PEU 356 Assignment 1

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1 3.10.1

1.1 Problem

The $u-, v-, z-$coordinate system frequently used in electrostatics and in hydrodynamics is defined by

$$xy = u, x^2 - y^2 = v, z = z$$

This $u-, v-, z-$system is orthogonal.

(a) In words, describe briefly the nature of each of the three families of coordinate surfaces.

(b) Sketch the system in the $xy-$plane showing the intersections of surfaces of constant u and surfaces of constant v with the $xy-$plane.

(c) Indicate the directions of the unit vectors $\hat{\mathbf{e}}_u$ and $\hat{\mathbf{e}}_v$ in all four quadrants.

(d) Finally, is this $u-, v-, z-$system right-handed ($\hat{\mathbf{e}}_u \times \hat{\mathbf{e}}_v = +\hat{\mathbf{e}}_z$) or left-handed ($\hat{\mathbf{e}}_u \times \hat{\mathbf{e}}_v = -\hat{\mathbf{e}}_z$)?

1.2 Solution

1.2.1 a

u curves are $\frac{n}{x}$ asymptotes, $u = 0$ is xz-plane, repeated for all values of z.

$$(x, \frac{u_i}{x}, z)$$

v curves are x shaped for $v = 0$, horizontal hyperbola for $v > 0$, vertical hyperbola for $v < 0$, repeated for all values of z.

$$(x, \pm\sqrt{x^2 - v_i}, z)$$

z curves are planes parallel to xy-plane.

$$(x, y, z_i)$$

1.2.2 b

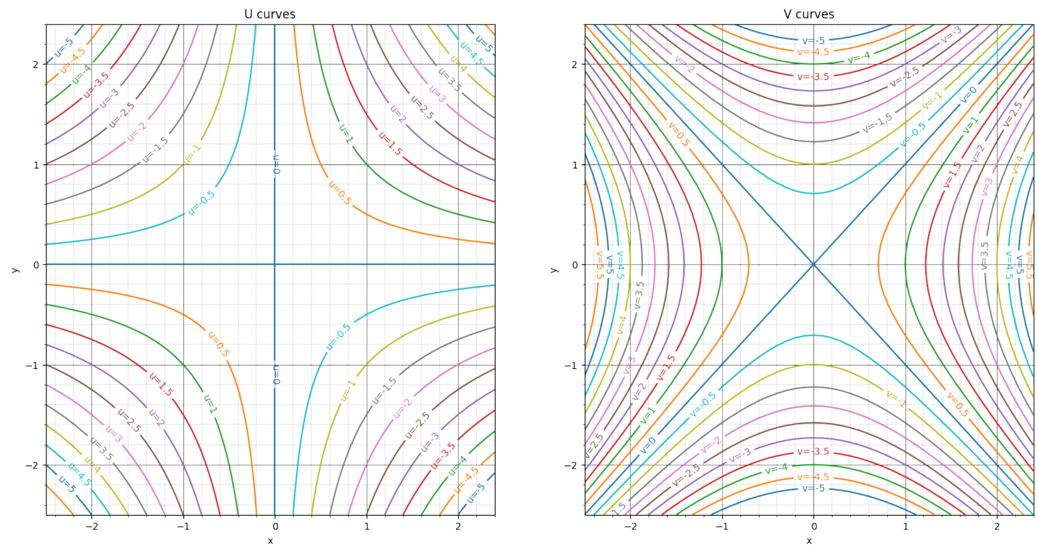

Figure 1: U and V curves.

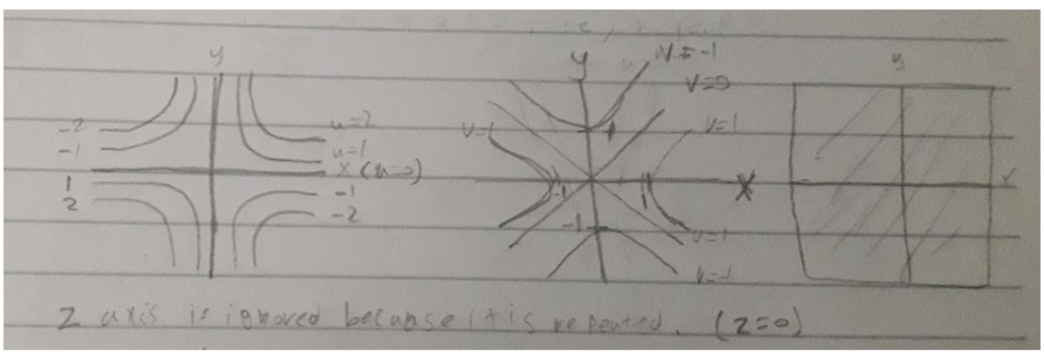

Figure 2: U and V curves sketch.

1.2.3 c

$$\nabla u = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} = \frac{\partial (xy)}{\partial x}\hat{i} + \frac{\partial (xy)}{\partial y}\hat{j} = y\hat{i} + x\hat{j}$$

$$\nabla v = \frac{\partial v}{\partial x}\hat{i} + \frac{\partial v}{\partial y}\hat{j} = \frac{\partial(x^2 - y^2)}{\partial x}\hat{i} + \frac{\partial(x^2 - y^2)}{\partial y}\hat{j} = 2x\hat{i} - 2y\hat{j}$$

$$\hat{\mathbf{e}}_u = \frac{\nabla u}{|\nabla u|} = \frac{y\hat{i} + x\hat{j}}{\sqrt{x^2 + y^2}}$$

$$\hat{\mathbf{e}}_v = \frac{\nabla v}{|\nabla v|} = \frac{2x\hat{i} - 2y\hat{j}}{\sqrt{4x^2 + 4y^2}}$$

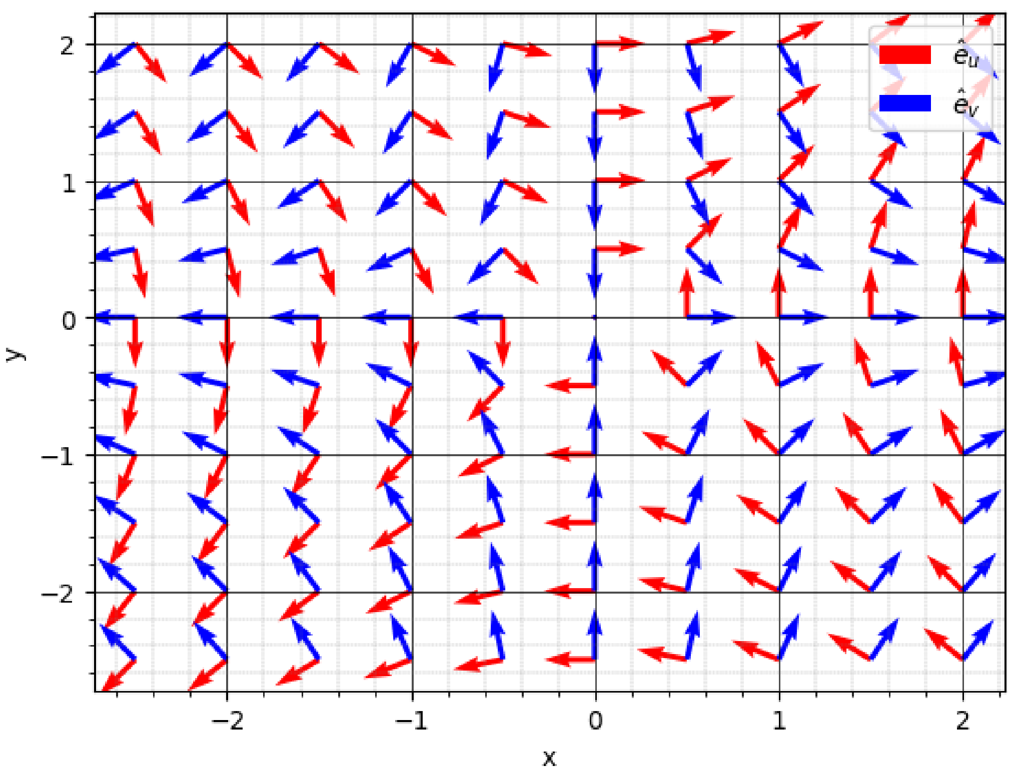

Figure 3: Basis Vectors Field Plot

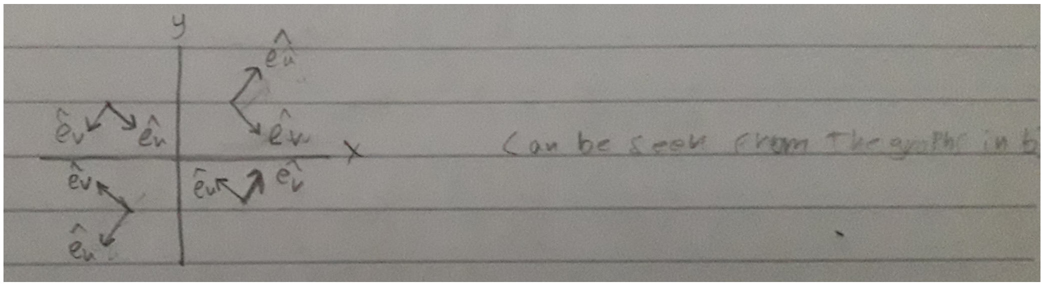

Figure 4: Basis Vectors Field Plot sketch.

1.2.4 d

$$\hat{\mathbf{e}}_u \times \hat{\mathbf{e}}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ \frac{x}{\sqrt{x^2+y^2}} & -\frac{y}{\sqrt{x^2+y^2}} & 0 \end{vmatrix}$$

$$= \left(-\frac{y}{\sqrt{x^2+y^2}} \frac{y}{\sqrt{x^2+y^2}} - \frac{x}{\sqrt{x^2+y^2}} \frac{x}{\sqrt{x^2+y^2}} \right) \hat{k} = -\hat{k}$$

From the previous result, we can see that the coordinate system is left-handed.

2 3.10.4

2.1 Problem

With $\hat{\mathbf{e}}_1$ a unit vector in the direction of increasing q_1, show that

(a) $\nabla \cdot \hat{\mathbf{e}}_1 = \frac{1}{h_1 h_2 h_3} \frac{\partial (h_2 h_3)}{\partial q_1}$

(b) $\nabla \times \hat{\mathbf{e}}_1 = \frac{1}{h_1} \left[\hat{\mathbf{e}}_2 \frac{1}{h_3} \frac{\partial h_1}{\partial q_3} - \hat{\mathbf{e}}_3 \frac{1}{h_2} \frac{\partial h_1}{\partial q_2} \right]$

Note that even though $\hat{\mathbf{e}}_1$ is a unit vector, its divergence and curl **do not necessarily vanish**.

2.2 Solution

$\hat{\mathbf{e}}_1 = \langle 1, 0, 0 \rangle$

2.2.1 a

$$\nabla \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (V_1 h_2 h_3) + \frac{\partial}{\partial q_2} (V_2 h_1 h_3) + \frac{\partial}{\partial q_3} (V_3 h_1 h_2) \right]$$

$$\nabla \cdot \hat{\mathbf{e}}_1 = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (1 * h_2 h_3) + \frac{\partial}{\partial q_2} (0 * h_1 h_3) + \frac{\partial}{\partial q_3} (0 * h_1 h_2) \right]$$

$$\nabla \cdot \hat{\mathbf{e}}_1 = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial q_1} (h_2 h_3)$$

2.2.2 b

$$\nabla \times \mathbf{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{\mathbf{e}}_1 h_1 & \hat{\mathbf{e}}_2 h_2 & \hat{\mathbf{e}}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$$

$$\nabla \times \hat{\mathbf{e}}_1 = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{\mathbf{e}}_1 h_1 & \hat{\mathbf{e}}_2 h_2 & \hat{\mathbf{e}}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 * 1 & h_2 * 0 & h_3 * 0 \end{vmatrix}$$

$$\nabla \times \hat{\mathbf{e}}_1 = \frac{1}{h_1 h_2 h_3} \left(\hat{\mathbf{e}}_2 h_2 \frac{\partial h_1}{\partial q_3} - \hat{\mathbf{e}}_3 h_3 \frac{\partial h_1}{\partial q_2} \right)$$

$$\nabla \times \hat{\mathbf{e}}_1 = \frac{1}{h_1} \left[\hat{\mathbf{e}}_2 \frac{1}{h_3} \frac{\partial h_1}{\partial q_3} - \hat{\mathbf{e}}_3 \frac{1}{h_2} \frac{\partial h_1}{\partial q_2} \right]$$

3 3.10.5

3.1 Problem

Show that a set of orthogonal unit vectors $\hat{\mathbf{e}}_i$ may be defined by

$$\hat{\mathbf{e}}_i = \frac{1}{h_i} \frac{\partial \mathbf{r}}{\partial q_i}$$

In particular, show that $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_i = 1$ leads to an expression for h_i in agreement with

$$h_i^2 = \left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2$$

The above equation for $\hat{\mathbf{e}}_i$ may be taken as a starting point for deriving

$$\frac{\partial \hat{\mathbf{e}}_i}{\partial q_j} = \hat{\mathbf{e}}_j \frac{1}{h_i} \frac{\partial h_j}{\partial q_i}, \quad i \neq j$$

$$\frac{\partial \hat{\mathbf{e}}_i}{\partial q_i} = -\sum_{j \neq i} \hat{\mathbf{e}}_j \frac{1}{h_j} \frac{\partial h_i}{\partial q_j}$$

3.2 Solution

$$\hat{\mathbf{e}}_i = \frac{1}{h_i} \frac{\partial \mathbf{r}}{\partial q_i}$$

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \frac{1}{h_i h_j} \left(\frac{\partial \mathbf{r}}{\partial q_i}\right) \cdot \left(\frac{\partial \mathbf{r}}{\partial q_j}\right) = \delta_{ij}$$

$$\frac{\partial \mathbf{r}}{\partial q_i} = \frac{\partial (x_j \hat{\varepsilon}_j)}{\partial q_i} = \frac{\partial x_j}{\partial q_i} \hat{\varepsilon}_j + x_j \frac{\partial \hat{\varepsilon}_j}{\partial q_i}$$

$$\because \hat{\varepsilon}_{ij} = \delta_{ij}$$

$$\therefore \frac{\partial \hat{\varepsilon}_j}{\partial q_i} = 0$$

$$\frac{\partial \mathbf{r}}{\partial q_i} = \frac{\partial x_j}{\partial q_i} \hat{\varepsilon}_j$$

$$\frac{\partial \mathbf{r}}{\partial q_i} \cdot \frac{\partial \mathbf{r}}{\partial q_j} = \frac{\partial x_k}{\partial q_i} \frac{\partial x_m}{\partial q_j} \left(\hat{\varepsilon}_k \cdot \hat{\varepsilon}_m\right)$$

$$= \frac{\partial x_k}{\partial q_i}\frac{\partial x_m}{\partial q_j}\delta_{km} = \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_j}$$

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \frac{1}{h_i h_j}\frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_j} = \delta_{ij}$$

$$\frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_j} = h_i h_j \delta_{ij}$$

$$i = j \implies \left(\frac{\partial x_k}{\partial q_i}\right)^2 = h_i^2$$

$$\because \frac{\partial (h_i \hat{\mathbf{e}}_i)}{\partial q_j} = \frac{\partial h_i}{\partial q_j}\hat{\mathbf{e}}_i + h_i \frac{\partial \hat{\mathbf{e}}_i}{\partial q_j}$$

$$\because \frac{\partial (h_i \hat{\mathbf{e}}_i)}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\frac{\partial \mathbf{r}}{\partial q_i}\right) = \frac{\partial^2 \mathbf{r}}{\partial q_j \partial q_i}$$

$$\because i \neq j \implies \frac{\partial q_i}{\partial q_j} = \frac{\partial q_j}{\partial q_i} = 0$$

$$\therefore \frac{\partial^2 \mathbf{r}}{\partial q_j \partial q_i} = \frac{\partial^2 \mathbf{r}}{\partial q_i \partial q_j}$$

$$= \frac{\partial (h_j \hat{\mathbf{e}}_j)}{\partial q_i}$$

$$\therefore \frac{\partial h_i}{\partial q_j}\hat{\mathbf{e}}_i + h_i \frac{\partial \hat{\mathbf{e}}_i}{\partial q_j} = \frac{\partial h_j}{\partial q_i}\hat{\mathbf{e}}_j + h_j \frac{\partial \hat{\mathbf{e}}_j}{\partial q_i}$$

$$\because \frac{\partial \hat{\mathbf{e}}_i}{\partial q_j} = f\left(q_j\right)\hat{\mathbf{e}}_j$$

We can now equate both pairs.

$$\therefore \frac{\partial \hat{\mathbf{e}}_i}{\partial q_j} = \frac{1}{h_i}\frac{\partial h_j}{\partial q_i}\hat{\mathbf{e}}_j, \quad i \neq j$$

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} \implies \frac{\partial\left(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j\right)}{\partial q_k} = \hat{\mathbf{e}}_i \cdot \frac{\partial \hat{\mathbf{e}}_j}{\partial q_k} + \frac{\partial \hat{\mathbf{e}}_i}{\partial q_k} \cdot \hat{\mathbf{e}}_j = 0$$

$$\hat{\mathbf{e}}_i \cdot \frac{\partial \hat{\mathbf{e}}_j}{\partial q_k} = -\frac{\partial \hat{\mathbf{e}}_i}{\partial q_k} \cdot \hat{\mathbf{e}}_j$$

We are interested in the case where $i = k$.

$$\hat{\mathbf{e}}_i \cdot \frac{\partial \hat{\mathbf{e}}_j}{\partial q_i} = -\frac{\partial \hat{\mathbf{e}}_i}{\partial q_i} \cdot \hat{\mathbf{e}}_j$$

From this relation, we can derive two statements.

$$\hat{\mathbf{e}}_i \cdot \frac{\partial \hat{\mathbf{e}}_i}{\partial q_i} = 0$$

Which tells us that the change of $\hat{\mathbf{e}}_i$ in the q_i direction is perpendicular to $\hat{\mathbf{e}}_i$ (if it exists).

$$\hat{\mathbf{e}}_j \cdot \frac{\partial \hat{\mathbf{e}}_i}{\partial q_i} = -\hat{\mathbf{e}}_i \cdot \frac{\partial \hat{\mathbf{e}}_j}{\partial q_i}, \quad i \neq j$$

$$\hat{\mathbf{e}}_j \cdot \frac{\partial \hat{\mathbf{e}}_i}{\partial q_j} = \frac{1}{h_i}\frac{\partial h_j}{\partial q_i}\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_j$$

$$= \frac{1}{h_i}\frac{\partial h_j}{\partial q_i}$$

$$\hat{\mathbf{e}}_j \cdot \frac{\partial \hat{\mathbf{e}}_i}{\partial q_i} = -\frac{1}{h_j}\frac{\partial h_i}{\partial q_j}$$

Now we just add all the components of $\frac{\partial \hat{\mathbf{e}}_i}{\partial q_i}$ together to get the final result.

$$\frac{\partial \hat{\mathbf{e}}_i}{\partial q_i} = -\sum_{j\neq i} \hat{\mathbf{e}}_j \frac{1}{h_j}\frac{\partial h_i}{\partial q_j}$$

4 3.10.28

4.1 Problem

Express $\partial/\partial x, \partial/\partial y, \partial/\partial z$ in spherical polar coordinates.

ANS.

$$\frac{\partial}{\partial x} = \sin\theta\cos\varphi\frac{\partial}{\partial r} + \cos\theta\cos\varphi\frac{1}{r}\frac{\partial}{\partial \theta} - \sin\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\varphi\frac{\partial}{\partial r} + \cos\theta\sin\varphi\frac{1}{r}\frac{\partial}{\partial \theta} + \cos\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{1}{r}\frac{\partial}{\partial \theta}$$

Hint. Equate ∇_{xyz} and $\nabla_{r\theta\varphi}$.

4.2 Solution

$$x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi, z = r\cos\theta$$

$$\nabla = \hat{\mathbf{e}}_1\frac{1}{h_1}\frac{\partial}{\partial q_1} + \hat{\mathbf{e}}_2\frac{1}{h_2}\frac{\partial}{\partial q_2} + \hat{\mathbf{e}}_3\frac{1}{h_3}\frac{\partial}{\partial q_3}$$

$$\nabla_{xyz} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\nabla_{r\theta\varphi} = \hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}$$

$$\nabla_{xyz} = \nabla_{r\theta\varphi}$$

$$\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}=\hat{\mathbf{e}}_r\frac{\partial}{\partial r}+\hat{\mathbf{e}}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}+\hat{\mathbf{e}}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\hat{\mathbf{e}}_r=\frac{\partial x_j}{\partial r}\hat{\varepsilon}_j+x_j\frac{\partial\hat{\varepsilon}_j}{\partial r}=\frac{\partial x}{\partial r}\hat{i}+\frac{\partial y}{\partial r}\hat{j}+\frac{\partial z}{\partial r}\hat{k}$$

$$=\sin\theta\cos\varphi\hat{i}+\sin\theta\sin\varphi\hat{j}+\cos\theta\hat{k}$$

$$\hat{\mathbf{e}}_\theta=\cos\theta\cos\varphi\hat{i}+\cos\theta\sin\varphi\hat{j}-\sin\theta\hat{k}$$

$$\hat{\mathbf{e}}_\varphi=-\sin\varphi\hat{i}+\cos\varphi\hat{j}$$

$$\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}=\left(\sin\theta\cos\varphi\hat{i}+\sin\theta\sin\varphi\hat{j}+\cos\theta\hat{k}\right)\frac{\partial}{\partial r}$$

$$+\left(\cos\theta\cos\varphi\hat{i}+\cos\theta\sin\varphi\hat{j}-\sin\theta\hat{k}\right)\frac{1}{r}\frac{\partial}{\partial\theta}$$

$$+\left(-\sin\varphi\hat{i}+\cos\varphi\hat{j}\right)\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$=\hat{i}\left(\sin\theta\cos\varphi\frac{\partial}{\partial r}+\cos\theta\cos\varphi\frac{1}{r}\frac{\partial}{\partial\theta}-\sin\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$+\hat{j}\left(\sin\theta\sin\varphi\frac{\partial}{\partial r}+\cos\theta\sin\varphi\frac{1}{r}\frac{\partial}{\partial\theta}+\cos\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$+\hat{k}\left(\cos\theta\frac{\partial}{\partial r}-\sin\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)$$

$$\frac{\partial}{\partial x}=\sin\theta\cos\varphi\frac{\partial}{\partial r}+\cos\theta\cos\varphi\frac{1}{r}\frac{\partial}{\partial\theta}-\sin\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\varphi\frac{\partial}{\partial r} + \cos\theta\sin\varphi\frac{1}{r}\frac{\partial}{\partial\theta} + \cos\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{1}{r}\frac{\partial}{\partial\theta}$$

5 3.10.29

5.1 Problem

Using results from Exercise 3.10.28, show that

$$-i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\frac{\partial}{\partial\varphi}$$

This is the quantum mechanical operator corresponding to the $z-$component of orbital angular momentum.

5.2 Solution

$$x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi$$

$$-i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$= -ir\sin\theta\cos\varphi\left(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \cos\theta\sin\varphi\frac{1}{r}\frac{\partial}{\partial\theta} + \cos\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$+ir\sin\theta\sin\varphi\left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \cos\theta\cos\varphi\frac{1}{r}\frac{\partial}{\partial\theta} - \sin\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$= ir\sin\theta\left(\sin\theta\sin\varphi\cos\varphi - \sin\theta\sin\varphi\cos\varphi\right)\frac{\partial}{\partial r}$$

$$+ir\sin\theta\left(\sin\varphi\cos\varphi - \sin\varphi\cos\varphi\right)\cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}$$

$$-i\left(\sin^2\varphi+\cos^2\varphi\right)\frac{\partial}{\partial\varphi}$$

$$=-i\frac{\partial}{\partial\varphi}$$

6 3.10.30

6.1 Problem

With the quantum mechanical orbital angular momentum operator defined as $\mathbf{L}=-i\left(\mathbf{r}\times\nabla\right)$, show that

(a) $L_x+iL_y=e^{i\varphi}\left(\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\varphi}\right)$

(b) $L_x-iL_y=-e^{-i\varphi}\left(\frac{\partial}{\partial\theta}-i\cot\theta\frac{\partial}{\partial\varphi}\right)$

6.2 Solution

$$\mathbf{L}=-i\left(\mathbf{r}\times\nabla\right)$$

$$=-i\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x & y & z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{vmatrix}$$

$$=-i\left(\hat{i}\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)+\hat{j}\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right)+\hat{k}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\right)$$

$$=i\left(z\frac{\partial}{\partial y}-y\frac{\partial}{\partial z}\right)\hat{i}+i\left(x\frac{\partial}{\partial z}-z\frac{\partial}{\partial x}\right)\hat{j}+i\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)\hat{k}$$

$$L_x=i\left(z\frac{\partial}{\partial y}-y\frac{\partial}{\partial z}\right)$$

$$L_y = i\left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right)$$

$$L_z = i\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$$

6.2.1 a

$$L_x + iL_y = i\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) - \left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right)$$

$$= z\frac{\partial}{\partial x} + iz\frac{\partial}{\partial y} - (x+iy)\frac{\partial}{\partial z}$$

$$= z\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) - (x+iy)\frac{\partial}{\partial z}$$

$$x + iy = r\sin\theta\cos\varphi + ir\sin\theta\sin\varphi = r\sin\theta\left(\cos\varphi + i\sin\varphi\right) = r\sin\theta e^{i\varphi}$$

$$\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} =$$

$$\sin\theta\cos\varphi\frac{\partial}{\partial r} + \cos\theta\cos\varphi\frac{1}{r}\frac{\partial}{\partial\theta} - \sin\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$+i\sin\theta\sin\varphi\frac{\partial}{\partial r} + i\cos\theta\sin\varphi\frac{1}{r}\frac{\partial}{\partial\theta} + i\cos\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$= \sin\theta\left(\cos\varphi + i\sin\varphi\right)\frac{\partial}{\partial r} + \cos\theta\left(\cos\varphi + i\sin\varphi\right)\frac{1}{r}\frac{\partial}{\partial\theta}$$

$$+ \left(i\cos\varphi - \sin\varphi\right)\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$= e^{i\varphi}\left(\sin\theta\frac{\partial}{\partial r} + \cos\theta\frac{1}{r}\frac{\partial}{\partial\theta} + i\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$L_x + iL_y =$$

$$r\cos\theta e^{i\varphi}\left(\sin\theta\frac{\partial}{\partial r} + \cos\theta\frac{1}{r}\frac{\partial}{\partial\theta} + i\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$-r\sin\theta e^{i\varphi}\left(\cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)$$

$$= re^{i\varphi}\left(\sin\theta\cos\theta\frac{\partial}{\partial r} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial\theta} + i\cot\theta\frac{1}{r}\frac{\partial}{\partial\varphi} - \sin\theta\cos\theta\frac{\partial}{\partial r} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial\theta}\right)$$

$$= e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

6.2.2 b

$$L_x - iL_y = i\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) + \left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right)$$

$$= z\left(i\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right) + (x - iy)\frac{\partial}{\partial z}$$

$$x - iy = r\sin\theta\cos\varphi - ir\sin\theta\sin\varphi = r\sin\theta(\cos\varphi - i\sin\varphi) = r\sin\theta e^{-i\varphi}$$

$$i\frac{\partial}{\partial y} - \frac{\partial}{\partial x} =$$

$$i\sin\theta\sin\varphi\frac{\partial}{\partial r} + i\cos\theta\sin\varphi\frac{1}{r}\frac{\partial}{\partial\theta} + i\cos\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$-\sin\theta\cos\varphi\frac{\partial}{\partial r}-\cos\theta\cos\varphi\frac{1}{r}\frac{\partial}{\partial\theta}+\sin\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$=-\sin\theta\left(\cos\varphi-i\sin\varphi\right)\frac{\partial}{\partial r}-\cos\theta\left(\cos\varphi-i\sin\varphi\right)\frac{1}{r}\frac{\partial}{\partial\theta}$$

$$+i\left(\cos\varphi-i\sin\varphi\right)\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$=-e^{-i\varphi}\left(\sin\theta\frac{\partial}{\partial r}+\cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}+i\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$L_x-iL_y=$$

$$-r\cos\theta e^{-i\varphi}\left(\sin\theta\frac{\partial}{\partial r}+\cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}+i\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$+r\sin\theta e^{-i\varphi}\left(\cos\theta\frac{\partial}{\partial r}-\sin\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)$$

$$=-re^{-i\varphi}\left(\sin\theta\cos\theta\frac{\partial}{\partial r}+\frac{\cos^2\theta}{r}\frac{\partial}{\partial\theta}+i\frac{\cot\theta}{r}\frac{\partial}{\partial\varphi}-\sin\theta\cos\theta\frac{\partial}{\partial r}+\frac{\sin^2\theta}{r}\frac{\partial}{\partial\theta}\right)$$

$$=-e^{-i\varphi}\left(\frac{\partial}{\partial\theta}-i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

7 3.10.31

7.1 Problem

Verify that $\mathbf{L}\times\mathbf{L}=i\mathbf{L}$ in spherical polar coordinates. $\mathbf{L}=-i\left(\mathbf{r}\times\nabla\right)$, the quantum mechanical orbital angular momentum operator.

Written in component form, this relation is

$$L_yL_z - L_zL_y = iL_x, \quad L_zL_x - L_xL_z = iL_y, \quad L_xL_y - L_yL_x = iL_z$$

Using the commutator notation, $[A, B] = AB - BA$, and the definition of the Levi-Civita symbol ε_{ijk}, the above can also be written

$$[L_i, L_j] = i\varepsilon_{ijk}L_k$$

where i, j, k are x, y, z in any order.

Hint. Use spherical polar coordinates for $\mathbf{L}$ but Cartesian components for the cross product.

7.2 Solution

$$\mathbf{L} = -i\left(\mathbf{r} \times \nabla\right) = i\left(\hat{\mathbf{e}}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} - \hat{\mathbf{e}}_\varphi \frac{\partial}{\partial\theta}\right)$$

$$\hat{\mathbf{e}}_r = \sin\theta\cos\varphi\hat{i} + \sin\theta\sin\varphi\hat{j} + \cos\theta\hat{k}$$

$$\hat{\mathbf{e}}_\theta = \cos\theta\cos\varphi\hat{i} + \cos\theta\sin\varphi\hat{j} - \sin\theta\hat{k}$$

$$\hat{\mathbf{e}}_\varphi = -\sin\varphi\hat{i} + \cos\varphi\hat{j}$$

$$\mathbf{J}_{ij} = \frac{\partial\hat{\mathbf{e}}_i}{\partial\hat{\mathbf{e}}_j}$$

$$\begin{pmatrix} \frac{\partial\hat{\mathbf{e}}_\theta}{\partial\theta} & \frac{\partial\hat{\mathbf{e}}_\theta}{\partial\varphi} \\ \frac{\partial\hat{\mathbf{e}}_\varphi}{\partial\theta} & \frac{\partial\hat{\mathbf{e}}_\varphi}{\partial\varphi} \end{pmatrix} = \begin{pmatrix} -\hat{\mathbf{e}}_r & \cos\theta\hat{\mathbf{e}}_\varphi \\ 0 & -\sin\theta\hat{\mathbf{e}}_r - \cos\theta\hat{\mathbf{e}}_\theta \end{pmatrix}$$

$$\mathbf{L} \times \mathbf{L} = \left(\hat{\mathbf{e}}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} - \hat{\mathbf{e}}_\varphi \frac{\partial}{\partial\theta}\right) \times \left(\hat{\mathbf{e}}_\varphi \frac{\partial}{\partial\theta} - \hat{\mathbf{e}}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$= \hat{\mathbf{e}}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} \times \hat{\mathbf{e}}_\varphi \frac{\partial}{\partial\theta} - \hat{\mathbf{e}}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} \times \hat{\mathbf{e}}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}$$

$$-\hat{\mathbf{e}}_\varphi \frac{\partial}{\partial\theta} \times \hat{\mathbf{e}_\varphi}\frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_\varphi \frac{\partial}{\partial\theta} \times \hat{\mathbf{e}}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}$$

$$= \left(\hat{\mathbf{e}}_\theta \times \frac{\partial\hat{\mathbf{e}}_\varphi}{\partial\varphi}\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} + \cancel{\left(\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_\varphi\right)\frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi\partial\theta}}$$

$$-\left(\hat{\mathbf{e}}_\varphi \times \frac{\partial\hat{\mathbf{e}}_\theta}{\partial\varphi}\right)\frac{1}{\sin^2\theta}\frac{\partial}{\partial\varphi} - \cancelto{0}{\left(\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_\theta\right)\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}}$$

$$-\cancelto{0}{\left(\hat{\mathbf{e}}_\varphi \times \frac{\partial\hat{\mathbf{e}}_\varphi}{\partial\theta}\right)\frac{\partial}{\partial\theta}} - \cancelto{0}{\left(\hat{\mathbf{e}}_\varphi \times \hat{\mathbf{e}}_\varphi\right)\frac{\partial^2}{\partial\theta^2}}$$

$$+\left(\hat{\mathbf{e}}_\varphi \times \frac{\partial\hat{\mathbf{e}}_\theta}{\partial\theta}\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} + \cancel{\left(\hat{\mathbf{e}}_\varphi \times \hat{\mathbf{e}}_\theta\right)\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi}}$$

$$= -\left(\hat{\mathbf{e}}_\theta \times \left(\sin\theta\hat{\mathbf{e}}_r + \cos\theta\hat{\mathbf{e}}_\theta\right)\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}$$

$$\cancelto{0}{-\left(\hat{\mathbf{e}}_\varphi \times \hat{\mathbf{e}}_\varphi\right)\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi}}$$

$$-\left(\hat{\mathbf{e}}_\varphi \times \hat{\mathbf{e}}_r\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}$$

$$= -\left(\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_r\right)\frac{\partial}{\partial\theta} + \cancelto{0}{\left(\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_\theta\right)\cot\theta\frac{\partial}{\partial\theta}} - \left(\hat{\mathbf{e}}_\varphi \times \hat{\mathbf{e}}_r\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}$$

$$= \hat{\mathbf{e}}_\varphi\frac{\partial}{\partial\theta} - \hat{\mathbf{e}}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}$$

$$= i\mathbf{L}$$

8 3.10.32

8.1 Problem

(a) Using

$$\nabla\psi(r,\theta,\varphi) = \hat{\mathbf{e}}_r\frac{\partial\psi}{\partial r} + \hat{\mathbf{e}}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\mathbf{e}}_\varphi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}$$

show that

$$\mathbf{L} = -i\left(\mathbf{r}\times\nabla\right) = i\left(\hat{\mathbf{e}}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} - \hat{\mathbf{e}}_\varphi\frac{\partial}{\partial\theta}\right)$$

(b) Resolving $\hat{\mathbf{e}}_\theta$ and $\hat{\mathbf{e}}_\varphi$ into Cartesian components, determine L_x, L_y, and L_z in terms of θ, φ, and their derivatives.

(c) From $L^2 = L_x^2 + L_y^2 + L_z^2$ show that

$$\mathbf{L}^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$= -r^2\nabla^2 + \frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

8.2 Solution

8.2.1 a

$$\mathbf{L} = -i\left(\mathbf{r}\times\nabla\right)$$

$$\mathbf{r} = r\hat{\mathbf{e}}_r$$

$$\nabla_{r\theta\varphi} = \hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\mathbf{L} = -i\mathbf{r}\times\left(\hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$= -ir\left((\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_r)\frac{\partial}{\partial r} + (\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_\theta)\frac{1}{r}\frac{\partial}{\partial\theta} + (\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_\varphi)\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$\because$ spherical polar coordinates are right-handed and orthonormal system.

$$\therefore \hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \varepsilon_{ijk}\hat{\mathbf{e}}_k$$

where ε_{ijk} is the Levi-Civita symbol. $\varepsilon_{ijk} = 1$ if i, j, k is an even permutation of r, θ, φ, -1 if it is an odd permutation, and 0 if any two indices are equal.

$$\therefore \mathbf{L} = -ir\left(0\frac{\partial}{\partial r} + 1\hat{\mathbf{e}}_\varphi\frac{1}{r}\frac{\partial}{\partial\theta} - 1\hat{\mathbf{e}}_\theta\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$= i\left(\hat{\mathbf{e}}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} - \hat{\mathbf{e}}_\varphi\frac{\partial}{\partial\theta}\right)$$

8.2.2 b

$$\hat{\mathbf{e}}_\theta = \cos\theta\cos\varphi\hat{i} + \cos\theta\sin\varphi\hat{j} - \sin\theta\hat{k}$$

$$\hat{\mathbf{e}}_\varphi = -\sin\varphi\hat{i} + \cos\varphi\hat{j}$$

$$\mathbf{L} = i\left(\cos\theta\cos\varphi\hat{i} + \cos\theta\sin\varphi\hat{j} - \sin\theta\hat{k}\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} + i\left(\sin\varphi\hat{i} - \cos\varphi\hat{j}\right)\frac{\partial}{\partial\theta}$$

$$= i\cos\varphi\cot\theta\hat{i}\frac{\partial}{\partial\varphi} + i\sin\varphi\cot\theta\hat{j}\frac{\partial}{\partial\varphi} - i\hat{k}\frac{\partial}{\partial\varphi} + i\sin\varphi\hat{i}\frac{\partial}{\partial\theta} - i\cos\varphi\hat{j}\frac{\partial}{\partial\theta}$$

$$= -i\hat{i}\left(-\cos\varphi\cot\theta\frac{\partial}{\partial\varphi} - \sin\varphi\frac{\partial}{\partial\theta}\right)$$

$$-i\hat{j}\left(\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right) - i\hat{k}\frac{\partial}{\partial\varphi}$$

$$L_x = i\left(\cos\varphi\cot\theta\frac{\partial}{\partial\varphi} + \sin\varphi\frac{\partial}{\partial\theta}\right)$$

$$L_y = i\left(\sin\varphi\cot\theta\frac{\partial}{\partial\varphi} - \cos\varphi\frac{\partial}{\partial\theta}\right)$$

$$L_z = -i\frac{\partial}{\partial\varphi}$$

8.2.3 c

$$L_x^2 = -\left(\cos\varphi\cot\theta\frac{\partial}{\partial\varphi} + \sin\varphi\frac{\partial}{\partial\theta}\right)^2$$

$$= -\cot^2\theta\cos\varphi\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial}{\partial\varphi}\right) - \sin^2\varphi\frac{\partial^2}{\partial\theta^2}$$

$$-\cos\varphi\cot\theta\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\theta}\right) - \sin\varphi\cos\varphi\frac{\partial}{\partial\theta}\left(\cot\theta\frac{\partial}{\partial\varphi}\right)$$

$$L_y^2 = -\left(\sin\varphi\cot\theta\frac{\partial}{\partial\varphi} - \cos\varphi\frac{\partial}{\partial\theta}\right)^2$$

$$= -\cot^2\theta\sin\varphi\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right) - \cos^2\varphi\frac{\partial^2}{\partial\theta^2}$$

$$+\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial}{\partial\theta}\right) + \sin\varphi\cos\varphi\frac{\partial}{\partial\theta}\left(\cot\theta\frac{\partial}{\partial\varphi}\right)$$

$$L_z^2 = -\frac{\partial^2}{\partial\varphi^2}$$

$$\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2$$

$$= -\cot^2\theta\cos\varphi\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial}{\partial\varphi}\right) - \sin^2\varphi\frac{\partial^2}{\partial\theta^2}$$

$$-\cos\varphi\cot\theta\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\theta}\right) - \sin\varphi\cos\varphi\frac{\partial}{\partial\theta}\left(\cot\theta\frac{\partial}{\partial\varphi}\right)$$

$$-\cot^2\theta\sin\varphi\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right) - \cos^2\varphi\frac{\partial^2}{\partial\theta^2}$$

$$+\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial}{\partial\theta}\right) + \sin\varphi\cos\varphi\frac{\partial}{\partial\theta}\left(\cot\theta\frac{\partial}{\partial\varphi}\right) - \frac{\partial^2}{\partial\varphi^2}$$

$$= -\cot^2\theta\left(\cos\varphi\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial}{\partial\varphi}\right) + \sin\varphi\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right)\right) - \frac{\partial^2}{\partial\theta^2} - \frac{\partial^2}{\partial\varphi^2}$$

$$+\cot\theta\left(\sin\varphi\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial}{\partial\theta}\right) - \cos\varphi\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\theta}\right)\right)$$

$$\cos\varphi\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial}{\partial\varphi}\right) = -\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} + \cos^2\varphi\frac{\partial^2}{\partial\varphi^2}$$

$$\sin\varphi\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right) = \sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} + \sin^2\varphi\frac{\partial^2}{\partial\varphi^2}$$

$$\sin\varphi\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial}{\partial\theta}\right) = -\sin^2\varphi\frac{\partial}{\partial\theta} + \sin\varphi\cos\varphi\frac{\partial^2}{\partial\varphi\partial\theta}$$

$$\cos\varphi\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\theta}\right) = \cos^2\varphi\frac{\partial}{\partial\theta} + \sin\varphi\cos\varphi\frac{\partial^2}{\partial\varphi\partial\theta}$$

$$\mathbf{L}^2 = -\cot^2\theta\left(-\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} + \cos^2\varphi\frac{\partial^2}{\partial\varphi^2} + \sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} + \sin^2\varphi\frac{\partial^2}{\partial\varphi^2}\right)$$

$$-\frac{\partial^2}{\partial\theta^2}-\frac{\partial^2}{\partial\varphi^2}$$

$$+\cot\theta\left(-\sin^2\varphi\frac{\partial}{\partial\theta}+\sin\varphi\cos\varphi\frac{\partial^2}{\partial\varphi\partial\theta}-\cos^2\varphi\frac{\partial}{\partial\theta}-\sin\varphi\cos\varphi\frac{\partial^2}{\partial\varphi\partial\theta}\right)$$

$$=-\cot^2\theta\frac{\partial^2}{\partial\varphi^2}-\frac{\partial^2}{\partial\theta^2}-\frac{\partial^2}{\partial\varphi^2}-\cot\theta\frac{\partial}{\partial\theta}$$

$$=-\left(1+\cot^2\theta\right)\frac{\partial^2}{\partial\varphi^2}-\frac{\partial^2}{\partial\theta^2}-\cot\theta\frac{\partial}{\partial\theta}$$

$$=-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$\nabla^2=\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

$$-r^2\nabla^2=-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$-r^2\nabla^2+\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)=-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$=\mathbf{L}^2$$

References

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