PEU 356 Assignment 8

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1 6.4.2

1.1 Problem

As a converse of the theorem that Hermitian matrices have real eigenvalues and that eigenvectors corresponding to distinct eigenvalues are orthogonal, show that if

(a) the eigenvalues of a matrix are real and

(b) the eigenvectors satisfy $\mathbf{x}_i^\dagger \mathbf{x}_j = \delta_{ij}$,

then the matrix is Hermitian.

1.2 Solution

Let M be a matrix with real eigenvalues and eigenvectors that satisfy $\mathbf{x}_i^\dagger \mathbf{x}_j = \delta_{ij}$. We want to show that M is Hermitian.

Let λ_i be the eigenvalues of M and $\mathbf{x}_i$ be the eigenvectors of M and U being the diagonalization matrix. Since the eigenvalues of M are real, we have

$$\lambda_i = \lambda_i^*$$

$$\langle \mathbf{x}_i | \mathbf{x}_j \rangle = \delta_{ij}$$

$$M|\mathbf{x}_i\rangle = \lambda_i |\mathbf{x}_i\rangle$$

$$UMU^{-1} = D$$

$$\because D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

$$\therefore D = D^\dagger$$

$$\therefore UMU^{-1} = \left(U^{-1}\right)^{\dagger} M^{\dagger} U^{\dagger}$$

Since M has orthogonal eigenvectors and the Diagonal matrix D's eigenvectors are the standard basis vectors which are orthogonal, U is a unitary matrix.

$$\therefore UMU^{-1} = UM^{\dagger}U^{-1}$$

$$\therefore M = M^{\dagger}$$

2 6.4.3

2.1 Problem

Show that a real matrix that is not symmetric cannot be diagonalized by an orthogonal or unitary transformation.

Hint. Assume that the nonsymmetric real matrix can be diagonalized and develop a contradiction.

2.2 Solution

Let A be a real matrix that is not symmetric, U be the orthogonal or unitary matrix that diagonalizes A, and D be the diagonal matrix.

$$UAU^\dagger = D$$

$$A = U^\dagger DU$$

$$A^\dagger = U^\dagger D^\dagger U$$

$$A^\dagger = U^\dagger D^* U$$

If we assume that eigenvalues of A are real, then the entries of D are real.

$$A^\dagger = U^\dagger DU$$

We know that A is not symmetric. Therefore, A is not equal to $A^\dagger$. This is a contradiction.

Note: If we assume an orthogonal transformation we don't need to assume that the eigenvalues are real.

3 6.4.5

3.1 Problem

A has eigenvalues λ_i and corresponding eigenvectors $|\mathbf{x}_i\rangle$. Show that A^{-1} has the same eigenvectors but with eigenvalues λ_i^{-1}.

3.2 Solution

$$A\,|\mathbf{x}_i\rangle = \lambda_i\,|\mathbf{x}_i\rangle$$

$$A^{-1}A\,|\mathbf{x}_i\rangle = A^{-1}\lambda_i\,|\mathbf{x}_i\rangle$$

$$|\mathbf{x}_i\rangle = \lambda_i A^{-1}\,|\mathbf{x}_i\rangle$$

Since A is invertible, we can be sure that $\lambda_i \neq 0$.

$$A^{-1}\,|\mathbf{x}_i\rangle = \lambda_i^{-1}\,|\mathbf{x}_i\rangle$$

4 6.4.6

4.1 Problem

A square matrix with zero determinant is labeled singular.

(a) If A is singular, show that there is at least one nonzero column vector **v** such that

$$A|\mathbf{v}\rangle = 0.$$

(b) If there is a nonzero vector $|\mathbf{v}\rangle$ such that

$$\mathrm{A}|\mathbf{v}\rangle = 0,$$

show that A is a singular matrix. This means that if a matrix (or operator) has zero as an eigenvalue, the matrix (or operator) has no inverse and its determinant is zero.

4.2 Solution

4.2.1 Part (a)

Let A be a singular matrix. Since A is singular, $\det A = 0$.

From the singular equation, we know that there is a nontrivial solution to the equation

$$\det(A - \lambda I) = 0$$

$$\lambda = 0$$

Therefore, there is at least one nonzero column vector **v** such that

$$A|\mathbf{v}\rangle = 0.$$

4.2.2 Part (b)

Let there be a nonzero vector $|\mathbf{v}\rangle$ such that

$$\mathrm{A}|\mathbf{v}\rangle = 0$$

This means that V is an eigenvector of A with eigenvalue 0.

Since there is an eigenvector of A with eigenvalue 0, $\det(\mathrm{A}) = 0$. Therefore, A is a singular matrix.

5 6.5.2

5.1 Problem

If A is a 2×2 matrix, show that its eigenvalues λ satisfy the secular equation

$$\lambda^2 - \lambda \, \mathrm{trace}(A) + \det(A) = 0.$$

5.2 Solution

Let A be a 2×2 matrix.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det(A) = ad - bc$$

$$\mathrm{trace}(A) = a + d$$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0$$

$$(a - \lambda)(d - \lambda) - bc = 0$$

$$\lambda^2 - (a + d)\lambda + ad - bc = 0$$

$$\lambda^2 - \mathrm{trace}(A)\lambda + \det(A) = 0$$

6 6.5.5

6.1 Problem

A is an n th-order Hermitian matrix with orthonormal eigenvectors $|\mathbf{x}_i\rangle$ and real eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$. Show that for a unit magnitude vector $|\mathbf{y}\rangle$,

$$\lambda_1 \leq \langle \mathbf{y}|\mathrm{A}|\mathbf{y}\rangle \leq \lambda_n.$$

6.2 Solution

We start by expanding the vector $|\mathbf{y}\rangle$ in terms of the eigenvectors of A.

$$|\mathbf{y}\rangle = \sum_{i=1}^{n} c_i |\mathbf{x}_i\rangle$$

$$\langle \mathbf{y}| = \sum_{i=1}^{n} c_i^* \langle \mathbf{x}_i|$$

$$\langle \mathbf{y}|\mathbf{y}\rangle = \sum_{i,j=1}^{n} c_i^* c_j \langle \mathbf{x}_i|\mathbf{x}_j\rangle$$

Since the eigenvectors are orthonormal, we have

$$\langle \mathbf{y}|\mathbf{y}\rangle = \sum_{i=1}^{n} |c_i|^2$$

Since the vector $|\mathbf{y}\rangle$ has unit magnitude, we have

$$\sum_{i=1}^{n} |c_i|^2 = 1$$

$$\langle \mathbf{y}|\mathrm{A}|\mathbf{y}\rangle = \sum_{i,j=1}^{n} c_i^* c_j \langle \mathbf{x}_i|\mathrm{A}|\mathbf{x}_j\rangle$$

$$\langle \mathbf{y}|\mathrm{A}|\mathbf{y}\rangle = \sum_{i,j=1}^{n} c_i^* c_j \lambda_j \langle \mathbf{x}_i|\mathbf{x}_j\rangle$$

$$\langle \mathbf{y}|\mathrm{A}|\mathbf{y}\rangle = \sum_{i=1}^{n} |c_i|^2 \lambda_i$$

We can substitute the value of λ_i in the above equation with the the smallest and largest eigenvalues.

$$(\langle \mathbf{y}|\mathrm{A}|\mathbf{y}\rangle)_{\mathrm{min}} = \lambda_1 \sum_{i=1}^{n} |c_i|^2 = \lambda_1$$

$$(\langle \mathbf{y}|\mathrm{A}|\mathbf{y}\rangle)_{\mathrm{max}} = \lambda_n \sum_{i=1}^{n} |c_i|^2 = \lambda_n$$

$$\lambda_1 \leq \langle \mathbf{y}|\mathrm{A}|\mathbf{y}\rangle \leq \lambda_n$$

7 6.5.8

7.1 Problem

A is a normal matrix with eigenvalues λ_n and orthonormal eigenvectors $|\mathbf{x}_n\rangle$. Show that A may be written as

$$\mathrm{A} = \sum_n \lambda_n \left|\mathbf{x}_n\right\rangle \left\langle \mathbf{x}_n\right| .$$

Hint. Show that both this eigenvector form of A and the original A give the same result acting on an arbitrary vector $|\mathbf{y}\rangle$.

7.2 Solution

Lets start by expanding the vector $|\mathbf{y}\rangle$ in terms of the eigenvectors of A.

$$|\mathbf{y}\rangle = \sum_n c_n |\mathbf{x}_n\rangle$$

$$\mathrm{A}|\mathbf{y}\rangle = \mathrm{A}\sum_n c_n |\mathbf{x}_n\rangle = \sum_n c_n \mathrm{A}|\mathbf{x}_n\rangle = \sum_n \lambda_n c_n |\mathbf{x}_n\rangle (1)$$

$$\langle \mathbf{x}_n|\mathbf{x}_m\rangle = \delta^n_m$$

$$\mathrm{A}|\mathbf{y}\rangle = \sum_n \lambda_n \left|\mathbf{x}_n\right\rangle \left\langle \mathbf{x}_n\right| \sum_m c_m |\mathbf{x}_m\rangle$$

$$= \sum_{n,m} \lambda_n c_m \left|\mathbf{x}_n\right\rangle \left\langle \mathbf{x}_n\right| \left.\mathbf{x}_m\right\rangle = \sum_n \lambda_n c_n |\mathbf{x}_n\rangle (2)$$

8 6.5.15

8.1 Problem

Two matrices U and H are related by

$$\mathrm{U} = e^{ia\mathrm{H}}$$

with a real.

(a) If H is Hermitian, show that U is unitary.

(b) If U is unitary, show that H is Hermitian. (H is independent of a.)

(c) If trace H $= 0$, show that $\det \mathrm{U} = +1$.

(d) If $\det \mathrm{U} = +1$, show that trace H $= 0$.

Hint. H may be diagonalized by a similarity transformation. Then U is also diagonal. The corresponding eigenvalues are given by $u_j = \exp(iah_j)$.

8.2 Solution

8.2.1 Part (a)

If H is Hermitian, then $\mathrm{H} = \mathrm{H}^\dagger$.

$$\mathrm{U}^\dagger \mathrm{U} = e^{-ia\mathrm{H}^\dagger} e^{ia\mathrm{H}} = e^{-ia\mathrm{H}+ia\mathrm{H}} = e^0 = I$$

$$\mathrm{U}^\dagger = \mathrm{U}^{-1}$$

8.2.2 Part (b)

If U is unitary, then $\mathrm{U}^\dagger \mathrm{U} = I$.

$$\mathrm{U}^\dagger \mathrm{U} = e^{-ia\mathrm{H}^\dagger} e^{ia\mathrm{H}} = e^{ia\left(\mathrm{H}-\mathrm{H}^\dagger\right)} = I = e^0$$

$$ia\left(\mathrm{H} - \mathrm{H}^\dagger\right) = 0$$

$$\mathrm{H} = \mathrm{H}^\dagger$$

8.2.3 Part (c)

$$\det(e^M) = e^{\mathrm{trace}(M)}$$

$$\det(\mathrm{U}) = e^{ia\cdot\mathrm{trace}(\mathrm{H})}$$

If trace $\mathrm{H} = 0$,

$$\det(\mathrm{U}) = e^{ia\cdot 0} = e^0 = 1$$

8.2.4 Part (d)

If $\det(\mathrm{U}) = +1$,

$$\det(\mathrm{U}) = e^{ia\cdot\mathrm{trace}(\mathrm{H})}$$

$$e^{ia\cdot\mathrm{trace}(\mathrm{H})} = 1$$

$$ia \cdot \mathrm{trace}(\mathrm{H}) = 0$$

$$\mathrm{trace}(\mathrm{H}) = 0$$

9 6.5.17

9.1 Problem

A matrix P is a projection operator satisfying the condition

$$\mathrm{P}^2 = \mathrm{P}.$$

Show that the corresponding eigenvalues $(\rho^2)_\lambda$ and ρ_λ satisfy the relation

$$\left(\rho^2\right)_\lambda = (\rho_\lambda)^2 = \rho_\lambda.$$

This means that the eigenvalues of P are 0 and 1.

9.2 Solution

$$P\left|\mathbf{x}_i\right\rangle = \lambda_i \left|\mathbf{x}_i\right\rangle$$

$$PP\left|\mathbf{x}_i\right\rangle = \lambda_i^2 \left|\mathbf{x}_i\right\rangle$$

Subtracting the above two equations, we get

$$\left(\lambda_i^2 - \lambda_i\right)\left|\mathbf{x}_i\right\rangle = 0$$

Since the eigenvectors are non-zero, we have

$$\lambda_i^2 - \lambda_i = 0$$

$$\lambda = 0, 1$$

References

[1] G.B. Arfken, H.J. Weber, and F.E. Harris. *Mathematical Methods for Physicists: A Comprehensive Guide*. Elsevier Science, 2013.

[2] M.H. El-Deeb. PEU-356 Assignments.