A Formula Sheet for Financial Economics

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Abstract

This document is meant to be used solely as a formula sheet. It contains very little in the way of explanation and is not meant to be used as a substitute for a financial economics text. It is aimed specifically at those students preparing for exam MFE offered by the Society of Actuaries, but it should be of some use to everyone studying financial economics. It covers the important formulas and methods used in put-call parity, option pricing using binomial trees, Brownian motions, stochastic calculus, stock price dynamics, the Sharpe ratio, the Black-Scholes equation, the Black-Scholes formula, option greeks, risk management techniques, estimations of volatilities and rates of appreciation, exotic options (asian, barrier, compound, gap, and exchange), simulation, interest rate trees, the Black model, and several interest rate models (Rendleman-Bartter, Vasicek, and Cox-Ingersoll-Ross)

1 Forwards, Puts, and Calls

1.1 Forwards

A forward contract is an agreement in which the buyer agrees at time t to pay the seller at time T and receive the asset at time T.

$$F_{t,T}(S) = S_t e^{r(T-t)} = S_t e^{r(T-t)} - FV_{t,T}(Dividends) = S_t e^{(r-\delta)(T-t)} \tag{1}$$

A prepaid forward contract is an agreement in which the buyer agrees at time t to pay the seller at time t and receive the asset at time T.

$$F_{t,T}^{P}(S) = S_t \text{ or } S_t - PV_{t,T}(Dividends) \text{ or } S_t e^{-\delta(T-t)} \tag{2}$$

1.2 Put-Call Parity

Call options give the owner the right, but not the obligation, to buy an asset at some time in the future for a predetermined strike price. Put options give the owner the right to sell. The price of calls and puts is compared in the following put-call parity formula for European options.

$$c(S_t, K, t, T) - p(S_t, K, t, T) = F_{t,T}^{P}(S) - Ke^{-r(T-t)} \tag{3}$$

1.3 Calls and Puts with Different Strikes

For European calls and puts, with strike prices K_1 and K_2 where $K_1 < K_2$, we know the following

$$0 \leq c(K_1) - c(K_2) \leq (K_2 - K_1)e^{-rT} \tag{4}$$

$$0 \leq p(K_2) - p(K_1) \leq (K_2 - K_1)e^{-rT} \tag{5}$$

For American options, we cannot be so strict. Delete the discount factor on the $(K_2 - K_1)$ term and then you're okay. Another important result arises for three different options with strike prices $K_1 < K_2 < K_3$

$$\frac{c(K_1) - c(K_2)}{K_2 - K_1} \geq \frac{c(K_2) - c(K_3)}{K_3 - K_2} \tag{6}$$

$$\frac{p(K_2) - p(K_1)}{K_2 - K_1} \leq \frac{p(K_3) - p(K_2)}{K_3 - K_2} \tag{7}$$

Exam MFE loves arbitrage questions. An important formula for determining arbitrage opportunities comes from the following equations.

$$K_2 = \lambda K_1 + (1 - \lambda)K_3 \tag{8}$$

$$\lambda = \frac{K_3 - K_2}{K_3 - K_1} \tag{9}$$

The coefficients in front of each strike price in equation 8 represent the number of options of each strike price to buy for two equivalent portfolios in an arbitrage-free market.

1.4 Call and Put Price Bounds

The following equations give the bounds on the prices of European calls and puts. Note that the lower bounds are no less than zero.

$$(F^P_{t,T}(S) - Ke^{-r(T-t)})_+ \leq c(S_t, K, t, T) \leq F^P_{t,T}(S) \tag{10}$$

$$(Ke^{-r(T-t)} - F^P_{t,T}(S))_+ \leq p(S_t, K, t, T) \leq Ke^{-r(T-t)} \tag{11}$$

We can also compare the prices of European and American options using the following inequalities.

$$c(S_t, K, t, T) \leq C(S_t, K, t, T) \leq S_t \tag{12}$$

$$p(S_t, K, t, T) \leq P(S_t, K, t, T) \leq K \tag{13}$$

1.5 Varying Times to Expiration

For American options only, when $T_2 > T_1$

$$C(S_t, K, t, T_2) \geq C(S_t, K, t, T_1) \leq S_t \tag{14}$$

$$P(S_t, K, t, T_2) \geq P(S_t, K, t, T_1) \leq S_t \tag{15}$$

1.6 Early Exercise for American Options

In the following inequality the two sides can be thought of the pros and cons of exercising the call early. The pros of exercising early are getting the stock's dividend payments. The cons are that we have to pay the strike earlier and therefore miss the interest on that money and we lose the put protection if the stock price should fall. So we exercise the call option if the pros are greater than the cons, specifically, we exercise if

$$PV_{t,T}(dividends) > p(S_t, K) + K(1 - e^{-r(T-t)}) \tag{16}$$

For puts, the situation is slightly different. The pros are the interest earned on the strike. The cons are the lost dividends on owning the stock and the call protection should the stock price rise. Explicitly, we exercise the put option early if

$$K(1 - e^{-r(T-t)}) > c(S_t, K) + PV_{t,T}(dividends) \tag{17}$$

2 Binomial Trees

2.1 The One-Period Replicating Portfolio

The main idea is to replicate the payoffs of the derivative with the stock and a risk-free bond.

$$\Delta e^{\delta h} S_0 u + B e^{rh} = C_u \tag{18}$$

$$\Delta e^{\delta h} S_0 d + B e^{rh} = C_d \tag{19}$$

To replicate the derivative we buy Δ shares and invest in B dollars.

$$\Delta = e^{-\delta h} \frac{C_u - C_d}{S_0(u-d)} \tag{20}$$

$$B = e^{-rh} \frac{uC_d - dC_u}{u-d} \tag{21}$$

Since we designed the portfolio to replicate the option they must, since there is no arbitrage, have the same time-0 price.

$$C_0 = \Delta S_0 + B \tag{22}$$

2.2 Risk-Neutral Probabilities

We define the risk-neutral probability of the stock price going up as follows

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} \tag{23}$$

Then the price of the option is

$$C_0 = e^{-rh}[p^* C_u - (1-p^*)C_d] \tag{24}$$

A key result of the risk-neutral world is that the expected price of the stock at future time t is

$$E^*[S_t] = p^* S_0 u + (1-p^*) S_0 d = S_0 e^{rt} \tag{25}$$

2.3 Multi-Period Trees

The single period binomial trees formulas can be used to go back one step at a time on the tree. Also, note that for a European option we can use this shortcut formula.

$$C_0 = e^{-2rh}[(p^*)^2 C_{uu} + 2p^*(1-p^*)C_{ud} + (1-p^*)^2 C_{dd}] \tag{26}$$

For American options, however, it's important to check the price of the option at each node of the tree. If the price of the option is less than the payout, then the option would be exercised and the price at that node should be the payoff at that point.

2.4 Trees from Volatilities

Assuming a forward tree

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} \tag{27}$$

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} \tag{28}$$

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} \tag{29}$$

Assuming a Cox-Ross-Rubinstein tree

$$u = e^{\sigma\sqrt{h}} \tag{30}$$

$$d = e^{-\sigma\sqrt{h}} \tag{31}$$

Assuming a Jarrow-Rudd (lognormal) forward tree

$$u = e^{(r-\delta-\frac{1}{2}\sigma^2)h+\sigma\sqrt{h}} \tag{32}$$

$$d = e^{(r-\delta-\frac{1}{2}\sigma^2)h-\sigma\sqrt{h}} \tag{33}$$

2.5 Options on Currencies

The easiest way to deal with these options is to treat the exchange rate as a stock where $x(t)$ is the underlying, r_f is the dividend rate and r is the risk-free rate. It is helpful to understand these following equation.

$$\pounds 1.00 = \$x(t) \tag{34}$$

Remember that if you treat $x(t)$ as the underlying asset than r is the risk-free rate for dollars, r_f is the risk-free rate for pounds, and the option is considered dollar-denominated.

$$c(K,T) - p(K,T) = x_0 e^{-r_f T} - K e^{-rT} \tag{35}$$

2.6 Options on Futures

The main difference between futures and the other previously discussed assets is that futures don't initially require any assets to change hands. The formulas are therefore adjusted as follows

$$\Delta = \frac{C_u - C_d}{F_0(u-d)} \tag{36}$$

$$B = C_0 = e^{-rh}\left(\frac{1-d}{u-d}C_u + \frac{u-1}{u-d}C_d\right) \tag{37}$$

Note that the previous equation becomes clearer when we define p^*, which, because it is sometimes possible to think of a forward as a stock with $\delta = r$, as

$$p^* = \frac{1-d}{u-d} \tag{38}$$

The other formulas all work the same way. Notably, the put-call parity formula becomes

$$c(K,T) - p(K,T) = F_0 e^{-rT} - Ke^{-rT} \tag{39}$$

2.7 True Probability Pricing

We've been assuming a risk-free world in the previous formulas as it makes dealing with some problems nicer. But it's important to examine the following real-world or true probability formulas.

$$E(S_t) = pS_0u + (1-p)S_0d = S_0e^{(\alpha-\delta)t} \tag{40}$$

$$p = \frac{e^{(\alpha-\delta)t} - d}{u-d} \tag{41}$$

It is possible to price options using real world probabilities. But r can no longer be used. Instead γ is the appropriate discount rate

$$e^{\gamma t} = \frac{S_0\Delta}{S_0\Delta + B}e^{\alpha t} + \frac{B}{S_0\Delta + B}e^{rt} \tag{42}$$

$$C_0 = e^{-\gamma t}[C_u + (1-p)C_d] \tag{43}$$

2.8 State Prices

State prices are so called because it's the cost of a security that pays one dollar upon reaching a particular state. Remember these following formulas for determining state prices

$$Q_H + Q_L = e^{-rt} \tag{44}$$

$$S_HQ_H + S_LQ_L = F^P_{0,t}(S) \tag{45}$$

$$C_0 = C_HQ_H + C_LQ_L \tag{46}$$

The above equations can easily be adapted for trinomial and higher order trees. The economic concept of utility also enters the stage in the following equations. Understand that, for example, U_H is the utility value in today's dollars attached to one dollar received in the up state.

$$Q_H = pU_H \tag{47}$$

$$Q_L = (1-p)U_L \tag{48}$$

This leads to the following result.

$$p^* = \frac{pU_H}{pU_H + (1-p)U_L} = \frac{Q_H}{Q_H + Q_L} \tag{49}$$

3 Continuous-Time Finance

More specifically, this section is going to cover Brownian motions, stochastic calculus and the lognormality of stock prices and introduce the Black-Scholes equation.

3.1 Standard Brownian Motion

The important properties of an SBM are as follows. One, $Z(t)$~$\mathrm{N}(0,t)$. Two, $\{Z(t)\}$ has independent increments. And three, $\{Z(t)\}$ has stationary increments such that $Z(t+s)-Z(t)$~$\mathrm{N}(0,s)$. Also useful is the fact that given a $Z(u): 0 \leq u \leq t$, $Z(t+s)$~$\mathrm{N}(Z(t),s)$.

3.2 Arithmetic Brownian Motion

We define $X(t)$ to be an arithmetic Brownian motion with drift coefficient μ and volatility σ if $X(t)=\mu t+\sigma Z(t)$. Note that an arithmetic Brownian motion with $\mu=0$ is called a driftless ABM. Finally, $X(t)$~$\mathrm{N}(\mu t,\sigma^2 t)$.

3.3 Geometric Brownian Motion

Arithmetic Brownian motions can be zero, though, and have a mean and variance that don't depend on the level of stock making them a poor model for stock prices. To solve these problems we consider a geometric Brownian motion.

$$Y(t)=Y(0)e^{X(t)}=Y(0)e^{[\mu t+\sigma Z(t)]} \tag{50}$$

The following equations can be used to find the moments of a GBM. In equation 51, let U be any normal random variable.

$$E(e^{kU})=e^{kE(U)+\frac{1}{2}k^2Var(U)} \tag{51}$$

So specifically for geometric Brownian motions

$$E[Y^k(t)]=Y^k(0)e^{(k\mu+\frac{1}{2}k^2\sigma^2)t} \tag{52}$$

Also note, then, that $Y(t)$ is lognormally distributed as follows

$$\ln Y(t)\text{~}\mathrm{N}(\ln Y(0)+\mu t,\sigma^2 t) \tag{53}$$

3.4 Ito's Lemma

First define X as a diffusion and present the following stochastic differential equation.

$$\mathrm{d}X(t)=a(t,X(t))\mathrm{d}t+b(t,X(t))\mathrm{d}Z(t) \tag{54}$$

Then for

$$Y(t) = f(t, X(t))\mathrm{d}t \tag{55}$$

We have

$$\mathrm{d}Y(t) = f_t(t, X(t)) + f_x(t, X(t))\mathrm{d}X(t) + \frac{1}{2} f_{xx}(t, X(t))[\mathrm{d}X(t)]^2 \tag{56}$$

where

$$[dX(t)]^2 = b^2(t, X(t))\mathrm{d}t \tag{57}$$

3.5 Stochastic Integrals

Recall the fundamental theorem of calculus.

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t a(s, X(s))\mathrm{d}s = a(t, X(t)) \tag{58}$$

The rule for stochastic integrals looks very similar.

$$\mathrm{d} \int_0^t b(s, X(s))\mathrm{d}Z(s) = b(t, X(t))\mathrm{d}Z(t) \tag{59}$$

3.6 Solutions to Some Common SDEs

For arithmetic Brownian motions, we can say the following

$$\mathrm{d}Y(t) = \alpha \mathrm{d}t + \sigma \mathrm{d}Z(t) \tag{60}$$

$$Y(t) = Y(0) + \alpha t + \sigma Z(t) \tag{61}$$

For geometric Brownian motions, there are several equivalent statements.

$$\mathrm{d}Y(t) = \mu Y(t)\mathrm{d}t + \sigma Y(t)\mathrm{d}Z(t) \tag{62}$$

$$\mathrm{d}[\ln Y(t)] = \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma \mathrm{d}Z(t) \tag{63}$$

$$Y(t) = Y(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma Z(t)} \tag{64}$$

And for Ornstein-Uhlenback processes, which will become very useful when we get to interest rate models, we know

$$\mathrm{d}Y(t) = \lambda[\alpha - Y(t)]\mathrm{d}t + \sigma \mathrm{d}Z(t) \tag{65}$$

$$Y(t) = \alpha + [Y(0) - \alpha]e^{-\lambda t} + \sigma \int_0^t e^{-\lambda(t-s)}\mathrm{d}Z(s) \tag{66}$$

3.7 Brownian Motion Variation

For any function Y(t), the kth-order variation is

$$\int_a^b |\mathrm{d}Y(t)|^k \tag{67}$$

For standard Brownian motions we can say that the total variation is infinity, the quadratic variation is $(b-a)$ and higher-order variations are zero. And for arithmetic Brownian motions we know that the total variation is infinity, the quadratic variation is $(b-a)\sigma^2$ and higher-order variations are zero.

3.8 Stock Prices as a GBM

A stock price that pays constant dividends at a rate δ, such that the rate of appreciation is $\alpha - \delta$, follows the following stochastic differential equation and solution

$$\frac{\mathrm{d}S(t)}{S(t)} = (\alpha - \delta)\mathrm{d}t + \sigma \mathrm{d}Z(t) \tag{68}$$

$$S(t) = S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})t+\sigma Z(t)} \tag{69}$$

$$\mathrm{d}[lnS(t)] = (\alpha - \delta - \frac{\sigma^2}{2})\mathrm{d}t + \sigma \mathrm{d}Z(t) \tag{70}$$

Therefore we can say that

$$S(t) \sim \mathrm{LN}(lnS(0) + (\alpha - \delta - \frac{\sigma^2}{2})t, \sigma^2 t) \tag{71}$$

3.9 Since Stock Prices are Lognormal

Stock prices being lognormal gives many convenient formulas. First, we can determine the percentiles of different values of the future stock price and therefore confidence intervals as follows

$$100p - \text{th percentile of } S(t) = S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})t+\sigma\sqrt{t}N^{-1}(p)} \tag{72}$$

$$100(1\text{ - }\beta)\% \text{ lognormal CI for } S(t) \text{ is } S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})t \pm z_{\beta/2}\sigma\sqrt{t}} \tag{73}$$

Similarly we know the probability that the future stock price will be above or below some value. We find this probability using the following two equations.

$$P(S(t) \leq K) = N(-\hat{d}_2) \tag{74}$$

$$\hat{d}_2 = \frac{ln\frac{S(0)}{K} + (\alpha - \delta - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} \tag{75}$$

And we can determine all the moments of the future stock price.

$$E[S^k(t)] = S^k(0)e^{k(\alpha-\delta)t+\frac{1}{2}k(k-1)(\sigma^2 t)} \tag{76}$$

Also very useful is the conditional expected price formulas.

$$E[S(t) \mid S(t) < K] = E[S(t)]\frac{N(-\hat{d}_1)}{N(-\hat{d}_2)} \tag{77}$$

$$E[S(t) \mid S(t) > K] = E[S(t)]\frac{N(\hat{d}_1)}{N(\hat{d}_2)} \tag{78}$$

where $\hat{d}_2$ is defined as before and $\hat{d}_1$ is

$$\hat{d}_1 = \frac{ln\frac{S(0)}{K} + (\alpha - \delta + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} \tag{79}$$

3.10 The Sharpe Ratio and Hedging

For any asset that has the dynamics $\frac{\mathrm{d}X(t)}{X(t)} = m\mathrm{d}t + s\mathrm{d}Z(t)$ and a continuously compounded dividend rate δ, the Sharpe ratio is defined as

$$\phi = \frac{m + \delta - r}{s} \tag{80}$$

Recalling that for a stock, $m = \alpha - \delta$ the Sharpe ratio of any asset written on a GBM is

$$\phi = \frac{\alpha - r}{\sigma} \tag{81}$$

The key thing to remember is that for any two assets with the same dynamics the Sharpe ratio must be the same. Using this fact we can derive the following hedging formulas. We can hedge a position of long one unit of X by buying N units of Y and investing W dollars.

$$N = -\frac{s_X X(t)}{s_Y Y(t)} \tag{82}$$

$$W = -X(t) - NY(t) \tag{83}$$

3.11 The Black-Scholes Equation

If we look at any derivative with value $V(S(t), t)$, use Itô's lemma to find $\mathrm{d}V(S(t), t)$, and put this into the Sharpe ratio formula we can derive the Black-Scholes equation.

$$rV = V_t + (r - \delta)SV_s + \frac{1}{2}\sigma^2 S^2 V_{ss} \tag{84}$$

3.12 Risk-Neutral Valuation and Power Contracts

Recall that to switch from the real world to the risk-neutral world we exchange α for r which leads to the following risk-neutral dynamics.

$$\frac{\mathrm{d}S(t)}{S(t)} = (r-\delta)\mathrm{d}t + \sigma \mathrm{d}[\tilde{Z}(t)] \tag{85}$$

$$\tilde{Z}(t) = Z(t) + \phi t \tag{86}$$

Using these risk-neutral equations we can show that

$$V(S(t),t) = e^{-r(T-t)}E^*[V(S(T),T) \mid S(T)] \tag{87}$$

This equation can be used to derive the following time-t price of a power contract. The payoff of a power contract is $S^a(T)$ at time T and the price is

$$F^p_{t,T}(S^a) = S^a(t)e^{(-r+a(r-\delta)+\frac{1}{2}a(a-1)\sigma^2)(T-t)} \tag{88}$$

4 The Black-Scholes Formula

4.1 Binary Options

Binary in this case is a fancy way of saying all or nothing.

Binary Option	Price
Cash-or-Nothing Call	$e^{-r(T-t)}N(d_2)$
Cash-or-Nothing Put	$e^{-r(T-t)}N(-d_2)$
Asset-or-Nothing Call	$S(t)e^{-r(T-t)}N(d_1)$
Asset-or-Nothing Put	$S(t)e^{-r(T-t)}N(-d_1)$

Where d_1 and d_2 are defined as before except that α is replaced with r.

4.2 The Black-Scholes Formula

With the binary option formulas in hand it's just a hop, skip, and a jump to the Black Scholes Formulas.

$$c(S(t),K,t) = S(t)e^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{89}$$

$$p(S(t),K,t) = Ke^{-r(T-t)}N(-d_2) - S(t)e^{-\delta(T-t)}N(-d_1) \tag{90}$$

It's helpful to note that, just like with the binomial formulas, for options on currencies we can replace δ with r_f and for options on futures we can replace δ with r. Another option is to use the more general prepaid forward version of the Black-Scholes option pricing formulas.

4.3 The Prepaid Forward Version

$$c(S(t),K,t) = F^P_{t,T}(S)N(d_1) - F^P_{t,T}(K)N(d_2) \tag{91}$$

$$p(S(t),K,t) = F^P_{t,T}(K)N(-d_2) - F^P_{t,T}(S)N(-d_1) \tag{92}$$

where

$$d_1 = \frac{ln\frac{F^P_{t,T}(S)}{F^P_{t,T}(K)} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \tag{93}$$

$$d_2 = \frac{ln\frac{F^P_{t,T}(S)}{F^P_{t,T}(K)} - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t} \tag{94}$$

So far an option on any asset, just plug in the correct prepaid forward price. As a reminder, for currencies this is $x(t)e^{-r_f(T-t)}$ and for futures this is $F(t)e^{-r(T-t)}$. In other words, we are simply replacing δ with the appropriate variable.

4.4 Greeks and Their Uses

Greeks are what are used to measure the risk of a derivative. The following three are the most important.

$$\Delta = V_S \ , \ \Gamma = V_{SS} = \Delta_S \ , \ \theta = V_t \tag{95}$$

The formula for the delta of a call and the delta of a put are useful to have memorized.

$$\Delta_c = e^{-\delta(T-t)} N(d_1) \tag{96}$$

$$\Delta_p = -e^{-\delta(T-t)} N(-d_1) \tag{97}$$

From the Black-Scholes equation we get the following relationship.

$$\theta + (r - \delta) S \Delta + \frac{1}{2} \sigma^2 S^2 \Gamma = rV \tag{98}$$

The following are the most important properties of the greeks. Δ is positive for calls and negative for puts. Γ is always positive. θ is negative for calls and puts generally, but can be positive for very in-the-money puts. Also, ν is always positive while ψ is negative for a call and positive for a put and ρ is positive for a call and negative for a put. Note that the signs of these greeks is assuming we are long.

We can approximate the price of a new derivative if we know the price of a slightly different derivative using the Delta-Gamma approximation.

$$V(S + \varepsilon, t) \approx V(S, t) + \Delta(S, t)\varepsilon + \frac{1}{2}\Gamma(S, t)\varepsilon^2 \tag{99}$$

This assumes an instantaneous change in the underlying price. A more useful formula is the Delta-Gamma-Theta Approximation.

$$V(S(t+h), t+h) \approx V(S(t), t) + \Delta(S(t), t)\varepsilon + \frac{1}{2}\Gamma(S(t), t)\varepsilon^2 + \theta(S(t), t)h \tag{100}$$

$$\varepsilon = S(t + h) - S(t) \tag{101}$$

To find the Delta, Gamma, or Theta for a portfolio, as opposed to a single security, simply add up the greeks of the options in the portfolio.

4.5 Elasticity

We can write the expected return and the volatility of a derivative.

$$m_V = \Omega\alpha + (1 - \Omega)r \tag{102}$$

$$s_V = \Omega\sigma = \frac{S\Delta\sigma}{V} \tag{103}$$

Ω is defined to be the elasticity. It's interpretation is the same as the economics interpretation. Also, we can conclude that if $\Omega > 1$ than the option is riskier than the underlying. Unlike the greeks, we cannot just sum the elasticities of the options in the portfolio to find the portfolio's elasticity.

$$\Omega = \Sigma \frac{w_i V_i}{P} \Omega_i \tag{104}$$

Where w_i is how many units of the i^{th} derivative are in the portfolio.

4.6 Greeks From Trees

$$\Delta = e^{-\delta h} \frac{C_u - C_d}{S_u - S_d} \tag{105}$$

$$\Gamma(S_h, h) = \frac{\Delta(Su, h) - \Delta(Sd, h)}{Su - Sd} \tag{106}$$

$$\theta(S, 0) \approx \frac{1}{2h}[C(Sud, 2h) - \varepsilon\Delta(S, 0) - \frac{1}{2}\varepsilon^2\Gamma(S, 0) - C(S, 0)] \tag{107}$$

4.7 Delta-Hedging

To Delta-Hedge a portfolio we want to make it such that the delta of the portfolio is zero. So if we are long one unit of an option written on X we purchase N units of the underlying and invest W dollars.

$$N = -\frac{s_X X(t)}{s_Y Y(t)} = -\frac{\frac{S\Delta\sigma}{V} V}{\sigma S} = -\Delta \tag{108}$$

$$W = -X(t) - NY(t) = -V + S\Delta \tag{109}$$

It is also possible to hedge on any other greek, simply enter into other positions until the value of the greek for the portfolio is zero. It is easy to determine the hedge profit and variance of the expected hedge profit.

$$\Gamma < 0 \Rightarrow S(t)(1 - \sigma\sqrt{h}) < S(t + h) < S(t)(1 + \sigma\sqrt{h}) \tag{110}$$

$$Var(\text{Profit}) = \frac{1}{2}\{[\Gamma(S(t), t)\sigma^2 S^2(t)]h\}^2 \tag{111}$$

4.8 Estimation of Volatilities & Appreciation Rates

Given a set of stock prices at different times, it is possible to estimate the volatility of the stock using the following methodology. First, define u_i as $ln\frac{S_i}{S_{(i-1)}}$, then find the average of the u_i's, then the find the sample variance, and with that estimate σ.

$$\bar{u} = \frac{1}{n}\Sigma u_i = \frac{1}{n} ln \frac{S_n}{S_0} \tag{112}$$

$$s_u^2 = \frac{1}{n-1}\Sigma(u_i - \bar{u})^2 \tag{113}$$

$$\hat{\sigma} = \frac{s_u}{\sqrt{h}} \tag{114}$$

It's also possible to calculate the expected rate of appreciation.

$$\hat{\alpha} = \frac{\bar{u}}{h} + \delta + \frac{\hat{\sigma}^2}{2} = \frac{lnS(T) - lnS(0)}{T} + \delta + \frac{\hat{\sigma}^2}{2} \tag{115}$$

To test for normality, you can draw a normal probability plot. To do this arrange the data in ascending order, give the ith order statistic the quantile $q = \frac{i-\frac{1}{2}}{n}$, convert the q's to z's where $z = N^{-1}(q)$, and plot the data. The straighter the plot, the more normal the data is.

5 Exotic Options & Simulations

5.1 Asian Options

Asian options are options that are based on averages in place of either the price or the strike. The average can be either an arithmetic average or a geometric average.

$$A(T) = \frac{1}{n}\Sigma S(ih) \tag{116}$$

$$G(T) = [\Pi S(ih)]^{\frac{1}{n}} \tag{117}$$

Then to price the option replace either the strike or the price with the appropriate path-dependent average, calculate the payoffs, and then discount them.

5.2 Barrier Options

Barrier options are options that become activated (knocked-in) or deactivated (knocked-out) if the stock price passes above or below a pre-determined barrier. The price of these options can either be calculated with a binomial model or with a parity equation. Knock-in option + Knock-out option = Ordinary Option.

5.3 Compound Options

Compound options are a pain in the butt to price and because of this the actuarial exam only asks that compound options be priced using the parity formula. Call on call + Put on call = Big Call - Ke^{-rt}. And similarly for puts.

5.4 Gap Options

Gap options are options whose strike for determining if the option is exercised are different than the strike used to determine the payoff of the option. The formulas don't need to be written as long as the original Black-Scholes formulas are understood. The strike used to calculate the value of d_1 and d_2 is the strike that determines if the option is exercised and the strike used to calculate the price of the option is the strike that determines the payoff.

5.5 Exchange Options

Exchange options can be priced using the following parity and duality equations.

$$c[S(t), Q(t), K, t, T] = Kp[S(t), Q(t), \frac{1}{K}, t, T] \tag{118}$$

$$p[S(t), Q(t), K, t, T] = Kc[S(t), Q(t), \frac{1}{K}, t, T] \tag{119}$$

$$c[S(t), Q(t), K, t, T] - p[S(t), Q(t), K, t, T] = F^P_{t,T}(S) - KF^P_{t,T}(Q) \tag{120}$$

In the above equations, S is the underlying asset and Q is the strike asset. Exchange options can also be priced using a formula very similar to the prepaid forward version of the Black-Scholes pricing equation. For example, the price of the option to get one unit of S in exchange for K units of Q can be written

$$F^P_{t,T}(S)N(d_1) - KF^P_{t,T}(Q)N(d_2) \tag{121}$$

$$d_1 = \frac{ln\frac{F^P_{t,T}(S)}{KF^P_{t,T}(Q)} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \tag{122}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{123}$$

$$\sigma^2 = \sigma_S^2 + \sigma_Q^2 - 2\rho\sigma_S\sigma_Q \tag{124}$$

Exchange options can be trivially applied to options that depend on a maximum or minimum of S or Q. The parity relationship is useful to know.

$$\max[S(T), KQ(T)] + \min[S(T), KQ(T)] = F^P_{t,T}(S) + KF^P_{t,T}(Q) \tag{125}$$

5.6 Monte-Carlo Simulations

The way we're going to simulate stocks is by taking advantage of the lognormality of stock prices. We'll use the following method: start with iid uniform numbers $u_1, u_2, ..., u_n$, calculate z's where $z_i = N^{-1}(u_1)$, convert these to $N(\mu, \sigma^2)$ random variables by letting $r_1 = \mu + \sigma z_1$.

To simulate a single stock price the following formula can be used.

$$S(T) = S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})T+\sigma\sqrt{T}z} \tag{126}$$

Or simulate the the stock price at some time T given the stock price at a closer future time t.

$$S(T) = S(t)e^{(\alpha-\delta-\frac{\sigma^2}{2})(T-t)+\sigma[Z(T)-Z(t)]} \tag{127}$$

Monte-Carlo simulation goes like this: simulate stock prices, calculate the payoff of the option for each of those simulated prices, find the average payoff, and then discount the average payoff. The variance of the Monte-Carlo estimate, where g_i is the ith simulated payoff, can be calculated. The variance is $e^{-2rT}\frac{s^2}{n}$ where

$$s^2 = \frac{1}{n-1}\Sigma[g(S_i) - \bar{g}]^2 \tag{128}$$

5.7 Variance Reduction

The first method of variance reduction is stratified sampling. To do this take the iid uniform numbers and instead of taking them as is make sure the correct number go into each group. As an example, given 20 variables re-define the first five to be distributed between 0 and .25 and then next five to be distributed between .25 and .5 and so on. Then proceed as before.

The other common technique involves finding an antithetic variate. To find this estimate use the normal distribution numbers as before to find V_1 and then flip the signs on all of them and do the process again to find V_2. Calculate the antithetic variate to be $V_3 = \frac{V_1+V_2}{2}$.

6 Interest Rate Trees

6.1 Bonds and Interest Rates

Recall that the price of an s-year zero is

$$P(0,S) = \frac{1}{[1+r(0,s)]^s} \text{ or } e^{-r(0,s)s} \tag{129}$$

After much algebra abuse we arrive at the following forward bond price formula.

$$F_{t,T}[P(T,T+s)] = \frac{P(t,T+s)}{P(t,T)} \tag{130}$$

Also recall that we can calculate the non-continuous annualized rate.

$$P(t,T)[1+r_t(T,T+s)]^{-s} = P(t,T+s) \tag{131}$$

6.2 Caplets and Caps

Just as with stock prices, trees for interest rates can be made. Interest rate caplets and caps (which are the sums of the appropriate caplets) are most easily understood by examples. The only thing to memorize is that the payoff of the caplet is $[r(t,T)-K]_+$. To calculate the price of the caplet make a table with the following columns.

Path	Probability	Time t_1 Payoff	Time t_2 Payoff	Contribution of Path

Then for each path find the total contribution and sum them to find the price of the caplet. Remembe rto either discount the contributions or the total to find the price.

6.3 Black-Derman-Toy Model

The BDT model is a commonly used interest rate model with the following features: $r_d = R_h$ and $r_u = R_h e^{2\sigma_1\sqrt{h}}$. And the next time step uses σ_2. The interest rates are all annual effective. Also, the way the rates are defined means that the following equality holds $\frac{r_{uu}}{r_{ud}} = \frac{r_{ud}}{r_{dd}}$.

The yield volatility for period-3 is

$$\frac{1}{2\sqrt{h}}\ln\frac{y_u}{y_d} \tag{132}$$

$$P(r_u,h,3h) = \frac{1}{1+r_u}\left(\frac{1}{2}\frac{1}{1+r_{uu}} + \frac{1}{2}\frac{1}{1+rud}\right) \tag{133}$$

$$y_u = [P(r_u,h,3h)]^{-\frac{1}{2h}} - 1 \tag{134}$$

6.4 The Black Formula

The Black formula is similar to the prepaid forward version of the Black-Scholes formula except that it uses forward prices.

$$c(S(0), K, T) = P(0,t)[F_{0,T}(S)N(d_1) - KN(d_2)] \tag{135}$$

$$p(S(0), K, T) = P(0,t)[KN(-d_2) - F_{0,T}(S)N(-d_1)] \tag{136}$$

$$d_1 = \frac{\ln\frac{F_{0,T}(S)}{K} + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}} \tag{137}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{138}$$

6.5 The Black Model for Bond Options

There is a put-call parity equation for zeros.

$$c(K,T) - p(K,T) = P(0,T+s) - KP(0,T) \tag{139}$$

We can further specify the Black Formula for Bond Options.

$$c(S(0), K, T) = P(0,t)[F)N(d_1) - KN(d_2)] \tag{140}$$

$$p(S(0), K, T) = P(0,t)[KN(-d_2) - FN(-d_1)] \tag{141}$$

$$F = F_{0,T}[P(T, T+s)] \tag{142}$$

$$d_1 = \frac{\ln\frac{F}{K} + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}} \tag{143}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{144}$$

$$\sigma^2 = \frac{1}{T}Var[lnP(T, T+s)] \tag{145}$$

And we can further specify the Black Formula for Caplets as well.

$$\text{caplet}(K, T, T+s) = P(0,T)\left(\frac{N(-d_2)}{1+K} - FN(-d_1)\right) \tag{146}$$

$$F = \frac{P(0,T+s)}{P(0,T)} \tag{147}$$

$$d_1 = \frac{\ln[F(1+K)] + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}} \tag{148}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{149}$$

6.6 Interest Rate Models

We've been assuming that the interest rates are known. Now we will model them instead and see how this affects our other models.

$$\mathrm{d}r(t) = a(r(t))\mathrm{d}t + \sigma(r(t))\mathrm{d}Z(t) \tag{150}$$

$$P(t,T) = e^{-\int_t^T r(s)ds} \tag{151}$$

And the dynamics of a derivative on a non-dividend paying stock are now

$$\frac{\mathrm{d}V(r(t),t)}{V(r(t),t)} = \alpha(r(t),t,T)\mathrm{d}t - q(r(t),t,T)\mathrm{d}Z(t) \tag{152}$$

And so the Sharpe Ratio is

$$\phi(r(t),t)\frac{\alpha(r(t),t,T) - r(t)}{q(r(t),t,T)} \tag{153}$$

And the following hedging rules apply. If you own one interest rate derivative V_1 buy N of V_2

$$N = -\frac{q_1(r(t),t,T_1)V_1(r(t),t)}{q_2(r(t),t,T_2)V_2(r(t),t)} \tag{154}$$

6.7 The Term Structure Equation

Using a process very similar for how we dealt with stocks and the Black-Scholes equation we derive the following equations.

$$V_t + [a(r) + \sigma(r)\phi(r,t)]V_r + \frac{[\sigma(r)]^2}{2}V_{rr} = rV \tag{155}$$

$$q(r(t),t,T) = -\frac{\sigma(r(t))V_r(r(t),t)}{V(r(t),t)} = -\Omega\sigma \tag{156}$$

6.8 Risk Neutral Valuation

Again, the thought process is similar. The formulas are adjusted appropriately, but "look" the same.

$$\mathrm{d}r(t) = [a(r) + \sigma(r)\phi(r(t),t)]\mathrm{d}t + \sigma(r)\mathrm{d}\tilde{Z}t \tag{157}$$

$$\tilde{Z}(t) = Z(t) - \int_0^t \phi(r(s),s)\mathrm{d}s \tag{158}$$

$$\mathrm{d}\tilde{Z}(t) = \mathrm{d}Z(t) - \phi(r(t),t)\mathrm{d}t \tag{159}$$

6.9 Greeks for Interest Rate Derivatives

The greeks all work the same and lead to the same approximation formulas. Some bonds do follow what's called an "affine" structure. A bond has an affine structure if it has a certain pricing formula.

$$P(r,t,T) = A(t,T)e^{-rB(t,T)} \tag{160}$$

The greeks for interest rate derivatives under an affine structure are easy to calculate.

$$\Delta = -B(t,T)P(r,t,T) \tag{161}$$

$$\Gamma = [B(t,T)]^2 P(r,t,T) \tag{162}$$

$$q(r,t,T) = \sigma(r)B(t,T) \tag{163}$$

6.10 Rendleman-Bartter

This is just a straightforward GBM.

$$\mathrm{d}r(t) = ar(t)\mathrm{d}t + \sigma r(t)\mathrm{d}Z(t) \tag{164}$$

$$r(t) = r(0)e^{(\frac{a-\sigma^2}{2})t+\sigma Z(t)} \tag{165}$$

6.11 Vasicek

This is an Ornstein-Uhlenback process.

$$\mathrm{d}r(t) = a[b - r(t)]\mathrm{d}t + \sigma \mathrm{d}Z(t) \tag{166}$$

$$r(t) = b + [r(0) - b]e^{-at} + \sigma \int_0^t e^{-a(t-s)} dZ(s) \tag{167}$$

This model has some nice formulas about its bond prices.

$$P(r,t,T) = A(T-t)e^{-rB(T-t)} \tag{168}$$

$$\bar{r} = b + \frac{\sigma\phi}{a} - \frac{\sigma^2}{2a^2} \tag{169}$$

$$B(T-t) = \frac{1 - e^{-a(T-t)}}{a} \tag{170}$$

$$q(r,t,T) = \sigma B(T-t) \tag{171}$$

$$\mathrm{d}r = [a(b-r)] + \sigma\phi]\mathrm{d}t + \sigma \mathrm{d}\tilde{Z} = a(b' - r)\mathrm{d}t + \sigma \mathrm{d}\tilde{Z} \tag{172}$$

$$b' = b + \frac{\sigma\phi}{a} \tag{173}$$

$$y(r,t,T) = -\frac{1}{T-t}\ln[P(r,t,T)] \tag{174}$$

6.12 Cox-Ingersoll-Ross

The short rate process and other useful formulas and equations are below.

$$\mathrm{d}r(t) = a[b - r(t)]\mathrm{d}t + \sigma\sqrt{r(t)}\mathrm{d}Z(t) \tag{175}$$

$$P(r,t,T) = A(T-t)e^{-rB(T-t)} \tag{176}$$

$$\gamma = \sqrt{(\alpha - \bar{\phi})^2 + 2\sigma^2} \tag{177}$$

$$q(r,t,T) = \sigma\sqrt{r}B(T-t) \tag{178}$$

$$\text{yield to maturity} = \frac{2ab}{a - \bar{\phi} + \gamma} \tag{179}$$