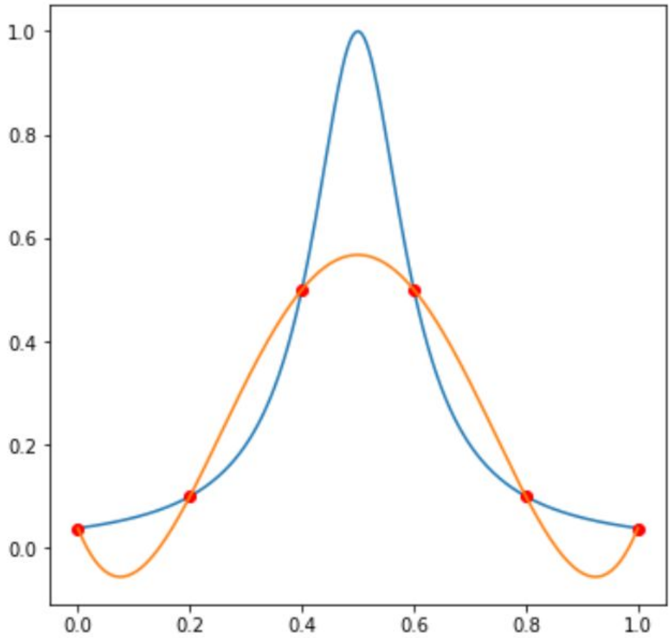
Interpolation



Interpolation: lagrange basis

Given (n+1) points $\{X_i\}^n_{i=0}$ in the interval [0,1], the **Lagrange interpolation operator** is:

$$\mathcal{L}^n : C^0([0,1]) \mapsto \mathcal{P}^n$$

such that:

$$(\mathcal{L}^n f)(x) = \sum_{i=0}^{n} f(X_i)\ell_i(x), \qquad i = 0, \ldots, n.$$

where:

$$\ell_i(x) := \prod_{i \neq j, j=0}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

Interpolation: lagrange basis

In this case:

$$V_{ij} := \ell_j(x_i)$$

therefore:

$$(\mathcal{L}^n u)(x_i) := \sum_{j=0}^{n} u(X_j)\ell_j(x_i) = \sum_j V_{ij} u(X_j)$$

Interpolation: lagrange basis

if **u** is a continuous function and **p̃** is the best approximation of **u** in $\mathcal{P}^n$

$$||\mathcal{L}^n \boldsymbol{u} - \boldsymbol{u}||_{L^\infty} \leq (1 + \Lambda)||\boldsymbol{u} - \tilde{\mathbf{p}}||_{L^\infty}$$

where Λ is the Lebesgue constant and $\Lambda = \sum_j |v_j|$

In the case of **equidistant nodes**, the Lebesgue constant **grows exponentially**.

On the other hand, the Lebesgue constant grows only **logarithmically** if Chebyshev nodes are used:

$$x_k = \cos\left(\frac{2k-1}{2n}\pi\right), \quad k = 1, \ldots, n. \qquad \text{in } (-1,1)$$

Interpolation: Runge phenomenon

if $\boldsymbol{u} \in C^0([a,b])$ is analytically extendible in an oval of radius R:

$$O(a,b,R) = \{z \in \mathbb{C} | dist(z,[a,b]) \leq R\}$$

then:

$$||\boldsymbol{u}^{n+1}||_{L^\infty} \leq \frac{(n+1)!}{R^{n+1}} ||\tilde{u}||_{L^\infty_{O(a,b,R)}}$$

where:

$$\tilde{u}|_{[a,b]} = \boldsymbol{u} \qquad \tilde{u} : \mathbb{C} \to \mathbb{R}$$

$$\Longrightarrow \quad ||\mathcal{L}^n \boldsymbol{u} - \boldsymbol{u}||_{L^\infty} \leq \left(\frac{(b-a)}{R}\right)^{n+1} ||\tilde{u}||_{L^\infty_{O(a,b,R)}}$$

Bernstain polynomial

Bernstein polynomial is a polynomial expressed as a linear combination of Bernstein basis polynomials.

The n+1 **Bernstein basis** polynomials of degree n are defined as

$$b_{\nu,n}(x) := \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu}, \quad \nu = 0, \ldots, n,$$

$$\begin{array}{llll}
b_{0,0}(x) = 1, & & & \\
b_{0,1}(x) = 1 - x, & b_{1,1}(x) = x & & \\
b_{0,2}(x) = (1-x)^2, & b_{1,2}(x) = 2x(1-x), & b_{2,2}(x) = x^2 & \\
b_{0,3}(x) = (1-x)^3, & b_{1,3}(x) = 3x(1-x)^2, & b_{2,3}(x) = 3x^2(1-x), & b_{3,3}(x) = x^3
\end{array}$$

Bernstein polynomial

The Bernstein basis polynomials of degree n form a basis for the vector space of polynomials of degree at most n with real coefficients.

A linear combination of Bernstein basis polynomials is called a Bernstein polynomial or polynomial in Bernstein form of degree n:

$$B_n(x) := \sum_{\nu=0}^{n} \beta_\nu b_{\nu,n}(x)$$

Let f be a continuous function on the interval [0, 1]. Consider the Bernstein polynomial

$$B_n(f)(x) = \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right) b_{\nu,n}(x).$$

It can be shown that $\lim_{n \to \infty} B_n(f) = f$