Q-1)

$Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

addition is denoted by $+$
multiplication is denoted by $*$ (to avoid confusion with x)

↳ write first order statements :

(a) $\boxed{\text{Addition is Commutative}}$

$$\forall x \forall y \; ((x \in Z) \land (y \in Z)) \longrightarrow ((x+y) = (y+x))$$

$\boxed{\text{Multiplication is Commutative}}$

$$\forall x \forall y \; ((x \in Z) \land (y \in Z)) \longrightarrow ((x * y) = (y * x))$$

(b) $\boxed{\text{Z is closed under addition}}$

$$\forall x \forall y \; ((x \in Z) \wedge (y \in Z)) \longrightarrow ((x+y) \in Z)$$

$\boxed{\text{Z is closed under multiplication}}$

$$\forall x \forall y \; ((x \in Z) \wedge (y \in Z)) \longrightarrow ((x * y) \in Z)$$

(c) $\boxed{\text{Multiplication distributes over addition}}$

$$\forall x \forall y \forall a \ \left(\left((x \in Z) \land (y \in Z) \land (a \in Z)\right) \longrightarrow \left((a * (x+y)) = ((a*x) + (a*y))\right)\right)$$

(d) $\boxed{\text{Multiplications are associative}}$

$$\forall x \forall y \forall a \ \left((x \in Z) \land (y \in Z) \land (a \in Z)\right) \longrightarrow \left((x * (y * a)) = ((x*y) * a)\right)$$

$\boxed{\text{Additions are associative}}$

$$\forall x \forall y \forall a \ \left((x \in Z) \land (y \in Z) \land (a \in Z)\right) \longrightarrow \left(((x+y)+a) = (x+(y+a))\right)$$

(e) $\boxed{\text{Multiplication have identity property}}$

$$\forall x \; (x \in Z) \longrightarrow ((x * 1) = x)$$

$\boxed{\text{Addition have identity property}}$

$$\forall x \; (x \in Z) \longrightarrow ((0 + x) = x)$$

Q-2

$$\boxed{\alpha = \forall x \left(P(x) \vee Q(x) \right)}$$

In α (alpha) world, it is one quantifier, i.e., for all x, ($P(x)$ or $Q(x)$)

$$\boxed{\beta = (\forall x\, P(x)) \vee (\forall x\, Q(x))}$$

In β world, it is two quantifiers.

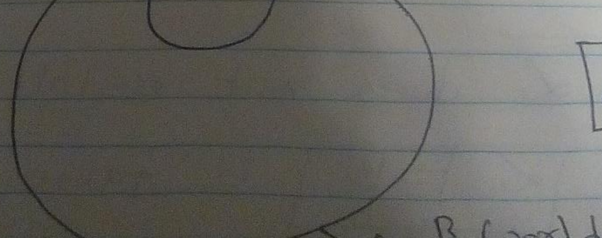
So making unique quantifiers for β:

$$\beta = (\forall x\, P(x)) \vee (\forall y\, Q(y))$$

i.e., for all x $P(x)$, or
for all y $Q(y)$

We are to prove/disprove if $\alpha \models \beta$

So, if $\alpha \models \beta$ then in a world, if α is true then β has to be true; otherwise $\alpha \models \beta$ is false claim;



$$\boxed{\alpha \models \beta}$$

Let's consider a world where domain of x and y is set of Integers, such that

$P(x)$: $(x=0)$

$Q(x)$: x is NOT zero, i.e., $\neg P(x)$

So,

$\alpha = \forall x \, (P(x) \vee Q(x))$, will always be true;

In the same world, let's consider expression for β:

$\beta = (\forall x \, P(x)) \vee (\forall y \, Q(y))$

here x & y are separately quantified variables but belongs to the same domain world. It is possible that in the same world, β is NOT true while α is true.

For example, in the above world, let $(x=2)$, then $P(x)$ is false, & $Q(x)$ is true, hence α is true.

let $(y=0)$ in this world, then $P(x)$ is false, & $Q(y)$ is false, hence β is false, while α is still true.

Hence, α does not entail β.

$\boxed{\alpha \models \beta}$ is false claim!

Assignment 4

Q-3) Following are the Notations :

MF : Millennium Falcon
HS : Han Solo
PL : Princess Leia
OWK : Obi-Wan Kenobi
RA : Rebel Alliance
CHB : Chewbacca

Representing each of the given sentences into First order Logic :

(a) Han Solo owns the Millenniom falcon.

Owns (HS, MF)

(b) Princess Leia is unhappy.

Unhappy (PL)

(c) Loves (PL, HS) : Princess Leia loves Han Solo

(d) For all x, if x owns the MF or x is unhappy then x visits OWK.

$\forall x$ Owns$(x, MF) \vee$ unhappy$(x) \longrightarrow$ visit(x, OWK)

where,
visit (x, OWK) : x visits OWK
Unhappy (x) : x is Unhappy
Owns (x, MF) : x owns MF

(e) For all x, if x visits OWK then x is wise.

wise(x) : x is wise

$\forall x$ visit(x, OWK) $\longrightarrow$ wise(x)

(f) For all x, if x owns MF and visits OWK, then OWK teaches x to use lightsaber.

teaches (OWK, x) : OWK teaches x to use light saber

$\forall x$ Owns(x, MF) $\wedge$ visit(x, OWK) $\longrightarrow$ teaches(OWK, x)

(g) for all x, if x is unhappy or owns MF, and if OWK teaches x to use lightsaber, then x joins Rebel Alliance

$\forall x$ (unhappy(x) $\vee$ owns(x, MF)) $\wedge$ teaches(OWK, x)

$\longrightarrow$ joins(x, RA)

(h) For all x & y, if x is unhappy and x loves y then x declares Love for y.

declaresLove (x,y) : x declares love for y

$\forall x \forall y$ unhappy$(x) \wedge$ loves$(x,y) \longrightarrow$ declaresLove(x,y)

(i) For all x & y, if OWK teaches x to use lightsaber and y declares Love for x & y is wise, then x has CHB as a friend.

HasFriend(x, CHB) : x has CHB as a friend

$\forall x \forall y$ Teaches$(OWK, x) \wedge$ declaresLove$(y,x) \wedge$ wise(y)
$\longrightarrow$ HasFriend(x, CHB)

Simplifying (d), (e), (f), (g) and (h) & (i),

(d1) $\forall x\ \text{Owns}(x, M) \longrightarrow \text{Visit}(x, OWK)$

(d2) $\forall x\ \text{Unhappy}(x) \longrightarrow \text{visit}(x, OWK)$

(e) $\forall x\ \text{visit}(x, OWK) \longrightarrow \text{Wise}(x)$

(f) $\forall x\ \text{Owns}(x, MF) \land \text{visit}(x, OWK) \longrightarrow \text{teaches}(OWK, x)$

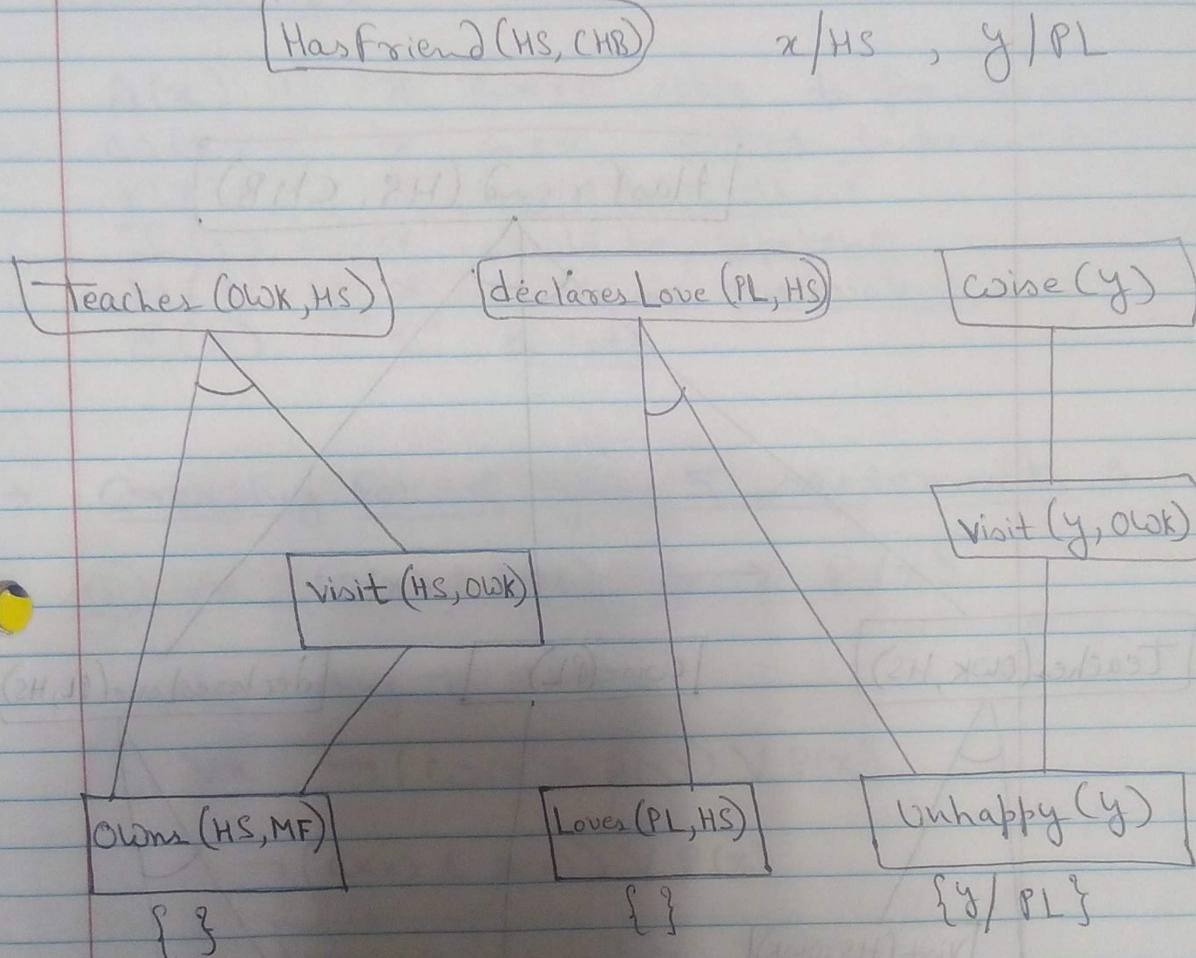
(g1) $\forall x\ \text{unhappy}(x) \land \text{teaches}(OWK, x) \longrightarrow \text{joins}(x, RA)$

(g2) $\forall x\ \text{Owns}(x, MF) \land \text{teaches}(OWK, x) \longrightarrow \text{joins}(x, RA)$

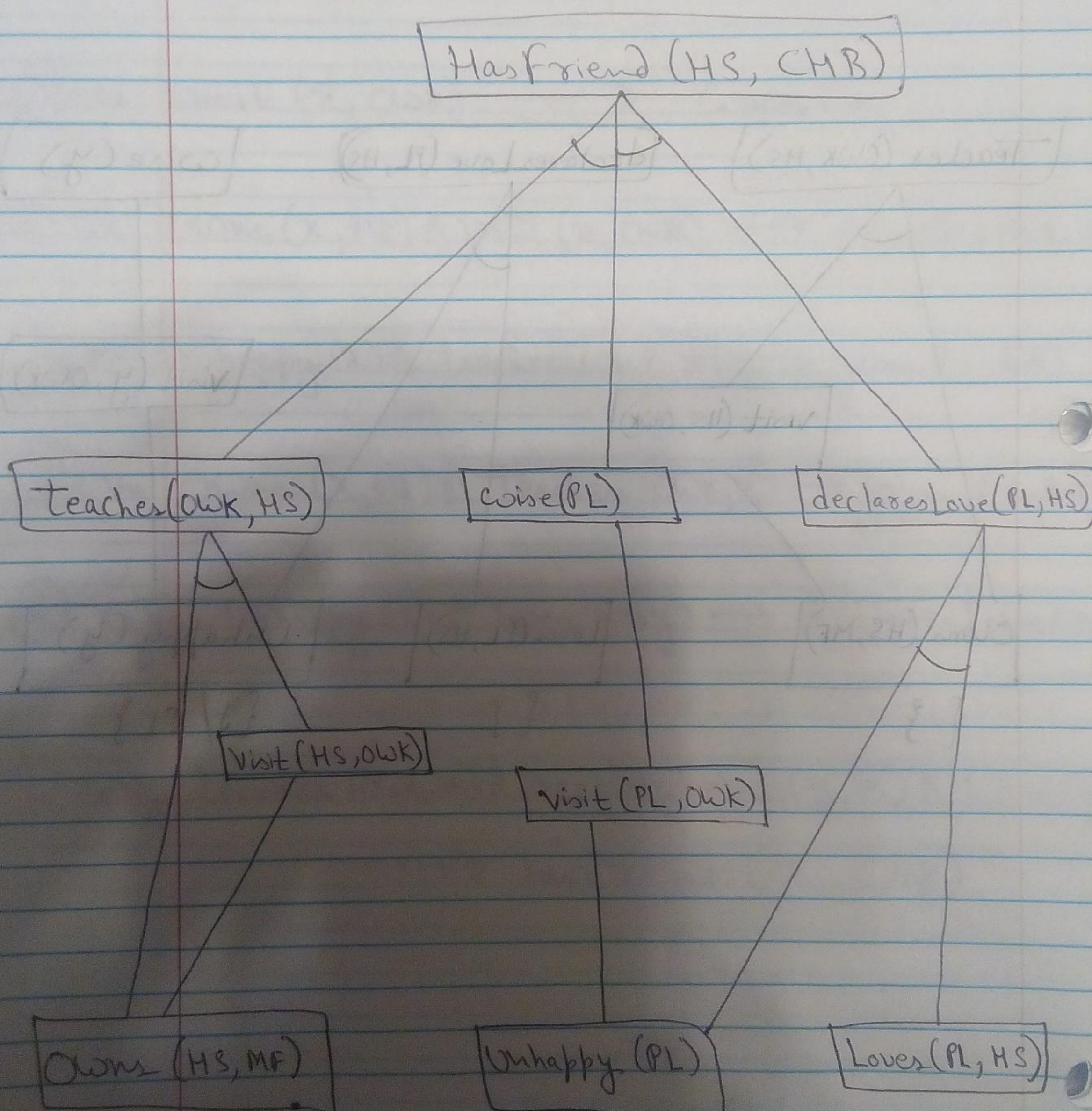
(h) $\forall x \forall y\ \text{unhappy}(x) \land \text{loves}(x, y) \longrightarrow \text{declaresLove}(x, y)$

(i) $\forall x \forall y\ \text{teaches}(OWK, x) \land \text{declaresLove}(y, x) \land \text{Wise}(y)$
$\longrightarrow \text{hasFriend}(x, CHB)$

3 (i) Backward Chaining:



(ii) Forward Chaining



(4) Assumption: domain of x is Zeus.

Notations:

$A(x)$: x were able to prevent evil
$W(x)$: x were willing to prevent evil
$P(x)$: x prevented evil
$I(x)$: x would be impotent
$M(x)$: x would be malevolent
$E(x)$: x exists

↳ Creating FOL of given 5 statements:

(i) $\forall x \quad W(x) \land A(x) \longrightarrow P(x)$

Converting to CNF:

$\forall x \quad \neg(W(x) \land A(x)) \lor P(x)$

$\neg(W(x) \land A(x)) \lor P(x)$

$$\boxed{\neg W(x) \lor \neg A(x) \lor P(x)}$$

(ii) $\forall x (\neg A(x)) \longrightarrow I(x)$

Conversion to CNF:

$\forall x \quad \neg(\neg A(x)) \lor I(x)$
$\forall x \quad A(x) \lor I(x)$

$$\boxed{A(x) \lor I(x)}$$

(iii) $\forall x \quad \neg W(x) \longrightarrow M(x)$

<u>CNF Conversion:</u>

$\forall x \quad \neg(\neg W(x)) \vee M(x)$

$\forall x \quad W(x) \vee M(x)$

$\boxed{W(x) \vee M(x)}$

(iv) $\forall x \quad \neg P(x)$

<u>CNF Conversion:</u>

$\boxed{\neg P(x)}$

(v) $\forall x \quad E(x) \longrightarrow \neg I(x) \wedge \neg M(x)$

<u>Conversion to CNF:</u>

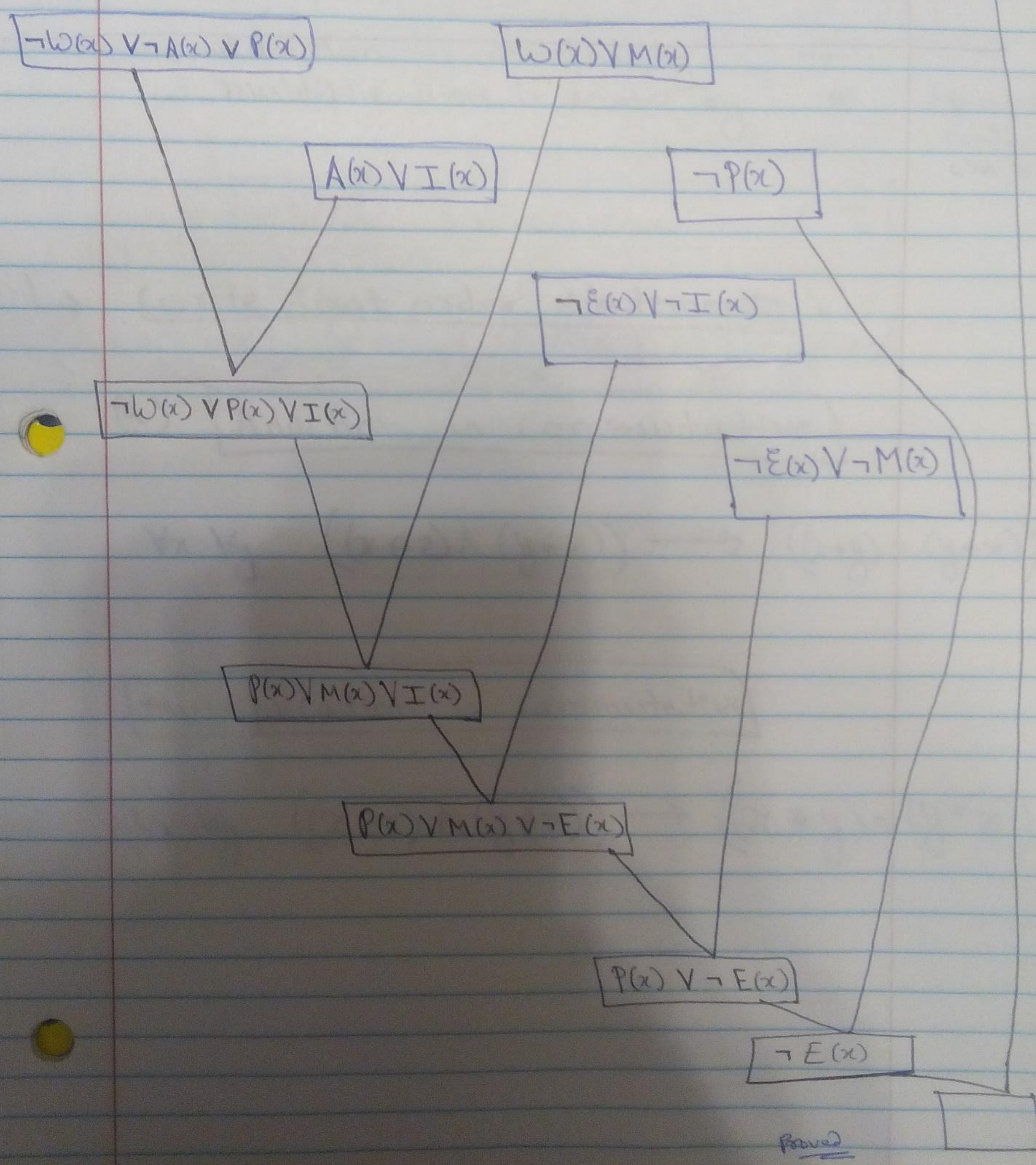
$(\neg E(x)) \vee (\neg I(x) \wedge \neg M(x))$

$\boxed{\neg E(x) \vee \neg I(x)} \wedge \boxed{\neg E(x) \vee \neg M(x)}$

To prove: Zeus does not exist.

For this consider its opposite: $\boxed{E(x)}$

Proof by Resolution:



Proved

In the end, an empty set is achieved; hence it is proved that $\neg E(x)$ is true, i.e., Zeus does not exist.

Q5) Show that the statements are valid.

(a) $\forall x \, (P(x) \rightarrow P(x))$

↳ using proof by Resolution;
↳ For this, first converting to CNF form:

$$\forall x \, (P(x) \longrightarrow P(x))$$

$$\forall x \, (\neg P(x) \vee P(x))$$

$$\boxed{\neg P(x) \vee P(x)}$$

↳ Now Prove that $\boxed{\neg P(x) \vee P(x)}$ is valid.

↳ Consider its negation:

$$\neg(\neg P(x) \vee P(x))$$

simplifying further: $(P(x) \wedge \neg P(x))$

so we have two sets:

$$\boxed{P(x)} \qquad \boxed{\neg P(x)}$$

$$\boxed{\quad\quad}$$

NULL SET

Since we receive a NULL/EMPTY set, it is proved that

$\boxed{\neg P(x) \vee P(x)}$ is valid,

or

$\boxed{\forall x \; P(x) \longrightarrow P(x)}$ is valid

FOL statement.

(b) $(\neg \exists x\, P(x)) \longrightarrow (\forall x\, \neg P(x))$

↳ using proof by resolution.

↳ for this, first converting to CNF :

$$(\neg \exists x\, P(x)) \longrightarrow (\forall x\, \neg P(x))$$

$$(\exists x\, P(x)) \vee (\forall x\, \neg P(x))$$

$$(\exists y\, P(y)) \vee (\forall x\, \neg P(x))$$

$$\forall x\, (P(G1) \vee \neg P(x))$$

$$\boxed{P(G1) \vee \neg P(x)}$$

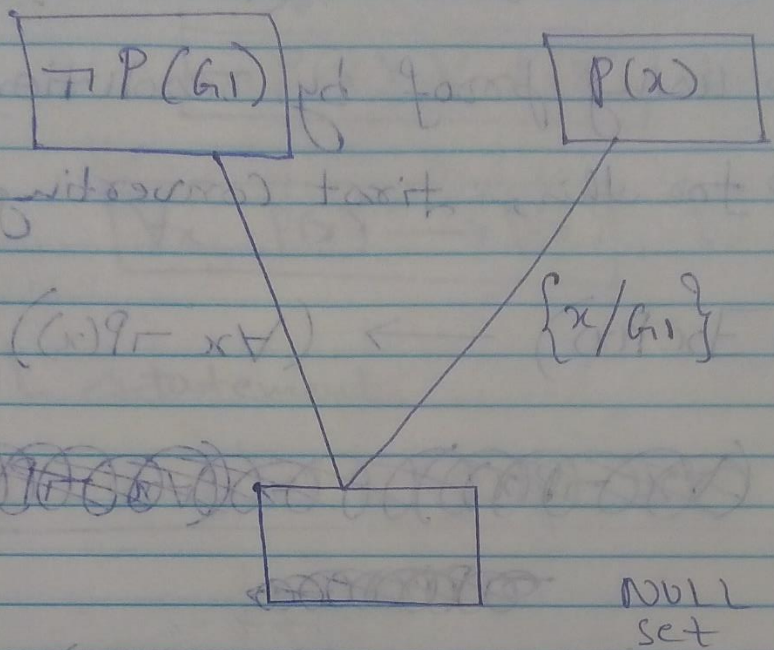
Here, G1 is skolem constant.

Consider it negation :

$$\neg (P(G1) \vee \neg P(x))$$

$$\boxed{\neg P(G1) \wedge P(x)}$$

We have two sets:



Since we receive a NULL / Empty set, it is proved that

$\boxed{P(G1) \vee \neg P(x)}$ is valid, or

$\boxed{(\neg \exists x\, P(x)) \longrightarrow (\forall x\, \neg P(x))}$ is a valid FOL statement.

(c)

$$(\forall x\, P(x) \vee Q(x)) \rightarrow (\forall x\, P(x)) \vee (\exists x\, Q(x))$$

Converting to CNF:

$$\neg(\forall x\,(P(x) \vee Q(x))) \vee (\forall x\, P(x)) \vee (\exists x\, Q(x))$$

$$(\exists x\,(\neg P(x) \wedge \neg Q(x))) \vee (\forall x\, P(x)) \vee (\exists x\, Q(x))$$

$$(\exists x\,(\neg P(x) \wedge \neg Q(x))) \vee (\forall y\, P(y)) \vee (\exists z\, Q(z))$$

Using skolemization, replace existentials by skolem function.

$$(\neg P(F(y)) \wedge \neg Q(F(y))) \vee (\forall y\, P(y)) \vee (Q(G(y)))$$

$$(\neg P(F(y)) \wedge \neg Q(F(y))) \vee (P(y)) \vee (Q(G(y)))$$

$$\boxed{(\neg P(F(y)) \vee P(y) \vee Q(G(y))) \wedge (\neg Q(F(y)) \vee P(y) \vee Q(G(y)))}$$

Using the negation part :

$$\neg\Big(\big(\neg P(F(y)) \vee P(y) \vee Q(G(y))\big) \wedge \big(\neg Q(F(y)) \vee P(y) \vee Q(G(y))\big)\Big)$$

~~FALSE~~

$$\big(P(F(y)) \wedge \neg P(y) \wedge \neg Q(G(y))\big) \vee \big(Q(F(y)) \wedge \neg P(y) \wedge \neg Q(G(y))\big)$$

Simplifying further...

$$(\text{FALSE} \wedge \neg Q(G(y))) \vee (\text{FALSE} \wedge \neg P(y))$$

$$(\text{FALSE}) \vee (\text{FALSE})$$

$$\boxed{(\text{FALSE})}$$

We see that the opposite of given statement is always FALSE.

Hence it is proved that the given statement is always true and is valid.
