THE BEILINSON FIBER SQUARE

ABSTRACT. Goodwillie originally proved that for an associative $\mathbf{Q}$-algebra R and nilpotent ideal I that the relative theory $\mathrm{K}(\mathrm{R},\mathrm{I})$ is equivalent to $\mathrm{HC}^{-}(\mathrm{R},\mathrm{I})$ via the Goodwillie-Jones trace map. The Dundas-Goodwillie-McCarthy theorem allows for general associative rings, but at the cost of replacing cyclic homology with topological cyclic homology. We will discuss a variant, originally due to Beilinson and refined by Antieau-Mathew-Morrow-Nikolaus, which works for commutative rings R henselian along (p) and the ideal $\mathrm{I} = (p)$. This still allows us to compute in terms of cyclic homology.

We'll then talk about an application of this result towards the p-adic variational Hodge conjecture. This generalizes a result of Bloch–Esnault–Kerz.

1. MAIN THEOREM

Let R be an associative ring and I a nilpotent ideal in R. The Dundas-Goodwillie-McCarthy theorem tells us that there is a cartesian square

$$\begin{tikzcd}
\mathrm{K}(\mathrm{R}) \arrow[r] \arrow[d, "\mathrm{tr}"] & \mathrm{K}(\mathrm{R}/\mathrm{I}) \arrow[d, "\mathrm{tr}"] \\
\mathrm{TC}(\mathrm{R}) \arrow[r] & \mathrm{TC}(\mathrm{R}/\mathrm{I})
\end{tikzcd}$$

That is, the cyclotomic trace induces an equivalence

$$\mathrm{K}(\mathrm{R},\mathrm{I}) \simeq \mathrm{TC}(\mathrm{R},\mathrm{I}).$$

Here, these denote the fibers of $\mathrm{K}(\mathrm{R}) \to \mathrm{K}(\mathrm{R}/\mathrm{I})$ and similarly for TC.

This is already extremely useful, but the problem is that TC can be a bit of a complicated invariant.

The theorem I'll be talking about allows us to understand K theory via HP and HC^{-}.

DEFINITION 1.1. Let R be an associative algebra over a base ring k. This is given by

$$\mathrm{HH}(\mathrm{R}) := \mathrm{R} \otimes^{\mathbb{L}}_{\mathrm{R}\otimes^{\mathbb{L}}\mathrm{R}^{\mathrm{op}}} \mathrm{R}.$$

This is an E_∞ algebra over k when R is commutative; we assume the base ring is $\mathbf{Z}$ if not specified.

This is basically THH, but given entirely in classical algebra: the same construction defines THH when done with the smash product of spectra, namely for an E_∞ ring spectrum R

we can define THH as $R \otimes_{R\otimes R^{op}} R$. This can also be viewed as $\mathrm{colim}_{S^1} R$ (one also regards this as $R \otimes S^1$), viewing S^1 as the suspension of $* \sqcup *$ to relate it to the previous definition.

We write $HC^- = HH(R)^{hS^1}$, $HP = HH(R)^{tS^1}$. Throughout, for various functors $X(-)$ from rings to spectra I will write $X(-;\mathbf{Z}_p)$ for the p-adic completion and $X(-;\mathbf{Q}_p)$ for the rationalization of the p-adic completion.

The theorem I will be talking about is the following:

> THEOREM 1.2 (Antieau, Mathew, Morrow, Nikolaus). Let R be an associative ring. There is a commutative diagram
>
> $$\begin{tikzcd} K(R;\mathbf{Q}_p) \arrow[r] \arrow[d, "\mathrm{tr}_{\mathrm{GJ}}"] & K(R/p;\mathbf{Q}_p) \arrow[d, "\mathrm{tr}_{\mathrm{cris}}"] \\ HC^-(R;\mathbf{Q}_p) \arrow[r] & HP(R;\mathbf{Q}_p) \end{tikzcd}$$
>
> This will be cartesian if you further assume that R is commutative and Henselian along (p). In particular, one gets for fibers
>
> $$K(R,(p);\mathbf{Q}_p) \simeq \Sigma\, HC(R;\mathbf{Q}_p).$$

A similar result was discovered by Beilinson under mild extra hypotheses (p-complete and bounded p-power torsion), for continuous K theory $\varprojlim K(R/p^n,(p))$. Under Beilinson's hypotheses, this recovers the above. However, the proof of the theorem will be much cleaner in this formulation, as we will be able to leverage results about TC.

One thing to note is that the hypothesis has changed here from nilpotence to $(R,(p))$ being a Henselian pair. This includes cases where (p) is locally nilpotent, but it also computes p-adically complete rings. In particular, we have extended the class of rings that the result applies to.

This advantage of this result is also that we are able to write everything purely in terms of HC, which is far easier. This mimics Goodwillie's original result, except that we don't enforce the strong condition that R to be a $\mathbf{Q}$-algebra.

I'll focus on a proof of this result, as well as an important application to the p-adic variational Hodge conjecture.

2. PROOF OF THE MAIN RESULT

As it turns out, the proof of this result is not actually too involved if you are willing to assume some results about relative K theory for Henselian pairs.

THEOREM 2.1 (Clausen-Mathew-Morrow). Let $(\mathrm{R},(p))$ be a Henselian pair. Then the trace induces an equivalence

$$\mathrm{K}(\mathrm{R},(p);\mathbf{Z}_p)\simeq\mathrm{TC}(\mathrm{R},(p);\mathbf{Z}_p).$$

In particular, we have a cartesian square

$$\begin{tikzcd}
\mathrm{K}(\mathrm{R};\mathbf{Q}_p) \arrow[r] \arrow[d, "\mathrm{tr}"] & \mathrm{K}(\mathrm{R}/p;\mathbf{Q}_p) \arrow[d, "\mathrm{tr}"] \\
\mathrm{TC}(\mathrm{R};\mathbf{Q}_p) \arrow[r] & \mathrm{TC}(\mathrm{R}/p;\mathbf{Q}_p).
\end{tikzcd}$$

The top of this square is what we want, but what we really need is to find a way to exit the realm of TC and enter HC on the bottom to allow for a more computable result.

To get these to appear, we need to do something involving tensoring with $\mathbf{Z}$. In particular, the rough idea we want to use is that the S^1-equivariant map

$$\mathrm{THH}(\mathrm{R};\mathbf{Z}_p)\otimes_{\mathbb{S}}\mathbf{Z}\to\mathrm{HH}(\mathrm{R};\mathbf{Z}_p)$$

is an equivalence after inverting p, which allows us to get HC^- and HP to appear out of the topological versions. In particular, we expect to get these out of applying $(-)^{hS^1}$ and $(-)^{tS^1}$ to this tensor product.

In what follows, we write $\mathrm{TC}^-(\mathrm{X}):=\mathrm{X}^{hS^1}$ and $\mathrm{TP}(\mathrm{X})=\mathrm{X}^{tS^1}$ for a cyclotomic spectrum $\mathrm{X}\in\mathrm{CycSp}$. Then set

$$\mathrm{TC}(\mathrm{X};\mathbf{Z}_p):=\mathrm{fib}(\mathrm{can}-\varphi:\mathrm{HC}^-(\mathrm{X};\mathbf{Z}_p)\to\mathrm{HP}(\mathrm{X};\mathbf{Z}_p)).$$

On the other hand, when we take a ring R and write $\mathrm{TC}(\mathrm{R};\mathbf{Z}_p)$ as shorthand for $\mathrm{TC}(\mathrm{THH}(\mathrm{R});\mathbf{Z}_p)$.

In the interest of taking $\mathrm{X}=\mathrm{THH}(\mathrm{R};\mathbf{Z}_p)$, the first fiber square we are interested in is the following, where $\mathbf{Z}$ means we take $\mathbf{Z}$ as a spectrum and equip it with trivial S^1 action. We make this a trivial cyclotomic spectrum by equipping it with the map

$$\mathbf{Z}\to\mathbf{Z}^{hC_p}\to\mathbf{Z}^{tC_p}$$

which gives the cyclotomic Frobenius.

THEOREM 2.2. Let X be a bounded below cyclotomic spectrum. There is a cartesian square

$$\begin{tikzcd}
\mathrm{TC}(\mathrm{X}\otimes_{\mathbb{S}}\mathbf{Z};\mathbf{Z}_p) \arrow[r] \arrow[d] & \mathrm{TC}(\mathrm{X}\otimes_{\mathbb{S}}\mathrm{THH}(\mathbf{F}_p);\mathbf{Z}_p) \arrow[d] \\
\mathrm{TC}^-(\mathrm{X}\otimes_{\mathbb{S}}\mathbf{Z};\mathbf{Z}_p) \arrow[r] & \mathrm{TC}^-(\mathrm{X}\otimes_{\mathbb{S}}\mathrm{THH}(\mathbf{F}_p);\mathbf{Z}_p)
\end{tikzcd}$$

Proof. For a bounded cyclotomic spectrum X we have

$$\mathrm{TC}(\mathrm{X}; \mathbf{Z}_p) \simeq \mathrm{eq}\left(\mathrm{TC}^{-}(\mathrm{X}; \mathbf{Z}_p) \rightrightarrows \mathrm{TP}(\mathrm{X}; \mathbf{Z}_p) \right)$$

The fact that the square is cartesian follows from

$$\mathrm{X} \otimes_{\mathbb{S}} \mathbf{Z} \to \mathrm{X} \otimes_{\mathbb{S}} \mathrm{THH}(\mathbf{F}_p)$$

is an equivalence on $\mathrm{TP}(\mathrm{X}; \mathbf{Z}_p)$. □

Once we have this, we upgrade to the following fiber square which gets us closed to fixing the bottom of our original square.

COROLLARY 2.3. Let X be a bounded below cyclotomic spectrum which is also p-complete. There exists natural map

$$\mathrm{TC}(\mathrm{X} \otimes_{\mathbb{S}} \mathrm{THH}(\mathbf{F}_p); \mathbf{Z}_p) \to (\mathrm{X} \otimes_{\mathbb{S}} \mathbf{Z})^{tS^1}.$$

Then using this, we have a square

$$\begin{tikzcd}
\mathrm{TC}(\mathrm{X} \otimes_{\mathbb{S}} \mathbf{Z}; \mathbf{Z}_p) \arrow[r] \arrow[d] & \mathrm{TC}(\mathrm{X} \otimes_{\mathbb{S}} \mathrm{THH}(\mathbf{F}_p); \mathbf{Z}_p) \arrow[d] \\
(\mathrm{X} \otimes_{\mathbb{S}} \mathbf{Z})^{hS^1} \arrow[r] & (\mathrm{X} \otimes_{\mathbb{S}} \mathbf{Z})^{tS^1}
\end{tikzcd}$$

where we have used this natural map on the right vertical arrow. This square is cartesian after we invert p.

Proof. Extend the previous square:

$$\begin{tikzcd}
\mathrm{TC}(\mathrm{X} \otimes_{\mathbb{S}} \mathbf{Z}; \mathbf{Z}_p) \arrow[r] \arrow[d] & \mathrm{TC}(\mathrm{X} \otimes_{\mathbb{S}} \mathrm{THH}(\mathbf{F}_p); \mathbf{Z}_p) \arrow[d] \\
(\mathrm{X} \otimes_{\mathbb{S}} \mathbf{Z})^{hS^1} \arrow[r] \arrow[d] & (\mathrm{X} \otimes_{\mathbb{S}} \mathrm{THH}(\mathbf{F}_p))^{hS^1} \arrow[d] \\
(\mathrm{X} \otimes_{\mathbb{S}} \mathbf{Z})^{tS^1} \arrow[r] & (\mathrm{X} \otimes_{\mathbb{S}} \mathrm{THH}(\mathbf{F}_p))^{tS^1}
\end{tikzcd}$$

Now use that the Tate construction is an equivalence on the bottom is an equivalence. This allows us to produce the desired commutative diagram, by using the fact that the boundary of this diagram is commutative.

To see it is cartesian after inverting p, it suffices to know that

$$(\mathrm{X} \otimes_{\mathbb{S}} \mathrm{THH}(\mathbf{F}_p))^{hS^1} \to (\mathrm{X} \otimes_{\mathbb{S}} \mathrm{THH}(\mathbf{F}_p))^{tS^1} \simeq (\mathrm{X} \otimes_{\mathbb{S}} \mathbf{Z})^{tS^1}$$

is an equivalence after inverting p. Then we use the previous theorem. □

Now we can prove what we actually want, which is the following square:

THEOREM 2.4. The square

$$\begin{tikzcd} \mathrm{TC}(\mathrm{R};\mathbf{Z}_p) \arrow[r] \arrow[d] & \mathrm{TC}(\mathrm{R}\otimes_{\mathbb{S}}\mathbf{F}_p;\mathbf{Z}_p) \arrow[d] \\ \mathrm{HC}^-(\mathrm{R};\mathbf{Z}_p) \arrow[r] & \mathrm{HP}(\mathrm{R};\mathbf{Z}_p). \end{tikzcd} \tag{1}$$

is commutative for R a ring, and cartesian after inverting p.

The point is that the top of this square agrees, after inverting p, with the bottom of our original square.

The previous corollary essentially already gives this statement, it just needs a bit of additional work. Take $\mathrm{X} = \mathrm{THH}(\mathrm{R};\mathbf{Z}_p)$ in the previous corollary, to obtain a square

$$\begin{tikzcd} \mathrm{TC}(\mathrm{THH}(\mathrm{R};\mathbf{Z}_p)\otimes_{\mathbb{S}}\mathbf{Z};\mathbf{Z}_p) \arrow[r] \arrow[d] & \mathrm{TC}(\mathrm{R}\otimes_{\mathbb{S}}\mathbf{F}_p;\mathbf{Z}_p) \arrow[d] \\ (\mathrm{THH}(\mathrm{R};\mathbf{Z}_p)\otimes_{\mathbb{S}}\mathbf{Z})^{hS^1} \arrow[r] & (\mathrm{THH}(\mathrm{R};\mathbf{Z}_p)\otimes_{\mathbb{S}}\mathbf{Z})^{tS^1} \end{tikzcd}$$

The $\mathbf{Z}_p$ is redundant in the top right, as we don't need to p-complete after. As $\mathrm{TC}(\mathrm{X}\otimes_{\mathbb{S}}\mathbf{Z};\mathbf{Z}_p) \simeq \mathrm{TC}(\mathrm{X};\mathbf{Z}_p)\otimes_{\mathbb{S}}\mathbf{Z}$, the top left can be changed to just have $\mathrm{TC}(\mathrm{THH}(\mathrm{R};\mathbf{Z}_p)\otimes_{\mathbb{S}}\mathbf{Z})$. This diagram is cartesian after inverting p, by the previous corollary.

Next, we make two observations for making this diagram what we want.

- For the bottom, there is a natural S^1-equivariant map of spectra
$$\mathrm{THH}(\mathrm{R};\mathbf{Z}_p)\otimes_{\mathbb{S}}\mathbf{Z} \to \mathrm{HH}(\mathrm{R};\mathbf{Z}_p)$$
which is an equivalence after inverting p. We get maps
$$(\mathrm{THH}(\mathrm{R};\mathbf{Z}_p)\otimes_{\mathbb{S}}\mathbf{Z})^{hS^1} \to \mathrm{HC}^-(\mathrm{R};\mathbf{Z}_p)$$
and $(\mathrm{THH}(\mathrm{R};\mathbf{Z}_p)\otimes_{\mathbb{S}}\mathbf{Z})^{tS^1} \to \mathrm{HP}(\mathrm{R};\mathbf{Z}_p)$. However, these need not be equivalences after inverting p.
- For the top, we have a natural map $\mathrm{THH}(\mathrm{R};\mathbf{Z}_p) \to \mathrm{THH}(\mathrm{R};\mathbf{Z}_p)\otimes_{\mathbb{S}}\mathbf{Z}$. In particular, there is a map
$$\mathrm{TC}(\mathrm{R};\mathbf{Z}_p) \to \mathrm{TC}(\mathrm{THH}(\mathrm{R};\mathbf{Z}_p)\otimes_{\mathbb{S}}\mathbf{Z})$$
which is an equivalence after inverting p.

We use these maps to extend the commutative diagram:

$$\begin{tikzcd}
\mathrm{TC}(\mathrm{R};\mathbf{Z}_p) \arrow[d] & \\
\mathrm{TC}(\mathrm{THH}(\mathrm{R};\mathbf{Z}_p)\otimes_{\mathbb{S}}\mathbf{Z}) \arrow[r] \arrow[d] & \mathrm{TC}(\mathrm{R}\otimes_{\mathbb{S}}\mathbf{F}_p) \arrow[d] \\
(\mathrm{THH}(\mathrm{R};\mathbf{Z}_p)\otimes_{\mathbb{S}}\mathbf{Z})^{hS^1} \arrow[r] \arrow[d] & (\mathrm{THH}(\mathrm{R};\mathbf{Z}_p)\otimes_{\mathbb{S}}\mathbf{Z})^{tS^1} \arrow[d] \\
\mathrm{HC}^-(\mathrm{R};\mathbf{Z}_p) \arrow[r] & \mathrm{HP}(\mathrm{R};\mathbf{Z}_p).
\end{tikzcd}$$

Now if we just use what we know so far about maps after inverting p, we see that using the equivalence with $\mathrm{TC}(\mathrm{R},\mathbf{Z}_p)$ to replace the top left we get the desired commututative diagram (1). We would need to know that the bottom square is cartesian after inverting p to conclude that (1) is cartesian.

On the bottom horizonal fibers, the induced map is

$$\Sigma(\mathrm{THH}(\mathrm{R};\mathbf{Z}_p)\otimes\mathbf{Z})_{hS^1}\to\Sigma(\mathrm{HH}(\mathrm{R};\mathbf{Z}_p))_{hS^1}.$$

This is an equivalence after inverting p due to $\mathrm{THH}(\mathrm{R};\mathbf{Z}_p)\otimes_{\mathbb{S}}\mathbf{Z}\to\mathrm{HH}(\mathrm{R};\mathbf{Z}_p)$ being an equivalence, and this is preserved after taking S^1 homotopy orbits.

We conclude that the bottom square is cartesian after inverting p, and we already knew the top square is cartesian after inverting p. We conclude that (1) is cartesian after inverting p.

Now we are ready for the proof of the main theorem: we do the same trick with stacking cartesian squares.

> THEOREM 2.5. Assume R is commutative and henselian along (p). There is a cartesian square
>
> $$\begin{tikzcd}
\mathrm{K}(\mathrm{R};\mathbf{Q}_p) \arrow[r] \arrow[d, "\mathrm{tr}_{\mathrm{GJ}}"] & \mathrm{K}(\mathrm{R}/p;\mathbf{Q}_p) \arrow[d, "\mathrm{tr}_{\mathrm{cris}}"] \\
\mathrm{HC}^-(\mathrm{R};\mathbf{Q}_p) \arrow[r] & \mathrm{HP}(\mathrm{R};\mathbf{Q}_p).
\end{tikzcd}$$

Proof. In this situation, we have now produced two cartesian square which stack on top of each other. Namely, we have

$$\begin{tikzcd}
\mathrm{K}(\mathrm{R};\mathbf{Q}_p) \arrow[r] \arrow[d, "\mathrm{tr}"] & \mathrm{K}(\mathrm{R}/p;\mathbf{Q}_p) \arrow[d, "\mathrm{tr}"] \\
\mathrm{TC}(\mathrm{R};\mathbf{Q}_p) \arrow[r] \arrow[d] & \mathrm{TC}(\mathrm{R}/p;\mathbf{Q}_p) \arrow[d] \\
\mathrm{HC}^-(\mathrm{R};\mathbf{Q}_p) \arrow[r] & \mathrm{HP}(\mathrm{R};\mathbf{Q}_p)
\end{tikzcd}$$

Here, we have used that $\mathrm{TC}(\mathrm{R} \otimes_{\mathbb{S}} \mathbf{F}_p; \mathbf{Q}_p) \simeq \mathrm{TC}(\mathrm{R}/p; \mathbf{Q}_p)$ is an equivalence. The rest is identical.

Thus, we deduce the existence of the desired fiber square. What remains is to check that the maps are actually the desired maps. On the left, we are using the cyclotomic trace first. Mapping to HC^-, we get the Goodwillie-Jones trace map.

On the right, we define the composition to be the p-adic chern character $\mathrm{tr}_{\mathrm{cris}}$. Later, we'll see that this actually makes sense. □

With this, we have proved the main theorem!

3. Applications

There is also a very interesting number theoretic consequence of this theorem. Suppose we have a smooth proper scheme

$$\mathrm{X} \to \mathrm{Spec}(\mathcal{O}_K)$$

where K is a p-adic field, with special fiber X_k and generic fiber X_K. We have a Chern character

$$\mathrm{ch} : \mathrm{K}_0(\mathrm{X};\mathbf{Q}) \to \mathrm{K}_0(\mathrm{X}_K;\mathbf{Q}) \to \bigoplus_{i\geq 0} \mathrm{H}^{2i}_{\mathrm{dR}}(\mathrm{X}_K/K).$$

The natural question, related to the Hodge conjecture, is to characterize exactly which classes in de Rham cohomology arise from the Chern class.

Idea We can attempt to reduce the question to characteristic p, where we only need to deal with the crystalline Chern character. In particular, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{K}_0(\mathrm{X};\mathbf{Q}) \arrow[r] \arrow[d, "\mathrm{ch}"] & \mathrm{K}_0(\mathrm{X}_k;\mathbf{Q}) \arrow[d, "\mathrm{ch}_{\mathrm{cris}}"] \\
\bigoplus_{i\geq 0} \mathrm{H}^{2i}_{\mathrm{dR}}(\mathrm{X}_K/K) \arrow[r, "\sim"] & \bigoplus_{i\geq 0} \mathrm{H}^{2i}_{\mathrm{cris}}(\mathrm{X}_k/\mathrm{W}(k)) \otimes_{\mathrm{W}(k)} K
\end{tikzcd}$$

On the right, we now have the crystalline Chern class. As usually defined, this is a map

$$\mathrm{K}_0(\mathrm{X}_k) \to \bigoplus_{i\geq 0} \mathrm{H}^{2i}_{\mathrm{cris}}(\mathrm{X}_k/\mathrm{W}(k)).$$

The idea is then the following: if we want to understand the image of ch, it suffices to understand the image of the reduction map $\mathrm{K}_0(\mathrm{X};\mathbf{Q}) \to \mathrm{K}_0(\mathrm{X}_k;\mathbf{Q})$, and also understand the crystalline Chern class. This would answer this form of the Hodge conjecture. The guess for the image is then as follows:

CONJECTURE 3.1 (p-adic variational Hodge conjecture). A class α in even degree de Rham cohomology $\mathrm{H}^{2*}_{\mathrm{dR}}(\mathrm{X}_K/K)$ is in the image of ch precisely if:

- The image of α in the above diagram under the de Rham-crystalline comparison lands in the image of the crystalline chern character.
- Additionally, the class α lies in $\bigoplus_{i\geq 0} \mathrm{Fil}^{\geq i}\mathrm{H}^{2i}_{\mathrm{dR}}(\mathrm{X}_K/K) \subseteq \bigoplus_{i\geq 0} \mathrm{H}^{2i}_{\mathrm{dR}}(\mathrm{X}_K/K)$. Here, we use the Hodge filtration.

At the moment, this is out of reach. However, the theorem we just proved can make some progress towards this question. To answer this, we equivalently need to understand when a class in $K_0(\mathrm{X}_k;\mathbf{Q})$ lifts to $K_0(\mathrm{X};\mathbf{Q})$. We can't quite do this, but we can when we use continuous K theory instead.

DEFINITION 3.2. Let $\mathrm{K}^{\mathrm{cts}}(\mathrm{X}) := \varprojlim \mathrm{K}(\mathrm{X}/\pi^n)$ for a uniformizer $\pi \in \mathcal{O}_K$.

REMARK 3.3. The big picture plan here is that you are supposed to perform another step, which is to relate this to lifting to $\mathrm{K}(\mathrm{X})$. The unfortunate truth is that this is not the same as continuous K theory in general, but in some cases it is the same. For example, if $\mathfrak{X}$ is a smooth formal scheme which is locally of the form $\mathrm{Spf}(\mathrm{R})$ where R is p-complete with bounded p-power torsion this should be true.

The target theorem is the following:

THEOREM 3.4. A class $x \in \mathrm{K}_0(\mathrm{X}_k;\mathbf{Q})$ lifts to $\mathrm{K}_0^{\mathrm{cts}}(\mathrm{X};\mathbf{Q})$ if and only if

$$\mathrm{ch}_{\mathrm{cris}}(x) \in \bigoplus_{i\geq 0} \mathrm{H}^{2i}_{\mathrm{cris}}(\mathrm{X}_k/\mathrm{W}(k))$$

is sent to $\bigoplus_{i\geq 0} \mathrm{Fil}^{\geq i}\mathrm{H}^{2i}_{\mathrm{dR}}(\mathrm{X}_K/K)$ under the crystalline-de Rham comparison.

The Beilinson fiber square we produced before is nearly what we want: we just need to apply the HKR isomorphism.

First, we derive a quick consequence of the main theorem.

THEOREM 3.5. Let $\mathfrak{X}$ be a formal scheme over $\mathrm{Spf}(\mathcal{O}_K)$. Assuming $\mathfrak{X}$ is smooth and proper, there are cartesian squares

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{cts}}(\mathfrak{X};\mathbf{Q}_p) \arrow[r] \arrow[d] & \mathrm{K}(\mathfrak{X}_k;\mathbf{Q}_p) \arrow[d] \\
\mathrm{TC}^{\mathrm{cts}}(\mathfrak{X};\mathbf{Q}_p) \arrow[r] \arrow[d] & \mathrm{TC}(\mathfrak{X}_k;\mathbf{Q}_p) \arrow[d] \\
\mathrm{HC}^{-,\mathrm{cts}}(\mathfrak{X}/\mathcal{O}_K;\mathbf{Q}_p) \arrow[r] & \mathrm{HP}^{\mathrm{cts}}(\mathfrak{X}/\mathcal{O}_K;\mathbf{Q}_p).
\end{tikzcd}$$

Proof. This follows from the extended diagram we drew before; we just appeal to the Zariski descent of all these objects to reduce to the case of $\mathrm{X} = \mathrm{Spf}(\mathrm{R})$. In that case, writing this as an ind-scheme of schemes whose underlying ring has p nilpotent (i.e. henselian along (p)), we deduce the result from the extended Beilinson fiber square we had earlier.

However, there is one catch. At the bottom, I have not written $\mathrm{HC}^{-,\mathrm{cts}}(\mathfrak{X};\mathbf{Q}_p)$ and similarly for HP, which is what this argument actually gives us.

Instead, this is done relative to $\mathcal{O}_K$. It turns out there is a homotopy cartesian diagram

$$\begin{tikzcd}
\mathrm{HC}^{-,\mathrm{cts}}(\mathfrak{X};\mathbf{Q}_p) \arrow[r] \arrow[d] & \mathrm{HP}^{\mathrm{cts}}(\mathfrak{X};\mathbf{Q}_p) \arrow[d] \\
\mathrm{HC}^{-,\mathrm{cts}}(\mathfrak{X}/\mathcal{O}_K;\mathbf{Q}_p) \arrow[r] & \mathrm{HP}^{\mathrm{cts}}(\mathfrak{X}/\mathcal{O}_K;\mathbf{Q}_p).
\end{tikzcd}$$

This allows us to conclude the actual result using the HKR isomorphism, and that $L_{\mathcal{O}_K/\mathrm{W}(k)}$ is quasi-isogenous to zero. □

Given this, the next step is to use the HKR isomorphism to reinterpret the crystalline Chern character we had before.

Now the main idea is the following:

LEMMA 3.6. We have isomorphisms

$$\mathrm{HP}^{\mathrm{cts}}(\mathfrak{X}/\mathcal{O}_K;\mathbf{Q}_p) \simeq \prod_{i\in\mathbf{Z}} \mathrm{R}\Gamma_{\mathrm{dR}}(\mathfrak{X}_K/K)[2i]$$

and

$$\mathrm{HC}^{-,\mathrm{cts}}(\mathfrak{X}/\mathcal{O}_K;\mathbf{Q}_p) \simeq \prod_{i\in\mathbf{Z}} \mathrm{Fil}^{\geq i}\mathrm{R}\Gamma_{\mathrm{dR}}(\mathfrak{X}_K/K)[2i].$$

Proof. This is the HKR isomorphism. ☐

With this, the theorem is proven assuming we can show that up to a scalar the map

$$\mathrm{K}_0(\mathfrak{X}_k;\mathbf{Q}_p) \to \pi_0 \operatorname{HP}^{\mathrm{cts}}(\mathfrak{X}/\mathcal{O}_K;\mathbf{Q}_p) \simeq \prod_{i\in\mathbf{Z}} \mathrm{H}^{2i}_{\mathrm{dR}}(\mathfrak{X}_K/K) \simeq \prod_{i\in\mathbf{Z}} \mathrm{H}^{2i}_{\mathrm{cris}}(\mathfrak{X}_k;\mathbf{Q}_p)\otimes_{\mathrm{W}(k)}K$$

is the crystalline Chern character. This can be checked for the universal case of BGL_n. This proves main claim.