Mathematical-Logic Notebook
Propositional Calculus

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Abstract

In this notebook, we only discuss about a simple logic system which we call by $\mathscr{P}$. Sometimes, in this note we also use notation $\mathcal{L}$ refers to a general logic system.

1 Systematic of $\mathcal{L}$-Logic

Suppose that we have logic system $\mathcal{L}$, then for $\mathcal{L}$ we have axiom, rules of inference, and proofs as syntax of $\mathcal{L}$, interpretation as Semantics of $\mathcal{L}$, the intersection of syntax and semantics is refered to well-formed-formulas and sometimes we denoted it as the element set of $\mathscr{S}$.

2 Formation Rules for $\mathscr{P}$

(a). Improper symbols: [ ] $\vee\wedge\neg$.
(b). Proper symbols: denumerably many propositional variables: p, q, r, s, $p_1, q_1, r_1, ...$

We shall denote a general formula where this formula is 'meaningfull expression' for $\mathscr{P}$. We define a wff (well-formed-formulas) inductively as follows:

(i). A standalone propositional variable is a wff.
(ii).If $\mathbf{A}$ defined as wff, then $\neg\mathbf{A}$ is wff.
(iii.) If $\mathbf{A}$ and $\mathbf{B}$ is a wff, then $[\mathbf{A}\wedge\mathbf{B}]$ is a wff.

After we understand what is point (i), (ii), (iii) try to explain, then we ready to take a look at the definition the set of wff $\mathscr{S}$

Definition [Well-Formed-Formulas]. The set of wff is the inersection of all set $\mathscr{S}$ of formulas such that:

(i). $\mathbf{p} \in \mathscr{S}$ for each propostional variable $\mathbf{p}$.
(ii). For each formulas $\mathbf{A}$, if $\mathbf{A} \in \mathscr{S}$ then $\neg\mathbf{A} \in \mathscr{S}$.

(iii). For all formulas, if $\mathbf{A} \in \mathscr{S}$ and $\mathbf{B} \in \mathscr{S}$, then $[\mathbf{A} \vee \mathbf{B}] \in \mathscr{S}$.

We shall define something inductively in order to simplicity and keep it understandable.

3 Principle of Inductive on the Construction of a Well Formed Formulas

Suppose that $\mathscr{R}$ be a property of wff $\mathbf{A}$, then.

(i). $\mathscr{R}(\mathbf{p})$ for each proposition variable $\mathbf{p}$.
(ii). If $\mathscr{R}(\mathbf{A})$, then $\mathscr{R}(\neg\mathbf{A})$.
(iii). If $\mathscr{R}(\mathbf{A})$ and $\mathscr{R}(\mathbf{B})$, then $\mathscr{R}([\mathbf{A} \vee \mathbf{B}])$.

3.1 Operator Substitution S

Suppose that $\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n}$ be a primitive symbols and let $\mathbf{Y_1}, \mathbf{Y_2}, ..., Y_n$ be a formulas, then

$$\mathbf{S}^{\mathbf{x_1},\ldots,\mathbf{x_n}}_{\mathbf{Y_1},\ldots,\mathbf{Y_n}}$$

is defined as a finite substitution θ such that

$$\theta\mathbf{x_i} = \mathbf{Y_i}$$

for $i = 1, 2, ..., n$

Suppose that $\mathbf{Z}$ be a formulas, then.

$$\mathbf{S}^{\mathbf{x_1},\ldots,\mathbf{x_n}}_{\mathbf{Y_1},\ldots,\mathbf{Y_n}}\mathbf{Z}$$

is the result of simultaneous substitution Y_1 for $x_1, ..., Y_n$ for x_n.

4 Supplement on Induction

In this section, we shall introduce the two most famous induction.

$$\vdash \forall P\,[[P(0) \wedge \forall n\,[P(n) \to P(n+1)]] \to \forall n P(n)] \tag{1}$$

$$\vdash \forall R\,[\forall n\,[(\forall j < n)R(j) \to R(n)] \to \forall n R(n)] \tag{2}$$

Equation (1) is known as principle of mathematical induction and equation (2) is known as principle of complete induction. In order to simplify writting this equation, we rewrite this principle as below.

$$\vdash \left[P(0).P(n) \overset{\forall n}{\to} P(n+1)\right] \overset{\forall P \forall n}{\to} P(n) \tag{3}$$

And

$$\vdash \left[(j < n)R(j) \overset{\forall n \forall j}{\to} R(n)\right] \overset{\forall R \forall n}{\to} R(n) \tag{4}$$

We shall discuss the principle of induction in a different papers. In this section, we only need to understand what is this principle stated.

5 Proposition

Here are discussion and proof of the useful propositon that we shall use.

Proposition. Every well-formed-formula is of the form $\mathbf{p}$ for some propositional variable p, or the form $\neg\mathbf{A}$ or $[\mathbf{A} \vee \mathbf{B}]$ for some well-formed-formula $\mathbf{A}$ and $\mathbf{B}$.

Proof.

By definition, $\mathbf{p} \in \mathscr{S}$ for some propositional variable. For $\mathbf{A} \in \mathscr{S}$, then $\neg\mathbf{A} \in \mathscr{S}$. And suppose that $\mathbf{A} \in \mathscr{S}$ and $\mathbf{B} \in \mathscr{S}$, then $[\mathbf{A} \vee \mathbf{B}] \in \mathscr{S}$.

■

Proposition. Suppose that θ is any substitution for variables and $\mathbf{A}$ and $\mathbf{B}$ is a well-formed-formula, then

(i).$(\theta\neg\mathbf{A}) = \neg\,(\theta\mathbf{A})$.
(ii).$(\theta\,[\mathbf{A} \vee \mathbf{B}]) = [(\theta\mathbf{A}) \vee (\theta\mathbf{B})]$

Proof.

Proof using principle of induction on the construction of well-formed-formulas and θ.

■

Proposition. Suppose that we are given following formulas tied with substitution operator.

$$\mathbf{S}^{p,q}_{[q\vee r],\neg p}\,[\neg\,[\neg p \vee q] \vee \neg\neg p] \tag{5}$$

Prove that there exist a formulas by evaluate the substitution-operator of the formulas above.

Proof.

Suppose that $\mathbf{A}$ will be a formulas by evaluated simultaneous substitution in equation (5), and for $\mathbf{A}$ we have

$$\begin{aligned}
\mathbf{S}^{p,q}_{[q\vee r],\neg p}\,[\neg\,[\neg p \vee q] \vee \neg\neg p] &= \mathbf{S}^{p,q}_{[q\vee r],\neg p}\,[\neg\,[\neg p \vee q] \vee p] \\
&= [[q \vee r] \wedge \neg\neg p] \vee [q \vee r] \\
&= [[q \vee r] \wedge [q \vee r] \vee [q \vee r]] \\
&= [q \vee r] \\
\because \mathbf{A} &= \mathbf{S}^{p,q}_{[q\vee r],\neg p}\,[\neg\,[\neg p \vee q] \vee \neg\neg p] \\
\therefore \mathbf{A} &= [q \vee r]
\end{aligned}$$

■

Proposition. Suppose that $\mathbf{p_1}, \mathbf{p_2}, ..., \mathbf{p_n}$ be a distinct variable and $\mathbf{B_1}, \mathbf{B2}, ..., \mathbf{B_n}$, and $\mathbf{A}$ be a wff. Then

$$\mathbf{S}^{\mathbf{B_1},\ldots,\mathbf{B_n}}_{\mathbf{p_1},\ldots,\mathbf{p_n}}\mathbf{Z}$$

is a wffs.