Bloch Sphere

General Equation : $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ — (*)

Euler Equation : $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$ — (Δ)

from (*) and (Δ) we have

$$\alpha = \cos\frac{\theta}{2} \quad , \beta = e^{i\phi}\sin\frac{\theta}{2}$$

① $|0\rangle$ (Comparing with (*))

$\rightarrow |\psi\rangle = 1|0\rangle + 0|1\rangle \Rightarrow \alpha = 1, \beta = 0$

$\Rightarrow \alpha = \cos\frac{\theta}{2} = 1 \Rightarrow \boxed{\theta = 0}$

$\beta = e^{i\phi}\sin\frac{\theta}{2} = 0 \Rightarrow e^{i\phi}\sin\left(\frac{0}{2}\right) = 0$

$\Rightarrow e^{i\phi} = 0$

$\Rightarrow \cos\phi + i\sin\phi = 0 \Rightarrow \cos\phi = 0 \Rightarrow \boxed{\phi = \frac{\pi}{2}}$

Bloch sphere coordinates $= [\theta, \phi, r]$

$= [0, \frac{\pi}{2}, 1]$ → unit radius

② $|1\rangle$

$\rightarrow |\psi\rangle = 0|0\rangle + 1|1\rangle \Rightarrow \alpha = 0, \beta = 1$

$\alpha = \cos\frac{\theta}{2} = 0 \Rightarrow \boxed{\theta = \pi}$

$$\beta = e^{i\phi}\sin\frac{\theta}{2} = 1 \Rightarrow e^{i\phi}\sin\left(\frac{\pi}{2}\right) = 1$$

$$\Rightarrow e^{i\phi} = 1$$

$$\Rightarrow \cos\phi + i\sin\phi = 1 \Rightarrow \cos\phi = 1 \Rightarrow \boxed{\phi = 0}$$

Bloch sphere coordinates = $[\pi, 0, 1]$

③ $\frac{1}{\sqrt{2}}[|0\rangle - i|1\rangle]$

$$\rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \left(-\frac{1}{\sqrt{2}}i\right)|1\rangle \Rightarrow \alpha = \frac{1}{\sqrt{2}},\ \beta = -\frac{1}{\sqrt{2}}i$$

$$\alpha = \cos\frac{\theta}{2} = \frac{1}{\sqrt{2}} \Rightarrow \boxed{\theta = \pi/2}$$

$$\beta = e^{i\phi}\sin\frac{\theta}{2} = -\frac{1}{\sqrt{2}}i \Rightarrow e^{i\phi}\sin\left(\frac{\pi/2}{2}\right) = -\frac{1}{\sqrt{2}}i$$

$$\Rightarrow e^{i\phi}\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}i$$

$$\Rightarrow e^{i\phi} = -i$$

$$\Rightarrow \cos\phi + i\sin\phi = 0 - i1 \Rightarrow \sin\phi = -1$$

$$\Rightarrow \phi = \sin^{-1}(-1)$$

$$\Rightarrow \boxed{\phi = -\pi/2}$$

Bloch Sphere coordinates = $\left[\frac{\pi}{2}, -\frac{\pi}{2}, 1\right]$

④ $\frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$

$$\rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}(i|0\rangle + 1|1\rangle) = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

multiplying by a global phase $e^{i\alpha}$

$$\Rightarrow e^{i\alpha}|\psi\rangle = e^{i\alpha}\cos\frac{\theta}{2}|0\rangle + e^{i(\alpha+\phi)}\sin\frac{\theta}{2}|1\rangle$$

$$\Rightarrow e^{i\alpha}\cos\frac{\theta}{2} = \frac{i}{\sqrt{2}}$$

If $\boxed{\theta = \frac{\pi}{2}}$

$$\Rightarrow e^{i\alpha} = i$$

$$e^{i(\alpha+\phi)}\sin\frac{\theta}{2} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow e^{i(\alpha+\phi)} = 1$$

(using $\theta = \pi/2$)

$$\Rightarrow e^{i\alpha}\cdot e^{i\phi} = 1$$

$$\Rightarrow e^{i\phi} = \frac{1}{i} \quad (\because e^{i\alpha} = i)$$

$$= \frac{i}{i^2} = -i$$

$$\Rightarrow \cos\phi + i\sin\phi = -i$$

$$\Rightarrow \sin\phi = -1$$

$$\Rightarrow \boxed{\phi = -\frac{\pi}{2}}$$

Bloch sphere coordinates $= \left[\frac{\pi}{2}, -\frac{\pi}{2}, 1\right]$

(5) $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$\rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \quad \Rightarrow \alpha = \frac{1}{\sqrt{2}},\ \beta = \frac{1}{\sqrt{2}}$$

$$\alpha = \cos\frac{\theta}{2} = \frac{1}{\sqrt{2}} \Rightarrow \boxed{\theta = \pi/2}$$

$$\beta = e^{i\phi}\sin\frac{\theta}{2} = \frac{1}{\sqrt{2}} \Rightarrow e^{i\phi}\sin\left(\frac{\pi/2}{2}\right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad e^{i\phi} = 1$$

$$\Rightarrow \cos\phi + i\sin\phi = 1$$

$$\Rightarrow \cos\phi = 1 \quad \Rightarrow \boxed{\phi = 0}$$

Bloch sphere coordinates $= \left[\frac{\pi}{2}, 0, 1\right]$