

Real Functions in Several Variables: Volume II

Continuous Functions in Several Variables

Leif Mejlbro



Leif Mejlbro

Real Functions in Several Variables

Volume-II Continuous Functions in Several Variables



Real Functions in Several Variables: Volume-II Continuous Functions in Several Variables

2nd edition

© 2015 Leif Mejlbro & bookboon.com

ISBN 978-87-403-0908-9

Contents

Volume I, Point Sets in \mathbb{R}^n	1
Preface	15
Introduction to volume I, Point sets in \mathbb{R}^n. The maximal domain of a function	19
1 Basic concepts	21
1.1 Introduction	21
1.2 The real linear space \mathbb{R}^n	22
1.3 The vector product	26
1.4 The most commonly used coordinate systems	29
1.5 Point sets in space	37
1.5.1 Interior, exterior and boundary of a set	37
1.5.2 Starshaped and convex sets	40
1.5.3 Catalogue of frequently used point sets in the plane and the space	41
1.6 Quadratic equations in two or three variables. Conic sections	47
1.6.1 Quadratic equations in two variables. Conic sections	47
1.6.2 Quadratic equations in three variables. Conic sectional surfaces	54
1.6.3 Summary of the canonical cases in three variables	66
2 Some useful procedures	67
2.1 Introduction	67
2.2 Integration of trigonometric polynomials	67
2.3 Complex decomposition of a fraction of two polynomials	69
2.4 Integration of a fraction of two polynomials	72
3 Examples of point sets	75
3.1 Point sets	75
3.2 Conics and conical sections	104
4 Formulae	115
4.1 Squares etc.	115
4.2 Powers etc.	115
4.3 Differentiation	116
4.4 Special derivatives	116
4.5 Integration	118
4.6 Special antiderivatives	119
4.7 Trigonometric formulae	121
4.8 Hyperbolic formulae	123
4.9 Complex transformation formulae	124
4.10 Taylor expansions	124
4.11 Magnitudes of functions	125
Index	127

Volume II, Continuous Functions in Several Variables	133
Preface	147
Introduction to volume II, Continuous Functions in Several Variables	151
5 Continuous functions in several variables	153
5.1 Maps in general	153
5.2 Functions in several variables	154
5.3 Vector functions	157
5.4 Visualization of functions	158
5.5 Implicit given function	161
5.6 Limits and continuity	162
5.7 Continuous functions	168
5.8 Continuous curves	170
5.8.1 Parametric description	170
5.8.2 Change of parameter of a curve	174
5.9 Connectedness	175
5.10 Continuous surfaces in \mathbb{R}^3	177
5.10.1 Parametric description and continuity	177
5.10.2 Cylindric surfaces	180
5.10.3 Surfaces of revolution	181
5.10.4 Boundary curves, closed surface and orientation of surfaces	182
5.11 Main theorems for continuous functions	185
6 A useful procedure	189
6.1 The domain of a function	189
7 Examples of continuous functions in several variables	191
7.1 Maximal domain of a function	191
7.2 Level curves and level surfaces	198
7.3 Continuous functions	212
7.4 Description of curves	227
7.5 Connected sets	241
7.6 Description of surfaces	245
8 Formulae	257
8.1 Squares etc.	257
8.2 Powers etc.	257
8.3 Differentiation	258
8.4 Special derivatives	258
8.5 Integration	260
8.6 Special antiderivatives	261
8.7 Trigonometric formulæ	263
8.8 Hyperbolic formulæ	265
8.9 Complex transformation formulæ	266
8.10 Taylor expansions	266
8.11 Magnitudes of functions	267
Index	269

Volume III, Differentiable Functions in Several Variables	275
Preface	289
Introduction to volume III, Differentiable Functions in Several Variables	293
9 Differentiable functions in several variables	295
9.1 Differentiability	295
9.1.1 The gradient and the differential	295
9.1.2 Partial derivatives	298
9.1.3 Differentiable vector functions	303
9.1.4 The approximating polynomial of degree 1	304
9.2 The chain rule	305
9.2.1 The elementary chain rule	305
9.2.2 The first special case	308
9.2.3 The second special case	309
9.2.4 The third special case	310
9.2.5 The general chain rule	314
9.3 Directional derivative	317
9.4 C^n -functions	318
9.5 Taylor's formula	321
9.5.1 Taylor's formula in one dimension	321
9.5.2 Taylor expansion of order 1	322
9.5.3 Taylor expansion of order 2 in the plane	323
9.5.4 The approximating polynomial	326
10 Some useful procedures	333
10.1 Introduction	333
10.2 The chain rule	333
10.3 Calculation of the directional derivative	334
10.4 Approximating polynomials	336
11 Examples of differentiable functions	339
11.1 Gradient	339
11.2 The chain rule	352
11.3 Directional derivative	375
11.4 Partial derivatives of higher order	382
11.5 Taylor's formula for functions of several variables	404
12 Formulæ	445
12.1 Squares etc.	445
12.2 Powers etc.	445
12.3 Differentiation	446
12.4 Special derivatives	446
12.5 Integration	448
12.6 Special antiderivatives	449
12.7 Trigonometric formulæ	451
12.8 Hyperbolic formulæ	453
12.9 Complex transformation formulæ	454
12.10 Taylor expansions	454
12.11 Magnitudes of functions	455
Index	457

Volume IV, Differentiable Functions in Several Variables	463
Preface	477
Introduction to volume IV, Curves and Surfaces	481
13 Differentiable curves and surfaces, and line integrals in several variables	483
13.1 Introduction	483
13.2 Differentiable curves	483
13.3 Level curves	492
13.4 Differentiable surfaces	495
13.5 Special C^1 -surfaces	499
13.6 Level surfaces	503
14 Examples of tangents (curves) and tangent planes (surfaces)	505
14.1 Examples of tangents to curves	505
14.2 Examples of tangent planes to a surface	520
15 Formulæ	541
15.1 Squares etc.	541
15.2 Powers etc.	541
15.3 Differentiation	542
15.4 Special derivatives	542
15.5 Integration	544
15.6 Special antiderivatives	545
15.7 Trigonometric formulæ	547
15.8 Hyperbolic formulæ	549
15.9 Complex transformation formulæ	550
15.10 Taylor expansions	550
15.11 Magnitudes of functions	551
Index	553
Volume V, Differentiable Functions in Several Variables	559
Preface	573
Introduction to volume V, The range of a function, Extrema of a Function in Several Variables	577
16 The range of a function	579
16.1 Introduction	579
16.2 Global extrema of a continuous function	581
16.2.1 A necessary condition	581
16.2.2 The case of a closed and bounded domain of f	583
16.2.3 The case of a bounded but not closed domain of f	599
16.2.4 The case of an unbounded domain of f	608
16.3 Local extrema of a continuous function	611
16.3.1 Local extrema in general	611
16.3.2 Application of Taylor's formula	616
16.4 Extremum for continuous functions in three or more variables	625
17 Examples of global and local extrema	631
17.1 MAPLE	631
17.2 Examples of extremum for two variables	632
17.3 Examples of extremum for three variables	668

17.4 Examples of maxima and minima	677
17.5 Examples of ranges of functions	769
18 Formulæ	811
18.1 Squares etc.	811
18.2 Powers etc.	811
18.3 Differentiation	812
18.4 Special derivatives	812
18.5 Integration	814
18.6 Special antiderivatives	815
18.7 Trigonometric formulæ	817
18.8 Hyperbolic formulæ	819
18.9 Complex transformation formulæ	820
18.10 Taylor expansions	820
18.11 Magnitudes of functions	821
Index	823
Volume VI, Antiderivatives and Plane Integrals	829
Preface	841
Introduction to volume VI, Integration of a function in several variables	845
19 Antiderivatives of functions in several variables	847
19.1 The theory of antiderivatives of functions in several variables	847
19.2 Templates for gradient fields and antiderivatives of functions in three variables	858
19.3 Examples of gradient fields and antiderivatives	863
20 Integration in the plane	881
20.1 An overview of integration in the plane and in the space	881
20.2 Introduction	882
20.3 The plane integral in rectangular coordinates	887
20.3.1 Reduction in rectangular coordinates	887
20.3.2 The colour code, and a procedure of calculating a plane integral	890
20.4 Examples of the plane integral in rectangular coordinates	894
20.5 The plane integral in polar coordinates	936
20.6 Procedure of reduction of the plane integral; polar version	944
20.7 Examples of the plane integral in polar coordinates	948
20.8 Examples of area in polar coordinates	972
21 Formulæ	977
21.1 Squares etc.	977
21.2 Powers etc.	977
21.3 Differentiation	978
21.4 Special derivatives	978
21.5 Integration	980
21.6 Special antiderivatives	981
21.7 Trigonometric formulæ	983
21.8 Hyperbolic formulæ	985
21.9 Complex transformation formulæ	986
21.10 Taylor expansions	986
21.11 Magnitudes of functions	987
Index	989

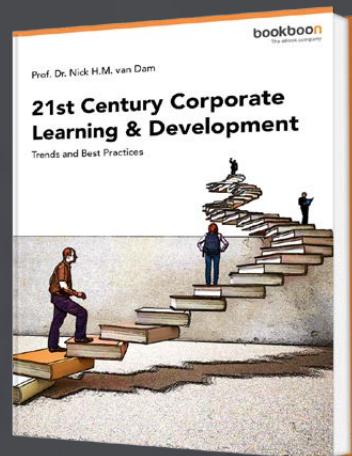
Volume VII, Space Integrals	995
Preface	1009
Introduction to volume VII, The space integral	1013
22 The space integral in rectangular coordinates	1015
22.1 Introduction	1015
22.2 Overview of setting up of a line, a plane, a surface or a space integral	1015
22.3 Reduction theorems in rectangular coordinates	1021
22.4 Procedure for reduction of space integral in rectangular coordinates	1024
22.5 Examples of space integrals in rectangular coordinates	1026
23 The space integral in semi-polar coordinates	1055
23.1 Reduction theorem in semi-polar coordinates	1055
23.2 Procedures for reduction of space integral in semi-polar coordinates	1056
23.3 Examples of space integrals in semi-polar coordinates	1058
24 The space integral in spherical coordinates	1081
24.1 Reduction theorem in spherical coordinates	1081
24.2 Procedures for reduction of space integral in spherical coordinates	1082
24.3 Examples of space integrals in spherical coordinates	1084
24.4 Examples of volumes	1107
24.5 Examples of moments of inertia and centres of gravity	1116
25 Formulæ	1125
25.1 Squares etc.	1125
25.2 Powers etc.	1125
25.3 Differentiation	1126
25.4 Special derivatives	1126
25.5 Integration	1128
25.6 Special antiderivatives	1129
25.7 Trigonometric formulæ	1131
25.8 Hyperbolic formulæ	1133
25.9 Complex transformation formulæ	1134
25.10 Taylor expansions	1134
25.11 Magnitudes of functions	1135
Index	1137
Volume VIII, Line Integrals and Surface Integrals	1143
Preface	1157
Introduction to volume VIII, The line integral and the surface integral	1161
26 The line integral	1163
26.1 Introduction	1163
26.2 Reduction theorem of the line integral	1163
26.2.1 Natural parametric description	1166
26.3 Procedures for reduction of a line integral	1167
26.4 Examples of the line integral in rectangular coordinates	1168
26.5 Examples of the line integral in polar coordinates	1190
26.6 Examples of arc lengths and parametric descriptions by the arc length	1201

27 The surface integral	1227
27.1 The reduction theorem for a surface integral	1227
27.1.1 The integral over the graph of a function in two variables	1229
27.1.2 The integral over a cylindric surface	1230
27.1.3 The integral over a surface of revolution	1232
27.2 Procedures for reduction of a surface integral	1233
27.3 Examples of surface integrals	1235
27.4 Examples of surface area	1296
28 Formulæ	1315
28.1 Squares etc.	1315
28.2 Powers etc.	1315
28.3 Differentiation	1316
28.4 Special derivatives	1316
28.5 Integration	1318
28.6 Special antiderivatives	1319
28.7 Trigonometric formulæ	1321
28.8 Hyperbolic formulæ	1323
28.9 Complex transformation formulæ	1324
28.10 Taylor expansions	1324
28.11 Magnitudes of functions	1325
Index	1327

Free eBook on Learning & Development

By the Chief Learning Officer of McKinsey

Download Now



Volume IX, Transformation formulæ and improper integrals	1333
Preface	1347
Introduction to volume IX, Transformation formulæ and improper integrals	1351
29 Transformation of plane and space integrals	1353
29.1 Transformation of a plane integral	1353
29.2 Transformation of a space integral	1355
29.3 Procedures for the transformation of plane or space integrals	1358
29.4 Examples of transformation of plane and space integrals	1359
30 Improper integrals	1411
30.1 Introduction	1411
30.2 Theorems for improper integrals	1413
30.3 Procedure for improper integrals; bounded domain	1415
30.4 Procedure for improper integrals; unbounded domain	1417
30.5 Examples of improper integrals	1418
31 Formulae	1447
31.1 Squares etc.	1447
31.2 Powers etc.	1447
31.3 Differentiation	1448
31.4 Special derivatives	1448
31.5 Integration	1450
31.6 Special antiderivatives	1451
31.7 Trigonometric formulae	1453
31.8 Hyperbolic formulae	1455
31.9 Complex transformation formulae	1456
31.10 Taylor expansions	1456
31.11 Magnitudes of functions	1457
Index	1459
Volume X, Vector Fields I; Gauß's Theorem	1465
Preface	1479
Introduction to volume X, Vector fields; Gauß's Theorem	1483
32 Tangential line integrals	1485
32.1 Introduction	1485
32.2 The tangential line integral. Gradient fields.	1485
32.3 Tangential line integrals in Physics	1498
32.4 Overview of the theorems and methods concerning tangential line integrals and gradient fields	1499
32.5 Examples of tangential line integrals	1502
33 Flux and divergence of a vector field. Gauß's theorem	1535
33.1 Flux	1535
33.2 Divergence and Gauß's theorem	1540
33.3 Applications in Physics	1544
33.3.1 Magnetic flux	1544
33.3.2 Coulomb vector field	1545
33.3.3 Continuity equation	1548
33.4 Procedures for flux and divergence of a vector field; Gauß's theorem	1549
33.4.1 Procedure for calculation of a flux	1549
33.4.2 Application of Gauß's theorem	1549
33.5 Examples of flux and divergence of a vector field; Gauß's theorem	1551
33.5.1 Examples of calculation of the flux	1551
33.5.2 Examples of application of Gauß's theorem	1580

34 Formulæ	1619
34.1 Squares etc.	1619
34.2 Powers etc.	1619
34.3 Differentiation	1620
34.4 Special derivatives	1620
34.5 Integration	1622
34.6 Special antiderivatives	1623
34.7 Trigonometric formulæ	1625
34.8 Hyperbolic formulæ	1627
34.9 Complex transformation formulæ	1628
34.10 Taylor expansions	1628
34.11 Magnitudes of functions	1629
Index	1631
Volume XI, Vector Fields II; Stokes's Theorem	1637
Preface	1651
Introduction to volume XI, Vector fields II; Stokes's Theorem; nabla calculus	1655
35 Rotation of a vector field; Stokes's theorem	1657
35.1 Rotation of a vector field in \mathbb{R}^3	1657
35.2 Stokes's theorem	1661
35.3 Maxwell's equations	1669
35.3.1 The electrostatic field	1669
35.3.2 The magnetostatic field	1671
35.3.3 Summary of Maxwell's equations	1679
35.4 Procedure for the calculation of the rotation of a vector field and applications of Stokes's theorem	1682

www.sylvania.com

We do not reinvent the wheel we reinvent light.

Fascinating lighting offers an infinite spectrum of possibilities: Innovative technologies and new markets provide both opportunities and challenges. An environment in which your expertise is in high demand. Enjoy the supportive working atmosphere within our global group and benefit from international career paths. Implement sustainable ideas in close cooperation with other specialists and contribute to influencing our future. Come and join us in reinventing light every day.

Light is OSRAM

OSRAM
SYLVANIA

Click on the ad to read more

35.5 Examples of the calculation of the rotation of a vector field and applications of Stokes's theorem	1684
35.5.1 Examples of divergence and rotation of a vector field	1684
35.5.2 General examples	1691
35.5.3 Examples of applications of Stokes's theorem	1700
36 Nabla calculus	1739
36.1 The vectorial differential operator ∇	1739
36.2 Differentiation of products	1741
36.3 Differentiation of second order	1743
36.4 Nabla applied on \mathbf{x}	1745
36.5 The integral theorems	1746
36.6 Partial integration	1749
36.7 Overview of Nabla calculus	1750
36.8 Overview of partial integration in higher dimensions	1752
36.9 Examples in nabla calculus	1754
37 Formulæ	1769
37.1 Squares etc.	1769
37.2 Powers etc.	1769
37.3 Differentiation	1770
37.4 Special derivatives	1770
37.5 Integration	1772
37.6 Special antiderivatives	1773
37.7 Trigonometric formulæ	1775
37.8 Hyperbolic formulæ	1777
37.9 Complex transformation formulæ	1778
37.10 Taylor expansions	1778
37.11 Magnitudes of functions	1779
Index	1781
Volume XII, Vector Fields III; Potentials, Harmonic Functions and Green's Identities	1787
Preface	1801
Introduction to volume XII, Vector fields III; Potentials, Harmonic Functions and Green's Identities	1805
38 Potentials	1807
38.1 Definitions of scalar and vectorial potentials	1807
38.2 A vector field given by its rotation and divergence	1813
38.3 Some applications in Physics	1816
38.4 Examples from Electromagnetism	1819
38.5 Scalar and vector potentials	1838
39 Harmonic functions and Green's identities	1889
39.1 Harmonic functions	1889
39.2 Green's first identity	1890
39.3 Green's second identity	1891
39.4 Green's third identity	1896
39.5 Green's identities in the plane	1898
39.6 Gradient, divergence and rotation in semi-polar and spherical coordinates	1899
39.7 Examples of applications of Green's identities	1901
39.8 Overview of Green's theorems in the plane	1909
39.9 Miscellaneous examples	1910

40 Formulæ	1923
40.1 Squares etc.	1923
40.2 Powers etc.	1923
40.3 Differentiation	1924
40.4 Special derivatives	1924
40.5 Integration	1926
40.6 Special antiderivatives	1927
40.7 Trigonometric formulæ	1929
40.8 Hyperbolic formulæ	1931
40.9 Complex transformation formulæ	1932
40.10 Taylor expansions	1932
40.11 Magnitudes of functions	1933
Index	1935



Deloitte.

Discover the truth at www.deloitte.ca/careers

© Deloitte & Touche LLP and affiliated entities.



Click on the ad to read more

Preface

The topic of this series of books on “*Real Functions in Several Variables*” is very important in the description in e.g. *Mechanics* of the real 3-dimensional world that we live in. Therefore, we start from the very beginning, modelling this world by using the coordinates of \mathbb{R}^3 to describe e.g. a motion in space. There is, however, absolutely no reason to restrict ourselves to \mathbb{R}^3 alone. Some motions may be rectilinear, so only \mathbb{R} is needed to describe their movements on a line segment. This opens up for also dealing with \mathbb{R}^2 , when we consider plane motions. In more elaborate problems we need higher dimensional spaces. This may be the case in *Probability Theory* and *Statistics*. Therefore, we shall in general use \mathbb{R}^n as our abstract model, and then restrict ourselves in examples mainly to \mathbb{R}^2 and \mathbb{R}^3 .

For rectilinear motions the familiar *rectangular coordinate system* is the most convenient one to apply. However, as known from e.g. *Mechanics*, circular motions are also very important in the applications in engineering. It becomes natural alternatively to apply in \mathbb{R}^2 the so-called *polar coordinates* in the plane. They are convenient to describe a circle, where the rectangular coordinates usually give some nasty square roots, which are difficult to handle in practice.

Rectangular coordinates and polar coordinates are designed to model each their problems. They supplement each other, so difficult computations in one of these coordinate systems may be easy, and even trivial, in the other one. It is therefore important always in advance carefully to analyze the geometry of e.g. a domain, so we ask the question: Is this domain best described in rectangular or in polar coordinates?

Sometimes one may split a problem into two subproblems, where we apply rectangular coordinates in one of them and polar coordinates in the other one.

It should be mentioned that in *real life* (though not in these books) one cannot always split a problem into two subproblems as above. Then one is really in trouble, and more advanced mathematical methods should be applied instead. This is, however, outside the scope of the present series of books.

The idea of polar coordinates can be extended in two ways to \mathbb{R}^3 . Either to *semi-polar* or *cylindrical coordinates*, which are designed to describe a cylinder, or to *spherical coordinates*, which are excellent for describing spheres, where rectangular coordinates usually are doomed to fail. We use them already in daily life, when we specify a place on Earth by its longitude and latitude! It would be very awkward in this case to use rectangular coordinates instead, even if it is possible.

Concerning the contents, we begin this investigation by modelling point sets in an n -dimensional Euclidean space E^n by \mathbb{R}^n . There is a subtle difference between E^n and \mathbb{R}^n , although we often identify these two spaces. In E^n we use *geometrical methods* without a coordinate system, so the objects are independent of such a choice. In the coordinate space \mathbb{R}^n we can use ordinary calculus, which in principle is not possible in E^n . In order to stress this point, we call E^n the “abstract space” (in the sense of calculus; not in the sense of geometry) as a warning to the reader. Also, whenever necessary, we use the colour black in the “abstract space”, in order to stress that this expression is theoretical, while variables given in a chosen coordinate system and their related concepts are given the colours blue, red and green.

We also include the most basic of what mathematicians call *Topology*, which will be necessary in the following. We describe what we need by a function.

Then we proceed with limits and continuity of functions and define continuous curves and surfaces, with parameters from subsets of \mathbb{R} and \mathbb{R}^2 , resp..

Continue with (partial) differentiable functions, curves and surfaces, the chain rule and Taylor's formula for functions in several variables.

We deal with maxima and minima and extrema of functions in several variables over a domain in \mathbb{R}^n . This is a very important subject, so there are given many worked examples to illustrate the theory.

Then we turn to the problems of integration, where we specify four different types with increasing complexity, *plane integral, space integral, curve (or line) integral and surface integral*.

Finally, we consider *vector analysis*, where we deal with vector fields, Gauß's theorem and Stokes's theorem. All these subjects are very important in theoretical Physics.

The structure of this series of books is that each subject is *usually* (but not always) described by three successive chapters. In the first chapter a brief theoretical theory is given. The next chapter gives some practical guidelines of how to solve problems connected with the subject under consideration. Finally, some worked out examples are given, in many cases in several variants, because the standard solution method is seldom the only way, and it may even be clumsy compared with other possibilities.

I have as far as possible structured the examples according to the following scheme:

A Awareness, i.e. a short description of what is the problem.

D Decision, i.e. a reflection over what should be done with the problem.

I Implementation, i.e. where all the calculations are made.

C Control, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

From high school one is used to immediately to proceed to **I. Implementation**. However, examples and problems at university level, let alone situations in real life, are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be executed.

This is unfortunately not the case with **C Control**, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of \wedge I shall either write "and", or a comma, and instead of \vee I shall write "or". The arrows \Rightarrow and \Leftrightarrow are in particular misunderstood by the students, so they should be totally avoided. They are not telegram short hands, and from a logical point of view they usually do not make sense at all! Instead, write in a plain language what you mean or want to do. This is difficult in the beginning, but after some practice it becomes routine, and it will give more precise information.

When we deal with multiple integrals, one of the possible pedagogical ways of solving problems has been to colour variables, integrals and upper and lower bounds in blue, red and green, so the reader by the colour code can see in each integral what is the variable, and what are the parameters, which

do not enter the integration under consideration. We shall of course build up a hierarchy of these colours, so the order of integration will always be defined. As already mentioned above we reserve the colour black for the theoretical expressions, where we cannot use ordinary calculus, because the symbols are only shorthand for a concept.

The author has been very grateful to his old friend and colleague, the late Per Wennerberg Karlsson, for many discussions of how to present these difficult topics on real functions in several variables, and for his permission to use his textbook as a template of this present series. Nevertheless, the author has felt it necessary to make quite a few changes compared with the old textbook, because we did not always agree, and some of the topics could also be explained in another way, and then of course the results of our discussions have here been put in writing for the first time.

The author also adds some calculations in MAPLE, which interact nicely with the theoretic text. Note, however, that when one applies MAPLE, one is forced first to make a geometrical analysis of the domain of integration, i.e. apply some of the techniques developed in the present books.

The theory and methods of these volumes on "Real Functions in Several Variables" are applied constantly in higher Mathematics, Mechanics and Engineering Sciences. It is of paramount importance for the calculations in *Probability Theory*, where one constantly integrate over some point set in space.

It is my hope that this text, these guidelines and these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
March 21, 2015

SIMPLY CLEVER

ŠKODA



We will turn your CV into
an opportunity of a lifetime



Do you like cars? Would you like to be a part of a successful brand?
We will appreciate and reward both your enthusiasm and talent.
Send us your CV. You will be surprised where it can take you.

Send us your CV on
www.employerforlife.com



Click on the ad to read more

Introduction to volume II, Continuous Functions in Several Variables

This is the second volume in the series of books on *Real Functions in Several Variables*. We start in Chapter 5 with the necessary theoretical background. Here we assume that the theory of volume I is known by the reader.

We introduce maps and functions, including vector functions, and we give some guidelines on how to visualize such functions. This is not always an easy task, because we easily are forced to consider graphs lying in spaces of dimension ≥ 4 , where very few human beings have a geometrical understanding of what is going on.

Then we introduce the *continuous functions*, starting with defining the basic concept of what we understand by taking a limit. We must apparently have some sense of “distance” in order to say that two points are close to each other. We therefore make use of the topological notions of norm and distance already introduced in volume I.

Continuous functions are then defined as functions, for which “the image points are lying close together, whenever the points themselves are close to each other”. We of course make this more precise in the text.

The first application of continuous functions is to introduce *continuous curves*. The safest description of such curves, though it is not always necessary, is to use a *parametric description* of them. This is also done in MAPLE, and at the same time we get a sense of direction of a motion along the curve from an initial point to a final point.

Then we use the continuous curves to define (curve) connected sets, which are the only connected sets we shall consider here. (There exist sets which are connected, but not curve connected; but they will not be of interest to us.) A set A is (curve) connected, if any two points \mathbf{x} and $\mathbf{y} \in A$ can always be connected with a continuous curve, which lies entirely in A . If $A \in \mathbb{R}^n$ is *open*, then any two points can always be connected by a continuous curve of a very special and convenient structure. The curve consists of concatenated line segments, where each of them is parallel to one of the axes in \mathbb{R}^n . This property will be very useful in the theory of integration later on.

If furthermore, two curves connecting any two given points \mathbf{x} and $\mathbf{y} \in A$ can be transformed continuously into each other without leaving A during this transformation process, then A in some sense “does not contain holes”, and A is called *simply connected*. As one would expect, simply connected sets have better properties than sets, which are only connected.

Once we have introduced continuous curves, using a parametric description, where the parameter set I of course is a one-dimensional interval, it is formally straightforward to replace this one-dimensional parameter interval I for a one-dimensional curve by a two-dimensional interval to get a two-dimensional surface. Then we discover that it is not essential that the parameter set indeed is an interval. A two-dimensional connected set will suffice.

The vague definition above of a surface is of course not precise, so we must first get rid of all pathological cases, but in general a continuous function $\mathbf{r} : E \rightarrow \mathbb{R}^n$, where E is a two-dimensional connected set, defines a two-dimensional surface \mathcal{F} in \mathbb{R}^n . If $n = 3$, we can visualize the process of the function \mathbf{r} as taking a two-dimensional plate of shape E and then bend, compress and stretch this plate, such that we in the end obtain the surface \mathcal{F} of the wanted shape in e.g. \mathbb{R}^3 .

The above gives the general idea, although matters are not always that easy.

A parameter set $E \subseteq \mathbb{R}^2$ may have a non-empty boundary ∂E . We would expect that it is mapped by \mathbf{r} into the “boundary” $\delta\mathcal{F}$ of the surface \mathcal{F} . Since topologically $\mathcal{F} = \partial\mathcal{F}$ is equal to its own boundary, we must describe, what is meant by the “boundary” of the different notation $\delta\mathcal{F}$ in \mathcal{F} . Usually, $\delta\mathcal{F} = \mathbf{r}(\partial E)$, but is easy to construct examples, where $\delta\mathcal{F} (\subseteq \mathbf{r}(\partial E))$ is not equal to $\mathbf{r}(\partial E)$.

Finally we recall (without proofs) the three main theorems for continuous functions, and we show some of their simplest implications, which will be used over and over again in the following volumes.

Chapter 6 on practical guidelines is very short in this volume.

Then follows a fairly long Chapter 7 with examples, following more or less the same structure as the theoretical Chapter 5, so the reader may consult both chapter, when reading this book.

Chapter 8 on Formulae is identical with Chapter 4 in volume I. It is convenient to have these formulae at the end of the books as reference, although many people alternatively may use MAPLE or MATHEMATICA instead.

The index is the same in all volumes, and it covers the whole text.

e-learning for kids

The number 1 MOOC for Primary Education
Free Digital Learning for Children 5-12
15 Million Children Reached

About e-Learning for Kids Established in 2004, e-Learning for Kids is a global nonprofit foundation dedicated to fun and free learning on the Internet for children ages 5 - 12 with courses in math, science, language arts, computers, health and environmental skills. Since 2005, more than 15 million children in over 190 countries have benefitted from eLessons provided by EFK! An all-volunteer staff consists of education and e-learning experts and business professionals from around the world committed to making difference. eLearning for Kids is actively seeking funding, volunteers, sponsors and courseware developers; get involved! For more information, please visit www.e-learningforkids.org.



Click on the ad to read more

5 Continuous maps and functions in several variables

5.1 Maps in general

We shall restrict ourselves to the concept of a map from a subset of \mathbb{R}^n into \mathbb{R}^m , i.e. a map is here defined on a set $D \subseteq \mathbb{R}^n$ in a coordinate space,

$$\mathbf{f} : D \rightarrow \mathbb{R}^m, \quad \mathbf{x} \mapsto \mathbf{f}(\mathbf{x}), \quad \text{where } D \subseteq \mathbb{R}^n.$$

This is the precise notation, but it is in general too complicated, so we shall allow ourselves to use a shorthand like

$$\mathbf{f} : D \rightarrow \mathbb{R}^m, \quad \text{where } D \subseteq \mathbb{R}^n.$$

If \mathbb{R}^m and \mathbb{R}^n already are given, we shall just write

$$\mathbf{f}(\mathbf{x}) \quad \text{for } \mathbf{x} \in D, \quad \text{or just } \mathbf{f}(\mathbf{x}) \text{ or } \mathbf{f}.$$

The notation

$$D \xrightarrow{\mathbf{f}} \mathbb{R}^m$$

may be useful, when we put several maps together into the same schematic structure in order to get a feeling of what is going on, when we e.g. form some compositions of maps.

The map $\mathbf{f} : D \rightarrow \mathbb{R}^m$ has its *domain* $D \subseteq \mathbb{R}^n$, and we call $\mathbf{f}(D) [\subseteq \mathbb{R}^m]$ its *range*. The map is said to be *surjective* $\mathbf{f} : D \rightarrow \mathbf{f}(D)$, i.e. every point of $\mathbf{f}(D)$ is the image of at least one point of D . If every point of $\mathbf{f}(D)$ is the image of precisely one point $\mathbf{x} \in D$, then \mathbf{f} is called *injective*.. If $\mathbf{f} : D \rightarrow \mathbb{R}^m$ is injective, then as seen above, it is both an injective and surjective map of D onto the range $\mathbf{f}(D)$, and we call in this case \mathbf{f} a *bijective map* or a *1-1 map*.

We shall use a little of our previously introduced *Topology*. We say that a map $\mathbf{f} : D \rightarrow \mathbb{R}^m$ is *bounded*, if there exists a ball B of finite radius in \mathbb{R}^m , such that $\mathbf{f}(D) \subseteq B$. The terminology agrees with what one would expect. A ball of finite radius must be bounded, and so is every subset of this ball.

It must be emphasized that a map $\mathbf{f} : D \rightarrow \mathbb{R}^m$ is specified by the operations defined by \mathbf{f} itself, *as well of its specified domain D!* If we for some reason extend the domain D to some other D_1 , in which the operations given by \mathbf{f} still make sense, or we let $D_1 \subset D$ be a real subset of D , so \mathbf{f} is defined by restriction to D_1 , then $\mathbf{f}_1 : D_1 \rightarrow \mathbb{R}^m$ is not considered as the same map as $\mathbf{f} \rightarrow \mathbb{R}^m$, although they are strongly related. We note the following important special cases: Given a map $\mathbf{f} : D \rightarrow \mathbb{R}^m$.

1) If $\mathbf{f}_1 : D_1 \rightarrow \mathbb{R}^m$ satisfies

$$D_1 \subset D \quad \text{and} \quad \mathbf{f}_1(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \text{ for all } \mathbf{x} \in D_1,$$

then (\mathbf{f}_1, D_1) is called a *restriction* of (\mathbf{f}, D) .

2) If $\mathbf{f}_1 : D_1 \rightarrow \mathbb{R}^m$ satisfies

$$D \subset D_1 \quad \text{and} \quad \mathbf{f}_1(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \text{ for all } \mathbf{x} \in D,$$

then (\mathbf{f}_1, D_1) is called an *extension* of (\mathbf{f}, D) .

There are of course other possibilities, but they are not as important as the two cases described above.

In practice we shall want to specify the map \mathbf{f} by its coordinates in $D \subseteq \mathbb{R}^n$. This may be written in the following way, or similarly,

$$\mathbf{f}(\mathbf{x}) = \cdots, \quad \text{where } \mathbf{x} \in \cdots,$$

where we for $\mathbf{x} \in \cdots$ write a specification of D using equations or inequalities between expressions in its coordinates.

One problem often occurs in practice. We may by some theoretical analysis have derived the structure of the map \mathbf{f} , but somehow we have not specified its domain D . Then the normal procedure is to analyze \mathbf{f} in order to find the *maximal domain*, in which \mathbf{f} can be defined. Some guidelines are given in Section 5.2 and Chapter 6. This maximal domain is defined by Mathematics alone. We may therefore later for physical reasons be forced to restrict this (mathematical) maximal domain, when we interpret the model in the real world. One example is that we may get a relation (a map) in which the temperature in Kelvin occurs. The maximal domain of the map may in a mathematical sense allow the temperature to be negative, which of course is not possible in Physics.

5.2 Functions in several variables

Assume that the map $\mathbf{f} : D \rightarrow \mathbb{R}$ maps into the real line \mathbb{R} , i.e. $m = 1$. In this case, when the range is one-dimensional it is customary to call \mathbf{f} a function, and we change the notation to $f : D \rightarrow \mathbb{R}$.

Let $f : D \rightarrow \mathbb{R}$ be a function, where the domain $D \subseteq \mathbb{R}^n$ is of dimension ≥ 2 . Then f is called a *function in several (real) variables*. In the present case we have n variables. Using the well-known theory of real functions in one real variable it is possible to derive simple properties of f by restricting f to one-dimensional subsets of D .

We shall in the following illustrate the question of maximal domain of a given function. This was introduced in Section 5.1 in general for maps.

- 1) Given $f_1(x, y) = \exp(x^2 + 2y^2)$ in \mathbb{R}^2 . Since \exp is defined for all $z \in \mathbb{R}$, and $z = x^2 + 2y^2 \in \mathbb{R}$ for all $(x, y) \in \mathbb{R}^2$, the maximal domain is \mathbb{R}^2 .
- 2) Given

$$f_2(x, y) = \sqrt{x} + \sqrt{y} + \frac{1}{xy}$$

in \mathbb{R}^2 . The square root \sqrt{z} is only defined in the real for $z \geq 0$, so we must require that both $x \geq 0$ and $y \geq 0$. However, a denominator must never be zero, so we also require that $xy \neq 0$, and we conclude that the maximal domain is the open first quadrant \mathbb{R}_+^2 .

- 3) Given $f_3(x, y) = \ln(x - 1) + \sqrt{2 - y}$ in \mathbb{R}^2 . The logarithm is only defined, if $z = x - 1 > 0$, i.e. $x > 1$, and the square root is only defined for $z = 2 - y \geq 0$, i.e. for $y \leq 2$. We conclude that the maximal domain of f_3 is $D_3 = [1, +\infty[\times]-\infty, 2]$, where we usually would prefer just to write $x > 1$ and $y \leq 2$ instead.

4) The function

$$f_4(x, y) = \frac{1}{x^2 + 2y^2 - 2x + 1}$$

in \mathbb{R}^2 is defined, when the denominator is $\neq 0$, i.e. when

$$0 \neq x^2 + 2y^2 - 2x + 1 = (x - 1)^2 + 2y^2.$$

The only requirement is that $(x, y) \neq (1, 0)$, so the maximal domain of f_4 is $\mathbb{R}^2 \setminus \{(1, 0)\}$.

5) Given in \mathbb{R}^2 the function

$$f_5(x, y) = \sqrt{4 - x^2 - y^2} + \sqrt{y}.$$

The requirements of the domain are $y \geq 0$ and $4 - x^2 - y^2 \geq 0$, i.e. $x^2 + y^2 \leq 4 = 2^2$, so the maximal domain D is the closed half-disc on Figure 5.1.

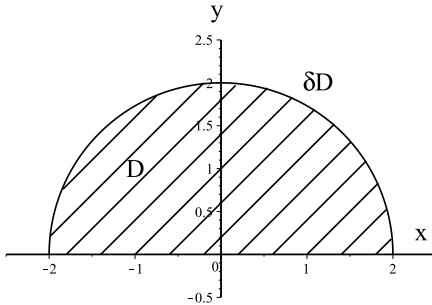


Figure 5.1: The maximal domain of f_5 is a closed half-disc.

Its boundary ∂D is composed of the line segment $[-2, 2]$ on the x -axis, where $y = 0$, and the half-circle $x^2 + y^2 = 2^2 = 4$, $y \geq 0$, in the upper half-plane, i.e. $y = +\sqrt{4 - x^2}$. The restriction of f_5 to ∂D is given by

$$\begin{cases} F_{5,1}(x) = f_5(x, 0) = \sqrt{4 - x^2}, & \text{for } x \in [-2, 2], \\ F_{5,2}(x) = f_5(x, \sqrt{4 - x^2}) & \text{for } x \in [-2, 2]. \end{cases}$$

It is a coincidence that $F_{5,1}$ and $F_{5,2}$ look the same. The reader should note the construction above, because such restrictions to the boundary will be very important in the following chapters, when we shall find the maximum and minimum of a function.

- 6) A commonly used restriction is the restriction of a function to a line. We may in \mathbb{R}^2 use the following parametric description,

$$\varphi(t) := (x_0 + \alpha t, y_0 + \beta t), \quad t \in \mathbb{R},$$

where $(\alpha, \beta) \neq (0, 0)$. If $\alpha = 0$ (and $\beta \neq 0$), we get the vertical line (parametric description)

$$\varphi(y) = (x_0, \beta y), \quad y \in \mathbb{R},$$

where we clearly cannot use x as a parameter. If $\alpha \neq 0$, we may for convenience choose $\alpha = 1$, so by some reformulation we get

$$\varphi(x) = (x, y_0 + \beta x), \quad x \in \mathbb{R}.$$

The parametric description in t above is the safest to apply. It is also used in MAPLE. If we use the other possibilities, there is an unexplainable tendency of forgetting the possibility of a vertical line.

Cynthia | AXA Graduate

AXA Global Graduate Program

Find out more and apply

redefining / standards **AXA**

A hand cursor icon points to the "Click on the ad to read more" button.

7) Consider in \mathbb{R}^2 the function

$$f_7(x, y) = \frac{x-y}{x}.$$

Its maximal domain in mathematical sense is given by $x \neq 0$, i.e. the maximal domain consists of all points in \mathbb{R}^2 , except for the points on the y -axis.

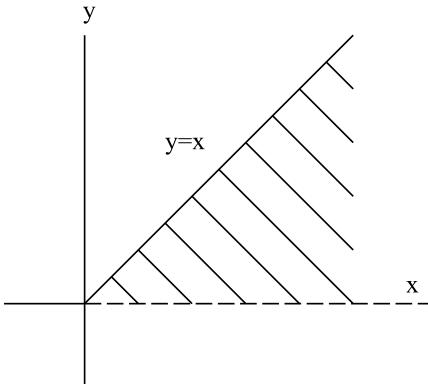


Figure 5.2: The thermodynamical domain of the function f_7 . This is clearly not equal to the maximal domain of f_7 in the mathematical sense.

We may interpret $f_7(x, y)$ in *Thermodynamics* as the theoretical efficiency of a given engine, which interacts with two heat reservoirs, a cold one of temperature y , and a warmer one of temperature x . Then we must require of thermodynamical reasons that

$$x > 0, \quad y > 0, \quad \text{and} \quad x \geq y,$$

because temperatures measured in Kelvin are always positive. This means that the *thermodynamical domain* is the restriction given in Figure 5.2.

5.3 Vector functions

Consider the map $\mathbf{f} : D \rightarrow \mathbb{R}^m$, $D \subseteq \mathbb{R}^n$, where $m > 1$. Then we call \mathbf{f} a *vector function*. It is written in the following way,

$$\mathbf{f} = (f_1, \dots, f_m), \quad \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_m(\mathbf{x})).$$

The functions f_1, \dots, f_m are called the *coordinate functions*. Using the ordinary orthonormal basis in \mathbb{R}^m and the inner (dot) product, the *projections* of $\mathbf{f}(\mathbf{x})$ onto the lines defined by the basis vectors are given by

$$f_1(\mathbf{x}) = \mathbf{e}_1 \cdot \mathbf{f}(\mathbf{x}), \dots, f_m(\mathbf{x}) = \mathbf{e}_m \cdot \mathbf{f}(\mathbf{x}).$$

The *maximal domain* of a vector function $\mathbf{f} = (f_1, \dots, f_m)$ is defined as the intersection of all the maximal domains of its coordinate functions f_1, \dots, f_m .

If $n = m > 1$, i.e. domain and range are of the same dimension > 1 , then the vector function $\mathbf{f} : D \rightarrow \mathbb{R}^m$ is called a *vector field*.

If $n = 1$, and all coordinate functions are differentiable in the variable $t \in D \subseteq \mathbb{R}$, then we define

$$\frac{d\mathbf{f}}{dt} := \left(\frac{df_1}{dt}, \dots, \frac{df_m}{dt} \right).$$

Similarly, if they are all integrable for $t \in [a, b]$,

$$\int_a^b \mathbf{f}(t) dt = \left(\int_a^b f_1(t) dt, \dots, \int_a^b f_m(t) dt \right).$$

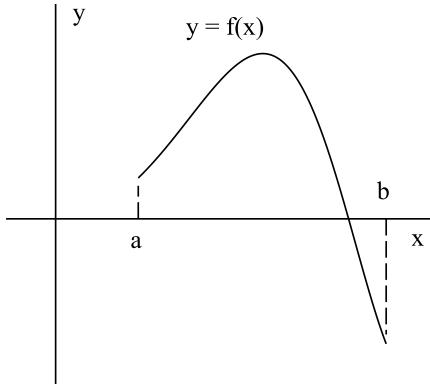


Figure 5.3: The graph of a function f defined in the interval $I = [a, b]$.

5.4 Visualization of functions

Nothing can be more instructive than an illustrative figure. In the case of describing a map we e.g. sketch its *graph*.

Let us first consider an ordinary function in one variable

$$f : I \rightarrow \mathbb{R}, \quad \text{where } I \subseteq \mathbb{R}.$$

Then its *graph* is defined as the set

$$\{(x, y) \in \mathbb{R}^2 \mid y = f(x), x \in I\} \subset \mathbb{R}^2.$$

In the given case, the graph is a curve in the plane \mathbb{R}^2 , cf. Figure 5.3.

A function $f : D \rightarrow \mathbb{R}$ in several variables has similarly given a graph. If e.g. $D \subseteq \mathbb{R}^2$, and $f : D \rightarrow \mathbb{R}$, then the *graph* of f is given by

$$\{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y), (x, y) \in D\}.$$

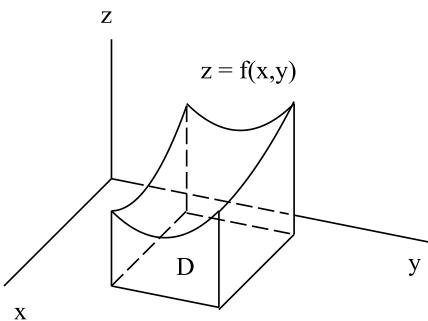


Figure 5.4: The graph of a function f defined in the interval $I = [a, b]$.

In this case the graph becomes a *surface* in \mathbb{R}^3 , cf. Figure 5.4

However, it is often difficult – even in MAPLE – to sketch the graph of a function in two variables, so instead one may introduce *level curves* of f . These are defined by fixing $z = \alpha$, where the constant α is a value of the range of f . Cf. Figure 5.5.

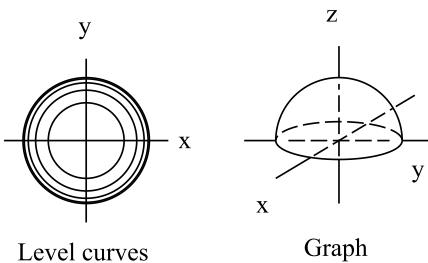


Figure 5.5: To the left we depict the level curves of the function $z = f(x, y) = 1 - x^2 - y^2$ for $\alpha = 0, 0.2, 0.4, 0.6$ and 0.8 . The level curves are not equally spaced. To the right we have for comparison sketched the graph of $z = 1 - x^2 - y^2$. The level curves are in the xy -plane, while the graph lies in the xyz -space. We note that when the level curves are close to each others, the graph is very steep.

If the domain D is of dimension 3 (or higher), the graph description of the function $f : D \rightarrow \mathbb{R}$ becomes impossible, because the graph is then at least a curved 3-dimensional space in the 4-dimensional \mathbb{R}^4 . The author has only met one person, who actually could argue geometrically in E^4 , namely his late professor in Geometry back in the 1970s. He told us young people that he could “see” some “vague

shadows" in E^4 . Not many people have this gift, so we must instead use the idea of level curves. We define in analogy with the above a *level surface* in the following way for a function $f : D \rightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}^3$,

$$\{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = \alpha, (x, y, z) \in D\}, \quad \alpha \in f(D) \text{ fixed.}$$

In general, the level surfaces may be complicated to sketch. However, the idea is not quite impossible in all cases.

Obviously, vector functions are far more difficult to visualize, unless one restricts oneself to only considering each coordinate function separately. Another possibility is to sketch the so-called *field lines*, which are curves which in each point take the value (a vector) of the vector function as its tangent.

I joined MITAS because
I wanted **real responsibility**

The Graduate Programme
for Engineers and Geoscientists
www.discovermitas.com



Month 16

I was a construction supervisor in the North Sea advising and helping foremen solve problems

Real work
International opportunities
Three work placements

5.5 Implicit given functions

We quite often end up – in particular in the applications in Physics – with an equation in some variables, which clearly are dependent of each other, but where it is not obvious which variable should be chosen as a function of the others, and where the function expression may be quite complicated. In order to explain this problem, let us for simplicity consider the case of three variables, which satisfy a relation like e.g.

$$(5.1) \quad F(x, y, z) = 0,$$

where $F : D \rightarrow \mathbb{R}$, $D \subseteq \mathbb{R}^3$, is a function in three variables. If F is continuous, then (5.1) describes a surface in \mathbb{R}^3 , cf. Section 5.4.

This surface is far from always a graph of a function. If e.g. $F(x, y, z) = x^2 + y^2 + z^2 - 1$, then (5.1) describes the unit sphere. When we solve the equation (5.1) with respect to e.g. z , we get two possible values,

$$x = \pm \sqrt{1 - x^2 - y^2} \quad \text{for } x^2 + y^2 \leq 1,$$

defined in the closed unit disc, and the “function” is not unique. But *locally* we can in the *open* unit disc choose one of the two possible signs and obtain a graph of a continuous function, e.g.

$$(5.2) \quad z = Z(x, y) = +\sqrt{1 - x^2 - y^2}, \quad \text{for } x^2 + y^2 < 1,$$

the graph of which is the open upper half of the unit sphere. (We may of course extend this function by continuity to the closed unit disc by adding $z = Z(x, y) = 0$ for $x^2 + y^2 = 1$ to the definition, but this is not the point here.)

The example of the unit sphere above illustrates the primitive and yet efficient way of isolating one of the variables as a function of the others. We fix a point (x, y) in the projection of the domain $D \subset \mathbb{R}^3$ onto \mathbb{R}^2 and then solve with respect to the remaining variable z . If there is just one solution, then we have found $z = Z(x, y)$ at this particular point (x, y) . If there are several possible values of z , then we must choose one of these. It is usually done, such that

$$(5.3) \quad z = Z(x, y)$$

is *locally continuous* in the neighbourhood of some given point (x_0, y_0) . In this case we say that z is *implicitly* given by (5.1), i.e. an expression of the type

$$F(x, y, z) = 0,$$

while (5.3), i.e.

$$z = Z(x, y) \quad \text{in a neighbourhood of } (x_0, y_0)$$

explicitly expresses z *locally* as a function in a neighbourhood of the given point (x_0, y_0) . In the *explicit case* $z = Z(x, y)$ is just an ordinary function in two variables.

Note in 5.3) the difference between z , which is a *variable*, and Z , which is a *function*, here in two variables. Strictly speaking, the two symbols z and Z must not be confused. They are related, but not identical. However, it is nevertheless customary to let z alone denote both the variable z and the function Z in order to avoid too many symbols.

5.6 Limits and continuity

The definition of a limit of a function in one variable is easy to generalize to limits of functions in several variables, when the *absolute value* $|\cdot|$ in \mathbb{R} is replaced by the previously introduced norm $\|\cdot\|$ in \mathbb{R}^n . We recall that $\|\cdot\|$ is here defined as the *Euclidean norm*, i.e.

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2} \quad \text{for } \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Let $\mathbf{x} \in \mathbb{R}^m$ be a fixed vector. By the symbol

$$\mathbf{x} \rightarrow \mathbf{x}_0$$

we shall understand that whenever we are given an $\varepsilon > 0$, then we restrict \mathbf{x} to the open ball $B(\mathbf{x}, \varepsilon)$, where

$$\|\mathbf{x} - \mathbf{x}_0\| < \varepsilon \quad \text{for all } \mathbf{x} \in B(\mathbf{x}, \varepsilon).$$

More generally, given a set $A \subseteq \mathbb{R}^m$, let $\mathbf{x}_0 \in \overline{A}$, i.e. the closure of A , where we assume that \mathbf{x}_0 is not an isolated point of \overline{A} . This means that

$$A \cap B(\mathbf{x}_0, r) \neq \emptyset \quad \text{for all radii } r > 0.$$

Then we say that

$$\mathbf{x} \rightarrow \mathbf{x}_0 \quad \text{in } A,$$

if

$$\|\mathbf{x} - \mathbf{x}_0\| \rightarrow 0 \quad \text{and } \mathbf{x} \in A \setminus \{\mathbf{x}_0\},$$

or, more explicitly, if for every given $\varepsilon > 0$, the point \mathbf{x} is restricted to the set

$$(A \cap B(\mathbf{x}_0, \varepsilon)) \setminus \{\mathbf{x}_0\}, \quad \text{on which } \|\mathbf{x} - \mathbf{x}_0\| < \varepsilon.$$

We assumed above that $\mathbf{x}_0 \in A$ was bounded, so we could apply balls of centre \mathbf{x}_0 and then shrink them by letting the radius $r \rightarrow 0+$. If A is unbounded, we also have to define, what is meant by $\mathbf{x} \rightarrow \infty$ on A , when $k \geq 2$. We define

$$\mathbf{x} \rightarrow \infty \quad \text{in } A, \quad \text{if } \|\mathbf{x}\| \rightarrow +\infty \text{ and } \mathbf{x} \in A.$$

Note the difference in notation between the symbol ∞ for the *unspecified infinity* and the *signed infinities* $+\infty$ and $-\infty$. The latter two are linked to the two directions of the real line $\mathbb{R} =]-\infty, +\infty[$. The unspecified infinity ∞ “lies far away in all possible directions at the same time”. A natural sequence of “neighbourhoods” of ∞ is given by e.g. $\mathbb{R}^m \setminus B[\mathbf{0}, n]$, $n \in \mathbb{N}$, where we let $n \rightarrow +\infty$, or similarly. When n increases, then clearly $\mathbb{R}^m \setminus B[\mathbf{0}, n]$ decreases, and points in $\mathbb{R}^m \setminus B[\mathbf{0}, n]$ satisfy $\|\mathbf{x}\| > n$.

Once these concepts have been specified we can build them together and e.g. define

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0, \mathbf{x} \in A} \mathbf{f}(\mathbf{x}) = \mathbf{a}, \quad \text{also written } \mathbf{f}(\mathbf{x}) \rightarrow \mathbf{a} \text{ for } \mathbf{x} \rightarrow \mathbf{x}_0 \text{ in } A.$$

This means that for every $\varepsilon > 0$ there exists a $\delta > 0$, such that

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{a}\| < \varepsilon, \quad \text{whenever } \|\mathbf{x} - \mathbf{x}_0\| < \delta \text{ and } \mathbf{x} \in A.$$

Similarly, for an unbounded set A ,

$$\lim_{\mathbf{x} \rightarrow \infty, \mathbf{x} \in A} \mathbf{f}(\mathbf{x}) = \mathbf{a}, \quad \text{also written } \mathbf{f}(\mathbf{x}) \rightarrow \mathbf{a} \text{ for } \mathbf{x} \rightarrow \infty \text{ in } A,$$

means that for every $\varepsilon > 0$ there is an $R > 0$, such that

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{a}\| < \varepsilon, \quad \text{whenever } \|\mathbf{x}\| > R \text{ and } \mathbf{x} \in A.$$

The rules of computation known from the 1-dimensional case, i.e. sum, difference, and if $m = 1$, product and quotient (provided that the denominator is always $\neq 0$) are easily extended to limits in several variables.

We also obtain some new rules of computation like e.g.: If (for images in the same \mathbb{R}^m)

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0, \mathbf{x} \in A} \mathbf{f}(\mathbf{x}) = \mathbf{a} \in \mathbb{R}^m \quad \text{and} \quad \lim_{\mathbf{x} \rightarrow \mathbf{x}_0, \mathbf{x} \in A} \mathbf{g}(\mathbf{x}) = \mathbf{b} \in \mathbb{R}^m,$$

then

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0, \mathbf{x} \in A} \{\mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x})\} = \mathbf{a} \cdot \mathbf{b},$$

where “.” is the *inner* (or dot) *product*.

When we restrict ourselves to \mathbb{R}^3 , i.e. choose $m = 3$, we get a similar result for the *vector* (or cross) *product*.

Another important result is that

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0, \mathbf{x} \in A} \mathbf{f}(\mathbf{x}) = \mathbf{a} = (a_1, \dots, a_m),$$

if and only if for all coordinate functions,

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0, \mathbf{x} \in A} f_1(\mathbf{x}) = a_1, \dots, \lim_{\mathbf{x} \rightarrow \mathbf{x}_0, \mathbf{x} \in A} f_m(\mathbf{x}) = a_m.$$

We shall briefly sketch some methods, which may show us, if a function $f(\mathbf{x})$ has a limit for $\mathbf{x} \rightarrow \mathbf{x}_0$, or if this is not the case. We shall illustrate the methods in RR^2 , where we for simplicity choose $\mathbf{x}_0 = \mathbf{0}$.

1) *A direct proof of convergence for $\mathbf{x} \rightarrow \mathbf{0}$ by comparing the magnitudes of the numerator and the denominator.* As an illustrative example we consider the function

$$f_1(x, y) = \frac{xy^2}{x^2 + y^2} \quad \text{for } (x, y) \neq (0, 0).$$

The numerator is a homogeneous monomial in (x, y) of degree $1 + 2 = 3$, while the denominator is a homogeneous polynomial in (x, y) of degree 2. Thus, if ϱ denotes the radius in polar coordinates,

then we have roughly ϱ^3 in the numerator and ϱ^2 in the denominator, so $f_1(x, y) \sim \varrho$, which tends towards 0 for $\varrho \rightarrow 0+$.

More precisely, in polar coordinates,

$$x = \varrho \cos \varphi \quad \text{and} \quad y = \varrho \sin \varphi,$$

so

$$f_1(x, y) = \frac{xy^2}{x^2 + y^2} = \frac{\varrho \cos \varphi \cdot \varrho^2 \sin^2 \varphi}{\varrho^2} = \varrho \cos \varphi \sin^2 \varphi \quad \text{for } \varrho > 0 \text{ and } \varphi \in \mathbb{R}.$$

To prove that $f_1(x, y) \rightarrow 0$ for $(x, y) \rightarrow (0, 0)$, i.e. for $\varrho \rightarrow 0+$, we simply use the definition and estimate,

$$|f_1(x, y) - 0| = |\varrho \cos \varphi \sin^2 \varphi - 0| \leq \varrho \rightarrow 0 \quad \text{for } \varrho \rightarrow 0+,$$

from which we conclude that $f_1(x, y) \rightarrow 0$ for $(x, y) \rightarrow (0, 0)$.

The advertisement features a background image of modern skyscrapers. On the left, the IE business school logo is displayed. In the center, a large white box contains the text: "93% OF MIM STUDENTS ARE WORKING IN THEIR SECTOR 3 MONTHS FOLLOWING GRADUATION". Below this, the program name "MASTER IN MANAGEMENT" is prominently displayed. To the right, a vertical column lists program details: "Length: 10 MONTHS", "Av. Experience: 1 YEAR", "Language: ENGLISH / SPANISH", "Format: FULL-TIME", and "Intakes: SEPT / FEB". At the bottom, three boxes highlight program features: "5 SPECIALIZATIONS PERSONALIZE YOUR PROGRAM", "#10 WORLDWIDE MASTER IN MANAGEMENT FINANCIAL TIMES", and "55 NATIONALITIES IN CLASS". The footer includes links to the website and admissions email, and social media icons for Facebook, Twitter, and LinkedIn, along with a "Follow us on IE MIM Experience" call to action.

www.ie.edu/master-management | mim.admissions@ie.edu | [f](#) [t](#) [li](#) Follow us on IE MIM Experience

- 2) A proof of divergence for $\mathbf{x} \rightarrow \mathbf{0}$ by comparing the magnitudes of the numerator and the denominator. If we change f_1 above to

$$f_2(x, y) = \frac{xy^2}{x^4 + y^4} \quad \text{for } (x, y) \neq (0, 0),$$

then the numerator is a monomial of degree 3, and the denominator is a homogeneous polynomial of degree 4. In this case we get $f_2(x, y) \sim 1/\varrho$, so we would expect divergence for $\varrho \rightarrow 0+$. To prove this we again apply polar coordinates, so

$$f_2(x, y) = \frac{xy^2}{x^4 + y^4} = \frac{\varrho^3 \cos \varphi \sin^2 \varphi}{\varrho^4 (\cos^4 \varphi + \sin^4 \varphi)} = \frac{1}{\varrho} \cdot \frac{\cos \varphi \sin^2 \varphi}{\cos^4 \varphi + \sin^4 \varphi}.$$

If $\varphi = n\pi/2$, $n \in \mathbb{Z}$, i.e. if (x, y) lies on either the x -axis or the y -axis, then clearly $f_2(x, y) = 0$, and in the limit $\varrho \rightarrow 0+$ we also get 0. If instead $\varphi \neq n\pi/2$, $n \in \mathbb{N}$, is kept fixed, then clearly $|f_2(x, y)| \rightarrow +\infty$ for $\varrho \rightarrow 0+$, so $f(x, y)$ is divergent for $(x, y) \rightarrow (0, 0)$. The argument above shows also that $f_2(x, y)$ does not diverge towards ∞ either.

- 3) Proof of divergence by restricting ourselves to straight lines. Consider again

$$f_2(x, y) = \frac{xy^2}{x^4 + y^4} \quad \text{for } (x, y) \neq (0, 0),$$

above. We have seen already that $f(0, y) = f(x, 0) = 0$, so along the axes we get the limit 0 at $(0, 0)$. A straight line through $(0, 0)$ is either given by the vertical y -axis, or it is described by the equation $y = \alpha x$ for some constant $\alpha \in \mathbb{R}$. Then by insertion for $(x, y) = (x, \alpha y)$ on this line,

$$f_2(x, \alpha x) = \frac{x^3 \alpha^2}{x^4 (1 + \alpha^4)} = \frac{1}{x} \cdot \frac{\alpha^2}{1 + \alpha^4}.$$

Choose any $\alpha \neq 0$, and the α -factor is a constant $\neq 0$, while $|1/x| \rightarrow +\infty$ for $x \rightarrow 0$, and $f_2(x, y)$ diverges for $(x, y) \rightarrow (0, 0)$.

Another illustrative example is the following, where both the numerator and the denominator are homogeneous polynomials of the same degree 2. We consider the function

$$f_3(x, y) = \frac{xy}{x^2 + y^2} \quad \text{for } (x, y) \neq (0, 0).$$

Clearly, $f_3(x, 0) = f_3(0, y) = 0$, so if the function converges, then the limit must necessarily be 0. This is not the case, for if we restrict ourselves to the straight line $y = \alpha x$ and exclude $(0, 0)$, then we get

$$f_3(x, \alpha x) = \frac{\alpha}{1 + \alpha^2},$$

which for $\alpha \neq 0$ is a constant $\neq 0$ along this straight line, so this must also be the limit along this line. But then we have found a different candidate of the limit, contradicting that the limit is unique. Hence, $f_3(x, y)$ is divergent for $(x, y) \rightarrow (0, 0)$.

A variant is of course to use polar coordinates, in which case

$$f_3(x, y) = \cos \varphi \sin \varphi = \frac{1}{2} \sin 2\varphi,$$

independent of ϱ , so along a straight half-line of angle φ the value of $f_3(x, y)$ is given by $(\sin 2\varphi)/2$, which is a nonconstant function in the angle φ , and we conclude again that $f_3(x, y)$ is divergent for $(x, y) \rightarrow (0, 0)$.

4) *Analysis of level curves.* In this case consider the function

$$f_4(x, y) = \frac{x}{x^2 + y^2} \quad \text{for } (x, y) \neq (0, 0).$$

Let us first try the already known methods. The numerator is homogeneous of degree 1, and the denominator is homogeneous of degree 2, so according to 2) we would expect divergence. Using polar coordinates we get

$$f_4(x, y) = \frac{x}{x^2 + y^2} = \frac{\rho \cos \varphi}{\rho^2} = \frac{1}{\rho} \cos \varphi.$$

Fix $\varphi \neq n\pi + \pi/2$, $n \in \mathbb{Z}$, so $\cos \varphi$ is a constant $\neq 0$. Then clearly

$$|f_4(x, y)| = \frac{1}{\rho} |\cos \varphi| \rightarrow +\infty \quad \text{for } \rho \rightarrow 0+,$$

so $f_4(x, y)$ is divergent for $(x, y) \rightarrow (0, 0)$, and the only possible limit is the unspecified ∞ . But since $f_4(0, y) = 0$ for all $y \neq 0$, this is not tending towards ∞ for $y \rightarrow 0$, so $f_4(x, y)$ is just divergent.

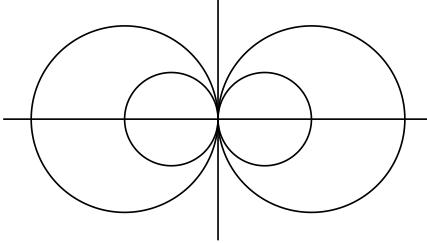


Figure 5.6: Some level curves of $f_4(x, y)$.

Alternatively we may analyze the level curves $f_4(x, y) = c$. If $c = 0$, then $x = 0$, so the level curves of f_4 corresponding to the value 0 are the positive and the negative y -axes.

If instead $c \neq 0$, and $(x, y) \neq (0, 0)$, then

$$f_4(x, y) = \frac{x}{x^2 + y^2} = c, \quad \text{if and only if} \quad x^2 + y^2 = \frac{1}{c}x,$$

which we rewrite as

$$\left(x - \frac{1}{2c}\right)^2 + y^2 = \frac{1}{4c^2}.$$

The level curve corresponding to the value $c \neq 0$ is therefore, with the exception of the point $(0, 0)$, the circle of centre $\left(\frac{1}{2c}, 0\right)$ and radius $\frac{1}{2|c|} > 0$. Cf. Figure 5.6. When we approach $(0, 0)$ along

the level curve (a circle or the y -axis) of constant c , we get the limit c at $(0, 0)$. Since $c \in \mathbb{R}$ is arbitrary, no unique limit exists, and $f_4(x, y)$ diverges for $(x, y) \rightarrow (0, 0)$.

- 5) *The possibility of restriction to other curves than straight lines.* The method above in 3), where we approach the point \mathbf{x}_0 along straight lines, is only applicable to prove that we have *divergence*. We shall below see that *even if the limit is the same on the restriction of all straight lines, this does not imply that the limit exists!* So the same limit on all straight lines is only a *necessary* and not a sufficient condition for that the limit exists.

Consider the function

$$f_5(x, y) = \frac{x^2 y}{x^4 + y^2} \quad \text{for } (x, y) \neq (0, 0).$$

If $x = 0$, i.e. we restrict ourselves to the y -axis, then

$$f_5(0, y) = 0 \rightarrow 0 \quad \text{for } y \rightarrow 0.$$

Then we restrict ourselves to the straight line of equation $y = \alpha x$, $\alpha \in \mathbb{R}$. Then

$$f_5(x, \alpha x) = \frac{x^2 \cdot \alpha x}{x^4 + \alpha^2 x^2} = \frac{\alpha x}{x^2 + \alpha^2}.$$

If $\alpha = 0$, then clearly

$$f_5(x, 0) = 0 \rightarrow 0 \quad \text{for } x \rightarrow 0.$$

If $\alpha \neq 0$, then

$$|f_5(x, \alpha x) - 0| = \left| \frac{\alpha x}{x^2 + \alpha^2} \right| \leq \left| \frac{\alpha x}{\alpha^2} \right| = \frac{1}{|\alpha|} \cdot |x| \rightarrow 0 \quad \text{for } x \rightarrow 0.$$

Thus we have proved that the limit of $f_5(x, y)$ exists on the restriction to every straight line through $(0, 0)$, when $(x, y) \rightarrow (0, 0)$, and the common value of these limits is 0, and the *necessary* condition is fulfilled.

It is *not sufficient!* To prove this we take a closer look on the denominator $x^4 + y^2$, which is not a homogeneous polynomial in (x, y) . The idea is to choose curves, on which x^4 and y^2 are comparable through the limit process. If we choose the curves $y = \alpha x^2$, $\alpha \in \mathbb{R} \setminus \{0\}$, i.e. a family of parabolas, then $x^4 + y^2 = x^4 \{1 + \alpha^2\}$, which is x^4 times a constant depending on α . Then we get by insertion for fixed α that

$$f_5(x, \alpha x^2) = \frac{x^2 \cdot \alpha x^2}{x^4 + \alpha^2 x^4} = \frac{\alpha}{1 + \alpha^2} \rightarrow \frac{\alpha}{1 + \alpha^2} \quad \text{for } x \rightarrow 0.$$

Hence, the limits exist for $(x, y) \rightarrow (0, 0)$ along these parabolas, but the values are different for different α , so we get lot of different candidates for the limit. This is not possible, because the limit – if it exists – is unique. Hence, the limit of $f_5(x, y)$ does not exist for $(x, y) \rightarrow (0, 0)$.

We emphasize that the methods described in 2)–5) can only be applied to prove *divergence*. To prove *convergence* we either use a direct proof using some estimate like

$$|f(\mathbf{x}) - a| \leq g(\mathbf{x}),$$

where we know – or prove – that $g(\mathbf{x}) \rightarrow 0$ for $\mathbf{x} \rightarrow \mathbf{x}_0$, or we prove that $f(\mathbf{x})$ is a (local) *contraction*. This means that there exists a constant $\alpha \in [0, 1[$, such that

$$|f(\mathbf{x}) - f(\mathbf{y})| < \alpha \|\mathbf{x} - \mathbf{y}\| \quad \text{for } \mathbf{x}, \mathbf{y} \text{ lying close to each other.}$$

5.7 Continuous functions

As in the one-dimensional case we use the concept of a limit, introduced in Section 5.6 to define continuity of a function in several variables.

Definition 5.1 Consider a (vector) function $\mathbf{f} : A \rightarrow \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$, and let $\mathbf{x}_0 \in A$ be a given point. We say that \mathbf{f} is continuous at \mathbf{x}_0 , if

$$\mathbf{f}(\mathbf{x}) \rightarrow \mathbf{f}(\mathbf{x}_0) \quad \text{for } \mathbf{x} \rightarrow \mathbf{x}_0 \text{ in } A.$$

We say that \mathbf{f} is continuous in a subset $B \subseteq A$, if \mathbf{f} is continuous at all points of B .

The traditional way of stating that \mathbf{f} is continuous at $\mathbf{x}_0 \in A$ is the following:

To every given $\varepsilon > 0$ we can find $\delta > 0$, such that

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)\| < \varepsilon, \quad \text{whenever } \mathbf{x} \in A \text{ and } \|\mathbf{x} - \mathbf{x}_0\| < \delta.$$

The usual rules of computation, known from real functions in one real variable, are easily carried over to our present case:

Given two (vector) functions $\mathbf{f}, \mathbf{g} : A \rightarrow \mathbb{R}^m$, and assume that they are both continuous at a given point $\mathbf{x}_0 \in A$. Then the *sum* and *difference* and *inner (dot) product* of \mathbf{f} and \mathbf{g} are all continuous, i.e.

$\mathbf{f} + \mathbf{g}$, $\mathbf{f} - \mathbf{g}$ and $\mathbf{f} \cdot \mathbf{g}$ are all continuous.

"I studied English for 16 years but...
...I finally learned to speak it in just six lessons"

Jane, Chinese architect

ENGLISH OUT THERE

Click to hear me talking before and after my unique course download



Click on the ad to read more

If $m = 3$, then the *vector (cross) product*

$$\mathbf{f} \times \mathbf{g} \quad \text{is continuous,} \quad (\text{in } \mathbb{R}^3).$$

If $m = 1$, then the *scalar product* (note, no notation of the scalar product)

$$fg \quad \text{is continuous,} \quad (\text{in } \mathbb{R}),$$

and also the *scalar quotient*

$$\frac{f}{g} \quad \text{is continuous at } \mathbf{x}_0, \text{ provided that } g(\mathbf{x}) \neq 0 \text{ in a neighbourhood of } \mathbf{x}_0.$$

Assume that $\mathbf{f} : A \rightarrow \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$, and $\mathbf{g} : B \rightarrow \mathbb{R}^n$, $B \subseteq \mathbb{R}^k$, are continuous in their respective domains. If furthermore, $\mathbf{g}(B) \subseteq A$, then the *composition*

$$\mathbf{f} \circ \mathbf{g} : B \rightarrow \mathbb{R}^m$$

exists and is continuous in $B \subseteq \mathbb{R}^k$. It is not hard to prove that *a vector function \mathbf{f} is continuous, if and only if all its coordinate functions are continuous*.

We defined in Section 5.6 the limit of a function $\mathbf{f}(\mathbf{x})$ for $\mathbf{x} \rightarrow \mathbf{x}_0$ in A , where we only required that $\mathbf{x}_0 \in \overline{A}$ is not an isolated point of the closure \overline{A} of A . Assume that $\mathbf{x}_0 \in \overline{A} \setminus A$ and that $\lim_{\mathbf{x} \rightarrow \mathbf{x}_0, \mathbf{x} \in A} \mathbf{f}(\mathbf{x}) = \mathbf{a}$ exists. Then we can extend the domain of \mathbf{f} to also including \mathbf{x}_0 , where the extension is defined by

$$\tilde{\mathbf{f}}(\mathbf{x}) = \begin{cases} \mathbf{f}(\mathbf{x}) & \text{for } \mathbf{x} \in A, \\ \lim_{\mathbf{x} \rightarrow \mathbf{x}_0, \mathbf{x} \in A} \mathbf{f}(\mathbf{x}) = \mathbf{a} & \text{for } \mathbf{x} = \mathbf{x}_0. \end{cases}$$

It follows immediately from this construction that if the extension is defined in $\mathbf{x}_0 \in \overline{A} \setminus A$, then the extension is automatically continuous at this point \mathbf{x}_0 .

We have already met an example of this type in Section 5.6, where we proved that

$$f_1(x, y) = \frac{xy^2}{x^2 + y^2} \quad \text{for } (x, y) \neq (0, 0),$$

has the limit

$$\lim_{(x,y) \rightarrow (0,0)} f_1(x, y) = 0.$$

Hence, the continuous extension of f_1 , defined in all of \mathbb{R}^2 , is given by

$$\tilde{f}_1(x, y) = \begin{cases} \frac{x^2y}{x^2 + y^2} & \text{for } (x, y) \neq (0, 0), \\ 0 & \text{for } (x, y) = (0, 0). \end{cases}$$

Sometimes one may be able to *factorize* the function under consideration and then cancel the common factor, which becomes zero in the limit in both the numerator and the denominator. One of the simplest examples is

$$f_6(x, y) = \frac{x^2 - y^2}{x - y} \quad \text{for } y \neq x.$$

In fact,

$$f_6(x, y) = \frac{x^2 - y^2}{x - y} = \frac{(x - y)(x + y)}{x - y} = x + y \quad \text{for } y \neq x.$$

Only the factor $x - y$ in both the numerator and the denominator is 0 at the exception set, and we cancel them by division in the set where $y \neq x$. Since the quotient $x + y$ also makes sense for $y = x$ (formally we take the limit to this set), the continuous extension of f_6 is defined by

$$\tilde{f}_6(x, y) = x + y \quad \text{for } (x, y) \in \mathbb{R}^2.$$

A more sophisticated example using the same idea is given by

$$f_7(x, y) = \frac{\sin(x + y)}{x + y} \quad \text{for } y \neq -x.$$

A common trick in mathematics is to give an “unpleasant expression” a new name. In this case we put $t := x + y$, and the restriction is then $t \neq 0$, in which case

$$f_7(x, y) = \frac{\sin t}{t}, \quad t = x + y \neq 0.$$

It is well-known from the theory of real functions in one real variable that

$$\lim_{t \rightarrow 0} \frac{\sin t}{t} = 1,$$

which means that $f_7(x, y)$ has the continuous extension to all of \mathbb{R}^2 ,

$$\tilde{f}_7(x, y) = \begin{cases} \frac{\sin(x + y)}{x + y} & \text{for } x + y \neq 0, \\ 1 & \text{for } x + y = 0. \end{cases}$$

5.8 Continuous curves

5.8.1 Parametric description

Intuitively, a continuous curve in \mathbb{R}^m is a path, along which e.g. a particle moves from an initial point to a final point, i.e. we have a sense of which direction the particle moves along the path. We coin these ideas in the following definition.

Definition 5.2 A continuous curve in \mathbb{R}^m is a continuous map $\mathbf{r} : I \rightarrow \mathbb{R}^m$ of a real interval $I \subseteq \mathbb{R}$. If I has the left end point a (including the possibility of $-\infty$) and the right end point b (including the possibility of $+\infty$), we call $\mathbf{r}(a)$ the initial point of the curve, and $\mathbf{r}(b)$ the final point of the curve.

The curve inherits the orientation of the interval I , so roughly speaking, “we are just taking the interval I , and then bend and stretch it” as described by the map $\mathbf{r} : I \rightarrow \mathbb{R}^m$.

Given a continuous curve $\mathbf{r} : I \rightarrow \mathbb{R}^m$. Its image is given by

$$\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^m \mid \mathbf{x} = \mathbf{r}(t), t \in I\} = \{\mathbf{r}(t) \mid t \in I\}.$$

This is often a better way to describe the curve than the formal definition above. Note, however, that it is always safe to use Definition 5.2 in the applications, and this is also the most common construction in MAPLE, where we e.g. in \mathbb{R}^2 write

$$[r_1(t), r_2(t), t = a..b],$$

where $(x, y) = \mathbf{r}(t)$, $t \in [a, b]$.

We call $\mathbf{x} = \mathbf{r}(t)$, $t \in I$, a *parametric description* of the curve \mathcal{K} , and t is the parameter, and I the parameter interval.

Given a continuous curve $\mathbf{r} : I \rightarrow \mathbb{R}^m$. Assume that n different parameters t_1, \dots, t_n , where $n \geq 2$, all are mapped into the same point on the curve,

$$\mathbf{r}(t_1) = \mathbf{r}(t_2) = \dots = \mathbf{r}(t_n) = \mathbf{u} \in \mathbb{R}^m.$$

Then we call the common point $\mathbf{u} \in \mathbb{R}^m$ a *multiple point* (of the curve). If $n = 2$, we may call it a *double point* instead.

Remark 5.1 Even if Definition 5.2 looks very straightforward, it is *not*. It was a shock for the mathematicians, when the Italian mathematician appr. 1900 constructed a continuous curve, which passed through all points in e.g. the unit square. And even worse a couple of years later, when Osgood modified Peano's construction obtaining a continuous curve without multiple points, which Peano's curve had, and of *positive area!* In particular, the unit one-dimensional interval $[0, 1]$, clearly of no area, was mapped continuously and bijectively onto a set of positive area. However, although such *space filling curves* are of interest in their own right, we shall not consider them further in this series of books. We shall be more interested in *differential curves*, for which such phenomena do not occur.
◊

Excellent Economics and Business programmes at:

 **university of
groningen** 



**“The perfect start
of a successful,
international career.”**

CLICK HERE
to discover why both socially
and academically the University
of Groningen is one of the best
places for a student to be

www.rug.nl/feb/education



Click on the ad to read more

Sometimes we may want to consider continuous curves, which are composed of axiparallel line segments. We therefore give such curves a name, namely *step lines*, because we step from one coordinate to the next one, when we run through the curve, only changing one coordinate at a time, which therefore locally can be used as a parameter. Cf. Figure 5.7.

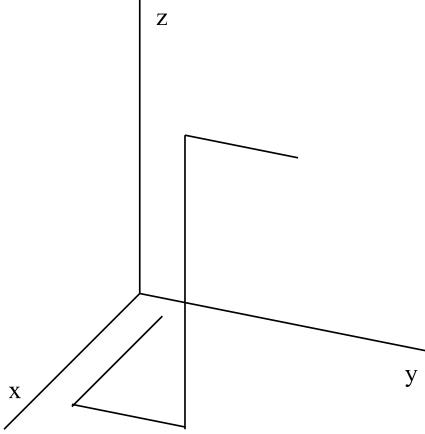


Figure 5.7: An example of a step line.

We list the most commonly used parametric descriptions of curves.

- 1) A *plane curve* of the equation

$$y = Y(x), \quad x \in I,$$

is already given by its parametric description, when we use $t = x \in I$ as its parameter. Its graph is

$$\mathcal{K} = \{(x, Y(x)) \mid x \in I\}.$$

- 2) A *straight line (segment)* in \mathbb{R}^m is given by the parametric description

$$\mathbf{x} = \mathbf{a} + \mathbf{v}t, \quad t \in I,$$

where \mathbf{a} and $\mathbf{v} \in \mathbb{R}^m$ are constant vectors, and $\mathbf{v} \neq \mathbf{0}$, and $I \subseteq \mathbb{R}$ is some given interval.

If $I = \mathbb{R}$, we get an *oriented line* in \mathbb{R}^m . If $I = [a, +\infty[,]a, +\infty[,]-\infty, b[$ or $] - \infty, b]$, we get an *oriented half line* in \mathbb{R}^m . Finally, if I is bounded, we get an *oriented line segment*. The orientation is inherited from the usual orientation of $I \subseteq \mathbb{R}$ with respect to the order relation \leq . The vector $\mathbf{v} \neq \mathbf{0}$ is called the *direction vector* of the line. This is quite often chosen as a unit vector,

- 3) A *circle* of radius $a > 0$ and centre $(0, 0) \in \mathbb{R}^2$ of equation

$$x^2 + y^2 = a^2,$$

is considered as a curve with the parametric description (in polar coordinates)

$$x = a \cos \varphi, \quad y = a \sin \varphi, \quad \varphi \in [0, 2\pi[,$$

or in MAPLE-notation,

$$[a \cdot \cos(t), b \cdot \sin(t), t = 0..2\pi].$$

The circle inherits its orientation from the interval $[0, 2\pi[$. In the present case it is also *positively oriented* in the plane \mathbb{R}^2 . This means that the curve moves counterclockwise around the centre $(0, 0)$.

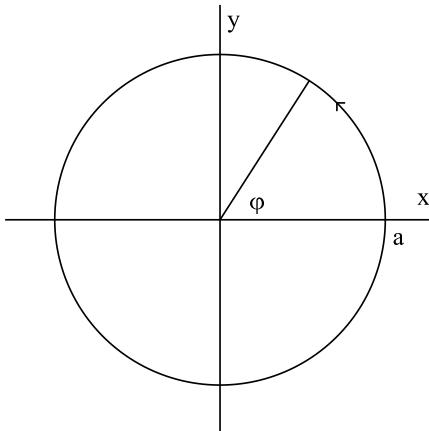


Figure 5.8: A circle of radius a as a curve of positive orientation in the plane \mathbb{R}^2 .

If we want a parameter description in the negative sense of the plane, just replace t by $-t$, so we get instead

$$x = a \cos \varphi, \quad y = -a \sin \varphi, \quad \varphi \in [0, 2\pi[.$$

The parameter descriptions above describe the circle run through just once (and without double points). Other choices of I are possible, like e.g. $]-\pi, \pi]$, where the initial point is $(-1, 0)$ on the negative x -axis. If $I = \mathbb{R}$, then the circle is run through infinitely many times, just to mention a few of the many possibilities.

The parameter $\varphi \in [0, 2\pi[$ can be interpreted as the angle of the radius vector from $(0, 0)$ to the point $(x, y) \neq (0, 0)$ under consideration.

- 4) A modification of the description of the circle above gives us the parametric description of an *ellipse* of the equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

In this case we just multiply the y -coordinate of the circle by the *affinity factor* b/a , so the basic parametric description of the ellipse becomes

$$x = a \cos \varphi, \quad y = b \sin \varphi, \quad \varphi \in [0, 2\pi[,$$

where $a, b > 0$ are the two half axes of the ellipse. Cf. Figure 5.9.

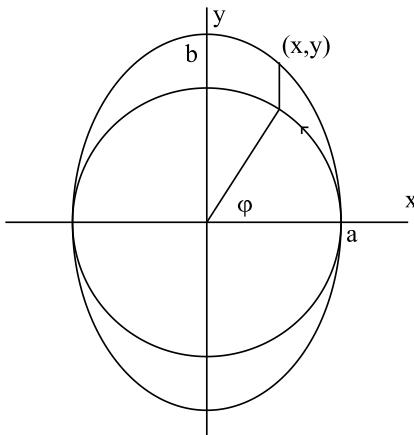


Figure 5.9: An ellipse of half axes a and b in the plane \mathbb{R}^2 is obtained from a circle by an application of an affinity.

5.8.2 Change of parameter of a curve

Given a continuous curve of parametric description

$$\mathbf{x} = \mathbf{r}(t), \quad t \in I.$$

As mentioned earlier, we may interpret the curve as the path of a particle. If we change the speed of this particle, we get another curve,

$$\mathbf{x} = \mathbf{r}_1(u), \quad u \in J.$$

The path itself is of course the same in the two cases, but the parameters do not match, so that is why we say that we have a different curve.

The change from \mathbf{r} to \mathbf{r}_1 is given by a uniquely determined function

$$\Phi : I \rightarrow J, \quad u = \Phi(t),$$

such that

$$\mathbf{r}_1(u) = \mathbf{r}(t) = \mathbf{r}(\Phi(u)) = (\mathbf{r} \circ \Phi)(u).$$

In fact, every point of I must by the monotony of the map correspond to precisely one point of J , and *vice versa*, and this gives us a bijective function $\Phi : I \rightarrow J$.

We call $\Phi : I \rightarrow J$ a *change of parameter*.

5.9 Connectedness

Using continuous curves we can introduce a new important topological concept, which will be used in the sequel. Given a set A , it is important that we can move from one point $\mathbf{x} \in A$ to another point $\mathbf{y} \in A$ along a continuous curve without leaving A during this motion. We coin this property in the following definition.

Definition 5.3 A set $A \subseteq \mathbb{R}^m$ is called connected, if any two points $\mathbf{x}, \mathbf{y} \in A$ can be connected with a continuous curve lying in A , i.e. we can find a continuous function $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^m$, such that

$$\mathbf{x} = \mathbf{r}(\mathbf{a}), \quad \mathbf{y} = \mathbf{r}(\mathbf{b}), \quad \{\mathbf{r}(t) \mid t \in [a, b]\} \subseteq A.$$

In particular *starshaped sets* A are connected, because there exists a point $\mathbf{x}_0 \in A$, which can be reached from any other point $\mathbf{x} \in A$ by a straight line segment in A . So when we construct a path from $\mathbf{x} \in A$ to $\mathbf{y} \in A$, we just take the detour via \mathbf{x}_0 .

In particular, a *convex set* A is connected, because the straight line segment between two points $\mathbf{x}, \mathbf{y} \in A$ also lies totally in A .

One can prove that if a subset $I \subseteq \mathbb{R}$ of the real line is connected, then it is an interval. This may seem obvious, and we have already tacitly used this property, when we described the process of changing parameters.

It will also be convenient to consider any set $A = \{\mathbf{x}_0\}$ consisting of just one point as connected.

American online LIGS University

is currently enrolling in the
Interactive Online BBA, MBA, MSc,
DBA and PhD programs:

- ▶ enroll by **September 30th, 2014** and
- ▶ save up to **16%** on the tuition!
- ▶ pay in 10 installments / 2 years
- ▶ Interactive Online education
- ▶ visit www.ligsuniversity.com to
find out more!

Note: LIGS University is not accredited by any nationally recognized accrediting agency listed by the US Secretary of Education.
[More info here.](#)



Click on the ad to read more

If A is an open and connected set, we call it an *open domain*. If we add some of its boundary points to A , we just call the result a *domain*. And if we add all of its boundary points, then we call it a *closed domain*.

We mention without proof the following theorem, which will be useful in the next volumes of this series. In particular in connection with line integrals.

Theorem 5.1 *Assume that A is an open domain. Then any two points $\mathbf{x}, \mathbf{y} \in A$ from A can be connected by a step line, i.e. a continuous curve consisting of only axiparallel line segments.*

Consider the two connected sets of Figure 5.10, i.e. a disc and an annulus. They clearly do not have the same topological shape, because the annulus contains a hole, which the disc does not. We therefore introduce the following:

Let A be a connected set. Let $\mathbf{x}, \mathbf{y} \in A$ be two points, connected with two continuous curves entirely in A ,

$$\mathbf{r}_0 : [0, 1] \rightarrow A, \quad \mathbf{r}_1 : [0, 1] \rightarrow A, \quad \text{where } \mathbf{r}_0(0) = \mathbf{r}_1(0) = \mathbf{x} \text{ and } \mathbf{r}_0(1) = \mathbf{r}_1(1) = \mathbf{y}.$$

Assume that we can change \mathbf{r}_0 continuously, so that we in the end get to \mathbf{r}_1 , i.e. we can deform the path of \mathbf{r}_0 continuously until we reach the path of \mathbf{r}_1 .

More precisely, we can find a family of maps

$$\mathbf{r}(t, \alpha) : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^m,$$

such that $\mathbf{r}(t, \alpha)$ is *continuous* in the variables $(t, \alpha) \in [0, 1] \times [0, 1]$ satisfying the conditions

$$\mathbf{r}(t, 0) = \mathbf{r}_0(t), \quad \mathbf{r}(t, 1) = \mathbf{r}_1(t), \quad \text{for all } t \in [0, 1].$$

We say that A is *simply connected*, if all curves $\mathbf{r}(\cdot, \alpha)$, $\alpha \in [0, 1]$ lie entirely in A . In some sense the set A does not have “holes”.

In \mathbb{R}^2 it is easy to understand, what a hole is. However, the reader must be careful in higher dimensions. If e.g. we just remove the centre of a solid ball in \mathbb{R}^3 , then the remaining set is still simply connected, even if one would believe that the removed point was a “hole”. Cf. Figure 5.11. However, if we remove all points of the z -axis, or even a tube as on Figure 5.11, then the remaining set is no longer simply connected. Consider e.g. two circles in this set, one circling around the z -axis, while the other one does not. Then one cannot change one of them continuously to the other one without cutting the z -axis, so we get outside A by this continuous transformation.

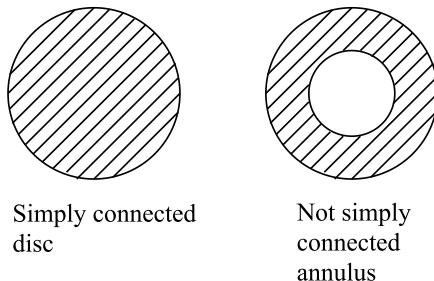


Figure 5.10: The disc to the left is simply connected, while the annulus to the right is not, though it of course is connected.

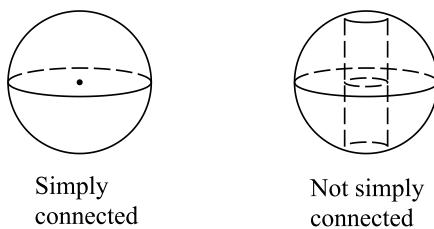


Figure 5.11: A simply connected and a not simply connected set in \mathbb{R}^3 .

5.10 Continuous surfaces in \mathbb{R}^3

Surfaces are like curves also important in the applications. We shall here for convenience restrict ourselves to surfaces in the 3-dimensional space \mathbb{R}^3 . The primitive idea is described in the following way: Take a plane plate and hammer it into a wanted shape. The hammering is then described by some continuous function.

5.10.1 Parametric description and continuity

We shall of course generalize the definition of a (1-dimensional) curve to a 2-dimensional surface. So instead of a 1-dimensional parameter interval $I \subseteq \mathbb{R}$ one is tempted to replace it by a 2-dimensional interval like $I \times J \subseteq \mathbb{R}^2$, where $I, J \subseteq \mathbb{R}$ are intervals. This actually is sufficient in many cases.

However, a closer look shows that we may allow more general 2-dimensional parameter sets $E \subseteq \mathbb{R}^2$. In fact, it suffices that E is connected, i.e. a domain in \mathbb{R}^2 . This is in agreement with our primitive idea in the introduction above, namely that E is some connected 2-dimensional plate, which should be bent and stretched or compressed to give the wanted surface in \mathbb{R}^3 .

Glancing at the previous definition of a curve we see that a surface should have the structure

$$\mathcal{F} = \{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \mathbf{r}(u, v), (u, v) \in E\}, \quad \text{where } E \subseteq \mathbb{R}^2.$$

Here, $\mathbf{r} : E \rightarrow \mathbb{R}^3$ is a continuous vector function in two variables.

The above illustrates the general idea of a parametric description of a surface, which we illustrate on Figure 5.12.

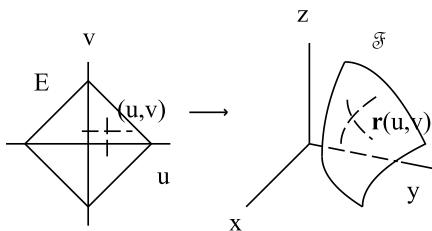


Figure 5.12: The parametric description $\mathbf{r} : E \rightarrow \mathbb{R}^3$ of a surface \mathcal{F} .

We call $\mathbf{x} = \mathbf{r}(u, v)$, $(u, v) \in E$ a *parametric description* of the surface \mathcal{F} . Let $(u, v) \in E$ be a point in the parameter domain. The vertical line segment in E through (u, v) is 1-dimensional. It is therefore mapped into a continuous curve on the surface \mathcal{F} . This curve is called the *parameter curve* on \mathcal{F} through (u, v) . Similarly, when we consider a horizontal line segment through $(u, v) \in E$.

The sloppy definition above of a surface includes some pathological cases, which we should avoid in practice. If e.g. $\mathbf{r}(u, v) = \mathbf{R}(u)$ is independent of v , then the “surface” generates to a curve, which one would not consider as a surface. Furthermore, since already curves can be space filling, the same is true for surfaces even for continuous parametric descriptions. Since we do not have the concept of a “null set” at hand, it is here not easy to give a precise definition of a surface, so we allow ourselves only to sketch the main points.

- 1) The *parametric map* $\mathbf{r} : E \rightarrow \mathbb{R}^3$ should not only be continuous. It should also be differentiable “almost everywhere”. (Differentiable functions are the subject of Volume III.) This only means that we allow some – though not too many – exceptional points, in which we do not have differentiability.
- 2) The *parametric curves* should at “almost every point $\mathbf{r}(u, v) \in \mathcal{F}$ have two parameter curves, which have linearly independent tangent vectors with respect to the parameters $(u, v) \in E$.

The simplest surfaces in \mathbb{R}^3 are probably the following:

- 1) *A plane in \mathbb{R}^3 .* Given two linearly independent vectors \mathbf{b}, \mathbf{c} in \mathbb{R}^3 , and let \mathbf{a} just be a point in \mathbb{R}^3 . Then

$$\mathbf{x} = \mathbf{r}(u, v) = \mathbf{a} + \mathbf{b}u + \mathbf{c}v, \quad (u, v) \in \mathbb{R}^2,$$

is a parametric description of a plane through the point $\mathbf{a} \in \mathbb{R}^3$.

- 2) *A graph of a function.* Assume that the surface \mathcal{F} is the graph of a function in two variables,

$$z = Z(x, y), \quad \text{for } (x, y) \in E.$$

Then this is clearly a parametric description. In fact, replace (x, y) with $(u, v) \in E$.

- 3) *A sphere of radius $a > 0$ and centre at $\mathbf{0}$.* In this case the most commonly used parametric description is

$$\mathbf{x} = (x, y, z) = \mathbf{r}(\theta, \varphi) = (a \sin \theta \cos \varphi, a \sin \theta \sin \varphi, a \cos \theta), \quad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi].$$

The construction is the following: Write the rectangular coordinates (x, y, z) as functions in the spherical coordinates (r, θ, φ) introduced in Chapter 1 (volume I), and then keep $r = a > 0$ fixed.

DON'T EAT YELLOW SNOW

What will your advice be?

Some advice just states the obvious. But to give the kind of advice that's going to make a real difference to your clients you've got to listen critically, dig beneath the surface, challenge assumptions and be credible and confident enough to make suggestions right from day one. At Grant Thornton you've got to be ready to kick start a career right at the heart of business.

Sound like you? Here's our advice: visit GrantThornton.ca/careers/students

Scan here to learn more about a career with Grant Thornton.

 **Grant Thornton**
An instinct for growth™





Click on the ad to read more

- 4) *An ellipsoidal surface.* In rectangular coordinates an ellipsoidal surface is given in its canonical form by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$$

When we modify the parametric description of the sphere above we get the following parametric description of the surface

$$(x, y, z) = (a \sin \theta \cos \varphi, b \sin \theta \sin \varphi, c \cos \theta), \quad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi].$$

In the next two sections we introduce other commonly occurring surfaces, which are also easily described.

5.10.2 Cylindric surfaces

A *cylindric surface* is the union of all straight lines, the *generators*, in a space, which are parallel and which all intersect a given curve. We shall here for convenience confine ourselves to the case, where the given curve lies in a plane, and the generators are all perpendicular to this plane, supplied with the extra assumption that the cylindric surface may consist of only *line segments* of the generators.

If the given curve \mathcal{L} lies in the xy -plane, the cylindric surface above \mathcal{L} is illustrated by taking a sheet of paper and fold it along the curve \mathcal{L} .

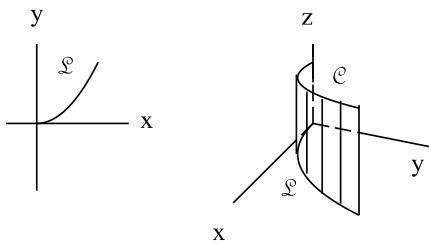


Figure 5.13: The given plane curve \mathcal{L} in the xy -plane and the corresponding perpendicular cylindric surface \mathcal{C} in the xyz -space.

The parametric description of a cylindric surface is constructed in the following way:

First assume that in the xy -plane the given curve \mathcal{L} has been given a parametric description of the form

$$\mathcal{L} = \{(x, y) \in \mathbb{R}^2 \mid x = X(t), y = Y(t), t \in I\}.$$

Then the cylindric surface \mathcal{C} is described by adding $J(t)$ as a z -interval above the point of the curve $(X(t), Y(t), 0)$ of the same parameter $t \in I$, hence

$$\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 \mid x = X(t), y = Y(t), z \in J(t), t \in I\}.$$

5.10.3 Surfaces of revolution

A *surface of revolution* is constructed in the following way:

Given an *axis of revolution* – usually chosen as the z -axis – and a so-called *meridian curve* \mathfrak{M} in the *meridian half-plane* $\{(\varrho, z) \mid \varrho \geq 0, z \in \mathbb{R}\}$.

Assume that the meridian curve has the following parametric description,

$$\mathfrak{M} = \{(\varrho, z) \mid \varrho = P(t) \geq 0, z = Z(t), t \in I\}.$$

When we rotate \mathfrak{M} in \mathbb{R}^3 around the z -axis, the surface of revolution \mathcal{O} is described in *semi-polar*, or *cylindric*, coordinates (cf. Chapter 1 in Volume I) by

$$\mathcal{O} : \quad \varrho = P(t) \geq 0 \quad \text{and} \quad z = Z(t), \quad \text{for } t \in I \text{ and } \varphi \in [0, 2\pi].$$

If we use rectangular coordinates, we of course get

$$\mathcal{O} : \quad x = P(t) \cos \varphi, y = P(t) \sin \varphi, z = Z(t), \quad \text{for } t \in I \text{ and } \varphi \in [0, 2\pi[,$$

cf. Figure 5.14.

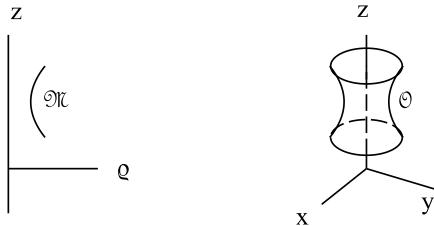


Figure 5.14: The meridian curve \mathfrak{M} in the meridian half-plane, and the corresponding surface of revolution \mathcal{O} in the space \mathbb{R}^3 .

If in particular \mathfrak{M} is a half-circle of radius $a > 0$ and centre at $\mathbf{0}$, then the surface of revolution becomes a sphere of centre $\mathbf{0}$ and radius a . A parametric description of \mathfrak{M} is

$$\mathfrak{M} : \quad \varrho = a \sin \theta, \quad z = a \cos \theta, \quad \theta \in [0, \pi],$$

so the parametric description of the sphere is the well-known description in spherical coordinates with $r = a$ fixed,

$$\mathcal{O} : \quad x = a \sin \theta \cos \varphi, \quad y = a \sin \theta \sin \varphi, \quad z = a \cos \theta, \quad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi[,$$

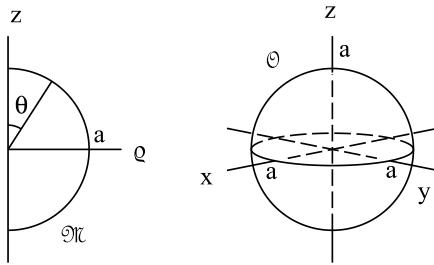


Figure 5.15: The meridian curve \mathfrak{M} is a half-circle, and the surface of revolution \mathcal{O} is a sphere in the xyz -space.

so this is a way to derive the spherical coordinates from the parametric description of \mathfrak{M} . Note that θ here is measured positively from the vertical z -axis towards the horizontal ϱ -axis, i.e. apparently in the negative orientation of the meridian half-plane. Cf. also Figure 5.15.

If instead the meridian curve is a circle lying in the *open* meridian half-plane, so it does not touch the axis of rotation, then its parametric description may be given by

$$\varrho = a + b \cos t, \quad z = b \sin t, \quad \text{for } t \in [0, 2\pi[, \quad \text{where } 0 < b < a,$$

cf. Figure 5.16.

The surface of revolution is a *torus* of parametric description in *semi-polar* or *cylindric, coordinates*

$$\mathcal{O}; \quad \varrho = a + b \cos t, \quad z = b \sin t, \quad \text{for } t \in [0, 2\pi[\text{ and } \varphi \in [0, 2\pi[.$$

Clearly, $(\varrho - a)^2 + z^2 = b^2$, which is an *equation* of the torus surface in semi-polar coordinates. The equation in *rectangular coordinates*, is

$$\left(\sqrt{x^2 + y^2} - a \right)^2 + z^2 = b^2, \quad \text{where } 0 < b < a,$$

because $x = \varrho \cos \varphi$ and $y = \varrho \sin \varphi$, so $\varrho = \sqrt{x^2 + y^2}$.

5.10.4 Boundary curves, closed surfaces and orientation of surfaces

When we consider a curve, then it is obvious that its initial point and final point – if they exist – are the points, where the curve stops in some sense. We note that a curve does not necessarily have initial and final points. One example is the unit circle, where we can continue moving along it without ever reaching one of its end points, because they do not exist.

Similarly, a surface \mathcal{F} may have a *boundary curve* $\delta\mathcal{F}$, where the surface \mathcal{F} in some sense stops. Also here we may expect cases, where such a boundary curve does not exist. We shall return to this later.

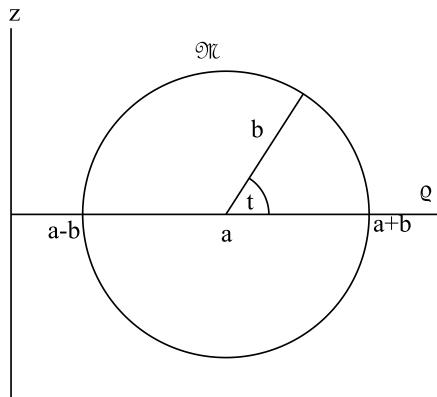


Figure 5.16: The meridian curve \mathfrak{M} is a circle in the open meridian half-plane. The surface of revolution \mathcal{O} is a *torus* in the xyz -space. A figure of the torus is given in MAPLE. However, for some obscure reason it has not been possible for the author to put it here.

We exclude here all space filling surfaces, so every surface \mathcal{F} under consideration will not have interior points in \mathbb{R}^3 , thus \mathcal{F} is equal to its *topological* boundary in \mathbb{R}^3 , i.e. $\mathcal{F} = \partial\mathcal{F}$. The boundary curves of surfaces we are considering here are intrinsic boundary curves with respect to the surface \mathcal{F} itself and they have nothing to do with the boundaries of sets in \mathbb{R}^3 . It is for this reason that we use the notation $\partial\mathcal{F}$ for such boundary curves.

..... Alcatel-Lucent 

www.alcatel-lucent.com/careers

What if you could build your future and create the future?



One generation's transformation is the next's status quo.
 In the near future, people may soon think it's strange that devices ever had to be "plugged in." To obtain that status, there needs to be "The Shift".



Click on the ad to read more

We get a hint of what is the meaning of $\delta\mathcal{F}$, when we consider the parametric domain $E \subseteq \mathbb{R}^2$ of $\mathcal{F} \subset \mathbb{R}^3$. Clearly, E has a usual *topological* boundary ∂E in \mathbb{R}^2 , and when we use the picture that E is hammered into the shape of \mathcal{F} in \mathbb{R}^3 by the application of the map $\mathbf{r} : E \rightarrow \mathbb{R}^3$, we would expect that $\delta\mathcal{F} = \mathbf{r}(\partial E)$. This is very often the case, though not always, which we shall show in the following.

Consider the unit sphere in spherical coordinates. Then the parameter domain is

$$E = [0, \pi] \times [0, 2\pi[,$$

which has the topological boundary

$$\partial E = \{0\} \times [0, 2\pi] \cup \{\pi\} \times [0, 2\pi] \cup [0, \pi] \times \{0\} \cup [0, \pi] \times \{2\pi\},$$

while the sphere \mathcal{F} does not have a boundary curve, $\delta\mathcal{F} = \emptyset$. By the “hammering” with the function \mathbf{r} we identify the two horizontal sides, $[0, \pi] \times \{0\}$ and $[0, \pi] \times \{2\pi\}$, leaving us with a cylinder. And then all points of $\{0\} \times [0, 2\pi]$ are identified, i.e. hammered and glued together to get the North Pole, and all points of $\{\pi\} \times [0, 2\pi]$ are identified i.e. hammered and glued together to get the South Pole. So all points of a possible boundary curve simply disappear by this process.

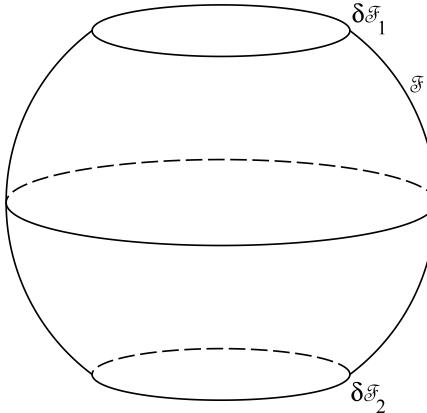


Figure 5.17: The boundary curve $\delta\mathcal{F} = \delta\mathcal{F}_1 \cup \delta\mathcal{F}_2$ is not connected. Its branches are two circles lying in parallel planes at different latitudes.

A boundary curve of a surface is not necessarily connected. If we cut the sphere with two parallel planes and let the surface \mathcal{F} be the part of the sphere, which lies between the two planes. Then the boundary curve consists of two parallel circles at different latitudes, cf. Figure 5.17.

Each connected component of $\delta\mathcal{F}$ is called a *branch* of $\delta\mathcal{F}$, and each branch is a continuous curve in space.

If a surface \mathcal{F} does not have a boundary curve in this sense, $\delta\mathcal{F} = \emptyset$, then we call \mathcal{F} a *closed surface*. We have already seen some closed surfaces; the *sphere*, the *ellipsoidal surface*, and the *torus*, all considered in the previous Section 5.10.3.

Assume that \mathcal{F} is a closed surface. If e.g. \mathcal{F} is the sphere, then it is obvious that we can talk of the inside and the outside of the sphere, so we can talk of a direction out of the ball, which has the sphere as its boundary. This is the general idea of the new concept *orientation*.

Not all surfaces have an inside and an outside, so we must find a means to decide when this is the case. First note that if the surface is divided into sufficiently small pieces, which overlap each others, then we *locally* can talk of two sides of the surface. We paint one of them red, and the other one blue, and then go to the neighbouring piece of the surface (with an overlap). Paint this neighbouring piece of the surface according to the colours in their overlap. Proceed in this way, until either all local pieces of surface have been painted, in which case we can define e.g. blue the inside and red the outside, and we have obtained an *orientation*. Or, we come to a piece of the surface, which according to this procedure should have each both sides painted both red and blue, which is not possible. In this case we say that the surface cannot be oriented.

The simplest example of a surface, which cannot be oriented, is the so-called *Möbius's strip*. Take a strip of paper and twist it once before gluing the ends of the strip together, cf. Figure 5.18.

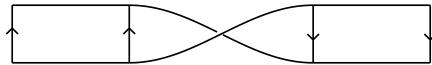


Figure 5.18: *Möbius's strip*. When the strip of paper is twisted once, we switch the local orientation, denoted by the arrows. When we glue the two ends together, we end up with a strip, which *globally* has only one surface!

we shall in this series of books on *Real Functions in Several Real Variables* only consider surfaces, which can be oriented.

5.11 Main theorems for continuous functions

We shall in this section quote (without proofs) the three main theorems for continuous functions, here restricted to the spaces \mathbb{R}^n . They will be very important in the applications in the sequel.

1st main theorem for continuous functions. Let $A \subseteq \mathbb{R}^n$ be a connected set, and let $\mathbf{f} : A \rightarrow \mathbb{R}^k$ be continuous. Then the range, $\mathbf{f}(A)$, is also connected.

It should be noted that even if $\mathbf{f} : A \rightarrow \mathbb{R}^k$ is continuous and $\mathbf{f}(A)$ is connected, we cannot conclude that the domain A itself is connected. Consider $f = \sin : A \rightarrow \mathbb{R}$, where $A \subseteq \mathbb{R}$. Then f is continuous and $\sin(A) = [-1, 1]$ is connected, while

$$A := \bigcup_{n \in \mathbb{Z}} \left[-\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi \right] \quad \text{is not connected.}$$

An important case is when $k = 1$, in which case $f(A)$ is a connected set in \mathbb{R} , i.e. an interval. We shall later use this observation over and over again.

Let $A \subset \mathbb{R}^n$. If A is bounded and closed, we call it *compact*. Compact sets are very important in Mathematics, and the next two main theorems are dealing with them.

2nd main theorem for continuous functions. *Let $A \subset \mathbb{R}^n$ be a compact set. If $\mathbf{f} : A \rightarrow \mathbb{R}^k$ is continuous, then its range $\mathbf{f}(A)$ is also compact.*

Again we consider the special case, when $k = 1$. If $f : A \rightarrow \mathbb{R}$ is continuous, and A is compact, then $f(A) \subset \mathbb{R}$ is also compact. In particular, $f(A)$ contains its upper and lower bounds. We therefore conclude that there must exist points $\mathbf{a}, \mathbf{b} \in A$, such that

$$f(\mathbf{a}) \leq f(\mathbf{x}) \leq f(\mathbf{b}) \quad \text{for all } \mathbf{x} \in A.$$

Clearly $f(\mathbf{a})$ is the *minimum*, and $f(\mathbf{b})$ is the *maximum* of f on A , so we can in principle find points in A , in which these extrema are attained. Unfortunately, the 2nd main theorem does not give any hint of how to find these points in A . We shall later give some results in this direction.

Finally, we turn to the 3rd main theorem for continuous functions. For some reason this is in general the most difficult one to understand for the reader. Let us start with the strict definition of *continuity* as it was given half a century ago,

$$(5.4) \quad \forall \varepsilon > 0 \forall \mathbf{x} \in A \exists \delta > 0 \forall \mathbf{y} \in A : \|\mathbf{x} - \mathbf{y}\| < \delta \Rightarrow \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| < \varepsilon.$$

Here, \forall is read “forall”, and \exists is read “there exists”.

Today one would use a lesser formal language like the following: First define the growth of the function by

$$\Delta \mathbf{f}(\mathbf{x}, \mathbf{h}) := \mathbf{f}(\mathbf{x} + \mathbf{h}) - \mathbf{f}(\mathbf{x}).$$

Then *continuity* at the *fixed point* $\mathbf{x} \in A$ means that $\Delta \mathbf{f}(\mathbf{x}, \mathbf{x}) \rightarrow \mathbf{0}$, when $\mathbf{h} \rightarrow \mathbf{0}$, which more explicitly means that *to every* $\varepsilon > 0$ *we can find* $\delta = \delta(\varepsilon, \mathbf{x}) > 0$, *depending on both* ε *and* \mathbf{x} , *such that*

$$\|\mathbf{h}\| < \delta \quad \text{implies that} \quad \|\Delta \mathbf{f}(\mathbf{x}, \mathbf{h})\| < \varepsilon.$$

It is not hard to show that this is the same as the more stringent definition (5.4).

In the applications we often need a stronger property of f than just continuity. It is important for many proofs that we can choose $\delta = \delta(\varepsilon) > 0$ above, *independently of the point* $\mathbf{x} \in A$. This means that δ is chosen after $\varepsilon > 0$, but before $\mathbf{x} \in A$. When a function f has this property, we call it *uniformly continuous*. The same pair (ε, δ) can be used everywhere in A , so the formal mathematical definition becomes

$$(5.5) \quad \forall \varepsilon > 0 \exists \delta > 0 \forall \mathbf{x} \in A \forall \mathbf{y} \in A : \|\mathbf{x} - \mathbf{y}\| < \delta \Rightarrow \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| < \varepsilon.$$

When we compare (5.4) and (5.5), we see that the difference is, at we in (5.4) write

$$\forall \varepsilon > 0 \forall \mathbf{x} \in A \exists \delta = \delta(\varepsilon, \mathbf{x}) > 0 \dots ,$$

i.e. the choice of δ depends on both ε and \mathbf{x} , while we in (5.5) have interchanged two groups of quantors, so

$$\forall \varepsilon > 0 \exists \delta = \delta(\varepsilon) > 0 \forall \mathbf{x} \in A \dots ,$$

i.e. $\forall x \in A$ follows *after* the specification of δ .

This makes a very big difference, and *uniform continuity* is clearly a stronger concept than just *continuity*.

3rd main theorem for continuous functions. *Let $A \subset \mathbb{R}^n$ be compact, and let $f : A \rightarrow \mathbb{R}^k$ be continuous. Then f is uniformly continuous.*

The latter two main theorems show that the compact sets (i.e. closed and bounded sets) in \mathbb{R}^n are very important. It is for that reason that they have been given a special name.

To show its importance we give a simple and useful consequence of the 3rd main theorem below.



Maastricht University *Leading in Learning!*

Join the best at the Maastricht University School of Business and Economics!

Top master's programmes

- 33rd place Financial Times worldwide ranking: MSc International Business
- 1st place: MSc International Business
- 1st place: MSc Financial Economics
- 2nd place: MSc Management of Learning
- 2nd place: MSc Economics
- 2nd place: MSc Econometrics and Operations Research
- 2nd place: MSc Global Supply Chain Management and Change

Sources: Keuzegids Master ranking 2013; Elsevier 'Beste Studies' ranking 2012;
Financial Times Global Masters in Management ranking 2012

Visit us and find out why we are the best!
Master's Open Day: 22 February 2014

Maastricht
University is
the best specialist
university in the
Netherlands
(Elsevier)

www.mastersopenday.nl



Click on the ad to read more

Theorem 5.2 Given a continuous function $f : I \times [a, b] \rightarrow \mathbb{R}$, where $[a, b]$ is a compact interval, while $I \neq \emptyset$ is just an interval. We define a function $F : I \rightarrow \mathbb{R}$ by

$$F(x) := \int_a^b f(x, t) dt, \quad \text{for } x \in I.$$

Then F is continuous on I .

PROOF. Choose any fixed $\mathbf{x} \in I^\circ$ (the interior of I), and then a compact interval

$$J = [x - c, x + c] \subset I^\circ,$$

which is possible, because $I^\circ \neq \emptyset$ is open. Then we have

$$\Delta := F(x + h) - F(x) = \int_a^b \{f(x + h, t) - f(x, t)\} dt \quad \text{for } |h| < c.$$

The restriction of the continuous function f to the compact set $J \times [a, b]$ is according to the 3rd main theorem *uniformly continuous*. Hence, to every given $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ depending only on ε , such that

$$|f(x, t) - f(y, u)| < \frac{\varepsilon}{b - a}, \quad \text{if } (x, t), (y, u) \in J \times [a, b] \text{ and } \|(x, t) - (y, u)\| < \delta.$$

Note that we only for technical reasons have divided ε by the length $b - a$ of the interval $[a, b]$.

The above is in particular true, if $u = t \in [a, b]$ and $y = x + h$, where $|h| < \min\{c, \delta\}$. When h is chosen in this way, then we get the estimates

$$|\Delta| \leq \int_a^b |f(x + h, t) - f(x, t)| dt \leq \int_a^b \frac{\varepsilon}{b - a} dt = \frac{\varepsilon}{b - a} \cdot (b - a) = \varepsilon,$$

and we have proved that F is continuous at every point in the interior of I . If $\mathbf{x} \in I$ is an end point, then just modify the J interval above. However, if the end point \mathbf{x} of I does not belong to I , we cannot conclude anything. However, Theorem 5.2 does not claim anything in this case. \square

6 A useful procedure

6.1 The domain of a function

Problem 6.1 Let the structure of a function $f(x, y, \dots)$ be given. Find the maximum domain of this function, based on this structure.

Procedure.

- 1) Divide the function into subfunctions according to the signs + and -, i.e. write $f = f_+ + f_-$, where $f_+ \geq 0$ and $f_- \leq 0$ (and $f_+ \cdot f_- = 0$).
- 2) Find the domain for each of the subfunctions (if possible, sketch a figure).
- 3) Then the domain of f is the intersection of the domains of subfunctions. (Sketch a figure).

If $\mathbf{f}(x, y, \dots)$ is a *vector function*, then apply the above separately for each coordinate function. The domain is the intersection of all the domains of the coordinate functions.

One should in particular be aware of the following rules:

- 1) **Never** divide by 0.

Analyze the set of zeros for the denominator, if it exists.

- 2) In real analysis, **never** take the square root of a negative number.

Find the set of zeros of the radicand of the square root. Check the sign in the domains which are bounded by this set of zeros.

- 3) In real analysis **never** take the logarithm of a negative number or of 0.

Find the set of zeros of the expression which we are going to take the logarithm of. Check the sign in the domains which are bounded by this set of zeros.

Remark 6.1 Experience tells that the square root is in particular difficult to handle. A professor once told me that “if one can handle the square root, then one can handle anything in mathematics!”. Notice that pocket calculators does not like square roots either. ◇

7 Examples of continuous functions in several variables

7.1 Maximal domain of a function

Example 7.1 Find and sketch in each of the following cases the domain of the given function.

$$1) f(x, y) = \ln|1 - x^2 - y^2|.$$

$$2) f(x, y) = \sqrt{-x^2 - y^2}.$$

$$3) f(x, y) = \ln(1 - x^2 - y^2) + \sqrt{(x - \frac{1}{2})(x^2 + y^2)}.$$

$$4) f(x, y) = \ln[y(x^2 + y^2 + 2y)].$$

$$5) f(x, y) = y\sqrt{2 - x^2} + \operatorname{Arctan} \frac{y}{x}.$$

$$6) f(x, y) = \sqrt{3 - x^2 - y^2} + 2 \operatorname{Arcsin}(x^2 - y^2).$$

$$7) f(x, y) = \operatorname{Arcsin}(2 - x^2 - y).$$

$$8) f(x, y) = \sqrt{xy - 1}.$$

$$9) f(x, y) = \sqrt{y + \sin x} + \sqrt{-y + \sin x}.$$

$$10) f(x, y) = x^y.$$

$$11) f(x, y) = \ln y + \ln(x^2 + y^2 + 2y).$$

A Domain of a function.

D Analyze the domain and the sketch the set.

I 1) The function $\ln|1 - x^2 - y^2|$ is defined for $|1 - x^2 - y^2| > 0$, i.e. for $x^2 + y^2 \neq 1$. The domain is \mathbb{R}^2 with the exception of the unit circle:

$$\mathbb{R}^2 \setminus \{(x, y) \mid x^2 + y^2 = 1\}.$$

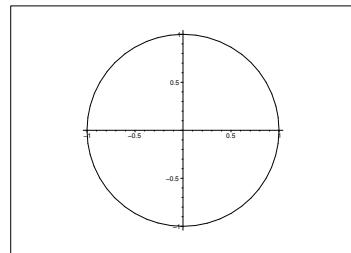


Figure 7.1: The domain of $f(x, y) = \ln|1 - x^2 - y^2|$

2) The requirement of the function $\sqrt{-x^2 - y^2}$ is that $-x^2 - y^2 \geq 0$, i.e. the domain is only the point $\{(0, 0)\}$.

3) The function $\ln(1 - x^2 - y^2) + \sqrt{(x - \frac{1}{2})(x^2 + y^2)}$ is defined for

$$1 - x^2 - y^2 > 0 \quad \text{and} \quad \left(x - \frac{1}{2}\right)(x^2 + y^2) \geq 0.$$

We first conclude that $x^2 + y^2 < 1$, so the domain must be contained in the open unit disc.

Then note that both requirements are fulfilled for $(x, y) = (0, 0)$, thus $(0, 0)$ belongs to the domain.

Finally, when $0 < x^2 + y^2 < 1$ we also have the requirement $x \geq \frac{1}{2}$.

Summarizing the domain is

$$\{(0, 0)\} \cup \{(x, y) \mid x \geq \frac{1}{2}, x^2 + y^2 < 1\}.$$

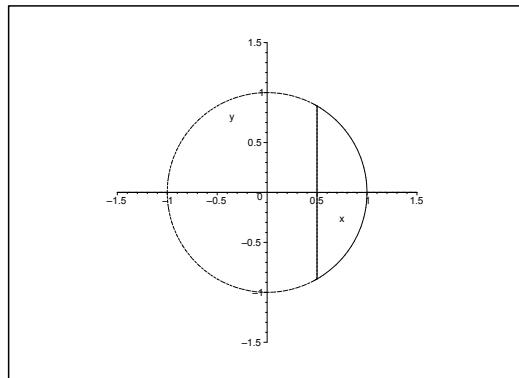


Figure 7.2: The domain of $f(x, y) = \ln(1 - x^2 - y^2) + \sqrt{(x - \frac{1}{2})(x^2 + y^2)}$

4) The function $\ln(y(x^2 + y^2 + 2y))$ is defined for

$$y(x^2 + y^2 + 2y) = y\{x^2 + (y+1)^2 - 1\} > 0.$$

Here we get two possibilities:

- a) When both $y > 0$ and $x^2 + (y+1)^2 > 1$, we see that we can reduce to $y > 0$, because then also $(y+1)^2 > 1$.
- b) The second possibility is that $y < 0$ and $x^2 + (y+1)^2 < 1$. In this case we reduce to $x^2 + (y+1)^2 < 1$, because this inequality determines an open disc in the lower half plane of centre $(0, -1)$ and radius 1, and $y < 0$ is automatically satisfied.

Summarizing we obtain the domain

$$\{(x, y) \mid y > 0\} \cup \{(x, y) \mid x^2 + (y+1)^2 < 1\},$$

i.e. the union of the upper half plane and the afore mentioned circle in the lower half plane.

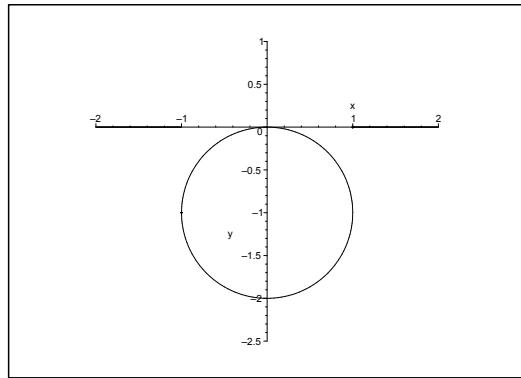


Figure 7.3: The domain of $f(x, y) = \ln[y(x^2 + y^2 + 2y)]$.

- 5) The function $y\sqrt{2-x^2} + \text{Arctan } \frac{y}{x}$ is defined for

$$2 - x^2 \geq 0 \quad \text{and} \quad x \neq 0,$$

i.e. the domain is the union of two vertical strips, which are neither open nor closed,

$$\{(x, y) \mid -\sqrt{2} \leq x < 0\} \cup \{(x, y) \mid 0 < x \leq \sqrt{2}\}.$$

This can also be written

$$[-\sqrt{2}, \sqrt{2}] \times \mathbb{R} \setminus \{(0)\} \times \mathbb{R}.$$

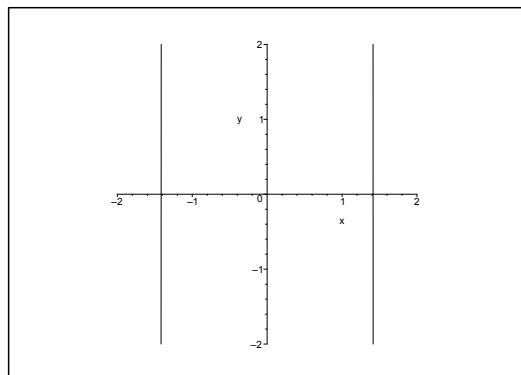


Figure 7.4: The domain of $f(x, y) = y\sqrt{2-x^2} + \text{Arctan } \frac{y}{x}$.

- 6) The function $\sqrt{3-x^2-y^2} + 2 \text{Arcsin}(x^2-y^2)$ is defined for

$$x^2 + y^2 \leq 3 \quad \text{and} \quad -1 \leq x^2 - y^2 \leq 1,$$

i.e. for

$$\sqrt{x^2 + y^2} \leq \sqrt{3}, \quad x^2 - y^2 \leq 1, \quad y^2 - x^2 \leq 1.$$

The domain is that component of the intersection with the disc which also contains the point $(0, 0)$.

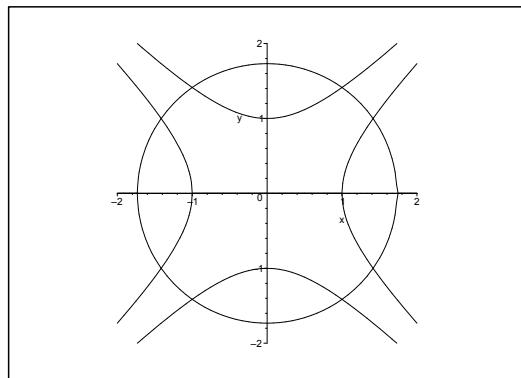


Figure 7.5: The domain of $f(x, y) = \sqrt{3 - x^2 - y^2} + 2 \operatorname{Arcsin}(x^2 - y^2)$.

- 7) The function $\operatorname{Arcsin}(2 - x^2 - y)$ is defined for

$$-1 \leq 2 - x^2 - y \leq 1,$$

i.e. when the following two conditions are fulfilled:

$$y \leq 3 - x^2 \quad \text{and} \quad y \geq 1 - x^2.$$

Empowering People. Improving Business.

BI Norwegian Business School is one of Europe's largest business schools welcoming more than 20,000 students. Our programmes provide a stimulating and multi-cultural learning environment with an international outlook ultimately providing students with professional skills to meet the increasing needs of businesses.

BI offers four different two-year, full-time Master of Science (MSc) programmes that are taught entirely in English and have been designed to provide professional skills to meet the increasing need of businesses. The MSc programmes provide a stimulating and multi-cultural learning environment to give you the best platform to launch into your career.

- MSc in Business
- MSc in Financial Economics
- MSc in Strategic Marketing Management
- MSc in Leadership and Organisational Psychology

www.bi.edu/master



Click on the ad to read more

Summarizing the domain becomes

$$\{(x, y) \mid 1 - x^2 \leq y \leq 3 - x^2\},$$

which is the closed set which lies between the two arcs of parabolas.

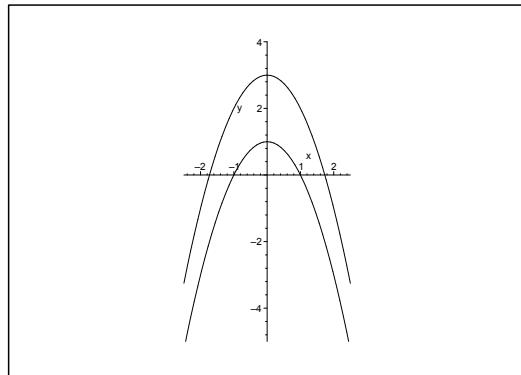


Figure 7.6: The domain of $f(x, y) = \text{Arcsin}(2 - x^2 - y)$.

- 8) The function $\sqrt{xy - 1}$ is defined for $xy \geq 1$ i.e. the sets in the first and third quadrant, which are bounded by the hyperbola $y = \frac{1}{x}$ and which is not close to any of the axes:

$$\{(x, y) \mid x > 0, y > 0, xy \geq 1\} \cup \{(x, y) \mid x < 0, y < 0, xy \geq 1\}.$$

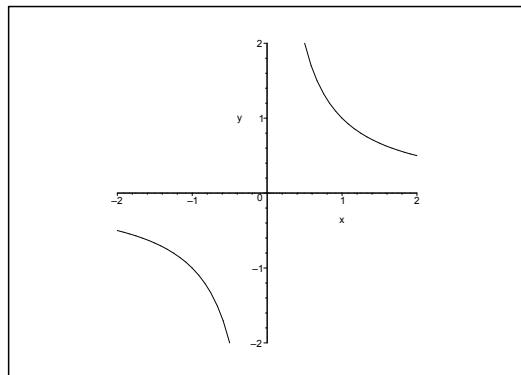


Figure 7.7: The domain of $f(x, y) = \sqrt{xy - 1}$.

- 9) The function $\sqrt{y + \sin x} + \sqrt{-y + \sin x}$ is defined when both

$$y + \sin x \geq 0 \quad \text{and} \quad -y + \sin x \geq 0,$$

i.e. when

$$-\sin x \leq y \leq \sin x.$$

Hence the condition $\sin x \geq 0$, i.e. $x \in [2p\pi, \pi + 2p\pi]$, $p \in \mathbb{Z}$, and the domain is

$$\bigcup_{p \in \mathbb{Z}} \{(x, y) \mid 2p\pi \leq x \leq 2p\pi + \pi, |y| \leq \sin x\}.$$

On the figure the domain is the union of every second of the connected subsets.

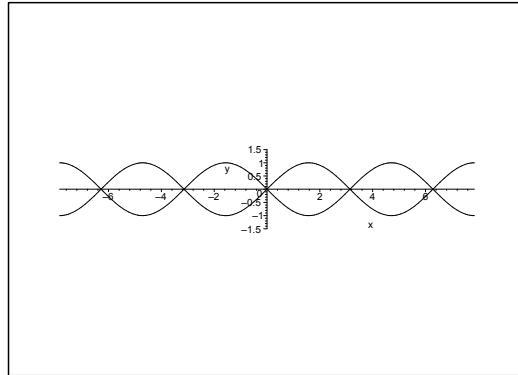


Figure 7.8: The domain of $f(x, y) = \sqrt{y + \sin x} + \sqrt{-y + \sin x}$.

- 10) This is a very difficult example. First notice that the function $f(x, y) = x^y$ is at least defined when $x, y > 0$.

When $x = 0$ the function is defined for every $y > 0$.

When $x < 0$, the function is defined for every $y = \frac{p}{2q+1}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}_0$.

When $y < 0$ is not a rational number of odd denominator, we must necessarily require that $x > 0$.

When $y = -\frac{p}{2q+1}$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, then x^y is also defined for $x < 0$, though not for $x = 0$.

REMARK. It is a matter of definition whether one can put $x^0 = 1$ for $x < 0$. This may be practical in some cases, though not in everyone. \diamond .

This domain is fairly complicated:

$$\{(x, y) \mid x > 0\} \cup \{(0, y) \mid y > 0\} \cup \bigcup_{p, q \in \mathbb{N}_0} \{(x, y) \mid x < 0, y = -p/(2q+1)\},$$

where one may discuss whether the point $(0, 0)$ should be included or not.

- 11) When the function $f(x, y) = \ln y + \ln(x^2 + y^2 + 2y)$ is defined, we must at least require that $y > 0$, because $\ln y$ in particular should be defined.

If on the other hand $y > 0$, then clearly also $x^2 + y^2 + 2y > 0$, no matter the choices of x and $y > 0$, thus $f(x, y)$ is defined for $y > 0$, i.e. in the upper half plane.

Example 7.2 Describe in each of the following cases the domain of the given function.

$$1) f(x, y, z) = \sqrt{1 - |x| - |y| - |z|}.$$

$$2) f(x, y, z) = \ln(\sqrt{1 - |x| - |y| - |z|}).$$

$$3) f(x, y, z) = \text{Arcsin}(x^2 + y^2 - 4).$$

$$4) f(x, y, z) = \sqrt[4]{x^2 + 4y^2 + 9z^2 - 1}.$$

$$5) f(x, y, z) = \text{Arctan} \frac{x+z}{y}.$$

$$6) f(x, y, z) = \exp(3x + 2y + 5z).$$

A Domain of functions in three variables.

D Analyze in each case the function. There will here be given no sketches.

I 1) The function $\sqrt{1 - |x| - |y| - |z|}$ is defined for $|x| + |y| + |z| \leq 1$,

$$\{(x, y, z) \mid |x| + |y| + |z| \leq 1\}.$$

This set is a closed tetrahedron in the space.

2) The function $\ln(\sqrt{1 - |x| - |y| - |z|})$ is defined in the corresponding *open* tetrahedron in space,

$$\{(x, y, z) \mid |x| + |y| + |z| < 1\}.$$

3) The function $\text{Arcsin}(x^2 + y^2 + z^2 - 4)$ is defined when

$$-1 \leq x^2 + y^2 + z^2 - 4 \leq 1,$$

i.e. in the shell

$$\left\{ (x, y, z) \mid (\sqrt{3})^2 \leq x^2 + y^2 + z^2 \leq (\sqrt{5})^2 \right\},$$

of centre $(0, 0, 0)$, inner radius $\sqrt{3}$ and outer radius $\sqrt{5}$.

4) The function $\sqrt[4]{x^2 + 4y^2 + 9z^2 - 1}$ is defined outside an ellipsoid,

$$\left\{ (x, y, z) \mid x^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{3}\right)^2 \geq 1 \right\},$$

where the half axes are 1 , $\frac{1}{2}$ and $\frac{1}{3}$.

5) The function $\text{Arctan} \frac{x+z}{y}$ is defined for $y \neq 0$.

6) The function $\exp(3x + 2y + 5z)$ is of course defined in the whole space \mathbb{R}^3 .

7.2 Level curves and level surfaces

Example 7.3 Let

$$f(x, y) = \ln(2 - 2x^2 - 3y^2) + 2 - 4x^2 - 6y^2, \quad (x, y) \in A.$$

- 1) Sketch the domain A .
- 2) Describe the level curves of the function. It is convenient to introduce a new variable u , such that $f(x, y) = F(u(x, y))$.
- 3) Sketch the level curve corresponding to $f(x, y) = 0$.
- 4) Find the range $f(A)$.

A Domain and level curves.

D Describe the set given by $2 - 2x^2 - 3y^2 > 0$, where $f(x, y)$ is defined. Then change the parameter to u .

I 1) The function is defined, if and only if

$$u = u(x, y) = 2 - 2x^2 - 3y^2 > 0,$$

i.e. for

$$\left(\frac{x}{1}\right)^2 + \left(\frac{y}{\sqrt{\frac{2}{3}}}\right)^2 < 1,$$

Need help with your dissertation?

Get in-depth feedback & advice from experts in your topic area. Find out what you can do to improve the quality of your dissertation!

Get Help Now

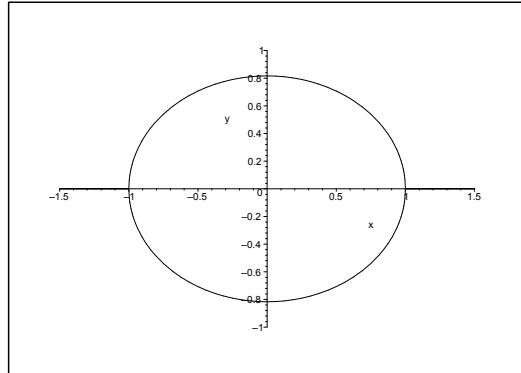


Go to www.helpmyassignment.co.uk for more info

Helpmyassignment

Click on the ad to read more

which describes an open ellipsoidal disc of centrum $(0, 0)$ and half axes 1 and $\sqrt{\frac{2}{3}}$.



2) If we define

$$u = u(x, y) = 2 - 2x^2 - 3y^2 > 0,$$

i.e. $u \in]0, 2]$, then

$$\begin{aligned} f(x, y) &= \ln(2 - 2x^2 - 3y^2) + 2 - 4x^2 - 6y^2 \\ &= \ln(2 - 2x^2 - 3y^2) + 2(2 - 2x^2 - 3y^2) - 2 \\ &= \ln u + 2u - u. \end{aligned}$$

This is clearly an increasing function in $u \in]0, 2]$. Every level curve

$$f(x, y) = \ln u + 2u - 2 = c$$

corresponds to

$$u = 2 - 2x^2 - 3y^2 = k \in]0, 2],$$

where k is unique according to the above.

Then by a rearrangement,

$$2x^2 + 3y^2 = 2 - k, \quad k \in]0, 2].$$

If $k = 2$, then the level "curve" degenerates to the point $(0, 0)$.

If $0 < k < 2$, then the level curve is an ellipse

$$\left(\frac{x}{\sqrt{\frac{2-k}{2}}}\right)^2 + \left(\frac{y}{\sqrt{\frac{2-k}{3}}}\right)^2 = 1$$

with the half axes $\sqrt{\frac{2-k}{2}}$ and $\sqrt{\frac{2-k}{3}}$.

3) When

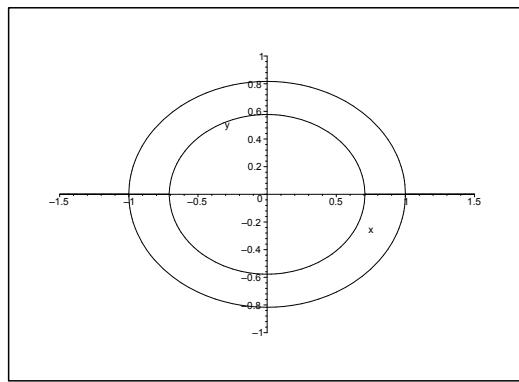
$$f(x, y) = \ln u + 2u - 2 = 0,$$

it follows that $u = 1$ is a solution. Since the function of u is strictly increasing, it follows that $u = 1$ is the only solution, so $k = 1$.

According to 2) the level curve $f(x, y) = 0$ is the ellipse

$$\left(\frac{x}{\sqrt{\frac{1}{2}}}\right)^2 + \left(\frac{y}{\sqrt{\frac{1}{3}}}\right)^2 = 1$$

of centre $(0, 0)$ and half axes $\sqrt{\frac{1}{2}}$ and $\sqrt{\frac{1}{3}}$.



4) We obtain the range by changing the variable to u ,

$$f(x, y) = F(u) = \ln u + 2u - 2, \quad u \in]0, 2],$$

because the value u is attained precisely on one level curve.

Since $F'(u) = \frac{1}{u} + 2$, we see that $F(u)$ is increasing.

When $u \rightarrow 0+$, we get $F(u) \rightarrow -\infty$. When $u = 2$, we get

$$F(u) = \ln 2 + 4 - 2 = 2 + \ln 2.$$

Since $F(u)$ is continuous, the connected interval $]0, 2]$ is mapped into the connected interval $]-\infty, 2 + \ln 2]$. Here we apply the third main theorem of continuous functions.

The range is $f(A) =]-\infty, 2 + \ln 2]$.

Example 7.4 Sketch for each for the functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ below the level curves given by $f(x, y) = C$ for the given values of the constant C .

- 1) $f(x, y) = x^2 + y^2, \quad C \in \{1, 2, 3, 4, 5\},$
- 2) $f(x, y) = x^2 - 4x + y^2, \quad C \in \{-3, -2, -1, 0, 1\},$
- 3) $f(x, y) = x^2 - 2y, \quad C \in \{-2, -1, 0, 1, 2\},$
- 4) $f(x, y) = \max\{|x|, |y|\}, \quad C \in \{1, 2, 3\},$
- 5) $f(x, y) = |x| + |y|, \quad C \in \{1, 2, 3\},$
- 6) $f(x, y) = (x^2 + y^2 + 1)^2 - 4x^2, \quad C \in \{\frac{1}{2}, 1, 3\},$
- 7) $f(x, y) = x^2 + y^2(1+x)^3, \quad C \in \{-4, 0, \frac{1}{4}, 1, 4\}.$

A Level curves.

D Whenever it is necessary, start by analyzing the given function.

I 1) The level curves are circles of centrum $(0, 0)$ and radii \sqrt{C} , i.e. $1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}$.

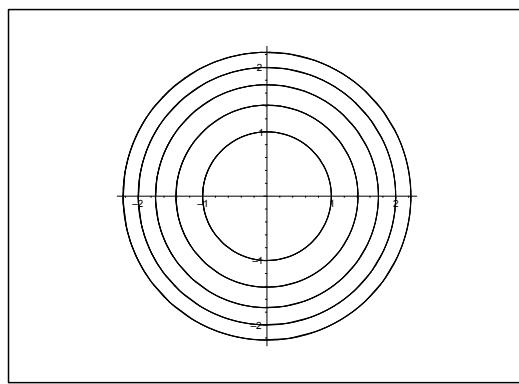


Figure 7.9: The level curves $x^2 + y^2 = C, C = 1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}$.

2) Since

$$f(x, y) = x^2 - 4x + y^2 = (x - 2)^2 + y^2 - 4,$$

we can also write the equation $f(x, y) = C$ of the level curves in the form

$$(x - 2)^2 + y^2 = 4 + C.$$

The level curves are circles of centre $(2, 0)$ and radius $\sqrt{4+C}$, i.e. $1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}$.

It follows that we obtain the same system as in 1), only translated to the centre $(2, 0)$.

3) The equation of the level curves $f(x, y) = C$ can also be written

$$y = \frac{1}{2}x^2 - \frac{C}{2}, \quad C \in \{-2, -1, 0, 1, 2\}.$$

These are parabolas of top points at $\left(0, -\frac{C}{2}\right)$.

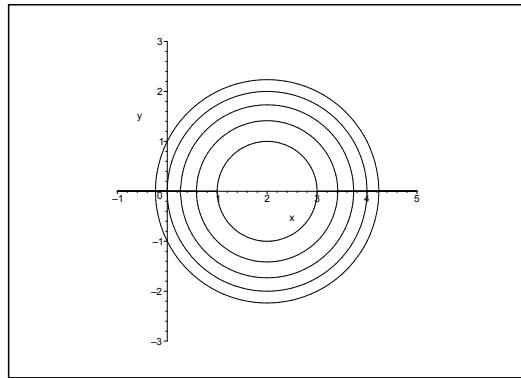


Figure 7.10: The level curves $x^2 - 4x + y^2 = C$, $C = -3, -2, -1, 0, 1$.

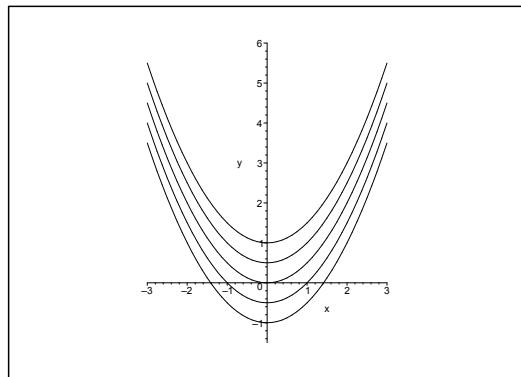


Figure 7.11: The level curves $x^2 - 2y = C$, $C = -2, -1, 0, 1, 2$.

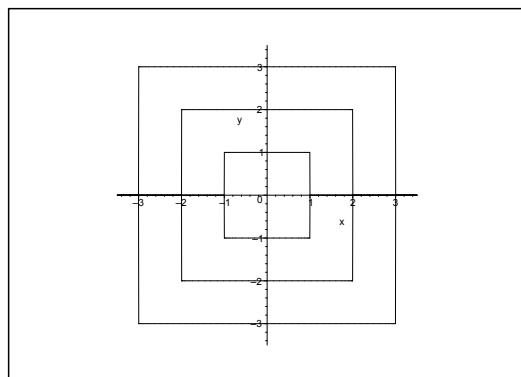


Figure 7.12: The level curves $\max\{|x|, |y|\} = C$, $C = 1, 2, 3$.

- 4) The level curves are the boundary of the squares of centre $(0, 0)$ and edge length $2C$.
- 5) The level curves are the boundaries of the squares of centre $(0, 0)$ and the corners $(\pm C, 0)$ and $(0, \pm C)$.

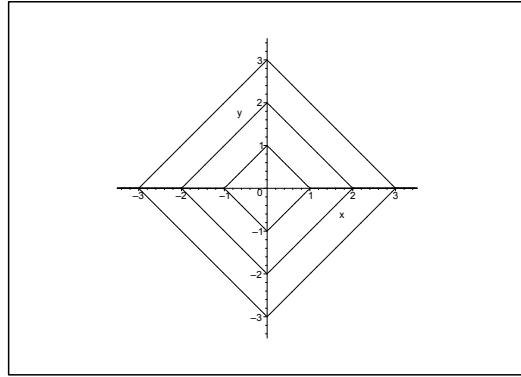


Figure 7.13: The level curves $|x| + |y| = C$, $C = 1, 2, 3$.

6) First note that

$$\begin{aligned} f(x, y) &= (x^2 + y^2 + 1)^2 - 4x^2 \\ &= (x^2 + y^2 + 1 - 2x)(x^2 + y^2 + 1 + 2x) \\ &= \{(x-1)^2 + y^2\}\{(x+1)^2 + y^2\}. \end{aligned}$$

The level curves $f(x, y) = C$ can then be interpreted as the curves composed of the points (x, y) , for which the product of the distances to $(1, 0)$ and $(-1, 0)$ is equal to \sqrt{C} .

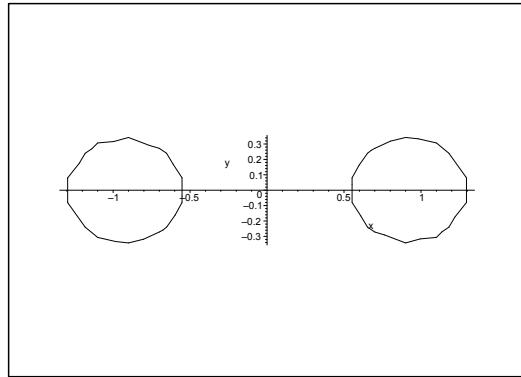


Figure 7.14: The level curve $(x^2 + y^2 + 1)^2 - 4x^2 = \frac{1}{2}$.

7) First note that when $x = -1$, then $f(-1, 0) = 1$. This means that we shall be particular careful in the case of $C = 1$.

Here we get five cases which are treated successively.

a) When $C = -4$, it follows from our first remark that $x \neq -1$. Clearly, $y \neq 0$, because $x^2 = -4$ does not have any real solution. The level curves are given by

$$y^2 = -\frac{4+x^2}{(1+x)^3} > 0.$$

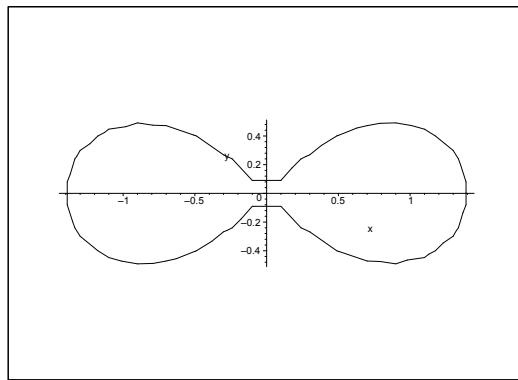


Figure 7.15: The level curve $(x^2 + y^2 + 1)^2 - 4x^2 = 1$. Though it cannot be seen (due to some error in the programme of sketching) the curves continue through $(0, 0)$.

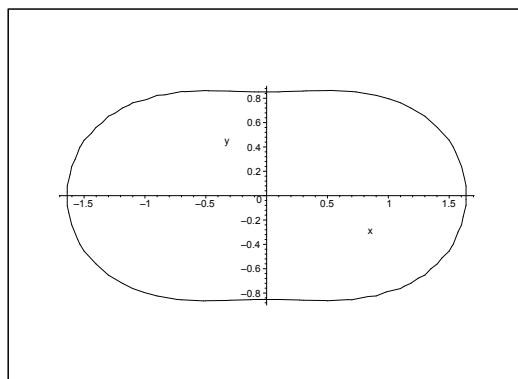


Figure 7.16: The level curve $(x^2 + y^2 + 1)^2 - 4x^2 = 3$.

Accordingly, $x < -1$, and

$$y^2 = -\frac{1}{(1+x)^2} \left(x - 1 + \frac{5}{1-x} \right),$$

i.e.

$$y = \pm \frac{1}{|1+x|} \sqrt{1-x - \frac{5}{1+x}} = \pm \frac{1}{|1+x|} \sqrt{2 + |1+x| + \frac{5}{|1+x|}},$$

for $x < -1$.

We get two level curves, which lie symmetrically to each other with respect to the X axis where the line $x = -1$ and the X axis are the asymptotes.

- b) When $C = 0$, we again find that $x \neq -1$. Note that if $y = 0$, then $x = 0$ is a solution, hence the point $(0, 0)$ belongs to the solutions. When $y \neq 0$, we get

$$y = \pm \left| \frac{x}{1+x} \right| \cdot \frac{1}{\sqrt{|1+x|}}, \quad x < -1.$$

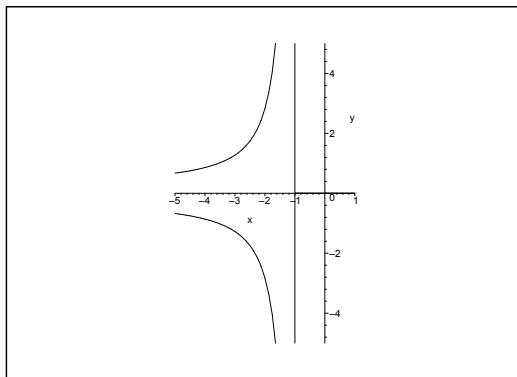


Figure 7.17: The level curves $x^2 + y^2(1+x)^3 = -4$.

The level “curves” are the point $(0,0)$ and two symmetric curves with respect to the X axis. These are closer the asymptotes than the level curves for $C = -4$.

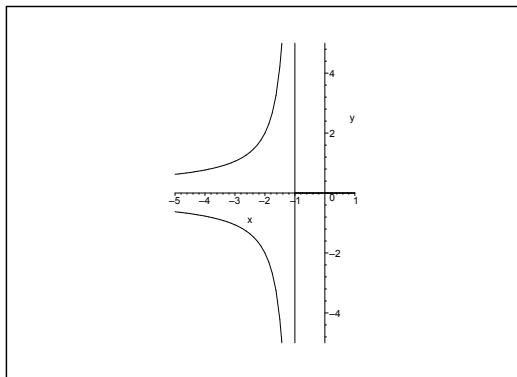


Figure 7.18: The level curves $x^2 + y^2(1+x)^3 = 0$, where the point $(0,0)$ should be added.

- c) If $C = \frac{1}{4}$, then $x \neq -1$, and

$$y^2 = \frac{\frac{1}{4} - x^2}{(1+x)^3} = -\frac{(x - \frac{1}{2})(x + \frac{1}{2})}{(x+1)^2} \geq 0.$$

We note that $y = 0$, if and only if $x = \pm \frac{1}{2}$.

Then the right hand side is positive, when either $|x| < \frac{1}{2}$ or $x < -1$.

The level curves are two symmetric curves for $x < -1$ with respect to the X axis, where the X axis and the line $x = -1$ are the asymptotes, supplied with a closed curve for $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

- d) For $C = 1$ we are in the exceptional case mentioned above where $x = -1$ is a level curve.

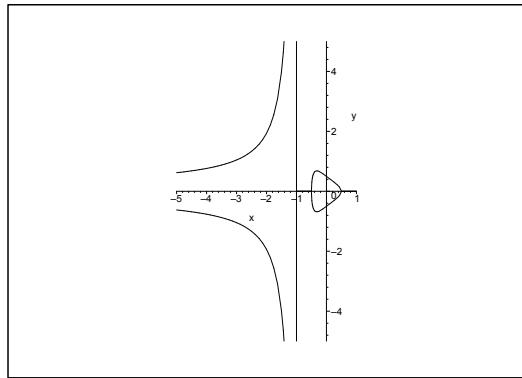


Figure 7.19: The level curves $x^2 + y^2(1+x)^3 = \frac{1}{4}$.

When $x \neq -1$, we get

$$y^2 = \frac{1-x^2}{(1+x)^3} = \frac{1-x}{(1+x)^2} \geq 0,$$

thus $x \leq 1$. When $x = 1$, we only get the solution $y = 0$, i.e. we get the point $(1, 0)$.

The level curves are the line $x = -1$, two symmetric curves with respect to the X axis for $x < -1$, and a curve with the X axis as an axis of symmetry for $x \in]-1, 1]$ and the line $x = -1$ as an asymptote.

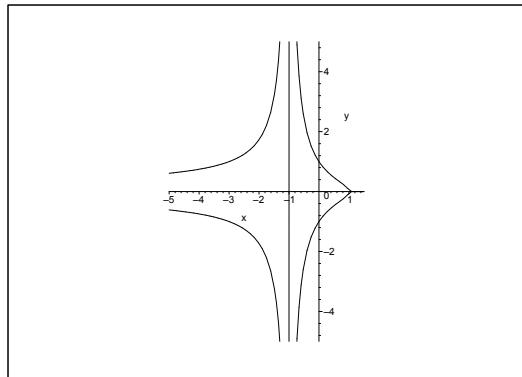


Figure 7.20: The level curves $x^2 + y^2(1+x)^3 = 1$.

e) When $C = 4$, we get

$$y^2 = \frac{4-x^2}{(1+x)^3} \geq 0.$$

It follows that $(\pm 2, 0)$ are solutions and that we only get solutions for either $x \leq -2$ or $-1 < x \leq 2$.

We obtain two curves, each symmetric with respect to the X axis. Furthermore, one of these curves has $x = -1$ as an asymptote.

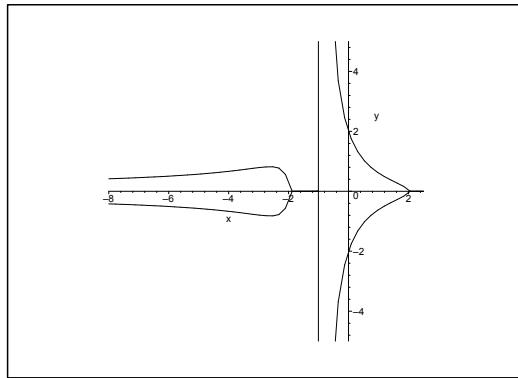


Figure 7.21: The level curves $x^2 + y^2(1+x)^3 = 4$.

Example 7.5 Describe the level surfaces for the following functions:

- 1) $f(x, y, z) = x$ for $(x, y, z) \in \mathbb{R}^3$,
- 2) $f(x, y, z) = \max\{|x|, |y|, |z|\}$ for $(x, y, z) \in \mathbb{R}^3$,
- 3) $f(x, y, z) = \sqrt{\max\{|x|, |y|, |z|\}}$ for $(x, y, z) \in \mathbb{R}^3$,
- 4) $f(x, y, z) = z - x^2 - y^2$ for $(x, y, z) \in \mathbb{R}^3$,
- 5) $f(x, y, z) = \frac{x^2 + y^2 + z^2 - a^2}{z}$ for $z \neq 0$.

A Level surfaces in space.

D Analyze the function. The sketches are left to the reader, because there are difficulties here with the MAPLE programs. (I am not clever enough to get the right drawings.)

I 1) Obviously, the level surfaces

$$f(x, y, z) = x = c$$

are all planes parallel to the YZ plane, where $c \in \mathbb{R}$.

- 2) The level surfaces are the boundaries of all cubes of centrum $(0, 0, 0)$ and edge length $2c$ for $c > 0$, supplied with the point $(0, 0, 0)$ when $c = 0$.

Only $c \geq 0$ is possible.

- 3) The level surfaces are the same as in 2), only the edge length is here $2c^2$ for $c > 0$. When $c = 0$ we obtain as before the point $(0, 0, 0)$.
- 4) Since $f(x, y, z) = z - x^2 - y^2 = c$ can also be written

$$z - c = x^2 + y^2,$$

we obtain all paraboloids of revolution with top point at $(0, 0, c)$, through the unit circle in the plane $z = 1 + c$ and with the Z axis as the axis of revolution.

5) First we rewrite

$$f(x, y, z) = \frac{x^2 + y^2 + z^2 - a^2}{z} = c, \quad z \neq 0,$$

to

$$x^2 + y^2 + z^2 - a^2 = cz, \quad z \neq 0,$$

i.e.

$$x^2 + y^2 + \left(z - \frac{c}{2}\right)^2 = a^2 + \frac{c^2}{4}, \quad z \neq 0 + .$$

The level surfaces are spheres of centrum $\left(0, 0, \frac{c}{2}\right)$ and radius $\sqrt{a^2 + \frac{c^2}{4}}$, with the exception of the points in the XY plane, i.e. with the exception of the circle

$$x^2 + y^2 = a^2, \quad z = 0.$$



Brain power

By 2020, wind could provide one-tenth of our planet's electricity needs. Already today, SKF's innovative know-how is crucial to running a large proportion of the world's wind turbines.

Up to 25 % of the generating costs relate to maintenance. These can be reduced dramatically thanks to our systems for on-line condition monitoring and automatic lubrication. We help make it more economical to create cleaner, cheaper energy out of thin air.

By sharing our experience, expertise, and creativity, industries can boost performance beyond expectations.

Therefore we need the best employees who can meet this challenge!

The Power of Knowledge Engineering

Plug into The Power of Knowledge Engineering.
 Visit us at www.skf.com/knowledge

SKF



Click on the ad to read more

Example 7.6 Consider the function $f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{e}$, $\mathbf{x} \in \mathbb{R}^k$, where \mathbf{e} is a constant unit vector.

- 1) Sketch the level curves of the function in the case of $k = 2$.
- 2) Describe the level surfaces of the function in the case of $k = 3$.

A Level curves and level surfaces.

D Sketch if possible a figure and analyze.

I 1) The level curves are all the straight lines ℓ , which are perpendicular to the line generated by the vector \mathbf{e} .

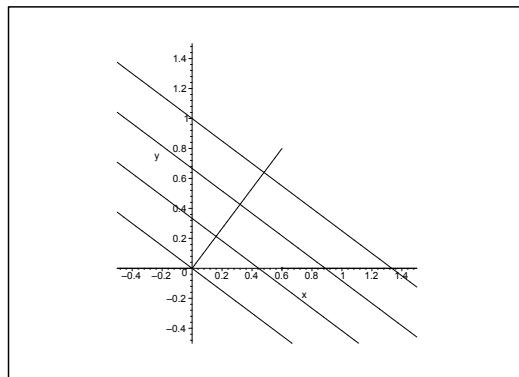


Figure 7.22: Some level curves when $\mathbf{e} = \left(\frac{3}{5}, \frac{4}{5}\right)$.

- 2) Analogously the level surfaces for $k = 3$ are all planes π , which are perpendicular to the line generated by the vector \mathbf{e} .

Example 7.7 Let a be a positive constant. Find the domain of the function

$$f(x, y, z) = \ln(a^2 - 3x^2 - y^2 - 2z^2).$$

The describe the level surfaces for f , and find the range of the function.

A Domain, level surfaces, range.

D Just follow the text.

I The function is defined for

$$3x^2 + y^2 + 2z^2 < a^2,$$

which describes the open ellipsoid with the half axes

$$\frac{a}{\sqrt{3}}, \quad a, \quad \frac{a}{\sqrt{2}}.$$

The level surfaces are all the ellipsoidal surfaces

$$3x^2 + y^2 + 2z^2 = b^2, \quad 0 < b < a,$$

with the half axes

$$\frac{b}{\sqrt{3}}, \quad b, \quad \frac{b}{\sqrt{2}}.$$

The value of the function on such a level surface is $\ln(a^2 - b^2)$.

The range of f is the same as the range of the function

$$g(t) = \ln(a^2 - t^2), \quad t \in [0, a[,$$

so the range is $]-\infty, 2 \ln a]$.

Example 7.8 Sketch the domain A of the function

$$f(x, y) = \ln(225 - 25x^2 - 9y^2).$$

Indicate the boundary ∂A of A , and sketch the level curve of f , which contains the point

$$(x, y) = \left(\frac{3}{2}, \frac{5}{2}\right).$$

A Domain and level curve.

D Since \ln is only defined on \mathbb{R}_+ , the domain is given by the requirement that the expression inside the \ln is positive.

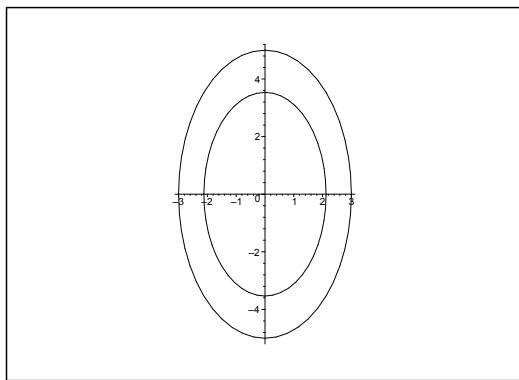


Figure 7.23: The domain A and the level curve through $\left(\frac{3}{2}, \frac{5}{2}\right)$.

I The function is defined for

$$225 - 25x^2 - 9y^2 > 0, \text{ i.e. for } (5x)^2 + (3y)^2 < 15^2,$$

hence

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 < 1.$$

The domain is an open ellipsoidal disc of centrum $(0,0)$ and half axes 3 and 5.

the level curve is given by

$$\ln(225 - 25x^2 - 9y^2) = f\left(\frac{3}{2}, \frac{5}{2}\right) = \ln\left\{225 - \left(5 \cdot \frac{3}{2}\right)^2 - \left(3 \cdot \frac{5}{2}\right)^2\right\},$$

i.e. by

$$225 - 25x^2 - 9y^2 = 225 \left(1 - \frac{1}{4} - \frac{1}{4}\right) = \frac{225}{2},$$

TURN TO THE EXPERTS FOR SUBSCRIPTION CONSULTANCY

Subscrybe is one of the leading companies in Europe when it comes to innovation and business development within subscription businesses.

We innovate new subscription business models or improve existing ones. We do business reviews of existing subscription businesses and we develop acquisition and retention strategies.

Learn more at [linkedin.com/company/subscrybe/](https://www.linkedin.com/company/subscrybe/) or contact
Managing Director Morten Suhr Hansen at mta@subscrybe.dk

SUBSCR✓BE - to the future



hence by a rearrangement,

$$(5x)^2 + (3y)^2 = \left(\frac{15}{\sqrt{2}}\right)^2.$$

This can also be written

$$\left(\frac{x}{\frac{3}{2}\sqrt{2}}\right)^2 + \left(\frac{y}{\frac{5}{2}\sqrt{2}}\right)^2 = 1.$$

Thus the level curve is an ellipse of centrum $(0,0)$ and half axes $\frac{3}{2}\sqrt{2} = \frac{3}{\sqrt{2}}$ and $\frac{5}{2}\sqrt{2} = \frac{5}{\sqrt{2}}$.

7.3 Continuous functions

Example 7.9 The range of each of the following functions in two variables is not the whole plane but $\mathbb{R}^2 \setminus M$, where $M \neq \emptyset$. Find the point set M in each case and explain why $f : \mathbb{R}^2 \setminus M \rightarrow \mathbb{R}$ is continuous. Finally, check whether the function has a continuous extension to either \mathbb{R}^2 or to $\mathbb{R}^2 \setminus L$, where $L \subset M$.

$$1) f(x, y) = \frac{x^2 - y^2}{x^2 + y^2},$$

$$2) f(x, y) = \frac{x^3 + y^3}{x^2 + y^2},$$

$$3) f(x, y) = \frac{x^2 y}{\sqrt{x^2 + y^2}},$$

$$4) f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}},$$

$$5) f(x, y) = \frac{3x - 2y}{2x - 3y},$$

$$6) f(x, y) = \frac{x^2 - y^2}{\text{Arctan}(x - y)},$$

$$7) f(x, y) = \frac{x^3 - y^3}{x - y},$$

$$8) f(x, y) = \frac{1 - e^{xy}}{xy}.$$

A Examination of functions, continuous extension.

D Find the set of exceptional points. Since the numerator and the denominator are continuous in \mathbb{R}^2 in all cases, it is only a matter of determining the zero set of the denominator. A possible continuous extension can only take place at points in which both the numerator and the denominator are zero, so this set should be examined too.

I 1) The denominator is clearly only zero at $(0, 0)$, so $M = \{(0, 0)\}$.

If we use polar coordinates, we get for $\varrho > 0$,

$$f(x, y) = \frac{\varrho^2 \cos^2 \varphi - \varrho^2 \sin^2 \varphi}{\varrho^2} = \cos^2 \varphi - \sin^2 \varphi = \cos 2\varphi,$$

and it is obvious that we cannot have a continuous extension to $(0, 0)$, because there is no restriction on φ .

2) Here also $M = \{(0, 0)\}$. By using polar coordinates we get

$$f(x, y) = \frac{\varrho^3 \cos^3 \varphi + \varrho^3 \sin^3 \varphi}{\varrho^2} = \varrho \{ \cos^3 \varphi + \sin^3 \varphi \},$$

which tends to 0 for $\varrho \rightarrow 0$. Hence, the function has a continuous extension to $(0, 0)$ given by $f(0, 0) = 0$.

3) Again $M = \{(0, 0)\}$. By using polar coordinates we get

$$f(x, y) = \frac{\varrho^3 \cos^2 \varphi \sin \varphi}{\varrho} = \varrho^2 \cos^2 \varphi \sin \varphi,$$

which tends to 0 for $\varrho \rightarrow 0$. Hence the function has a continuous extension given by $f(0, 0) = 0$.

4) Also here $M = \{(0, 0)\}$. Again by polar coordinates,

$$f(x, y) = \frac{\varrho^2 \sin \varphi \cos \varphi}{\varrho} = \varrho \sin \varphi \cos \varphi \rightarrow 0 \quad \text{for } \varrho \rightarrow 0.$$

By continuous extension we get $f(0, 0) = 0$.

5) Here

$$M = \{(x, y) \mid 2x = 3y\} = \left\{ (x, y) \mid y = \frac{2}{3}x \right\}.$$

The only possibility of a continuous extension must take place on that subset where the numerator is also zero, i.e. on $\{(0, 0)\}$. Using polar coordinates we get

$$f(x, y) = \frac{3 \cos \varphi - 2 \sin \varphi}{2 \cos \varphi - 3 \sin \varphi},$$

which clearly does not have a limit, when $\varrho \rightarrow 0$, and $\varphi \in [0, 2\pi]$. In this case we do not have a continuous extension.

6) Here $M = \{(x, y) \mid y = x\}$. Since

$$f(x, y) = \frac{x + y}{\arctan(x - y)}, \quad (x, y) \notin M,$$

where

$$\frac{\arctan t}{t} \rightarrow 1 \quad \text{for } t \rightarrow 0,$$

it is possible to extend the function to all of M by

$$f(x, x) = 2x, \quad (x, x) \in M.$$

7) Here we also have $M = \{(x, y) \mid y = x\}$. We get by a division

$$f(x, y) = \frac{x^3 - y^3}{x - y} = x^2 + xy + y^2, \quad (x, y) \notin M.$$

Clearly, the latter expression can be continuously extended to all of \mathbb{R}^2 . On M we get

$$f(x, x) = 3x^2, \quad (x, x) \in M.$$

8) Here $M = \{(x, y) \mid x = 0 \text{ or } y = 0\}$, i.e. the union of the coordinate axes.

Since

$$\frac{1 - e^t}{t} = -\frac{e^t - e^0}{t - 0} \rightarrow -1 \quad \text{for } t \rightarrow 0,$$

it follows from an application of the substitution $t = xy$ that f can be extended to the axes by

$$f(0, y) = f(x, 0) = -1.$$

What do you want to do?

No matter what you want out of your future career, an employer with a broad range of operations in a load of countries will always be the ticket. Working within the Volvo Group means more than 100,000 friends and colleagues in more than 185 countries all over the world. We offer graduates great career opportunities – check out the Career section at our web site www.volvologroup.com. We look forward to getting to know you!

VOLVO
 AB Volvo (publ)
www.volvologroup.com

VOLVO TRUCKS | RENAULT TRUCKS | MACK TRUCKS | VOLVO BUSES | VOLVO CONSTRUCTION EQUIPMENT | VOLVO PENTA | VOLVO AERO | VOLVO IT
 VOLVO FINANCIAL SERVICES | VOLVO 3P | VOLVO POWERTRAIN | VOLVO PARTS | VOLVO TECHNOLOGY | VOLVO LOGISTICS | BUSINESS AREA ASIA



Click on the ad to read more

Example 7.10 In each of the following cases one shall find the domain D of the given function f , and explain why f is continuous. Then show that f has a continuous extension to a point set B , where $B \supset D$.

$$1) f(x, y) = \frac{x + y - 1}{\sqrt{x} - \sqrt{1-y}},$$

$$2) f(x, y) = (x + y) \ln \sinh(x + y),$$

$$3) f(x, y) = \frac{\operatorname{Arcsin}(xy - 2)}{\operatorname{Arctan}(3xy - 6)},$$

$$4) f(x, y) = \exp\left(-\frac{1}{(x - y)^2}\right).$$

A Examination of functions and continuous extensions.

D Find the point set where the numerator and the denominator are defined and continuous.

Then check a possible extension to the set where both the numerator and the denominator are zero.

I 1) The numerator is defined in \mathbb{R}^2 . The numerator is defined and continuous when $x \geq 0$ and $1 - y \geq 0$, i.e. for $y \leq 1$.

The denominator is zero, when $\sqrt{x} = \sqrt{1-y}$ for $x \geq 0$ and $y \leq 1$. A squaring shows that the denominator is zero when

$$x + y = 1, \quad x \geq 0, \quad y \leq 1,$$

and we see that the numerator is zero on the same set. We see that the domain is

$$D = \{(x, y) \mid x \geq 0, y \leq 1, x + y \neq 1\} = D_1 \cup D_2.$$

In the two subdomains D_1 (the “lower triangular domain”) and D_2 (the “upper triangular domain”) both the numerator and the denominator are continuous, and the denominator is not zero in these two sets, so the function is continuous on D .

It has already above been given a hint that there is a possible continuous extension to the line $x + y = 1$ for $x \geq 0$ and $y \leq 1$, because both the numerator and the denominator are here 0. We get by a simple rearrangement for $(x, y) \in D$, i.e. in particular for $x + y \neq 1$, that

$$f(x, y) = \frac{x - (1 - y)}{\sqrt{x} - \sqrt{1-y}} = \frac{(\sqrt{x})^2 - (\sqrt{1-y})^2}{\sqrt{x} - \sqrt{1-y}} = \sqrt{x} + \sqrt{1-y}.$$

This expression is continuous on the set

$$\{(x, y) \mid x \geq 0, y \leq 1\},$$

and we have found our continuous extension of the original function.

2) Here $f(x, y)$ is defined and continuous for $\sinh(x + y) > 0$, i.e. when $x + y > 0$, and the domain is

$$D = \{(x, y) \mid x + y > 0\}.$$

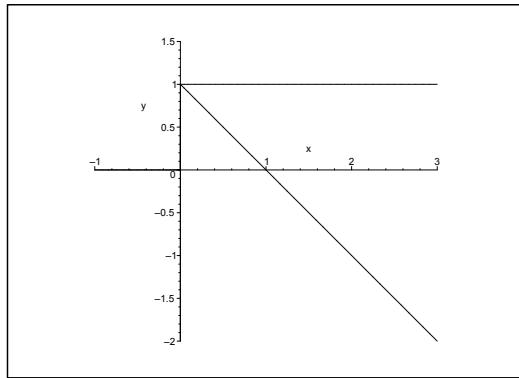


Figure 7.24: The domain of $f(x, y) = \frac{x + y - 1}{\sqrt{x} - \sqrt{1 - y}}$.

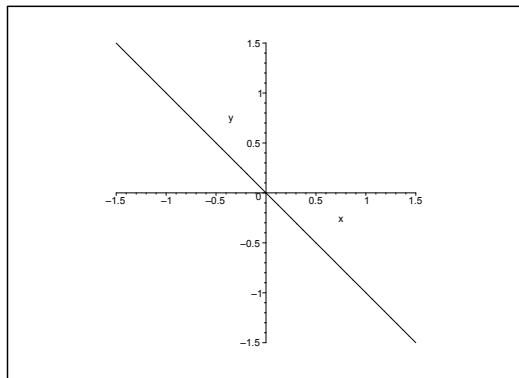


Figure 7.25: The domain of $f(x, y) = (x + y) \ln \sinh(x + y)$ lies above the oblique line.

By putting $t = x + y > 0$ we exploit that $f(x, y)$ actually is a function in $x + y$. Then

$$f(x, y) = g(t) = t \ln \sinh t = \frac{t}{\sinh t} \{\sinh t \cdot \ln \sinh t\}.$$

Here, $\frac{t}{\sinh t} \rightarrow 1$ for $t \rightarrow 0+$, and $\sinh t \cdot \ln \sinh t \rightarrow 0$ for $\sinh t \rightarrow 0+$, i.e. for $t \rightarrow 0+$. We therefore conclude for $z = \sinh t$ that

$$\lim_{t \rightarrow 0+} t \ln \sinh t = 0.$$

Then by the substitution $t = x + y$,

$$(x + y) \ln \sinh(x + y) \rightarrow 0 \quad \text{for } x + y \rightarrow 0+.$$

Hence, the function can be extended continuously to the set

$$\overline{D} = \{(x, y) \mid x + y \geq 0\},$$

where we for $x + y = 0$ put

$$\bar{f}(x, -x) = 0, \quad x \in \mathbb{R}.$$

- 3) The numerator $\text{Arcsin}(xy - 2)$ is defined and continuous, when $-1 \leq xy - 2 \leq 1$, i.e. when $1 \leq xy \leq 3$.

The denominator $\text{Arctan}(3xy - 6)$ is defined and continuous for every $(x, y) \in \mathbb{R}^2$.

The denominator is zero for $xy = 2$, and we see that the numerator is zero on the same set.

Thus the domain is

$$D = \{(x, y) \mid 1 \leq xy < 2 \text{ or } 2 < xy \leq 3\}.$$

We see that the domain has four connected components.

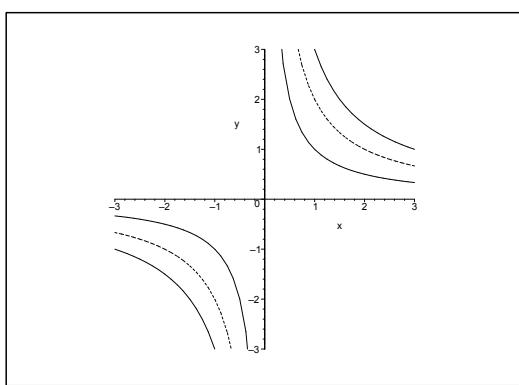


Figure 7.26: The domain of $f(x, y) = \frac{\text{Arcsin}(xy - 2)}{\text{Arctan}(3xy - 6)}$ is the union of the sets which lie between the hyperbolas in the first and third quadrant, with the exception of the dotted hyperbola in the “middle” of each set.

Since both the numerator and the denominator are zero on the exceptional hyperbola of the equation $xy = 2$, there is a possibility of a continuous extension to this hyperbola. We shall now examine this possibility.

First note that

$$\frac{\text{Arcsin } t}{\text{Arctan } 3t} = \frac{1}{3} \cdot \frac{\text{Arcsin } t}{t} \cdot \frac{3t}{\text{Arctan } 3t} \rightarrow \frac{1}{3} \quad \text{for } t \rightarrow 0.$$

Then by the substitution $t = xy - 2$,

$$f(x, y) = \frac{\text{Arcsin}(xy - 2)}{\text{Arctan}(3xy - 6)} \rightarrow \frac{1}{3} \quad \text{for } xy \rightarrow 2.$$

Hence, we can extend f continuously to the set

$$B = \{(x, y) \mid 1 \leq xy \leq 3\}$$

by putting

$$\bar{f}(x, y) = \begin{cases} \frac{\text{Arcsin}(xy - 2)}{\text{Arctan}(3xy - 6)} & \text{for } xy \in [1, 3] \setminus \{2\}, \\ \frac{1}{3} & \text{for } xy = 2. \end{cases}$$

- 4) The function is defined and continuous for $y \neq x$, so the domain is given by

$$D = \{(x, y) \mid y \neq x\}.$$

Since

$$\lim_{t \rightarrow 0} \exp\left(-\frac{1}{t^2}\right) = 0,$$

it follows by the substitution $t = x - y$ that $f(x, y)$ can be extended to all of \mathbb{R}^2 by the continuous extension

$$\bar{f}(x, y) = \begin{cases} \exp\left(-\frac{1}{(x-y)^2}\right) & \text{for } y \neq x, \\ 0 & \text{for } y = x. \end{cases}$$

Challenge the way we run

EXPERIENCE THE POWER OF FULL ENGAGEMENT...

**RUN FASTER.
RUN LONGER..
RUN EASIER...**

READ MORE & PRE-ORDER TODAY
WWW.GAITEYE.COM

Example 7.11 Sketch in each of the cases below the domain of the given function or vector function. Then examine whether the (vector) function has a limit for $(x, y) \rightarrow (0, 0)$, and find this limit when it exists.

$$1) f(x, y) = \frac{\sin(xy)}{x},$$

$$2) f(x, y) = \frac{1}{x} \sin y,$$

$$3) f(x, y) = x \sin \frac{1}{y},$$

$$4) \mathbf{f}(x, y) = \left(\frac{\ln(1 + x^2 + y^2)}{\sqrt{x^2 + y^2}}, \frac{\ln x + \ln y}{\ln(xy)} \right),$$

$$5) \mathbf{f}(x, y) = \left(\frac{x \sin y}{\sqrt{x^2 + y^2}}, \frac{x^2 y^2 + x^2 + y^2}{x^2 + 3y^2} \right),$$

$$6) f(x, y) = \left(\frac{x}{x+y}, \sqrt{x+y} \right).$$

A Domains; limits.

D Analyze the function; take the limit.

I 1) The function is defined for $x \neq 0$, i.e. everywhere except on the Y axis,

$$D = \{(x, y) \mid x \neq 0\}.$$

There is of course no need to sketch the domain in this case.

By using polar coordinates we get from $x = \rho \cos \varphi \neq 0$ in D that $\rho > 0$ and $\cos \varphi \neq 0$. This shows that in D ,

$$|f(x, y)| = \left| \frac{\sin(\rho^2 \cos \varphi \sin \varphi)}{\rho \cos \varphi} \right| \leq \frac{\rho^2 |\cos \varphi| |\sin \varphi|}{\rho |\cos \varphi|} = \rho |\sin \varphi|,$$

which tends to 0 for $\rho \rightarrow 0+$, hence

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0.$$

ALTERNATIVELY one can use directly that

$$|f(x, y) - 0| = \left| \frac{\sin(xy)}{x} \right| \leq \frac{|xy|}{|x|} = |y| \rightarrow 0$$

for $|y| \leq \sqrt{x^2 + y^2} \rightarrow 0$.

2) The domain is the same as in 1).

The limit does not exist, because e.g.

$$f(x, x) = \frac{\sin x}{x} \rightarrow 1 \quad \text{for } x \rightarrow 0,$$

$$f(x, -x) = -\frac{\sin x}{x} \rightarrow -1 \quad \text{for } x \rightarrow 0.$$

- 3) The function is defined for $y \neq 0$, i.e. at the points outside the X axis. There is no need either to sketch this set.

The limit is 0, because

$$|f(x, y) - 0| = |x| \cdot \left| \sin \frac{1}{y} \right| \leq |x| \rightarrow 0 \quad \text{for } (x, y) \rightarrow (0, 0).$$

- 4) The vector function is defined (and continuous), when

- a) $1 + x^2 + y^2 > 0$ (always fulfilled),
- b) $x^2 + y^2 > 0$ (i.e. $(x, y) \neq (0, 0)$),
- c) $x > 0$,
- d) $y > 0$,
- e) $xy > 0$,
- f) $xy \neq 1$.

Summarizing we see that the domain is the open first quadrant, with the exception of a branch of a hyperbola,

$$D = \{(x, y) \mid x > 0, y > 0\} \setminus \{(x, y) \mid xy = 1\}.$$

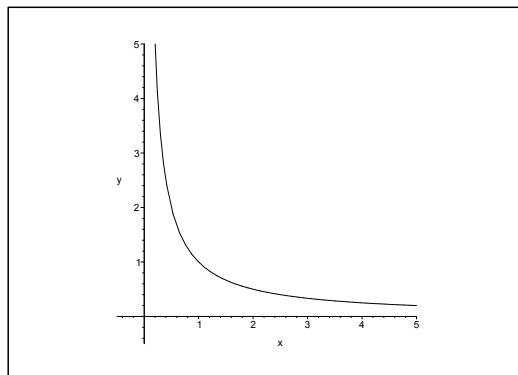


Figure 7.27: The vector function is defined in the first quadrant with the exception of the branch of the hyperbola.

Since

$$\begin{aligned} \frac{\ln(1 + x^2 + y^2)}{\sqrt{x^2 + y^2}} &= \frac{1}{\sqrt{x^2 + y^2}} \{(x^2 + y^2) + (x^2 + y^2)\varepsilon(x^2 + y^2)\} \\ &= \sqrt{x^2 + y^2}\{1 + \varepsilon(x^2 + y^2)\} \rightarrow 0 \end{aligned}$$

for $(x, y) \rightarrow (0, 0)$, and

$$\frac{\ln x + \ln y}{\ln(xy)} = \frac{\ln(xy)}{\ln(xy)} = 1 \quad \text{for } (x, y) \in D,$$

we conclude that

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ (x,y) \in D}} f(x,y) = (0,1).$$

- 5) The vector function is defined for $(x,y) \neq (0,0)$.

Let us estimate the first coordinate function,

$$\left| \frac{x \sin t}{\sqrt{x^2 + y^2}} \right| = \frac{|x|}{\sqrt{x^2 + y^2}} |\sin y| \leq 1 \cdot |\sin y| \rightarrow 0$$

for $(x,y) \rightarrow (0,0)$. We see that the first coordinate function converges towards 0 by the limit.

In the examination of the second coordinate function we use polar coordinates $0 < \varphi < \frac{\pi}{2}$, $\varrho > 0$. We get by insertion

$$\frac{x^2y^2 + x^2 + y^2}{x^2 + 3y^2} = \frac{\varrho^4 \cos^2 \varphi \cdot \sin^2 \varphi + \varrho^2}{\varrho^2(1 + 2 \sin^2 \varphi)} = \frac{1}{1 + 2 \sin^2 \varphi} + \varrho^2 \cdot \frac{\sin^2 \varphi \cos^2 \varphi}{1 + 2 \sin^2 \varphi}.$$

This e-book
is made with
SetaPDF



PDF components for **PHP** developers

www.setasign.com



Click on the ad to read more

The latter term converges towards 0 for $\varrho \rightarrow 0$; but the first term depends on φ and not on ϱ .

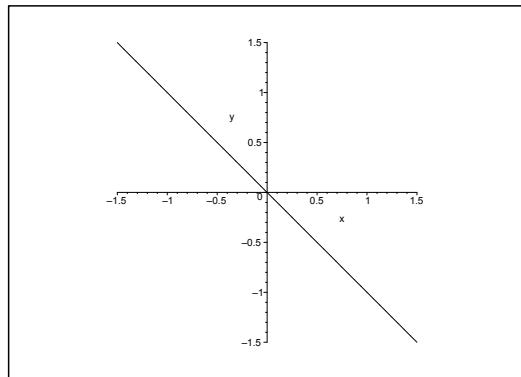


Figure 7.28: **Example 7.11.5.** The domain is the half plane which lies above the line.

Since the second coordinate function cannot be extended continuously to $(0, 0)$, neither can the vector function itself be extended continuously to $(0, 0)$.

- 6) The vector function

$$\mathbf{f}(x, y) = \left(\frac{x}{x+y}, \sqrt{x+y} \right)$$

is defined for $x + y \neq 0$ and $x + y \geq 0$, so the domain is

$$\{(x, y) \mid x + y > 0\}.$$

The first coordinate function does not have a limit for $(x, y) \rightarrow (0, 0)$ in the domain. In fact if we in particular restrict ourselves to the positive X axis where $y = 0$, then

$$\lim_{x \rightarrow 0^+} f_1(x, 0) = \lim_{x \rightarrow 0^+} \frac{x}{x+0} = 1.$$

If we instead restrict ourselves to the positive Y axis we get

$$\lim_{y \rightarrow 0^+} f_1(0, y) = \lim_{y \rightarrow 0^+} \frac{0}{0+y} = 0.$$

Since $1 \neq 0$, the limit does not exist.

Example 7.12 Let $f : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$ be given by

$$f(x, y) = \frac{x^2y^2}{x^2y^2 + (x - y)^2}.$$

Show that

$$\lim_{x \rightarrow 0} \left(\lim_{y \rightarrow 0} f(x, y) \right) = \lim_{y \rightarrow 0} \left(\lim_{x \rightarrow 0} f(x, y) \right) = 0,$$

and that f nevertheless does not have a limit for $(x, y) \rightarrow (0, 0)$.

A Limits.

D Calculate the successive limits and finally the limit along the line $y = x$.

I If $x \neq 0$, then

$$x^2y^2 \rightarrow 0 \text{ and } x^2y^2 + (x - y)^2 \rightarrow x^2 \neq 0 \quad \text{for } y \rightarrow 0,$$

hence

$$\lim_{y \rightarrow 0} f(x, y) = 0 \quad \text{for } x \neq 0.$$

Note also that

$$\lim_{y \rightarrow 0} f(0, y) = \lim_{y \rightarrow 0} \frac{0}{y^2} = 0.$$

Since $f(x, y) = f(y, x)$, it follows immediately that

$$\lim_{x \rightarrow 0} \left(\lim_{y \rightarrow 0} f(x, y) \right) = \lim_{y \rightarrow 0} \left(\lim_{x \rightarrow 0} f(x, y) \right) = 0.$$

Then consider the limit $(x, y) \rightarrow (0, 0)$ along the line $y = x$. This is given by

$$\lim_{x \rightarrow 0} f(x, x) = \lim_{x \rightarrow 0} \frac{x^4}{x^4 + 0^2} = 1 \neq 0.$$

We conclude that f does not have a limit for $(x, y) \rightarrow (0, 0)$.

Example 7.13 Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} \sin \frac{1}{x} \sin y, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Prove that $f(x, y) \rightarrow 0$ for $(x, y) \rightarrow (0, 0)$; and that we nevertheless do not have

$$\lim_{x \rightarrow 0} \left(\lim_{y \rightarrow 0} f(x, y) \right) = \lim_{y \rightarrow 0} \left(\lim_{x \rightarrow 0} f(x, y) \right).$$

A Limits.

D Use the definition of a limit in 1), and the rules of calculations in 2).

I If $x \neq 0$, then

$$|f(x, y) - f(0, 0)| = \left| \sin \frac{1}{x} \right| \cdot |\sin y| \leq |\sin y| \rightarrow 0 \quad \text{for } (x, y) \rightarrow (0, 0),$$

and it follows trivially for $x = 0$ that

$$|f(0, y) - f(0, 0)| = 0 \rightarrow 0 \quad \text{for } (x, y) \rightarrow (0, 0).$$

We conclude that

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0.$$

Then it follows immediately that

$$\lim_{y \rightarrow 0} f(x, y) = \begin{cases} \lim_{y \rightarrow 0} \sin \frac{1}{x} \cdot \sin y = 0, & \text{for } x \neq 0, \\ \lim_{y \rightarrow 0} 0 = 0, & \text{for } x = 0, \end{cases}$$

thus

$$\lim_{x \rightarrow 0} \left(\lim_{y \rightarrow 0} f(x, y) \right) = 0.$$

On the other hand, $\sin \frac{1}{x} \cdot \sin y$ for $y \neq p\pi$, $p \in \mathbb{Z}$, does not have a limit for $x \rightarrow 0$, so

$$\lim_{y \rightarrow 0} \left(\lim_{x \rightarrow 0} f(x, y) \right)$$

is not defined.

Example 7.14 Find the domain A of

$$f(x, y) = \frac{xy}{x + y}.$$

Show that f cannot be continuously extended to a point set $B \supset A$.
Then let

$$D = \{(x, y) \mid 0 \leq x, 0 \leq y, x^2 + y^2 > 0\},$$

and consider the function $g : D \rightarrow \mathbb{R}$ given by

$$g(x, y) = \frac{xy}{x + y}.$$

Sketch D , and prove that g has a continuous extension to the point set $D \cup \{(0, 0)\}$. Compare with the formula of the resulting resistance of a connection in parallel of two resistances.

A Domain; continuous extension; limit.

D Find the point set, in which the denominator is 0, and then indicate A . Examine the limit in D .

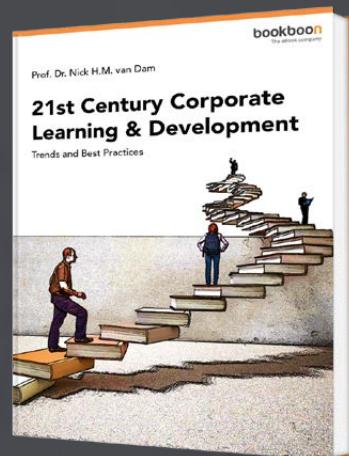
I Clearly,

$$A = \{(x, y) \mid y \neq -x\}.$$

Free eBook on Learning & Development

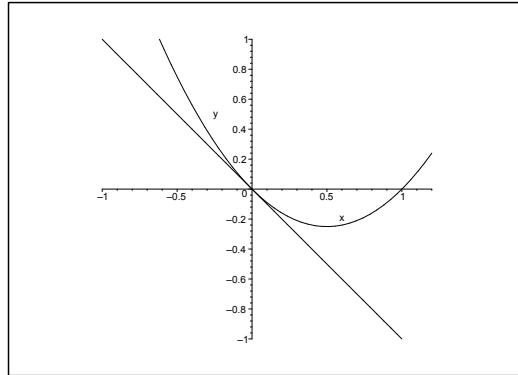
By the Chief Learning Officer of McKinsey

Download Now



Click on the ad to read more

Furthermore, $(0, 0)$ is the only point in which both the numerator and the denominator are zero, so there is only a possibility of a continuous extension to the set $A \cup \{(0, 0)\}$.



When we restrict ourselves to the curve $y = -x + x^2$, $x \neq 0$, we get

$$\lim_{x \rightarrow 0} f(x, -x + x^2) = \lim_{x \rightarrow 0} \frac{-x^2 + x^3}{x^2} = -1.$$

On the other hand, it is obvious that $f(x, 0) = 0 \rightarrow 0$ for $x \rightarrow 0$, so we get two different limits by approaching $(0, 0)$ along two different curves. Hence, the limit does not exist, and f cannot be extended continuously.

The set D is the closed first quadrant with the exception of the point $(0, 0)$. Since $x \geq 0$ and $y \geq 0$ in D , we have the estimate

$$0 < \max\{x, y\} \leq x + y \quad \text{for every } (x, y) \in D,$$

and hence

$$|g(x, y) - 0| = \left| \frac{x}{x+y} \right| \cdot |y| \leq |y| \rightarrow 0 \quad \text{for } (x, y) \rightarrow (0, 0) \text{ in } D.$$

This shows that g can be extended continuously to $(0, 0)$, when we define $g(0, 0) = 0$.

By the rearrangement

$$\frac{1}{g(x, y)} = \frac{x+y}{xy} = \frac{1}{x} + \frac{1}{y}, \quad x > 0, \quad y > 0,$$

we get the connection to the formula of the resulting resistance for a connection in parallel. From the above follows that

$$g(x, y) = \frac{xy}{x+y}$$

in D° can be extended to $D^\circ \cup \{(0, 0)\}$.

7.4 Description of curves

Example 7.15 In the following there are given some curves. In each case one shall find an equation of the curve by eliminating the parameter t . Indicate the name of the curve.

1) $\mathbf{r}(t) = \left(a \frac{1-t^2}{1+t^2}, b \frac{2t}{1+t^2} \right)$, for $t \in \mathbb{R}$.

2) $\mathbf{r}(t) = \left(\frac{a}{\cos t}, b \tan t \right)$, for $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \cup \left[\frac{\pi}{2}, \frac{3\pi}{2} \right]$.

3) $\mathbf{r}(t) = (at^2, 2at)$, for $t \in \mathbb{R}$.

4) $\mathbf{r}(t) = (a \sin t, a \cos 2t)$, for $t \in [-\pi, \pi]$.

A Description of curves.

D Eliminate the parameter.

I 1) It follows from $x = a \frac{1-t^2}{1+t^2}$ and $y = b \frac{2t}{1+t^2}$ that

$$\frac{x}{a} = \frac{1-t^2}{1+t^2} \quad \text{and} \quad \frac{y}{b} = \frac{2t}{1+t^2},$$

where the idea is that the two right hand sides are independent of the arbitrary constants a and b .

We get by squaring and adding

$$\left(\frac{x}{a} \right)^2 + \left(\frac{y}{b} \right)^2 = \left(\frac{1-t^2}{1+t^2} \right)^2 + \left(\frac{2t}{1+t^2} \right)^2 = \frac{(1-2t^2+t^4)+4t^2}{(1+t^2)^2} = \frac{1+2t^2+t^4}{1+2t^2+t^4} = 1.$$

Thus the curve is a subset of an ellipse of centre $(0, 0)$ and the half axes a and b .

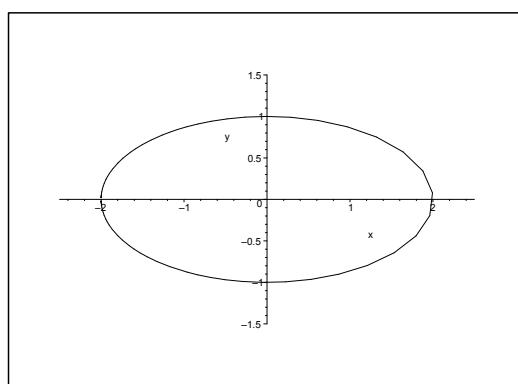


Figure 7.29: the curve for $a = 2$ and $b = 1$.

Then

$$\frac{1-t^2}{1+t^2} = -1 + \frac{2}{1+t^2} \leq 1$$

with equality for $t = 0$, so $\frac{1-t^2}{1+t^2}$ runs through the interval $]-1, 1]$ (twice), when t runs through \mathbb{R} . Since $\frac{2t}{1+t^2}$ changes its sign for $t = 0$, we conclude that the arc of the curve is the ellipse with the exception of the point $(-a, 0)$.

- 2) It follows from $x = \frac{a}{\cos t}$ and $y = b \tan t$ that

$$\frac{x}{a} = \frac{1}{\cos t} \quad \text{and} \quad \frac{y}{b} = \frac{\sin t}{\cos t},$$

so the parameter t is eliminated by

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = \frac{1 - \sin^2 t}{\cos^2 t} = 1.$$



www.sylvania.com

We do not reinvent the wheel we reinvent light.

Fascinating lighting offers an infinite spectrum of possibilities: Innovative technologies and new markets provide both opportunities and challenges. An environment in which your expertise is in high demand. Enjoy the supportive working atmosphere within our global group and benefit from international career paths. Implement sustainable ideas in close cooperation with other specialists and contribute to influencing our future. Come and join us in reinventing light every day.

Light is OSRAM

OSRAM
SYLVANIA



Click on the ad to read more

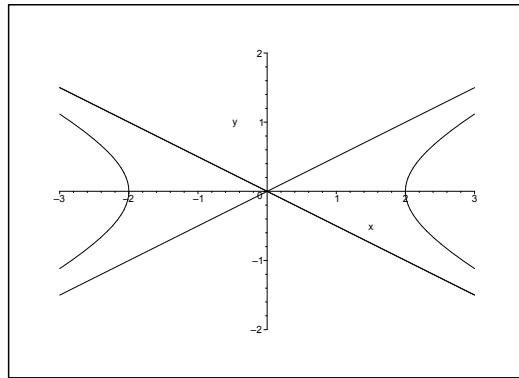


Figure 7.30: The curves for $a = 2$ and $b = 1$.

This describes an hyperbola of the half axes a and b and of centre $(0,0)$. The two intervals $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and $\left]\frac{\pi}{2}, \frac{3\pi}{2}\right[$ corresponds to the two branches.

- 3) Here, $x = at^2$ and $y = 2at$, so $t = \frac{y}{2a}$. Then by insertion,

$$x = at^2 = \frac{1}{4a} y^2,$$

which is the equation of a parabola with top point $(0,0)$ and the X axis as its axis.

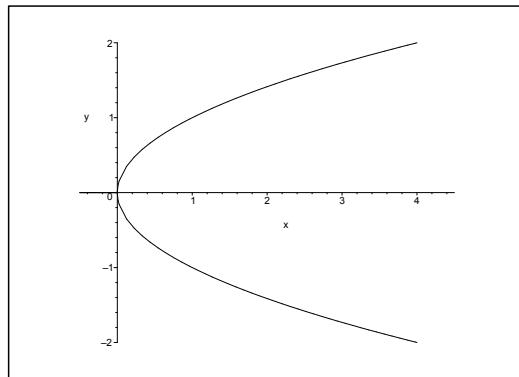


Figure 7.31: The curve for $a = \frac{1}{4}$.

- 4) When

$$(x, y) = \mathbf{r}(t) = (a \sin t, a \cos 2t), \quad t \in [-\pi, \pi],$$

and $a > 0$, it follows that

$$y = a \cos 2t = a(1 - 2 \sin^2 t) = 1 - \frac{2}{a}(a \sin t)^2 = a - \frac{2}{a} x^2,$$

i.e.

$$y = a - \frac{2}{a} x^2, \quad x \in [-\pi, \pi],$$

which is a part of a parabolic arc. Note that we use that

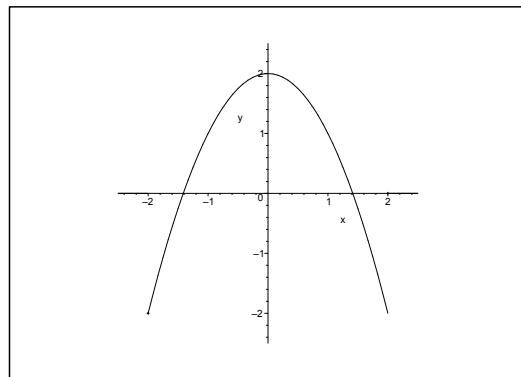


Figure 7.32: The curve for $a = 2$.

$$|x| = |a \sin t| \leq a,$$

when we find the domain $[-a, a]$, where we can have both $x = -a$ and $x = a$.



Deloitte.

Discover the truth at www.deloitte.ca/careers

© Deloitte & Touche LLP and affiliated entities.

Example 7.16 Prove that the curve given by

$$\mathbf{r}(t) = (3t + t^2, t - t^2, 3 - 5t + t^2), \quad t \in \mathbb{R},$$

lies in a plane, and find an equation of this plane.

A Space curve lying in a plane.

D Put the coordinate functions of the curve into the general equation of a plane and find the coefficients.

I In general the equation of a plane is given by

$$ax + by + cz = k.$$

Then by insertion of

$$(x, y, z) = (3t + t^2, t - t^2, 3 - 5t + t^2),$$

we get

$$\begin{aligned} k &= a(3t + t^2) + b(t - t^2) + c(3 - 5t + t^2) \\ &= t^2(a - b + c) + t(3a + b - 5c) + 3c, \quad t \in \mathbb{R}. \end{aligned}$$

This should hold for every t , so we must necessarily have

$$\begin{cases} a - b + c = 0, \\ 3a + b - 5c = 0, \\ 3c = k. \end{cases}$$

It follows that if $k = 0$, then we only get $(a, b, c) = (0, 0, 0)$ as a solution.

By choosing $k \neq 0$, e.g. $k = 3$, we get $c = 1$, and then by insertion

$$\begin{cases} a - b = -c = -1, \\ 3a + b = 5c = 5. \end{cases}$$

An addition shows that $4a = 4$, i.e. $a = 1$, and it follows that $b = 2$.

Hence an equation of the plane is

$$x + 2y + z = 3,$$

and we have at the same time proved that the curve lies in this plane.

Example 7.17 Prove that the curve given by

$$\mathbf{r}(t) = \left(2t\sqrt{1-t}, 2(1-t)\sqrt{t}, 1-2t \right), \quad t \in [0, 1],$$

lies on a sphere of centre $(0, 0, 0)$.

A A space curve lying on a sphere.

D Put the coordinate functions into the equation of the sphere and find its radius r .

I The general equation of a sphere of centrum $(0, 0, 0)$ is

$$x^2 + y^2 + z^2 = r^2.$$

By putting

$$x = 2r\sqrt{1-t}, \quad y = 2(1-t)\sqrt{t}, \quad z = 1-2t,$$

we get

$$\begin{aligned} x^2 + y^2 + z^2 &= 4t^2(1-t) + 4(1-t)^2t + (1-2t)^2 \\ &= 4t(1-t)\{t+(1-t)\} + (1-2t)^2 \\ &= (4t - 4t^2) + (1 - 4t + 4t^2) = 1, \end{aligned}$$

and we conclude that the curve lies on the unit sphere.

Example 7.18 Prove that the curve given by

$$\mathbf{r}(t) = (a(1 - \sin t) \cos t, b(\sin t + \cos^2 t), c \cos t), \quad t \in [-\pi, \pi],$$

lies on an hyperboloid.

A A space curve lying on an hyperboloid.

D Calculate $\left(\frac{x}{a}\right)^2$, $\left(\frac{y}{b}\right)^2$ and $\left(\frac{z}{c}\right)^2$, which are three expressions which are independent of the constants a , b and c . Then compare.

I We calculate

$$\begin{aligned} \left(\frac{x}{a}\right)^2 &= (1 - \sin t)^2 \cos^2 t = (1 - 2 \sin t + \sin^2 t) \cos^2 t \\ &= \cos^2 t - 2 \sin t \cos^2 t + \sin^2 t \cos^2 t, \\ \left(\frac{y}{b}\right)^2 &= (\sin t + \cos^2 t)^2 = \sin^2 t + 2 \sin t \cos^2 t + \cos^4 t \\ \left(\frac{z}{c}\right)^2 &= \cos^2 t. \end{aligned}$$

Hence

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 + \cos^2 t = 1 + \left(\frac{z}{c}\right)^2,$$

and by a rearrangement

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2,$$

and we conclude that the curve lies on an hyperboloid with one sheet.

Example 7.19 Sketch the so-called cycloid given by

$$\mathbf{r}(t) = (a(t - \sin t), a(1 - \cos t)), \quad t \in \mathbb{R}.$$

A Sketch of a curve.

D If one does not have MAPLE at hand, start by finding some points of the curve. One may exploit the geometrical meaning of

$$\mathbf{r}(t) = a(t, 1) - a(\sin t, \cos t), \quad t \in \mathbb{R},$$

where the former term on the right hand side is a rectilinear and even motion, while the latter term is a circular motion. Thus the curve describes the motion of a point on a wheel, which is rolling along the X axis.

I Clearly, $\mathbf{r}(t)$ is periodical of period 2π , so it suffices to sketch one period and a little bit of the neighbouring periods.

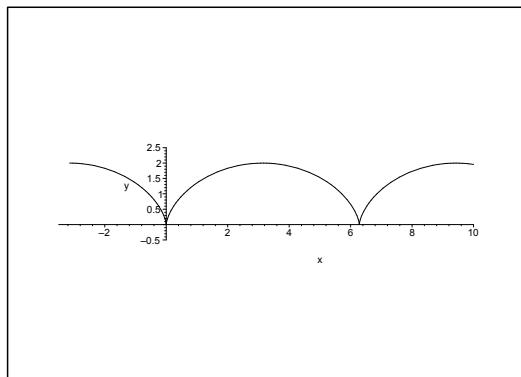


Figure 7.33: The cycloid for $a = 1$.

Example 7.20 Find in each of the cases below an equation of the given curve by eliminating the parameter t , and then sketch the curve.

1) $\mathbf{r}(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3} \right)$, for $t \in \mathbb{R} \setminus \{-1\}$.

2) $\mathbf{r}(t) = (\cos t, \sin t \cos t)$, for $t \in \mathbb{R}$.

3) $\mathbf{r}(t) = (a \cos^3 t, a \sin^3 t)$, for $t \in [-\pi, \pi]$.

4) $\mathbf{r}(t) = (a(1-3t^2), at(3-t^2))$, for $t \in \mathbb{R}$.

A Description of curves.

D Eliminate the parameter.

I 1) When $t \neq -1$, we get

$$x = \frac{3t}{1+t^3} \quad \text{and} \quad y = \frac{3t^2}{1+t^3}.$$

For $t = 0$ we get the point $(x, y) = (0, 0)$.

For $t \neq 0$ and $t \neq -1$ we get $t = \frac{y}{x}$, where $x \neq 0$ and $y \neq 0$, so by insertion

$$x = \frac{3t}{1+t^3} = \frac{3y/x}{1+(y/x)^3} = \frac{3x^2y}{x^3+y^3}.$$

SIMPLY CLEVER

ŠKODA



We will turn your CV into
an opportunity of a lifetime



Do you like cars? Would you like to be a part of a successful brand?
 We will appreciate and reward both your enthusiasm and talent.
 Send us your CV. You will be surprised where it can take you.

Send us your CV on
www.employerforlife.com



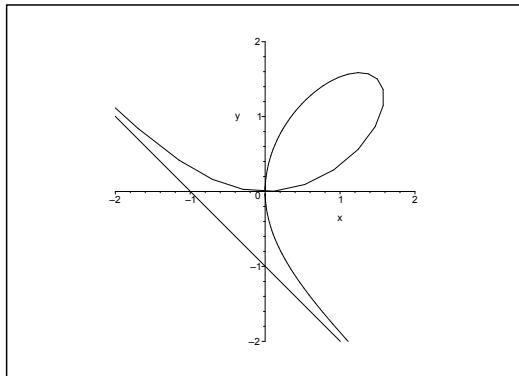
Click on the ad to read more

When $x \neq 0$ this is reduced to

$$x^3 + y^3 = 3xy.$$

Finally, we see that $(x, y) = (0, 0)$, which corresponds to $t = 0$, also satisfies this equation, so we can remove the restriction.

Note that the line $y = x$ is an axis of symmetry.

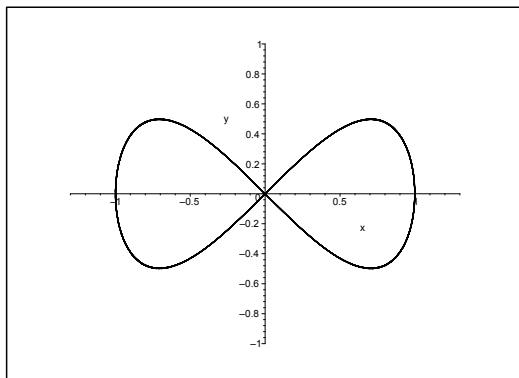


2) Here, $x = \cos t$ and $y = \sin t \cos t$, hence

$$y^2 = \sin^2 t \cos^2 t = (1 - \cos^2 t) \cos^2 t = (1 - x)x^2,$$

or written more conveniently,

$$y^2 = (1 - x^2)x^2, \quad \text{hence } y = \pm|x|\sqrt{1 - x^2}, \quad x \in [-1, 1].$$



3) From

$$x = a \cos^3 t, \quad y = a \sin^3 t,$$

we get by elimination

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \{ \cos^2 t + \sin^2 t \} = a^{\frac{2}{3}}.$$

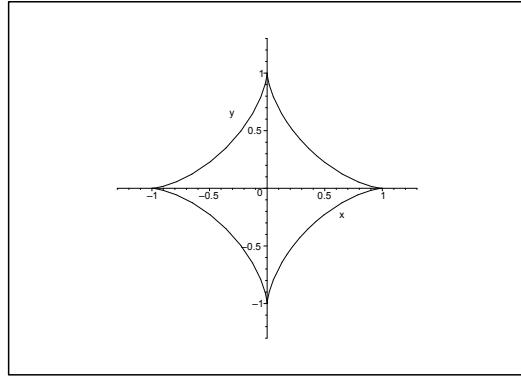


Figure 7.34: The curve for $a = 1$.

Note that

$$\mathbf{r}'(t) = 3a \sin t \cdot \cos(-\cos t, \sin t)$$

is $\mathbf{0}$ for $t = p \cdot \frac{-\pi}{2}$, $p = -2, -1, 0, 1, 2$, corresponding to the cusps on the curve.

- 4) First note that $\mathbf{r}'(t) = a(-6t, 3 - 3t^2)$, so $y'(t) = 0$ for $t = \pm 1$. It follows from

$$x(t) = a(1 - 3t^2) \quad \text{and} \quad y(t) = at(3 - t^2)$$

that $x(t)$ is largest for $t = 0$, corresponding to $x(t) \leq x(0) = a$. For this value the point on the curve is $\mathbf{r}(0) = (a, 0)$.

Furthermore, we see that the X axis is an axis of symmetry.

Note

- a) that $x(t) = 0$ for $t = \pm \frac{1}{\sqrt{3}}$, corresponding to

$$(x, y) = \left(0, \pm \frac{8a}{3\sqrt{3}}\right),$$

- b) that the curve has a horizontal tangent for $y'(t) = 0$, i.e. for $t = \pm 1$, corresponding to

$$(x, y) = (-2a, \pm 2a),$$

- c) and that $y(t) = 0$ for $t = 0$ and $t = \pm \sqrt{3}$, corresponding to

$$(0, 0) \quad \text{and} \quad (-8a, 0).$$

- d) that y and t have the same sign for $0 < |t| < \sqrt{3}$, and opposite sign for $|t| > \sqrt{3}$. The latter means that we are allowed to square by the elimination of t .

It follows from

$$\frac{x}{a} = 1 - 3t^2 \quad \text{and} \quad \frac{y}{a} = t(3 - t^2)$$

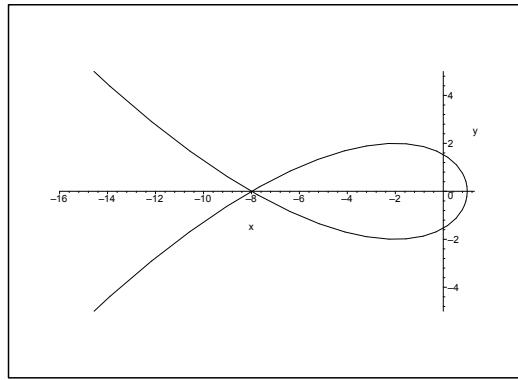


Figure 7.35: The curve for $a = 1$.

that

$$t^2 = \frac{1}{3} \left\{ 1 - \frac{x}{a} \right\},$$

so we finally get by a squaring,

$$\frac{y^2}{a^2} = t^2 (3 - t^2)^2 = \frac{1}{3} \left\{ 1 - \frac{x}{a} \right\} \left(3 - \frac{1}{3} \left\{ 1 - \frac{x}{a} \right\} \right)^2 = \frac{1}{27} \left(1 - \frac{x}{a} \right) \left(8 + \frac{x}{a} \right)^2,$$

thus

$$y^2 = \frac{1}{27a} (a - x)(8a + x)^2.$$

Note that $|y|$ tends faster towards $+\infty$ than $|x|$ for $|t| \rightarrow +\infty$.

Example 7.21 Sketch the point set A in the first quadrant of the plane, which is bounded by the three curves given by

$$\mathbf{r}(t) = (\cos t, 1 + \sin t), \quad t \in \left[0, \frac{\pi}{2} \right],$$

$$\mathbf{r}(t) = (1 + \cos t, \sin t), \quad t \in \left[0, \frac{\pi}{2} \right],$$

$$\mathbf{r}(t) = (2 \cos t, 2 \sin t), \quad t \in \left[0, \frac{\pi}{2} \right].$$

A A set bounded by given curves.

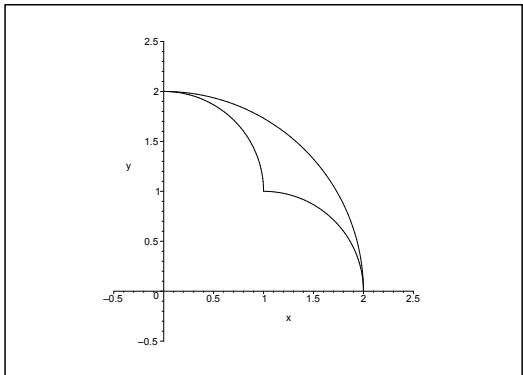
D Identify the curves and sketch the set.

I All three curves are quarter circles, which follows from

$$\mathbf{r}_1(t) = (0, 1) + (\cos t, \sin t), \quad t \in \left[0, \frac{\pi}{2} \right],$$

$$\mathbf{r}_2(t) = (1, 0) + (\cos t, \sin t), \quad t \in \left[0, \frac{\pi}{2} \right],$$

$$\mathbf{r}_3(t) = (0, 0) + 2(\cos t, \sin t), \quad t \in \left[0, \frac{\pi}{2} \right].$$



The advertisement features a woman teacher smiling and interacting with two young students (a boy and a girl) who are looking at a laptop screen. The background is yellow with orange swirling graphics. The e-Learning for Kids logo is in the top left. In the bottom right, there's a green callout box with text and a hand cursor icon.

About e-Learning for Kids Established in 2004, e-Learning for Kids is a global nonprofit foundation dedicated to fun and free learning on the Internet for children ages 5 - 12 with courses in math, science, language arts, computers, health and environmental skills. Since 2005, more than 15 million children in over 190 countries have benefitted from eLessons provided by EFK! An all-volunteer staff consists of education and e-learning experts and business professionals from around the world committed to making difference. eLearning for Kids is actively seeking funding, volunteers, sponsors and courseware developers; get involved! For more information, please visit www.e-learningforkids.org.

• The number 1 MOOC for Primary Education
• Free Digital Learning for Children 5-12
• 15 Million Children Reached



Click on the ad to read more

Example 7.22 Let α be a non-negative constant, and let the curve \mathcal{K} be given by the equation

$$\varrho = \frac{c}{1 + \alpha \cos \varphi}, \quad \varphi \in I,$$

where I is a symmetric interval around the point 0, which is as big as possible. Prove that \mathcal{K} is (a part of) a conical section.

A Conical section in polar coordinates.

D Multiply by the denominator and reduce to rectangular coordinates, where c as usual denotes some positive constant.

I Since $\varrho \geq 0$ and $c > 0$, we must have $1 + \alpha \cos \varphi > 0$. Therefore, in order to find I we must find possible zeros of the denominator, i.e. we shall examine the equation

$$1 + \alpha \cos \varphi = 0, \quad \text{i.e. } \cos \varphi = -\frac{1}{\alpha}.$$

Since $\alpha \geq 0$, we have to distinguish between the cases

$$\alpha = 0, \quad 0 < \alpha < 1, \quad \alpha = 1, \quad \alpha > 1.$$

- 1) If $\alpha = 0$, then $\varrho = c$, which is the polar equation of a circle of radius $c > 0$. The circle is clearly a conical section, and $I = \mathbb{R}$.
- 2) If $0 < \alpha < 1$, then $1 + \alpha \cos \varphi \geq 1 - \alpha > 0$ for every φ , and the denominator is always positive, and we get $I = \mathbb{R}$. When we multiply by the denominator we get by using rectangular coordinates,

$$c = \varrho + \alpha \varrho \cos \varphi = \alpha x,$$

hence by a rearrangement,

$$\sqrt{x^2 + y^2} = c - \alpha x \geq 0.$$

We get in particular the condition $x < \frac{c}{\alpha}$, which should be checked at the very end of this example.

When this restriction is satisfied we can square, obtaining

$$x^2 + y^2 = c^2 - 2\alpha cx + \alpha^2 x^2.$$

Then by a rearrangement,

$$(1 - \alpha^2)x^2 + 2\alpha cx + y^2 = c^2, \quad 0 < \alpha < 1,$$

i.e.

$$(1 - \alpha^2) \left\{ x^2 + \frac{2\alpha c}{1 - \alpha^2} x + \left(\frac{\alpha c}{1 - \alpha^2} \right)^2 \right\} + y^2 = c^2 + \frac{\alpha^2 c^2}{1 - \alpha^2} = \frac{c^2}{1 - \alpha^2}.$$

This can be written in the following canonical way

$$\left\{ \frac{x + \frac{\alpha c}{1 - \alpha^2}}{\frac{c}{1 - \alpha^2}} \right\}^2 + \left\{ \frac{y}{\frac{\alpha c}{\sqrt{1 - \alpha^2}}} \right\}^2 = 1.$$

This is the equation of an ellipse, hence a conical section of

$$\text{centre: } \left(-\frac{\alpha c}{1-\alpha^2}, 0 \right) \quad \text{and half axes: } \frac{c}{1-\alpha^2} \quad \text{and} \quad \frac{c}{\sqrt{1-\alpha^2}}.$$

Note that

$$-\frac{\alpha c}{1-\alpha^2} + \frac{c}{1-\alpha^2} = \frac{c}{1+\alpha} < \frac{c}{\alpha} \quad \text{for } 0 < \alpha < 1,$$

and we conclude that the earlier restriction for the squaring is automatically fulfilled.

- 3) If $\alpha = 1$, the denominator is $1 + \cos \varphi = 0$ for $\varphi = \text{an odd multiple of } \pi$, and > 0 otherwise.
 The searched for interval is $I =]-\pi, \pi[$.

When we multiply with the denominator we get

$$c = \varrho + \varrho \cos \varphi = \sqrt{x^2 + y^2} + x,$$

hence

$$\sqrt{x^2 + y^2} = c - x \geq 0, \quad \text{i.e. } x \leq c.$$

Assuming this we get by squaring,

$$x^2 + y^2 = c^2 - 2cx + x^2,$$

so after some reduction we obtain the equation of the parabola

$$x = -\frac{1}{2c}y^2 + \frac{c}{2}.$$

Clearly, this expression is $\leq c$, so \mathcal{K} is the whole of the parabola, and a parabola is also a conical section.

- 4) If $\alpha > 1$, then $1 + \alpha \cos \varphi = 0$ for

$$\cos \varphi = -\frac{1}{\alpha} \in]-1, 0[,$$

i.e. the largest possible symmetric domain interval I is

$$I = \left] -\arccos\left(-\frac{1}{\alpha}\right), \arccos\left(-\frac{1}{\alpha}\right) \right[.$$

In this interval we get as in 2) that

$$\sqrt{x^2 + y^2} = c - \alpha x \geq 0, \quad \text{i.e. } x \leq \frac{c}{\alpha},$$

and the calculations are then continued in the usual way under this assumption by a squaring,

$$x^2 + y^2 = c^2 + \alpha^2 x^2 - 2\alpha cx \quad \text{for } x \leq \frac{c}{\alpha}.$$

Then by a rearrangement,

$$(1 - \alpha^2) \left\{ x^2 + \frac{2\alpha c}{1 - \alpha^2} x + \left(\frac{\alpha c}{1 - \alpha^2} \right)^2 \right\} + y^2 = \frac{c^2}{1 - \alpha^2} < 0,$$

hence by norming

$$\left\{ \frac{x - \frac{\alpha c}{c^2 - 1}}{\frac{c}{c^2 - 1}} \right\}^2 - \left\{ \frac{y}{\frac{\sqrt{\alpha^2 - 1}}{c}} \right\}^2 = 1.$$

Thus, for $\varphi \in I$ we get an arc of an hyperbola, which again is a conical section.

7.5 Connected sets

Example 7.23 Examine if the point set

$$A = \{(x, y) \mid (x^2 + y^2 + 2x)(y^2 - x) < 0\}$$

is connected.

A Connected set.

D First find the boundary curves of A . Sketch a figure.

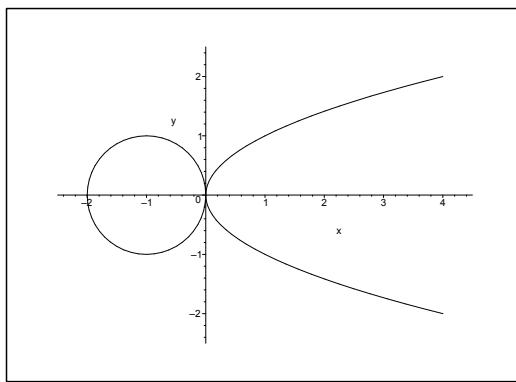


Figure 7.36: The set A consists of the points which either lies inside the circle or inside the parabola.

I Since $(x^2 + y^2 + 2x)(y^2 - x)$ is continuous in \mathbb{R}^2 , the boundary ∂A is given by

$$0 = (x^2 + y^2 + 2x)(y^2 - x) = \{(x+1)^2 + y^2 - 1\}(y^2 - x),$$

i.e. the boundary is composed of the circle of equation

$$(x+1)^2 + y^2 = 1$$

of centre $(-1, 0)$ and radius 1, and the parabola of equation $x = y^2$. The plane \mathbb{R}^2 is in this way divided into three subregions in which $f(x, y)$ due to the continuity must have a fixed sign in each of these.

The set A is characterized by the condition $f(x, y) < 0$.

Inserting the centre $(-1, 0)$ of the circle we get

$$f(-1, 0) = -1 \cdot 1 = -1 < 0,$$

so by the continuity it follows that the open disc is contained in A .

The point $(1, 0)$ lies inside the parabola, and the value is

$$f(1, 0) = 3 \cdot (-1) = -3 < 0,$$

so the interior of the parabola is also a subset of A .

This is sufficient to declare that the set is not connected, because it is impossible to connect $(-1, 0) \in A$ with $(1, 0) \in A$ by any continuous curve without intersecting at least one of the zero curves, which do *not* lie in A . We therefore conclude that A is not connected.

REMARK. Since $(0, 1)$ is a point in the latter component, and

$$f(0, 1) = 1 \cdot 1 = 1 > 0,$$

the third component of \mathbb{R}^2 does not contain any point from A , and A consists of precisely the union of the open disc and the open interior of the parabola. However, one was never asked this question. ◇

Cynthia | AXA Graduate

AXA Global Graduate Program

Find out more and apply

redefining / standards AXA

Click on the ad to read more

Example 7.24 Give an example of a point set which fulfils the following condition: A is not connected, but its closure \overline{A} is connected.

- A** Connected sets.
- D** Analyze the concept of connected sets and give examples.
- I** According to **Example 7.27** below an extreme example is $A = \mathbb{Q}$, which is not connected in \mathbb{R} , while $\overline{A} = \mathbb{R}$ is connected.

A simpler example is $A = \mathbb{R} \setminus \{0\}$ where $\overline{A} = \mathbb{R}$.

Another example is given by **Example 7.23**, because one by the closure also include the point $(0, 0)$, which can be reached by a continuous curve from both components.

Example 7.25 Show by an example that two connected point sets A and B do not necessarily have a connected intersection.

- A** Connected sets.
- D** Sketch an “amoeba” in the plane.
- I** Sketch two “half moons” which only intersect in their tips, we see that the intersection has got two components, and the intersection is not connected. Clearly, each “half moon” is connected.

The sketches are left to the reader.

Example 7.26 Examine if the domain of the function

$$f(x, y) = \text{Arcsin}(x^2 + y^2 - 3)$$

is simply connected.

- A** Simply connected sets.
- D** Find the domain and analyze.
- I** The function $f(x, y)$ is defined for

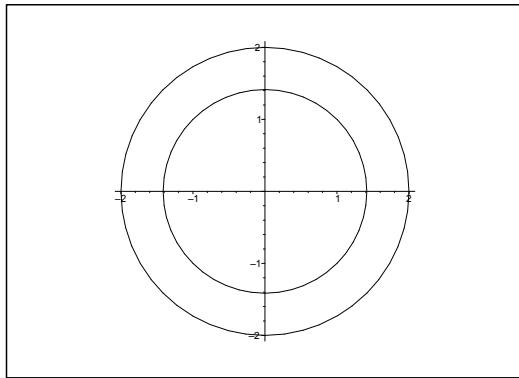
$$-1 \leq x^2 + y^2 - 3 \leq 1,$$

i.e. for

$$2 \leq x^2 + y^2 \leq 4.$$

This set is an annulus (a set containing a “hole”) of inner radius $\sqrt{2}$ and outer radius 2.

This set is clearly not simply connected.



Example 7.27 Prove that the set of rational numbers is not connected. Formulate a similar result for a set in the plane.

A Connected sets.

D Analyze the definition of connected sets.

I Let $x, y \in \mathbb{Q}$, where e.g. $x < y$. Every continuous curve in \mathbb{R} , which connects x and y , will then contain the interval $[x, y]$, which also contains irrational numbers, i.e. points outside \mathbb{Q} . We conclude that \mathbb{Q} is not connected.

The set $\{(x, y) \mid x \in \mathbb{Q}, y \in \mathbb{Q}\}$ is not connected in the plane.

Example 7.28 Check in each of the cases below if the domain of the given function is connected.

- 1) $f(x, y, z) = \ln|1 - x^2 - y^2 - z^2|$.
- 2) $f(x, y, z) = \ln(1 - x^2 - y^2 - z^2)$.
- 3) $f(x, y, z) = \sqrt{y^2 - x^2} + \sqrt{z^2 - 1}$.
- 4) $f(x, y, z) = \sqrt{y - x} + \sqrt{z - 1}$.
- 5) $f(x, y, z) = \ln(1 - y^2) + \sqrt{x^2 - 4} + \sqrt{9 - x^2}$.

A Connected domains.

D First find the domain. Then analyze.

- 1) The function is defined for $x^2 + y^2 + z^2 \neq 1$, i.e. everywhere with the exception of the unit sphere. The set can obviously be divided into two connected components, so it is not connected.
- 2) In this case the domain is the open unit ball, which is connected.
- 3) It suffices to realize that the domain has one part lying in the half space $z \geq 1$ and another part in the half space $z \leq -1$ and no point in between. Hence the set is not connected.
- 4) The domain is given by $y \geq x$ and $z \geq 1$, i.e. the union of two *half spaces* (convex sets) and thus connected.

- 5) The function is independent of z , and defined for

$$1 - y^2 > 0, \quad x^2 - 4 \geq 0, \quad 9 - x^2 \geq 0,$$

so the domain is

$$[-3, -2] \times] -1, 1[\times \mathbb{R} \cup [2, 3] \times] -1, 1[\times \mathbb{R}.$$

This set contains two connected components, hence it is not itself connected.

7.6 Description of surfaces

Example 7.29 In the following there are given some surfaces in the form $\mathbf{x} = \mathbf{r}(u, v)$, $(u, v) \in \mathbb{R}^2$. Find in each of these cases an equation of the surface by eliminating the parameters (u, v) , and then describe the type of the surface.

- 1) $\mathbf{r}(u, v) = (u, u + 2v, v - u)$.
- 2) $\mathbf{r}(u, v) = (u, \sin v, 3 \cos v)$.
- 3) $\mathbf{r}(u, v) = (u \cos v, u \sin v, u^2 \sin 2v)$.
- 4) $\mathbf{r}(u, v) = (a(\cos v - u \sin v), b(\sin v + u \cos v), cu)$.
- 5) $\mathbf{r}(u, v) = (u \cos v, 2u \sin v, u^2)$.
- 6) $\mathbf{r}(u, v) = (u + v, u - v, 4v^2)$.
- 7) $\mathbf{r}(u, v) = (u + v, u^2 + v^2, u^3 + v^3)$.

A Description of surfaces.

D Eliminate (u, v) to obtain some known relationship between x, y, z .

- I** 1) Here

$$x = u, \quad y = u + 2v, \quad z = v - u,$$

hence

$$y - 2z = u + 2v - 2v + 2u = 3u = 3x,$$

or

$$3x - y + 2z = 0.$$

This is the equation of a plane through $(0, 0, 0)$ with the normal vector $(3, -1, 2)$.

- 2) Here

$$x = u, \quad y = \sin v, \quad z = 3 \cos v,$$

i.e.

$$y^2 + \left(\frac{z}{3}\right)^2 = 1, \quad x = u, \quad u \in \mathbb{R}.$$

This is a cylindric surface with the X axis as its axis and the ellipse of centrum $(0, 0)$ and half axes 1 and 3 in the YZ plane as the generating curve.

3) It follows from

$$x = u \cos v, \quad y = u \sin v, \quad z = u^2 \sin 2v$$

that

$$2xy = 2u^2 \cos v \cdot \sin v = u^2 \sin 2v = z,$$

i.e.

$$z = 2xy,$$

which describes an hyperbolic paraboloid.

4) Here

$$\frac{x}{a} = \cos v - u \sin v, \quad \frac{y}{b} = \sin v + u \cos v, \quad \frac{z}{c} = u,$$

hence

$$\left(\frac{x}{a}\right)^2 = \cos^2 v - 2u \sin v \cdot \cos v + u^2 \sin^2 v,$$

$$\left(\frac{y}{b}\right)^2 = \sin^2 v + 2u \sin v \cdot \cos v + u^2 \cos^2 v,$$

I joined MITAS because
I wanted **real responsibility**

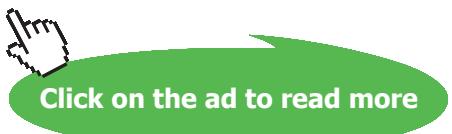


The Graduate Programme
for Engineers and Geoscientists
www.discovermitas.com

Real work
International opportunities
Three work placements



Month 16
I was a construction supervisor in the North Sea advising and helping foremen solve problems



and accordingly

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 + u^2 = 1 + \left(\frac{z}{c}\right)^2.$$

This is the equation of an hyperboloid with one sheet.

5) It follows from

$$x = u \cos v, \quad \frac{y}{2} = u \sin v, \quad z = u^2$$

that

$$x^2 + \left(\frac{y}{2}\right)^2 = u^2 = z,$$

which is the equation of an elliptic paraboloid.

6) It follows from

$$x = u + v, \quad y = u - v, \quad z = 4v^2$$

that $2v = x - y$, i.e.

$$z = 4v^2 = (x - y)^2.$$

This is the equation of a cylindric surface with the line $y = x$ as its axis and a parabola as its generating curve.

7) It follows from

$$x = u + v, \quad y = u^2 + v^2, \quad z = u^3 + v^3$$

that

$$2z = 2(u^3 + v^3) = (u + v)(2u^2 - 2uv + 2v^2) = x(2y - 2uv),$$

where

$$2uv = (u + v)^2 - (u^2 + v^2) = x^2 - y.$$

Then by insertion,

$$2z = x(3y - x^2).$$

This equation contains terms of first, second and third order.

Example 7.30 Sketch the following cylindric surfaces.

1) $x = \cos \varphi, y = \sin \varphi, \varphi \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right], z \in [1, 2\varphi].$

2) $xy = 1, y \in \left[\frac{1}{2}, 2\right], z \in [0, x].$

3) $y = e^{-x}, z \in [y, 1].$

4) $x = y^2, z \in [x, y].$

A Cylindric surfaces.

D First sketch the projection onto the XY plane.

I 1) Here we get a circular arc in the XY - plane.

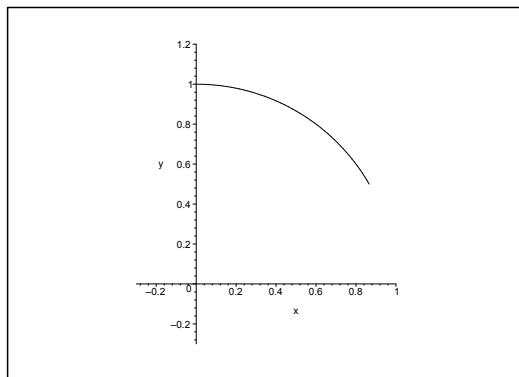


Figure 7.37: The projection onto the XY plane.

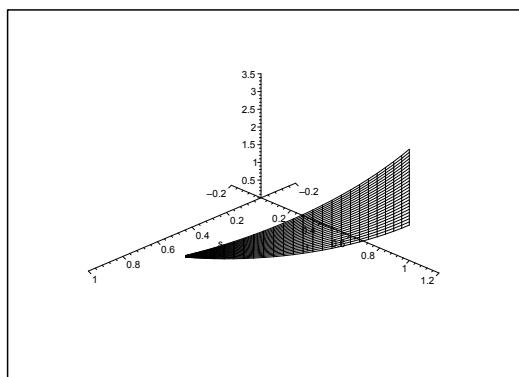


Figure 7.38: The cylindric surface of 1).

- 2) The projection onto the XY plane is an arc of an hyperbola, lying in the first quadrant. Note that $x \in [\frac{1}{2}, 2]$.

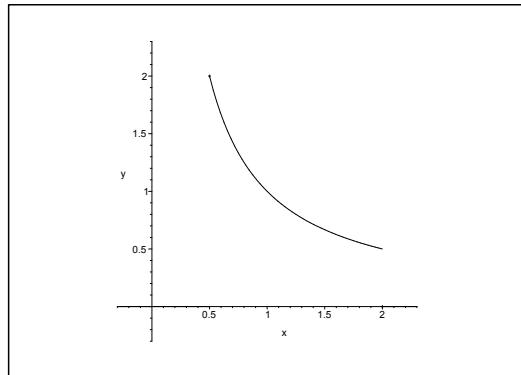


Figure 7.39: The projection onto the XY plane.

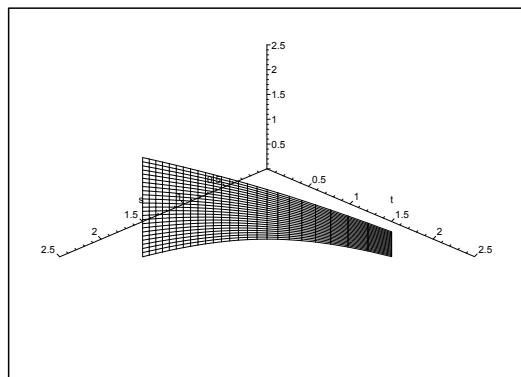


Figure 7.40: The cylindric surface of 2).

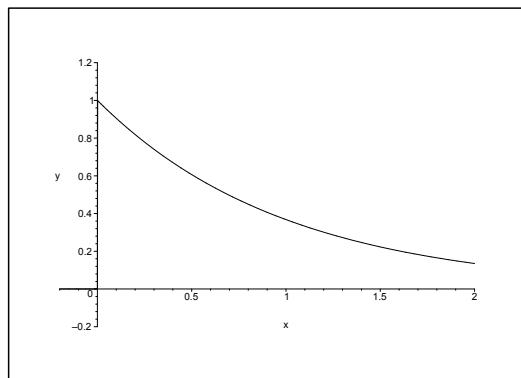


Figure 7.41: The projection onto the XY plane.

- 3) Since $y \leq 1$, we must have $x \geq 0$.
- 4) From $x = y^2 \leq z \leq y$ we get the condition $0 \leq y \leq 1$. On the figure the surface looks wrong. There may here be an error in the MAPLE programme, though I am not sure.

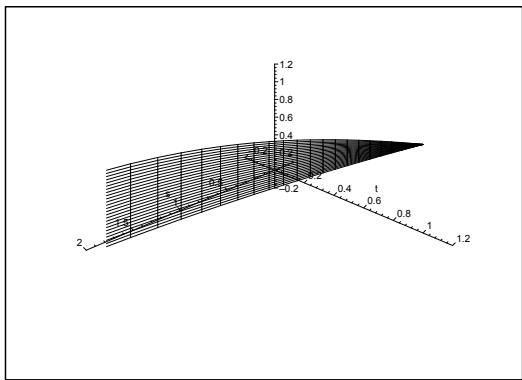


Figure 7.42: The cylindric surface of 3).

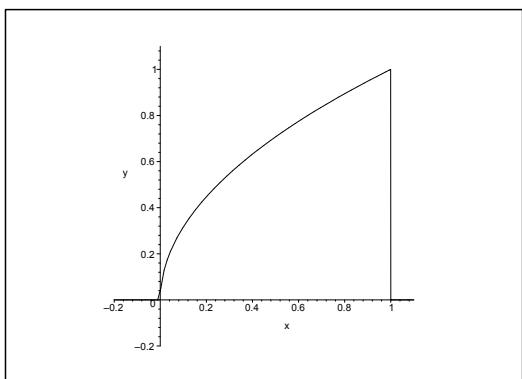


Figure 7.43: The projection onto the XY plane.

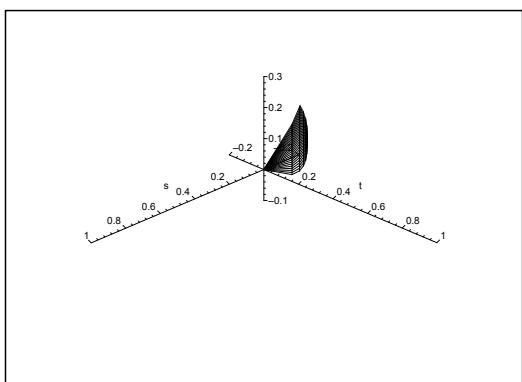


Figure 7.44: The cylindric surface of 4).

Example 7.31 In the following there are given some equations of meridian curves. Set up in each case an equation of the corresponding surface of revolution \mathcal{O} and find the name of \mathcal{O} .

- 1) $z = \varrho$.
- 2) $\varrho = |z|$.
- 3) $\varrho = a$.
- 4) $z^2 + 2\varrho^2 = 2az$.
- 5) $z^2 - \varrho^2 = a^2$.
- 6) $\varrho^2 - z^2 = a^2$.

A Surfaces of revolution with a given meridian curve.

D First sketch the meridian curve in the PZ half plane.

I 1) This is a cone of vertex $(0, 0, 0)$.

The advertisement features a background image of modern skyscrapers. On the left, the IE business school logo is displayed. The central text highlights that 93% of MIM students are working in their sector 3 months following graduation. Below this, the program name "MASTER IN MANAGEMENT" is prominently displayed. To the left of the main text, there is a list of benefits: "STUDY IN THE CENTER OF MADRID AND TAKE ADVANTAGE OF THE UNIQUE OPPORTUNITIES THAT THE CAPITAL OF SPAIN OFFERS", "PROPEL YOUR EDUCATION BY EARNING A DOUBLE DEGREE THAT BEST SUITS YOUR PROFESSIONAL GOALS", and "STUDY A SEMESTER ABROAD AND BECOME A GLOBAL CITIZEN WITH THE BEYOND BORDERS EXPERIENCE". To the right, detailed information about the program is provided: Length: 10 MONTHS, Av. Experience: 1 YEAR, Language: ENGLISH / SPANISH, Format: FULL-TIME, Intakes: SEPT / FEB. At the bottom, three boxes highlight "5 SPECIALIZATIONS PERSONALIZE YOUR PROGRAM", "#10 WORLDWIDE MASTER IN MANAGEMENT FINANCIAL TIMES", and "55 NATIONALITIES IN CLASS". Contact information and social media links are also included at the very bottom.

www.ie.edu/master-management | mim.admissions@ie.edu | [f](#) [t](#) [g](#) Follow us on IE MIM Experience



Click on the ad to read more

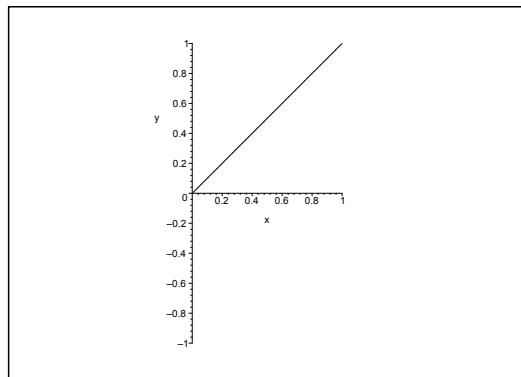


Figure 7.45: The meridian curve of 1).

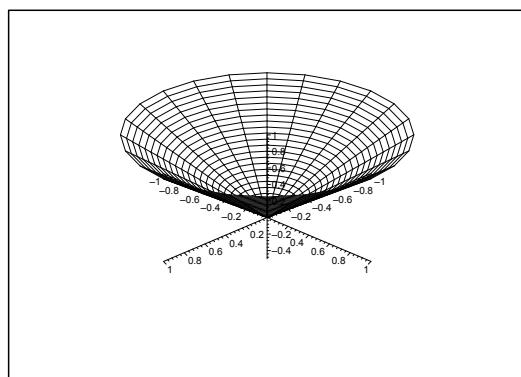


Figure 7.46: The surface of 1).

2) This is a double cone of vertex $(0, 0, 0)$.

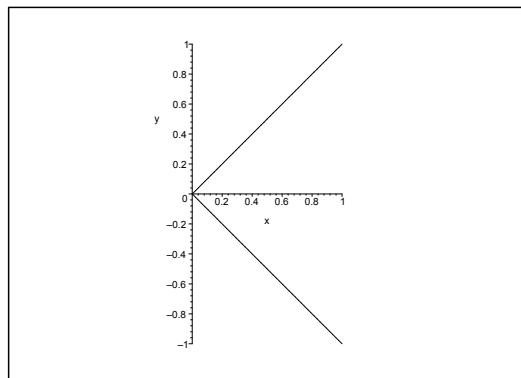


Figure 7.47: The meridian curve of 2).

3) This is clearly a cylinder.

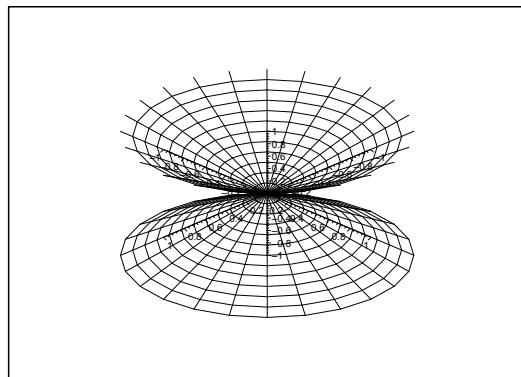


Figure 7.48: The surface of 2).

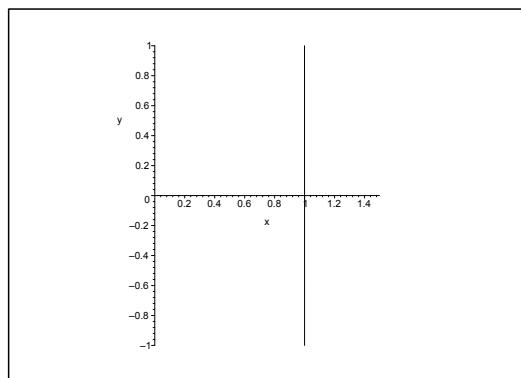


Figure 7.49: The meridian curve of 3).

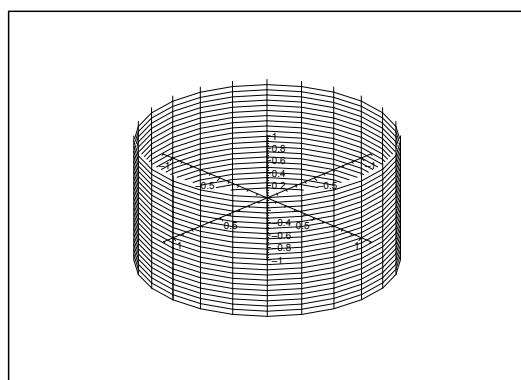


Figure 7.50: The surface of 3).

- 4) It follows by a small rearrangement that the equation is equivalent to

$$(z - a)^2 + 2\rho^2 = a^2,$$

i.e. in the canonical form

$$\left(\frac{\varrho}{a}\right)^2 + \left(\frac{z-a}{a}\right)^2 = 1, \quad \varrho \geq 0.$$

The meridian curve is an half ellipse in the PZ half plane of centre $(0, a)$ and half axes $\frac{a}{\sqrt{2}}$ and a .

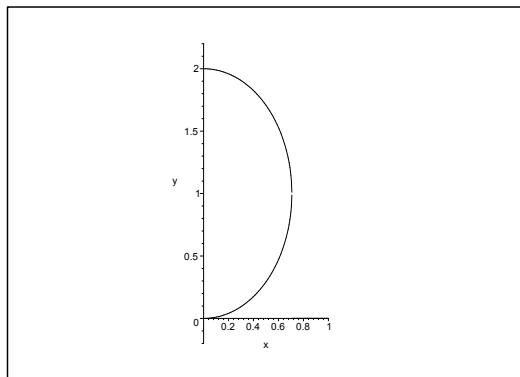


Figure 7.51: The meridian curve of 4).

The surface of revolution is the surface of an ellipsoid of centre $(0, 0, a)$ and half axes $\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}$ and a . Notice that one of the top points lies at $(0, 0, 0)$. Also note that the scales are different on the axes on the figure.

**"I studied English for 16 years but...
...I finally learned to speak it in just six lessons"**

Jane, Chinese architect

ENGLISH OUT THERE

Click to hear me talking before and after my unique course download

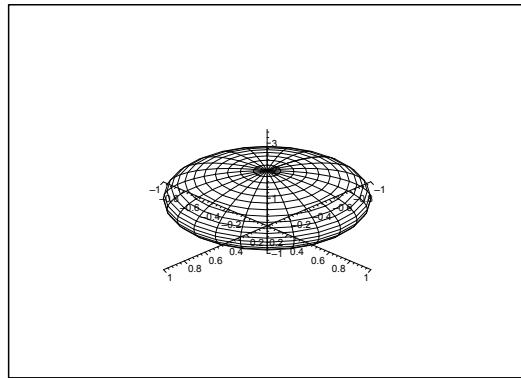


Figure 7.52: The surface of 4).

- 5) In this case the meridian curves consist of two halves of branches of an hyperbola.

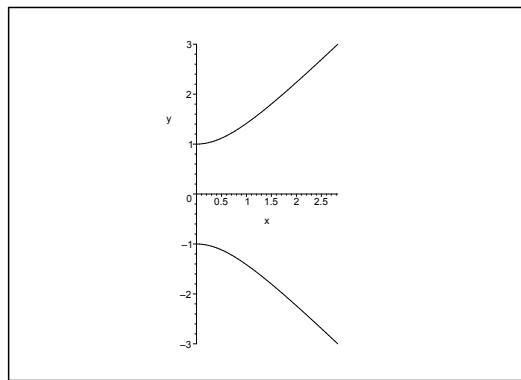


Figure 7.53: The meridian curves of 5).

By the revolution we get an hyperboloid with two sheets. Only the upper sheet is sketched on the figure (and we use different scales on the axes). There is a similar surface in the lower half space.

- 6) The curve $\rho^2 - z^2 = a^2$, $\rho \geq 0$, is a branch of an hyperbola with its top point at $(a, 0)$ and its half axes a and a . The surface of revolution is an hyperboloid with one sheet and of centre $(0, 0, 0)$ and with the Z axis as its axes of revolution and with the half axes a , a , a .

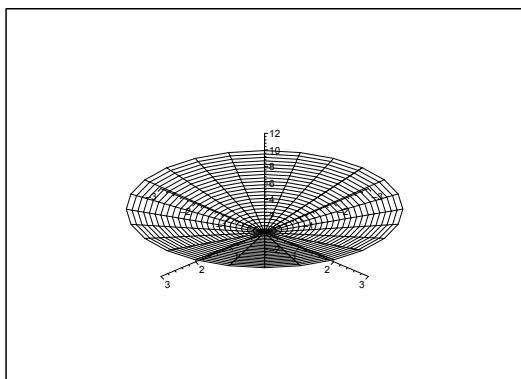


Figure 7.54: The upper surface of 5).

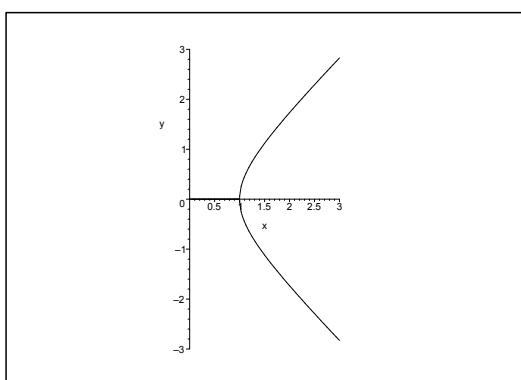


Figure 7.55: The meridian curve of 6).

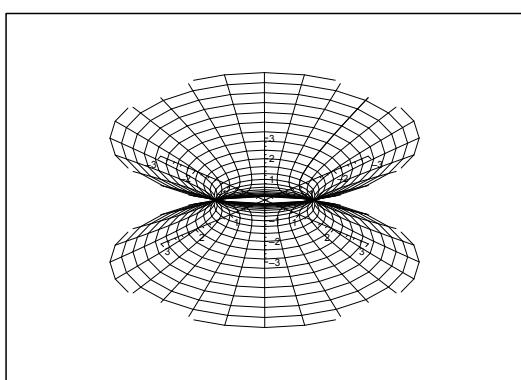


Figure 7.56: The surface of 6).

8 Formulæ

Some of the following formulæ can be assumed to be known from high school. It is highly recommended that one *learns most of these formulæ in this appendix by heart*.

8.1 Squares etc.

The following simple formulæ occur very frequently in the most different situations.

$$\begin{aligned} (a+b)^2 &= a^2 + b^2 + 2ab, & a^2 + b^2 + 2ab &= (a+b)^2, \\ (a-b)^2 &= a^2 + b^2 - 2ab, & a^2 + b^2 - 2ab &= (a-b)^2, \\ (a+b)(a-b) &= a^2 - b^2, & a^2 - b^2 &= (a+b)(a-b), \\ (a+b)^2 &= (a-b)^2 + 4ab, & (a-b)^2 &= (a+b)^2 - 4ab. \end{aligned}$$

8.2 Powers etc.

Logarithm:

$$\begin{aligned} \ln |xy| &= \ln |x| + \ln |y|, & x, y \neq 0, \\ \ln \left| \frac{x}{y} \right| &= \ln |x| - \ln |y|, & x, y \neq 0, \\ \ln |x^r| &= r \ln |x|, & x \neq 0. \end{aligned}$$

Power function, fixed exponent:

$$(xy)^r = x^r \cdot y^r, x, y > 0 \quad (\text{extensions for some } r),$$

$$\left(\frac{x}{y} \right)^r = \frac{x^r}{y^r}, x, y > 0 \quad (\text{extensions for some } r).$$

Exponential, fixed base:

$$\begin{aligned} a^x \cdot a^y &= a^{x+y}, & a > 0 & \quad (\text{extensions for some } x, y), \\ (a^x)^y &= a^{xy}, & a > 0 & \quad (\text{extensions for some } x, y), \end{aligned}$$

$$a^{-x} = \frac{1}{a^x}, a > 0, \quad (\text{extensions for some } x),$$

$$\sqrt[n]{a} = a^{1/n}, a \geq 0, \quad n \in \mathbb{N}.$$

Square root:

$$\sqrt{x^2} = |x|, \quad x \in \mathbb{R}.$$

Remark 8.1 It happens quite frequently that students make errors when they try to apply these rules. They must be mastered! In particular, as one of my friends once put it: “If you can master the square root, you can master everything in mathematics!” Notice that this innocent looking square root is one of the most difficult operations in Calculus. Do not forget the *absolute value!* ◇

8.3 Differentiation

Here are given the well-known rules of differentiation together with some rearrangements which sometimes may be easier to use:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x),$$

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) = f(x)g(x) \left\{ \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} \right\},$$

where the latter rearrangement presupposes that $f(x) \neq 0$ and $g(x) \neq 0$.

If $g(x) \neq 0$, we get the usual formula known from high school

$$\left\{ \frac{f(x)}{g(x)} \right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

It is often more convenient to compute this expression in the following way:

$$\left\{ \frac{f(x)}{g(x)} \right\} = \frac{d}{dx} \left\{ f(x) \cdot \frac{1}{g(x)} \right\} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f(x)}{g(x)} \left\{ \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)} \right\},$$

where the former expression often is *much easier* to use in practice than the usual formula from high school, and where the latter expression again presupposes that $f(x) \neq 0$ and $g(x) \neq 0$. Under these assumptions we see that the formulæ above can be written

$$\frac{\{f(x)g(x)\}'}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

$$\frac{\{f(x)/g(x)\}'}{f(x)/g(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Since

$$\frac{d}{dx} \ln |f(x)| = \frac{f'(x)}{f(x)}, \quad f(x) \neq 0,$$

we also name these the *logarithmic derivatives*.

Finally, we mention the rule of **differentiation of a composite function**

$$\{f(\varphi(x))\}' = f'(\varphi(x)) \cdot \varphi'(x).$$

We first differentiate the function itself; then the insides. This rule is a 1-dimensional version of the so-called *Chain rule*.

8.4 Special derivatives.

Power like:

$$\frac{d}{dx} (x^\alpha) = \alpha \cdot x^{\alpha-1}, \quad \text{for } x > 0, \text{ (extensions for some } \alpha\text{).}$$

$$\frac{d}{dx} \ln |x| = \frac{1}{x}, \quad \text{for } x \neq 0.$$

Exponential like:

$$\begin{aligned}\frac{d}{dx} \exp x &= \exp x, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} (a^x) &= \ln a \cdot a^x, && \text{for } x \in \mathbb{R} \text{ and } a > 0.\end{aligned}$$

Trigonometric:

$$\begin{aligned}\frac{d}{dx} \sin x &= \cos x, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \cos x &= -\sin x, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \tan x &= 1 + \tan^2 x = \frac{1}{\cos^2 x}, && \text{for } x \neq \frac{\pi}{2} + p\pi, p \in \mathbb{Z}, \\ \frac{d}{dx} \cot x &= -(1 + \cot^2 x) = -\frac{1}{\sin^2 x}, && \text{for } x \neq p\pi, p \in \mathbb{Z}.\end{aligned}$$

Hyperbolic:

$$\begin{aligned}\frac{d}{dx} \sinh x &= \cosh x, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \cosh x &= \sinh x, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \tanh x &= 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \coth x &= 1 - \coth^2 x = -\frac{1}{\sinh^2 x}, && \text{for } x \neq 0.\end{aligned}$$

Inverse trigonometric:

$$\begin{aligned}\frac{d}{dx} \arcsin x &= \frac{1}{\sqrt{1-x^2}}, && \text{for } x \in]-1, 1[, \\ \frac{d}{dx} \arccos x &= -\frac{1}{\sqrt{1-x^2}}, && \text{for } x \in]-1, 1[, \\ \frac{d}{dx} \arctan x &= \frac{1}{1+x^2}, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \text{arccot } x &= \frac{1}{1+x^2}, && \text{for } x \in \mathbb{R}.\end{aligned}$$

Inverse hyperbolic:

$$\begin{aligned}\frac{d}{dx} \text{arsinh } x &= \frac{1}{\sqrt{x^2+1}}, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \text{arcosh } x &= \frac{1}{\sqrt{x^2-1}}, && \text{for } x \in]1, +\infty[, \\ \frac{d}{dx} \text{artanh } x &= \frac{1}{1-x^2}, && \text{for } |x| < 1, \\ \frac{d}{dx} \text{arcoth } x &= \frac{1}{1-x^2}, && \text{for } |x| > 1.\end{aligned}$$

Remark 8.2 The derivative of the trigonometric and the hyperbolic functions are to some extent exponential like. The derivatives of the inverse trigonometric and inverse hyperbolic functions are power like, because we include the logarithm in this class. ◇

8.5 Integration

The most obvious rules are dealing with linearity

$$\int \{f(x) + \lambda g(x)\} dx = \int f(x) dx + \lambda \int g(x) dx, \quad \text{where } \lambda \in \mathbb{R} \text{ is a constant,}$$

and with the fact that differentiation and integration are “inverses to each other”, i.e. modulo some arbitrary constant $c \in \mathbb{R}$, which often tacitly is missing,

$$\int f'(x) dx = f(x).$$

If we in the latter formula replace $f(x)$ by the product $f(x)g(x)$, we get by reading from the right to the left and then differentiating the product,

$$f(x)g(x) = \int \{f(x)g(x)\}' dx = \int f'(x)g(x) dx + \int f(x)g'(x) dx.$$

Hence, by a rearrangement

The rule of partial integration:

$$\int f'(x)g(x) dx = f(x)g(x) - \int f(x)g'(x) dx.$$

The differentiation is moved from one factor of the integrand to the other one by changing the sign and adding the term $f(x)g(x)$.

Remark 8.3 This technique was earlier used a lot, but is almost forgotten these days. It must be revived, because MAPLE and pocket calculators apparently do not know it. It is possible to construct examples where these devices cannot give the exact solution, unless you first perform a partial integration yourself. ◇

Remark 8.4 This method can also be used when we estimate integrals which cannot be directly calculated, because the antiderivative is not contained in e.g. the catalogue of MAPLE. The idea is by a succession of partial integrations to make the new integrand smaller. ◇

Integration by substitution:

If the integrand has the special structure $f(\varphi(x)) \cdot \varphi'(x)$, then one can change the variable to $y = \varphi(x)$:

$$\int f(\varphi(x)) \cdot \varphi'(x) dx = " \int f(\varphi(x)) d\varphi(x) " = \int_{y=\varphi(x)} f(y) dy.$$

Integration by a monotonous substitution:

If $\varphi(y)$ is a *monotonous* function, which maps the y -interval *one-to-one* onto the x -interval, then

$$\int f(x) dx = \int_{y=\varphi^{-1}(x)} f(\varphi(y))\varphi'(y) dy.$$

Remark 8.5 This rule is usually used when we have some “ugly” term in the integrand $f(x)$. The idea is to put this ugly term equal to $y = \varphi^{-1}(x)$. When e.g. x occurs in $f(x)$ in the form \sqrt{x} , we put $y = \varphi^{-1}(x) = \sqrt{x}$, hence $x = \varphi(y) = y^2$ and $\varphi'(y) = 2y$. \diamond

8.6 Special antiderivatives

Power like:

$$\int \frac{1}{x} dx = \ln|x|, \quad \text{for } x \neq 0. \quad (\text{Do not forget the numerical value!})$$

$$\int x^\alpha dx = \frac{1}{\alpha+1} x^{\alpha+1}, \quad \text{for } \alpha \neq -1,$$

$$\int \frac{1}{1+x^2} dx = \arctan x, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{1-x^2} dx = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|, \quad \text{for } x \neq \pm 1,$$

$$\int \frac{1}{1-x^2} dx = \operatorname{Artanh} x, \quad \text{for } |x| < 1,$$

$$\int \frac{1}{1-x^2} dx = \operatorname{Arcoth} x, \quad \text{for } |x| > 1,$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x, \quad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = -\arccos x, \quad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{x^2+1}} dx = \operatorname{Arsinh} x, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2+1}} dx = \ln(x + \sqrt{x^2+1}), \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{x}{\sqrt{x^2-1}} dx = \sqrt{x^2-1}, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2-1}} dx = \operatorname{Arcosh} x, \quad \text{for } x > 1,$$

$$\int \frac{1}{\sqrt{x^2-1}} dx = \ln|x + \sqrt{x^2-1}|, \quad \text{for } x > 1 \text{ eller } x < -1.$$

There is an error in the programs of the pocket calculators TI-92 and TI-89. The *numerical signs* are missing. It is obvious that $\sqrt{x^2-1} < |x|$ so if $x < -1$, then $x + \sqrt{x^2-1} < 0$. Since you cannot take the logarithm of a negative number, these pocket calculators will give an error message.

Exponential like:

$$\int \exp x \, dx = \exp x, \quad \text{for } x \in \mathbb{R},$$

$$\int a^x \, dx = \frac{1}{\ln a} \cdot a^x, \quad \text{for } x \in \mathbb{R}, \text{ and } a > 0, a \neq 1.$$

Trigonometric:

$$\int \sin x \, dx = -\cos x, \quad \text{for } x \in \mathbb{R},$$

$$\int \cos x \, dx = \sin x, \quad \text{for } x \in \mathbb{R},$$

$$\int \tan x \, dx = -\ln |\cos x|, \quad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \cot x \, dx = \ln |\sin x|, \quad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos x} \, dx = \frac{1}{2} \ln \left(\frac{1 + \sin x}{1 - \sin x} \right), \quad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin x} \, dx = \frac{1}{2} \ln \left(\frac{1 - \cos x}{1 + \cos x} \right), \quad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos^2 x} \, dx = \tan x, \quad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin^2 x} \, dx = -\cot x, \quad \text{for } x \neq p\pi, \quad p \in \mathbb{Z}.$$

Hyperbolic:

$$\int \sinh x \, dx = \cosh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \cosh x \, dx = \sinh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \tanh x \, dx = \ln \cosh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \coth x \, dx = \ln |\sinh x|, \quad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh x} \, dx = \operatorname{Arctan}(\sinh x), \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\cosh x} \, dx = 2 \operatorname{Arctan}(e^x), \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh x} \, dx = \frac{1}{2} \ln \left(\frac{\cosh x - 1}{\cosh x + 1} \right), \quad \text{for } x \neq 0,$$

$$\int \frac{1}{\sinh x} dx = \ln \left| \frac{e^x - 1}{e^x + 1} \right|, \quad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh^2 x} dx = \tanh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh^2 x} dx = -\coth x, \quad \text{for } x \neq 0.$$

8.7 Trigonometric formulæ

The trigonometric formulæ are closely connected with circular movements. Thus $(\cos u, \sin u)$ are the coordinates of a point P on the unit circle corresponding to the angle u , cf. figure A.1. This geometrical interpretation is used from time to time.

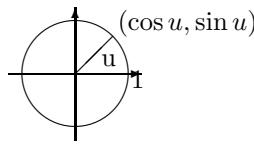


Figure 8.1: The unit circle and the trigonometric functions.

The fundamental trigonometric relation:

$$\cos^2 u + \sin^2 u = 1, \quad \text{for } u \in \mathbb{R}.$$

Using the previous geometric interpretation this means according to *Pythagoras's theorem*, that the point P with the coordinates $(\cos u, \sin u)$ always has distance 1 from the origo $(0, 0)$, i.e. it is lying on the boundary of the circle of centre $(0, 0)$ and radius $\sqrt{1} = 1$.

Connection to the complex exponential function:

The *complex exponential* is for imaginary arguments defined by

$$\exp(iu) := \cos u + i \sin u.$$

It can be checked that the usual functional equation for \exp is still valid for complex arguments. In other word: The definition above is extremely conveniently chosen.

By using the definition for $\exp(iu)$ and $\exp(-iu)$ it is easily seen that

$$\cos u = \frac{1}{2}(\exp(iu) + \exp(-iu)),$$

$$\sin u = \frac{1}{2i}(\exp(iu) - \exp(-iu)),$$

Moivre's formula: We get by expressing $\exp(inu)$ in two different ways:

$$\exp(inu) = \cos nu + i \sin nu = (\cos u + i \sin u)^n.$$

Example 8.1 If we e.g. put $n = 3$ into Moivre's formula, we obtain the following typical application,

$$\begin{aligned} \cos(3u) + i \sin(3u) &= (\cos u + i \sin u)^3 \\ &= \cos^3 u + 3i \cos^2 u \cdot \sin u + 3i^2 \cos u \cdot \sin^2 u + i^3 \sin^3 u \\ &= \{\cos^3 u - 3 \cos u \cdot \sin^2 u\} + i\{3 \cos^2 u \cdot \sin u - \sin^3 u\} \\ &= \{4 \cos^3 u - 3 \cos u\} + i\{3 \sin u - 4 \sin^3 u\} \end{aligned}$$

When this is split into the real- and imaginary parts we obtain

$$\cos 3u = 4 \cos^3 u - 3 \cos u, \quad \sin 3u = 3 \sin u - 4 \sin^3 u. \quad \diamond$$

Addition formulæ:

$$\begin{aligned} \sin(u+v) &= \sin u \cos v + \cos u \sin v, \\ \sin(u-v) &= \sin u \cos v - \cos u \sin v, \\ \cos(u+v) &= \cos u \cos v - \sin u \sin v, \\ \cos(u-v) &= \cos u \cos v + \sin u \sin v. \end{aligned}$$

Products of trigonometric functions to a sum:

$$\begin{aligned} \sin u \cos v &= \frac{1}{2} \sin(u+v) + \frac{1}{2} \sin(u-v), \\ \cos u \sin v &= \frac{1}{2} \sin(u+v) - \frac{1}{2} \sin(u-v), \\ \sin u \sin v &= \frac{1}{2} \cos(u-v) - \frac{1}{2} \cos(u+v), \\ \cos u \cos v &= \frac{1}{2} \cos(u-v) + \frac{1}{2} \cos(u+v). \end{aligned}$$

Sums of trigonometric functions to a product:

$$\begin{aligned} \sin u + \sin v &= 2 \sin\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right), \\ \sin u - \sin v &= 2 \cos\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right), \\ \cos u + \cos v &= 2 \cos\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right), \\ \cos u - \cos v &= -2 \sin\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right). \end{aligned}$$

Formulæ of halving and doubling the angle:

$$\sin 2u = 2 \sin u \cos u,$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2 \cos^2 u - 1 = 1 - 2 \sin^2 u,$$

$$\sin \frac{u}{2} = \pm \sqrt{\frac{1 - \cos u}{2}} \quad \text{followed by a discussion of the sign,}$$

$$\cos \frac{u}{2} = \pm \sqrt{\frac{1 + \cos u}{2}} \quad \text{followed by a discussion of the sign,}$$

8.8 Hyperbolic formulæ

These are very much like the trigonometric formulæ, and if one knows a little of Complex Function Theory it is realized that they are actually identical. The structure of this section is therefore the same as for the trigonometric formulæ. The reader should compare the two sections concerning similarities and differences.

The fundamental relation:

$$\cosh^2 x - \sinh^2 x = 1.$$

Definitions:

$$\cosh x = \frac{1}{2} (\exp(x) + \exp(-x)), \quad \sinh x = \frac{1}{2} (\exp(x) - \exp(-x)).$$

“Moivre’s formula”:

$$\exp(x) = \cosh x + \sinh x.$$

This is trivial and only rarely used. It has been included to show the analogy.

Addition formulæ:

$$\begin{aligned}\sinh(x+y) &= \sinh(x)\cosh(y) + \cosh(x)\sinh(y), \\ \sinh(x-y) &= \sinh(x)\cosh(y) - \cosh(x)\sinh(y), \\ \cosh(x+y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y), \\ \cosh(x-y) &= \cosh(x)\cosh(y) - \sinh(x)\sinh(y).\end{aligned}$$

Formulæ of halving and doubling the argument:

$$\begin{aligned}\sinh(2x) &= 2\sinh(x)\cosh(x), \\ \cosh(2x) &= \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1, \\ \sinh\left(\frac{x}{2}\right) &= \pm\sqrt{\frac{\cosh(x)-1}{2}} \quad \text{followed by a discussion of the sign,} \\ \cosh\left(\frac{x}{2}\right) &= \sqrt{\frac{\cosh(x)+1}{2}}.\end{aligned}$$

Inverse hyperbolic functions:

$$\begin{aligned}\text{Arsinh}(x) &= \ln\left(x + \sqrt{x^2 + 1}\right), \quad x \in \mathbb{R}, \\ \text{Arcosh}(x) &= \ln\left(x + \sqrt{x^2 - 1}\right), \quad x \geq 1, \\ \text{Artanh}(x) &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad |x| < 1, \\ \text{Arcoth}(x) &= \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \quad |x| > 1.\end{aligned}$$

8.9 Complex transformation formulæ

$$\begin{aligned}\cos(ix) &= \cosh(x), & \cosh(ix) &= \cos(x), \\ \sin(ix) &= i \sinh(x), & \sinh(ix) &= i \sin x.\end{aligned}$$

8.10 Taylor expansions

The generalized binomial coefficients are defined by

$$\binom{\alpha}{n} := \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{1 \cdot 2 \cdots n},$$

with n factors in the numerator and the denominator, supplied with

$$\binom{\alpha}{0} := 1.$$

The Taylor expansions for *standard functions* are divided into *power like* (the radius of convergency is finite, i.e. = 1 for the standard series) and *exponential like* (the radius of convergency is infinite).

Power like:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1,$$

$$(1+x)^n = \sum_{j=0}^n \binom{n}{j} x^j, \quad n \in \mathbb{N}, x \in \mathbb{R},$$

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad \alpha \in \mathbb{R} \setminus \mathbb{N}, |x| < 1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad |x| < 1,$$

$$\text{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad |x| < 1.$$

Exponential like:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \quad x \in \mathbb{R}$$

$$\exp(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n, \quad x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \quad x \in \mathbb{R},$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \quad x \in \mathbb{R},$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \quad x \in \mathbb{R},$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \quad x \in \mathbb{R}.$$

8.11 Magnitudes of functions

We often have to compare functions for $x \rightarrow 0+$, or for $x \rightarrow \infty$. The simplest type of functions are therefore arranged in an hierarchy:

- 1) logarithms,
- 2) power functions,
- 3) exponential functions,
- 4) faculty functions.

When $x \rightarrow \infty$, a function from a higher class will always dominate a function from a lower class. More precisely:

A) A *power function* dominates a *logarithm* for $x \rightarrow \infty$:

$$\frac{(\ln x)^\beta}{x^\alpha} \rightarrow 0 \quad \text{for } x \rightarrow \infty, \quad \alpha, \beta > 0.$$

B) An *exponential* dominates a *power function* for $x \rightarrow \infty$:

$$\frac{x^\alpha}{a^x} \rightarrow 0 \quad \text{for } x \rightarrow \infty, \quad \alpha, a > 1.$$

C) The *faculty function* dominates an *exponential* for $n \rightarrow \infty$:

$$\frac{a^n}{n!} \rightarrow 0, \quad n \rightarrow \infty, \quad n \in \mathbb{N}, \quad a > 0.$$

D) When $x \rightarrow 0+$ we also have that a *power function* dominates the *logarithm*:

$$x^\alpha \ln x \rightarrow 0-, \quad \text{for } x \rightarrow 0+, \quad \alpha > 0.$$

Index

- absolute value 162
- acceleration 490
- addition 22
- affinity factor 173
- Ampère-Laplace law 1671
- Ampère-Maxwell's law 1678
- Ampère's law 1491, 1498, 1677, 1678, 1833
- Ampère's law for the magnetic field 1674
- angle 19
- angular momentum 886
- angular set 84
- annulus 176, 243
- anticommutative product 26
- antiderivative 301, 847
- approximating polynomial 304, 322, 326, 336, 404, 488, 632, 662
- approximation in energy 734
- Archimedes's spiral 976, 1196
- Archimedes's theorem 1818
- area 887, 1227, 1229, 1543
- area element 1227
- area of a graph 1230
- asteroid 1215
- asymptote 51
- axial moment 1910
- axis of revolution 181
- axis of rotation 34, 886
- axis of symmetry 49, 50, 53
- barycentre 885, 1910
- basis 22
- bend 486
- bijective map 153
- body of revolution 43, 1582, 1601
- boundary 37–39
- boundary curve 182
- boundary curve of a surface 182
- boundary point 920
- boundary set 21
- bounded map 153
- bounded set 41
- branch 184
- branch of a curve 492
- Brownian motion 884
- cardioid 972, 973, 1199, 1705
- Cauchy-Schwarz's inequality 23, 24, 26
- centre of gravity 1108
- centre of mass 885
- centrum 66
- chain rule 305, 333, 352, 491, 503, 581, 1215, 1489, 1493, 1808
- change of parameter 174
- circle 49
- circular motion 19
- circulation 1487
- circulation theorem 1489, 1491
- circumference 86
- closed ball 38
- closed differential form 1492
- closed disc 86
- closed domain 176
- closed set 21
- closed surface 182, 184
- closure 39
- clothoid 1219
- colour code 890
- compact set 186, 580, 1813
- compact support 1813
- complex decomposition 69
- composite function 305
- conductivity of heat 1818
- cone 19, 35, 59, 251
- conic section 19, 47, 54, 239, 536
- conic sectional conic surface 59, 66
- connected set 175, 241
- conservation of electric charge 1548, 1817
- conservation of energy 1548, 1817
- conservation of mass 1548, 1816
- conservative force 1498, 1507
- conservative vector field 1489
- continuity equation 1548, 1569, 1767, 1817
- continuity 162, 186
- continuous curve 170, 483
- continuous extension 213
- continuous function 168
- continuous surfaces 177
- contraction 167
- convective term 492
- convex set 21, 22, 41, 89, 91, 175, 244
- coordinate function 157, 169
- coordinate space 19, 21

- Cornu's spiral 1219
Coulomb field 1538, 1545, 1559, 1566, 1577
Coulomb vector field 1585, 1670
cross product 19, 163, 169, 1750
cube 42, 82
current density 1678, 1681
current 1487, 1499
curvature 1219
curve 227
curve length 1165
curved space integral 1021
cusp 486, 487, 489
cycloid 233, 1215
cylinder 34, 42, 43, 252
cylinder of revolution 500
cylindric coordinates 15, 21, 34, 147, 181, 182, 289, 477, 573, 841, 1009, 1157, 1347, 1479, 1651, 1801
cylindric surface 180, 245, 247, 248, 499, 1230
degree of trigonometric polynomial 67
density 885
density of charge 1548
density of current 1548
derivative 296
derivative of inverse function 494
Descartes'a leaf 974
dielectric constant 1669, 1670
difference quotient 295
differentiability 295
differentiable function 295
differentiable vector function 303
differential 295, 296, 325, 382, 1740, 1741
differential curves 171
differential equation 369, 370, 398
differential form 848
differential of order p 325
differential of vector function 303
diffusion equation 1818
dimension 1016
direction 334
direction vector 172
directional derivative 317, 334, 375
directrix 53
Dirichlet/Neumann problem 1901
displacement field 1670
distribution of current 886
divergence 1535, 1540, 1542, 1739, 1741, 1742
divergence free vector field 1543
dodecahedron 83
domain 153, 176
domain of a function 189
dot product 19, 350, 1750
double cone 252
double point 171
double vector product 27
eccentricity 51
eccentricity of ellipse 49
eigenvalue 1906
elasticity 885, 1398
electric field 1486, 1498, 1679
electrical dipole moment 885
electromagnetic field 1679
electromagnetic potentials 1819
electromotive force 1498
electrostatic field 1669
element of area 887
elementary chain rule 305
elementary fraction 69
ellipse 48–50, 92, 113, 173, 199, 227
ellipsoid 56, 66, 110, 197, 254, 430, 436, 501, 538, 1107
ellipsoid of revolution 111
ellipsoidal disc 79, 199
ellipsoidal surface 180
elliptic cylindric surface 60, 63, 66, 106
elliptic paraboloid 60, 62, 66, 112, 247
elliptic paraboloid of revolution 624
energy 1498
energy density 1548, 1818
energy theorem 1921
entropy 301
Euclidean norm 162
Euclidean space 19, 21, 22
Euler's spiral 1219
exact differential form 848
exceptional point 594, 677, 920
expansion point 327
explicit given function 161
extension map 153
exterior 37–39
exterior point 38
extremum 580, 632
Faraday-Henry law of electromagnetic induction 1676
Fick's first law of diffusion 297

- Fick's law 1818
field line 160
final point 170
fluid mechanics 491
flux 1535, 1540, 1549
focus 49, 51, 53
force 1485
Fourier's law 297, 1817
function in several variables 154
functional matrix 303
fundamental theorem of vector analysis 1815

Gaussian integral 938
Gauß's law 1670
Gauß's law for magnetism 1671
Gauß's theorem 1499, 1535, 1540, 1549, 1580, 1718,
 1724, 1737, 1746, 1747, 1749, 1751, 1817,
 1818, 1889, 1890, 1913
Gauß's theorem in \mathbb{R}^2 1543
Gauß's theorem in \mathbb{R}^3 1543
general chain rule 314
general coordinates 1016
general space integral 1020
general Taylor's formula 325
generalized spherical coordinates 21
generating curve 499
generator 66, 180
geometrical analysis 1015
global minimum 613
gradient 295, 296, 298, 339, 847, 1739, 1741
gradient field 631, 847, 1485, 1487, 1489, 1491,
 1916
gradient integral theorem 1489, 1499
graph 158, 179, 499, 1229
Green's first identity 1890
Green's second identity 1891, 1895
Green's theorem in the plane 1661, 1669, 1909
Green's third identity 1896
Green's third identity in the plane 1898

half-plane 41, 42
half-strip 41, 42
half disc 85
harmonic function 426, 427, 1889
heat conductivity 297
heat equation 1818
heat flow 297
height 42
helix 1169, 1235

Helmholtz's theorem 1815
homogeneous function 1908
homogeneous polynomial 339, 372
Hopf's maximum principle 1905
hyperbola 48, 50, 51, 88, 195, 217, 241, 255, 1290
hyperbolic cylindric surface 60, 63, 66, 105, 110
hyperbolic paraboloid 60, 62, 66, 246, 534, 614,
 1445
hyperboloid 232, 1291
hyperboloid of revolution 104
hyperboloid of revolution with two sheets 111
hyperboloid with one sheet 56, 66, 104, 110, 247,
 255
hyperboloid with two sheets 59, 66, 104, 110, 111,
 255, 527
hysteresis 1669

identity map 303
implicit given function 21, 161
implicit function theorem 492, 503
improper integral 1411
improper surface integral 1421
increment 611
induced electric field 1675
induction field 1671
infinitesimal vector 1740
infinity, signed 162
infinity, unspecified 162
initial point 170
injective map 153
inner product 23, 29, 33, 163, 168, 1750
inspection 861
integral 847
integral over cylindric surface 1230
integral over surface of revolution 1232
interior 37–40
interior point 38
intrinsic boundary 1227
isolated point 39
Jacobian 1353, 1355

Kronecker symbol 23

Laplace equation 1889
Laplace force 1819
Laplace operator 1743
latitude 35
length 23
level curve 159, 166, 198, 492, 585, 600, 603

- level surface 198, 503
limit 162, 219
line integral 1018, 1163
line segment 41
Linear Algebra 627
linear space 22
local extremum 611
logarithm 189
longitude 35
Lorentz condition 1824
- Maclaurin's trisectrix 973, 975
magnetic circulation 1674
magnetic dipole moment 886, 1821
magnetic field 1491, 1498, 1679
magnetic flux 1544, 1671, 1819
magnetic force 1674
magnetic induction 1671
magnetic permeability of vacuum 1673
magnostatic field 1671
main theorems 185
major semi-axis 49
map 153
MAPLE 55, 68, 74, 156, 171, 173, 341, 345, 350, 352–354, 356, 357, 360, 361, 363, 364, 366, 368, 374, 384–387, 391–393, 395–397, 401, 631, 899, 905–912, 914, 915, 917, 919, 922–924, 926, 934, 935, 949, 951, 954, 957–966, 968, 971–973, 975, 1032–1034, 1036, 1037, 1039, 1040, 1042, 1053, 1059, 1061, 1064, 1066–1068, 1070–1072, 1074, 1087, 1089, 1091, 1092, 1094, 1095, 1102, 1199, 1200
matrix product 303
maximal domain 154, 157
maximum 382, 579, 612, 1916
maximum value 922
maximum-minimum principle for harmonic functions 1895
Maxwell relation 302
Maxwell's equations 1544, 1669, 1670, 1679, 1819
mean value theorem 321, 884, 1276, 1490
mean value theorem for harmonic functions 1892
measure theory 1015
Mechanics 15, 147, 289, 477, 573, 841, 1009, 1157, 1347, 1479, 1651, 1801, 1921
meridian curve 181, 251, 499, 1232
meridian half-plane 34, 35, 43, 181, 1055, 1057, 1081
- method of indefinite integration 859
method of inspection 861
method of radial integration 862
minimum 186, 178, 579, 612, 1916
minimum value 922
minor semi-axis 49
mmf 1674
Möbius strip 185, 497
Moivre's formula 122, 264, 452, 548, 818, 984, 1132, 1322, 1454, 1626, 1776, 1930
monopole 1671
multiple point 171
- nabla 296, 1739
nabla calculus 1750
nabla notation 1680
natural equation 1215
natural parametric description 1166, 1170
negative definite matrix 627
negative half-tangent 485
neighbourhood 39
neutral element 22
Newton field 1538
Newton-Raphson iteration formula 583
Newton's second law 1921
non-oriented surface 185
norm 19, 23
normal 1227
normal derivative 1890
normal plane 487
normal vector 496, 1229
- octant 83
Ohm's law 297
open ball 38
open domain 176
open set 21, 39
order of expansion 322
order relation 579
ordinary integral 1017
orientation of a surface 182
orientation 170, 172, 184, 185, 497
oriented half line 172
oriented line 172
oriented line segment 172
orthonormal system 23
- parabola 52, 53, 89–92, 195, 201, 229, 240, 241
parabolic cylinder 613

- parabolic cylindric surface 64, 66
paraboloid of revolution 207, 613, 1435
parallelepipedum 27, 42
parameter curve 178, 496, 1227
parameter domain 1227
parameter of a parabola 53
parametric description 170, 171, 178
parfrac 71
partial derivative 298
partial derivative of second order 318
partial derivatives of higher order 382
partial differential equation 398, 402
partial fraction 71
Peano 483
permeability 1671
piecewise C^k -curve 484
piecewise C^n -surface 495
plane 179
plane integral 21, 887
point of contact 487
point of expansion 304, 322
point set 37
Poisson's equation 1814, 1889, 1891, 1901
polar coordinates 15, 19, 21, 30, 85, 88, 147, 163,
172, 213, 219, 221, 289, 347, 388, 390,
477, 573, 611, 646, 720, 740, 841, 936,
1009, 1016, 1157, 1165, 1347, 1479, 1651,
1801
polar plane integral 1018
polynomial 297
positive definite matrix 627
positive half-tangent 485
positive orientation 173
potential energy 1498
pressure 1818
primitive 1491
primitive of gradient field 1493
prism 42
Probability Theory 15, 147, 289, 477, 573, 841,
1009, 1157, 1347, 1479, 1651, 1801
product set 41
projection 23, 157
proper maximum 612, 618, 627
proper minimum 612, 613, 618, 627
pseudo-sphere 1434
Pythagoras's theorem 23, 25, 30, 121, 451, 547,
817, 983, 1131, 1321, 1453, 1625, 1775,
1929
quadrant 41, 42, 84
quadratic equation 47
range 153
rectangle 41, 87
rectangular coordinate system 29
rectangular coordinates 15, 21, 22, 147, 289, 477,
573, 841, 1009, 1016, 1079, 1157, 1165,
1347, 1479, 1651, 1801
rectangular plane integral 1018
rectangular space integral 1019
rectilinear motion 19
reduction of a surface integral 1229
reduction of an integral over cylindric surface 1231
reduction of surface integral over graph 1230
reduction theorem of line integral 1164
reduction theorem of plane integral 937
reduction theorem of space integral 1021, 1056
restriction map 153
Riccati equation 369
Riesz transformation 1275
Rolle's theorem 321
rotation 1739, 1741, 1742
rotational body 1055
rotational domain 1057
rotational free vector field 1662
rules of computation 296
saddle point 612
scalar field 1485
scalar multiplication 22, 1750
scalar potential 1807
scalar product 169
scalar quotient 169
second differential 325
semi-axis 49, 50
semi-definite matrix 627
semi-polar coordinates 15, 19, 21, 33, 147, 181,
182, 289, 477, 573, 841, 1009, 1016, 1055,
1086, 1157, 1231, 1347, 1479, 1651, 1801
semi-polar space integral 1019
separation of the variables 853
signed curve length 1166
signed infinity 162
simply connected domain 849, 1492
simply connected set 176, 243
singular point 487, 489
space filling curve 171
space integral 21, 1015

- specific capacity of heat 1818
sphere 35, 179
spherical coordinates 15, 19, 21, 34, 147, 179, 181, 289, 372, 477, 573, 782, 841, 1009, 1016, 1078, 1080, 1081, 1157, 1232, 1347, 1479, 1581, 1651, 1801
spherical space integral 1020
square 41
star-shaped domain 1493, 1807
star shaped set 21, 41, 89, 90, 175
static electric field 1498
stationary magnetic field 1821
stationary motion 492
stationary point 583, 920
Statistics 15, 147, 289, 477, 573, 841, 1009, 1157, 1347, 1479, 1651, 1801
step line 172
Stokes's theorem 1499, 1661, 1676, 1679, 1746, 1747, 1750, 1751, 1811, 1819, 1820, 1913
straight line (segment) 172
strip 41, 42
substantial derivative 491
surface 159, 245
surface area 1296
surface integral 1018, 1227
surface of revolution 110, 111, 181, 251, 499
surjective map 153
- tangent 486
tangent plane 495, 496
tangent vector 178
tangent vector field 1485
tangential line integral 861, 1485, 1598, 1600, 1603
Taylor expansion 336
Taylor expansion of order 2, 323
Taylor's formula 321, 325, 404, 616, 626, 732
Taylor's formula in one dimension 322
temperature 297
temperature field 1817
tetrahedron 93, 99, 197, 1052
Thermodynamics 301, 504
top point 49, 50, 53, 66
topology 15, 19, 37, 147, 289, 477, 573, 841, 1009, 1157, 1347, 1479, 1651, 1801
torus 43, 182–184
transformation formulæ 1353
transformation of space integral 1355, 1357
transformation theorem 1354
trapeze 99
- triangle inequality 23, 24
triple integral 1022, 1053
- uniform continuity 186
unit circle 32
unit disc 192
unit normal vector 497
unit tangent vector 486
unit vector 23
unspecified infinity 162
- vector 22
vector field 158, 296, 1485
vector function 21, 157, 189
vector product 19, 26, 30, 163, 169, 1227, 1750
vector space 21, 22
vectorial area 1748
vectorial element of area 1535
vectorial potential 1809, 1810
velocity 490
volume 1015, 1543
volumen element 1015
- weight function 1081, 1229, 1906
work 1498
- zero point 22
zero vector 22
- (r, s, t)-method 616, 619, 633, 634, 638, 645–647, 652, 655
 C^k -curve 483
 C^n -functions 318
1-1 map 153