Simple Forms for the Distance-Redshift Relation in Dark Energy Cosmologies

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ABSTRACT

I show that the relationship between distance and redshift in the cosmological model with matter and a cosmological constant can be expressed in closed form using the incomplete beta function (also known as the beta distribution). This special function is commonly implemented in numerical packages, including in Excel, so the resulting cosmological expression can be quickly computed. Furthermore, the results generalize to dark energy models with a constant equation-of-state parameter. Numerical packages often provide an inverse for the incomplete beta function, allowing a closed-form result for the redshift given the coordinate distance. Additional results for nearly flat dark energy models and for the time-redshift relation are provided. While the calculations are routinely handled by simple numerical integration and interpolation techniques, these results in terms of incomplete beta functions may be particularly useful for beginning students, for whom the numerical techniques would be a distraction to starting to work with the standard cosmological model.

1. Introduction

One of the most common calculations in a cosmological model is the relation between distance and redshift in the Friedman-Robertson-Walker metric. The light reaching us today from a redshift z has traveled a distance in comoving coordinates

$$r(z) = \int_{t_{\rm emit}}^{t_{\rm observe}} \frac{dt}{a(t)} = \int_0^z \frac{c\,dz}{H(z)} \tag{1}$$

where $a(t)$ is the scale factor $1/(1+z)$, c is the speed of light, and $H(z)$ is the Hubble parameter $(1/a)(da/dt)$. In the common cosmological model with a cosmological constant, we have the simple form

$$H(z) = H_0 \left[\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_K(1+z)^2 + \Omega_\Lambda\right]^{1/2} \tag{2}$$

where H_0 is today's Hubble constant and $\Omega_K = 1 - \Omega_r - \Omega_m - \Omega_\Lambda = 1$. The other Ω's are the ratios of the present-day density of radiation, matter, and the cosmological constant to the critical density $3H_0^2/8\pi G$. Once one has computed $r(z)$, other useful quantities such as the angular diameter distance $D_A(z)$ and luminosity distance $D_L(z)$ can be quickly computed as

$$D_L(z) = (1+z)^2 D_A(z) = (1+z)S[r(z)] \tag{3}$$

where $S(r) = r$ for flat models ($\Omega_K = 0$), $S(r) = R_c \sin(r/R_c)$ for closed models ($\Omega_K < 0$), $S(r) = R_c \sinh(r/R_c)$ for open models ($\Omega_K > 0$), and $R_c = (c/H_0)/\sqrt{|\Omega_K|}$. We note that $c/H_0 = 2997.92h^{-1}$ Mpc for $H_0 = 100h$ km/s/Mpc. See Hogg (1999) for more explanation of these cosmological distances. We note that the assumption of the Friedman-Robertson-Walker metric assumes a homogeneous cosmology; there is also extensive study of inhomogeneous cases (e.g., Dyer & Roeder 1974).

The above equations have a number of widely used closed-form solutions in special cases, such as matter-only Einstein-de Sitter cosmologies ($\Omega_r = \Omega_K = \Omega_\Lambda = 0$), matter+curvature cosmologies ($\Omega_r = \Omega_\Lambda = 0$), and matter-radiation cosmologies ($\Omega_K = \Omega_\Lambda = 0$). For example, the famous form of Mattig (1958) gives $D_L(z)$ for the matter-curvature case.

With the discovery of the acceleration of the expansion history of the Universe and the repeated confirmation of the now-standard cosmological model with $\Omega_m \sim 0.3$ and a dark energy similar to a cosmological constant, these closed-form solutions are not as useful, as none admit a cosmological constant. More complicated closed-form solutions in terms of elliptic functions do exist for the cosmological constant cases (Edwards 1972, 1973; Dabrowski & Stelmach 1986, 1987; Kantowski et al. 2000; Mészáros & Řípa 2013), including for some inhomogeneous cases. However, the unfamiliarity of elliptic functions has caused these to be rarely used.

We also are interested in more complicated dark energy models, in which the dark energy density evolves with time. A famous example is when the density evolves as a power-law in scale factor, leading to the $H(z)$

$$H(z) = H_0 \left[\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_K(1+z)^2 + \Omega_{\rm DE}(1+z)^{3(1+w)}\right]^{1/2}. \tag{4}$$

Here, we have introduced the common notation for the equation of state parameter w, which we take to be a constant. The cosmological constant is the case $w = -1$ in this parameterization. Closed-form solutions for this case were presented by Doran et al. (2001), Giovi & Amendola (2001), and Gruppuso & Finelli (2006) in terms of hypergeometric functions, but again these are more complicated and rarely used.

Instead, most researchers use numerical integration of equation (1). The integrand is smooth and easy to integrate, particularly if one changes the integration variable so as to cure the integrable singularity at $a = 0$. When one needs to evaluate the result for many redshift, one can tabulate a grid of solutions and numerically interpolate.

The purpose of this paper is to present convenient closed-form solutions in terms of incomplete beta functions for the cosmological constant and even constant equation-of-state cases. The incomplete beta function, also known as the beta distribution, is a common special function in statistics and as a result is widely found in numerical packages. For example, it is implemented in Excel, Mathematica, MatLab, SciPy, the GNU Science Library, and Numerical Recipes. In addition to the distance-redshift relation, these same substitutions give simple expressions in terms of incomplete beta functions for the time-redshift relation.

As we will see, the solvable cases are not fully general. However, they do include a number of important cases, including the now-favored cosmology with matter and a cosmological constant and the leading generalization thereof in terms of a constant equation of state and/or small spatial curvature. We do not expect these beta-function solutions to replace the use of numerical integrations, as those are more general and flexible. But we do believe that closed-form solutions can be of value, particularly to students. While tabulating and interpolating a numerical integration is not a hurdle for more experienced researchers, it can be a substantial diversion for beginners.

Solutions involving incomplete beta functions were presented for the growth function in the cosmological constant model by Bildhauer, Buchert, & Kasai (1992) and extended to the case of non-zero curvature by Hamilton (2001). Indeed, it is the solution from Bildhauer, Buchert, & Kasai (1992) that first led us to the application presented here. As discussed by Linder & Jenkins (2003), the generalization of the growth function to cases of $w \neq -1$ does not follow the same integral solution, and hence the generalizations we find for the distance-redshift relation cannot be immediately applied to the growth function.

We will present the primary results for the distance-redshift relation for cosmological constant in §2 and for more general dark energy models in §3. We then turn to a number of more exotic cases in §4 to §7. We present results for the time-redshift relation in §8.

2. The Distance-Redshift Relation for the Cosmological Constant model

We start with the case of a Universe with matter and a cosmological constant. This yields the integral

$$r(z) = \frac{c}{H_0} \int_0^z \frac{dz}{\sqrt{\Omega_m (1+z)^3 + \Omega_\Lambda}}. \tag{5}$$

We restrict to the case of $\Omega_\Lambda > 0$. The cubic in the radicand is the challenge to a conventional integration. We change variables to $a = 1/(1+z)$, yielding

$$\frac{H_0}{c} r(z) = \int_a^1 \frac{\sqrt{a}}{\sqrt{\Omega_m + \Omega_\Lambda a^3}} \frac{da}{a}. \tag{6}$$

Note that to preserve a little more generality, we leave Ω_m and Ω_Λ explicitly in the problem, not using the idea that they sum to unity in the conventional interpretation.

Now we use the transformation

$$x = \frac{\Omega_\Lambda a^3}{\Omega_m + \Omega_\Lambda a^3}. \tag{7}$$

This can be rearranged to find

$$a^3 = \frac{x}{1-x} \frac{\Omega_m}{\Omega_\Lambda}, \tag{8}$$

which gives the differential relation

$$3a^2 da = \frac{dx}{(1-x)^2} \frac{\Omega_m}{\Omega_\Lambda}, \tag{9}$$

which can be manipulated to

$$\frac{da}{a} = \frac{dx}{3x(1-x)}. \tag{10}$$

Our radical becomes

$$\sqrt{\Omega_m + \Omega_\Lambda a^3} = \sqrt{\frac{\Omega_m}{1-x}} \tag{11}$$

Note that at $z = 0$, we have $a = 1$ and $x = x_0 = \Omega_\Lambda/(\Omega_m + \Omega_\Lambda)$. More generally, we have

$$x(z) = \frac{\Omega_\Lambda}{\Omega_m(1+z)^3 + \Omega_\Lambda} \tag{12}$$

The variable x has a simple interpretation: it is the ratio of the cosmological constant density to the total (matter plus cosmological constant) density at the redshift z. In other words, it is the Ω_Λ that an observer at redshift z would report.

Inserting this transformation and manipulating, we have

$$\begin{aligned}
\frac{H_0}{c} r(z) &= \int_{x(z)}^{x_0} \sqrt{\frac{1-x}{\Omega_m}} \sqrt{a} \frac{dx}{3x(1-x)} && (13)\\
&= \frac{1}{3\sqrt{\Omega_m}} \int_{x(z)}^{x_0} \sqrt{1-x} \left(\frac{\Omega_m}{\Omega_\Lambda} \frac{x}{1-x} \right)^{1/6} \frac{dx}{3x(1-x)} && (14)\\
&= \frac{1}{3\Omega_m^{1/3}\Omega_\Lambda^{1/6}} \int_{x(z)}^{x_0} x^{1/6}(1-x)^{1/3} \frac{dx}{x(1-x)}. && (15)
\end{aligned}$$

This yields our first result

$$r(z) = \frac{c}{3H_0\Omega_m^{1/3}\Omega_\Lambda^{1/6}} \left[B_{x_0}\left(\frac{1}{6}, \frac{1}{3}\right) - B_{x(z)}\left(\frac{1}{6}, \frac{1}{3}\right) \right]. \tag{16}$$

Here we have introduced the incomplete beta function:

$$B_x(m,n) = \int_0^x t^{m-1}(1-t)^{n-1} \, dt. \tag{17}$$

The incomplete beta function is a common special function. In the statistics literature, for the domain $0 \le x \le 1$, it is the cumulative probability distribution of the well-used beta distribution. Most special function packages implement the normalized version

$$I_x(m,n) = \frac{B_x(m,n)}{B(m,n)}, \tag{18}$$

where the beta function $B(m,n) = \int_0^1 dt\, t^{m-1}(1-t)^{n-1}$ has a simple solution $\Gamma(m)\Gamma(n)/\Gamma(m+n)$ in terms of the gamma function (assuming $m > 0$ and $n > 0$). These special functions can be computed quickly and rapidly in most numerical packages. For example, in Excel, we can compute $I_x(m,n)$

as BETADIST(x, m, n) and the beta function normalization $B(m, n)$ as EXP(GAMMALN(m) + GAMMALN(n) − GAMMALN$(m + n)$). In Mathematica, $B_x(m, n)$ is available as Beta(x, m, n).

In numerical packages that allow manipulation of vectors, such as SciPy, the beta distribution implementations typically operate cleanly on vectors, so that one can compute the distances to many redshifts in one line of code.

Interestingly, these packages also have common implementations of the inverse of the incomplete beta function. This means that we can easily construct redshifts from distances, a process that otherwise requires a numerical search. In particular, we have

$$B_{x(z)}\left(\frac{1}{6}, \frac{1}{3}\right) = B_{x_0}\left(\frac{1}{6}, \frac{1}{3}\right) - 3\Omega_m^{1/3}\Omega_\Lambda^{1/6}\frac{rH_0}{c}. \tag{19}$$

Changing to the normalized form, we have

$$I_{x(z)}\left(\frac{1}{6}, \frac{1}{3}\right) = I_{x_0}\left(\frac{1}{6}, \frac{1}{3}\right) - \frac{3\Omega_m^{1/3}\Omega_\Lambda^{1/6}}{B\left(\frac{1}{6}, \frac{1}{3}\right)}\frac{rH_0}{c}. \tag{20}$$

Given r, we can solve for $x(z)$ and then find z using equation (8), i.e., $1+z = [(x^{-1}-1)\Omega_\Lambda/\Omega_m]^{1/3}$. For example, in Excel, the inverse of $I_x(m, n)$ is given by BETAINV(x, m, n).

Comparing these results to the on-line Cosmology Calculator[1] (Wright 2006), we find good agreement. There are 0.1%-level deviations because the Cosmology Calculator includes the standard radiation density of $\Omega_r = 4.2 \times 10^{-5} h^{-2}$, whereas we have set $\Omega_r = 0$. We could improve this using the methods of §7.

Gruppuso & Finelli (2006) point out that their hypergeometric function solution can be written as an incomplete beta function; however, they adopt an argument that generally falls outside of the $0 \le x \le 1$ domain of the beta distribution and hence can't use the usual numerical implementations. This limits the evaluation of their functions to approximation by Taylor series, whereas the choice of argument presented above allow one to use the more accurate and widely available methods.

3. Solutions for Dark Energy Models with Constant Equation of State

These same solutions can be generalized cleanly to cases in which the dark energy density scales as a power-law in redshift. Here, we have the base equation

$$r(z) = \frac{c}{H_0}\int_0^z \frac{dz}{\sqrt{\Omega_m(1+z)^3 + \Omega_{\rm DE}(1+z)^{3(1+w)}}} = \frac{c}{H_0}\int_a^1 \frac{\sqrt{a}}{\sqrt{\Omega_m + \Omega_{\rm DE}a^{-3w}}}\frac{da}{a}, \tag{21}$$

where w is a constant and we require $\Omega_{\rm DE} > 0$. Again, $w = -1$ for the cosmological constant, and we will find a practical restriction to the case $w < -1/3$, although the transformation itself does not require that.

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[1] http://www.astro.ucla.edu/∼wright/CosmoCalc.html

We use the transformation

$$x = \frac{\Omega_{\rm DE} a^{-3w}}{\Omega_m + \Omega_{\rm DE} a^{-3w}}. \tag{22}$$

This yields

$$a^{-3w} = \frac{x}{1-x}\frac{\Omega_m}{\Omega_{\rm DE}}, \tag{23}$$

$$\sqrt{\Omega_m + \Omega_{\rm DE} a^{-3w}} = \sqrt{\frac{\Omega_m}{1-x}}, \tag{24}$$

and

$$\frac{da}{a} = \frac{dx}{-3wx(1-x)}. \tag{25}$$

Inserting this, we find

$$\begin{aligned} r(z) &= \frac{c}{-3wH_0\sqrt{\Omega_m}}\left(\frac{\Omega_m}{\Omega_{\rm DE}}\right)^{-1/6w}\int_{x(z)}^{x_0} x^{-\frac{1}{6w}}(1-x)^{\frac{1}{2}+\frac{1}{6w}}\frac{dx}{x(1-x)} & (26)\\ &= \frac{2mc}{H_0\sqrt{\Omega_m}}\left(\frac{\Omega_m}{\Omega_{\rm DE}}\right)^m\left[B_{x_0}\left(m,\frac{1}{2}-m\right) - B_{x(z)}\left(m,\frac{1}{2}-m\right)\right]. & (27)\end{aligned}$$

where $x_0 = \Omega_{\rm DE}/(\Omega_m + \Omega_{\rm DE})$ and $m = -1/6w$.

This expression has all of the same simple properties as the cosmological constant case. For example, it can be inverted in the same manner as equation (20) to yield the redshift corresponding to a given distance.

We note that in the $\Omega_{\rm DE} \to 0$ limit, x_0 and $x(z)$ both approach zero. In this limit, $B_x(m,n) = x^m/m$. This yields that

$$m\left(\frac{\Omega_m}{\Omega_{\rm DE}}\right)^m B_{x(z)}\left(m,\frac{1}{2}-m\right) \to (1+z)^{-1/2}, \tag{28}$$

thereby recovering the Einstein-de Sitter result of

$$r(z) = \frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right). \tag{29}$$

Of course, numerical implementations will not successfully evaluate equation (26) for $\Omega_{\rm DE} = 0$. One must either use equation (29) or use a small but non-zero $\Omega_{\rm DE}$.

There is an important practical restriction that most numerical implementations of the incomplete beta function $I_x(m,n)$ require that the two parameters m and n be positive. This corresponds to requiring that the integral be finite in the $x \to 0$ and $x \to 1$ limit, i.e., that the complete beta function give a finite answer. This is despite the fact that the actual range of integration in equation (26) includes neither singularity.

The requirement that $n > 0$ limits the utility of the result to the case of $w < -1/3$. Physically, this means that we are limited to cases that asymptotically accelerate in the future, so that the amount of comoving distance that a future-facing light ray traverses is finite.

However, the requirement of $n > 0$ is actually avoidable; it results from the fact that the numerical algorithms divide by the complete beta function in order to return a normalized form of the beta distribution. Our application ends up multiplying by this same factor. One can rewrite the algorithms to remove this issue. For example, Numerical Recipes (Press et al. 1992) presents the solution as

$$B_x(m,n) = \frac{x^m(1-x)^n}{m}\mathrm{betacf(m,n,x)} \tag{30}$$

where betacf(m,n,x) is a function that implements a continued fraction approximation. This routine returns correct answers even if $n \leq 0$; one still does require $m > 0$. Also, although the convergence of the fraction is faster if $x < (1+m)/(2+m+n)$, the result is still correct even if this is not so. Convergence only slows when we drive $x \to 1$, which corresponds to future times ($a \gg 1$) in our application. The GNU Science Library code is based on the same expressions and can easily be modified to avoid the unneeded normalization. Hence, with some simple recoding one can use $n < 0$ cases, which allows this solution to work for all $w < 0$.

4. Models with Negative Dark Energy

Flat cosmological models with negative dark energy ($\Omega_{\rm DE} < 0$ and $\Omega_m > 1$) and constant equations of state can also be solved with a related transformation. In these models, the dark energy is producing additional deceleration, so that the Universe reaches a maximum size at epoch $a_{\rm max} = (-\Omega_m/\Omega_{\rm DE})^{-1/3w}$.

Starting from equation (21) and use

$$y = -\frac{\Omega_{\rm DE}a^{-3w}}{\Omega_m}, \tag{31}$$

which implies

$$\frac{da}{a} = \frac{1}{-3w}\frac{dy}{y}, \tag{32}$$

$$\sqrt{\Omega_m + \Omega_{\rm DE}a^{-3w}} = \sqrt{\Omega_m(1-y)}, \tag{33}$$

and $y_0 = \Omega_{\rm DE}/\Omega_m$. This maps the scale factor range $a = 0$ to $a_{\rm max}$ to the range $y = 0$ to 1. We find

$$r(z) = \frac{c}{-3wH_0\sqrt{\Omega_m}}\left(\frac{\Omega_m}{-\Omega_{\rm DE}}\right)^m \int_{y(z)}^{y_0} y^m(1-y)^{1/2}\frac{dy}{y(1-y)} \tag{34}$$

with $m = -1/6w$. This can be expressed as an incomplete beta function

$$r(z) = \frac{2mc}{H_0\sqrt{\Omega_m}}\left(\frac{\Omega_m}{-\Omega_{\rm DE}}\right)^m \left[B_{y_0}\left(m,\frac{1}{2}\right) - B_{y(z)}\left(m,\frac{1}{2}\right)\right], \tag{35}$$

similar in form to the result for $\Omega_{\rm DE} > 0$. As before, this result can be inverted to derive redshifts from distances.

The results in the rest of this paper could be derived for the negative dark energy case as well; however, we omit them as today's data strongly prefers $\Omega_m < 1$ and $\Omega_{\rm DE} > 0$.

5. General Binomial Cases

Having generalized one term in the Hubble parameter to an arbitrary power in §3, we can actually generalize both terms, i.e., solve cases of the form

$$r(z) = \frac{c}{H_0}\int_0^z \frac{dz}{\sqrt{\Omega_1(1+z)^{3(1+w_1)} + \Omega_2(1+z)^{3(1+w_2)}}} \tag{36}$$

for constants w_1, w_2, Ω_1, and Ω_2 (with both $\Omega > 0$). Defining $w = w_2 - w_1$, we find

$$r(z) = \frac{c}{H_0}\int_a^1 \frac{a^{\frac{1+3w_1}{2}}}{\sqrt{\Omega_1 + \Omega_2 a^{-3w}}}\frac{da}{a}. \tag{37}$$

Again we use the transformation

$$x = \frac{\Omega_2 a^{-3w}}{\Omega_1 + \Omega_2 a^{-3w}}. \tag{38}$$

This yields

$$a^{-3w} = \frac{x}{1-x}\frac{\Omega_1}{\Omega_2}, \tag{39}$$

$$\sqrt{\Omega_1 + \Omega_2 a^{-3w}} = \sqrt{\frac{\Omega_1}{1-x}}, \tag{40}$$

and

$$\frac{da}{a} = \frac{dx}{-3wx(1-x)}. \tag{41}$$

Inserting this, we find

$$\begin{aligned} r(z) &= \frac{c}{-3wH_0\sqrt{\Omega_1}}\left(\frac{\Omega_1}{\Omega_2}\right)^{-\frac{1+3w_1}{6w}}\int_{x(z)}^{x_0} x^{-\frac{1+3w_1}{6w}}(1-x)^{\frac{1}{2}+\frac{1+3w_1}{6w}}\frac{dx}{x(1-x)} && (42)\\ &= \frac{c}{-3wH_0\sqrt{\Omega_1}}\left(\frac{\Omega_1}{\Omega_2}\right)^{m}\left[B_{x_0}\left(m, \frac{1}{2}-m\right) - B_{x(z)}\left(m, \frac{1}{2}-m\right)\right]. && (43)\end{aligned}$$

where $x_0 = \Omega_2/(\Omega_1+\Omega_2)$ and $m = -(1+3w_1)/6w$. Again, this form has all of the same properties as above, including having a simple inverse.

As before, the numerical packages require $m > 0$ and $1/2 - m > 0$. These produce the requirements that $w_1 > -1/3$ and $w_2 < -1/3$, exactly as one would expect from the requirement that the integral in equation (36) have integrable singularities. There is no additional requirement on $w = w_2 - w_1$.

6. Nearly flat models

Current cosmological data argues that the spatial curvature of the Universe is small, i.e., that any curvature scale is substantially larger than the current Hubble radius. This corresponds to values of Ω_K near zero. However, we remain interested in small differences from zero, with either sign.

Taking a model with dark energy with $\Omega_{\rm DE} > 0$, $w < -1/3$, and a small curvature, we have

$$r(z) = \frac{c}{H_0}\int_0^z \frac{dz}{\sqrt{\Omega_m(1+z)^3+\Omega_K(1+z)^2+\Omega_{\rm DE}(1+z)^{3(1+w)}}} \quad (44)$$

$$= \frac{c}{H_0}\int_a^1 \frac{\sqrt{a}}{\sqrt{\Omega_m+\Omega_K a+\Omega_{\rm DE}a^{-3w}}}\frac{da}{a}, \quad (45)$$

While this integral does not have an exact solution with incomplete beta function methods of this paper, we can consider expanding the integrand as a Taylor series in Ω_K around the flat ($\Omega_K = 0$) case.

Writing $E_{\rm flat} = \sqrt{\Omega_m + \Omega_{\rm DE}a^{-3w}}$, the Taylor series is

$$\left(\Omega_m+\Omega_K a+\Omega_{\rm DE}a^{-3w}\right)^{-1/2} = \frac{1}{E_{\rm flat}} - \frac{1}{2}\frac{\Omega_K a}{E_{\rm flat}^3} + \frac{1}{2}\frac{3}{4}\frac{(\Omega_K a)^2}{E_{\rm flat}^5} \quad (46)$$

$$\ldots + \frac{1}{p!}(-1)^p\frac{1}{2}\frac{3}{2}\cdots\frac{2p-1}{2}\left(\frac{\Omega_K a}{E_{\rm flat}^2}\right)^p\frac{1}{E_{\rm flat}} \quad (47)$$

We can then use the same substitutions as §3 to do the integral for $r(z)$. This yields

$$r(z) = \frac{c}{-3wH_0\sqrt{\Omega_m}}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\left(\frac{1}{2}\frac{3}{2}\cdots\frac{2p-1}{2}\right)\left(\frac{\Omega_K}{\Omega_m}\right)^p\left(\frac{\Omega_m}{\Omega_{\rm DE}}\right)^{m_p} \quad (48)$$

$$\times\left[B_{x_0}\left(m_p,\frac{1}{2}+p-m_p\right) - B_{x(z)}\left(m_p,\frac{1}{2}+p-m_p\right)\right]. \quad (49)$$

where $m_p = -(1+2p)/6w$. This series converges quickly for $\Omega_K/\Omega_m \ll 1$. Note that it can no longer be simply inverted, although for small values of Ω_K/Ω_m, one could plan to iterate to a solution.

Recall that once $r(z)$ is computed, one must still compute the $S(r)$ functions from equation (3) to get angular diameter and luminosity distances.

Note that when doing this Taylor expansion, we hold the values of Ω_m and $\Omega_{\rm DE}$ fixed at their non-flat values. This means that the $\Omega_m + \Omega_{\rm DE} \neq 1$ and that $x_0 = \Omega_{\rm DE}/(\Omega_m + \Omega_{\rm DE}) \neq \Omega_{\rm DE}$ in this case.

Hamilton (2001) presents a related solution for the growth function in non-flat cosmological constant models, using the same method of Taylor expanding in Ω_K. He points out that in the cosmological constant case, there are recursion relations for the incomplete beta function that allow more rapid calculation of the higher terms in the series. This optimization could also be applied here for the case of $w = -1$, but it doesn't work for general w as it relies on the sequence of m_p to yield integer offsets from earlier values. Fortunately, for small Ω_K, one simply doesn't need many terms and computing the incomplete beta functions is not a burden.

Although we omit the derivation, this same expansion can also work for terms of the form $\Omega_K(1+z)^{3(1+v)}$, where $0 > v > w$. The above case is $v = -1/3$. The formula for $r(z)$ changes only in that $m_p = -(1-6vp)/6w$. One could generalize to Taylor expansions of multiple additional monomial terms with small coefficients.

7. Including Radiation as a Perturbation

The methods of the last section can also be applied to include the effects at low redshift of a small amount of radiation. The standard model has a population of photons and low-mass neutrinos that yield $\Omega_r h^2 = 4.2 \times 10^{-5}$. The radiation will have $v = +1/3$, so the first-order term from equation (48) produces a change in $r(z)$ of

$$\Delta r(z) = \frac{c}{6wH_0\sqrt{\Omega_m}} \left(\frac{\Omega_r}{\Omega_m}\right) \left(\frac{\Omega_m}{\Omega_{\rm DE}}\right)^{m_1} \left[B_{x_0}\left(m_1, \frac{3}{2} - m_1\right) - B_{x(z)}\left(m_1, \frac{3}{2} - m_1\right)\right] \tag{50}$$

where $m_1 = 1/6w$. When using this, one must remember that the non-zero value of Ω_r implies for a flat cosmology that $\Omega_{\rm DE} = 1 - \Omega_m - \Omega_r$.

The problem with this expansion is that $m_1 < 0$, which indicates that the integral has a non-integrable singularity at early times. This is the result of the fact that our approximation of the radiation as a small perturbation breaks down at early times. However, the quantity in square brackets is the result of the integral

$$\int_{x(z)}^{x_0} x^{m_1}(1-x)^{\frac{3}{2}-m_1} \frac{dx}{x(1-x)}. \tag{51}$$

So long as one avoids evaluating the integral at $x = 0$, this is non-singular. For low redshifts, well after matter-radiation equality and where the Taylor series may be stopped at first order, we can evaluate the integral approximately, e.g., as

$$\left[\frac{1}{m_1} x^{m_1} \left(1 - \frac{m_1}{m_1+1}\frac{1-2m_1}{2} x\right)\right]_{x(z)}^{x_0}. \tag{52}$$

Alternatively, computing this integral as $B_{x_0}(m_1, (3/2) - m_1) - B_{x(z)}(m_1, (3/2) - m_1)$ using the Numerical Recipes betacf() code described in equation (30) does return accurate answers.

We find that this correction is of order 0.01% out to $z = 1$, 0.05% to $z = 10$, and 0.5% to $z = 1000$, so one can neglect the radiation entirely in many applications.

8. Redshift-Time Relations in Dark Energy Models

These same transformations can be helpful in other cosmological calculations. For example, the age of the Universe at a given redshift is

$$t(z) = \int_z^\infty \frac{dz}{(1+z)H(z)}. \tag{53}$$

This is just like equation 1 save for an extra power of a in the numerator. For the constant equation of state model in §3, this changes equation (21) to be

$$t(z) = \frac{1}{H_0} \int_0^a \frac{a^{3/2}}{\sqrt{\Omega_m + \Omega_{\rm DE} a^{-3w}}} \frac{da}{a}. \tag{54}$$

Using the same substitutions, we find for $\Omega_{\rm DE} > 0$

$$t(z) = \frac{2m}{3H_0\sqrt{\Omega_m}} \left(\frac{\Omega_m}{\Omega_{\rm DE}}\right)^m B_x\left(m, \frac{1}{2} - m\right) \tag{55}$$

where $m = -1/2w$.

Here, the requirement of the numerical packages is that $0 < m < 1/2$, which requires $w < -1$. Unfortunately, this means that the cosmological constant case $w = -1$ will not yield an answer. Physically, the reason is that models with $w \geq -1$ will have exist for infinite time, whereas models with $w < -1$ expand to infinite size at some finite time in the future (the Big Rip).

As described at the end of §3, this limitation of positive parameters can be relaxed with some simple recoding. This allows the solution to be used for any $w < 0$. Note that the cosmological case also has the closed-form solution (Weinberg 1972, 1989)

$$t(z) = \frac{2}{3H_0} \frac{1}{\sqrt{1-\Omega_m}} \sinh^{-1}\left((1+z)^{-3/2}\sqrt{\frac{1-\Omega_m}{\Omega_m}}\right). \tag{56}$$

9. Conclusions

Many cosmological calculations start with use of the model-dependent relations between redshift, distance, and time. We have shown that for models combining matter and dark energy with a constant equation of state, including the important case of a cosmological constant, these quantities can be expressed in closed form using the incomplete beta function, a commonly available special function. These formulae provide a convenient access point for cosmological calculations, as an alternative to the numerical integrations that are commonly used. We don't argue that these numerical integrations and interpolations thereof are difficult, but they do represent an additional code complexity that can be avoided.

We note that the beta distribution is widely available, often in a form that will act on vectors. Its inverse is often also available, allowing quick inversion of the distance-redshift relation. For beginning students and laboratory projects, these functions are available in Excel, Matlab, and SciPy. Hence, there is no reason to avoid using the standard cosmological model, with a cosmological constant, or more general dark energy models in such exercises for lack of a convenient solution.

DJE thanks David Hogg, Eric Linder, and Zack Slepian for useful conversations. DJE is supported by grant DE-SC0013718 from the U.S. Department of Energy.

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A. Excel Implementation

Excel is commonly used in undergraduate laboratory settings. To be explicit, one can implement the distance-redshift relation for the constant equation of state case (Eq. 26) in Excel via the following:

We load the value of Ω_m in cell B1 and the value of the equation of state w in cell B2. We compute the quantities $\Omega_{\rm DE}$ and m as

$$B3 = 1 - B1 \quad \text{(A1)}$$

$$B4 = -1.0/6.0/B2. \quad \text{(A2)}$$

Then with the redshift in cell A10, we can compute the distance $r(z)$ as

```
2*B4*2997.92/SQRT(B1)*(B1/B3)^B4*
(BETADIST(B3/(B1+B3),B4,0.5-B4)-BETADIST(B3/(B3+B1*(1+A10)^(-3*B2)),B4,0.5-B4))
*EXP(GAMMALN(B4)+GAMMALN(0.5-B4)-GAMMALN(0.5))
```

Alternatively, with the distance in cell A11, we can compute the redshift $z(r)$ as

```
-1+POWER(B3/B1*(-1+1/BETAINV(BETADIST(B3/(B3+B1),B4,0.5-B4)
-A11*SQRT(B1)/2/2997.92/B4*(B3/B1)^B4
/EXP(GAMMALN(B4)+GAMMALN(0.5-B4)-GAMMALN(0.5)),B4,0.5-B4)),2*B4)
```