① Pure state

$$|\Psi_{pr}\rangle = \frac{1}{2}\left(|0_A 0_B\rangle + |0_A 1_B\rangle + |1_A 0_B\rangle + |1_A 1_B\rangle\right)$$

$\Rightarrow$

$$\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$$

$$= \frac{1}{2}\left[|0_A 0_B\rangle + |0_A 1_B\rangle + |1_A 0_B\rangle + |1_A 1_B\rangle\right]$$

$$\frac{1}{2}\left[\langle 0_A 0_B| + \langle 0_A 1_B| + \langle 1_A 0_B| + \langle 1_A 1_B|\right]$$

$$= \frac{1}{4}\begin{bmatrix} |0_A 0_B\rangle\langle 0_A 0_B| + |0_A 0_B\rangle\langle 0_A 1_B| + |0_A 0_B\rangle\langle 1_A 0_B| \\ + |0_A 0_B\rangle\langle 1_A 1_B| + |0_A 1_B\rangle\langle 0_A 0_B| + |0_A 1_B\rangle\langle 0_A 1_B| \\ + |0_A 1_B\rangle\langle 1_A 0_B| + |0_A 1_B\rangle\langle 1_A 1_B| \\ + |1_A 0_B\rangle\langle 0_A 0_B| + |1_A 0_B\rangle\langle 0_A 1_B| + |1_A 0_B\rangle\langle 1_A 0_B| \\ + |1_A 0_B\rangle\langle 1_A 1_B| + |1_A 1_B\rangle\langle 0_A 0_B| + |1_A 1_B\rangle\langle 0_A 1_B| \\ + |1_A 1_B\rangle\langle 1_A 0_B| + |1_A 1_B\rangle\langle 1_A 1_B| \end{bmatrix}$$

Say we calculate for subsystem B.

$$\rho_B = Tr_A(\rho_{AB})$$

Solving the way given in Qiskit Textbook we get,

$$
\begin{aligned}
= \frac{1}{4} \Big[& Tr\left(|0_A\rangle\langle 0_A|\right)|0_B\rangle\langle 0_B| + Tr\left(|0_A\rangle\langle 0_A|\right)|0_B\rangle\langle 1_B| \\
& + Tr\left(|0_A\rangle\langle 1_A|\right)|0_B\rangle\langle 0_B| + Tr\left(|0_A\rangle\langle 1_A|\right)|0_B\rangle\langle 1_B| \\
& + Tr\left(|0_A\rangle\langle 0_A|\right)|1_B\rangle\langle 0_B| + Tr\left(|0_A\rangle\langle 0_A|\right)|1_B\rangle\langle 1_B| \\
& + Tr\left(|0_A\rangle\langle 1_A|\right)|1_B\rangle\langle 0_B| + Tr\left(|0_A\rangle\langle 1_A|\right)|1_B\rangle\langle 1_B| \\
& + Tr\left(|1_A\rangle\langle 0_A|\right)|0_B\rangle\langle 0_B| + Tr\left(|1_A\rangle\langle 0_A|\right)|0_B\rangle\langle 1_B| \\
& + Tr\left(|1_A\rangle\langle 1_A|\right)|0_B\rangle\langle 0_B| + Tr\left(|1_A\rangle\langle 1_A|\right)|0_B\rangle\langle 1_B| \\
& + Tr\left(|1_A\rangle\langle 0_A|\right)|1_B\rangle\langle 0_B| + Tr\left(|1_A\rangle\langle 0_A|\right)|1_B\rangle\langle 1_B| \\
& + Tr\left(|1_A\rangle\langle 1_A|\right)|1_B\rangle\langle 0_B| + Tr\left(|1_A\rangle\langle 1_A|\right)|1_B\rangle\langle 1_B| \Big]
\end{aligned}
$$

$\langle 0|1\rangle = 0$ $\quad$ $\langle 0|0\rangle = 1$, using this we get.

$$
\begin{aligned}
= \frac{1}{4} \Big[& |0_B\rangle\langle 0_B| + |0_B\rangle\langle 1_B| \\
& + 0 + 0 \\
& + |1_B\rangle\langle 0_B| + |1_B\rangle\langle 1_B| \\
& + 0 + 0 \\
& + 0 + 0 \\
& + |0_B\rangle\langle 0_B| + |0_B\rangle\langle 1_B| \\
& + 0 + 0 \\
& + |1_B\rangle\langle 0_B| + |1_B\rangle\langle 1_B| \Big]
\end{aligned}
$$

$$
= \frac{1}{4}\Big[2\,|0_B\rangle\langle 0_B| + 2\,|0_B\rangle\langle 1_B| + 2\,|1_B\rangle\langle 0_B| + 2\,|1_B\rangle\langle 1_B| \Big]
$$

$$= \frac{1}{2}\left[|0_B\rangle\langle 0_B| + |0_B\rangle\langle 1_B| + |1_B\rangle\langle 0_B| + |1_B\rangle\langle 1_B|\right]$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \boxed{\begin{array}{l} \rho_B = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ \rho_A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{array}}$$

Simillarly,

2) $\rho_{mr} = \frac{1}{4}|\phi^+\rangle\langle\phi^+| + \frac{3}{4}|\phi^-\rangle\langle\phi^-|$

$$= \frac{1}{4}\left(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)\right)$$

$$+ \frac{3}{4}\left(\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)\frac{1}{\sqrt{2}}(\langle 00| - \langle 11|)\right)$$

$$= \frac{1}{8}\left[(|0_A 0_B\rangle + |1_A 1_B\rangle)(\langle 0_A 0_B| + \langle 1_A 1_B|)\right]$$

$$+ \frac{3}{8}\left[(|0_A 0_B\rangle - |1_A 1_B\rangle)(\langle 0_A 0_B| - \langle 1_A 1_B|)\right]$$

$$= \frac{1}{8}\Big[|0_A 0_B\rangle\langle 0_A 0_B| + |0_A 0_B\rangle\langle 1_A 1_B| + |1_A 1_B\rangle\langle 0_A 0_B| + |1_A 1_B\rangle\langle 1_A 1_B| \Big]$$

$$+ \frac{3}{8}\Big[|0_A 0_B\rangle\langle 0_A 0_B| - |0_A 0_B\rangle\langle 1_A 1_B| - |1_A 1_B\rangle\langle 0_A 0_B| + |1_A 1_B\rangle\langle 1_A 1_B| \Big]$$

$$= \frac{1}{2}\Big[|0_A 0_B\rangle\langle 0_A 0_B| \Big] - \frac{1}{2}\Big[|0_A 0_B\rangle\langle 1_A 1_B| \Big] - \frac{1}{2}\Big[|1_A 1_B\rangle\langle 0_A 0_B| \Big] + \frac{1}{2}\Big[|1_A 1_B\rangle\langle 1_A 1_B| \Big]$$

$$= \frac{1}{2}\Big[|0_A 0_B\rangle\langle 0_A 0_B| - |0_A 0_B\rangle\langle 1_A 1_B| - |1_A 1_B\rangle\langle 0_A 0_B| + |1_A 1_B\rangle\langle 1_A 1_B| \Big]$$

We calculate reduced density matrix for subsystem B.

$$\rho_B = \frac{1}{2}\Big[\mathrm{Tr}(|0_A\rangle\langle 0_A|)\, |0_B\rangle\langle 0_B| - \mathrm{Tr}(|0_A\rangle\langle 1_A|)\, |0_B\rangle\langle 1_B| - \mathrm{Tr}(|1_A\rangle\langle 0_A|)\, |1_B\rangle\langle 0_B| + \mathrm{Tr}(|1_A\rangle\langle 1_A|)\, |1_B\rangle\langle 1_B| \Big]$$

$$= \frac{1}{2}\Big[|0_B\rangle\langle 0_B| + |1_B\rangle\langle 1_B| \Big] = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$