# THE ARC TOPOLOGY + PERFECTOIDS

## 1. Almost purity

Before we start, I want to explain what a perfectoid ring is in full generality.

Definition 1.1. Let $K$ be a non-archimedean field. A Banach $K$-algebra is a $K$-algebra R equipped with a function $|\cdot| \to \mathrm{R}_{\geq 0}$ with the following properties:

- $|\cdot|$ extends the valuation on $K$.
- If $|f| = 0$, then $f = 0$.
- We have $|fg| \leq |f||g|$.
- We have $|f+g| \leq \max(|f|, |g|)$.
- R is complete wrt the metric $d(f,g) := |f-g|$.

Morphisms are just continuous $K$-algebra maps (so the preserve the topology induced by the norm, but not necessarily the norm itself).

In this setting, if $K$ is a perfectoid field, we have the following more classical definition of a perfectoid ring:

Definition 1.2. A perfectoid ring R over a perfectoid field $K$ is a uniform Banach $K$-algebra R (meaning $\mathrm{R}^\circ$ is open and bounded) and there is a pseudouniformizer (topologically nilpotent unit) $\pi \in \mathrm{R}$ such that $\pi^p | p$ and the Frobenius map

$$\mathrm{R}^\circ/\pi \to \mathrm{R}^\circ/\pi^p$$

is an isomorphism.

Remark 1.3. There is a more general definition that uses uniform Tate rings (and similarly for everything with adic spaces). In particular, complete Tate $K$-algebras R are equivalent to the category of Banach $K$-algebras.

Any Banach $K$-algebra is a complete Tate ring, by setting $\mathrm{R}_0$ to be the unit ball and picking a topologically nilpotent unit $\pi$ in $K^\times$.

Conversely, define a Banach norm, for example $|r| = \inf_{n \in \mathbf{Z}: \pi^n r \in \mathrm{R}_0} 2^n$.

The definition does not depend on the choice of psuedouniformizer.

> LEMMA 1.4. In the previous definition, we can replace the conditions after uniform Banach $K$-algebra with $\mathrm{R}^\circ/p \twoheadrightarrow \mathrm{R}^\circ/p$. In particular, the choice of psuedouniformizer doesn't matter.

*Proof.* The map $\mathrm{R}^\circ/\pi \to \mathrm{R}^\circ/\pi^p$ is automatically injective when $\pi^p|p$. Given an element $x \in \mathrm{R}^\circ$, if $x \pmod{\pi}$ is sent to zero under Frobenius, $x^p = \pi^p y$ for some $y \in \mathrm{R}^\circ$. But then $x/\pi \in \mathrm{R}$ lies in $\mathrm{R}^\circ$, as the $p$th power (namely $y$) does. It follows that $x \pmod{\pi}$ was zero. Note that also that if we only assume $\mathrm{R}^\circ/p \twoheadrightarrow \mathrm{R}^\circ/p$, such $\pi$ exists since $K$ is perfectoid.

Hence, the task is to show $\mathrm{R}^\circ/\pi \twoheadrightarrow \mathrm{R}^\circ/\pi^p$ if and only if $\mathrm{R}^\circ/p \twoheadrightarrow \mathrm{R}^\circ/p$. One direction is easy: if we get surjectivity modulo $p$, then $\mathrm{R}^\circ/\pi \twoheadrightarrow \mathrm{R}^\circ/\pi^p$. Indeed, we have a square

$$\begin{array}{ccc} \mathrm{R}^\circ/p & \xrightarrow{\Phi} & \mathrm{R}^\circ/p \\ \downarrow & & \downarrow \\ \mathrm{R}^\circ/\pi & \xrightarrow{\Phi} & \mathrm{R}^\circ/\pi^p \end{array}$$

Now every map is surjective except the bottom, so the claim follows.

In the other direction, successive approximation using surjectivity of the bottom arrow shows any $x \in \mathrm{R}^\circ$ has the form

$$x = \sum_{i \geq 0} \pi^{pi} x_i^p$$

for $x_i \in \mathrm{R}^\circ$. It follows

$$x - (x_0 + \pi x_1 + \pi^2 x_2 + \ldots)^p \in p\mathrm{R}^\circ$$

by expanding. Thus, we can get any element of $\mathrm{R}^\circ/p$ from Frobenius: take your desired element, find a lift $x$, use the above to find $x_0 + \pi x_1 + \pi^2 x_2 + \ldots$, then reduce this modulo $p$. $\square$

> EXAMPLE 1.5. A nice example is $\mathbf{Q}_p(p^{1/p^\infty})^\wedge\langle \mathrm{T}^{1/p^\infty}\rangle$. Here, $\mathbf{Q}_p(p^{1/p^\infty})^\wedge$ is a perfectoid field since the value group is clearly not discrete. The valuation ring is $\mathbf{Z}_p[p^{1/p^\infty}]$, and modulo $p$ it's semiperfect since
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> $$\mathbf{Z}_p[p^{1/p^\infty}] \simeq (\mathbf{Z}_p[t^{1/p^\infty}]/(t-p))^\wedge$$
>
> and reducing this mod $p$ gives $\mathbf{F}_p[t^{1/p^\infty}]/t$ which is semiperfect hence Frobenius is surjective.

> The Banach algebra structure is given by taking $\mathrm{R}^\circ = \mathbf{Z}_p(p^{1/p^\infty})^\wedge\langle \mathrm{T}^{1/p^\infty}\rangle$ and picking a psuedouniformizer $\pi$. Then equip $\mathrm{R} = \mathrm{R}^\circ[1/\pi]$ with $|r| = \inf_{n\in\mathbf{Z}:\pi^n r\in\mathrm{R}^\circ} 2^n$.
>
> Then $\mathrm{R}^\circ$ is open and powerbounded, so it remains only to check surjectivity of Frobenius on $\mathrm{R}/p$. A similar calculation shows we get another semiperfect ring.

The target theorem is the following.

> THEOREM 1.6 (Almost purity). Let R be perfectoid ring. For any finite étale R-algebra S, we know S is perfectoid and the algebra $\mathrm{S}^\circ$ is almost finite étale over $\mathrm{R}^\circ$.
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> More generally, take a perfectoid affinoid $K$-algebra $(\mathrm{R}, \mathrm{R}^+)$. Then $\mathrm{R}^{+a}_{\text{fét}} \simeq \mathrm{R}_{\text{fét}}$.

Let me give an example due to Scholze which illustrates the reason for the name.

Consider the rings

$$\mathrm{R}_m = \mathbf{Z}_p[p^{1/p^m}, \mathrm{T}^{\pm 1/p^m}].$$

Then $\mathrm{R}_0 = \mathbf{Z}_p[\mathrm{T}^\pm]$ is a torus. These are all smooth over $\mathbf{Z}_p[p^{1/p^m}]$. Suppose we are handed some finite normal $\mathrm{R}_0$-algebra $\mathrm{S}_0$, and we know that $\mathrm{S}_0[1/p]$ is étale. The thing that prevents $\mathrm{S}_0$ from being étale is that there is some possible ramification on the special fiber.

The idea is to attempt to fix this by using the ramified tower $\mathrm{R}_m$, and setting $\mathrm{S}_m := \mathrm{Norm}(\mathrm{S}_0 \otimes_{\mathrm{R}_0} \mathrm{R}_m)$, where here we take the normalization of this tensor product.

This almost gets rid of the ramification when we look at $\mathrm{S}_m \to \mathrm{R}_m$. At the generic point of the special fiber, the local ring is a DVR. The discriminant of the extension of DVRs becomes small as $m \to \infty$.

If this ramification actually becomes trivial for some $m$, then Zariski-Nagata purity saves the day. Indeed, the ramification locus of $\mathrm{S}_m$ over $\mathrm{R}_m$ has to be pure of codimension one, and we know what these points are. They are either characteristic zero in which case $\mathrm{S}_0[1/p]$ being étale takes care of it, or they are the generic point of the special fiber. We have just ruled out the latter, so there must be no ramification at all.

Almost purity, as stated above, tells us that it is always the case that $\mathrm{S}_\infty$ becomes almost étale over $\mathrm{R}_\infty$.

Now, how do we prove this theorem? There is an almost isomorphism $\mathrm{R}^+ \to \mathrm{R}^\circ$, so it suffices to prove it for $\mathrm{R}^\circ$. So far, we have completed the following diagram for R perfectoid:

$$\begin{tikzcd} \mathrm{R}_{\text{fét}} & \mathrm{R}^{\circ a}_{\text{fét}} \arrow[l] \arrow[r, "\sim"] & (\mathrm{R}^{\circ a}/\pi)_{\text{fét}} \arrow[d, "\sim"] \\ \mathrm{R}^{\flat}_{\text{fét}} & \mathrm{R}^{\flat\circ a}_{\text{fét}} \arrow[l, "\sim"'] \arrow[r, "\sim"] & (\mathrm{R}^{\flat\circ a}/\pi^{\flat})_{\text{fét}} \end{tikzcd}$$

where the arrow on the bottom left is because we know almost purity in characteristic $p$. What remains is almost purity, or rather showing the top morphism is actually an isomorphism.

From tilting, we know already $\mathrm{R}^{\flat}_{\text{fét}} \to \mathrm{R}_{\text{fét}}$ is fully faithful. Also, we know the theorem in the case of a field. Thus, the real content of the theorem is that the functor is actually essentially surjective in general outside of the case of a field.

The idea of the proof is to use the geometry of the perfectoid space $\mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$ to localize and reduce to the case of a field, where we already know the full story.

This crucially uses rigid analytic geometry: if we were to try the same strategy with $\mathrm{Spec}\,\mathrm{R}$, the stalks would be far from correct since we need to get an actual field after completion. There are not enough opens in the Zariski topology to accomplish this.

For this, we will need some definitions in rigid analytic geometry.

## 2. Perfectoid spaces: crash course

The formalism for perfectoid spaces I will use here is Huber's adic spaces, because this is largely what has been adopted in many papers that use these objects.

Let $K$ be a non-archimedean field. The idea of rigid analytic spaces or more generally adic spaces is to try to emulate what happens with complex analytic spaces over $\mathbf{C}$ in a non-archimedean setting. There are several models of rigid-analytic geometry one can use. I'll be using adic spaces since Scholze uses these for perfectoid spaces.

On important requirement is that there should be an analytification functor

$$\mathrm{Var}_K \to \{\text{rigid analytic spaces}/K\}$$

and we should also expect some form of GAGA to hold when the variety is proper. Another important thing is that unlike algebraic geometry, for $f \in \Gamma(\mathrm{X}^{\mathrm{ad}}, \mathcal{O}_{\mathrm{X}^{\mathrm{ad}}})$ we should be able to make sense of not just the vanishing locus of $f$ but also the set $\{x \in \mathrm{X}^{\mathrm{ad}} : |f(x)| \leq 1\}$. As we'll see, this is essentially the added content that makes it analytic when compared to algebraic geometry.

This idea means that we ought to be able to attach, for any $x \in \mathrm{X}^{\mathrm{ad}}$, a valuation function

$$f \mapsto |f(x)|$$

on functions. Let me be precise about what valuation means - these are really more like seminorms, but this is the usual terminology.

DEFINITION 2.1. Let $\mathrm{R}$ be a ring. A valuation on $\mathrm{R}$ is a multiplicative map

$$|\cdot| : \mathrm{R} \to \Gamma \cup 0$$

where $\Gamma$ is a totally ordered abelian group (written multiplicatively). We ask that $|0| = 0, |1| = 1$ and also

$$|x + y| \leq \max(|x|, |y|)$$

for all $x, y \in \mathrm{R}$.

We say two valuations are equivalent if there is an isomorphism of the totally ordered abelian groups taking one to the other.

If $\mathrm{R}$ has a topology, which will be the case in our situation, we ask that $\{x \in \mathrm{R} : |x| < \gamma\}$ is always open.

Our next task is to define the equivalent of affine schemes in the rigid analytic world, affinoids. In the adic space formalism, these are specified by a pair of rings $(\mathrm{R}, \mathrm{R}^+)$ to which we associate a space $\mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$ of certain continuous valuations on $\mathrm{R}$.

We will restrict ourselves to the case of algebras over a non-archimedean field, as the full generality of Huber's theory isn't needed here.

DEFINITION 2.2. A Tate $K$-algebra $\mathrm{R}$ is a topological $K$-algebra $\mathrm{R}$ for which there exists a subring $\mathrm{R}_0 \subset \mathrm{R}$ such that $a\mathrm{R}_0$ for $a \in K^\times$ forms a basis of open neighborhoods of 0.

An affinoid $K$-algebra is a pair $(\mathrm{R}, \mathrm{R}^+)$ consisting of a Tate $K$-algebra $\mathrm{R}$ and an open integrally closed subring $\mathrm{R}^+ \subset \mathrm{R}^\circ$. A morphism of affinoid $K$-algebras is a $K$-algebra map $\mathrm{R} \to \mathrm{S}$ carrying $\mathrm{R}^+$ to $\mathrm{S}^+$.

The typical example takes $\mathrm{R}$ to be a quotient of $K\langle \mathrm{T}_i \rangle$, and $\mathrm{R}^+ = \mathrm{R}^\circ = K^\circ\langle \mathrm{T}_i \rangle$. In everything that follows, $|f(x)|$ denotes the valuation $x$ applied to the function $f \in \mathrm{R}$. We use $[x]$ to denote the equivalence class of a valuation.

DEFINITION 2.3. Given an affinoid $K$-algebra $(\mathrm{R}, \mathrm{R}^+)$, we give $\mathrm{Spa}(\mathrm{R}, \mathrm{R}^+) := \{[x] : |f(x)| \leq 1, f \in \mathrm{R}^+\}$ the topology with basis given by the open rational subsets

$$\mathrm{U}\left(\frac{f_1, \ldots, f_n}{g}\right) := \{x \in \mathrm{Spa}(\mathrm{R}, \mathrm{R}^+) : |f_i(x)| \leq |g(x)| \neq 0\}.$$

Here, $f_i$ generate the unit ideal. Essentially, we ask that $|g(x)| \neq 0$ is open (like $D(g)$ in algebraic geometry) and $|f(x)| \leq 1$ is open (a feature we want in rigid geometry).

We will eventually want to make this the underlying topological space in an upgraded notion of a locally ringed space; the role is analogous to $\operatorname{Spec} \mathrm{R}$ in algebraic geometry.

When first seeing a definition of adic spaces, it makes sense that $\mathrm{R}$ appears but the role of $\mathrm{R}^+$ is confusing because it is unclear what exactly it is doing, or why it needs to be an open integrally closed subring of $\mathrm{R}^\circ$.

I want to first demystify the conditions on $\mathrm{R}^+$.

> Lemma 2.4. There is a bijection between sets of equivalence classes $F_S = \bigcap_{f\in S}\{[x] : |f(x)| \le 1\}$ for arbitrary subsets $S \subset \mathrm{R}^\circ$ and open and integrally closed subrings $\mathrm{R}^+ \subseteq \mathrm{R}^\circ$.
>
> We send
> $$\mathrm{R}^+ \mapsto \operatorname{Spa}(\mathrm{R}, \mathrm{R}^+) = \{[x] : |f(x)| \le 1, f \in \mathrm{R}^+\},$$
> and
> $$F_S \mapsto \{f \in \mathrm{R} : |f(x)| \le 1 \text{ for all } x \in F_S\}.$$

Thus, we can think of the set $\operatorname{Spa}(\mathrm{R}, \mathrm{R}^+)$ as imposing the condition $|f(x)| \le 1$ for $f$ in a particular subset on the space of valuations. Taking $\mathrm{R}^+ = \mathrm{R}^\circ$ means we impose the most conditions. It is not reasonable to ask for such a bound when $f$ is not power bounded. In summary, $\mathrm{R}^+$ is really just encoding which functions $f \in \mathrm{R}$ are $\le 1$, or the functions to the unit disk.

The conditions on $\mathrm{R}^+$ (being an open and integrally closed) arise naturally as a result of this lemma.

I also want to explain why it is necessary to allow possibilities for $\mathrm{R}^+$ other than $\mathrm{R}^\circ$. Indeed, if you look at classical rigid geometry we only use this. One important reason is that even when defining $\operatorname{Spa}(\mathrm{R}, \mathrm{R}^\circ)$ we would want the open rational subsets to again be affinoids.

Having defined $\operatorname{Spa}(\mathrm{R}, \mathrm{R}^+)$ and motivated $\mathrm{R}^+$, we will formally define an affinoid adic space.

> Definition 2.5. Let $\mathrm{U}\left(\frac{f_1,\ldots,f_n}{g}\right)$ be a rational open in $\operatorname{Spa}(\mathrm{R}, \mathrm{R}^+)$. Let
> $$\mathrm{B} \subseteq \mathrm{R}\left[\frac{f_i}{g}\right]$$

be the integral closure of $R^+[\frac{f_i}{g}]$ in $R[\frac{f_i}{g}]$. Topologizing $R[\frac{f_i}{g}]$ by making $aR_0[\frac{f_i}{g}]$ for $a \in K^\times$ a basis of opens at 0, we get an affinoid $K$-algebra $(R[\frac{f_i}{g}], B)$. Upon completion, we get $(R\langle \frac{f_i}{g} \rangle, \widehat{B})$.

Define presheaves $\mathcal{O}_X(U), \mathcal{O}_X^+(U)$ on $X = \mathrm{Spa}(R, R^+)$ by

$$(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) = \left(R\left\langle \frac{f_i}{g} \right\rangle, \widehat{B}\right).$$

In particular, $\mathcal{O}_X(U)$ allows all convergent series in $f_i/g$ to be considered as functions on it.

Now for some standard facts:

THEOREM 2.6. Let $X = \mathrm{Spa}(R, R^+)$. Under a Noetherian condition, $\mathcal{O}_X(U)$ and $\mathcal{O}_X^+(U)$ are sheaves (for example, if R is a quotient of $K\langle T_1, \dots T_n \rangle$).

For a rational open, we have $U \simeq \mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$. The stalks $\mathcal{O}_{X,x}$ are local rings, making $(X, \mathcal{O}_X)$ into a locally ringed space.

We have now equipped equipped X with a locally ringed space structure! To account for the sheaf $\mathcal{O}_X^+$, we make a modification.

DEFINITION 2.7. The category $\mathcal{V}$ consists of locally ringed topological spaces $(X, \mathcal{O}_X)$ where X is a sheaf of complete topological $K$-algebras along with a continuous valuation

$$f \mapsto |f(x)|$$

on $\mathcal{O}_{X,x}$ for every $x \in X$.

A morphism is a morphism of locally topologically ringed spaces which are continuous $K$-algebra morphisms on $\mathcal{O}_X$ and compatible with the valuations.

The data of $\mathcal{O}_X^+$ is given by the valuations: we always have

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) : |f(x)| \leq 1, x \in U\}.$$

Thus, $(R, R^+)$ naturally gives an object in $\mathcal{V}$, which we call affinoid adic spaces.

DEFINITION 2.8. An adic space is an object of $\mathcal{V}$ locally isomorphic to an affinoid adic space $\mathrm{Spa}(R, R^+)$.

After this formalism, it's important go through some examples.

EXAMPLE 2.9 (The disk). Consider the adic space $\mathrm{Spa}(\mathbf{C}_p\langle \mathrm{T}\rangle, \mathcal{O}_{\mathbf{C}_p}\langle \mathrm{T}\rangle)$.

Let me point out some obvious points of this space. For any $x \in \mathbf{C}_p$ with $|x| \leq 1$, we obtain a valuation

$$f \mapsto |f(x)|_{\mathbf{C}_p} \in \mathrm{R}_{>0}$$

by literally evaluating $f \in \mathbf{C}_p\langle \mathrm{T}\rangle$ at $x$. This corresponds to a maximal ideal of $\mathbf{C}_p\langle \mathrm{T}\rangle$, by taking the kernel of $f \mapsto f(x) \in \mathbf{C}_p$. These classical points are why this is called the (closed) adic unit disk.

Next, we can see the use of having valuations which are rank $> 1$. You might attempt to decompose this as a topological space via

$$\mathrm{Spa}(\mathbf{C}_p\langle \mathrm{T}\rangle, \mathcal{O}_{\mathbf{C}_p}\langle \mathrm{T}\rangle) = \{x : |\mathrm{T}(x)| = 1\} \cup \{x : \cup_{\varepsilon>0}|\mathrm{T}(x)| < 1 - \varepsilon\}$$

which are both open. Geometrically, this is breaking a closed disk into the open disk and the boundary.

If we only had these classical points, as in rigid analytic geometry, the disk would fail to be connected (this is part of why there we cannot use an honest topology). However, points corresponding to rank $> 1$ valuations fix the issue. In general there are 5 types of points, the 5th one having value group $\mathrm{R}_{>0} \times \gamma^{\mathbf{Z}}$ with lexicographic ordering and $\gamma > 1$. For $x \in \mathbf{C}_p$ with $|x| \leq 1$ and $r \in (0, 1]$, define when $f = \sum_{n\geq 0} a_n(\mathrm{T} - x)^n$ the valuation

$$|f(x_{r^-})| = \max_n |a_n| r^n \gamma^{-n} \in \mathrm{R}_{>0} \in \gamma^{\mathbf{Z}}.$$

There is also $x_{r^+}$, where we take positive powers of $\gamma$.

This indeed gives a valuation, and we will note that $0_{1^-}$ does not lie in either open. Putting $f = \mathrm{T}$, $|\mathrm{T}(0_{1^-})| = (1, \gamma^{-1})$. This does not equal 1, but it is also not $\leq 1 - \varepsilon$ due to the rank two ordering.

Note that if you take $K = \mathbf{C}_p$, the functor of points of $\mathrm{Spa}(\mathbf{C}_p\langle \mathrm{T}\rangle, \mathcal{O}_{\mathbf{C}_p}\langle \mathrm{T}\rangle)$ spits out $\mathrm{R}^+$ (which justifies the earlier slogan).

We can make adic spaces out of perfectoid rings as well, which gives perfectoid spaces:

DEFINITION 2.10. Let $K$ be a perfectoid field. A perfectoid space X is an adic space which is locally isomorphic to an affinoid perfectoid space $\mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$, which simply means R is a perfectoid ring over $K$.

Note that $\mathrm{R}^+$ is almost equal to $\mathrm{R}^\circ$, as in this case

$$\mathfrak{m}\mathrm{R}^\circ \subseteq \mathrm{R}^+ \subseteq \mathrm{R}^\circ.$$

This is because topologically nilpotent elements lie in $\mathrm{R}^+$ due to it being integrally closed.

It is also a theorem you have to prove that the structure sheaf is a sheaf here, because it falls outside of the Noetherian hypothesis we stated earlier that ensures this.

> EXAMPLE 2.11. Affine examples of perfectoid rings are clear.

## 3. Proof of almost purity

Now we're ready to use this new geometry to help localize the proof of almost purity to the case of fields.

> DEFINITION 3.1. A map $f : (\mathrm{A}, \mathrm{A}^+) \to (\mathrm{B}, \mathrm{B}^+)$ of affinoid $K$-algebras finite étale if $\mathrm{A} \to \mathrm{B}$ is, and $\mathrm{B}^+$ is the integral closure of $f(\mathrm{A}^+)$ in $\mathrm{B}$.

This extends to adic spaces in the obvious way by taking a cover by affinoids.

> DEFINITION 3.2. Suppose we are working with perfectoid $K$-algebras, where $K$ is also perfectoid. Then a map is strongly finite étale if additionally $\mathrm{B}^+$ is almost finite étale over $\mathrm{A}^+$.
>
> This also globalizes to perfectoid spaces in the obvious way.

This definition will later be redundant after we know almost purity.

> LEMMA 3.3. If $\mathrm{Y}$ is affinoid perfectoid and $f : \mathrm{X} \to \mathrm{Y}$ is strongly finite étale, then $\mathrm{X}$ is affinoid perfectoid and also
> $$(\mathcal{O}_\mathrm{Y}(\mathrm{Y}), \mathcal{O}_\mathrm{Y}^+(\mathrm{Y})) \to (\mathcal{O}_\mathrm{X}(\mathrm{X}), \mathcal{O}_\mathrm{X}^+(\mathrm{X}))$$
> is strongly finite étale.

I'm not proving this, but I do want to just remark that this is not immediately trivial since it relies on a result of Elkik-Gabber-Romero which we'll use later that requires some work to prove.

The following is now basically immediate:

> COROLLARY 3.4. Let $\mathrm{X} = \mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$ be an affinoid perfectoid space. The following are true:

- We have $\mathrm{sf\acute{e}t}(X) \simeq R^{+a}_{\mathrm{f\acute{e}t}}$. In particular, we can tilt the category of strongly finite étale maps.
- The functor $\mathrm{sf\acute{e}t}(X) \to R_{\mathrm{f\acute{e}t}}$ is fully faithful.
- For rational subsets $U \subseteq X$, the assignment
$$U \mapsto \mathrm{sf\acute{e}t}(U)$$
is a sheaf of categories.

*Proof.* For the first assertion, the previous lemma shows that we can check strongly finite étale on global sections in the affinoid case (which a priori needs to be check on a cover by affinoids). In particular, strongly finite étale maps are the same as almost finite étale covers of $R^+$ (the condition on $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ being étale is automatic from $\mathcal{O}_Y^+(Y) \to \mathcal{O}_X^+(X)$ being almost finite étale). It is tiltable because after the equivalence we can work in the almost setting, where this is already done.

For the second assertion, after we have the first this reduces to the full faithfulness we already knew in almost purity.

For the final assertion, this is again precisely the previous lemma. □

The strategy is now to show that there is an equivalence of sheaves of categories
$$\eta : \mathrm{sf\acute{e}t}(U) \to \mathcal{O}_X(U)_{\mathrm{f\acute{e}t}}$$
given by inverting $\pi$.

Knowing this is fully faithful for each individual $U$ and that $\mathrm{sf\acute{e}t}(U)$ is a sheaf of categories on $K$-rational subsets of $X$, we are then reduced to verifying that as a presheaf $\mathcal{O}_X(U)_{\mathrm{f\acute{e}t}}$ is separated and that $\eta$ is an equivalence on stalks.

Here, $\mathcal{F}$ separated means that
$$\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$$
is injective. The main content is checking an equivalence on stalks, which we will do by reducing to almost purity for fields.

Before starting, recall the following result which is crucial to the reduction to the case of a field when computing stalks of these sheaves.

THEOREM 3.5 (Elkik-Gabber-Romero). Let $R$ be a flat $K^\circ$-algebra Henselian along $(\pi)$. Then
$$R[1/\pi]_{\mathrm{f\acute{e}t}} \simeq \widehat{R}[1/\pi]_{\mathrm{f\acute{e}t}}$$
where on the right we have $\pi$-adically completed $R$.

This is a nontrivial theorem, and I won't discuss the proof here. I will note that a related result, the Fujiwara-Gabber theorem, tells us that the étale cohomology will agree. The proof of this can be achieved by showing étale cohomology satisfies arc-descent, and proving a rigidity result.

It is tempting to apply this to the category of finite étale covers and attempt to show arc-descent for it, but the proof crucially uses the fact that equivalences in the derived $\infty$-category $\mathrm{D}(\Lambda)$ can be checked after pullbacks which won't work a categorical level higher. Nevertheless, it remains true.

Finishing the proof of almost purity requires a stalk calculation in adic spaces. It is here that we see why $\operatorname{Spec} \mathrm{R}$ really fails as this task:

> LEMMA 3.6. Let $\mathrm{X}$ be an adic space over $K$, and suppose $\varpi \in K$ is topologically nilpotent. Then for $x \in \mathrm{X}$ the $\varpi$-adic completion of $\mathcal{O}^+_{\mathrm{X},x}$ is the $\varpi$-adic completion of $k(x)^+$.

This is really surprising if you're thinking in terms of algebraic geometry, since this really shouldn't be close to a field. The reason can be seen via example, say with $\mathbb{D} = \mathrm{Spa}(\mathbf{C}_p\langle T\rangle, \mathcal{O}_{\mathbf{C}_p}\langle \mathrm{T}\rangle)$. The element $p$ in the base field $\mathbf{C}_p$ is topologically nilpotent. At 0, there is a map

$$\mathcal{O}^+_{\mathbb{D},0} \to k(0)^+ = \mathcal{O}_{\mathbf{C}_p}$$

which has dense image. The claim is then that in the kernel of this map, say I, the elements are $p$-divisible. We can evaluate elements of the stalk at 0 to get the map $f \mapsto |f(0)| \in k(0)^+ = \mathcal{O}_{\mathbf{C}_p}$. The element $\mathrm{T}$ is going to be in the kernel, but the surprising this is that so is $\mathrm{T}/p$.

Indeed, take $\mathcal{O}^+_{\mathrm{X}}(\mathrm{U})$ for $\mathrm{U} = \{x : |\mathrm{T}(x)| \leq |p|\}$. Then $\mathrm{T} \in p\mathcal{O}^+_{\mathrm{X}}(\mathrm{U})$, as $\mathcal{O}^+_{\mathrm{X}}(\mathrm{U}) \supseteq \mathcal{O}_{\mathbf{C}_p}[\mathrm{T}/p]$ (it is the integral closure). It follows that since $\mathrm{U}$ exists, the element $\mathrm{T}/p$ makes sense in the stalk. Further, $\mathrm{T}/p$ lies in the kernel! This general sort of argument shows the kernel is $p$-divisible. This means that after $p$-adic completion the kernel gets killed, so the dense image claim just becomes an equivalence.

This behavior is not present in schemes, since such open sets do not exist and so will not be part of the stalk. Having understood this surprising point about stalks, it's now more believable that we can deduce almost purity from the field case by looking at stalks.

> THEOREM 3.7. The morphism $\eta$ is an equivalence of sheaves of categories on rational subsets of $\mathrm{X} = \mathrm{Spa}(\mathrm{R}, \mathrm{R}^+)$.

*Proof.* As discussed, it suffices to check the equivalence on stalks. Precisely, for any $x \in \mathrm{X}$ we must show

$$\mathrm{colim}_{x\in\mathrm{U}}\,\mathcal{O}_{\mathrm{X}}^{+}(\mathrm{U})^{a}_{\text{fét}} \simeq \mathrm{colim}_{x\in\mathrm{U}}\,\mathcal{O}_{\mathrm{X}}(\mathrm{U})_{\text{fét}}$$

via inverting $\pi$.

Via tilting and almost purity in characteristic $p$,

$$\mathrm{colim}_{x\in\mathrm{U}}\,\mathcal{O}_{\mathrm{X}}^{+}(\mathrm{U})^{a}_{\text{fét}} \simeq \mathrm{colim}_{x^\flat\in\mathrm{U}^\flat}\,\mathcal{O}_{\mathrm{X}^\flat}^{+}(\mathrm{U}^\flat)^{a}_{\text{fét}} \simeq \mathrm{colim}_{x^\flat\in\mathrm{U}^\flat}\,\mathcal{O}_{\mathrm{X}^\flat}(\mathrm{U}^\flat)_{\text{fét}}.$$

Now observe that

$$\mathrm{colim}_{x^\flat\in\mathrm{U}^\flat}\,\mathcal{O}_{\mathrm{X}^\flat}^{+}(\mathrm{U}^\flat)$$

is Henselian along $\pi^\flat$. Moreover, it has completion equal to $\widehat{k(x^\flat)^{+}}$, as we saw earlier.

Then it follows by Elkik-Gabber-Ramero that

$$\mathrm{colim}_{x^\flat\in\mathrm{U}^\flat}\,\mathcal{O}_{\mathrm{X}^\flat}(\mathrm{U}^\flat)_{\text{fét}} \simeq \widehat{k(x^\flat)}_{\text{fét}}.$$

Therefore, we see $\mathrm{colim}_{x\in\mathrm{U}}\,\mathcal{O}_{\mathrm{X}}^{+}(\mathrm{U})^{a}_{\text{fét}} \simeq \widehat{k(x^\flat)}_{\text{fét}}$.

Similarly, we can also do the untilted version for $\mathrm{colim}_{x\in\mathrm{U}}\,\mathcal{O}_{\mathrm{X}}(\mathrm{U})_{\text{fét}}$ via Elkik-Gabber-Romero, which gives $\widehat{k(x)}_{\text{fét}} \simeq \widehat{k(x^\flat)}_{\text{fét}}$ via tilting for perfectoid fields.

Tracing through these equivalences, since almost purity in characteristic $p$ inverts $\pi^\flat$ the overall equivalence is given by inverting $\pi$. □

This completes the proof of almost purity after taking global sections.

## 4. Motivation for the arc topology

## 5. The arc topology

## 6. Arc descent for étale cohomology

## 7. Applications