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Formulação de Lagrange-Euler

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1 Introdução

$$\tau = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}$$

onde $L = K - P$ é denominado Lagrangeano, K é a energia cinética e P é a energia potencial.

A energia cinética de cada elo pode ser calculada por

$$K_i = \frac{1}{2} m_i V_i^T V_i = \frac{1}{2} m_i \operatorname{Tr}\left(V_i V_i^T\right)$$

e a energia potencial pode ser computada por

$$P_i = -m_i g^{T\,0} P_{ci} = -m_i g^{T\,0} T_i {}^i P_{ci}$$

$$K = \sum_{i=1}^{n} K_i$$

$$P = \sum_{i=1}^{n} P_i$$

2 Velocidade do Centro de Massa dos Elos

$${}^0V_i = \frac{d}{dt} {}^0P_{ci} = \frac{d}{dt}\left({}^0T_i {}^iP_{ci}\right) = \frac{d}{dt} {}^0T_i {}^iP_{ci}$$

$${}^0V_i = \left({}^0\dot{T}_1 {}^1T_i + {}^0T_1 {}^1\dot{T}_2 {}^2T_i + \cdots + {}^0T_{i-1} {}^{i-1}\dot{T}_i\right) {}^iP_{ci}$$

$$^0V_i = \left(\frac{\partial^0 T_1}{\partial q_1} \dot{q}_1 {}^1T_i + {}^0T_1 \frac{\partial^1 T_2}{\partial q_2} \dot{q}_2 {}^2T_i + \cdots + {}^0T_{i-1} \frac{\partial^{i-1} T_i}{\partial q_i} \dot{q}_i \right) {}^iP_{ci}$$

$$^0V_i = \left(\sum_{j=1}^{i} \frac{\partial^0 T_i}{\partial q_j} \dot{q}_j \right) {}^iP_{ci}$$

Tem-se das convenções de Denavit-Hartenberg que:

$$^{i-1}T_i = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i \operatorname{sen}\theta_i & \operatorname{sen}\alpha_i \operatorname{sen}\theta_i & a_i \cos\theta_i \\ \operatorname{sen}\theta_i & \cos\alpha_i \cos\theta_i & -\operatorname{sen}\alpha_i \cos\theta_i & a_i \operatorname{sen}\theta_i \\ 0 & \operatorname{sen}\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Portanto, para junta rotacional, ou seja $q_i = \theta_i$:

$$\frac{\partial^{i-1} T_i}{\partial \theta_i} = \begin{bmatrix} -\operatorname{sen}\theta_i & -\cos\alpha_i \cos\theta_i & \operatorname{sen}\alpha_i \cos\theta_i & -a_i \operatorname{sen}\theta_i \\ \cos\theta_i & -\cos\alpha_i \operatorname{sen}\theta_i & \operatorname{sen}\alpha_i \operatorname{sen}\theta_i & a_i \cos\theta_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

que pode ser escrito na forma

$$\frac{\partial^{i-1} T_i}{\partial \theta_i} = Q_i {}^{i-1}T_i$$

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$$Q_i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Para junta prismática, ou seja, $q_i = d_i$, tem-se

$$\frac{\partial^{i-1} T_i}{\partial d_i} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

que pode ser escrito na forma

$$\frac{\partial^{i-1} T_i}{\partial \theta_i} = Q_i {}^{i-1}T_i$$

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$$Q_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pode-se agora calcular

$$U_{ij} = \frac{\partial {}^0T_i}{\partial q_j} = \begin{cases} {}^0T_{j-1} Q_j {}^{j-1}T_i & \text{, para } j \leq i \\ 0 & \text{, para } j > i \end{cases}$$

de onde pode-se obter

$${}^0V_i = \left(\sum_{j=1}^{i} U_{ij} \dot{q}_j \right) {}^iP_{ci}$$

3 Energia Cinética dos Elos

$$dK_i = \frac{1}{2} \operatorname{Tr} \left({}^0V_i {}^0V_i^T \right) dm_i$$

$$\begin{aligned} dK_i &= \frac{1}{2} \operatorname{Tr} \left[\sum_{p=1}^{i} U_{ip} \dot{q}_p {}^iP_{ci} \left(\sum_{r=1}^{i} U_{ir} \dot{q}_r {}^iP_{ci} \right)^T \right] dm_i \\ &= \frac{1}{2} \operatorname{Tr} \left[\sum_{p=1}^{i} \sum_{r=1}^{i} U_{ip} \dot{q}_p {}^iP_{ci} {}^iP_{ci}^T \dot{q}_r U_{ir}^T \right] dm_i \\ &= \frac{1}{2} \operatorname{Tr} \left[\sum_{p=1}^{i} \sum_{r=1}^{i} U_{ip} \left({}^iP_{ci} {}^iP_{ci}^T dm_i \right) U_{ir}^T \dot{q}_p \dot{q}_r \right] \end{aligned}$$

Como U_{ij} e $\dot{q}_i$ são independentes da distribuição de massa do elo i, pode-se escrever

$$K_i = \int dK_i = \frac{1}{2} \operatorname{Tr} \left[\sum_{p=1}^{i} \sum_{r=1}^{i} U_{ip} \int {}^iP_{ci} {}^iP_{ci}^T dm_i U_{ir}^T \dot{q}_p \dot{q}_r \right]$$

Definindo-se

$$J_i = \int {}^iP_{ci} {}^iP_{ci}^T dm_i = \begin{bmatrix} \int x_i^2 dm_i & \int x_i y_i dm_i & \int x_i z_i dm_i & \int x_i dm_i \\ \int x_i y_i dm_i & \int y_i^2 dm_i & \int y_i z_i dm_i & \int y_i dm_i \\ \int x_i z_i dm_i & \int y_i z_i dm_i & \int z_i^2 dm_i & \int z_i dm_i \\ \int x_i dm_i & \int y_i dm_i & \int z_i dm_i & \int dm_i \end{bmatrix}$$

pode-se escrever

$$\begin{aligned} K &= \sum_{i=1}^{n} K_i = \frac{1}{2} \sum_{i=1}^{n} \text{ Tr } \left(\sum_{p=1}^{i} \sum_{r=1}^{i} U_{ip} J_i U_{ir}^T \dot{q}_p \dot{q}_r \right) \\ &= \frac{1}{2} \sum_{i=1}^{n} \sum_{p=1}^{i} \sum_{r=1}^{i} \text{ Tr } \left(U_{ip} J_i U_{ir}^T \right) \dot{q}_p \dot{q}_r \end{aligned}$$

Considerando-se a definição do tensor de inércia

$$I_{ij} = \int \left[\delta_{ij} \left(\sum_k x_k^2 \right) - x_i x_j \right] dm$$

pode-se obter J_i a partir de

$$J_i = \begin{bmatrix} \frac{-I_{xx}+I_{yy}+I_{zz}}{2} & I_{xy} & I_{xz} & m_i x_{ci} \\ I_{xy} & \frac{I_{xx}-I_{yy}+I_{zz}}{2} & I_{yz} & m_i y_{ci} \\ I_{xz} & I_{yz} & \frac{I_{xx}+I_{yy}-I_{zz}}{2} & m_i z_{ci} \\ m_i x_{ci} & m_i y_{ci} & m_i z_{ci} & m_i \end{bmatrix}$$

4 Energia Potencial dos Elos

$$P = -\sum_{i=1}^{n} m_i g^{T0} T_i^i P_{ci}$$

5 Lagrangeano

$$L = \frac{1}{2} \sum_{i=1}^{n} \sum_{p=1}^{i} \sum_{r=1}^{i} \text{ Tr } \left(U_{ip} J_i U_{ir}^T \right) \dot{q}_p \dot{q}_r + \sum_{i=1}^{n} m_i g^{T0} T_i^i P_{ci}$$

6 Torque

$$\tau_i = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}$$

$$\frac{\partial L}{\partial \dot{q}_i} = \sum_{j=i}^{n} \sum_{k=1}^{j} \text{ Tr } \left(U_{jk} J_j U_{ji}^T \right) \dot{q}_k$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \sum_{j=i}^{n} \sum_{k=1}^{j} \text{ Tr } \left(U_{jk} J_j U_{ji}^T \right) \ddot{q}_k + \sum_{j=i}^{n} \sum_{k=1}^{j} \sum_{m=1}^{j} \frac{\partial}{\partial q_m} \text{ Tr } \left(U_{jk} J_j U_{ji}^T \right) \dot{q}_k \dot{q}_m$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \sum_{j=i}^{n}\sum_{k=1}^{j} \text{ Tr } \left(U_{jk}J_jU_{ji}^T\right)\ddot{q}_k + \sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j} \text{ Tr } \left(\frac{\partial U_{jk}}{\partial q_m}J_jU_{ji}^T\right)\dot{q}_k\dot{q}_m$$

Definindo

$$U_{jkm} = \frac{\partial U_{jk}}{\partial q_m} = \begin{cases} {}^0T_{k-1}Q_k{}^{k-1}T_{m-1}Q_m{}^{m-1}T_j & , j \geq m \geq k \\ {}^0T_{m-1}Q_m{}^{m-1}T_{k-1}Q_k{}^{k-1}T_j & , j \geq k \geq m \\ 0 & , j < k \text{ ou } j < m \end{cases}$$

pode-se escrever

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \sum_{j=i}^{n}\sum_{k=1}^{j} \text{ Tr } \left(U_{jk}J_jU_{ji}^T\right)\ddot{q}_k + \sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j} \text{ Tr } \left(U_{jkm}J_jU_{ji}^T\right)\dot{q}_k\dot{q}_m$$

Tem-se ainda que

$$\frac{\partial L}{\partial q_i} = \sum_{j=i}^{n} m_j g^T U_{ji}{}^jP_{cj}$$

e portanto

$$\begin{aligned} \tau_i &= \sum_{j=i}^{n}\sum_{k=1}^{j} \text{ Tr } \left(U_{jk}J_jU_{ji}^T\right)\ddot{q}_k \\ &+ \sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j} \text{ Tr } \left(U_{jkm}J_jU_{ji}^T\right)\dot{q}_k\dot{q}_m \\ &- \sum_{j=i}^{n} m_j g^T U_{ji}{}^jP_{cj} \end{aligned}$$

que pode ser escrito de forma mais compacta como

$$\tau_i = \sum_{k=1}^{n} M_{ik}\ddot{q}_k + \sum_{k=1}^{n}\sum_{m=1}^{n} V_{ikm}\dot{q}_k\dot{q}_m + G_i$$

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$$M_{ik} = \sum_{j=\max(i,k)}^{n} \text{ Tr } \left(U_{jk}J_jU_{ji}^T\right)$$

$$V_{ikm} = \sum_{j=\max(i,k,m)}^{n} \mathrm{Tr}\ \left(U_{jkm} J_j U_{ji}^T \right)$$

$$G_i = \sum_{j=i}^{n} -m_j g^T U_{ji}{}^j P_{cj}$$

ou ainda, na forma matricial

$$\tau = M(q)\ddot{q} + V(q, \dot{q}) + G(q)$$

7 Modelo no Espaço de Estados

$$\ddot{q} = M^{-1}(q) \left[\tau - V(q, \dot{q}) - G(q) \right]$$

$$\ddot{q} = M^{-1}(q) \left[\tau - V'(q, \dot{q})\dot{q} - G'(q)q \right]$$

$$\begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}(q)G'(q) & -M^{-1}(q)V'(q, \dot{q}) \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}(q) \end{bmatrix} \tau$$