Chapter 21 Solusion

github.com/frc123/CLRS

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21.1

21.1-1

Edge processed	Collection of disjoint sets										
initial sets	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$	$\{f\}$	$\{g\}$	$\{h\}$	$\{i\}$	$\{j\}$	$\{k\}$
(d, i)	$\{a\}$	$\{b\}$	$\{c\}$	$\{d, i\}$	$\{e\}$	$\{f\}$	$\{g\}$	$\{h\}$		$\{j\}$	$\{k\}$
(f, k)	$\{a\}$	$\{b\}$	$\{c\}$	$\{d, i\}$	$\{e\}$	$\{f, k\}$	$\{g\}$	$\{h\}$		$\{j\}$	
(g, i)	$\{a\}$	$\{b\}$	$\{c\}$	$\{d, g, i\}$	$\{e\}$	$\{f, k\}$		$\{h\}$		$\{j\}$	
(b, g)	$\{a\}$	$\{b, d, g, i\}$	$\{c\}$		$\{e\}$	$\{f, k\}$		$\{h\}$		$\{j\}$	
(a, h)	$\{a, h\}$	$\{b, d, g, i\}$	$\{c\}$		$\{e\}$	$\{f, k\}$				$\{j\}$	
(i, j)	$\{a, h\}$	$\{b, d, g, i, j\}$	$\{c\}$		$\{e\}$	$\{f, k\}$					
(d, k)	$\{a, h\}$	$\{b, d, f, g, i, j, k\}$	$\{c\}$		$\{e\}$						
(b, j)	$\{a, h\}$	$\{b, d, f, g, i, j, k\}$	$\{c\}$		$\{e\}$						
(d, f)	$\{a, h\}$	$\{b, d, f, g, i, j, k\}$	$\{c\}$		$\{e\}$						
(g, j)	$\{a, h\}$	$\{b, d, f, g, i, j, k\}$	$\{c\}$		$\{e\}$						
(a, e)	$\{a, e, h\}$	$\{b, d, f, g, i, j, k\}$	$\{c\}$								

21.1-2

Proof. By contents in B.4, we know that the connected components of a graph are the equivalence classes of vertices under the "is reachable from" relation. The collection of the disjoint sets is exactly the quotient set of $G.V$ by the "is reachable from" relation. It is not hard to find out that CONNECTED-COMPONENTS construct such the quotient set since the procedure unions vertices based on all edges, and edges connect two reachable vertices with the smallest length of the path (recall that a equivalence relation must be transitive). Two vertices are in the same connected component if and only if they are reachable from each other. $\square$

21.1-3

FIND-SET: $2 \cdot |E|$
UNION: $|V| - k$

21.2

21.2-1

```
struct Set
{
    Node *head;
    Node *tail;
    int size;
};

struct Node
{
    int key;
    Set *set;
    Node *next;
};

void MakeSet(Node *x)
{
    x->next = nullptr;
    x->set = new Set;// need to be freed
    x->set->head = x;
    x->set->tail = x;
    x->set->size = 1;
}

Node* FindSet(Node *x)
{
    return x->set->head;
}

void Union(Node *x, Node *y)
{
    Node *node;
    if (x->set->size < y->set->size)
    {
        Union(y, x);
    }
    else
```

```
    {
        node = y->set->head;
        x->set->size += y->set->size;
        x->set->tail->next = node;
        x->set->tail = y->set->tail;
        delete y->set;
        while (node)
        {
            node->set = x->set;
            node = node->next;
        }
    }
}
```

21.2-2

collection before line 3:

$$\{\{x_1\},\{x_2\},\{x_3\},\{x_4\},\{x_5\},\{x_6\},\{x_7\},\{x_8\},\{x_9\},\{x_{10}\},\{x_{11}\},\{x_{12}\},\{x_{13}\},\{x_{14}\},\{x_{15}\},\{x_{16}\}\}$$

collection before line 5:

$$\{\{x_1,x_2\},\{x_3,x_4\},\{x_5,x_6\},\{x_7,x_8\},\{x_9,x_{10}\},\{x_{11},x_{12}\},\{x_{13},x_{14}\},\{x_{15},x_{16}\}\}$$

collection before line 7:

$$\{\{x_1,x_2,x_3,x_4\},\{x_5,x_6,x_7,x_8\},\{x_9,x_{10},x_{11},x_{12}\},\{x_{13},x_{14},x_{15},x_{16}\}\}$$

collection before line 8:

$$\{\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8\},\{x_9,x_{10},x_{11},x_{12}\},\{x_{13},x_{14},x_{15},x_{16}\}\}$$

collection before line 9:

$$\{\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8\},\{x_9,x_{10},x_{11},x_{12},x_{13},x_{14},x_{15},x_{16}\}\}$$

collection before line 10:

$$\{\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10},x_{11},x_{12},x_{13},x_{14},x_{15},x_{16}\}\}$$

Hence FIND-SET(x_2) and FIND-SET($x_1 1$) return a pointer points to x_1.

21.2-3

Lemma 1. Using the linked-list representation of disjoint sets and the weighted-union heuristic, a sequence of h UNION operations on a disjoint set that has never been operated UNION takes $O(h \lg h)$ time.

Proof. We claim that, after h UNION operations, the largest set has at most $h+1$ members. Notice that the number of sets decreases by one each time UNION is called. Suppose that we have n sets in which each set contains one member in the beginning. After the h UNION operations, we have $(n-h)$ sets. Note that each set must contain one member. In order to maximize the number of members in the the largest set, we let $(n-h-1)$ sets contains one member, and let the remaining set contains all remaining members. Then the remaining set contains

$$n-(n-h-1)=h+1$$

members.

We claim that each object's pointer back to its set object is updated at most $\lceil \lg h \rceil$ times over all the UNION operations. Let x be an arbitrary object. By the similar approach in the proof of Theorem 21.1, we know that for any $k \leq h+1$, after x's pointers has been updated $\lceil \lg k \rceil$ times, the resulting set must have at least k members. Since the largest set has at most $h+1$ members, each object's pointer is updated at most $\lg(h+1)$ times over all the UNION operations.

We claim that there are h elements have been updated their pointers back to their set objects at least once. Consider a set contains k members. Then within this set, there are $(k-1)$ members have been updated their pointers back to their set objects at least once since there must exists exactly $(k-1)$ members updated their pointers from the initial pointer to the current one. Let $\mathcal{S}$ be our collection of sets. Then after the h UNION operations, the number of elements have been updated their pointers is

$$\sum_{A \in \mathcal{S}}(|A|-1)=\sum_{A \in \mathcal{S}}|A|-|\mathcal{S}|=n-(n-h)=h$$

Since each object's pointer is updated at most $\lceil \lg h \rceil$ times and there are h elements have been updated their pointers, we conclude h UNION operations on a disjoint set that has never been operated UNION takes $O(h \lg h)$ time. □

Claim 2. The amortized time of MAKE-SET and FIND-SET is $O(1)$, and the amortized time of UNION is $O(\lg n)$.

Proof. Suppose that we performed h UNION operations. Since n MAKE-SET operations are performed, we know $(m-n-h)$ FIND-SET operations are performed By the lemma, we know that the total actual cost of UNION is $O(h \lg h)$. Hence the total actual cost of the sequence is

$$O(\underbrace{n}_{\text{MAKE-SET}}+\underbrace{(m-n-h)}_{\text{FIND-SET}}+\underbrace{h \lg h}_{\text{UNION}})=O(m-h+h \lg h)$$

The total amortized cost of the sequence is

$$O(\underbrace{n}_{\text{MAKE-SET}}+\underbrace{(m-n-h)}_{\text{FIND-SET}}+\underbrace{h \lg n}_{\text{UNION}})=O(m-h+h \lg n)$$

Since $h<n$, we have showed the claim successfully. □

21.2-4

In the ith UNION operation, we call UNION(x_{i+1}, x_i). At this time, the size of set contains x_i contains i members, and the size of set contains x_{i+1} contains 1 members. Then we notice, for all $i \geq 2$, we append the list contains x_{i+1} onto the list contains x_i with the weighted-union heuristic, and this only takes $\Theta(1)$ time for each operation. We operate n times MAKE-SET and $(n-1)$ times UNION, so the sequence takes $\Theta(n + (n-1)) = \Theta(n)$ time.

21.2-5

```
struct Node
{
    int key;
    Node *next;
    // let the tail element be the set's representative
    union
    {
        Node *tail;// for non-tail elements
        Node *head;// for the tail element
    } representative;
    int size;// only for the tail element
};

void MakeSet(Node *x)
{
    x->next = nullptr;
    x->representative.head = x;
    x->size = 1;
}

Node* FindSet(Node *x)
{
    return x->next ? x->representative.tail : x;
}

void Union(Node *x, Node *y)
{
    Node **node, *x_head, *y_head, *x_representative, *y_representative;
    if (x->representative.tail->size < y->representative.tail->size)
    {
        Union(y, x);
```

```
    }
    else
    {
        x_representative = FindSet(x);
        y_representative = FindSet(y);
        x_head = x_representative->representative.head;
        y_head = y_representative->representative.head;
        x_representative->size += y_representative->size;
        node = &y_head;
        while (*node)
        {
            (*node)->representative.tail = x_representative;
            node = &((*node)->next);
        }
        *node = x_head;
        x_representative->representative.head = y_head;
    }
}
```

21.2-6

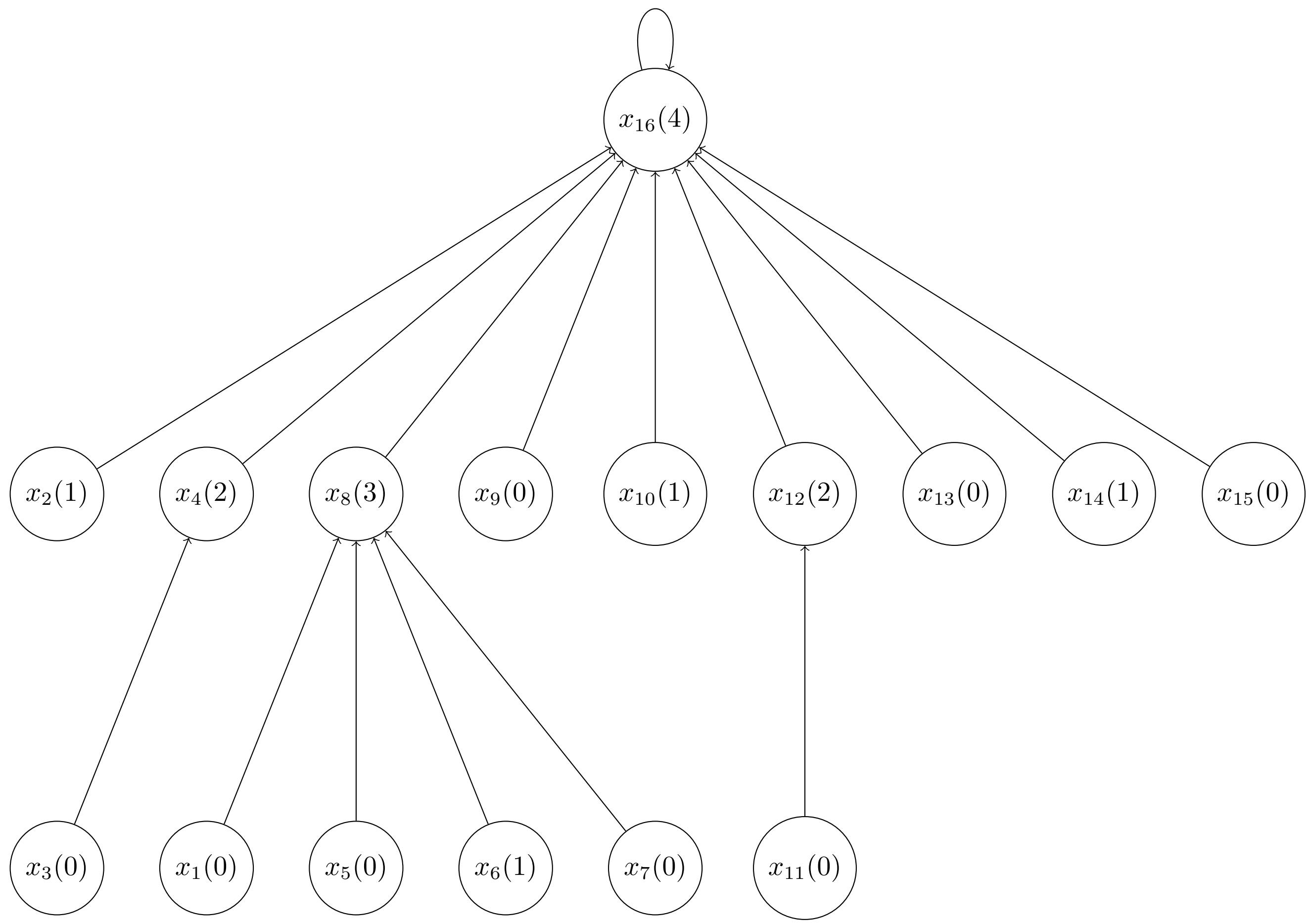
```
struct Set
{
    Node *head;
    int size;
};

struct Node
{
    int key;
    Set *set;
    Node *next;
};

void MakeSet(Node *x)
{
    x->next = nullptr;
    x->set = new Set;// need to be freed
    x->set->head = x;
    x->set->size = 1;
```

```
}

Node* FindSet(Node *x)
{
    return x->set->head;
}

void Union(Node *x, Node *y)
{
    Node **node, *x_second;
    if (x->set->size < y->set->size)
    {
        Union(y, x);
    }
    else
    {
        x->set->size += y->set->size;
        x_second = x->next;
        x->next = y->set->head;
        node = &(x->next);
        delete y->set;
        while (*node)
        {
            (*node)->set = x->set;
            node = &((*node)->next);
        }
        *node = x_second;
    }
}
```

21.3

21.3-1

The data structure in the end (rank of each node is in the parentheses):



Hence FIND-SET(x_2) and FIND-SET($x_1$1) return a pointer points to $x_1$6.

21.3-2

```
Node* FindSet(Node *x)
{
    Node *representative, *tmp;
    representative = x;
    while (representative != representative->p)
    {
        representative = representative->p;
    }
    while (x->p != representative)
    {
        tmp = x->p;
        x->p = representative;
        x = tmp;
    }
```

```
    return representative;
}
```

21.3-3

Note that we are proving that the upper bound $O(m \lg n)$ is tight (least upper bound), instead of prove it is a tight bound $\Theta(m \lg n)$. In order to prove it, we just need to find an example that takes $\Omega(m \lg n)$ time, which is what the question is asking for. We want our sequence to take as much as possible time. WLOG, assume that $n = 2^k$ for some $k \in \mathbb{N}$. Consider the following sequence:

$$\begin{aligned}
&\langle \text{MAKE-SET}(x_1), \text{MAKE-SET}(x_2), \cdots, \text{MAKE-SET}(x_n), \\
&\ \text{UNION}(x_1, x_2), \text{UNION}(x_3, x_4), \cdots, \text{UNION}(x_{n-1}, x_n), \\
&\ \text{UNION}(x_1, x_3), \text{UNION}(x_5, x_7), \cdots, \text{UNION}(x_{n-3}, x_{n-1}), \\
&\ \text{UNION}(x_1, x_5), \text{UNION}(x_9, x_13), \cdots, \text{UNION}(x_{n-7}, x_{n-3}), \\
&\qquad\qquad \vdots \\
&\ \text{UNION}(x_1, x_{n/2+1}), \\
&\ \text{FIND-SET}(x_1) \cdots \qquad \text{(until all } m \text{ operations are performed)}\rangle
\end{aligned}$$

We performed n MAKE-SET, $(n-1)$ UNION, and $(m-2n+1)$ FIND-SET. We observed we performed $n/2$ UNION(x_i, x_{i+1}), $n/4$ UNION(x_i, x_{i+2}), $n/8$ UNION(x_i, x_{i+4}), $\cdots$. We conclude we performed $n/2^j$ UNION$(x_i, x_{i+2^{j-1}})$ for all $j = \{1, 2, \cdots, \lg n\}$. For each j, the height of each tree increases by 1. Hence after all $(n-1)$ UNION operations, the height of the tree (for the only set) is $\lg n$. Note that x_1 is the deepest element in the tree. Then, each of FIND-SET(x_1) operation takes $\Theta(\lg n)$ time, and we perform $(m-2n+1)$ times FIND-SET(x_1) operation. Hence all of FIND-SET(x_1) take $(m-2n+1)\Theta(\lg n) = \Theta(m \lg n)$ time. We successfully find a sequence that takes $\Omega(m \lg n)$ time.

21.3-4

We just need to modify LINK procedure to maintain the data structure.

```
void Link(Node *x, Node *y)
{
    Node *y_next;
    if (x->rank > y->rank)
    {
        y->p = x;
    }
    else
    {
        x->p = y;
        if (x->rank == y->rank)
```

```
            ++y->rank;
    }
    // maintain the circular list
    y_next = y->next;
    y->next = x->next;
    x->next = y_next;
}

std::list<Node*> PrintSet(Node *x)
{
    Node *node;
    std::list<Node*> result;
    result.push_back(x);
    for (node = x->next; node != x; node = node->next)
    {
        result.push_back(node);
    }
    return result;
}
```

21.3-5

Let a_i be the number of nodes with depth greater than 0 (i.e. non-root) in the forest after the ith operation. Let b_i be the number of nodes with depth greater than 1 (i.e. non-root and non-child-of-root) in the forest after the ith operation. Suppose that we start to perform FIND-SET operations in the kth operation. Let the potential function be

$$\Phi(D_i) \begin{cases} a_i & \text{if } i < k \\ b_i & \text{if } i \geq k \end{cases},$$

where D_i is the disjoint forest after the ith operation. Let $\Phi(D_0) = 0$. Observed each of MAKE-SET and LINK takes $O(1)$ time. Denote $depth_i(x)$ as the depth of the node x in the tree after the ith operation. Then FIND-SET(x) moves $\max(0, depth_{i-1}(x) - 1)$ nodes to be the children of the root node. Denote c_i as the cost of the ith operation. Then we assume

$$c_i = \begin{cases} 1 & \text{if MAKE-SET is performed in the } i\text{th operation,} \\ 1 & \text{if LINK is performed in the } i\text{th operation,} \\ \max(1, depth_{i-1}(x)) & \text{if FIND-SET}(x) \text{ is performed in the } i\text{th operation} \end{cases}$$

Case 1. MAKE-SET is performed in the ith operation.

Then $a_i = a_{i-1}$, so

$$\hat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 0 = 1$$

Case 2. LINK(x, y) is performed in the ith operation.

Note that x and y must be root nodes of different tree. This operation make either x or y be a non-root node. Then $a_i = a_{i-1} + 1$, so

$$\hat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 1 = 2$$

Case 3. FIND-SET(x) is performed in the ith operation.

Note that $b_i \le a_i$ for all i. If $i = k$, then

$$\Phi(D_i) - \Phi(D_{i-1}) = b_i - a_{i-1} \le b_i - b_{i-1}$$

If $i > k$, then

$$\Phi(D_i) - \Phi(D_{i-1}) = b_i - b_{i-1}$$

Note that

$$b_i - b_{i-1} = -(\max(1, \mathit{depth}_{i-1}(x)) - 1) = 1 - \max(1, \mathit{depth}_{i-1}(x))$$

Hence

$$\hat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1}) \le c_i + b_i - b_{i-1} = \max(1, \mathit{depth}_{i-1}(x)) + (1 - \max(1, \mathit{depth}_{i-1}(x))) = 1$$

We have shown that amortized of each operation is $O(1)$ with the path-compression heuristic, no matter whether we use union by rank or not. Hence the sequence takes $O(m)$ time with the path-compression heuristic, no matter whether we use union by rank or not.

21.4

21.4-1

Proof. We prove by induction on the number of operations.

(Base) Initially, there is no element in the disjoint sets, so it is trivial.

(Induction) Denote D_i be the data structure after ith operation. Denote $p_i(x)$ be $x.p$ after ith operation. Denote $\mathit{rank}_i(x)$ be $x.\mathit{rank}$ after ith operation. Suppose that, for all $x \in D_{i-1}$, ① $\mathit{rank}_{i-1}(x) < \mathit{rank}_{i-1}(p_{i-1}(x))$ if $x \ne p_{i-1}(x)$, ② $\mathit{rank}_{i-1}(x) = \mathit{rank}_{i-2}(x)$ if $x \ne p_{i-1}(x)$, and ③ $\mathit{rank}_{i-1}(p_{i-1}(x)) \ge \mathit{rank}_{i-2}(p_{i-2}(x))$.

Case 1. MAKE-SET(y) is performed in the ith operation.

Then all structures of trees in D_{i-1} remain same in D_i, and ①②③ are vacuously true for y.

Case 2. FIND-SET(y) is performed in the ith operation.

Since no rank changes in FIND-SET, ② holds. Let z be the root of the tree contains y. Let A be the set contains all the nodes on the simple path from y to z except z in D_{i-1}. Let $w \in A$ be arbitrary choice. We have $w \ne p_{i-1}(w)$. By ①, since $p_i(w) = z$, we have

$$\mathit{rank}_{i-1}(w) < \mathit{rank}_{i-1}(p_{i-1}(w)) \le \mathit{rank}_{i-1}(z) = \mathit{rank}_{i-1}(p_i(w))$$

Since no rank changes in FIND-SET, we have $rank_i(w) < rank_i(p_i(w))$ and $rank_i(p_i(w)) \geq rank_{i-1}(p_{i-1}(w))$. For all elements not in A, their p are not changed in the ith operation. Thus, we conclude ①③ holds.

Case 3. UNION(y, z) is performed in the ith operation. Let f be the root of the tree contains y, and let g be the root of the tree contains z. UNION(y, z) is acutally perfroming the following sequence

$$\langle \text{FIND-SET}(y), \text{FIND-SET}(z), \text{LINK}(f, g)\rangle$$

By case 2, we know IH holds after FIND-SET. Then we assume ①②③ holds for $(i-1)$th operation and we are performing LINK(f, g) in the ith operation. There are three subcases: $rank_{i-1}(f) > rank_{i-1}(g)$, $rank_{i-1}(f) < rank_{i-1}(g)$, and $rank_{i-1}(f) = rank_{i-1}(g)$; we notice the first two are similar, so we ignore the first subcase. First, we suppose that $rank_{i-1}(f) < rank_{i-1}(g)$. Notice that f is the only node that p attribute is changed in the ith operation. We have $p_{i-1}(f) = f$ and $p_i(f) = g$. Then

$$rank_{i-1}(p_{i-1}(f)) = rank_{i-1}(f) < rank_{i-1}(g) = rank_{i-1}(p_i(f))$$

Since no rank changes in this subcase, we have $rank_i(f) < rank_i(p_i(f))$ and $rank_i(p_i(f)) \geq rank_{i-1}(p_{i-1}(f))$. Thus, we conclude ①②③ holds. Now, we suppose that $rank_{i-1}(f) = rank_{i-1}(g)$. By the similar approach to the last subcase, we have

$$rank_{i-1}(p_{i-1}(f)) = rank_{i-1}(f) = rank_{i-1}(g) = rank_{i-1}(p_i(f))$$

By line 5 of the procedure, we have $rank_i(g) = rank_{i-1}(g) + 1$, and ranks of all nodes except g remain same in the ith operation. Then

$$rank_{i-1}(p_{i-1}(f)) = rank_i(f) = rank_{i-1}(f) = rank_{i-1}(g) = rank_i(g) - 1 < rank_i(g) = rank_i(p_i(f))$$

Hence ①②③ holds for f. Since $p_{i-1}(g) = p_i(g) = g$, we have

$$rank_{i-1}(p_{i-1}(g)) = rank_{i-1}(g) = rank_i(g) - 1 < rank_i(g) = rank_i(p_i(g))$$

Hence ①②③ holds for g (①② are vacuously true). For all nodes other than f and g, rank and p attributes remain same in ith operation. Thus, we conclude ①②③ holds. □

21.4-2

Lemma 3. If there exist a node has rank of k, then there exists at least 2^k nodes in the forest.

Proof. *(Base)* If there exist a node has rank of 0, there is at least one node in the forest.

(Induction) Suppose that the lemma is true for $k = h$ for some $h \geq 0$. Consider $k = h + 1$. If there exist a node has rank of $h + 1$, then, by line 4 of LINK, there must were at least two node have rank of h before. By the inductive hypothesis, there exists at least $2 \cdot 2^k = 2^{k+1}$ nodes in the forest. □

Claim 4. In an n disjoint sets using union by rank and path compression heuristic, every node has rank at most $\lfloor \lg n \rfloor$.

Proof. Suppose that their exist a node has rank of $\lfloor \lg n \rfloor + 1$, for the purpose of contradiction. By the lemma, there exists at least $2^{\lfloor \lg n \rfloor + 1}$ nodes in the forest.

$$2^{\lfloor \lg n \rfloor + 1} > 2^{\lg n} = n$$

Contradiction. □

21.4-3

We need $\lceil \lg(k+1) \rceil$ bits to store value k. Hence $\lceil \lg(\lfloor \lg n \rfloor + 1) \rceil$ bits are necesary to store $x.rank$.

21.4-4

Proof. Taking advantage of Lemma 21.7, we convert the sequence into a sequence of m Make-Set, Link, and Find-Set. Without path compression, rank of each node is exactly the height of the node. Observed Make-Set and Link take $O(1)$ time, and Find-Set(x) takes $O(x.rank)$ time. Hence the sequence run in $O(m \lg n)$ time. □

21.4-5

No. Consider $x.rank = 1$, $x.p.rank = 7$, and $x.p.p.rank = 8$. Clearly, $x.rank > 0$ and $x.p$ is not a root. Since

$$A_2(x.rank) = A_2(1) = 7$$

and

$$A_3(x.rank) = A_3(1) = 2047 \quad ,$$

we have

$$A_2(x.rank) \le x.p.rank < A_3(x.rank) \quad ,$$

so

$$\text{level}(x) = 2 \quad .$$

Since

$$A_0(x.p.rank) = A_0(7) = 8$$

and

$$A_1(x.p.rank) = A_1(7) = 15 \quad ,$$

we have

$$A_0(x.p.rank) \le x.p.p.rank < A_1(x.p.rank) \quad ,$$

so

$$\text{level}(x.p) = 0 \quad .$$

Therefore,

$$\text{level}(x) > \text{level}(x.p) \quad .$$

21.4-6

Since $A_3(1) = 2047$, $\alpha'(n) \leq 3$ implies

$$A_3 = 2047 \geq \lg(n+1) \quad .$$

Then

$$n \leq 2^{2047} - 1 = \frac{2^{2048}}{2} - 1 = \frac{(2^4)^{512}}{2} - 1 = 16^{511} - 1 \gg 10^{80} \quad .$$

Replace all $\alpha(n)$ with $\alpha'(n)$ in the argument. The only modification we need to make is at the bound (21.1). We claim that

$$0 \leq \text{level}(x) < \alpha'(n) \quad .$$

We need to modify the argument to prove $\text{level}(x) < \alpha'(n)$.

$$\begin{aligned}
A_{\alpha'(n)}(x.rank) &\geq A_{\alpha'(n)}(1) && \text{(bacause } A_k(j) \text{ is strictly increasing)} \\
&\geq \lg(n+1) && \text{(by the definition of } \alpha'(n)) \\
&> \lfloor \lg(n) \rfloor \\
&\geq x.p.rank && \text{(by exercise 21.4-2)}
\end{aligned}$$

Updating...