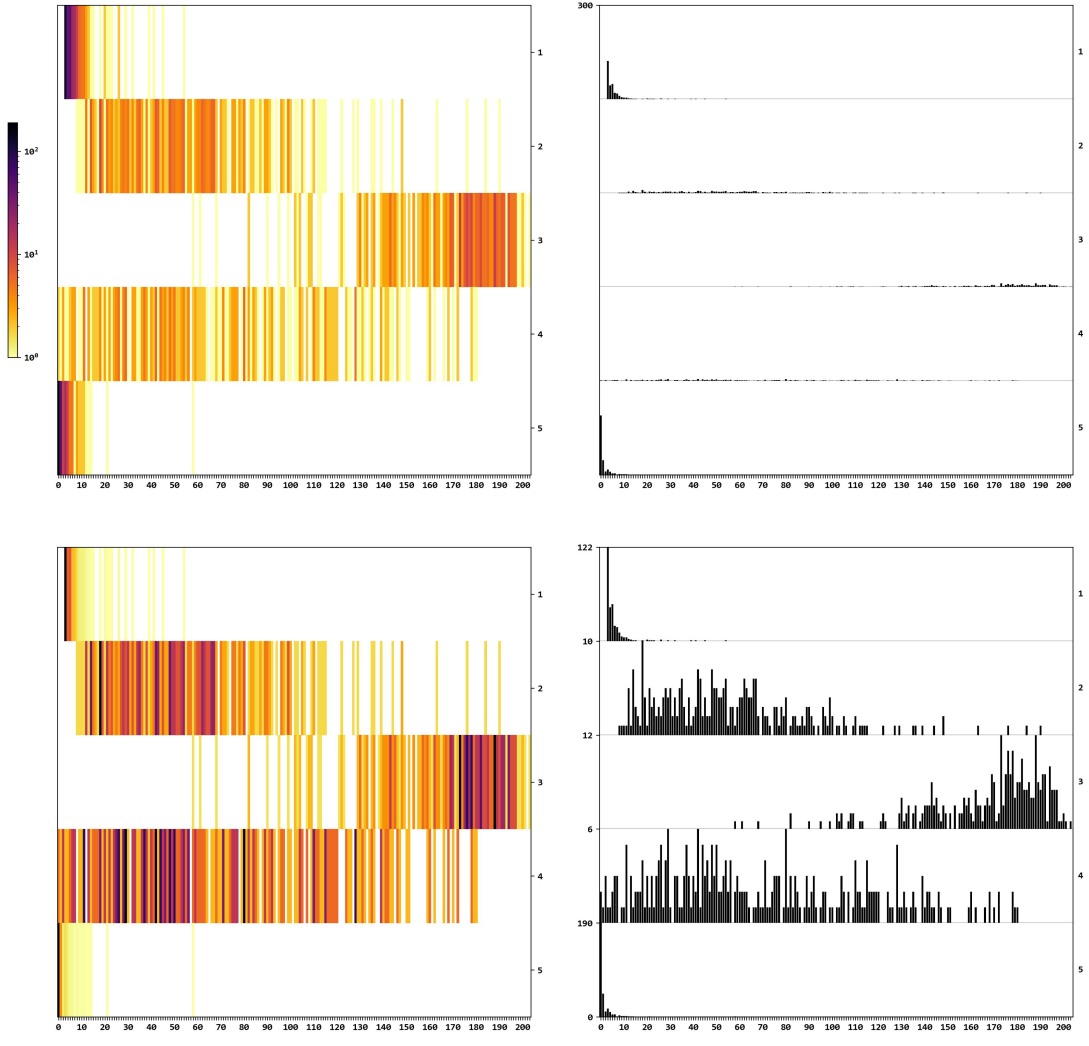
A real “small-world” network.



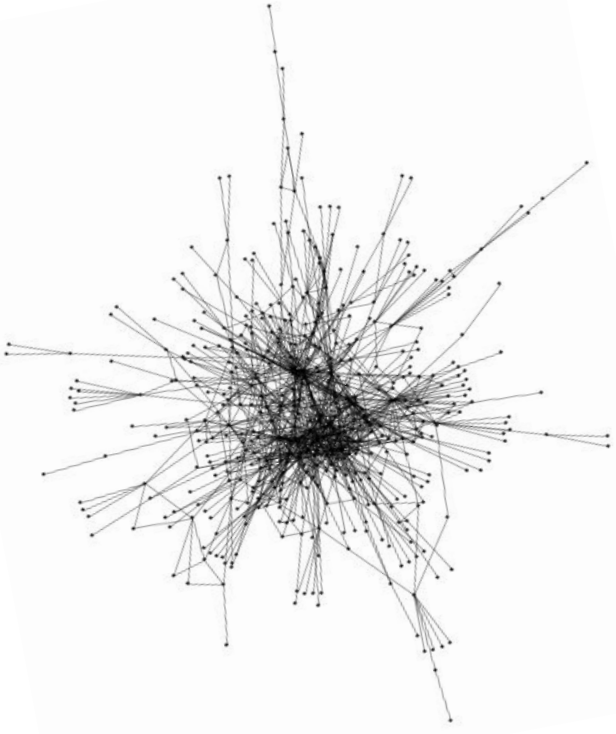
Graph	Barabási–Albert Network nx.barabasi_albert_graph(300,3)
Nodes	300
Edges	891



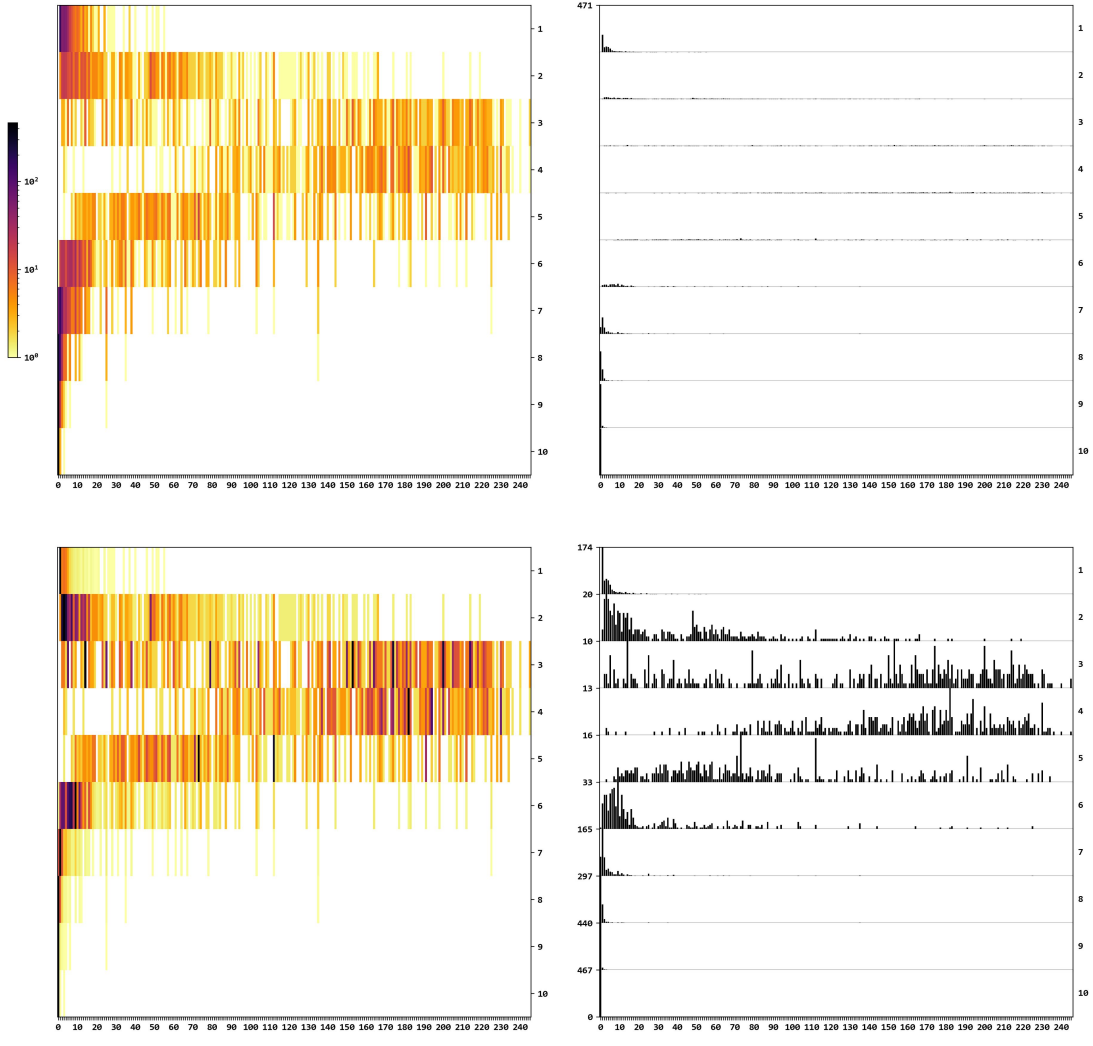
A model “small-world” network.



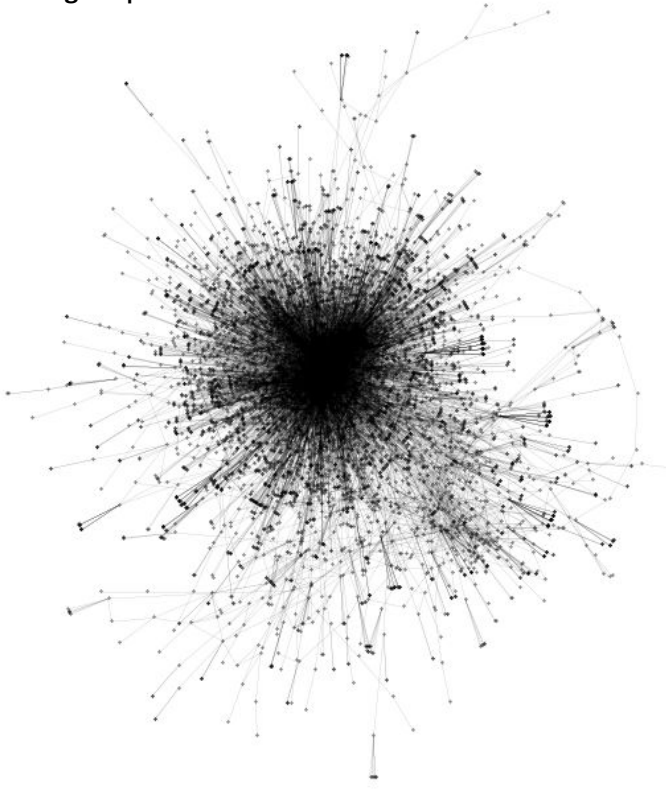
Graph	Semantic Network "Fortran" direditedock/infobox_interlinker
Nodes	471
Edges	1169



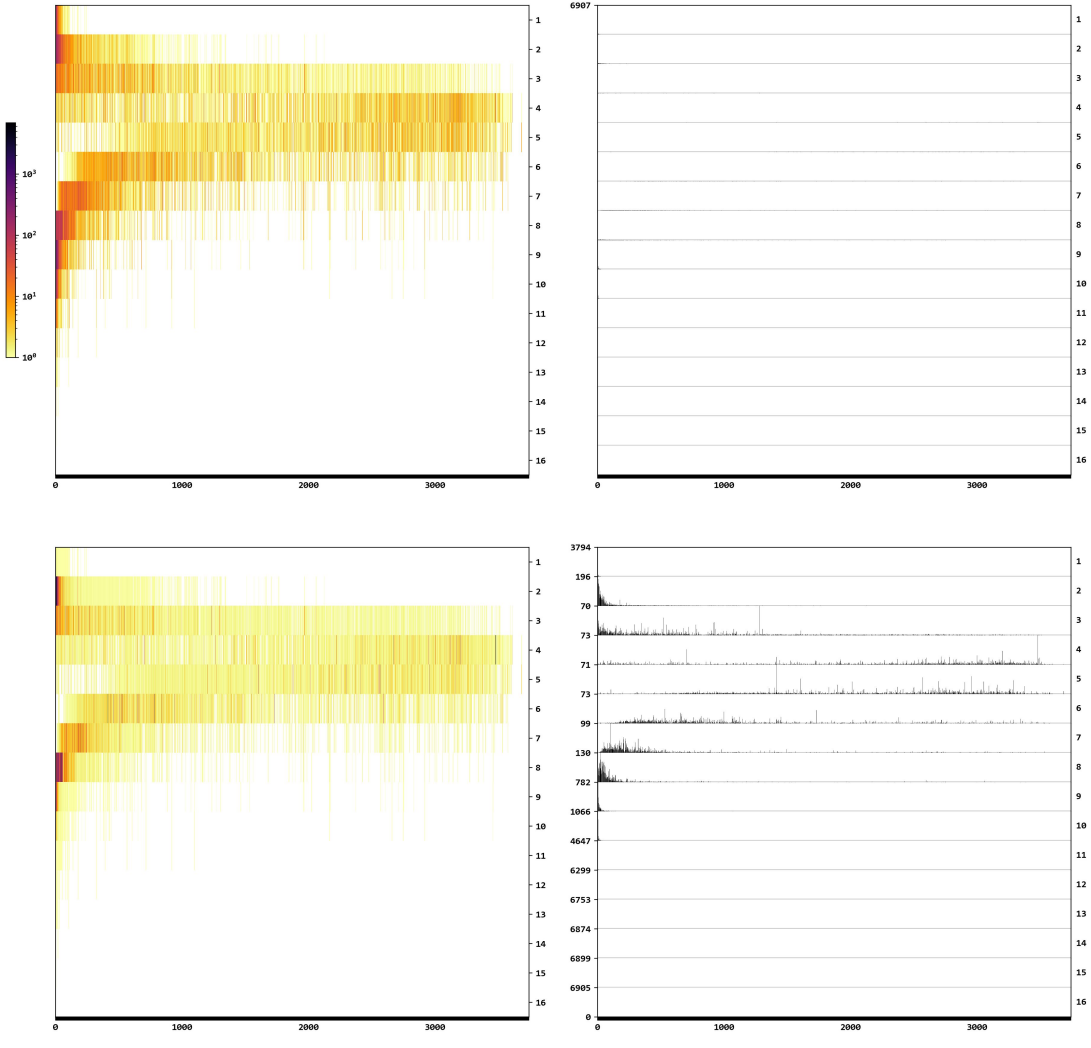
A scraped network of Wikipedia infoboxes.



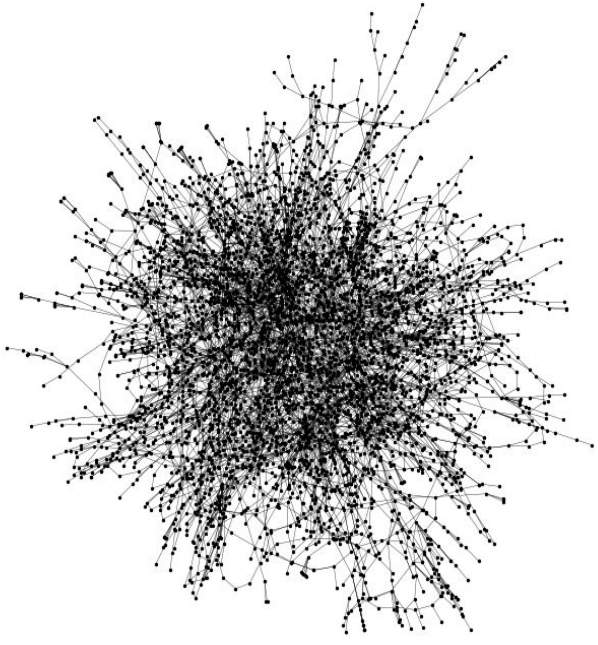
Graph	Semantic Network "Carl_Jung" direditedock/infobox_interlinker
Nodes	6907
Edges	16592

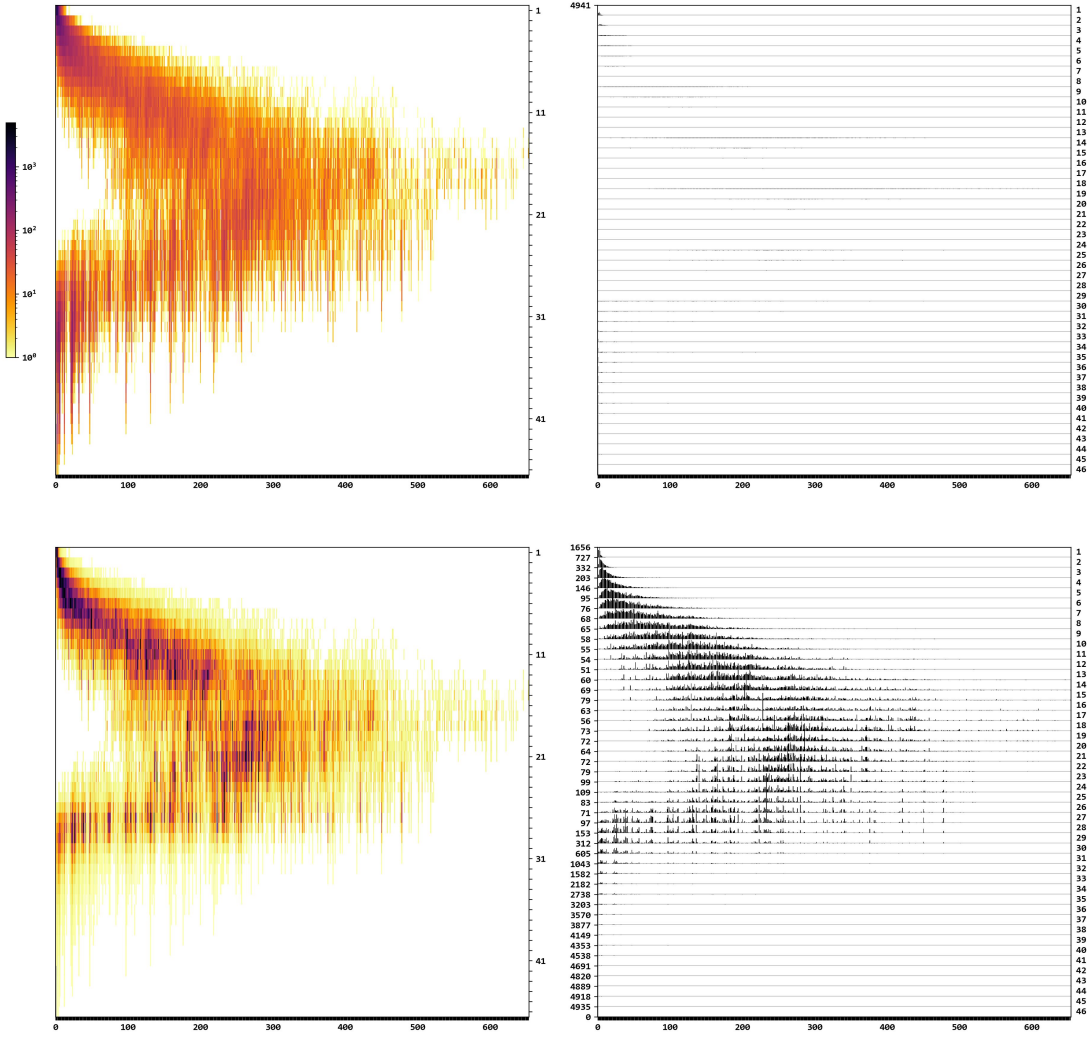


Another scraped network of Wikipedia infoboxes.



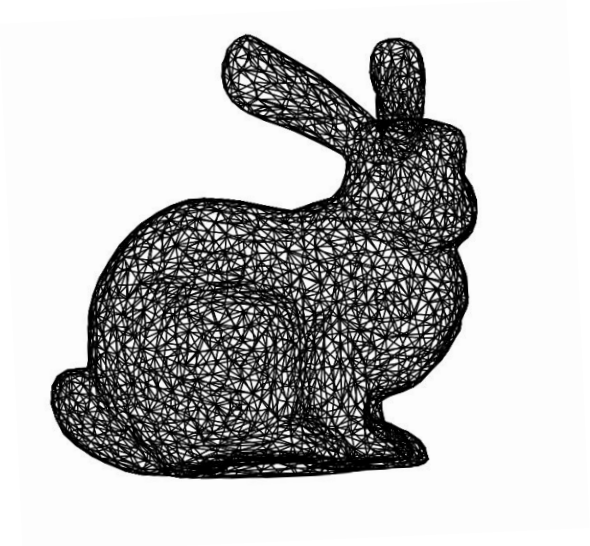
Graph	Western States Power Grid personal.umich.edu/~mejn/netdata
Nodes	4941
Edges	6594

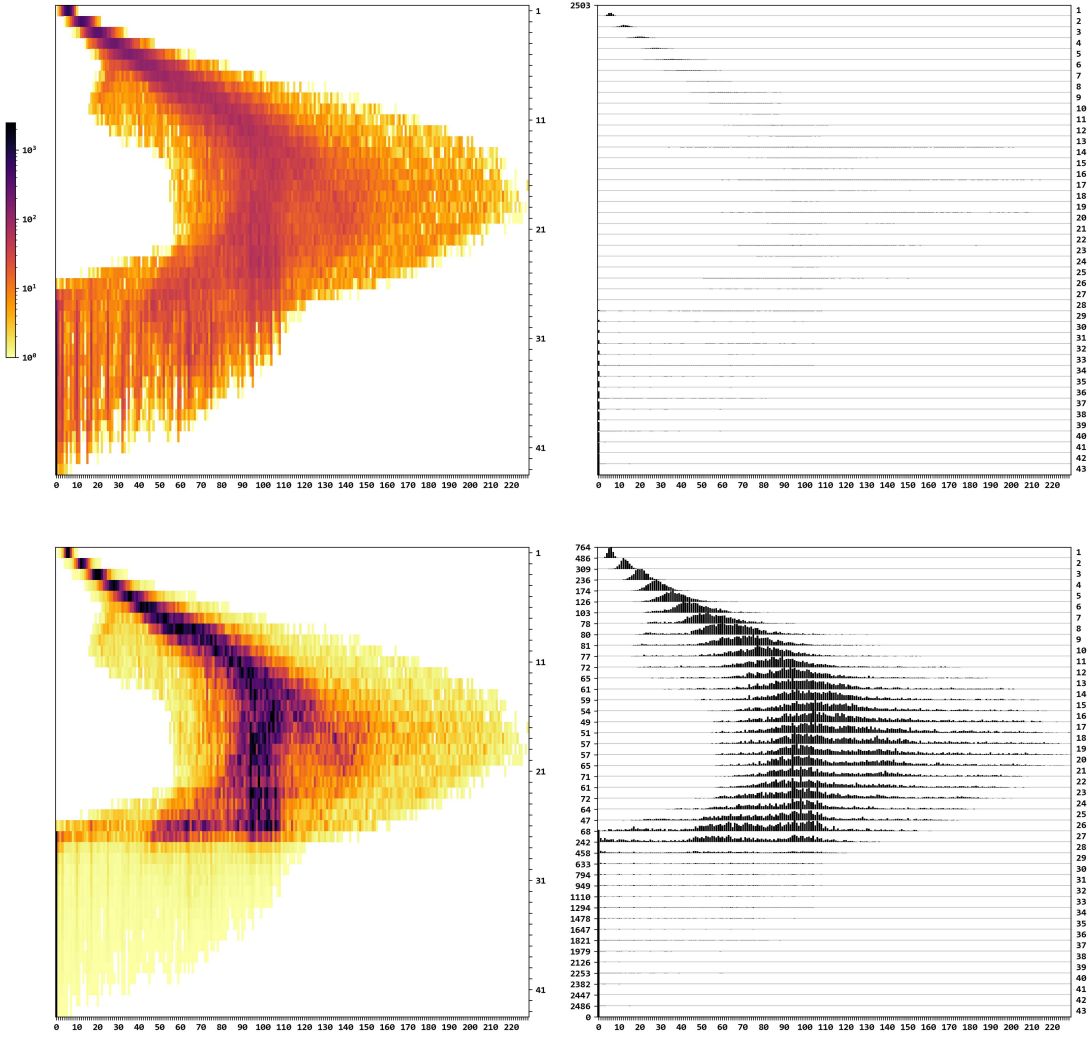




A real-world grid-like network.

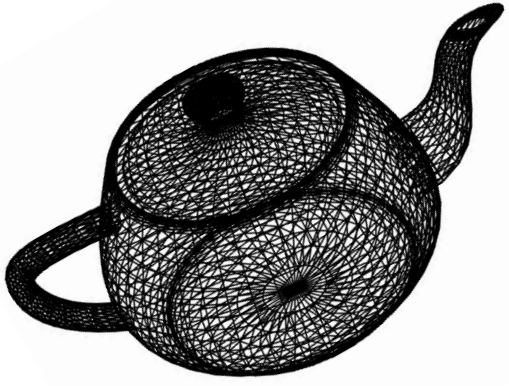
Graph	Stanford Bunny (edgelist)
Nodes	2593
Edges	7048

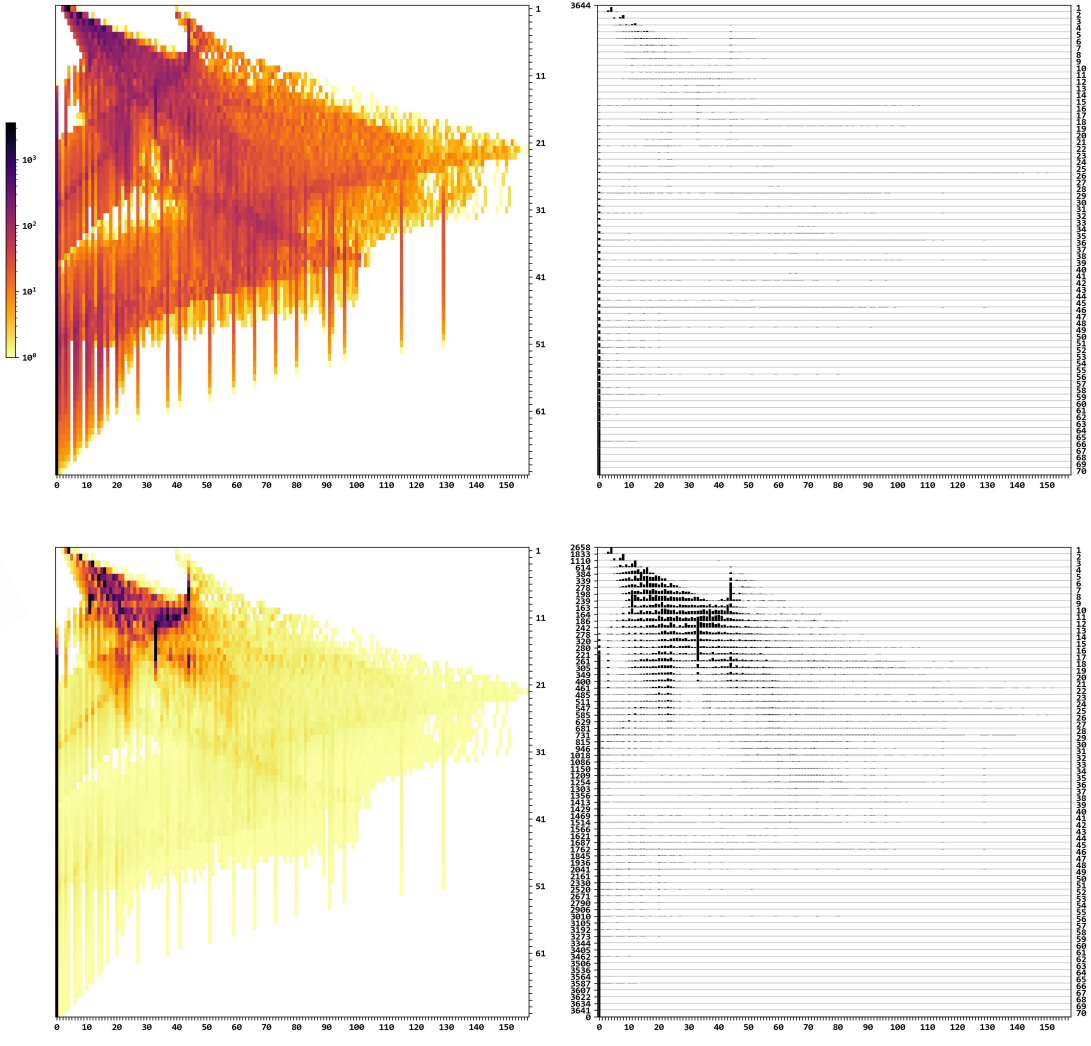




A famous example from computer graphics.

Graph	Utah Teapot (edgelist)
Nodes	3644
Edges	6870





Another famous example from computer graphics.

IN SUMMARY

`BMatrix_Explainer` is a collection of Python scripts for the exploration and analysis of networks through the B-Matrix data abstraction.

REQUIRED PYTHON LIBRARIES

`numpy`
`networkx`
`matplotlib`

MAIN SCRIPTS

`algorithm_BMatrix.py`

Modified from `github.com/bagrow/portraits`, exports the B-Matrix as a numpy array from a networkx graph object as input.

`plot_BMatrix_colormap.py`

Exports the canonical heatmap network portrait visualization, with the addition of optional row-normalization.

`plot_BMatrix_histogram.py`

Exports an alternative monochromatic bar chart histogram visualization, also with optional row-normalization.