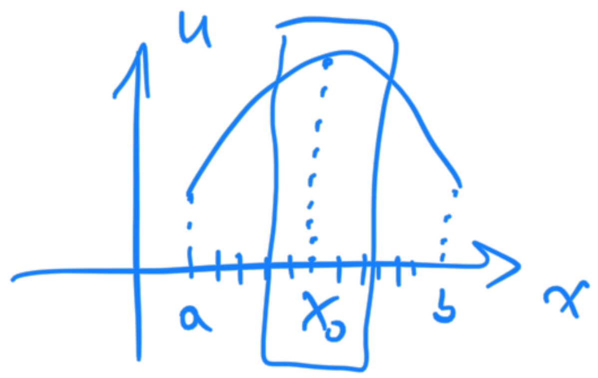
1) FINITE DIFFERENCE METHOD

2) FINITE ELEMENTS METHOD

$$\begin{cases} u'' = 1 & \text{in } x \in (a, b) \\ u(a) = \alpha & \alpha, \beta \in R \\ u(b) = \beta \end{cases}$$

$x_0 \in (a, b)$

$a = x_0 < x_1 < \dots < x_i < \dots < x_n = b$

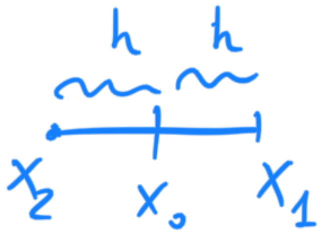


n = number of points

$$h = \frac{b-a}{n-1}$$

$X_1 = X_0 + h$

$X_2 = X_0 - h$



$$u'(X_0) \simeq F\{u(X_0), u(X_2), u(X_1)\}$$

1) $u(X_0) = u(X_0)$

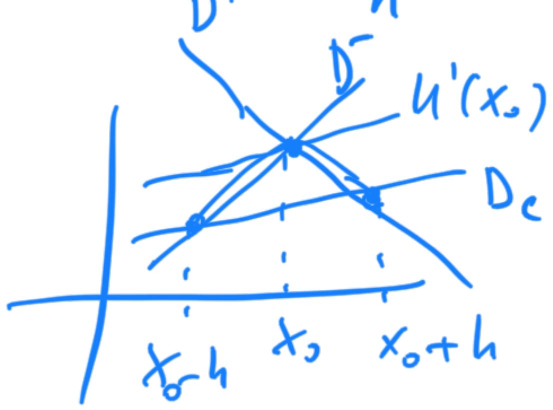
2) $u(X_0 + h) = u(X_0) + u'(X_0)h + \frac{u''(X_0)}{2}h^2 +$
$+ \frac{u'''(X_0)}{6}h^3 + O(h^4)$

3) $u(X_0 - h) = u(X_0) - u'(X_0)h + u''(X_0)h^2 -$

$$- \frac{u'''(x_0)}{6} h^3 + O(h^4)$$

$$\underbrace{\frac{u(x_0+h) - u(x_0)}{h} = u'(x_0) + \underbrace{\frac{u''(x_0)}{2} h +}_{} + \frac{u'''(x_0)}{6} h^2 + O(h^3)}_{}$$

$$\Rightarrow D^+ = \frac{u(x_0+h) - u(x_0)}{h} = \underline{u'(x_0)} + O(h)$$



$$D^- = \frac{u(x_0) - u(x_0-h)}{h} = u'(x_0) + O(h)$$

$$D^c = \frac{u(x_0+h) - u(x_0-h)}{2h} = u'(x_0) + O(h^2)$$

$$D_c^2(u(x_0)) = D^+(D^- u(x_0)) =$$

$$= D^+\left(\frac{u(x_0) - u(x_0-h)}{h}\right) = \frac{1}{h}\left\{D^+ u(x_0) - \right.$$

$$\left. - D^+ u(x_0-h)\right\} =$$

$$= \frac{1}{h}\left\{\frac{u(x_0+h) - u(x_0)}{h} - \frac{u(x_0) - u(x_0-h)}{h}\right\} =$$

$$= \frac{1}{h^2}\left\{ \underbrace{u(x_0+h)}_{\text{Taylor exp.}} - 2u(x_0) + \underbrace{u(x_0-h)}_{\text{Taylor exp}} \right\}$$

$$\Rightarrow \quad \frac{1}{h^2}\left\{ u(x_0+h) - 2u(x_0) + u(x_0-h) \right\} = u''(x_0) + O(h^2)$$

$$\Rightarrow \quad \boxed{u^{k}(x_0) = \sum_{i=0}^{2} u(x_i)\,\alpha_i}$$

$x_2 = x_0 - h$

$x_1 = x_0 + h$

x_0

UNKOWN COEFFICIENTS METHOD

$$u'(x_0) \cong \alpha u(x_0) + \beta u(x_0+h)$$

$$\alpha u(x_0) = \alpha u(x_0)$$

$$\beta u(x_0+h) = \beta u(x_0) + \beta u'(x_0) h + \beta \frac{u''(x_0)}{2} h^2 + \beta \frac{u'''(x_0)}{6} h^3 + O(h^4)$$

$$\alpha u(x_0) + \beta u(x_0+h) = \overbrace{(\alpha+\beta)}^{=0} u(x_0) + \overbrace{\beta u'(x_0) h}^{=1} + \beta \frac{u''(x_0)}{2} h^2 + \beta \frac{u'''(x_0)}{6} h^3 + O(h^4)$$

$$\begin{cases} \alpha + \beta = 0 \\ \alpha h = 1 \end{cases} \Rightarrow \begin{cases} \alpha = -\frac{1}{h} \\ \beta = 1 \end{cases}$$

$\beta h - 1 \qquad h$

$$u'(x_0) \simeq \alpha u(x_0) + \beta u(x_0+h) = \frac{u(x_0+h) - u(x_0)}{h}$$

$$u''(x_0) \simeq \alpha u(x_0) + \beta u(x_0+h) + \gamma u(x_0-h)$$

$$\alpha u(x_0) = \alpha u(x_0)$$

$$\beta u(x_0+h) = \beta u(x_0) + \beta u'(x_0)h + \beta \frac{u''(x_0)h^2}{2} + \dots$$

$$\gamma u(x_0-h) = \gamma u(x_0) - \gamma u'(x_0)h + \gamma \frac{u''(x_0)}{2} h^2 + \dots$$

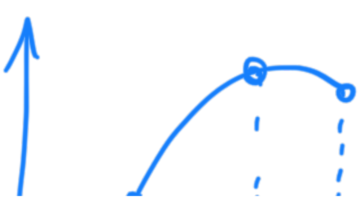
$$\begin{cases} \alpha + \beta + \gamma = 0 \\ \beta h - \gamma h = 0 \\ \beta \frac{h^2}{2} + \frac{\gamma h^2}{2} = 1 \end{cases}$$

$$u'(x_0) \simeq \alpha u(x_0) + \beta u(x_0+h) + \gamma u(x_0+2h)$$

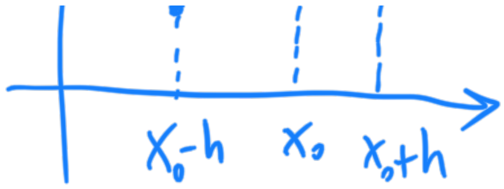
$$u'(x_0) = \frac{u(x_0+h) - u(x_0)}{h}$$

$$u'(x_0) = \frac{\cancel{u(x_0)} + u'(x_0)h + \frac{u''(x_0)h^2}{2} + O(h^3) - \cancel{u(x_0)}}{h} =$$

$$= u'(x_0) + u''(x_0)h + \dots = u'(x_0) + O(h)$$



Let's try to interpolate the tree points



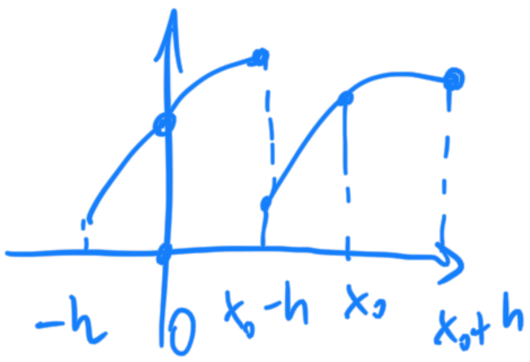
$(x_0-h,\ u(x_0-h))$, $(x_0, u(x_0))$ and

$(x_0+h,\ u(x_0+h))$

$(1, x, x^2)$ $\qquad y = ax^2 + bx + c$

such that $\begin{cases} y(x_0-h) = u(x_0-h) \\ y(x_0) = u(x_0) \\ y(x_0+h) = u(x_0+h) \end{cases}$

$$x^* = x - x_0 \implies \begin{cases} x_0 \to 0 \\ x_0+h \to h \\ x_0-h \to -h \end{cases}$$



$$\begin{cases} ax^2 + bx + c\,\big|_{x=0} = u(0) \\ ax^2 + bx + c\,\big|_{x=h} = u(h) \\ ax^2 + bx + c\,\big|_{x=-h} = u(-h) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} c = u(0) \\ ah^2 + bh + u(0) = u(h) \\ ah^2 - bh + u(0) = u(-h) \end{cases}$$

$$\implies a = \frac{u(h) - u(0) - bh}{h^2}$$

$$\cancel{h^2}\ \frac{u(h)-u(0)-bh}{\cancel{h^2}} - bh + u(0) = u(-h)$$

$$u(h) - \cancel{u(0)} - bh - bh + \cancel{u(0)} = u(-h)$$

$$-2bh = u(-h) - u(h)$$

$$b = \frac{u(-h)-u(h)}{-2h} = \frac{u(h)-u(-h)}{2h}$$

$$a = \frac{u(h)-u(0)-bh}{h^2} = \frac{u(h)-u(0)-\frac{1}{2}(u(h)-u(-h))}{h^2}$$

$$= \frac{\frac{1}{2}u(h) - u(0) + \frac{1}{2}u(-h)}{h^2} = \frac{u(h)-2u(0)+u(-h)}{2h^2}$$

$$\boxed{y = \frac{u(h)-2u(0)+u(-h)}{2h^2}x^2 + \frac{u(h)-u(-h)}{2h}x + u(0)}$$

$$\boxed{y' = \frac{u(h)-2u(0)+u(-h)}{h^2}x + \frac{u(h)-u(-h)}{2h}}$$

$$\boxed{y'' = \frac{u(h)-2u(0)+u(-h)}{h^2}} \Longleftrightarrow D_c^2(u(x_0))$$

$$y'(0) = \frac{u(h)-u(-h)}{2h}$$

HERMITE INTERPOLATION

$u(x_0)$, $u(x_0-h)$, $u(x_0+h)$ $\Rightarrow$ We have considered a polynomial of order 2

$$\underbrace{u(x_0),\ u(x_0-h),\ u(x_0+h),\ u'(x_0-h),\ u'(x_0+h)}$$

5 information $\Rightarrow$ We can try to interpolate with a polynome of order 4

$$y = ax^4 + bx^3 + cx^2 + dx + e$$

$$y' = 4ax^3 + 3bx^2 + 2cx + d$$

$$\underbrace{\begin{cases} y(0) = e = u(0) \\ y(h) = ah^4 + bh^3 + ch^2 + dh + u(0) = u(h) \\ y(-h) = ah^4 - bh^3 + ch^2 - dh + u(0) = u(-h) \\ y'(h) = 4ah^3 - 3bh^2 + 2ch + d = \hat{y}'(h) \\ y'(-h) = -4ah^3 + 3bh^2 - 2ch + d = \hat{y}'(-h) \end{cases}}$$

$$\hat{y}(x_0) = \frac{y(x_0+h) - y(x_0)}{h} \simeq y'(x_0)$$

$$y'(0) \simeq \hat{y}'(0) \overset{d}{=} -\frac{1}{4}\left(\hat{y}'(h) + \hat{y}'(-h)\right) + \frac{3}{4}\left(y(h) - y(-h)\right)$$

$$\hat{y}'(h) = -\frac{1}{4}\left(\hat{y}'(2h) + \hat{y}'(0)\right) + \frac{3}{4}\left(y(2h) - y(0)\right)$$

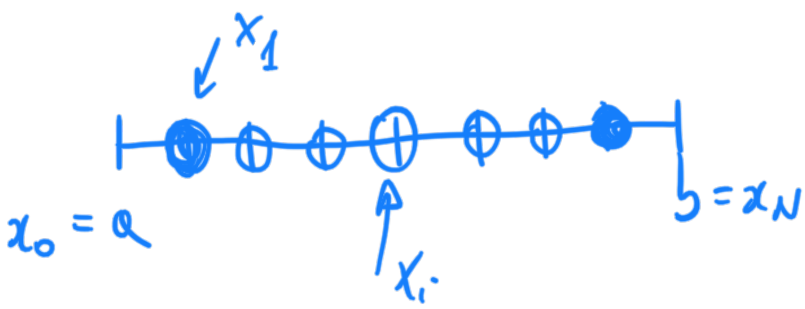
$$\Downarrow$$

$$A\hat{y}' = By$$

$$\underline{\hat{y}}' = \underline{\underline{C}}\;\underline{y} \qquad \Downarrow \qquad \underline{y}' = \underline{\underline{A}}^{-1}\,\underline{\underline{B}}\;\underline{y}$$

COMPACT FINITE DIFFERENCE SCHEMES

$$\begin{cases} -u'' = f & \text{in } x \in (a,b) \\ u(a) = \alpha & \alpha, \beta \in \mathbb{R} \\ u(b) = \beta \end{cases}$$



$a = x_0 < x_1 < \dots < x_i < \dots < x_N = b$

$|x_{i+1} - x_i| = h \quad \forall i$

On x_i we need to enforce the diff. equation above

$u(x_i) = u_i$

$$u'' \simeq D^2(u(x_i)) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

$$-u'' = f \implies -\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f_i$$

$\downarrow$ ANALYTICAL $\qquad\qquad \downarrow$ numerical (in x_i)

$$x_1 \implies -\frac{u_2 - 2u_1 + u_0}{h^2} = f_1$$

$$x_2 \Rightarrow \quad -\frac{u_3 - 2u_2 + u_1}{h^2} = f_2$$

$$\vdots$$

$$\begin{bmatrix} 1 & 0 & 0 & \dots & & 0 \\ -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} & 0 & \dots & 0 \\ 0 & -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} & \dots & \\ & & & & & \\ 0 & 0 & 0 & \dots & & 1 \end{bmatrix} \begin{pmatrix} \cancel{u_0} \\ u_1 \\ \vdots \\ u_i \\ \vdots \\ \cancel{u_N} \end{pmatrix} = \begin{pmatrix} \alpha \\ f_1 \\ \vdots \\ f_i \\ \vdots \\ \beta \end{pmatrix}$$

$$x_1 \Rightarrow \quad -\frac{u_2 - 2u_1 + \alpha}{h^2} = f_1 \qquad (1)$$

$$x_{N-1} \Rightarrow \quad -\frac{\beta - 2u_{N-1} + u_{N-2}}{h^2} = f_{N-1} \qquad (2)$$

$$(1) \qquad -\frac{u_2 - 2u_1}{h^2} = f_1 + \frac{\alpha}{h^2}$$

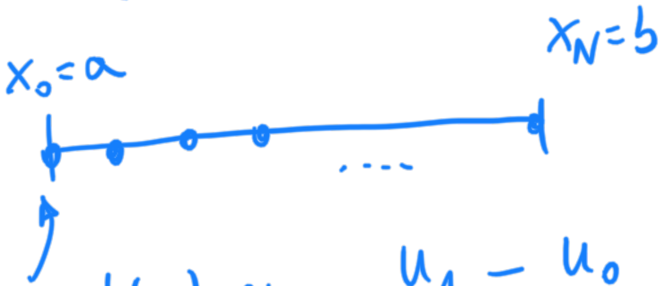
$$(2) \qquad -\frac{u_{N-2} - 2u_{N-1}}{h^2} = f_{N-1} + \frac{\beta}{h^2}$$

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ \dots & & & & \\ \dots & & & & \\ 0 & 0 & \dots & 2 & -1 \end{bmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} f_1 + \frac{\alpha}{h^2} \\ f_2 \\ \vdots \\ f_{N-1} + \frac{\beta}{h^2} \end{pmatrix}$$

Now we try to consider a Neuman BC on the left extreme of the interval

$$\begin{cases} \boxed{-u'' = f} \end{cases}$$

$$\begin{cases} \dots \\ u'(a) = \alpha \\ u(b) = \beta \end{cases}$$



$$u'(a) \simeq \frac{u_1 - u_0}{h} = \alpha \quad (D^+)$$

$$\begin{bmatrix} -1/h & 1/h & 0 \dots 0 \\ & --- & \\ & --- & \\ & & \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_i \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} \alpha \\ -- \\ -- \end{pmatrix}$$

$$u'(a) \simeq \alpha u_0 + \beta u_1 + \gamma u_2$$

Extra point tecnique

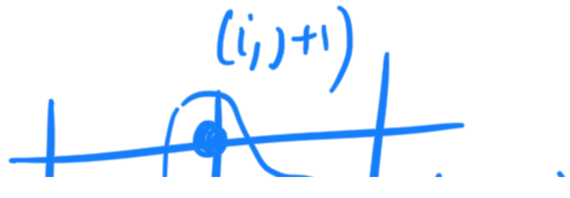
$$u'(a) \simeq \frac{u_1 - u_{-1}}{2h} = \alpha$$

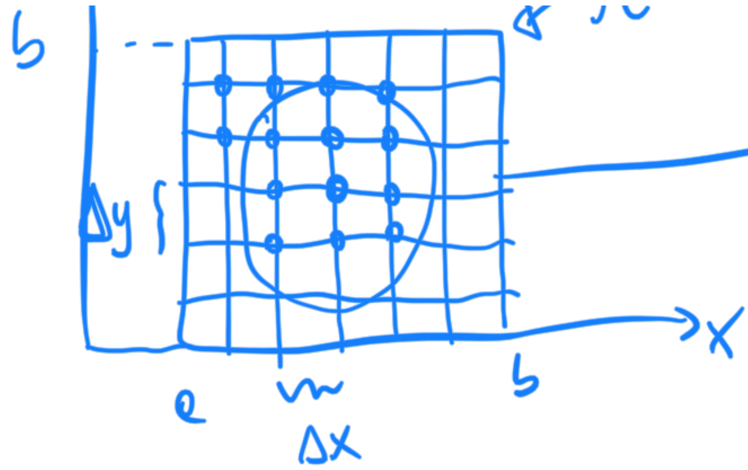
$$-\frac{u_1 - 2u_0 + u_{-1}}{h^2} = f_0$$

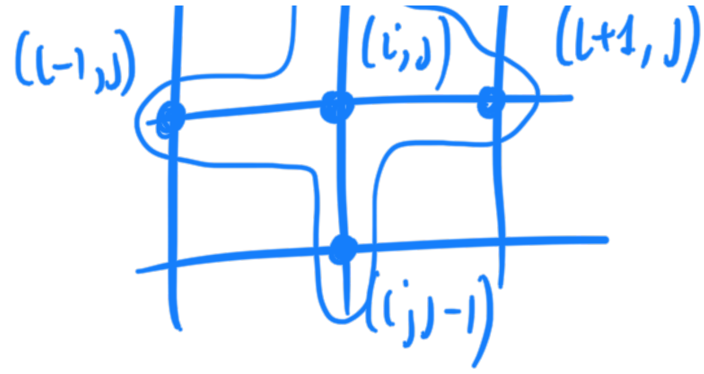
$$\begin{cases} -\nabla^2 u = f & \text{in } \Omega \subseteq R^2 \\ u = u_D \in R & \Omega = [a,b]^2 \\ \quad \text{on } \partial\Omega & \end{cases}$$

ϕy

$\in \Omega$







$\Delta x = \Delta y = h$

$$-\nabla^2 u = f \Rightarrow -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(i+1,j) + u(i-1,j) - 2u(i,j)}{h^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u(i,j+1) + u(i,j-1) - 2u(i,j)}{h^2}$$

In the generic point (i,j) $\big((x_i, y_j)\big)$ we have

$$-\frac{u(i+1,j) + u(i-1,j) - 2u(i,j)}{h^2} - \frac{u(i,j+1) + u(i,j-1) - 2u(i,j)}{h^2} = f(i,j)$$

$$\Rightarrow -\frac{u(i+1,j) + u(i-1,j) + u(i,j+1) + u(i,j-1) - 4u(i,j)}{h^2}$$

$$= f(i,j)$$

$$\begin{cases} a u'' + b u' + c u = f & \text{in } (0,1) \\ u(0) = \alpha \\ u(1) + u'(1) = \beta \end{cases}$$

1) Consider a backward FD for u', a central FD for u'' such that the order of accuracy is $O(h^2)$

2) Try to make an extension to a 2D framework.

FINITE ELEMENTS

$$\begin{cases} -u'' = 1 & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

$v \in V$ $\quad v$ = test functions

$v \in C^\infty(0,1)$

$$-u'' = 1 \implies -u''v = 1\cdot v$$

$$\implies \int_0^1 -u''v = \int_0^1 1\cdot v$$

$$\implies \int_0^1 u'v' - \cancel{u'v\Big|_{x=0}^{x=1}} = \int_0^1 1\cdot v$$

$$v(0) = v(1) = 0$$

$$\boxed{\int_0^1 u'v' = \int_0^1 v}$$

$$V = \left\{ v \in H^1(0,1) \;:\; v(0) = v(1) = 0 \right\} = H^1_0(0,1)$$

$$a(u,v) = \int_0^1 u'v'$$

$$b(v) = \int_0^1 v$$

$$\boxed{\underbrace{a(u,v)}_{\text{biliner form}} = \underbrace{b(v)}_{\text{liner form}}}$$