PEU 356 Assignment 2

Mohamed Hussien El-Deeb (201900052)

April 16, 2024

Contents

1 4.1.1 **3**
- 1.1 Problem 3
- 1.2 Solution 3

2 4.1.2 **4**
- 2.1 Problem 4
- 2.2 Solution 4

3 4.1.5 **5**
- 3.1 Problem 5
- 3.2 Solution 5

4 4.1.6 **7**
- 4.1 Problem 7
- 4.2 Solution 7

5 4.1.7 **8**
- 5.1 Problem 8
- 5.2 Solution 8

6 4.1.8 **9**
- 6.1 Problem 9
- 6.2 Solution 9

7 4.1.9 **10**

7.1 Problem 10
7.2 Solution 10

1 4.1.1

1.1 Problem

Show that if all the components of any tensor of any rank vanish in one particular coordinate system, they vanish in all coordinate systems.

Note. This point takes on special importance in the four-dimensional (4-D) curved space of general relativity. If a quantity, expressed as a tensor, exists in one coordinate system, it exists in all coordinate systems and is not just a consequence of a **choice** of a coordinate system (as are centrifugal and Coriolis forces in Newtonian mechanics).

1.2 Solution

The transformation law for a mixed tensor of arbitrary rank is given by

$$(\mathbf{T}')^{\alpha\ldots}{}_{\beta\ldots} = \mathbf{T}^{i\ldots}{}_{j\ldots} \frac{\partial (x')^{\alpha}}{\partial x^{i}} \frac{\partial x^{j}}{\partial (x')^{\beta}} \cdots$$

If all the components of a tensor vanish in one particular coordinate system, then the multiplication of the components by the transformation law will also vanish in the new coordinate system. Therefore, the components of the tensor will vanish in all coordinate systems.

$$\mathbf{0} = \mathbf{0} \frac{\partial (x')^{\alpha}}{\partial x^{i}} \frac{\partial x^{j}}{\partial (x')^{\beta}} \cdots$$

2 4.1.2

2.1 Problem

The components of tensor **A** are equal to the corresponding components of tensor **B** in one particular coordinate system denoted, by the superscript 0; that is,

$$\mathbf{A}_{ij}^{0} = \mathbf{B}_{ij}^{0}.$$

Show that tensor **A** is equal to tensor **B**, $\mathbf{A}_{ij} = \mathbf{B}_{ij}$, in all coordinate systems.

2.2 Solution

The transformation law for a covariant tensor of second rank is given by

$$\mathbf{A}_{ij}^{0} = \frac{\partial x^{\alpha}}{\partial (x^{0})^{i}} \frac{\partial x^{\beta}}{\partial (x^{0})^{j}} \mathbf{A}_{\alpha\beta}$$

$$\mathbf{B}_{ij}^{0} = \frac{\partial x^{\alpha}}{\partial (x^{0})^{i}} \frac{\partial x^{\beta}}{\partial (x^{0})^{j}} \mathbf{B}_{\alpha\beta}$$

$$\because \mathbf{A}_{ij}^{0} = \mathbf{B}_{ij}^{0}$$

$$\therefore \frac{\partial x^{\alpha}}{\partial (x^{0})^{i}} \frac{\partial x^{\beta}}{\partial (x^{0})^{j}} \mathbf{A}_{\alpha\beta} = \frac{\partial x^{\alpha}}{\partial (x^{0})^{i}} \frac{\partial x^{\beta}}{\partial (x^{0})^{j}} \mathbf{B}_{\alpha\beta}$$

$$\implies \mathbf{A}_{\alpha\beta} = \mathbf{B}_{\alpha\beta} \equiv \mathbf{A}_{ij} = \mathbf{B}_{ij}$$

3 4.1.5

3.1 Problem

The 4-D fourth-rank Riemann-Christoffel curvature tensor of general relativity, $\mathbf{R}_{iklm}$, satisfies the symmetry relations

$$\mathbf{R}_{iklm} = -\mathbf{R}_{ikml} = -\mathbf{R}_{kilm}$$

With the indices running from 0 to 3, show that the number of independent components is reduced from 256 to 36 and that the condition

$$\mathbf{R}_{iklm} = \mathbf{R}_{lmik}$$

further reduces the number of independent components to 21. Finally, if the components satisfy an identity $\mathbf{R}_{iklm} + \mathbf{R}_{ilmk} + \mathbf{R}_{imkl} = 0$, show that the number of independent components is reduced to 20.

Note. The final three-term identity furnishes new information only if all four indices are different.

3.2 Solution

Given identities,

$$[\mathbf{1}] \quad \mathbf{R}_{iklm} = -\mathbf{R}_{kilm}$$

$$[\mathbf{2}] \quad \mathbf{R}_{iklm} = -\mathbf{R}_{ikml}$$

$$[\mathbf{3}] \quad \mathbf{R}_{iklm} = \mathbf{R}_{lmik}$$

$$[\mathbf{4}] \quad \mathbf{R}_{iklm} = -(\mathbf{R}_{ilmk} + \mathbf{R}_{imkl})$$

Number of dimensions,

$$n = 4$$

General number of indices,

$$n^4 = 4^4 = 256$$

We could treat $\mathbf{R}_{iklm}$ as a 4-D matrix of 4-D matrices,

$$\mathbf{R}_{iklm} \equiv \mathbf{R}_{AB} \quad \text{where} \quad A_{ik} \quad \text{and} \quad B_{lm}$$

Each of the 4-D matrices A, B is antisymmetric and thus has 6 independent components, this follows from identities [1] and [2],

$$N(A) = N(B) = N = n(n-1)/2 = 4(4-1)/2 = 6$$

If we stop here we get $N(A) * N(B) = 6 * 6 = 36$ independent components. However, we can further reduce the number of independent components by using identity [3],

$$N[N+1]/2 = 6(6+1)/2 = 21$$

Since the final identity [4] furnishes new information only if all four indices are different we need to only consider the ${}^4C_4 = 1$ constraint.

Subtracting that from 21 results in a final count of 20 independent components.

4 4.1.6

4.1 Problem

$\mathbf{T}_{iklm}$ is antisymmetric with respect to all pairs of indices. How many independent component has it (in 3-D space)?

4.2 Solution

$$-\mathbf{T}_{iklm} = \mathbf{T}_{kilm} = \mathbf{T}_{lkim} = \mathbf{T}_{mkli} = \mathbf{T}_{ilkm} = \mathbf{T}_{imlk} = \mathbf{T}_{ikml}$$

Since we are in 3-D space, a 4th rank tensor is bound to have a repeated index which results in forcing the whole tensor to be 0 meaning that we have 0 independent components.

5 4.1.7

5.1 Problem

If $\mathbf{T}_{\dots i}$ is a tensor of rank n, show that $\partial \mathbf{T}_{\dots i}/\partial x^j$ is a tensor of rank $n+1$ (Cartesian coordinates).

Note. In non-Cartesian coordinate systems the coefficients a_{ij} are, in general, functions of the coordinates, and the derivatives the components of a tensor of rank n do not form a tensor except in the special case $n=0$. In this case the derivative does yield a covariant vector (tensor of rank 1).

5.2 Solution

$$(\mathbf{T}')_{\dots i} = \underbrace{\cdots \frac{\partial x^\alpha}{\partial (x')^i}}_{n} \mathbf{T}_{\dots \alpha}$$

$$\frac{\partial \left((\mathbf{T}')_{\dots i}\right)}{\partial (x')^j} = \frac{\partial \left(\cdots \frac{\partial x^\alpha}{\partial (x')^i} \mathbf{T}_{\dots\alpha}\right)}{\partial (x')^j} = \cdots \frac{\partial x^\alpha}{\partial (x')^i} \frac{\partial \left(\mathbf{T}_{\dots\alpha}\right)}{\partial (x')^j} + \frac{\partial \left(\cdots \frac{\partial x^\alpha}{\partial (x')^i}\right)}{\partial (x')^j} \mathbf{T}_{\dots\alpha}$$

Since we are in Cartesian $\frac{\partial x^i}{\partial x^j} = \delta^i_j$,

$$= \cdots \frac{\partial x^\alpha}{\partial (x')^i} \frac{\partial x^\beta}{\partial (x')^j} \frac{\partial \left(\mathbf{T}_{\dots\alpha}\right)}{\partial x^\beta} + \frac{\partial x^\beta}{\partial (x')^j} \underbrace{\frac{\partial \left(\cdots \frac{\partial x^\alpha}{\partial (x')^i}\right)}{\partial x^\beta}}_{\cancelto{0}{\cdots + \cdots \frac{\partial}{\partial (x')^i}\left(\frac{\partial x^\alpha}{\partial x^\beta}\right)}} \mathbf{T}_{\dots\alpha}$$

$$\frac{\partial \left((\mathbf{T}')_{\dots i}\right)}{\partial (x')^j} = \underbrace{\cdots \frac{\partial x^\alpha}{\partial (x')^i} \frac{\partial x^\beta}{\partial (x')^j}}_{n+1} \frac{\partial \left(\mathbf{T}_{\dots\alpha}\right)}{\partial x^\beta}$$

6 4.1.8

6.1 Problem

If $\mathbf{T}_{ijk\ldots}$ is a tensor of rank n, show that $\sum_j \partial \mathbf{T}_{ijk\ldots}/\partial x^j$ is a tensor of rank $n-1$ (Cartesian coordinates).

6.2 Solution

$$(\mathbf{T}')_{ijk\ldots} = \underbrace{\frac{\partial x^\alpha}{\partial (x')^i}\frac{\partial x^\beta}{\partial (x')^j}\frac{\partial x^\gamma}{\partial (x')^k}\cdots}_{n}\mathbf{T}_{\alpha\beta\gamma\ldots}$$

$$\sum_j \frac{\partial\left((\mathbf{T}')_{ijk\ldots}\right)}{\partial (x')^j} = \sum_j \frac{\partial\left(\frac{\partial x^\alpha}{\partial (x')^i}\frac{\partial x^\beta}{\partial (x')^j}\frac{\partial x^\gamma}{\partial (x')^k}\cdots \mathbf{T}_{\alpha\beta\gamma\ldots}\right)}{\partial (x')^j}$$

$$= \sum_j \left(\frac{\partial x^\alpha}{\partial (x')^i}\frac{\partial x^\beta}{\partial (x')^j}\frac{\partial x^\gamma}{\partial (x')^k}\cdots \frac{\partial\left(\mathbf{T}_{\alpha\beta\gamma\ldots}\right)}{\partial (x')^j} + \cancelto{0}{\frac{\partial\left(\frac{\partial x^\alpha}{\partial (x')^i}\frac{\partial x^\beta}{\partial (x')^j}\frac{\partial x^\gamma}{\partial (x')^k}\cdots\right)}{\partial (x')^j}}\mathbf{T}_{\alpha\beta\gamma\ldots}\right)$$

The second term was cancelled using the same argument from the previous question,

$$\sum_j \frac{\partial\left((\mathbf{T}')_{ijk\ldots}\right)}{\partial (x')^j} = \frac{\partial x^\alpha}{\partial (x')^i}\frac{\partial x^\gamma}{\partial (x')^k}\cdots\sum_j\left(\cancelto{\delta_j^\eta}{\sum_\eta \frac{\partial x^\beta}{\partial (x')^j}\frac{\partial x^\eta}{\partial (x')^j}}\right)\frac{\partial\left(\mathbf{T}_{\alpha\beta\gamma\ldots}\right)}{\partial x^\eta}$$

$$\sum_j \frac{\partial\left((\mathbf{T}')_{ijk\ldots}\right)}{\partial (x')^j} = \underbrace{\frac{\partial x^\alpha}{\partial (x')^i}\frac{\partial x^\gamma}{\partial (x')^k}\cdots}_{n-1}\sum_j \frac{\partial\left(\mathbf{T}_{\alpha\beta\gamma\ldots}\right)}{\partial x^j}$$

7 4.1.9

7.1 Problem

The operator

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

may be written as

$$\sum_{i=1}^{4} \frac{\partial^2}{\partial {x_i}^2},$$

using $x_4 = ict$. This is the 4-D Laplacian, sometimes called the d'Alembertian and denoted by $\square^2$. Show that it is a **scalar** operator, that is, invariant under Lorentz transformations, i.e., under rotations in the space of vectors (x^1, x^2, x^3, x^4).

7.2 Solution

$$\square^2 = \sum_{i=1}^{4} \frac{\partial^2}{\partial {x_i}^2}$$

$$(\square')^2 = \sum_{i=1}^{4} \frac{\partial^2}{\partial {(x')_i}^2}$$

$$= \sum_{i=1}^{4} \cancelto{\delta^i_j}{\left(\sum_{j=1}^{4} \frac{\partial x_j}{\partial (x')_i} \frac{\partial x_j}{\partial (x')_i} \right)} \frac{\partial^2}{\partial {x_j}^2}$$

$$(\square')^2 = \sum_{i=1}^{4} \frac{\partial^2}{\partial {x_i}^2} = \square^2$$

Another solution is to realize that $\square^2$ is $\partial^\mu \cdot \partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right) \cdot \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$

References

[1] G.B. Arfken, H.J. Weber, and F.E. Harris. *Mathematical Methods for Physicists: A Comprehensive Guide*. Elsevier Science, 2013.

[2] M.H. El-Deeb. PEU-356 Assignments.