Interest Rate Swaps and Their Derivatives: A Practitioner's Guide
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APPENDIX A
Taylor Series Expansion

In this appendix, we review the Taylor Series expansion formula from ordinary analysis. This expansion is commonly used to relate sensitivities (risk, PV01, convexity) to profit and loss (P&L) for financial instruments (bonds, swaps, ...), as shown in Chapters 1 and 6. The much-dreaded Ito's Lemma used in Chapters 10 and 11 is basically Taylor Series expansion in a stochastic setting, and can be easily used in practice via a *multiplication table*.

FUNCTION OF ONE VARIABLE

For a function of one variable, $f(x)$, the Taylor Series formula is:

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + 1/2\, f''(x)(\Delta x)^2 + \ldots + 1/n!\, f^{(n)}(x)(\Delta x)^n + \ldots.$$

where $f'(x)$ is the first derivative, $f''(x)$ the second derivative, $f^{(n)}(x)$ the n-th derivative, and so on. In practice, we usually just use the first two derivatives, and ignore the effect of the remaining *higher-order* terms:

$$f(x+\Delta x) - f(x) = f'(x)\Delta x + 1/2\, f''(x)(\Delta x)^2 + \text{Higher Order Terms}$$

For example, considering the Price-Yield formula for bonds, we have:

$$\begin{aligned} P(y+\Delta y) - P(y) &\approx P'(y)\Delta y + 1/2\, P''(y)(\Delta y)^2 \\ &= \text{PV01} \times \frac{\Delta y}{0.0001} + 1/2 \times \text{Convexity} \times (\Delta y)^2 \end{aligned}$$

FUNCTION OF SEVERAL VARIABLES

A similar formula holds for functions of several variables $f(x_1, \ldots, x_n)$. This is usually written as

$$\begin{aligned} f(x_1 + \Delta x_1, \ldots, x_n + \Delta x_n) = {} & f(x_1, \ldots, x_n) \\ & + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n)\Delta x_i \\ & + 1/2 \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n)\frac{\partial f}{\partial x_j}(x_1, \ldots, x_n)\Delta x_i \Delta x_j \\ & + \text{Higher Order Terms} \end{aligned}$$

For example, using Black's Formula, the expected P&L of an option is usually computed by considering the first-order terms and only one second-order term (gamma), ignoring all others:

$$\begin{aligned} C(F + \Delta F, \sigma + \Delta\sigma, t + \Delta t) - C(F, \sigma, t) \approx {} & \frac{\partial C}{\partial F}\Delta F + \frac{\partial C}{\partial \sigma}\Delta\sigma + \frac{\partial C}{\partial t}\Delta t + 1/2\frac{\partial^2 C}{\partial F^2}(\Delta F)^2 \\ = {} & \text{Delta} \times \Delta F + 1/2 \times \text{Gamma} \times (\Delta F)^2 \\ & + \text{Vega} \times \Delta\sigma + \text{Theta} \times \Delta t \end{aligned}$$

ITO'S LEMMA: TAYLOR SERIES FOR DIFFUSIONS

Ito's Lemma is basically Taylor series expansions for stochastic diffusions. For a given diffusion $X(t, \omega)$ driven by

$$dX(t, \omega) = \mu(t, \omega)dt + \sigma(t, \omega)dB(t, \omega)$$

consider a function $f(t, X(t, \omega))$. Ito's Lemma allows one to compute the diffusion for $f(t, X)$ by following Taylor series expansion for two variables, and employing the following simple *multiplication rule*:[1]

$\times$	dt	$dB(t, \omega)$
dt	0	0
$dB(t, \omega)$	0	dt

In particular, it means that we only need to keep first-order terms and only one second-order term ($dB \times dB = dt$), ignoring all other terms.

Starting with

$$dX(t, \omega) = \mu(t, \omega)dt + \sigma(t, \omega)dB(t, \omega)$$

we proceed formally with Taylor Series for a function of two variables $f(t, X)$, and ignore all terms with order higher than 2, or any term with $(dt)^2$ or $dt \times dB$:

$$\begin{aligned} df(t, X(t, \omega)) &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX(t, \omega) + 1/2\frac{\partial^2 f}{\partial X^2}(dX(t, \omega))^2 \\ &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}[\mu(t, \omega)dt + \sigma(t, \omega)dB(t, \omega)] + 1/2\frac{\partial^2 f}{\partial X^2}\sigma^2(t, \omega)dt \\ &= \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X}\mu(t, \omega) + 1/2\frac{\partial^2 f}{\partial X^2}\sigma^2(t, \omega)\right]dt + \frac{\partial f}{\partial X}\sigma(t, \omega)dB(t, \omega) \end{aligned}$$

The most common application of Ito's Lemma in finance is to start with the following dynamics for proportional (percent changes) of an asset:

$$\frac{dA(t, \omega)}{A(t, \omega)} = \mu dt + \sigma dB(t, \omega)$$

where the drift μ and volatility σ are constant numbers. Therefore,

$$dA(t, \omega) = \mu(t, \omega)dt + \sigma(t, \omega)dB(t, \omega)$$

where

$$\mu(t, \omega) = \mu \times A(t, \omega)$$
$$\sigma(t, \omega) = \sigma \times A(t, \omega)$$

Considering $f(t, A(t, \omega)) = \ln A(t, \omega)$, we notice $\frac{\partial}{\partial t} f = 0$ since f is not a direct function of t, and recalling

$$\frac{d}{dx}\ln(x) = 1/x, \qquad \frac{d^2}{dx^2}\ln(x) = -1/x^2$$

from ordinary calculus, Ito's Lemma gives us:

$$
\begin{aligned}
d\ln(A(t,\omega)) &= df(t,\omega) \\
&= \left[\frac{1}{A(t,\omega)} \times \mu \times A(t,\omega) - 1/2\frac{1}{A^2(t,\omega)} \times \sigma^2 \times A^2(t,\omega)\right] dt \\
&+ \left[\frac{1}{A(t,\omega)} \times \sigma \times A(t,\omega)\right] dB(t,\omega) \\
&= (\mu - \sigma^2/2)dt + \sigma dB(t,\omega)
\end{aligned}
$$

Integrating both sides, we have

$$
\begin{aligned}
\ln A(t,\omega) - \ln A(0,\omega) &= (\mu - \sigma^2/2)t + \sigma \int_0^t dB(t,\omega) \\
&= (\mu - \sigma^2/2)t + \sigma(B(t,\omega) - B(0,\omega)) \\
&= (\mu - \sigma^2/2)t + \sigma B(t,\omega)
\end{aligned}
$$

since a Brownian motion is started at 0, $B(0,\omega) = 0$. Recalling that a standard Brownian motion is Normally distributed, $B(t,\omega) \sim N(0,t)$, we get:

$$A(t,\omega) = A(0,\omega)e^{(\mu - 1/2\sigma^2)t + \sigma N(0,t)}$$

that is, $A(t,\omega)/A(0)$ is Log-Normal: $A(t,\omega)/A(0) \sim LN((\mu - \sigma^2/2)t, \sigma^2 t)$, $EA(t,\omega) = A(0)e^{\mu t}$. Note that if the process for A is drift less, that is, $\mu = 0$, then $dA(t,\omega) = \sigma A(t,\omega)dB(t,\omega)$, and $EA(t,\omega) = A(0)$. In this case, A(t) has zero expected change and is a martingale.