Path tracking for an automated vehicle

Multivariable control and
coordination systems (EE–477)

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1 Modeling, linearization, and discretization

This first part of the case study will only examine the open-loop control of the vehicle. In the second part, a controller will be developed.

The model of the vehicle is given as a continous non-linear model. Dealing with non-linear model is difficult, thus it is linearized using a small signal approach. More specifically, the model is linearized around a nominal trajectory. If the car stay close to the nominal trajectory, the linearized model should give a good enough approximation. The linearized model is then discretized using Euler's approximation.

1.1 Rewrite the non-linear model in the standard form $\dot{x} = f(x, u)$.

The non-linear continous vehicle's model is formulated as follows:

$$f(x,u) = \begin{bmatrix} \dot{s} \\ \dot{d} \\ \dot{\theta}_e \\ \dot{v} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \frac{v\cos(\theta_e)}{1-d\kappa(s)} \\ v\sin(\theta_e) \\ \frac{v}{L}\tan(\phi/16) - \kappa(s)\dot{s} \\ \sigma_v(v_{ref} - v) \\ \sigma_\phi(\phi_{ref} - \phi) \end{bmatrix} = \begin{bmatrix} \frac{v\cos(\theta_e)}{1-d\kappa(s)} \\ v\sin(\theta_e) \\ \frac{v}{L}\tan(\phi/16) - \kappa(s)\frac{v\cos(\theta_e)}{1-d\kappa(s)} \\ \sigma_v(v_{ref} - v) \\ \sigma_\phi(\phi_{ref} - \phi) \end{bmatrix} \tag{1}$$

Some remarks about the model:

- s is the distance along the reference trajectory. When the car goes forward, it is expected to increase. The reference trajectory is given as $(x_\pi(s), y_\pi(s), \theta_\pi(s))$.
- The position of the car is taken on the middle of the vehicle's rear bumper (see figure 3).
- d is the lateral error from the vehicle's position to the reference trajectory.
- The states are defined to be $x = [s, d, \theta_e, v, \phi]$.
- The inputs are defined to be $u = [v_{ref}, \phi_{ref}]$.

Figure 1: Vehicle's coordinate system

1.2 Given the chosen state space, why it wouldn't be appropriate linearizing the system around a nominal point.

By definition, an operating point is the state where the derivative $f(x, u) = \dot{x} = 0$. It is possible to find such a state by specifying conditions. However, the vehicle will unlikely stay around an operating point. s, the distance, will increase when travelling, thus go further away from the operating point and increase the error of the linearized model.

The solution is to linearize around a nominal trajectory. This will ensure that the linearized model has a good approximation at all time.

1.3 Calculate the nominal trajectory representing the constant-speed perfect tracking situation

The nominal trajectory is defined as the trajectory where the vehicle pefectly follows the path. The vehicle's speed is defined to be v_{ref} and the path curvature is a constant κ_{ref}. This allows to define the following conditions on the non-linear continous model:

$$\bar{v} = v_{ref} \tag{2a}$$

$$\kappa(s) = \kappa_{ref} \tag{2b}$$

$$\dot{\bar{\theta}}_e = \bar{\theta}_e = 0 \tag{2c}$$

$$\dot{\bar{d}} = \bar{d} = 0 \tag{2d}$$

Conditions (2c) and (2d) follow naturally from the fact that the vehicle has no tracking error. Using these conditions, the nominal trajectory can be deduced from the non-linear continous model (see Eq. 1):

$$f(\bar{x},\bar{u}) = \begin{bmatrix} \dot{\bar{s}} \\ \dot{\bar{d}} \\ \dot{\bar{\theta}}_e \\ \dot{\bar{v}} \\ \dot{\bar{\phi}} \end{bmatrix} = \begin{bmatrix} \frac{\bar{v}\cos(\bar{\theta_e})}{1-\bar{d}\kappa(s)} \\ \bar{v}\sin(\bar{\theta}_e) \\ \frac{\bar{v}}{L}\tan(\bar{\phi}/16) - \kappa(s)\dot{\bar{s}} \\ \sigma_v(v_{ref}-\bar{v}) \\ \sigma_\phi(\phi_{ref}-\bar{\phi}) \end{bmatrix} = \begin{bmatrix} v_{ref} \\ 0 \\ 0 \\ 0 \\ 0 \\ \sigma_\phi(\phi_{ref}-\bar{\phi}) \end{bmatrix} \tag{3}$$

Some additionnal computations are done to find the nominal trajectory.

- $\bar{s}$ is found by integrating $\dot{\bar{s}}$. It is assumed that $\bar{s}$ starts at 0, at initial time ($t = 0$).
- $\bar{\phi}$ is found using the state equation of θ_e.

The nominal trajectory is then deduced to be the following:

$$\bar{x}(t) = \begin{bmatrix} \bar{s} \\ \bar{d} \\ \bar{\theta}_e \\ \bar{v} \\ \bar{\phi} \end{bmatrix} = \begin{bmatrix} v_{ref}\cdot t \\ 0 \\ 0 \\ v_{ref} \\ 16\tan^{-1}(\kappa_{ref}L) \end{bmatrix} \tag{4}$$

1.3.1 Linearize the system around such a nominal trajectory

Once the nominal trajectory has been found, the non-linear continous model is derived with respect to each state variables to find the linear continous model.

$$\frac{\partial f}{\partial x}(x,u) = \begin{bmatrix} 0 & \kappa_{ref}\frac{v\cos(\theta_e)}{(1-d\kappa_{ref})^2} & -\frac{v\sin(\theta_e)}{1-d\kappa_{ref}} & \frac{cos(\theta_e)}{1-d\kappa_{ref}} & 0 \\ 0 & 0 & v\cos(\theta_e) & sin(\theta_e) & 0 \\ 0 & -\kappa^2(s)\frac{v\cos(\theta_e)}{(1-d\kappa_{ref})^2} & \kappa_{ref}\frac{v\sin(\theta_e)}{1-d\kappa_{ref}} & \frac{1}{L}\tan(\phi/16) - \kappa_{ref}\frac{cos(\theta_e)}{1-d\kappa_{ref}} & \frac{v}{16L}(1+\tan^2(\phi/16)) \\ 0 & 0 & 0 & -\sigma_v & 0 \\ 0 & 0 & 0 & 0 & -\sigma_\phi \end{bmatrix} \tag{5}$$

$$\frac{\partial f}{\partial u}(x,u) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \sigma_v & 0 \\ 0 & \sigma_\phi \end{bmatrix} \tag{6}$$

$f(x,u)$ is evaluated at the nominal trajectory to find the linear continous model around the nominal trajectory:

$$\frac{\partial f}{\partial x}(\bar{x},\bar{u}) = \begin{bmatrix} 0 & \kappa_{ref}\cdot v_{ref} & 0 & 1 & 0 \\ 0 & 0 & v_{ref} & 0 & 0 \\ 0 & -\kappa_{ref}^2\cdot v_{ref} & 0 & 0 & \frac{v_{ref}}{16L}(1+(\kappa_{ref}L)^2) \\ 0 & 0 & 0 & -\sigma_v & 0 \\ 0 & 0 & 0 & 0 & -\sigma_\phi \end{bmatrix} \tag{7}$$

$$\frac{\partial f}{\partial u}(\bar{x},\bar{u}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \sigma_v & 0 \\ 0 & \sigma_\phi \end{bmatrix} \tag{8}$$

1.4 Discretize the system using Euler approximation

The linearized model with a state-space representation as the following form:

$$\dot{\tilde{x}} = \frac{\partial f}{\partial x}(\bar{x}, \bar{u})\tilde{x} + \frac{\partial f}{\partial u}(\bar{x}, \bar{u})\tilde{u} \tag{9}$$

Where $\tilde{x}$ and $\tilde{u}$ designate the deviations from the nominal trajectory defined by:

$$\tilde{x} = x - \bar{x} \tag{10}$$

$$\tilde{u} = u - \bar{u} \tag{11}$$

Euler's approximation is one the simplest integration method which will assume that the derivative is constant over a time step. The next state is found by multiplying the time step h and the derivative.

$$\tilde{x}(k+1) = \left[\frac{\partial f}{\partial x}(\bar{x}, \bar{u})\tilde{x}(k) + \frac{\partial f}{\partial u}(\bar{x}, \bar{u})\tilde{u}(k)\right] \cdot h + \tilde{x}(k) \tag{12}$$

By rearranging the terms.

$$\tilde{x}(k+1) = \underbrace{\left[\frac{\partial f}{\partial x}(\bar{x}, \bar{u})h + I\right]}_{\Phi} \tilde{x}(k) + \underbrace{\left[\frac{\partial f}{\partial u}(\bar{x}, \bar{u})h\right]}_{\Gamma} \tilde{u}(k) \tag{13}$$

$$\Phi = \begin{bmatrix} 1 & \kappa_{ref} \cdot v_{ref}h & 0 & h & 0 \\ 0 & 1 & v_{ref}h & 0 & 0 \\ 0 & -\kappa_{ref}^2 \cdot v_{ref}h & 1 & 0 & \frac{v_{ref}}{16L}(1 + (\kappa_{ref}L)^2)h \\ 0 & 0 & 0 & -\sigma_v h + 1 & 0 \\ 0 & 0 & 0 & 0 & -\sigma_\phi h + 1 \end{bmatrix} \tag{14}$$

$$\Gamma = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \sigma_v h & 0 \\ 0 & \sigma_\phi h \end{bmatrix} \tag{15}$$

2 Model implementation and simulation

This part will cover the implementation of the model discussed in section 1. Three models are built and compared:

1. Continous non-linear model
2. Continous linearized model
3. Discrete linearized model

2.1 Provide the arrays Φ and Γ obtained using the Euler approximation method and the Matlab command c2d

The matrices Φ and Γ are computed for the discretized system using the equations 14 and 15 for the Euler approximation. The other constants are: $\kappa_{ref} = 1e^{-10}$, $L = 4$, $\sigma_v = 1$, $\sigma_\phi = 5$, $v_{ref} = 5$, $h = 0.1$. The resulting matrices Φ and Γ are compared against matrices obtained from zero-order hold.

	Euler's approximation	Matlab's **c2d** zero-order hold
Φ	$\begin{bmatrix} 1 & 0 & 0 & 0.1 & 0 \\ 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0.125 \\ 0 & 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0.095 & 0 \\ 0 & 1 & 0.5 & 0 & 0.027 \\ 0 & 0 & 1 & 0 & 0.098 \\ 0 & 0 & 0 & 0.905 & 0 \\ 0 & 0 & 0 & 0 & 0.607 \end{bmatrix}$
Γ	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.1 & 0 \\ 0 & 0.5 \end{bmatrix}$	$\begin{bmatrix} 0.005 & 0 \\ 0 & 0.005 \\ 0 & 0.027 \\ 0.095 & 0 \\ 0 & 0.393 \end{bmatrix}$

Table 1: Numerical results of Φ and Γ

These results confirm that the Euler's approximation seems correct. In the discretized system, the matrices computed using zero-order hold will be used as there are more accurate.

2.2 Open-loop simulation

With the help of Simulink, the system is simulated. Given the nominal trajectory for the states and the inputs, an input sequence is given which then simulates the 3 models in open-loop. The results are compared afterwards.

2.2.1 Provide a screenshot of the block diagram implemented in `open_loop_experiments.slx`

The block diagram implementation in Simulink is as follows:

Figure 2: Simulink implementation for part 1

Some remarks about the Simulink implementation:

- The input control is a discrete sequence.
- Zero-order holds blocks are put in front of continous time models to feed a continous input to the system from discrete inputs. The input is held constant during a sampling period.
- Linearized models need the deviation from the nominal trajectory and give deviation from the nominal trajectory as output.
- The output of the systems is assumed to be all the states. C is the identity matrix. $D = 0$.

2.2.2 Provide a screenshot of your implementation of the `Non linear model - continuous time block`

The implementation of the non-linear continous block is given here:

Figure 3: Block diagram for the non-linear continous system

This is the implementation of the equation 1. It is the most accurate model of the vehicle.

2.2.3 Report the simulation results, including information concerning the initial state you used, the sequence of inputs you applied, and the simulated results

The initial state is set at the nominal trajectory. This will ensure that at least at the beginning, the linearized models gives a good approximation.

$$x(0) = \bar{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ v_{ref} \\ 16\tan^{-1}(\kappa_{ref}L) \end{bmatrix} \tag{16}$$

The nominal trajectory is computed using the numerical values:

$$\bar{x}(t) = \begin{bmatrix} 5t \\ 0 \\ 0 \\ v_{ref} \\ 0 \end{bmatrix} \tag{17}$$

The simulation is run for $t = [0, 10]$ with a sampling time of $h = 10^{-3}$.

The input control is composed of two variables $u = [v_r eg,$