PEU 356 Assignment 5

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1 5.1.11

1.1 Problem

Using conventional vector notation, evaluate $\sum_j |\hat{\mathbf{e}}_j\rangle \langle \hat{\mathbf{e}}_j \mid \mathbf{a}\rangle$, where $\mathbf{a}$ is an arbitrary vector in the space spanned by the $\hat{\mathbf{e}}_j$.

1.2 Solution

$$\sum_j |\hat{\mathbf{e}}_j\rangle \langle \hat{\mathbf{e}}_j \mid \mathbf{a}\rangle = \sum_j |\hat{\mathbf{e}}_j\rangle (\hat{\mathbf{e}}_j \cdot \mathbf{a}) = \sum_j (\hat{\mathbf{e}}_j \cdot \mathbf{a})\, \hat{\mathbf{e}}_j$$

This is the projection of $\mathbf{a}$ onto the basis vectors $\hat{\mathbf{e}}_j$.

2 5.1.12

2.1 Problem

Letting $\mathbf{a} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2$ and $\mathbf{b} = b_1\hat{\mathbf{e}}_1 + b_2\hat{\mathbf{e}}_2$ be vectors in $\mathbb{R}^2$, for what values of k, if any, is

$$\langle \mathbf{a} \mid \mathbf{b} \rangle = a_1 b_1 - a_1 b_2 - a_2 b_1 + k a_2 b_2$$

a valid definition of a scalar product?

2.2 Solution

$$\langle \mathbf{a} \mid \mathbf{a} \rangle = (a_1 - a_2)^2 + (k-1)a_2^2 \geq 0$$

For this to be true, $k \geq 1$.

3 5.2.2

3.1 Problem

Apply the Gram-Schmidt procedure to form the first three Laguerre polynomials:

$$u_n(x) = x^n, \quad n = 0, 1, 2, \ldots, \quad 0 \leq x < \infty, \quad w(x) = e^{-x}.$$

The conventional normalization is

$$\int_0^\infty L_m(x) L_n(x) e^{-x} dx = \delta_{mn}.$$

ANS. $\quad L_0 = 1, \quad L_1 = (1 - x), \quad L_2 = \frac{2-4x+x^2}{2}.$

3.2 Solution

$$L_n = x^n - \sum_{k=0}^{n-1} \langle x^n \mid L_k \rangle \tilde{L_k}$$

$$\tilde{L_k} = \frac{L_k}{\langle L_k \mid L_k \rangle}$$

$$\hat{L_k} = \frac{L_k}{\sqrt{\langle L_k \mid L_k \rangle}}$$

$$L_0 = 1$$

$$\langle 1 \mid 1 \rangle = \int_0^\infty e^{-x} dx = 1$$

$$\tilde{L_0} = \frac{L_0}{\langle L_0 \mid L_0 \rangle} = \frac{1}{\langle 1 \mid 1 \rangle} = 1$$

$$\langle x \mid 1 \rangle = \int_0^\infty x e^{-x} dx = 1$$

$$L_1 = x - \langle x \mid 1 \rangle \, 1 = x - 1$$

$$\langle x-1 \mid x-1 \rangle = \int_0^\infty (x-1)^2 e^{-x} dx = 1$$

$$\tilde{L_1} = \hat{L_1} = x - 1$$

$$\left\langle x^2 \mid 1 \right\rangle = \int_0^\infty x^2 e^{-x} dx = 2$$

$$\left\langle x^2 \mid x-1 \right\rangle = \int_0^\infty x^2 (x-1) e^{-x} dx = 4$$

$$L_2 = x^2 - 2 - 4(x-1) = x^2 - 4x + 2$$

$$\langle L_2 \mid L_2 \rangle = \int_0^\infty \left(x^2 - 4x + 2\right)^2 e^{-x} dx = 4$$

$$\hat{L_2} = \frac{x^2 - 4x + 2}{2}$$

4 5.2.4

4.1 Problem

Using the Gram-Schmidt orthogonalization procedure, construct the lowest three Hermite polynomials:

$$u_n(x) = x^n, \quad n = 0, 1, 2, \ldots, \quad -\infty < x < \infty, \quad w(x) = e^{-x^2}.$$

For this set of polynomials the usual normalization is

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) w(x) dx = \delta_{mn} 2^m m! \pi^{1/2}$$

ANS. $H_0 = 1, \quad H_1 = 2x, \quad H_2 = 4x^2 - 2.$

4.2 Solution

$$H_n = x^n - \sum_{k=0}^{n-1} \langle x^n \mid H_k \rangle \tilde{H}_k$$

$$\tilde{H}_k = \frac{H_k}{\langle H_k \mid H_k \rangle}$$

$$\hat{H}_k = \sqrt{\frac{2^k k! \sqrt{\pi}}{\langle H_k \mid H_k \rangle}} H_k$$

$$H_0 = 1$$

$$\langle H_0 \mid H_0 \rangle = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\tilde{H}_0 = \frac{H_0}{\langle H_0 \mid H_0 \rangle} = \frac{1}{\langle 1 \mid 1 \rangle} = \frac{1}{\sqrt{\pi}}$$

$$\hat{H}_0 = 1$$

$$H_0 \equiv \hat{H}_0$$

$$\langle x \mid H_0 \rangle = \int_{-\infty}^{\infty} x e^{-x^2} dx = 0$$

$$H_1 = x - \langle x \mid H_0 \rangle \tilde{H}_0 = x$$

$$\langle H_1 \mid H_1 \rangle = \int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\tilde{H}_1 = \frac{2x}{\sqrt{\pi}}$$

$$\hat{H}_1 = 2x$$

$$H_1 \equiv \hat{H}_1$$

$$\left\langle x^2 \mid H_0 \right\rangle = \int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\left\langle x^2 \mid H_1 \right\rangle = 2 \int_{-\infty}^{\infty} x^3 e^{-x^2} dx = 0$$

$$H_2 = x^2 - \left\langle x^2 \mid H_0 \right\rangle \tilde{H}_0 - \left\langle x^2 \mid H_1 \right\rangle \tilde{H}_1 = x^2 - \frac{1}{2}$$

$$\langle H_2 \mid H_2 \rangle = \int_{-\infty}^{\infty} (x^2 - \frac{1}{2})^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\hat{H}_2 = 4x^2 - 2$$

5 5.3.1

5.1 Problem

Show (without introducing matrix representations) that the adjoint of the adjoint of an operator restores the original operator, i.e., that $\left(A^\dagger\right)^\dagger = A$.

5.2 Solution

$$\langle\psi\,|A|\,\phi\rangle = \left\langle A^\dagger\psi \mid \phi\right\rangle = \left\langle\phi\left|A^\dagger\right|\psi\right\rangle^* = \left\langle \left(A^\dagger\right)^\dagger\psi \mid \phi\right\rangle^* = \left\langle\psi\left|\left(A^\dagger\right)^\dagger\right|\phi\right\rangle$$

6 5.3.2

6.1 Problem

U and V are two arbitrary operators. Without introducing matrix representations of these operators, show that

$$(UV)^\dagger = V^\dagger U^\dagger.$$

Note the resemblance to adjoint matrices.

6.2 Solution

$$\langle \psi \left| UV \right| \phi \rangle = \left\langle U^\dagger \psi \left| V \right| \phi \right\rangle = \left\langle V^\dagger U^\dagger \psi \mid \phi \right\rangle$$

$$\langle \psi \left| UV \right| \phi \rangle = \left\langle (UV)^\dagger \psi \mid \phi \right\rangle$$

$$\left\langle (UV)^\dagger \psi \mid \phi \right\rangle = \langle V^\dagger U^\dagger \psi \mid \phi \rangle$$

$$(UV)^\dagger = V^\dagger U^\dagger$$

7 5.4.4

7.1 Problem

The operator $\mathcal{L}$ is Hermitian. Show that $\langle \mathcal{L}^2 \rangle \geq 0$, meaning that for all ψ in the space in which $\mathcal{L}$ is defined, $\langle \psi \left| \mathcal{L}^2 \right| \psi \rangle \geq 0$.

7.2 Solution

$$\left\langle \psi \left| \mathcal{L}^2 \right| \psi \right\rangle = \langle \psi \left| \mathcal{L}\mathcal{L} \right| \psi \rangle = \left\langle \mathcal{L}^\dagger \psi | \mathcal{L}\psi \right\rangle = \langle \mathcal{L}\psi | \mathcal{L}\psi \rangle$$

By definition of inner product,

$$\langle \mathcal{L}\psi | \mathcal{L}\psi \rangle \geq 0$$

References

[1] M.H. El-Deeb. PEU-356 Assignments.