Linear system: direct methods

Direct methods for solving linear systems

The aim is to solve a linear system of the form:

$$A\mathbf{x} = \mathbf{b} \qquad \det(A) \neq 0$$

$$\downarrow$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \; i = 1, \ldots, n.$$

Cramer' rule → (n+1)! operations

Forward substitutions algorithm

Lower triangular system:

$$L\mathbf{y} = \mathbf{b} \longrightarrow \begin{cases} l_{11}y_1 & = & b_1 \\ l_{21}y_1 + l_{22}y_2 & = & b_2 \\ \vdots & & \\ l_{n1}y_1 + l_{n2}y_2 + \ldots + l_{nn}y_n & = & b_n \end{cases}$$

$$y_1 = b_1/l_{11} \longrightarrow y_i = \frac{1}{l_{ii}}\left(b_i - \sum_{j=1}^{i-1} l_{ij}y_j\right)$$

1 operation

1+2(i-1) operation $\longrightarrow$ n^2

Backward substitutions algorithm

Upper triangular system:

$$U\mathbf{x} = \mathbf{y} \longrightarrow \begin{cases} u_{11}x_1 + \ldots + u_{1,n-1}x_{n-1} + u_{1n}x_n & = & y_1 \\ & \vdots & \\ u_{n-1,n-1}x_{n-1} + u_{n-1,n}x_n & = & y_{n-1} \\ u_{nn}x_n & = & y_n \end{cases}$$

$$x_n = y_n / u_{nn} \longrightarrow x_i = \frac{1}{u_{ii}}\left(y_i - \sum_{j=i+1}^{n} u_{ij}x_j\right).$$

1 operation

1+2(i-1) operation $\longrightarrow$ n^2

Now assume that you can write

$$A = LU.$$

$2n^3/3 \rightarrow$ Gauss method

$$A\mathbf{x} = \mathbf{b} \quad \Leftrightarrow \quad LU\mathbf{x} = \mathbf{b} \quad \Leftrightarrow \quad \begin{cases} L\mathbf{y} = \mathbf{b}, \\ U\mathbf{x} = \mathbf{y}. \end{cases}$$

$2n^2$

Gauss method

It transforms the system in an equivalent upper triangular system, easy to solve:

$$A\mathbf{x} = \mathbf{b} \longrightarrow U\mathbf{x} = \widehat{\mathbf{b}}$$

Define $$l_{i1} = \frac{a_{i1}^{(1)}}{a_{11}^{(1)}}, \quad i = 2, 3, \ldots, n,$$

and modify as:

$$a_{ij}^{(2)} = a_{ij}^{(1)} - l_{i1} a_{1j}^{(1)}, \quad i, j = 2, \ldots, n,$$

$$b_i^{(2)} = b_i^{(1)} - l_{i1} b_1^{(1)}, \quad i = 2, \ldots, n,$$

$$
\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \dots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \dots & a_{2n}^{(2)} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2}^{(2)} & \dots & a_{nn}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(2)} \\ \vdots \\ b_n^{(2)} \end{bmatrix}
$$

After n-1 operations we obtain an upper triangular system if : $a_{ii}^{(i)} \neq 0$ for $i = 1, \dots, k-1$

$$
\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \dots & \dots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & & & a_{2n}^{(2)} \\ \vdots & & \ddots & & \vdots \\ 0 & & & \ddots & \vdots \\ 0 & & & & a_{nn}^{(n)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(2)} \\ \vdots \\ \vdots \\ b_n^{(n)} \end{bmatrix}
$$

General formulas

$$l_{ik} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}, \quad i = k+1, \ldots, n,$$

n-k operations

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - l_{ik} a_{kj}^{(k)}, \quad i, j = k+1, \ldots, n,$$

$2(n-k)^2$ operations

$$b_i^{(k+1)} = b_i^{(k)} - l_{ik} b_k^{(k)}, \quad i = k+1, \ldots, n.$$

$2(n-k)$ operations

Cholesky factorization

If the matrix A is symmetric and positive definite, there exists a unique upper triangular matrix R with positive diagonal elements such that $A = R^T R$.

$$r_{11} = \sqrt{a_{11}}$$

$$r_{ji} = \frac{1}{r_{jj}} \left(a_{ij} - \sum_{k=1}^{j-1} r_{ki} r_{kj} \right), \quad j = 1, \ldots, i-1,$$

$$i = 2, \ldots, n$$

$$r_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} r_{ki}^2}$$

$n^3/3$ operations

Exercise

Consider PDE problem:

$$-u_{xx}(x) = f(x) \quad \text{in } \Omega = (0,1)$$
$$u(x) = 0, \quad \text{on } \partial\Omega = \{0,1\}$$

The physical interpretation of this problem is related to the modelling of an elastic string, which occupies at rest the space [0,1] and is fixed at the two extremes. The unknown u(x) represents the displacement of the string at the point x , and the right-hand side models a prescribed force f(x) on the string.

For the numerical discretization of the problem, we consider a **Finite Difference** (FD) Approximation. Let n be an integer, consider a uniform subdivision of the interval $(0,1)$ using n equispaced points, denoted by $\{x_i\}_{i=0\dots n}$. Moreover, let u_i be the FD approximation of $u(x_i)$, and similarly $f_i \approx f(x_i)$.

In order to formulate the discrete problem, we consider a FD approximation of the left-hand side, as follows

$$-u_{xx}(x_i) \approx \frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2}$$

being $h = 1/n-1$ the size of each subinterval (x_i, x_{i+1}).

The problem that we need to solve is

$$
\begin{aligned}
u_i &= 0 & i &= 0, \\
\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} &= f_i & i &= 1, \ldots, n-1, \\
u_i &= 0 & i &= n.
\end{aligned}
$$

$$\downarrow$$

$$\mathbf{A}\mathbf{u} = \mathbf{f}. \qquad f(x) = x(1-x)$$

The exact solution is: $u(x) = u_{\text{ex}}(x) = \dfrac{x^4}{12} - \dfrac{x^3}{6} + \dfrac{x}{12}$