BIPARTITE SUBGRAPH DECOMPOSITION FOR CRITICALLY SAMPLED WAVELET FILTERBANKS ON ARBITARY GRAPHS

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Introduction

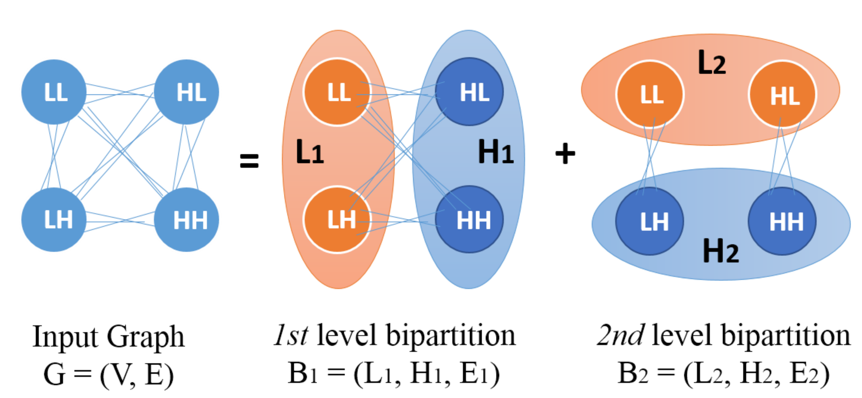
Why do we care about graph wavelet filterbanks?

- Graph: widely adopted to describe data structures e.g. social, sensor networks, etc.
- Key problem: design of graph wavelet filterbanks to account for vertex correlations and achieve compact representation

Why do we need bipartite subgraph decomposition?

- Limit of recent works: GraphBior [1], applicable only to bipartite graph-signals
- Non-bipartite graph needs to be decomposed into bipartite subgraphs, as shown in *Fig. 1*
- Previous methods [2][3][4] lack criteria directly related to compact representation
- Proposed method: a) minimize the mid-frequency multiplicity; b) maximize the structure preservation

Fig. 1 Example of Bipartite Subgraph Decomposition



Graph Wavelet Filterbanks

Graph Spectrum and Spectral Filter

- Laplacian matrix L = D - W, where D is degree matrix, W is adjacency matrix
- Normalized form $\mathcal{L} = D^{-1/2} L D^{-1/2}$: eigenvalues $\{\lambda_i\}$ within range [0, 2], interpreted as graph spectrum
- Spectral Filter: defined with spectral kernel h(λ)

Critically Sampled Wavelet Filterbanks — for Bipartite Graph-Signal

- Flowchart *Fig. 2*: decompose ***f*** into low-pass and high-pass components
- **H** and **G**: cancel frequency folding, with spectral kernels in *Fig. 3*
- λ = 1: minimal energy discrimination
- Bipartite subgraph decomposition is required to apply the above filterbanks

Fig.2 Two-channel wavelet filterbanks on bipartite graph B = (L, H, E)

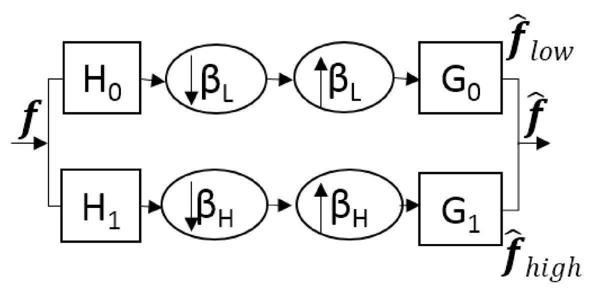
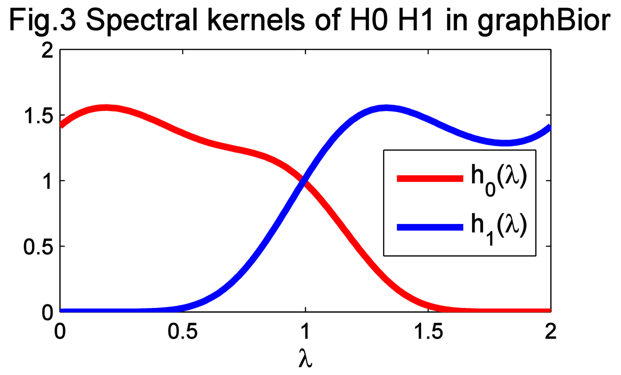


Fig.3 Spectral kernels of H0 H1 in graphBior



Proposed Method (MFS)

- Goal: compact representation of signals in original graph G projected to wavelet domain of bipartite subgraph G'
- Criteria: minimum mid-frequency multiplicity & maximum structure preservation

Minimum Mid-Frequency Multiplicity

Fig.4 How mid-frequency multiplicity affects compact representation

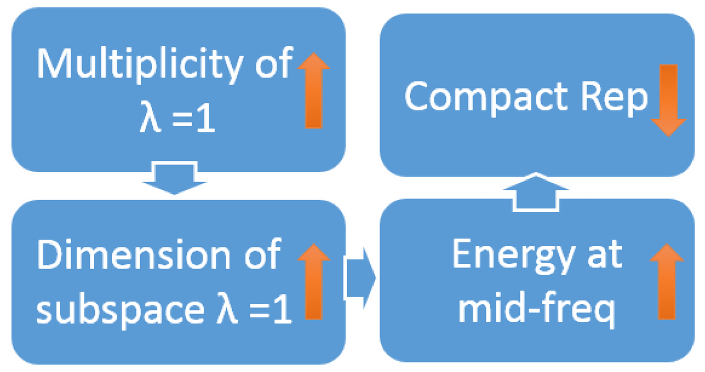
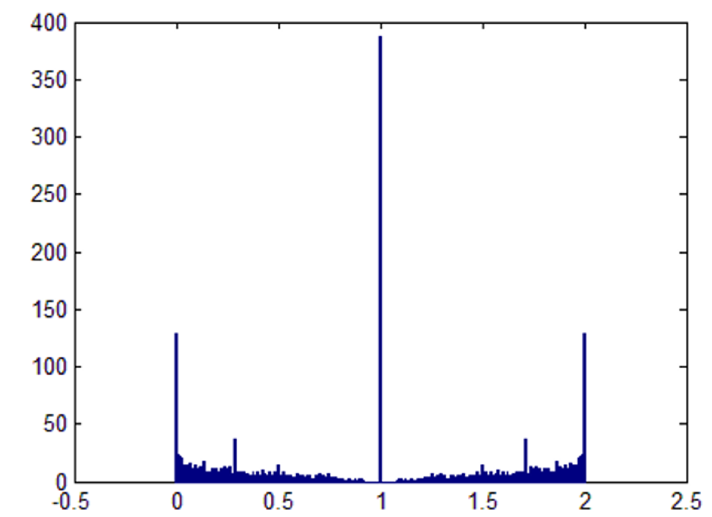


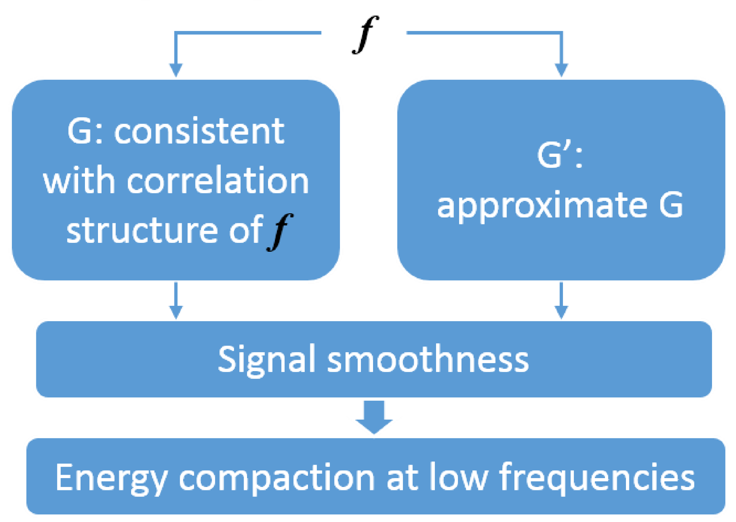
Fig. 5 Eigenvalue distribution of 1st level bipartite graph for Minnesota traffic graph using Harary's [2]: high multiplicity of λ=1 is exhibited



- Multiplicity of λ = 1 is equivalent to null(W)
- Measurement: rank(W)

Maximum Structure Preservation

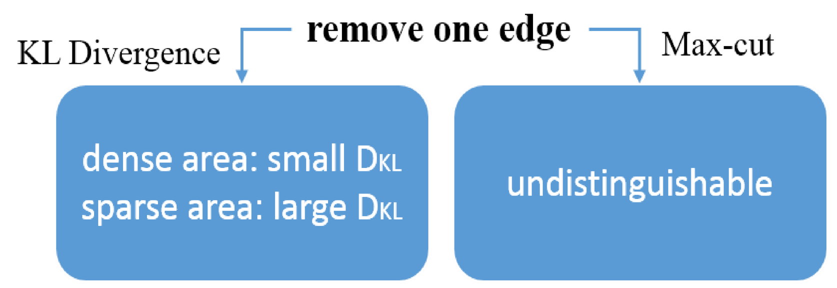
Fig. 6 How structure preservation leads to compact representation



Measurement for Structure Preservation

- [3][4] use max-cut as the measurement
- In our work, KL Divergence is adopted

Fig. 7 KL Divergence vs Max-cut



- Original graph $\mathcal{N}(\mu, \Sigma)$, graph with e_{12} removed $\mathcal{N}_R(\mu_R, \Sigma_R)$, KL divergence given by:

$$D_{KL}(\mathcal{N}||\mathcal{N}_R) = \frac{1}{2}\Big(\mathrm{tr}(\mathbf{\Sigma}_R^{-1}\mathbf{\Sigma}) + (\mu_R - \mu)^T \mathbf{\Sigma}_R^{-1}(\mu_R - \mu) - N + \ln\Big(\frac{|\mathbf{\Sigma}_R|}{|\mathbf{\Sigma}|}\Big)\Big)$$

When vertex degree d_1 d_2 are large, σ_1 σ_2 are small, then D_{KL} is small.

$$D_{KL}(\mathcal{N}||\mathcal{N}_R) \approx \frac{1}{2}\Big(-\frac{\delta}{1+\delta}(\sigma_1+\sigma_2) - \ln\big(1 - \frac{\delta}{1+\delta}(\sigma_1+\sigma_2)\big)\Big).$$

References: [1] S.N. and A.O. "Compact support biorthogonal wavelet filterbanks for arbitrary undirected graphs," TSP'13.

[2] S.N. and A.O. "Perfect reconstruction two-channel wavelet filter banks for graph structured data," TSP'12.

[3] S.N. and A.O. "Multi-dimensional separable critically sampled wavelet filterbanks on arbitrary graphs," ICASSP'12.

[4] H.N. and M.Do "Downsampling of signals on graphs via maximum spanning trees," TSP'15.

Algorithm (MFS)

- maximizing rank(W) ≠ minimizing D_{KL}
- Heuristic algorithm:

Algorithm 1 Bipartite Subgraph Decomposition Optimizing Mid-frequency and Structure

Input: graph $\mathcal{G}$, decomposition level k
Output: edge-disjoint bipartite graphs $\mathcal{B}_1,\ldots,\mathcal{B}_k$

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1: for i = 1:k do
2:     Find connected components in G.
3:     For each component, put the starting vertex in set 1.
4:     Use breadth-first search to explore other vertices, and choose
   the proper set by jointly comparing rank(W_{1,2}) and D_KL.
5:     After all vertices are discovered, bipartite graph B_i is given.
6:     Update G by removing edges in B_i.
7: end for
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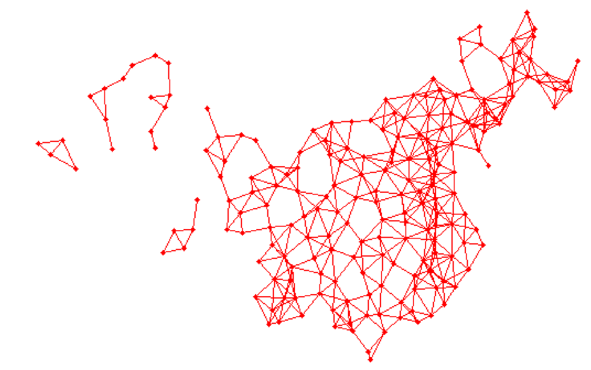
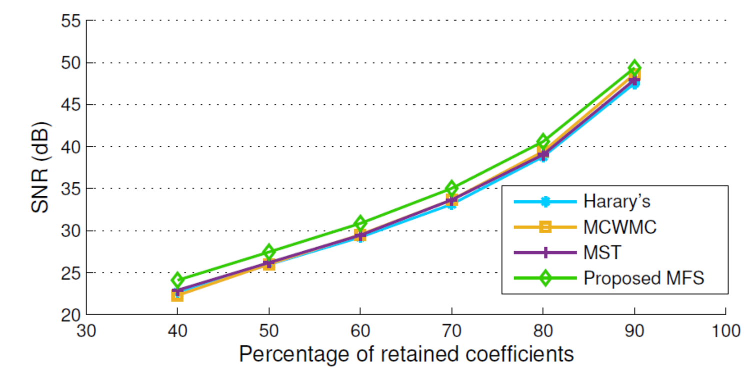
Experiments

- Given input graph, first do bipartite subgraph decomposition, then apply GraphBior and reconstruct the signal with n% largest wavelet coefficients
- China temperature graph: monthly average temperature from Oct.09 to May12, vertices connected to neighbors with distance < threshold T





- Table 1 Average gain of proposed MFS over competing schemes in SNR(dB) for graphs with different connections: column 2~5, threshold from T to 1.4 T; column 6~8, vertices connected to knn with k = 7, 8, 9.

	T	0.8T	1.2T	1.4T	k=7	k=8	k=9
Harary's[2]	1.65	1.43	0.82	0.82	0.76	0.64	1.34
MCWMC[3]	1.35	0.74	1.17	1.24	1.56	1.62	2.06
MST[4]	1.35	0.16	2.24	1.38	0.93	0.64	1.91

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