In [1]:

```python
from qiskit import QuantumCircuit
import qiskit.quantum_info as qi
```

Exercise 1.1

Find the corresponding density matrix for the following pure states. Use Qiskit to prepare the states, and verify your results using the `quantum_info` module:

1. $\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$

1. $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix} - i\begin{pmatrix}0\\1\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\i\end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$$

$$\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$$

$$= \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix}1\\-i\end{bmatrix}\begin{bmatrix}1 & i\end{bmatrix}\right)$$

$$\rho_{AB} = \frac{1}{2}\begin{bmatrix}1 & i\\-i & 1\end{bmatrix}$$

In [2]:

```python
qc_AB = QuantumCircuit(1)
qc_AB.h(0)
qc_AB.s(0)
qc_AB.z(0)
qc_AB.draw()
```

Out[2]:

```
q_0: ┤ H ├┤ S ├┤ Z ├
```

In [3]:

```python
psi_AB = qi.Statevector.from_instruction(qc_AB)
psi_AB.draw('latex', prefix='|\\psi_{AB}\\rangle = ')
```

Out[3]:

$$|\psi_{AB}\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}i \end{bmatrix}$$

In [4]:

```
rho_AB = qi.DensityMatrix.from_instruction(qc_AB)
rho_AB.draw('latex', prefix='\\rho_{AB} = ')
```

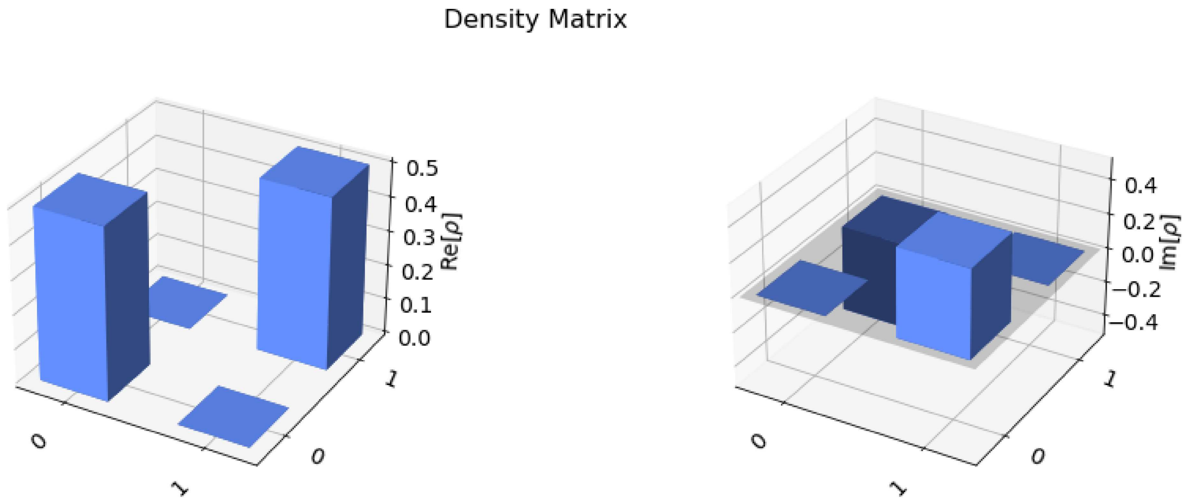
Out[4]:

$$\rho_{AB} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}i \\ -\frac{1}{2}i & \frac{1}{2} \end{bmatrix}$$

In [5]:

```
from qiskit.visualization import plot_state_city
plot_state_city(rho_AB.data, title='Density Matrix')
```

Out[5]:



$$|\psi_{AB}\rangle = \frac{1}{2}\left[|00\rangle + |11\rangle + |01\rangle + |10\rangle\right]$$

$$= \frac{1}{2}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix} + \begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$$

$$= \left(\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)\left(\frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}\right)$$

$$= \frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\end{bmatrix}$$

In [6]:

```
qc_AB = QuantumCircuit(2)
qc_AB.h(0)
qc_AB.h(1)
qc_AB.draw()
```

Out[6]:

```
     ┌───┐
q_0: ┤ H ├
     ├───┤
q_1: ┤ H ├
     └───┘
```

In [7]:

```
psi_AB = qi.Statevector.from_instruction(qc_AB)
psi_AB.draw('latex', prefix='|\\psi_{AB}\\rangle = ')
```

Out[7]:

$$|\psi_{AB}\rangle = \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

In [8]:

```
rho_AB = qi.DensityMatrix.from_instruction(qc_AB)
rho_AB.draw('latex', prefix='\\rho_{AB} = ')
```

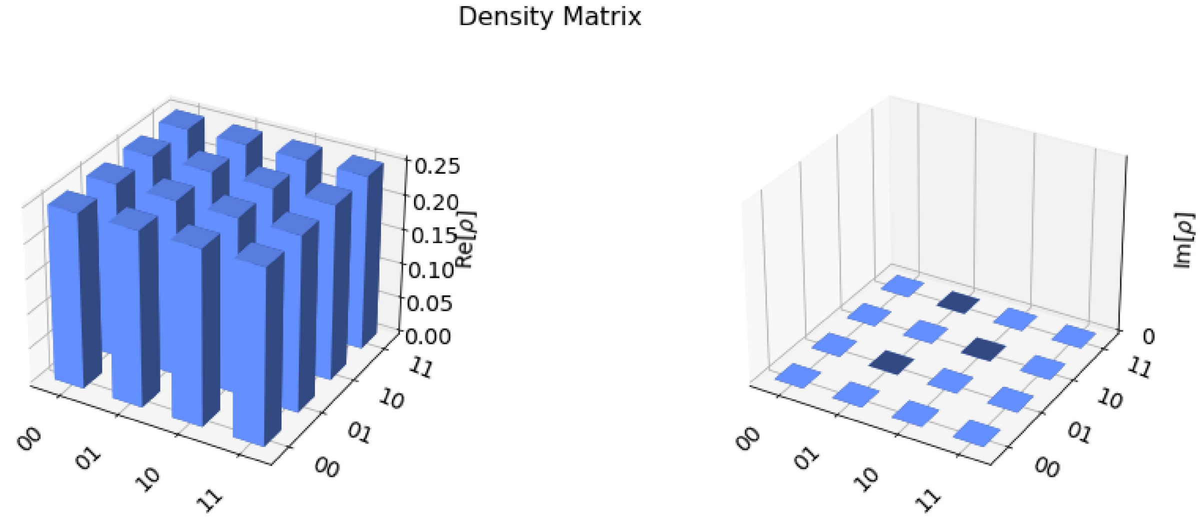
Out[8]:

$$\rho_{AB} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

In [9]:

```
from qiskit.visualization import plot_state_city
plot_state_city(rho_AB.data, title='Density Matrix')
```

Out[9]:



Exercise 2.2

Derive and compare the density matrices for:

1. A pure one-qubit state in an equal superposition of $|0\rangle$ and $|1\rangle$:

$$\rho_p = |\psi_p\rangle\langle\psi_p|, \text{ with } |\psi_p\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

1. A mixed one-qubit state in an equal mixture of $|0\rangle$ and $|1\rangle$:

$$\rho_m = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

① $|\psi_p\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ Pure state

$$|\psi_p\rangle = \frac{1}{\sqrt{2}}\left(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

Density matrix

$$\rho_p = |\psi_p\rangle\langle\psi_p| = \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}\right)$$

$$\rho_p = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

② $\rho_m = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$ Mixed state

$$= \frac{1}{2}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Exercise 4.1

Calculate the reduced density matrices for each of the following composite states.

1. The pure state: $|\psi_{pr}\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

1. The mixed state: $\rho_{mr} = \frac{1}{4}|\Phi^+\rangle\langle\Phi^+| + \frac{3}{4}|\Phi^-\rangle\langle\Phi^-|$, where $|\Phi^+\rangle$ and $|\Phi^-\rangle$ are the Bell states:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

1) For mathematical workout please see the pdf attached in the repo, titled 'Reduced_Density_Unsolved'

In [15]:

```
qc_AB = QuantumCircuit(2)
qc_AB.h(0)
```

```python
qc_AB.h(1)
qc_AB.draw()
```

Out[15]:

```
     ┌───┐
q_0: ┤ H ├
     ├───┤
q_1: ┤ H ├
     └───┘
```

In [16]:

```python
psi_AB = qi.Statevector.from_instruction(qc_AB)
psi_AB.draw('latex', prefix='|\\psi_{AB}\\rangle = ')
```

Out[16]:

$$|\psi_{AB}\rangle = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

In [17]:

```python
rho_AB = qi.DensityMatrix.from_instruction(qc_AB)
rho_AB.draw('latex', prefix='\\rho_{AB} = ')
```

Out[17]:

$$\rho_{AB} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

In [18]:

```python
rho_B = qi.partial_trace(rho_AB,[0])
rho_A = qi.partial_trace(rho_AB,[1])

display(rho_B.draw('latex', prefix=" \\rho_{B} = "),
        rho_A.draw('latex', prefix=" \\rho_{A} = "))
```

$$\rho_B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\rho_A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

2) Please see the pdf attached in the repo, titled 'Reduced_Density_Unsolved'

In [ ]:

```python

```