PEU 356 Assignment 4

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1 4.3.7

1.1 Problem

Verify that $V_{i;j} = g_{ik} V^k_{;j}$ by showing that

$$\frac{\partial V_i}{\partial q^j} - V_k \Gamma^k_{ij} = g_{ik} \left[\frac{\partial V^k}{\partial q^j} + V^m \Gamma^k_{mj} \right].$$

1.2 Solution

$$\frac{\partial V_i}{\partial q^j} = \frac{\partial \left(g_{ik} V^k \right)}{\partial q^j} = g_{ik} \frac{\partial V^k}{\partial q^j} + V^k \frac{\partial g_{ik}}{\partial q^j}$$

$$\because \frac{\partial g_{ik}}{\partial q^j} = \varepsilon_i \cdot \frac{\partial \varepsilon_k}{\partial q^j} + \varepsilon_k \cdot \frac{\partial \varepsilon_i}{\partial q^j}$$

$$\therefore \frac{\partial V_i}{\partial q^j} = g_{ik} \frac{\partial V^k}{\partial q^j} + \left(\varepsilon_i \cdot \frac{\partial \varepsilon_k}{\partial q^j} \right) V^k + \left(\varepsilon_k \cdot \frac{\partial \varepsilon_i}{\partial q^j} \right) V^k$$

$$= g_{ik} \frac{\partial V^k}{\partial q^j} + \left(\varepsilon^l \cdot \frac{\partial \varepsilon_k}{\partial q^j} \right) V^k g_{il} + \left(\varepsilon^k \cdot \frac{\partial \varepsilon_i}{\partial q^j} \right) V_k$$

$$\frac{\partial V_i}{\partial q^j} = g_{ik} \frac{\partial V^k}{\partial q^j} + V^k \Gamma^l_{kj} g_{il} + V_k \Gamma^k_{ij}$$

$$\frac{\partial V_i}{\partial q^j} - V_k \Gamma^k_{ij} = g_{ik} \frac{\partial V^k}{\partial q^j} + V^k \Gamma^l_{kj} g_{il}$$

$$\frac{\partial V_i}{\partial q^j} - V_k \Gamma^k_{ij} = g_{ik} \frac{\partial V^k}{\partial q^j} + V^l \Gamma^k_{lj} g_{ik}$$

$$\frac{\partial V_i}{\partial q^j} - V_k \Gamma^k_{ij} = g_{ik} \left[\frac{\partial V^k}{\partial q^j} + V^m \Gamma^k_{mj} \right].$$

2 4.3.8

2.1 Problem

From the circular cylindrical metric tensor g_{ij}, calculate the Γ^k_{ij} for circular cylindrical coordinates. Note. There are only three nonvanishing Γ.

2.2 Solution

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Gamma^n_{ij} = \frac{1}{2} g^{nk} \left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k} \right)$$

$$\Gamma^n_{ij} = \frac{1}{2} g^{nn} \left(\frac{\partial g_{in}}{\partial q^j} + \frac{\partial g_{jn}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^n} \right)$$

$$\frac{\partial g_{22}}{\partial q^1} = \frac{\partial \rho^2}{\partial \rho} = 2\rho$$

$$\Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{\rho}$$

$$\Gamma^1_{22} = -\rho$$

3 4.3.10

3.1 Problem

Show that for the metric tensor $g_{ij;k} = g^{ij}_{;k} = 0$.

3.2 Solution

$$g_{ij;k} = \frac{\partial g_{ij}}{\partial k} - \Gamma^{\alpha}_{ik} g_{\alpha j} - \Gamma^{\alpha}_{jk} g_{i\alpha}$$

$$= \frac{\partial g_{ij}}{\partial k} - \frac{1}{2} g_{j\alpha} g^{\alpha\beta} \left(\frac{\partial g_{\beta k}}{\partial i} + \frac{\partial g_{\beta i}}{\partial k} - \frac{\partial g_{ik}}{\partial \beta} \right)$$

$$- \frac{1}{2} g_{i\alpha} g^{\alpha\beta} \left(\frac{\partial g_{\beta k}}{\partial j} + \frac{\partial g_{\beta j}}{\partial k} - \frac{\partial g_{jk}}{\partial \beta} \right)$$

$$= \frac{\partial g_{ij}}{\partial k} - \frac{1}{2} \left(\frac{\partial g_{jk}}{\partial i} + \frac{\partial g_{ji}}{\partial k} - \frac{\partial g_{ik}}{\partial j} \right) - \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial j} + \frac{\partial g_{ij}}{\partial k} - \frac{\partial g_{jk}}{\partial i} \right) = 0$$

Contravariant g is just a function of covariant g, so the same applies for $g^{ij}_{;k}$. if covariant g does not depend on k, then neither does contravariant g.

4 4.3.12

4.1 Problem

The covariant vector A_i is the gradient of a scalar. Show that the difference of covariant derivatives $A_{i;j} - A_{j;i}$ vanishes.

4.2 Solution

$$A_{i;j} = \frac{\partial A_i}{\partial q^j} - A_k \Gamma^k_{ij}$$

$$A_i = \frac{\partial \phi}{\partial q^i}$$

$$A_{i;j} = \frac{\partial^2 \phi}{\partial q^j \partial q^i} - \frac{\partial \phi}{\partial q^k} \Gamma^k_{ij}$$

$$A_{j;i} = \frac{\partial^2 \phi}{\partial q^i \partial q^j} - \frac{\partial \phi}{\partial q^k} \Gamma^k_{ji}$$

Since the partial derivatives commute and the Christoffel symbols are symmetric in their lower indices, $A_{i;j} - A_{j;i} = 0$.

5 4.4.1

5.1 Problem

Assuming the functions u and v to be differentiable,

(a) Show that a necessary and sufficient condition that $u(x, y, z)$ and $v(x, y, z)$ are related by some function $f(u, v) = 0$ is that $(\nabla u) \times (\nabla v) = 0$;

(b) If $u = u(x, y)$ and $v = v(x, y)$, show that the condition $(\nabla u) \times (\nabla v) = 0$ leads to the 2-D Jacobian

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0.$$

5.2 Solution

5.2.1 Part (a)

$$f = 0$$

$$\nabla f = \frac{\partial f}{\partial u} \nabla u + \frac{\partial f}{\partial v} \nabla v = 0$$

$$\frac{\partial f}{\partial u} \nabla u = -\frac{\partial f}{\partial v} \nabla v$$

$$\nabla u = c \nabla v$$

$$(\nabla u) \times (\nabla v) = c(\nabla v) \times (\nabla v) = 0$$

5.2.2 Part (b)

$$(\nabla u) \times (\nabla v) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & 0 \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & 0 \end{vmatrix}$$

$$\implies \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\implies \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

6 4.4.2

6.1 Problem

A 2-D orthogonal system is described by the coordinates q_1 and q_2. Show that the Jacobian J satisfies the equation

$$J \equiv \frac{\partial(x,y)}{\partial(q_1,q_2)} \equiv \frac{\partial x}{\partial q_1}\frac{\partial y}{\partial q_2} - \frac{\partial x}{\partial q_2}\frac{\partial y}{\partial q_1} = h_1 h_2.$$

Hint. It's easier to work with the square of each side of this equation.

6.2 Solution

$$\left(\frac{\partial x}{\partial q_1}\frac{\partial y}{\partial q_2} - \frac{\partial x}{\partial q_2}\frac{\partial y}{\partial q_1}\right)^2 = \left(\frac{\partial x}{\partial q_1}\frac{\partial y}{\partial q_2}\right)^2 + \left(\frac{\partial x}{\partial q_2}\frac{\partial y}{\partial q_1}\right)^2 - 2\frac{\partial x}{\partial q_1}\frac{\partial x}{\partial q_2}\frac{\partial y}{\partial q_2}\frac{\partial y}{\partial q_1}$$

$$h_i{}^2 = \left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2$$

$$h_1{}^2 h_2{}^2 = \left(\left(\frac{\partial x}{\partial q_1}\right)^2 + \left(\frac{\partial y}{\partial q_1}\right)^2\right)\left(\left(\frac{\partial x}{\partial q_2}\right)^2 + \left(\frac{\partial y}{\partial q_2}\right)^2\right)$$

$$= \left(\frac{\partial x}{\partial q_1}\frac{\partial y}{\partial q_2}\right)^2 + \left(\frac{\partial x}{\partial q_2}\frac{\partial y}{\partial q_1}\right)^2 - 2\frac{\partial x}{\partial q_1}\frac{\partial x}{\partial q_2}\frac{\partial y}{\partial q_2}\frac{\partial y}{\partial q_1}$$

$$+\left(\frac{\partial x}{\partial q_1}\frac{\partial x}{\partial q_2}\right)^2 + \left(\frac{\partial y}{\partial q_1}\frac{\partial y}{\partial q_2}\right)^2 + 2\frac{\partial x}{\partial q_1}\frac{\partial x}{\partial q_2}\frac{\partial y}{\partial q_2}\frac{\partial y}{\partial q_1}$$

$$= \left(\frac{\partial x}{\partial q_1}\frac{\partial y}{\partial q_2} - \frac{\partial x}{\partial q_2}\frac{\partial y}{\partial q_1}\right)^2 + \left(\frac{\partial x}{\partial q_1}\frac{\partial x}{\partial q_2} + \frac{\partial y}{\partial q_1}\frac{\partial y}{\partial q_2}\right)^2$$

$$\frac{\partial x}{\partial q_1}\frac{\partial x}{\partial q_2} + \frac{\partial y}{\partial q_1}\frac{\partial y}{\partial q_2} = \varepsilon_1 \cdot \varepsilon_2 = 0$$

$$
{h_1}^2{h_2}^2 = \left(\frac{\partial x}{\partial q_1}\frac{\partial y}{\partial q_2} - \frac{\partial x}{\partial q_2}\frac{\partial y}{\partial q_1}\right)^2
$$

$$
h_1 h_2 = \frac{\partial x}{\partial q_1}\frac{\partial y}{\partial q_2} - \frac{\partial x}{\partial q_2}\frac{\partial y}{\partial q_1}
$$

7 4.4.3

7.1 Problem

For the transformation $u = x + y, v = x/y$, with $x \geq 0$ and $y \geq 0$, find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$

(a) By direct computation,

(b) By first computing J^{-1}.

7.2 Solution

7.2.1 Part (a)

$$x = \frac{uv}{1+v} \quad y = \frac{u}{1+v}$$

$$\frac{\partial x}{\partial u} = \frac{v}{1+v} \quad \frac{\partial x}{\partial v} = \frac{u}{(1+v)^2}$$

$$\frac{\partial y}{\partial u} = \frac{1}{1+v} \quad \frac{\partial y}{\partial v} = -\frac{u}{(1+v)^2}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{v}{1+v} & \frac{u}{(1+v)^2} \\ \frac{1}{1+v} & -\frac{u}{(1+v)^2} \end{vmatrix} = -\frac{uv}{(1+v)^3} - \frac{u}{(1+v)^3} = -\frac{u}{(1+v)^2}$$

7.2.2 Part (b)

$$J^{-1} = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1/y & -x/y^2 \end{vmatrix} = -\frac{x}{y^2} - \frac{1}{y} = -\frac{x+y}{y^2} = -\frac{u}{\frac{u^2}{(1+v)^2}}$$

$$J^{-1} = -\frac{(1+v)^2}{u}$$

$$J = -\frac{u}{(1+v)^2}$$

References

[1] G.B. Arfken, H.J. Weber, and F.E. Harris. *Mathematical Methods for Physicists: A Comprehensive Guide*. Elsevier Science, 2013.

[2] M.H. El-Deeb. PEU-356 Assignments.