

CH3050 PROCESS DYNAMICS & CONTROL ASSIGNMENT - 2

- ① a) Let ρ be density of reactor contents, F_i be the flow rate in, C_A be the concentration of A in reactor, C_{Ai} be concentration of A at the inlet, V be volume of reactor, T_i be temperature at inlet and T be temperature of reactor.
- Also, for convenience, let $k = \alpha e^{-B/T}$ $\alpha = 2.4 \times 10^5$
 $B = -2.5 \times 10^4$
- Assumptions

- ① Reactor is uniformly mixed ensuring same T and C_A everywhere
- ② $\Delta H_{\text{reaction}}$ is constant over the prevailing temperatures
- ③ Physical properties such as ρ, c_p are also constant
- ④ Incompressible flow with no hold-up/accumulation
 $(F_{\text{in}} = F_{\text{out}} = F_i)$

Mass balance of A :

$$V \frac{dC_A}{dt} = F_i (C_{Ai} - C_A) - k C_A$$

$$\Rightarrow \frac{dC_A}{dt} = - \left(\frac{F_i}{V} + \alpha e^{-B/T} \right) C_A + \frac{F_i C_{Ai}}{V} \quad \text{--- (1)}$$

Energy balance around the reactor

$$\rho V C_p \frac{\partial T}{\partial t} = \rho F_i C_p (T_i - T) + \left((-\Delta H_r) k C_A \right) V$$

$$\Rightarrow \frac{\partial T}{\partial t} = \frac{F_i}{V} T_i - \frac{F_i}{V} T + \frac{(-\Delta H_r) \alpha e^{-B/T}}{\rho C_p} \quad \text{--- ②}$$

c) Hand calculations (linearisation & finding transfer function)

$$\text{Let } \frac{dC_A}{dt} = f(C_A, T, C_{A_i})$$

(recognise that $x = \begin{bmatrix} C_A \\ T \end{bmatrix}; u = C_{A_i}$)

$$f(C_A, T, C_{A_i}) \approx f \Big|_{\text{steady-state}} \Big|_{C_{A,ss}, T_{ss}, C_{A_i,ss}} + \frac{\partial f}{\partial C_A} (C_A - C_{A,ss}) + \frac{\partial f}{\partial T} (T - T_{ss}) + \frac{\partial f}{\partial C_{A_i}} (C_{A_i} - C_{A_i,ss})$$

At steady state $\frac{dC_A}{dt} = 0 \Rightarrow f = 0$

Also, denote deviation variables with a $\tilde{~}$ for convenience.

$$\Rightarrow \frac{d\tilde{C}_A}{dt} = - \left(\frac{F_i}{V} + \alpha e^{-B/T_{ss}} \right) \tilde{C}_A + \left(\frac{-\alpha B e^{-B/T_{ss}} C_{A,ss}}{T_{ss}^2} \right) \tilde{T} + \frac{F_i}{V} \tilde{C}_{A_i} \quad \text{--- ③}$$

Doing a similar Taylor expansion for $\frac{dT}{dt}$ around steady state,

$$\frac{dT}{dt} = - \left(\frac{\Delta H_R}{\rho C_p} e^{-\beta/T_{ss}} \right) \tilde{C}_A + \left[\frac{-F/V}{\rho C_p T_{ss}} - \frac{\Delta H_R \beta C_{ss}}{\rho C_p T_{ss}^2} e^{-\beta/T_{ss}} \right] \tilde{T} \quad (4)$$

C_{ss} and T_{ss} are found using findop in MATLAB.

$$C_{ss} = 0.0193 \text{ mol/l} \quad T_{ss} = 558.564^\circ \text{R} = 98.89^\circ \text{F}$$

Notice that coefficients are just constant values, so

$$\text{let } \frac{d\tilde{x}}{dt} = a_{11} \tilde{C}_A + a_{12} \tilde{T} + b_1 \tilde{C}_I$$

$$\frac{d\tilde{T}}{dt} = a_{21} \tilde{C}_A + a_{22} \tilde{T}$$

Take Laplace transform on both sides

$$\Rightarrow s \tilde{C}_A(s) = a_{11} \tilde{C}_A(s) + a_{12} \tilde{T}(s) + b_1 \tilde{C}_I(s) \quad (5)$$

$$s \tilde{T}(s) = a_{21} \tilde{C}_A(s) + a_{22} \tilde{T}(s) \quad (6)$$

(\because initial conditions are 0, $\mathcal{L}\left\{\frac{d\tilde{x}}{dt}\right\} = s\tilde{x}(s)$)

$$\Rightarrow (s - a_{11}) \tilde{C}_A(s) - a_{12} \tilde{T}(s) = b_1 \tilde{C}_I(s) \quad (7)$$

$$\text{and } \tilde{C}_A(s) = \frac{(s - a_{22})}{a_{21}} \tilde{T}(s) \quad (8)$$

Subst (8) in (7) to get

$$T(s) \left[\frac{(s-a_{11})(s-a_{22})}{a_{21}} - a_{12} \right] = b_1 C_A(s)$$

$$\Rightarrow T(s) = \frac{a_{21} b_1 C_A(s)}{(s-a_{11})(s-a_{22}) - a_{12} a_{21}}$$

$$\Rightarrow \frac{T(s)}{C_A(s)} = \frac{a_{21} b_1}{s^2 - (a_{11} + a_{22})s + a_{11}a_{22} - a_{12}a_{21}} \quad \text{--- (9)}$$

Subst. back in (8),

$$\frac{C_A(s)}{C_A(s)} = \frac{(s-a_{22}) b_1}{s^2 - (a_{11} + a_{22})s + a_{11}a_{22} - a_{12}a_{22}} \quad \text{--- (10)}$$

Substituting the values, we get

$$\frac{T(s)}{C_A(s)} = \frac{0.1283}{s^2 - 0.6996s + 0.114} \quad \text{and} \quad \frac{C_A(s)}{C_A(s)} = \frac{0.01607s - 1.19 \times 10^{-4}}{s^2 - 0.6996s + 0.114}$$

Both the transfer functions match.
The transfer functions obtained in MATLAB.

$$(3) a) i) G(s) = \frac{s+1}{(s+3)(s+2)(s+5)} = \frac{s+1}{(s+3)(s+2)(s+5)}$$

$$\Rightarrow \frac{A_1}{(s+2)} + \frac{A_2}{(s+3)} + \frac{A_3}{(s+5)} = \frac{s+1}{(s+3)(s+2)(s+5)}$$

$$\Rightarrow A_1(s+3)(s+5) + A_2(s+2)(s+5) + A_3(s+3)(s+2) = (s+1)$$

$$s = -2 \Rightarrow A_1 = \frac{-4}{(-2)(-3)} = \frac{1}{3}$$

$$s = -3 \Rightarrow A_2 = \frac{-2}{(-2)(+5)} = \frac{1}{5}$$

$$s = -5 \Rightarrow A_3 = \frac{-4}{(-5)(-2)} = -\frac{2}{5}$$

$$\therefore G(s) = \frac{1}{s+3} - \frac{1}{3(s+2)} - \frac{2}{5(s+5)}$$

$$= \frac{1}{s+3} - \left(\frac{1}{3}\right) \frac{1}{s+2} - \left(\frac{2}{5}\right) \left(\frac{1}{s+5}\right)$$

$$X_1(s) = \frac{U(s)}{s+3}; \quad X_2(s) = \frac{U(s)}{s+5}; \quad X_3(s) = \frac{U(s)}{s+2}$$

Inverse LT.

$$\Rightarrow \dot{x}_1 = -3x_1 + u(t); \quad \dot{x}_2 = -5x_2 + u(t); \quad \dot{x}_3 = -2x_3 + u(t)$$

$$\Rightarrow \dot{x}_1 = -3x_1 + u(t); \quad \dot{x}_2 = -5x_2 + u(t); \quad \dot{x}_3 = -2x_3 + u(t)$$

$$\Rightarrow \dot{x}_1 = -3x_1 + u(t); \quad \dot{x}_2 = -5x_2 + u(t); \quad \dot{x}_3 = -2x_3 + u(t)$$

$$\therefore \boxed{A = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -2 \end{pmatrix} ; B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

Note that $G(s) = \frac{X_1(s)}{U(s)} - \frac{2}{3} \frac{X_2(s)}{U(s)} - \frac{1}{3} \frac{X_3(s)}{U(s)}$

$$\Rightarrow Y(s) = X_1(s) - \frac{2}{3} X_2(s) - \frac{1}{3} X_3(s)$$

$$\therefore y(t) = x_1(t) - \frac{2}{3} x_2(t) - \frac{1}{3} x_3(t)$$

$$\boxed{C = \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}}$$

ii) $G(s) = \frac{Y(s)}{U(s)} = \frac{s+1}{(s^3+10s^2+31s+30)}$

$$\Rightarrow (s^3+10s^2+31s+30)Y(s) = (s+1)U(s)$$

Take inverse Laplace Transform.

$$\Rightarrow \frac{d^3y}{dt^3} + 10\frac{d^2y}{dt^2} + 31\frac{dy}{dt} + 30y = \frac{du}{dt} + u$$

[initial conditions are 0
since it uses deviation
variables]

$$\Rightarrow \frac{d^3y}{dt^3} = -10\frac{d^2y}{dt^2} - 31\frac{dy}{dt} + \frac{du}{dt} + u - 30y$$

Integrate 1

$$\Rightarrow \frac{d^2y}{dt^2} = -10\frac{dy}{dt} - 31y + u + \int (u - 3y) dt$$

Integrate 2

Again integrate wrt u ,

$$\Rightarrow \frac{dy}{dt} = -10y + \int \left[(u - 31y) + \int (u - 30y) dt \right] dt$$

Integrate wrt time,

$$\Rightarrow y = \int \left[-10y + \int \left\{ u - 31y + \left(\int (u - 30y) dt \right) \right\} dt \right] dt$$

$$\text{Let } x_1 = \int (u - 30y) dt$$

$$x_2 = \int (u - 31y + \int (u - 30y) dt) dt$$

$$x_3 = y$$

$$\Rightarrow \dot{x}_1 = u - 30y = -30x_3 + u$$

$$\dot{x}_2 = u - 31y + x_1 = x_1 - 31x_3 + u$$

$$\dot{x}_3 = -10y + x_2 = x_2 - 10x_3$$

$$\therefore \begin{matrix} A = \begin{bmatrix} 0 & 0 & -30 \\ 1 & 0 & -31 \\ 0 & 1 & -10 \end{bmatrix} & B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

Notice the eigen values of A in second part
 is same as eigen values of A in first part (\therefore
 same system \Rightarrow same poles \Rightarrow same $\text{eig}(A)$
 $[-3, -5 \text{ and } -2]$

and also if $x = Tw$ then

$$T^{-1} A T = \Lambda \text{ new.}$$

We can get this form from eigen value decomposition ($\therefore V^{-1} A V = \Lambda$)

$$\text{So } T = V^{-1}$$

$$= \begin{bmatrix} 6.1237 & -18.371 & 55.1135 \\ 1.3123 & -6.562 & 32.8084 \\ -5.626 & 11.353 & -22.206 \end{bmatrix}$$

Eigen vector & inverse obtained from MATLAB.

Note that MATLAB gives $D = \begin{pmatrix} -2 & & \\ & -3 & \\ & & -5 \end{pmatrix}$

so I had to rearrange the V matrix so that
 it matches my $A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$$\cancel{x = Tw} \cdot x = Tw$$

x are states in SS2

w - states in SS1

③ b) $G_{11}(s) = \frac{4s-11}{(s-1)(s-3)}$

$= -\frac{3}{2} \frac{1}{(s-1)} + \frac{11}{2} \frac{1}{s-3}$ — ①

$G_{12}(s) = \frac{10s}{(s-2)(s-3)} = -20 \frac{1}{s-2} + 30 \frac{1}{(s-3)}$ — ②

We notice that both the transfer functions have a contribution from $\frac{1}{s-3}$ term.
 So we can have the subsystems obeying the dynamics of $\frac{1}{s-3}$. ($= \frac{X(s)}{U(s)}$) same for both cases since we need a minimal realization (we need minimum no. of states to preserve observability & controllability)

So let $\frac{X_1(s)}{U(s)} = \frac{1}{s-1}$

$\frac{X_2(s)}{U(s)} = \frac{1}{s-2}$

$\frac{X_3(s)}{U(s)} = \frac{1}{s-3}$

taking inverse L.T.

$\Rightarrow \dot{x}_1 = -x_1 + u; \dot{x}_2 = -2x_2 + u; \dot{x}_3 = -3x_3 + u$

$y_1 = -\frac{3}{2}x_1 + \frac{11}{2}x_3; y_2 = -20x_2 + 30x_3$

(obtained by writing eqns ① & ② in terms of $\frac{Y(s)}{U(s)}$ & $\frac{X_k(s)}{U(s)}$ with $U(s)$ cancelling on both sides & then taking mark Laplace transfer.)

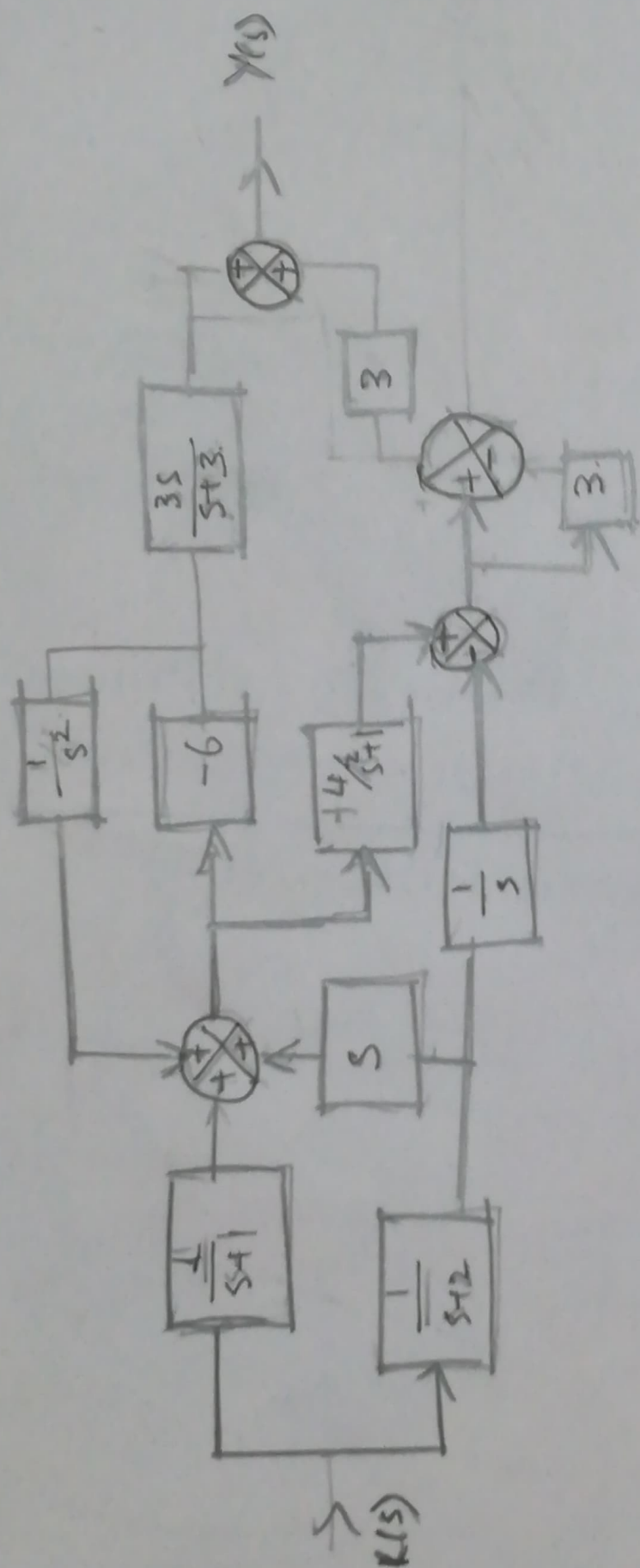
$$\therefore A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; C = \begin{bmatrix} -\frac{3}{2} & 0 & \frac{11}{2} \\ 0 & -20 & 30 \end{bmatrix}$$

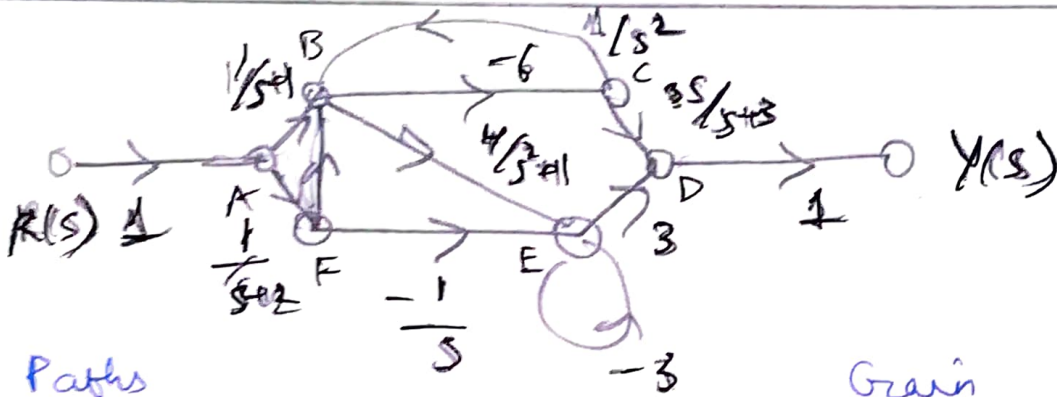
rank of controllability matrix = 3 rank of observability matrix = 3 = no. of states (evaluated in MATLAB). \therefore system is observable & controllable

\Rightarrow the obtained realisation is indeed the minimal order realisation

④)



4
ii)



Paths

$$P_1 : RABCDY$$

Gain

$$\frac{-18s}{(s+1)(s+3)}$$

$$P_2 : RABEDY$$

$$\frac{12}{(s^2+1)(s+1)}$$

$$P_3 : RAFBEDY$$

$$\frac{12s}{(s+2)(s^2+1)}$$

$$P_4 : RAFEDY$$

$$\frac{-3}{s(s+2)}$$

$$P_5 : RAFBCDY$$

$$\frac{-18s}{(s+2)(s+3)}$$

$$L_1 : BCB$$

$$-6/s^2$$

$$L_2 : E \rightarrow E$$

$$-3$$

$$\Delta = 1 - L_1 - L_2 + L_1 L_2 \quad (\because L_1, L_2 \text{ are not touching})$$

$$= 1 + \frac{6}{s^2} + 3 + \frac{18}{s^2} = 4 + \frac{24}{s^2}$$

$$\Delta_1 = \Delta \big|_{L_1=0} \text{ (only } L_1 \text{ touches } P_1) = 4$$

$$\Delta_2 = \Delta \big|_{L_1=L_2=0} \text{ (} L_1 \text{ \& } L_2 \text{ touch } P_2) = 1$$

$$\Delta_3 = \Delta \big|_{L_1=L_2=0} \text{ (} L_1 \text{ \& } L_2 \text{ touch } P_3) = 1$$

$$\Delta_4 = \Delta \big|_{L_2=0} \text{ (} L_2 \text{ touches } P_4) = \frac{1+6}{5}$$

$$\Delta_5 = \Delta \big|_{L_1=0} \text{ (} L_1 \text{ touches } P_5) = 4.$$

$$\therefore \Delta_1 P_1 = \frac{-72s}{(s+1)(s+3)} \quad \Delta_2 P_2 = \frac{12}{(s^2+1)(s+1)}$$

$$P_3 \Delta_3 = \frac{12s}{(s^2+3)(s+2)} \quad P_4 \Delta_4 = \frac{-3\left(1+\frac{6}{s}\right)}{s(s+2)}$$

$$P_5 \Delta_5 = \frac{-72s}{(s+2)(s+3)}$$

Using the eqn for gain 1

$$\frac{Y(s)}{R(s)} = \frac{\sum_{i=1}^5 P_i \Delta_i}{\Delta}$$

$$\Rightarrow \frac{Y(s)}{R(s)} = \frac{-72s}{(s+1)(s+3)} + \frac{12}{(s^2+1)(s+1)} + \frac{12s}{(s^2+1)(s+2)} - 3\left(1 - \frac{6}{s^2}\right)$$

$$\frac{-72s}{(s+2)(s+3)}$$

$$4 + \frac{24}{s^2}$$

$$= \frac{-72s^4(s^2+1)(s+2) + 12s^3(s+2)(s+3) - 3(s^2+1)(s+6)(s+3) + 12s^4(s+3)(s+1) - 72s^4(s^2+1)(s+1)}{4(s^2+6)s(s^2+1)(s+1)(s+2)(s+3)}$$

$$= \frac{-72s^4(s^2+1)(s+2) + 12s^3(s+2)(s+3) - 3(s^2+1)(s+6)(s+3) + 12s^4(s+3)(s+1) - 72s^4(s^2+1)(s+1)}{4(s^2+6)s(s^2+1)(s+1)(s+2)(s+3)}$$