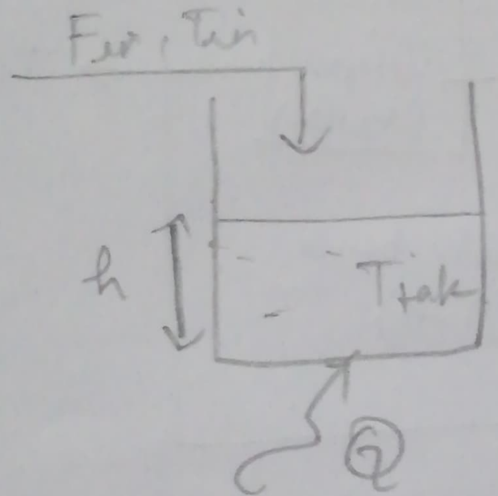
CH3050 ASSIGNMENT - 1

① a) <u>Fed Batch</u>

We can imagine the system to have water flowing in, with heat being supplied to the stored water



There is no flow out because it is a storage geyser

b) <u>Variables</u>: F_{in}, h_{tank}, T_{tank}, T_{in}, Q

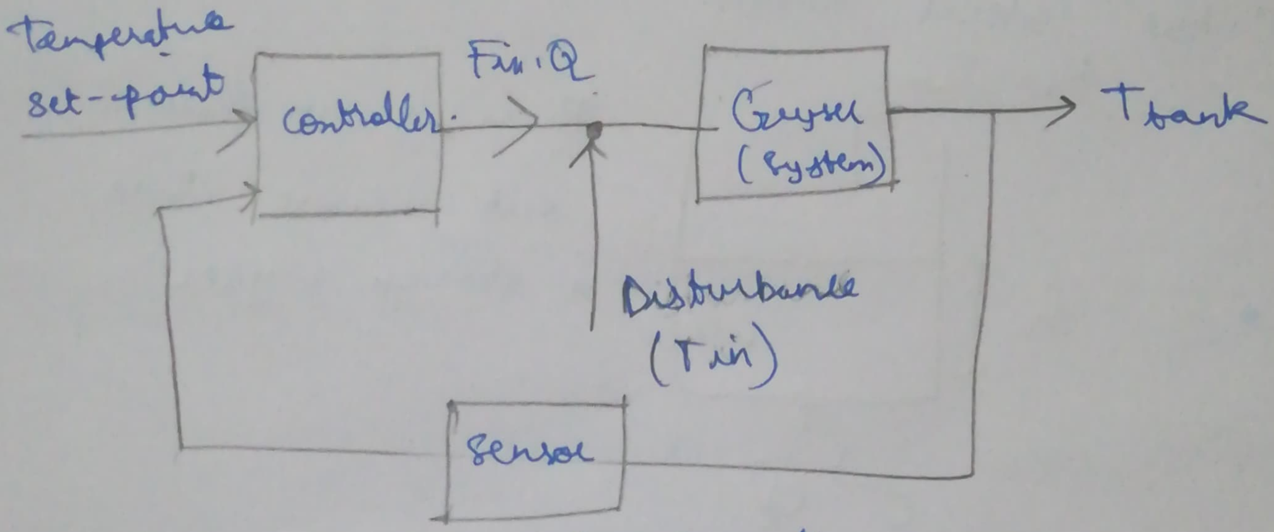
→ Manipulated variables: ~~F~~ F_{in}, Q (where F_{in} is flow in, and Q is the heat added) by the

I assume that Q can be modified

→ Disturbance variables: T_{in} (Inlet temperature of water is dictated by outside conditions)

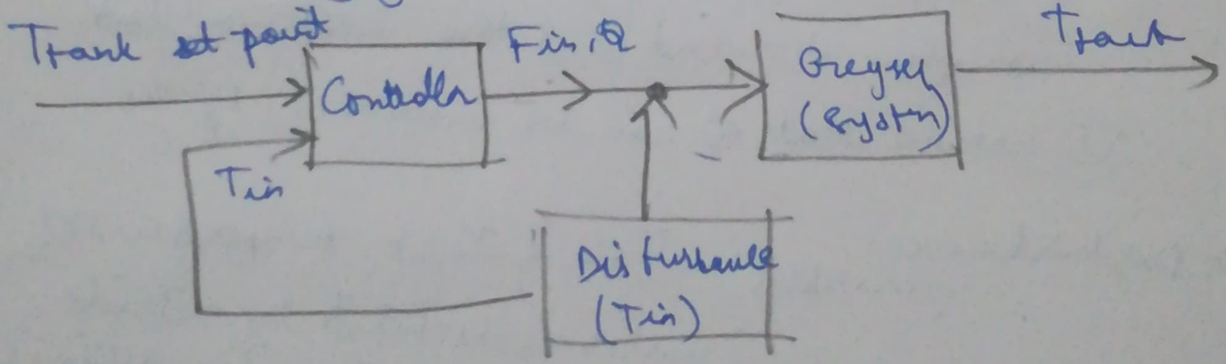
→ Controlled variables: T_{tank} (temperature of water in the tank.

c) Feedback system: Measure the temperature of tank (T_{tank}) and manipulate F_{in} and Q accordingly. F_{in} manipulated by a valve
Q . by using a rheostate



(eg. thermocouple

d) Feedforward system: We measure T_{in} (the disturbance) and accordingly change the manipulated variable.



2) a) At steady state $\frac{dy}{dt} = \frac{d^2y}{dt^2} = 0$

$\Rightarrow \cancel{y(t)}$ $y_{ss} = \frac{b_0}{a_0} u_{ss}$. —— 1

Consider the ODE

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y(t) = b_0 u(t)$$

$$\frac{d^2}{dt^2}(y - y_{ss}) + a_1 \frac{d}{dt}(y - y_{ss}) + a_0 (y(t) - y_{ss}) + a_0 y_{ss} = b_0 (u(t) - u_{ss}) + b_0 u_{ss}$$ —— 2

$\left(\because \frac{dy_{ss}}{dt} = 0 \rightarrow y_{ss} \text{ is a constant}\right)$

But 1 $\Rightarrow a_0 y_{ss} = b_0 u_{ss}$

$$\therefore \frac{d^2 \tilde{y}}{dt} + a_1 \frac{d\tilde{y}}{dt} + a_0 \tilde{y}(t) = b_0 \tilde{u}(t)$$ —— 3

(This is expected because the ODE is linear)

b) $\cancel{y_{ss} = \frac{b_0}{a_0} u_{ss} \Rightarrow y_{ss} = \frac{3}{15} \times u_{ss}}$

3 $\Rightarrow$ $a_0 \tilde{y}_{ss} = b_0 \tilde{u}_{ss}$ (@ steadystate)

$\Rightarrow \tilde{y}_{ss}\, 2a_0 = b_0 \tilde{u}$

$\Rightarrow \tilde{u} = \frac{2 \times 15}{3}$

$\Rightarrow$ $\boxed{\Delta u = 10 \text{ units}}$

c) Substituting $\tilde{u}(t) = K_c(2 - \tilde{y}(t))$ in eqn ③,

$$\frac{d^2\tilde{y}}{dt^2} + a_1\frac{d\tilde{y}}{dt} + a_0\tilde{y}(t) = b_0K_c(2 - \tilde{y}(t))$$

$$\Rightarrow \frac{d^2\tilde{y}}{dt^2} + a_1\frac{d\tilde{y}}{dt} + \tilde{y}(t)(a_0 + b_0K_c) = 2b_0K_c \quad \text{——— ④}$$

Now to answer whether the system will achieve the control objective is equal to asking whether ~~the system~~ the system found in 4 saturates (whether it is stable)

To analyse the stability we can see the poles of this new system.

Also $\left.\frac{d^2\tilde{y}}{dt^2}\right|_{t=0} = 0$ & $\left.\frac{d\tilde{y}}{dt}\right|_{t=0} = 0$ & $\left.\tilde{y}(t)\right|_{t=0} = 0$

because the system is initially assumed to be in steady state.

So $\mathcal{L}\left\{\frac{d^2\tilde{y}}{dt^2}\right\} = s^2\tilde{Y}(s)$

and $\mathcal{L}\left\{\frac{dy}{dt}\right\} = s\tilde{Y}(s)$

~~a)~~ taking Laplace transform,

$$s^2 \tilde{Y}(s) + a_1 s \tilde{Y}(s) + a_0 \tilde{Y}(s) = \frac{2b_0 K_c}{s}$$
$$+ b_0 K_c \tilde{Y}(s)$$

$$\Rightarrow \tilde{Y}(s) \cancel{+} = \frac{2 b_0 K_c}{s(s^2 + a_1 s + a_0 \cancel{)} + b_0 K_c)}$$

To apply
~~By~~ Final value thm, we need the conditions
$s\tilde{Y}(s)$ to be stable.

$\Rightarrow$ roots of $s^2 + a_1 s + a_0 + b_0 K_c < 0$

$$\Rightarrow \frac{-a_1 \pm \sqrt{a_1^2 - 4(a_0 + b_0 K_c)}}{2} < 0$$

~~Note that~~ $\Rightarrow 0 \leq a_1^2 - 4(a_0 + b_0 K_c) < a_1^2$

$$\Rightarrow K_c \leq \frac{a_1^2 - 4a_0}{4b_0} \text{ and}$$
$$K_c > -\frac{a_0}{b_0}$$

$\Rightarrow K_c \leq \frac{1}{3}$ and $K_c > -5$

$\Rightarrow -5 < K_c \leq 1/3 \Rightarrow 0 < K_c \leq 1/3$

Under such conditions we apply FVT

to get $\lim_{t \to \infty} \tilde{y}(t) = \lim_{s \to 0} s\tilde{Y}(s)$

$$\rightarrow \lim_{t\to\infty} \tilde{y}(t) = \lim_{s\to 0} \cancel{s} \times \frac{2 b_0 K_c}{\cancel{s}(s^2+a_1 s+a_0+b_0 K_c)}$$

$$= 2$$

$\therefore$ Required objective is attained for $0 < K_c \leq 1/3$

d) ③ $\Rightarrow \frac{d^2\tilde{y}}{dt^2} + a_1 \frac{d\tilde{y}}{dt} + a_0 \tilde{y}(t) = b_0 \tilde{u}(t)$

Differentiate wrt time,

$$\frac{d^3\tilde{y}}{dt^3} + a_1 \frac{d^2\tilde{y}}{dt^2} + a_0 \frac{d\tilde{y}}{dt} = b_0 \frac{d\tilde{u}}{dt}$$

$$= b_0 \left(K_c\left(-\frac{d\tilde{y}}{dt}\right) + K_I (2-\tilde{y})\right)$$

For notational convenience, ~ is dropped

$$\rightarrow \frac{d^3y}{dt^3} + (a_1 \cancel{+b_0K_c}) \frac{d^2y}{dt^2} + \underbrace{(a_0+b_0K_c)}\frac{dy}{dt} + b_0 K_I y = 2 K_I b_0$$

Similar to previous part, take Laplace transform to get,

$$(s^3 + a_1 s^2 + (a_0+b_0K_c)s + b_0 K_I) Y(s) = \frac{2 K_I b_0}{s}$$

$$\Rightarrow Y(s) = \frac{2 K_I b_0}{s(s^3 + a_1 s^2 + (a_0 + b_0 K_c) s + b_0 K_I)}$$

~~Let~~ We want the poles $(sY(s))$ to be stable

$$\Rightarrow \text{roots}(s^3 + a_1 s^2 + (a_0 + b_0 K_c) s + b_0 K_I) < 0$$

$$\Rightarrow \text{roots}(s^3 + 8s^2 + (15 + 3K_c) s + 3K_I) < 0 \quad (5)$$

Assuming roots obey this property, apply FVT

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} s \times \frac{2 K_I b_0}{s(s^3 + a_1 s^2 + (a_0 + b_0 K_c) s + b_0 K_I)}$$

$$= 2$$

$\therefore$ objective is achieved.

Conclusion: The objective can't be achieved for any $\{K_c, K_I\}$, it is achieved only when (5) is satisfied.

eg. $K_c = K_I = 1$, roots are $-3.9 + 1.14j$, $-3.9 - 1.14j$, -0.181

$\Rightarrow$ objective is not achieved.

But for $K_c = 0.25$, $K_I = 0.5$

roots are -4.75, -3.14, -0.1

$\Rightarrow$ objective is achieved.

③
a) $$\frac{dw}{dt} = -\left(\frac{L+Va}{M}\right)w + \frac{Va}{M}z \quad \text{— ①}$$

$a = 0.5$
$z_f = 0.1$
$M = 40$

$$\frac{dz}{dt} = \frac{L}{M}w - \left(\frac{L+Va}{M}\right)z + \frac{Vz_f}{M} \quad \text{— ②}$$

At steady state, $L = L_{ss} = 4$, $V = V_{ss} = 200$
$= 80$

$$\frac{dw}{dt} = 0, \quad \frac{dz}{dt} = 0$$

① $\Rightarrow$ $-6.5w + 2.5z = 0$ — ③

② $\Rightarrow$ $4w - 6.5z = -0.5$ — ④

Solving ③ & ④,

$w_{ss} = 0.0388$ and $z_{ss} = 0.1008$

$\Rightarrow$ steady state values: $w = 0.0388$
$z = 0.1008$

b) In this model, states $(x) = \begin{bmatrix} w \\ z \end{bmatrix}$

Inputs $(u) = \begin{bmatrix} L \\ V \end{bmatrix}$ Let $\frac{dw}{dt} = f(\cdot)$ & $\frac{dz}{dt} = g(\cdot)$

$$\frac{dw}{dt} = \frac{dw}{dt}\Big|_{@\text{ steady state}} + \frac{\partial f}{\partial L}(L - L_{ss}) + \frac{\partial f}{\partial V}(V - V_{ss}) + \frac{\partial f}{\partial w}(w - w_{ss}) + \frac{\partial f}{\partial z}(z - z_{ss})$$

$$a_{11} = \frac{\partial f}{\partial w} = \frac{-(Va + L)}{M}$$

$$a_{12} = \frac{\partial f}{\partial z} = \frac{V}{M}$$

$$a_{21} = \frac{\partial g}{\partial w} = \frac{L}{M} \;;\; a_{22} = -\left(\frac{L + Va}{M}\right)$$

$$b_{11} = \frac{\partial f}{\partial L} = \frac{-w}{M} \;;\; b_{12} = \frac{\partial f}{\partial y} = \frac{-w}{M} + \frac{az}{M}$$

$$b_{21} = \frac{\partial g}{\partial L} = \frac{w}{M} \;;\; b_{22} = \frac{-az}{M} + \frac{zf}{M}$$

$$\therefore A = \begin{bmatrix} -6.5 & 2.5 \\ 4 & -6.5 \end{bmatrix}$$

(all values evaluated at steady state)

$$B = \begin{bmatrix} -0.0019 & 0.0016 \\ -0.0031 & 0.0025 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \frac{dw}{dt} \\ \frac{dz}{dt} \end{bmatrix} = A \begin{bmatrix} w - w_{ss} \\ z - z_{ss} \end{bmatrix} + B \begin{bmatrix} L - L_{ss} \\ V - V_{ss} \end{bmatrix}$$

Let $y = \begin{bmatrix} w - w_{ss} \\ z - z_{ss} \end{bmatrix}$

$\Rightarrow y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4.433 \\ 2-2y \end{bmatrix}$

c) eigenvalues : -3.338, -9.162

eigenvectors : $\begin{bmatrix} 0.6202 \\ 0.7845 \end{bmatrix}$ & $\begin{bmatrix} -0.6202 \\ 0.7845 \end{bmatrix}$

Since $|\lambda_2| > |\lambda_1|$

$\therefore$ ~~-9.162~~

fastest dir : eigenvector 2 : $\begin{pmatrix} -0.6202 \\ 0.7845 \end{pmatrix}$

slowest dir : eigenvector 1 : $\begin{pmatrix} 0.6202 \\ 0.7845 \end{pmatrix}$

4) a) $L\{x(t)\} =$

For $0 \le t < 3$

$L(t-2) = \int_0^3 t e^{-st} dt - 2\int_0^3 e^{-st} dt$

$= \frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2} - \frac{2}{s}\left(1 - e^{-3st}\right)\Big|_0^3$

$= \cancel{\frac{1-2s-(1+se^{-3s})}{s^2}}$

$= \frac{te^{-st}}{-s}\Big|_0^3 - \int_0^3 \frac{e^{-st}}{s} - \frac{2}{s}\left(1 - e^{-3st}\right)$

$= \frac{-3e^{-3s}}{s} + \frac{e^{-3s}}{s^2} - \frac{1}{s^2} - \frac{2}{s}\left(1 - e^{-3s}\right)$

$$\Rightarrow X(s) = \boxed{\frac{-(1+s)e^{-3s} - 2s + 1}{s^2}} \qquad 0 \le t < 3$$

$3 \le t \le 4$

$$X(s) = \int_3^4 e^{-st}\, dt$$

$$\boxed{= \frac{e^{-3s} - e^{-4s}}{s}}$$

$4 \le t < 5$

$$X(s) = \int_4^5 -\cos(3\pi(t-4))\, e^{-st}\, dt$$

$$\Rightarrow X(s) = \cancel{\cos(3\pi} \; \frac{-\cos(3\pi(t-4))\, e^{-st}}{-s}\Bigg|_4^5 + \int 3\pi \sin(3\pi(t-4))\, \frac{e^{-st}}{-s}\, dt$$

$$= \frac{-\cos(3\pi(t-4))}{-s} e^{-st} + \left(\frac{3\pi \sin(3\pi(t-4))\, e^{-st}}{-s^2} \right|_4^5$$

$$- \int \left(\frac{9\pi^2 \cos(3\pi(t-4))\, e^{-st}}{-s^2} \right) dt$$

$$\downarrow$$

$$X(s) \times \frac{9\pi^2}{s^2}$$

$$\Rightarrow X(s)\left(1 + \frac{9\pi^2}{s^2}\right) = \frac{-e^{-5s}}{s} + \frac{-3\pi e^{-4s}}{s^2}$$

$$\Rightarrow \boxed{X(s) = \frac{\left(e^{-5s} + 3\pi e^{-4s}\right)}{\frac{9\pi^2}{s^2} + 1}}$$

$t \geq 5$

$$\mathcal{L}\{e^{-2t}\, x(s)\} = \int_5^{\infty} e^{-2(t-5)} \cos(5\pi(t-5))\, e^{-st}\, dt$$

$$= \int_5^{\infty} e^{-t(s+2)-5} \cos(5\pi(t-5))\, dt$$

By parts

$$\Rightarrow X(s) = \left. \frac{e^{-t(s+2)+10} \cos(5\pi(t-5))}{-(s+2)} \right|_5^{\infty} - 5\pi \int \frac{\sin(5\pi(t-5))\, e^{-t(s+2)-5}}{s+2}\, dt$$

$$= \frac{e^{-5(s+2)+10}}{s+2} - 5\pi \left[\frac{\sin(5\pi(t-5))}{(s+2)^2} + \frac{5\pi}{4} \int \frac{\cos(5\pi(t-5))\, e^{-t(s+2)-5}}{(s+2)^2}\, dt \right]$$

$$\Rightarrow X(s)\left(1 + \frac{25\pi^2}{(s+2)^2}\right) = e^{-5s}$$

$$\Rightarrow \boxed{X(s) = \frac{e^{-5s}}{1 + \frac{25\pi^2}{(s+2)^2}}}$$

④ b) $\dfrac{s-2}{T^2 s\left(s^2 + \frac{2 \xi s}{T} + 1\right)}$

Roots of $s^2 + \dfrac{\xi s}{T} + \dfrac{1}{T^2}$:

$$\alpha = -\frac{\xi}{T} + \frac{\sqrt{\xi^2 - 1}}{T}$$

$$\beta = -\frac{\xi}{T} - \frac{\sqrt{\xi^2 - 1}}{T}$$

i) ~~Real~~ $\xi > 1$, real unique roots

Let $\dfrac{A_1}{s} + \dfrac{A_2}{s-\alpha} + \dfrac{A_3}{s-\beta} = X(s) = \dfrac{s-2}{T^2 s\left(s^2 + \frac{2 \xi s}{T} + 1\right)}$

$$\Rightarrow A_1(s-\alpha)(s-\beta) + A_2 s(s-\beta) + A_3 s(s-\alpha) = \frac{s-2}{T^2}$$

Put $s = 0$,

$$A_1 = \frac{-2}{T^2 \alpha \beta}$$

Put $s = \alpha$,

$$A_2 = \frac{\alpha - 2}{T^2 \alpha (\alpha - \beta)}$$

Put $s = \beta$,

$$A_3 = \frac{\beta - 2}{T^2 \beta (\beta - \alpha)}$$

$$\therefore \quad X(s) = \frac{A_1}{s} + \frac{A_2}{s-\alpha} + \frac{A_3}{s-\beta}$$

$$= \left(\frac{-2}{T^2 \alpha \beta}\right) \times \frac{1}{s} + \left(\frac{\alpha - 2}{T^2 (\alpha)(\alpha-\beta)}\right) \frac{1}{s-\alpha} + \left(\frac{\beta - 2}{T^2 \beta (\beta - \alpha)}\right) \frac{1}{s-\beta}.$$

Take Laplace inverse.

& use the property

$$\mathcal{L}^{-1}\{F(s) + G(s)\} = f(t) + g(t)$$

$$\Rightarrow x(t) = \frac{-2}{T^2 \alpha \beta} \exp(0t) + \frac{\alpha - 2}{T^2 (\alpha)(\alpha - \beta)} \exp(\alpha t) + \frac{\beta - 2}{T^2 \beta (\beta - \alpha)} \exp(\beta t)$$

$$\alpha \beta = \frac{1}{T^2} \; ; \; \alpha - \beta = \frac{2\sqrt{\zeta^2 - 1}}{T}$$

$$\Rightarrow x(t) = -2 + \frac{(\alpha - 2)\beta}{T^2 (\alpha \beta)(\alpha - \beta)} \exp(\alpha t) + \frac{(\beta - 2)\alpha}{T^2 \alpha \beta (\beta - \alpha)} \exp(\beta t)$$

$$\Rightarrow x(t) = -2 + \frac{(\alpha - 2)\beta \exp(\alpha t)}{\alpha - \beta} + \frac{(\beta - 2)\alpha \exp(\beta t)}{(\beta - \alpha)} \quad ★$$

$$= -2 + \frac{1}{T}\frac{\exp(\alpha t)}{2\sqrt{\zeta^2-1}} - 2\left(\frac{-\zeta - \sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}\right)\exp(\alpha t)$$

$$- \frac{1}{T}\frac{\exp(\beta t)}{2\sqrt{\zeta^2-1}} + \frac{2\left(-\zeta + \sqrt{\zeta^2-1}\right)\exp(\beta t)}{2\sqrt{\zeta^2-1}}$$

$$\Rightarrow x(t) = -2 + \frac{1}{2T\sqrt{\zeta^2-1}}\exp\left(\left(\frac{-\zeta + \sqrt{\zeta^2-1}}{T}\right)t\right) + \left(\frac{\zeta + \sqrt{\zeta^2-1}}{\sqrt{\zeta^2-1}}\right)\exp\left(\left(\frac{-\zeta + \sqrt{\zeta^2-1}}{T}\right)t\right) - \cancel{\frac{1}{T}}$$

$$- \frac{1}{2T\sqrt{\zeta^2-1}}\exp\left(\left(\frac{-\zeta \mp \sqrt{\zeta^2-1}}{T}\right)t\right) + \left(\frac{-\zeta + \sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}\right)\exp\left(\left(\frac{-\zeta - \sqrt{\zeta^2-1}}{T}\right)t\right)$$

$$\Rightarrow \boxed{x(t) = -2 + \frac{1}{2T\sqrt{\zeta^2-1}}\exp\left(\left(\frac{-\zeta + \sqrt{\zeta^2-1}}{T}\right)t\right) - \frac{1}{2T\sqrt{\zeta^2-1}}\exp\left(\left(\frac{-\zeta - \sqrt{\zeta^2-1}}{T}\right)t\right) + \left(\frac{\zeta + \sqrt{\zeta^2-1}}{\sqrt{\zeta^2-1}}\right)\exp\left(\left(\frac{-\zeta + \sqrt{\zeta^2-1}}{T}\right)t\right) + \left(\frac{-\zeta + \sqrt{\zeta^2-1}}{\sqrt{\zeta^2-1}}\right)\exp\left(\left(\frac{-\zeta - \sqrt{\zeta^2-1}}{T}\right)t\right)}$$

ii) $\zeta = 1$

$$X(s) = \frac{A_1}{s} + \frac{A_2}{s-\alpha} + \frac{A_3}{(s-\alpha)^2}.$$

$$\alpha = \beta = -\frac{\zeta}{T} = -1/T$$

$$\Rightarrow A_1(s-\alpha)^2 + A_2(s-\alpha)s + A_3 s = \frac{s-2}{T^2}$$

$$2A_1(s-\alpha) + A_2\,(2s-\alpha) = \frac{1}{T^2} \Rightarrow A_1 = -A_2$$

$$\Rightarrow A_2 = \frac{2}{\alpha^2 T^2} \;;\; A_3 = \frac{\alpha-2}{T^2\alpha}$$

$$A_1 = \frac{-2}{T^2\alpha^2}$$

$$\therefore X(s) = \left(\frac{-2}{T^2\alpha^2}\right)\left(\frac{1}{s}\right) + \left(\frac{2}{\alpha^2 T^2}\right)\frac{1}{(s-\alpha)} + \left(\frac{\alpha-2}{\alpha T^2}\right)\frac{1}{(s-\alpha)^2}$$

take inverse Laplace transform.

$$\Rightarrow x(t) = \frac{-2}{T^2\alpha^2}\exp(0t) + \frac{2}{\alpha^2 T^2}\exp(\alpha t) + \left(\frac{\alpha-2}{T^2}\right)\frac{t}{\alpha}\exp(\alpha t)$$

Subst. α 1

$$x(t) = -2 + \frac{2}{\alpha^2 T^2}\left(\exp\left(-\frac{t}{T}\right)\right) + \frac{t}{T^2}\exp\left(-\frac{t}{T}\right) + \frac{2t}{T}\exp\left(-\frac{t}{T}\right)$$

$$\boxed{\Rightarrow x(t) = -2 \left(\exp\left(-\frac{t}{T}\right)\right)\left(\frac{t}{T^2} + \frac{2t}{T} + 2\right)}$$

iii) $\xi < 1$, complex roots

Simplification
~~Soln~~ same as.. in (i) till the eqn marked as '$\circledast$'

Now we know that the solution will be real so we just take the real part by simplifying using euler's formula.

$$\alpha - \beta = \frac{2j\sqrt{1-\xi^2}}{T}$$

$$\cancel{x(t) = -L + (\alpha\beta)\exp(\alpha t)}$$

$$\alpha\beta = \frac{1}{T^2} \qquad = \frac{2\sqrt{1-\xi^2}}{T}\exp\left(\frac{-\pi}{2}j\right)$$

$$\alpha = \frac{1}{T}\exp\left(\tan^{-1}\left(-\frac{\sqrt{-\xi^2+1}}{T}\right) \times j\right)$$

$$\beta = \frac{1}{T}\exp\left(\tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{T}\right)j\right)$$

$$\therefore x(t) = -2 + \frac{\alpha\beta \exp(\alpha t)}{\alpha-\beta} - \frac{2\beta \exp(\alpha t)}{\alpha-\beta}$$

$$+ \frac{\alpha\beta \exp(\beta t)}{\beta-\alpha} - \frac{2\alpha}{(\beta-\alpha)}\exp(\beta t)$$

term 1 : $\dfrac{\frac{\alpha\beta}{T^2}}{\alpha-\beta}\exp(\alpha t)$

$$= \frac{1}{2T^2 \frac{\sqrt{1-\zeta^2}}{T}\exp\left(-\frac{\pi}{2}j\right)}\exp\left(-\frac{\zeta t}{T}\right)\exp\left(j\left(\sqrt{1-\zeta^2}\right)t\right)$$

$$= \frac{1}{2T\sqrt{1-\zeta^2}}\exp\left(\frac{-\zeta t}{T}\right)\exp\left(j\left(\frac{\sqrt{1-\zeta^2}}{T}t+\frac{\pi}{2}\right)\right)$$

term2 : $\dfrac{-2\beta}{\alpha-\beta}\exp(\alpha t)$

$$= \frac{\frac{1}{T}-\beta}{2\sqrt{1-\zeta^2}}\exp\left(\frac{-\zeta t}{T}\right)\exp\left(j\left(\frac{\pi}{2}+\frac{\sqrt{1-\zeta^2}}{T}\right)t\right)$$

$$= \frac{1}{\zeta\sqrt{1-\zeta^2}}\exp\left(\frac{-\zeta t}{T}\right)\exp\left(j\left(\left(\frac{\pi}{2}+\frac{\sqrt{1-\zeta^2}}{T}\right)t+\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)\right)$$

term 3 : $\dfrac{\alpha\beta}{\beta-\alpha}\exp\left(-\dfrac{\zeta t}{T}\right)\exp\left(-j\left(\sqrt{1-\zeta^2}\,t\right)\right)$

$$= \frac{-1}{2T\sqrt{1-\zeta^2}}\exp\left(\frac{-\zeta t}{T}\right)\exp\left(j\left(-\sqrt{1-\zeta^2}\frac{t}{T}+\frac{\pi}{2}\right)\right)$$

$\Rightarrow x(t) =$

$$-2 + \frac{\exp\left(\frac{-\varphi}{T}\right)}{T\sqrt{1-\varphi^2}}\left(\sin\left(\frac{\sqrt{1-\varphi^2}\,t}{T}\right) + \sin\left(\frac{\sqrt{1-\varphi^2}\,t}{T} + \tan^{-1}\left(\frac{\sqrt{1-\varphi^2}}{\varphi}\right)\right)\right)$$

Didnot show some steps to reduce pdf size.

Will send if required. (I have written but I am not including in scan.)