Homework 1

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Problem 1 (Golan 12). For a field $\mathbb{F} = \langle F, +, \cdot, -, 0, 1\rangle$, show that the function $a \mapsto a^{-1}$ is a permutation of the set $F \setminus \{0_F\}$.

Solution.

Problem 2 (Golan 16). Let z_1, z_2, and z_3 be complex numbers satisfying $|z_i| = 1$ for $i = 1, 2, 3$. Show that $|z_1 z_2 + z_1 z_3 + z_2 z_3| = |z_1 + z_2 + z_3|$.

Solution.

Problem 3 (Golan 22 *Abel's inequality*). Let $z_1, \dots, z_n$ be a list of complex numbers and, for each $1 \leq k \leq n$, let $s_k = \sum_{i=1}^{k} z_i$. For real numbers $a_1, \dots, a_n$ satisfying $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, show that

$$\left| \sum_{i=1}^{k} a_i z_i \right| \leq a_1 \left(\max_{1 \leq k \leq n} |s_k| \right). \tag{1}$$

Solution.

Problem 4 (Golan 24). If p is a prime positive integer, find all subfields of $GF(p)$.

Solution.

Problem 5. Write down the definition of a *module* as a (universal) algebra, $\mathbf{M} = \langle M, F\rangle$. That is, describe the set F of operations and give the conditions that they should satisfy in order for $\mathbf{M}$ to agree with the classical definition of a module over a ring.

[*Hint:* Let $\mathbf{R} = \langle R, +, \cdot, -, 0, 1\rangle$ be a ring and, for each $r \in R$, define a scalar multiply operation $f_r \in F$.]

Solution.

Problem 6. Let $\mathbf{R} = \langle R, +, \cdot, -, 0, 1 \rangle$ be a ring.
1. Define *left ideal* of $\mathbf{R}$.
2. Let $\mathscr{A} = \{A_i : i \in \mathscr{I}\}$ be a family of left ideals of $\mathbf{R}$. Prove that $\bigcap \mathscr{A}$ is a left ideal.

Solution.

Problem 7. Let $\mathbf{R}$ be a ring and fix $a, b \in R$. Prove that if $1 - ba$ is left invertible, then $1 - ab$ is also left invertible. What is the inverse?

[*Hint:* Consider the left ideal $R(1 - ab)$. It contains the left ideal $Rb(1 - ab) = Rb$ and therefore contains 1. Verify these statements, then try to compute the inverse of $1 - ab$, as follow. (Ask for more hints as needed.)]

Solution.