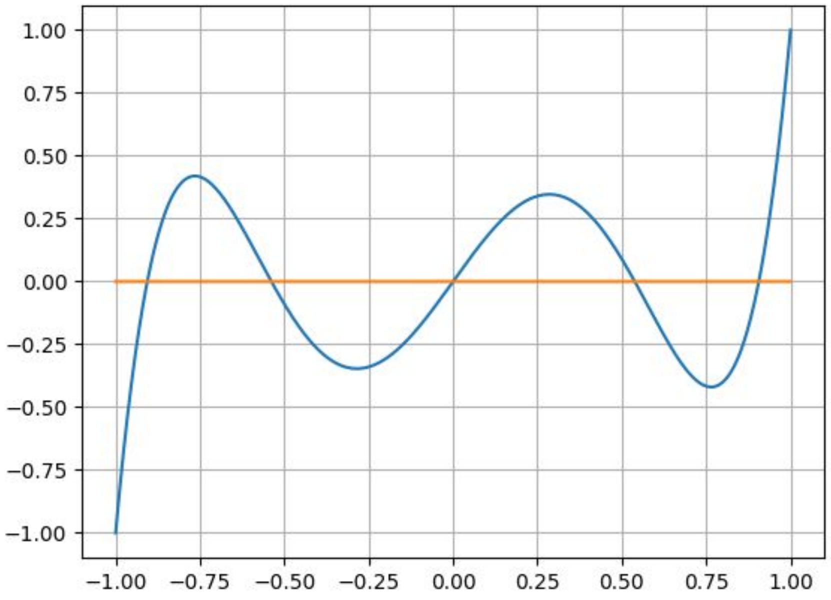
# Nonlinear Equations



# Fixed point iterations

Given a function f, the root of f can be found as follow:

$$f(x) = 0 \rightarrow x - \phi(x) = 0$$
$$x^{k+1} = \phi(x^k)$$

We want to solve $f = \frac{t}{8}\left(63t^4 - 70t^2 + 15\right) = 0$

Then we can focus our attention on the roots finding of $f_1 = 63t^4 - 70t^2 + 15.$

We recast $f_1(t)$ in term of t-φ(t).

# Fixed point iterations

We consider different ways to go:

1. divide by 70t $$\frac{63}{70}t^3 - t + \frac{15}{70t} = 0 \implies \phi_1 = \frac{63}{70}t^3 + \frac{15}{70t}$$
2. divide by $63t^3$ $$t - \frac{70}{63t} + \frac{15}{63t^3} = 0 \implies \phi_2 = \frac{70}{63t} - \frac{15}{63t^3}$$
3. multiply by t/15 $$\frac{63}{15}t^5 - \frac{70}{15}t^3 + t = 0 \implies \phi_3 = -\frac{63}{15}t^5 + \frac{70}{15}t^3$$
4. $$70t^2 = 63t^4 + 15 \implies t = \sqrt{\frac{63t^4 + 15}{70}} \implies \phi_4 = \sqrt{\frac{63t^4 + 15}{70}}$$

# Recall

**Proposition 1.** *(Global convergence)*
*1. Assume that* $\phi(x)$ *is continuous on* $[a,b]$ *and such that* $\phi(x) \in [a,b]$ *for all* $x \in [a,b]$*; then there exists at least one fixed point* $\alpha \in [a,b]$ *of* $\phi$.

*2. If* $\exists L < 1$ *such that* $|\phi(x_1) - \phi(x_2)| \leq L|x_1 - x_2|\ \forall x_1, x_2 \in [a,b]$,

*then there exists a unique fixed point* $\alpha \in [a,b]$ *and the sequence* $x^{(k+1)} = \phi(x^{(k)})$, $k \geq 0$ *converges to* $\alpha$, *for any initial guess* $x^{(0)} \in [a,b]$.

# Recall

**Remark 1.**
*If* $\phi(x)$ *is differentiable in* $[a, b]$ *and*

$\exists K < 1$ *such that* $|\phi'(x)| \leq K \ \forall x \in [a, b]$,

*then the condion 2 of the proposal* (1) *is satisfied. This assumption is stronger, but is more often used in practice because it is easier to check.*

**Proposition 2.** *(Local convergence - Theorem 2.1 in the book)*
*Let $\phi$ be a continuous and differentiable function on $[a, b]$ and $\alpha$ be a fixed point of $\phi$. If $\mid \phi'(\alpha) \mid < 1$, then there exists $\delta > 0$ such that, for all $x^{(0)}$, $\mid x^{(0)} - \alpha \mid \leq \delta$, the sequence $\{x^{(k)}\}$ defined by $x^{(k+1)} = \phi(x^{(k)})$ converges to $\alpha$ when $k \to \infty$.*
*Moreover, it holds*

$$\boxed{\lim_{k\to\infty} \frac{x^{(k+1)} - \alpha}{x^{(k)} - \alpha} = \phi'(\alpha).}$$

Note that, if $0 <\mid \phi'(\alpha) \mid < 1$, then for any constant $C$ such that $|\phi'(\alpha)| < C < 1$, if $k$ is large enough, we have:

$$\mid x^{(k+1)} - \alpha \mid \leq C \mid x^{(k)} - \alpha \mid .$$

# Recall

**Proposition 3.** *(Proposition 2.2 in the book)*
*Let $\phi$ be a twice differentiable on $[a, b]$ and $\alpha$ be a fixed point of $\phi$. Let us consider that $x^{(0)}$ converges locally. If $\phi'(\alpha) = 0$ and $\phi''(\alpha) \neq 0$, then the fixed point interations converges with order 2 and*

$$\boxed{\lim_{k \to \infty} \frac{x^{(k+1)} - \alpha}{(x^{(k)} - \alpha)^2} = \frac{\phi''(\alpha)}{2}.}$$

# Least squares

Consider n+1 points x0, x1, …, xn and n+1 values y0, y1, …, yn.
If n is large, interpolation (polynomial of degree n) can show large oscillations.

Instead of interpolating the values, it is possible to define a **polynomial of degree m** < n that approximate the data at best.

The **least square polynomial** is the polynomial of degree m that **minimize distance to the data**:

$$\sum_{i=0}^{n} |y_i - \tilde{f}_m(x_i)|^2 \leq \sum_{i=0}^{n} |y_i - p_m(x_i)|^2 \qquad \forall p_m(x) \in \mathbb{P}_m$$

# Least squares

If the polynomial is:

$$\tilde{f}_m(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m$$

We can define the function:

$$\Phi(a_0, a_1, \ldots, a_m) = \sum_{i=0}^{n} \left| y_i - \left(a_0 + a_1 x_i + a_2 x_i^2 + \ldots + a_m x_i^m\right) \right|^2$$

Then the coefficients are the solutions of:

$$\frac{\partial \Phi}{\partial a_k} = 0, \qquad k = 0, \ldots, m,$$

# Least squares

The solution of the previous problem is equivalent to solve:

$$B^T B\mathbf{a} = B^T \tilde{\mathbf{y}}.$$

where

$$B = \begin{pmatrix} 1 & x_0 & \dots & x_0^m \\ 1 & x_1 & \dots & x_1^m \\ \vdots & & & \vdots \\ 1 & x_n & \dots & x_n^m \end{pmatrix}$$

# Least squares: exercises

The result of census of the population of Switzerland between 1900 and 2010 (in thousands) is summarized in the table.

- Is it possible to estimate the number of inhabitants of Switzerland during the year when there has not been census, for example in 1945 and 1975?

- Is it possible to predict the number of inhabitants of Switzerland in 2020?

| year | population |
|---|---|
| 1900 | 3315 |
| 1910 | 3753 |
| 1920 | 3880 |
| 1930 | 4066 |
| 1941 | 4266 |
| 1950 | 4715 |
| 1960 | 5429 |
| 1970 | 6270 |
| 1980 | 6366 |
| 1990 | 6874 |
| 2000 | 7288 |
| 2010 | 7783 |