EQUILATERAL TRIANGLES

and

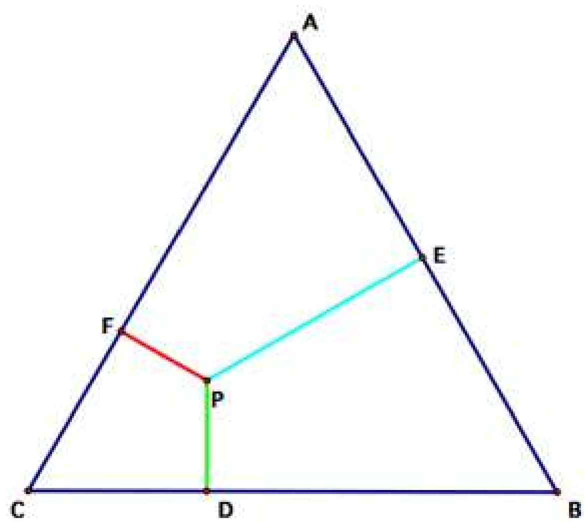
"VIVIANI'S THEORM"

Let ABC be an equilateral triangle and P by an arbitrary point inside or on the triangle. Construct PD, PE, PF, as the segments from P perpendicular to the sides BC, AC, AB respectively.
Develop 4 different proofs that the sum of PD + PE + PF is equal to the length of an altitude of the equilateral triangle.

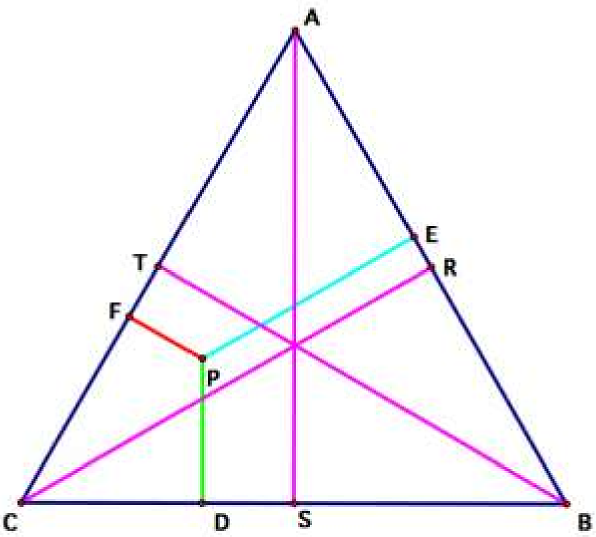
Discuss and contrast the pedagogical value of each proof.

When first investigating this problem I was sure there must be someone who discovered this relationship, and must have his or her name attached to a theorem. After little investigation I found that in fact an Italian mathematician and scientist named Vincenzo Viviani, a pupil of Torricelli and a disciple of Galileo, is credited with this theorem.
Viviani's theorem states that the sum of the distances from a point to the sides of an equilateral triangle equals the length of its altitude, and the theorem can be extended to equilateral polygons and equiangular polygons.
CLICK HERE for more information on Vincenzo Viviani

Here are the proofs...........



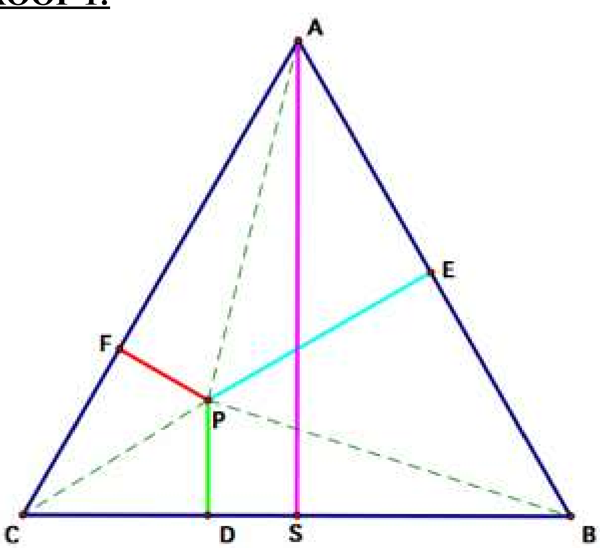
I want to preface the following proofs with the fact that in each of the proofs I let the length of the altitude be equal to the segment AS, which is the altitude from the vertex A to side BC. The following is a short proof that all the altitudes of an equilateral triangle are equal in length, so that when drawing the conclusion that PD+PE+PF = AS, we can conclude that the sum is equal to any of the altitudes of the triangle.



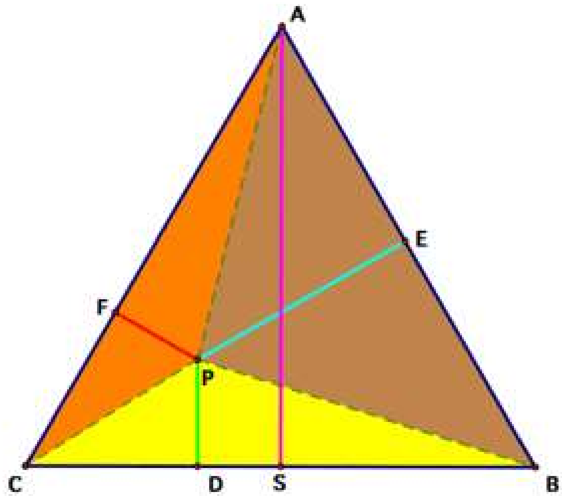
Let AS, BT, CR be the altitudes of triangle ABC
We then know…..

△BCR≅△ACR≅△BAS≅△CAS≅△CBT≅△ABT by ASA or HA.
Therefore CR≅AS≅BT by corresponding parts of congruent triangles are congruent.

PROOF 1:



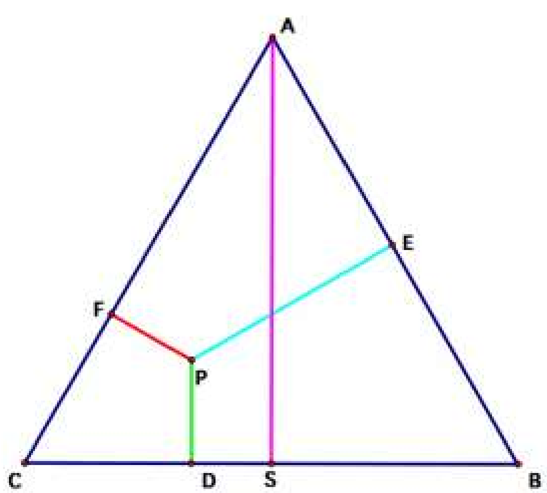
Construct segments from P to each vertices of the equilateral triangle.

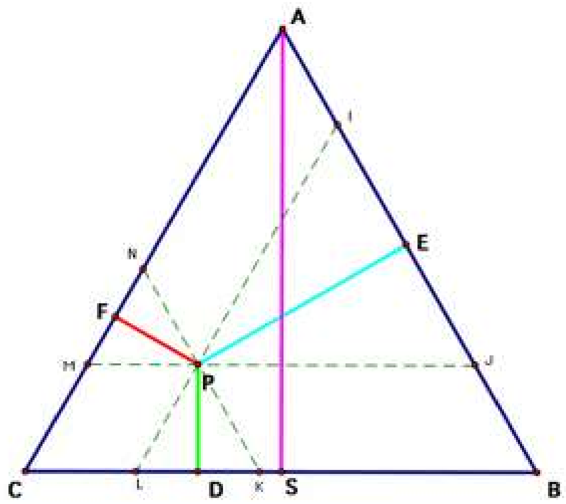


Area of triangle APB = ½ area of rectangle with sides AB, PE (triangles with same base and the third vertex on the same line parallel to this base have equal areas)
Area of triangle BPC = ½ area of rectangle with sides BC, PD
Area of triangle APC = ½ area of rectangle with sides CA, PF
But the sum of triangles APB, BPC, APC = triangle ABC
And AB, BC, AC are equal
Therefore, area of triangle ABC = ½ area of rectangle with sides BC, (PE + PD + PF);
But area of triangle ABC = ½ area of rectangle with sides BC, AS
So,
AS = (PE + PD + PF), which is the sum of our perpendiculars from point P to each side.

Discussion: In our first proof the pedagogical advantages are the fact that the proof incorporates the concept of area. It also used one of my favorite theorems, the fact that triangles with the same base and their 3rd vertices on the same line parallel to the base have equal area. This theorem is used when we set the area of a triangle = to ½ the area of a square. This proof is great to use when looking to dive deeper into the topic of area, and to remind or reintroduce the theorem above.

PROOF 2:





Draw segments IL parallel to AC, MJ parallel to CB, and KN parallel to AB.
Triangles LPK, MNP, and PIJ are similar to triangle ABC and therefore are equilateral triangles.
We can now set up some proportions using our similar triangles:

$$\frac{PD}{AS}=\frac{LK}{CB} \qquad \frac{PF}{AS}=\frac{MN}{CA} \qquad \frac{PE}{AS}=\frac{IJ}{AB}$$

Let us add these three equalities:

$$\frac{PD}{AS}+\frac{PF}{AS}+\frac{PE}{AS}=\frac{LK}{CB}+\frac{MN}{CA}+\frac{IJ}{AB}$$

Note that CB=CA=AB because ABC is equilateral triangle….

$$\frac{PD}{AS}+\frac{PF}{AS}+\frac{PE}{AS}=\frac{LK}{CB}+\frac{MN}{CB}+\frac{IJ}{CB}$$

Also, MN = MP (equilateral triangle) = CL (MPLC is a parallelogram)
And, IJ = PJ (equilateral triangle) = KB (JPKB is a parallelogram)
So,…

$$\frac{PD}{AS}+\frac{PF}{AS}+\frac{PE}{AS}=\frac{LK}{CB}+\frac{CL}{CB}+\frac{KB}{CB}$$

$$CL + LK + KB = CB$$

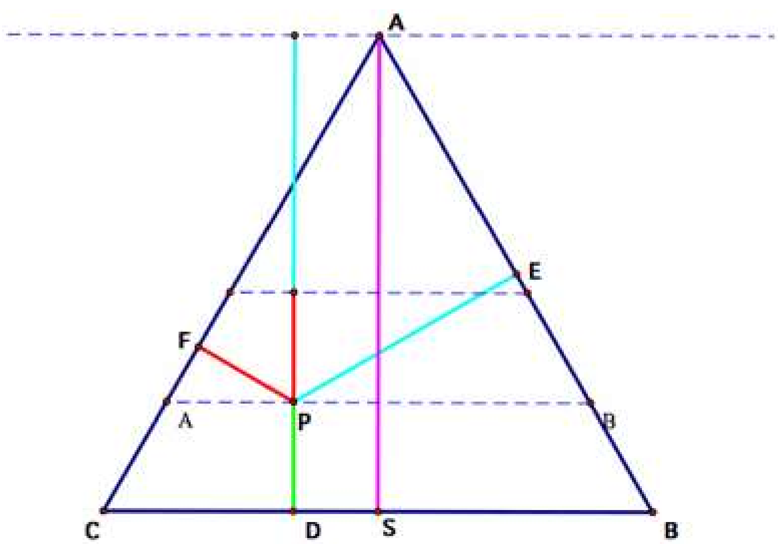
Therefore,

$$\frac{PD}{AS}+\frac{PF}{AS}+\frac{PE}{AS}=1$$

and, we can conclude that PD + PF + PE = AS.

Discussion: Our second proof uses two main concepts, similar triangles and their properties, and the properties of parallelograms. When discussing the topic of similar triangles in a geometry class this would be a fantastic problem to give to challenge your students, asking them specifically to use similar triangles to proof it. It is also important that the students recall and can incorporate the properties of parallelograms, this might be a nice way to review/remind your students of those concepts that they have previously studied.

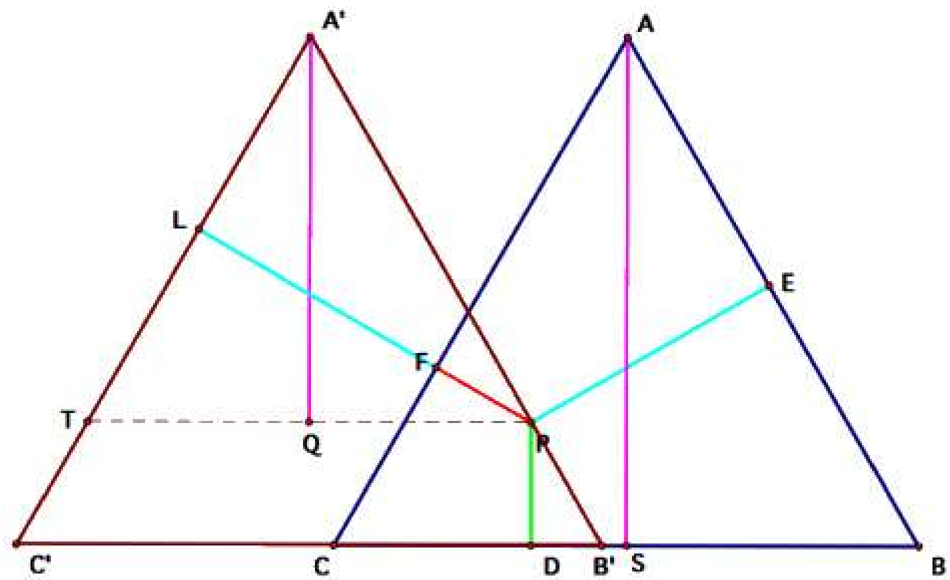
PROOF 3:



This proof utilizes rotation and translation.
First, rotate segment PF -60 degrees about point P.
Second, rotate segment PE 60 degrees about point P and then translate vertically by the length of segment PF (this can be done quite easily with GSP and using a marked vector)
From this one can see that the PD + PF + PE = AS (which is the length of the altitude of our equilateral triangle).

Discussion: For the third and fourth proof I have them grouped together in terms of pedagogical value. Both these proofs use the idea of manipulating our perpendicular segments to form a straight line, "stacking them one next to, or on top of, each other", that can then be easily compared to the length of our altitude. Proof 4 is a little harder, and most likely more difficult for the students to produce, but still results in the same comparison in the end as proof 3. I think this line of reasoning is probably the easiest for the students to understand, and doesn't require much manipulation or use of other concepts beyond translations, reflections, and rotations, (proof 4 does use some knowledge of properties of parallel lines and similar triangle – although the proof could be completed strictly using reflection, rotation and translation). I think this would be the first proof I would show a class when discussing the problem, and then ask them to find other ways to prove the same thing.

PROOF 4:

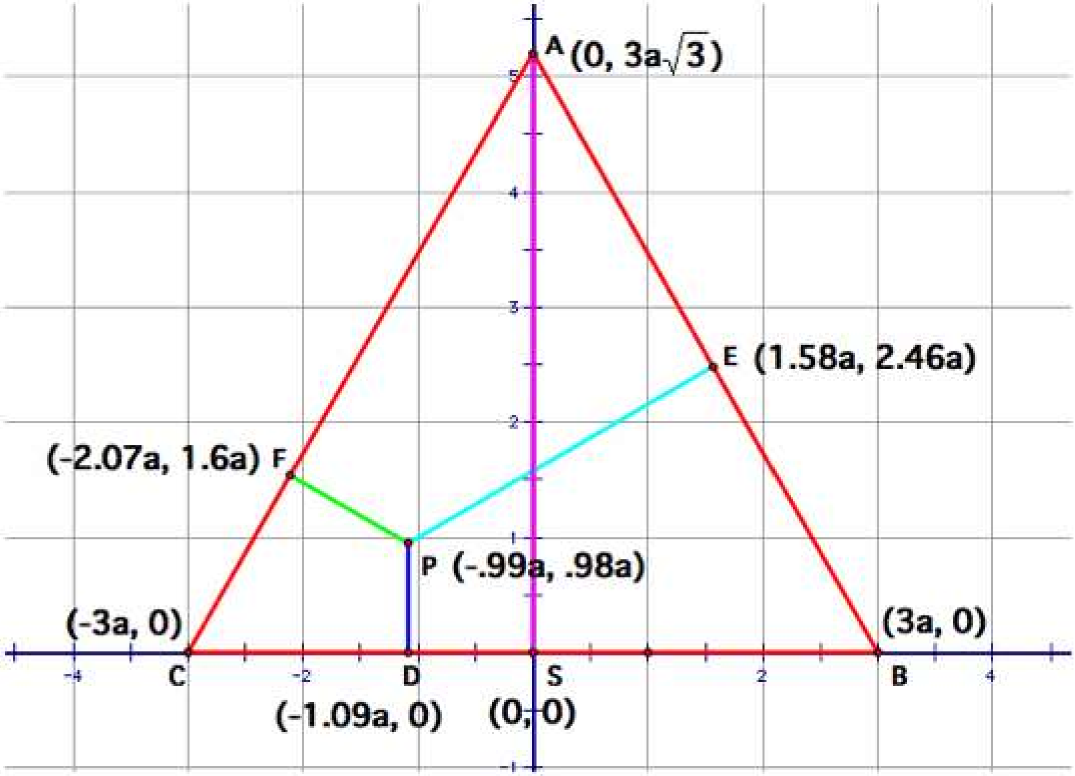


Consider a shifted copy A'B'C' of Triangle ABC, such that P lies on B'A'. Construct PT such that PT is parallel to C'B'. Let Q be the foot of the perpendicular from A' onto PT, and L be the foot of the perpendicular from P onto A'C'. Then

$$
\begin{aligned}
AS &= A'Q + PD \\
&= PL + PD \text{ (similar triangle with ratio of sides 1:1, therefore equilateral triangle, altitudes are equal in length)} \\
&= FL + PF + PD \\
&= PE + PF + PD.
\end{aligned}
$$

PROOF 5:

With this proof I will use the coordinate plane and the distance formula to show that the sum of our perpendiculars from P are equal to length of Altitude.



Let a be our unit measure
Construct equilateral triangle with side lengths of 6a, arbitrary number just helps make our diagram easier.
Plot any point P interior to triangle ABC.
Construct perpendiculars from P to sides of ABC, we get PD, PE, PF
Label coordinates of points D,E,F,S with respect to unit measure = to a.
Use distance formula to find measures of AS, PD, PE, PF
Compare sum of PD, PE, PF to measure of AS….

$$|PE| = \sqrt{(1.58a+.99a)^2 + (2.46a-.98a)^2} = \sqrt{(2.57a)^2 + (1.48a)^2} = \sqrt{6.60a^2 + 2.19a^2} = \sqrt{8.79a^2} \sim 2.9a$$

$$|PF| = \sqrt{(-.99a+2.07a)^2 + (.98a-1.6a)^2} = \sqrt{(1.08a)^2 + (-.62a)^2} = \sqrt{1.17a^2 + .38a^2} = \sqrt{1.55a^2} \sim 1.2a$$

$$|PD| = \sqrt{(0+1.09a)^2 + (0)^2} = \sqrt{(1.09a)^2} = \sqrt{1.19a^2} \sim 1.1a$$

$$|AS| = \sqrt{(0-0)^2+(3a\sqrt{3}-0)^2} = \sqrt{0+27a^2} = 3a\sqrt{3} \sim 5.2a$$

$$|PE|+|PF|+|PD| = 2.9a+1.2a+1.1a = 5.2a = |AS|$$

Click here to explore this proof further in GSP

Discussion: The fifth proof is an interesting one because it actually combines algebra with geometry. The proof uses the distance formula often introduced to students in Algebra I, and then sometimes incorporated in geometry, but not often. I think this proof has great value. Often times our students belief that mathematics has strict boundaries between the branches, and that problems are classified as algebra problems or geometry problems. What this does is show and reinforce to our students the important understanding that mathematics is not a subject with black and white areas, in fact the beauty of mathematics is that problems can often be solved many different ways and as we continue in our study of mathematics we get grayer and grayer until there are not boundaries between branches.