SHTUKAS

1. Shtukas

First, I want to motivate the concept of a shtuka from a number theory perspective. In order to get a some sort of map from the automorphic side to the spectral side, the way of doing this over $\mathbf{Q}$ is to look at schemes over number fields which have both an action of $\mathrm{G}(\mathbf{A}_f)$ and an action of G_K for a number field K. More specifically, we need some specific control over these actions, which comes in the form of relating the action of the Hecke algebra $\mathbf{T}$ to the Galois action of $\mathrm{Frob}_p \in \mathrm{G}_K$.

This can also be done in the function field setting, where the resulting objects are a great deal simpler. We have a correspondence

$$\begin{tikzcd} & \mathrm{Hecke}_{\mathrm{X}^{\mathrm{I}}} \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_{\mathrm{G}} & & \mathrm{Bun}_{\mathrm{G}} \end{tikzcd}$$

which gives an action of Hecke operators on G-bundles. In particular, Hecke operators act by modifications of G-bundles.

Since the Hecke stack has a map $\pi : \mathrm{Hecke}_{\mathrm{X}^{\mathrm{I}}} \to \mathrm{X}^{\mathrm{I}}$, one idea might be to use $\pi_!$ in some way to produce a sheaf on X^{I} with an action of the Hecke algebra. The naive expectation might be that we can get a Galois action via $\pi_1^{\text{ét}}(\mathrm{X})^{\mathrm{I}}$, but actually this fails quite badly: there is a difference between $\pi_1^{\text{ét}}(\mathrm{X}^{\mathrm{I}})$ and $\pi_1^{\text{ét}}(\mathrm{X})^{\mathrm{I}}$, for example take $\mathrm{X} = \mathbf{A}^1_{\overline{\mathbf{F}}_p}$. What is needed to fix this issue is the idea of a partial Frobenius: if we add compatibility with partial Frobenii on X^{I}, Drinfeld's lemma gives us an actual Galois action. There is not too much hope in this situation because the action of Frobenius on a G-bundle and modifications of G-bundles have nothing to do with each other, so we shouldn't expect any such compatibility.

We define a stack $\mathrm{Sht}_{\mathrm{G,I}}$ as a fiber product

$$\begin{tikzcd} \mathrm{Sht}_{\mathrm{I}} \arrow[r] \arrow[d] & \mathrm{Hecke}_{\mathrm{X}^{\mathrm{I}}} \arrow[d, "\overleftarrow{h},\overrightarrow{h}"] \\ \mathrm{Bun}_{\mathrm{G}} \arrow[r, "\mathrm{Frob},\mathrm{Id}"] & \mathrm{Bun}_{\mathrm{G}} \times \mathrm{Bun}_{\mathrm{G}} \end{tikzcd}$$

so that we've forced modifications of G-bundles and Frobenius to have something to do with each other. This comes equipped with a map $\pi' : \mathrm{Sht}_{\mathrm{I}} \to \mathrm{X}^{\mathrm{I}}$ via the Hecke stack. As a basic example, we know that

$$\mathrm{Funct}_c(\mathrm{Bun}_{\mathrm{G}}(\mathbf{F}_q)) = \mathrm{Autom}$$

which clearly has a Hecke action. This is coming from $\mathrm{Sht}_{\emptyset}$: we have a cartesian square

$$\begin{tikzcd} \mathrm{Sht}_\emptyset = \mathrm{Bun}_{\mathrm{G}}(\mathbf{F}_q) \arrow[r] \arrow[d] & \mathrm{Bun}_{\mathrm{G}} \arrow[d] \\ \mathrm{Bun}_{\mathrm{G}} \arrow[r, "\mathrm{Frob},\mathrm{Id}"] & \mathrm{Bun}_{\mathrm{G}} \times \mathrm{Bun}_{\mathrm{G}}. \end{tikzcd}$$

Taking $\pi'_!\overline{\mathbf{Q}}_\ell$, we get Autom.

In general, we use the sheaves $\mathcal{S}_{\mathrm{V}}$ on the Hecke stack coming from classical geometric Satake. Namely, for such a $\mathrm{V} \in \mathsf{Rep}(\check{\mathrm{G}})^{\otimes \mathrm{I}}$ we obtain a sheaf $\mathcal{S}_{\mathrm{V}} \in \mathsf{Shv}(\mathrm{Hecke}_{\mathrm{X}^{\mathrm{I}}})$. This induces a sheaf $\mathcal{S}'_{\mathrm{V}}$ on $\mathrm{Sht}_{\mathrm{I}}$, and we then define

$$\mathrm{Sht}_{\mathrm{I},\mathrm{V}} := \pi'_!\mathcal{S}'_{\mathrm{V}} \in \mathsf{Shv}(\mathrm{X}^{\mathrm{I}}).$$

It is now known by a theorem on Cong Xue that these lie in $\mathsf{QLisse}(\mathrm{X}^{\mathrm{I}})$. Furthermore, they now come equipped equivariance with partial Frobenii on X^{I} so that we actually get a Galois action.

The sheaves $\mathrm{Sht}_{\mathrm{I},\mathrm{V}}$ can be assembled into a sheaf $\mathsf{DrinfSht}$ in $\mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}})$. There is an alternative construction Drinf of this sheaf as a categorical trace, which we'll talk about.

These are really the main player in everything done in global function field Langlands. By adding level structures, these are essential in the proof for GL_n. Lafforgue utilizes these, again with level structure, to construct *excursion operators*. These correspond to generators of an algebra $\mathcal{B}$ related to $\mathcal{E}xc = \Gamma(\mathrm{LocSys}^{\mathrm{arithm}}, \mathcal{O})$ (the discrete, underived version of this). This algebra acts on automorphic forms via the excursion operators, and then automorphic forms get decomposed into $\mathcal{B}$-eigenspaces. By construction, the characters of this algebra correspond to L-parameters, and compatibility of excursion and Hecke operators gives the desired compatibility of the decomposition

$$C_c^{\mathrm{cusp}}(\mathrm{Bun}_{\mathrm{G},\mathrm{N}}(\mathbf{F}_q)/\Xi, \overline{\mathbf{Q}}_\ell) = \bigoplus_{\sigma:\mathrm{Gal}(\overline{F}/F)\to\check{\mathrm{G}}(\overline{\mathbf{Q}}_\ell)} \mathfrak{h}_\sigma$$

with Satake parameters. The sheaf $\mathsf{DrinfSht} \simeq \mathsf{Drinf}$ captures an unramified version of this via the action of $\mathcal{E}xc$ acting on its global sections, which are going to be Autom.

2. Construction of DrinfSht

Our first task is to assemble these elements of $\mathsf{QLisse}(\mathrm{X}^{\mathrm{I}})$ into a single object $\mathsf{DrinfSht} \in \mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}})$.

> Lemma 2.1. The category $\mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{restr}})$ are *canonically* self-dual, induced by the pairing
>
> $$\mathscr{F}_1, \mathscr{F}_2 \mapsto \Gamma_!(\mathrm{LocSys}^{\mathrm{restr}}, \mathscr{F}_1 \otimes \mathscr{F}_2).$$
>
> The same holds for $\mathrm{LocSys}^{\mathrm{arithm}}$, but with Γ.

Proof. This holds using Γ for any quasicompact stack which is locally almost of finite type. When we are in a situation where things are not quasicompact, we make a replacement $\Gamma_!$ to still get a duality. □

Using the canonical self-duality, we will be able to construct objects from collections of functors resembling shtukas. Some definitions are in order first.

Let $\mathrm{I} \in \mathsf{FinSet}$, and $\mathrm{V} \in \mathsf{Rep}(\check{\mathrm{G}})^{\otimes \mathrm{I}}$. We will produce an object

$$\mathcal{E}_{\mathrm{V}}^{\mathrm{I}} \in \mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}) \otimes \mathsf{QLisse}(\mathrm{X})^{\otimes \mathrm{I}}.$$

Let us explain how this is produced in general. If $\mathcal{H}$ is a dualizable gentle Tannakian category, we look at the map

$$\mathsf{Rep}(\check{\mathrm{G}})^{\otimes \mathrm{I}} \to \mathsf{QCoh}(\mathsf{Maps}(\mathsf{Rep}(\check{\mathrm{G}}), \mathcal{H}))^{\otimes \mathrm{I}} \otimes \mathcal{H}^{\otimes \mathrm{I}}$$

and applying the tensor product functor we land in $\mathsf{QCoh}(\mathsf{Maps}(\mathsf{Rep}(\check{\mathrm{G}})), \mathcal{H}) \otimes \mathcal{H}^{\otimes \mathrm{I}}$.

We obtain the first functor by passing to the limit on the functors

$$\mathsf{Rep}(\check{\mathrm{G}}) \to \mathsf{QCoh}(S) \otimes \mathcal{H}$$

for $S \in \mathsf{Sch}^{\mathrm{aff}}_{/\mathsf{Maps}(\mathsf{Rep}(\check{\mathrm{G}}), \mathcal{H})}$, which will be $\mathsf{QCoh}(\mathsf{Maps}(\mathsf{Rep}(\check{\mathrm{G}}), \mathcal{H})) \otimes \mathcal{H}$ by virtue of $\mathcal{H}$ being dualizable.

Now apply this formalism for QLisse, and we have explained how to produce $\mathcal{E}^{\mathrm{I}}$. As constructed, this is a functor and not an object; $\mathcal{E}^{\mathrm{I}}_{\mathrm{V}}$ is the value on V.

THEOREM 2.2. The functor

$$\mathrm{coLoc} : \mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}) \to \mathsf{Maps}(\mathsf{Rep}(\check{\mathrm{G}})^{\otimes \mathsf{FinSet}}, \mathsf{QLisse}(\mathrm{X})^{\otimes \mathsf{FinSet}})$$

is an equivalence.

This sends $\mathscr{F}$ to the functors

$$\mathrm{V} \mapsto (\Gamma_!(\mathrm{LocSys}^{\mathrm{restr}}, -) \otimes \mathrm{Id})(\mathcal{E}^{\mathrm{I}}_{\mathrm{V}} \otimes \mathscr{F})$$

from $\mathsf{Rep}(\check{\mathrm{G}})^{\otimes \mathrm{I}} \to \mathsf{QLisse}(\mathrm{X})^{\otimes \mathrm{I}}$.

In particular, we can take the collection of such functors given by cohomology of shtukas. This produces the desired object. There is some sleight of hand here: the actual objects we got lived in $\mathsf{QLisse}(\mathrm{X}^{\mathrm{I}})$, not $\mathsf{QLisse}(\mathrm{X})^{\mathrm{I}}$. Fortunately for us, these are isomorphic.

We are not yet done: we have a quasicoherent sheaf on the wrong stack. We need to figure out how to descend these.

LEMMA 2.3. The data of an isomorphism $\mathscr{F} \simeq i_*\mathscr{F}$ where $i : \mathrm{LocSys}^{\mathrm{arithm}} \to \mathrm{LocSys}^{\mathrm{restr}}$ is the natural map is equivalent to the structure of partial Frobenii on the functors

$$\mathrm{V} \mapsto (\Gamma_!(\mathrm{LocSys}^{\mathrm{restr}}, -) \otimes \mathrm{Id})(\mathcal{E}^{\mathrm{I}}_{\mathrm{V}} \otimes \mathscr{F})$$

from $\mathsf{Rep}(\check{\mathrm{G}})^{\otimes \mathrm{I}} \to \mathsf{QLisse}(\mathrm{X})^{\otimes \mathrm{I}}$.

By construction, we took these to arise from $\mathrm{Sht}_{\mathrm{I},\mathrm{V}}$. These are equipped with partial Frobenii (as we know from number theory), so we actually get a sheaf $\mathsf{DrinfSht}$ on $\mathrm{LocSys}^{\mathrm{arithm}}$.

This sheaf acts as a sort of universal shtuka, as it assembles all shtukas into a single object and allows us to recover them through the tautological objects $\mathcal{E}^{\mathrm{I}}_{\mathrm{V}}$. Moreover, note that if I put $\mathrm{I} = \emptyset$ and V as the trivial representation, I get $\mathrm{Funct}_c(\mathrm{Bun}_{\mathrm{G}}(\mathbf{F}_q)) = \mathrm{Autom}$. In particular, global sections of this sheaf give us unramified automorphic forms.

3. Trace and Drinf

Let us now define another sheaf, called Drinf. We will define objects

$$\widetilde{\mathsf{Sht}}_{\mathrm{I,V}} \in \mathsf{QLisse}(\mathrm{X}^{\mathrm{I}})$$

by considering the Hecke functors

$$\mathrm{H}(\mathrm{V}, -) : \mathsf{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{\mathrm{G}}) \to \mathsf{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{\mathrm{G}}) \otimes \mathsf{QLisse}(\mathrm{X}^{\mathrm{I}}).$$

Now precomposing with Frob_*, we obtain a functor

$$F : \mathbf{C} \to \mathbf{C} \otimes \mathbf{D}$$

where $\mathbf{C} = \mathsf{Shv}_{\mathrm{Nilp}}\mathrm{Bun}_{\mathrm{G}}$ and $\mathbf{D} = \mathsf{QLisse}(\mathrm{X}^{\mathrm{I}})$. As $\mathsf{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{\mathrm{G}})$ is dualizable, we obtain a categorical trace of this functor F as

$$\mathsf{Vect}_{\overline{\mathbf{Q}}_\ell} \to \mathbf{C}^\vee \otimes \mathbf{C} \to \mathbf{C}^\vee \otimes \mathbf{C} \otimes \mathbf{D} \to \mathbf{D}.$$

This construction results in $\widetilde{\mathsf{Sht}}_{\mathrm{I,V}} \in \mathsf{QLisse}(\mathrm{X}^{\mathrm{I}})$.

This produces a sheaf Drinf on $\mathrm{LocSys}^{\mathrm{restr}}$. However, if we want to see it actually descends to $\mathrm{LocSys}^{\mathrm{arithm}}$, it is better construct it directly as an enhanced trace.

We define

$$\mathsf{Drinf} := \mathrm{Tr}^{\mathrm{enh}}_{\mathrm{Frob}^*, \mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{restr}})}(\mathrm{Frob}_*, \mathsf{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{\mathrm{G}})).$$

This is a 2-categorical construction called enhanced trace, where we are regarding $\mathsf{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{\mathrm{G}})$ as a module category over $\mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{restr}})$. First, to the pair $(\mathrm{Frob}^*, \mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}))$ we associate the trace

$$\mathrm{Tr}(\mathrm{Frob}^*, \mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{restr}})) \simeq \mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}).$$

Let me explain what this means. In general, given a symmetric monoidal category $\mathbf{A}$ with endofunctor $F_{\mathbf{A}}$, define

$$\mathrm{Tr}(F_{\mathbf{A}}, \mathbf{A}) := F_{\mathbf{A}} \otimes_{\mathbf{A} \otimes \mathbf{A}^{\mathrm{op}}} \mathbf{A}$$

where we view the endofunctor $F_{\mathbf{A}}$ as an $\mathbf{A} \otimes \mathbf{A}^{\mathrm{op}}$-bimodule category (precisely, $\mathbf{A}$ with an action on one side twisted by $F_{\mathbf{A}}$). For the previous pair, the result of this construction is $\mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}})$.

Now to the pair $(\mathrm{Frob}_*, \mathsf{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{\mathrm{G}}))$, using that $\mathsf{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{\mathrm{G}})$ is a module category over $\mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{restr}})$, we can associate a class in this Hochschild homology which is going to be Drinf.

For the construction of this class, let us move to the general setting where we have pairs $(\mathbf{A}, F_{\mathbf{A}})$ and $(\mathbf{M}, F_{\mathbf{M}})$ where $\mathbf{M}$ is a dualizable module category over a rigid symmetric monoidal category $\mathbf{A}$ and the second part of the pairs is an endofunctor (which is symmetric monoidal in the case of $\mathbf{A}$). We ask for a compatibility datum, namely that the diagram

$$\begin{tikzcd} \mathbf{A} \otimes \mathbf{M} \arrow[r] \arrow[d, "F_{\mathbf{A}} \otimes F_{\mathbf{M}}"] & \mathbf{M} \arrow[d, "F_{\mathbf{M}}"] \\ \mathbf{A} \otimes \mathbf{M} \arrow[r] & \mathbf{M} \end{tikzcd}$$

commutes. Viewing a module category as a 1-morphism

$$\mathrm{T} : \mathsf{DGCat} \to \mathbf{A} - \mathsf{Mod}$$

this data is a natural transformation

$$\alpha : \mathrm{T} \circ \mathrm{id} \to \mathbf{F}_{\mathbf{A}} \circ \mathrm{T}.$$

In such a situation, categorical constructions provide us with a morphism

$$\mathrm{Tr}(\mathrm{id}, \mathsf{DGCat}) \to \mathrm{Tr}(F_{\mathbf{A}}, \mathbf{A} - \mathsf{Mod}).$$

That is, a morphism from $\mathsf{Vect} \to \mathrm{Tr}(F_{\mathbf{A}}, \mathbf{A} - \mathsf{Mod})$; we get an object.

With our given pairs, this is the class in $\mathrm{Tr}(\mathrm{Frob}^*, \mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{restr}})) \simeq \mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}})$ we want to associate.

One can prove that we recover $\widetilde{\mathsf{Sht}}_{\mathrm{I,V}}$ in the following way.

> Theorem 3.1. The sheaf $\mathsf{Drinf} \in \mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}})$ has the property
> $$(\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{\mathrm{G}}}, -) \otimes \mathrm{Id})(\mathsf{Drinf} \otimes \mathcal{E}^{\mathrm{I}}_{\mathrm{V}}) \simeq \widetilde{\mathrm{Sht}}_{\mathrm{I,V}}$$

Here, by abuse of notation we write $\mathcal{E}^{\mathrm{I}}_{\mathrm{V}}$ for the restriction to $\mathrm{LocSys}^{\mathrm{arithm}}$. This result shows that these sheaves in $\mathsf{QLisse}(\mathrm{X}^{\mathrm{I}})$ are equipped with partial Frobenii, which means that if we had done the original construction we would be able to descend it to $\mathrm{LocSys}^{\mathrm{arithm}}$. Note here that we also use Γ, as here the stack ends up being quasicompact (although this is nontrivial to show).

4. A (fixable) fake proof

The following is now a theorem:

> Theorem 4.1. There is a canonical isomorphism
> $$\mathsf{Drinf} \simeq \mathsf{DrinfSht}$$
> in $\mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}})$.

> Remark 4.2. Once we have this isomorphism, using the tautological objects $\mathcal{E}^{\mathrm{I}}_{\mathrm{V}}$ on both must produce the same results. We know that Drinf gives $\widetilde{\mathrm{Sht}}_{\mathrm{I,V}}$, and for $\mathsf{DrinfSht}$ this gives $\mathrm{Sht}_{\mathrm{I,V}}$. We know we get $\mathrm{Sht}_{\mathrm{I,V}}$ on $\mathrm{LocSys}^{\mathrm{restr}}$, and the partial Frobenii descend it to $\mathrm{LocSys}^{\mathrm{arithm}}$ and $\mathcal{E}^{\mathrm{I,arithm}}_{\mathrm{V}}$ extracts the same element of $\mathsf{QLisse}(\mathrm{X}^{\mathrm{I}})$.
>
> The canonical identification of these gives the trace conjecture.

This is a nontrivial result, but I'd like to explain why you should believe these quasicoherent sheaves on $\mathrm{LocSys}^{\mathrm{arithm}}$ are isomorphic following Gaitsgory's argument.

This argument crucially assumes we live in a simpler world that allows

$$\mathsf{Shv}(\mathrm{Bun}_{\mathrm{G}} \times \mathrm{Bun}_{\mathrm{G}}) \simeq \mathsf{Shv}(\mathrm{Bun}_{\mathrm{G}}) \otimes \mathsf{Shv}(\mathrm{Bun}_{\mathrm{G}}).$$

This is not actually true, and one needs to use $\mathsf{Shv}_{\mathrm{Nilp}}$ instead to make this work. The correct argument also uses a self-duality of $\mathsf{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{\mathrm{G}})$ instead of $\mathsf{Shv}(\mathrm{Bun}_{\mathrm{G}})$, which is also more difficult. Both of these results need Beilinson's spectral projector to be done correctly.

We will pretend the following are true:

- $\mathsf{Shv}(\mathrm{Bun}_{\mathrm{G}} \times \mathrm{Bun}_{\mathrm{G}}) \simeq \mathsf{Shv}(\mathrm{Bun}_{\mathrm{G}}) \otimes \mathsf{Shv}(\mathrm{Bun}_{\mathrm{G}})$.

- $\mathrm{LocSys}^{\mathrm{restr}}$ is a quasicompact algebraic stack (I think we don't actually need this; it's just to avoid $\Gamma_!$ for the self-duality). In Gaitsgory's argument he actually pretends that all étale local systems assemble into an algebraic stack, but we don't need to go this far in our assumptions.
- Redefine $\mathsf{Drinf} := \mathrm{Tr}^{\mathrm{enh}}_{\mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{restr}})}(\mathrm{Frob}^!, \mathsf{Shv}(\mathrm{Bun}_{\mathrm{G}}))$, so we can still get compactly supported functions on $\mathrm{Bun}_{\mathrm{G}}(\mathbf{F}_q)$ on global sections. In particular, we want to get
$$\mathrm{Tr}(\mathrm{Frob}^!, \mathsf{Shv}(\mathrm{Bun}_{\mathrm{G}})) = \mathrm{Funct}_c(\mathrm{Bun}_{\mathrm{G}}(\mathbf{F}_q))$$
for the usual trace.

The argument I'll explain follows essentially the same overall structure as the correct one, just with less technical details as we are allowed to make these assumptions.

The basic idea is that both Drinf and $\mathsf{DrinfSht}$ satisfy
$$\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}, \mathcal{E}_{\mathrm{V},x} \otimes \mathscr{F}) \simeq \mathrm{H}^\bullet(\mathrm{Bun}_{\mathrm{G}} \times_{\mathrm{Frob}_*, \mathrm{Bun}_{\mathrm{G}} \times \mathrm{Bun}_{\mathrm{G}}} \mathrm{Hecke}_x, \mathcal{S}'_{\mathrm{V}}).$$
For $\mathsf{DrinfSht}$, this is by definition in our setting.

Here, $\mathcal{E}_{\mathrm{V},x}$ is originally produced on $\mathrm{LocSys}^{\mathrm{restr}}$ by using the inclusion of a rational point $x \to \mathrm{X}$, and the representation V of $\check{\mathrm{G}}$ gives us an element in QCoh for $\check{\mathrm{G}}$-local systems on x. We then restrict to get the desired sheaf. The sheaf $\mathcal{S}_{\mathrm{V}'}$ is given by geometric Satake, and originally lives on the Hecke stack at x before we pull it back to the fiber product.

This property uniquely characterizes a sheaf in $\mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{restr}})$. Indeed, given any sheaf the functor $\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}, - \otimes \mathscr{F})$ gives a functor to Vect. Then canonical self-duality gives us a corresponding object, which is then $\mathscr{F}$. Since $\mathcal{E}_{\mathrm{V},x}$ generate (by this I mean everything can be written as colimits involving them, which suffices if we want a cocontinuous functor to use duality on), this describes the entire functor.

Let's see this for Drinf. The first thing to note is that $\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}, \mathcal{E}_{\mathrm{V},x} \otimes \mathsf{Drinf}) : \mathsf{Vect} \to \mathsf{Vect}$ can be decomposed further by first unraveling the definition of $\mathsf{Drinf} : \mathsf{Vect} \to \mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}})$.

Notation. In the following we will be abbreviating $\mathsf{Shv}(\mathrm{Bun}_{\mathrm{G}})$ as $\mathrm{D}(\mathrm{Bun}_{\mathrm{G}})$, and will just write LocSys for the restricted version. This is mostly just so things have a remote chance of fitting on a page.

The idea is that if we have defined this via enhanced trace, we are realizing $\mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}})$ as $\mathrm{Tr}(\mathrm{Frob}^*, \mathsf{QCoh}(\mathrm{LocSys}))$. That is, we write it as
$$\mathsf{QCoh}(\mathrm{LocSys}) \otimes_{\mathrm{Frob}, \mathsf{QCoh}(\mathrm{LocSys} \times \mathrm{LocSys}), \Delta} \mathsf{QCoh}(\mathrm{LocSys})$$
where the Frob means we use the graph of Frobenius for the map, and Δ means we use the diagonal for the map to take the tensor product over.

Further breaking down the definition of Drinf, the enhanced trace constructs an element of
$$\mathrm{Tr}(\mathrm{Frob}^*, \mathsf{QCoh}(\mathrm{LocSys})) = \mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}).$$
via the compatibility datum of $\mathrm{Frob}^!$ on $\mathsf{Shv}(\mathrm{Bun}_{\mathrm{G}})$ as a module category over $\mathsf{QCoh}(\mathrm{LocSys})$. The following is simply unwinding the definition of this element from the compatibility datum.

The first functor we'll use to make Drinf is
$$\mathsf{Vect} \to \mathrm{D}(\mathrm{Bun}_{\mathrm{G}}) \otimes_{\mathsf{QCoh}(\mathrm{LocSys})} \mathrm{D}(\mathrm{Bun}_{\mathrm{G}})$$

via the unit of the self-duality datum of $\mathrm{D}(\mathrm{Bun_G})$ as a module category over $\mathsf{QCoh}(\mathrm{LocSys})$.

Then we follow this with an isomorphism

$\mathrm{D}(\mathrm{Bun_G}) \otimes_{\mathsf{QCoh}(\mathrm{LocSys})} \mathrm{D}(\mathrm{Bun_G}) \simeq (\mathrm{D}(\mathrm{Bun_G}) \otimes \mathrm{D}(\mathrm{Bun_G})) \otimes_{\mathsf{QCoh}(\mathrm{LocSys}\times\mathrm{LocSys})} \mathsf{QCoh}(\mathrm{LocSys})$.

There is a map

$$\Phi : (\mathrm{D}(\mathrm{Bun_G}) \otimes \mathrm{D}(\mathrm{Bun_G})) \to \mathsf{QCoh}(\mathrm{LocSys}).$$

This is given by

$$\Gamma(\mathrm{LocSys}, \mathcal{E} \otimes \Phi(D_1, D_2)) := \mathrm{H}^\bullet(\mathrm{Bun_G}, \mathrm{Frob}^!(D_1) \otimes^! (\mathcal{E} * D_2)).$$

Canonical self-duality on $\mathrm{LocSys}^{\mathrm{restr}}$ means this actually makes sense as a map, which again formally follows from this being a reasonable stack.

Applying this map, we then have a map

$$\begin{array}{c}
(\mathrm{D}(\mathrm{Bun_G}) \otimes \mathrm{D}(\mathrm{Bun_G})) \otimes_{\mathsf{QCoh}(\mathrm{LocSys}\times\mathrm{LocSys})} \mathsf{QCoh}(\mathrm{LocSys}) \\
\big\downarrow \Phi \\
\mathsf{QCoh}(\mathrm{LocSys}) \otimes_{\mathrm{Frob},\mathsf{QCoh}(\mathrm{LocSys}\times\mathrm{LocSys}),\Delta} \mathsf{QCoh}(\mathrm{LocSys}) \\
\big\downarrow \sim \\
\mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}})
\end{array}$$

To make the identification with $\mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}})$ we have written it as $\mathrm{Tr}(\mathrm{Frob}^*, \mathsf{QCoh}(\mathrm{LocSys}))$, which is precisely the second to last term when you apply the Hochschild homology definition.

This in total defines a functor

$$\mathsf{Vect} \to \mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}})$$

matching Drinf. We can then tensor with $\mathcal{E}_{\mathrm{V},x}$ and take global sections to get the overall result $\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}, \mathcal{E}_{\mathrm{V},x} \otimes \mathsf{Drinf}) : \mathsf{Vect} \to \mathsf{Vect}$ as a chain of simpler compositions.

Now note that starting at $(\mathrm{D}(\mathrm{Bun_G}) \otimes \mathrm{D}(\mathrm{Bun_G})) \otimes_{\mathsf{QCoh}(\mathrm{LocSys}\times\mathrm{LocSys})} \mathsf{QCoh}(\mathrm{LocSys})$ in this chain of compositions, we can replace the rest of the compositions by

$$\begin{array}{c}
(\mathrm{D}(\mathrm{Bun_G}) \otimes \mathrm{D}(\mathrm{Bun_G})) \otimes_{\mathsf{QCoh}(\mathrm{LocSys}\times\mathrm{LocSys})} \mathsf{QCoh}(\mathrm{LocSys}) \\
\big\downarrow \\
D(\mathrm{Bun_G}) \otimes \mathrm{D}(\mathrm{Bun_G}) \\
\big\downarrow \Phi \\
\mathsf{QCoh}(\mathrm{LocSys}) \\
\big\downarrow \Gamma(\mathrm{LocSys}, \mathcal{E}_{\mathrm{V},x} \otimes -) \\
\mathsf{Vect}
\end{array}$$

This helps: taking the full composition from $\mathsf{Vect} \to \mathrm{D}(\mathrm{Bun_G}) \otimes \mathrm{D}(\mathrm{Bun_G})$, this now is the absolute unit of self-duality for $\mathrm{Bun_G}$ and breaks down as

$$\mathsf{Vect} \longrightarrow \mathrm{D}(\mathrm{Bun_G}) \xrightarrow{\Delta} \mathrm{D}(\mathrm{Bun_G} \times \mathrm{Bun_G}) \simeq \mathrm{D}(\mathrm{Bun}_G) \otimes \mathrm{D}(\mathrm{Bun_G}).$$

where the first map corresponds to $\omega_{\mathrm{Bun_G}}$ (this is where $\overline{\mathbf{Q}}_\ell$ is sent; it is the dualizing sheaf or unit of $\otimes^!$ in the constructible setting). Thus,

$$\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}, \mathcal{E}_{\mathrm{V},x} \otimes \mathsf{Drinf}) : \mathsf{Vect} \to \mathsf{Vect}$$

can be identified with

$$\mathsf{Vect} \longrightarrow \mathrm{D}(\mathrm{Bun_G}) \xrightarrow{\Delta} \mathrm{D}(\mathrm{Bun_G}) \otimes \mathrm{D}(\mathrm{Bun_G}) \xrightarrow{\Phi} \mathsf{QCoh}(\mathrm{LocSys}) \longrightarrow \mathsf{Vect}$$

where the last map is $\Gamma(\mathrm{LocSys}, \mathcal{E}_{\mathrm{V},x} \otimes -)$, and we abuse notation with Δ by using the isomorphism $\mathrm{D}(\mathrm{Bun_G} \times \mathrm{Bun_G}) \simeq \mathrm{D}(\mathrm{Bun}_G) \otimes \mathrm{D}(\mathrm{Bun_G})$.

In this, the only complicated map is Φ. However, its defining property makes this far easier, as we have essentially given its definition by saying what it gives when composed with the map $\Gamma(\mathrm{LocSys}, \mathcal{E}_{\mathrm{V},x} \otimes -)$. Applying this, the final component becomes

$$\mathrm{D}(\mathrm{Bun_G}) \otimes \mathrm{D}(\mathrm{Bun_G}) \xrightarrow{\mathrm{Id} \otimes \mathrm{H}_{\mathrm{V},x}} \mathrm{D}(\mathrm{Bun_G}) \otimes \mathrm{D}(\mathrm{Bun_G}) \xrightarrow{\mathrm{Frob}^! \otimes \mathrm{Id}} \mathrm{D}(\mathrm{Bun_G}) \otimes \mathrm{D}(\mathrm{Bun_G})$$

followed by $\Delta^!_{\mathrm{Bun_G}}$, which lands in $\mathrm{D}(\mathrm{Bun_G})$, and then $\mathrm{H}^\bullet$ to get to Vect. This is simply unwinding the definition of Φ: the first part is using that $\mathcal{E}_{\mathrm{V},x} * D_2$ is just the Hecke action, the second part gives us the result $\mathrm{Frob}^!(D_1)$, and $\Delta^!$ takes the shriek tensor product. Finally, $\mathrm{H}^\bullet$ completes the definition of Φ.

At this point, we can now apply base change for the square giving the fiber product $\mathrm{Bun_G} \times_{\mathrm{Frob}, \mathrm{Bun_G} \times \mathrm{Bun_G}} \mathrm{Hecke}_x$, now that we have reduced everything to

$$\mathrm{H}^\bullet(\mathrm{Bun_G} \times_{\mathrm{Frob}, \mathrm{Bun_G} \times \mathrm{Bun_G}} \mathrm{Hecke}_x, \mathcal{S}'_{\mathrm{V}}) \simeq \mathrm{H}^\bullet(\mathrm{Bun_G}, \Delta^!(\mathrm{Frob}^! \otimes \mathrm{H}_{\mathrm{V},x})(\Delta_* \omega)).$$

Thus, Drinf satisfies the desired characterization.

We can see $\mathscr{F} = \mathsf{DrinfSht}$ also satisfies

$$\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}, \mathcal{E}_{\mathrm{V},x} \otimes \mathscr{F}) \simeq \mathrm{H}^\bullet(\mathrm{Bun_G} \times_{\mathrm{Frob}, \mathrm{Bun_G} \times \mathrm{Bun_G}} \mathrm{Hecke}_x, \mathcal{S}'_{\mathrm{V}})$$

which is by definition in this setting. Thus, the two sheaves are isomorphic.