Linear system: iterative methods

Iterative methods to solve linear systems

The aim is to solve a linear system of the form:

$$A\mathbf{x} = \mathbf{b} \qquad\qquad \lim_{k\to\infty} \mathbf{x}^{(k)} = \mathbf{x}$$

A general way of setting up an iterative method is based on the decomposition of the matrix A:

$$A = P - (P - A)$$

$$A\mathbf{x} = \mathbf{b} \quad \Leftrightarrow \quad P\mathbf{x} = (P - A)\mathbf{x} + \mathbf{b}$$

$$\mathbf{x} = B\mathbf{x} + \mathbf{g}. \qquad\qquad \mathbf{x}^{(k+1)} = B\mathbf{x}^{(k)} + \mathbf{g}, \quad k \geq 0$$

Exercise

Consider PDE problem:

$$
\begin{aligned}
-u_{xx}(x) = f(x) \quad \text{in } \Omega = (0,1) \\
u(x) = 0, \quad \text{on } \partial\Omega = \{0,1\}
\end{aligned}
$$

The physical interpretation of this problem is related to the modelling of an elastic string, which occupies at rest the space [0,1] and is fixed at the two extremes. The unknown u(x) represents the displacement of the string at the point x , and the right-hand side models a prescribed force f(x) on the string.

For the numerical discretization of the problem, we consider a **Finite Difference** (FD) Approximation. Let n be an integer, consider a uniform subdivision of the interval $(0,1)$ using n equispaced points, denoted by $\{x_i\}_{i=0\ldots n}$. Moreover, let u_i be the FD approximation of $u(x_i)$, and similarly $f_i \approx f(x_i)$.

In order to formulate the discrete problem, we consider a FD approximation of the left-hand side, as follows

$$-u_{xx}(x_i) \approx \frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2}$$

being $h = 1/n-1$ the size of each subinterval (x_i, x_{i+1}) .

The problem that we need to solve is

$$
\begin{aligned}
u_i &= 0 & i &= 0, \\
\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} &= f_i & i &= 1, \ldots, n-1, \\
u_i &= 0 & i &= n.
\end{aligned}
$$

$$\downarrow$$

$$A\mathbf{u} = \mathbf{f}. \qquad f(x) = x(1-x)$$

The exact solution is: $u(x) = u_{\text{ex}}(x) = \frac{x^4}{12} - \frac{x^3}{6} + \frac{x}{12}$

Jacobi

$$P = D = diag(a_{11}, a_{22}, \ldots, a_{nn})$$

$$D\mathbf{x}^{(k+1)} = \mathbf{b} - (A - D)\mathbf{x}^{(k)} \qquad k \geq 0.$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1, j \neq i}^{n} a_{ij} x_j^{(k)} \right), \quad i = 1, \ldots, n.$$

Gauss-Seidel

$$P = D - E$$

$$\begin{cases} E_{ij} = -a_{ij} & \text{if } i > j \\ E_{ij} = 0 & \text{if } i \leq j \end{cases}$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right), \quad i = 1, \ldots, n.$$

Gradient method

$$\mathbf{r}^k = \mathbf{b} - A\mathbf{x}^k$$

$$\alpha^k = \frac{\mathbf{r}^{k^T}\mathbf{r}^k}{\mathbf{r}^{k^T} A\mathbf{r}^k}$$

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \alpha^k\mathbf{r}^k$$

Preconditioned:

$$P\mathbf{z}^k = \mathbf{r}^k$$

$$\alpha^k = \frac{\mathbf{z}^{k^T}\mathbf{r}^k}{\mathbf{z}^{k^T} A\mathbf{z}^k}$$

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \alpha^k\mathbf{z}^k$$

$$\mathbf{r}^{k+1} = \mathbf{r}^k - \alpha^k A\mathbf{z}^k$$

Conjugate gradient method

$$\alpha^k = \frac{\mathbf{p}^{k^T}\mathbf{r}^k}{\mathbf{p}^{k^T} A\mathbf{p}^k}$$

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \alpha^k \mathbf{p}^k$$

$$\mathbf{r}^{k+1} = \mathbf{r}^k - \alpha^k A\mathbf{p}^k$$

$$\beta^k = \frac{(A\mathbf{p}^k)^T\mathbf{r}^{k+1}}{(A\mathbf{p}^k)^T\mathbf{p}^k}$$

$$\mathbf{p}^{k+1} = \mathbf{r}^{k+1} - \beta^k \mathbf{p}^k$$

Preconditioned:

$$\alpha^k = \frac{\mathbf{p}^{k^T}\mathbf{r}^k}{(A\mathbf{p}^k)^T\mathbf{p}^k}$$

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \alpha^k \mathbf{p}^k$$

$$\mathbf{r}^{k+1} = \mathbf{r}^k - \alpha^k A\mathbf{p}^k$$

$$P\mathbf{z}^{k+1} = \mathbf{r}^{k+1}$$

$$\beta_k = \frac{\mathbf{z}^{(k+1)^T}\mathbf{r}^{(k+1)}}{\mathbf{z}^{(k)^T}\mathbf{r}^{(k)}}$$

$$\mathbf{p}^{k+1} = \mathbf{z}^{k+1} - \beta^k \mathbf{p}^k$$