PEU 356 Assignment 7

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Contents

1 Question 1

1.1 Problem

Find the eigenvalues and corresponding normalized eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Orthogonalize any degenerate eigenvectors.

ANS. $\lambda = -1, -1, 2.$

1.2 Solution

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} \lambda^2 & -\lambda \\ 1 & -\lambda \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ -1 & \lambda \end{vmatrix} + \begin{vmatrix} 1 & -\lambda \\ 1 & 1 \end{vmatrix} = 0$$

$$\lambda^3 - 3\lambda - 2 = 0$$

$$(\lambda + 1)(\lambda + 1)(\lambda - 2) = 0$$

$$\lambda = -1, -1, 2$$

For $\lambda = -1$

$$A\vec{v} = -\vec{v}$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$$

$$y + z = -x$$

$$x + z = -y$$

$$x + y = -z$$

$$\vec{v} = \begin{pmatrix} x \\ y \\ -x - y \end{pmatrix}$$

$$\vec{v} = x \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\vec{v_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \vec{v_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

For $\lambda = 2$

$$A\vec{v} = 2\vec{v}$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$y + z = 2x$$

$$x + z = 2y$$

$$x + y = 2z$$

$$y = z = x$$

$$\vec{v} = x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{v_3} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

2 Question 2

2.1 Problem

Find the eigenvalues and corresponding normalized eigenvectors of the matrix

$$A = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

Orthogonalize any degenerate eigenvectors.

ANS. $\lambda = 1, 1, 6$.

2.2 Solution

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} 5-\lambda & 0 & 2 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(1-\lambda)(2-\lambda) - 4(1-\lambda) = 0$$

$$(1-\lambda)((5-\lambda)(2-\lambda) - 4) = 0$$

$$\lambda_1 = 1$$

$$(5-\lambda)(2-\lambda) = 4$$

$$\lambda_2 = 1, \quad \lambda_3 = 6$$

For $\lambda = 1$

$$A\vec{v} = \vec{v}$$

$$\begin{pmatrix} 5 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$y = y$$

$$z = -2x$$

$$\vec{v} = \begin{pmatrix} x \\ y \\ -2x \end{pmatrix}$$

$$\vec{v} = x \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{v_1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \quad \vec{v_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda = 6$

$$A\vec{v} = 6\vec{v}$$

$$\begin{pmatrix} 5 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix}$$

$$2z = x$$

$$y = y$$

$$\vec{v} = \begin{pmatrix} 2z \\ y \\ z \end{pmatrix}$$

$$\vec{v} = z \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{v_3} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

3 Question 3

3.1 Problem

Describe the geometric properties of the surface

$$x^2 + 2xy + 2y^2 + 2yz + z^2 = 1.$$

How is it oriented in 3-D space? Is it a conic section? If so, which kind?

3.2 Solution

Writing the quadratic form equation in matrix form, we get,

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)((2-\lambda)(1-\lambda) - 2) = 0$$

$$\lambda_1 = 1$$

$$(2-\lambda)(1-\lambda) = 2$$

$$\lambda_2 = 3, \quad \lambda_3 = 0$$

$$a^2 + 3b^2 = 1$$

Since we have two positive eigenvalues and one zero eigenvalue, the surface is an elliptic cylinder.

The direction is the eigenvector corresponding to the zero eigenvalue.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x = -y$$

$$x = z$$

$$\vec{v} = x \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\vec{v_3} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Note that this is not an equation of a conic section, but rather would produce a conic section if a plane with normal vector $\vec{v_3}$ and passes by the origin intersects the surface.

To get the transformation matrix we need to find the other two normalized eigenvectors.

For $\lambda = 1$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$y = 0$$

$$x + z = 0$$

$$\vec{v} = x \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\vec{v_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For $\lambda = 3$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

$$y = 2x$$

$$z = x$$

$$\vec{v} = x \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\vec{v_2} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$$

4 Question 4

4.1 Problem

A system of N degrees of freedoms is subject to the following potential

$$V = \frac{1}{2} V_{ij} x_i x_j, \quad i, j = 1, \ldots, N.$$

(i) Derive Newton's second law for the system.

(ii) Assume we are looking for solutions of the form

$$X(t) = X(0) \sin \omega t, \quad X(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{pmatrix}.$$

Show that $X(0)$ satisfies an eigenvalue problem.

(iii) What are the conditions on the eigenvalues of V_{ij} in order to have normal modes (i.e., oscillatory SII motion with a fixed frequency).

4.2 Solution

(i) Newton's second law for the system is given by

$$(m_k \ddot{x}_k)_k = -\frac{\partial V}{\partial x_k}$$

$$(m_k \ddot{x}_k)_k = -\frac{1}{2} V_{ij} \frac{\partial}{\partial x_k} (x_i x_j)$$

$$(m_k \ddot{x}_k)_k = -\frac{1}{2} V_{ij} \left(\delta_{ik} x_j + \delta_{jk} x_i \right)$$

$$(m_i \ddot{x}_i)_i = -\frac{1}{2} \left(V_{ij} + V_{ji} \right) x_i \mid\mid (m_i \ddot{x}_i)_i = -\frac{1}{2} \big(\left(V + V^T \right) \vec{x} \big)_i$$

(ii)

$$\overrightarrow{x(t)} = \sin{(\omega t)}\overrightarrow{x_0}$$

$$\overrightarrow{\ddot{x}(t)} = -\omega^2 \sin{(\omega t)}\overrightarrow{x_0}$$

$$\left(m_i \ddot{x}_i\right)_i = -\frac{1}{2}\left(V_{ij} + V_{ji}\right) x_i$$

$$\omega^2 m_i x_i = \frac{1}{2}\left(V_{ij} + V_{ji}\right) x_i$$

$$\omega^2 m_i = \frac{1}{2}\left(V_{ij} + V_{ji}\right)$$

$$\omega = \pm\sqrt{\frac{\sum_j V_{0j} + V_{j0}}{2m_0}}$$

If we assume m is a constant and V is symmetric, then

$$\omega^2 m\vec{x} = V\vec{x}$$

Therefore, $\vec{x}$ satisfies an eigenvalue problem.

$$\omega = \pm\frac{1}{\sqrt{m}}\sqrt{\sum_j V_{0j}}$$

(iii)

The eigenvalues of V_{ij} must be positive in order to have normal modes.

$$\omega^2 \geq 0$$

5 Question 5

5.1 Problem

Consider the previous example with the following potential

$$V = x_1 x_2 + x_1 x_3 + x_2 x_3.$$

Find the normal modes.

5.2 Solution

$$V = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\lambda \end{vmatrix} = 0$$

$$\lambda \left(\lambda^2 - \frac{1}{4} \right) - \frac{1}{2} \left(\frac{\lambda}{2} + \frac{1}{4} \right) - \frac{1}{2} \left(\frac{\lambda}{2} + \frac{1}{4} \right) = 0$$

$$(\lambda - 1) (2\lambda + 1)^2 = 0$$

$$\lambda = 1, -\frac{1}{2}, -\frac{1}{2}$$

For $\lambda = 1$

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$y + z = 2x$$

$$x + z = 2y$$

$$x + y = 2z$$

$$x = y = z$$

$$\vec{v} = x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{v_1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda = -\frac{1}{2}$

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$x + y = -z$$

$$\vec{v} = \begin{pmatrix} x \\ y \\ -x - y \end{pmatrix}$$

$$\vec{v} = x \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\vec{v_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \vec{v_3} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

References

[1] G.B. Arfken, H.J. Weber, and F.E. Harris. *Mathematical Methods for Physicists: A Comprehensive Guide*. Elsevier Science, 2013.

[2] M.H. El-Deeb. PEU-356 Assignments.