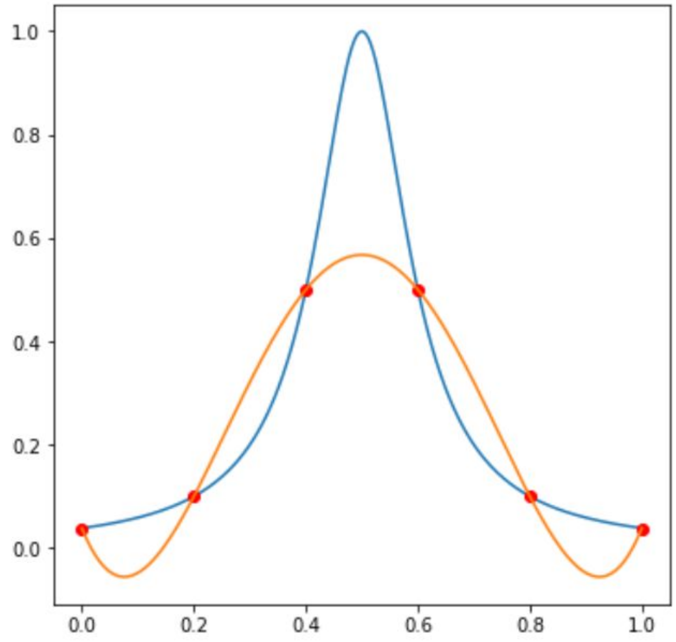
Interpolation



Interpolation

Given (n+1) points $\{X_i\}^n_{i=0}$ in the interval [0,1], the **Lagrange interpolation operator** is:

$$\mathcal{L}^n : C^0([0,1]) \mapsto \mathcal{P}^n$$

such that:

$$(\mathcal{L}^n f)(X_i) = f(X_i), \qquad i = 0, \ldots, n.$$

$C^0([0,1])$ infinitely dimensional space

$\mathcal{P}^n$ finite dimensional space of polynomials of order n

Interpolation: monomial basis

Such a space has dimension n+1 , and can be constructed using n+1 linear combinations of linear independent polynomials of order ≤n, for example the monomials:

$$\mathcal{P}^n = \text{span}\{v_i := x^i\}_{i=0}^n$$

To recall: If we want to construct the Lagrange interpolation of a given function on equispaced n+1 points in [0,1], then we are actively looking for an element of $\mathcal{P}^n$ that coincides with the function at these given points.

Interpolation: problem formulation

Given a basis $\{v_i\}^n_{i=0}$, any element of $\mathcal{P}^n$ can be written as a linear combination of the basis, i.e.,

$$\forall p \in \mathcal{P}^n, \quad \exists! \{p^i\}_{i=0}^n \quad | \quad p(x) = \sum_{i=0}^{n} p^i v_i(x)$$

If we want to solve the interpolation problem above, then we need to find the coefficients p^i of the polynomial p that interpolates at the points X_i:

$$v_j(X_i)p^j = f(X_i), \qquad \Longleftrightarrow \qquad [[V]][p] = [F]$$

Interpolation: Lebesgue constant

Remember this inequality and the Lebesgue function:

$$||\mathcal{L}^n \boldsymbol{u}||_{L^\infty} \leq ||V^{-1}||_{l^\infty} ||\Lambda||_{L^\infty} ||\boldsymbol{u}||_{L^\infty}$$

where: $\Lambda = \sum_j |v_j|$

Interpolation: lagrange basis

Given (n+1) points $\{X_i\}^n_{i=0}$ in the interval [0,1], the **Lagrange interpolation operator** is:

$$\mathcal{L}^n : C^0([0,1]) \mapsto \mathcal{P}^n$$

such that:

$$(\mathcal{L}^n f)(x) = \sum_{i=0}^{n} f(X_i)\ell_i(x), \qquad i = 0, \ldots, n.$$

where:

$$\ell_i(x) := \prod_{i \neq j, j=0}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

Interpolation: lagrange basis

In this case:

$$V_{ij} := \ell_j(x_i)$$

therefore:

$$(\mathcal{L}^n u)(x_i) := \sum_{j=0}^{n} u(X_j)\ell_j(x_i) = \sum_j V_{ij} u(X_j)$$

Interpolation: Runge phenomenon

The **Weierstrass approximation theorem** states that for every continuous function f(x) defined on an interval [a,b], there exists a set of polynomial functions $P_n(x)$ for n=0, 1, 2, …, each of degree at most n, that approximates f(x) with uniform convergence over [a,b] as n tends to infinity, that is,

$$\lim_{n\to\infty}\left(\sup_{a\le x\le b}|f(x)-P_n(x)|\right)=0.$$

The theorem only states that a set of polynomial functions exists, without providing a general method of finding one. The $P_n(x)$ produced in this manner may in fact diverge away from f(x) as n increases;

Interpolation: find lagrange from monomial basis

We could compute a new basis for which the interpolation matrix V is the identity, in order to reduce the problems with the condition numbers of the matrix.
Ve would like to build the new basis in such a way that they satisfy:

$$[V]_{ij} := \ell_i(X_j) = \delta_{ij}$$

The coefficients (in this basis) of the polynomial are given by:

$$[p] = [V]^{-1}[F] = [F]$$

Interpolation: find lagrange from monomial basis

How do we compute the lagrange basis, once we have the monomial basis?

In the previous computation, we constructed the polynomial p(x) evaluated on x.
Let's call this vector [P]:

$$p(x) = \sum_{i=0}^{n} p^i v_i(x) \quad \Longrightarrow \quad [P] = [B][p] = [B][V]^{-1}[F]$$

Therefore, the equivalent of the matrix B for the lagrange basis must be

$$[B_\ell] := [B][V]^{-1} \quad \Longrightarrow \quad \ell_i = \sum_j [V]_{ij}^{-T} v_j$$

Interpolation: Runge phenomenon

if $\boldsymbol{u} \in C^0([a,b])$ is analytically extendible in an oval of radius R:

$$O(a,b,R) = \{z \in \mathbb{C} | dist(z,[a,b]) \leq R\}$$

then:

$$||\boldsymbol{u}^{n+1}||_{L^\infty} \leq \frac{(n+1)!}{R^{n+1}} ||\tilde{u}||_{L^\infty_{O(a,b,R)}}$$

where:

$$\tilde{u}|_{[a,b]} = \boldsymbol{u} \qquad \tilde{u} : \mathbb{C} \to \mathbb{R}$$

$$\Longrightarrow \quad ||\mathcal{L}^n \boldsymbol{u} - \boldsymbol{u}||_{L^\infty} \leq \left(\frac{(b-a)}{R}\right)^{n+1} ||\tilde{u}||_{L^\infty_{O(a,b,R)}}$$