$$\sum \mathcal{H} \cdot \mathbf{n} S = \sum \mathbf{T}^{-1} \mathbf{F}(\mathbf{T}\mathbf{U}) S \tag{1}$$

$$\mathbf{T}^{-1} \mathbf{F}(\mathbf{T}\mathbf{U}^{n+1}) S = \mathbf{T}^{-1} \mathbf{F}(\mathbf{T}\mathbf{U}^{n}) S + \mathbf{T}^{-1} \frac{\partial \mathbf{F}(\mathbf{T}\mathbf{U}^n)}{\partial (\mathbf{T}\mathbf{U}^n)} \Delta(\mathbf{T}\mathbf{U}^n) S \tag{2}$$

For simplicity, drop $()^n$.

$$\mathbf{T}^{-1} \frac{\partial \mathbf{F}(\mathbf{T}\mathbf{U})}{\partial (\mathbf{T}\mathbf{U})} \Delta(\mathbf{T}\mathbf{U}) = \mathbf{T}^{-1} \left[\frac{\partial \mathbf{F}(\mathbf{T}\mathbf{U}_L)}{\partial (\mathbf{T}\mathbf{U}_L)} \Delta(\mathbf{T}\mathbf{U}_L) + \frac{\partial \mathbf{F}(\mathbf{T}\mathbf{U}_R)}{\partial (\mathbf{T}\mathbf{U}_R)} \Delta(\mathbf{T}\mathbf{U}_R) \right] \tag{3}$$

$$\mathbf{T}^{-1} \frac{\partial \mathbf{F}(\mathbf{T}\mathbf{U})}{\partial (\mathbf{T}\mathbf{U})} \Delta(\mathbf{T}\mathbf{U}) = \mathbf{T}^{-1} \left[\frac{\mathbf{A}_L + |\mathbf{A}|}{2} \Delta(\mathbf{T}\mathbf{U}_L) + \frac{\mathbf{A}_R - |\mathbf{A}|}{2} \Delta(\mathbf{T}\mathbf{U}_R) \right] \tag{4}$$

$$\mathbf{T}^{-1} \frac{\partial \mathbf{F}(\mathbf{T}\mathbf{U})}{\partial (\mathbf{T}\mathbf{U})} \Delta(\mathbf{T}\mathbf{U}) = \mathbf{T}^{-1} \left[\frac{\mathbf{A}_L + |\mathbf{A}|}{2} \mathbf{T}\Delta\mathbf{U}_L + \frac{\mathbf{A}_R - |\mathbf{A}|}{2} \mathbf{T}\Delta\mathbf{U}_R \right] \tag{5}$$

$$\mathbf{T}^{-1} \frac{\partial \mathbf{F}(\mathbf{T}\mathbf{U})}{\partial (\mathbf{T}\mathbf{U})} \Delta(\mathbf{T}\mathbf{U}) S = \mathbf{M}_L \Delta\mathbf{U}_L - \mathbf{M}_R \Delta\mathbf{U}_R \tag{6}$$

$$\mathbf{M}_L = \left(\mathbf{T}^{-1} \frac{\mathbf{A}_L + |\mathbf{A}|}{2} \mathbf{T} \right) \Delta\mathbf{U}_L S \tag{7}$$

$$\mathbf{M}_R = -\left(\mathbf{T}^{-1} \frac{\mathbf{A}_R - |\mathbf{A}|}{2} \mathbf{T} \right) \Delta\mathbf{U}_R S \tag{8}$$

Set $()$ to $()^n$ back again.

$$\sum \mathbf{T}^{-1} \mathbf{F}(\mathbf{T}\mathbf{U}^{n+1}) S = \sum \mathbf{T}^{-1} \mathbf{F}(\mathbf{T}\mathbf{U}^{n}) S + \sum \mathbf{M}_L \Delta\mathbf{U}_L^n - \sum \mathbf{M}_R \Delta\mathbf{U}_R^n \tag{9}$$

$$\sum \mathbf{R}(\mathbf{U}^{n+1}) = \sum \mathbf{T}^{-1} \mathbf{F}(\mathbf{T}\mathbf{U}^{n}) S + \sum \mathbf{M}_L \Delta\mathbf{U}_L^n - \sum \mathbf{M}_R \Delta\mathbf{U}_R^n \tag{10}$$

A cell can be a left or right cell.

$$\sum \mathbf{R}(\mathbf{U}_i^{n+1}) = \sum \mathbf{T}^{-1} \mathbf{F}(\mathbf{T}\mathbf{U}_i^{n}) S + \sum \mathbf{M}_i \Delta\mathbf{U}_i^n + \sum \mathbf{M}_j \Delta\mathbf{U}_j^n \tag{11}$$

Consider a generic temporal discretization:

$$\alpha V \frac{\Delta \mathbf{U}}{\Delta t} - \mathbf{T} + \sum \mathbf{R}(\mathbf{U}_i^{n+1}) = 0 \tag{12}$$

where, α is a coefficient depending on temporal discretization and $\mathbf{T}$ is higher order terms of the temporal discretization. Rearranging the equation:

$$\alpha V \frac{\Delta \mathbf{U}}{\Delta t} = -\sum \mathbf{R}(\mathbf{U}_i^{n+1}) + \mathbf{T} \tag{13}$$

$$\left(\alpha \frac{V}{\Delta t} + \sum \mathbf{M}_i \right) \Delta\mathbf{U}_i = -\sum \mathbf{R}(\mathbf{U}_i) + \sum \mathbf{M}_j \Delta\mathbf{U}_j + \mathbf{T} \tag{14}$$

where, for each cell-face,

$$\mathbf{M}_i = \begin{cases} \mathbf{M}_L & \text{if } i \text{ is on the left} \\ -\mathbf{M}_R & \text{if } i \text{ is on the right} \end{cases} \tag{15}$$

$$\mathbf{M}_j = \begin{cases} -\mathbf{M}_R & \text{if } i \text{ is on the left} \\ \mathbf{M}_L & \text{if } i \text{ is on the right} \end{cases} \tag{16}$$

For example, consider three-time level backward Euler discretization.

$$V \frac{3\mathbf{U}^{n+1} - 4\mathbf{U}^n + \mathbf{U}^{n-1}}{2\Delta t} = -\sum \mathbf{R}(\mathbf{U}_i^{n+1}) \tag{17}$$

Implicit Scheme	α	$\mathbf{T}$
Euler	1	$[0]^T$
Three-time level	$\frac{3}{2}$	$\frac{1}{2}\frac{V}{\Delta t}(\mathbf{U}^n - \mathbf{U}^{n-1})$

Separate the first and higher order terms of temporal discretization.

$$V\frac{3\mathbf{U}^{n+1} - 4\mathbf{U}^n + \mathbf{U}^{n-1}}{2\Delta t} = \frac{1}{2}\frac{V}{\Delta t}[3\Delta\mathbf{U} - (\mathbf{U}^n - \mathbf{U}^{n-1})] \tag{18}$$

$$\frac{3}{2}\frac{V}{\Delta t}\Delta\mathbf{U} = -\sum \mathbf{R}(\mathbf{U}_i^{n+1}) + \frac{1}{2}\frac{V}{\Delta t}(\mathbf{U}^n - \mathbf{U}^{n-1}) \tag{19}$$

$$\left(\frac{3}{2}\frac{V}{\Delta t} + \mathbf{M}_i\right)\Delta\mathbf{U}_i = -\sum \mathbf{R}(\mathbf{U}_i^n) + \sum \mathbf{M}_j\Delta\mathbf{U}_j + \frac{1}{2}\frac{V}{\Delta t}(\mathbf{U}^n - \mathbf{U}^{n-1}) \tag{20}$$