

PEU 356 Assignment 1

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1 3.10.1

1.1 Problem

The u –, v –, z –coordinate system frequently used in electrostatics and in hydrodynamics is defined by

$$xy = u, x^2 - y^2 = v, z = z$$

This u –, v –, z –system is orthogonal.

- In words, describe briefly the nature of each of the three families of coordinate surfaces.
- Sketch the system in the xy –plane showing the intersections of surfaces of constant u and surfaces of constant v with the xy –plane.
- Indicate the directions of the unit vectors $\hat{\mathbf{e}}_u$ and $\hat{\mathbf{e}}_v$ in all four quadrants.

- (d) Finally, is this u -, v -, z -system right-handed ($\hat{\mathbf{e}}_u \times \hat{\mathbf{e}}_v = +\hat{\mathbf{e}}_z$) or left-handed ($\hat{\mathbf{e}}_u \times \hat{\mathbf{e}}_v = -\hat{\mathbf{e}}_z$)?

1.2 Solution

1.2.1 a

u curves are $\frac{n}{x}$ asymptotes, $u = 0$ is xz -plane, repeated for all values of z .

$$(x, \frac{u_i}{x}, z)$$

v curves are x shaped for $v = 0$, horizontal hyperbola for $v > 0$, vertical hyperbola for $v < 0$, repeated for all values of z .

$$(x, \pm\sqrt{x^2 - v_i}, z)$$

z curves are planes parallel to xy -plane.

$$(x, y, z_i)$$

1.2.2 b

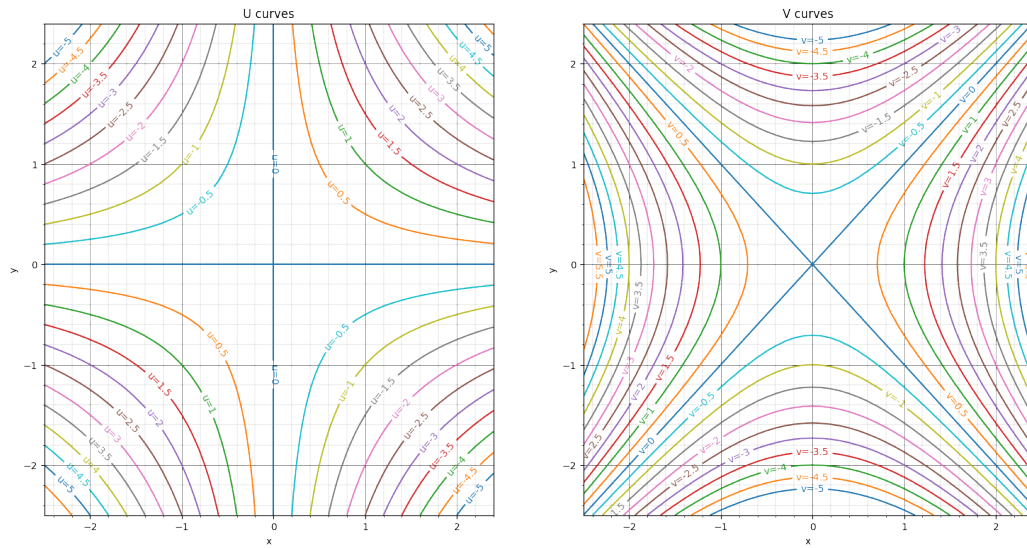


Figure 1: U and V curves.

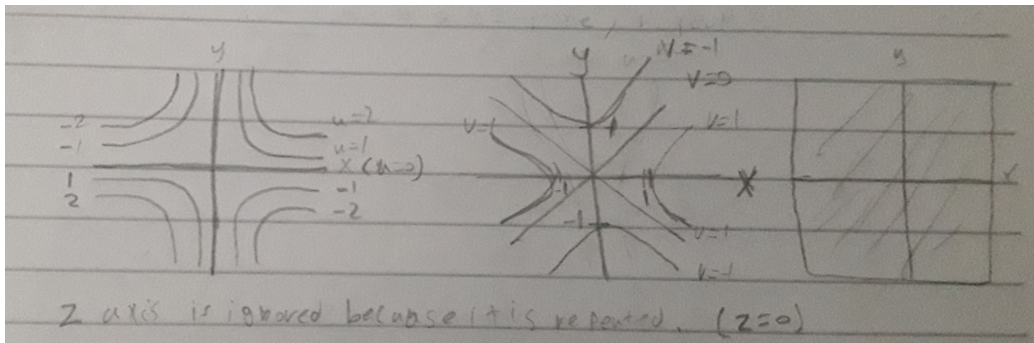


Figure 2: U and V curves sketch.

1.2.3 c

$$\nabla u = \frac{\partial u}{\partial x} \hat{i} + \frac{\partial u}{\partial y} \hat{j} = \frac{\partial(xy)}{\partial x} \hat{i} + \frac{\partial(xy)}{\partial y} \hat{j} = y \hat{i} + x \hat{j}$$

$$\nabla v = \frac{\partial v}{\partial x} \hat{i} + \frac{\partial v}{\partial y} \hat{j} = \frac{\partial(x^2 - y^2)}{\partial x} \hat{i} + \frac{\partial(x^2 - y^2)}{\partial y} \hat{j} = 2x\hat{i} - 2y\hat{j}$$

$$\hat{\mathbf{e}}_u = \frac{\nabla u}{|\nabla u|} = \frac{y\hat{i} + x\hat{j}}{\sqrt{x^2 + y^2}}$$

$$\hat{\mathbf{e}}_v = \frac{\nabla v}{|\nabla v|} = \frac{2x\hat{i} - 2y\hat{j}}{\sqrt{4x^2 + 4y^2}}$$

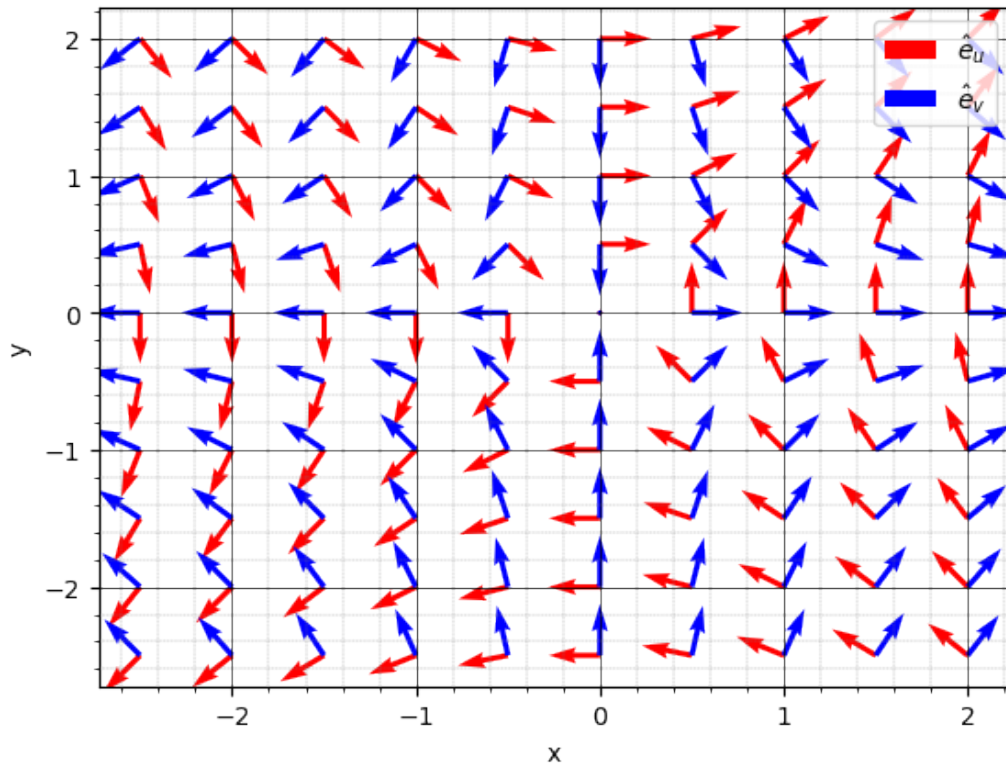


Figure 3: Basis Vectors Field Plot

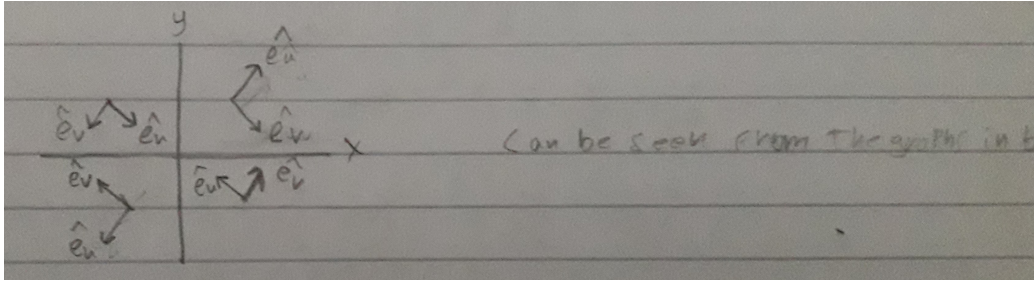


Figure 4: Basis Vectors Field Plot sketch.

1.2.4 d

$$\begin{aligned}\hat{\mathbf{e}}_u \times \hat{\mathbf{e}}_v &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ \frac{x}{\sqrt{x^2+y^2}} & -\frac{y}{\sqrt{x^2+y^2}} & 0 \end{vmatrix} \\ &= \left(-\frac{y}{\sqrt{x^2+y^2}} \frac{y}{\sqrt{x^2+y^2}} - \frac{x}{\sqrt{x^2+y^2}} \frac{x}{\sqrt{x^2+y^2}} \right) \hat{k} = -\hat{k}\end{aligned}$$

From the previous result, we can see that the coordinate system is left-handed.

2 3.10.4

2.1 Problem

With $\hat{\mathbf{e}}_1$ a unit vector in the direction of increasing q_1 , show that

- (a) $\nabla \cdot \hat{\mathbf{e}}_1 = \frac{1}{h_1 h_2 h_3} \frac{\partial(h_2 h_3)}{\partial q_1}$
- (b) $\nabla \times \hat{\mathbf{e}}_1 = \frac{1}{h_1} \left[\hat{\mathbf{e}}_2 \frac{1}{h_3} \frac{\partial h_1}{\partial q_3} - \hat{\mathbf{e}}_3 \frac{1}{h_2} \frac{\partial h_1}{\partial q_2} \right]$

Note that even though $\hat{\mathbf{e}}_1$ is a unit vector, its divergence and curl **do not necessarily vanish**.

2.2 Solution

$$\hat{\mathbf{e}}_1 = \langle 1, 0, 0 \rangle$$

2.2.1 a

$$\nabla \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (V_1 h_2 h_3) + \frac{\partial}{\partial q_2} (V_2 h_1 h_3) + \frac{\partial}{\partial q_3} (V_3 h_1 h_2) \right]$$

$$\nabla \cdot \hat{\mathbf{e}}_1 = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (1 * h_2 h_3) + \frac{\partial}{\partial q_2} (0 * h_1 h_3) + \frac{\partial}{\partial q_3} (0 * h_1 h_2) \right]$$

$$\nabla \cdot \hat{\mathbf{e}}_1 = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial q_1} (h_2 h_3)$$

2.2.2 b

$$\nabla \times \mathbf{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{\mathbf{e}}_1 h_1 & \hat{\mathbf{e}}_2 h_2 & \hat{\mathbf{e}}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$$

$$\nabla \times \hat{\mathbf{e}}_1 = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{\mathbf{e}}_1 h_1 & \hat{\mathbf{e}}_2 h_2 & \hat{\mathbf{e}}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 * 1 & h_2 * 0 & h_3 * 0 \end{vmatrix}$$

$$\nabla \times \hat{\mathbf{e}}_1 = \frac{1}{h_1 h_2 h_3} \left(\hat{\mathbf{e}}_2 h_2 \frac{\partial h_1}{\partial q_3} - \hat{\mathbf{e}}_3 h_3 \frac{\partial h_1}{\partial q_2} \right)$$

$$\nabla \times \hat{\mathbf{e}}_1 = \frac{1}{h_1} \left[\hat{\mathbf{e}}_2 \frac{1}{h_3} \frac{\partial h_1}{\partial q_3} - \hat{\mathbf{e}}_3 \frac{1}{h_2} \frac{\partial h_1}{\partial q_2} \right]$$

3 3.10.5

3.1 Problem

Show that a set of orthogonal unit vectors $\hat{\mathbf{e}}_i$ may be defined by

$$\hat{\mathbf{e}}_i = \frac{1}{h_i} \frac{\partial \mathbf{r}}{\partial q_i}$$

In particular, show that $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_i = 1$ leads to an expression for h_i in agreement with

$$h_i^2 = \left(\frac{\partial x}{\partial q_i} \right)^2 + \left(\frac{\partial y}{\partial q_i} \right)^2 + \left(\frac{\partial z}{\partial q_i} \right)^2$$

The above equation for $\hat{\mathbf{e}}_i$ may be taken as a starting point for deriving

$$\frac{\partial \hat{\mathbf{e}}_i}{\partial q_j} = \hat{\mathbf{e}}_j \frac{1}{h_i} \frac{\partial h_j}{\partial q_i}, \quad i \neq j$$

$$\frac{\partial \hat{\mathbf{e}}_i}{\partial q_i} = - \sum_{j \neq i} \hat{\mathbf{e}}_j \frac{1}{h_j} \frac{\partial h_i}{\partial q_j}$$

3.2 Solution

$$\hat{\mathbf{e}}_i = \frac{1}{h_i} \frac{\partial \mathbf{r}}{\partial q_i}$$

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \frac{1}{h_i h_j} \left(\frac{\partial \mathbf{r}}{\partial q_i} \right) \cdot \left(\frac{\partial \mathbf{r}}{\partial q_j} \right) = \delta_{ij}$$

$$\frac{\partial \mathbf{r}}{\partial q_i} = \frac{\partial (x_j \hat{\mathbf{e}}_j)}{\partial q_i} = \frac{\partial x_j}{\partial q_i} \hat{\mathbf{e}}_j + x_j \frac{\partial \hat{\mathbf{e}}_j}{\partial q_i}$$

$$\because \hat{\mathbf{e}}_{ij} = \delta_{ij}$$

$$\therefore \frac{\partial \hat{\mathbf{e}}_j}{\partial q_i} = 0$$

$$\frac{\partial \mathbf{r}}{\partial q_i} = \frac{\partial x_j}{\partial q_i} \hat{\mathbf{e}}_j$$

$$\frac{\partial \mathbf{r}}{\partial q_i} \cdot \frac{\partial \mathbf{r}}{\partial q_j} = \frac{\partial x_k}{\partial q_i} \frac{\partial x_m}{\partial q_j} (\hat{\mathbf{e}}_k \cdot \hat{\mathbf{e}}_m)$$

$$= \frac{\partial x_k}{\partial q_i} \frac{\partial x_m}{\partial q_j} \delta_{km} = \frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_j}$$

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \frac{1}{h_i h_j} \frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_j} = \delta_{ij}$$

$$\frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_j} = h_i h_j \delta_{ij}$$

$$i = j \implies \left(\frac{\partial x_k}{\partial q_i} \right)^2 = h_i^2$$

$$\therefore \frac{\partial(h_i \hat{\mathbf{e}}_i)}{\partial q_j} = \frac{\partial h_i}{\partial q_j} \hat{\mathbf{e}}_i + h_i \frac{\partial \hat{\mathbf{e}}_i}{\partial q_j}$$

$$\therefore \frac{\partial(h_i \hat{\mathbf{e}}_i)}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\frac{\partial \mathbf{r}}{\partial q_i} \right) = \frac{\partial^2 \mathbf{r}}{\partial q_j \partial q_i}$$

$$\therefore i \neq j \implies \frac{\partial q_i}{\partial q_j} = \frac{\partial q_j}{\partial q_i} = 0$$

$$\therefore \frac{\partial^2 \mathbf{r}}{\partial q_j \partial q_i} = \frac{\partial^2 \mathbf{r}}{\partial q_i \partial q_j}$$

$$= \frac{\partial(h_j \hat{\mathbf{e}}_j)}{\partial q_i}$$

$$\therefore \frac{\partial h_i}{\partial q_j} \hat{\mathbf{e}}_i + h_i \frac{\partial \hat{\mathbf{e}}_i}{\partial q_j} = \frac{\partial h_j}{\partial q_i} \hat{\mathbf{e}}_j + h_j \frac{\partial \hat{\mathbf{e}}_j}{\partial q_i}$$

$$\therefore \frac{\partial \hat{\mathbf{e}}_i}{\partial q_j} = f(q_j) \hat{\mathbf{e}}_j$$

We can now equate both pairs.

$$\therefore \frac{\partial \hat{\mathbf{e}}_i}{\partial q_j} = \frac{1}{h_i} \frac{\partial h_j}{\partial q_i} \hat{\mathbf{e}}_j, \quad i \neq j$$

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} \implies \frac{\partial (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j)}{\partial q_k} = \hat{\mathbf{e}}_i \cdot \frac{\partial \hat{\mathbf{e}}_j}{\partial q_k} + \frac{\partial \hat{\mathbf{e}}_i}{\partial q_k} \cdot \hat{\mathbf{e}}_j = 0$$

$$\hat{\mathbf{e}}_i \cdot \frac{\partial \hat{\mathbf{e}}_j}{\partial q_k} = -\frac{\partial \hat{\mathbf{e}}_i}{\partial q_k} \cdot \hat{\mathbf{e}}_j$$

We are interested in the case where $i = k$.

$$\hat{\mathbf{e}}_i \cdot \frac{\partial \hat{\mathbf{e}}_j}{\partial q_i} = -\frac{\partial \hat{\mathbf{e}}_i}{\partial q_i} \cdot \hat{\mathbf{e}}_j$$

From this relation, we can derive two statements.

$$\hat{\mathbf{e}}_i \cdot \frac{\partial \hat{\mathbf{e}}_i}{\partial q_i} = 0$$

Which tells us that the change of $\hat{\mathbf{e}}_i$ in the q_i direction is perpendicular to $\hat{\mathbf{e}}_i$ (if it exists).

$$\hat{\mathbf{e}}_j \cdot \frac{\partial \hat{\mathbf{e}}_i}{\partial q_i} = -\hat{\mathbf{e}}_i \cdot \frac{\partial \hat{\mathbf{e}}_j}{\partial q_i}, \quad i \neq j$$

$$\hat{\mathbf{e}}_j \cdot \frac{\partial \hat{\mathbf{e}}_i}{\partial q_j} = \frac{1}{h_i} \frac{\partial h_j}{\partial q_i} \hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_j$$

$$= \frac{1}{h_i} \frac{\partial h_j}{\partial q_i}$$

$$\hat{\mathbf{e}}_j \cdot \frac{\partial \hat{\mathbf{e}}_i}{\partial q_i} = -\frac{1}{h_j} \frac{\partial h_i}{\partial q_j}$$

Now we just add all the components of $\frac{\partial \hat{\mathbf{e}}_i}{\partial q_i}$ together to get the final result.

$$\frac{\partial \hat{\mathbf{e}}_i}{\partial q_i} = - \sum_{j \neq i} \hat{\mathbf{e}}_j \frac{1}{h_j} \frac{\partial h_i}{\partial q_j}$$

4 3.10.28

4.1 Problem

Express $\partial/\partial x, \partial/\partial y, \partial/\partial z$ in spherical polar coordinates.

ANS.

$$\frac{\partial}{\partial x} = \sin \theta \cos \varphi \frac{\partial}{\partial r} + \cos \theta \cos \varphi \frac{1}{r} \frac{\partial}{\partial \theta} - \sin \varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial y} = \sin \theta \sin \varphi \frac{\partial}{\partial r} + \cos \theta \sin \varphi \frac{1}{r} \frac{\partial}{\partial \theta} + \cos \varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial z} = \cos \theta \frac{\partial}{\partial r} - \sin \theta \frac{1}{r} \frac{\partial}{\partial \theta}$$

Hint. Equate ∇_{xyz} and $\nabla_{r\theta\varphi}$.

4.2 Solution

$$x = r \sin \theta \cos \varphi, y = r \sin \theta \sin \varphi, z = r \cos \theta$$

$$\nabla = \hat{\mathbf{e}}_1 \frac{1}{h_1} \frac{\partial}{\partial q_1} + \hat{\mathbf{e}}_2 \frac{1}{h_2} \frac{\partial}{\partial q_2} + \hat{\mathbf{e}}_3 \frac{1}{h_3} \frac{\partial}{\partial q_3}$$

$$\nabla_{xyz} = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

$$\nabla_{r\theta\varphi} = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}$$

$$\nabla_{xyz} = \nabla_{r\theta\varphi}$$

$$\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}$$

$$\hat{\mathbf{e}}_r = \frac{\partial x_j}{\partial r} \hat{\mathbf{e}}_j + x_j \frac{\partial \hat{\mathbf{e}}_j}{\partial r} = \frac{\partial x}{\partial r} \hat{i} + \frac{\partial y}{\partial r} \hat{j} + \frac{\partial z}{\partial r} \hat{k}$$

$$= \sin \theta \cos \varphi \hat{i} + \sin \theta \sin \varphi \hat{j} + \cos \theta \hat{k}$$

$$\hat{\mathbf{e}}_\theta = \cos \theta \cos \varphi \hat{i} + \cos \theta \sin \varphi \hat{j} - \sin \theta \hat{k}$$

$$\hat{\mathbf{e}}_\varphi = -\sin \varphi \hat{i} + \cos \varphi \hat{j}$$

$$\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} = \left(\sin \theta \cos \varphi \hat{i} + \sin \theta \sin \varphi \hat{j} + \cos \theta \hat{k} \right) \frac{\partial}{\partial r}$$

$$+ \left(\cos \theta \cos \varphi \hat{i} + \cos \theta \sin \varphi \hat{j} - \sin \theta \hat{k} \right) \frac{1}{r} \frac{\partial}{\partial \theta}$$

$$+ \left(-\sin \varphi \hat{i} + \cos \varphi \hat{j} \right) \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}$$

$$= \hat{i} \left(\sin \theta \cos \varphi \frac{\partial}{\partial r} + \cos \theta \cos \varphi \frac{1}{r} \frac{\partial}{\partial \theta} - \sin \varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right)$$

$$+ \hat{j} \left(\sin \theta \sin \varphi \frac{\partial}{\partial r} + \cos \theta \sin \varphi \frac{1}{r} \frac{\partial}{\partial \theta} + \cos \varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right)$$

$$+ \hat{k} \left(\cos \theta \frac{\partial}{\partial r} - \sin \theta \frac{1}{r} \frac{\partial}{\partial \theta} \right)$$

$$\frac{\partial}{\partial x} = \sin \theta \cos \varphi \frac{\partial}{\partial r} + \cos \theta \cos \varphi \frac{1}{r} \frac{\partial}{\partial \theta} - \sin \varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial y} = \sin \theta \sin \varphi \frac{\partial}{\partial r} + \cos \theta \sin \varphi \frac{1}{r} \frac{\partial}{\partial \theta} + \cos \varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial z} = \cos \theta \frac{\partial}{\partial r} - \sin \theta \frac{1}{r} \frac{\partial}{\partial \theta}$$

5 3.10.29

5.1 Problem

Using results from [Exercise 3.10.28](#), show that

$$-i \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = -i \frac{\partial}{\partial \varphi}$$

This is the quantum mechanical operator corresponding to the z -component of orbital angular momentum.

5.2 Solution

$$x = r \sin \theta \cos \varphi, y = r \sin \theta \sin \varphi$$

$$\begin{aligned} & -i \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \\ &= -ir \sin \theta \cos \varphi \left(\sin \theta \sin \varphi \frac{\partial}{\partial r} + \cos \theta \sin \varphi \frac{1}{r} \frac{\partial}{\partial \theta} + \cos \varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right) \\ &+ ir \sin \theta \sin \varphi \left(\sin \theta \cos \varphi \frac{\partial}{\partial r} + \cos \theta \cos \varphi \frac{1}{r} \frac{\partial}{\partial \theta} - \sin \varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right) \\ &= ir \sin \theta (\sin \theta \sin \varphi \cos \varphi - \sin \theta \sin \varphi \cos \varphi) \frac{\partial}{\partial r} \\ &+ ir \sin \theta (\sin \varphi \cos \varphi - \sin \varphi \cos \varphi) \cos \theta \frac{1}{r} \frac{\partial}{\partial \theta} \end{aligned}$$

$$\begin{aligned}
& -i (\sin^2 \varphi + \cos^2 \varphi) \frac{\partial}{\partial \varphi} \\
& = -i \frac{\partial}{\partial \varphi}
\end{aligned}$$

6 3.10.30

6.1 Problem

With the quantum mechanical orbital angular momentum operator defined as $\mathbf{L} = -i(\mathbf{r} \times \nabla)$, show that

$$\begin{aligned}
\text{(a)} \quad L_x + iL_y &= e^{i\varphi} \left(\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \varphi} \right) \\
\text{(b)} \quad L_x - iL_y &= -e^{-i\varphi} \left(\frac{\partial}{\partial \theta} - i \cot \theta \frac{\partial}{\partial \varphi} \right)
\end{aligned}$$

6.2 Solution

$$\begin{aligned}
\mathbf{L} &= -i(\mathbf{r} \times \nabla) \\
&= -i \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} \\
&= -i \left(\hat{i} \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) + \hat{j} \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) + \hat{k} \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \right) \\
&= i \left(z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} \right) \hat{i} + i \left(x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x} \right) \hat{j} + i \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) \hat{k} \\
L_x &= i \left(z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} \right)
\end{aligned}$$

$$L_y = i \left(x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x} \right)$$

$$L_z = i \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right)$$

6.2.1 a

$$\begin{aligned} L_x + iL_y &= i \left(z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} \right) - \left(x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x} \right) \\ &= z \frac{\partial}{\partial x} + iz \frac{\partial}{\partial y} - (x + iy) \frac{\partial}{\partial z} \\ &= z \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) - (x + iy) \frac{\partial}{\partial z} \end{aligned}$$

$$x + iy = r \sin \theta \cos \varphi + ir \sin \theta \sin \varphi = r \sin \theta (\cos \varphi + i \sin \varphi) = r \sin \theta e^{i\varphi}$$

$$\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} =$$

$$\begin{aligned} &\sin \theta \cos \varphi \frac{\partial}{\partial r} + \cos \theta \cos \varphi \frac{1}{r} \frac{\partial}{\partial \theta} - \sin \varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \\ &+ i \sin \theta \sin \varphi \frac{\partial}{\partial r} + i \cos \theta \sin \varphi \frac{1}{r} \frac{\partial}{\partial \theta} + i \cos \varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \\ &= \sin \theta (\cos \varphi + i \sin \varphi) \frac{\partial}{\partial r} + \cos \theta (\cos \varphi + i \sin \varphi) \frac{1}{r} \frac{\partial}{\partial \theta} \\ &\quad + (i \cos \varphi - \sin \varphi) \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \end{aligned}$$

$$= e^{i\varphi} \left(\sin \theta \frac{\partial}{\partial r} + \cos \theta \frac{1}{r} \frac{\partial}{\partial \theta} + i \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right)$$

$$L_x + iL_y =$$

$$r \cos \theta e^{i\varphi} \left(\sin \theta \frac{\partial}{\partial r} + \cos \theta \frac{1}{r} \frac{\partial}{\partial \theta} + i \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right)$$

$$-r \sin \theta e^{i\varphi} \left(\cos \theta \frac{\partial}{\partial r} - \sin \theta \frac{1}{r} \frac{\partial}{\partial \theta} \right)$$

$$= r e^{i\varphi} \left(\sin \theta \cos \theta \frac{\partial}{\partial r} + \frac{\cos^2 \theta}{r} \frac{\partial}{\partial \theta} + i \cot \theta \frac{1}{r} \frac{\partial}{\partial \varphi} - \sin \theta \cos \theta \frac{\partial}{\partial r} + \frac{\sin^2 \theta}{r} \frac{\partial}{\partial \theta} \right)$$

$$= e^{i\varphi} \left(\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \varphi} \right)$$

6.2.2 b

$$L_x - iL_y = i \left(z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} \right) + \left(x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x} \right)$$

$$= z \left(i \frac{\partial}{\partial y} - \frac{\partial}{\partial x} \right) + (x - iy) \frac{\partial}{\partial z}$$

$$x - iy = r \sin \theta \cos \varphi - ir \sin \theta \sin \varphi = r \sin \theta (\cos \varphi - i \sin \varphi) = r \sin \theta e^{-i\varphi}$$

$$i \frac{\partial}{\partial y} - \frac{\partial}{\partial x} =$$

$$i \sin \theta \sin \varphi \frac{\partial}{\partial r} + i \cos \theta \sin \varphi \frac{1}{r} \frac{\partial}{\partial \theta} + i \cos \varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}$$

$$\begin{aligned}
& -\sin \theta \cos \varphi \frac{\partial}{\partial r} - \cos \theta \cos \varphi \frac{1}{r} \frac{\partial}{\partial \theta} + \sin \varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \\
& = -\sin \theta (\cos \varphi - i \sin \varphi) \frac{\partial}{\partial r} - \cos \theta (\cos \varphi - i \sin \varphi) \frac{1}{r} \frac{\partial}{\partial \theta} \\
& \quad + i (\cos \varphi - i \sin \varphi) \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \\
& = -e^{-i\varphi} \left(\sin \theta \frac{\partial}{\partial r} + \cos \theta \frac{1}{r} \frac{\partial}{\partial \theta} + i \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right) \\
& L_x - iL_y = \\
& -r \cos \theta e^{-i\varphi} \left(\sin \theta \frac{\partial}{\partial r} + \cos \theta \frac{1}{r} \frac{\partial}{\partial \theta} + i \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right) \\
& \quad + r \sin \theta e^{-i\varphi} \left(\cos \theta \frac{\partial}{\partial r} - \sin \theta \frac{1}{r} \frac{\partial}{\partial \theta} \right) \\
& = -re^{-i\varphi} \left(\sin \theta \cos \theta \frac{\partial}{\partial r} + \frac{\cos^2 \theta}{r} \frac{\partial}{\partial \theta} + i \frac{\cot \theta}{r} \frac{\partial}{\partial \varphi} - \sin \theta \cos \theta \frac{\partial}{\partial r} + \frac{\sin^2 \theta}{r} \frac{\partial}{\partial \theta} \right) \\
& = -e^{-i\varphi} \left(\frac{\partial}{\partial \theta} - i \cot \theta \frac{\partial}{\partial \varphi} \right)
\end{aligned}$$

7 3.10.31

7.1 Problem

Verify that $\mathbf{L} \times \mathbf{L} = i\mathbf{L}$ in spherical polar coordinates. $\mathbf{L} = -i(\mathbf{r} \times \nabla)$, the quantum mechanical orbital angular momentum operator.

Written in component form, this relation is

$$L_y L_z - L_z L_y = iL_x, \quad L_z L_x - L_x L_z = iL_y, \quad L_x L_y - L_y L_x = iL_z$$

Using the commutator notation, $[A, B] = AB - BA$, and the definition of the Levi-Civita symbol ε_{ijk} , the above can also be written

$$[L_i, L_j] = i\varepsilon_{ijk}L_k$$

where i, j, k are x, y, z in any order.

Hint. Use spherical polar coordinates for \mathbf{L} but Cartesian components for the cross product.

7.2 Solution

$$\mathbf{L} = -i(\mathbf{r} \times \nabla) = i \left(\hat{\mathbf{e}}_\theta \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} - \hat{\mathbf{e}}_\varphi \frac{\partial}{\partial \theta} \right)$$

$$\hat{\mathbf{e}}_r = \sin \theta \cos \varphi \hat{i} + \sin \theta \sin \varphi \hat{j} + \cos \theta \hat{k}$$

$$\hat{\mathbf{e}}_\theta = \cos \theta \cos \varphi \hat{i} + \cos \theta \sin \varphi \hat{j} - \sin \theta \hat{k}$$

$$\hat{\mathbf{e}}_\varphi = -\sin \varphi \hat{i} + \cos \varphi \hat{j}$$

$$\mathbf{J}_{ij} = \frac{\partial \hat{\mathbf{e}}_i}{\partial \hat{\mathbf{e}}_j}$$

$$\begin{pmatrix} \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} & \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \varphi} \\ \frac{\partial \hat{\mathbf{e}}_\varphi}{\partial \theta} & \frac{\partial \hat{\mathbf{e}}_\varphi}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} -\hat{\mathbf{e}}_r & \cos \theta \hat{\mathbf{e}}_\varphi \\ 0 & -\sin \theta \hat{\mathbf{e}}_r - \cos \theta \hat{\mathbf{e}}_\theta \end{pmatrix}$$

$$\begin{aligned} \mathbf{L} \times \mathbf{L} &= \left(\hat{\mathbf{e}}_\theta \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} - \hat{\mathbf{e}}_\varphi \frac{\partial}{\partial \theta} \right) \times \left(\hat{\mathbf{e}}_\varphi \frac{\partial}{\partial \theta} - \hat{\mathbf{e}}_\theta \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \right) \\ &= \hat{\mathbf{e}}_\theta \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \times \hat{\mathbf{e}}_\varphi \frac{\partial}{\partial \theta} - \hat{\mathbf{e}}_\theta \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \times \hat{\mathbf{e}}_\theta \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \end{aligned}$$

$$\begin{aligned}
& -\hat{\mathbf{e}}_\varphi \frac{\partial}{\partial \theta} \times \hat{\mathbf{e}}_\varphi \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_\varphi \frac{\partial}{\partial \theta} \times \hat{\mathbf{e}}_\theta \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \\
& = \left(\hat{\mathbf{e}}_\theta \times \frac{\partial \hat{\mathbf{e}}_\varphi}{\partial \varphi} \right) \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} + \cancel{(\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_\varphi) \frac{1}{\sin \theta} \frac{\partial^2}{\partial \varphi \partial \theta}} \\
& - \left(\hat{\mathbf{e}}_\varphi \times \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \varphi} \right) \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \varphi} - \cancel{(\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_\theta) \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}} \\
& - \cancel{\left(\hat{\mathbf{e}}_\varphi \times \frac{\partial \hat{\mathbf{e}}_\varphi}{\partial \theta} \right) \frac{\partial}{\partial \theta}} - \cancel{(\hat{\mathbf{e}}_\varphi \times \hat{\mathbf{e}}_\varphi) \frac{\partial^2}{\partial \theta^2}} \\
& + \left(\hat{\mathbf{e}}_\varphi \times \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} \right) \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} + \cancel{(\hat{\mathbf{e}}_\varphi \times \hat{\mathbf{e}}_\theta) \frac{1}{\sin \theta} \frac{\partial^2}{\partial \theta \partial \varphi}} \\
& = -(\hat{\mathbf{e}}_\theta \times (\sin \theta \hat{\mathbf{e}}_r + \cos \theta \hat{\mathbf{e}}_\theta)) \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \\
& - \cancel{(\hat{\mathbf{e}}_\varphi \times \hat{\mathbf{e}}_\varphi) \frac{\cos \theta}{\sin^2 \theta} \frac{\partial}{\partial \varphi}} \\
& - (\hat{\mathbf{e}}_\varphi \times \hat{\mathbf{e}}_r) \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \\
& = -(\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_r) \frac{\partial}{\partial \theta} + \cancel{(\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_\theta) \cot \theta \frac{\partial}{\partial \theta}} - (\hat{\mathbf{e}}_\varphi \times \hat{\mathbf{e}}_r) \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \\
& = \hat{\mathbf{e}}_\varphi \frac{\partial}{\partial \theta} - \hat{\mathbf{e}}_\theta \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \\
& = i\mathbf{L}
\end{aligned}$$

8 3.10.32

8.1 Problem

(a) Using

$$\nabla\psi(r, \theta, \varphi) = \hat{\mathbf{e}}_r \frac{\partial\psi}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial\psi}{\partial\theta} + \hat{\mathbf{e}}_\varphi \frac{1}{r \sin\theta} \frac{\partial\psi}{\partial\varphi}$$

show that

$$\mathbf{L} = -i(\mathbf{r} \times \nabla) = i \left(\hat{\mathbf{e}}_\theta \frac{1}{\sin\theta} \frac{\partial}{\partial\varphi} - \hat{\mathbf{e}}_\varphi \frac{\partial}{\partial\theta} \right)$$

(b) Resolving $\hat{\mathbf{e}}_\theta$ and $\hat{\mathbf{e}}_\varphi$ into Cartesian components, determine L_x , L_y , and L_z in terms of θ , φ , and their derivatives.

(c) From $L^2 = L_x^2 + L_y^2 + L_z^2$ show that

$$\begin{aligned} \mathbf{L}^2 &= -\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \\ &= -r^2 \nabla^2 + \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) \end{aligned}$$

8.2 Solution

8.2.1 a

$$\mathbf{L} = -i(\mathbf{r} \times \nabla)$$

$$\mathbf{r} = r\hat{\mathbf{e}}_r$$

$$\nabla_{r\theta\varphi} = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_\varphi \frac{1}{r \sin\theta} \frac{\partial}{\partial\varphi}$$

$$\mathbf{L} = -i\mathbf{r} \times \left(\hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_\varphi \frac{1}{r \sin\theta} \frac{\partial}{\partial\varphi} \right)$$

$$= -ir \left((\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_r) \frac{\partial}{\partial r} + (\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_\theta) \frac{1}{r} \frac{\partial}{\partial \theta} + (\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_\varphi) \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right)$$

\therefore spherical polar coordinates are right-handed and orthonormal system.

$$\therefore \hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \varepsilon_{ijk} \hat{\mathbf{e}}_k$$

where ε_{ijk} is the Levi-Civita symbol. $\varepsilon_{ijk} = 1$ if i, j, k is an even permutation of r, θ, φ , -1 if it is an odd permutation, and 0 if any two indices are equal.

$$\begin{aligned} \therefore \mathbf{L} &= -ir \left(0 \frac{\partial}{\partial r} + 1 \hat{\mathbf{e}}_\varphi \frac{1}{r} \frac{\partial}{\partial \theta} - 1 \hat{\mathbf{e}}_\theta \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right) \\ &= i \left(\hat{\mathbf{e}}_\theta \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} - \hat{\mathbf{e}}_\varphi \frac{\partial}{\partial \theta} \right) \end{aligned}$$

8.2.2 b

$$\hat{\mathbf{e}}_\theta = \cos \theta \cos \varphi \hat{i} + \cos \theta \sin \varphi \hat{j} - \sin \theta \hat{k}$$

$$\hat{\mathbf{e}}_\varphi = -\sin \varphi \hat{i} + \cos \varphi \hat{j}$$

$$\mathbf{L} = i \left(\cos \theta \cos \varphi \hat{i} + \cos \theta \sin \varphi \hat{j} - \sin \theta \hat{k} \right) \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} + i \left(\sin \varphi \hat{i} - \cos \varphi \hat{j} \right) \frac{\partial}{\partial \theta}$$

$$= i \cos \varphi \cot \theta \hat{i} \frac{\partial}{\partial \varphi} + i \sin \varphi \cot \theta \hat{j} \frac{\partial}{\partial \varphi} - i \hat{k} \frac{\partial}{\partial \varphi} + i \sin \varphi \hat{i} \frac{\partial}{\partial \theta} - i \cos \varphi \hat{j} \frac{\partial}{\partial \theta}$$

$$= -i \hat{i} \left(-\cos \varphi \cot \theta \frac{\partial}{\partial \varphi} - \sin \varphi \frac{\partial}{\partial \theta} \right)$$

$$-i \hat{j} \left(\cos \varphi \frac{\partial}{\partial \theta} - \sin \varphi \cot \theta \frac{\partial}{\partial \varphi} \right) - i \hat{k} \frac{\partial}{\partial \varphi}$$

$$L_x = i \left(\cos \varphi \cot \theta \frac{\partial}{\partial \varphi} + \sin \varphi \frac{\partial}{\partial \theta} \right)$$

$$L_y = i \left(\sin \varphi \cot \theta \frac{\partial}{\partial \varphi} - \cos \varphi \frac{\partial}{\partial \theta} \right)$$

$$L_z = -i \frac{\partial}{\partial \varphi}$$

8.2.3 c

$$L_x^2 = - \left(\cos \varphi \cot \theta \frac{\partial}{\partial \varphi} + \sin \varphi \frac{\partial}{\partial \theta} \right)^2$$

$$= - \cot^2 \theta \cos \varphi \frac{\partial}{\partial \varphi} \left(\cos \varphi \frac{\partial}{\partial \varphi} \right) - \sin^2 \varphi \frac{\partial^2}{\partial \theta^2}$$

$$- \cos \varphi \cot \theta \frac{\partial}{\partial \varphi} \left(\sin \varphi \frac{\partial}{\partial \theta} \right) - \sin \varphi \cos \varphi \frac{\partial}{\partial \theta} \left(\cot \theta \frac{\partial}{\partial \varphi} \right)$$

$$L_y^2 = - \left(\sin \varphi \cot \theta \frac{\partial}{\partial \varphi} - \cos \varphi \frac{\partial}{\partial \theta} \right)^2$$

$$= - \cot^2 \theta \sin \varphi \frac{\partial}{\partial \varphi} \left(\sin \varphi \frac{\partial}{\partial \varphi} \right) - \cos^2 \varphi \frac{\partial^2}{\partial \theta^2}$$

$$+ \sin \varphi \cot \theta \frac{\partial}{\partial \varphi} \left(\cos \varphi \frac{\partial}{\partial \theta} \right) + \sin \varphi \cos \varphi \frac{\partial}{\partial \theta} \left(\cot \theta \frac{\partial}{\partial \varphi} \right)$$

$$L_z^2 = - \frac{\partial^2}{\partial \varphi^2}$$

$$\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2$$

$$\begin{aligned}
&= -\cot^2 \theta \cos \varphi \frac{\partial}{\partial \varphi} \left(\cos \varphi \frac{\partial}{\partial \varphi} \right) - \sin^2 \varphi \frac{\partial^2}{\partial \theta^2} \\
&\quad - \cos \varphi \cot \theta \frac{\partial}{\partial \varphi} \left(\sin \varphi \frac{\partial}{\partial \theta} \right) - \sin \varphi \cos \varphi \frac{\partial}{\partial \theta} \left(\cot \theta \frac{\partial}{\partial \varphi} \right) \\
&\quad - \cot^2 \theta \sin \varphi \frac{\partial}{\partial \varphi} \left(\sin \varphi \frac{\partial}{\partial \varphi} \right) - \cos^2 \varphi \frac{\partial^2}{\partial \theta^2} \\
&\quad + \sin \varphi \cot \theta \frac{\partial}{\partial \varphi} \left(\cos \varphi \frac{\partial}{\partial \theta} \right) + \sin \varphi \cos \varphi \frac{\partial}{\partial \theta} \left(\cot \theta \frac{\partial}{\partial \varphi} \right) - \frac{\partial^2}{\partial \theta^2} - \frac{\partial^2}{\partial \varphi^2} \\
&= -\cot^2 \theta \left(\cos \varphi \frac{\partial}{\partial \varphi} \left(\cos \varphi \frac{\partial}{\partial \varphi} \right) + \sin \varphi \frac{\partial}{\partial \varphi} \left(\sin \varphi \frac{\partial}{\partial \varphi} \right) \right) - \frac{\partial^2}{\partial \theta^2} - \frac{\partial^2}{\partial \varphi^2} \\
&\quad + \cot \theta \left(\sin \varphi \frac{\partial}{\partial \varphi} \left(\cos \varphi \frac{\partial}{\partial \theta} \right) - \cos \varphi \frac{\partial}{\partial \varphi} \left(\sin \varphi \frac{\partial}{\partial \theta} \right) \right) \\
&\quad \cos \varphi \frac{\partial}{\partial \varphi} \left(\cos \varphi \frac{\partial}{\partial \varphi} \right) = -\sin \varphi \cos \varphi \frac{\partial}{\partial \varphi} + \cos^2 \varphi \frac{\partial^2}{\partial \varphi^2} \\
&\quad \sin \varphi \frac{\partial}{\partial \varphi} \left(\sin \varphi \frac{\partial}{\partial \varphi} \right) = \sin \varphi \cos \varphi \frac{\partial}{\partial \varphi} + \sin^2 \varphi \frac{\partial^2}{\partial \varphi^2} \\
&\quad \sin \varphi \frac{\partial}{\partial \varphi} \left(\cos \varphi \frac{\partial}{\partial \theta} \right) = -\sin^2 \varphi \frac{\partial}{\partial \theta} + \sin \varphi \cos \varphi \frac{\partial^2}{\partial \varphi \partial \theta} \\
&\quad \cos \varphi \frac{\partial}{\partial \varphi} \left(\sin \varphi \frac{\partial}{\partial \theta} \right) = \cos^2 \varphi \frac{\partial}{\partial \theta} + \sin \varphi \cos \varphi \frac{\partial^2}{\partial \varphi \partial \theta}
\end{aligned}$$

$$\mathbf{L}^2 = -\cot^2 \theta \left(-\sin \varphi \cos \varphi \frac{\partial}{\partial \varphi} + \cos^2 \varphi \frac{\partial^2}{\partial \varphi^2} + \sin \varphi \cos \varphi \frac{\partial}{\partial \varphi} + \sin^2 \varphi \frac{\partial^2}{\partial \varphi^2} \right)$$

$$\begin{aligned}
& -\frac{\partial^2}{\partial \theta^2} - \frac{\partial^2}{\partial \varphi^2} \\
& + \cot \theta \left(-\sin^2 \varphi \frac{\partial}{\partial \theta} + \sin \varphi \cos \varphi \frac{\partial^2}{\partial \varphi \partial \theta} - \cos^2 \varphi \frac{\partial}{\partial \theta} - \sin \varphi \cos \varphi \frac{\partial^2}{\partial \varphi \partial \theta} \right) \\
& = -\cot^2 \theta \frac{\partial^2}{\partial \varphi^2} - \frac{\partial^2}{\partial \theta^2} - \frac{\partial^2}{\partial \varphi^2} - \cot \theta \frac{\partial}{\partial \theta} \\
& = -(1 + \cot^2 \theta) \frac{\partial^2}{\partial \varphi^2} - \frac{\partial^2}{\partial \theta^2} - \cot \theta \frac{\partial}{\partial \theta} \\
& = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \\
& \nabla^2 = \frac{1}{r^2 \sin \theta} \left[\sin \theta \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2}{\partial \varphi^2} \right] \\
& -r^2 \nabla^2 = -\frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \\
& -r^2 \nabla^2 + \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \\
& = \mathbf{L}^2
\end{aligned}$$

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