Bloch Sphere

General Equation : $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ ——— (*)

Euler Equation : $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$ ——— (Δ)

from (*) and (Δ) we have

$$\boxed{\alpha = \cos\frac{\theta}{2}, \quad \beta = e^{i\phi}\sin\frac{\theta}{2}}$$

Comparing with (*)

(1) $|0\rangle$

$\rightarrow |\psi\rangle = 1|0\rangle + 0|1\rangle \Rightarrow \alpha = 1, \beta = 0$

$\Rightarrow \alpha = \cos\frac{\theta}{2} = 1 \Rightarrow \boxed{\theta = 0}$

$\beta = e^{i\phi}\sin\frac{\theta}{2} = 0 \Rightarrow e^{i\phi}\sin\left(\frac{0}{2}\right) = 0$

$\Rightarrow e^{i\phi} = 0$

$\Rightarrow \cos\phi + i\sin\phi = 0 \Rightarrow \cos\phi = 0 \Rightarrow \boxed{\phi = 0}$

Bloch sphere coordinates $= [\theta, \phi, r]$

$= [0, 0, 1]$ unit radius

(2) $|1\rangle$

$\rightarrow |\psi\rangle = 0|0\rangle + 1|1\rangle \Rightarrow \alpha = 0, \beta = 1$

$\alpha = \cos\frac{\theta}{2} = 0 \Rightarrow \boxed{\theta = \pi}$

$\beta = e^{i\phi} \sin\frac{\theta}{2} = 1 \Rightarrow e^{i\phi}\sin\left(\frac{\pi}{2}\right) = 1$

$\Rightarrow e^{i\phi} = 1$

$\Rightarrow \cos\phi + i\sin\phi = 1 \Rightarrow \cos\phi = 1 \Rightarrow \boxed{\phi = 0}$

Bloch sphere coordinates $= [\pi, 0, 1]$

③ $\frac{1}{\sqrt{2}}[|0\rangle - i|1\rangle]$

$\rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \left(-\frac{1}{\sqrt{2}}i\right)|1\rangle \Rightarrow \alpha = \frac{1}{\sqrt{2}}, \beta = -\frac{1}{\sqrt{2}}i$

$\alpha = \cos\frac{\theta}{2} = \frac{1}{\sqrt{2}} \Rightarrow \boxed{\theta = \pi/2}$

$\beta = e^{i\phi}\sin\frac{\theta}{2} = -\frac{1}{\sqrt{2}}i \Rightarrow e^{i\phi}\sin\left(\frac{\pi/2}{2}\right) = -\frac{1}{\sqrt{2}}i$

$\Rightarrow e^{i\phi}\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}i$

$\Rightarrow e^{i\phi} = -i$

$\Rightarrow \cos\phi + i\sin\phi = 0 - i1 \Rightarrow \sin\phi = -1$

$\Rightarrow \phi = \sin^{-1}(1)$

$\Rightarrow \boxed{\phi = -\pi/2}$

Bloch Sphere coordinates $= \left[\frac{\pi}{2}, -\frac{\pi}{2}, 1\right]$

④ $\frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$

$\rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}(i|0\rangle + 1|1\rangle) = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$

Multiplying by a global phase $e^{i\alpha}$

$\Rightarrow e^{i\alpha}|\psi\rangle = e^{i\alpha}\cos\frac{\theta}{2}|0\rangle + e^{i(\alpha+\phi)}\sin\frac{\theta}{2}|1\rangle$

$\Rightarrow e^{i\alpha}\cos\frac{\theta}{2} = \frac{i}{\sqrt{2}}$

If $\boxed{\theta = \frac{\pi}{2}}$

$\Rightarrow e^{i\alpha} = i$

$e^{i(\alpha+\phi)}\sin\frac{\theta}{2} = \frac{1}{\sqrt{2}}$

$\Rightarrow e^{i(\alpha+\phi)} = 1$

{using $\theta = \pi/2$}

$\Rightarrow e^{i\alpha}\cdot e^{i\phi} = 1$

$\Rightarrow e^{i\phi} = \frac{1}{i}$ $\{\because e^{i\alpha} = i\}$

$= \frac{i}{i^2} = -i$

$\Rightarrow \cos\phi + i\sin\phi = -i$

$\Rightarrow \sin\phi = -1$

$\Rightarrow \boxed{\phi = -\frac{\pi}{2}}$

Bloch sphere co-ordinates $= \left[\frac{\pi}{2}, -\frac{\pi}{2}, 1\right]$

⑤ $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$\rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ $\Rightarrow \alpha = \frac{1}{\sqrt{2}}, \beta = \frac{1}{\sqrt{2}}$

$\alpha = \cos\frac{\theta}{2} = \frac{1}{\sqrt{2}}$ $\Rightarrow \boxed{\theta = \pi/2}$

$\beta = e^{i\phi}\sin\frac{\theta}{2} = \frac{1}{\sqrt{2}}$ $\Rightarrow e^{i\phi}\sin\left(\frac{\pi/2}{2}\right) = \frac{1}{\sqrt{2}}$

$$\Rightarrow \quad e^{i\phi} = 1$$

$$\Rightarrow \cos\phi + i\sin\phi = 1$$

$$\Rightarrow \cos\phi = 1 \quad \Rightarrow \boxed{\phi = 0}$$

Bloch sphere coordinates $= \left[\frac{\pi}{2}, 0, 1\right]$