$$I^n : u \in C^0([0,1])$$

(f)

$$\begin{array}{ccc} & u & \\ \swarrow & & \searrow \\ \underline{P} \in \mathbb{R}^n & & w \in C^0([0,1]) \end{array}$$

$$\underbrace{w = \sum_{i=1}^{n} P_i V_i}_{\text{I perspective}} \qquad \underbrace{\qquad\qquad\qquad}_{\text{II perspective}}$$

I perspective

$$\frac{\| f(x+\delta x) - f(x) \|_{\mathcal{X}}}{\| \delta x \|_{\mathcal{X}}} =$$

$$= \frac{\| I^n(u+\delta u) - I^n(u) \|_{\ell^\infty}}{\quad} =$$

$$\|\delta u\|_{L_\infty}$$

$$= \frac{\| \cancel{I^n(u)} + I^n(\delta u) - \cancel{I^n(u)} \|_{\ell_\infty}}{\|\delta u\|_{L_\infty}} =$$

$$= \frac{\| I^n(\delta u)\|_{\ell_\infty}}{\|\delta u\|_{L_\infty}} = \frac{\| \underline{\underline{V}}^{-1} \underline{\delta u} \|_{\ell_\infty}}{\|\delta u\|}$$

$$\underline{\underline{V}}\, \underline{p} = \underline{u} \implies \boxed{\underline{p} = \underline{\underline{V}}^{-1} \underline{u}}$$

$$\downarrow$$

$$\underline{\delta p} = \underline{\underline{V}}^{-1} \underline{\delta u}$$

$$\leq \frac{\|\underline{\underline{V}}^{-1}\|_{\ell_\infty} \cancel{\|\delta u\|}_{L_\infty}}{\cancel{\|\delta u\|}_{L_\infty}}$$

$$\boxed{K_{abs} = \|\underline{\underline{V}}^{-1}\|_{\ell_\infty}}$$

$$\boxed{\underline{\underline{V}} = \underline{\underline{I}}}$$

$$u = \sum_{i=1}^{n} P_i V_i$$

$$\boxed{V_i = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - a_i}{a_j - a_i}}$$

$a_i =$
= interpolation points

II perspective

$$\frac{\| f(x+\delta x) - f(x) \|}{\| \delta x \|} =$$

$$= \frac{\| I^n(u+\delta u) - I^n(u) \|_{L^\infty}}{\| \delta u \|_{L^\infty}} =$$

$$= \frac{\| I^n(\delta u) \|_{L^\infty}}{\| \delta u \|_{L^\infty}} =$$

$$I^{\cdot\cdot}(u) = \sum_{i=1} p_i v_i - \sum_{i=1} u_i \cdot x_i$$

$$= \frac{\left\| \sum_{i=1}^{n} \delta u_i \, l_i \right\|_{L^\infty}}{\|\delta u\|_{L^\infty}} \leq$$

$$\leq \frac{\left\| \sum_{i=1}^{n} |l_i| \right\|_{L^\infty} \cancel{\|\delta u\|_{L^\infty}}}{\cancel{\|\delta u\|_{L^\infty}}} =$$

$$= \underbrace{\left\| \sum_{i=1}^{n} |l_i| \right\|}_{}_{L^\infty} = \|\Lambda\|_{L^\infty}$$

$$\boxed{k_{abs} = \|\Lambda\|_{L^\infty},}$$

We try to study the behaviour of Λ

1) $\forall A^n \in R^n$

$\exists\, c > 0$:

$$: \boxed{\| \Lambda^n \|_{L^\infty} \geq \frac{2}{\pi} \log(n-1) - c}$$

2) $\forall A^n \in R^n$

$\exists f$:

$$\lim_{n \to \infty} \| I^n(f) - f \| = \infty$$

$$\| \underset{\underset{p}{\shortparallel}}{I^n(f)} - f \|_{L^\infty}$$

p the best approximation

$\Longleftrightarrow \quad \forall q \in P^n \quad \| p - f \|_{L^\infty} \leq$

$\leq \| q - f \|_{L^\infty}$

$$\boxed{\| I^n(f) - f \|_{L^\infty}} =$$

$$= \| I^n(f) - p + p - f \|_{L^\infty}$$

$$= \| \underbrace{I^n(f) - I^n(p)}_{} + \underbrace{p - f}_{} \|_{L^\infty}$$

$$\boxed{I^n(p) = p}$$

$$\leq \| I^n(f - p) \|_{L^\infty} + \| p - f \|_{L^\infty}$$

$$\leq \| I^n \| \, \| p - f \|_{L^\infty} + \| p - f \|_{L^\infty} =$$

$$= \underbrace{\left(1 + \| I^n \| \right)}_{} \| p - f \|_{L^\infty}$$

$$\Longrightarrow \| q - f \|_{L^\infty} \leq$$

$$\leq \left(1 + \| I^n \| \right) \| p - f \|_{L^\infty}$$

$$\| I^n \| = \sup_{\|u\| \neq 0} \frac{\| I^n u \|}{\| u \|} =$$

$$= \| 1 \|_{L^\infty}$$

<u>Therem</u> :

$f \in C^{n+1}([0,1])$, $\{a_i\}_{i=0}^n$ int. points

in $(0,1)$, $\forall x \in (0,1)$ $\underline{\exists \beta}$:

$$\underbrace{\left(f - I(f) \right)}_{\substack{\shortparallel \\ P}} \{\beta\} =$$

$$= \boxed{\frac{f^{n+1}(\beta)}{(n+1)!} \, w(x)}$$

w is the so-called characteristic polynomial

$\shortparallel$

$$\boxed{w(x) = \prod_{i=0}^{n} (x - a_i)} \qquad \boxed{f(a_i) = p(a_i)}$$

$\forall x \in (0,1)$

$$\boxed{G(t) = \left(f(t) - p(t) \right) w(x) - \left(f(x) - p(x) \right) w(t)}$$

$G(t)$ exhibits $n+2$ roots

(solutions of $G(t) = 0$)

$$\Rightarrow \quad G^{n}(t) \text{ exhibits } 2 \text{ zeros}$$

$$G^{n}(t) = A(t)$$

$\exists \; s_1, s_2 : \boxed{A(s_1) = A(s_2)} = 0$

$$A'(t) = \underbrace{G^{n+1}(t)}_{} = 0$$

$$G^{n+1}(t) = f^{n+1}(t) \cdot w(x) -$$

$$- \left(\underbrace{f(x) - p(x)} \right)(n+1)! = 0$$

for $t = \beta$

$$\Rightarrow \quad f - p = \frac{f^{n+1} \, w}{(n+1)!}$$

$$\|f - p\|_{L^\infty} \leq \frac{1}{(n+1)!} \underbrace{\|w\|_{L^\infty}}_{(b-a)^n} \underbrace{\|f^{n+1}\|_{L^\infty}}$$