HONDA-TATE THEORY

DYLAN PENTLAND

ABSTRACT. Over the complex numbers, it is (relatively) easy to understand the category of abelian varieties using polarizable integral Hodge structures of type (-1,0),(0,-1). Over a finite field this becomes much more difficult but is actually still possible!

Honda-Tate theory gives a complete description of the isogeny category of abelian varieties over $\mathbf{F}_q$. We'll go over what exactly this description entails and give a brief explanation about how the proof goes. At the end, I'll talk about a recent result of Centeleghe and Stix which gives a description of the category of abelian varieties over $\mathbf{F}_q$ which heavily uses this theory.

CONTENTS

1. MOTIVATION AND STATEMENTS OF HONDA-TATE THEORY

Let's start by recalling what an abelian variety is before we jump into anything.

> DEFINITION 1.1. An abelian variety over a field k is a smooth connected projective group variety over k.

As the name suggests, the group law on these is always commutative. In this talk we'll take $k = \mathbf{F}_q$ always, unless otherwise noted. The end goal of this talk will be to explain how recent results of Centeleghe and Stix have given a description of the category of abelian varieties over $\mathbf{F}_q$.

Over $\mathbf{C}$, we can easily describe abelian varieties. Let $\mathsf{AV}_{\mathbf{C}}$ be the category of abelian varieties over $\mathbf{C}$. There is an equivalence

$$\mathsf{AV}_{\mathbf{C}} \xrightarrow{\simeq} \{\text{polarizable } \mathbf{Z} - \text{Hodge structures of type } (-1,0),(0,-1)\}$$

via $\mathrm{A} \mapsto \mathrm{H}_1(\mathrm{A}(\mathbf{C}), \mathbf{Z})$. Thus, we can understand them entirely in terms of linear algebraic data. Over a finite field, this is not really possible. There's a result of Deligne that takes this idea of linear algebraic data as far as you can in positive characteristic on the full subcategory of ordinary abelian varieties over $\mathbf{F}_q$. In general, we need something more non-commutative.

What is easier in characteristic p is to describe what's called the isogeny category. Honda-Tate theory gives us a way to describe abelian varieties over $\mathbf{F}_q$ up to isogeny. This is a powerful tool, and can be used to understand the entire category of abelian varieties over $\mathbf{F}_q$.

> Definition 1.2. The category $\mathsf{AV}^0_{\mathbf{F}_q}$ of abelian varieties up to isogeny over $\mathbf{F}_q$ has objects given by abelian varieties over $\mathbf{F}_q$, and morphisms $\mathrm{Hom}(\mathrm{A}, \mathrm{B}) \otimes \mathbf{Q}$.

Over $\mathbf{C}$, this is also easy to describe: we use rational Hodge structures instead. How do we describe such a category when we work over $\mathbf{F}_q$? A major tool that helps us is the fact that it is a semisimple. The Poincaré decomposition theorem tells us that

$$\mathrm{A} \sim \prod_w \mathrm{A}_w$$

where A_w are simple abelian varieties. Thus, as an abelian category $\mathsf{AV}^0_{\mathbf{F}_q}$ is semisimple. It then suffices to understand the simple objects and their endomorphism algebras.

We'll start with simple objects.

> Definition 1.3. A Weil q-integer is an algebraic integer π such that under every embedding $\sigma : \mathbf{Q}(\pi) \to \mathbf{C}$ we have $|\sigma(\pi)| = q^{1/2}$. The set of these, up to having the same miminal polynomial, is denoted by $\mathrm{W}(q)/\sim$.

Let π_A denote the Frobenius endomorphism in $\mathrm{End}^0(\mathrm{A})$. For a simple abelian variety, this is a division ring, and $\mathbf{Q}(\pi_\mathrm{A})$ is a finite extension as π_A lies in the center. In particular, we may regard π_A as an algebraic integer.

> Theorem 1.4. There is a bijection
> $$\mathrm{HT} : \mathrm{A} \mapsto \pi_\mathrm{A} \in \mathrm{End}(\mathrm{A}) \otimes \mathbf{Q}$$
> from the set of isogeny classes of simple abelian varieties $\mathrm{A}/\mathbf{F}_q$ and the set of $\mathrm{W}(q)/\sim$ of Weil q-integers up to conjugacy.

Understanding the isogeny category has almost been accomplished by the theorem above. The last component is the following result determining the endomorphism algebras.

THEOREM 1.5. Let A be a simple abelian variety over $\mathbf{F}_q$. The endomorphism algebra of A in $\mathsf{AV}^0_{\mathbf{F}_q}$ is given by a division algebra D whose center is $\mathbf{Q}(\pi_{\mathrm{A}})$.

We can specify this precisely using class field theory, namely using the exact sequence

$$0 \longrightarrow \mathrm{Br}(K) \longrightarrow \bigoplus_v \mathrm{Br}(K_v) \xrightarrow{\Sigma} \mathbf{Q}/\mathbf{Z} \longrightarrow 0,$$

and then the Brauer class specified by the local invariants will contain exactly one division algebra.

To specify the division algebra D over $\mathbf{Q}(\pi_{\mathrm{A}})$ it then suffices to describe the local invariants $\mathrm{inv}_v(\mathrm{D})$. These are 0 for $v \nmid p\infty$, or is complex, and $\frac{1}{2}$ for v real. For finite places, we have

$$\mathrm{inv}_v(\mathrm{D}) = \frac{v(\pi_{\mathrm{A}})[\mathbf{Q}(\pi_{\mathrm{A}})_v : \mathbf{Q}_p]}{v(q)}$$

when $v \mid p$. It splits at all places not dividing p.

With this, we have entirely determined the category $\mathsf{AV}^0_{\mathbf{F}_q}$. This has some nice applications already.

EXAMPLE 1.6. We can completely classify the isogeny classes of elliptic curves over $\mathbf{F}_q$. Here, π_{E} is completely determined by the trace of Frobenius. For example, over $\mathbf{F}_p$ every trace of Frobenius within the Weil bounds is realized; there is a corresponding isogeny class of elliptic curve. We get one supersingular isogeny class, and $2\lfloor 2\sqrt{p} \rfloor$ ordinary ones. For q this is slightly more complicated, but can again be made explicit.

2. A little bit about the proof

The first major component of the proof comes from Tate's theorem. Set $\mathrm{T}_\ell(\mathrm{A}) := \varprojlim \mathrm{A}[\ell^n]$, where $\ell \neq p$ is prime.

The ℓ-adic Tate module contains the same information as étale cohomology, in the sense that

$$\mathrm{H}^1_{\text{ét}}(\mathrm{A}_{\overline{\mathbf{F}}_q}, \underline{\mathbf{Z}_\ell}) = \mathrm{Hom}(\pi_1(\mathrm{A}), \mathbf{Z}_\ell).$$

Then, $\pi_1(\mathrm{A})^{\mathrm{ab}} \otimes \mathbf{Z}_\ell \simeq \mathrm{T}_\ell(\mathrm{A})$ (torsion in abelian varieties classifies finite covering spaces; more precisely by this we really mean the maximal pro-ℓ quotient). So, as representations they are dual; no information has been lost. Indeed, for an abelian variety we have as graded $\mathbf{Z}_\ell$-algebras with a Galois action that

$$\mathrm{H}^\bullet_{\text{ét}}(\mathrm{A}_{\overline{\mathbf{F}}_q}, \underline{\mathbf{Z}_\ell}) = \bigwedge^{\bullet}[\mathrm{T}_\ell(\mathrm{A})^\vee].$$

This tells us that studying the Galois action on $\mathrm{T}_\ell(\mathrm{A})$ tells us the entire Galois action.

Over any field, we have an injection

$$\mathrm{Hom}(\mathrm{A}, \mathrm{B}) \otimes \mathbf{Z}_\ell \to \mathrm{Hom}_{\mathbf{Z}_\ell}(\mathrm{T}_\ell(\mathrm{A}), \mathrm{T}_\ell(\mathrm{B}))$$

of $\mathbf{Z}_\ell$-modules. For example, over $\mathbf{C}$ there is an obvious injection

$$\mathrm{Hom}(\mathrm{A}, \mathrm{B}) \to \mathrm{Hom}(\pi_1(\mathrm{A}(\mathbf{C})), \pi_1(\mathrm{B}(\mathbf{C}))).$$

This is because the actual homomorphisms are morphisms of integral Hodge structures, where the Hodge structure comes from $\mathrm{H}_1(\mathrm{A}(\mathbf{C}), \mathbf{Z})$. But this amounts to a regular homomorphism between these with extra conditions. This is one example of a refinement of this injection to be an isomorphism: we take homomorphisms of Hodge structures. These can be seen as morphisms representations of the Deligne torus $\mathrm{Res}^{\mathbf{C}}_{\mathbf{R}}\mathbb{G}_m$.

Over $\mathbf{F}_q$, there is a good refinement of this map that makes it an isomorphism.

Theorem 2.1 (Tate). Let A, B be abelian varieties over $\mathbf{F}_q$. There is an isomorphism

$$\mathrm{Hom}(\mathrm{A}, \mathrm{B}) \otimes \mathbf{Z}_\ell \simeq \mathrm{Hom}_{\mathrm{G}_{\overline{\mathbf{F}}_q}}(\mathrm{T}_\ell(\mathrm{A}), \mathrm{T}_\ell(\mathrm{B})).$$

On a projective algebraic variety over $\mathbf{F}_q$, Deligne showed that the eigenvalues of Frobenius acting on $\mathrm{H}^i_{\text{ét}}(\mathrm{X}_{\overline{\mathbf{F}}_q}, \underline{\mathbf{Q}_\ell})$ have absolute value $q^{i/2}$. In general it's very difficult to go the other way and produce a variety with prescribed Frobenius eigenvalues. These tell us a lot of information: if we can understand the characteristic polynomial of Frobenius on these étale cohomology groups, we recover the Zeta function.

When $\mathrm{A} = \mathrm{B}$ is a simple abelian variety, this tells us that the Weil q-integer π_{A} coming from $\pi_{\mathrm{A}} \in \mathrm{End}(\mathrm{A}) \otimes \mathbf{Q}_\ell$ can be recovered from the Frobenius on $\mathrm{T}_\ell(\mathrm{A}) \otimes \mathbf{Q}_\ell$. In particular, the Frobenius eigenvalues on $\mathrm{H}^1_{\text{ét}}(\mathrm{A}, \underline{\mathbf{Q}_\ell})$ should have absolute value $q^{1/2}$, so we see now how Honda-Tate theory relates to the Weil conjectures. One can see this more directly through the Rosati involution.

Note that proving these in the case of abelian varieties is very easy: we can do everything in terms of the ℓ-adic Tate module, and the Grothendieck-Lefschetz trace formula has an easy subcase here as well.

A thing that's important to emphasize here is that the full statement of Honda-Tate theory gives us a *converse* to the Weil conjectures on abelian varieties. In particular, it allows us to construct a simple abelian variety from $\pi_A \in W(q)/\sim$. But the Frobenius endomorphism π_A's corresponding action, lying in $\mathrm{End}_{G_{\mathbf{F}_q}}(T_\ell(A) \otimes \mathbf{Q}_\ell) \simeq \mathrm{End}^0_{\mathbf{F}_q}(A) \otimes \mathbf{Q}_\ell$, is the dual to the action on the first étale cohomology; as we saw, the full étale cohomology ring is an exterior on this. Thus, specifying π_A allows us to specify the eigenvalues of Frobenius on $H^\bullet_{\text{ét}}(A, \underline{\mathbf{Q}_\ell})$, and to also say which ones actually occur.

Apart from allowing us to connect back to the Weil conjectures, this also allows us to precisely write down when two abelian varieties over $\mathbf{F}_q$ are isogenous.

Corollary 2.2. Two abelian varieties A, B over $\mathbf{F}_q$ are isogenous if and only the characteristic polynomials χ_A, χ_B of Frobenius on the ℓ-adic Tate modules are equal. More generally, A is $\mathbf{F}_q$-isogenous to an abelian subvariety of B if and only if $\chi_A | \chi_B$.

Moreover, for a simple abelian variety over $\mathbf{F}_q$ the polynomial χ_A is a power of an irreducible polynomial.

In particular, we get injectivity of the Honda-Tate map: $\mathbf{F}_q$-simple abelian varieties have characteristic polynomials which are powers of an irreducible polynomial. If we have $\pi_A = \pi_B$, then they are powers of the *same* irreducible polynomial; it follows if the characteristic polynomials are not equal, then one divides the other. But this gives a contradiction using the above corollary, as the abelian varieties are simple. So they must in fact be isogenous.

Surjectivity is a bit harder. One way of doing this is to construct abelian varieties over extensions of $\mathbf{Q}_p$, and show they have good reductions giving us enough Weil q-integers to conclude the result.

3. Moving past the rational story

Honda-Tate theory gives us a description of $\mathsf{AV}^0_{\mathbf{F}_q}$, but actually today (as of 2021) we can describe the entire category $\mathsf{AV}_{\mathbf{F}_q}$.

A first approximation of this result was given by Deligne in 1969 for ordinary abelian varieties, shortly after Honda-Tate theory was established.

Define $\mathcal{L}_q$ to be the category of pairs (T, Fr) where:

- T is a finitely generated free $\mathbf{Z}$-module, and $\mathrm{Fr} \in \mathrm{End}(\mathrm{T})$.
- The endomorphism $\mathrm{Fr} \otimes \mathbf{Q}$ is semisimple, with eigenvalues of absolute value $\sqrt{q}$.
- At least half of the roots of the characteristic polynomial of Fr in $\overline{\mathbf{Q}}_p$ are p-adic units.
- There is a Vischiebung V such that $\mathrm{Fr} \circ \mathrm{V} = q$.

Morphisms are morphisms of free $\mathbf{Z}$-modules respecting Fr: $\varphi \circ \mathrm{Fr} = \mathrm{Fr}' \circ \varphi$.

Theorem 3.1. Let $\mathsf{AV}^{\mathrm{ord}}_{\mathbf{F}_q}$ be the category of ordinary abelian varieties over $\mathbf{F}_q$. Fix an embedding $\varepsilon : \mathrm{W}(\overline{\mathbf{F}}_q) \to \mathbf{C}$. Let $\mathrm{A}^{\#}$ be the Serre-Tate canonical lifting of A to $\mathrm{W}(\overline{\mathbf{F}}_q)$. Put

$$\mathrm{T}(\mathrm{A}) := \mathrm{H}_1(\mathrm{A}^{\#} \otimes_{\varepsilon} \mathbf{C}).$$

Additionally, set Fr(A) to be the endomorphism of T(A) induced by the corresponding lift of Frobenius. Then the map

$$\mathrm{A} \mapsto (\mathrm{T}(\mathrm{A}), \mathrm{Fr}(\mathrm{A}))$$

is an equivalence of categories $\mathsf{AV}^{\mathrm{ord}}_{\mathbf{F}_q} \simeq \mathcal{L}_q$.

This requires several bad choices which we would like to avoid. This theorem attempts to capture the spirit of what happens over $\mathbf{C}$, namely that the polarizable integral Hodge structure $\mathrm{H}_1(\mathrm{A}, \mathbf{Z})$ gives an equivalence of categories between abelian varieties over $\mathbf{C}$ and polarizable integral Hodge structures of type (-1,0),(0,-1). The idea is that for *ordinary* abelian varieties, we can use the same strategy with T(A) except add the data of Frobenius.

However, in general such an association is bound to fail on the entire category $\mathsf{AV}_{\mathbf{F}_q}$. For example, with supersingular elliptic curves the endomorphism algebra does not admit 2-dimensional $\mathbf{Q}$-representations, so any attempt to capture morphisms is bound to fail. Indeed, such elliptic curves are precisely when the ordinary condition fails.

The result of Centeleghe and Stix allows us to recover this result and get far more generality. Unfortunately, the functor is still not without choices.

There is always some $\mathbf{F}_q$-isogeny

$$\mathrm{A} \to \prod_{\pi \in \mathrm{W}(q)/\sim} \mathrm{A}_{\pi}^{n_\pi}$$

by semisimplicity of the isogeny category where A_π are simple abelian varieties. Call the set of π that appear in the decomposition the *Weil support*. For any subset $w \subseteq \mathrm{W}(q)/\sim$, denote by AV_w the full subcategory of $\mathsf{AV}_{\mathbf{F}_q}$ whose objects are abelian varieties with Weil support contained in w.

Theorem 3.2 (Centeleghe-Stix). There is an ind-representable anti-equivalence of categories

$$\mathrm{T} : \mathsf{AV}_w \to \mathsf{Mod}_{\mathbf{Z}-\mathrm{tf}}(\mathcal{S}_w)$$

for a certain non-commutative pro-ring $\mathcal{S}_w$. Here, this denotes *left* modules. They are required to be free of finite rank over $\mathbf{Z}$.

The pro-ring $\mathcal{S}_w$ is the endomorphism ring of the representing pro-object. This is where the choice comes in. Taking w to be the set of all Weil numbers, we can describe the entire category $\mathsf{AV}_{\mathbf{F}_q}$. Or, taking ordinary Weil numbers, we recover Deligne's result.