e. $T(n) = 7T(\frac{n}{2}) + n^2 \implies a = 7$ and $b = 2$ in the master theorem.
Since $f(n) = n^2 = O(n^{\log_2 7-\epsilon})$, where $\epsilon = 0.1 > 0$. By the case 1, we have: $T(n) = \Theta(n^{\log_2 7})$.

f. For all odd $n = 2k+1 > 2$, $T(n) = T(1) + \sum_{i=1}^{k} (2i+1)^2 = T(1) + \sum_{i=1}^{k} 4i^2 + \sum_{i=1}^{k} 4i + \sum_{i=1}^{k} 1$. Since $\sum_{i=1}^{k} 4i^2 = \theta(n^3)$, and $T(1), \sum_{i=1}^{k} 4i$, $\sum_{i=1}^{k} 1$ are all in $O(n^3)$, it implies $T(n) = \theta(n^3)$ for odd n. We can obtain the same result for even n in similar way. Finally we know $T(n) = \theta(n^3)$.

4 Problem 4-3

a. Since $\log_3 4 > 1.26$ and $n \lg n = o(n^{1.26})$, we have $T(n) = 4T(n/3) + n \lg n = \Theta(n^{\log_3 4})$ by Master Theorem.

h. Let $c = T(1)$. We have

$$T(n) = T(n-1) + \lg n = c + \sum_{i=2}^{n} \lg i \leq c + n \lg n = O(n \lg n) \text{ and}$$

$$T(n) = c + \sum_{i=2}^{n} \lg i \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \lg i \geq \left(n - \left\lceil \frac{n}{2} \right\rceil\right) \lg \left(\frac{n}{2}\right) \geq \frac{n}{3}(\lg n - 1) = \Omega(n \lg n).$$

So $T(n) = \Theta(n \lg n)$.

5 Problem 4-5

a. Assume there are 3 chips A, B and C. Since at least 2 chips are bad, there are four possibilities,

- All chips are bad.
- A is good.
- B is good.
- C is good.

If the good chip exists, then it must say “the other is bad” in any pairwise test, since there is at most one good chip. When the bad chips always say “the other chips is bad,” *all chips* must say “the other is bad” in *all pairwise tests*. In this scenario, no strategy based on pairwise tests can distinguish the four possibilities, because there is only one possible outcome.

b. The following algorithm finds one good chip if there are more than $n/2$ good chips.

1. If there is only one chip, then it must be good.
2. Split the chips into 2-chip pairs. If the number of chips is odd, then let c denote the lonely chip.
3. Test each pair. If the result is good-good, then remove arbitrary one, otherwise remove both chips.
4. Go to 1.

The first three steps of the algorithm above uses $\lfloor \frac{n}{2} \rfloor$ pairwise tests, since we can only split n chips into $\lfloor \frac{n}{2} \rfloor$ pairs. Moreover, there will be at most $\lceil \frac{n}{2} \rceil$ chips remaining after these steps, since we remove at least one chip per pair. The rest is to show that at least half of the remaining chips are good.

If the result of pairwise test is good-good, then the chips are either both good or both bad, otherwise at least one of the chips is bad. Assume

- x pairs consist of two good chips.
- y pairs consist of a good chip and a bad chip.
- z pairs consist of two bad chips.
- There are g good chips and b bad chips.

There are three possibilities,

- If n is even, then $g = 2x + y \geq b = y + 2z$. This implies $x \geq y$. In this case, x good chips and y bad chips remain.
- If n is odd and c is bad, then $g = 2x + y \geq b = y + 2z + 1$. This implies $x \geq z + 1$, since x and z are integers. In this case, x good chips and $z + 1$ bad chips remain.
- If n is odd and c is good, then $g = 2x + y + 1 \geq b = y + 2z$. This implies $x + 1 \geq z$, since x and z are integers. In this case, $x + 1$ good chips and z bad chips remain.

We conclude that at least half of remaining chips are still good after the first three step.

c. Use the result of b. Find a good chip G in $T_1(n) = T_1(\lceil n/2 \rceil) + O(n) = O(n)$. Then test G with all the other chips in $T_2(n) = n - 1$. G says "the other is good" only when tested with good chips.

6 Problem 4-6

a. By the definition of Monge array. If an $m \times n$ array is Monge, then the inequality:

$$A[i, j] + A[i + 1, j + 1] \leq A[i, j + 1] + A[i + 1, j].$$

for $i = 1, 2, ..., m - 1$ and $j = 1, 2, ..., n - 1$ is trivially hold.

Conversely if an $m \times n$ array has above property. we proof that for any i, j that $1 \leq i < m, 1 \leq j < n$, the following inequality hold:

$$A[i, j] + A[i + x, j + y] \leq A[i, j + y] + A[i + x, j]$$

for $1 \leq x \leq m - i, 1 \leq y \leq n - j$. That is, the array is Monge.

For x=y=1, the property hold by the assumption.

If for some $x' < m$ and $y' \leq n$. The property hold for all $1 \leq x \leq x', 1 \leq y \leq y'$. Then we proof it also hold for $x = x' + 1, y = y'$.

Consider some i, j that $i < m - x'$ and $j \leq n - y'$. By the assumption, we have:

$$A[i, j] + A[i + x', j + y'] \leq A[i, j + y'] + A[i + x', j]$$

and

$$A[i + x', j] + A[i + x' + 1, j + y'] \leq A[i + x', j + y'] + A[i + x' + 1, j]$$