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   (d) ballistic versus cruise missiles — does the rotation of the Earth matter for ICBMs?

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   (f) swords — kinetic energy versus momentum, impulse of striking

2 Schedule

Parallel each week:

1. Theme area

2. Skill drills

3. Derivations

3 Kinematics of Points

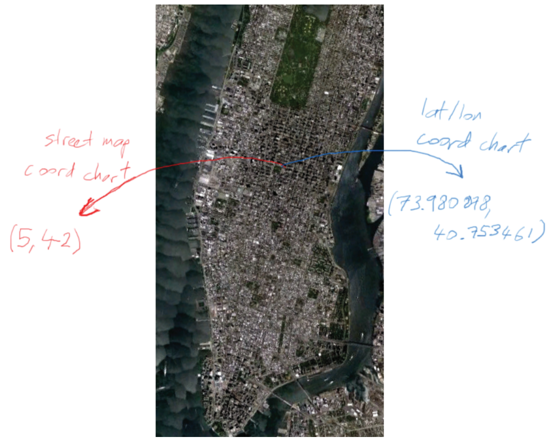
3.1 Positions and Coordinates

A *position* in 2D or 3D space is a single location, which may or may not be occupied by a physical object. Positions exist before we measure or describe them in any way, but to do calculations we need to introduce coordinates for positions.

A *coordinate chart* is a map that takes a position in space and tells us what its *coordinates* are. Coordinates serve to label positions. For example, the coordinates "5th Ave. and 42nd St." label the intersection next to the New York Public Library in the street map coordinate chart. The coordinates of a position are a list of scalars that act as a label for the position, such as $(x = 5, y = 42)$ for this position in New York.

It is common to have several different coordinate charts in use for the same positions in space. For example, "40°45′12.46″N, 73°58′51.16″W" are also coordinates for this same intersection, but this time in the latitude/longitude coordinate chart, which we could write as the pair of scalars $(\lambda = -73.980878, \phi = 40.753461)$[1].

[1] Here we converted degrees/minutes/seconds to decimal notation, wrote the coordinates in the East-West/North-South order to be horizontal/vertical, and realized that directions West are negative



We can also think of a coordinate chart as mapping a list of coordinates to a particular point, which is how we often illustrate a coordinate chart graphically. For example, if we know a latitude and longitude then we can locate the corresponding point on the Earth. The *origin* of a coordinate chart is the point with all zero coordinates. Different coordinate charts may share an origin, or they may have different origins.

If we have a point that moves around in space, then its coordinates will change with time. It is also possible for the coordinate chart itself to be moving in time, so that even if a point remains stationary in space, its coordinates might be changing.

Example: Polar and Cartesian coordinates. Show polar and Cartesian coordinate lines, with different origins, moving particle, show coords in both systems as functions of time. Ask: can the point move somewhere so that both $(r, \theta) = (0, 0)$ and $(x, y) = (0, 0)$ simultaneously?

Example: Moving coordinate charts. Same as previous example, but moving coordinate charts with a stationary point.

When we describe a coordinate chart mathematically, we do so by relating it to another coordinate chart (typically Cartesian).

3.1.1 Cartesian Coordinates

3.1.2 Polar Coordinates

$$x = r\cos\theta \qquad\qquad r = \sqrt{x^2 + y^2} \tag{1}$$

$$y = r\sin\theta \qquad\qquad \theta = \operatorname{atan2}(y, x) \tag{2}$$

3.1.3 Visualizing coordinate charts.

To visualize a coordinate chart we will often draw the lines on which just one of the coordinates is changing.

Example: Visualizing a coordinate chart. Have an active canvas on which we can draw with the mouse by clicking (just enable drawing points? or also lines?). Have a readout showing the coordinates of the mouse at all times. Use polar coordinates, which an origin that is not centered on the canvas (or maybe some more complex coordinate system?). Start with a blank canvas. Ask: draw points that have first coordinate equal to 2. Then draw the points with first coordinate equal to 1. Do the same for points with second coordinate equal to 1, 2, 3, 4, 5, 6. What type of coordinate system is this? Show the standard polar grid.

3.2 Vectors and Bases

A *vector* is an arrow with a length and a direction. Just like positions, vectors exist before we measure or describe them. Unlike positions, vectors can mean many different things, such as position vectors, velocities, etc. Vectors are not anchored to particular positions in space, so we can slide a vector around and locate it at any position.

Advanced aside: Some textbooks differentiate between *free vectors*, which are free to slide around, and *bound vectors*, which are anchored in space.

Notation: We will write $\boldsymbol{v}$ to indicate a vector. The length of $\boldsymbol{v}$ is written v.

Example V.PI: show two vectors v and w and draw them at several different positions each.

Vectors can be multiplied by a scalar number, which multiplies their length. Vectors can also be added together, using the *parallelogram law of addition.*

Example V.PA: parallelogram law of vector addition.

To describe vectors mathematically, we write them as a combination of *basis vectors.* An *orthonormal basis* is a set of two (in 2D) or three (in 3D) basis vectors which are *orthogonal* (have 90° angles between them) and normal (have length equal to one)[2]. Any other vector can be written as a *linear combination* of the basis vectors:

$$\boldsymbol{v} = v_1\,\hat{\imath} + v_2\,\hat{\jmath} \tag{3}$$

The numbers v_1 and v_2 are called the *components* of $\boldsymbol{v}$ in the $\hat{\imath}$, $\hat{\jmath}$ basis.

Example V.BS: Show a vector $\boldsymbol{v}$ and two different bases $\hat{\imath}$, $\hat{\jmath}$ and $\hat{a}$, $\hat{b}$ and show the components. From now on, always have $\hat{\imath}$, $\hat{\jmath}$ being the regular axis-aligned basis and $\hat{a}$, $\hat{b}$ being rotated by $\pi/4$.

Image quiz: Show the same basis twice, once as a regular pair connected at the origin, one where the basis vectors are disconnected. Ask if these are the same basis.

3.2.1 Length of Vectors

The length of a vector $\boldsymbol{v}$ is written either $\|\boldsymbol{v}\|$ or just plain v. The length can be computed using *Pythagorus' theorem*:

$$\boldsymbol{v} = v_1\,\hat{i} + v_2\,\hat{j} \tag{4}$$

$$v = \|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2} \tag{5}$$

Important point: The formula $v = \sqrt{v_1^2 + v_2^2}$ only works if the components are from a single orthonormal basis.

Example: If $\boldsymbol{v} = 3\,\hat{\imath} + 4\,\hat{a}$, what is v? Answer: not $\sqrt{3^2 + 4^2}$. Instead, it is ....

Example: Triangle inequality. If $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$, how are the lengths u, v, w related? Show picture. Answer: certainly it is not true that $u = v + w$. Instead it is true that $u \leq v + w$. There is no way that u can be longer than the combined lengths of v and w, and it will only be equal if v and w are in the same direction.

[2] We will not be using non-orthogonal or non-normal bases in this course.

3.2.2 Unit vectors

Defines the direction of $\boldsymbol{v}$.

$$\hat{v} = \frac{\boldsymbol{v}}{v} \tag{6}$$

Any vector can be written as:

$$\boldsymbol{v} = v\hat{v} \tag{7}$$

Here v is the length, $\hat{v}$ is the direction unit vector.

3.2.3 Dot Product

$$\boldsymbol{u} \cdot \boldsymbol{v} = uv\cos\theta \tag{8}$$

Squared length:

$$v^2 = \boldsymbol{v} \cdot \boldsymbol{v} \tag{9}$$

Length and angle from dot product:

$$v = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} \tag{10}$$

$$\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{uv} \tag{11}$$

Example: Parallelogram area. $\boldsymbol{u} \cdot \boldsymbol{v}$ is the area of the parallelogram formed by $\boldsymbol{u}$ and $\boldsymbol{v}$.

Example: Finding great-circle distance between cities. Take lat/long for Urbana and Perth, say. We want to know great-circle distance. Convert lat/long/R to xyz position vectors from earth center. Consider great circle plane and great circle, that contains these two vectors. The angle on this plane is the angle between the vectors. Find this from the dot product. Find distance with $s = R\theta$. Make the note that this is very hard to do directly from lat/long without switching to xyz and using vectors.

Example V.OR: Finding orthogonal vectors. If $\boldsymbol{u} = u_1\,\hat{\imath} + u_2\,\hat{\jmath}$, then $\boldsymbol{v} = -u_2\,\hat{\imath} + u_1\,\hat{\jmath}$ is orthogonal, as is $\boldsymbol{w} = u_2\,\hat{\imath} - u_1\,\hat{\jmath} = -\boldsymbol{v}$. This doesn't work in 3D!

3.2.4 Cross Product

$$\begin{aligned}
\boldsymbol{w} &= \boldsymbol{u} \times \boldsymbol{v} && (12)\\
&= (u_2v_3 - u_3v_2)\,\hat{\imath} + (u_3v_1 - u_1v_3)\,\hat{\jmath} + (u_1v_2 - u_2v_1)\,\hat{k} && (13)\\
w &= uv\sin\theta && (14)\\
\boldsymbol{w} \cdot \boldsymbol{u} = \boldsymbol{w} \cdot \boldsymbol{v} &= 0 && (15)
\end{aligned}$$

Example: Satellite positions. Take ISS. Assume we have xyz position $\boldsymbol{r}$ of ISS at time zero, and also we know xyz normal vector $\boldsymbol{n}$, all in an inertial frame. We also know orbital period T. We want to work out where the ISS is in 1 hour, say. Do this by finding a basis consisting of $\boldsymbol{r}, \boldsymbol{n}, \boldsymbol{a} = \boldsymbol{n} \times \boldsymbol{r}$. In this basis the ISS moves in the $\boldsymbol{r}$-$\boldsymbol{a}$ plane with angular velocity $2\pi/T$ starting from $(R, 0, 0)$. Find position in this basis at later time. Convert back to xyz.

Useful formulas:

- Antisymmetry:

$$\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u} \tag{16}$$

Derivation: Write out both sides in components.

- $\boldsymbol{u} \times \boldsymbol{u} = 0$ (Derivation: from antisymmetry)

- Bilinearity:

$$\begin{aligned} \boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) &= \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c} && (17) \\ (\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} &= \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c} && (18) \\ \boldsymbol{a} \times (\beta \boldsymbol{b}) &= \beta(\boldsymbol{a} \times \boldsymbol{b}) = (\beta \boldsymbol{a}) \times \boldsymbol{b} && (19) \end{aligned}$$

Derivation: Write out both sides in components.

- Scalar triple product: $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$ is the volume of the parallelepiped defined by $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. It satisfies the scalar triple product formula:

$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{u}) = \boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v}) \tag{20}$$

Derivation: write first two expressions out in components and check that they give the same result:

$$\begin{aligned} \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) &= (u_1\,\hat{\imath} + u_2\,\hat{\jmath} + u_3\,\hat{k}) \cdot \big((v_1\,\hat{\imath} + v_2\,\hat{\jmath} + v_3\,\hat{k}) \times (w_1\,\hat{\imath} + w_2\,\hat{\jmath} + w_3\,\hat{k})\big) && (21) \\ &= (u_1\,\hat{\imath} + u_2\,\hat{\jmath} + u_3\,\hat{k}) \cdot \big((v_2w_3 - v_3w_2)\,\hat{\imath} + (v_3w_1 - v_1w_3)\,\hat{\jmath} + (v_1w_2 - v_2w_1)\,\hat{k}\big) && (22) \\ &= u_1v_2w_3 - u_1v_3w_2 + u_2v_3w_1 - u_2v_1w_3 + u_3v_1w_2 - u_3v_2w_1 && (23) \\ \boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{u}) &= (v_1\,\hat{\imath} + v_2\,\hat{\jmath} + v_3\,\hat{k}) \cdot \big((w_1\,\hat{\imath} + w_2\,\hat{\jmath} + w_3\,\hat{k}) \times (u_1\,\hat{\imath} + u_2\,\hat{\jmath} + u_3\,\hat{k})\big) && (24) \\ &= (v_1\,\hat{\imath} + v_2\,\hat{\jmath} + v_3\,\hat{k}) \cdot \big((w_2u_3 - w_3u_2)\,\hat{\imath} + (w_3u_1 - w_1u_3)\,\hat{\jmath} + (w_1u_2 - w_2u_1)\,\hat{k}\big) && (25) \\ &= v_1w_2u_3 - v_1w_3u_2 + v_2w_3u_1 - v_2w_1u_3 + v_3w_1u_2 - v_3w_2u_1 && (26) \end{aligned}$$

The third expression is also the same.

- Vector triple product is $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$. Vector triple product expansion:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} \tag{27}$$

Derivation: In components:

$$
\begin{aligned}
\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= (a_1\,\hat{\imath} + a_2\,\hat{\jmath} + a_3\,\hat{k}) \times \Big((b_1\,\hat{\imath} + b_2\,\hat{\jmath} + b_3\,\hat{k}) \times (c_1\,\hat{\imath} + c_2\,\hat{\jmath} + c_3\,\hat{k}) \Big) && (28)\\
&= (a_1\,\hat{\imath} + a_2\,\hat{\jmath} + a_3\,\hat{k}) \times \Big((b_2c_3 - b_3c_2)\,\hat{\imath} + (b_3c_1 - b_1c_3)\,\hat{\jmath} + (b_1c_2 - b_2c_1)\,\hat{k} \Big) && (29)\\
&= \Big(a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3) \Big)\,\hat{\imath} && (30)\\
&\quad + \Big(a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1) \Big)\,\hat{\jmath} && (31)\\
&\quad + \Big(a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2) \Big)\,\hat{k} && (32)\\
&= (a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3)\,\hat{\imath} && (33)\\
&\quad + (a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1)\,\hat{\jmath} && (34)\\
&\quad + (a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2)\,\hat{k} && (35)\\
&= (a_1b_1c_1 + a_2b_1c_2 + a_3b_1c_3 - a_1b_1c_1 - a_2b_2c_1 - a_3b_3c_1)\,\hat{\imath} && (36)\\
&\quad + (a_1b_2c_1 + a_2b_2c_2 + a_3b_2c_3 - a_1b_1c_2 - a_2b_2c_2 - a_3b_3c_2)\,\hat{\jmath} && (37)\\
&\quad + (a_1b_3c_1 + a_2b_3c_2 + a_3b_3c_3 - a_1b_1c_3 - a_2b_2c_3 - a_3b_3c_3)\,\hat{k} && (38)\\
&= (a_1b_1 + a_2c_2 + a_3c_3)b_1\,\hat{\imath} + (a_1c_1 + a_2c_2 + a_3c_3)b_2\,\hat{\jmath} + (a_1c_1 + a_2c_2 + a_3c_3)b_3\,\hat{k} && (39)\\
&\quad - (a_1b_1 + a_2b_2 + a_3b_3)c_1\,\hat{\imath} - (a_1b_1 + a_2b_2 + a_3b_3)c_2\,\hat{\jmath} - (a_1b_1 + a_2b_2 + a_3b_3)c_3\,\hat{k} && (40)\\
&= (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} && (41)
\end{aligned}
$$

- Cross product orthogonality:

$$\boldsymbol{a} \times \boldsymbol{b} \text{ is orthogonal to } \boldsymbol{a} \text{ and } \boldsymbol{b} \tag{42}$$

Derivation: follows immediately from the scalar triple product formula:

$$\boldsymbol{a} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{a}) = 0 \tag{43}$$

and similarly for $\boldsymbol{b}$.

- Binet-Cauchy identity:

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}) \tag{44}$$

Derivation: From the scalar triple product formula and the vector triple product expansion:

$$
\begin{aligned}
(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) &= \boldsymbol{c} \cdot \big(\boldsymbol{d} \times (\boldsymbol{a} \times \boldsymbol{b}) \big) && (45)\\
&= \boldsymbol{c} \cdot \big((\boldsymbol{d} \cdot \boldsymbol{b})\boldsymbol{a} - (\boldsymbol{d} \cdot \boldsymbol{a})\boldsymbol{b} \big) && (46)\\
&= (\boldsymbol{d} \cdot \boldsymbol{b})(\boldsymbol{c} \cdot \boldsymbol{a}) - (\boldsymbol{d} \cdot \boldsymbol{a})(\boldsymbol{c} \cdot \boldsymbol{b}) && (47)
\end{aligned}
$$

- Lagrange's identity:

$$\|\boldsymbol{a} \times \boldsymbol{b}\|^2 = \|\boldsymbol{a}\|^2 \|\boldsymbol{b}\|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2 \tag{48}$$

Derivation: Follows immediately from the Binet-Cauchy identity.

- Cross product length:

$$\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\| \|\boldsymbol{b}\| \sin\theta \tag{49}$$

Derivation: From Lagrange's identity:

$$\begin{aligned}
\|\boldsymbol{a} \times \boldsymbol{b}\|^2 &= \|\boldsymbol{a}\|^2\|\boldsymbol{b}\|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2 && (50)\\
&= \|\boldsymbol{a}\|^2\|\boldsymbol{b}\|^2 - (\|\boldsymbol{a}\|\|\boldsymbol{b}\| \cos\theta)^2 && (51)\\
&= \|\boldsymbol{a}\|^2\|\boldsymbol{b}\|^2(1 - \cos^2\theta) && (52)\\
&= \|\boldsymbol{a}\|^2\|\boldsymbol{b}\|^2 \sin^2\theta && (53)
\end{aligned}$$

- Jacobi's identity:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = 0 \tag{54}$$

Derivation: Using the vector triple product expansion:

$$\begin{aligned}
&\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) && (55)\\
&\quad = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} + (\boldsymbol{b} \cdot \boldsymbol{a})\boldsymbol{c} - (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a} + (\boldsymbol{c} \cdot \boldsymbol{b})\boldsymbol{a} - (\boldsymbol{c} \cdot \boldsymbol{a})\boldsymbol{b} && (56)\\
&\quad = 0 && (57)
\end{aligned}$$

- Quadruple vector product expansion:

$$(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{a} \times \boldsymbol{c}) = \big(\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})\big)\boldsymbol{a} \tag{58}$$

Derivation: Take $\boldsymbol{d} = (\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{a} \times \boldsymbol{c})$. Then $\boldsymbol{d}$ is in the $\boldsymbol{a}, \boldsymbol{b}$ plane and in the $\boldsymbol{a}, \boldsymbol{c}$ plane, so it is a scalar multiple of $\boldsymbol{a}$. We use the scalar triple product formula and the vector triple product expansion to compute:

$$\begin{aligned}
\boldsymbol{d} \cdot \boldsymbol{a} &= \boldsymbol{a} \cdot \big((\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{a} \times \boldsymbol{c})\big) && (59)\\
&= (\boldsymbol{a} \times \boldsymbol{c}) \cdot \big(\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{b})\big) && (60)\\
&= (\boldsymbol{a} \times \boldsymbol{c}) \cdot \big((\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{a})\boldsymbol{b}\big) && (61)\\
&= -(\boldsymbol{a} \cdot \boldsymbol{a})\boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{c}) && (62)\\
&= (\boldsymbol{a} \cdot \boldsymbol{a})\big(\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})\big) && (63)
\end{aligned}$$

Then

$$\boldsymbol{d} = \mathrm{Proj}(\boldsymbol{d}, \boldsymbol{a}) = \left(\frac{\boldsymbol{d} \cdot \boldsymbol{a}}{\boldsymbol{a} \cdot \boldsymbol{a}}\right)\boldsymbol{a} = \big(\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})\big)\boldsymbol{a} \tag{64}$$

3.2.5 Projection and Rejection

$$\begin{aligned}
\mathrm{Proj}(\boldsymbol{u}, \boldsymbol{v}) &= (\boldsymbol{u} \cdot \hat{v})\hat{v} = \left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{v^2}\right)\boldsymbol{v} = \left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}}\right)\boldsymbol{v} && (65)\\
\mathrm{Rej}(\boldsymbol{u}, \boldsymbol{v}) &= \boldsymbol{u} - \mathrm{Proj}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{u} - (\boldsymbol{u} \cdot \hat{v})\hat{v} = \boldsymbol{u} - \left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{v^2}\right)\boldsymbol{v} = \boldsymbol{u} - \left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}}\right)\boldsymbol{v} && (66)
\end{aligned}$$

3.2.6 Changing bases

Important point: Vector expressions are true no matter which basis we write the vectors in, even if they are written in different bases.

Example V.CB: $\boldsymbol{w} = \boldsymbol{u} + 2\boldsymbol{v}$, draw the image. This is true no matter which bases are being used for $\boldsymbol{u}$ and $\boldsymbol{v}$. For example, even if we use different bases: $\boldsymbol{u} = 3\,\hat{\imath} + 2\,\hat{\jmath}$ and $\boldsymbol{v} = 5\,\hat{a} - \hat{b}$, then $\boldsymbol{w} = 3\,\hat{\imath} + 2\,\hat{\jmath} + 10\,\hat{a} - 2\,\hat{b}$ is true. Of course, we would normally convert these to a single basis, but we don't have to.

To change the basis that a vector is written in, we need to know how the basis vectors are related. For example, if we have $\boldsymbol{v} = 3\,\hat{\imath} + 2\,\hat{\jmath}$ and we want to write this in the $\hat{a}$, $\hat{b}$ basis, then we need to know $\hat{\imath}$, $\hat{\jmath}$ in terms of $\hat{a}$, $\hat{b}$. For example,

$$\hat{\imath} = \frac{1}{\sqrt{2}}\,\hat{a} - \frac{1}{\sqrt{2}}\,\hat{b} \tag{67}$$

$$\hat{\jmath} = \frac{1}{\sqrt{2}}\,\hat{a} + \frac{1}{\sqrt{2}}\,\hat{b} \tag{68}$$

Then we can substitute and re-arrange:

$$\begin{aligned} \boldsymbol{v} &= 3\,\hat{\imath} + 2\,\hat{\jmath} && (69)\\ &= 3\left(\frac{1}{\sqrt{2}}\,\hat{a} - \frac{1}{\sqrt{2}}\,\hat{b}\right) + 2\left(\frac{1}{\sqrt{2}}\,\hat{a} + \frac{1}{\sqrt{2}}\,\hat{b}\right) && (70)\\ &= \left(\frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}}\right)\hat{a} + \left(-\frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}}\right)\hat{b} && (71)\\ &= \frac{5}{\sqrt{2}}\,\hat{a} - \frac{1}{\sqrt{2}}\,\hat{b} && (72) \end{aligned}$$

If we want to convert back the other way then we would need to know $\hat{a}$, $\hat{b}$ in terms of $\hat{\imath}$, $\hat{\jmath}$. We can find this by solving for $\hat{a}$, $\hat{b}$ above.

Example: $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v}$, same as above: $\boldsymbol{u} = 3\,\hat{\imath} + 2\,\hat{\jmath}$ and $\boldsymbol{v} = 5\,\hat{a} - \hat{b}$. Then $\boldsymbol{w} = 15\,\hat{\imath} \times \hat{a} - 3\,\hat{\imath} \times \hat{b} + 10\,\hat{\jmath} \times \hat{a} - 2\,\hat{\jmath} \times \hat{b}$. Work out what these are individually. Take $\hat{a}$, $\hat{b}$ rotated by $\pi/4$ from $\hat{\imath}$, $\hat{\jmath}$, so we can do the cross products by hand or something?

3.2.7 Time-dependent vectors and bases

We can have dynamic vectors which change over time, so their components also change. Alternatively, we can have a fixed vector but dynamic basis.

Example V.CV: Show a lengthening and rotating vector and how its components change over time in a fixed basis.

Example V.RB: Show a fixed vector, but a rotating basis, and how the components change over time.

3.2.8 Basis vectors from different coordinate systems

polar coordinates:

$$\hat{r} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \tag{73}$$

$$\hat{\theta} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} \tag{74}$$

3.3 Calculus and Vectors

Time-dependent vectors can be differentiated in exactly the same way that we differentiate scalar functions. For a time-dependent vector $\boldsymbol{v}(t)$, the derivative $\dot{\boldsymbol{v}}(t)$ is

$$\dot{\boldsymbol{v}}(t) = \frac{d}{dt}\boldsymbol{v}(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{v}(t + \Delta t) - \boldsymbol{v}(t)}{\Delta t} \tag{75}$$

Example V.DV: Vector derivatives. Show a vector $\boldsymbol{v}(t)$ with a fixed origin and the end tracing out a curved path (path is shown as a dotted line). Show $\boldsymbol{v}(t)$ animated. Then show $\boldsymbol{v}(t + \Delta t)$ and the construction for the derivative, resulting in the derivative vector.

Notation: We will use either the dot notation $\dot{\boldsymbol{v}}(t)$ or the full derivative notation $\frac{d\boldsymbol{v}(t)}{dt}$, depending on which is clearer and more convenient. We will often not write the time dependency explicitly, so we might write just $\dot{\boldsymbol{v}}$ or $\frac{d\boldsymbol{v}}{dt}$.

In a fixed basis we differentiate a vector by differentiating each component:

$$\boldsymbol{v}(t) = v_1(t)\,\hat{\imath} + v_2(t)\,\hat{\jmath} \tag{76}$$

$$\dot{\boldsymbol{v}}(t) = \dot{v}_1(t)\,\hat{\imath} + \dot{v}_2(t)\,\hat{\jmath} \tag{77}$$

Example: Differentiating vectors component-wise (fixed basis). Show the sample example as above, but now show the v_1 and v_2 components of $\boldsymbol{v}$. Turn on derivatives of each component and show how they sum to give the vector derivative.

Derivation:

$$\begin{aligned}
\boldsymbol{v}(t) &= v_1(t)\,\hat{\imath} + v_2(t)\,\hat{\jmath} && (78)\\
\dot{\boldsymbol{v}}(t) &= \lim_{\Delta t \to 0} \frac{\boldsymbol{v}(t + \Delta t) - \boldsymbol{v}(t)}{\Delta t} && (79)\\
&= \lim_{\Delta t \to 0} \frac{(v_1(t + \Delta t)\,\hat{\imath} + v_2(t + \Delta t)\,\hat{\jmath}) - (v_1(t)\,\hat{\imath} + v_2(t)\,\hat{\jmath})}{\Delta t} && (80)\\
&= \lim_{\Delta t \to 0} \frac{(v_1(t + \Delta t) - v_1(t))\,\hat{\imath} + (v_2(t + \Delta t) - v_2(t))\,\hat{\jmath}}{\Delta t} && (81)\\
&= \left(\lim_{\Delta t \to 0} \frac{v_1(t + \Delta t) - v_1(t)}{\Delta t}\right)\hat{\imath} + \left(\lim_{\Delta t \to 0} \frac{v_2(t + \Delta t) - v_2(t)}{\Delta t}\right)\hat{\jmath} && (82)\\
&= \dot{v}_1(t)\,\hat{\imath} + \dot{v}_2(t)\,\hat{\jmath} && (83)
\end{aligned}$$

We can also differentiate vector expressions, such as dot products and cross products. These act like multiplication, so the *product rule* applies:

$$\frac{d}{dt}(\boldsymbol{v} \cdot \boldsymbol{u}) = \dot{\boldsymbol{v}} \cdot \boldsymbol{u} + \boldsymbol{v} \cdot \dot{\boldsymbol{u}} \tag{84}$$

$$\frac{d}{dt}(\boldsymbol{v} \times \boldsymbol{u}) = \dot{\boldsymbol{v}} \times \boldsymbol{u} + \boldsymbol{v} \times \dot{\boldsymbol{u}} \tag{85}$$

Example: Differentiating complex vector expressions. What is $\frac{d}{dt}\big(\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{v} \times \boldsymbol{w})\big)$?

$$\begin{aligned}
\frac{d}{dt}\big(\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{v} \times \boldsymbol{w})\big) &= \dot{\boldsymbol{u}} \cdot (\boldsymbol{v} + \boldsymbol{v} \times \boldsymbol{w}) + \boldsymbol{u} \cdot \frac{d}{dt}(\boldsymbol{v} + \boldsymbol{v} \times \boldsymbol{w}) && (86)\\
&= \dot{\boldsymbol{u}} \cdot (\boldsymbol{v} + \boldsymbol{v} \times \boldsymbol{w}) + \boldsymbol{u} \cdot (\dot{\boldsymbol{v}} + \dot{\boldsymbol{v}} \times \boldsymbol{w} + \boldsymbol{v} \times \dot{\boldsymbol{w}}) && (87)
\end{aligned}$$

We can break a vector derivative down into the change in length and the change in direction:

$$\boldsymbol{v} = v\hat{v} \tag{88}$$

$$\dot{\boldsymbol{v}} = \dot{v}\hat{v} + v\dot{\hat{v}} \tag{89}$$

To compute the derivative of length and direction we can use:

$$\dot{v} = \dot{\boldsymbol{v}} \cdot \hat{v} \tag{90}$$

$$\dot{\hat{v}} = \frac{1}{v}\operatorname{Rej}(\dot{\boldsymbol{v}}, \hat{v}) \tag{91}$$

This means that

$$\dot{v}\hat{v} = (\dot{\boldsymbol{v}} \cdot \hat{v})\hat{v} = \operatorname{Proj}(\dot{\boldsymbol{v}}, \boldsymbol{v}) \tag{92}$$

$$v\dot{\hat{v}} = v\frac{1}{v}\operatorname{Rej}(\dot{\boldsymbol{v}}, \hat{v}) = \operatorname{Rej}(\dot{\boldsymbol{v}}, \hat{v}) \tag{93}$$

So the decomposition is

$$\dot{\boldsymbol{v}} = \underbrace{\dot{v}\hat{v}}_{\operatorname{Proj}(\dot{\boldsymbol{v}},\boldsymbol{v})} + \underbrace{v\dot{\hat{v}}}_{\operatorname{Rej}(\dot{\boldsymbol{v}},\hat{v})} \tag{94}$$

Derivation: For length:

$$v = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} \tag{95}$$

$$\frac{d}{dt}v = \frac{d}{dt}\big((\boldsymbol{v} \cdot \boldsymbol{v})^{1/2}\big) \tag{96}$$

$$\dot{v} = \frac{1}{2}(\boldsymbol{v} \cdot \boldsymbol{v})^{-1/2}(\dot{\boldsymbol{v}} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \dot{\boldsymbol{v}}) \tag{97}$$

$$= \frac{1}{2\sqrt{v^2}}(2\dot{\boldsymbol{v}} \cdot \boldsymbol{v}) \tag{98}$$

$$= \dot{\boldsymbol{v}} \cdot \hat{v} \tag{99}$$

Derivation: For direction:

$$\hat{v} = \frac{\boldsymbol{v}}{v} \tag{100}$$

$$\frac{d}{dt}\hat{v} = \frac{d}{dt}\left(\frac{\boldsymbol{v}}{v}\right) \tag{101}$$

$$\dot{\hat{v}} = \frac{\dot{\boldsymbol{v}}v - \boldsymbol{v}\dot{v}}{v^2} \tag{102}$$

$$= \frac{\dot{\boldsymbol{v}}}{v} - \frac{\dot{\boldsymbol{v}} \cdot \hat{v}}{v^2}\boldsymbol{v} \tag{103}$$

$$= \frac{1}{v}\big(\dot{\boldsymbol{v}} - (\dot{\boldsymbol{v}} \cdot \hat{v})\hat{v}\big) \tag{104}$$

$$= \frac{1}{v}\operatorname{Rej}(\dot{\boldsymbol{v}}, \hat{v}) \tag{105}$$

Example V.DD: Changing length and direction. Show the same vector derivative example as before, but decompose the derivative into $\hat{v}$ and $\dot{\hat{v}}$ components. Show the projections.

3.4 Rotations and Angular Velocity

A *rotation* of a vector is a change which only alters the direction, not the length, of a vector. A rotation consists of a *rotation axis* and a *rotation rate*. By taking the rotation axis as a direction and the rotation rate as a length, we can write the rotation as a vector, known as the *angular velocity vector* $\boldsymbol{\omega}$. We use the *right-hand-rule* to describe the direction of rotation.

Example: Show a cloud of vectors rotating in 3D around an angular velocity vector. Maybe have the angular velocity vector change in length, showing the rotation slowing down and speeding up. Maybe also have it change in direction, showing the rotation axis change.

In 2D the angular velocity can be thought of as a scalar (positive for counter-clockwise, negative for clockwise). This scalar is just the out-of-plane component of the full angular velocity vector. We can draw the angular velocity as either a vector pointing out of the plane, or as a circle-arrow in the plane, which is simpler for 2D diagrams.

Example: Show a set of 2D vectors rotating in the plane, with the angular velocity shown as an out-of-plane vector and an in-plane circle-arrow.

If a vector $\boldsymbol{v}$ is rotating with angular velocity vector $\boldsymbol{\omega}$, the rate of change of $\boldsymbol{v}$ is

$$\dot{\boldsymbol{v}} = \boldsymbol{\omega} \times \boldsymbol{v} \tag{106}$$

We will see below that this is correct.

3.4.1 Properties of rotations

- Vector derivatives due to rotation are orthogonal to the vector:

$$\dot{\boldsymbol{v}} \cdot \boldsymbol{v} = 0 \tag{107}$$

Derivation: Using the scalar triple product formula:

$$\boldsymbol{v} \cdot \dot{\boldsymbol{v}} = \boldsymbol{v} \cdot \big(\boldsymbol{\omega} \times \boldsymbol{v}\big) \qquad = \boldsymbol{\omega} \cdot \big(\boldsymbol{v} \times \boldsymbol{v}\big) = 0 \tag{108}$$

- If we have two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ both rotating with an angular velocity $\boldsymbol{\omega}$, then $\boldsymbol{u} \cdot \boldsymbol{v}$ is constant:

$$\frac{d}{dt}\big(\boldsymbol{u} \cdot \boldsymbol{v}\big) = 0 \tag{109}$$

Derivation: Using the scalar triple product formula:

$$\begin{aligned} \frac{d}{dt}\big(\boldsymbol{u} \cdot \boldsymbol{v}\big) &= \dot{\boldsymbol{u}} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \dot{\boldsymbol{v}} && (110) \\ &= (\boldsymbol{\omega} \times \boldsymbol{u}) \cdot \boldsymbol{v} + \boldsymbol{u} \cdot (\boldsymbol{\omega} \times \boldsymbol{v}) && (111) \\ &= \boldsymbol{v} \cdot (\boldsymbol{\omega} \times \boldsymbol{u}) + \boldsymbol{v} \cdot (\boldsymbol{u} \times \boldsymbol{\omega}) && (112) \\ &= \boldsymbol{v} \cdot (\boldsymbol{\omega} \times \boldsymbol{u}) - \boldsymbol{v} \cdot (\boldsymbol{\omega} \times \boldsymbol{u}) && (113) \\ &= 0 && (114) \end{aligned}$$

- Lengths of vectors are preserved by rotation:

$$v = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \text{constant} \tag{115}$$

Derivation: Use derivative of dot product formula.

- Angles between vectors are preserved by rotation:

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{uv}\right) = \text{constant} \tag{116}$$

  Derivation: Use derivative of dot product formula.

- If $\boldsymbol{v}$ is parallel to $\boldsymbol{\omega}$ then $\dot{\boldsymbol{v}} = 0$.

This shows that the rotation formula $\dot{\boldsymbol{v}} = \boldsymbol{\omega} \times \boldsymbol{v}$ does indeed perform a rigid rotation about $\boldsymbol{\omega}$.
The units of $\boldsymbol{\omega}$ are rad/s.

Example V.AF: Show the relationship between rad/s, Hz, and period T. See `https://en.wikipedia.org/wiki/Angular_frequency` for some nice examples.

$$\boldsymbol{\omega} = \frac{\dot{\boldsymbol{v}} \times \boldsymbol{v}}{v^2} = \frac{\dot{\boldsymbol{v}} \times \boldsymbol{v}}{\boldsymbol{v}\cdot\boldsymbol{v}} \tag{117}$$

$$\frac{d}{dt}\,\hat{v} = \boldsymbol{\omega} \times \,\hat{v} \tag{118}$$

$$\frac{d}{dt}\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{v} \tag{119}$$

$$\frac{d}{dt}\boldsymbol{v} = \dot{v}\,\hat{v} + \boldsymbol{\omega} \times \boldsymbol{v} \tag{120}$$

3.5 Position, Velocity, and Acceleration Vectors

3.5.1 Position vectors

Two points A and B can be used to define a vector $\boldsymbol{r}_{AB} = \overrightarrow{AB}$ from A to B. We call this the *relative position* of B from A. If we start from the origin O, so we have $\boldsymbol{r}_{OA} = \overrightarrow{OA}$, then we call this the *position vector* of position A. When it is clear, we will write $\boldsymbol{r}_P$ for this position vector, or sometimes even just $\boldsymbol{r}$.

Important point: The position vector $\boldsymbol{r}_{OP}$ of a point P depends on which origin we are using. Using a different origin will result in a different position vector for the same point.

Example V.PV: Show two different origins with the same $\hat{\imath}, \hat{\jmath}$ basis at both, and a point P, and show the position vector relative to each origin.

Important point: We can write any position vector in any basis.

Example V.PB: Mix and match bases. Show two origins, one with $\hat{\imath}, \hat{\jmath}$ basis and one with $\hat{a}, \hat{b}$ basis. Show a point P and the two position vectors. Show that either position vector can be written in terms of either basis.

3.5.2 Transformation of position vectors

The position vectors of a point from two different origins differ by the offset vector between the origins. That is:

$$\overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P} \tag{121}$$

$$\boldsymbol{r}_{OP} = \boldsymbol{r}_{OO'} + \boldsymbol{r}_{O'P} \tag{122}$$

Example V.OO: Origin offset formula. Show two origins and the two position vectors to a point P. Show that the relative offset relation holds.

Philosophical point: Coordinates are for positions, bases are for vectors? Coordinates can be non-linear, bases are always linear? Position vectors relate these two things? We use the term coordinates for positions and components for vectors.

Example V.PG: Pantograph. Consider a four-bar pantogram, as in `https://en.wikipedia.org/wiki/Pantograph`. Take O to be the fixed point, T to be the tracing point, and D to be the drawing point. Show that $\overrightarrow{OD} = \alpha \overrightarrow{OT}$ for some scalar α. Why does this mean that the pantograph works? What determines α.

3.6 Linear and Circular Motion

Constant acceleration:

$$v = v_0 + a_0 t \tag{123}$$

$$x = x_0 + v_0 t + a_0 t^2 \tag{124}$$

$$v = \omega r \tag{125}$$

$$\boldsymbol{a} = \alpha\,\hat{\theta} - r\omega^2\,\hat{r} \tag{126}$$

3.7 Tangential/Normal Coordinates

3.8 Cylindrical coordinates

Coordinates:

$$(r, \theta, z) \tag{127}$$

To Cartesian:

$$x = r\cos\theta \tag{128}$$

$$y = r\sin\theta \tag{129}$$

$$z = z \tag{130}$$

From Cartesian:

$$r = \sqrt{x^2 + y^2} \tag{131}$$

$$\theta = \operatorname{atan2}(y, x) \tag{132}$$

$$z = z \tag{133}$$

Basis vectors:

$$\hat{r} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \tag{134}$$

$$\hat{\theta} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} \tag{135}$$

$$\hat{z} = \hat{k} \tag{136}$$

Angular velocity:

$$\boldsymbol{\omega} = \dot{\theta}\,\hat{k} \tag{137}$$

Basis vector derivatives:

$$\dot{\hat{r}} = \boldsymbol{\omega} \times \hat{r} = \dot{\theta}\,\hat{\theta} \tag{138}$$
$$\dot{\hat{\theta}} = \boldsymbol{\omega} \times \hat{\theta} = -\dot{\theta}\,\hat{r} \tag{139}$$
$$\dot{\hat{k}} = \boldsymbol{\omega} \times \hat{k} = 0 \tag{140}$$

Position, velocity, and acceleration:

$$\boldsymbol{r} = r\,\hat{r} + z\,\hat{k} \tag{141}$$
$$\boldsymbol{v} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} + \dot{z}\,\hat{k} \tag{142}$$
$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\theta} + \ddot{z}\,\hat{k} \tag{143}$$

3.9 Spherical coordinates

Coordinates:

$$(r, \theta, \phi) \tag{144}$$

To Cartesian:

$$x = r\cos\theta\cos\phi \tag{145}$$
$$y = r\sin\theta\cos\phi \tag{146}$$
$$z = r\sin\phi \tag{147}$$

From Cartesian:

$$r = \sqrt{x^2 + y^2 + z^2} \tag{148}$$
$$\theta = \operatorname{atan2}(y, x) \tag{149}$$
$$\phi = \operatorname{atan2}(z, r) \tag{150}$$

Basis vectors:

$$\hat{r} = \cos\theta\cos\phi\,\hat{\imath} + \sin\theta\cos\phi\,\hat{\jmath} + \sin\phi\,\hat{k} \tag{151}$$
$$\hat{\theta} = -\sin\theta\cos\phi\,\hat{\imath} + \cos\theta\cos\phi\,\hat{\jmath} \tag{152}$$
$$\hat{\phi} = -\cos\theta\sin\phi\,\hat{\imath} - \sin\theta\sin\phi\,\hat{\jmath} + \cos\phi\,\hat{k} \tag{153}$$

Angular velocity:

$$\boldsymbol{\omega} = \dot{\theta}\,\hat{k} = \dot{\theta}(\sin\phi\,\hat{r} + \cos\phi\,\hat{\phi}) \tag{154}$$

Basis vector derivatives:

$$\dot{\hat{r}} = \boldsymbol{\omega} \times \hat{r} = \dot{\theta}\cos\phi\,\hat{\theta} + \dot{\phi}\,\hat{\phi} \tag{155}$$
$$\dot{\hat{\theta}} = \boldsymbol{\omega} \times \hat{\theta} = -\dot{\theta}\cos\phi\,\hat{r} + \dot{\theta}\sin\phi\,\hat{\phi} \tag{156}$$
$$\dot{\hat{\phi}} = \boldsymbol{\omega} \times \hat{\phi} = -\dot{\phi}\,\hat{r} - \dot{\theta}\sin\theta\,\hat{\theta} \tag{157}$$

Position, velocity, and acceleration:

$$\boldsymbol{r} = r\,\hat{r} \tag{158}$$
$$\boldsymbol{v} = \dot{r}\,\hat{r} + r\dot{\theta}\cos\phi\,\hat{\theta} + r\dot{\phi}\,\hat{\phi} \tag{159}$$
$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2\cos^2\phi - r\dot{\phi}^2)\,\hat{r} \tag{160}$$
$$+ (r\ddot{\theta}\cos\phi + 2\dot{r}\dot{\theta}\cos\phi - 2r\dot{\theta}\dot{\phi}\sin\phi)\,\hat{\theta} \tag{161}$$
$$+ (r\ddot{\phi} + 2\dot{r}\dot{\phi} + r\dot{\theta}^2\sin\phi\cos\phi)\,\hat{\phi} \tag{162}$$

4 Non-inertial Coordinate Frames

A frame is an origin with an (ordered) orthonormal basis (we might want to say that the origin and basis can be time-dependent, but "time-dependent" doesn't mean anything here, because who is to say which frame is the fixed frame?).

This defines a Galilean (rigid affine — orthonormal rotation plus translation) transform between any two frames by mapping their origins and basis vectors to each other (these Galilean transforms can be time-dependent — this is the right way to think about where time-dependency enters).

We can speak of vectors, basis vectors, etc, without worrying about which frame we are in, because all observers agree on positions, lengths, and directions (this is not relativity).

Time derivatives of scalars are defined without reference to frames (because all frames in Newtonian mechanics use the same time clock).

Time derivatives of vectors are defined with respect to a frame, because different frames disagree about which vectors are constant and which are changing.

The time derivative of a vector with respect to a frame is defined by assuming that the basis vectors of the frame are stationary for this time derivative (we do not need to worry about whether the origin is stationary, because vectors don't care about position anyway, just length and direction).

The vector derivative definition is that if $\{\hat{\imath}, \hat{\jmath}\}$ is the frame basis and $\boldsymbol{v} = v_1\hat{\imath} + v_2\hat{\jmath}$, then $\dot{\boldsymbol{v}} = \dot{v}_1\hat{\imath} + \dot{v}_2\hat{\jmath}$.

All frames agree on which physical forces act on a given mass.

An inertial frame is one in which $F = ma$ with only physical forces F.

A non-inertial frame is one in which $F \neq ma$ if only physical forces F are considered. We can make $F = ma$ by adding fictitious forces (centrifugal, etc).

Theorem: Any two inertial frames differ by a uniform translational motion.

Discussion: A weird thing here is that although everyone agrees about "positions" at a single instant of time, we don't agree about which positions are the same at different times. That is, if we have position A at time 1 and position B at time 2, you might think they are the same position at two different times, while I might think they are different positions. There is fundamentally no way to resolve this, as there is no preferred origin in the universe. Given two vectors V and W at times 1 and 2, however, we can use a method to always agree on whether V and W are equal or not, by agreeing to compare them against a non-rotating basis. We might choose different non-rotating bases, but our bases will not rotate with respect to each other and so we will agree about whether V and W are equal or not. We will also agree about whether their lengths are equal (no length-dilation effects in Newtonian mechanics).

For TAM212, I think we should only define "inertial vector derivatives" at the start of the semester, which are calculated with any non-rotating basis. Discussion of rotating frames and fictitious forces should be left until late in the semester.