

Chapter 14 Solusion

<https://github.com/frc123/CLRS>

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14.1

14.1-1

recursion	$x.key$	r	i
1	26	13	10
2	17	8	10
3	21	3	2
4	19	1	2
5	20	1	1

The result is 20.

14.1-2

iteration	$y.key$	r
1	35	1
2	38	1
3	30	3
4	41	3
5	26	16

The result is 16.

14.1-3

```
1     template <class Key, class T>
2     typename OrderStatisticsTree<Key, T>::Node* OrderStatisticsTree<Key, T>::Select
3         (Node* subtree_root_node, size_t rank)
4     {
5         size_t root_rank;
6         root_rank = subtree_root_node->left->size + 1;
7         while (rank != root_rank)
8         {
9             if (rank < root_rank)
```

```
10         {
11             subtree_root_node = subtree_root_node->left;
12         }
13         else
14         {
15             subtree_root_node = subtree_root_node->right;
16             rank -= root_rank;
17         }
18         root_rank = subtree_root_node->left->size + 1;
19     }
20     return subtree_root_node;
21 }
```

14.1-4

```
1     template <class Key, class T>
2     size_t OrderStatisticsTree<Key, T>::Rank(Node* node)
3     {
4         if (node == root_)
5             return node->left->size + 1;
6         else if (node == node->parent->left)
7             return Rank(node->parent) - node->right->size - 1;
8         else
9             return Rank(node->parent) + node->left->size + 1;
10    }
```

14.1-5

```
1   $r = \text{OS-RANK}(T, x)$ 
2   $\text{succ} = \text{OS-SELECT}(T.\text{root}, r + i)$ 
```

The result is *succ*.

14.1-6

For $\text{RB-INSERT}(T, z)$, set $z.\text{rank} = 1$, and change the while loop to the following code:

```
1  while  $x \neq T.\text{nil}$ 
2       $y = x$ 
3      if  $z.\text{key} < x.\text{key}$ 
4           $x.\text{rank} = x.\text{rank} + 1$ 
5           $x = x.\text{left}$ 
6      else  $x = x.\text{right}$ 
```

For $\text{RB-DELETE}(T, z)$, add the following code right before line 18 (in the else branch):

$y.\text{rank} = z.\text{rank}$

And invoke $\text{RB-DELETE-FIX-RANK}(T, x)$ right before line 21.

$\text{RB-DELETE-FIX-RANK}(T, x)$

```
1  while  $x \neq T.\text{root}$ 
2      if  $x == x.p.\text{left}$ 
3           $x.p.\text{rank} = x.p.\text{rank} - 1$ 
4       $x = x.p$ 
```

For $\text{LEFT-ROTATE}(T, x)$, add the following code to the end of the procedure:

$y.\text{rank} = y.\text{rank} + x.\text{rank}$

For $\text{RIGHT-ROTATE}(T, y)$, add the following code to the end of the procedure:

$y.\text{rank} = y.\text{rank} - x.\text{rank}$

14.1-7

```
1  template <typename Key>
2  size_t CountInversions(std::vector<Key> array)
3  {
4      size_t inversions, i, rank;
5      OrderStatisticsTree<Key, int> tree;
6      std::pair<typename OrderStatisticsTree<Key, int>::Iterator, bool> insert_result;
7      inversions = 0;
8      for (i = 0; i < array.size(); ++i)
9      {
10         insert_result = tree.Insert({array[i], 0});
11         rank = tree.Rank(insert_result.first);
12         inversions += (1 + i - rank);
13     }
14     return inversions;
15 }
```

14.1-8

```
1  size_t CountIntersections(std::vector<Chord> chords)
2  {
3      size_t intersections, i, rank_a, rank_b, rank_diff_1, rank_diff_2;
```

```
4      OSTree tree;
5      std::pair<typename OSTree::Iterator, bool> insert_result_a, insert_result_b;
6      intersections = 0;
7      for (i = 0; i < chords.size(); ++i)
8      {
9          insert_result_a = tree.Insert({chords[i].endpoint_a, 0});
10         insert_result_b = tree.Insert({chords[i].endpoint_b, 0});
11         rank_a = tree.Rank(insert_result_a.first);
12         rank_b = tree.Rank(insert_result_b.first);
13         if (rank_a > rank_b) std::swap(rank_a, rank_b);
14         // rank_a must smaller than rank_b
15         rank_diff_1 = rank_b - rank_a - 1;
16         rank_diff_2 = tree.Size() - rank_b + rank_a - 1;
17         intersections += std::min(rank_diff_1, rank_diff_2);
18     }
19     return intersections;
20 }
```

14.2

14.2-1

Add *prev* and *succ* attributes to each node in the tree. Let *prev* points to predecessor of the node, and let *succ* points to successor of the node. Let *T.nil.succ* points to the minimum element in the tree, and let *T.nil.prev* points to the maximum element in the tree. A circular doubly linked list is formed.

In order to maintain these informations, we just need to modify RB-INSERT and RB-DELETE.

For RB-INSERT(*T*, *z*), modify line 9 - 13 to the following code:

```
1  if  $y == T.nil$ 
2       $T.root = z$ 
3       $z.succ = T.nil$ 
4       $z.prev = T.nil$ 
5  elseif  $z.key < y.key$ 
6       $y.left = z$ 
7       $z.succ = y$ 
8       $z.prev = y.prev$ 
9  else  $y.right = z$ 
10      $z.prev = y$ 
11      $z.succ = y.succ$ 
12  $z.succ.prev = z$ 
13  $z.prev.succ = z$ 
```

For $RB-DELETE(T, z)$, add the following code right before line 21:

```
1   $z.prev.succ = z.succ$ 
2   $z.succ.prev = z.prev$ 
```

14.2-2

We can maintain black-heights of nodes without affecting the asymptotic performance since a change to $x.bh$ propagates only to ancestors of x in the tree.

For $RB-INSERT(T, z)$, add the following code right before line 17:

$$z.bh = 1$$

For $RB-INSERT-FIXUP(T, z)$, add the following code right before line 8 (in the if branch) (case 1):

$$z.p.p.bh = z.p.p.bh + 1$$

For $RB-DELETE-FIXUP(T, z)$, add the following code right before line 11 (in the if branch) (case 2):

$$x.p.bh = x.p.bh - 1$$

And add the following code right before line 21 (in the else branch) (case 4):

$$\begin{aligned} x.p.bh &= x.p.bh - 1 \\ x.p.p.bh &= x.p.p.bh + 1 \end{aligned}$$

We cannot maintain depths of nodes without affecting the asymptotic performance since a change to $x.bh$ propagates to descendants of x in the tree.

14.2-3

After the rotation on x is performed, run the following code:

```
1   $x.p.f = x.f$ 
2   $x.f = x.left.f \otimes x.right.f \otimes x.a$ 
```

Apply to the *size* attributes in order-statistic trees, we just need to change f to *size*, change \otimes to $+$, and attribute a of each node will be 1; the code will be:

```
1   $x.p.size = x.size$ 
2   $x.size = x.left.size + x.right.size + 1$ 
```

14.2-4

The following procedure takes $\Theta(m + \lg n)$ time (to understand this asymptotic performance, refer to theorem 12.1 and exercise 12.2-8):

```
RB-ENUMERATE( $x, a, b$ )
1  if  $a \leq x.key$  and  $x.key \leq b$ 
2      OUTPUT( $x$ )
3  if  $a \leq x.key$  and  $x.left \neq T.nil$ 
4      RB-ENUMERATE( $x.left, a, b$ )
5  if  $x.key \leq b$  and  $x.right \neq T.nil$ 
6      RB-ENUMERATE( $x.right, a, b$ )
```

Note that we need to implement $\text{RB-ENUMERATE}(x, a, b)$ in $\Theta(m + \lg n)$ time, so it does not meet the requirement of the question if we implement the procedure in the following ways (augment the tree in the way of exercise 14.2-1) since it takes $\Theta(m)$ time only:

```
RB-ENUMERATE( $T, a, b$ )
1   $k = a$ 
2  OUTPUT( $k$ )
3  repeat
4       $k = k.succ$ 
5      OUTPUT( $k$ )
6  until  $k == b$ 
```

14.3

14.3-1

Add the following lines to the end of $\text{LEFT-ROTATE}(T, x)$:

```
1   $y.max = x.max$ 
2   $x.max = \max(x.int.high, x.left.max, x.right.max)$ 
```

14.3-2

INTERVAL-SEARCH(T, i)

```
1   $x = T.root$ 
2  while  $x \neq T.nil$  and  $i$  does not overlap  $x.int$ 
3      if  $x.left \neq T.nil$  and  $x.left.max > i.low$ 
4           $x = x.left$ 
5      else  $x = x.right$ 
6  return  $x$ 
```

14.3-3

Iterative version:

INTERVAL-SEARCH-MIN(T, i)

```
1   $x = T.root$ 
2   $smallest = T.nil$ 
3  while  $x \neq T.nil$ 
4      if  $x.left \neq T.nil$  and  $x.left.max \geq i.low$ 
5          if  $i$  overlaps  $x.int$ 
6               $smallest = x$ 
7               $x = x.left$ 
8      else if  $i$  overlaps  $x.int$ 
9          return  $x$ 
10      $x = x.right$ 
11 return  $smallest$ 
```

Recursive version:

Invoke INTERVAL-SEARCH-MIN($T, T.root, i$)

INTERVAL-SEARCH-MIN(T, x, i)

```
1  if  $x.left \neq T.nil$  and  $x.left.max \geq i.low$ 
2       $smallest = \text{INTERVAL-SEARCH-MIN}(T, x.left, i)$ 
3      if  $smallest \neq T.nil$ 
4          return  $smallest$ 
5      elseif  $i$  overlaps  $x.int$ 
6          return  $x$ 
7      else return  $T.nil$ 
8  else if  $i$  overlaps  $x.int$ 
9      return  $x$ 
10 else return  $\text{INTERVAL-SEARCH-MIN}(T, x.right, i)$ 
```

14.3-4

LIST-OVERLAP-INTERVAL(T, x, i)

```
1  if  $x.int$  overlaps  $i$ 
2      OUTPUT( $x$ )
3  if  $x.left \neq T.nil$  and  $x.left.max \geq i.low$ 
4      LIST-OVERLAP-INTERVAL( $T, x.left, i$ )
5  if  $x.right \neq T.nil$  and  $x.right.max \geq i.low$  and  $x.int.low < i.high$ 
6      LIST-OVERLAP-INTERVAL( $T, x.right, i$ )
```

14.3-5

INTERVAL-SEARCH-EXACTLY(T, i)

```
1   $x = T.root$ 
2  while  $x \neq T.nil$  and  $(x.int.low \neq i.low$  or  $x.int.high \neq i.high)$ 
3      if  $i.high > x.max$ 
4           $x = T.nil$ 
5      elseif  $i.low < x.low$ 
6           $x = x.left$ 
7      elseif  $i.low > x.low$ 
8           $x = x.right$ 
9      else  $x = T.nil$ 
10 return  $x$ 
```

14.3-6

Consider to augment the red-black tree by adding attribute *min-gap*, *min*, and *max* to every node.

min-gap is the minimum gap in the subtree rooted at the node.

min is the minimum key in the subtree rooted at the node.

max is the maximum key in the subtree rooted at the node.

When MIN-GAP(Q) is called, we just need to return $Q.root.min-gap$, which takes $O(1)$ time.

Let x be arbitrary node in the red-black tree Q . In order to maintain *min-gap* in $O(\lg n)$, we want $x.min-gap$ depends on only the information in nodes x , $x.left$, and $x.right$.

$$x.min-gap = \min(x.left.min-gap, x.right.min-gap, x.key - x.left.max, x.right.min - x.key)$$

$$Q.nil.min-gap = \infty$$

$$Q.nil.min = \infty$$

$$Q.nil.max = -\infty$$

Notice when there is no left subtree of x ($x.left == Q.nil$), $x.prev$ will be the ancestor of x , and the gap between $x.prev$ and x will be compared by $x.prev$, so the gap will not be neglected. (recall that $x.min-gap$ only contains the minimum gap in the subtree rooted at x)

We can maintain *min* and *max* in $O(\lg n)$ also since $x.min$ and $x.max$ depends only on only the information in nodes x , $x.left$, and $x.right$.

$x.min = \min(x.left.min, x.key)$
 $x.max = \max(x.right.max, x.key)$

Notice that if $x.left \neq Q.nil$, $x.left.min < x.key$ must be true; if $x.right \neq Q.nil$, $x.right.max > x.key$ must be true.

14.3-7

```
1  bool DetermineOverlapRectangles(const std::vector<Rectangle>& rectangles)
2  {
3      bool found_overlap;
4      RectangleWithXCoordIndex *alloc_ptr, *now;
5      std::allocator<RectangleWithXCoordIndex> alloc;
6      std::vector<RectangleWithXCoordIndex*> heap;
7      Tree tree; // interval tree
8      found_overlap = false;
9      // sort the x-coordinates with the minimum heap
10     alloc_ptr = ConstructHeap(rectangles, heap);
11     while (heap.size() > 0)
12     {
13         now = HeapExtractMin(heap);
14         if (now->coord_pos == RectangleWithXCoordIndex::LEFT)
15         {
16             if (tree.Find(now->rectangle->y_int) != tree.End())
17             {
18                 found_overlap = true;
19                 break;
20             }
21             now->relate.right->relate.left =
22                 tree.Insert({now->rectangle->y_int, 0}).first;
23         }
24         else
25         {
26             tree.Delete(now->relate.left);
27         }
28     }
29     DestructHeap(alloc_ptr, rectangles.size());
30     return found_overlap;
31 }
```

Chapter 14 Problems

14-1

(a)

Proof. Let $[a, b]$ be the interval of maximum overlap. In other word, $[a, b]$ is the intersection of the overlap segments. This says all points in $[a, b]$ have the same number of overlap segments. Notice a and b must be one of the endpoint of a segment that is included in the intersection. Hence there will always be a point of maximum overlap that is an endpoint of the segments. \square

(b)

```
1      struct Node
2      {
3          Node* parent;
4          Node* left;
5          Node* right;
6          enum { BLACK, RED } color;
7          T mapped_value; // only lower endpoint
8          const Key key;
9          enum { LOWER = +1, HIGHER = -1 } endpoint_type;
10         Node* related_endpoint; // other endpoint of the current interval
11         /**
12             * x.type_sum = x.left.type_sum + x.endpoint_type + x.right.type_sum
13             */
14         int type_sum;
15         /**
16             * x.left_max_overlap_num = x.left.max_overlap_num
17             * x.overlap_num = x.left.type_sum + x.endpoint_type
18             * x.right_max_overlap_num = x.overlap_num + x.right.max_overlap_num
19             * x.max_overlap_num =
20             *      max(x.left_max_overlap_num, x.overlap_num, x.right_max_overlap_num)
21             *
22             * Notice that overlap_num only consider local overlap
23             *      (in the subtree root at the node)
24             * This says overlap_num is relatively
25             */
26         // node with endpoint which has maximum number of overlap segments
27         Node* max_overlap_node;
28         // maximum number of overlap segments
29         int max_overlap_num;
```

```
30
31     Node() : key(Key()) {}
32     Node(const Key& key) : key(key) {}
33 };
34
35 template <class Key, class T>
36 bool IntervalTreePOM<Key, T>::MaintainAugmentedAttributesOfSingleNode(Node *node)
37 {
38     int type_sum, max_overlap_num, overlap_num, right_max_overlap_num;
39     Node *max_overlap_node;
40     bool modified;
41     /* maintain type_sum */
42     type_sum = node->left->type_sum + node->endpoint_type + node->right->type_sum;
43     /* maintain max_overlap */
44     overlap_num = node->left->type_sum + node->endpoint_type;
45     if (node->left != nil_ && node->left->max_overlap_num > overlap_num)
46     {
47         max_overlap_num = node->left->max_overlap_num;
48         max_overlap_node = node->left->max_overlap_node;
49     }
50     else
51     {
52         max_overlap_num = overlap_num;
53         max_overlap_node = node;
54     }
55     if (node->right != nil_)
56     {
57         right_max_overlap_num = overlap_num + node->right->max_overlap_num;
58         if (right_max_overlap_num > max_overlap_num)
59         {
60             max_overlap_num = right_max_overlap_num;
61             max_overlap_node = node->right->max_overlap_node;
62         }
63     }
64     /* do modify */
65     modified = false;
66     if (node->type_sum != type_sum)
67     {
68         node->type_sum = type_sum;
69         modified = true;
```

```
70     }
71     if (node->max_overlap_num != max_overlap_num)
72     {
73         node->max_overlap_num = max_overlap_num;
74         modified = true;
75     }
76     if (node->max_overlap_node != max_overlap_node)
77     {
78         node->max_overlap_node = max_overlap_node;
79         modified = true;
80     }
81     return modified;
82 }
83
84 // start to check augmented attributes from node->parent to root_
85 template <class Key, class T>
86 void IntervalTreePOM<Key, T>::FixAugmentedAttributes(Node* node)
87 {
88     Key max;
89     while (node != root_)
90     {
91         node = node->parent;
92         if (MaintainAugmentedAttributesOfSingleNode(node) == false) break;
93     }
94 }
95
96 template <class Key, class T>
97 std::pair<typename IntervalTreePOM<Key, T>::Iterator, bool>
98     IntervalTreePOM<Key, T>::Insert(const ValueType& value)
99 {
100     Node **node_ptr, *lower_node, *higher_node, *parent;
101     node_ptr = FindNodePtrByLowerKey(value.interval.low, &parent);
102     if (*node_ptr == nil_)
103     {
104         /* insert lower endpoint */
105         lower_node = new Node(value.interval.low);
106         *node_ptr = lower_node;
107         lower_node->parent = parent;
108         lower_node->left = nil_;
109         lower_node->right = nil_;
```

```
110         lower_node->color = Node::RED;
111         lower_node->mapped_value = value.mapped_value;
112         lower_node->endpoint_type = Node::LOWER;
113         MaintainAugmentedAttributesOfSingleNode(lower_node);
114         FixAugmentedAttributes(lower_node);
115         InsertFixup(lower_node);
116         /* insert higher endpoint */
117         node_ptr = FindLeafNodePtrToInsertByKey(value.interval.high, &parent);
118         higher_node = new Node(value.interval.high);
119         *node_ptr = higher_node;
120         lower_node->related_endpoint = higher_node;
121         higher_node->related_endpoint = lower_node;
122         higher_node->parent = parent;
123         higher_node->left = nil_;
124         higher_node->right = nil_;
125         higher_node->color = Node::RED;
126         higher_node->endpoint_type = Node::HIGHER;
127         MaintainAugmentedAttributesOfSingleNode(higher_node);
128         FixAugmentedAttributes(higher_node);
129         InsertFixup(higher_node);
130         return std::make_pair(Iterator(lower_node, this), true);
131     }
132     else
133     {
134         return std::make_pair(Iterator(nil_, this), false);
135     }
136 }
137
138 template <class Key, class T>
139 void IntervalTreePOM<Key, T>::Delete(Iterator pos)
140 {
141     DeleteNode(pos.node->related_endpoint);
142     DeleteNode(pos.node_);
143 }
144
145 template <class Key, class T>
146 void IntervalTreePOM<Key, T>::DeleteNode(Node *node)
147 {
148     Node *replaced, *replaced_replaced;
149     bool is_black_deleted;
```

```
150     replaced = node;
151     is_black_deleted = replaced->color == Node::BLACK;
152     if (replaced->left == nil_)
153     {
154         replaced_replaced = replaced->right;
155         Transplant(replaced, replaced_replaced);
156         FixAugmentedAttributes(replaced_replaced); // replaced_replaced will NOT be checked
157     }
158     else if (replaced->right == nil_)
159     {
160         replaced_replaced = replaced->left;
161         Transplant(replaced, replaced_replaced);
162         FixAugmentedAttributes(replaced_replaced); // replaced_replaced will NOT be checked
163     }
164     else
165     {
166         replaced = TreeMinimum(node->right);
167         is_black_deleted = replaced->color == Node::BLACK;
168         replaced_replaced = replaced->right;
169         if (replaced->parent == node)
170         {
171             replaced_replaced->parent = replaced;
172         }
173         else
174         {
175             Transplant(replaced, replaced_replaced);
176             replaced->right = node->right;
177             replaced->right->parent = replaced;
178         }
179         Transplant(node, replaced);
180         replaced->left = node->left;
181         replaced->left->parent = replaced;
182         replaced->color = node->color;
183         FixAugmentedAttributes(replaced_replaced); // replaced_replaced will NOT be checked
184         FixAugmentedAttributes(replaced->right); // so replaced will be checked
185     }
186     if (is_black_deleted)
187         DeleteFixup(replaced_replaced);
188     delete node;
189 }
```

```
190
191     template <class Key, class T>
192     Key IntervalTreePOM<Key, T>::FindPOM()
193     {
194         return root_->max_overlap_node->key;
195     }
```

14-2

(a)

```
1     /**
2     * running time:  $O(mn)$ 
3     * caller is responsible to deallocate the return value
4     *  $n, m$  must greater than 0
5     */
6     std::vector<int>* GetJosephusPermutation(int n, int m)
7     {
8         Node *nodes, **ptr;
9         std::vector<int>* result;
10        int i;
11        /* allocate nodes */
12        nodes = new Node[n];
13        /* build circular singly linked list */
14        for (i = 0; i < n - 1; ++i)
15        {
16            nodes[i].key = i + 1;
17            nodes[i].next = &(nodes[i + 1]);
18        }
19        nodes[i].key = i + 1;
20        nodes[i].next = nodes; // &(nodes[0])
21        /* allocate vector */
22        result = new std::vector<int>;
23        result->reserve(n);
24        /* output */
25        /* find the m-th key */
26        ptr = &(nodes[n - 1].next);
27        for (i = 1; i < m; ++i) ptr = &((*ptr)->next);
28        while (true)
29        {
30            /* add the key into the permutation */
```

```
31         result->push_back((*ptr)->key);
32         /* remove the key */
33         if (*ptr == (*ptr)->next) break;
34         *ptr = (*ptr)->next;
35         /* find the next m-th key */
36         for (i = 1; i < m; ++i) ptr = &((*ptr)->next);
37     }
38     /* deallocate nodes */
39     delete[] nodes;
40     /* return */
41     return result;
42 }
```

(b)

```
1  /**
2   * running time: O(nlgn)
3   * caller is responsible to deallocate the return value
4   * n, m must greater than 0
5   */
6  std::vector<int>* GetJosephusPermutation(int n, int m)
7  {
8      int i, rank;
9      OrderStatisticsTree<int, int> ostree;
10     OrderStatisticsTree<int, int>::Iterator it;
11     std::vector<int>* result;
12     /* build order statistics tree: O(nlgn) */
13     for (i = 1; i <= n; ++i) ostree.Insert(std::make_pair(i, 0));
14     /* allocate vector */
15     result = new std::vector<int>;
16     result->reserve(n);
17     /* output: O(nlgn) */
18     rank = m;
19     for (i = n; i > 0; --i) // n times
20     {
21         rank = ((rank - 1) % i) + 1;
22         it = ostree.Select(rank); // O(lgn)
23         result->push_back(it->first);
24         ostree.Delete(it); // O(lgn)
25         rank += (m - 1);

```



```
26         }  
27     return result;  
28 }
```