The First Derivative of tan(x)

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Abstract

We already know the first derivative for tan(x) is $sec^2(x)$, but only few people know and understand where is it comefrom. In this paper, we will discuss the quotient-rules and using it to prove the first derivate of $f(x) = tan(x)$.

1 Derivative of f(x) = u(x)/v(x)

Prove that if given $f(x) = u(x)/v(x)$ then

$$f^{'}(x) = \frac{u^{'}(x)v(x) - u(x)v^{'}(x)}{v^2(x)} \tag{1}$$

Proof.

The definition of derivative for a given function f(x) is defined as.

$$f^{'}(x) \stackrel{\text{def}}{=} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2}$$

for f(x) = u(x)/v(x) then

$$f(x+h) = \frac{u(x+h)}{v(x+h)} \tag{3}$$

By plugins f(x+h) from equation (3) to equation (2) we obtain

$$
\begin{aligned}
f^{'}(x) &= \lim_{h\to 0}\frac{(\frac{u(x+h)}{v(x+h)}) - \frac{u(x)}{v(x)}}{h} \\
&= \lim_{h\to 0}\frac{u(x+h)}{hv(x+h)} - \frac{u(x)}{hv(x)} \\
&= \lim_{h\to 0}\frac{hu(x+h)v(x) - hu(x)v(x+h)}{h^2v(x+h)v(x)} \\
&= \lim_{h\to 0}\frac{u(x+h)v(x) - u(x)v(x+h)}{hv(x+h)v(x)} \\
&= \frac{1}{v(x)}\lim_{h\to 0}\frac{u(x+h)v(x) - u(x)v(x) + u(x)v(x) - u(x)v(x+h)}{hv(x+h)} \\
&= \frac{1}{v(x)}\lim_{h\to 0}\frac{[u(x+h) - u(x)]v(x) - u(x)[v(x+h) - v(x)]}{hv(x+h)} \\
&= \frac{1}{v(x)}\lim_{h\to 0}\left\{\left[\frac{u(x+h) - u(x)}{h}\right]\frac{v(x)}{v(x+h)} - \frac{u(x)}{v(x+h)}\left[\frac{v(x+h) - v(x)}{h}\right]\right\} \\
&= \frac{1}{v(x)}\left\{\left[\lim_{h\to 0}\frac{u(x+h) - u(x)}{h}\right]\lim_{h\to 0}\frac{v(x)}{v(x+h)} - \lim_{h\to 0}\frac{u(x)}{v(x+h)}\left[\lim_{h\to 0}\frac{v(x+h) - v(x)}{h}\right]\right\}
\end{aligned}
$$

By using definition of derivative in (2) and evaluate the $\lim_{h\to 0}$, we obtain.

$$
\begin{aligned}
f^{'}(x) &= \frac{1}{v(x)}\left[\frac{u^{'}(x)v(x)}{v(x+0)} - \frac{u(x)v^{'}(x)}{v(x+0)}\right] \\
&= \frac{1}{v(x)}\left[\frac{u^{'}(x)v(x) - u(x)v^{'}(x)}{v(x)}\right] \\
&= \frac{u^{'}(x)v(x) - u(x)v^{'}(x)}{v^2(x)} \\
\therefore f^{'}(x) &= \frac{u^{'}(x)v(x) - u(x)v^{'}(x)}{v^2(x)}
\end{aligned}
$$

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2 Derivative of tan(x)

Suppose that we are given a function such that f(x) = tan(x). Show that the first derivative of f(x) is defined as $f'(x) = sec^2(x)$.

Proof.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$for f(x) = tan(x)$ then $f(x+h) = tan(x+h)$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{tan(x+h) - tan(x)}{h} \\
\because tan(x+h) &= \frac{tan(x) + tan(h)}{1 - tan(x)tan(h)} \\
f'(x) &= \lim_{h\to 0} \frac{\frac{tan(x)+tan(h)}{1-tan(x)tan(h)} - tan(x)}{h} \\
&= \lim_{h\to 0} \frac{tan(x) + tan(h)}{h(1 - tan(x)tan(h))} - \frac{tan(x)}{h} \\
&= \lim_{h\to 0} \frac{h[tan(x) + tan(h)] - tan(x)h(1 - tan(x)tan(h))}{h^2(1 - tan(x)tan(h))} \\
&= \lim_{h\to 0} \frac{(tan(x) + tan(h)) - tan(x)[1 - tan(x)tan(h)]}{h[1 - tan(x)tan(h)]} \\
&= \lim_{h\to 0} \frac{(tan(x) + tan(h)) - [tan(x) - tan^2(x)tan(h)]}{h[-tan(x)tan(h)]} \\
&= \lim_{h\to 0} \frac{tan(h) + tan^2(x)tan(h)}{h[1 - tan(x)tan(h)]} \\
&= \lim_{h\to 0} \frac{tan(h)[1 + tan^2(x)]}{h[1 - tan(x)tan(h)]} \\
&= \lim_{h\to 0} \frac{1 + tan^2(x)}{1 - tan(x)tan(h)} \lim_{h\to 0} \frac{tan(h)}{h} \\
\because \lim_{h\to 0} \frac{tan(h)}{h} &= 1
\end{aligned}
$$

By evaluate the $\lim_{h\to 0}$, we have

$$
\begin{aligned}
f'(x) &= \frac{1 + tan^2(x)}{1 - tan(x)tan(0)} (1) \\
\because tan(0) &= 0 \\
f'(x) &= \frac{1 + tan^2(x)}{1 - 0} \\
&= 1 + tan^2(x) \\
\because sec^2(x) &= 1 + tan^2(x) \\
\therefore f'(x) &= sec^2(x)
\end{aligned}
$$

Alternatively, we can find the first derivative of tan(x) by using quotient-rules that we already prove in equation (1).

With the help of the identity of trigonometry, we know that tan(x) is defined such that.

$$tan(x) = \frac{sin(x)}{cos(x)} \tag{4}$$

By using quotient-rules in equation (1), for $f(x) = u(x)/v(x)$ we have

$$f^{'}(x) = \frac{u^{'}(x)v(x) - u(x)v^{'}(x)}{v^2(x)}$$

For identity of tan(x) in (4), assume that.

$$\begin{aligned} u(x) &= sin(x) & v(x) &= cos(x) \\ u^{'}(x) &= cos(x) & v^{'}(x) &= -sin(x) \end{aligned}$$

Assign $u(x), v(x), u^{'}(x)$, and $v^{'}(x)$ in $f^{'}(x)$, hence.

$$\begin{aligned} f'(x) &= \frac{cos(x)cos(x) - sin(x)[-sin(x)]}{cos^2(x)} \\ &= \frac{cos^2(x) + sin^2(x)}{cos^2(x)} \\ \because sin^2(x) + cos^2(x) &= 1 \\ f^{'}(x) &= \frac{1}{cos^2(x)} \\ \because sec^2(x) &= \left(\frac{1}{cos(x)}\right)^2 \\ \therefore f^{'}(x) &= sec^2(x) \end{aligned}$$

$\blacksquare$

3 Conclusion

We already see that it is not as simple as we use the result of it. The purpose of proving the quotient-rule and using it to prove the first derivative of tan(x) is to improve our mind, and to improve our ability to do an analytical proof if we are given the definition, fact, and information what we know about the system.

References

[1] James Stewart, "Calculus, Fourth Edition", A Division of International Thomson Publishing Inc,1998.

[2] Edwin J. Purcell, "Calculus with Analytic Geometry", Fifth Edition, Prentice-Hall1987