$$\mathcal{H} \cdot \mathbf{n} = \mathbf{T}^{-1}\mathbf{F}(\mathbf{T}\mathbf{U}) \tag{1}$$

$$\mathbf{T}^{-1}\mathbf{F}(\mathbf{T}\mathbf{U}^{n+1}) = \mathbf{T}^{-1}\mathbf{F}(\mathbf{T}\mathbf{U}^{n}) + \mathbf{T}^{-1}\frac{\partial \mathbf{F}(\mathbf{T}\mathbf{U})}{\partial(\mathbf{T}\mathbf{U})}\Delta(\mathbf{T}\mathbf{U}) \tag{2}$$

$$\mathbf{T}^{-1}\frac{\partial \mathbf{F}(\mathbf{T}\mathbf{U})}{\partial(\mathbf{T}\mathbf{U})}\Delta(\mathbf{T}\mathbf{U}) = \mathbf{T}^{-1}\left[\frac{\partial \mathbf{F}(\mathbf{T}\mathbf{U}_L)}{\partial(\mathbf{T}\mathbf{U}_L)}\Delta(\mathbf{T}\mathbf{U}_L) + \frac{\partial \mathbf{F}(\mathbf{T}\mathbf{U}_R)}{\partial(\mathbf{T}\mathbf{U}_R)}\Delta(\mathbf{T}\mathbf{U}_R)\right] \tag{3}$$

$$\mathbf{T}^{-1}\frac{\partial \mathbf{F}(\mathbf{T}\mathbf{U})}{\partial(\mathbf{T}\mathbf{U})}\Delta(\mathbf{T}\mathbf{U}) = \mathbf{T}^{-1}\left[\frac{\mathbf{A}_L + |\mathbf{A}|}{2}\Delta(\mathbf{T}\mathbf{U}_L) + \frac{\mathbf{A}_R - |\mathbf{A}|}{2}\Delta(\mathbf{T}\mathbf{U}_R)\right] \tag{4}$$

$$\mathbf{T}^{-1}\frac{\partial \mathbf{F}(\mathbf{T}\mathbf{U})}{\partial(\mathbf{T}\mathbf{U})}\Delta(\mathbf{T}\mathbf{U}) = \mathbf{T}^{-1}\left[\frac{\mathbf{A}_L + |\mathbf{A}|}{2}\mathbf{T}\Delta\mathbf{U}_L + \frac{\mathbf{A}_R - |\mathbf{A}|}{2}\mathbf{T}\Delta\mathbf{U}_R\right] \tag{5}$$

$$\mathbf{T}^{-1}\frac{\partial \mathbf{F}(\mathbf{T}\mathbf{U})}{\partial(\mathbf{T}\mathbf{U})}\Delta(\mathbf{T}\mathbf{U}) = \mathbf{N}_L\Delta\mathbf{U}_L - \mathbf{N}_R\Delta\mathbf{U}_R \tag{6}$$

where,

$$\mathbf{N}_L = (\mathbf{T}^{-1}\frac{\mathbf{A}_L + |\mathbf{A}|}{2}\mathbf{T})\Delta\mathbf{U}_L \tag{7}$$

$$\mathbf{N}_R = -(\mathbf{T}^{-1}\frac{\mathbf{A}_R - |\mathbf{A}|}{2}\mathbf{T})\Delta\mathbf{U}_R \tag{8}$$

Consider first order forward Euler temporal discretization,

$$V\frac{\mathbf{U}^{n+1} - \mathbf{U}^{n}}{\Delta t} = -\sum \mathbf{T}^{-1}\mathbf{F}(\mathbf{T}\mathbf{U})S \tag{9}$$

$$V\frac{\Delta\mathbf{U}}{\Delta t} = -\sum \mathbf{T}^{-1}\mathbf{F}(\mathbf{T}\mathbf{U})S \tag{10}$$

A cell can be either on the left or on the right of a face. If a cell is on the left of a face,

$$V\frac{\Delta\mathbf{U}_L}{\Delta t} = -\mathbf{N}_L\Delta\mathbf{U}_L S + \mathbf{N}_R\Delta\mathbf{U}_R S \tag{11}$$

Define $\mathbf{M}_{LK} = \mathbf{N}_{LK}S$.

$$\left(\frac{V}{\Delta t} + \mathbf{M}_L\right)\Delta\mathbf{U}_L = \mathbf{M}_R\Delta\mathbf{U}_R \tag{12}$$

If a cell is on the right of a face, flux through a face is opposite to that for the left cell, that is, $-\mathbf{T}^{-1}\mathbf{F}(\mathbf{T}\mathbf{U}) \rightarrow \mathbf{T}^{-1}\mathbf{F}(\mathbf{T}\mathbf{U})$.

$$V\frac{\Delta\mathbf{U}_R}{\Delta t} = \mathbf{N}_L\Delta\mathbf{U}_L S - \mathbf{N}_R\Delta\mathbf{U}_R S \tag{13}$$

$$\left(\frac{V}{\Delta t} + \mathbf{M}_R\right)\Delta\mathbf{U}_R = \mathbf{M}_L\Delta\mathbf{U}_L \tag{14}$$

In general,

$$\left(\frac{V}{\Delta t} + \mathbf{M}_i\right)\Delta\mathbf{U}_i = \sum \mathbf{M}_j\Delta\mathbf{U}_j \tag{15}$$

Now consider second order three-time level backward Euler discretization.

$$V\frac{3\mathbf{U}^{n+1} - 4\mathbf{U}^{n} + \mathbf{U}^{n-1}}{2\Delta t} = -\sum \mathbf{T}^{-1}\mathbf{F}(\mathbf{T}\mathbf{U})S \tag{16}$$

$$
\begin{aligned}
V\frac{3\mathbf{U}^{n+1}-4\mathbf{U}^{n}+\mathbf{U}^{n-1}}{2\Delta t} &= \frac{V}{\Delta t}\left(\frac{3}{2}\mathbf{U}^{n+1}-2\mathbf{U}^{n}+\frac{1}{2}\mathbf{U}^{n-1}\right) \\
&= \frac{3}{2}\frac{V}{\Delta t}(\mathbf{U}^{n+1}-\mathbf{U}^{n})-\frac{1}{2}\frac{V}{\Delta t}(\mathbf{U}^{n}-\mathbf{U}^{n-1}) \\
&= \frac{3}{2}\frac{V}{\Delta t}\Delta\mathbf{U}-\frac{1}{2}\frac{V}{\Delta t}(\mathbf{U}^{n}-\mathbf{U}^{n-1})
\end{aligned} \tag{17}
$$

$$
\frac{3}{2}\frac{V}{\Delta t}\Delta\mathbf{U} = -\sum \mathbf{T}^{-1}\mathbf{F}(\mathbf{T}\mathbf{U})S+\frac{1}{2}\frac{V}{\Delta t}(\mathbf{U}^{n}-\mathbf{U}^{n-1}) \tag{18}
$$

In general,

$$
\left(\frac{3}{2}\frac{V}{\Delta t}+\mathbf{M}_i\right)\Delta\mathbf{U}_i = \sum \mathbf{M}_j\Delta\mathbf{U}_j+\frac{1}{2}\frac{V}{\Delta t}(\mathbf{U}^{n}-\mathbf{U}^{n-1}) \tag{19}
$$