PEU 356 Assignment 6

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May 26, 2024

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1.1 Problem

(a) Given (in $\mathbb{R}^3$) the basis $\varphi_1 = x, \varphi_2 = y, \varphi_3 = z$, consider the basis transformation $x \to z, y \to y, z \to -x$. Find the 3×3 matrix $\cup$ for this transformation.

(b) This transformation corresponds to a rotation of the coordinate axes. Identify the rotation and reconcile your transformation matrix with an appropriate matrix $S(\alpha, \beta, \gamma)$ is of the form,

$$\begin{pmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha & -\cos\gamma\sin\beta \\ -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\sin\beta \\ \sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{pmatrix}$$

(c) Form the column vector c representing (in the original basis) $f = 2x - 3y + z$, find the result of applying U to c, and show that this is consistent with the basis transformation of part (a).

Note. You do not need to be able to form scalar products to handle this exercise; a knowledge of the linear relationship between the original and transformed functions is sufficient.

1.2 Solution

(a)

$$\cup = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

(b)

$$\cup\vec{v} = \vec{v}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$z = x = 0, \quad y = y$$

$$\hat{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{r}' = \mathbf{r}\cos\boldsymbol{\Phi} + \mathbf{r} \times \hat{\mathbf{n}}\sin\boldsymbol{\Phi} + \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{r})(1 - \cos\boldsymbol{\Phi})$$

.

$$\mathbf{r}' = (r_x\hat{\imath} + r_y\hat{\jmath} + r_z\hat{k})\cos\boldsymbol{\Phi} + (r_x\hat{k} - r_z\hat{\imath})\sin\boldsymbol{\Phi} + r_y(1 - \cos\boldsymbol{\Phi})\hat{\jmath}$$

$$\mathbf{r}' = \langle r_x\cos\boldsymbol{\Phi} - r_z\sin\boldsymbol{\Phi}, r_y, r_x\sin\boldsymbol{\Phi} + r_z\cos\boldsymbol{\Phi}\rangle$$

$$T = \begin{pmatrix} \cos\boldsymbol{\Phi} & 0 & -\sin\boldsymbol{\Phi} \\ 0 & 1 & 0 \\ \sin\boldsymbol{\Phi} & 0 & \cos\boldsymbol{\Phi} \end{pmatrix}$$

$$\boldsymbol{\Phi} = \frac{3\pi}{2}$$

(C)

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$$

$$f = 2\varphi_1 - 3\varphi_2 + 1\varphi_3 \to f' = 1\varphi_1' - 3\varphi_2' - 2\varphi_3'$$

It is consistent with the basis transformation of part (a).

2 5.5.4

2.1 Problem

The unitary transformation U that converts an orthonormal basis $\{\varphi_i\}$ into the basis $\{\varphi'_i\}$ and the unitary transformation V that converts the basis $\{\varphi'_i\}$ into the basis $\{\chi_i\}$ have matrix representations

$$\mathrm{U} = \begin{pmatrix} i\sin\theta & \cos\theta & 0 \\ -\cos\theta & i\sin\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathrm{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & i\sin\theta \\ 0 & \cos\theta & -i\sin\theta \end{pmatrix}.$$

Given the function $f(x) = 3\varphi_1(x) - \varphi_2(x) - 2\varphi_3(x)$,

(a) By applying U, form the vector representing $f(x)$ in the $\{\varphi'_i\}$ basis and then by applying V form the vector representing $f(x)$ in the $\{\chi_i\}$ basis. Use this result to write $f(x)$ as a linear combination of the χ_i.

(b) Form the matrix products UV and VU and then apply each to the vector representing $f(x)$ in the $\{\varphi_i\}$ basis. Verify that the results of these applications differ and that only one of them gives the result corresponding to part (a).

2.2 Solution

$$\vec{f} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$

(a)

$$\begin{pmatrix} i\sin\theta & \cos\theta & 0 \\ -\cos\theta & i\sin\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3i\sin\theta - \cos\theta \\ -3\cos\theta - i\sin\theta \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & i\sin\theta \\ 0 & \cos\theta & -i\sin\theta \end{pmatrix} \begin{pmatrix} 3i\sin\theta - \cos\theta \\ -3\cos\theta - i\sin\theta \\ -2 \end{pmatrix} =$$

$$\begin{pmatrix} 3i\sin\theta - \cos\theta \\ \cos\theta(-3\cos\theta - i\sin\theta) - 2i\sin\theta \\ \cos\theta(-3\cos\theta - i\sin\theta) + 2i\sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} 3i\sin\theta - \cos\theta \\ -3\cos^2\theta - i\sin\theta\cos\theta - 2i\sin\theta \\ -3\cos^2\theta - i\sin\theta\cos\theta + 2i\sin\theta \end{pmatrix}$$

$$f(x) = (3i\sin\theta - \cos\theta)\,\chi_1(x) +$$

$$(\cos\theta(-3\cos\theta - i\sin\theta) - 2i\sin\theta)\,\chi_2(x) +$$

$$(\cos\theta(-3\cos\theta - i\sin\theta) + 2i\sin\theta)\,\chi_3(x)$$

(b)

UV was already implicitly calculated in part (a).

$$VU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & i\sin\theta \\ 0 & \cos\theta & -i\sin\theta \end{pmatrix} \begin{pmatrix} i\sin\theta & \cos\theta & 0 \\ -\cos\theta & i\sin\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} i\sin\theta & \cos\theta & 0 \\ -\cos^2\theta & i\sin\theta\cos\theta & i\sin\theta \\ -\cos^2\theta & i\sin\theta\cos\theta & -i\sin\theta \end{pmatrix}$$

$$\begin{pmatrix} i\sin\theta & \cos\theta & 0 \\ -\cos^2\theta & i\sin\theta\cos\theta & i\sin\theta \\ -\cos^2\theta & i\sin\theta\cos\theta & -i\sin\theta \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 3i\sin\theta - \cos\theta \\ -3\cos^2 - i\sin\theta\cos\theta + 2i\sin\theta \\ -3\cos^2 - i\sin\theta\cos\theta - 2i\sin\theta \end{pmatrix}$$

There is a slight difference between the two results, and only the result of UV corresponds to part (a).

$$\left(UV\vec{f}\right)_2 = \left(VU\vec{f}\right)_3, \quad \left(UV\vec{f}\right)_3 = \left(VU\vec{f}\right)_2$$

3 5.6.1

3.1 Problem

(a) Using the two spin functions $\varphi_1 = \alpha$ and $\varphi_2 = \beta$ as an orthonormal basis (so $\langle \alpha \mid \alpha \rangle = \langle \beta \mid \beta \rangle = 1, \langle \alpha \mid \beta \rangle = 0$), and the relations

$$S_x\alpha = \frac{1}{2}\beta, \quad S_x\beta = \frac{1}{2}\alpha, \quad S_y\alpha = \frac{1}{2}i\beta, \quad S_y\beta = -\frac{1}{2}i\alpha, \quad S_z\alpha = \frac{1}{2}\alpha, \quad S_z\beta = -\frac{1}{2}\beta,$$

construct the 2×2 matrices of S_x, S_y, and S_z.

(b) Taking now the basis $\varphi_1' = C(\alpha+\beta), \varphi_2' = C(\alpha-\beta)$:

(i) Verify that φ_1' and φ_2' are orthogonal,

(ii) Assign C a value that makes φ_1' and φ_2' normalized,

(iii) Find the unitary matrix for the transformation $\{\varphi_i\} \to \{\varphi_i'\}$.

(c) Find the matrices of S_x, S_y, and S_z in the $\{\varphi_i'\}$ basis.

3.2 Solution

(a)

$$S_x|\alpha\rangle = \frac{1}{2}|\beta\rangle \to S_x|\alpha\rangle\langle\alpha| = \frac{1}{2}|\beta\rangle\langle\alpha| \longrightarrow (1)$$

$$S_x|\beta\rangle = \frac{1}{2}|\alpha\rangle \to S_x|\beta\rangle\langle\beta| = \frac{1}{2}|\alpha\rangle\langle\beta| \longrightarrow (2)$$

$$(1)+(2) \longrightarrow S_x(|\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta|) = \frac{1}{2}(|\beta\rangle\langle\alpha| + |\alpha\rangle\langle\beta|)$$

$$\because \sum_i |\varphi_i\rangle\langle\varphi_i| = I$$

$$S_x = \frac{1}{2}(|\beta\rangle\langle\alpha| + |\alpha\rangle\langle\beta|)$$

$$S_x = \begin{pmatrix} \alpha_1\beta_1 & \frac{\alpha_1\beta_2+\alpha_2\beta_1}{2} \\ \frac{\alpha_1\beta_2+\alpha_2\beta_1}{2} & \alpha_2\beta_2 \end{pmatrix}$$

$$S_y|\alpha\rangle = \frac{i}{2}|\beta\rangle \rightarrow S_y|\alpha\rangle\langle\alpha| = \frac{i}{2}|\beta\rangle\langle\alpha| \longrightarrow (1)$$

$$S_y|\beta\rangle = -\frac{i}{2}|\alpha\rangle \rightarrow S_y|\beta\rangle\langle\beta| = -\frac{i}{2}|\alpha\rangle\langle\beta| \longrightarrow (2)$$

$$(1)+(2) \longrightarrow S_y(|\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta|) = \frac{i}{2}(|\beta\rangle\langle\alpha| - |\alpha\rangle\langle\beta|)$$

$$S_y = \frac{i}{2}(|\beta\rangle\langle\alpha| - |\alpha\rangle\langle\beta|)$$

$$S_y = \begin{pmatrix} 0 & \frac{i(\alpha_1\beta_2-\alpha_2\beta_1)}{2} \\ \frac{i(\alpha_2\beta_1-\alpha_1\beta_2)}{2} & 0 \end{pmatrix}$$

$$S_z|\alpha\rangle = \frac{1}{2}|\alpha\rangle \rightarrow S_z|\alpha\rangle\langle\alpha| = \frac{1}{2}|\alpha\rangle\langle\alpha| \longrightarrow (1)$$

$$S_z|\beta\rangle = -\frac{1}{2}|\beta\rangle \rightarrow S_z|\beta\rangle\langle\beta| = -\frac{1}{2}|\beta\rangle\langle\beta| \longrightarrow (2)$$

$$(1)+(2) \longrightarrow S_z(|\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta|) = \frac{1}{2}(|\alpha\rangle\langle\alpha| - |\beta\rangle\langle\beta|)$$

$$S_z = \frac{1}{2}(|\alpha\rangle\langle\alpha| - |\beta\rangle\langle\beta|)$$

$$S_z = \begin{pmatrix} \frac{\alpha_1^2-\beta_1^2}{2} & \frac{\alpha_1\alpha_2-\beta_1\beta_2}{2} \\ \frac{\alpha_1\alpha_2-\beta_1\beta_2}{2} & \frac{\alpha_2^2-\beta_2^2}{2} \end{pmatrix}$$

(b)

(i)

If we assume $C = \frac{1}{\sqrt{2}}U$ where U is a unitary matrix.

$$C^\dagger C = \left(\frac{1}{\sqrt{2}}U\right)^\dagger \frac{1}{\sqrt{2}}U = U^\dagger \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}U = \frac{1}{2}U^\dagger U = \frac{1}{2}I$$

$$\langle \varphi_1' | \varphi_2' \rangle = \langle C^\dagger C(\alpha + \beta) | \alpha - \beta \rangle = \frac{1}{2}\langle \alpha + \beta | \alpha - \beta \rangle$$

$$= \frac{1}{2}\left(\langle \alpha|\alpha\rangle - \langle \alpha|\beta\rangle + \langle \beta|\alpha\rangle - \langle \beta|\beta\rangle\right) = 0$$

Note: if we want C to be constant U would just be the identity matrix.

(ii)

$$C = \frac{1}{\sqrt{2}}U$$

Where U is a unitary matrix.

$$\langle \varphi_1' | \varphi_1' \rangle = \langle C(\alpha + \beta) | C(\alpha + \beta) \rangle = \langle C^\dagger C(\alpha + \beta) | (\alpha + \beta) \rangle$$

$$= \frac{1}{2}\langle (\alpha + \beta) | (\alpha + \beta) \rangle = \frac{1}{2}(\langle \alpha|\alpha\rangle + \langle \beta|\beta\rangle + \langle \alpha|\beta\rangle + \langle \beta|\alpha\rangle) = 1$$

$$\langle \varphi_2' | \varphi_2' \rangle = \langle C(\alpha - \beta) | C(\alpha - \beta) \rangle = \langle C^\dagger C(\alpha - \beta) | (\alpha - \beta) \rangle$$

$$= \frac{1}{2}\langle (\alpha - \beta) | (\alpha - \beta) \rangle = \frac{1}{2}(\langle \alpha|\alpha\rangle + \langle \beta|\beta\rangle - \langle \alpha|\beta\rangle - \langle \beta|\alpha\rangle) = 1$$

(iii)

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Note: there is a mistake in the question, T is not a unitary matrix. If we want T to be unitary, then $\varphi_2 = -C(\alpha - \beta)$, there should be a negative sign that was not included. This will not have an effect on the rest of the answers.

If we follow the fix T would be a unitary matrix.

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(c)

We simply need to substitute $\alpha_i \to \frac{1}{\sqrt{2}}(\alpha_i + \beta_i)$ and $\beta_i \to \frac{1}{\sqrt{2}}(\alpha_i - \beta_i)$ into the matrices of S_x, S_y, and S_z.

$$S_x = \begin{pmatrix} \alpha_1\beta_1 & \frac{\alpha_1\beta_2+\alpha_2\beta_1}{2} \\ \frac{\alpha_1\beta_2+\alpha_2\beta_1}{2} & \alpha_2\beta_2 \end{pmatrix}$$

$$S_y = \begin{pmatrix} 0 & \frac{i(\alpha_1\beta_2-\alpha_2\beta_1)}{2} \\ -\frac{i(\alpha_1\beta_2-\alpha_2\beta_1)}{2} & 0 \end{pmatrix}$$

$$S_z = \begin{pmatrix} \frac{\alpha_1^2-\beta_1^2}{2} & \frac{\alpha_1\alpha_2-\beta_1\beta_2}{2} \\ \frac{\alpha_1\alpha_2-\beta_1\beta_2}{2} & \frac{\alpha_2^2-\beta_2^2}{2} \end{pmatrix}$$

$$S_x{}' = \begin{pmatrix} \frac{1}{\sqrt{2}}(\alpha_1+\beta_1)\frac{1}{\sqrt{2}}(\alpha_1-\beta_1) & \frac{\frac{1}{\sqrt{2}}(\alpha_1+\beta_1)\frac{1}{\sqrt{2}}(\alpha_2-\beta_2)+\frac{1}{\sqrt{2}}(\alpha_2+\beta_2)\frac{1}{\sqrt{2}}(\alpha_1-\beta_1)}{2} \\ \frac{\frac{1}{\sqrt{2}}(\alpha_1+\beta_1)\frac{1}{\sqrt{2}}(\alpha_2-\beta_2)+\frac{1}{\sqrt{2}}(\alpha_2+\beta_2)\frac{1}{\sqrt{2}}(\alpha_1-\beta_1)}{2} & \frac{1}{\sqrt{2}}(\alpha_2+\beta_2)\frac{1}{\sqrt{2}}(\alpha_2-\beta_2) \end{pmatrix}$$

$$S_x{}' = \begin{pmatrix} \frac{1}{2}(\alpha_1^2-\beta_1^2) & \frac{\alpha_1\alpha_2-\beta_1\beta_2}{2} \\ \frac{\alpha_1\alpha_2-\beta_1\beta_2}{2} & \frac{1}{2}(\alpha_2^2-\beta_2^2) \end{pmatrix}$$

$$\alpha_1\beta_2 - \alpha_2\beta_1 \to \frac{1}{2}\left((\alpha_1+\beta_1)(\alpha_2-\beta_2) - (\alpha_2+\beta_2)(\alpha_1-\beta_1)\right)$$

$$= \alpha_2\beta_1 - \alpha_1\beta_2$$

$$S_y{}' = \begin{pmatrix} 0 & -\frac{i(\alpha_1\beta_2-\alpha_2\beta_1)}{2} \\ \frac{i(\alpha_1\beta_2-\alpha_2\beta_1)}{2} & 0 \end{pmatrix}$$

$$S_z{}' = \begin{pmatrix} \alpha_1\beta_1 & \frac{\alpha_1\beta_2+\alpha_2\beta_1}{2} \\ \frac{\alpha_1\beta_2+\alpha_2\beta_1}{2} & \alpha_2\beta_2 \end{pmatrix}$$

4 5.6.2

4.1 Problem

For the basis $\varphi_1 = Cxe^{-r^2}, \varphi_2 = Cye^{-r^2}, \varphi_3 = Cze^{-r^2}$, where $r^2 = x^2 + y^2 + z^2$, with the scalar product defined as an unweighted integral over $\mathbb{R}^3$ and with C chosen to make the φ_i normalized:

(a) Find the 3×3 matrix of $L_x = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$;

(b) Using the transformation matrix U $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -i/\sqrt{2} \\ 0 & 1/\sqrt{2} & i/\sqrt{2} \end{pmatrix}$, find the transformed matrix of L_x;

(c) Find the new basis functions φ_i' defined by the transformation U, and write explicitly (in terms of x, y, and z) the functional forms of $L_x\varphi_i', i = 1, 2, 3$.

Hint. Use $\int e^{-r^2} d^3r = \pi^{3/2}, \int x^2 e^{-r^2} d^3r = \frac{1}{2}\pi^{3/2}$; the integrals are over $\mathbb{R}^3$.

4.2 Solution

(a)

$$A_{nm} = \left\langle \phi_n | \hat{A} | \phi_m \right\rangle$$

$$L_x = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$\langle \phi_n | L_x | \phi_m \rangle = -i\left\langle \phi_n \left| y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right| \phi_m \right\rangle$$

$$\langle \phi_n | \phi_m \rangle = \int \phi_n^* \phi_m d^3r$$

$$\langle \phi_n | L_x | \phi_m \rangle = -i\int \phi_n^* \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \phi_m d^3r$$

$$= -i|C|^2 \int x_n e^{-r^2} \left(x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} \right) x_m e^{-r^2} d^3r$$

$$= -i|C|^2 \int x_n e^{-r^2} \left(x_2 \frac{\partial \left(x_m e^{-r^2} \right)}{\partial x_3} - x_3 \frac{\partial \left(x_m e^{-r^2} \right)}{\partial x_2} \right) d^3r$$

$$= -i|C|^2 \int x_n e^{-r^2} \left(x_2 \left(\delta_3^m e^{-r^2} + x_m \frac{\partial e^{-r^2}}{\partial x_3} \right) - x_3 \left(\delta_2^m e^{-r^2} + x_m \frac{\partial e^{-r^2}}{\partial x_2} \right) \right) d^3r$$

$$\frac{\partial e^{-r^2}}{\partial x_i} = -2r \frac{\partial r}{\partial x_i} e^{-r^2} = -2x_i e^{-r^2}$$

$$= -i|C|^2 \int x_n e^{-2r^2} \left(x_2 \left(\delta_3^m - 2x_m x_3 \right) - x_3 \left(\delta_2^m - 2x_m x_2 \right) \right) d^3r$$

$$L_{nm} = i|C|^2 \int e^{-2r^2} x_n \left(x_3 \delta_2^m - x_2 \delta_3^m \right) d^3r$$

$$L_{nm} = i|C|^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2{x_1}^2} e^{-2{x_2}^2} e^{-2{x_3}^2} x_n \left(x_3 \delta_2^m - x_2 \delta_3^m \right) dx_1 dx_2 dx_3$$

From this we can see that only the terms with $n = 2, m = 3$ and $n = 3, m = 2$ will survive. Because they are the product of two even functions, the rest are have odd symmetry and will integrate to zero.

$$L_{23} = -i|C|^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2{x_1}^2} e^{-2{x_2}^2} e^{-2{x_3}^2} {x_2}^2 dx_1 dx_2 dx_3$$

$$= -i|C|^2 \int_{-\infty}^{\infty} e^{-2{x_1}^2} dx_1 \int_{-\infty}^{\infty} {x_2}^2 e^{-2{x_2}^2} dx_2 \int_{-\infty}^{\infty} e^{-2{x_3}^2} dx_3$$

$$= -i|C|^2 \left(\frac{\pi^{\frac{3}{2}}}{8\sqrt{2}}\right)$$

$$|C|^2 = \frac{8\sqrt{2}}{\pi^{\frac{3}{2}}}$$

$$L_{23} = -i$$

$$L_{32} = i\frac{8\sqrt{2}}{\pi^{\frac{3}{2}}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2x_1{}^2} e^{-2x_2{}^2} e^{-2x_3{}^2} x_3{}^2 dx_1 dx_2 dx_3$$

$$= i\frac{8\sqrt{2}}{\pi^{\frac{3}{2}}} \int_{-\infty}^{\infty} e^{-2x_1{}^2} dx_1 \int_{-\infty}^{\infty} e^{-2x_2{}^2} dx_2 \int_{-\infty}^{\infty} x_3{}^2 e^{-2x_3{}^2} dx_3$$

$$= i\frac{8\sqrt{2}}{\pi^{\frac{3}{2}}} \left(\frac{\pi^{\frac{3}{2}}}{8\sqrt{2}}\right) = i$$

$$L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

(b)

U is a unitary matrix. because $UU^\dagger = I$.

$$U^{-1} = U^\dagger$$

$$UL_xU^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(c)

$$x \to x, \quad y \to \frac{1}{\sqrt{2}}(y+iz), \quad z \to \frac{1}{\sqrt{2}}(y-iz)$$

$$\varphi_1^{'} = Cxe^{-r^2}$$

$$\varphi_2^{'} = \frac{C}{\sqrt{2}}(y+iz)e^{-r^2}$$

$$\varphi_3^{'} = \frac{C}{\sqrt{2}}(y-iz)e^{-r^2}$$

Lx transforms as follows:

$$x \to 0, \quad y \to -iz, \quad z \to iy$$

$$L_x \varphi_1' = 0$$

$$L_x \varphi_2' = -\frac{C}{\sqrt{2}}(y+iz)e^{-r^2}$$

$$L_x \varphi_3' = \frac{C}{\sqrt{2}}(y-iz)e^{-r^2}$$

5 5.7.1

5.1 Problem

Using the formal properties of unitary transformations, show that the commutator $[x, p] = i$ is invariant under unitary transformation of the matrices representing x and p.

5.2 Solution

$$[x, p] = i$$

$$UxU^{\dagger} = x'$$

$$UpU^{\dagger} = p'$$

$$[x', p'] = UxU^{\dagger}UpU^{\dagger} - UpU^{\dagger}UxU^{\dagger}$$

$$= UxpU^{\dagger} - UpxU^{\dagger} = U[x, p]U^{\dagger} = UiU^{\dagger} = i$$

6 5.7.2

6.1 Problem

The Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

have commutator $[\sigma_1, \sigma_2] = 2i\sigma_3$. Show that this relationship continues to be valid if these matrices are transformed by

$$\mathbf{U} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

6.2 Solution

$$\mathbf{U}\sigma_1\mathbf{U}^\dagger = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \sin 2\theta & \cos 2\theta \\ \cos 2\theta & -\sin 2\theta \end{pmatrix}$$

$$\mathbf{U}\sigma_2\mathbf{U}^\dagger = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} i\sin\theta & -i\cos\theta \\ i\cos\theta & i\sin\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{U}\sigma_3\mathbf{U}^\dagger = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix}$$

$$\left[\mathbf{U}\sigma_1\mathbf{U}^\dagger, \mathbf{U}\sigma_2\mathbf{U}^\dagger\right] = \begin{pmatrix} \sin 2\theta & \cos 2\theta \\ \cos 2\theta & -\sin 2\theta \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \sin 2\theta & \cos 2\theta \\ \cos 2\theta & -\sin 2\theta \end{pmatrix}$$

$$= \begin{pmatrix} i\cos 2\theta & -i\sin 2\theta \\ -i\sin 2\theta & -i\cos 2\theta \end{pmatrix} - \begin{pmatrix} -i\cos 2\theta & i\sin 2\theta \\ i\sin 2\theta & i\cos 2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 2i\cos 2\theta & -2i\sin 2\theta \\ -2i\sin 2\theta & -2i\cos 2\theta \end{pmatrix}$$

$$= 2i\mathbf{U}\sigma_3\mathbf{U}^\dagger$$

References

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