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Robot Kinematics: Position Analysis

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2.1 INTRODUCTION

In this chapter, we will study forward and inverse kinematics of robots. The forward kinematics will enable us to determine where the robot's end (hand) will be if all joint variables are known. Inverse kinematics will enable us to calculate what each joint variable must be if we desire that the hand be located at a particular point and have a particular orientation. Using matrices, we will first establish a way of describing objects, locations, orientations, and movements. Then we will study the forward and inverse kinematics of different configurations of robots, such as Cartesian, cylindrical, and spherical coordinates. Finally, we will use the Denavit–Hartenberg representation to derive forward and inverse kinematic equations of all possible configurations of robots.

It is important to realize that in reality, manipulator-type robots are delivered with no end effector. In most cases, there may be a gripper attached to the robot. However, depending on the actual application, different end effectors are attached to the robot by the user. Obviously, the end effector's size and length determine where the end of the robot is. For a short end effector, the end will be at a different location than for a long end effector. In this chapter, we will assume that the end of the robot is a plate to which the end effector can be attached, as necessary. We will call this the "hand" or the "end-plate" of the robot. If necessary, we can always add the length of the end effector to the robot for determining the location and orientation of the end effector.

2.2 ROBOTS AS MECHANISMS

Manipulator-type robots have multiple degrees of freedom (DOF), are three-dimensional, are open loop, and are chain mechanisms.

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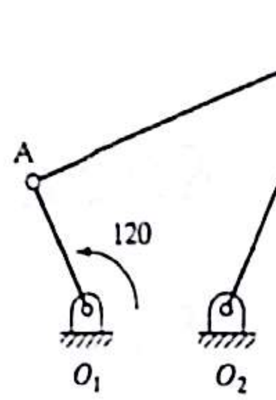

Figure 2.1 A one-degree-of-freedom closed-loop four-bar mechanism.

In a one-degree-of-freedom system, when the variable is set to a particul value, the mechanism is totally set, and all its other variables are known. For e ample, in the four-bar mechanism of Figure 2.1, when the crank is set to 120°, th angles of the coupler link and the rocker arm are also known. However, in a mu tiple-degree-of-freedom mechanism, all input variables must be individually set i order to know the remaining parameters. Robots are such machines, where eac joint variable must be known in order to know where the hand of the robot is.

Robots are three-dimensional machines if they are to move in space. Althoug it is possible to have a two-dimensional multiple-degree-of-freedom robot, they ar not common.

Robots are open-loop mechanisms. Unlike mechanisms that are closed loo (e.g., four-bar mechanisms), even if all joint variables are set to particular value there is no guarantee that the hand will be at the given location. This is because i there is any deflection in any joint or link, it will change the location of all sub sequent links without feedback. For example, in the four-bar mechanism of Fig ure 2.2, if the link AB deflects, it will affect link O_2B, whereas in an open-loo system such as the robot, the deflections will move all succeeding members withou any feedback. As a result, in open-loop systems, either all joint and link parameter must continuously be measured or the end of the system must be monitored for th kinematic position of the machine to be known. This difference can be expressed b comparing the vector equations describing the relationship between different link of the two mechanisms as follows:

$$\overline{O_1A} + \overline{AB} = \overline{O_1O_2} + \overline{O_2B}, \tag{2.1}$$

$$\overline{O_1A} + \overline{AB} + \overline{BC} = \overline{O_1C}. \tag{2.2}$$

As you can see, if there is a deflection in link AB, link O_2B will move accordingly. However, the two sides of Equation (2.1) have changed corresponding to th changes in the links. On the other hand, if link AB of the robot deflects, all subsequent links will move too, but unless O_1C is measured by other means, the chang will not be known.

To remedy this problem in open-loop robots, the position of the hand is constantly measured with devices such as a camera, the robot is made into a closed-loo system with external means such as the use of secondary arms or laser beams [1,2,3] or, as is the standard practice, the robot's links and joints are made excessivel strong to eliminate all deflections. The last method renders the robot very heavy

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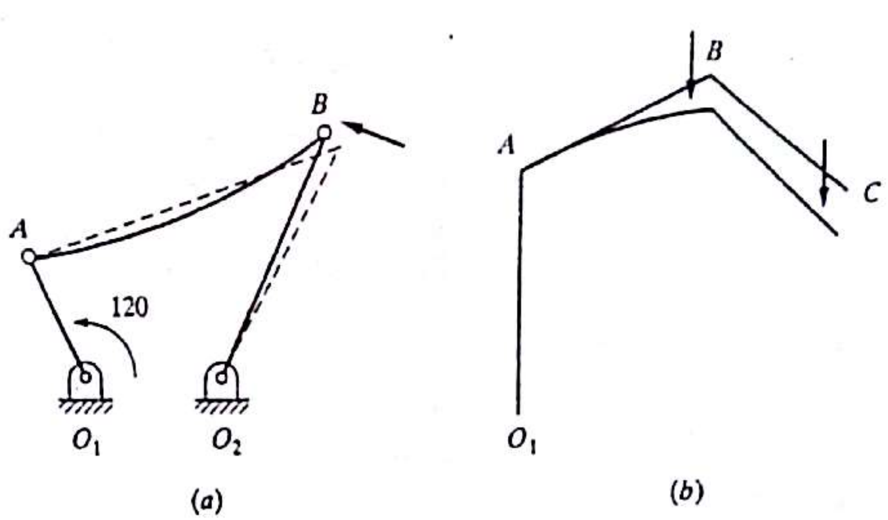

Figure 2.2 (a) Closed-loop versus (b) open-loop mechanisms.

massive, and slow, and its specified payload is very low compared with what it actually can carry.

2.3 MATRIX REPRESENTATION

Matrices can be used to represent points, vectors, frames, translations, rotations, and transformations, as well as objects and other kinematic elements in a frame. We will use this representation throughout this book to derive equations of motion for robots.

2.3.1 Representation of a Point in Space

A point P in space (Figure 2.3) can be represented by its three coordinates relative to a reference frame:

$$P = a_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + c_z\hat{\mathbf{k}}, \tag{2.3}$$

where a_x, b_y, and c_z are the three coordinates of the point represented in the reference frame. Obviously, other coordinate representations can also be used to describe the location of a point in space.

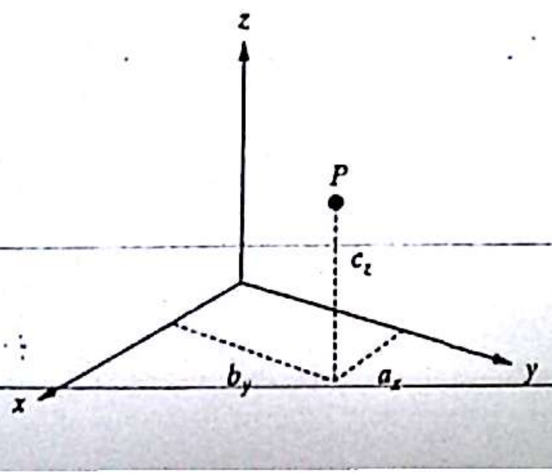

Figure 2.3 Representation of a point in space.

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2.3.2 Representation of a Vector in Space

A vector can be represented by three coordinates of its tail and of its head. If th vector starts at a point A and ends at point B, then it can be represented by $\overline{P}_{AB}$ = $(B_x - A_x)\hat{\mathrm{i}} + (B_y - A_y)\hat{\mathrm{j}} + (B_z - A_z)\hat{\mathrm{k}}$. Specifically, if the vector starts at the origi (Figure 2.4), then:

$$\overline{P} = a_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + c_z\hat{\mathrm{k}}, \qquad (2.4$$

where a_x, b_y, and c_z are the three components of the vector in the reference frame In fact, point P in the previous section is in reality represented by a vector con nected to it at point P and expressed by the three components of the vector.

The three components of the vector can also be written in a matrix form, as i Equation (2.5). This format will be used throughout this book to represent all kine matic elements:

$$\overline{P} = \begin{bmatrix} a_x \\ b_y \\ c_z \end{bmatrix}. \qquad (2.5$$

This representation can be slightly modified to also include a scale factor w such that if x, y, and z are divided by w, they will yield a_x, b_y, and c_z. Thus, the vecto can be written as

$$\overline{P} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, \quad \text{where } a_x = \frac{x}{w}, b_y = \frac{y}{w}, \text{ etc.} \qquad (2.6$$

Variable w may be any number, and as it changes, it can change the overall size o the vector. This is similar to zooming a picture in computer graphics. As the value o w changes, the size of the vector changes accordingly. If w is greater than unity, al vector components enlarge; if w is less than unity, all vector components become smaller. This is also used in computer graphics for changing the size of pictures anc drawings.

If w is unity, the size of the components remain unchanged. However, if $w = 0$ then a_x, b_y, and c_z will be infinity. In this case, x, y, and z (as well as a_x, b_y, and c_z) wil

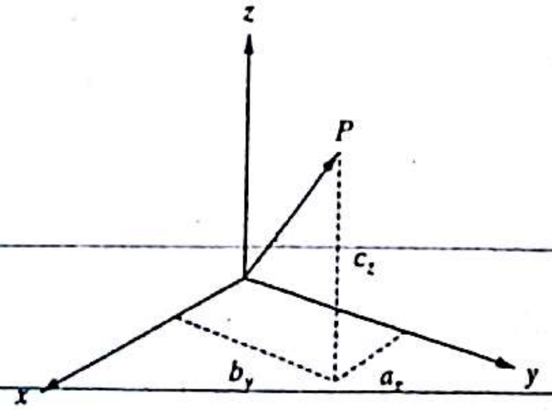

Figure 2.4 Representation of a vector in space.

represent a vector whose length is infinite, but nonetheless, is in the direction represented by the vector. This means that a **directional vector** can be represented by a scale factor of $w = 0$, where the length is not of importance, but the direction is represented by the three components of the vector. This will be used throughout this book to represent directional vectors.

Example 2.1

A vector is described as $\overline{P} = 3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$. Express the vector in matrix form:

(1) With a scale factor of 2.
(2) If it were to describe a direction as a unit vector.

Solution The vector can be expressed in matrix form with a scale factor of 2, as well as 0 for direction, as

$$\overline{P} = \begin{bmatrix} 6 \\ 10 \\ 4 \\ 2 \end{bmatrix} \quad \text{and} \quad \overline{P} = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 0 \end{bmatrix}.$$

However, in order to make the vector into a unit vector, we will normalize the length such that the new length will be equal to unity. To do this, each component of the vector will be divided by the square root of the sum of the squares of the three components:

$$\lambda = \sqrt{p_x^2 + p_y^2 + p_z^2} = 6.16, \quad \text{where } p_x = \frac{3}{6.16} = 0.487, p_y = \frac{5}{6.16}, \text{ etc.,}$$

$$\text{and} \quad \overline{P}_{\text{unit}} = \begin{bmatrix} 0.487 \\ 0.811 \\ 0.324 \\ 0 \end{bmatrix}.$$

2.3.3 Representation of a Frame at the Origin of a Fixed-Reference Frame

A frame centered at the origin of a reference frame is represented by three vectors, usually mutually perpendicular to each other, called unit vectors $\overline{n}, \overline{o}, \overline{a}$, for *normal, orientation*, and *approach* vectors (Figure 2.5). Each unit vector is represented by its

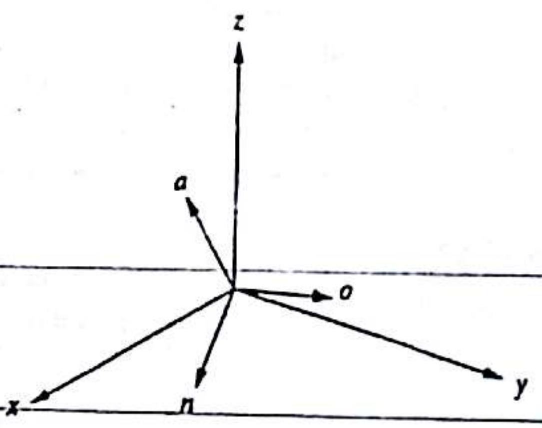

Figure 2.5 Representation of a frame at the origin of the reference frame.

three components in the reference frame as in Section 2.3.2. Thus, a frame F can be represented by three vectors in a matrix form as:

$$F = \begin{bmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix} \tag{2.7}$$

2.3.4 Representation of a Frame in a Fixed Reference Frame

If a frame is not at the origin (or, in fact, even if it is at the origin) then the location of the origin of the frame relative to the reference frame must also be expressed. In order to do this, a vector will be drawn between the origin of the frame and the origin of the reference frame describing the location of the frame (Figure 2.6). This vector is expressed through its components relative to the reference frame. Thus, the frame can be expressed by three vectors describing its directional unit vectors, as well as a fourth vector describing its location as follows:

$$F = \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.8}$$

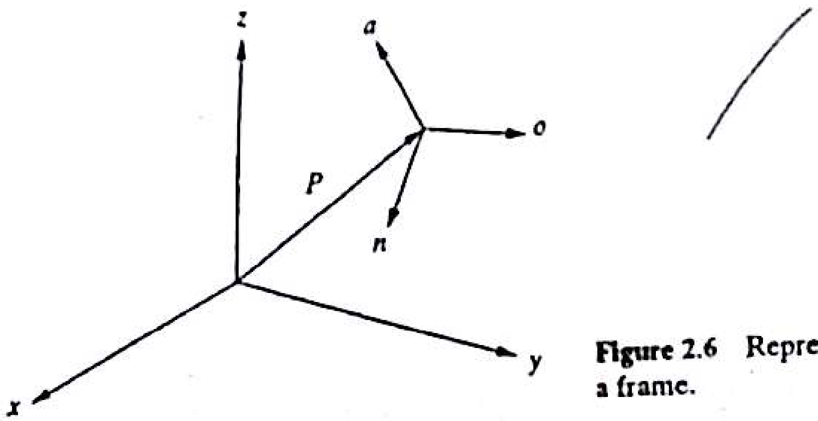

Figure 2.6 Representation of a frame in a frame.

As shown in Equation (2.8), the first three vectors are directional vectors with $w = 0$, representing the directions of the three unit vectors of the frame $\bar{n}, \bar{o}, \bar{a}$, while the fourth vector with $w = 1$ represents the location of the origin of the frame relative to the reference frame. Unlike the unit vectors, the length of vector P is important to us, and thus we use a scale factor of one. A frame may also be represented with a 3×4 matrix without the scale factors, but it is not commonly done this way.

Example 2.2

The frame F shown in Figure 2.7 is located at 3,5,7 units, with its n-axis parallel to x, its o-axis at 45° relative to the y-axis, and its a-axis at 45° relative to the z-axis. The frame can be described by:

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$$F = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0.707 & -0.707 & 5 \\ 0 & 0.707 & 0.707 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

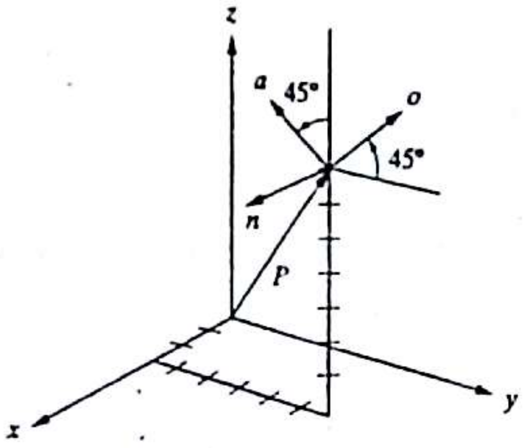

Figure 2.7 An example of representation of a frame in space.

2.3.5 Representation of a Rigid Body

An object can be represented in space by attaching a frame to it and representing the frame in space. Since the object is permanently attached to this frame, its position and orientation relative to this frame is always known. As a result, so long as the frame can be described in space, the object's location and orientation relative to the fixed frame will be known (Figure 2.8). As before, a frame in space can be represented by a matrix, where the origin of the frame, as well as the three vectors representing its orientation relative to the reference frame, are expressed. Thus,

$$F_{\text{object}} = \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.9}$$

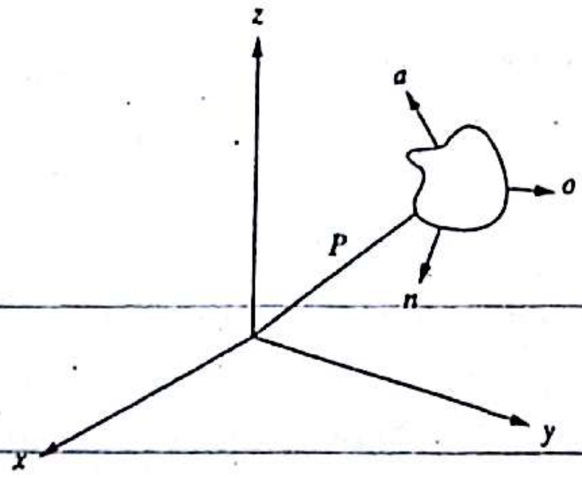

Figure 2.8 Representation of an object in space.

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As was discussed in Chapter 1, a point in space has only three degrees of freedom; it can only move along the three reference axes. However, a rigid body in space has 6 degrees of freedom, meaning that not only it can move along three axes of X, Y, and Z, but can also rotate about these three axes. Thus, all that is needed to completely define an object in space is 6 pieces of information describing the location of the origin of the object in the reference frame relative to the three reference axes, as well as its orientation about the three axes. However, as can be seen in Equation (2.9), 12 pieces of information are given, 9 for orientation, and 3 for position. (This excludes the scale factors on the last row of the matrix, because they do not add to this information.) Obviously, there must be some constraints present in this representation to limit the preceding to 6. Thus, we need 6 constraint equations to reduce the amount of information from 12 to 6 pieces. The constraints come from the known characteristics of the frame, which we have not used yet:

- the three unit vectors $\bar{n}, \bar{o}, \bar{a}$ are mutually perpendicular, and
- each unit vector's length must be equal to unity.

These constraints translate into the following six constraint equations:

(1) $\bar{n} \cdot \bar{o} = 0$. (The dot product of $\bar{n}$ and $\bar{o}$ vectors must be zero.)
(2) $\bar{n} \cdot \bar{a} = 0$.
(3) $\bar{a} \cdot \bar{o} = 0$.
(4) $|n| = 1$. (The magnitude of the length of the vector must be 1.)
(5) $|o| = 1$.
(6) $|a| = 1$.

As a result, the values representing a frame in a matrix must be such that the foregoing equations are true. Otherwise, the frame will not be correct. Alternatively, the first three equations in Equation (2.10) can be replaced by a cross product of the three vectors as follows:

$$\bar{n} \times \bar{o} = \bar{a}. \tag{2.11}$$

Example 2.3

For the following frame, find the values of the missing elements, and complete the matrix representation of the frame:

$$F = \begin{bmatrix} ? & 0 & ? & 5 \\ 0.707 & ? & ? & 3 \\ ? & ? & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution Obviously, the values 5,3,2 representing the position of the origin of the frame do not affect the constraint equations. You also notice that only three values for directional vectors are given. This is all that is needed. Using Equation (2.10), we get

$$n_x o_x + n_y o_y + n_z o_z = 0, \quad \text{or} \quad n_x(0) + 0.707(o_y) + n_z(o_z) = 0,$$
$$n_x a_x + n_y a_y + n_z a_z = 0 \quad \text{or} \quad n_x(a_x) + 0.707(a_y) + n_z(0) = 0,$$

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$$a_x o_x + a_y o_y + a_z o_z = 0, \quad \text{or} \quad a_x(0) + a_y(o_y) + 0(o_z) = 0,$$

$$n_x^2 + n_y^2 + n_z^2 = 1, \quad \text{or} \quad n_x^2 + 0.707^2 + n_z^2 = 1,$$

$$o_x^2 + o_y^2 + o_z^2 = 1, \quad \text{or} \quad 0^2 + o_y^2 + o_z^2 = 1,$$

$$a_x^2 + a_y^2 + a_z^2 = 1, \quad \text{or} \quad a_x^2 + a_y^2 + 0^2 = 1.$$

Simplifying these equations yields

$$0.707\, o_y + n_z o_z = 0,$$

$$n_x a_x + 0.707\, a_y = 0.$$

$$a_y o_y = 0,$$

$$n_x^2 + n_z^2 = 0.5,$$

$$o_y^2 + o_z^2 = 1,$$

$$a_x^2 + a_y^2 = 1.$$

Solving these six equations yields $n_x = \pm 0.707$, $n_z = 0$, $o_y = 0$, $o_z = 1$, $a_x = \pm 0.707$, and $a_y = -0.707$. Please notice that both n_x and a_x must have the same sign. The reason for multiple solutions is that with the given parameters, it is possible to have two sets of mutually perpendicular vectors in opposite directions. The final matrix will be:

$$F = \begin{bmatrix} 0.707 & 0 & 0.707 & 5 \\ 0.707 & 0 & -0.707 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{or} \quad F = \begin{bmatrix} -0.707 & 0 & -0.707 & 5 \\ 0.707 & 0 & -0.707 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

As can be seen, both matrices satisfy all the requirements set by the constraint equations. It is important to realize that the values represented by the three direction vectors are not arbitrary, but are bound by these equations. Thus, you may not arbitrarily use any desired values in the matrix.

The same problem may be solved by taking the cross products of $\bar{n}$ and $\bar{o}$ and setting it equal to $\bar{a}$ as $\bar{n} \times \bar{o} = \bar{a}$, or

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ n_x & n_y & n_z \\ o_x & o_y & o_z \end{vmatrix} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}},$$

or $$\hat{\mathbf{i}}(n_y o_z - n_z o_y) - \hat{\mathbf{j}}(n_x o_z - n_z o_x) + \hat{\mathbf{k}}(n_x o_y - n_y o_x) = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}.$$

Substituting the values into this equation yields:

$$\hat{\mathbf{i}}(0.707 o_z - n_z o_y) - \hat{\mathbf{j}}(n_x o_z) + \hat{\mathbf{k}}(n_x o_y) = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + 0\hat{\mathbf{k}}.$$

Solving the three simultaneous equations gives:

$$0.707\, o_z - n_z o_y = a_x,$$

$$-n_x o_z = a_y,$$

$$n_x o_y = 0,$$

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which replace the three equations for the dot products. Together with the three unit vector length constraint equations, there are six equations, which ultimately result in the same values for the unknown parameters. Please verify that you get the same results.

2.4 HOMOGENEOUS TRANSFORMATION MATRICES

For a variety of reasons, it is desirable to keep transformation matrices in square form, either 3×3 or 4×4. First, as we will see later, it is much easier to calculate the inverse of square matrices than rectangular matrices. Second, in order to multiply two matrices, their dimensions must match, such that the number of columns of the first matrix must be the same as the number of rows of the second matrix, as in $(m \times n)$ and $(n \times p)$, which results in a matrix of $(m \times p)$ dimensions. If two matrices A and B are square with $(m \times m)$ and $(m \times m)$ dimensions, one may multiply A by B or B by A, both resulting in the same $(m \times m)$ dimensions. However, if the two matrices are not square, with $(m \times n)$ and $(n \times p)$ dimensions, respectively, A can be multiplied by B, but B may not be multiplied by A, and the result of AB has a dimension different from A and B. Since we will have to multiply many matrices together in different orders to find the equations of motion of the robots, we desire square matrices.

To keep representation matrices square, if we represent both orientation and position in the same matrix, we will add the scale factors to the matrix to make it a 4×4 matrix. If we represent the orientation alone, we may either drop the scale factors and use 3×3 matrices, or add a 4th column with zeros for position in order to keep the matrix square. Matrices of this form are called homogeneous matrices, and we write them as follows:

$$F = \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.12}$$

2.5 REPRESENTATION OF TRANSFORMATIONS

A transformation is defined as making a movement in space. When a frame (a vector, an object, or a moving frame) moves in space relative to a fixed reference frame, we can represent this motion in a form similar to a frame representation. This is because a transformation itself is a change in the state of a frame (representing the change in its location and orientation), and thus it can be represented as a frame. A transformation may be in one of the following forms:

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- A pure translation,
- A pure rotation about an axis,
- A combination of translations or rotations.

To see how these can be represented, we will study each one separately.

2.5.1 Representation of a Pure Translation

If a frame (which may also be representing an object) moves in space without any change in its orientation, the transformation is a pure translation. In this case, the directional unit vectors remain in the same direction and thus do not change. All that changes is the location of the origin of the frame relative to the reference frame, as shown in Figure 2.9. The new location of the frame relative to the fixed reference frame can be found by adding the vector representing the translation to the vector representing the original location of the origin of the frame. In matrix form, the new frame representation may be found by premultiplying the frame with a matrix representing the transformation. Since the directional vectors do not change in a pure translation, the transformation T will simply be:

$$T = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.13}$$

where d_x, d_y, and d_z are the three components of a pure translation vector $\bar{d}$ relative to the x-, y-, and z-axes of the reference frame. As you can see, the first three columns represent no rotational movement (equivalent to unity), while the last column represents the translation. The new location of the frame will be

$$F_{new} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & P_x + d_x \\ n_y & o_y & a_y & P_y + d_y \\ n_z & o_z & a_z & P_z + d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.14}$$

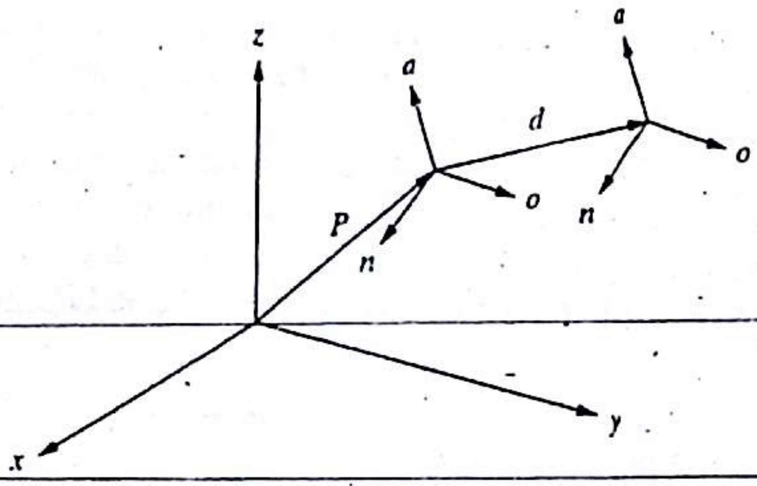

Figure 2.9 Representation of a pure translation in space.

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This equation is also symbolically written as

$$F_{new} = \text{Trans}\,(d_x,d_y,d_z) \times F_{old}. \tag{2.15}$$

First, as you see, by premultiplying the transformation matrix with the frame matrix, the new location can be found. This, in one form or another, is true for all transformations, as we will see later. Second, you notice that the directional vectors remain the same after a pure translation, but that as vector addition of $\bar{d}$ and $\bar{P}$ would result, the new location of the frame is $\bar{d} + \bar{P}$. Third, you also notice how homogeneous transformation matrices facilitate the multiplication of matrices, resulting in the same dimensions as before.

Example 2.4

A frame *F* has been moved nine units along the *x*-axis and five units along the *z*-axis of the reference frame. Find the new location of the frame:

$$F = \begin{bmatrix} 0.527 & -0.574 & 0.628 & 5 \\ 0.369 & 0.819 & 0.439 & 3 \\ -0.766 & 0 & 0.643 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution Using Equation (2.14) or Equation (2.15), we get

$$F_{new} = \text{Trans}\,(d_x,d_y,d_z) \times F_{old} = F_{new} = \text{Trans}\,(9,0,5) \times F_{old}$$

and

$$F = \begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0.527 & -0.574 & 0.628 & 5 \\ 0.369 & 0.819 & 0.439 & 3 \\ -0.766 & 0 & 0.643 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.527 & -0.574 & 0.628 & 14 \\ 0.369 & 0.819 & 0.439 & 3 \\ -0.766 & 0 & 0.643 & 13 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2.5.2 Representation of a Pure Rotation about an Axis

To simplify the derivation of rotations about an axis, let's first assume that the frame is at the origin of the reference frame and parallel to it. We will later expand the results to other rotations, as well as combination of rotations.

Let's assume that a frame $(\bar{n},\bar{o},\bar{a})$, located at the origin of the reference frame $(\bar{x},\bar{y},\bar{z})$, will rotate through an angle of θ about the *x*-axis of the reference frame. Let's also assume that attached to the rotating frame $(\bar{n},\bar{o},\bar{a})$ is a point *P*, with coordinates P_x, P_y, and P_z relative to the reference frame and P_n, P_o, and P_a relative to the moving frame. As the frame rotates about the *x*-axis, point *P* attached to the frame, will also rotate with it. Before rotation, the coordinates of the point in both frames are the same. (Remember that the two frames are at the same location and are parallel to each other.) After rotation, the P_n, P_o, and P_a coordinates of the point remain

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the same in the rotating frame $(\bar{n},\bar{o},\bar{a})$, but P_x, P_y, and P_z will be different in the $(\bar{x},\bar{y},\bar{z})$ frame (Figure 2.10). We desire to find the new coordinates of the point relative to the fixed-reference frame after the moving frame has rotated.

Now let's look at the same coordinates in 2-D as if we were standing on the x-axis. The coordinates of point P are shown before and after rotation in Figure 2.11. The coordinates of point P relative to the reference frame are P_x, P_y,

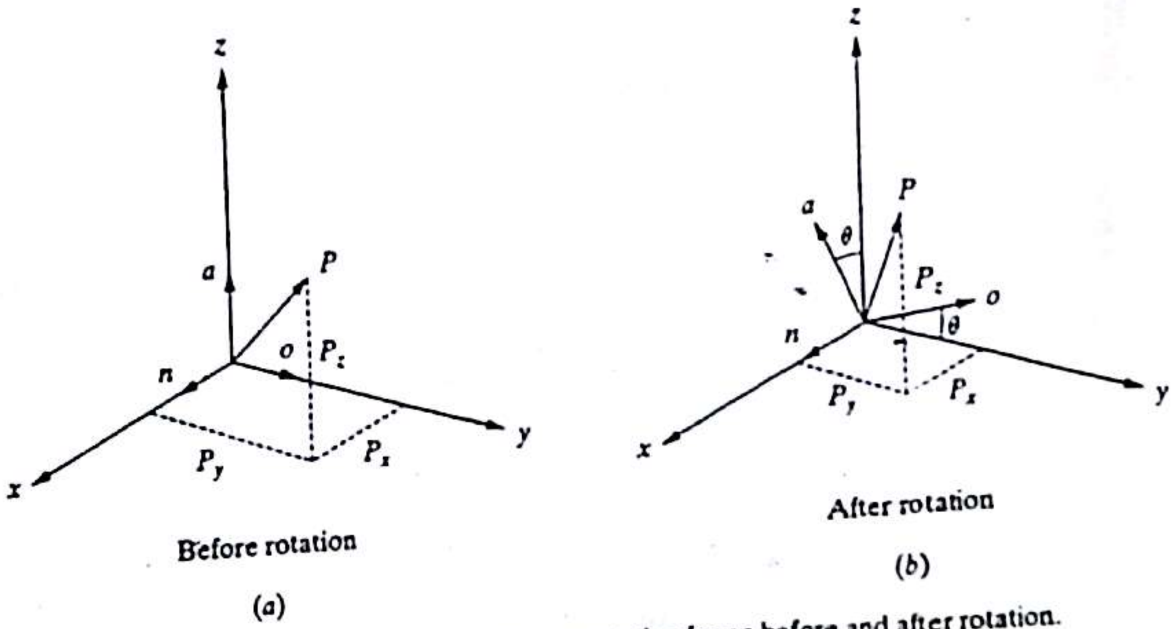

Figure 2.10 Coordinates of a point in a rotating frame before and after rotation.

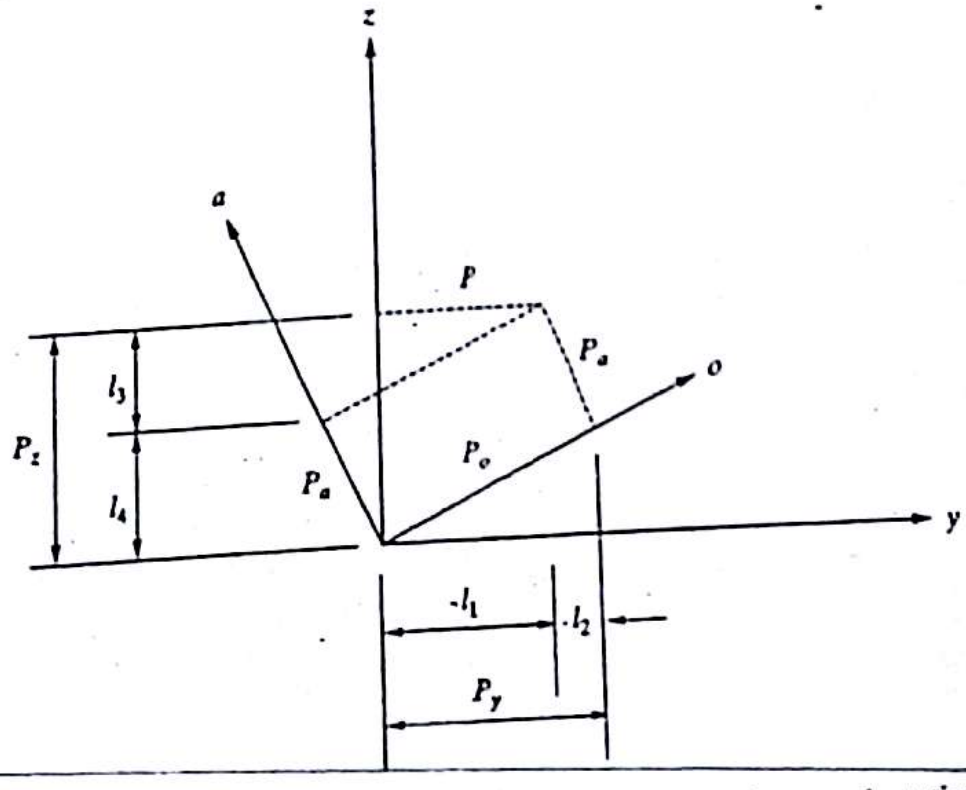

Figure 2.11 Coordinates of a point relative to the reference frame and rotating frame as viewed from the x-axis.

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and P_z, while its coordinates relative to the rotating frame (to which the point is attached) remain as P_n, P_o, and P_a.

From Figure 2.11, you will see that the value of P_x does not change as the frame rotates about the x-axis, but the values of P_y and P_z do change. Please verify that

$$\begin{aligned} P_x &= P_n, \\ P_y &= l_1 - l_2 = P_o \cos\theta - P_a \sin\theta, \\ P_z &= l_3 + l_4 = P_o \sin\theta + P_a \cos\theta, \end{aligned} \tag{2.16}$$

which is in matrix form

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} P_n \\ P_o \\ P_a \end{bmatrix}. \tag{2.17}$$

This means that the coordinates of the point (or vector) P in the rotated frame must be pre-multiplied by the rotation matrix, as shown, to get the coordinates in the reference frame. This rotation matrix is only for a pure rotation about the x-axis of the reference frame and is denoted as

$$P_{xyz} = \text{Rot}(x,\theta) \times P_{noa}. \tag{2.18}$$

Please also notice that the first column of the rotation matrix in Equation (2.16), which expresses the location relative to the x-axis, has 1,0,0 values, indicating that the coordinate along the x-axis has not changed.

Desiring to simplify writing of these matrices, it is customary to designate $C\theta$ to denote $\cos\theta$ and $S\,\theta$ to denote $\sin\theta$. Thus, the rotation matrix may be also written as

$$\text{Rot}(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix}. \tag{2.19}$$

You may want to do the same for the rotation of a frame about the y- and z-axes of the reference frame. Please verify that the results are

$$\text{Rot}(y,\theta) = \begin{bmatrix} C\theta & 0 & S\theta \\ 0 & 1 & 0 \\ -S\theta & 0 & C\theta \end{bmatrix} \quad \text{and} \quad \text{Rot}(z,\theta) = \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.20}$$

Equation (2.18) can also be written in a conventional form, which assists in easily following the relationship between different frames. Denoting the transformation as ${}^{U}T_{R}$ (and reading it as the transformation of frame R relative to frame U (for Universe)), denoting P_{noa} as ${}^{R}P$ (P relative to frame R), and denoting P_{xyz} as ${}^{U}P$ (P relative to frame U), Equation (2.18) simplifies to

$${}^{U}P = {}^{U}T_{R} \times {}^{R}P. \tag{2.21}$$

As you notice, canceling the R's gives the coordinates of point P relative to U. The same notation will be used throughout this book to relate to multiple transformations.

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Example 2.5

A point $P\ (2,3,4)^T$ is attached to a rotating frame. The frame rotates 90° about the x-axis of the reference frame. Find the coordinates of the point relative to the reference frame after the rotation, and verify the result graphically.

Solution Of course, since the point is attached to the rotating frame, the coordinates of the point relative to the rotating frame remain the same after the rotation. The coordinates of the point relative to the reference frame will be

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix} \cdot \begin{bmatrix} P_n \\ P_o \\ P_a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}.$$

As you notice in Figure 2.12, the coordinates of point P relative to the reference frame after rotation are 2, −4, 3, as obtained by the preceding transformation.

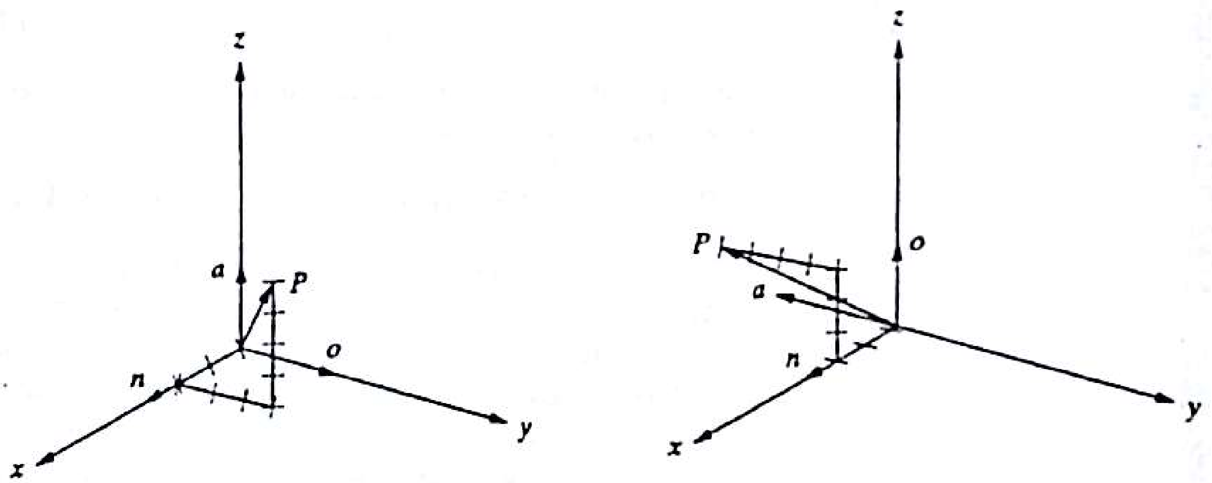

Figure 2.12 Rotation of a frame relative to the reference frame.

2.5.3 Representation of Combined Transformations

Combined transformations consist of a number of successive translations and rotations about the fixed reference frame axes or the moving current frame axes. Any transformation can be resolved into a set of translations and rotations in a particular order. For example, one may rotate a frame about the x-axis, translate about the x-, y-, and z-axes, and then rotate about the y-axis in order to accomplish the transformation that is needed. As we will see later, this order is very important, and if the order of two successive transformations is changed, the result may be completely different.

To see how combined transformations are handled, let's assume that a frame $(\bar{n},\bar{o},\bar{a})$ is subjected to the following three successive transformations relative to the reference frame (x,y,z):

(1) Rotation of α degrees about the x-axis,

(2) Followed by a translation of $[l_1,l_2,l_3]$ (relative to the x-, y-, and z-axes, respectively),

(3) Followed by a rotation of β degrees about the y-axis

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Also, let's say that a point P_{noa} is attached to the rotating frame at the origin of the reference frame. As the frame $(\bar{n},\bar{o},\bar{a})$ rotates or translates relative to the reference frame, the point P within the frame moves as well, and the coordinates of the point relative to the reference frame change. After the first transformation, as we saw in the previous section, the coordinates of point P relative to the reference frame can be calculated by

$$P_{1,xyz} = \mathrm{Rot}(x,\alpha) \times P_{noa}, \tag{2.22}$$

where $P_{1,xyz}$ is the coordinates of the point after the first transformation relative to the reference frame. The coordinates of the point relative to the reference frame at the conclusion of the second transformation will be:

$$P_{2,xyz} = \mathrm{Trans}(l_1,l_2,l_3) \times P_{1,xyz} = \mathrm{Trans}(l_1,l_2,l_3) \times \mathrm{Rot}(x,\alpha) \times P_{noa}.$$

Similarly, after the third transformation, the coordinates of the point relative to the reference frame will be

$$P_{xyz} = P_{3,xyz} = \mathrm{Rot}(y,\beta) \times P_{2,xyz} = \mathrm{Rot}(y,\beta) \times \mathrm{Trans}(l_1,l_2,l_3) \times \mathrm{Rot}(x,\alpha) \times P_{noa}$$

As you see, the coordinates of the point relative to the reference frame at the conclusion of each transformation is found by premultiplying the coordinates of the point by each transformation matrix. Of course, as shown in Appendix A, the order of matrices cannot be changed. Please also notice that for each transformation relative to the reference frame, the matrix is premultiplied. Thus, the order of matrices *written* is the opposite of the order of transformations performed.

Example 2.6

A point $P(7,3,2)^T$ is attached to a frame $(\bar{n},\bar{o},\bar{a})$ and is subjected to the transformations described next. Find the coordinates of the point relative to the reference frame at the conclusion of transformations.

(1) Rotation of 90° about the z-axis,
(2) Followed by a rotation of 90° about the y-axis,
(3) Followed by a translation of $[4, -3, 7]$.

Solution The matrix equation representing the transformation is

$$P_{xyz} = \mathrm{Trans}(4,-3,7)\mathrm{Rot}(y,90)\mathrm{Rot}(z,90)P_{noa} =$$

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ 1 \end{bmatrix}$$

As you notice, the first transformation of 90° about the z-axis rotates the $(\bar{n},\bar{o},\bar{a})$ frame as shown in Figure 2.13, followed by the second rotation about the y-axis, followed by the translation relative to the reference frame x-, y-, z-axes. The point P in the frame can then be found relative to the $\bar{n}$-, $\bar{o}$-, $\bar{a}$-axes, as shown. The final coordinates of the point can be traced on the x-, y-, z-axes to be $4 + 2 = 6$, $-3 + 7 = 4$, and $7 + 3 = 10$. Please make sure that you follow this graphically.

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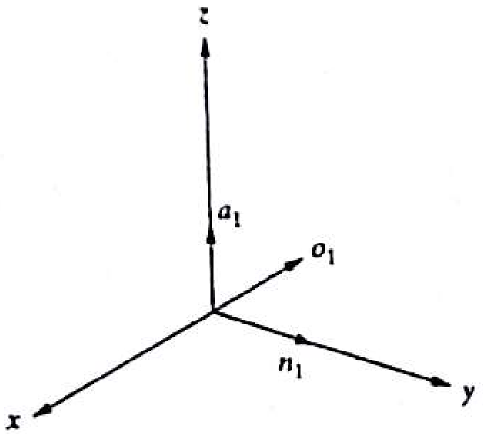
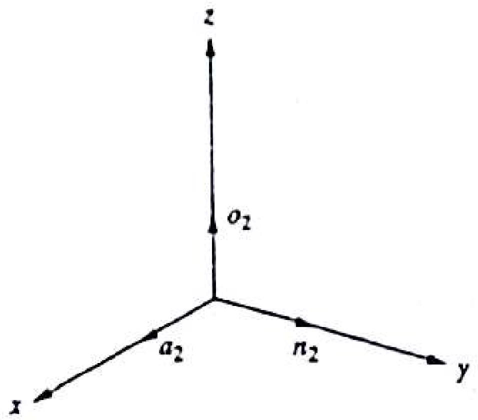
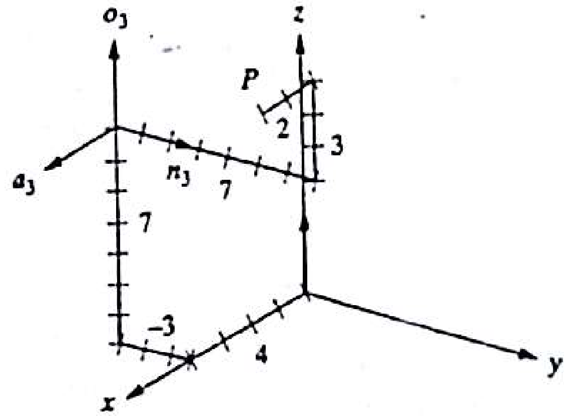


After the first transformation



After the second transformation



After the third transformation

Figure 2.13 Effects of three successive transformations.

Example 2.7

In this case, assume that the same point $P(7,3,2)^T$, attached to a frame $(\bar{n},\bar{o},\bar{a})$, is subjected to the same transformations, but that the transformations are performed in a different order, as shown. Find the coordinates of the point relative to the reference frame at the conclusion of transformations:

(1) A rotation of 90° about the z-axis,
(2) Followed by a translation of [4, −3, 7],
(3) Followed by a rotation of 90° about the y-axis.

Solution The matrix equation representing the transformation is

$$P_{xyz} = \mathrm{Rot}(y,90)\mathrm{Trans}(4,-3,7)\mathrm{Rot}(z,90)P_{noa} =$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ -1 \\ 1 \end{bmatrix}$$

As you see, although the transformations are exactly the same as in Example 2.6, since the order of transformations is changed, the final coordinates of the point

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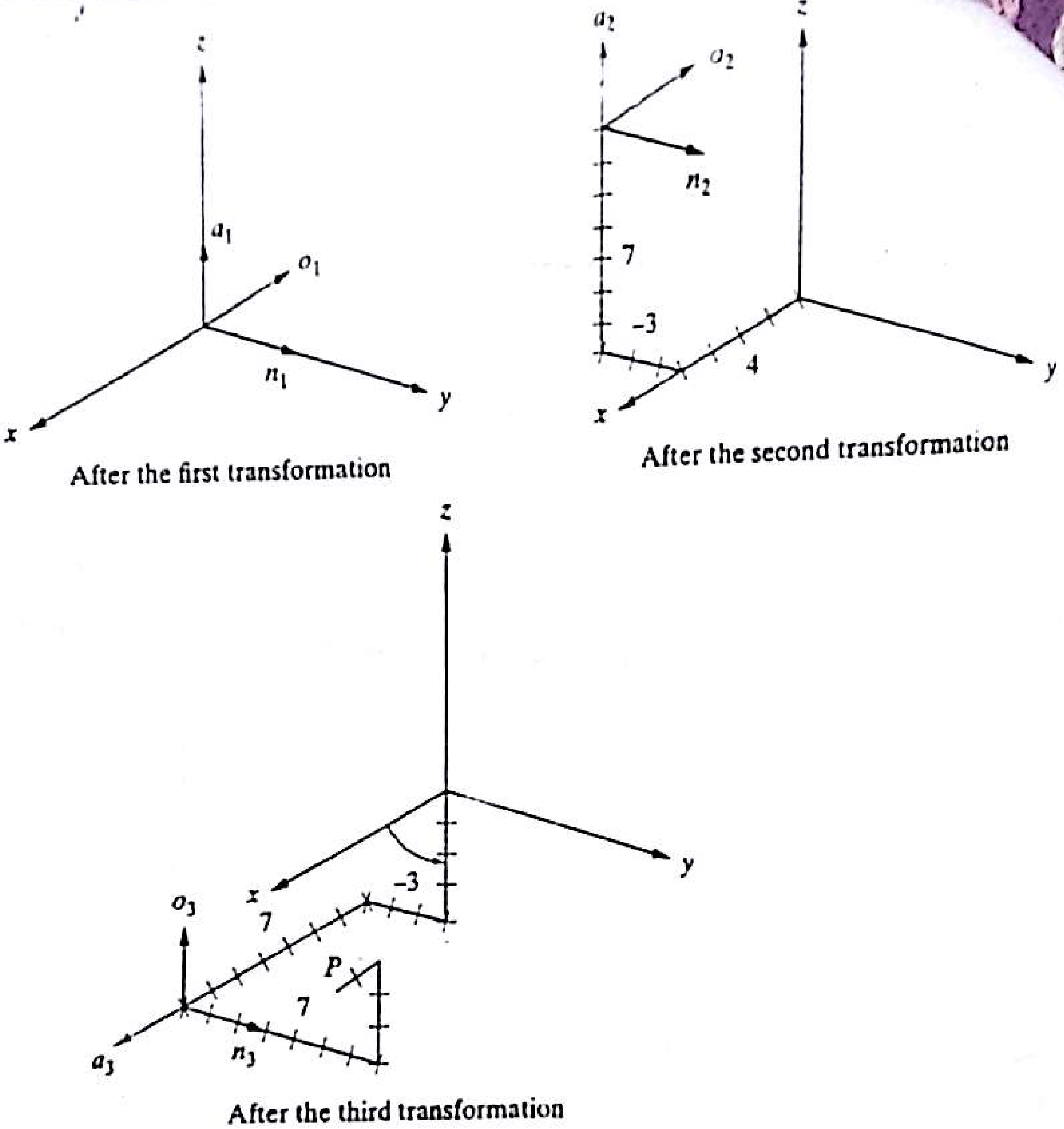

Figure 2.14 Changing the order of transformations will change the final result.

are completely different from the previous example. This can clearly be demonstrated graphically as in Figure 2.14. In this case you see that although the first transformation creates exactly the same change in the frame, the second transformation's result is very different, as the translation relative to the reference frame axes will move the rotating $(\bar{n},\bar{o},\bar{a})$ frame outwardly. As a result of the third transformation, this frame will rotate about the reference frame y-axis, thus rotating downwardly. The location of point P, attached to the frame is also shown.

Please verify that the coordinates of this point relative to the reference frame are $7 + 2 = 9$, $-3 + 7 = 4$, and $-4 + 3 = -1$, which is the same as the analytical result.

2.5.4 Transformations Relative to the Rotating Frame

All transformations we have discussed so far have been relative to the fixed reference frame. This means that all translations, rotations, and distances (except for the location of a point relative to the moving frame) have been measured relative to the reference frame axes. However, it is in fact possible to make transformations relative to the axes of a moving or current frame. This means that, for example, a rota-

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tion of 90° may be made relative to the $\bar{n}$-axis of the moving frame (also referred to as the current frame), and not the x-axis of the reference frame. To calculate the changes in the coordinates of a point attached to the current frame relative to the reference frame, the transformation matrix is postmultiplied instead. Please note that since the position of a point or an object attached to a moving frame is always measured relative to that moving frame, the position matrix describing the point or object is also always postmultiplied.

Example 2.8

Assume that the same point as in Example 2.7 is now subjected to the same transformations, but all relative to the current moving frame, as listed next. Find the coordinates of the point relative to the reference frame after transformations are completed:

(1) A rotation of 90° about the $\bar{a}$-axis,
(2) Then a translation of [4, -3, 7] along $\bar{n},\bar{o},\bar{a}$
(3) Followed by a rotation of 90° about the $\bar{o}$-axis.

Solution In this case, since the transformations are made relative to the current frame, each transformation matrix is post multiplied. As a result, the equation representing the coordinates is

$$P_{xyz} = \mathrm{Rot}(\bar{a},90)\mathrm{Trans}(4,-3,7)\mathrm{Rot}(\bar{o},90)P_{noa} =$$

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 1 \end{bmatrix}.$$

As expected, the result is completely different from the other cases, both because the transformations are made relative to the current frame and because the order of the matrices is now different. Figure 2.15 shows the results graphically. Please notice how the transformations are accomplished relative to the current frames.

Please notice how the 7,3,2 coordinates of point P in the current frame will result in 0,6,0 coordinates relative to the reference frame.

Example 2.9

A frame B was rotated about the x-axis 90°; it was then translated about the current a-axis 3 inches before being rotated about the z-axis 90°. Finally, it was translated about current o-axis 5 inches.

(a) Write an equation describing the motions.
(b) Find the final location of a point $P(1,5,4)$ attached to the frame relative to the reference frame.

Solution In this case, motions alternate relative to the reference frame and current frame.

(a) Pre- or postmultiplying each motion's matrix accordingly, we get

$$^{U}T_{B} = \mathrm{Rot}(z,90)\mathrm{Rot}(n,90)\mathrm{Trans}(0,0,3)\mathrm{Trans}(0,5,0).$$

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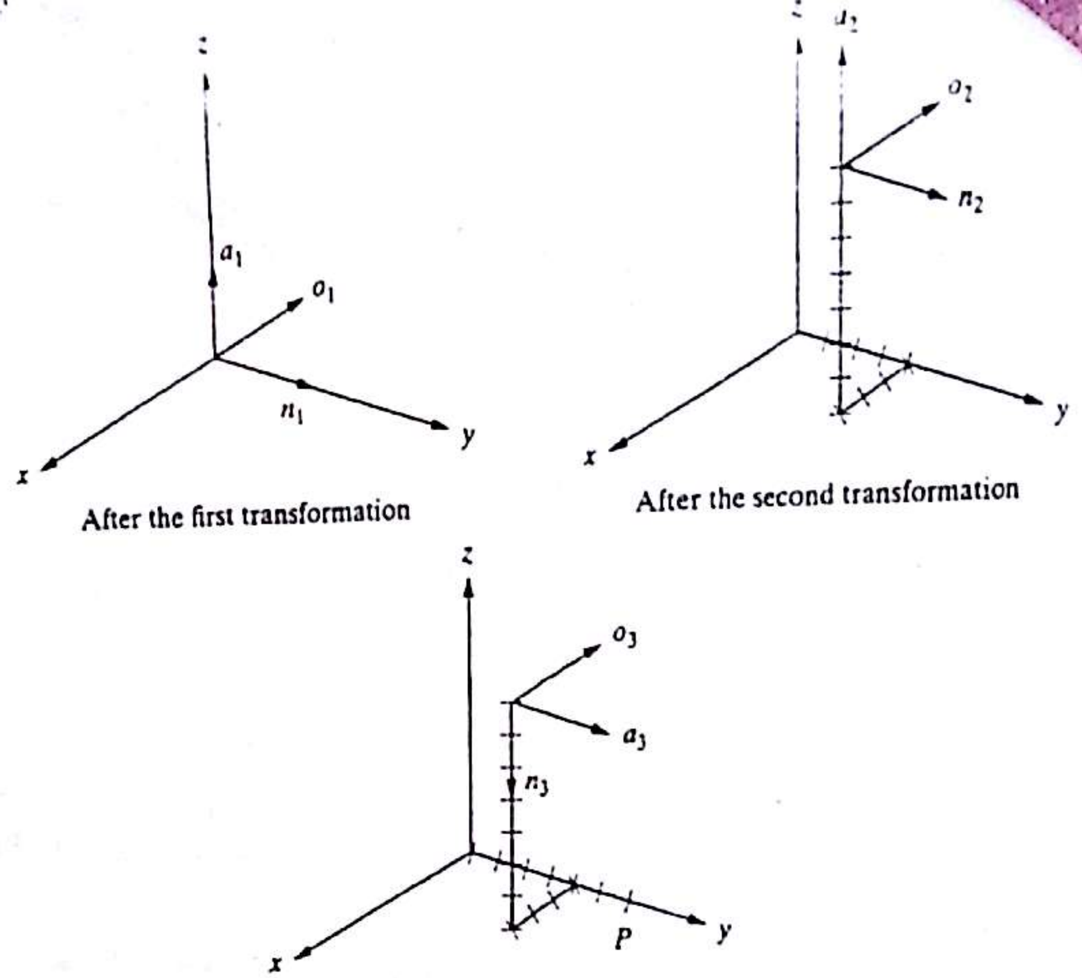

After the first transformation

After the second transformation

After the third transformation

Figure 2.15 Transformations relative to the current frames.

(b) Substituting the matrices and multiplying them, we get

$$
{}^{U}P = {}^{U}T_{B} \times {}^{B}P = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & 0 & 1 & 7 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

2.6 INVERSE OF TRANSFORMATION MATRICES

As was mentioned earlier, there are many situations where the inverse of a matrix will be needed in robotic analysis. One situation where transformation matrices may be involved can be seen in the next example. Suppose that the robot in Figure 2.16 is to be moved towards part P in order to drill a hole in the part. The robot's base position relative to the reference frame U is described by a frame R, the robot's hand is described by frame H, and the end effector (let's say the end of the drill bit that will be used to drill the hole) is described by frame E. The part's position is also described by frame P. The location of the point where the hole will be

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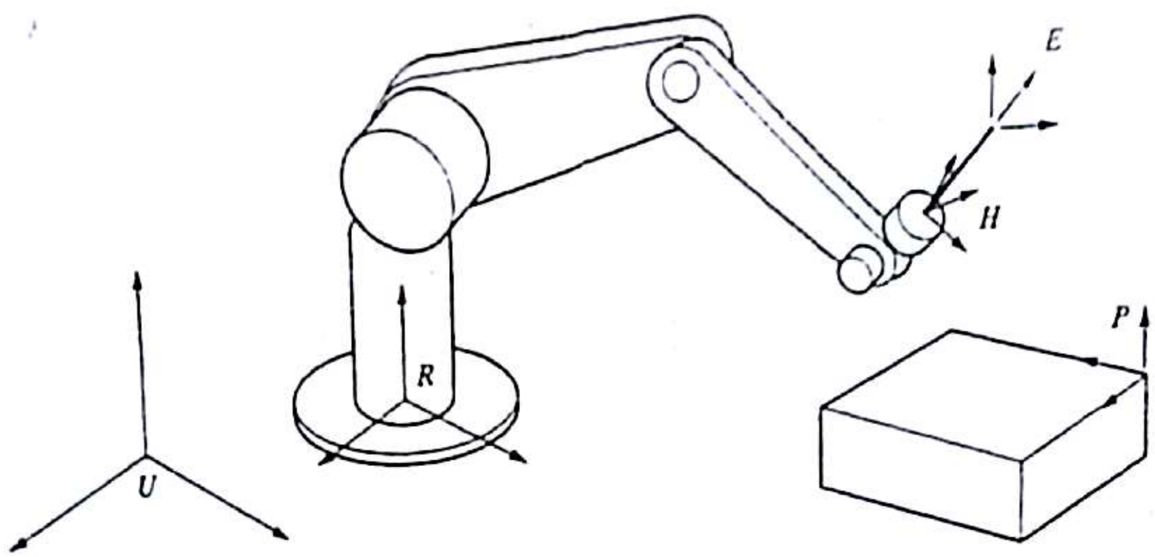

Figure 2.16 The Universe, robot, hand, part, and end effector frames.

drilled can be related to the reference frame U through two independent paths: one through the part and one through the robot. Thus, we can write

$$^{U}T_E = {}^{U}T_R\,{}^{R}T_H\,{}^{H}T_E = {}^{U}T_P\,{}^{P}T_E, \tag{2.24}$$

which means that the location of point E on the part can be achieved by moving from U to P, and from P to E, or it can alternatively be achieved by a transformations from U to R, from R to H, and from H to E.

In reality, the transformation ${}^{U}T_R$, or the transformation of frame R relative to the U (Universe reference frame) is known, since the location of the robot's base must be known in any setup. For example, if a robot is installed in a work cell, the location of the robot's base will be known, since it is bolted to a table. Even if the robot is mobile or attached to a conveyor belt, its location at any instant will be known, since a controller must be following the position of the robot's base at all times. The ${}^{H}T_E$, or the transformation of the end effector relative to the robot's hand is also known, since any tool used at the end effector is a known tool, and its dimensions and configuration are known. ${}^{U}T_P$, or the transformation of the part relative to the universe, is also known, since we must know where the part is located if we are to drill a hole in it. This location is known by putting the part in a jig, through the use of a camera and vision system, through the use of a conveyor belt and sensors, or other similar devices. ${}^{P}T_E$ is also known, since we need to know where the hole is to be drilled on the part. Consequently, the only unknown transformation is ${}^{R}T_H$, or the transformation of the robot's hand relative to the robot's base. This means that we need to find out what the robot's joint variables (the angle of the revolute joints and the length of the prismatic joints of the robot) must be in order to place the end effector at the hole for drilling. As you see, it is necessary to calculate this transformation, which will tell us what needs to be accomplished. The transformation will later be used to actually solve for joint angles and link lengths.

To calculate this matrix, unlike in an algebraic equation, we cannot simply divide the right side by the left side of the equation. We need to pre- or postmultiply by inverses of appropriate matrices to eliminate them. As a result, we will have

$$({}^{U}T_R)^{-1}({}^{U}T_R\,{}^{R}T_H\,{}^{H}T_E)({}^{H}T_E)^{-1} = ({}^{U}T_R)^{-1}\,({}^{U}T_P\,{}^{P}T_E)\,({}^{H}T_E)^{-1}, \tag{2.25}$$

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or since $({}^{U}T_R)^{-1}({}^{U}T_R)=1$, and $({}^{H}T_E)({}^{H}T_E)^{-1} = 1$, the left side of Equation simplifies to ${}^{R}T_H$, and we get

$$ {}^{R}T_H = {}^{U}T_R^{-1}\,{}^{U}T_P\,{}^{P}T_E\,{}^{H}T_E^{-1}. \tag{2.26}$$

We can check the accuracy of this equation by realizing that $({}^{H}T_E)^{-1}$ is the same as ${}^{E}T_H$. Thus, the equation can be rewritten as

$$ {}^{R}T_H = {}^{U}T_R^{-1}\,{}^{U}T_P\,{}^{P}T_E\,{}^{H}T_E^{-1} = {}^{R}T_U\,{}^{U}T_P\,{}^{P}T_E\,{}^{E}T_H = {}^{R}T_H. \tag{2.27}$$

It is now clear that we need to be able to calculate the inverse of transformation matrices for robot kinematic analysis as well.

To see what transpires, let's calculate the inverse of a simple rotation matrix about the x-axis. Please review the process for calculation of square matrices in Appendix A. The rotation matrix about the x-axis is

$$\text{Rot}(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix}. \tag{2.28}$$

Please recall that the following steps must be taken to calculate the inverse of a matrix:

- Calculate the determinant of the matrix.
- Transpose the matrix.
- Replace each element of the transposed matrix by its own minor (adjoint matrix).
- Divide the converted matrix by the determinant.

Applying the process to the rotation matrix, we get

$$\Delta = 1(C^2\theta + S^2\theta) + 0 = 1,$$

$$\text{Rot}(x,\theta)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & S\theta \\ 0 & -S\theta & C\theta \end{bmatrix}.$$

Now calculate each minor. As an example, the minor for the 2,2 element will be $C\theta - 0 = C\theta$, the minor for 1,1 element will be $C^2\theta + S^2\theta = 1$, etc. As you notice, the minor for each element will be the same as the element itself. Thus,

$$\text{Rot}(x,\theta)^T_{\text{minor}} = \text{Rot}(x,\theta)^T.$$

Since the determinant of the original rotation matrix is unity, dividing the $\text{Rot}(x,\theta)^T_{\text{minor}}$ matrix by the determinant will yield the same result. Thus, the inverse of a rotation matrix about the x-axis is the same as its transpose, or

$$\text{Rot}(x,\theta)^{-1} = \text{Rot}(x,\theta)^T. \tag{2.29}$$

Of course, you would get the same result with the second method mentioned in Appendix A. A matrix with this characteristic is called a unitary matrix. It turns

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out that all rotation matrices are unitary matrices. Thus, all we need to do to calculate the inverse of a rotation matrix is to transpose it. Please verify that rotation matrices about the y- and z-axes are also unitary in nature.

Please beware that only rotation matrices are unitary. If a matrix is not a simple rotation matrix, it may not be unitary.

The preceding result is also true only for a simple 3×3 rotation matrix without representation of a location. For a homogenous 4×4 transformation matrix, it can be shown that the matrix inverse can be written by dividing the matrix into two portions: The rotation portion of the matrix can be simply transposed, as it is still unitary. The position portion of the homogeneous matrix is the negative of the dot product of vector $\bar{P}$ with each of $\bar{n}, \bar{o}, \bar{a}$ the vectors as follows:

$$T = \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T^{-1} = \begin{bmatrix} n_x & n_y & n_z & -\bar{P}\cdot\bar{n} \\ o_x & o_y & o_z & -\bar{P}\cdot\bar{o} \\ a_x & a_y & a_z & -\bar{P}\cdot\bar{a} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.30}$$

As shown, the rotation portion of the matrix is simply transposed, the position portion is replaced by the negative of the dot products, and the last row (scale factors) are not affected. This is very helpful, since we will need to calculate inverses of transformation matrices, but directly calculating inverse of a 4×4 matrix is a lengthy process.

Example 2.10

Calculate the matrix representing $\mathrm{Rot}(x, 40°)^{-1}$.

Solution The matrix representing a 40° rotation about the x-axis is

$$\mathrm{Rot}(X, 40°) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.766 & -0.643 & 0 \\ 0 & 0.643 & 0.766 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The inverse of this matrix is

$$\mathrm{Rot}(X, 40°)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.766 & 0.643 & 0 \\ 0 & -0.643 & 0.766 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

As you notice, since the position vector of the matrix is zero, its dot product with the $\bar{n}, \bar{o}, \bar{a}$ vectors is also zero.

Example 2.11

Calculate the inverse of the following transformation matrix:

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$$T = \begin{bmatrix} .5 & 0 & .866 & 3 \\ .866 & 0 & -5 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution Based on the preceding calculations, the inverse of the transformation is

$$T^{-1} = \begin{bmatrix} 0.5 & 0.866 & 0 & -(3 \times 0.5 + 2 \times 0.866 + 5 \times 0) \\ 0 & 0 & 1 & -(3 \times 0 + 2 \times 0 + 5 \times 1) \\ 0.866 & -0.5 & 0 & -(3 \times 0.866 + 2 \times -0.5 + 5 \times 0) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.866 & 0 & -3.23) \\ 0 & 0 & 1 & -5 \\ 0.866 & -0.5 & 0 & -1.598) \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

You may want to verify that $T\,T^{-1}$ will be an identity matrix.

Example 2.12

In a robotic setup, a camera is attached to the fifth link of a robot with six degrees of freedom. The camera observes an object and determines its frame relative to the camera's frame. Using the following information, determine the necessary motion the end effector has to make to get to the object:

$$^{5}T_{cam} = \begin{bmatrix} 0 & 0 & -1 & 3 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad ^{5}T_{H} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0. & 0 & 1 \end{bmatrix}$$

$$^{cam}T_{obj} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad ^{H}T_{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution Referring to Equation (2.24), we can write a similar equation that relates the different transformations and frames together as

$$^{R}T_{5} \times {}^{5}T_{H} \times {}^{H}T_{E} \times {}^{E}T_{obj} = {}^{R}T_{5} \times {}^{5}T_{cam} \times {}^{cam}T_{obj}.$$

Since $^{R}T_{5}$ appears on both sides of the equation, we can simply neglect it. All other matrices, with the exception of $^{E}T_{obj}$, are known. Therefore,

$$^{E}T_{obj} = {}^{H}T_{E}^{-1} \times {}^{5}T_{H}^{-1} \times {}^{5}T_{cam} \times {}^{cam}T_{obj} = {}^{E}T_{H} \times {}^{H}T_{5} \times {}^{5}T_{cam} \times {}^{cam}T_{obj},$$

where $$^{H}T_{E}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad ^{5}T_{H}^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

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Substituting the matrices and the inverses in the foregoing equation results in:

$$
{}^{E}T_{obj} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 3 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

$$
\text{or} \quad {}^{E}T_{obj} = \begin{bmatrix} -1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

2.7 FORWARD AND INVERSE KINEMATICS OF ROBOTS

Suppose that we have a robot whose configuration is known. This means that all the link lengths and joint angles of the robot are known. Calculating the position and orientation of the hand of the robot is called forward kinematic analysis. In other words, if all robot joint variables are known, using forward kinematic equations, one can calculate where the robot is at any instant. However, if one desires to place the hand of the robot at a desired location and orientation, one has to know how much each link length or joint angle of the robot must be such that at those values, the hand will be at the desired position and orientation. This is called inverse kinematic analysis. This means that instead of substituting the known robot variables in the forward kinematic equations of the robot, we need to find the inverse of these equations to enable us to find the necessary joint values to place the robot at the desired location and orientation. In reality, what is more important is the inverse kinematic equations, since the robot controller will calculate the joint values using these equations, and it will run the robot to the desired position and orientation. We will first develop the forward kinematic equations of robots, and then, using these equations, we will calculate the inverse kinematic equations.

For forward kinematics, we will have to develop a set of equations that relate to the particular configuration of a robot (the way it is put together) such that by substituting the joint and link variables in these equations, we may calculate the position and orientation of the robot. These equations will then be used to derive the inverse kinematic equations.

You may recall from Chapter 1 that in order to position and orientate a rigid body in space, we attach a frame to the body and then describe the position of the origin of the frame and the orientation of its three axes. This requires a total of six degrees of freedom, or, alternatively, six pieces of information, to completely define the position and orientation of the body. Here too, if we want to define or find the position and orientation of the hand of the robot in space, we will attach a frame to it and will define the position and orientation of the hand frame of the robot. The means by which the robot accomplishes this determines what the forward kinematic equations are. In other words, depending on the configuration of the links and joints

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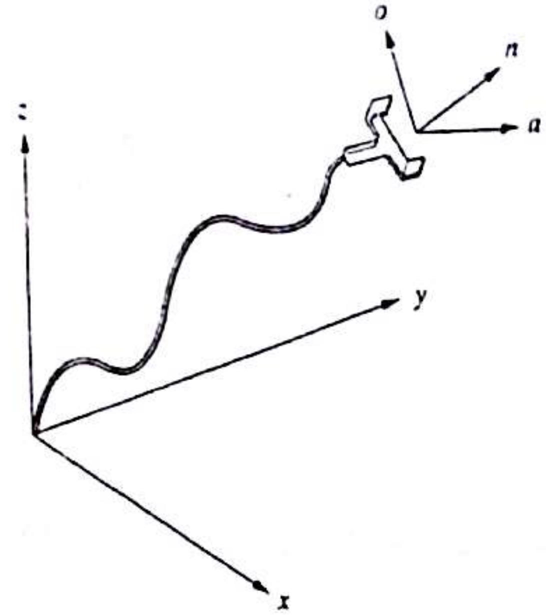

Figure 2.17 The hand frame of the robot relative to the reference frame.

of the robot, a particular set of equations will relate the hand frame of the robot to the reference frame. Figure 2.17 shows a hand frame, the reference frame, and their relative position and orientation. The undefined connection between the two frames is related to the configuration of the robot. Of course, there may be many different possibilities for this configuration, and we will later see how we can develop the equations relating the two frames, depending on the robot configuration.

In order to simplify the process, we will analyze the position and orientation issues separately. First, we will develop the position equations; then we will do the same for orientation. Later, we will combine the two for a complete set of equations. Finally, we will see about the use of the Denavit–Hartenberg representation, which can model any robot configuration.

2.7.1 Forward and Inverse Kinematic Equations for Position

In this section, we will study the forward and inverse kinematic equations for position. As was mentioned earlier, the position of the origin of a frame attached to a rigid body has three degrees of freedom and thus can be completely defined by three pieces of information. As a result, the position of the origin of the frame may be defined in any customary coordinates, and the positioning of the robot may be accomplished through motions related to any customary coordinate frames. For example, one may position a point in space based on Cartesian coordinates, which means that there will be three linear movements relative to the $\bar{x}$-, $\bar{y}$-, and $\bar{z}$-axes. Alternatively, it may be accomplished through spherical coordinates, which means that there will be one linear motion and two rotary motions to accomplish the position. The following possibilities will be discussed:

(a) Cartesian (gantry, rectangular) coordinates.
~~**(b)** Cylindrical coordinates.~~
(c) Spherical coordinates.
(d) Articulated (anthropomorphic, or all-revolute) coordinates.

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2.7.1(a) Cartesian (Gantry, Rectangular) Coordinates

In this case, there will be three linear movements along the three major x-,y-,z-axes. In this type of a robot, all actuators are linear (such as a hydraulic ram or a linear power screw), and the positioning of the hand of the robot is accomplished by moving the three linear joints along the three axes (Figure 2.18). A gantry robot is basically a Cartesian coordinate robot, except that the robot is usually attached to a rectangular frame upside down. The IBM 7565 robot is a gantry Cartesian robot.

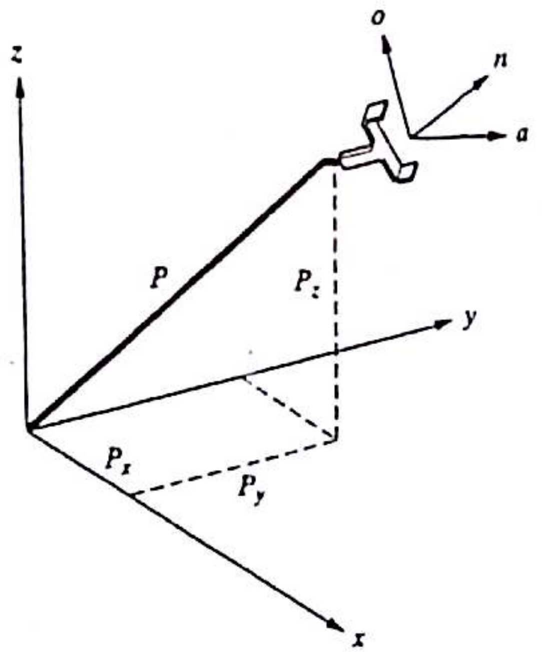

Figure 2.18 Cartesian coordinates.

Of course, since there are no rotations, the transformation matrix representing this motion to point P is a simple translation transformation matrix, as shown next. Please note that here we are only referring to the position of the origin of the frame, and not its orientation. The transformation matrix representing the forward kinematic equation of the position of the hand of the robot in a Cartesian coordinate system will be

$$ {}^{R}T_{P} = T_{\text{cart}} = \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y \\ 0 & 0 & 1 & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.31} $$

where ${}^{R}T_{P}$ is the transformation between the reference frame and the origin of the hand P, and T_{cart} denotes Cartesian transformation matrix. For inverse kinematic solution, simply set the desired position equal to P.

Example 2.13

It is desired to position the origin of the hand frame of a Cartesian robot at point $P = [3,4,7]^T$. Calculate the necessary Cartesian coordinate motions that need to be made.

Solution Setting the forward kinematic equation, represented by the ${}^{R}T_{P}$ matrix of Equation (2.31), with the desired position will yield the following result:

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$$
{}^{R}T_{P} = \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y \\ 0 & 0 & 1 & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{or} \quad P_x = 3, P_y = 4, P_z = 7.
$$

2.7.1(b) Cylindrical Coordinates

A cylindrical coordinate system includes two linear translations and one rotation. The sequence is a translation of r along the x-axis, a rotation of α about the z-axis, and a translation of l along the z-axis, as shown in Figure 2.19. Since these transformations are all relative to the axes of the universe reference frame, the total transformation caused by the three transformations that relates the origin of the hand frame to the reference frame can be found by premultiplying by each matrix as follows:

$$
{}^{R}T_{P} = T_{\text{cyl}}(r,\alpha,l) = \text{Trans}(0,0,l)\text{Rot}(z,\alpha)\text{Trans}(r,0,0), \tag{2.32}
$$

$$
{}^{R}T_{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} C\alpha & -S\alpha & 0 & 0 \\ S\alpha & C\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^{R}T_{P} = T_{\text{cyl}} = \begin{bmatrix} C\alpha & -S\alpha & 0 & rC\alpha \\ S\alpha & C\alpha & 0 & rS\alpha \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.33}
$$

The first three columns represent the orientation of the frame after this series of transformations. However, at this point, we are only interested in the position of

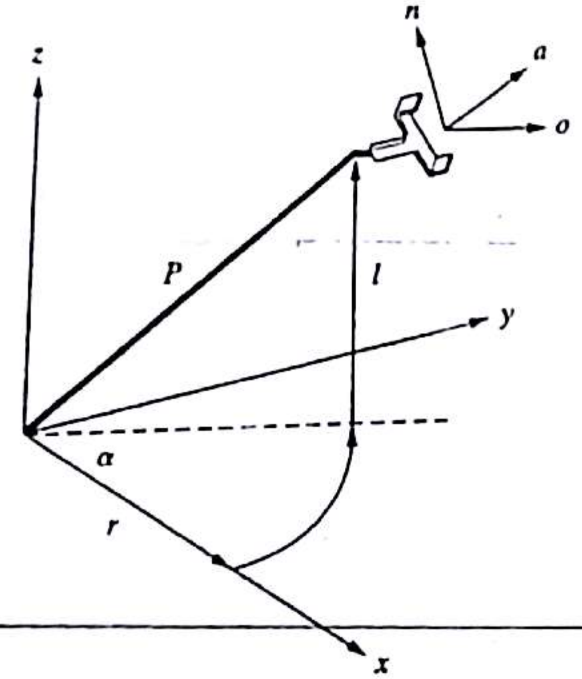

Figure 2.19 Cylindrical Coordinates.

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the origin of the frame, or the last column. Obviously, in cylindrical coordinate movements, due to the rotation α about the z-axis, the orientation of the moving frame will change. This orientation change will be discussed later.

One may, in fact, "unrotate" the frame back to being parallel to the original reference frame by rotating the $\bar{n},\bar{o},\bar{a}$ frame about the a-axis an angle of $-a$, which is equivalent of post-multiplying the cylindrical coordinate matrix by a rotation matrix of Rot$(\bar{a}, -\alpha)$. As a result, the frame will be at the same location, but will be parallel to the reference frame again, as follows:

$$T_{cyl} \times \mathrm{Rot}(z, -\alpha) = \begin{bmatrix} C\alpha & -S\alpha & 0 & rC\alpha \\ S\alpha & C\alpha & 0 & rS\alpha \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} C(-\alpha) & -S(-\alpha) & 0 & 0 \\ S(-\alpha) & C(-\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & rC\alpha \\ 0 & 1 & 0 & rS\alpha \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

As you can see, the location of the origin of the moving frame has not changed, but it was "unrotated" back to being parallel to the reference frame. Please notice that the last rotation was performed about the local a-axis in order to not cause any change in the location of the frame, but only in its orientation.

Example 2.14

Suppose that we desire to place the origin of the hand frame of a cylindrical robot at $[3,4,7]^T$. Calculate the joint variables of the robot.

Solution Setting the components of the location of the origin of the frame from the T_{cyl} matrix of Equation (2.33) to the desired values, we get

$$l = 7,$$
$$rC\alpha = 3,$$
$$rS\alpha = 4, \text{ and thus } \tan\alpha = \frac{4}{3} \text{ and } \alpha = 53.1^\circ.$$

Substituting α into either equation will yield $r = 5$. The final answer is

$$r = 5 \text{ units}, \ \alpha = 53.1^\circ, \text{ and } l = 7 \text{ units}.$$

Note: As discussed in Appendix A, it is necessary to ensure that the angles calculated in robot kinematics are in correct quadrants. In this example, you notice that since $rC\alpha$ and $rS\alpha$ are both positive and that the length r is always positive, that the sin and cos are also both positive. Thus, the angle α is in the first quadrant and is correctly 53.1°.

2.7.1(c) Spherical Coordinates

Spherical coordinate systems consist of one linear motion and two rotations. The sequence is a translation of r along the z-axis, a rotation of β about the y-axis, and a

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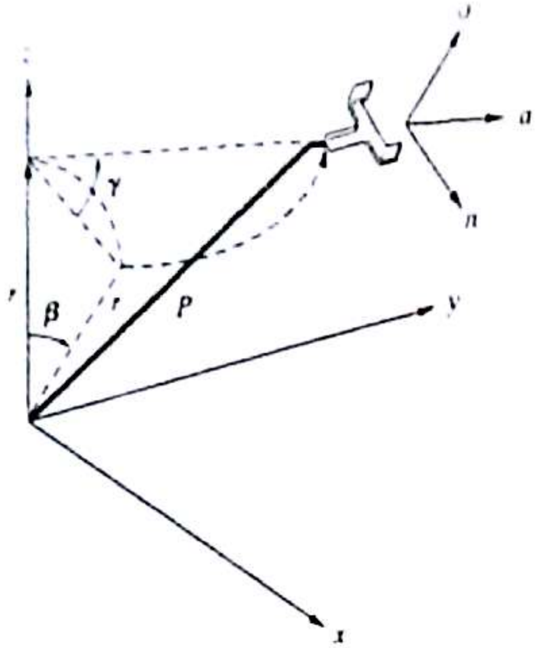

Figure 2.20 Spherical coordinates.

rotation of γ about the z-axis as shown in Figure 2.20. Since these transformations are all relative to the axes of the Universe reference frame, the total transformation caused by these three transformations, which relates the origin of the hand frame to the reference frame, can be found by premultiplying by each matrix as follows:

$$^{R}T_{P} = T_{sph}(r,\beta,\gamma) = \text{Rot}(z,\gamma)\text{Rot}(y,\beta)\text{Trans}(0,0,r) \tag{2.34}$$

$$^{R}T_{P} = \begin{bmatrix} C\gamma & -S\gamma & 0 & 0 \\ S\gamma & C\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} C\beta & 0 & S\beta & 0 \\ 0 & 1 & 0 & 0 \\ -S\beta & 0 & C\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{R}T_{P} = T_{sph} = \begin{bmatrix} C\beta \cdot C\gamma & -S\gamma & S\beta \cdot C\gamma & rS\beta \cdot C\gamma \\ C\beta \cdot S\gamma & C\gamma & S\beta \cdot S\gamma & rS\beta \cdot S\gamma \\ -S\beta & 0 & C\beta & rC\beta \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.35}$$

Once again, the first three columns represent the orientation of the frame after this series of transformations, while the last column is the position of the origin. We will further discuss the orientation part of the matrix later.

Here, too, one may "unrotate" the final frame to make it parallel to the reference frame. This exercise is left for you to find the correct sequence of movements to get the right answer.

The inverse kinematic equations for spherical coordinates are more complicated than the simple Cartesian or cylindrical coordinates, because the two angles β and γ are coupled. Let's see how this could be done through an example.

Example 2.15

Suppose that we now desire to place the origin of the hand of a spherical robot at $[3,4,7]^T$. Calculate the joint variables of the robot.

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Solution Setting the components of the location of the origin of the frame from T_{sph} matrix of Equation (2.35) to the desired values, we get

$$rS\beta C\gamma = 3,$$
$$rS\beta S\gamma = 4,$$
$$rC\beta = 7.$$

From the third equation, we determine that $C\beta$ is positive, but there is no such information about $S\beta$. Dividing the first two equations by each other, we get the following double results. Please notice that there are two possible solutions, because we do not know what the actual sign of $S\beta$ is. As a result, the following approach presents two possible solutions, and we will have to check the final results later to ensure that they are correct:

$$\begin{array}{llll}
\tan\gamma = \frac{4}{3} \quad \rightarrow & \gamma = 53.1^\circ, & \text{or} & 233.1^\circ, \\
\text{then} & S\gamma = 0.8, & \text{or} & -0.8, \\
\text{and} & C\gamma = 0.6, & \text{or} & -0.6, \\
\text{and} & rS\beta = \frac{3}{0.6} = 5, & \text{or} & -5, \\
\text{and since} & rC\beta = 7, \beta = 35.5^\circ, & \text{or} & -35.5^\circ, \\
\text{and} & r = 8.6. & &
\end{array}$$

You may check both answers and verify that they both satisfy all position equations. If you also follow these angles about the given axes in three dimensions you will get to the same point physically. However, you must notice that only one set of answers will also satisfy the orientation equations. In other words, the two foregoing answers will result in the same position, but at different orientations. Since we are not concerned with the orientation of the hand frame at this point, both position answers are correct. In fact, since we may not specify any orientation for a three-degree-of-freedom robot, we cannot determine which of the two answers relates to a specific orientation anyway.

2.7.1(d) Articulated Coordinates

Articulated coordinates consist of three rotations, as shown in Figure 2.21. We will develop the matrix representation for this later, when we discuss the Denavit-Hartenberg representation.

2.7.2 Forward and Inverse Kinematic Equations for Orientation

Suppose that the moving frame attached to the hand of the robot has already moved to a desired position, but is still parallel to the reference frame, or that it is in an ori-

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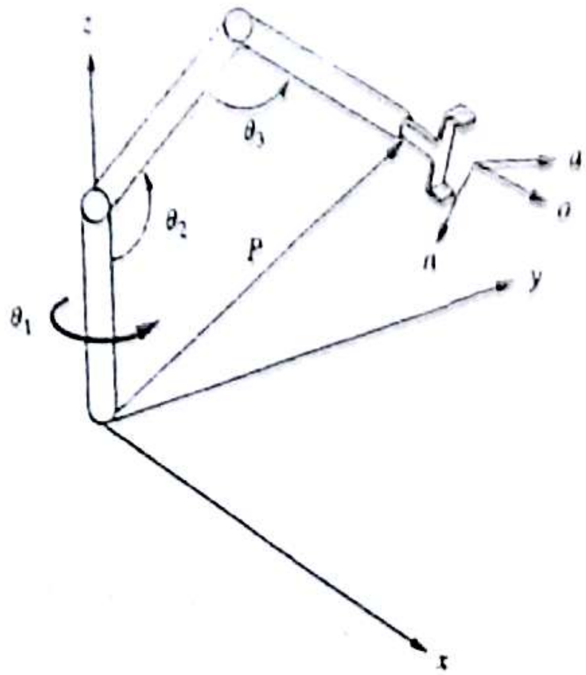

Figure 2.21 Articulated Coordinates

entation other than what is desired. The next step will be to rotate the frame appropriately in order to achieve a desired orientation without changing its position. The appropriate sequence of rotations depends on the design of the wrist of the robot,and the way the joints are assembled together. We will consider the following three common configurations:

(a) Roll, Pitch, Yaw (RPY) angles.
(b) Euler angles
(c) Articulated joints

2.7.2(a) Roll, Pitch, Yaw (RPY) Angles

This is a sequence of three rotations about current $\bar{a},\bar{o},\bar{n}$ axes respectively, which will orientate the hand of the robot to a desired orientation. The assumption here is that the current frame is parallel to the reference frame and thus its orientation is the same as the reference frame before the RPY movements. If the current moving frame is not parallel to the reference frame, then the final orientation of the robot's hand will be a combination of the previous orientation, postmultiplied by the RPY

It is very important to realize that since we do not want to cause any change in the position of the origin of the moving frame (we have already placed it at the desired location and only want to rotate it to the desired orientation), the movements relating to RPY rotations are relative to the current moving axes. Otherwise, as we saw before, the moving frame will change position. Thus, all matrices relating to the orientation change due to RPY (as well as other rotations) will be postmultiplied.

Referring to Figure 2.22, we see that the RPY sequence of rotations consists of the following:

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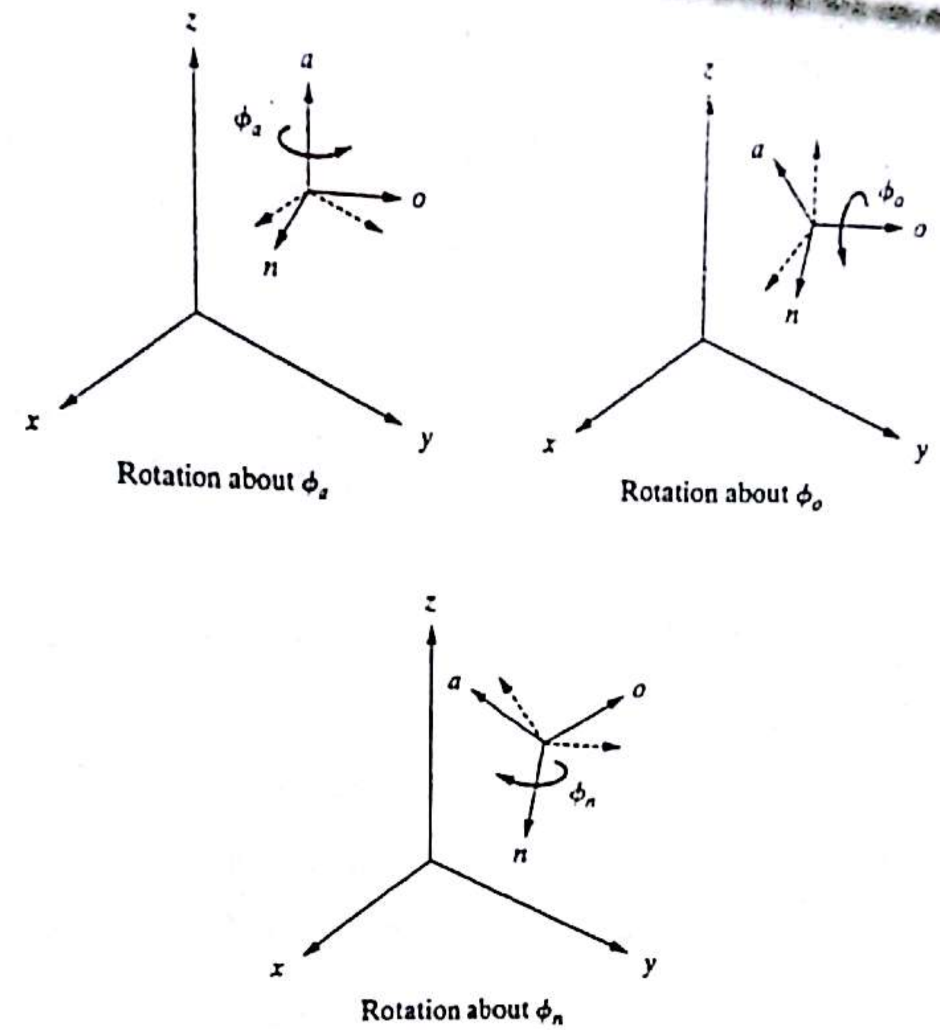

Figure 2.22 RPY rotations about the current axes.

Rotation of ϕ_a about the $\bar{a}$-axis (z-axis of the moving frame) called Roll,
Rotation of ϕ_o about the $\bar{o}$-axis (y-axis of the moving frame) called Pitch,
Rotation of ϕ_n about the $\bar{n}$-axis (x-axis of the moving frame) called Yaw.

The matrix representing the RPY orientation change will be

$$\mathrm{RPY}(\phi_a,\phi_o,\phi_n) = \mathrm{Rot}(a,\phi_a)\mathrm{Rot}(o,\phi_o),\mathrm{Rot}(n,\phi_n) =$$

$$\begin{bmatrix} C\phi_a C\phi_o & C\phi_a S\phi_o S\phi_n - S\phi_a C\phi_n & C\phi_a S\phi_o C\phi_n + S\phi_a S\phi_n & 0 \\ S\phi_a C\phi_o & S\phi_a S\phi_o S\phi_n + C\phi_a C\phi_n & S\phi_a S\phi_o C\phi_n - C\phi_a S\phi_n & 0 \\ -S\phi_o & C\phi_o S\phi_n & C\phi_o C\phi_n & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (2.36)$$

This matrix represents the orientation change caused by the RPY alone. The location and the final orientation of the frame relative to the reference frame will be the product of the two matrices representing the position change and the RPY. For example, suppose that a robot is designed based on spherical coordinates and RPY. Then the robot may be represented by

$${}^{R}T_H = T_{\mathrm{sph}}(r,\beta,\gamma) \times \mathrm{RPY}(\phi_a,\phi_o,\phi_n).$$

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The inverse kinematic solution for the RPY is more complicated than spherical coordinates, because here there are three coupled angles where we need to have information about the sines and the cosines of all three angles individually to solve for them. To solve for these sines and cosines, we will have to decouple these angles. To do this, we will premultiply both sides of Equation by the inverse of $\mathrm{Rot}(a,\phi_a)$:

$$\mathrm{Rot}(a,\phi_a)^{-1}\mathrm{RPY}(\phi_a,\phi_o,\phi_n) = \mathrm{Rot}(o,\phi_o),\mathrm{Rot}(n,\phi_n). \tag{2.37}$$

Assuming that the final desired orientation achieved by RPY is represented by the $(\bar{n},\bar{o},\bar{a})$ matrix, we have

$$\mathrm{Rot}(a,\phi_a)^{-1}\begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathrm{Rot}(o,\phi_o),\mathrm{Rot}(n,\phi_n). \tag{2.38}$$

Multiplying the matrices, we get

$$\begin{bmatrix} n_xC\phi_a + n_yS\phi_a & o_xC\phi_a + o_yS\phi_a & a_xC\phi_a + a_yS\phi_a & 0 \\ n_yC\phi_a - n_xS\phi_a & o_yC\phi_a - o_xS\phi_a & a_yC\phi_a - a_xS\phi_a & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C\phi_o & S\phi_oS\phi_n & S\phi_oC\phi_n & 0 \\ 0 & C\phi_n & -S\phi_n & 0 \\ -S\phi_o & C\phi_oS\phi_n & C\phi_oC\phi_n & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.39}$$

Please remember that the $\bar{n},\bar{o},\bar{a}$ components in Equation (2.38) represent the final desired values, which are normally given or known. The values of the RPY angles are the unknown variables.

Equating the different elements of the right- and the left-hand sides of Equation (2.39) will result in the following (please refer to Appendix A for explanation of ATAN2 function):

From the 2,1 elements, we get

$$n_yC\phi_a - n_xS\phi_a = 0 \rightarrow \phi_a = \mathrm{ATAN2}(n_y,n_x) \quad \text{and} \quad \phi_a = \mathrm{ATAN2}(-n_y,-n_x). \tag{2.40}$$

From the 3,1 and 1,1 elements, we get

$$\begin{aligned} &S\phi_o = -n_z, \\ &C\phi_o = n_xC\phi_a + n_yS\phi_a \rightarrow \phi_0 = \mathrm{ATAN2}(-n_z,(n_xC\phi_a + n_yS\phi_a)). \end{aligned} \tag{2.41}$$

Finally, from 2,2 and 2,3, elements, we get

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$$C\phi_n = o_y C\phi_a - o_x S\phi_a,$$

$$S\phi_n = -a_y C\phi_a + a_x S\phi_a \rightarrow$$

$$\phi_n = \text{ATAN2}[(-a_y C\phi_a + a_x S\phi_a), (o_y C\phi_a - o_x S\phi_a)]. \tag{2.42}$$

Example 2.16

The desired final position and orientation of the hand of a Cartesian–RPY robot is given next. Find the necessary roll, pitch, and yaw angles and displacements:

$$^RT_P = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.354 & -0.674 & 0.649 & 4.33 \\ 0.505 & 0.722 & 0.475 & 2.50 \\ -0.788 & 0.160 & 0.595 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution From the preceding equations, we find two sets of answers:

$$\phi_a = \text{ATAN2}(n_y, n_x) = \text{ATAN2}(0.505, 0.354) = 55°, \text{ or } 235°,$$

$$\phi_o = \text{ATAN2}(-n_z, (n_x C\phi_a + n_y S\phi_a)) = \text{ATAN2}(0.788, 0.616) = 52°, \text{ or } 128°,$$

$$\phi_n = \text{ATAN2}((-a_y C\phi_a - a_x S\phi_a), (o_y C\phi_a - o_x S\phi_a))$$
$$= \text{ATAN2}(0.259, 0.966) = 15°, \text{ or } 195°,$$

$$p_x = 4.33, \qquad p_y = 2.5, \qquad p_z = 8 \text{ units.}$$

Example 2.17

For the same position and orientation as in Example 2.16, find all necessary joint variables if the robot is cylindrical–RPY.

Solution In this case, we will use

$$^RT_P = \begin{bmatrix} 0.354 & -0.674 & 0.649 & 4.33 \\ 0.505 & 0.722 & 0.475 & 2.50 \\ -0.788 & 0.160 & 0.595 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} = T_{cyl}(r, \alpha, l) \times \text{RPY}(\phi_a, \phi_o, \phi_n).$$

The right-hand side of this equation now involves four angles that are coupled and must be decoupled as previously shown. However, since the rotation of α about z-axis for the cylindrical coordinates does not affect the a-axis, it remains parallel to the z-axis. As a result, the rotation of ϕ_a about the a-axis for RPY will simply be added to α. This means that the angle we found for ϕ_a is in fact the summation of $\phi_a + \alpha$. (See Figure 2.23.) Using the position information given, we find the solution of Example 2.16, and referring to Equation (2.33), we will get

$$rC\alpha = 4.33, \quad rS\alpha = 2.5 \quad \rightarrow \quad \alpha = 30°,$$

$$\phi_a + \alpha = 55° \quad \rightarrow \quad \phi_a = 25°,$$

$$S\alpha = 0.5 \quad \rightarrow \quad r = 5.

$p_z = 3 \qquad \rightarrow \qquad l = 8.$

As in Example 2.16, $\qquad \rightarrow \qquad \phi_o = 52°, \qquad \phi_n = 15°.$

Of course, a similar solution may be found for the second set of answers.

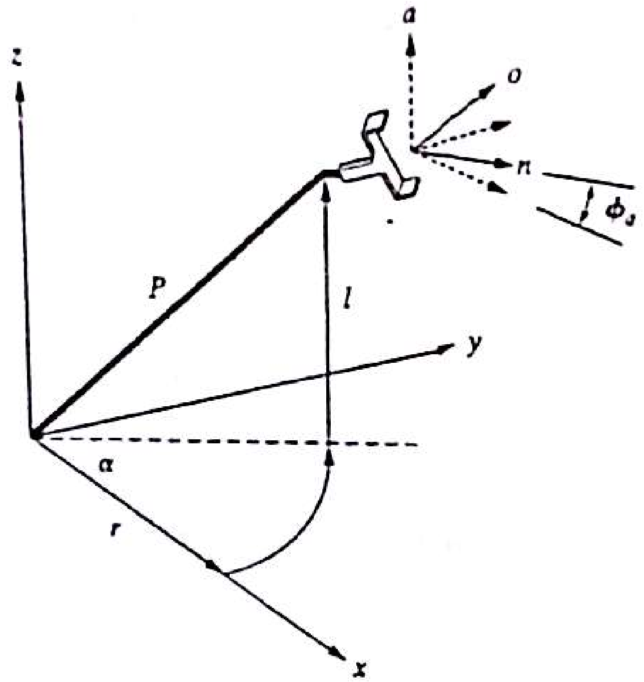

Figure 2.23 Cylindrical and RPY coordinates of Example 2.17.

2.7.2(b) Euler Angles

Euler angles are very similar to RPY, except that the last rotation is also about the current a-axis. (See Figure 2.24.) We still need to make all rotations relative to the current axes to prevent any change in the position of the robot. Thus, the rotations representing the Euler angles are as follows:

Rotation of ϕ about the $\bar{a}$-axis (z-axis of the moving frame) followed by,
Rotation of θ about the $\bar{o}$-axis (y-axis of the moving frame) followed by,
Rotation of ψ about the $\bar{a}$-axis (z-axis of the moving frame).

Then the matrix representing the Euler angles orientation change is

$$\mathrm{Euler}(\phi,\theta,\psi) = \mathrm{Rot}(a,\phi)\mathrm{Rot}(o,\theta),\mathrm{Rot}(a,\psi) =$$

$$\begin{bmatrix} C\phi C\theta C\psi - S\phi S\psi & -C\phi C\theta S\psi - S\phi C\psi & C\phi S\theta & 0 \\ S\phi C\theta C\psi + C\phi S\psi & -S\phi C\theta S\psi + C\phi C\psi & S\phi S\theta & 0 \\ -S\theta C\psi & S\theta S\psi & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (2.43)$$

Once again, this matrix represents the orientation change caused by the Euler angles alone. The location and the final orientation of the frame relative to the reference frame will be the product of the two matrices representing the position change and the Euler angles.

The inverse kinematic solution for the Euler angles can be found in a manner very similar to RPY. We will premultiply the two sides of the Euler equation by

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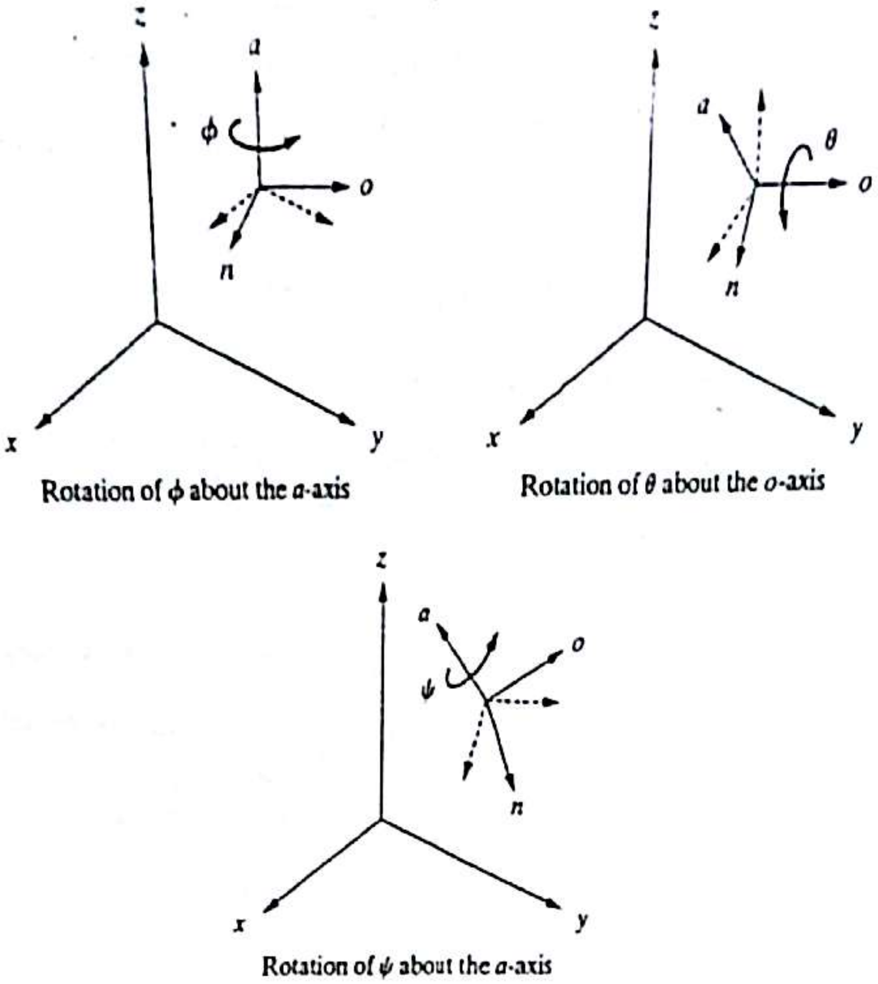

Figure 2.24 Euler rotations about the current axes.

$\text{Rot}^{-1}(a,\phi)$ to eliminate ϕ from one side. By equating the elements of the two sides to each other, we find the following equations (assuming that the final desired orientation achieved by Euler angles is represented by $(\bar{n},\bar{o},\bar{a})$ matrix ,

$$\text{Rot}^{-1}(a,\phi) \times \begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C\theta C\psi & -C\theta S\psi & S\theta & 0 \\ S\psi & C\psi & 0 & 0 \\ -S\theta C\psi & S\theta S\psi & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.44}$$

or

$$\begin{bmatrix} n_xC\phi + n_yS\phi & o_xC\phi + o_yS\phi & a_xC\phi + a_yS\phi & 0 \\ -n_xS\phi + n_yC\phi & -o_xS\phi + o_yC\phi & -a_xS\phi + a_yC\phi & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} C\theta C\psi & -C\theta S\psi & S\theta & 0 \\ S\psi & C\psi & 0 & 0 \\ -S\theta C\psi & S\theta S\psi & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.45}$$

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Please remember that the $\bar{n},\bar{o},\bar{a}$ the components in Equation (2.44) represent the final desired values, which are normally given or known. The values of the Euler angles are the unknown variables.

Equating the different elements of the right- and the left-hand sides of Equation (2.45) will result in the following:

From the 2,3 elements, we get

$$-a_x S\phi + a_y C\phi = 0 \rightarrow \phi = \text{ATAN2}(a_y,a_x), \quad \text{or} \quad \phi = \text{ATAN2}(-a_y,-a_x). \tag{2.46}$$

With ϕ evaluated, all the elements of the left-hand side of Equation (2.45) are known. From the 2,1 and 2,2 elements, we get

$$\begin{aligned} S\psi &= -n_x S\phi + n_y C\phi, \\ C\psi &= -o_x S\phi + o_y C\phi \quad \rightarrow \quad \psi = \text{ATAN2}(-n_x S\phi + n_y C\phi, -o_x S\phi + o_y C\phi). \end{aligned} \tag{2.47}$$

Finally, from 1,3 and 3,3 elements, we get

$$\begin{aligned} S\theta &= a_x C\phi + a_y S\phi, \\ C\theta &= a_z \quad \rightarrow \quad \theta = \text{ATAN2}(a_x C\phi + a_y S\phi, a_z). \end{aligned} \tag{2.48}$$

Example 2.18

The desired final orientation of the hand of a Cartesian–Euler robot is given. Find the necessary Euler angles:

$$^{R}T_H = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.579 & -0.548 & -0.604 & 5 \\ 0.540 & 0.813 & -0.220 & 7 \\ 0.611 & -0.199 & 0.766 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution From the foregoing equations, we find

$$\phi = \text{ATAN2}(a_y,a_x) = \text{ATAN2}(-0.220,-0.604) = 20^\circ \text{ or } 200^\circ.$$

Realizing that both the sines and cosines of 20° and 200° can be used for the remainder, we find that

$$\psi = \text{ATAN2}(-n_x S\phi + n_y C\phi, -o_x S\phi + o_y C\phi) = (0.31, 0.952) = 18^\circ, \text{ or } 198^\circ,$$

$$\theta = \text{ATAN2}(a_x C\phi + a_y S\phi, a_z) = \text{ATAN2}(-0.643, 0.766) = -40^\circ, \text{ or } 40^\circ.$$

2.7.2(c) Articulated Joints

Articulated joints consist of three rotations other than the the ones just presented, which are based on the design of the joints. As we did in section 2.7.1(d), we will develop the matrix representing articulated joints later, when we discuss the Denavit–Hartenberg representation.

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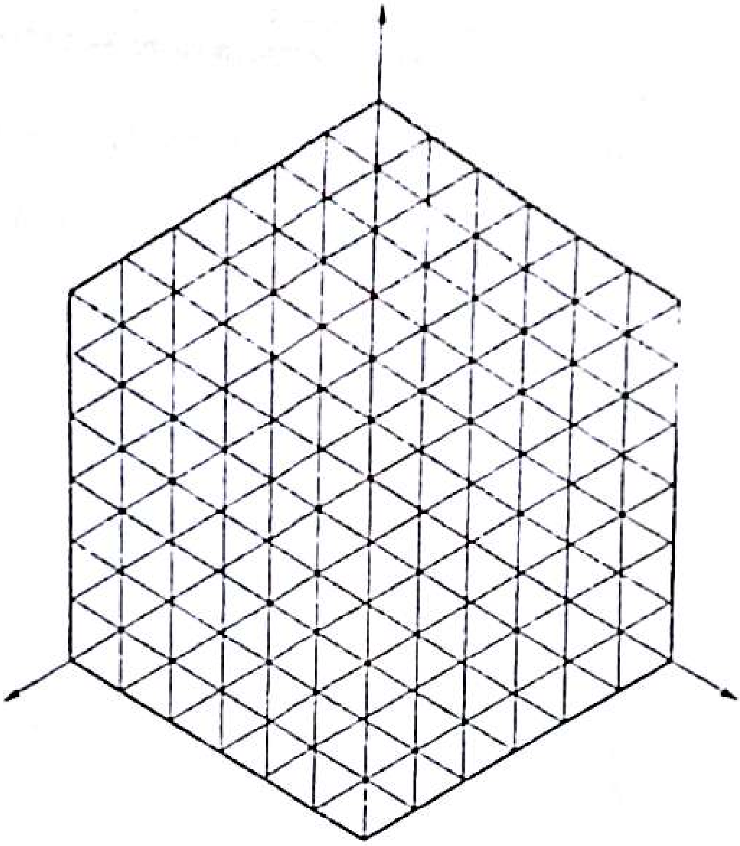

PROBLEMS

1. Suppose that instead of a frame, a point $P = (3,5,7)^T$ in space was translated a distance of $d = (2,3,4)^T$. Find the new location of the point relative to the reference frame.
2. The following was moved a distance of $d = (5,2,6)^T$:

$$B = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Find the new location of the frame relative to the reference frame.

3. For the following frame, find the values of the missing elements and complete the matrix representation of the frame:

$$F = \begin{bmatrix} ? & 0 & -1 & 5 \\ ? & 0 & 0 & 3 \\ ? & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

4. Derive the matrix that represents a pure rotation about the y-axis of the reference frame.
5. Derive the matrix that represents a pure rotation about the z-axis of the reference frame.

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6. Verify that the rotation matrices about the reference frame axes follow the required constraint equations set by orthogonality and length requirements of directional unit vectors.

7. Find the coordinates of point $P(2,3,4)^T$ relative to the reference frame after a rotation of 45about the x-axis.

8. Find the coordinates of point $P(3,5,7)^T$ relative to the reference frame after a rotation of 30° about the z-axis.

9. A point P in space is defined as ${}^BP = [5,3,4]^T$ relative to frame B and is attached to the origin of the reference frame A and is parallel to it. Apply the following transformations to frame B and find ${}^AP$. Using the three-dimensional grid provided in this chapter, plot the transformations and the result and verify it. Also verify graphically that you would not get the same results if you apply the transformations relative to the current frame:
   - Rotate 90° about the x-axis.
   - Then translate 3 units about the y-axis, 6 units about the z-axis, and 5 units about the x-axis.
   - Then, rotate 90° about the z-axis.

10. A point P in space is defined as ${}^BP = [2,3,5]^T$ relative to frame B, which is attached to the origin of the reference frame A and is parallel to it. Apply the following transformations to frame B and find ${}^AP$. Using the three-dimensional grid, plot the transformations and the result and verify it:
    - Rotate 90° about the x-axis.
    - Then, rotate 90° about the local a-axis.
    - Then translate 3 units about the y-axis, 6 units about the z-axis, and 5 units about the x-axis.

11. Show that rotation matrices about the y-axis and the z-axis are unitary.

12. Calculate the inverse of the following transformation matrices:

$$T_1 = \begin{bmatrix} 0.527 & -0.574 & 0.628 & 2 \\ 0.369 & 0.819 & 0.439 & 5 \\ -0.766 & 0 & 0.643 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0.92 & 0 & 0.39 & 5 \\ 0 & 1 & 0 & 6 \\ -0.39 & 0 & 0.92 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

13. Write the correct sequence of movements that must be made in order to "unrotate" the spherical coordinates and to make it parallel to the reference frame. About what axes are these rotations supposed to be?

14. A spherical coordinate system is used to describe the position of the hand of a robot. In a certain situation, the hand is later "unrotated" back to be parallel to the reference frame, and the matrix representing it is described as

$$T_{sph} = \begin{bmatrix} 1 & 0 & 0 & 3.1375 \\ 0 & 1 & 0 & 2.195 \\ 0 & 0 & 1 & 3.214 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

    (a) Find the necessary values of r, β, γ to achieve this location.

    (b) Find the components of the original matrix $\bar{n}, \bar{o}, \bar{a}$ vectors for the hand before it was unrotated.

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15. Suppose that a robot is made of a Cartesian and RPY combination of joints. Find the necessary RPY angles to achieve the following:

$$T = \begin{bmatrix} 0.527 & -0.574 & 0.628 & 4 \\ 0.369 & 0.819 & 0.439 & 6 \\ -0.766 & 0 & 0.643 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

16. Suppose that a robot is made of a Cartesian and Euler combination of joints. Find the necessary Euler angles to achieve the following:

$$T = \begin{bmatrix} 0.527 & -0.574 & 0.628 & 4 \\ 0.369 & 0.819 & 0.439 & 6 \\ -0.766 & 0 & 0.643 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

17. A frame ^{U}B was moved along its own n-axis a distance of 5 units and then rotated about its o-axis an angle of 60°, followed by a rotation of about the z-axis; it was then translated about its a-axis 3 units and finally rotated about x-axis 45°.
   - **(a)** Calculate the total transformation performed.
   - **(b)** What angles and movements would we have to make if we were to create the same location and orientation using Cartesian and RPY configurations?

18. Frames describing the base of a robot and an object are given relative to the Universe frame.
   - **(a)** Find a transformation $^{R}T_{H}$ of the robot configuration if the hand of the robot is to be placed on the object.
   - **(b)** By inspection, show whether this robot can be a three-axis spherical robot, and, if so, find α, β, γ.
   - **(c)** Assuming that the robot is a six-axis robot with Cartesian and Euler coordinates, find $x, y, z, \phi, \theta, \psi$.

$$^{U}T_{obj} = = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad ^{U}T_{R} = \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

19. A robot arm with three degrees of freedom has been designed for applying paint on flat walls as shown in Figure P.2.19.
   - **(a)** Assign coordinate frames as necessary based on D-H representation.
   - **(b)** Fill out the parameters table.
   - **(c)** Find the $^{U}T_{H}$ matrix.

20. The robot shown in Figure P.2.20 has two degrees of freedom, and the transformation matrix $^{0}T_{H}$ is given in symbolic form, as well as in numerical form for a specific location. The length of each link l_1 and l_2 is 1 ft.
   - **(a)** Derive the inverse kinematic equations for θ_1 and θ_2 in symbolic form.
   - **(b)** Calculate the values of θ_1 and θ_2 for the given location.

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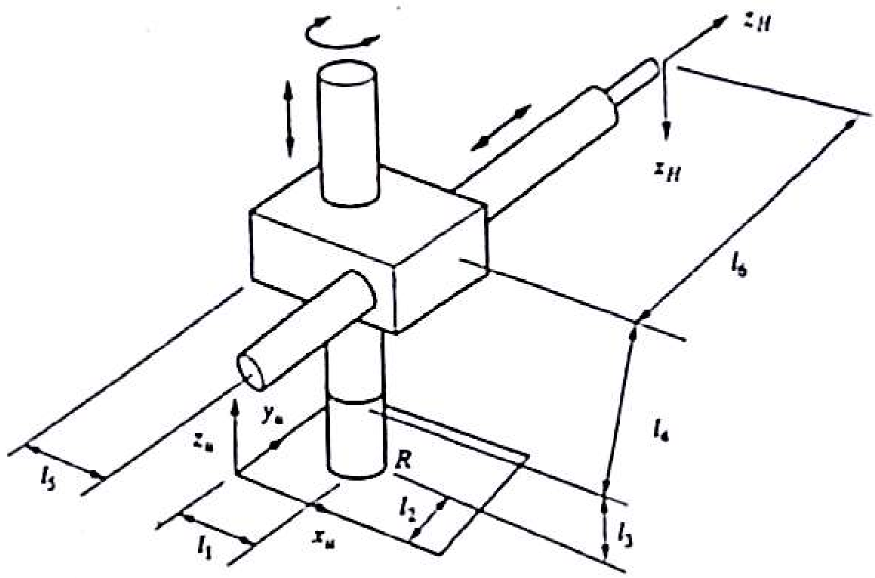

Figure P.2.19

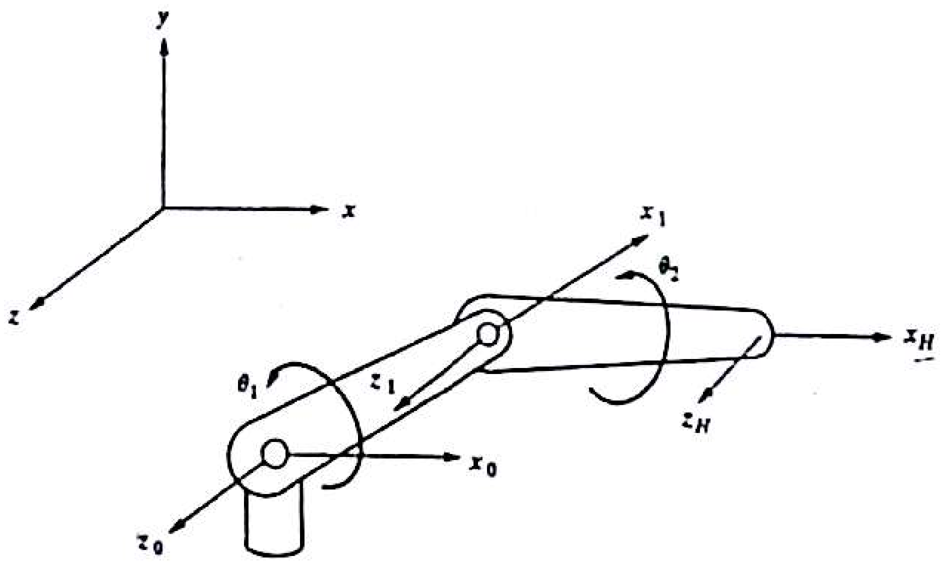

Figure P.2.20

$$
{}^0T_H = \begin{bmatrix} C_{12} & -S_{12} & 0 & l_2C_{12} + l_1C_1 \\ S_{12} & C_{12} & 0 & l_2S_{12} + l_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.2924 & -0.9563 & 0 & 0.6978 \\ 0.9563 & -0.2924 & 0 & 0.8172 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

21. For the SCARA-type robot shown in Figure P.2.21,
    (a) Assign the coordinate frames based on D-H representation.
    (b) Fill out the parameters table.
    (c) Write all the A matrices.
    (d) Write the ${}^UT_H$ matrix in terms of the A matrices.

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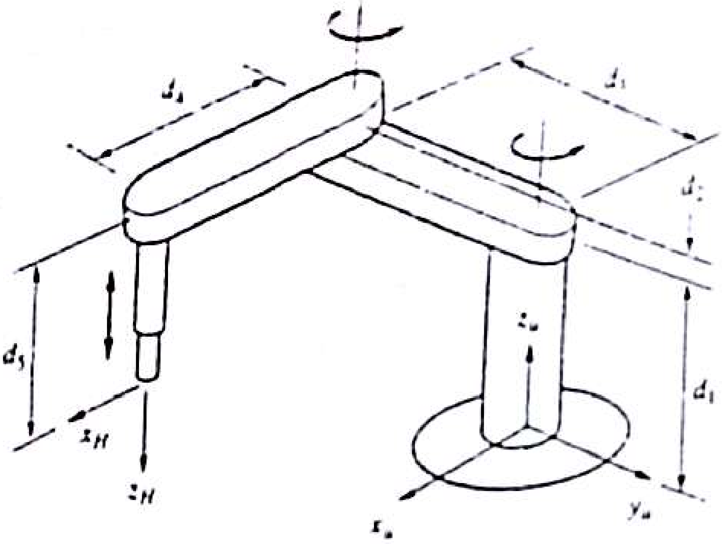

Figure P.2.21

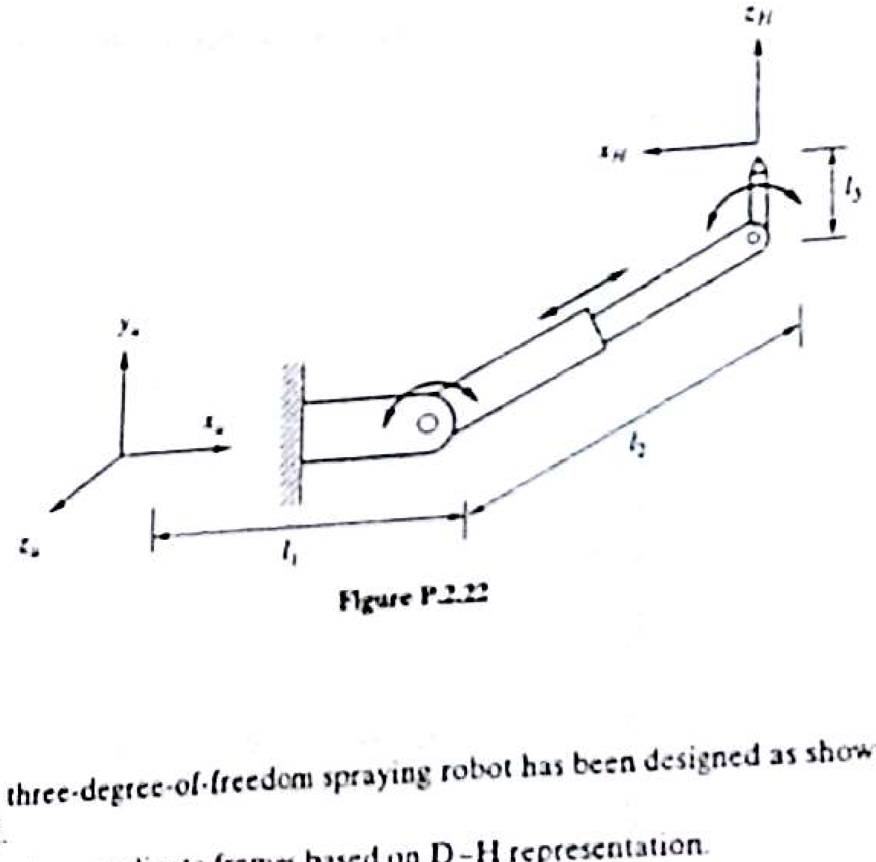

Figure P.2.22

22. A special three-degree-of-freedom spraying robot has been designed as shown in Figure P.2.22.
    (a) Assign the coordinate frames based on D-H representation.
    (b) Fill out the parameters table.
    (c) Write all the A matrices.
    (d) Write the ${}^{U}T_{H}$ matrix in terms of the A matrices.
23. For the Unimation Puma 562, six-axis robot shown in Figure P.2.23,
    (a) Assign the coordinate frames based on D-H representation.
    (b) Fill out the parameters table.

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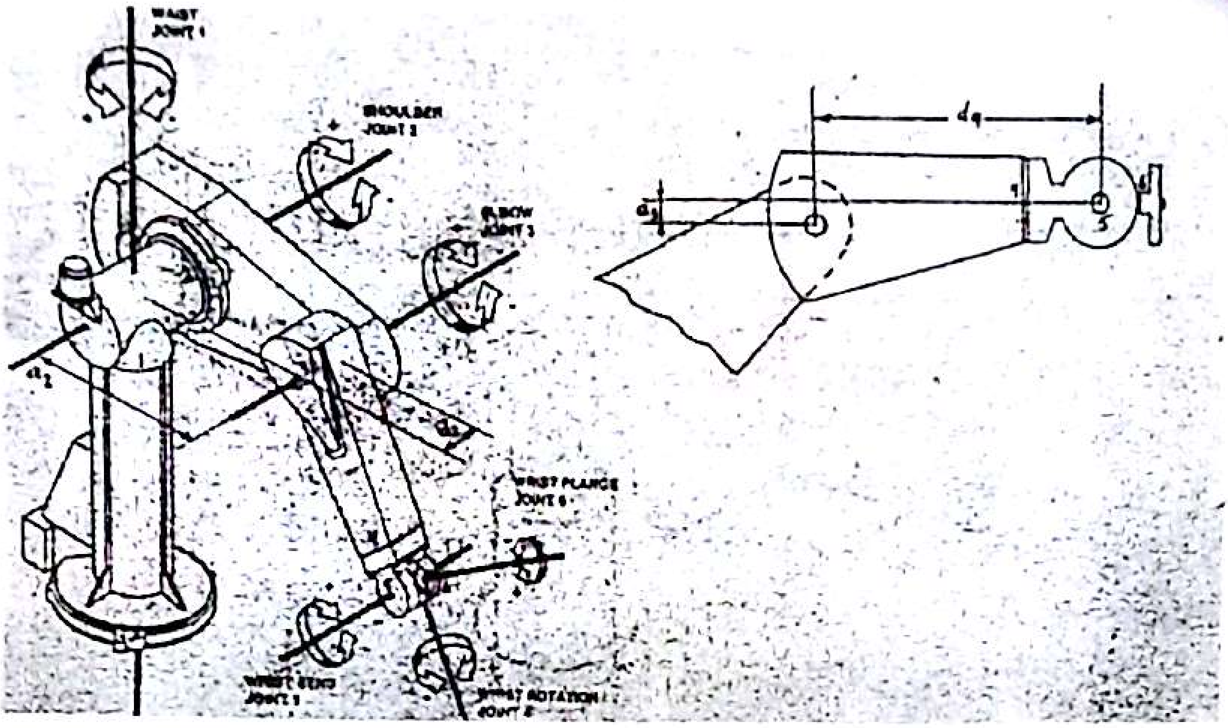

Figure P.2.23 Picture of Puma 562 is reprinted with permission from Staubli Robotics.

#	θ	d	a	α
1				
2				
3				
4				
5				
6				

(c) Write all the A matrices.

(d) Find the ${}^{R}T_{H}$ matrix for the following values:

$$\text{Base height} = 27 \text{ in}, d_2 = 6 \text{ in}, a_2 = 15 \text{ in}, a_3 = 1 \text{ in}, d_4 = 18 \text{ in}, d_6 = 5 \text{ in},$$
$$\theta_1 = 0°, \theta_2 = 45°, \theta_3 = 0°, \theta_4 = 0°, \theta_5 = -45°, \theta_6 = 0°$$

24. For the four-degree-of-freedom robot depicted in Figure P.2.24,

(a) Assign appropriate frames for D–H representation.

(b) Fill out the parameters table.

(c) Write an equation in terms of A matrices that shows how ${}^{U}T_{H}$ can be calculated.

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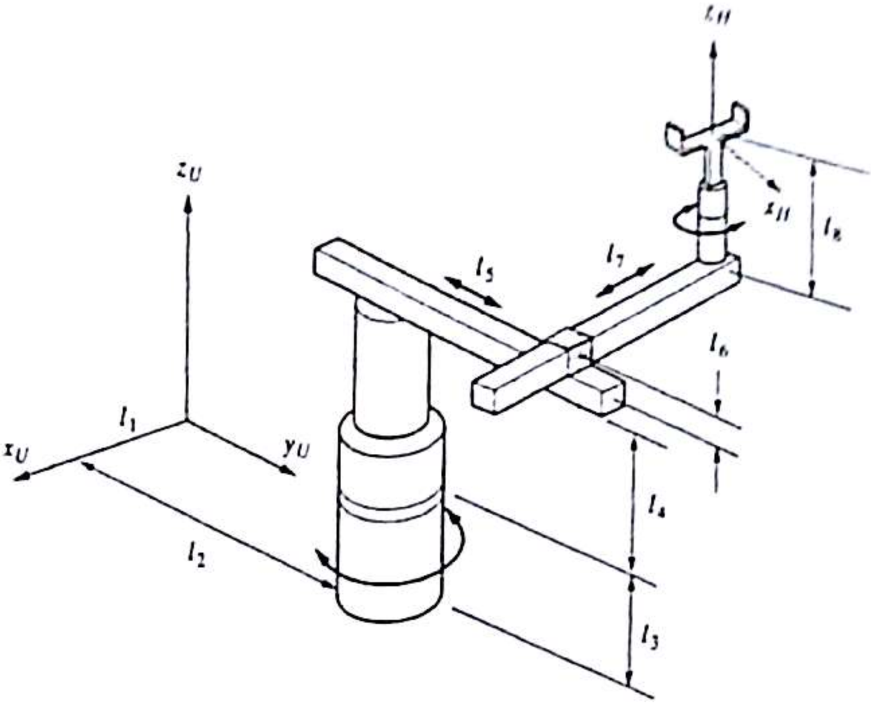

Figure P.2.24

#	θ	d	a	α
1				
2				
3				
4				

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Congratulations!
Now you know a lot
about Robotics !

— Now go, make one =

Muntaka 
25.5.17

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