University Physics, 6th ed (1982)
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ISBN 0-201-07195-9

Chapter 1
**Questions**
**1-1** What are the units of the number $\pi$?

The number $\pi$ is dimensionless. It is the ratio of the circumference of a circle to its diameter, so the units cancel.

**1-2** The rate of climb of a mountain trail was described as 150 meters per kilometer. How can this be expressed as a number with no units?

We can write this as a slope, equal to the rise over run. The units would cancel and a dimensionless quantity would remain:

$$rate = \frac{rise}{run} = \frac{150\,m}{1000\,m} = \frac{150}{1000} = 0.150$$

**1-3** Suppose you are asked to compute the cosine of 3 meters. Is this possible?

The value you pass to the cosine function would have to be dimensionless, otherwise values that are logically equivalent (3 meters is about equal to 3 yards, for example) would yield different results.

For this question to make sense, I think the value would have to refer to a length of arc. We need radians to compute the cosine, so we would need to know the radius of the corresponding circle. Dividing the arc length by the radius gives us a proper dimensionless quantity.

Suppose the radius of the circle is $4\,m$, and $3\,m$ represents a portion of its circumference. That gives us:

$$\theta = \frac{s}{r} = \frac{3}{4} = 0.75\,radians$$

$$\cos\theta = \cos(0.75) \approx 0.73$$

Otherwise, if $3\,m$ is just a generic length, I don't know that it would be meaningful to compute its cosine.

**1-4** Hydrologists describe the rate of volume flow of rivers in "second-feet". Is this unit technically correct? If not, what would be a correct unit?

The units of flow rate would be $ft^3/sec$, so "second-foot" is just a convenient short-hand. It's telling you what is the unit of volume (1 cubic foot) and the unit of time (1 second). If you were to tack on another time, this would indicate a volume of water. For example, a "second-foot day" is the volume of water that flows at a rate of 1 cubic feet per second for 24 hours. This is analogous to a "kilowatt-hour", which is the amount of energy that flows at a rate of 1000 Joules per second for 1 hour.

**1-5** Does a vector having zero length have a direction?

The answer is probably no, because the direction would be ambiguous. A displacement of 0 meters in one direction is no different from a vector having 0 displacement in any other direction. Indeed, we often write a vector whose magnitude is 0 as a scalar, without attaching any unit vectors.

However, suppose we have an object with a velocity $\vec{v} = v\,\hat{i}$, which is decelerating (due to friction, say), so that after some time its velocity becomes zero. I wouldn't argue that it's wrong to say "its velocity in the x-direction is now 0". Context counts for something.

**1-6** What is your weight in newtons?

We still use Ye Olde English units here in the US, which means I'd normally say my weight is 175 lbs. But what do we mean by this?

Weight is the force due to gravity. When someone says "I weigh 175 lbs" they mean *pounds of force*, abbreviated $lbf$. But force refers to the acceleration of a mass. I know the acceleration due to gravity: it's $32\,ft/s^2$. So what is my mass?

If something weighs $1\,lbf$ here on earth, where the acceleration is $32\,ft/s$, then how much mass is that? We define $1\,lbm$ as the mass that weighs $1\,lbf$ on the surface of the earth. So if I weigh $175\,lbf$, then my mass is $175\,lbm$.

But suppose I want to convert between English units and standard units. What is my mass then? By definition, $1\,lbm$ is equal to $0.454\,kg$. So if I weigh $175\,lbf$, then my mass in English units is $175\,lbm$, and so my mass in standard units is $175\,lbm \cdot 0.454\frac{kg}{lbm} = 79.5\,kg$.

Now we already know that the acceleration due to gravity is $9.8\,m/s^2$, so my weight in standard units would be $79.5 \cdot 9.8 = 779.5\,N$.

**1-7** What is your height in centimeters?

Oh this one's easy. My height in English units is $6\,ft$. There are 12 inches to a foot, and 2.54 centimeters to an inch, so that's $6 \cdot 12 \cdot 2.54 \approx 183\,cm$.

**1-8** What physical phenomena (other than a pendulum or cesium clock) could be used to define a time standard?

You would need something periodic, such as a pendulum. For very course time, you could use the phases of the moon, or position of stars in the sky.

**1-9** Could some atomic quantity be used for a definition of a unit of mass? What advantages or disadvantages would this have compared to the one-kilogram platinum cylinder kept at Sèvres?

This seems like it would be too small to be practical. And would atomic interactions change the mass?

**1-10** How could you measure the thickness of a sheet of paper with an ordinary ruler?

I would stack a set of sheets together, such as a ream (500 sheets), measure the height of the stack, and then divide by the number of sheets.

**1-11** Can two vectors having different lengths have a vector sum of zero? What length restrictions are required for three vectors to have a vector sum of zero?

In the case of two vectors, the lengths would have to be the same, to cancel (one vector would be the negative of the other). For three vectors, they would have to form the sides of a triangle.

**1-12** What is the displacement when a car travels from the north side of a circular race track of radius $500\,m$ to the south side? When it makes one complete circle around the track?

The distance from the top of the track to the bottom equals the diameter, which is $1000\,m$. Displacement is a vector quantity, so you'd write $-1000\,\hat{j}$ or $1000\,(-\hat{j})$. When you have made one complete circle, then you're back where you started, so the displacement would be 0.

**1-13** What are the units of volume? If a student tells you a cylinder of radius $r$ and height $h$ has volume given by $\pi r^3 h$, explain why this cannot be right.

The units of volume are $length^3$. The formula given has dimension $length^4$, which would be incorrect (not a volume, anyway).

**1-14** An angle (measured in radians) is a number with no units, since it is a ratio of two lengths. Think of other geometrical or physical quantities that are unitless.

Exponents are unitless. For example, the charge on a capacitor in an electric circuit would be something like $Q(t) = Q_0(1 - e^{-t/\tau})$, where $\tau$ is a time constant equal to $RC$. The ratio $-t/\tau$ is unitless.

**1-15** Can a vector quantity ever have components different from zero but a magnitude of zero?

I don't think so, no, because the magnitude is the square root of the sum of the squares of the component values. The sum of squares is never zero, so the magnitude would have to be positive.

**1-16** Once sometimes speaks of the "direction of time," evolving from past to future. Does this mean that time is a vector quantity?

No, time is not a vector quantity. Direction here doesn't refer to some spatial dimension.

**1-17** Is the scalar product of two vectors commutative? Explain.

Yes, the scalar product is commutative. It is defined as $A \cdot B = |A||B|\cos(\theta)$, so the order doesn't matter here, because you're just multiplying magnitudes. (To be precise: you're multiplying the magnitude of the projection of one vector onto another, by the magnitude of the other vector.)

**1-18** What is the scalar product of a vector with itself? The vector product?

The scalar product of a vector with itself equals the square of the magnitude of the vector. The vector product of a vector with itself would be 0, because there is no perpendicular component when the vectors are parallel.

**Problems**

**1-1** Starting with the definition of $1\,in. = 2.54\,cm$, compute the number of kilometers in one mile, to five significant figures.

$1 mile \cdot \frac{5280\,ft}{mile} \cdot \frac{12in.}{foot} \cdot \frac{2.54\,cm}{in.} \cdot \frac{1\,m}{100\,cm} \cdot \frac{1\,km}{1000\,m} = 1.609344\,km.$

To five significant figures, that's $1.6093\,km$. See problem 1-11.

**1-2** The density of water is $1\,g \cdot cm^{-3}$. What is this value in kilograms per cubic meter?

$\frac{1\,g}{cm^3} \cdot \frac{1\,kg}{1000\,g} \cdot \left(\frac{100\,cm}{m}\right)^3 = \frac{1000\,kg}{m^3}$

**1-3** Convert the following speeds, as indicated.

a) $60\,mi \cdot hr^{-1}$ to feet per second;

b) $100\,km \cdot hr^{-1}$ to meters per second.

For part (a), $\frac{60\,mi}{hr} \cdot \frac{5280\,ft}{mi} \cdot \frac{1\,hr}{3600\,sec} = \frac{88\,ft}{sec}.$

**1-4** What is the mass in kilograms of a person weighing $170\,lb$ ?

$170\,lbm \cdot \frac{0.454\,kg}{lbm} = 77\,kg$

**1-5** Compute the number of seconds in a day, and in a year (365 da).

$1\,day \cdot \frac{24\,hr}{day} \cdot \frac{3600\,sec}{hr} = 86,400\,sec$

$1\,year \cdot \frac{365\,da}{yr} \cdot \frac{24\,hr}{day} \cdot \frac{3600\,sec}{hr} = 31,536,000\,sec$

**1-6** What is the percent error in each of the following approximations to $\pi$ ?

a) 22/7

b) 355/113

For part (a), $\frac{22/7 - \pi}{\pi} \cdot 100 = 0.04\%$

For part (b), $\frac{355/133-\pi}{\pi} \cdot 100 = 8.5 \times 10^{-6}\%$

**1-7** What is the percent error in the approximate statement $1\,yr = \pi \times 10^7\,s$?

$\frac{\pi \times 10^7 - 31,536,000}{31,536,000} \cdot 100 = 0.38\%$

**1-8** Estimate the percent error in measuring

a) a distance of about $50\,cm$ with a meter stick;

b) a mass of about $1\,g$ with a chemical balance;

c) a time interval of about $4\,min$ with a stopwatch.

For (a), if we assume that the meter stick has mm markings, then we can probably measure to about $0.5\,mm$; that gives us $\frac{0.05}{50} \cdot 100 = 0.1\%$

For (b), if we assume the chemical balance has a measuring precision of $1\,mg$, that gives us $\frac{0.001}{1} \cdot 100 = 0.1\%$

For (c), if we assume a stopwatch that can measure time to 1/10th of a second, that's $\frac{0.1}{240} \cdot 100 = 0.04\%$. The answer key says the value is 0.05%, but for that to be true, the measuring precision would be about $0.12\,s$, which would be odd. A more likely possibility is that they used $200\,s$ as the value of the denominator by mistake, which gives you 0.05% exactly.

**1-9** The mass of the earth is $5.98 \times 10^{24}\,kg$, and its radius is $6.38 \times 10^6\,m$. Compute the density of the earth, using powers-of-ten notation and the correct number of significant figures.

The density is $\frac{mass}{volume}$. The volume of a sphere is $\frac{4\pi r^3}{3}$. Plugging in the numbers that's $\frac{5.98 \times 10^{24}}{\frac{4\pi(6.38 \times 10^6)^3}{3}} = 5.50 \times 10^3\,kg/m^3$.

**1-10** The piston displacement of a certain automobile engine is given as 2.0 liters. Using only the facts that 1 liter $= 1000\,cm^3$ and $1\,in. = 2.54\,cm$, express this volume in cubic inches.

$2\,liters \cdot \frac{1000\,cm^3}{1\,liter} \cdot \left(\frac{1\,in}{2.54\,cm}\right)^3 = 1.2 \times 10^2\,in^3$

**1-11** Using the definition $1\,in. = 2.54\,cm$, compute the number of kilometers in one mile; comment on the precision of your result.

$1\,mile \cdot \frac{5280\,ft}{mi} \cdot \frac{12\,in}{ft} \cdot \frac{2.54\,cm}{in} \cdot \frac{1\,m}{100\,cm} \cdot \frac{1\,km}{1000\,m} = 1.609344\,km$. This is a definition, not a measure, so we can use exact precision here. See problem 1-1.

**1-12** An angle is given, to one significant figure, as $5°$, meaning that its value is between $4.5°$ and $5.5°$. Find the corresponding range of possible values of the cosine of the angle. Is this a case where there are more significant figures in the result than in the input data?

The values are: $\cos(4.5°) = 0.9969, \cos(5.0°) = 0.9962, \cos(5.5°) = 0.9954$; in order to distinguish the cosine values, we must carry 4 significant figures in the result, which has more significant figures than the input.

**1-13** Two points $P_1$ and $P_2$ are described by their x- and y-coordinates, $(x_1, y_1)$ and $(x_2, y_2)$, respectively. Show that the components of the displacement **A** from $P_1$ to $P_2$ are $A_x = x_2 - x_1$ and $A_y = y_2 - y_1$. Also derive expressions for the magnitude and direction of this displacement.

Let $\vec{R}_1$ be the displacement vector for $P_1$ and $\vec{R}_2$ be the displacement vector for $P_2$. Vector **A** is the difference between these vectors, $\vec{A} = \vec{R}_2 - \vec{R}_1$. That gives us components $A_x = R_{2,x} - R_{1,x} = x_2 - x_1$, and $A_y = R_{2,y} - R_{1,y} = y_2 - y_1$.

The magnitude is just the distance between two points in a plane, which is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. The direction is $\arctan\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$.

In Mathematica, you can use Norm[] or EuclideanDistance[] to find the magnitude, and ArcTan[] to find the direction. (Vectors are represented as lists.)

**1-14** When two vectors **A** and **B** are drawn from a common point, the angle between them is $\theta$. Show that the magnitude of their vector sum is $\sqrt{A^2 + B^2 + 2AB\cos\theta}$.

Consider the triangle formed from vectors **A** and **B**. Let C be the length of the third side, opposite $\theta$ (though C is not really relevant for this problem). Also, let $\alpha$ be the angle opposite side A and $\beta$ be the angle opposite side B. The sum of the interior angles of a triangle is $180°$, so we have $\alpha + \beta + \theta = 180°$.

If we arrange vectors **A** and **B** head-to-tail, we get another triangle; call the length of its third side D (this is the value the problem is looking for). In this case, the angle opposite D is $\alpha + \beta$. The Law of Cosines for this triangle is $D^2 = A^2 + B^2 - 2AB\cos(\alpha + \beta)$. We know that $\alpha + \beta = 180° - \theta$, so we can write this as $D^2 = A^2 + B^2 - 2AB\cos(180° - \theta)$.

The cosine of a difference, $\cos(x - y)$, is $\cos x \cos y + \sin x \sin y$, so the cosine term above simplifies to $\cos(180° - \theta) = \cos 180° \cos\theta + \sin 180° \sin\theta = -\cos\theta$. Our cosine law becomes $D^2 = A^2 + B^2 - 2AB(-\cos\theta) = A^2 + B^2 + 2AB(\cos\theta)$. Taking the square root of both sides gives us: $D = \sqrt{A^2 + B^2 + 2AB\cos\theta}$.

**1-15** Find the magnitude and direction of the vector represented by each of the following pairs of components:

a) $A_x = 3\,cm$, $A_y = -4\,cm$;
b) $A_x = -5\,m$, $A_y = -12\,m$;
c) $A_x = -2\,km$, $A_y = 3\,km$.

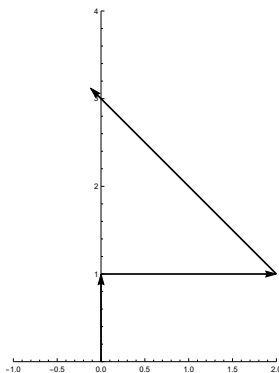For (a), $|\mathbf{A}| = \sqrt{3^2 + (-4)^2} = 5\,cm$, $\theta = \arctan(3, -4) = -53.1°$.

For (b), $|\mathbf{A}| = \sqrt{(-5)^2 + (-12)^2} = 13\,m$, $\theta = \arctan(-5, -12) = -112.6°$.

For (c), $|\mathbf{A}| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13} = 3.61$, $\theta = \arctan(-2, 3) = 123.7°$.

**1-16** A delivery truck drives 1 mile north, then 2 miles east, then 3 miles northwest. Determine the resultant displacement

a) by drawing a scale diagram;
b) by using components.

Here's the plot for part (a):



From the plot we can see that the magnitude of the resultant vector is a little longer than $3\,mi$, with a direction a little greater than $90°$.

We can compute the exact values for magnitude and direction by determining the components of the resultant vector, as suggested for part (b). The component in particular direction is the sum of the displacements in that direction for each leg of path.

For the x-direction, that's $0 + 2 + 3 \cdot \cos 135° = 2 - \frac{\sqrt{3}}{2} \approx -0.12$. For the y-direction, that's $1 + 0 + 3 \cdot \sin 135° = 1 + \frac{\sqrt{3}}{2} \approx 3.12$. The magnitude is $\sqrt{(2 - \frac{\sqrt{3}}{2})^2 + (1 + \frac{\sqrt{3}}{2})^2} \approx 3.12\,mi$, while the direction is $\arctan\left(2 - \frac{\sqrt{3}}{2}, 1 + \frac{\sqrt{3}}{2}\right) \approx 92.23°$.

The answer key gives $2.20°$ W of N (same as $92.20°$) as the answer. You can account for the difference by using the approximate values for the x and y components when computing the arctan.

**1-17** A bug starts at the center of a 12-inch phonograph record and crawls along a straight radial line to the edge. While this is happening the record turns through an angle of $45°$. Draw a sketch of the situation and describe the magnitude and direction of the bug's displacement.

In the time that the bug travels a radial distance of $6\,in$, the phonograph record has rotated $45°$, so we can express the radial distance away from the center as a function of $\theta$: $r(\theta) = \frac{6}{45°}\theta$. This allows us to make a polar plot as follows:
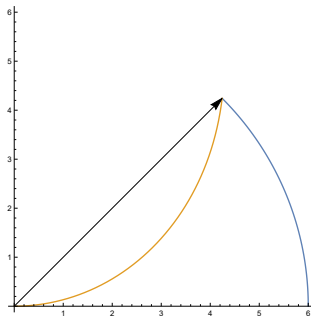


The final displacement of the bug has a magnitude of $6\,in$ and a direction of $45°$.

**1-18** A cave explorer is surveying a cave. He follows a passage $100\,m$ straight east, then $50\,m$ in a direction $30°$ west of north, then $150\,m$ at $45°$ west of south. After a fourth unmeasured displacement, he finds himself back where he started. Using a scale drawing, determine the fourth displacement (magnitude and direction).

The path of the cave explorer is as follows:
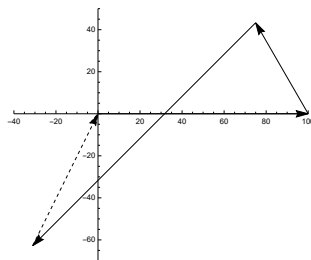


We can determine the total displacement after the 3rd leg of the path by computing the total displacement in each direction. The total x-displacement is $100 + 50 \cos 120° + 150 \cos 225° = 75 - 75\sqrt{2} \approx -31.1$, and the total y-displacment is $0 + 50 \sin 120° + 150 \sin 225° = -75\sqrt{2} + 25\sqrt{3} \approx -62.8$.

It doesn't matter from where we started, since we are told that we return to that same point. For the 4th leg to undo the displacements done by the first 3 legs, we take the negative of cumulative displacement, so the x-component of the 4th leg is 31.1 and its y-component is 62.8. In vector form that's $\vec{R}_4 = 31.1\hat{i} + 62.8\hat{j}$.

The magnitude is $\sqrt{31.1^2 + 62.8^2} = 70.0\,m$, with direction $\arctan(31.1, 62.8) = 63.7°$ (counterclockwise from the horizontal, the same as $90 - 63.7 = 26.3°$ E of N).

**1-19** Find graphically the magnitude and direction of the vector sum of the three forces in Fig. 1-14 (of the book). Use the polygon method. Check the precision of your result by using the component method.

Let's first arrange the displacement vectors head-to-tail, as follows:
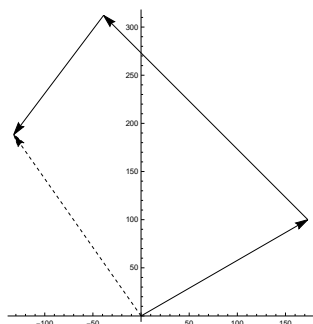


As usual, to determine the components of the resultant vector, we sum the components of the vectors. For the x-component, that's $200\cos 30° + 300\cos 135° + 155\cos 233° = -132.2$, and for the y-component, that's $200\sin 30° + 300\sin 135° + 155\sin 233° = 188.3$.
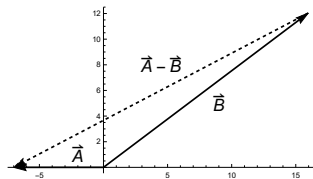
The magnitude is $\sqrt{(-132.2)^2 + 188.3^2} = 230.1\,N$, with direction $\arctan(-132.2, 188.3) = 125.1°$ (same as $180 - 125.1 = 54.9°$ measured clockwise from negative horizonal).

**1-20** Find graphically the vector sum **A + B** and the vector difference **A - B** in Fig. 1-15 (of the book).

For the case of a sum of vectors, we arrange the vectors head-to-tail:
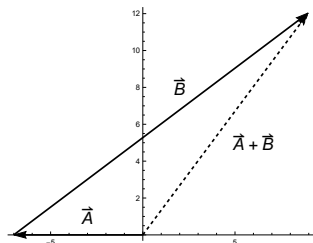


For the difference of vectors, we arrange the vectors tail-to-tail:

**1-21** Find the vector sum **A** + **B** and the vector difference **A** - **B** in Fig. 1-15 (of the book), using components.

The components of **A** + **B** are $-7 + 20\cos 37° = 9.0$ (in the x-direction) and $0 + 20\sin 37° = 12.0$ (in the y-direction). The magnitude is $\sqrt{9^2 + 12^2} = 15.0\,N$, and the direction is $\arctan(9, 12) = 53.3°$.

The components of **A** - **B** are $-7 - 20\cos 37° = -23.0$ and $0 - 20\sin 37° = -12.0$. The magnitude of the resultant is $\sqrt{(-23)^2 + (-12)^2} = 25.9\,N$, with direction $\arctan(-23, -12) = -152.3°$ (same as $180 - 152.3 = 27.3°$ below the –x-axis).
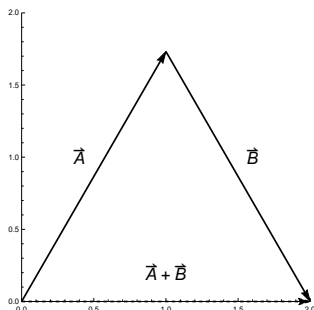
**1-22** Vector **A** is 2 in. long and is $60°$ above the x-axis in the first quadrant. Vector **B** is 2 in. long and is $60°$ below the x-axis in the fourth quadrant. Find graphically

a) the vector sum **A** + **B**, and
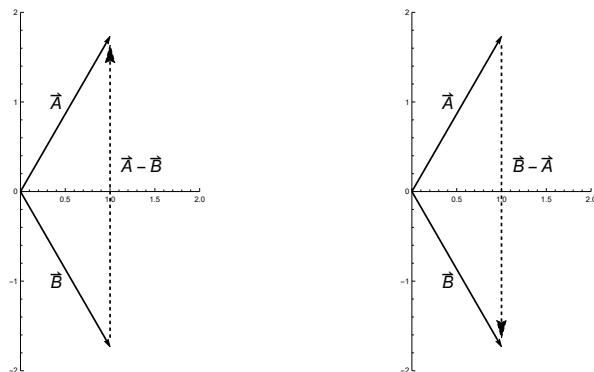
b) the vector differences **A** - **B** and **B** - **A**.

Here's the graph of vectors **A** and **B**:
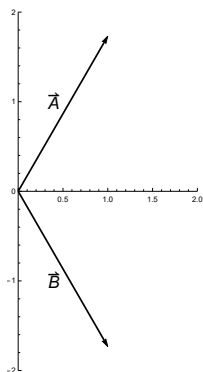


For part (a), we arrange the vectors head-to-tail. The resultant vector is:

For part (b), we arrange the vectors tail-to-tail. The resultant vector connects the heads:



**1-23** Obtain the vector sum and differences requested in Problem 1-22, using the method of components.

For the vector sum $\mathbf{A} + \mathbf{B}$, the x-component is $2\cos 60° + 2\cos(-60°) = 1+1 = 2$, and the y-component is $2\sin 60° + 2\sin(-60°) = \sqrt{3} - \sqrt{3} = 0$. The magnitude of the resultant is $\sqrt{2^2 + 0^2} = 2\,in$, with direction $\arctan(2,0) = 0°$.

For the vector difference $\mathbf{A}$ - $\mathbf{B}$, the x-component is $2\cos 60° - 2cos(-60°) = 1 - 1 = 0$, and the y-component is $2\sin 60° - 2\sin(-60°) = \sqrt{3} - (-\sqrt{3}) = 2\sqrt{3}$. The magnitude is $\sqrt{0^2 + (2\sqrt{3})^2} = 2\sqrt{3}\,in$, with direction $\arctan(0, 2\sqrt{3}) = 90°$.

For the vector difference $\mathbf{B}$ - $\mathbf{A}$, the x-component is $2\cos(-60°) - 2cos60° = 1-1 = 0$, and the y-component is $2\sin(-60°) - 2\sin 60° = -\sqrt{3} - \sqrt{3} = -2\sqrt{3}$. The magnitude is $\sqrt{0^2 + (-2\sqrt{3})^2} = 2\sqrt{3}\,in$, with direction $\arctan(0, -2\sqrt{3}) = -90°$.

**1-24** Vector $\mathbf{A}$ has components $A_x = 2$ cm, $A_y = 3$ cm, and vector $\mathbf{B}$ has components $B_x = 4$ cm, $B_y = -2$ cm. Find

a) the components of the vector sum $\mathbf{A} + \mathbf{B}$;
b) the magnitude and direction of $\mathbf{A} + \mathbf{B}$;
c) the components of the vector difference $\mathbf{A}$ - $\mathbf{B}$;
d) the magnitude and direction of $\mathbf{A}$ - $\mathbf{B}$.

For (a), the x-component is $2 + 4 = 6$, and the y-component is $3 + (-2) = 1$.

For (b), the magnitude is $\sqrt{6^2 + 1^2} = \sqrt{37} \approx 6.1\,cm$, and the direction is $\arctan(6, 1) = 9.5°$.

For (c), the x-component is $2 - 4 = -2$, and the y-component is $3 - (-2) = 5$.

For (d), the magnitude is $\sqrt{(-2)^2 + 5^2} = \sqrt{29} \approx 5.4\,cm$, and the direction is $\arctan(-2, 5) = 111.8°$.

**1-25** A car drives 5 mi east, then 4 mi south, then 2 mi west. Find the magnitude and direction of the resultant displacement.

The x-component is $5 + 0 - 2 = 3$, and the y-component is $0 - 4 + 0 = -4$. The magnitude is $\sqrt{3^2 + (-4)^2} = 5\,mi$, and the direction is $\arctan(3, -4) = -53.1°$ (same as $53.1°$ south of east).

**1-26** A sailboat sails 2 km east, then 4 km southeast, then an additional distance in an unknown direction. Its final position is 5 km directly east of the starting point. Find the magnitude and direction of the third leg of the journey.

We are told the final displacement of the sailboat. The final displacement is the sum of the displacements from each leg, so we can work backwards to find the displacement due to the third leg.

The displacement in the x-direction is $2 + 4\cos(-45°) + x = 5$; solving for x yields the value $3 - 2\sqrt{2} \approx 0.172$.

The displacement in the y-direction is $0 + 4\sin(-45°) + y = 0$; solving for y yields the value $2\sqrt{2} \approx 2.828$.

We have each component, so we can find the magnitude and direction in the normal way. The magnitude is $\sqrt{(3 - 2\sqrt{2})^2 + (2\sqrt{2})^2} = 2.83\,\text{km}$, having a direction $\arctan(3 - 2\sqrt{2}, 2\sqrt{2}) = 86.5°$ (same as $90 - 86.5 = 3.5°$ east of north).

The answer booklet gives $3.4°$ east of north as the answer, which is the result obtained if you compute the arctan using component values having fewer than 3 digits of precision.

**1-27** Vector **M**, of magnitude $5\,\text{cm}$, is at $36.9°$ counterclockwise from the $+$x-axis. It is added to vector **N**, and the resultant is a vector of magnitude $5\,\text{cm}$, at $53.1°$ counterclockwise from the $+$x-axis. Find

a) the components of **N**;

b) the magnitude and direction of **N**.

We are told that $\mathbf{R} = \mathbf{M} + \mathbf{N}$, so **N** is just the vector difference $\mathbf{R}$ - $\mathbf{M}$. The x-component of **N** is $5\cos 53.1° - 5\cos 36.9° = $ -1 cm, and the y-component is $5\sin 53.1° - 5\sin 36.9° = 1\,\text{cm}$.
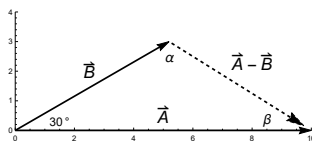
The magnitude is $\sqrt{(-1)^2 + 1^2} = 1.41\,\text{cm}$, with direction $\arctan(-1, 1) = 135°$.

The exact angles are $2\arctan\frac{1}{2}$ (for $53.1°$) and $\frac{\pi}{2} - 2\arctan\frac{1}{2}$ (for $36.9°$).

**1-28** A vector **A** of length 10 units makes an angle of $30°$ with a vector **B** of length 6 units. Find the magnitude of the vector difference **A** - **B** and the angle it makes with vector **A**:

a) by the triangle method;

b) by the method of rectangular resolution.

Here's a diagram of the vectors:



The problem is asking us to find the angle between **A** and **A** - **B**; that's angle $\beta$ in the diagram, the angle opposite vector **B**.

For part (a), we can use the Law of Cosines to find the length of the difference vector (call it **C**), and then use the Law of Sines to find the angle.

Per the Law of Cosines, the length $C = \sqrt{A^2 + B^2 - 2AB\cos 30°} = \sqrt{10^2 + 6^2 - 2 \cdot 10 \cdot 6 \cdot \cos 30°} = \sqrt{136 - 60\sqrt{3}} \approx 5.66$.

Per Law of Sines, the relationship among the lengths and angles is $\frac{\sin\beta}{B} = \frac{\sin 30°}{C}$, so solving for $\beta$, that's $\beta = \arcsin\left(\frac{B\cdot\sin 30°}{C}\right) = \arcsin\left(\frac{6\cdot\sin 30°}{5.66}\right) = 32.0°$.

For part (b), we use vector composition. The x-component of the vector difference is 10 - 6 $\cos 30° = 10 - 3\sqrt{3} \approx 4.8$, and the y-component is $0 - 6\sin 30° = $ -3. It has magnitude is $\sqrt{4.8^2 + (-3)^2} = 5.66$ and direction $\arctan(4.8, -3) = -32.0°$.

That angle value is measured from the positive +x-axis, but want the interior angle of the triangle ($\beta$ in the diagram above), so that's 32.0°.

**1-29** Two vectors **A** and **B** have the same magnitude. Under what circumstances does the vector sum **A + B** have the same magnitude as **A** and **B**? When does the vector difference **A - B** have this magnitude?

The vectors form the sides of an equilateral triangle. The vector sum **A + B** looks like this:
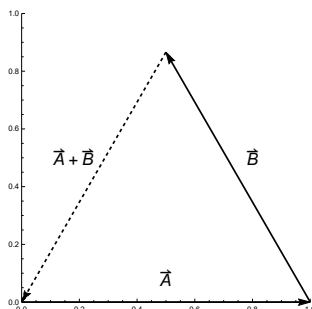


The interior angles are all 60°, so the direction of vector **B**, measured from the +x-axis, would be 120°. We can prove this mathematically using the Law of Cosines. All the magnitudes are the same, so we can write $A^2 = A^2 + A^2 - 2AA\cos\theta$. Dividing by $A^2$ and simplifying, that's $2\cos\theta = 1$, so $\theta = \arccos(1/2) = 60°$.

For the vector difference **A - B**, we arrange the vectors tail-to-tail:
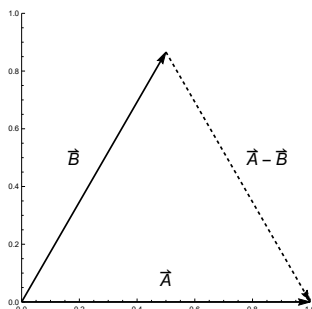


Here the direction for **B** is the same as interior angle, 60°.

**1-30** Given two vectors $\mathbf{A} = 2\hat{i} + 3\hat{j}$ and $\mathbf{B} = \hat{i} - 2\hat{j}$, obtain the following:
a) Find the magnitude of each vector.
b) Write an expression for the vector sum, using unit vectors.
c) Find the magnitude and direction of the vector sum.
d) Write an expression for the vector difference **A - B**, using unit vectors.
e) Find the magnitude and direction of the vector difference **A - B**.

The magnitude of **A** is $\sqrt{2^2 + 3^2} = \sqrt{13}$, and **B** is $\sqrt{1^2 + (-2)^2} = \sqrt{5}$.

The x-component of the sum is $2 + 1 = 3$, and the y-component is $3 - 2 = 1$. Written in vector form, that's $\mathbf{A} + \mathbf{B} = 3\hat{i} + \hat{j}$.

The magnitude of the vector sum is $\sqrt{3^2 + 1^2} = \sqrt{10}$, and its direction is $\arctan(3, 1) = 18.4°$.

The x-component of the difference is $2 - 1 = 1$, and the y-component is $3 - (-2) = 5$. In vector form that's $\mathbf{A} - \mathbf{B} = \hat{i} + 5\hat{j}$.

The magnitude of the difference is $\sqrt{1^2 + 5^2} = \sqrt{26}$, and the direction is $\arctan(1, 5)$ = 78.7°.

**1-31** Given two vectors $\mathbf{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\mathbf{B} = \hat{i} - 2\hat{j} + 3\hat{k}$, obtain the following:

a) Find the magnitude of each vector.

b) Write an expression for the vector sum, using unit vectors.

c) Find the magnitude of the vector sum.

d) Write an expression for the vector difference $\mathbf{A}$ - $\mathbf{B}$, using unit vectors.

e) Find the magnitude of the vector difference $\mathbf{A}$ - $\mathbf{B}$. Is this the same as the magnitude of $\mathbf{B}$ - $\mathbf{A}$? Explain.

The magnitude of $\mathbf{A}$ is $\sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$, and for $\mathbf{B}$ it's $\sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$.

The components of the vector sum are $2 + 1 = 3$, $3 - 2 = 1$, and $4 + 3 = 7$. The vector form of the sum is $\mathbf{A} + \mathbf{B} = 3\hat{i} + \hat{j} + 7\hat{k}$.

The magnitude of the vector sum is $\sqrt{3^2 + 1^1 + 7^2} = \sqrt{59}$.

The components of the vector difference are $2 - 1 = 1$, $3 - (-2) = 5$, and $4 - 3 = 1$. In vector form that's $\mathbf{A}$ - $\mathbf{B} = \hat{i} + 5\hat{j} + \hat{k}$.

The magnitude of the vector difference $\mathbf{A}$ - $\mathbf{B}$ is $\sqrt{1^2 + 5^2 + 1^2} = \sqrt{27} = 3\sqrt{3}$. This is the same as the magnitude of $\mathbf{B}$ - $\mathbf{A}$, since it differs from $\mathbf{A}$ - $\mathbf{B}$ only in its direction.

**1-32** Write out a multiplication table for the scalar products of all possible pairs of unit vectors, such as $\mathbf{i} \cdot \mathbf{i} = ?$, $\mathbf{i} \cdot \mathbf{j} = ?$, and so on.

Here's a table of scalar products:

| Unit Vector | Dot Product |
|:---:|:---:|
| $\hat{i} \cdot \hat{i}$ | 1 |
| $\hat{i} \cdot \hat{j}$ | 0 |
| $\hat{i} \cdot \hat{k}$ | 0 |
| $\hat{j} \cdot \hat{i}$ | 0 |
| $\hat{j} \cdot \hat{j}$ | 1 |
| $\hat{j} \cdot \hat{k}$ | 0 |
| $\hat{k} \cdot \hat{i}$ | 0 |
| $\hat{k} \cdot \hat{j}$ | 0 |
| $\hat{k} \cdot \hat{k}$ | 1 |

**1-33** Write out a multiplication table for the vector products of all possible pairs of unit vectors, such as $\mathbf{i} \times \mathbf{i} = ?$, $\mathbf{i} \times \mathbf{j} = ?$, and so on, using a righthanded coordinate system.

Here's a table of vector products:

| Unit Vectors | Vector Product |
|---|---|
| $\hat{i} \times \hat{i}$ | 0 |
| $\hat{i} \times \hat{j}$ | $\hat{k}$ |
| $\hat{i} \times \hat{k}$ | $-\hat{j}$ |
| $\hat{j} \times \hat{i}$ | $-\hat{k}$ |
| $\hat{j} \times \hat{j}$ | 0 |
| $\hat{j} \times \hat{k}$ | $\hat{i}$ |
| $\hat{k} \times \hat{i}$ | $\hat{j}$ |
| $\hat{k} \times \hat{j}$ | $-\hat{i}$ |
| $\hat{k} \times \hat{k}$ | 0 |

**1-34** Find the angle between the diagonal of a cube and an edge, using a method similar to that in the example of Sec. 1-8. Is this the same as the angle between the diagonal of a face and an edge?

The dot product $\mathbf{A} \cdot \mathbf{B}$ is the sum of the product of like components, $A_x \cdot B_x + A_y \cdot B_y + A_z \cdot B_z$. Here, $\mathbf{A}$ is the diagonal of a cube, $\hat{i} + \hat{j} + \hat{k}$. We can use any edge, so let $\mathbf{B} = \hat{i}$. The dot product would have as its scalar value $1 \cdot 1 + 1 \cdot 0 + 1 \cdot 0 = 1$.

But the dot product is defined as $\mathbf{A} \cdot \mathbf{B} = |A||B| \cos\theta$. The magnitude of $\mathbf{A}$ is $|A| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$, while the magnitude of $\mathbf{B}$ is $|B| = \sqrt{1^2} = 1$. Equating then dividing by the magnitudes, that's $\cos\theta = \frac{1}{\sqrt{3} \cdot 1}$, so $\theta = \arccos \frac{1}{\sqrt{3}} = 54.7°$.

This value is different from the angle between the diagonal of a face and an edge, which would be $45°$.

**1-35** Find the scalar product of the two vectors given in Problem 1-31.

$\mathbf{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, and $\mathbf{B} = \hat{i} - 2\hat{j} + 3\hat{k}$, so all we need to do is take the sum of the product of like components. $\mathbf{A} \cdot \mathbf{B} = 2 \cdot 1 + 3 \cdot (-2) + 4 \cdot 3 = 2 - 6 + 12 = 8$.

**1-36** Find the vector product of the two vectors given in Problem 1-31. What is the magnitude of this vector product?

Let $\mathbf{C} = \mathbf{A} \times \mathbf{B}$. We compute the components per eqn. (1-29) in the book. $C_x = A_y B_z - A_z B_y = 3 \cdot 3 - 4 \cdot (-2) = 9 + 8 = 17$. $C_y = A_z B_x - A_x B_z = 4 \cdot 1 - 2 \cdot 3 = 4 - 6 = -2$. $C_z = A_x B_y - A_y B_x = 2 \cdot (-2) - 3 \cdot 1 = -4 - 3 = -7$.

In vector form, that's $\mathbf{C} = 17\hat{i} - 2\hat{j} - 7\hat{k}$. The magnitude is $\sqrt{17^2 + (-2)^2 + (-7)^2} = \sqrt{342}$.

**1-37** Obtain a unit vector perpendicular to the two vectors given in Problem 1-31.

To obtain the unit vector, we divide the vector cross product by its magnitude: $\pm \frac{1}{\sqrt{342}}(17\hat{i} - 2\hat{j} - 7\hat{k})$. Here we use $\pm$ to indicate that either direction qualifies as a (unit) vector perpendicular to the plane of the two vectors $\mathbf{A}$ and $\mathbf{B}$.

**1-38** Find the angle between the two vectors $\mathbf{A} = 3\hat{i}+4\hat{j}+5\hat{k}$ and $\mathbf{B} = 3\hat{i}+4\hat{j}-5\hat{k}$.

We use the formula for the dot product: $AB\cos\theta = \mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$. Here the dot product is $3 \cdot 3 + 4 \cdot 4 + 5 \cdot (-5) = 9 + 16 - 25 = 0$, so $\theta = \arccos 0 = 90°$.

**1-39** Consider the two repeated vector products $\mathbf{i} \times (\mathbf{i} \times \mathbf{j})$ and $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j}$.
a) Are these two products equal?
b) Can you generalize this result for this type of repeated product?

No, they're not equal: $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = \mathbf{-j}$, but $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = 0$. The vector cross product is not associative, so in general they won't be equal.

**1-40** Prove that for any three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$.

Let $\mathbf{D} = \mathbf{B} \times \mathbf{C}$. Then $D_x = B_y C_z - B_z C_y$, $D_y = B_z C_x - B_x C_z$, and $D_z = B_x C_y - B_y C_x$. The scalar product is $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{A} \cdot \mathbf{D} = A_x D_x + A_y D_y + A_z D_z = A_x(B_y C_z - B_z C_y) + A_y(B_z C_x - B_x C_z) + A_z(B_x C_y - B_y C_x) = A_x B_y C_z - A_x B_z C_y + A_y B_z C_x - A_y B_x C_z + A_z B_x C_y - A_z B_y C_x$.

We now group according to components of $\mathbf{C}$: $A_y B_z C_x - A_z B_y C_x + A_z B_x C_y - A_x B_z C_y + A_x B_y C_z - A_y B_x C_z = (A_y B_z - A_z B_y)C_x + (A_z B_x - A_x B_z)C_y + (A_x B_y - A_y B_x)C_z$. But the parenthesized terms are just the components $\mathbf{A} \times \mathbf{B}$, and the respective products are the dot product, $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$.