INDIAN INSTITUTE OF TECHNOLOGY MADRAS
Department of Chemical Engineering

CH3050 Process Dynamics and Control
Jan-May 2021 Assignment 5 Solutions

1

A process $G_p(s)$ is in feedback control with a P-controller using a measuring element $G_{\text{sens}}(s)$.

$$G(s) = \frac{s^2 - 4s + 8}{s(s+1)(s+3)};\ G_{\text{sens}}(s) = \frac{1}{s+10}$$

(a) Sketch the root locus of a feedback compensated closed-loop system consisting of as the proportional controller gain K_c varies from 0 to $+\infty$. Compute the asymptotes angles, centroid, angles of arrival, break-in and entry points.

(b) Generate the root locus on the computer and verify your sketch (do not reverse the order of parts (b) and (a) for your own benefit!).

(c) Determine the ultimate gain by hand (show your calculations) and verify your answer. with the computer generated plot.

(d) Find the value of K_c such that the closed-loop response to a set-point change has the minimum settling time.

(e) If a PI-controller $G_c(s) = K_c + \frac{K_I}{s}$ was used instead, find the ultimate value of KI with the value of Kc fixed to what you obtained in (1d)

(a & b)

Answer: Given the tranfer function of the system

$$G(s) = \frac{s^2 - 4s + 8}{s(s+1)(s+3)}$$

with the sensor dynamics $H(s) = \dfrac{K_c}{s+10}$, the loop transfer function can be written as

$$G(s)H(s) = \frac{K_c(s^2 - 4s + 8)}{s(s+1)(s+3)(s+10)}$$

Assymptotic angles

$$\begin{aligned}\phi_k &= \frac{(2k+1)\pi}{\#poles - \#zeros}, \forall k = 0(i)(\#poles - \#zeros - 1)\\ &\implies \frac{\pi}{2}, \frac{3\pi}{2}\end{aligned}$$

Centroid:

$$\text{centroid} = \frac{\sum poles - \sum zeros}{\#poles - \#zeros}$$
$$\implies \text{centroid} = -9$$

Break away points:
At break away points, the gain attain it's local maximum as the system changes from being over-danped to critically damped.

$$|K_c| = \frac{1}{G(s)}$$
$$|K_c| = \frac{s(s+1)(s+3)(s+10)}{s^2-4s+8}$$
$$\left.\frac{dK_c}{ds}\right|_{s=s*} = 0$$
$$(s^*, k^*) = (-7.09, 5.93) \text{ and } (-0.384, 0.614)$$

Angle of arrival:
The contribution by the pole at origin is= $\tan^{-1}(1)$
The contribution by the pole at $s=-1$ is= $\tan^{-1}(\frac{-2}{3})$
The contribution by the pole at $s=-3$ is= $\tan^{-1}(\frac{2}{5})$
The contribution by the pole at $s=-10$ is= $\tan^{-1}(\frac{2}{12})$
The contribution by the zero at $s=2-2j$ is= 90
So, the angle of arrival at the complex zero is $180+90-(\tan^{-1}(1)+\tan^{-1}(\frac{2}{3})+\tan^{-1}(\frac{2}{5})+\tan^{-1}(\frac{2}{12})) = 160.0462$

(c)

Ultimate gain refers to the maximum value of the gain for which the closed-loop system remains stable. This can be obtained by calculating the gain associated with the frequency at which the closed-loop poles lie on the $j\omega$ axis.

$$\left.(s^2-4s+8)+K_c s(s+1)(s+3)(s+10)\right|_{s=j\omega, K=K_C^*} = 0 \tag{1}$$

$$\left.Re((s^2-4s+8)+K_c^* s(s+1)(s+3)(s+10)\right|_{s=j\omega} = 0 \text{ and} \tag{2}$$

$$\left.Im((s^2-4s+8)+K_c^* s(s+1)(s+3)(s+10)\right|_{s=j\omega} = 0 \tag{3}$$

From (2) and (3) the value of ω and K_c^* can be found as ± 0.894 and 4.7 respectively.

(e)

For a PI controller with the canonical form $G_c(s) = \left(K_c + \frac{K_I}{s}\right)$, the characterstic equation of the corresponding closed-loop system can be written as

$$\begin{aligned} 1 + G_p(s)G_c(s)H(s) &= 0 \\ \implies s^2(s+10)(s+3)(s+1) + K_c s(s^2-4s+8) + K_I(s^2-4s+8) &= 0 \\ \implies G_L(s) = \frac{(s^2-4s+8)}{K_c s(s^2-4s+8) + s^2(s+1)(s+3)(s+10))} \end{aligned}$$

So the effective open-loop transfer with K_I as tuning parameter is

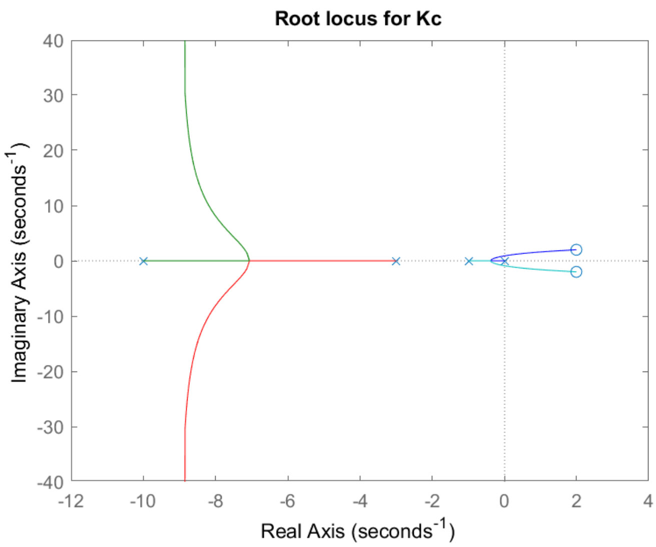
$$G_L(s) = \frac{(s^2-4s+8)}{K_c s(s^2-4s+8) + s^2(s+1)(s+3)(s+10))}$$

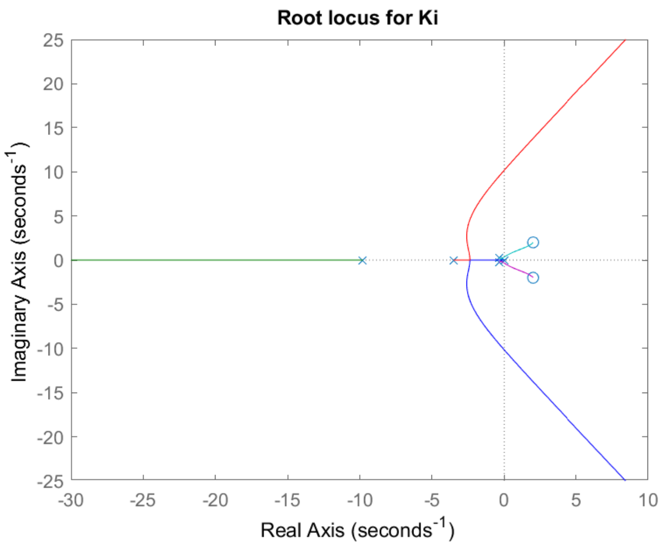
The ultimate gain in this case, can be calculated as 1520 units.

MATLAB code:

```
%question 1.4
sys=tf([1 -4 8],[1 14 43 30 0]);

rlocus(sys)
title('Root locus for Kc')
j=1;
K=0.05:0.005:4.45;
t_set=zeros(length(K),1);
for i=0.05:0.005:4.45
sys_cl=i*sys/(i*sys+1);
S=stepinfo(sys_cl);
t_set(j)=S.SettlingTime;
j=j+1;
end
K(find(t_set==min(t_set)))
clear all

%Question 1.5
s=tf("s")
G=(s^2-4*s+8)/(s^2*(s+1)*(s+3)*(s+10)+0.855*s*(s^2-4*s+8))
figure
rlocus(G)
title('Root locus for Ki')

ans =

0.8550


s =

s
```

```
Continuous-time transfer function.


G =

s^2 - 4 s + 8
---------------------------------------------
s^5 + 14 s^4 + 43.85 s^3 + 26.58 s^2 + 6.84 s

Continuous-time transfer function.
```





2

A process with the transfer function $G(s) = \frac{2(s+4)}{10s^2+7s+1}e^{-2s}$ is placed in feedback with a controller $G_c(s)$

(a) Suppose G_c is a P-controller. Design K_c s.t. the gain margin is 8.2 dB. Report the corresponding PM.

(b) DelayNaJaane, who is in charge of the controller design, is uncertain about the delay but would like to know the extent of delay for which the control system can robustly remain stable. What is the maximum delay uncertainty with the value of K_c chosen in (2a)?

(c) Using the K_c value in part (2a), now design a PI controller of the form $Gc(s) = K_c\left(1 + \frac{1}{\tau_I s}\right)$ s.t. the phase margin is 60. Report the corresponding GM.

(d) Evaluate the sensitivity function of the feedback system with the above settings. Verify numerically that indeed Bode's sensitivity integral holds (up to the numerical approximation).

(a)

Given process transfer function

$$G(s) = \frac{2(s+4)e^{-2s}}{10s^2 + 7s + 1}$$

with a P type controller of gain K_c, the closed-loop transfer function can be written as

$$G_{CL}(s) = 1 + K_c \frac{2(s+4)e^{-2s}}{10s^2 + 7s + 1}$$

Given, the control objective of a GM of 8.2 db

$$\tan^{-1}(\frac{\omega}{4}) - [\tan^{-1}(5\omega) + \tan^{-1}(2\omega)] - 2\omega \Big|_{\omega=\omega_{pc}} = -\pi \tag{4}$$

$$\implies \omega_{pc} = 0.588 \tag{5}$$

The required value of the controller gain ($\hat{K}_C$) for the system to have a GM of 8.2 db$=2.5704$ units is

$$\hat{K}_C = \frac{1}{|G(jw_{pc})| \times 2.5704} = 0.2312$$

(b)

Delay margin (assume D) amounts to calculating the additional delay required for the gain and phase crossover frequencies to converge.

$$0.2312|G(j\omega)H(j\omega)| \Big|_{\omega=\omega_{gc}} = 1 \tag{6}$$

$$\implies \omega_{gc} = 0.261 \tag{7}$$

$$\tan^{-1}(\frac{\omega}{4}) - [\tan^{-1}(5\omega) + \tan^{-1}(2\omega)] - 2\omega - D\omega \Big|_{\omega=\omega_{gc}} = -\pi \tag{8}$$

$$D\omega_{gc} = 1.287 \tag{9}$$

$$D = 4.929 \tag{10}$$

(c)

Using the PID tuning toolbox in MATLAB, the required value of τ_I for the system to have a PM of 60 is obtained as

$$\tau_I^* = 18.79$$

(d)

```
sys=tf([2 8],[10 7 1],'IODelay',2)
Pi=tf([0.2312*18.79 0.2312],[18.79 0])
G_L=1/(1+sys*Pi)
[mag,~,freq]=bode(G_L);
Sens_val=trapz(freq, log((mag(:))))

sys =

2 s + 8
exp(-2*s) * ----------------
10 s^2 + 7 s + 1

Continuous-time transfer function.


Pi =

4.344 s + 0.2312
----------------
18.79 s

Continuous-time transfer function.


G_L =

A =
x1         x2         x3
x1    -0.7462    -0.2874  -0.009844
x2          1          0          0
x3          0          1          0

B =
u1
x1  0.5
x2    0
x3    0

C =
x1         x2         x3
y1  -0.09248   -0.3748  -0.01969

D =
u1
y1   1

(values computed with all internal delays set to zero)

Internal delays (seconds): 2
```

```
Continuous-time state-space model.


Sens_val =

0.0557
```

As it can be seen that the Bode's sensitivity integral value is almost negligible. This demonstrates that Bode's sensitivity integral holds true.

3

A process has the transfer function $G(s) = \frac{2(s+2)}{s^2-2s+3}e^{-s}$

(a) Using Pade's first-order approximation, design a P controller (call it G_{c_1}) such that the real part of a dominant pole of the stable closed-loop system has the real part of -0.2.

(b) Design another P controller (call it G_{c_2}) using the Nyquist diagram such that the gain margin is 10.5 dB. Calculate the offset in output to a step-type set-point change for this value of K_c.

(c) Using SIMULINK, compare the performances of above two controllers for step-type setpoint change and disturbance. Would the performance of first controller improve if we had taken into account the delay using a Pad´e's second-order approximation?

(a)

Given the transfer function

$$G_p(s) = \frac{2(s+2)}{s^2+2s-3}e^{-s}$$

Using Pade's first order approximation,

$$G(s) = K\frac{2(s+2)}{s^2+2s-3} \times \frac{(2-s)}{(s+2)}$$

The characteristic equation can be written as

$$(s^2+2s-3) + 2K(2-s) = 0 \tag{11}$$
$$s^2 + (2-2K)s - 3 + 4K = 0 \tag{12}$$

For the roots of (12) to have the real part of -0.2

$$\frac{2-2K}{2} = 0.2$$
$$\implies K = 0.8$$

(b)

As it can be seen from the Nyquist plot of $G_p(s)$ (figure 1) that the point $-1+j0$ has a clockwise encirclement. This along with one unstable open-loop pole confirms that the number of closed-loop poles with positive real parts are two. The only way to make the closed-loop system stable is to place $-1+j0$ in the encirclement bounded by $w = 0$ and $w = 0.78$ because in that scenario the critical point undergoes an encirclement in the anti-clockwise direction which makes the total effective encirclement to be zero considering the single open-loop unstable pole. In that case, the conditions on K can be written as

$$0.7499 \leq K \leq 0.9155$$

Now, the minimum value of K_c that provides a gain margin closest to 10.5 dB can be obtained as 0.76.

MATLAB Code

```
sys=tf([2 4],[1 2 -3],'IODelay',1);
t=0.76:0.001:0.77;
p=zeros(length(t),1);
j=1;
for k=0.76:0.001:0.77
S = allmargin(k*sys);
p(j)= abs(S.GainMargin(4)-10.4);
j=j+1;
end
kc=t(find(p==min(p)))
Skc = allmargin(kc*sys)
Skc.GainMargin(4)

kc =

0.7600


Skc =

struct with fields:

GainMargin: [1×48 double]
GMFrequency: [1×48 double]
PhaseMargin: 1.5754
PMFrequency: 0.1768
DelayMargin: 0.1555
DMFrequency: 0.1768
Stable: 1


ans =

9.4363
```

(c)

The first order Pade's approximation provides a gain of 16 whereas for Pade's second order approximation with the controller gain set to $k = 0.791$ (refer to figure 2) the corresponding gain is $\approxeq 20$. The same for the second controller can be found as 65. This is because the controller value $(K = 0.76)$ that satisfies the gain margin criterion of 10.5dB results a closed loop system with very low relative stability (Note the lower limit of K for a stable closed loop system is 0.755).

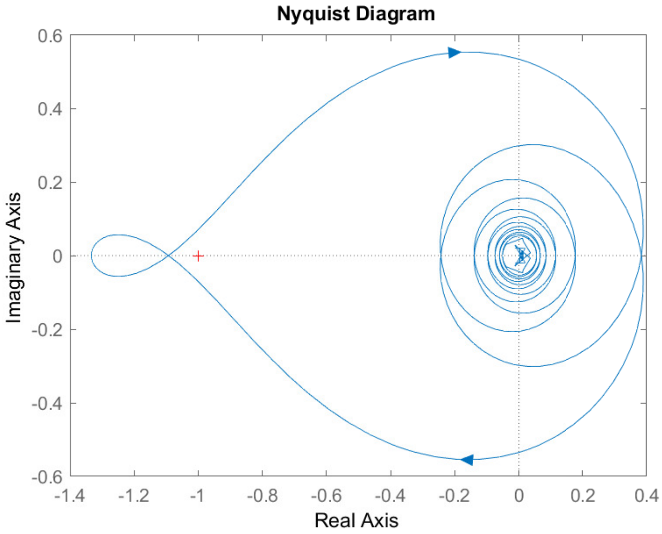


Figure 1: Nyquist plot for $G_p(s)$

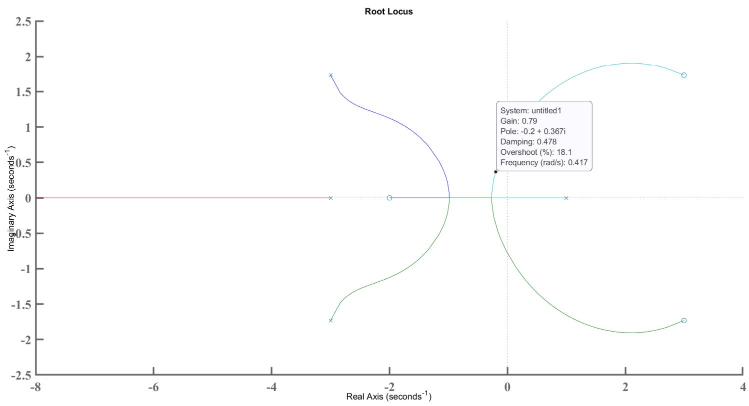
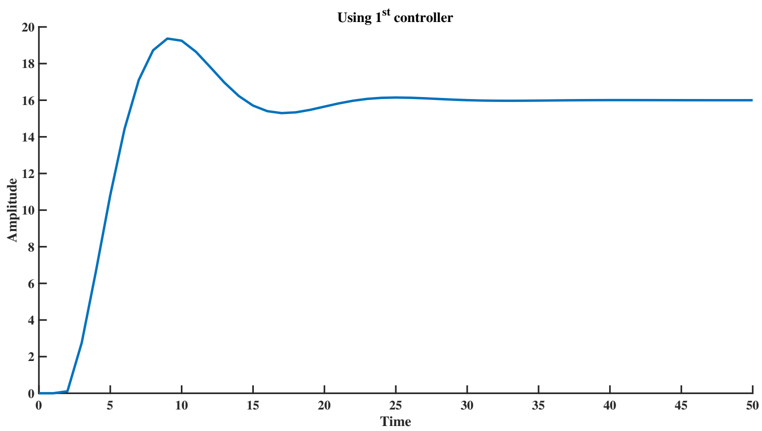
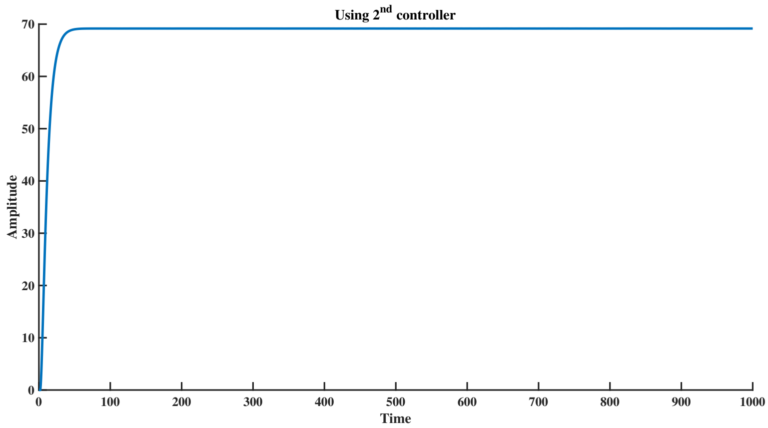
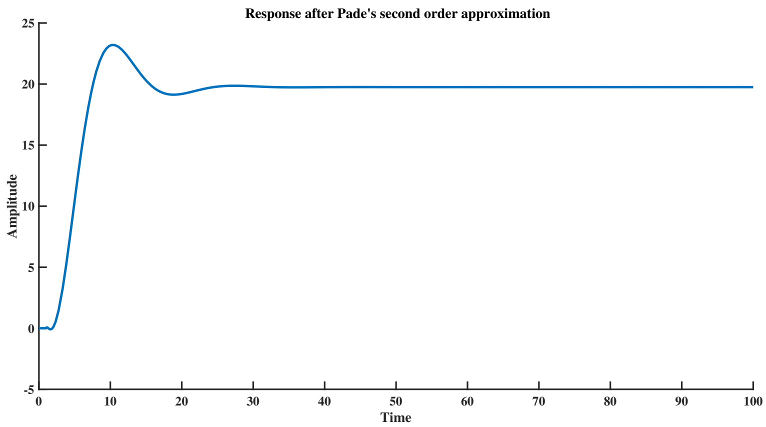


Figure 2: Root locus of after Pade's 2^{nd} order approximation




Using 2nd controller
Amplitude
Time
0
10
20
30
40
50
60
70
0
100
200
300
400
500
600
700
800
900
1000


Response after Pade's second order approximation
Amplitude
Time
-5
0
5
10
15
20
25
0
10
20
30
40
50
60
70
80
90
100