$\mathcal{X}$ = input space

$\mathcal{Y}$ = output space

f = algorithms

$$\boxed{y = f(x)}$$

1) $\forall x \in \mathcal{X} \ \exists y \in \mathcal{Y} : \ y = f(x)$

2) $\exists!\, y$

Stability (absolute)

$\forall\, \delta x : x + \delta x \in \mathcal{X} \quad \exists\, k_{abs}$

$$: \quad \|\delta y\|_{\mathcal{Y}} \leq k_{abs} \|\delta x\|_{\mathcal{X}}$$

$$10^{2} \qquad 10^{16} \qquad 10^{-14}$$

Stability (relative)

$$\forall x \in X \quad : \quad \|x\|_X \neq 0 \quad \|f(x)\|_Y \neq 0$$

$$\Rightarrow \quad \exists\, K_{rel} : \quad \frac{\|\delta y\|_Y}{\|y\|_Y} \leq K_{rel} \frac{\|\delta x\|}{\|x\|}$$

$$\|\delta y\| = \| f(x+\delta x) - f(x) \|$$

$$y = f(x) \qquad \cancel{y} + \delta y - \cancel{y}$$

Absolute stability :

$$\|f(x+\delta x) - f(x)\|_Y \leq K_{abs} \underbrace{\|\delta x\|_X}$$

$$\Downarrow$$

$$\frac{\| f(x+\delta x) - f(x) \|}{\|\delta x\|} \leq K_{abs}$$

$$K_{abs} = \sup_{x \in X} \sup_{\delta x} \frac{\| f(x+\delta x) - f(x)\|}{\|\delta x\|}$$

Example

$\mathcal{X} = R^2$ $\qquad x \in \mathcal{X}$

$\mathcal{Y} = R$ $\qquad x = (x_1, x_2) \in \mathcal{X}$

$y = x_1 + x_2 \in \mathcal{Y}$

$\delta x = (\delta x_1, \delta x_2) \in \mathcal{X}$

$\delta y = \delta x_1 + \delta x_2$

$$\| f(x+\delta x) - f(x) \| =$$

$$= \| \delta y \| = \| \delta x_1 + \delta x_2 \|$$

$$\| x \|_{\mathcal{X}} = \| x \|_{\ell^1} = |x_1| + |x_2|$$

$$\ell_p\text{-norms} = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

$p = 1$

$$\| y \|_{\mathcal{Y}} = |y|$$

$$\frac{\| f(x+\delta x) - f(x) \|}{\| \delta x \|} =$$

$$= \frac{\| \delta y \|}{\| \delta x \|} = \frac{| \delta x_1 + \delta x_2 |}{|\delta x_1| + |\delta x_2|}$$

$$\leq \frac{\cancel{|\delta x_1| + |\delta x_2|}}{\cancel{|\delta x_1| + |\delta x_2|}} = 1$$

$$\boxed{\kappa_{abs} = 1}$$

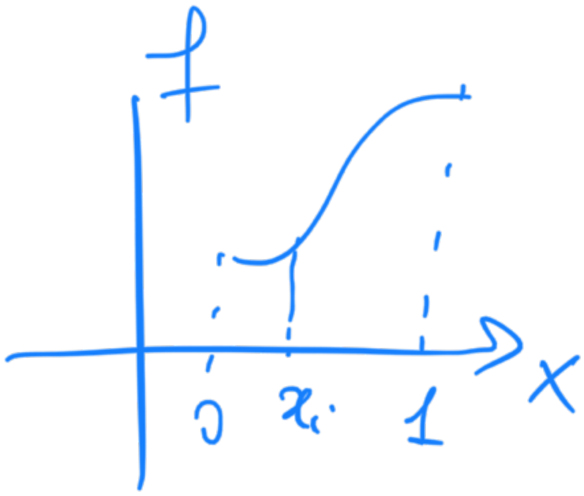
$$\frac{\dfrac{\| \delta y \|}{\| y \|}}{\dfrac{\| \delta x \|}{\| x \|}} = \frac{\| \delta y \|}{\| y \|} \; \frac{\| x \|}{\| \delta x \|} \leq$$

$$\leq 1$$

$$\leq 1 \cdot \frac{\| x \|}{\| y \|} = \frac{|x_1| + |x_2|}{|x_1 + x_2|}$$

$$\boxed{\sup_{\forall x_1, x_2 \in X} \frac{|x_1| + |x_2|}{|x_1 + x_2|} =: K_{rel}}$$

$|x_1| \sim |x_2| \qquad x_1, x_2 \leq 0$



$f \in C^0([0,1])$

$x_i \in [0,1]$

$h \in R^+$

$$f'(x_i) \cong \frac{f(x_i + h) - f(x_i)}{h}$$

NUMERICAL APPROX. OF AN ABSTRACT PROBLEM

$y = f(x) \longrightarrow$

x_n

y_n depending on n

f_n

$x_h \longrightarrow x \qquad X_n$

$y_n \to y$ $\quad$ Y_n

$f_n \to f$

Numerical approxim. is well posed $\Longleftrightarrow$ $y_n = f_n(x_n)$ is well posed $\forall n$

It is <u>consistent</u> when $x \in X^n$ $\forall n$

$$\lim_{n\to\infty} \| f_n(x) - f(x) \| = 0$$

It is <u>convergent</u> when $(x_n \to x)$

$$\lim_{n\to\infty} \| f_n(x_n) - f(x) \| = 0$$

$f :$ $g \in C^0([0,1]) \to$

$\to$ $g'(\xi) = f(g(\xi))$

$\xi \in [0,1]$

FD

CFD

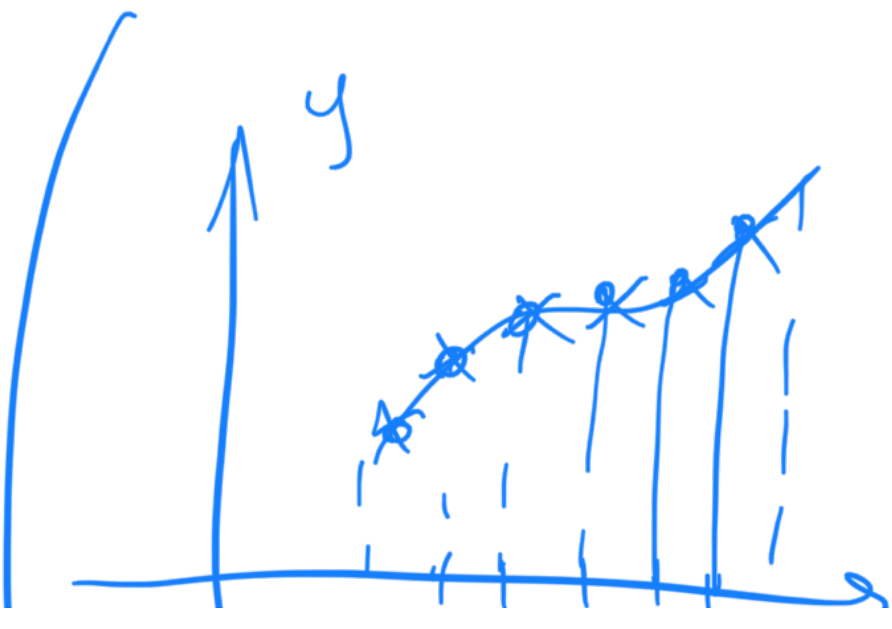
$$f_n(g) = \frac{g(\xi + \frac{1}{n}) - g(\xi)}{\frac{1}{n}}$$

$h = \frac{1}{n}$

$$\lim_{n \to \infty} \| f_n(g) - f(g) \| = 0$$

$\Rightarrow$ the n. a. is consistent

Lax theorem



$\{x_i\}_{i=1}^{n}$

$0 \qquad 1 \qquad x$

$|x_{i+1} - x_i| = h = \frac{1}{n}$

N.A is consistent

<u>convergence</u> $\Longleftrightarrow$ <u>stability</u>

$$\| f_n(g) - f(g) \| =$$

$$= \left\| \frac{g(s+\frac{1}{n}) - g(s)}{\frac{1}{n}} - g'(s) \right\|$$

$$= \left\| \left[\underbrace{g(s+\tfrac{1}{n})} - g(s) \right] n - g'(s) \right\| =$$

$$= \left\| \left[\cancel{g(s)} + g'(s)\cancel{h} + \frac{g''(s)}{2} h^{2} + \right. \right.$$

$$+ \sum_{k=3}^{\infty} \frac{g^{k}(s)}{k!} h^{k-1} - \cancel{g(s)} \Big] \cancel{n} -$$

$$- g'(s) \Big\|$$

$$\boxed{h = \frac{1}{n}}$$

$$\| f^n(g) - f(g) \| =$$

$$= \left\| \frac{g(s+\frac{1}{n}) - g(s)}{\frac{1}{n}} - g'(s) \right\|$$

$$= \left\| \frac{\cancel{g(s)} + g'(s)\cdot\frac{1}{n} + \frac{g''(s)}{2!}\frac{1}{n^2} + \dots - \cancel{g(s)}}{\frac{1}{n}} \right.$$

$$\left. - g'(s) \right\| = \left\| \frac{g''(s)}{2!}\frac{1}{n^2} \cdot \cancel{n} + \dots \right\| =$$

$$= \left\| \boxed{\frac{g''(s)}{2!}\frac{1}{n}} + O\left(\frac{1}{n^2}\right) \right\|$$

truncation error

$$\| f^n(g) - f(g) \| =$$

$$= \left\| \frac{g(s+h) - g(s-h)}{} - g'(s) \right\|$$

$$= \left\| \frac{\qquad}{2h} - g(\,)\right\|$$

$=$

Operational norm

$X, \quad Y, \quad f \qquad\qquad y = f(x)$

$$\| \ \|_* = \sup_{\substack{x \in X \\ \|x\| \neq 0}} \frac{\| f(x)\|_p}{\|x\|_p}$$

$$A : R^n \longrightarrow R^m$$

$$X = R^n \qquad Y = R^m$$

$$* = P \implies \|A\|_* = \|A\|_p$$

$$\implies \boxed{\|Ax\|_p \leq \|A\|_p \|x\|_p}$$

$$\|A\|_p = \sup_{\substack{x \in X \\ \|x\| \neq 0}} \frac{\|Ax\|_p}{\|x\|_p}$$

Example

$n = m$ $\qquad X = Y = R^n$

$\boxed{y = Ax}$ $\qquad A \in R^{n \times n}$

$\exists K_{abs} : \forall x \in X, \ \forall \delta x \in X$

$: x + \delta x \in X \Rightarrow$

$$\Rightarrow \frac{\|f(x+\delta x) - f(x)\|_p}{\|\delta x\|_p} \leq K_{abs}$$

$f(x) = Ax$

$f(x+\delta x) = A(x + \delta x) = Ax + A\delta x$

$$\Rightarrow \frac{\|\cancel{Ax} + A\delta x - \cancel{Ax}\|_p}{} =$$

$$\|\delta x\|_p$$

$$= \frac{\|A\,\delta x\|_p}{\|\delta x\|_p} \leq \frac{\|A\|_p \,\cancel{\|\delta x\|_p}}{\cancel{\|\delta x\|_p}}$$

$$\Longrightarrow \boxed{K_{abs} = \|A\|_p}$$

$$\underbrace{\frac{\|\delta y\|}{\|y\|} \cdot \frac{\|x\|}{\|\delta x\|}} \leq \|A\|_p \frac{\|x\|_p}{\|y\|_p}$$

$$y = f(x) \qquad\qquad y = Ax$$

$$\Downarrow$$

$$x = f^{-1}(y) \qquad\qquad x = A^{-1} y$$

$$\Longrightarrow \frac{\|A\|_p \, \|A^{-1}\|_p \, \cancel{\|y\|_p}}{\cancel{\|y\|_p}}$$

$$\Longrightarrow K_{rel} = \|A\|_p \, \|A^{-1}\|_p$$

$$\begin{cases} K_{abs} = \|A\|_p \\ \boxed{K_{rel} = \|A\|_p \, \|A^{-1}\|_p} \end{cases}$$

INTERPOLATION PROBLEM

We are going to approximate the elements of a space of infinite dimension by using a finite dimensional subspace (the subspace is generated / spanned by a set of linearly independent vectors)

$$V = C^0([0,1])$$

$$P^n \subseteq V$$

$$\forall p \in P^n \quad \exists \{w_i\}_{i=1}^{n} :$$

$$: \quad p = \sum_{i=1}^{n} w_i \tilde{v}_i$$

$\hat{v}_i$ basis of P^n

$$P^n = \text{span}\{\tilde{v}_i\}_{i=1}^{n}$$

$$P^3 = \text{span}\{1, x, x^2, x^3\}$$

$$p^3(x) = ax^3 + bx^2 + cx + d =$$

$$= \sum_{i=0}^{3} w_i \tilde{v}_i$$

$w_0 \to d$ $\qquad$ $w_3 \to a$

$w_1 \to c$

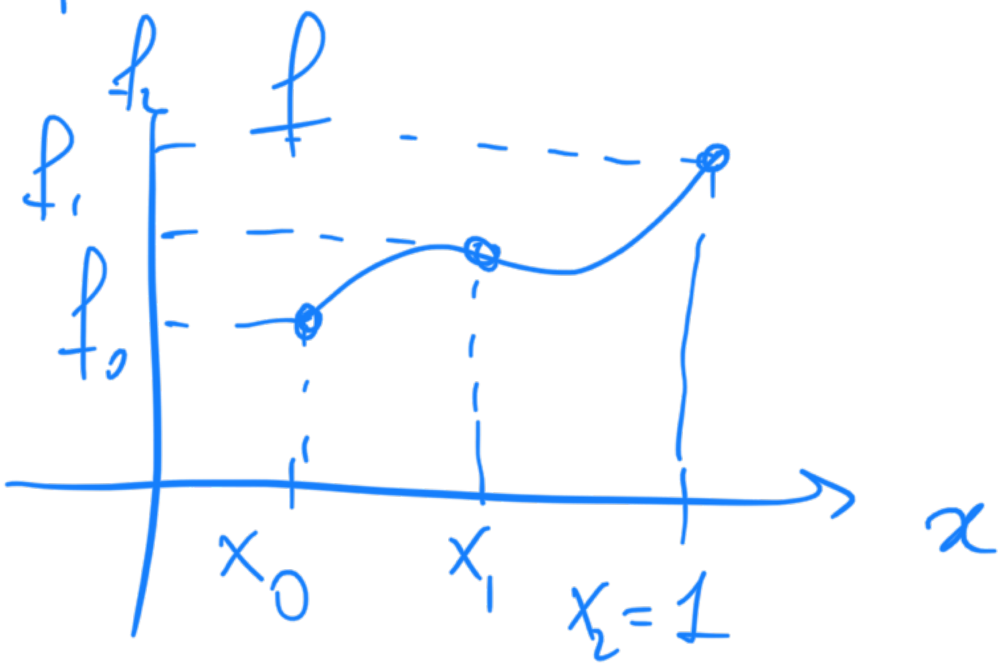
$W_2 \to b$

$$\tilde{V}_i = x^{i-1}$$

$\{x_i\}_{i=1}^{n}$ interpolation points

$\{f(x_i)\}_{i=1}^{n}$ values assumed by our function f on int. points

$$p(x_i) = f(x_i) \quad \forall i$$



$$P_2(x) = ax^2 + bx + c$$

$$\begin{cases} P_2(x_0) = f_0 \\ P_2(x_1) = f_1 \\ P_2(x_2) = f_2 \end{cases}$$

$$\Downarrow$$

$$\underbrace{\begin{bmatrix} x_0^2 & x_0 & 1 \\ x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \end{bmatrix}}_{\underline{\underline{V}}} \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{\underline{a}} = \underbrace{\begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix}}_{\underline{b}}$$

Vandermonde matrix

$$\underline{a} = \underline{\underline{V}}^{-1} \underline{b}$$

CONDITIONING NUMBER OF THE POLYNOMIAL INTERPOLATION

$$
\begin{array}{rccc}
f: & C^0([0,1]) & \longrightarrow & \mathbb{R}^n \\
 & \parallel & & \parallel \\
 & X & & Y
\end{array}
$$

$$
\begin{array}{rccc}
f: & C^0([0,1]) & \longrightarrow & C^0([0,1]) \\
 & \parallel & & \parallel
\end{array}
$$