

# Complex Functions

## Examples c-9

The Argument Principle and Many-valued Functions

Leif Mejlbro



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Complex Functions Examples c-9 - The Argument Principle and  
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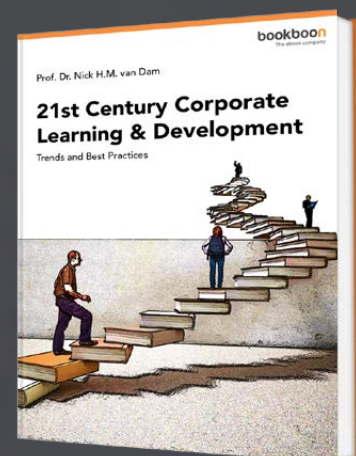
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## Introduction

This is the ninth book containing examples from the *Theory of Complex Functions*. We shall here treat the important *Argument Principle*, which e.g. is applied in connection with *Criteria of Stability* in *Cybernetics*. Finally, we shall also consider the *Many-valued functions* and their pitfalls.

Even if I have tried to be careful about this text, it is impossible to avoid errors, in particular in the first edition. It is my hope that the reader will show some understanding of my situation.

Leif Mejlbro  
27th June 2008



# 1 Some theoretical background

## 1.1 The argument principle

Let  $f : M \subset \mathbb{C}$  be a function defined on a set  $M$ . We define  $\arg f$  as any function on  $M$ , which for every  $t \in M$  is one of the values from the set  $\arg f(t)$ . Then

**Theorem 1.1** *Let  $f : [a, b] \rightarrow \mathbb{C} \setminus \{0\}$  be a continuous complex function on an interval  $[a, b]$ , which is different from zero. Then  $f$  has a continuous argument function  $\arg f$ .*

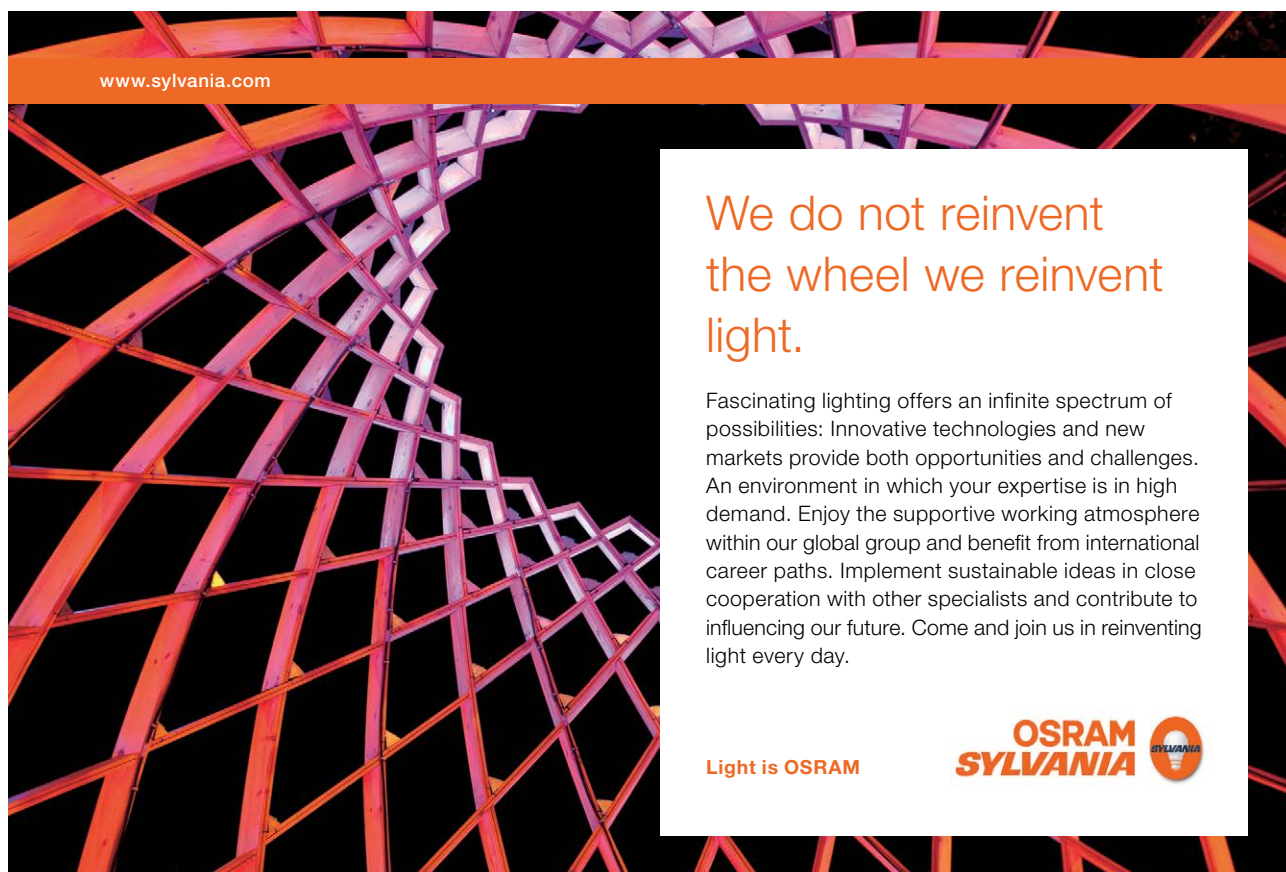
Clearly, we get all continuous argument functions from one continuous argument function  $\theta = \arg f$  by an addition of a multiple of  $2\pi$ . It follows that the difference

$$(1) \arg f(b) - \arg f(a)$$

has the same value for every continuous argument function  $\arg f$ . This difference (1) is called the *argument variation* of  $f$  on the interval  $[a, b]$ .

In particular, if  $f(a) = f(b)$ , then the continuous curve with the parametric description  $z = f(t)$ ,  $t \in [a, b]$ , is a closed curve, which does not pass through 0, and the argument variation is a multiple of  $2\pi$ ,

$$\arg f(b) - \arg f(a) = 2n\pi, \quad \text{for some } n \in \mathbb{Z}.$$




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This number  $n \in \mathbb{Z}$  is only defined for closed curves. It is called the *winding number around 0* of the curve or the function. It is geometrically interpreted as the number of times (with respect to the orientation of the plane), which the curve winds around 0, where negative windings of course cancel positive windings.

The importance of the *winding number around 0* is shown by the following theorem:

**Theorem 1.2** Let  $f : [a, b] \rightarrow \mathbb{C} \setminus \{0\}$  and  $g : [a, b] \rightarrow \mathbb{C}$  be two given continuous complex functions, for which  $f(a) = f(b)$  and  $g(a) = g(b)$ . Assume that we have for every  $t \in [a, b]$  that

$$|g(t)| < |f(t)|.$$

Then the two functions  $f$  and  $f + g$  have the same winding number  $n$  around 0.

According to this theorem one may allow small perturbations a closed curves without changing the winding number.

**Definition 1.1** Given an analytic function  $f : \Omega \rightarrow \mathbb{C}$ , which only has poles as its singularities, and which is not the zero function. We define the logarithmic derivative of  $f$  as the uniquely determined analytic function

$$\frac{f'(z)}{f(z)}, \quad \text{defined in } \{z \in \Omega \mid f(z) \neq 0\}.$$

We see that if  $\text{Log } f(z)$  is defined, then

$$\frac{d}{dz} \text{Log } f(z) = \frac{f'(z)}{f(z)}.$$

It is well-known that  $\text{Log } f(z)$  needs not be defined globally, while  $\frac{f'(z)}{f(z)}$  is always uniquely defined, if only  $f(z) \neq 0$ .

We have the following important result, which in particular is used in the applications of *Cybernetics*.

**Theorem 1.3** THE ARGUMENT PRINCIPLE. Given an open domain  $\Omega \subseteq \mathbb{C}$  and an analytic function  $f : \Omega \rightarrow \mathbb{C}$ , which is not the zero function, and which only has poles in  $\Omega$  as its singularities. Let  $C$  be a simple closed curve in  $\Omega$ , which does not pass through any zero or pole of  $f$ , and let  $\omega \subset \mathbb{C}$  be the bounded domain lying inside the curve  $C$ . The total number of zeros of  $f$  in  $\omega$  is denoted  $N$ , and the total number of poles of  $f$  in  $\omega$  is denoted  $P$ , all counted by multiplicity. Then the winding number around 0 for the closed curve  $f(C)$  in the  $w$ -plane is equal to the difference  $N - P$ , i.e.

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = N - P = \text{winding number of } f(C) \text{ around } w_0 = 0.$$

When we combine the argument principle with the previous theorem, we get

**Theorem 1.4** ROUCHÉ'S THEOREM. *Given an open domain  $\Omega \subseteq \mathbb{C}$ , and given two analytic functions  $f, g : \Omega \rightarrow \mathbb{C}$ . Let  $C$  be a simple closed curve in  $\Omega$ , and assume that neither  $f$  nor  $g$  have singularities inside  $C$ . Assume that we have the estimate*

$$|f(z)| > |g(z)| \quad \text{for every } z \text{ on the curve } C.$$

*Then the two functions  $f$  and  $g$  have the same number of zeros (counted by multiplicity) inside the curve  $C$ .*

## 1.2 Stability criteria

Considering the question of stability of mechanical or electrical systems with respect to oscillations, it is of great importance to decide if all zeros of some polynomial lie in a left half plane. This polynomial is typically the *numerator* of the so-called *transfer function*. This polynomial  $P(z)$  will usually have *complex* coefficients. However, if we instead consider the polynomial  $P(z)\overline{P}(z)$ , where the coefficients of the latter factor are the complex conjugated of the coefficients of the former factor, then this new factor will have the complex conjugated roots of the roots of  $P(z)$ . We see that all roots of  $P(z)$  lie in the left half plane, if and only if all roots of the polynomial  $P(z)\overline{P}(z)$  with real coefficients also lie in the left half plane. We can therefore in the following restrict ourselves to only considering polynomials of real coefficients.

We now introduce the following:

**Definition 1.2** *We call a polynomial  $P(z)$  of real coefficients a Hurwitz-polynomial, if all its zeros lie in the open left half plane.*

Concerning Hurwitz-polynomials we have the following simple result:

**Theorem 1.5** *A necessary condition for a polynomial*

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

*of real coefficients to be a Hurwitz-polynomial is that all its coefficients  $a_0, \dots, a_n$  have the same sign. In case of  $n = 1$  or  $n = 2$  this condition is also sufficient.*

In general this only gives us a *necessary* condition, which is not sufficient. This follows from the example

$$16z^3 + 8z^2 + 9z + 17 = (z + 1) \left( z - \frac{1}{4} - i \right) \left( z - \frac{1}{4} + i \right)$$



We can do better with

**Theorem 1.6** *A polynomial  $P(z)$  of real coefficients is a Hurwitz-polynomial, if and only if*

$$\begin{cases} |P(z)| > |P(-z)| & \text{for } \operatorname{Re} z > 0, \\ P(iy) \neq 0 & \text{for every } y \in \mathbb{R}. \end{cases}$$

The standard method for proving that a given polynomial with real coefficients is a Hurwitz-polynomial, is the following

**Theorem 1.7** HURWITZ'S CRITERION (1895). *Given a polynomial*

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

*with positive coefficient. Then  $P(z)$  is a Hurwitz-polynomial, if and only if the following determinant inequalities are all fulfilled,*

$$D_1 = a_1 > 0, \quad D_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0, \quad D_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0, \quad \dots,$$

$$D_n = \begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & \cdots & a_n \end{vmatrix} > 0,$$

where we put  $a_k = 0$  for  $k > n$ .

Clearly, this criterion may in practice be rather difficult to use, when the degree of the polynomial is large. Fortunately we have an alternative criterion, by which on by iteration is able to reduce the degree with (at least) 1 in every step:

**Theorem 1.8** SCHUR'S CRITERION. *A polynomial  $P(z)$  with real coefficients is a Hurwitz-polynomial, if and only if its coefficients all have the same sign, and the polynomial*

$$Q(z) = \frac{P(1)P(z) - P(-1)P(-z)}{z+1}$$

*is also a Hurwitz-polynomial.*

We shall finally mention

**Theorem 1.9** NYQUIST'S CRITERION (1932). *Given a rational function  $H(z)$ , where  $H(iy) \neq 1$  on the imaginary axis. We denote by  $P$  the number of poles for  $\frac{1}{1+H(z)}$  in the right half plane (counted by the multiplicity). Then the control system, which corresponds to the transfer function*

$$f(z) = \frac{H(z)}{1+H(z)},$$

*is stable, if and only if  $\frac{1}{1+H(z)}$  has the winding number  $a$   $P$  around the point  $w = 1$ , when  $z$  runs through the imaginary axis.*

supplied with

**Theorem 1.10** NYQUIST-MICHAÏLOW'S CRITERION. *Given a rational function  $G(z)$  with no zeros on the imaginary axis and with  $N$  zeros in the right half plane. Let  $K$  be a so-called amplification factor.*

*Then the control system, which corresponds to the transfer function*

$$f(z) = \frac{1}{K \cdot G(z)} + 1,$$

*is stable, if and only if  $\frac{1}{G(z)}$  has the winding number  $N$  around the point  $w = -K$ , when  $z$  runs through the imaginary axis from  $-i\infty$  to  $+i\infty$ . Here,  $N$  denotes the number of zeros of  $G(z)$  in the right half plane.*



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### 1.3 Inverse functions

Given an analytic function  $f(z)$  in an open domain  $\Omega$ , and assume that  $f : \Omega \rightarrow f(\Omega)$  is *one-to-one*. Although it is far from trivial, one can then prove that  $f'(z) \neq 0$  everywhere in  $\Omega$ , so the inverse map exists and is an analytic function with the derivative

$$\frac{d}{dw} f^{\circ-1}(w) = \frac{1}{f'(f^{\circ-1}(w))}.$$

Clearly, one can then always *locally* determine the inverse  $f^{\circ-1}(w)$  of an analytic function  $f(z)$  in a neighbourhood of any point, in which  $f'(z) \neq 0$ . Then question is now, when these local inverses can be glued together to a *global* analytic function. We know already from the example of the many-valued function  $\log w$  that this is not always possible.

Since the topic here is the *inverse function*, we shall for convenience interchange the variables  $z$  and  $w$ . An obvious definition is

**Definition 1.3** Let  $f : \Omega \rightarrow \mathbb{C}$  be analytic in an open domain  $\Omega$  in the  $w$ -plane. A point  $w_0 \in \Omega$  is said to be a *singular point* of the function  $f(w)$ , if  $f'(w_0) = 0$ .

When  $w_0 \in \Omega$  is a singular point, we call the image  $z_0 = f(w_0)$  a *branch point* of the corresponding Riemann-surface.

Another type of branch points are the *logarithmic branch points*.

**Definition 1.4** We say that a point  $z_0$  is a *logarithmic branch point* of the function  $f(w)$ , defined on the open set  $\Omega$ , if the following two conditions hold,

- 1) The point  $z_0$  is isolated in  $\mathbb{C} \setminus f(\Omega)$ .
- 2) There does not exist any larger open domain  $\Omega_1 \supset \Omega$  with corresponding analytic function  $f_1 : \Omega_1 \rightarrow \mathbb{C}$ , such that

$$f_1(w) = f(w) \quad \text{for } w \in \Omega, \quad \text{and} \quad z_0 \in f_1(\Omega_1).$$

Condition 2) above can also be expressed in the following way:

- 2a) For given any open set  $\Omega_1 \supset \Omega$  and any analytic function  $f_1 : \Omega \rightarrow \mathbb{C}$  we have that either  $f_1$  is not an analytic extension of  $f$ , or – if it is – then the point  $z_0 \notin f_1(\Omega_1)$ .

Then we can describe the following procedure for determining the inverse function, i.e. for solving the equation

$$z = f(w), \quad w \in \Omega.$$

- 1) First find the *singular points*  $w_0 \in \Omega$ , i.e. the solutions of the equation

$$f'(w) = 0.$$

- 2) Then for every solution  $w_0$  of this equation, the image  $z_0 = f(w_0)$  is a *branch point*, where we do not have any one-to-one connection between any of the neighbourhoods of  $w_0$  and  $z_0$ .

- 3) Then check the image  $f(\Omega)$  for the possibility of one (or more) *branch points*.
- 4) When we have found the image  $f(\Omega)$ , then we shall find the largest possible *open* domain  $\omega \subseteq \Omega$ , for which  $f : \omega \rightarrow f(\omega)$  is one-to-one. We call any such maximum open domain a *fundamental domain*.
- 5) If the *boundary* of a fundamental domain  $\omega$  is composed of piecewise differentiable curves contained in  $\Omega$ , then the boundary of  $f(\Omega) \setminus f(\omega)$  is consisting of curves between the branch points. We notice that these can be ordinary branch points or logarithmic branch points. We call any such curve in the  $z$ -plane a *branch cut*, and the uniquely determined inverse map  $f^{\circ-1} : f(\omega) \rightarrow \omega$  onto the image of a fundamental domain is called a *branch* of the inverse map  $f^{\circ-1}$ .
- 6) Whenever possible we try directly to find an expression in  $\Omega$  of the many-valued function  $f^{\circ-1}(z)$ .
- 7) By means of the fundamental domains, the branch cuts, some pieces of paper, a pair of scissors, some glue or tape, and possibly also some patience it is possible to construct a model of the corresponding *Riemann-surface*.

**Remark 1.1** Contrarily to the branch points, the *fundamental domains*, the *branch cuts* and the *branches* are not uniquely determined.  $\diamond$

Even if the Riemann-surfaces in general are difficult to handle, they have nevertheless found their way into the theory of *Stability*.

Assume that  $f(z)$  is a polynomial or a rational function. Then there exists a Riemann-surface  $\mathcal{R}$ , corresponding to  $f(z)$ , such that  $f$  maps the  $z$ -plane onto the Riemann-surface  $\mathcal{R}$ . The criterion of stability can then be reduced to the condition that the image  $f(\overline{\Omega})$  of the closed right half plane

$$\overline{\Omega} = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\}$$

does not contain points which lie above the point  $w = 0$ . If the image  $f(\overline{\Omega})$  of the right half plane into the Riemann-surface  $\mathcal{R}$  is bounded by a curve  $\Gamma$  (where  $\Gamma$  is the image of the imaginary axis), then we get the following stability criterion:

**Theorem 1.11** *If the domain  $f(\Omega)$  to the right of the curve  $\Gamma$  on the corresponding Riemann-surface  $\mathcal{R}$  does not contain any point above  $w = 0$  (and the curve  $\Gamma$  does not pass through such a point), then  $f(z)$  is the transfer function for a stable system. If these conditions are not satisfied, then the system is unstable.*

It is of course in practice very difficult to investigate  $\Gamma$  on the Riemann-surface  $\mathcal{R}$  itself. Instead we consider the *projection*  $\gamma$  of  $\Gamma$  onto the  $w$ -plane. We obtain a parametric description by putting  $z = iy$  into the expression of  $f(z)$  and then separate the real and the imaginary part. Then we get the parametric description:

$$\gamma : \quad u = u(y), \quad v = v(y), \quad y \in \mathbb{R}.$$

We lose some information by this projection and the condition that the point  $w = 0$  must not lie to the right of the curve  $\gamma$ , is then only *necessary* and *not sufficient*, which is illustrated by the example

$$f(z) = z^3 - z^2 + 2z - 3.$$

The coefficients have different signs, so the corresponding system cannot be stable. On the other hand, it is easy to prove that  $w = 0$  does not lie to the right of the curve  $\gamma$ , demonstrating that the condition is not sufficient.

## 2 The argument variation

**Example 2.1** Prove that the polynomial  $z^n - a$  has  $n$  roots in the unit disc for every fixed  $a \in \mathbb{C}$ , where  $0 < |a| < 1$ .

We put  $f(z) = z^n$  and  $g(z) = -a$  and let  $C$  denote the unit circle  $|z| = 1$  run through in the positive sense. On  $C$  we have the following estimate

$$|f(z)| = |z^n| = 1 > |a| = |g(z)| \quad \text{for } |z| = 1,$$

so it follows from *Rouché's theorem* that  $f(z) = z^n$  and  $f(z) + g(z) = z^n - a$  have the same number of zeros inside  $C$ , i.e.  $n$ . According to the solution formula for the binomial equation  $z^n = a$ , which also could have been applied here, all  $n$  roots are different, when  $0 < |z| < 1$ , so they are all *simple*. In particular we see that by adding any small constant  $a \neq 0$  to  $f(z) = z^n$  the  $n$  identical roots  $z = 0$  are branching away in  $n$  different directions.

**Example 2.2** Compute the line integral

$$\frac{1}{2\pi i} \oint_{|z|=2} \frac{10z^9 + 25 \sinh z + \frac{1}{2} \exp\left(\frac{z}{2}\right) \exp\left(\exp\left(\frac{z}{2}\right)\right)}{z^{10} + 25 \cosh z + \exp\left(\exp\left(\frac{z}{2}\right)\right)} dz.$$

This example shall only demonstrate how one may use the argument principle to calculate “impossible” line integrals of the type

$$\frac{1}{2\pi i} \oint_C \frac{h'(z)}{h(z)} dz.$$

Clearly, the usual method of inserting some parametric description is doomed to fail.

Instead we put

$$h(z) = z^{10} + 25 \cosh z + \exp\left(\exp\left(\frac{z}{2}\right)\right), \quad \text{where} \quad h'(z) = 10z^9 + 25 \sinh z + \frac{1}{2} \exp\left(\frac{z}{2}\right) \exp\left(\exp\left(\frac{z}{2}\right)\right).$$

Thus the integral is of the form

$$\frac{1}{2\pi i} \oint_C \frac{h'(z)}{h(z)} dz = N_h - P_h = N_h,$$

where we immediately see that  $h(z)$  does not have any poles, so  $P_h = 0$ .

We shall only find the *number* of zeros of  $h(z)$  inside  $|z| = 2$ .

In order to get an idea of what to do we estimate each term of the denominator  $h(z)$  along the circle  $|z| = 2$ . This gives

$$|z^{10}| = 2^{10} = 1024,$$

$$|25 \cosh z| = 25 \left| \frac{1}{2} \{e^z + e^{-z}\} \right| \leq \frac{25}{2} \cdot 2 |e^2| \leq 200,$$

and

$$\left| \exp \left( \exp \left( \frac{z}{2} \right) \right) \right| \leq e^e < 3^3 = 27.$$

We see that the first estimate will give the biggest number, so if we put

$$f(z) = z^{10} \quad \text{and} \quad g(z) = 25 \cosh z + \exp \left( \exp \left( \frac{z}{2} \right) \right),$$

then it follows for  $|z| = 2$  that

$$|f(z)| = |z^{10}| = 1024 > 200 + 27 > |25 \cosh z| + \left| \exp \left( \exp \left( \frac{z}{2} \right) \right) \right| \geq |g(z)|.$$

Then *Rouché's theorem* implies that  $f(z)$  and  $f(z) + g(z) = h(z)$  have the same number of zeros inside  $|z| = 2$ , i.e.

$$N_h = N_f = 10.$$

Then

$$\frac{1}{2\pi i} \oint_{|z|=2} \frac{10z^9 + 25 \sinh z + \frac{1}{2} \exp \left( \frac{z}{2} \right) \exp \left( \exp \left( \frac{z}{2} \right) \right)}{z^{10} + 25 \cosh z + \exp \left( \exp \left( \frac{z}{2} \right) \right)} dz = 10.$$

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**Example 2.3** Find the number of zeros of the analytic function

$$z^2 + 5 \sin z \quad \text{in the rectangle} \quad \left\{ z = x + iy \mid -\frac{\pi}{2} < x < \frac{\pi}{2} \text{ and } -1 < y < 1 \right\}.$$

Clearly,  $z = 0$  is a simple zero, so there exist zeros in this open set. The question is, if there are others.

We have in general,

$$|\sin z|^2 = \sin^2 x + \sinh^2 y,$$

from which we get the estimate

$$|\sin z| \geq \left| \sin \left( \pm \frac{\pi}{2} \right) \right| = 1,$$

on the vertical segments  $\pm \frac{\pi}{2} + iy$ ,  $-1 < y < 1$  of the rectangle. On the horizontal segments,  $x \pm i$ ,  $|x| < \frac{\pi}{2}$ , so here we have with  $y = \pm 1$  the estimate

$$|\sin z| \geq \sinh 1 > 1.$$

Choosing  $f(z) = 5 \sin z$ , estimating *downwards* we get on the boundary  $C$  of the rectangle that

$$|f(z)| = |5 \sin z| \geq 5 \quad \text{for } z \in C.$$

Then the remaining term  $g(z) = z^2$  is estimated *upwards* on the boundary curve  $C$ , i.e. by a small geometric argument,

$$|g(z)| = |z^2| \leq \left| \frac{\pi}{2} + i \right|^2 = \frac{\pi^2}{4} + 1 < \frac{10}{4} + 1 < 5 \quad \text{for } z \in \mathbb{C}.$$

Then it follows by *Rouché's theorem* that  $f(z) = 5 \sin z$  and  $f(z) + g(z) = z^2 + 5 \sin z$  have the same number of zeros in the open rectangle. Since  $f(z) = 5 \sin z$  only has the simple zeros  $\{p\pi \mid p \in \mathbb{Z}\}$  in  $\mathbb{C}$ , we conclude that  $z = 0$  is the only possible zero for  $z^2 + 5 \sin z$  inside  $C$ .

**Example 2.4** Give a coarse estimate of where the roots of the polynomial

$$z^3 + 2z^2 - 50z + 100$$

are lying with respect to origo.

We first notice that the polynomial has degree three, so it follows from the *Fundamental Theorem of Algebra* that the polynomial has three zeros in all of  $\mathbb{C}$ . The question here is how close they are to the origo.

We shall first prove that all zeros lie inside the circle  $|z| = 9$ . Choosing

$$f_1(z) = z^3 \quad \text{and} \quad g_1(z) = 2z^2 - 50z + 100,$$

we get the estimates

$$|f_1(z)| = 9^3 = 729 \quad \text{and} \quad |g_1(z)| \leq 2 \cdot 9^2 + 50 \cdot 9 + 100 = 712 < |f_1(z)| \quad \text{for } |z| = 9,$$

so we conclude from *Rouché's theorem* that there are three zeros (and hence all of them) inside  $|z| = 9$ .

Then we prove that we inside  $|z| = 4$  only have one zero. In this case we choose

$$f_2(z) = -50z \quad \text{og} \quad g_2(z) = z^3 + 2z^2 + 100,$$

and we get the estimates

$$|f_2(z)| = |-50z| = 200 \quad \text{and} \quad |g_2(z)| = |z^3 + 2z^2 + 100| \leq 64 + 32 + 100 = 196 < |f_2(z)| \quad \text{for } |z| = 4,$$

so  $f_2(z)$  is dominating, what the number of zeros is concerned. Since  $-50z$  has just one zero inside  $|z| = 4$  (this zero is 0), the polynomial has also just one zero inside  $|z| = 4$ . Notice, however, that this zero does not lie in the “neighbourhood” of 0. We shall namely prove at last that there is no zero inside the circle  $|z| = \frac{7}{4}$ .

Choosing

$$f_3(z) = 100 \quad \text{and} \quad g_3(z) = z^3 + 2z^2 - 50z,$$

we get the estimate

$$|g_3(z)| = |z^3 + 2z^2 - 50z| \leq \left(2 + \frac{7}{4}\right) \left(\frac{7}{4}\right)^2 + 50 \cdot \frac{7}{4} = \frac{15}{4} \cdot \frac{49}{16} + \frac{7}{4} \cdot 50 < \left(\frac{1}{4} + \frac{7}{4}\right) \cdot 50 = 100 = |f_3(z)|$$

for  $|z| = \frac{7}{4}$ . Hence, the polynomial  $f_3(z) + g_3(z)$  has the same number of zeros inside  $|z| = \frac{7}{4}$  than the constant 100, i.e. none.

As a control we also find the roots of the polynomial  $z^3 + 2z^2 - 50z + 100$  by either using Cardano's formula or by using a pocket calculator. Approximative values are

$$-8,889\,794\,306, \quad 2,658\,473\,477, \quad 4,231\,320\,828.$$

All three of them satisfy the estimates  $\frac{7}{4} = 1,75 < z_0 < 9$ , and only the middle one fulfils  $\frac{7}{4} = 1,75 < z_0 < 4$ .

**Remark 2.1** If we instead had considered the polynomial  $z^3 + 2z^2 + 50z + 100$  with the plus sign everywhere, then – apart from the trivial change of sign – repeat the whole argument above, so the zeros of this polynomial also lie in the same annuli. However, here we can make a shortcut and directly find the zeros, because we have the splitting into factors

$$z^3 + 2z^2 + 50z + 100 = (z + 2)(z^2 + 50),$$

proving that the roots are  $-2, 5\sqrt{2}i$  and  $-5\sqrt{2}i$ . On a figure one may see that this change of sign dramatically changes the placements of the roots, since we in the former case have three real roots, and in the latter case two complex conjugated roots and one real root.  $\diamond$

**Example 2.5** Compute the line integral

$$\oint_{|z|=1} \frac{z-1}{e^z-3z} dz.$$

Putting  $f(z) = e^z - 3z$ , it follows that

$$|e^z| \leq e < 3 = |3z| \quad \text{for } |z| = 1,$$

and we conclude from *Rouché's theorem* that  $f(z)$  has only a *simple* zero  $z_0$  inside  $|z| = 1$ . Then it is obvious that

$$\exp(z_0) = 3z_0,$$

which we shall exploit in the following.

Then  $f'(z) = e^z - 3$ , where

$$|f'(z)| \geq 3 - |e^z| \geq 3 - e > 0, \quad \text{for } |z| \leq 1.,$$

so  $f'(z) \neq 0$  inside  $|z| = 1$ .

We proceed with a rather tricky argument,

$$\begin{aligned} \oint_{|z|=1} \frac{z-1}{e^z-3z} dz &= \oint_{|z|=1} \frac{z-1}{e^z-3z} \cdot \frac{e^z-3}{e^z-3z} dz = \oint_{|z|=1} \frac{z-1}{e^z-3z} \cdot \frac{f'(z)}{f(z)} dz \\ &= 2\pi i \operatorname{res} \left( \frac{z-1}{e^z-3z} \cdot \frac{f'(z)}{f(z)}; z_0 \right) = 2\pi i \frac{z_0-1}{\exp z_0-3} \cdot 1 = 2\pi i \cdot \frac{z_0-1}{3z_0-3} = \frac{2\pi i}{3}, \end{aligned}$$

because it is easily proved that in general,

$$\operatorname{res} \left( \varphi(z) \cdot \frac{f'(z)}{f(z)}; z_0 \right) = \varphi(z_0) \cdot m,$$

assuming that  $\varphi(z)$  is analytic in a neighbourhood of  $z_0$  and that  $f(z)$  has a *zero* of order  $m$ , if  $m > 0$ , or a *pole* of order  $-m = |z|$ , if  $m < 0$ .

ALTERNATIVELY we may apply Rule II, because the solution  $z_0$  of  $\exp z_0 = 3z_0$  in the unit disc is a simple pole, hence

$$\begin{aligned} \oint_{|z|=1} \frac{z-1}{e^z-3z} dz &= 2\pi i \cdot \lim_{z \rightarrow z_0} \frac{z-1}{\frac{d}{dz} \{e^z-3z\}} = 2\pi i \cdot \lim_{z \rightarrow z_0} \frac{z-1}{e^z-3} \\ &= 2\pi i \cdot \frac{z_0-1}{\exp z_0-3} = 2\pi i \cdot \frac{z_0-1}{3z_0-3} = \frac{2\pi i}{3}. \end{aligned}$$

**Remark 2.2** It is remarkable that even if one cannot directly find the value of the integral by inserting a parametric description, and even if one cannot explicitly find the pole  $z_0$ , we are nevertheless able to compute the line integral, because  $z_0$  is cancelled in the calculations.  $\diamond$

**Example 2.6** Compute

$$(a) \oint_{|z|=3} \frac{z^2}{z^3-2} dz, \quad (b) \oint_{|z|=2} \frac{2z^3+1}{z^4+2z^2+1} dz, \quad (c) \oint_{|z|=2} \frac{2z^3+1}{z^4+2z+1} dz.$$

(a) Putting  $f(z) = z^3 - 2$ , it is seen that  $f$  has three zeros inside  $|z| = 3$  and no pole. Hence by the *argument principle*,

$$\oint_{|z|=3} \frac{z^2}{z^3-2} dz = \frac{1}{3} \oint_{|z|=3} \frac{3z^2}{z^3-2} dz = \frac{1}{3} \oint_{|z|=3} \frac{f'(z)}{f(z)} dz = \frac{2\pi i}{3} (3 - 0) = 2\pi i.$$

ALTERNATIVELY, use the residuum at  $\infty$ , because there is no pole for  $|z| \geq 3$ :

$$\oint_{|z|=3} \frac{z^2}{z^3-2} dz = - \oint_{|z|=3}^* \frac{z^2}{z^3-2} dz = -2\pi i \operatorname{res} \left( \frac{z^2}{z^3-2}; \infty \right) = -2\pi i \left\{ - \lim_{z \rightarrow \infty} \frac{z^3}{z^3-2} \right\} = 2\pi i.$$



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(b) It follows from  $z^4 + 2z^2 + 1 = (z^2 + 1)^2$  that

$$\frac{2z^3 + 1}{z^4 + 2z^2 + 1}$$

has two double poles inside  $|z| = 2$  and no pole outside. We cannot directly apply the argument principle. We have, however, a zero at  $\infty$ , so by the residuum at  $\infty$  we get

$$\begin{aligned} \oint_{|z|=2} \frac{2z^3 + 1}{z^4 + 2z^2 + 1} dz &= - \oint_{|z|=2}^* \frac{2z^3 + 1}{z^4 + 2z^2 + 1} dz = -2\pi i \operatorname{res} \left( \frac{2z^3 + 1}{z^4 + 2z^2 + 1}; \infty \right) \\ &= -2\pi i \left\{ - \lim_{z \rightarrow \infty} z \cdot \frac{2z^3 + 1}{z^4 + 2z^2 + 1} \right\} = 4\pi i. \end{aligned}$$

(c) Put  $f(z) = z^4 + 2z + 1$ . We have on  $|z| = 2$  the estimate

$$|z|^4 = 2^4 = 16 > 2 \cdot 2 + 1 = 5 \geq |2z + 1| \quad \text{for } |z| = 2,$$

so  $w = f(z)$  has the winding number 4 around 0 for  $|z| = 2$ . Then it follows from the *argument principle* that

$$\oint_{|z|=2} \frac{2z^3 + 1}{z^4 + 2z + 1} dz = \frac{1}{2} \oint_{|z|=1} \frac{f'(z)}{f(z)} dz = \frac{1}{2} \cdot 2\pi i \cdot 4 = 4\pi i.$$

ALTERNATIVELY we get

$$|2z + 1| \leq 2r + 1 < r^4 = |z^4| \quad \text{for } |z| = r \geq 2,$$

thus

$$|z^4 + 2z + 1| \geq |z^4| - |2z + 1| > 0 \quad \text{for } |z| \geq 2.$$

Hence, the function

$$\frac{2z^3 + 1}{z^4 + 2z + 1}$$

does not have poles outside  $|z| = 2$ , and since  $\infty$  is a simple zero, we get

$$\begin{aligned} \oint_{|z|=2} \frac{2z^3 + 1}{z^4 + 2z + 1} dz &= - \oint_{|z|=2}^* \frac{2z^3 + 1}{z^4 + 2z + 1} dz = -2\pi i \operatorname{res} \left( \frac{2z^3 + 1}{z^4 + 2z + 1}; \infty \right) \\ &= -2\pi i \cdot \left\{ - \lim_{z \rightarrow \infty} \frac{z(2z^3 + 1)}{z^4 + 2z + 1} \right\} = 2\pi i \cdot 2 = 4\pi i. \end{aligned}$$

**Example 2.7** Compute the complex line integral

$$\frac{1}{2\pi i} \oint_{|z|=1} \frac{10z + e^z + \cos z}{5z^2 + e^z + \sin z} dz.$$

Put  $f(z) = 5z^2$  and  $g(z) = e^z + \sin z$ . We have for  $|z| = 1$  the estimates

$$|g(z)| \leq |e^x e^{iy}| + |\sin z| \leq e^1 + \cosh 1 < e + 2 < 5 = |f(z)|.$$

Hence

$$|f(z)| > |g(z)| \quad \text{for } |z| = 1.$$

Since  $f(z)$  and  $g(z)$  are analytic, we conclude from Rouché's theorem that

$$\begin{aligned} \frac{1}{2\pi i} \oint_{|z|=1} \frac{10z + e^z + \cos z}{5z^2 + e^z + \sin z} dz &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{f'(z) + g'(z)}{f(z) + g(z)} dz = \frac{1}{2\pi i} \oint_{|z|=1} \frac{f'(z)}{f(z)} dz \\ &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{10z}{5z^2} dz = 2 \cdot \frac{1}{2\pi i} \oint_{|z|=1} \frac{1}{z} dz = 2. \end{aligned}$$

ALTERNATIVELY,

$$\frac{1}{2\pi i} \oint_{|z|=1} \frac{f'(z)}{f(z)} dz$$

is equal to the number of zeros counted by multiplicity of  $f(z) = 5z^2$  inside  $|z| = 1$ , where we use that  $f(z)$  does not have poles. We immediately see that this number is 2.

**Example 2.8** Given a polynomial  $f(z)$ . Prove that

$$\frac{1}{2\pi i} \oint_{|z|=R} \frac{f'(z)}{f(z)} z dz$$

is the sum of all roots of  $f$ , when  $R$  is chosen sufficiently large.

Let  $z_0$  denote any root of the polynomial of multiplicity  $n_0$ , thus

$$f(z) = (z - z_0)^{n_0} P(z),$$

where  $P(z_0) \neq 0$ . Then

$$\begin{aligned} \frac{f'(z)}{f(z)} \cdot z &= \frac{n_0 (z - z_0)^{n_0-1} P(z) z}{(z - z_0)^{n_0} P(z)} + \frac{(z - z_0)^{n_0} P'(z) \cdot z}{(z - z_0)^{n_0} P(z)} = n_0 \cdot \frac{z}{z - z_0} + \frac{P'(z)}{P(z)} \cdot z \\ &= n_0 \cdot \frac{z - z_0 + z_0}{z - z_0} + \frac{P'(z)}{P(z)} \cdot z \frac{n_0 z_0}{z - z_0} + g(z), \end{aligned}$$

where  $g(z)$  is analytic in a neighbourhood of  $z_0$ . It follows that

$$\operatorname{res} \left( \frac{f'(z)}{f(z)} \cdot z; z_0 \right) = n_0 z_0.$$



This holds for every zero of the denominator, so if  $f(z)$  has the structure

$$f(z) = A(z - z_1)^{n_1} \cdots (z - z_k)^{n_k},$$

and if  $R$  is chosen so large that all roots lie inside  $|z| = R$ , then

$$\frac{1}{2\pi i} \oint_{|z|=R} \frac{f'(z)}{f(z)} z dz = \sum_{j=1}^k \operatorname{res} \left( \frac{f'(z)}{f(z)} \cdot z; z_j \right) = \sum_{j=1}^k n_j z_j,$$

which is precisely the sum of all roots counted by multiplicity.

**Example 2.9** Prove that the equation  $e^z - 3z^7 = 0$  has seven zeros in the unit disc  $|z| < 1$ .

Put  $f(z) = -3z^7$  and  $g(z) = e^z$ . If  $|z| = 1$ , then we have the estimates

$$|f(z)| = |-3z^7| = 3|z|^7 = 3,$$

and

$$|g(z)| = |e^x e^{iy}| = e^x \leq e^1 < 3 = |f(z)|.$$

We conclude from Rouché's theorem that

$$f(z) = -3z^7 \quad \text{and} \quad f(z) + g(z) = e^z - 3z^7$$

have the same number of zeros inside  $|z| = 1$ .

Since  $f(z) = -3z^7$  has the 7 identical roots lying in  $z = 0$ , we conclude that  $e^z - 3z^7$  has 7 zeros inside  $|z| = 1$ .

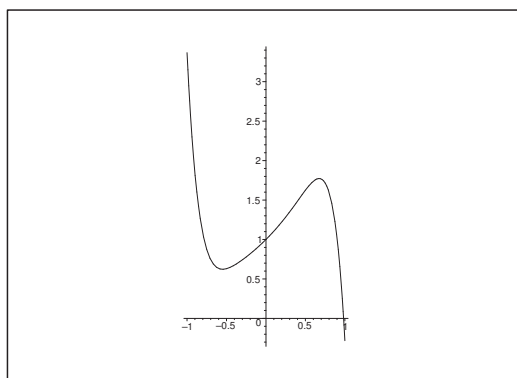


Figure 1: The graph of  $f(x) = e^x - 3x^7$ ,  $x \in [-1, 1]$ .

By considering the graph we conclude that the equation has precisely one real root. An approximative value is

$$z = 0,983\,720\,9.$$

**Example 2.10** Prove for a given  $a > e$  that the equation

$$e^z = a z^n$$

has  $n$  solutions in the unit disc.

Putting  $f(z) = a z^n$  and  $g(z) = -e^z$ , we get for  $|z| = 1$  that


$$|g(z)| = |e^z| \leq e < a = |a z^n| = |f(z)|, \quad \text{for } |z| = 1,$$

and we conclude from Rouché's theorem that

$$f(z) + g(z) = a z^n - e^z \quad \text{and} \quad f(z) = a z^n$$

have the same number of zeros in the unit.


Since  $f(z)$  has an  $n$ -tuple zero at 0, this number is  $n$ . The claim follows from that  $a z^n - e^z = 0$  is equivalent to  $e^z = a z^n$ .



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**Example 2.11** *Prove that the equation*

$$e^{z+2} - 2z^7 = 0$$

*has no solution in the unit disc.*

This example is trivial, because we have for  $|z| \leq 1$ ,

$$|e^{z+2} - 2z^7| \geq e^1 - 2 > 0,$$

thus

$$e^{z+2} - 2z^7 \neq 0 \quad \text{for every } z \text{ where } |z| < 1.$$

**Example 2.12** *Given  $\lambda \in \mathbb{R}$ ,  $\lambda > 1$ . Prove that the equation  $ze^{\lambda-z} = 1$  has precisely one root in the unit disc  $|z| < 1$ . This root is real and positive.*

When we multiply by  $e^z \neq 0$  and rearrange, we get the equation

$$ze^\lambda - e^z = 0.$$

When  $|z| = 1$ , we get the estimate

$$|ze^\lambda| = e^\lambda > e \geq |e^z| = e^x.$$

Hence we conclude by Rouché's theorem that the functions

$$ze^\lambda - e^z \quad \text{and} \quad ze^\lambda$$

have the same number of zeros in  $z$  in the unit disc  $|z| \leq 1$ , i.e. precisely one.

Now

$$f(0) = -1 \quad \text{and} \quad f(1) = e^\lambda - e^1 > 0,$$

where  $f(z) = ze^\lambda - e^z$  is analytic and in particular real and continuous, when  $z$  is real. Thus there exists at least one real zero in the interval  $]0, 1[$ . On the other hand we have just proved above that we have precisely one complex zero in the unit disc, and since we have found a real zero, this is the only zero in the unit disc.

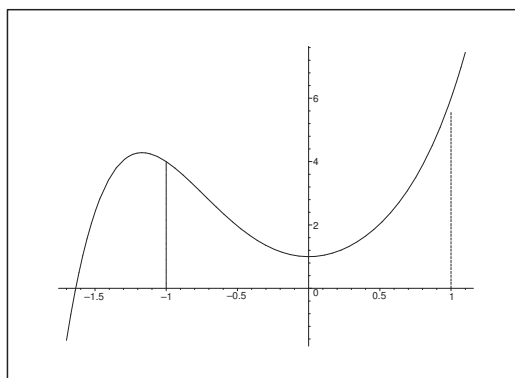
**Example 2.13** *Find the number of roots of the polynomial  $1 + 4z^2 + z^5$  inside the unit disc? Are they real? Does the polynomial have real roots?*

Put  $f(z) = 4z^2$  and  $g(z) = 1 + z^5$ . If  $|z| = 1$ , then we get the estimate

$$|g(z)| = |1 + z^5| \leq 1 + 1 = 2 < 4 = |4z^2| = |f(z)|.$$

Thus we conclude by Rouché's theorem that

$$f(z) + g(z) = 1 + 4z^2 + z^5 \quad \text{and} \quad f(z) = 4z^2$$

Figure 2: The graph of  $f(x) = 1 + 4x^2 + x^5$ .

have the same number of zeros inside the unit disc, i.e. 2.

Then consider the real function

$$f(x) = 1 + 4x^2 + x^5, \quad x \in [-1, 1].$$

If  $x = 0$ , then  $f(0) = 1 > 0$ , and if  $x \in [-1, 1] \setminus \{0\}$ , then

$$f(x) = 1 + 4x^2 + x^5 > 1 + 0 - 1 = 0,$$

thus  $f(x) \neq 0$  for  $x \in [-1, 1]$ . We therefore conclude that no zero in the unit disc is real.

Finally, consider

$$f(x) = 1 + 4x^2 + x^5, \quad \text{for } x \in \mathbb{R}.$$

Then clearly,

$$f(x) \rightarrow -\infty \quad \text{for } x \rightarrow -\infty,$$

and

$$f(x) \rightarrow +\infty \quad \text{for } x \rightarrow +\infty.$$

Since  $f(x)$  is real and continuous,  $f(x)$  has (at least) one real zero.

**Example 2.14** Prove for  $n > 2$  that all roots of

$$4z^n - z^2 - z - 1 = 0$$

lie in the unit disc.

Put  $f(z) = 4z^n$  and  $g(z) = -z^2 - z - 1$ . If  $|z| = 1$ , then we get the estimates

$$|g(z)| = |-z^2 - z - 1| \leq 1 + 1 + 1 = 3 < 4 = |4z^n|,$$

and it follows from Rouché's theorem that

$$f(z) + g(z) = 4z^n - z^2 - z - 1 \quad \text{and} \quad f(z) = 4z^n$$

have the same number of zeros inside  $|z| = 1$ , i.e.  $n$  zeros.

**Remark 2.3** The argument is unchanged for  $n = 2, 1$  and  $0$ , and the conclusion is in general that we have  $n$  roots. It is, however, obvious in the latter three cases that we can reduce the equation, so we do not have to use Rouché's theorem in these cases.  $\diamond$

**Example 2.15** Find  $r > 0$ , such that the polynomial

$$z^3 - 4z^2 + z - 4$$

has precisely two roots inside the circle  $|z| = r$ .

It follows by inspection that

$$z^3 - 4z^2 + z - 4 = (z - 4)(z^2 + 1),$$

so the roots are  $4$  and  $\pm i$ , and it follows that every  $r \in ]1, 4[$  can be used.

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If we instead try to apply Rouché's theorem, then it is natural to put

$$f(z) = -4z^2 \quad \text{and} \quad g(z) = z^3 + z - 4.$$

We get the following estimates for  $|z| = r$ ,

$$|g(z)| \leq r^3 + r + 4 \quad \text{and} \quad |f(z)| = 4r^2.$$

(Note the “change of sign” by the estimate of  $|g(z)|$ ).

If we want to be absolutely certain that we have exactly two zeros inside  $|z| = r$ , then we shall choose  $r$ , such that

$$r^3 + r + 4 < 4r^2.$$

Due to the terms  $r^3$  and 4 on the left hand side it is obvious that we must have  $r < 4$  and  $r > 1$ . If we try  $r = 2$ , we get

$$r^3 + r + 4 = 8 + 2 + 4 = 14 < 16 = 4r^2,$$

so we may use  $r = 2$ .

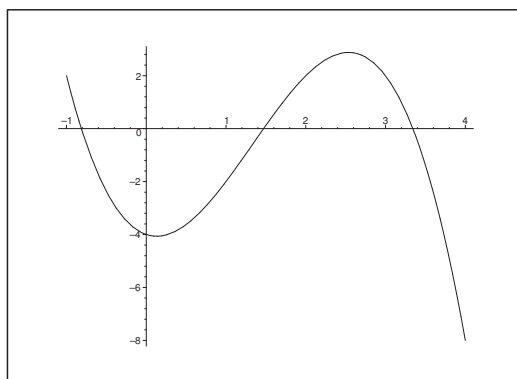


Figure 3: The graph of  $4r^2 - r^3 - r - 4$ .

**Remark 2.4** If we try to find the zeros of  $4r^2 - r^3 - r - 4$ , then we have one in the interval  $]1, 2[$ , and one in  $]3, 4[$ , cf. the figure. We cannot get the precise result  $r \in ]1, 4[$  by only using Rouché's theorem. This is due to the fact that we have estimated  $|g(z)|$  above, and we are now computing the exact number.  $\diamond$



**Example 2.16** Find the number of roots of

$$z^8 - 5z^4 - 2z - 1$$

inside  $|z| = 1$ .

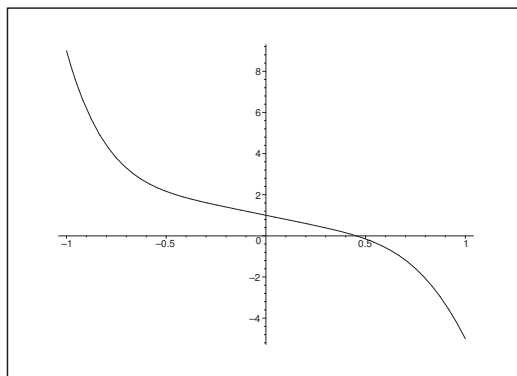


Figure 4: The graph of  $x^8 - 5x^5 - 2x + 1$  for  $x \in [-1, 1]$ .

Since the boundary curve is given by  $|z| = 1$ , it is almost obvious that we shall seek the numerically larger coefficient and isolate the corresponding term. Hence, we try to apply Rouché's theorem on the functions

$$f(z) = -5z^5 \quad \text{and} \quad g(z) = z^8 - 2z + 1,$$

where

$$f(z) + g(z) = z^8 - 5z^5 - 2z + 1.$$

If  $|z| = 1$ , then we have the estimates

$$|f(z)| = |-5z^5| = 5,$$

and

$$|g(z)| = |z^8 - 2z + 1| \leq |z|^8 + 2|z| + 1 = 4 < 5 = |f(z)|.$$

(Notice that we in the estimation of  $g(z)$  write plus everywhere between the terms.) This proves that  $f(z)$  is dominating. Now,  $f(z)$  has a five-tuple zero at  $z = 0$  and no other zero inside  $|z| = 1$ , so it follows from Rouché's theorem that at

$$z^8 - 5z^5 - 2z + 1$$

has 5 roots inside  $|z| = 1$ .

**Remark 2.5** The polynomial has real coefficients, so the non-real roots are pairwise complex conjugated. The unit disc  $\{z \in \mathbb{C} \mid |z| \leq 1\}$  is symmetric with respect to the  $x$ -axis, so it follows that we must have 1, 3 or 5 roots inside  $|z| = 1$ . When we consider the graph of the restriction to the real interval  $[-1, 1]$ , we see that we have precisely one root inside  $|z| = 1$ .  $\diamond$

**Example 2.17** Find the number of roots of the polynomial

$$z^{87} + 36z^{57} + 71z^4 + z^3 - z + 1$$

(a) in the domain  $|z| < 1$ , (b) in the domain  $|z| < 2$ .

**Remark 2.6** Due to the large exponents a graphical solution of the problem will be very difficult, even in MAPLE.  $\diamond$

(a) Put

$$f_1(z) = 71z^4 \quad \text{and} \quad g_1(z) = z^{87} + 36z^{57} + z^3 - z + 1.$$

If  $|z| = 1$ , then we get the estimate

$$|g_1(z)| \leq 1 + 36 + 1 + 1 + 1 = 40 < 71 = |f_1(z)|.$$

Using Rouché's theorem we conclude that  $f_1(z)$  and  $f_1(z) + g_1(z)$  have the same number of roots inside  $|z| = 1$ . Since  $f_1(z) = 71z^4$  only has the four-tuple root  $z = 0$ , we conclude that

$$z^{87} + 36z^{57} + 71z^4 + z^3 - z + 1$$

has four roots in the set  $\{z \in \mathbb{C} \mid |z| < 1\}$ .

(b) Put

$$f_2(z) = z^{87} \quad \text{and} \quad g_2(z) = 36z^{57} + 71z^4 + z^3 - z + 1,$$

If  $|z| = 2$ , then we have the estimates

$$|g_2(z)| \leq 36 \cdot 2^{57} + 71 \cdot 2^4 + 2^3 + 2 + 1 < 64 \cdot 2^{57} = 2^{63} < 287 = |f_2(z)|,$$

so all roots lie inside  $|z| = 2$ . Hence the number of roots is 87.

**Remark 2.7** Put

$$f_3(z) = 1 \quad \text{and} \quad g_3(z) = z^{87} + 36z^{57} + 71z^4 + z^3 - z + 1.$$

Then we prove in the same way that

$$|g_3(z)| < |f_3(z)| = 1 \quad \text{for } |z| = 0, 3,$$

cf. the figure, thus  $f_3(z)$  and  $f_3(z) + g_3(z)$  have the same number of roots inside  $|z| = \frac{3}{10}$ , i.e. no zero at all. Hence there must lie 4 roots in the annulus

$$\left\{ z \in \mathbb{C} \mid \frac{3}{10} < |z| < 1 \right\},$$

Furthermore, all 87 roots lie inside  $|z| = 1, 13$  (which can even be improved to  $|z| = 1, 1271$  by using a pocket calculator). In fact, if  $|z| = 1, 13$ , then we have the estimates

$$|z^{87}| > 41\,400 > 40\,800,$$

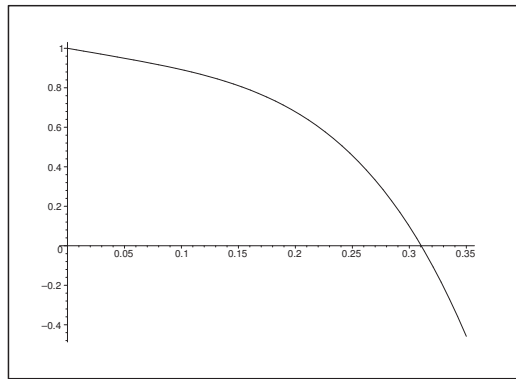


Figure 5: The graph of  $1 - \{r^{87} + 36r^{57} + 71r^4 + r^3 + r + 1\}$ .

and

$$|36z^{57}| + |71z^4| + |z^3| + |z| + 1 < 38\,800 + 116 + 2 + 1 + 1 < 40\,800,$$

and then we apply Rouché's theorem as above. We see in particular that we have  $87 - 4 = 83$  roots in the annulus

$$\left\{ z \in \mathbb{C} \mid 1 < |z| < \frac{113}{100} \right\}.$$



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If we consider  $f_4(z) = 36z^{57}$  instead, then we prove in the same way that there are 57 roots in the disc

$$\left\{ z \in \mathbb{C} \mid |z| < \frac{111}{100} \right\},$$

so we have  $57 - 4 = 53$  roots in the annulus

$$\left\{ z \in \mathbb{C} \mid 1 < |z| < \frac{111}{100} \right\},$$

and  $87 - 57 = 30$  roots in the annulus

$$\left\{ z \in \mathbb{C} \mid \frac{111}{100} < |z| < \frac{113}{100} \right\}.$$

There is only one real root, and one can with some difficulty prove that it is  $\approx -1,012$ . This example shows that it is not an easy task to find the zeros of polynomials of a large degree. There exists, however, a method, by which one can separate roots which lie close to each other and then find the roots. However, this lies beyond the scope of these notes.  $\diamond$

**Example 2.18** Find the number of zeros of the polynomial

$$2z^5 - 6z^2 + z + 1$$

in the closed annulus  $1 \leq |z| \leq 2$ .

First consider the boundary  $|z| = 2$ . We put

$$f_1(z) = 2z^5 \quad \text{and} \quad g_1(z) = -6z^2 + z + 1,$$

and find that

$$|g_1(z)| \leq 6 \cdot 4 + 2 + 1 = 27 < 64 = |f_1(z)|, \quad \text{for } |z| = 2.$$

Using Rouché's theorem we see that  $2z^5 - 6z^2 + z + 1$  and  $f_1(z) = 2z^5$  must have the same number of zeros inside  $|z| = 2$ , thus  $2z^5 - 6z^2 + z + 1$  has all its five zeros lying inside the disc  $\{z \in \mathbb{C} \mid |z| \leq 2\}$ .

Then we consider  $|z| = 1$ . Here we put

$$f_2(z) = -6z^2 \quad \text{and} \quad g_2(z) = 2z^5 + z + 1.$$

Then we have the estimates

$$|g_2(z)| \leq 2 + 1 + 1 = 4 < 6 = |f_2(z)|, \quad \text{for } |z| = 1.$$

An application of Rouché's theorem gives that  $2z^5 - 6z^2 + z + 1$  and  $f_2(z) = -6z^2$  have the same number of zeros inside  $|z| = 1$ , hence  $2z^5 - 6z^2 + z + 1$  has two zeroes lying in  $\{z \in \mathbb{C} \mid |z| < 1\}$ .

Summing up it follows from the above that  $2z^5 - 6z^2 + z + 1$  has  $5 - 2 = 3$  zeros in the closed annulus

$$\{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}.$$

**Remark 2.8** The degree of the polynomial is so small that modern computers easily can find the roots. We have approximatively the roots

$$-0,33172, \quad 0,51413, \quad 1,32755, \quad -0,75498 \pm 1,27999i.$$

We see that all roots lie inside  $|z| = \frac{3}{2}$ . By an elaboration of the estimate above it is possible by Rouché's theorem to prove that all roots satisfy  $|z| < \frac{16}{10}$ , but one cannot derive in this way that even  $|z| < \frac{3}{2}$ .

An analogous argument shows that there are no root inside  $|z| = \frac{3}{10}$ .  $\diamond$

**Example 2.19** Prove that the equation  $e^z = 2z + 1$  has precisely one solution in the disc  $|z| < 1$ .

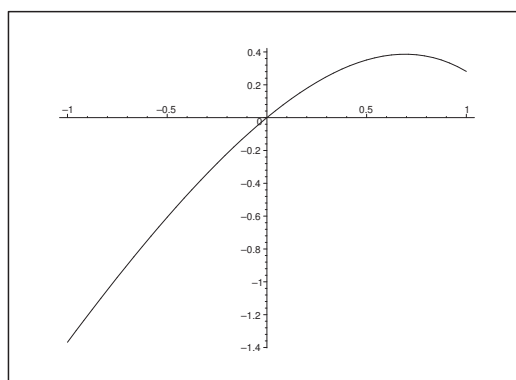


Figure 6: The graph of  $2x + 1 - e^x$  for  $x \in [-1, 1]$ .

Clearly,  $z = 0$  is a solution. Then it follows by the figure that this is the only *real* solution in the interval  $[-1, 1]$ , i.e. in the unit disc. We shall prove that there does not exist any other *complex* solution.

Put

$$f(z) = 2z + 1 \quad \text{and} \quad g(z) = -e^z.$$

The idea is of course to apply Rouché's theorem to prove that

$$f(z) + g(z) = 2z + 1 - e^z \quad \text{and} \quad f(z) = 2z + 1$$

have the same number of zeros inside  $|z| = 1$ , i.e. one. Now,

$$|f(z)| \geq |2z| - 1 = 1 \quad \text{and} \quad |g(z)| < e,$$

so it is not possible to use the usual trick. Instead we are forced to the following pointwise estimate.

Let  $|z| = 1$ , i.e.  $x^2 + y^2 = 1$ . Then

$$g(z) = e^x, \quad x \in [-1, 1],$$

and

$$\begin{aligned} |f(z)| &= |2z + 1| = \sqrt{(2x + 1)^2 + 4y^2} = \sqrt{4x^2 + 4x + 1 + 4y^2} \\ &= \sqrt{4(x^2 + y^2) + 4x + 1} = \sqrt{5 + 4x}, \quad \text{for } x^2 + y^2 = 1. \end{aligned}$$

The claim will be proved, if we can prove that

$$e^x < \sqrt{5 + 4x} \quad \text{for } x \in [-1, 1],$$

or, in an equivalent formulation,

$$e^{2x} < 5 + 4x \quad \text{for } x \in [-1, 1].$$

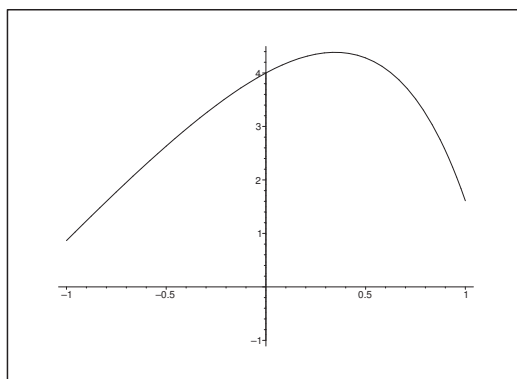


Figure 7: The graph of  $\varphi(x) = 5 + 4x - e^{2x}$ ,  $x \in [-1, 1]$ .

Put

$$\varphi(x) = 5 + 4x - e^{2x}, \quad \text{for } x \in [-1, 1].$$

Then

$$\varphi'(x) = 4 - 2e^{2x} = 2(2 - e^{2x}),$$

and we conclude that  $\varphi(x)$  is increasing in  $\left[-1, \frac{1}{2} \ln 2\right]$  and decreasing in  $\left[\frac{1}{2} \ln 2, 1\right]$ . We conclude from

$$\varphi(-1) = 0, \quad \varphi(1) = 9 - e^2 > 0,$$

that  $\varphi(x) > 0$ . This implies that

$$e^{2x} < 5 + 4x, \quad \text{for } x \in [-1, 1],$$

as wanted. Then by the results above,

$$|f(z)| > |g(z)| \quad \text{for } |z| = 1,$$

and it follows from Rouché's theorem that the equation  $e^z = 2z + 1$  has precisely one solution in the open unit disc  $|z| < 1$ .



**Example 2.20** Given an analytic function  $f$  which maps the closed unit disc  $|z| \leq 1$  onto a set  $D$ , which is contained in the interior of this disc. Prove that  $f$  has precisely one fix point in  $|z| \leq 1$ , which means that there exists precisely one  $z$ ,  $|z| \leq 1$ , such that  $f(z) = z$ .

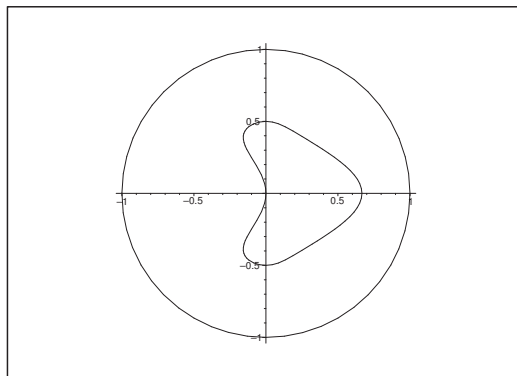


Figure 8: An example of  $D$ .

Consider the function

$$g(z) = z - f(z).$$

Since boundary is mapped into boundary, we have  $|f(z)| < 1$  for  $|z| = 1$ , and the winding number for  $g(z)$  around 0 is therefore by the *argument principle* and *Rouché's theorem* equal to the winding number of  $z$  around 0 along  $|z| = 1$ , i.e. is 1.

Since  $g(z)$  does not have poles in the unit disc, there must be precisely one zero of  $g(z)$  in the unit disc  $|z| \leq 1$ , i.e. there exists exactly one  $z_0$  where  $|z_0| < 1$ , such that

$$g(z_0) = z_0 - f(z_0) = 0,$$

hence by a rearrangement,

$$f(z_0) = z_0,$$

and  $z_0$  is a fix point of  $f(z)$ .

**Example 2.21** Find the location of the zeros of the polynomial  $z^4 - z + 5$ .

The polynomial  $z^4 - z + 5$  has real coefficients, so its roots are either real or pairwise complex conjugated. It is seen from the graph that the polynomial cannot have real roots in the interval  $[-2, 2]$ .

If  $|z| = 1$ , then we choose  $f(z) = 5$  and  $g(z) = z^4 - z$ . Then we get the estimate

$$|g(z)| = |z^4 - z| \leq 2 < 5 = |f(z)| \quad \text{for } |z| = 1.$$

Since  $f(z) = 5$  does not have zeros, we conclude from Rouché's theorem that  $z^4 - z + 5$  does not have any zero in the unit disc  $\{z \in \mathbb{C} \mid |z| \leq 1\}$ .

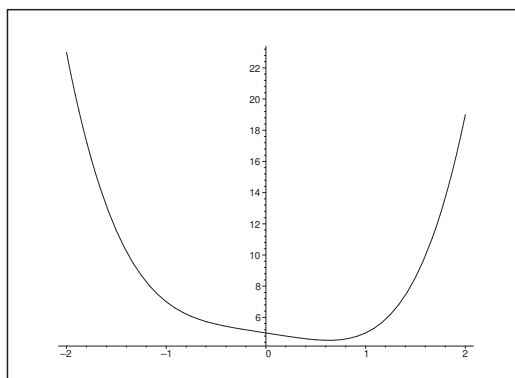


Figure 9: The graph of  $x^4 - x + 5$  for  $x \in [-2, 2]$ .

If  $|z| = 2$ , we instead choose  $f(z) = z^4$  and  $g(z) = -z + 5$ . Then we get the estimate

$$|g(z)| = |-z + 5| \leq |-z| + 5 = 7 < 16 = 2^4 = |f(z)| \quad \text{for } |z| = 2.$$

Hence, by Rouché's theorem, all four roots lie inside  $|z| = 2$ . By an elaboration of the estimate one can show that all roots lie inside  $|z| = 1, 7$ .

Summing up the results above we conclude that all roots lie in the annulus

$$\{z \in \mathbb{C} \mid 1 < |z| < 1, 7\},$$

and that they are pairwise complex conjugated.

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**Example 2.22** Apply Rouché's theorem to find the quadrants, in which the zeros of  $z^4 + iz^2 + 2$  are lying, and find the number of zeros which lie inside circles of centrum 0 and various radii. We may check our results by finding the roots directly.

**Remark 2.9** The main purpose of this example is of course to show a new technique. Therefore, we have chosen an example, which can be solved explicitly, so one may check the results.  $\diamond$

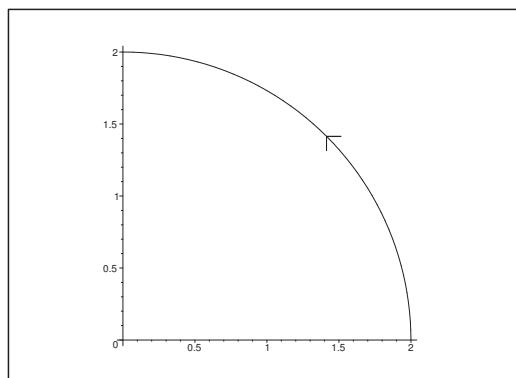


Figure 10: The closed curve  $C_R$  for  $RH = 2$ .

Consider the first quadrant. Let  $C_R$  be the closed curve, which is sketched on the figure with  $R = 2$ . It  $z = it$  runs through the vertical segment from  $iR$  to 0, this curve is mapped into the graph of

$$f(it) = t^4 - it^2 + 2, \quad t \in [0, R],$$

i.e. into a parabola.

If  $z = t$  runs through the real interval  $[0, R]$ , the image curve is the graph of

$$f(t) = t^4 + it^2 + 2, \quad t \in [0, R],$$

i.e. a parabola.

If  $z = Re^{i\theta}$ ,  $\theta \in [0, \frac{\pi}{2}]$ , runs throught the circular arc, then the image curve is the graph of

$$f(Re^{i\theta}) = R^4 e^{4i\theta} + iR^2 e^{2i\theta} + 2,$$

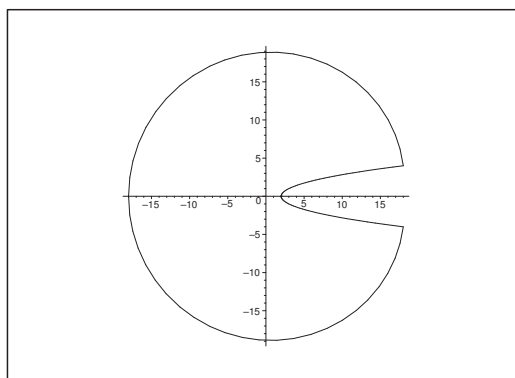
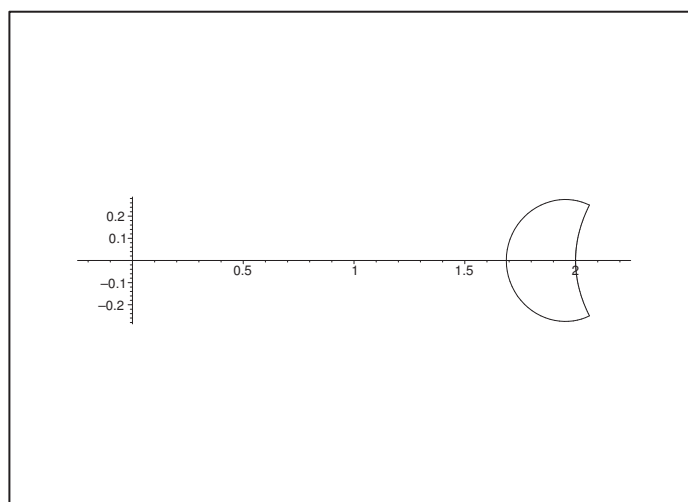
where

$$\operatorname{Im} \{ f(Re^{i\theta}) \} = R^4 \sin 4\theta + R^2 \cos 2\theta = (2R^2 \sin 2\theta + 1) R^2 \cos 2\theta,$$

which is only zero for  $\theta = \frac{\pi}{4}$ . This corresponds to

$$f\left(R \exp\left(i\frac{\pi}{4}\right)\right) = R^4 e^{i\pi} + iR^2 \exp\left(i\frac{\pi}{2}\right) + 2 = 2 - R^2 - R^4 < 0 \quad \text{for } R > 1.$$

Now  $\sin 2\theta > 0$  for  $\theta \in ]0, \frac{\pi}{2}[$ , so this is the only crossing of the real axis for  $R > 1$ , and it follows by the figure that the winding number around 0 is 1, when  $R > 1$ .

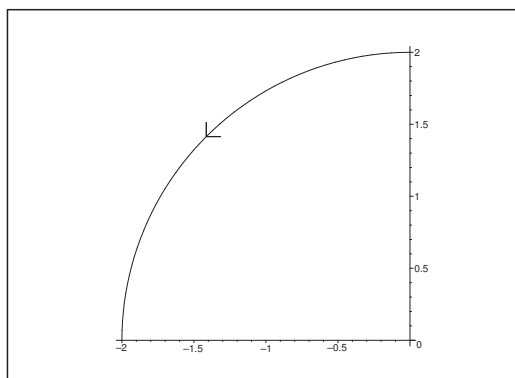
Figure 11: The image curve  $f(C_R)$  for  $R = 2$ .Figure 12: The image curve  $f(C_R)$  for  $R = \frac{1}{2}$ .

An analogous analysis shows that the winding number is 0 around 0, when  $R < 1$ , cf. the figure. Summing up we conclude from Rouché's theorem that there is precisely one zero in the first quadrant, and that it is even given by  $|z| = 1$  and  $\text{Arg } z = \frac{\pi}{4}$ , i.e.

$$z = \frac{1}{\sqrt{2}}(1 + i).$$

In the second quadrant we consider the curve  $C_R$  as shown on the figure for  $R = 2$ . The axes are mapped into the same curves as above, only taken in the opposite direction. On the circular arc we also get as before,

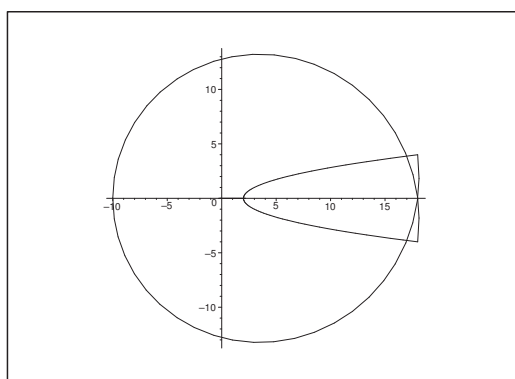
$$f(Re^{i\theta}) = R^4 e^{i4\theta} + i R^2 e^{2i\theta} + i R^2 e^{2i\theta} + 2, \quad \theta \in \left[\frac{\pi}{2}, \pi\right].$$

Figure 13: The curve  $C_R$  in the second quadrant for  $R = 2$ .

Here

$$\operatorname{Im} \{f(R e^{i\theta})\} = (2R^2 \sin 2\theta + 1) R^2 \cos 2\theta$$

is equal to 0, if either  $\theta = \frac{3\pi}{4}$ , or  $\sin 2\theta = -\frac{1}{2R^2}$ , for  $R > \frac{1}{\sqrt{2}}$ .

Figure 14: The image curve  $f(C_R)$  of the curve in the second quadrant for  $R = 2 > \frac{1}{\sqrt{2}}$ .

If  $\theta = \frac{3\pi}{4}$ , and  $R > \sqrt{2}$ , then

$$f\left(R \exp\left(i \frac{3\pi}{4}\right)\right) = R^4 e^{3i\pi} + i R^2 e^{3i \frac{\pi}{2}} + 2 = -R^4 + R^2 + 2 < 0.$$

If  $R > \sqrt{2}$  and  $\sin 2\theta = -\frac{1}{2R^2}$ , then

$$\operatorname{Im} \{f(R e^{i\theta})\} = 0,$$

hence

$$\begin{aligned} f(Re^{i\theta}) &= \operatorname{Re}\{f(Re^{i\theta})\} = R^4 \cos 4\theta - R^2 \sin 2\theta + 2 = R^4 (1 - 2 \sin^2 2\theta) - R^2 \sin 2\theta + 2 \\ &= R^4 \left\{1 - 2 \cdot \frac{2}{4R^4}\right\} - R^2 \cdot \frac{1}{2R^2} + 2 = R^4 - \frac{1}{2} - \frac{1}{2} + 2 = R^4 + 1. \end{aligned}$$

A sketch of the image curve for  $R > \sqrt{2}$  is given on the figure with  $R = 2$ . The winding number is 1, so we have one root in the second quadrant for  $|z| \geq \sqrt{2}$ .

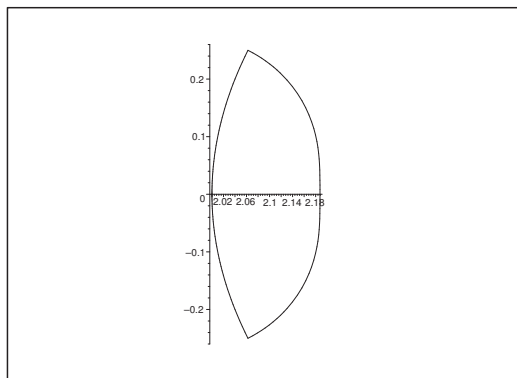


Figure 15: The image curve  $f(C_R)$  of the curve in the second quadrant for  $R = \frac{1}{2} < \sqrt{2}$ .

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An analogous analysis shows that if  $0 < R < \sqrt{2}$ , then the winding number is 1 around 0. Hence we conclude that there is precisely one root in the second quadrant, and it fulfils  $|z| = \sqrt{2}$ . The argument variation shows that the zero must satisfy  $\text{Arg } z = \frac{3\pi}{4}$ , so we have even found the root

$$z = \sqrt{2} \cdot e^{i \frac{3\pi}{4}} = -1 + i.$$

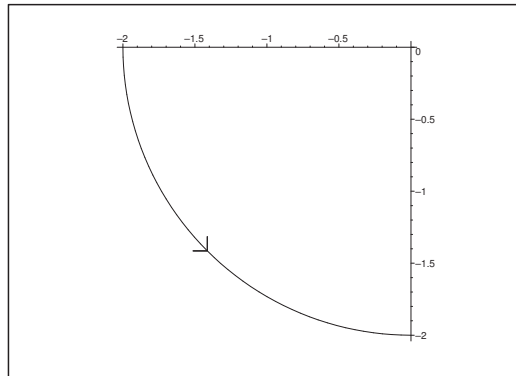


Figure 16: The curve  $C_R$  in the third quadrant for  $R = 2$ .

In the third quadrant the images of the axes are the usual parabolic arcs where the direction is as in the first quadrant. It follows from

$$f(Re^{i\theta}) = R^4 e^{4i\theta} + i R^2 e^{2i\theta} + 2, \quad \theta \in \left[\pi, \frac{3\pi}{2}\right],$$

and

$$\text{Im} \{f(Re^{i\theta})\} = (2R^2 \sin 2\theta + 1) R^2 \cos 2\theta,$$

where  $\sin 2\theta > 0$ , that the image is the same as in the first quadrant, so there is only one zero in the third quadrant, and it lies on the circle  $|z| = 1$  and satisfies  $\text{Arg } z = -\frac{3\pi}{4}$ , so

$$z = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}.$$

Finally, the discussion of the fourth quadrant is identical with the discussion of the second quadrant, so we have here precisely one root which lies on the circle  $|z| = \sqrt{2}$  and satisfies  $\text{Arg } z = -\frac{\pi}{4}$ , thus

$$z = 1 - i.$$

**Remark 2.10** Finally, we solve the equation completely by noting that it is an equation of second order in  $z^2$ ,

$$z^4 + i z^2 + 2 = (z^2)^2 + i (z^2) + 2 = 0.$$

Hence

$$z^2 = \frac{-i \pm \sqrt{-1-8}}{2} = \frac{-i \pm 3i}{2} = \begin{cases} i, \\ -2i. \end{cases}$$



The solutions are then

$$z = \pm \frac{1}{\sqrt{2}}(1 + i) \quad \text{and} \quad z = \pm(1 - i),$$

i.e. the same results as found previously in a somewhat harder way.  $\diamond$

**Remark 2.11** Even if this method is a little difficult, it may be successful in cases, when one cannot find the exact solutions. It is of course a coincidence that we here can find the roots by either of the two methods.  $\diamond$

**Example 2.23** Find the number of zeros of

$$z^4 + z^3 + 5z^2 + 2z + 4$$

in the first quadrant.

HINT: Use the argument principle on a curve  $C_R$ , which is composed of the line segments from  $iR$  to  $0$ , from  $0$  to  $R$  and the circular arc  $Re^{i\theta}$ ,  $0 < \theta < \frac{\pi}{2}$ , for  $R$  sufficiently large.

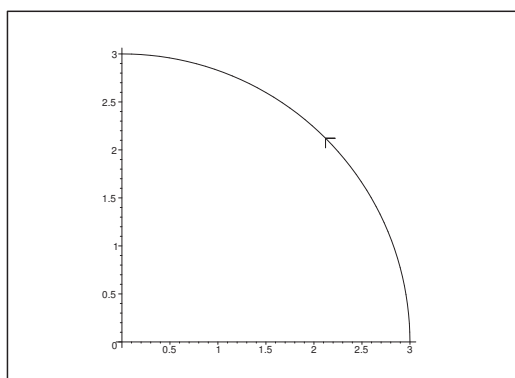


Figure 17: The closed curve  $C_R$  in the first quadrant for  $R = 3$ .

We shall find the winding number around 0 for the image curve  $f(C_R)$ .

We have on the real axis,

$$f(t) = t^4 + t^3 + 5t^2 + 2t + 4 > 0 \quad \text{for } t \in [0, R].$$

Then we get on the circular arc  $z = Re^{i\theta}$ ,  $\theta \in [0, \frac{\pi}{2}]$ , when  $R$  is sufficiently large, i.e. when

$$R^3 + 5R^2 + 2R + 4 < R^4,$$

e.g. when  $R = 3$ , that the curve  $f(Re^{i\theta})$  is a perturbation of the curve

$$(Re^{i\theta})^4 = R^4 e^{4i\theta}, \quad \theta \in [0, \frac{\pi}{2}],$$

thus  $f(Re^{i\theta})$  starts at a point on  $\mathbb{R}_+$ , proceeds into the first quadrant, through the second quadrant and further through the third quadrant, and it ends in the fourth quadrant (possibly with a continuation into the first quadrant).

Finally,

$$f(it) = (it)^4 + (it)^3 + 5(it)^2 + 2it + 4 = t^4 - 5t^2 + 4 + it(2 - t^2), \quad t \in [0, R],$$

so it follows that  $f(it) = 0$  for  $t = 0$  or  $t = \pm\sqrt{2}$ . Hence, if  $R > \sqrt{2}$ , then  $f(it)$  is real at an interior point of  $]0, R[$  only when  $R = \sqrt{2}$ , corresponding to the intersection point

$$f(i\sqrt{2}) = (\sqrt{2})^4 - 5(\sqrt{2})^2 + 4 = 4 - 5 \cdot 2 + 4 = -2$$

on the real axis.

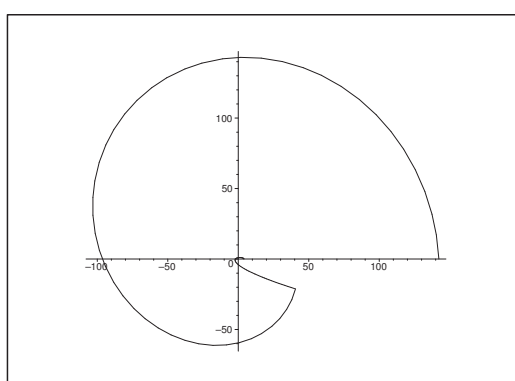


Figure 18: The curve  $f(C_R)$  for  $R = 3$ .

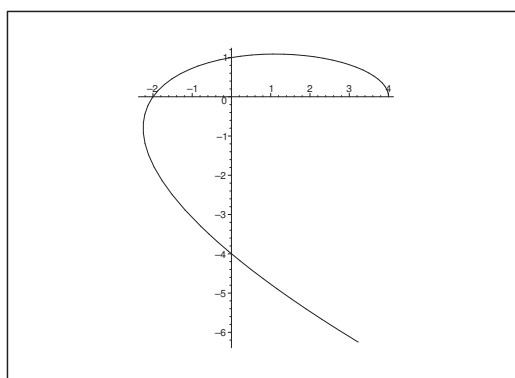


Figure 19: That part of the curve  $f(C_R)$  for  $R = 3$ , which lies in the neighbourhood of 0.

Since  $f(C_R)$  is a closed curve, it is in principle like the sketch on the figure for  $R = 3$ . We see that it is difficult to see what happens in a neighbourhood of 0, so we also give a closeup of the curve in a

neighbourhood of 0. It is seen that the winding number around 0 is 0, no matter how big  $R$  is. We therefore conclude that the polynomial does not have roots in the first quadrant.

**Remark 2.12** If we apply *Hurwitz's criterion* on the problem,

$$\begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ - & - & - & - \\ 2 & 5 & 1 & 1 \\ - & - & - & - \\ 0 & 4 & 2 & 5 \\ - & - & - & - \\ 0 & 0 & 0 & 4 \end{vmatrix},$$

then

$$D_1 = 1 > 0, \quad D_2 = 5 - 2 = 3 > 0, \quad D_3 = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 5 & 1 \\ 0 & 4 & 2 \end{vmatrix} = 2 > 0,$$

and  $D_4 = 8 > 0$ , and we conclude that all roots lie in the left half plane.

This also follows from that there are no roots in the first quadrant. In fact, since the polynomial has *real* coefficients, it follows by a conjugation that there are no root in the fourth quadrant either, so all roots must lie in the left half plane.  $\diamond$

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**Example 2.24** Prove that the equation  $z + e^{-z} = \lambda$  has precisely one solution (which of course is real) in the right half plane, when  $\lambda > 1$ .

HINT: Consider a half circle in the right half plane of centrum at 0 and radius  $R > \lambda + 1$ .

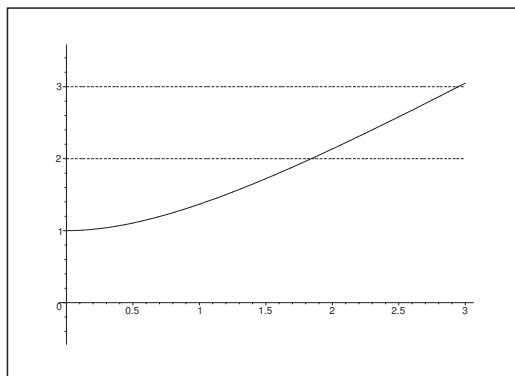


Figure 20: The graph of  $\varphi(x) = x + e^{-x}$  with  $\lambda = 2$  and  $\lambda = 3$ .

The real function

$$\varphi(x) = x + e^{-x}, \quad x \in \mathbb{R}_+,$$

has the derivative

$$\varphi'(x) = 1 - e^{-x} > 0, \quad \text{for } x \in \mathbb{R}_+,$$

so  $\varphi$  is increasing on  $\mathbb{R}_+$ . Now,  $\varphi(0) = 1$  and  $\varphi(x) \rightarrow +\infty$  for  $x \rightarrow +\infty$ , so it follows from the continuity of  $\varphi$  that every value  $\lambda \in ]1, +\infty[$  is attained exactly once on  $\mathbb{R}_+$ .

Then we shall prove that there do not exist any other *complex* solutions for  $\operatorname{Re}(z) > 0$ .

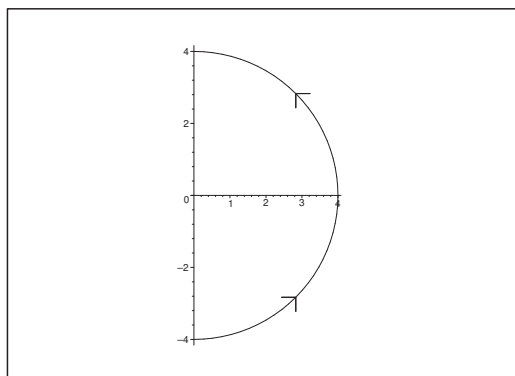


Figure 21: The curve  $C_R$  for  $R = 4$ .

Put  $f(z) = z + e^{-z} - \lambda$ , where  $\lambda > 1$ , and denote by  $C_R$  the boundary of the half disc in the right half plane of centrum 0 and radius  $R$ , where  $R > \lambda + 1$ , cf. the figure with  $R = 4$ .

The image curve of the vertical line segment of the parametric description

$$z(t) = i(R - t), \quad t \in [0, 2R],$$

is

$$f(i(R - t)) = i(R - t) + e^{-i(R-t)} - \lambda = \cos(R - t) - \lambda + i\{R - t - \sin(R - t)\}, \quad t \in [0, 2R].$$

It follows from  $\lambda > 1 \geq \cos(R - t)$ , that  $\operatorname{Re}\{f(i(R - t))\} < 0$ , so the image curve lies in the left half plane.

The image curve of the circular arc of the parametric description

$$z(\theta) = R e^{i\theta}, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

where  $R > \lambda + 1$ , is given by

$$f(R e^{i\theta}) = R e^{i\theta} + \exp(-R\{\cos \theta + i \sin \theta\}) - \lambda.$$

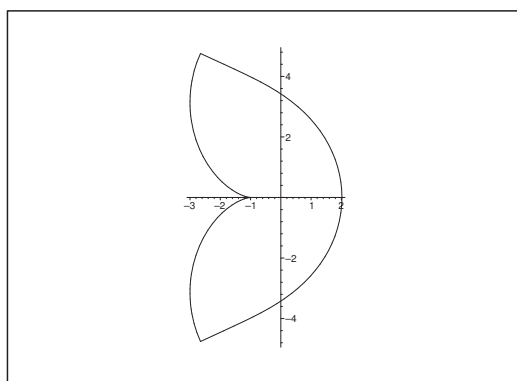


Figure 22: The image curve  $f(C_R)$  for  $\lambda = 2$  and  $R = 4$ .

The intersection points with the  $x$ -axis are given by

$$\operatorname{Im}\{f(R e^{i\theta})\} = R \cdot \sin \theta - e^{-R \cos \theta} \cdot \sin(R \cdot \sin \theta) = 0,$$

with the trivial solution  $\theta = 0$ . Since

$$|R \cdot \sin \theta| \geq |\sin(R \cdot \sin \theta)| > |e^{-R \cos \theta} \cdot \sin(R \cdot \sin \theta)|,$$

when  $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$ , it follows that  $\operatorname{Im}\{f(R e^{i\theta})\}$  has the same sign as  $R \cdot \sin \theta$ ,  $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ . Hence,  $\theta = 0$  gives us the only intersection point with the  $x$ -axis. This is given by

$$f(R) = R + e^{-R} - \lambda > R + 1 - \lambda > 0,$$

so the figure gives with  $\lambda = 2$  and  $R = 4$  the principal picture of the image curve  $f(C_R)$ . We conclude that  $f(C_R)$  for  $R > \lambda + 1$  has the winding number 1 around 0. Then it follows from the *argument principle* that  $z + e^{-z} = \lambda$  has precisely one solution in the right half plane, and the claim is proved.

**Example 2.25** *Prove that*

$$f(z) = a - 8z^2 + z^4 + e^{-z} \quad \text{for } a \in [1, +\infty[,$$

*has precisely two zeros in the right half plane.*

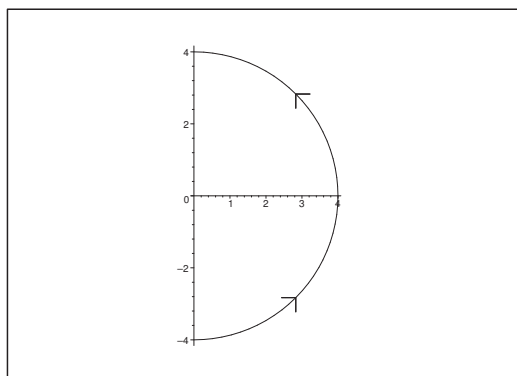


Figure 23: The closed curve  $C_R$  in the right half plane for  $R = 4$ .

Let  $C_R$  denote the closed curve in the right half plane consisting of a half circle of centrum at 0 and radius  $R$ , and a line segment on the imaginary axis. Then a parametric description of the line segment on the imaginary axis is given by

$$f(it) = a + 8 + t^4 + e^{-it} = 8 + a + t^4 + \cos t - i \sin t, \quad t \in [-R, R],$$

(but of course run through in the opposite direction).

Since  $\operatorname{Re}\{f(it)\} = 8 + a + t^4 + \cos t > 0$ , this curve segment lies entirely in the right half plane.

Then consider the circular arc of the parametric description

$$z(\theta) = R e^{i\theta}, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

where  $R$  is chosen large. It follows from

$$f(R e^{i\theta}) = a - 8R^2 e^{2i\theta} + R^4 e^{4i\theta} + e^{-R \cos \theta - i R \sin \theta},$$

that we have for  $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ , where  $\cos \theta > 0$ , the estimates

$$\begin{aligned} R^4 &= |R^4 e^{4i\theta}| > a + 8R^2 + e^{-R} \\ &= |a - 8R^2 e^{2i\theta} + e^{-R \cos \theta - i R \sin \theta}|, \end{aligned}$$

for  $R$  sufficiently large.

We conclude that  $f(R e^{i\theta})$  for  $R$  “large” can be considered as a small perturbation of

$$f_1(R e^{i\theta}) = R^4 e^{4i\theta}, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

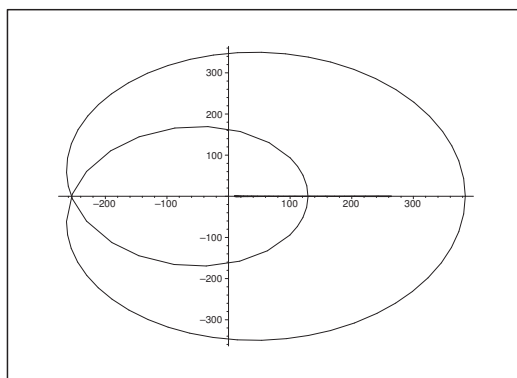


Figure 24: The image curve  $f(C_R)$  for  $a = 1$  and  $R = 4$ .

It follows from

$$4\theta \in [-2\pi, 2\pi] = 2[-\pi, \pi] \quad \text{for } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

that the latter curve has the winding number 2 around 0. Hence, the same is true for  $f(Re^{i\theta})$ . We conclude that  $8z^2 + z^4 + e^{-z}$  has precisely two zeros in the right half plane for  $a \in [1, +\infty[$ .

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**Example 2.26** Find the number of zeros in the right half plane of the polynomial

$$1 + z^2 - z^3.$$

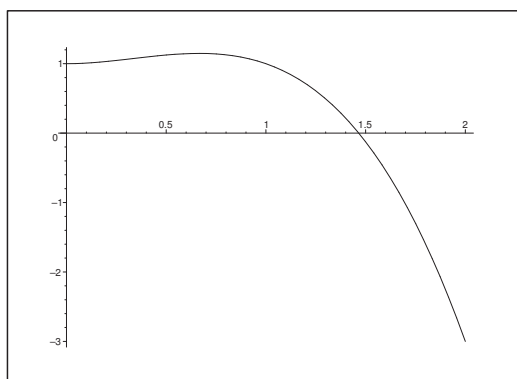


Figure 25: The graph of  $1 + x^2 - x^3$ ,  $x \in [0, 2]$ .

There is of course nothing wrong in first make a real analysis of the function. We see that we have a real root in the right half plane  $\approx 1.5$ . Then let  $C_R$  denote the curve, which is indicated on the figure in the special case  $R = 4$ .

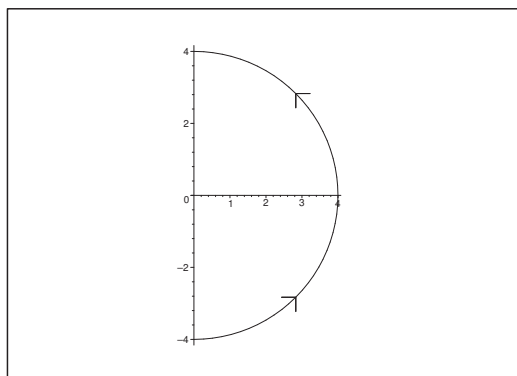


Figure 26: The closed curve  $C_R$  in the right half plane for  $R = 4$ .

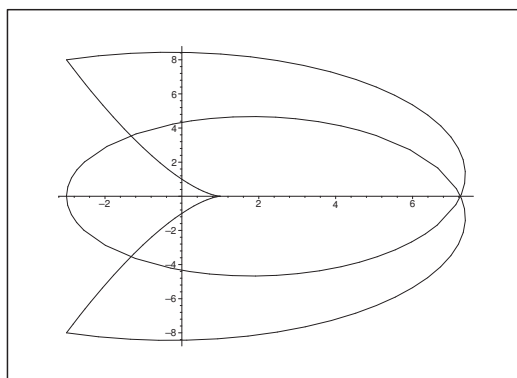
The task is to find the winding number of the image curve  $f(C_R)$  around 0 when  $R$  is large.

If  $z = it$ ,  $t \in [-R, R]$ , then

$$f(it) = 1 - t^2 + it^3, \quad t \in [-R, R].$$

Now,  $|z|^3 > 1 + |z|^2$  for  $|z| = R$  large, so the image curve by  $f$  of  $Re^{i\theta}$  is a perturbation of

$$-R^3 e^{3i\theta}, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Figure 27: The image curve  $f(C_R)$  for  $R = 2$ .

A principal sketch of the image curve  $f(C_R)$  is given on the figure as an example, where  $R = 2$ . This curve has the winding number 1 around 0, so we have only one zero in the right half plane. The polynomial has real coefficients and it is of odd degree, so this zero is real, and we have already found it on the graph in the beginning of the example.

**Example 2.27** 1) Prove that all roots of the polynomial

$$P(z) = (1 + 2i)z^4 - iz^3 + 3z + 4 + 3i$$

belong to the open disc  $|z| < 2$ .

- 2) Prove that both the real and the imaginary part of  $P(z)$  are positive on both the  $x$  and the  $y$  axes, where  $z = x + iy$ .
- 3) Prove that  $P(z)$  has a root in each of the quadrants of the  $z$ -plane.

1) Put

$$f(z) = (1 + 2i)z^4 \quad \text{and} \quad g(z) = -iz^3 + 3z + 4 + 3i.$$

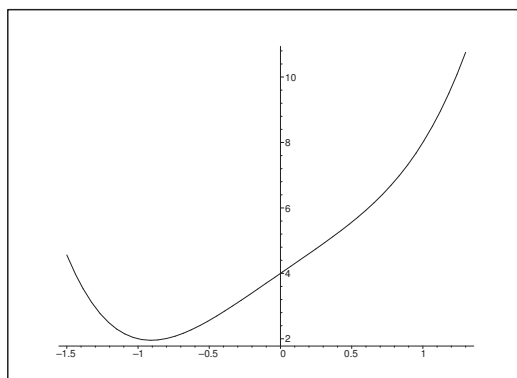
If  $|z| = 2$ , then

$$|f(z)| = |(1 + 2i)z^4| = \sqrt{5} \cdot 2^4 = 16\sqrt{5}, \quad |z| = 2,$$

and

$$|g(z)| \leq |-iz^3| + |3z| + 4 + |3i| = 8 + 6 + 4 + 3 = 21 < 32 < 16\sqrt{5} = |f(z)|.$$

Then it follows from Rouché's theorem that  $f(z)$  and  $P(z) = f(z) + g(z)$  have the same number of zeros inside the circle  $|z| = 2$ , i.e. four zeros. Then by the *Fundamental theorem of Algebra* all roots of the polynomial lie in the disc  $|z| < 2$ .

Figure 28: The graph of  $\varphi(x) = x^4 + 3x + 4$ .

2) By a computation we get for  $z = x \in \mathbb{R}$  that

$$P(x) = (1 + 2i)x^4 - ix^3 + 3x + 4 + 3i = \{x^4 + 3x + 4\} + i\{2x^4 - x^3 + 3\}.$$

Clearly,

$$\varphi(x) = x^4 + 3x + 4 > 0 \quad \text{for } x \geq -\frac{4}{3}.$$

Since

$$\varphi'(x) = 4x^3 + 3 \leq -4 \cdot \left(\frac{4}{3}\right)^3 + 3 < 0 \quad \text{for } x \leq -\frac{4}{3},$$

the function is decreasing for  $x \leq -\frac{4}{3}$ , so we conclude that  $\varphi(x) > 0$  on  $\mathbb{R}$ .

Clearly,

$$\psi(x) = 2x^4 - x^3 + 3 > 0 \quad \text{for } x \leq \sqrt[3]{3}.$$

If  $x > \sqrt[3]{3}$ , then

$$\psi(x) = x^3(2x - 1) + 3 > 0.$$

Hence, we conclude that  $\operatorname{Re}\{P(x)\}$  and  $\operatorname{Im}\{P(x)\}$  both are positive, when  $x \in \mathbb{R}$ , so the curve  $P(\mathbb{R})$  lies in the first quadrant.

If  $z = iy$ ,  $y \in \mathbb{R}$ , then

$$P(iy) = (1 + 2i)y^4 - i(iy)^3 + 3iy + 4 + 3i = \{y^4 - y^3 + 4\} + i\{2y^4 + 3y + 3\}.$$

We have

$$\varphi(y) = y^4 - y^3 + 4 > 0 \quad \text{for } |y| \geq 1,$$

because  $y^4 > |y|^3$  for  $|y| > 1$ . Furthermore,

$$\varphi(y) > 0 \quad \text{for } |y| \leq 1,$$

because  $|y|^3 < 4$  for  $|y| < 1$ .

If we put  $\psi(y) = 2y^4 + 3y + 3$ , then  $\psi'(y) = 8y^3 + 3$ , and we have locally (actually globally) a minimum corresponding to

$$y = -\sqrt[3]{\frac{3}{8}} \in ]-1, 1[.$$

Now

$$\psi(y) = 2y^4 + 3(y+1) > 0 \quad \text{for } y = -\sqrt[3]{\frac{3}{8}},$$

so we conclude that  $\psi(y) > 0$  for every  $y \in \mathbb{R}$ .

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Summing up we get that  $\operatorname{Re}\{P(iy)\}$  and  $\operatorname{Im}\{P(iy)\}$  are both positive for  $y \in \mathbb{R}$ , so the curve  $P(i\mathbb{R})$  lies in the first quadrant.

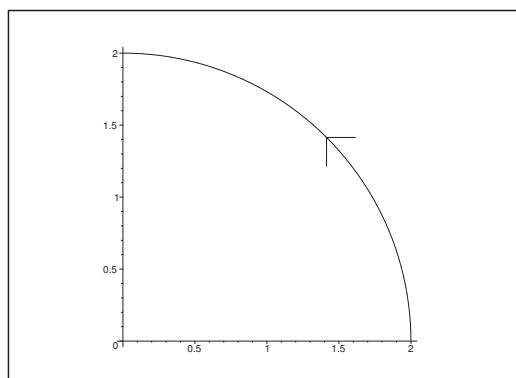


Figure 29: The closed curve  $C$ .

- 3) Since  $P(z) = f(z) + g(z)$ , where  $|g(z)| < |f(z)|$  on  $|z| = 2$ , we see that  $P(z)$  is only a perturbation of  $f(z) = (1 + 2i)z^4$  along  $|z| = 2$ . In particular, if  $C$  denotes the boundary of the quarter disc  $|z| \leq 2$ ,  $x \geq 0$ ,  $y \geq 0$ , then  $P(z)$  taken along  $|z| = 2$  starts in the first quadrant, goes once through the second, the third and the fourth quadrant, and finally ends in the first quadrant. Then it follows from (2) that the images of the line segments on the axes lie in the first quadrant. We conclude that the winding number is 1 around 0, so there is precisely one root in the first quadrant according to the argument principle.

Since the investigations in all the other quadrants are the same, we conclude that there lies one root in each quadrant.

**Example 2.28** Compute – possible by using Rouché’s theorem – the complex line integral

$$\oint_{|z|=1} \frac{3z^2 + 4 + e^z}{z^3 + 4z + e^z} dz.$$

If we put  $h(z) = z^3 + 4z + e^z$ , then  $h(z)$  is analytic in  $\mathbb{C}$ , and  $h'(z) = 3z^2 + 4 + e^z$ . Then

$$\oint_{|z|=1} \frac{3z^2 + 4 + e^z}{z^3 + 4z + e^z} dz = \oint_{|z|=1} \frac{h'(z)}{h(z)} dz = 2\pi i \{\text{number of zeros of } h(z) \text{ in } |z| < 1\}.$$

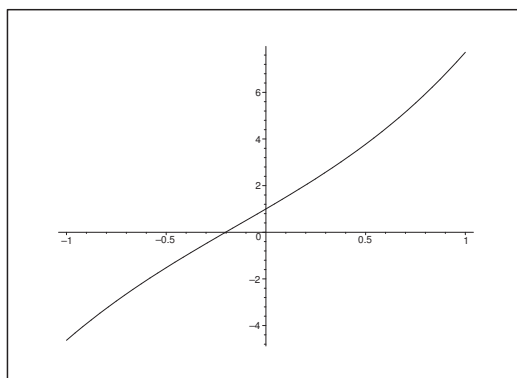
Then put  $f(z) = 4z$  and  $g(z) = z^3 + e^z$ . We have the following estimates on  $|z| = 1$ ,

$$|g(z)| \leq |z|^3 + |e^z| \leq 1 + e < 4 = |4z| = |f(z)|, \quad |z| = 1.$$

Hence, we conclude from Rouché’s theorem that  $h(z)$  and  $f(z)$  have the same number of zeros inside  $|z| = 1$ .

Since  $f(z) = 4z$  has only the simple zero  $z = 0$  inside  $|z| = 1$ , we conclude that this number is 1, so

$$\oint_{|z|=1} \frac{3z^2 + 4 + e^z}{z^3 + 4z + e^z} dz = 2\pi i.$$

Figure 30: The graph of  $x^3 + 4x + e^x$ ,  $x \in [-1, 1]$ 

**Remark 2.13** It follows by considering the graph that the zero of the denominator inside  $|z| = 1$  is real. The approximate

$$z_0 \approx -0,202\,172\,6$$

is then found by means of a pocket calculator.  $\diamond$

**Example 2.29** Compute – possible by using Rouché’s theorem – the complex line integral

$$\oint_{|z|=1} \frac{15z^2 + 2z + 1 - \sin z}{5z^3 + z^2 + z + \cos z} dz.$$

The integral is of the type

$$\oint_{|z|=1} \frac{h'(z)}{h(z)} dz,$$

and since  $h(z) = 5z^3 + z^2 + z + \cos z$  is analytic everywhere, the value of the line integral is according the *argument principle* equal to  $2\pi i$  times the number of zeros of  $h(z)$  inside  $|z| = 1$ .

In order to find this number we apply *Rouché’s theorem* with

$$f(z) = 5z^3 \quad \text{and} \quad g(z) = z^2 + z + \cos z.$$

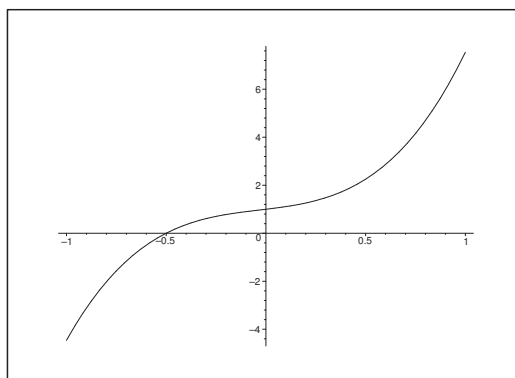
When  $|z| = 1$ , then  $|f(z)| = 5$  and

$$|g(z)| \leq 1 + 1 + |\cos z| \leq 2 + \cosh 1 < 2 + 2 = 4 < 5 = |f(z)|,$$

where we have used the estimate  $|\cos z| \leq \cosh 1$  for  $|z| \leq 1$ .

By *Rouché’s theorem*, the number of zeros of  $h(z) = f(z) + g(z)$  inside  $|z| = 1$  is equal to the number of zeros (counted by multiplicity) of  $f(z) = 5z^3$  inside  $|z| = 1$ , thus equal to 3. Hence by insertion,

$$\oint_{|z|=1} \frac{15z^2 + 2z + 1 - \sin z}{5z^3 + z^2 + z + \cos z} dz = 2\pi i \cdot 3 = 6\pi i.$$

Figure 31: The graph of  $5x^3 + x^2 + x + \cos x$ ,  $x \in [-1, 1]$ .

**Remark 2.14** It is seen by considering the graph that only one of the three roots inside  $|z| = 1$  is real.  $\diamond$

**Example 2.30** Given the polynomial

$$P(z) = z^3 + 6iz - 3.$$

- (a) Sketch the image of the  $y$ -axis by the map  $P$  and find the increase of the argument of  $P(z)$  on the  $y$ -axis, when this is run through from  $i\infty$  to  $-i\infty$ .
- (b) Let  $C_R$  denote the half circle of the parametric description

$$z(\theta) = Re^{i\theta}, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$$

and let  $\Delta_R \arg P$  denote the increase of the argument of  $P(z)$ , when  $z$  runs through the half circle from the point corresponding to the parameter  $\theta = -\frac{\pi}{2}$  to the point corresponding to the parameter  $\theta = \frac{\pi}{2}$ . Find

$$\lim_{R \rightarrow +\infty} \Delta_R \arg P.$$

- (c) The equation  $P(z) = 0$  has three complex roots. Use the argument principle to find the number of roots of positive real part – i.e. the number of roots lying in the right half plane.

- (a) A parametric description of the imaginary axis run through from  $i\infty$  to  $-i\infty$ , is e.g. given by

$$z(t) = -it, \quad t \in \mathbb{R}.$$

We get by insertion,

$$w = u + iv = P(-it) = (-it)^3 + 6i(-it) - 3 = it^3 + 6t - 3,$$



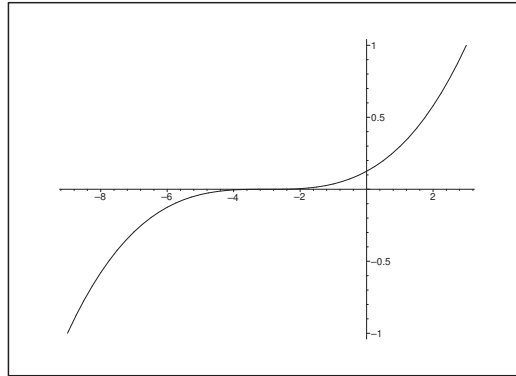


Figure 32: The image of the imaginary axis.

thus

$$(2) \quad u(t) = 6t - 3 \quad \text{and} \quad v(t) = t^3,$$

hence  $t = \frac{u+3}{6}$ , and therefore

$$v = \left( \frac{u+3}{6} \right)^3.$$



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The direction of the curve is given by (2), i.e. we start in the third quadrant, continue through the second quadrant and end in the first quadrant. Notice in particular that 0 lies below the curve.

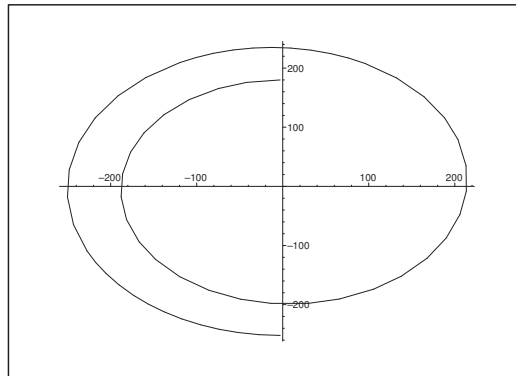


Figure 33: The image of the half circle  $C_R$ .

- (b) If  $R$  is large, then  $z^3$  dominates in  $P(z)$ , so the curve starts at  $P(-iR)$  in the first quadrant, follows  $R^3 e^{3i\theta}$  one and a half times around 0, and ends at  $P(iR)$  in the third quadrant. In particular,

$$\lim_{R \rightarrow +\infty} \Delta_R \arg P = \lim_{R \rightarrow +\infty} \Delta_R \arg \left( z^3 \left\{ 1 + \frac{6i}{z^2} - \frac{3}{z^2} \right\} \right) = \lim_{R \rightarrow +\infty} \Delta_R (z^3) = 3\pi.$$

- (c) When the two curves are put together, and since the image of the imaginary axis lies *above* 0, the winding number around 0 must be 1, when  $R$  is sufficiently large. We therefore conclude from the argument principle that there is one root in the right half plane.

**Remark 2.15** The roots of

$$z^3 + 6iz - 3 = z^3 + pz + q,$$

can in principle be found by *Cardano's formula*, where  $p = 6i$  and  $q = -3$ . In fact,

$$z = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} = \sqrt[3]{\frac{3}{2} + \frac{1}{2}\sqrt{9-32i}} + \sqrt[3]{\frac{3}{2} - \frac{1}{2}\sqrt{9-32i}},$$

where the cubic roots are chosen, such that

$$\sqrt[3]{\frac{3}{2} + \frac{1}{2}\sqrt{9-32i}} \cdot \sqrt[3]{\frac{3}{2} - \frac{1}{2}\sqrt{9-32i}} = -2i.$$

In *practice* this solution formula is usually inappropriate, because for example,

$$\sqrt{9-32i} = \pm \left\{ \sqrt{\frac{\sqrt{1105}+9}{2}} - i\sqrt{\frac{\sqrt{1105}-9}{2}} \right\}.$$

This shows why one in practice should avoid the application of *Cardano's formula*.  $\diamond$

**Example 2.31** *Given*

$$\varphi(z) = e^z - 3z^2.$$

Prove that  $\varphi(z)$  has two zeros  $z_1$  and  $z_2$  in the domain  $\{z \in \mathbb{C} \mid |z| < 1\}$ , and prove that  $z_1$  and  $z_2$  are both simple and real. (The exact values of  $z_1$  and  $z_2$  are not required).

Compute the values of

$$\operatorname{res}\left(\frac{z(z-2)}{e^z - 3z^2}; z_j\right), \quad j = 1, 2,$$

by eliminating  $z_j$  by means of the equation  $\varphi(z_j) = 0$ .

Finally, compute the complex line integral

$$\oint_{|z|=1} \frac{z(z-2)}{e^z - 3z^2} dz.$$

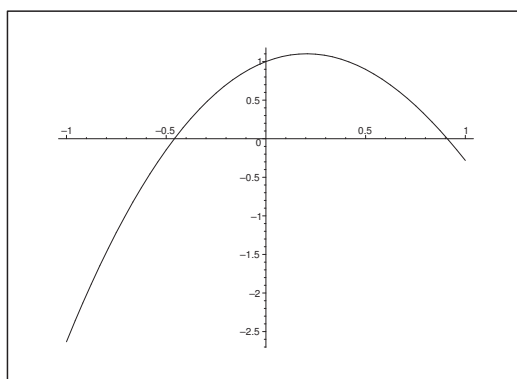


Figure 34: The graph of  $\varphi(x) = e^x - 3x^2$ ,  $x \in [-1, 1]$ .

1) We first apply *Rouché's theorem*. The functions

$$f(z) = -3z^2 \quad \text{and} \quad g(z) = e^z$$

are both analytic in  $\mathbb{C}$ . We get for  $|z| = 1$  the estimates,

$$|g(z)| = |e^z| = e^x \leq e^1 < 3 = |f(z)|, \quad |z| = 1,$$

and then it follows from *Rouché's theorem* that  $f(z) = -3z^2$  and  $\varphi(z) = e^z - 3z^2$  have the same number of zeros inside  $|z| = 1$ , i.e. 2.

When we consider the graph, we see that  $\varphi(x) = e^x - 3x^2 = 0$  for two *different* real values in  $] -1, 1[$ , and since we according to the above have precisely two zeros inside  $|z| = 1$ , the roots are real and simple.

**Remark 2.16** One can approximately find the numerical values

$$z_1 \approx -0,4590 \quad \text{og} \quad z_2 \approx 0,9100. \quad \diamond$$

**Remark 2.17** It is not so important for the rest of the example that the zeros are real. The important thing is that they are *simple*, so  $\varphi'(z_0) \neq 0$ , when  $\varphi(z_0) = 0$ . Thus, we assume

$$\varphi(z_0) = e^{z_0} - 3z_0^2 = 0,$$

and we find

$$\varphi'(z_0) = e^{z_0} - 6z_0 = (e^{z_0} - 3z_0) + (3z_0^2 - 6z_0) = 3z_0(z_0 - 2).$$

Therefore, if both  $\varphi(z_0) = 0$  and  $\varphi'(z_0) = 0$ , then we necessarily have either  $z_0 = 0$  (which is not possible) or  $z_0 = 2$  (which is not possible either), because it is easily shown that

$$\varphi(0) = 1 \neq 0 \quad \text{and} \quad \varphi(2) = e^2 - 12 \neq 0.$$

Hence we conclude that  $\varphi(z)$  and  $\varphi'(z)$  are never zero simultaneously, and it follows that every zero of  $\varphi(z)$  is *simple*.  $\diamond$

2) Since  $\varphi(z)$  only has *simple* zeros  $z_j$ , and since we in each of these have

$$e^{z_j} = 3z_j^2,$$

we conclude from RULE II that

$$\operatorname{res} \left( \frac{z(z-2)}{e^z - 3z^2}; z_j \right) = \operatorname{res} \left( \frac{z(z-2)}{\varphi(z)}; z_j \right) = \frac{z_j(z_j-2)}{\varphi'(z_j)} = \frac{z_j^2 - 2z_j}{e^{z_j} - 6z_j} = \frac{z_j^2 - 2z_j}{3z_j^2 - 6z_j} = \frac{1}{3}.$$

3) Finally, it follows from *Cauchy's integral formula* that

$$\oint_{|z|=1} \frac{z(z-2)}{e^z - 3z^2} dz = 2\pi i \left\{ \operatorname{res} \left( \frac{z(z-2)}{e^z - 3z^2}; z_1 \right) + \operatorname{res} \left( \frac{z(z-2)}{e^z - 3z^2}; z_2 \right) \right\} = 2\pi i \left( \frac{1}{3} + \frac{1}{3} \right) = \frac{4\pi i}{3}.$$

**Example 2.32** Let  $K$  denote the square with the corners

$$1 + i, \quad -1 + i, \quad -1 - i, \quad 1 - i,$$

run through in this sequence.

1) Prove the estimate  $|z^2 + \sin z| < 4$  for  $z \in K$ .

2) Compute – e.g. by an application of Rouché's theorem – the complex line integral

$$\oint_K \frac{16z^3 + 2z + \cos z}{4z^4 + z^2 + \sin z} dz.$$

1) It follows from

$$\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$$

that

$$|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = \cosh^2 y - \cos^2 x,$$

hence

$$|z^2 + \sin z| \leq |z|^2 + |\sin z| < 2 + \cosh 1 < 4$$

for every  $z \in K$ , and the estimate is proved.

2) If we put  $f(z) = 4z^4$  and  $g(z) = z^2 + \sin z$ , then

$$|f(z)| \geq 4 \quad \text{and} \quad |g(z)| < 4 \quad \text{for } z \in K.$$

Thus by Rouché's theorem,

$$\oint_K \frac{16z^3 + 2z + \cos z}{4z^4 + z^2 + \sin z} dz = \oint_K \frac{f'(z) + g'(z)}{f(z) + g(z)} dz = \oint_K \frac{f'(z)}{f(z)} dz = \oint_K \frac{4}{z} dz = 8i\pi.$$



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### 3 Stability criteria

**Example 3.1** Check if  $P(z) = z^3 + 2z^2 + 3z + 1$  is a Hurwitz polynomial.

By using *Schur's criterion* we get the polynomial of second degree,

$$Q(z) = \frac{1}{z^1} \{7(z^3 + 2z^2 + 3z + 1) - (-1)(-z^3 + 2z^2 - 3z + 1)\} = 6z^2 + 10z + 8.$$

Since  $Q(z)$  is a polynomial of second degree, it is a Hurwitz polynomial, because all its coefficients are positive. Since  $P(z)$  also has only positive coefficients, it follows from *Schur's criterion* that  $P(z)$  is a Hurwitz polynomial.

If we instead apply *Hurwitz's criterion*, then we get the determinants

$$D_1 = 2, \quad D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5, \quad D_3 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 5.$$

All these determinants are positive, so we conclude that the polynomial is a Hurwitz polynomial, and all its roots lie in the left half plane.

**Example 3.2** Check if the polynomial  $P(z) = z^3 + 2z^2 + z + 3$  is a Hurwitz polynomial.

When we apply *Schur's criterion*, we get

$$Q(z) = \frac{1}{z+1} \{7(z^3 + 2z^2 + z + 3) - 3(-z^3 + 2z^2 - 3z + 1)\} = 10z^2 - 2z + 12.$$

Here  $Q(z)$  has terms of both positive and negative coefficients, so  $Q(z)$  is not a Hurwitz polynomial. We therefore conclude that  $P(z)$  is not a Hurwitz polynomial.

If we instead apply *Hurwitz's criterion*, we immediately get that

$$D_2 = \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = -1 < 0.$$

Hence there is at least one root in the closed right half plane  $\operatorname{Re} z \geq 0$ .

When we insert the parametric description  $z = iy$  of the imaginary axis, then

$$P(iy) = (3 - 2y^2) + iy(1 - y^2),$$

which is not zero for any  $y \in \mathbb{R}$ . Therefore, there is at least one root in the open right half plane. Clearly, such a root cannot be real, and since the coefficients of the polynomial are real, the complex conjugated of the root must also be a root. By using a pocket calculator we find the approximate values

$$-2,174\,559\,41 \quad \text{og} \quad 0,087\,279\,7 \pm 1,171\,312\,1\,i.$$

**Example 3.3** Find the set of values of the parameter  $c$ , such that the polynomial

$$f(z) = z^4 + 2z^3 + 3z^2 + 4z + c$$

has only roots in the left half plane.

We shall use *Hurwitz's criterion*. It follows from

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = 3, \quad a_3 = 4, \quad a_4 = c,$$

that  $D_4$  is given by

$$D_4 = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 & 0 \\ 4 & 3 & 2 & 1 \\ 0 & c & 4 & 3 \\ 0 & 0 & 0 & c \end{vmatrix}.$$

It follows easily that

$$D_1 = 2 > 0, \quad D_2 = 2 \cdot 3 - 1 \cdot 4 = 2 > 0, \quad D_4 = c \cdot D_3.$$

When we apply *Hurwitz's criterion*, it follows that  $f(z)$  has its roots in the left half plane, if and only if  $c > 0$  and  $D_3 > 0$ . Since

$$D_3 = \begin{vmatrix} 2 & 1 & 0 \\ 34 & 3 & 2 \\ 0 & c & 4 \end{vmatrix} = 24 - 16 - 4c = 8 - 4c = 4(2 - c),$$

the condition becomes  $c \in ]0, 2[$ .

**Remark 3.1** Of continuity reasons two of the roots for  $c = 2$  must lie on the imaginary axis. When  $c = 2$ , we get more specifically,

$$\begin{aligned} z^4 + 2z^3 + 3z^2 + 4z + 2 &= (z^4 + 2z^3 + z^2) + (2z^2 + 4z + 4) \\ &= (z^2 + 2)(z^2 + 2z + 1) = (z - i\sqrt{2})(z + i\sqrt{2})(z + 1)^2, \end{aligned}$$

and the claim follows.

ALTERNATIVELY we put  $z = i\lambda$ ,  $\lambda \in \mathbb{R}$ , and separate into the real and the imaginary part,

$$(i\lambda)^4 + 2(i\lambda)^3 + 3(i\lambda)^2 + 4(i\lambda) + 2 = \lambda^4 - 3\lambda^2 + 2 + 2i(-\lambda^3 + 2\lambda).$$

This expression is zero, if and only if

$$\lambda^4 - 3\lambda^2 + 2 = 0 \quad \text{and} \quad \lambda(\lambda^2 - 2) = 0,$$

thus if and only if  $\lambda^2 = 2$ , hence  $\lambda = \pm\sqrt{2}$ . Then we conclude that

$$(z - i\sqrt{2})(z + i\sqrt{2}) = z^2 + 2$$

is a divisor, and it is easy to find all roots.  $\diamond$

**Example 3.4** Find the set of parameters  $c$ , for which the polynomial

$$f(z) = 3z^3 + 2z^2 + z + c$$

has only roots in the left half plane.

We shall use *Hurwitz's criterion* where

$$a_0 = 3, \quad a_1 = 2, \quad a_2 = 1, \quad a_3 = c.$$

Then

$$D_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 0 \\ c & 1 & 2 \\ 0 & 0 & c \end{vmatrix} = c \cdot D_2.$$

It follows from

$$D_1 = 2 > 0 \quad \text{and} \quad D_2 = 2 - 3c > 0 \quad \text{for } c < \frac{2}{3},$$

that  $f(z)$  has all its roots lying in the left half plane, if and only if  $c \in \left] 0, \frac{2}{3} \right[$ .

**Remark 3.2** When  $c = \frac{2}{3}$ , then by the continuity at least one root must lie on the imaginary axis.

Now

$$3z^2 + 2z^2 + z + \frac{2}{3} = (3z^3 + z) + 2\left(z^2 + \frac{1}{3}\right) = (3z + 2)\left(z^2 + \frac{1}{3}\right),$$

so when  $c = \frac{2}{3}$  all roots are given by

$$z = -\frac{2}{3}, \quad z = \pm \frac{i}{\sqrt{3}}.$$

ALTERNATIVELY, put  $z = i\lambda$ ,  $\lambda \in \mathbb{R}$  into the polynomial to get

$$3(i\lambda)^3 + 2(i\lambda)^2 + i\lambda + \frac{2}{3} = -2\lambda^2 + \frac{2}{3} + i\lambda(1 - 3\lambda^2).$$

This expression is zero, if and only if both

$$2\lambda^2 = \frac{2}{3} \quad \text{and} \quad \lambda(1 - 3\lambda^2) = 0,$$

thus if and only if  $\lambda^2 = \frac{1}{3}$ , hence  $\lambda = \pm \frac{1}{\sqrt{3}}$ , and the roots on the imaginary axis are  $\pm \frac{i}{\sqrt{3}}$ . If we then divide by

$$\left(z - \frac{i}{\sqrt{3}}\right)\left(z + \frac{i}{\sqrt{3}}\right) = z^2 + \frac{1}{3}$$

we get the real root  $z = -\frac{2}{3}$ .  $\diamond$



**Example 3.5** Find the set of the parameters  $a$  and  $b$ , such that the polynomial

$$f(z) = 4z^4 + z^3 + z^2 + az + b$$

has only roots in the left half plane. Sketch the solution set in the  $(a, b)$ -plane.

It follows from

$$a_0 = 4, \quad a_1 = 1, \quad a_2 = 1, \quad a_3 = a, \quad a_4 = b,$$

that

$$D_4 = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 0 & 0 \\ a & 1 & 1 & 4 \\ 0 & b & a & 1 \\ 0 & 0 & 0 & b \end{vmatrix}.$$

According to *Hurwitz's criterion* we get the conditions

$$D_1 = 1 > 0, \quad D_2 = 1 - 4a > 0, \quad D_4 = b \cdot D_3 > 0,$$

and

$$D_3 = \begin{vmatrix} 1 & 4 & 0 \\ a & 1 & 1 \\ 0 & b & a \end{vmatrix} = a - b - 4a^2 > 0.$$

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These requirements are fulfilled, if and only if  $a, b > 0$  and  $a < \frac{1}{4}$ , and

$$b < a - 4a^2 = a(1 - 4a).$$

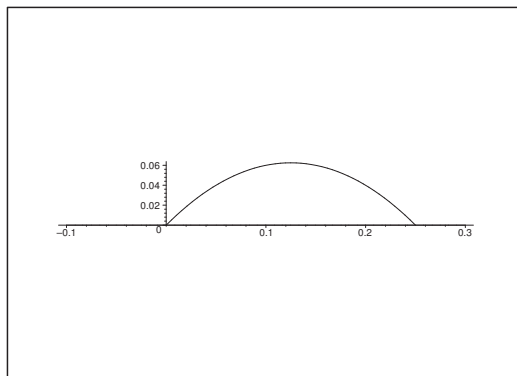


Figure 35: The domain of stability in the  $(a, b)$ -plane.

It follows from the inequalities above that the domain of stability in the  $(a, b)$ -plane is given by

$$\left\{ (a, b) \mid a \in \left[ 0, \frac{1}{4} \right], 0 < b < a - 4a^2 \right\},$$

which is bounded by the parabola  $b = a - 4a^2$  and the  $a$ -axis.

**Remark 3.3** When  $(a, b)$  lies on the boundary  $b = a - 4a^2$ , then by the continuity at least one of the roots must lie on the imaginary axis. It follows by a reduction that

$$\begin{aligned} 4z^4 + z^3 + z^2 + az + a - 4a^2 &= 4(z^4 - a^2) + (z^3 + az) + (z^2 + 1) \\ &= 4(z^2 + a)(z^2 - 1) + z(z^2 + a) + (z^2 + a) = (z^2 + a)(4z^2 - 4a + z + 1) \\ &= (z^2 + a)(4z^2 + z + 1 - 4a), \end{aligned}$$

hence the roots are

$$z = \pm i\sqrt{a} \quad \text{and} \quad z = \frac{-1 \pm \sqrt{1 - 16(1 - 4a)}}{8} = \frac{-1 \pm \sqrt{64a - 15}}{8}. \quad \diamond$$

**Example 3.6** *Prove that*

$$z^3 + (3 - i)z^2 + (5 - 2i)z + 3 - i$$

*has all its roots lying in the left half plane.*

Since the polynomial has *complex* coefficients, we cannot apply Hurwitz's criterion directly. However, since the conjugated polynomial has the complex conjugated roots, the polynomial

$$\begin{aligned} P(z)\overline{P}(z) &= \{z^3 + (3 - i)z^2 + (5 - 2i)z + 3 - i\} \{z^3 + (3 + i)z^2 + (5 + 2i)z + 3 + i\} \\ &= z^6 + 6z^5 + (10 + 9 + 1)z^4 + \{6 + 2 \operatorname{Re}\{(3 - i)(5 + 2i)\}\}z^3 \\ &\quad + 2\{2 \operatorname{Re}\{(3 - i)(3 + i)\} + 25 + 4\}z^2 + 2 \operatorname{Re}\{(5 - 2i)(3 + i)\}z + 9 + 1 \\ &= z^6 + 6z^5 + 20z^4 + \{6 + 2(15 + 2)\}z^3 + \{2 \cdot 10 + 29\}z^2 + 2(15 + 2)z + 10 \\ &= z^6 + 6z^5 + 20z^4 + 40z^3 + 49z^2 + 34z + 10 \end{aligned}$$

must have its roots lying in the left half plane, if and only if

$$P(z) = z^3 + (3 - i)z^2 + (5 - 2i)z + 3 - i$$

has its roots lying in the left half plane. Hence we can ALTERNATIVELY check

$$f(z) = z^6 + 6z^5 + 20z^4 + 40z^3 + 49z^2 + 34z + 10,$$

where

$$a_0 = 1, a_1 = 6, a_2 = 20, a_3 = 40, a_4 = 49, a_5 = 34, a_6 = 10.$$

Here we have a couple of possible solutions, of which we start with *Hurwitz's criterion*, which gives a lot of computation, because the determinants are so large.

**Hurwitz's criterion.** First we write down

$$D_6 = \begin{vmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \\ 0 & a_6 & a_5 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_6 & a_5 & a_4 \\ 0 & 0 & 0 & 0 & 0 & a_6 \end{vmatrix} = \begin{vmatrix} 6 & 1 & 0 & 0 & 0 & 0 \\ 40 & 20 & 6 & 1 & 0 & 0 \\ 34 & 49 & 40 & 20 & 6 & 1 \\ 0 & 10 & 34 & 49 & 40 & 20 \\ 0 & 0 & 0 & 10 & 34 & 49 \\ 0 & 0 & 0 & 0 & 0 & 10 \end{vmatrix},$$

where we shall find the signs of all the minorants. It follows immediately that

$$D_1 = 6, \quad D_2 = 120 - 40 = 80 \quad \text{and} \quad D_6 = 10 \cdot D_5,$$

so it "only" remains to check that both

$$D_3 > 0, \quad D_4 > 0 \quad \text{and} \quad D_5 > 0.$$

We get by a computation,

$$\begin{aligned} D_3 &= \begin{vmatrix} 6 & 1 & 0 \\ 40 & 20 & 6 \\ 34 & 49 & 40 \end{vmatrix} = 4 \begin{vmatrix} 3 & 1 & 0 \\ 10 & 10 & 3 \\ 17 & 49 & 40 \end{vmatrix} = 4 \begin{vmatrix} 3 & 1 & 0 \\ 10 & 10 & 3 \\ 14 & 48 & 40 \end{vmatrix} = 8 \begin{vmatrix} 3 & 1 & 0 \\ 10 & 10 & 3 \\ 7 & 24 & 20 \end{vmatrix} \\ &= 8 \begin{vmatrix} 3 & 1 & 0 \\ 10 & 10 & 3 \\ 10 & 25 & 20 \end{vmatrix} = 40 \begin{vmatrix} 3 & 1 & 0 \\ 10 & 10 & 3 \\ 2 & 5 & 4 \end{vmatrix} = 40(120 + 6 - 40 - 45) = 40 \cdot 41 > 0. \end{aligned}$$

Furthermore,

$$\begin{aligned}
 D_4 &= \begin{vmatrix} 6 & 1 & 0 & 0 \\ 40 & 20 & 6 & 1 \\ 34 & 49 & 40 & 20 \\ 0 & 10 & 34 & 49 \end{vmatrix} = 4 \begin{vmatrix} 3 & 1 & 0 & 0 \\ 20 & 20 & 3 & 1 \\ 17 & 49 & 20 & 20 \\ 0 & 10 & 17 & 49 \end{vmatrix} = 4 \begin{vmatrix} 3 & 1 & 0 & 0 \\ 20 & 30 & 20 & 50 \\ 20 & 50 & 20 & 20 \\ 0 & 10 & 17 & 49 \end{vmatrix} \\
 &= 400 \begin{vmatrix} 3 & 1 & 0 & 0 \\ 2 & 3 & 2 & 5 \\ 2 & 5 & 2 & 2 \\ 0 & 10 & 17 & 49 \end{vmatrix} = 400 \left\{ 3 \begin{vmatrix} 3 & 2 & 5 \\ 5 & 2 & 2 \\ 10 & 17 & 49 \end{vmatrix} - \begin{vmatrix} 2 & 2 & 5 \\ 2 & 2 & 2 \\ 0 & 17 & 49 \end{vmatrix} \right\} \\
 &= 400 \left\{ 3(6 \cdot 49 + 40 + 25 \cdot 17 - 100 - 490 - 102) - \begin{vmatrix} 2 & 2 & 5 \\ 0 & 0 & -3 \\ 0 & 17 & 49 \end{vmatrix} \right\} \\
 &= 400\{3(294 + 40 + 425 - 100 - 490 - 102) - 2 \cdot 3 \cdot 17\} \\
 &= 1200\{759 - 692 - 34\} = 1200 \cdot 33 > 0.
 \end{aligned}$$

Finally,

$$\begin{aligned}
 D_5 &= \begin{vmatrix} 6 & 1 & 0 & 0 & 0 \\ 40 & 20 & 6 & 1 & 0 \\ 34 & 49 & 40 & 20 & 6 \\ 0 & 10 & 34 & 49 & 40 \\ 0 & 0 & 0 & 10 & 34 \end{vmatrix} = 8 \begin{vmatrix} 3 & 1 & 0 & 0 & 0 \\ 20 & 20 & 3 & 1 & 0 \\ 17 & 49 & 20 & 20 & 3 \\ 0 & 10 & 17 & 49 & 20 \\ 0 & 0 & 0 & 10 & 17 \end{vmatrix} \\
 &= 8 \begin{vmatrix} 3 & 1 & 0 & 0 & 0 \\ 20 & 20 & 3 & 1 & 0 \\ 20 & 50 & 20 & 20 & 3 \\ 20 & 30 & 20 & 50 & 20 \\ 0 & 0 & 0 & 10 & 17 \end{vmatrix} = 80 \begin{vmatrix} 3 & 1 & 0 & 0 & 0 \\ 20 & 20 & 3 & 1 & 0 \\ 20 & 50 & 20 & 30 & 20 \\ 2 & 3 & 2 & 5 & 2 \\ 0 & 0 & 0 & 10 & 17 \end{vmatrix} \\
 &= 800 \begin{vmatrix} 3 & 1 & 0 & 0 & 0 \\ 20 & 20 & 3 & 1 & 0 \\ 2 & 5 & 2 & 3 & 2 \\ 2 & 3 & 2 & 5 & 2 \\ 0 & 0 & 0 & 10 & 17 \end{vmatrix} = 800 \begin{vmatrix} 3 & 1 & 0 & 0 & 0 \\ 20 & 20 & 3 & 1 & 0 \\ 0 & 2 & 0 & -2 & 0 \\ 2 & 3 & 2 & 5 & 2 \\ 0 & 0 & 0 & 10 & 17 \end{vmatrix},
 \end{aligned}$$

hence

$$\begin{aligned}
 D_5 &= 1600 \begin{vmatrix} 3 & 1 & 0 & 0 & 0 \\ 20 & 20 & 3 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 2 & 3 & 2 & 5 & 2 \\ 0 & 0 & 0 & 10 & 17 \end{vmatrix} = 1600 \begin{vmatrix} 3 & 1 & 0 & 0 & 0 \\ 20 & 21 & 3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 2 & 8 & 2 & 5 & 2 \\ 0 & 10 & 0 & 10 & 17 \end{vmatrix} \\
 &= 1600 \begin{vmatrix} 3 & 1 & 0 & 0 \\ 20 & 21 & 3 & 0 \\ 2 & 8 & 2 & 2 \\ 0 & 10 & 0 & 17 \end{vmatrix} = 3200 \begin{vmatrix} 3 & 1 & 0 & 0 \\ 20 & 21 & 3 & 0 \\ 1 & 4 & 1 & 1 \\ 0 & 10 & 0 & 17 \end{vmatrix} \\
 &= 3200 \left\{ 3 \begin{vmatrix} 21 & 3 & 0 \\ 4 & 1 & 1 \\ 10 & 0 & 17 \end{vmatrix} - 1 \begin{vmatrix} 20 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 17 \end{vmatrix} \right\} \\
 &= 3200 \left\{ 9 \begin{vmatrix} 7 & 1 & 0 \\ 4 & 1 & 1 \\ 10 & 0 & 17 \end{vmatrix} - 17^2 \right\} = 3200 \left\{ 9 \begin{vmatrix} 7 & 1 & 0 \\ -3 & 0 & 1 \\ 10 & 0 & 17 \end{vmatrix} - 289 \right\} \\
 &= 3200 \left\{ 9 \cdot (-1) \begin{vmatrix} -3 & 1 \\ 10 & 17 \end{vmatrix} - 289 \right\} = 3200 \{ 9(51 + 10) - 289 \} = 3200 \cdot 260 > 0,
 \end{aligned}$$

and we have  $D_5 > 0$ . Since this implies that  $D_6 > 0$ , we have proved that all minorants are positive, and it follows from *Hurwitz's criterion* that all roots are lying in the open left half plane.



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**Schur's criterion.** ALTERNATIVELY we apply *Schur's criterion* on the polynomial

$$f(z) = z^6 + 6z^5 + 20z^4 + 40z^3 + 49z^2 + 34z + 10.$$

Since all coefficients are positive, this polynomial is a Hurwitz polynomial, if and only if

$$\frac{f(1)f(z) - f(-1)f(-z)}{z+1}$$

is a Hurwitz polynomial. A computation gives

$$f(z) = z^6 + 6z^5 + 20z^4 + 40z^3 + 49z^2 + 34z + 10 = \frac{1}{z+1} \{160f(z) - 0 \cdot f(-z)\} = 160 \cdot \frac{f(z)}{z+1},$$

so  $z = -1$  is a root. Using division by polynomials we get

$$\frac{f(z)}{z+1} = z^5 + 5z^4 + 15z^3 + 25z^2 + 24z + 10.$$

Since  $f(z) = P(z)\overline{P}(z)$ , we again obtain  $z = -1$  as a root, so by another division by polynomials,

$$f_1(z) = \frac{f(z)}{(z+1)^2} = z^4 + 4z^3 + 11z^2 + 14z + 10.$$

Since all coefficients are positive, and  $f_1(1) = 40$  and  $f_1(-1) = 4$ , it follows that  $f_1(z)$  is a Hurwitz polynomial, if and only if  $f_2(z)$  is a Hurwitz polynomial, where

$$\begin{aligned} f_2(z) &= \frac{f_1(1)f_1(z) - f_1(-1)f_1(-z)}{z+1} = \frac{4}{z+1} \{10f_1(z) - f_1(-z)\} \\ &= \frac{4}{z+1} \{10z^4 + 40z^3 + 110z^2 + 140z + 100 - z^4 + 4z^3 - 11z^2 + 14z - 10\} \\ &= \frac{4}{z+1} \{9z^4 + 44z^3 + 99z^2 + 154z + 90\} = 4 \{9z^3 + 35z^2 + 64z + 90\}. \end{aligned}$$

Putting

$$f_3(z) = 9z^3 + 35z^2 + 64z + 90,$$

we get  $f_3(1) = 198$  and  $f_3(-1) = 52$ , so  $f_2(z)$  and  $f_3(z)$  are Hurwitz polynomials, if and only if

$$\begin{aligned} f_4(z) &= \frac{f_3(1)f_3(z) - f_3(-1)f_3(-z)}{z+1} = \frac{2}{z+1} \{99f_3(z) - 26f_3(-z)\} \\ &= \frac{2}{z+1} \{99(9z^3 + 35z^2 + 64z + 90) - 26(-9z^3 + 35z^2 - 64z + 90)\} \\ &= \frac{2}{z+1} \{125 \cdot 9z^3 + 73 \cdot 35z^2 + 125 \cdot 64z + 73 \cdot 90\} \\ &= \frac{10}{z+1} \{9 \cdot 25z^3 + 7 \cdot 73z^2 + 25 \cdot 64z + 18 \cdot 73\} \\ &= \frac{10}{z+1} \{225z^3 + 511z^2 + 1600z + 1314\} = 10 \{225z^2 + 286z + 1314\} \end{aligned}$$

is a Hurwitz polynomial. Since  $f_4(z)$  is a polynomial of second degree of positive coefficients, it is a Hurwitz polynomial, and we conclude that the original polynomial

$$z^3 + (3-i)z^2 + (5-2i)z + 3-i$$

has all its roots lying in the open left half plane.

**Remark 3.4** By chance we found by *Schur's criterion* that  $z = -1$  is a root. Then by division by a polynomial,

$$z^3 + (3 - i)z^2 + (5 - 2i)z + 3 - i = (z + 1)(z^2 + (2 - i)z + 3 - i).$$

The roots of  $z^2 + (2 - i)z + 3 - i$  are given by

$$z = \frac{-2 + i \pm \sqrt{(-2 + i)^2 - 4(3 - i)}}{2} = \frac{-2 + i \pm \sqrt{3 - 4i - 12 + 4i}}{2} = \frac{-2 + i \pm \sqrt{-9}}{2} = \frac{-2 + i \pm 3i}{2},$$

hence  $z^3 + (3 - i)z^2 + (5 - 2i)z + 3 - i$  has the three roots

$$-1, \quad -1 + 2i \quad \text{and} \quad -1 - i,$$

which all lie in the left half plane.  $\diamond$

**Example 3.7** Given the transfer function  $f(z) = \frac{z^2 - 1}{2z^2 - 5}$  of a control system. Check if the control system is stable.

We shall first put  $f(z)$  on the form  $\frac{H(z)}{1 + H(z)}$ , i.e.

$$H(z) = \frac{-\frac{z^2 - 1}{2z^2 - 5}}{\frac{z^2 - 1}{2z^2 - 5} - 1} = \frac{-z^2 + 1}{z^2 - 1 - 2z^2 + 5} = \frac{-z^2 + 1}{-z^2 + 4} = \frac{z^2 - 1}{z^2 - 4}.$$

By *Nyquist's criterion* we shall then consider

$$\frac{1}{1 + H(z)} = \frac{1}{1 + \frac{z^2 - 1}{z^2 - 4}} = \frac{z^2 - 4}{z^2 - 4 + z^2 - 1} = \frac{z^2 - 4}{2z^2 - 5}.$$

The poles of this function are  $\pm\sqrt{\frac{5}{2}}$ , of which only one lies in the right half plane, i.e.  $P = 1$ .

Then let  $z = it$ ,  $t \in \mathbb{R}$ , run through the imaginary axis. Then

$$w = \frac{1}{1 + H(it)} = \frac{1}{1 + \frac{-t^2 - 1}{-t^2 - 4}} = \frac{1}{1 + \frac{t^2 + 1}{t^2 + 4}}, \quad t \in \mathbb{R},$$

so the image curve runs through the interval  $\left[\frac{1}{2}, \frac{4}{5}\right]$  twice, when  $z = it$  runs through the imaginary axis. In particular, the winding number is  $0 \neq 1$  around  $w = 1$ , so the control system is unstable.

**Example 3.8** Find the set of  $a \in \mathbb{R}_+$ , for which the polynomial

$$P(z) = az^4 + 4z^3 + 3z^2 + 2z + 1$$

has all its roots in the open left half plane.

Find an  $a \in \mathbb{R}$ , such that  $P(z) = 0$  has a solution on the imaginary axis, and then find in this case all its roots.

1) We note that it is given that  $a > 0$ . We shall use *Hurwitz's criterion*, where

$$a_0 = a, \quad a_1 = 4, \quad a_2 = 3, \quad a_3 = 2, \quad a_4 = 1.$$

It follows that

$$D_4 = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{vmatrix} = \begin{vmatrix} 4 & a & 0 & 0 \\ 2 & 3 & 4 & a \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

Hence,  $D_1 = 4 > 0$ ,

$$D_2 = 12 - 2a > 0 \quad \text{for } a < 6,$$

and

$$D_3 = 24 - 4a - 16 = 8 - 4a > 0 \quad \text{for } a < 2,$$

and  $D_4 = 1 \cdot D_3 > 0$  for  $a < 2$ . Summing up, the condition is that  $a \in ]0, 2[$ .

2) Assume that  $z = iy$ ,  $y \in \mathbb{R}$ , is a root. Then we get by insertion,

$$P(iy) = ay^4 - 4iy^3 - 3y^2 + 2iy + 1 = \{ay^4 - 3y^2 + 1\} + 2iy\{-2y^2 + 1\} = 0.$$

When we separate the real and the imaginary part, we get for the imaginary part the possibilities  $y = 0$  (which must be rejected, because the real part then is  $1 \neq 0$ ) and  $y^2 = \frac{1}{2}$ . When we put  $y^2 = \frac{1}{2}$  into the condition for the real part, then

$$0 = ay^4 - 3y^2 + 1 = \frac{1}{4}a - 3 \cdot \frac{1}{2} + 1 = \frac{1}{4}a - \frac{1}{2},$$

so  $a = 2$ , which we also could have obtained by the continuity, because the roots are lying in the open left half plane, if and only if  $a \in ]0, a[$ .

When we insert this  $a$ -value, we get after a reduction that

$$P(z) = 2z^4 + 4z^3 + 3z^2 + 2z + 1 = (2z^2 + 1)(z^2 + 2z + 1) = (2z^2 + 1)(z + 1)^2,$$

so the roots are  $z = -1$  (a double root) and  $z = \pm i \frac{1}{\sqrt{2}}$ .

ALTERNATIVELY, both  $z = iy$  and  $z = -iy$  are simultaneously roots, because the polynomial has real coefficients. This means that

$$(z - iy)(z + iy) = z^2 + y^2$$



must be a divisor in the polynomial. We get by the division,

$$az^4 + 4z^3 + 3z^2 + 2z + 1 = (z^2 + y^2)(az^2 + 4z + (3 - ay^2)) + (2 - 4y^2)z + (ay^4 - 3y^2 + 1).$$

Since the remainder part must be 0 for every  $z \in \mathbb{C}$ , when  $\pm iy$  are roots, we conclude that

$$y^2 = \frac{1}{2} \quad \text{and} \quad ay^4 - 3y^2 + 1 = a \cdot \frac{1}{4} - \frac{3}{2} + 1 = 0,$$

so we get again  $a = 2$ , and then we proceed as above.

**Example 3.9** *Given*

$$P_\lambda(z) = z^4 + 4z^3 + 8z^2 + \lambda z + 12.$$

*Find the largest possible open interval  $]a, b[ \subseteq \mathbb{R}$ , for which all roots of  $P_\lambda(z)$  lie in the open left half plane, when  $\lambda \in ]a, b[$ .*

*Then find all roots of  $P_a(z)$ .*

We must obviously require that  $\lambda > 0$ . We shall apply *Hurwitz's criterion*. We first identify the coefficients,

$$P_\lambda(z) = 1 \cdot z^4 + 4 \cdot z^3 + 8 \cdot z^2 + \lambda \cdot z + 12 = a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4.$$

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Then

$$D_4 = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{vmatrix} = \begin{vmatrix} 4 & 1 & 0 & 0 \\ \lambda & 8 & 4 & 1 \\ 0 & 12 & \lambda & 8 \\ 0 & 0 & 0 & 12 \end{vmatrix}.$$

The minorants are then computed

$$D_1 = 4, \quad D_2 = \begin{vmatrix} 4 & 1 \\ \lambda & 8 \end{vmatrix} = 32 - \lambda, \quad D_3 = \begin{vmatrix} 4 & 1 & 0 \\ \lambda & 8 & 4 \\ 0 & 12 & \lambda \end{vmatrix} = 32\lambda - \lambda^2 - 192,$$

and  $D_4 = 12 \cdot D_3$ , so  $D_4 > 0$ , if and only if  $D_3 > 0$ . It is trivial that  $D_1 > 0$ . Furthermore,

$$D_2 = 32 - \lambda > 0 \quad \text{for } \lambda < 32,$$

and

$$D_3 = -\lambda^2 + 32\lambda - 192 = -(\lambda - 8)(\lambda - 24) > 0 \quad \text{for } \lambda \in ]8, 24[.$$

Summing up we see that all roots of  $P_\lambda(z)$  lie in the open left half plane for

$$\lambda \in ]8, 24[ = ]8, 24[.$$

Then we shall find all roots of  $P_a(z) = P_8(z)$ . We get by some small rearrangements,

$$\begin{aligned} P_8(z) &= z^4 + \underline{4z^3} + 8z^2 + \underline{8z} + 12 = 4z(z^2 + 2) + z^4 + 8z^2 + 12 \\ &= 4z(z^2 + 2) + (z^2 + 2)(z^2 + 6) = (z^2 + 2)(z^2 + 4z + 6) \\ &= (z^2 + 1)(\{z + 2\}^2 + 1), \end{aligned}$$

which shows that the roots are

$$z = \pm i\sqrt{2} \quad \text{and} \quad z = -2 \pm i\sqrt{2}.$$

ALTERNATIVELY,  $\lambda = 8$  is a limiting case, so at least one root must lie on the imaginary axis. If we put  $z = iy$ , then

$$\begin{aligned} P_8(iy) &= y^4 - 4iy^3 - 8y^2 + 8iy + 12 = (y^4 - 8y^2) - 4iy(y^2 - 2) \\ &= (y^2 - 2)(y^2 - 6) - 4iy(y^2 - 2) = (y^2 - 2)\{y^2 - 6 + 8iy\}, \end{aligned}$$

and we conclude that  $y = \pm\sqrt{2}$ , thus  $z = \pm i\sqrt{2}$ , is a solution, and we get by a division by

$$(z - i\sqrt{2})(z + i\sqrt{2}) = z^2 + 2$$

the remaining roots  $z = -2 \pm i\sqrt{2}$ .

**Remark 3.5** For completeness we also compute  $P_{24}(z)$ . We get

$$\begin{aligned} P_{24}(z) &= z^4 + \underline{4z^3} + 8z^2 + \underline{24z} + 12 = (z^4 + 8z^2 + 12) + 4(z^2 + 6)z \\ &= (z^2 + 2)(z^2 + 6) + 4z(z^2 + 6) = (z^2 + 6)(z^2 + 4z + 2) \\ &= (z - i\sqrt{6})(z + i\sqrt{6})(\{z + 2\}^2 - 2), \end{aligned}$$

so the roots are

$$z = \pm i\sqrt{6} \quad \text{and} \quad z = -2 \pm \sqrt{2}. \quad \diamond$$

**Example 3.10** Given the polynomial

$$P_{\lambda,\mu}(z) = z^4 + 2z^3 + \lambda z^2 + \mu z + 1$$

which depends on the parameters  $\lambda, \mu \in \mathbb{R}_+$ .

(a) For every fixed  $\mu \in \mathbb{R}_+$ , find the largest possible open half line

$$]a(\mu), +\infty[ \subseteq \mathbb{R}_+,$$

such that all the roots of  $P_{\lambda,\mu}(z)$  are lying in the open left half plane  $\operatorname{Re}(z) < 0$ , when  $\lambda \in ]a(\mu), +\infty[$ .

(b) We define by the above a function  $a(\mu)$ ,  $\mu \in \mathbb{R}_+$ . Prove that  $a(\mu)$  has a minimum for some  $\mu_0 \in \mathbb{R}_+$ , and then find  $\mu_0$ .

(c) Solve the equation

$$P_{a(\mu_0),\mu_0}(z) = 0.$$

Clearly, this example is inviting an application of *Routh-Hurwitz's criterion*. Let  $\mu \in \mathbb{R}_+$  be fixed. Putting

$$P_{\lambda,\mu}(z) = 1 \cdot z^4 + 2 \cdot z^3 + \lambda \cdot z^2 + \mu \cdot z + 1 = a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4,$$

we see that the corresponding matrix is

$$\begin{pmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ \mu & \lambda & 2 & 1 \\ 0 & 1 & \mu & \lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Since all coefficients have the same sign, it follows from the above that

$$D_1 = 2 > 0, \quad D_2 = \begin{vmatrix} 2 & 1 \\ \mu & \lambda \end{vmatrix} = 2\lambda - \mu, \quad D_4 = D_3,$$

and

$$D_3 = \begin{vmatrix} 2 & 1 & 0 \\ \mu & \lambda & 2 \\ 0 & 1 & \mu \end{vmatrix} = 2\lambda\mu - 4 - \mu^2.$$

It follows from  $D_1, \dots, D_4 > 0$ , that the conditions are

$$\lambda > \frac{\mu}{2} \quad \text{og} \quad \lambda > \frac{\mu}{2} + \frac{2}{\mu} \quad \left( > \frac{\mu}{2} \right).$$

Then, by *Routh-Hurwitz's criterion*,

$$a(\mu) = \frac{\mu}{2} + \frac{2}{\mu},$$

and we see that all roots lie in the open left half plane, if

$$\lambda \in \left] \frac{\mu}{2} + \frac{2}{\mu}, +\infty \right[.$$

(b) Putting  $t = \frac{\mu}{2} \in \mathbb{R}_+$ , and considering the function

$$\varphi(t) = t + \frac{1}{t} \quad \text{for } t \in \mathbb{R}_+,$$

it follows that

$$\varphi(t) \rightarrow +\infty \quad \text{for } t \rightarrow 0+ \text{ and for } t \rightarrow +\infty,$$

and

$$\varphi'(t) = 1 - \frac{1}{t^2} = 0 \quad \text{only for } t = 1 \in \mathbb{R}_+.$$

The corresponding point must then be a minimum,

$$\mu_0 = 2 \quad \text{and} \quad a(\mu_0) = a(2) = 2.$$

(c) It follows by *inspection* that if  $\mu_0 = 2$  and  $a(\mu_0) = 2$  then

$$\begin{aligned} P_{2,2}(z) &= z^4 + 2z^3 + 2z^2 + 2z + 1 = (z^4 + 2z^2 + z^2) + (z^2 + 2z + 1) \\ &= (z^2 + 1)(z^2 + 2z + 1) = (z + 1)^2(z^2 + 1). \end{aligned}$$

This polynomial is 0 for  $z = -1$  (a double root) and for  $z = \pm i$  (simple roots).

ALTERNATIVELY we guess that we must have roots on the imaginary axis. Hence, if we put  $z = iy$ ,  $y \in \mathbb{R}$ , then

$$P_{2,2} = y^4 - 2iy^3 - 2y^2 + 2iy + 1 = (y^4 - 2y^2 + 1) - 2iy(y^2 - 1) = (y^2 - 1)\{y^2 - 2iy - 1\} = 0,$$

which for  $y \in \mathbb{R}$  only is fulfilled for  $y = \pm 1$ , thus  $z = \pm i$ . When we divide by  $z^2 + 1$  we get  $z^2 + 2z + 1$ , which has  $z = -1$  as a double root.

**Remark 3.6** It is here possible in general to find the roots of  $P_{a(\mu),\mu}(z)$ ,  $\mu \in \mathbb{R}_+$ . The calculations are easy,

$$\begin{aligned} P_{a(\mu),\mu}(z) &= z^4 + 2z^3 + \left\{ \frac{2}{\mu}\mu + \frac{\mu}{2} \right\} z^2 + \mu z + 1 = \left\{ z^2 + 2z + \frac{2}{\mu} \right\} z^2 + \frac{\mu}{2} \left\{ z^2 + 2z + \frac{2}{\mu} \right\} \\ &= \left( z^2 + 2z + \frac{2}{\mu} \right) \left( z^2 + \frac{\mu}{2} \right). \end{aligned}$$

Since  $\mu > 0$ , we always get to imaginary roots,

$$\pm i\sqrt{\frac{\mu}{2}}.$$

If  $\mu \in ]0, \mu[$ , then we furthermore get the two complex conjugated roots

$$z = -1 \pm i\sqrt{\frac{2}{\mu} - 1}, \quad \mu \in ]0, 2[.$$

If  $\mu = 2$ , then we get the double root  $z = -1$  (cf. the above).

When  $\mu > 2$ , we have the additional two real roots

$$z = -1 \pm \sqrt{1 - \frac{2}{\mu}}, \quad \mu \in ]2, +\infty[. \quad \diamond$$

**Example 3.11** Find the largest open interval  $]a, b[$ , such that

$$z^4 + 4z^3 + 8z^2 + \lambda z + 15$$

is a Hurwitz polynomial for every  $\lambda \in ]a, b[$ .

A necessary condition is of course that  $\lambda > 0$ . We derive from

$$P_\lambda(z) = 1 \cdot z^4 + 4 \cdot z^3 + 8 \cdot z^2 + \lambda \cdot z + 15 = a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4,$$

the corresponding matrix

$$\begin{pmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 0 & 0 \\ \lambda & 8 & 4 & 1 \\ 0 & 15 & \lambda & 8 \\ 0 & 0 & 0 & 15 \end{pmatrix}.$$



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Then by *Hurwitz's criterion*,

$$D_1 = 4 > 0, \quad D_2 = 32 - \lambda > 0, \quad D_3 = 32\lambda - \lambda^2 - 16 \cdot 15 > 0, \quad D_4 = 15 \cdot D_3 > 0.$$

(The latter condition is fulfilled if only  $D_3 > 0$ , so we may neglect it). It follows from  $D_2 > 0$  that  $\lambda < 32$ , and then we conclude from

$$-\lambda^2 + 32\lambda - 16 \cdot 15 = -(\lambda - 16)^2 + 16 = -(\lambda - 12)(\lambda - 20),$$

that  $D_3 > 0$ , if and only if  $\lambda \in ]12, 20[$ .

Summing up we get

$$]a, b[ = ]12, 20[.$$

ALTERNATIVELY, though not very smart in this case, we apply *Schur's criterion* twice. The details are left to the reader.

**Remark 3.7** When  $\lambda = a = 12$ , we get the factorization,

$$z^4 + 4z^3 + 8z^2 + 12z + 15 = (z^2 + 3)(z^2 + 4z + 5),$$

which shows that the roots are

$$z = \pm i\sqrt{3} \quad \text{and} \quad z = -2 \pm i.$$

If  $\lambda = b = 20$ , then we get the factorization

$$z^4 + 4z^3 + 8z^2 + 20z + 15 = (z^2 + 5)(z^2 + 4z + 3),$$

showing that the roots are

$$z = \pm i\sqrt{3} \quad \text{og} \quad z = -3, \quad z = -1. \quad \diamond$$

## 4 The infinitely-valued function $\log z$

**Example 4.1** Compute all complex values of  $1^z$ ,  $z = x + iy$ .

It follows from  $\log 1 = 0 + 2ip\pi$ ,  $p \in \mathbb{Z}$ , that

$$\begin{aligned} 1^z &= \exp(\{x \cdot 0 - y \cdot 2p\pi\} + i\{x \cdot 2p\pi + y \cdot 0\}) \\ &= e^{-2p\pi y} \cdot \{\cos(2xp\pi) + i \cdot \sin(2xp\pi)\}, \quad p \in \mathbb{Z}. \end{aligned}$$

If we put  $y = 0$ , i.e.  $z = x \in \mathbb{R}$ , we conclude from the above that

$$1^x = \cos(2xp\pi) + i \cdot \sin(2xp\pi) = e^{2ixp\pi}, \quad p \in \mathbb{Z}.$$

Notice that if  $x$  is not rational,  $x \in \mathbb{R} \setminus \mathbb{Q}$ , then one can prove that the symbol  $1^x$  represents a point set which is dense on the unit circle.

Note, however, that if we choose  $x = \frac{1}{n}$ ,  $n \in \mathbb{N}$ , then the expression is reduced to the usual  $n$ -th unit roots.

**Example 4.2** Compute all values of

$$(a) \operatorname{Log}(1+i)^2, \quad (b) \operatorname{Log}(1+i)^{10i}, \quad (c) \log e, \quad (d) \operatorname{Log} e.$$

(a) We shall always compute from the inside to the outside,

$$\operatorname{Log}(1+i)^2 = \operatorname{Log} 2i = \ln 2 + i \operatorname{Arg}(2i) = \ln 2 + i \frac{\pi}{2}.$$

(b) Here we first apply the definition of  $a^b$ , and then we compute outwards from the inside,

$$\begin{aligned} \operatorname{Log}(1+i)^{10i} &= \operatorname{Log}\{\exp(10i \cdot \log(1+i))\} = \operatorname{Log}\left\{\exp\left(10i \cdot \left\{\frac{1}{2} \ln 2 + i \arg(1+i)\right\}\right)\right\} \\ &= \operatorname{Log}\left\{\exp\left(10i \cdot \left\{\frac{1}{2} \ln 2 + i \left(\frac{\pi}{4} - 2p\pi\right)\right\}\right)\right\} \\ &= \operatorname{Log}\left\{\exp\left(-\frac{5\pi}{2} + 20p\pi + i \cdot 5 \ln 2\right)\right\}. \end{aligned}$$

There is a trap here, because

$$5 \ln 2 \approx 3.44 > \pi,$$

so  $5 \ln 2$  is *not* the principal argument. Hence we get

$$\operatorname{Log}(1+i)^{10i} = -\frac{5\pi}{2} + 20p\pi + i \{5 \ln 2 - 2\pi\}, \quad p \in \mathbb{Z}.$$

**Remark 4.1** The example should be surprising for two reasons. First we shall find out, if  $5 \ln 2$  is a principal value or not. (And it is not!) Secondly, we see that the principal logarithm of a many-valued function again may be many-valued.  $\diamond$

(c)

$$\log e = \ln e + i \arg e = 1 + 2ip\pi, \quad p \in \mathbb{Z}.$$

(d)

$$\operatorname{Log} e = \ln e + i \operatorname{Arg} e = 1 + i \cdot 0 = 1.$$

**Example 4.3** Find all values of

$$(a) \log 2, \quad (b) \log i, \quad (c) \log(1 - i).$$

(a)

$$\log 2 = \log \{2 \cdot e^{2ip\pi}\} = \ln 2 + 2ip\pi, \quad p \in \mathbb{Z}.$$

(b)

$$\log i = \ln |i| + i \arg i = i \left( \frac{\pi}{2} + 2p\pi \right), \quad p \in \mathbb{Z}.$$

(c)

$$\log(1 - i) = \ln |1 - i| + i \arg(1 - i) = \frac{1}{2} \ln 2 + i \left( -\frac{\pi}{4} + 2p\pi \right), \quad p \in \mathbb{Z}.$$



## 5 The many-valued functions $a^z$ and $z^a$

**Example 5.1** Compute all the complex values of  $z^i$  for  $z \in \mathbb{C} \setminus \{0\}$ .

We get by insertion into the formula of the many-valued function  $z^a$  for  $a = i$  that

$$\begin{aligned} z^i &:= \exp(i\{\ln|z| + i(\operatorname{Arg} z - 2p\pi)\}) \\ &= e^{-\operatorname{Arg} z + 2p\pi i} \cot\{\cos(\ln|z|) + i \cdot \sin(\ln|z|)\}, \quad p \in \mathbb{Z}. \end{aligned}$$

We have here used that we also have  $\arg = \{\operatorname{Arg} z - 2p\pi \mid p \in \mathbb{Z}\}$  which give a nicer final result.

**Example 5.2** Find all values of

$$(a) \ i^i, \quad (b) \ 1^i, \quad (c) \ 1^{\sqrt{2}}.$$

Here we shall stick very close to the definitions.

(a)

$$i^i := \exp(i \log i) = \exp\left(i \left\{ \ln|i| + i \left( \frac{\pi}{2} - 2p\pi \right) \right\}\right) = \exp\left(-\frac{\pi}{2} + 2p\pi\right), \quad p \in \mathbb{Z},$$

which is a sequence of real numbers!

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(b)

$$1^i := \exp(i \log 1) = \exp(i \cdot \{0 - 2ip\pi\}) = e^{2ip\pi}, \quad p \in \mathbb{Z},$$

which is also a sequence of real numbers.

(c)

$$1^{\sqrt{2}} := \exp(\sqrt{2} \cdot \log 1) = \exp(\sqrt{2} \cdot 2p\pi i) = \cos(2\sqrt{2}p\pi) + i \sin(2\sqrt{2}p\pi), \quad p \in \mathbb{Z}.$$

This set is dense on the unit circle.

**Remark 5.1** The examples above show that concerning many-valued functions one cannot rely on one's intuition.  $\diamond$

**Example 5.3** Given  $c \in \mathbb{C} \setminus \{0\}$  and  $z = x + iy \in \mathbb{C}$ .

(a) Find all values of  $\log(c^z)$ .(b) Find all values of  $z \log c$ .(c) For which values of  $z$  and  $c$  do we have  $\log(c^z) = z \log c$ ?(a) Let  $p, q \in \mathbb{Z}$  denote free parameters. Then

$$\begin{aligned} \log(c^z) &= \log(e^{z \cdot \log c}) = z \cdot \log c + 2ip\pi = (x + iy)\{\ln|c| + i \operatorname{Arg} c + 2iq\pi\} + 2ip\pi \\ &= x \ln|c| - y \operatorname{Arg} c - 2yq\pi + i\{x \operatorname{Arg} c + y \ln|c| + 2xq\pi + 2p\pi\}. \end{aligned}$$

(b) It follows from the computation above that  $z \cdot \log c$  is obtained by putting  $p = 0$ , thus

$$z \log c = x \ln|c| - y \operatorname{Arg} c - 2yq\pi + i\{x \operatorname{Arg} c + y \ln|c| + 2xq\pi\}, \quad q \in \mathbb{Z}.$$

(c) A necessary (though not sufficient) condition for

$$\log(c^z) = z \log c,$$

is that the sets

$$\{2\pi(xq + p) \mid p, q \in \mathbb{Z}\} \quad \text{and} \quad \{2\pi xq \mid q \in \mathbb{Z}\}$$

are identical.

Hence a necessary and sufficient condition is that

$$\mathbb{Z} \subseteq \{xq \mid q \in \mathbb{Z}\}, \quad \text{thus } x \in \mathbb{Q}.$$

If  $y \neq 0$ , then the real parts are identical, if and only if we choose the same  $q \in \mathbb{Z}$ . Then, concerning the imaginary parts, we are forced to choose  $p = 0$ , and  $p \in \mathbb{Z}$  is no longer a free parameter. Therefore,  $y = 0$ .

When this is the case, i.e.  $z = x \in \mathbb{Q}$ , then

$$\log(x^x) = x \ln|c| + i\{x \operatorname{Arg} c + (xq + p) \cdot 2\pi\}, \quad p, q \in \mathbb{Z},$$

and

$$x \log c = x \ln |c| + i\{x \operatorname{Arg} c + xq \cdot 2\pi\}, \quad p \in \mathbb{Z}.$$

Since  $\{xq + p \mid p, q \in \mathbb{Z}\}$  considered as a set (not counted by multiplicity) is identical with  $\{xq \mid q \in \mathbb{Z}\}$ , the two expressions are identical, if and only if

$$z = x \in \mathbb{Q},$$

while  $c \in \mathbb{C} \setminus \{0\}$  can be chosen arbitrarily.

**Example 5.4** Compute all values of

$$(a) \log(1+i)^{\pi i}, \quad (b) (-i)^{-i}, \quad (c) 3^{\pi}, \quad (d) 2^{\pi i}.$$

(a) We first compute  $(1+i)^{\pi i}$ . Here,

$$\begin{aligned} (1+i)^{\pi i} &= \exp(\pi i \log(1+i)) = \exp\left(\pi i \left\{\frac{1}{2} \ln 2 + i\left(\frac{\pi}{4} - 2p\pi\right)\right\}\right) \\ &= \exp\left(-\frac{\pi^2}{4} + 2p\pi^2 + i\frac{\pi}{2} \ln 2\right), \quad p \in \mathbb{Z}, \end{aligned}$$

hence

$$\log(1+i)^{\pi i} = \left(2p - \frac{1}{4}\right)\pi^2 + i\pi\left(\frac{1}{2} \ln 2 + 2q\right), \quad p, q \in \mathbb{Z}.$$

**Remark 5.2** We see that we have *two* independent parameters  $p, q \in \mathbb{Z}$ .  $\diamond$

(b) A computation gives here

$$\begin{aligned} (-i)^{-i} &= \exp(-i\{\ln |-i| + i \arg(-i)\}) = \exp(-i \cdot i \cdot \arg(-i)) \\ &= \exp(\arg(-i)) = \exp\left(\left(2p - \frac{1}{2}\right)\pi\right), \quad p \in \mathbb{Z}, \end{aligned}$$

i.e. a set of real positive numbers.

(c) Here,

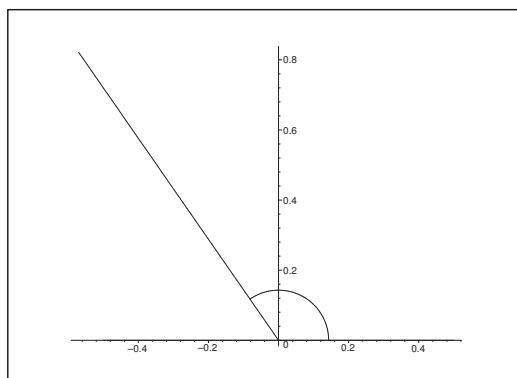
$$3^{\pi} = \exp(\pi \ln 3) = \exp(\pi\{\ln 3 + 2ip\pi\}) = e^{\pi^3} \{\cos(2p\pi^2) + i \sin(2p\pi^2)\}, \quad p \in \mathbb{Z}.$$

It is easily seen that this set is dense on the circle of centrum 0 and radius  $e^{\pi \ln 3}$ .

(d) Here,

$$2^{\pi i} = \exp(\pi i \log 2) = \exp(\pi i\{\ln 2 - 2ip\pi\}) = e^{2p\pi^2} \{\cos(\pi \ln 2) + i \sin(\pi \ln 2)\}, \quad p \in \mathbb{Z},$$

which represent infinitely many numbers on the half line from 0 and forming the angle  $\pi \ln 2$  with the real axis.

Figure 36: The points of **(d)** lie on a half line.

**Example 5.5** Compute all values of

$$(a) (1+i)^{1+i}, \quad (b) (1+i)^i(1+i)^{-i}, \quad (c) i^2.$$

**(a)** It follows from the definition and a computation that

$$\begin{aligned} (1+i)^{1+i} &:= \exp((1+i)\log(1+i)) = \exp\left((1+i)\left(\frac{1}{2}\ln 2 + i\left\{\frac{\pi}{4} + 2p\pi\right\}\right)\right) \\ &= \exp\left(\frac{1}{2}\ln 2 - \frac{\pi}{4} - 2p\pi + i\left\{\frac{1}{2}\ln 2 + \frac{\pi}{4} + 2p\pi\right\}\right) \\ &= \sqrt{2} \exp\left(-\frac{\pi}{4} - 2p\pi\right) \times \left\{\cos\left(\frac{1}{2}\ln 2 + \frac{\pi}{4}\right) + i \sin\left(\frac{1}{2}\ln 2 + \frac{\pi}{4}\right)\right\}. \end{aligned}$$

**(b)** By the definition,

$$\begin{aligned} (1+i)^i(1+i)^{-i} &:= \exp(i\log(1+i)) \cdot \exp(-i\log(1+i)) \\ &= \exp\left(i\left\{\frac{1}{2}\ln 2 + i\left(\frac{\pi}{4} + 2p\pi\right)\right\}\right) \exp\left(-i\left\{\frac{1}{2}\ln 2 + i\left(\frac{\pi}{4} + 2q\pi\right)\right\}\right) \\ &= \exp\left(-\frac{\pi}{4} - 2p\pi + i\frac{1}{2}\ln 2 - i\frac{1}{2}\ln 2 + \frac{\pi}{4} + 2q\pi\right) \\ &= \exp(2(q-p)\pi) = \exp(2n\pi), \quad n \in \mathbb{Z}. \end{aligned}$$

**(c)** At this stage the reader should be suspicious concerning many-valued functions. Nevertheless, this example should not give any problem,

$$i^2 = -1.$$

**Example 5.6** Compute all values of

$$(a) 2^i, \quad (b) \sqrt[4]{16}, \quad (c) (\sqrt{3} - i)^i, \quad (d) |(-1)^i|.$$

(a)

$$2^i := \exp(i \log 2) = \exp(i (\ln 2 - 2ip\pi)) = e^{2p\pi} \{\cos(\ln 2) + i \sin(\ln 2)\}, \quad p \in \mathbb{Z}.$$

(b)

$$\sqrt[4]{16} := \exp\left(\frac{1}{4} \log 16\right) = \exp\left(\frac{1}{4} (4 \ln 2 + 2ip\pi)\right) = \exp\left(\ln 2 + ip \frac{\pi}{2}\right) = \{2, 2i, -2, -2i\}.$$

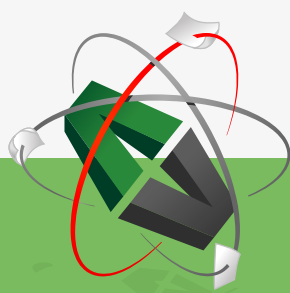
(c)

$$\begin{aligned} (\sqrt{3} - i)^i &:= \exp(i \log(\sqrt{3} - i)) = \exp\left(i \left\{ \ln 2 + i \left(-\frac{\pi}{6} - 2ip\pi\right) \right\}\right) \\ &= \exp\left(\frac{\pi}{6} + 2p\pi\right) \cdot \{\cos(\ln 2) + i \sin(\ln 2)\}, \quad p \in \mathbb{Z}. \end{aligned}$$

(d)

$$|(-1)^i| = |\exp(i\{-i\pi - 2ip\pi\})| = |\exp(\pi + 2p\pi)| = e^{(2p+1)\pi}, \quad p \in \mathbb{Z}.$$

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**Remark 5.3** The strange thing here is that also

$$(-1)^i = \exp(i\{-i\pi - 2ip\pi\}) = \exp(\pi + 2p\pi) = e^{(2p+1)\pi}, \quad p \in \mathbb{Z},$$

hence  $(-1)^i = |(-1)^i|$ .  $\diamond$

**Example 5.7** Put  $z = r e^{i\theta}$ . Compute all values of

$$(a) \operatorname{Re}(z^i), \quad (b) \operatorname{Im}(z^i), \quad (c) |z^i|.$$

It follows from

$$z^i := \exp(i\{\ln r + i\theta - 2ip\pi\}) = \exp(-\theta + 2p\pi + i \ln r),$$

that

(a)

$$\operatorname{Re}(z^i) = e^{\theta+2p\pi} \cos(\ln r), \quad p \in \mathbb{Z},$$

(b)

$$\operatorname{Im}(z^i) = e^{-\theta+2p\pi} \sin(\ln r), \quad p \in \mathbb{Z},$$

(c)

$$|z^i| = e^{-\theta+2p\pi}, \quad p \in \mathbb{Z}.$$

**Example 5.8** Put  $z = x + iy$ . Compute all values of

$$(a) \operatorname{Re}(i^z), \quad (b) \operatorname{Im}(i^z), \quad (c) |i^z|.$$

It follows from

$$\begin{aligned} i^z &:= \exp(z \cdot \log i) = \exp\left((x + iy)i \left\{\frac{\pi}{2} + 2p\pi\right\}\right) \\ &= \exp\left(-u \left\{\frac{1}{2} + 2p\right\} \pi + i x \left\{\frac{1}{2} + 2p\right\} \pi\right), \quad p \in \mathbb{Z}, \end{aligned}$$

that

(a)

$$\operatorname{Re}(i^z) = \exp\left(-y\pi \left\{2p + \frac{1}{2}\right\}\right) \cdot \cos\left(x\pi \left\{2p + \frac{1}{2}\right\}\right), \quad p \in \mathbb{Z},$$

(b)

$$\operatorname{Im}(i^z) = \exp\left(-y\pi \left\{2p + \frac{1}{2}\right\}\right) \cdot \sin\left(x\pi \left\{2p + \frac{1}{2}\right\}\right), \quad p \in \mathbb{Z},$$

(c)

$$|i^z| = \exp\left(-y\pi\left\{2p + \frac{1}{2}\right\}\right), \quad p \in \mathbb{Z}.$$

**Example 5.9** Find all the solutions of the following equations

$$(a) \operatorname{Log} z = \frac{1}{4}\pi i, \quad (b) e^z = i, \quad (c) e^z = 1 + i\sqrt{3}.$$

(a) We first note that  $\frac{1}{4}\pi i$  lies in the image of the principal logarithm, so there exists a solution. This is given by

$$z = e^{\operatorname{Log} z} = \exp\left(i\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1 + i).$$

(b)

$$z = \log i = i\left\{\frac{\pi}{2} + 2p\pi\right\}, \quad p \in \mathbb{Z}.$$

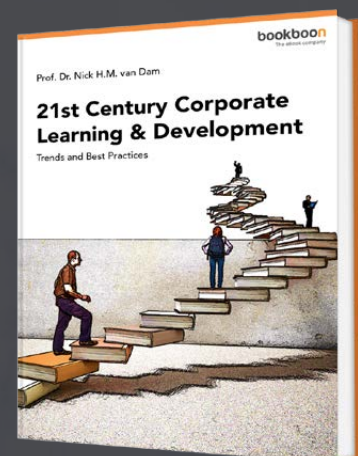
(c)

$$z = \log(1 + i\sqrt{3}) = \ln 2 + i\left\{\frac{\pi}{3} + 2p\pi\right\}, \quad p \in \mathbb{Z}.$$

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## 6 The Arcus Functions and the Area Functions

**Example 6.1** Find all solutions of the following equations,

$$(a) \cos z = \sin z, \quad (b) \sin z = 2, \quad (c) \sin z = \cosh 4.$$

(a) It follows by a rearrangement and a multiplication by

$$\frac{1}{\sqrt{2}} = \sin \frac{\pi}{4} = \cos \frac{\pi}{4}$$

using the *addition formula* for  $\sin z$  that

$$0 = \frac{1}{\sqrt{2}} \sin z - \frac{1}{\sqrt{2}} \cos z = \sin z \cdot \cos \left(-\frac{\pi}{4}\right) + \cos z \cdot \sin \left(-\frac{\pi}{4}\right) = \sin \left(z - \frac{\pi}{4}\right).$$

The zeros of  $\sin w$  are  $w = p\pi$ ,  $p \in \mathbb{Z}$ , so the set of solutions is

$$z = \frac{\pi}{4} + p\pi, \quad p \in \mathbb{Z}.$$

ALTERNATIVELY we get by *Euler's formula*,

$$\frac{1}{2} (e^{iz} + e^{-iz}) = \frac{1}{2i} (e^{iz} - e^{-iz}).$$

A multiplication by  $2ie^{iz} \neq 0$  gives

$$ie^{2iz} + i = e^{2iz} - 1,$$

thus

$$(1 - i)e^{2iz} = 1 + i,$$

and hence

$$e^{2iz} = \frac{1+i}{1-i} = \frac{i(-i+1)}{1-i} = i.$$

Taking the complex logarithm we get

$$2iz = \log i = i \frac{\pi}{2} + 2ip\pi, \quad p \in \mathbb{Z},$$

and the set of solutions is given by

$$z = \frac{\pi}{4} + p\pi, \quad p \in \mathbb{Z}.$$

ALTERNATIVELY it follows from  $\sin^2 z + \cos^2 z = 1$  that  $\sin z = \cos z \neq 0$  for every solution of the equation. Hence it is equivalent to the equation

$$\tan z = 1.$$

Then finally,

$$z = \arctan 1 = \frac{1}{2i} \log \left( \frac{1+i}{1-i} \right) = \frac{1}{2i} \log i = \frac{1}{2i} \left\{ i \frac{\pi}{2} + 2ip\pi \right\} = \frac{\pi}{4} + p\pi, \quad p \in \mathbb{Z}.$$



(b) The solution of the equation  $\sin z = 2$  is given by a solution formula,

$$\begin{aligned} z &= \arcsin 2 = \frac{1}{i} \log(2i \pm \sqrt{1-4}) = \frac{1}{i} \log(i\{2 \pm \sqrt{3}\}) \\ &= \frac{1}{i} \left\{ \ln(2 \pm \sqrt{3}) + i \left( \frac{\pi}{2} + 2p\pi \right) \right\} = \frac{\pi}{2} + 2p\pi \mp \ln(2 + \sqrt{3}), \quad p \in \mathbb{Z}. \end{aligned}$$

(c) The solution of  $\sin z = \cosh 4$  is also given by a solution formula,

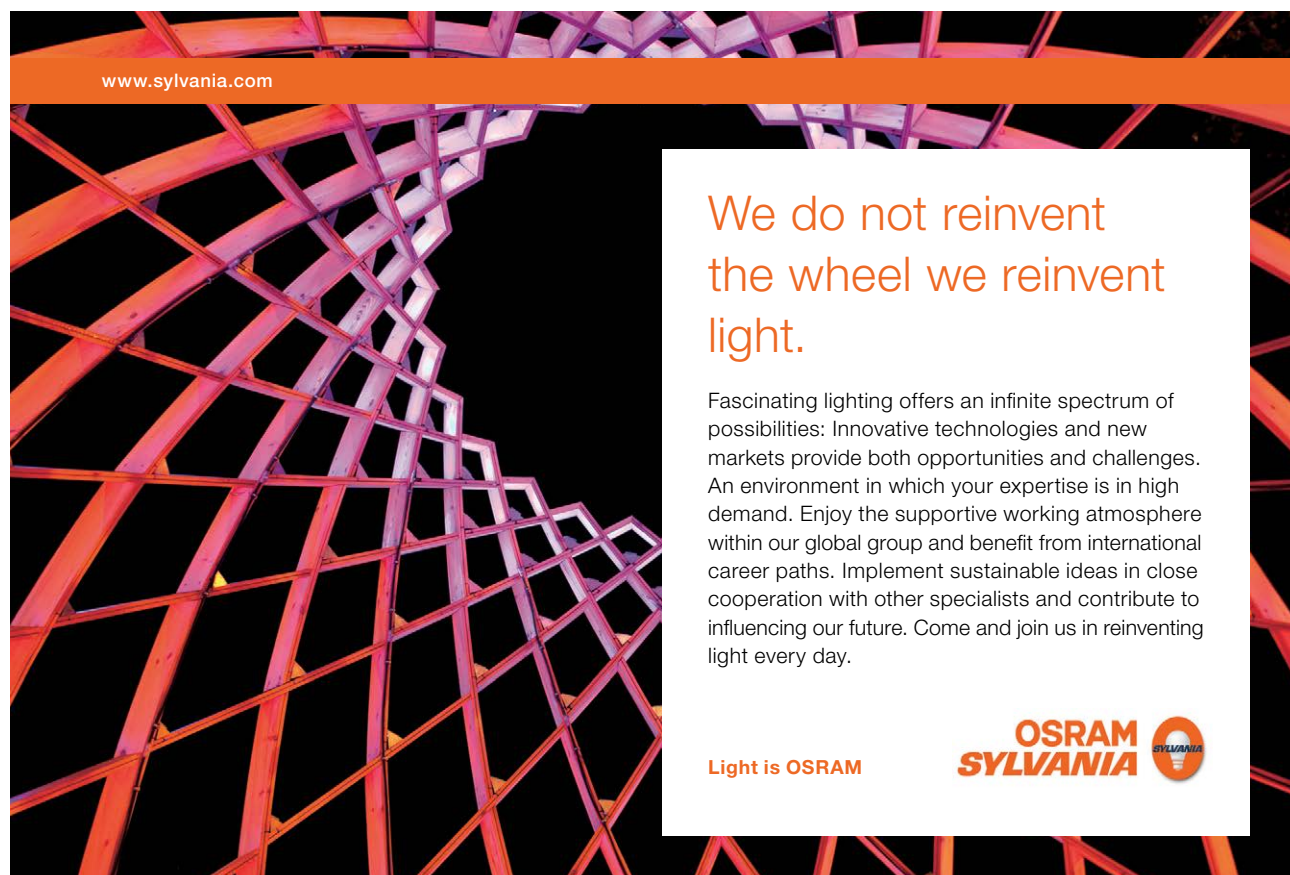
$$\begin{aligned} z &= \arcsin(\cosh 4) = \frac{1}{i} \log(i \cosh 4 \pm \sqrt{1 - \cosh^2 4}) \\ &= \frac{1}{i} \log(i \cosh 4 \pm \sqrt{-\sinh^2 4}) = \frac{1}{i} \log(i \cosh 4 \pm i \sinh 4) \\ &= \frac{1}{i} \log(i\{\cosh 4 \pm \sinh 4\}). \end{aligned}$$

Now

$$\cosh 4 + \sinh 4 = e^4 \quad \text{and} \quad \cosh 4 - \sinh 4 = e^{-4},$$

hence by insertion,

$$z = \frac{1}{i} \log(i e^{\pm 4}) = \frac{1}{i} \left\{ \pm 4 + i \frac{\pi}{2} + 2ip\pi \right\} = \frac{\pi}{2} + 2p\pi \mp 4i, \quad p \in \mathbb{Z}.$$




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**Example 6.2** Find all values of

(a)  $\arcsin i$ , (b)  $\arcsin 1$ , (c)  $\arcsin 100$ .

(a) By insertion into the solution formula we get

$$\begin{aligned}\arcsin i &= \frac{1}{i} \log(-1 \pm \sqrt{1+1}) = \frac{1}{i} \log(-1 \pm \sqrt{2}) \\ &= \begin{cases} \frac{1}{i} \{\ln(\sqrt{2}-1) + 2ip\pi\} &= 2p\pi + i \ln(\sqrt{2}+1), \\ \frac{1}{i} \{\ln(\sqrt{2}+1) + i(2p+1)\pi\} &= (2p+1)\pi - i \ln(\sqrt{2}-1), \end{cases}\end{aligned}$$

where  $p \in \mathbb{Z}$ .

(b) Here we get as expected,

$$\arcsin 1 = \frac{1}{i} \log(i + \sqrt{1-1}) = \frac{1}{i} \log i = \frac{1}{i} \left\{ i \left( \frac{\pi}{2} + 2p\pi \right) \right\} = \frac{\pi}{2} + 2p\pi, \quad p \in \mathbb{Z}.$$

(c) By using the solution formula,

$$\begin{aligned}\arcsin 100 &= \frac{1}{i} \log(i \cdot 100 \pm \sqrt{1-100^2}) = \frac{1}{i} \log(i\{100 \pm 3\sqrt{1111}\}) = \\ &= -i \left\{ \pm \ln(100 + 3\sqrt{1111}) + i \left( \frac{\pi}{2} + 2p\pi \right) \right\} \\ &= \frac{\pi}{2} + 2p\pi \mp i \ln(100 + 3\sqrt{1111}), \quad p \in \mathbb{Z}.\end{aligned}$$

**Example 6.3** Solve the equations

(a)  $\cos^2 z = -1$ , (b)  $\sin z + \cos z = i$ , (c)  $\cos z = 2$ .

Whenever one considers trigonometric equations one shall in general first try to reduce them as much as possible.

(a) Here it is tempting first to find the square root, so we shall solve the equation  $\cos z = \pm i$ . This is of course possible, but we shall get some problems with the double  $\pm$ . Hence, it is easier first to notice that

$$\cos 2z = 2 \cos^2 z - 1 = 2 \cdot (-1) - 1 = -3.$$

Then we get directly by the solution formula,

$$\begin{aligned}z &= \frac{1}{2} \arccos(-3) = \frac{1}{2i} \log(-3 \pm i\sqrt{1-9}) = \frac{1}{2i} \log(-\{3 \pm 2\sqrt{2}\}) \\ &= \frac{1}{2i} \{\ln(3 \pm 2\sqrt{2}) + i(\pi + 2p\pi)\} = \frac{1}{2i} \left\{ \ln((\sqrt{2} \pm 1)^2) + i(\pi + 2p\pi) \right\} \\ &= \frac{\pi}{2} + p\pi \mp i \ln(\sqrt{2} + 1), \quad p \in \mathbb{Z}.\end{aligned}$$

(b) Since

$$\sin z + \cos z = \sqrt{2} \cdot \sin\left(z + \frac{\pi}{4}\right) = i,$$

the equation is reduced to

$$\sin\left(z + \frac{\pi}{4}\right) = \frac{i}{\sqrt{2}}.$$

Then by a rearrangement of the solution formula and a reduction,

$$\begin{aligned} z &= -\frac{\pi}{4} + \arcsin\left(\frac{i}{\sqrt{2}}\right) = -\frac{\pi}{4} + \frac{1}{i} \log\left(-\frac{1}{\sqrt{2}} \pm \sqrt{1 + \frac{1}{2}}\right) \\ &= -\frac{\pi}{4} + \frac{1}{i} \log\left(\frac{-1 \pm \sqrt{3}}{\sqrt{2}}\right) = \begin{cases} -\frac{\pi}{4} + \frac{1}{i} \left\{ \ln\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right) + 2ip\pi \right\} \\ -\frac{\pi}{4} + \frac{1}{i} \left\{ \ln\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right) + i(\pi + 2p\pi) \right\} \end{cases} \\ &= \begin{cases} -\frac{\pi}{4} + 2p\pi + i \ln\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right) \\ -\frac{\pi}{4} + \pi + 2p\pi - i \ln\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right) \end{cases} = \frac{\pi}{4} + 2p\pi \pm \left\{ \frac{\pi}{2} - i \ln\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right) \right\} \\ &= \frac{\pi}{4} + 2p\pi \pm \left\{ \frac{\pi}{2} - \frac{i}{2} \ln(2 + \sqrt{3}) \right\}, \quad p \in \mathbb{Z}, \end{aligned}$$

where we have used that

$$\ln\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right) = \frac{1}{2} \ln\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^2 = \frac{1}{2} \ln\left(\frac{4+2\sqrt{3}}{2}\right) = \frac{1}{2} \ln(2 + \sqrt{3}).$$

(c) It follows directly by the solution formula that

$$\begin{aligned} z &= \arccos 2 = \frac{1}{i} \log(2 \mp i\sqrt{1-4}) = \frac{1}{i} \log(2 \mp \sqrt{3}) = \frac{1}{i} \{\mp \ln(2 + \sqrt{3}) + 2ip\pi\} \\ &= 2p\pi \pm i \ln(2 + \sqrt{3}), \quad p \in \mathbb{Z}. \end{aligned}$$

**Example 6.4** Solve the equations

$$(a) \cos z = \sqrt{3}, \quad (b) \cos z = 4, \quad (c) \cos z = \frac{3}{4} + \frac{i}{4}.$$

(a) The solution is

$$\begin{aligned} z &= \arccos \sqrt{3} = \frac{1}{i} \log(\sqrt{3} \pm i\sqrt{1-3}) = \frac{1}{i} \log(\sqrt{3} \mp \sqrt{2}) \\ &= \frac{1}{i} \{\mp \ln(\sqrt{3} + \sqrt{2}) + 2ip\pi\} = 2p\pi \pm i \ln(\sqrt{3} + \sqrt{2}), \quad p \in \mathbb{Z}. \end{aligned}$$

(b) The solution is

$$\begin{aligned} z &= \arccos 4 = \frac{1}{i} \log(4 \pm i\sqrt{1-16}) = \frac{1}{i} \log(4 \mp \sqrt{15}) \\ &= \frac{1}{i} \{\mp \ln(4 + \sqrt{15}) + 2ip\pi\} = 2p\pi \pm i \ln(4 + \sqrt{15}), \quad p \in \mathbb{Z}. \end{aligned}$$

The solution is

$$\begin{aligned} z &= \arccos \left( \frac{3}{4} + \frac{i}{4} \right) = \frac{1}{i} \log \left( \frac{3+i}{4} \pm i\sqrt{1 - \frac{(3+i)^2}{16}} \right) \\ &= \frac{1}{i} \log \left( \frac{3+i}{4} \pm \frac{i}{4} \sqrt{16 - (9 - 1 + 6i)} \right) = \frac{1}{i} \log \left( \frac{3+i}{4} \pm \frac{i}{4} \sqrt{8 - 6i} \right) \\ &= \frac{1}{i} \log \left( \frac{3+i}{4} \pm \frac{i}{4} (3-i) \right) = \frac{1}{i} \log \left( \frac{3+1}{4} \pm \frac{1+3i}{4} \right) \\ &= \left\{ \begin{array}{l} \frac{1}{i} \log(1+i) = \frac{1}{i} \left\{ \frac{1}{2} \ln 2 + i \frac{\pi}{4} + 2ip\pi \right\} \\ \frac{1}{i} \log \left( \frac{1-i}{2} \right) = \frac{1}{i} \left\{ -\frac{1}{2} \ln 2 - i \frac{\pi}{4} + 2ip\pi \right\} \end{array} \right\} \\ &= 2p\pi \pm \left\{ \frac{\pi}{4} - \frac{i}{2} \ln 2 \right\}, \quad p \in \mathbb{Z}. \end{aligned}$$

**Example 6.5** Solve the equation

$$(1 + i\sqrt{3}) \cos z - (\sqrt{3} - i) \sin z = 4e^{-iz}.$$

Using that

$$(1 + i\sqrt{3}) \cos z - (\sqrt{3} - i) \sin z = (1 + i\sqrt{3}) \{\cos z + i \sin z\} = (1 + i\sqrt{3})e^{iz},$$

the equation is reduced to

$$(1 + i\sqrt{3})e^{iz} = 4e^{-iz}.$$

Hence, the equation is equivalent to

$$e^{2iz} = \frac{4}{1+i\sqrt{3}} = \frac{4}{1+3} (1-i\sqrt{3}) = 1-i\sqrt{3} = 2 \exp\left(-i\frac{\pi}{3}\right),$$

så vi finder, at

$$2iz = \log\left(2 \exp\left(-i\frac{\pi}{3}\right)\right) = \ln 2 - i\frac{\pi}{3} + 2ip\pi, \quad p \in \mathbb{Z},$$

so

$$z = p\pi - \frac{\pi}{6} - i\frac{1}{2} \ln 2, \quad p \in \mathbb{Z}.$$



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**Example 6.6** Prove that the complex formula

$$(3) \quad \arccos z = \frac{1}{i} \operatorname{Log} \left( z + i \exp \left( \frac{1}{2} \operatorname{Log} (1 - z^2) \right) \right)$$

for  $z = t \in ]-1, 1[$  is equal to the usual real function  $\operatorname{Arccos} t$ .

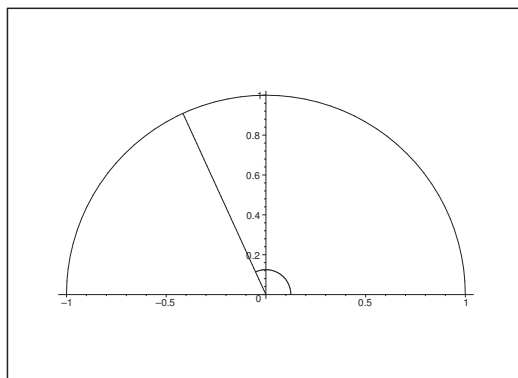


Figure 37: The point on the circle is  $t + i\sqrt{1-t^2}$ ,  $t \in ]-1, 1[$ , and the corresponding angle is  $\operatorname{Arccos} t$ .

Let  $z = t \in ]-1, 1[$ . When we compute the right hand side of (3), we get

$$\begin{aligned} \frac{1}{i} \operatorname{Log} \left( t + i \exp \left\{ \frac{1}{2} \operatorname{Log} (1 - t^2) \right\} \right) &= \frac{1}{i} \operatorname{Log} \left( t + i\sqrt{1-t^2} \right) \\ &= \frac{1}{i} \left\{ \frac{1}{2} \ln (t^2 + 1 - t^2) + i \operatorname{Arg} \left( t + i\sqrt{1-t^2} \right) \right\} \\ &= \operatorname{Arg} \left( t + i\sqrt{1-t^2} \right) = \operatorname{Arccos} \left( \frac{t}{\sqrt{t^2 + 1 - t^2}} \right) = \operatorname{Arccos} t, \end{aligned}$$

because  $\sqrt{1-t^2} > 0$  for  $t \in ]-1, 1[$ .

**Example 6.7** Solve the equations

$$(a) \quad e^z = -1 + i, \quad (b) \quad \tan z = i, \quad (c) \quad \cos^2 z = -9,$$

(a) The solution is

$$z = \log(-1 + i) = \frac{1}{2} \ln 2 + i \left\{ \frac{3\pi}{4} + 2p\pi \right\}, \quad p \in \mathbb{Z}.$$

(b) The set of solutions is empty, because  $\tan z = i$  implies that

$$\frac{1}{\cos^2 z} = 1 + \tan^2 z = 1 + i^2 = 0,$$

which is not possible for any  $z \in \mathbb{C}$ .

(c) Using that

$$\cos 2z = 2 \cos^2 z - 1 = -19,$$

we get the solution

$$\begin{aligned} z &= \frac{1}{2} \arccos(-19) = \frac{1}{2} \cdot \frac{1}{i} \log \left( -19 \pm i \sqrt{1 - (-19)^2} \right) \\ &= \frac{1}{2i} \log(-19 \pm i \sqrt{1 - 361}) = \frac{1}{2i} \log(-19 \pm i \cdot 6i \sqrt{10}) \\ &= \frac{1}{2i} \log(-\{19 \pm 6\sqrt{10}\}) = \frac{1}{2i} \{\ln(19 \pm 6\sqrt{10}) + i(\pi + 2p\pi)\} \\ &= \frac{\pi}{2} + p\pi + \frac{1}{2i} \ln \left( \{\sqrt{10} \pm 3\}^2 \right) \\ &= \frac{\pi}{2} + p\pi \mp i \ln(3 + \sqrt{10}), \quad p \in \mathbb{Z}. \end{aligned}$$

**Example 6.8** Solve the equations

$$\tan z = 3i \quad \text{and} \quad \tanh z = 3.$$

(a) The solution is

$$\begin{aligned} z &= \arctan(3i) = \frac{1}{2i} \log \left( \frac{1-3}{1+3} \right) = \frac{1}{2i} \log \left( -\frac{1}{2} \right) \\ &= \frac{1}{2i} \{-\ln 2 + i(\pi + 2p\pi)\} = \frac{\pi}{2} p\pi + i \frac{1}{2} \ln 2, \quad p \in \mathbb{Z}. \end{aligned}$$

(b) The solution is

$$\begin{aligned} z &= \operatorname{artanh} 3 = \frac{1}{2} \log \left( \frac{1+3}{1-3} \right) = \frac{1}{2} \log(-2) \\ &= \frac{1}{2} \{\ln 2 + i(\pi + 2p\pi)\} = \frac{1}{2} \ln 2 + i \left\{ \frac{\pi}{2} + p\pi \right\}, \quad p \in \mathbb{Z}. \end{aligned}$$

**Remark 6.1** It follows by a comparison that

$$\arctan(3i) = i \operatorname{artanh} 3. \quad \diamond$$

**Example 6.9** *Prove the complex formula*

$$(4) \operatorname{Arctan} z = \frac{1}{2i} \operatorname{Log} \left( \frac{1+iz}{1-iz} \right) = \frac{1}{2i} \operatorname{Log} \left( \frac{i-z}{i+z} \right)$$

is equal to usual real function  $\operatorname{Arctan} t$ .

Put  $z = t \in \mathbb{R}$ . When we compute the right hand of formula (4), we get

$$\begin{aligned} \frac{1}{2i} \operatorname{Log} \left( \frac{1+it}{1-it} \right) &= \frac{1}{2i} \left\{ \ln \left| \frac{1+it}{1-it} \right| + i \operatorname{Arg} \left( \frac{1+it}{1-it} \right) \right\} = \frac{1}{2i} \left\{ \ln 1 + i \operatorname{Arg} \left( \frac{(1+it)^2}{1+t^2} \right) \right\} \\ &= \frac{1}{2} \operatorname{Arg}(1+it)^2. \end{aligned}$$

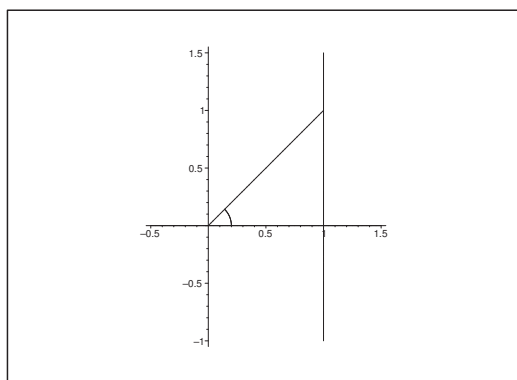


Figure 38: The line of the parametric description  $1 + it$ ,  $t \in \mathbb{R}$ .

Since  $1 + it$  lies in the *right* half plane, we have

$$\operatorname{Arg}(1 + it) \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[ ,$$

hence  $2 \operatorname{Arg}(1 + it) \in ] -\pi, \pi[$  is a principal value. Then we can continue in the following way,

$$\frac{1}{2i} \operatorname{Log} \left( \frac{1+it}{1-it} \right) = \frac{1}{2} \operatorname{Arg}(1+it)^2 = \frac{1}{2} \cdot 2 \operatorname{Arg}(1+it) = \operatorname{Arctan} \frac{t}{1} = \operatorname{Arctan} t,$$

and the claim is proved.



**Example 6.10** Find all values of

(a)  $\arctan(1 + 2i)$ , (b)  $\operatorname{arcosh}(2i)$ , (c)  $\operatorname{artanh}(1 - i)$ .

(a) We get by insertion into the formula and a computation that

$$\begin{aligned}\arctan(1 + 2i) &= \frac{1}{2i} \log \left( \frac{i - 1 - 2i}{i + 1 + 2i} \right) = \frac{1}{2i} \log \left( \frac{-1 - i}{1 + 3i} \right) = \frac{1}{2i} \log \left( -\frac{(1 + i)(1 - 3i)}{10} \right) \\ &= \frac{1}{2i} \log \left( -\frac{1}{10} \{1 + 3 + i(1 - 3)\} \right) \\ &= \frac{1}{2i} \log \left( -\frac{1}{5} (2 - i) \right) = \frac{1}{2i} \log \left( -\frac{2}{5} + \frac{i}{5} \right) \\ &= \frac{1}{2i} \left( \ln \frac{1}{\sqrt{5}} + i \left\{ \pi + \operatorname{Arctan} \left( -\frac{1}{2} \right) + 2p\pi \right\} \right) \\ &= -\frac{1}{2} \operatorname{Arctan} \frac{1}{2} + \frac{\pi}{2} + p\pi + \frac{i}{4} \ln 5, \quad p \in \mathbb{Z}.\end{aligned}$$

(b) Here we get

$$\operatorname{Arcosh}(2i) = \log(2i \pm \sqrt{-4 - 1}) = \log(i\{2 \pm \sqrt{5}\}) = \begin{cases} \ln(2 + \sqrt{5}) + i \left\{ \frac{\pi}{2} + 2p\pi \right\}, \\ -\ln(2 + \sqrt{5}) + i \left\{ \frac{3\pi}{2} + 2p\pi \right\}, \end{cases} \quad p \in \mathbb{Z}.$$

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(c) Finally,

$$\begin{aligned}\operatorname{artanh}(1-i) &= \frac{1}{2} \log \left( \frac{1+1-i}{1-1+i} \right) = \frac{1}{2} \log \left( \frac{2-i}{i} \right) = \frac{1}{2} \log(-1-2i) \\ &= \frac{1}{2} \{ \ln \sqrt{5} + i(\operatorname{Arctan} 2 + \pi + 2p\pi) \} \\ &= \frac{1}{4} \ln 5 + i \left\{ \frac{1}{2} \operatorname{Arctan} 2 + \frac{\pi}{2} + p\pi \right\}, \quad p \in \mathbb{Z}.\end{aligned}$$

**Example 6.11** Find all real numbers  $t \in \mathbb{R}$  for which the complex formula

$$(5) \quad \operatorname{Arccot} z = \frac{1}{2i} \operatorname{Log} \left( \frac{z+i}{z-i} \right)$$

is equal to the usual real function  $\operatorname{Arccot} t$ .

Put  $z = t \in \mathbb{R}$ , and then compute the right hand side of formula (5),

$$\frac{1}{2i} \operatorname{Log} \left( \frac{t+i}{t-i} \right) = \frac{1}{2i} \left\{ \ln \left| \frac{t+i}{t-i} \right| + i \operatorname{Arg} \left( \frac{t+i}{t-i} \right) \right\} = \frac{1}{2} \operatorname{Arg} \left( \frac{t+i}{t-i} \right).$$

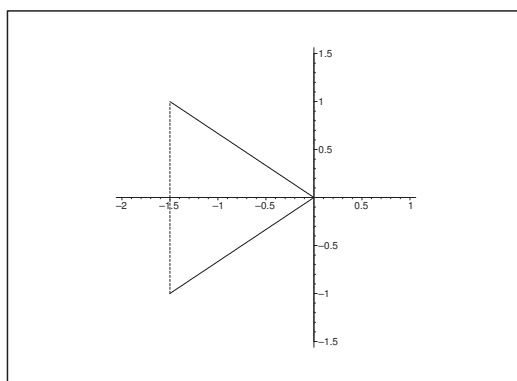


Figure 39: The points  $t-i$  and  $t$  and the corresponding angles for  $t = -1, 5$ .

Since  $t$  is real, it follows by considering the figure that

$$\operatorname{Arg}(t+i) = -\operatorname{Arg}(t-i), \quad t \in \mathbb{R},$$

and it is tempting to expect that

$$\operatorname{Arg}(t+i) = \operatorname{Arccot} t.$$

However, this is *not* true for all  $t$ . In fact, it follows by a computation that

$$\frac{1}{2i} \operatorname{Log} \left( \frac{t+i}{t-i} \right) = \frac{1}{2} \operatorname{Arg} \left( \frac{t+i}{t-i} \right) = \frac{1}{2} \operatorname{Arg} \left\{ \frac{(t+i)^2}{t^2+1} \right\} = \frac{1}{2} \operatorname{Arg} \{ (t+i)^2 \} = \frac{1}{2} \operatorname{Arg} (t^2 - 1 + 2it).$$

If  $t < 0$ , then  $t^2 - 1 + 2it$  lies in the lower half plane, so

$$\frac{1}{2} \operatorname{Arg}(t^2 - 1 + 2it) \in \left] -\frac{\pi}{2}, 0 \right[ ,$$

and since  $\operatorname{Arccot} t \in ]0, \pi[$ , this is not possible.

If  $t = 0$ , then of course

$$\frac{1}{2} \operatorname{Arg}(-1) = \frac{\pi}{2} = \operatorname{Arccot} 0.$$

If  $t > 0$ , then both  $t + i$  and  $t - i$  lie in the right half plane, so

$$\begin{aligned} \frac{1}{2i} \operatorname{Log} \left( \frac{t+i}{t-i} \right) &= \frac{1}{2} \operatorname{Arg} \left( \frac{t+i}{t-i} \right) = \frac{1}{2} \{ \operatorname{Arg}(t+i) - \operatorname{Arg}(t-i) \} \\ &= \operatorname{Arg}(t+i) = \operatorname{Arccot} t. \end{aligned}$$

Summing up we have

$$\frac{1}{2i} \operatorname{Log} \left( \frac{t+i}{t-i} \right) = \operatorname{Arccot} t \quad \text{for } t \text{ real and } t \geq 0.$$

**Example 6.12** Construct the Riemann surface of  $\arctan z$ .

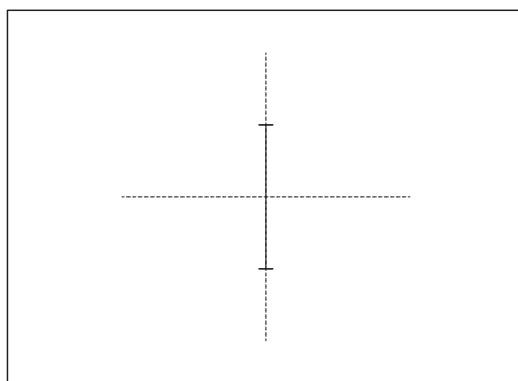


Figure 40: One branch with the branch cut along the imaginary axis from  $-i$  to  $i$ .

It follows from

$$\arctan z = \frac{1}{2i} \log \left( \frac{1+iz}{1-iz} \right), \quad z \in \mathbb{C} \setminus \{i, -i\},$$

that the branch points are  $i$  and  $-i$ , and that they are both logarithmic.

The surface is obtained by cutting infinitely many planes along the line segment between  $-i$  and  $i$  and then glue them together along the cut, such that the left hand edge (where  $\operatorname{Re}(z) < 0$  and  $-1 < \operatorname{Im}(z) < 1$ ) is glued to the right edge (where  $\operatorname{Re}(z) > 0$  and  $-1 < \operatorname{Im}(z) < 1$ ) in the plane above the first plane. Continue in this way infinitely often upwards as well as downwards.

**Example 6.13** Prove that the complex formula

$$(6) \operatorname{Arsinh} z = \operatorname{Log} \left( z + \exp \left( \frac{1}{2} \operatorname{Log} (z^2 + 1) \right) \right)$$

agrees with the usual real funktion  $\operatorname{Arsinh} t$  for  $z = t \in \mathbb{R}$ .

This is almost trivial, because we have for  $z = t \in \mathbb{R}$ ,

$$\begin{aligned} \operatorname{Log} \left( t + \exp \left\{ \frac{1}{2} \operatorname{Log} (t^2 + 1) \right\} \right) &= \operatorname{Log} \left( t + \exp \left\{ \ln \sqrt{t^2 + 1} \right\} \right) = \operatorname{Log} \left( t + \sqrt{t^2 + 1} \right) \\ &= \ln \left( t + \sqrt{t^2 + 1} \right) = \operatorname{Arsinh} t, \end{aligned}$$

where the latter follows from the usual real analysis.

**Example 6.14** Solve each of the equations

$$\tan z = 2i \quad \text{and} \quad \tan z = 1 - i.$$

(a) The solution is

$$\begin{aligned} z &= \arctan(2i) = \frac{1}{2i} \log \left( \frac{1-2}{1+2} \right) = \frac{1}{2i} \log \left( -\frac{1}{3} \right) = \frac{1}{2i} \{ -\ln 3 + i(\pi + 2p\pi) \} \\ &= \frac{\pi}{2} + p\pi + \frac{i}{2} \ln 3, \quad p \in \mathbb{Z}. \end{aligned}$$

(b) The solution is

$$\begin{aligned} z &= \arctan(1-i) = \frac{1}{2i} \log \left( \frac{1+i+1}{1-i-1} \right) = \frac{1}{2i} \log \left( \frac{2+i}{-i} \right) = \frac{1}{2i} \log(-1+2i) \\ &= \frac{1}{2i} \left\{ \ln \sqrt{5} + i \left( \operatorname{Arccot} \left( -\frac{1}{2} \right) + 2p\pi \right) \right\} = \frac{1}{2} \operatorname{Arccot} \left( -\frac{1}{2} \right) + p\pi - \frac{i}{4} \ln 5, \quad p \in \mathbb{Z}. \end{aligned}$$

**Example 6.15** An ocean is considered as a plane of usual rectangular coordinates. A navigable ship  $A$ , which sails with the constant speed  $a$ , receives at the position  $(x_0, y_0)$  the following SOS message:

“Ship  $B$  at the position  $(x_1, y_1)$ . All instruments destroyed. We sail with the speed  $b$  along a straight line though unknown direction.”

Assuming that  $a > b$ , prove that  $A$  can find  $B$  (i.e. sail in such a way that  $A$  at some later time is at the same position as  $B$ ).

Assume that  $(x_1, y_1) = 0$ . If ship  $A$  sails directly towards 0, then it will be at time  $t$  at the distance  $|z_0| - t \cdot a$  from 0. Hence, the two ships have the same distance from 0, when

$$|z_0| - t \cdot a = t \cdot b,$$

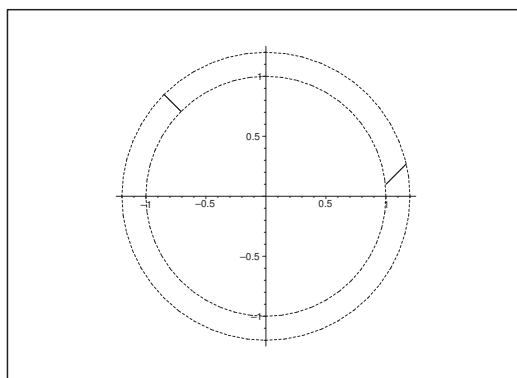


Figure 41:  $B$  starts at 0 and then sails along a radius (above to the left). The distance between the circles is  $b$ . Since  $a > b$ , then let  $A$  sail in a direction, such that  $A$  is always on the same circle as  $B$ .

i.e. to time  $t = \frac{|z_0|}{a+b}$ , where the distance to the centrum is  $\frac{b|z_0|}{a+b}$ . Then  $A$  changes its direction, such that  $A$  and  $B$  always have the same distance to 0 (i.e. such that they always lie on the same circle of centrum 0). Now,  $a > b$ , so  $A$  has a positive variation of the angle with respect to 0, and this variation of the angle will eventually go through the whole interval  $[0, 2\pi]$ . Since  $a > b$ , this is possible, when  $A$  runs through some part of a logarithmic spiral. Since  $B$  has the variation of the angle 0, the points  $A$  and  $B$  will eventually be at the same position, so it is actually possible for  $A$  to find  $B$  under the given circumstances.






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**Example 6.16** *Prove the formula*

$$\arccos z = -i \log \left( z + \sqrt{z^2 - 1} \right).$$

Put

$$z = \cos w = \frac{1}{2} (e^{iw} + e^{-iw}).$$

When we multiply by  $2e^{iw} \neq 0$  and then rearrange the equation, we get

$$(e^{iw})^2 - 2ze^{iw} + 1 = 0,$$

which is an equation of second degree in  $e^{iw}$ . The solution is

$$e^{iw} = z \pm \sqrt{z^2 - 1},$$

hence finally,

$$w = -i \log \left( z \pm \sqrt{z^2 - 1} \right).$$

**Example 6.17** *Prove the formula*

$$\operatorname{arcosh} z = \log \left( z + \sqrt{z^2 - 1} \right).$$

Put

$$z = \cosh w = \frac{1}{2} (e^w + e^{-w}).$$

When we multiply by  $2e^w$  and then rearrange the equation, we get

$$(e^w)^2 - 2ze^w + 1 = 0,$$

which is an equation of second degree in  $e^w$ . Its solution is

$$e^w = z \pm \sqrt{z^2 - 1},$$

and hence

$$w = \operatorname{arcosh} z = \log \left( z \pm \sqrt{z^2 - 1} \right).$$

**Example 6.18** *Prove the formula*

$$\operatorname{arsinh} z = \log \left( z + \sqrt{z^2 + 1} \right).$$

Put

$$z = \sinh w = \frac{1}{2} (e^w - e^{-w}).$$

When we multiply by  $2e^w$ , and then rearrange, we get

$$(e^w)^2 - 2ze^w - 1 = 0,$$

which is an equation of second degree in  $e^w$ . Its solution is

$$e^w = z \pm \sqrt{z^2 + 1},$$

thus

$$w = \operatorname{arsinh} z = \log \left( z \pm \sqrt{z^2 + 1} \right).$$

**Example 6.19** *Prove the formula*

$$\arctan z = \frac{i}{2} \log \frac{i+z}{i-z}.$$

Put

$$z = \tan w = \frac{1}{i} \frac{e^{2iw} - 1}{e^{2iw} + 1}.$$

When we solve this equation with respect to  $e^{2iw}$ , we get

$$e^{2iw} = \frac{-iz - 1}{iz - 1} = \frac{i - z}{i + z},$$

and thus

$$w = \arctan z = \frac{1}{2i} \log \left( \frac{i - z}{i + z} \right) = \frac{i}{2} \log \left( \frac{i + z}{i - z} \right).$$

**Example 6.20** *Prove the formula*

$$\operatorname{arccot} z = \frac{i}{2} \log \frac{z-i}{z+i}.$$

Put

$$z = \cot w = i \cdot \frac{e^{2iw} + 1}{e^{2iw} - 1}.$$

Solving this with respect to  $e^{2iw}$  gives

$$e^{2iw} = \frac{z+i}{z-i} = \left( \frac{z-i}{z+i} \right)^{-1},$$

and thus

$$w = \operatorname{arccot} z = -\frac{1}{2i} \log \left( \frac{z-i}{z+i} \right) = \frac{i}{2} \log \left( \frac{z-i}{z+i} \right).$$

**Example 6.21** *Prove the formula*

$$\operatorname{artanh} z = \frac{1}{2} \log \frac{1+z}{1-z}.$$

Put

$$z = \tanh w = \frac{e^{2w} - 1}{e^{2w} + 1}.$$

Solving this with respect to  $e^{2w}$  gives

$$e^{2w} = \frac{-z-1}{z-1} = \frac{1+z}{1-z},$$

hence

$$w = \operatorname{artanh} z = \frac{1}{2} \log \frac{1+z}{1-z}.$$

**Example 6.22** *Prove the formula*

$$\operatorname{arcoth} z = \frac{1}{2} \log \frac{z+1}{z-1}.$$

Put

$$z = \coth w = \frac{e^{2w} + 1}{e^{2w} - 1}.$$

Solving this with respect to  $e^{2w}$  gives

$$e^{2w} = \frac{z+1}{z-1},$$

thus

$$w = \operatorname{arcoth} z = \frac{1}{2} \log \frac{z+1}{z-1}.$$



**Example 6.23** Find all complex numbers  $z \in \mathbb{C}$ , which satisfy the equation

$$2 \cos 2z - 5 \cos z + 2 = 0.$$

This is a complex trigonometric equation, which can be solved in various ways.

**First variant.** The equation is solved as a pure trigonometric equation by using the formula

$$\cos 2z = 2 \cos^2 z - 1,$$

which also holds for complex numbers. By putting this into the equation we get


$$\begin{aligned} 0 &= 2 \cos 2z - 5 \cos z + 2 = 2 \{2 \cos^2 z - 1\} - 5 \cos z + 2 = 4 \cos^2 z - 5 \cos z \\ &= 4 \cos z \left\{ \cos z - \frac{5}{4} \right\}. \end{aligned}$$

Thus we get the two equations

$$\cos z = 0 \quad \text{and} \quad \cos z = \frac{5}{4}.$$

The solution of  $\cos z = 0$  is given by


$$z = \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z}.$$



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The latter equation  $\cos z = \frac{5}{4}$  does indeed make sense in the Theory of Complex Functions. According to a formula,

$$\begin{aligned} z &= \arccos \frac{5}{4} = \frac{1}{i} \log \left( \frac{5}{4} \pm i \sqrt{1 - \left( \frac{5}{4} \right)^2} \right) = \frac{1}{i} \log \left( \frac{5}{4} \pm i \sqrt{1 - \frac{25}{16}} \right) = \frac{1}{i} \log \left( \frac{5}{4} \pm i \sqrt{-\frac{9}{16}} \right) \\ &= \frac{1}{i} \log \left( \frac{5}{4} \pm i^2 \cdot \frac{3}{4} \right) = \frac{1}{i} \log \left( \frac{5}{4} \mp \frac{3}{4} \right). \end{aligned}$$

It follows from

$$\frac{5}{4} - \frac{3}{4} = \frac{1}{2} \quad \text{and} \quad \frac{5}{4} + \frac{3}{4} = 2,$$

that

$$\ln \left( \frac{5}{4} - \frac{3}{4} \right) = -\ln \left( \frac{5}{4} + \frac{3}{4} \right) = -\ln 2,$$

and

$$\arg \left( \frac{5}{4} - \frac{3}{4} \right) = \arg \left( \frac{5}{4} + \frac{3}{4} \right) = \{2p\pi \mid p \in \mathbb{Z}\},$$

hence

$$\begin{aligned} z &= \frac{1}{i} \log \left( \frac{5}{4} \mp \frac{3}{4} \right) = \frac{1}{i} \left\{ \ln \left( \frac{5}{4} \mp \frac{3}{4} \right) + i \arg \left( \frac{5}{4} \mp \frac{3}{4} \right) \right\} = \frac{1}{i} \{ \mp \ln 2 + i 2p\pi \} \\ &= \frac{1}{i} \{ \pm i^2 \ln 2 + i 2p\pi \} = 2p\pi \pm i \ln 2, \quad p \in \mathbb{Z}. \end{aligned}$$

The complete solution is given by

$$z = \begin{cases} \frac{\pi}{2} + p\pi, & p \in \mathbb{Z}, \\ 2q\pi \pm i \ln 2, & q \in \mathbb{Z}. \end{cases}$$

**Second variant.** ALTERNATIVELY we apply *Euler's formulæ*, i.e.

$$\cos z = \frac{1}{2} \{ e^{iz} + e^{-iz} \}.$$

This is the STANDARD METHOD, if the given equation already contains  $e^{iz}$ . Then substitute  $w = e^{iz}$ , and we get after a reduction an algebraic equation in  $w$ , which is then solved.

In the particular case we have

$$\cos 2z = \frac{1}{2} \{ e^{2iz} + e^{-2iz} \} = \frac{1}{2} \left\{ w^2 + \frac{1}{w^2} \right\},$$

and

$$\cos z = \frac{1}{2} \{ e^{iz} + e^{-iz} \} = \frac{1}{2} \left\{ w + \frac{1}{w} \right\},$$

hence by insertion,

$$0 = 2 \cos 2z - 5 \cos z + 2 = 2 \cdot \frac{1}{2} \left\{ w^2 + \frac{1}{w^2} \right\} - 5 \cdot \frac{1}{2} \left\{ w + \frac{1}{w} \right\} + 2.$$

When we multiply by  $2w^2 = 2e^{2iz} \neq 0$ , we get

$$\begin{aligned} 0 &= 2 \{w^4 + 1\} - 5 \{w^3 + w\} + 4w^2 = 2 \{w^4 + 2w^2 + 1\} - 5w \{w^2 + 1\} \\ &= 2(w^2 + 1)^2 - 5w(w^2 + 1) = (w^2 + 1) \{2w^2 - 5w + 2\}. \end{aligned}$$

This equation has the solutions  $w = \pm i$  and

$$w = \frac{5 \pm \sqrt{5^2 - 4 \cdot 2 \cdot 2}}{4} = \frac{5 \pm 3}{4} = \begin{cases} 2, \\ \frac{1}{2}. \end{cases}$$

Since  $w = e^{iz}$ , we shall then solve the four equations

$$(1) e^{iz} = i, \quad (2) e^{iz} = -i, \quad (3) e^{iz} = 2, \quad (4) e^{iz} = \frac{1}{2}.$$

1) If

$$e^{iz} = i = \exp\left(i \frac{\pi}{2}\right),$$

we get by the logarithm,

$$iz = i \cdot \frac{\pi}{2} + 2ip\pi, \quad p \in \mathbb{Z},$$

hence

$$z = \frac{\pi}{2} + 2p\pi, \quad p \in \mathbb{Z}.$$

2) If

$$e^{iz} = -i = \exp\left(-i \frac{\pi}{2}\right),$$

we get by the logarithm,

$$iz = -i \cdot \frac{\pi}{2} + 2ip\pi, \quad p \in \mathbb{Z},$$

hence

$$z = -\frac{\pi}{2} + 2p\pi, \quad p \in \mathbb{Z}.$$

3) If

$$e^{iz} = 2 = e^{\ln 2},$$

we get by the logarithm,

$$iz = \ln 2 + 2ip\pi, \quad p \in \mathbb{Z},$$

hence

$$z = 2p\pi - i \ln 2, \quad p \in \mathbb{Z}.$$

4) If

$$e^{iz} = \frac{1}{2} = e^{-\ln 2},$$

we get by the logarithm,

$$iz = -\ln 2 + 2ip\pi, \quad p \in \mathbb{Z},$$

hence

$$z = 2p\pi + i \ln 2, \quad p \in \mathbb{Z}.$$

**Remark 6.2** Notice that an easier variant is to note in (1) that  $w^2 + 1 = 0$  is equivalent to  $e^{2iz} + 1 = 0$ , thus

$$2iz = \log(-1) = i\pi + 2ip\pi, \quad p \in \mathbb{Z},$$

and hence

$$z = \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z}. \quad \diamond$$

**Example 6.24** Find all complex solutions of the equation

$$2e^{3iz}(1 + 2 \cos 2z) \sin z = -63i.$$

**First method.** We find by *Euler's formulæ*,

$$-63i = 2e^{3iz}(1 + 2 \cos 2z) \sin z = 2e^{3iz}(1 + e^{2iz} + e^{-2iz}) \cdot \frac{1}{2i}(e^{iz} - e^{-iz}),$$

hence by a multiplication by  $i$ ,

$$\begin{aligned} (7) \quad 63 &= e^{3iz}(e^{2iz} + 1 + e^{-2iz})(e^{iz} - e^{-iz}) = \{e^{2iz}(e^{2iz} + 1 + e^{-2iz})\} \{e^{iz}(e^{iz} - e^{-iz})\} \\ &= (e^{4iz} + e^{2iz} + 1)(e^{2iz} - 1) = e^{6iz} - 1, \end{aligned}$$

thus

$$(8) \quad e^{6iz} = 63 + 1 = 64 = 2^6,$$

and by the logarithm,

$$6iz = \log(2^6) = 6 \ln 2 + 2ip\pi, \quad p \in \mathbb{Z},$$

so the solution is

$$z = p \cdot \frac{\pi}{3} - i \ln 2, \quad p \in \mathbb{Z}.$$

**Remark 6.3** The computation (7) can be performed in many ways. The chosen one is neither the longest nor the shortest one.  $\diamond$

**Second method.** It is also possible to solve the equation by using some elementary trigonometry.

It follows from

$$\cos 2z \cdot \sin z = \frac{1}{2} \sin 3z - \frac{1}{2} \sin z,$$

fas

$$(1 + 2 \cos 2z) \sin z = \sin z + \sin 3z - \sin z = \sin 3z,$$

that

$$(9) \quad 2e^{3iz} \cdot \sin 3z = -63i.$$

Then by *Euler's formulæ*,

$$2e^{3iz} \cdot \frac{1}{2i} (e^{3iz} - e^{-3iz}) = \frac{1}{i} (e^{6iz} - 1) = -63i,$$

and we find again

$$e^{6iz} = 64.$$

Then proceed from (8) above.

**Remark 6.4** Since  $z$  is *complex*, we no longer have that  $\sin 3z$  is the imaginary part of  $e^{3iz}$ . (In spite of its name the imaginary part is always *real*, while  $\sin 3z$  is not real for all  $z \in \mathbb{C}$ ). Hence we can no longer immediately separate the original equation or equation (9) into its real and imaginary parts.  $\diamond$

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**Example 6.25** Find all complex solutions of the equation

$$e^{2iz}(2 \cos z - 3) + 2e^{iz} - \cos 2z = 2(4 + \sin^2 z).$$

By using *Euler's formulæ* we get the equation

$$e^{2iz}(e^{iz} - 3 + e^{-iz}) + 2e^{iz} - \frac{1}{2}e^{2iz} - \frac{1}{2}e^{-2iz} = 2\left\{4 + \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2\right\},$$

hence by a computation

$$e^{3iz} - 3e^{2iz} + e^{iz} + 2e^{iz} - \frac{1}{2}e^{2iz} - \frac{1}{2}e^{-2iz} = 2\left\{4 - \frac{1}{4}(e^{2iz} - 2 + e^{-2iz})\right\},$$

thus

$$e^{3iz} - 3e^{2iz} + e^{iz} + 2e^{iz} - \frac{1}{2}e^{2iz} - \frac{1}{2}e^{-2iz} = 8 - \frac{1}{2}e^{2iz} + 1 - \frac{1}{2}e^{-2iz},$$

which is reduced to

$$(10) \quad e^{3iz} - 3e^{2iz} + 3e^{iz} - 1 = 8.$$

This equation can also be written

$$(11) \quad (e^{iz} - 1)^3 = 2^3.$$

**Remark 6.5** Since

$$-\cos 2z = 2 \sin^2 z - 1,$$

it is of course faster to get to (10) by adding

$$\cos 2z - 1 = -2 \sin^2 z$$

to the original equation.  $\diamond$

**First variant.** We solve (11) by taking its cubic root. We get the following three possibilities,

$$\begin{cases} e^{iz} - 1 = 2, \\ e^{iz} - 1 = 2\left\{-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right\}, \\ e^{iz} - 1 = 2\left\{-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right\}, \end{cases} \quad \text{thus} \quad \begin{cases} e^{iz} = 3, \\ e^{iz} = i\sqrt{3}, \\ e^{iz} = -i\sqrt{3}. \end{cases}$$

By using the logarithm,

$$iz = \begin{cases} \log 3 & = \ln 3 + i2p\pi, \\ \log(i\sqrt{3}) & = \frac{1}{2} \ln 3 + i\left(\frac{\pi}{2} + 2p\pi\right) \\ \log(-i\sqrt{3}) & = \frac{1}{2} \ln 3 + i\left(-\frac{\pi}{2} + 2p\pi\right), \end{cases} \quad p \in \mathbb{Z},$$

where the latter two families of solutions may be joined in

$$iz = \frac{1}{2} \ln 3 + i \left( \frac{\pi}{2} + p\pi \right), \quad p \in \mathbb{Z}.$$

Hence, the solution is given by

$$z = \begin{cases} 2p\pi - i \ln 3, \\ \frac{\pi}{2} + p\pi - i \frac{1}{2} \ln 3, \end{cases} \quad p \in \mathbb{Z}.$$

**Second variant.** If one only derived (10) – and not (11) – then we may instead proceed in the following way,

First put  $w = e^{iz}$  into (10) to get the polynomial of entire coefficients

$$w^3 - 3z^2 - 3w^2 + 3w - 9 = 0.$$

Since the constant term is  $-9$  and the coefficient of  $w^3$  is 1, and since the polynomial has entire coefficients, any possible rational root must belong to the set

$$\{\pm 1, \pm 3, \pm 9\}.$$

A simple check shows that  $w = 3$  is a root, and then we get by a simple division that

$$w^3 - 3z^2 - 3w^2 + 3w - 9 = (w - 3)(w^2 + 3)0.$$

The problem is then reduced to solving the two equations

$$(a) \quad w = e^{iz} = 3 \quad \text{and} \quad (b) \quad w^2 = e^{2iz} = -3.$$

(a) We get by the logarithm,

$$iz = \ln 3 + 2ip\pi, \quad \text{dvs.} \quad z = 2p\pi - i \ln 3, \quad p \in \mathbb{Z}.$$

(b) We get by the logarithm,

$$2iz = \ln 3 + i\pi + 2ip\pi, \quad \text{dvs.} \quad z = \frac{\pi}{2} + p\pi - i \frac{1}{2} \ln 3, \quad p \in \mathbb{Z}.$$

Hence,

$$z = \begin{cases} 2p\pi - i \ln 3, \\ \frac{\pi}{2} + p\pi - i \frac{1}{2} \ln 3, \end{cases} \quad p \in \mathbb{Z}.$$

**Example 6.26** Find all solutions  $z \in \mathbb{C}$  of the equation

$$\cos 2z - \cos z = i e^{2iz} \sin z.$$

Putting  $w = e^{iz} \neq 0$ , it follows by Euler's formula,

$$\cos 2z = \frac{1}{2} \left\{ w^2 + \frac{1}{w^2} \right\}, \quad \cos z = \frac{1}{2} \left\{ w + \frac{1}{w} \right\}, \quad \sin z = \frac{1}{2i} \left\{ w - \frac{1}{w} \right\}.$$

When these expressions are put into the equation, we get

$$\frac{1}{2} \left\{ w^2 + \frac{1}{w^2} \right\} - \frac{1}{2} \left\{ w + \frac{1}{w} \right\} = i w^2 \cdot \frac{1}{2i} \left\{ w - \frac{1}{w} \right\}.$$

Then multiply by  $2w^2 \neq 0$  to get

$$w^4 + 1 - w^3 - w = w^5 - w^3,$$

which is rearranged and reduced to

$$0 = w^5 - w^4 + w - 1 = (w - 1)(w^4 + 1) = (e^{iz} - 1)(e^{4iz} + 1).$$

Hence the solution of the equation is given by

$$z = \frac{1}{i} \log 1 = \frac{1}{i} \cdot 2ip\pi = 2p\pi, \quad p \in \mathbb{Z},$$

and

$$z = \frac{1}{4i} \log(-1) = \frac{1}{4i} \cdot (2p+1)i\pi = \frac{\pi}{4} + p \cdot \frac{\pi}{2}, \quad p \in \mathbb{Z}.$$

ALTERNATIVELY,

$$\begin{aligned} \cos z + i e^{2iz} \sin z &= \frac{1}{2} (e^{iz} + e^{-iz}) + i e^{2iz} \cdot \frac{1}{2i} (e^{iz} - e^{-iz}) = \frac{1}{2} \{e^{iz} + e^{-iz} + e^{3iz} - e^{iz}\} \\ &= e^{iz} \cdot \frac{1}{2} (e^{2iz} + e^{-2iz}) = e^{iz} \cos 2z, \end{aligned}$$

and we reformulate the equation in the following way

$$e^{iz} \cos 2z - \cos 2z = (e^{iz} - 1) \cos 2z = 0,$$

from which we get the two possibilities

$$e^{iz} = 1 \quad \text{or} \quad \cos 2z = 0,$$

corresponding to the solutions

$$z = 2p\pi, \quad p \in \mathbb{Z}, \quad \text{or} \quad z = \frac{\pi}{4} + p \cdot \frac{\pi}{2}, \quad p \in \mathbb{Z}.$$



**Example 6.27** Find all solutions of the equation

$$e^{iz} \cos z - 4i \sin z + 1 = 2e^{-2iz}.$$

It follows by *Euler's formulæ* that the equation is equivalent to

$$e^{iz} \cdot \frac{1}{2} (e^{iz} + e^{-iz}) - 4i \cdot \frac{1}{2i} (e^{iz} - e^{-iz}) + 1 = 2e^{-2iz},$$

hence by a multiplication by  $2e^{2iz}$  ( $\neq 0$  for all  $z \in \mathbb{C}$ ) we get the equivalent equation

$$e^{2iz} (e^{2iz} + 1) - 4e^{iz} (e^{2iz} - 1) + 2e^{2iz} = 4.$$

Then by a reduction,

$$e^{4iz} - 4e^{3iz} + 3e^{2iz} + 4e^{iz} - 4 = 0.$$

Putting  $w = e^{iz}$ , we obtain the equation of fourth degree,

$$\begin{aligned} 0 &= w^4 - 4w^3 + 3w^2 + 4w - 4 = w^4 - 4w^3 + (4w^2 - w^2) + 4w - 4 \\ &= (w^4 - 4w^3 + 4w^2) - (w^2 - 4w + 4) = (w^2 - 1)(w^2 - 4w + 4) \\ &= (w - 1)(w + 1)(w - 2)^2, \end{aligned}$$

which has the simple roots  $-1$  and  $1$ , and the double root  $2$ .



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ALTERNATIVELY one may also find the roots as one of the rational possibilities

$$\{\pm 1, \pm 2, \pm 4\}.$$

This is left to the reader.

It follows from  $w = e^{iz}$  that we have

1) the equation  $e^{iz} = 1$ , hence

$$iz = \log 1 = 2ip\pi, \quad \text{or} \quad z = 2p\pi, \quad p \in \mathbb{Z},$$

2) the equation  $e^{iz} = -1$ , hence

$$iz = \log(-1) = i\pi + 2ip\pi, \quad \text{or} \quad z = \pi + 2p\pi, \quad p \in \mathbb{Z},$$

3) the equation  $e^{iz} = 2$ , hence

$$iz = \log 2 = \ln 2 + 2ip\pi, \quad \text{or} \quad z = 2p\pi - i \ln 2, \quad p \in \mathbb{Z}.$$

Summing up we see that the solution is given by

$$z = \begin{cases} p\pi, & p \in \mathbb{Z}, \\ 2p\pi - i \ln 2, & p \in \mathbb{Z}. \end{cases}$$

ALTERNATIVELY and with more calculations we see that the equation is equivalent to

$$\begin{aligned} 0 &= e^{iz} \cos z - 4i \sin z + 1 - 2e^{-2iz} \\ &= (\cos z + i \sin z) \cos z - 4i \sin z + 1 - 2 \cos 2z + 2i \sin 2z \\ &= \cos^2 z + i \sin z \cdot \cos z - 4i \sin z + 1 - 2(1 - 2 \sin^2 z) + 4i \sin z \cdot \cos z \\ &= 1 - \sin^2 z + i \sin z \{\cos z - 4 + 4 \cos z\} + 1 - 2 + 4 \sin^2 z \\ &= 3 \sin^2 z + i \sin z \{5 \cos z - 4\} = i \sin z \{-3i \sin z + 5 \cos z - 4\} \\ &= i \cdot \sin z \cdot \left\{ -\frac{3i}{2i} (e^{iz} - e^{-iz}) + \frac{5}{2} (e^{iz} + e^{-iz}) - 4 \right\} \\ &= i \cdot \sin z \cdot \left\{ -\frac{3}{2} e^{iz} + \frac{3}{2} e^{-iz} + \frac{5}{2} e^{iz} + \frac{5}{2} e^{-iz} - 4 \right\} = i \cdot \sin z \cdot \{e^{iz} - 4 + 4e^{-iz}\} \\ &= i e^{-iz} \sin z \cdot \{(e^{iz})^2 - 4e^{iz} + 4\} = i e^{-iz} \sin z \cdot \{e^{iz} - 2\}^2. \end{aligned}$$

This is 0, if either  $\sin z = 0$ , or  $e^{iz} = 2$ , hence the solutions are

$$\begin{cases} \sin z = 0 : & z = p\pi, \\ e^{iz} = 2 : & iz = \log 2 = \ln 2 + 2ip\pi, \end{cases} \quad p \in \mathbb{Z},$$

and the conclusion is

$$z = \begin{cases} p\pi, & p \in \mathbb{Z}, \\ 2p\pi - i \ln 2, & p \in \mathbb{Z}. \end{cases}$$

**Example 6.28** Find all solutions  $z \in \mathbb{C}$  of the equation

$$4i \cos z - 2e^{iz} \sin z + 3i = ie^{-iz}.$$

Apply Euler's formulae to get

$$\begin{aligned} 0 &= 4i \cos z - 2e^{iz} \sin z + 3i - ie^{-iz} = 4i \cdot \frac{1}{2} (e^{iz} + e^{-iz}) - 2e^{iz} \cdot \frac{1}{2i} (e^{iz} - e^{-iz}) + 3i - ie^{-iz} \\ &= 2i (e^{iz} + e^{-iz}) + ie^{iz} (e^{iz} - e^{-iz}) + 3i - ie^{-iz} = ie^{-iz} \{2e^{2iz} + 2 + e^{3iz} - e^{iz} + 3e^{iz} - 1\} \\ (12) &= ie^{-iz} \{(e^{iz})^3 + 2(e^{iz})^2 + 2e^{iz} + 1\} = ie^{-iz} (e^{iz} + 1) \{(e^{iz})^2 + e^{iz} + 1\}. \end{aligned}$$

Then either

$$e^{iz} = -1,$$

i.e.

$$iz = \log(-1) = i(\pi + 2p\pi), \quad p \in \mathbb{Z},$$

and thus

$$z = \pi + 2p\pi, \quad p \in \mathbb{Z},$$

or

$$e^{iz} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = \exp\left(\pm i \frac{2\pi}{3}\right),$$

i.e.

$$z = \pm \frac{2\pi}{3} + 2p\pi, \quad p \in \mathbb{Z}.$$

Hence all solutions are *real* and given by

$$z = \begin{cases} \pi + 2p\pi, \\ \frac{2\pi}{3} + 2p\pi, \\ \frac{4\pi}{3} + 2p\pi, \end{cases} \quad p \in \mathbb{Z}.$$

Notice that (12) can also be written

$$\begin{aligned} 0 &= ie^{-iz} (e^{iz} + 1) \{(e^{iz})^2 + e^{iz} + 1\} = ie^{i\frac{z}{2}} (e^{i\frac{z}{2}} + e^{-i\frac{z}{2}}) \{e^{iz} + 1 + e^{-iz}\} \\ &= 2ie^{i\frac{z}{2}} \cos \frac{z}{2} \cdot (1 + 2 \cos z) = 4ie^{i\frac{z}{2}} \cos \frac{z}{2} \cdot \left\{ \cos z + \frac{1}{2} \right\}, \end{aligned}$$

so we shall only solve

$$\cos \frac{z}{2} = 0,$$

which has the solutions  $\frac{z}{2} = \frac{\pi}{2} + p\pi$ , i.e.

$$z = \pi + 2p\pi, \quad p \in \mathbb{Z},$$

and

$$\cos z = -\frac{1}{2},$$

which has the solutions

$$z = \pm \frac{2\pi}{3} + 2p\pi, \quad p \in \mathbb{Z},$$

and we have found the same solutions as previously.

A THIRD VARIANT is given by the following,

$$\begin{aligned} 0 &= 4i \cos z - 2e^{iz} \sin z + 3i - ie^{-iz} \\ &= 4i \cos z - 2(\cos z + i \sin z) \sin z + 3i - i(\cos z - i \sin z) \\ &= 4i \cos z - 2 \cos z \sin z - 2i \sin^2 z + 2i + i - i \cos z - \sin z \\ &= 4i \cos z + 2i \cos^2 z + i(1 - \cos z) - \sin z(1 + 2 \cos z) \\ &= i(2 \cos^2 z + 3 \cos z + 1) - \sin z(1 + 2 \cos z) \\ &= i(2 \cos z + 1)(\cos z + 1) + i(1 + 2 \cos z) \cdot i \sin z \\ &= i(2 \cos z + 1)(\cos z + 1 + i \cdot \sin z) \\ &= i(2 \cos z + 1)(e^{iz} + 1), \end{aligned}$$

and the task is reduced to the solution of the two equations

$$\cos z = -\frac{1}{2} \quad \text{or} \quad e^{iz} = -1,$$

from which follows that

$$z = \begin{cases} \frac{2\pi}{3} + 2p\pi, \\ \frac{4\pi}{3} + 2p\pi, \\ \pi + 2p\pi, \end{cases} \quad p \in \mathbb{Z}.$$

**Example 6.29** Find all solutions  $z \in \mathbb{C}$  of the equation

$$\tan^2 z + (2 - 4i) \tan z = 3 + 6i.$$

Clearly, this equation is an equation of second degree in  $\tan z$ , so we get

$$\begin{aligned} \tan z &= -1 + 2i \pm \sqrt{(-1 + 2i)^2 + 3 + 6i} = -1 + 2i \pm \sqrt{1 - 4 - 4i + 3 + 6i} \\ &= -1 + 2i \pm \sqrt{2i} = -1 + 2i \pm \sqrt{(1 + i)^2} = -1 + 2i \pm (1 + i) \\ &= \begin{cases} -1 + 2i + 1 + 1 &= 3i, \\ -1 + 2i - 1 - i &= -2 + i. \end{cases} \end{aligned}$$

We shall only solve the two equations

$$(a) \tan z = 3i \quad \text{and} \quad (b) \tan z = -2 + i.$$

It follows from the formula of arctan that

(a)

$$\begin{aligned} z &= \arctan(3i) = \frac{1}{2i} \log \left( \frac{1-3}{1+3} \right) = \frac{1}{2i} \log \left( -\frac{1}{2} \right) = \frac{1}{2i} \{-\ln 2 + i(\pi + 2p\pi)\} \\ &= \frac{\pi}{2} + p\pi + i \frac{1}{2} \ln 2, \quad p \in \mathbb{Z}, \end{aligned}$$

(b)

$$\begin{aligned} z &= \arctan(-2+i) = \frac{1}{2i} \log \left( \frac{1-2i-1}{1+2i+1} \right) = \frac{1}{2i} \log \left( \frac{-2i}{2+2i} \right) = \frac{1}{2i} \log \left( \frac{-i}{1+i} \cdot \frac{1-i}{1-i} \right) \\ &= \frac{1}{2i} \log \left( \frac{1}{2} \{-1-i\} \right) = \frac{1}{2i} \log \left( \frac{1}{\sqrt{2}} \exp \left( i \frac{3\pi}{4} \right) \right) = \frac{1}{2i} \left\{ -\frac{1}{2} \ln 2 + i \left( \frac{3\pi}{4} + 2p\pi \right) \right\} \\ &= \frac{3\pi}{8} + p\pi + \frac{i}{4} \ln 2, \quad p \in \mathbb{Z}. \end{aligned}$$



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Hence,

$$z = \begin{cases} \frac{\pi}{2} + p\pi + i \frac{1}{2} \ln 2, & p \in \mathbb{Z}, \\ \frac{3\pi}{8} + p\pi + i \frac{1}{4} \ln 2, & p \in \mathbb{Z}. \end{cases}$$

ALTERNATIVELY, the equation is written as an equation in  $w = e^{2iz}$ . It follows from

$$\tan z = \frac{\sin z}{\cos z} = \frac{1}{i} \cdot \frac{e^{2iz} - 1}{e^{2iz} + 1} = \frac{1}{i} \cdot \frac{w - 1}{w + 1},$$

that

$$\begin{aligned} 0 &= \tan^2 z + (2 - 4i) \tan z - 3 - 6i = -\left(\frac{w-1}{w+1}\right)^2 + (2-4i) \cdot \frac{1}{i} \cdot \frac{w-1}{w+1} - 3 - 6i \\ &= -\frac{1}{(w+1)^2} \{(w-1)^2 - (2-4i) \cdot (-i) \cdot (w-1)(w+1) + (3+6i)(w+1)^2\} \\ &= -\frac{1}{(w+1)^2} \{(w-1)^2 + (4+2i)(w^2-1) + (3+6i)(w+1)^2\} \\ &= -\frac{1}{(w+1)^2} \{w^2 - 2w + 1 + (4+2i)w^2 - 4 - 2i + (3+6i)(w^2 + 2w + 1)\} \\ &= -\frac{1}{(w+1)^2} \{(1+4i+3+6i)w^2 + (-2+6+12i)w + (1-4-2i+3+6i)\} \\ &= -\frac{1}{(w+1)^2} \{(8+8i)w^2 + (4+12i)w + 4i\} \\ &= -\frac{4}{(w+1)^2} \{2(1+i)w^2 + (1+3i)w + i\} \\ &= -\frac{4}{(w+1)^2} \cdot \frac{1}{1-i} \cdot \{2 \cdot 2w^2 + (4+2i)w + 1+i\}. \end{aligned}$$

Since  $w = -1$  is not a zero of the numerator, we shall only find the solutions of the equation of second degree

$$4w^2 + (4+2i)w + 1+i = 0.$$

These are given by the usual solution formula

$$\begin{aligned} e^{2iz} = w &= \frac{-4-2i \pm \sqrt{(-4-2i)^2 - 4 \cdot 4(1+i)}}{8} = \frac{-4-2i \pm \sqrt{16+16i-4-16-16i}}{8} \\ &= \frac{-4-2i \pm \sqrt{-4}}{8} = \frac{-4-2i \pm 2i}{8} = \frac{-2-i \pm i}{4} = \begin{cases} -\frac{1}{2}, \\ -\frac{1+i}{2}, \end{cases} \end{aligned}$$

so

(a)

$$z = \frac{1}{2i} \log \left( -\frac{1}{2} \right) = \frac{1}{2i} \{-\ln 2 + i(\pi + 2p\pi)\} = \frac{\pi}{2} + p\pi + i \frac{1}{2} \ln 2, \quad p \in \mathbb{Z},$$

(b)

$$\begin{aligned}
 z &= \frac{1}{2i} \log \left( -\frac{1+i}{2} \right) = \frac{1}{2i} \log \left( \frac{1}{\sqrt{2}} \exp \left( -i \frac{3\pi}{4} \right) \right) = \frac{1}{2i} \left\{ -\frac{1}{2} \ln 2 + i \left( -\frac{3\pi}{4} + 2p\pi \right) \right\} \\
 &= -\frac{3\pi}{8} + p\pi + \frac{i}{4} \ln 2, \quad p \in \mathbb{Z}.
 \end{aligned}$$

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## 7 The inverse of an algebraic expression.

**Example 7.1** Find  $w$  as a many-valued function of  $z$ , when

$$3w^4 - 4w^3 - 6w^2 + 12w = z.$$

This example illustrates in all details what may happen, when one tries to find the inverse many-valued function.

We see that  $\Omega = \mathbb{C}$  and that the function  $f : \Omega \rightarrow \mathbb{C}$  is given by

$$f(w) = 3w^4 - 4w^3 - 6w^2 + 12w.$$

According to the *Fundamental Theorem of Algebra* the equation

$$f(w) = 3w^4 - 4w^3 - 6w^2 + 12w = z$$

has four solutions  $w_1, w_2, w_3, w_4$ , and since these four solutions usually are different, the inverse  $f^{\circ-1}(z)$  will in general be four-valued, and we conclude that the corresponding Riemann surface is composed of four branches.

Then we find the *singular points* in the  $w$ -plane, i.e. we shall find the points  $w$ , for which  $f'(w) = 0$ . We get the equation

$$f'(w) = 12(w^3 - w^2 - w + 1) = 12(w^2 - 1)(w - 1) = 12(w - 1)^2(w + 1) = 0.$$

It follows that the singular points are  $w_1 = 1$  and  $w_2 = 2$ , corresponding to the *branch points* in the  $z$ -plane

$$z_1 = f(w_1) = 5 \quad \text{and} \quad z_2 = f(w_2) = -11.$$

We have in general that if  $w_0 \in \Omega$  is a zero of order  $n - 1$  of the derivative  $f'(w)$ , i.e.

$$f^{(n)}(w_0) = f^{(1+n-1)}(w_0)$$

is the first *derivative*, which is different from zero (note that the value of the function itself  $f(w_0)$  here is of no importance), then the structure of the Riemann surface is in a neighbourhood of the branch point  $z_0 = f(w_0)$  of the same structure as the Riemann surface of  $\sqrt[n]{z - z_0}$ . Hence, we conclude that we have a *cubic root structure* around the point  $z_1 = 5$  and a *square root structure* around the point  $z_2 = -11$ .

ALTERNATIVELY we may give the following direct proof: By definition,  $w = 1$  is a root, so we get by a factor expansion of

$$0 = f(w) - f(w_1) = f(w) - 5,$$

that

$$f(w) - 5 = 3w^4 - 4w^3 - 6w^2 + 12w - 5 = (3w + 5)(w - 1)^3 = 0.$$

In a small neighbourhood of the point  $w_1 = 1$  the factor  $3w + 5$  is very close to 8, so we conclude that we must have a structure as a cubic root.



Analogously we get for  $0 = f(w) - f(w_2) = f(w) + 11$  that

$$f(w) + 11 = 3w^4 - 4w^3 - 6w^2 + 12w + 11 = (3w^2 - 10w + 11)(w + 1)^2,$$

and since the factor  $3w^2 - 10w + 11$  is close to 24 in a small neighbourhood of  $w_2 = -1$ , the square root structure follows.

Usually we start with the fundamental domains and then find the branch cuts from these. However, since the structures of the square root and the cubic root are well-known, we shall instead fix the branch cuts, i.e.  $] -\infty, -11]$  of the square root, and  $[5, +\infty[$  of the cubic root.

We have now four  $z$ -planes, which all must contain at least one branch cut. It follows by a small analysis, that since we shall distribute  $5 = 3+2$  branch cuts among four planes, two of the planes must only have the branch cut  $[5, +\infty[$  of the cubic root, one plane must have the branch cut  $] -\infty, -11]$  from the square root, and finally, one  $z$  plane is equipped with both branch cuts. Then we can use a pair of scissors, some paper and some glue to construct the Riemann surface (which of course must have some self intersections).

We shall still find the *fundamental domains* in the  $w$ -plane. This must be bounded by curves, which by the function  $z = f(w)$  is mapped either into the branch cut of the square root  $] -\infty, -11]$  (this is true for two curves from  $w = -1$  to  $\infty$ ) or into the branch cut of the cubic root  $[5, +\infty[$ . This is true for three curves from  $w = +1$  to  $\infty$ , where the half tangents of these curves shall form the angle  $\frac{2\pi}{3}$  with respect to each other in a neighbourhood of  $w = 1$ , because we here have approximately the same structure as the fundamental domains of  $(w - 1)^3 = a$ .

Let  $w = u + iv (= w(t))$  be any such curve. Then  $f(w)$  is either contained in  $] -\infty, -11]$  or in  $[5, +\infty[$ . In particular,  $\text{Im } f(w) = 0$ , so

$$0 = \text{Im} \{3w^4 - 4w^3 - 6w^2 + 12w\} = 4v \{3u^3 - 3uv^2 - 3u^2 + v^2 - 3u + 3\}.$$

It follows immediately that  $v = 0$ . However, the interval  $] -11, 5[$  is not contained in the image, so only the curve  $v = 0, u \geq +1$ , can be used, and this is mapped into  $[5, +\infty[$ .

We shall now neglect this possibility, so we assume that  $v \neq 0$  in the following. When we solve the equation with respect to  $v^2$ , then

$$(13) \quad v^2 = \frac{3(u-1)^2(u+1)}{3u-1}, \quad u \neq \frac{1}{3}.$$

A routine check shows that if  $u = \frac{1}{3}$ , then we only get the “solution”  $v = 0$ , which is rejected, because we assumed that  $v \neq 0$ . Hence, none of the solution curves can intersect the line  $u = \frac{1}{3}$ .

An analysis of the sign of (13) shows that if  $v \neq 0$ , then we have the following possibilities:

$$1) \quad u \in ] -\infty, -1[, \quad 2) \quad u \in \left] \frac{1}{3}, 1 \right[, \quad 3) \quad u \in ] 1, +\infty[.$$

It is easy to prove in 1) that the image curve is the branch cut  $] -\infty, -11]$  in the  $z$ -plane.

Similarly, we get in case 2) that the image curve is the branch cut  $] 5, +\infty[$ .

In case 3), however, we get a “false solution”, because the image curve becomes  $] -\infty, 5[$ , which has not been defined as a branch cut.

By taking the square root of (13) we obtain the four remaining curves.

The only thing which is missing is to find the correspondence between the four planes with their branch cuts and the fundamental domains. We shall first choose the  $z$ -plane, which has two branch cuts, i.e. in which both  $w = -1$  and  $w = +1$  are branch points. Then put the other  $z$ -plane also containing the square root branch cut below this plane and glue them together. Analogously, put the latter two  $z$ -planes with only the branch cut of the cubic root above the chosen plane and glue them together as a cubic root. Finally, if we abstract ourselves from the self intersections of this surface, we have constructed the corresponding Riemann surface of the inverse map  $w = f^{\circ-1}(z)$ .

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## 8 Simple example of potential flows

We collect in this chapter the simplest examples of the potentials, which are met in the literature. In the first three of them, the (*source* (and the *drain*), the *curl* and the *curl source*, the potential is a many-valued function. Finally, the *dipole*, is here derived from the joint system of a *source* and a *drain* of the same strength, followed by taking some limit. By this limit the potential of the dipole becomes a usual analytic function.

### Example 8.1 Sources and drains.

The potential of a *source* at  $z = 0$  is a (many-valued) function of the type

$$F(z) = \frac{N}{2\pi} \log z + c, \quad z \in \mathbb{C} \setminus \{0\}, \quad c \in \mathbb{C},$$

where we call  $N \in \mathbb{R} \setminus \{0\}$  the *strength of the source*. If  $N < 0$ , the source is called a *drain*.

By separating  $F(z) = \varphi(x, y) + i \cdot \psi(x, y)$  into its real and imaginary parts we get

$$\varphi(x, y) = \frac{N}{2\pi} \ln |z| + c_1, \quad \psi(x, y) = \frac{N}{2\pi} \arg z + c_2.$$

The *equipotential curves*  $\varphi(x, y) = k$  are circles  $|z| = C$ , where  $C = \exp\left(\frac{2\pi}{N}\{k - c_1\}\right)$ , and the *flow lines*  $\psi(x, y) = k$  are the half lines  $\arg z = \alpha$ . We should of course here write  $\alpha \in \arg z$  instead. However, the not so correct notation will hardly cause any misunderstanding.

If  $N > 0$ , we interpret the model as the description of a point source at  $z = 0$ , from which there is sent a radial flow. In instead  $N < 0$ , all flow arrows are pointing at  $z = 0$ , which explains why we here call it a *drain* instead.

### Example 8.2 Curls.

The potential of a *curl* around  $z = 0$  is a (many-valued) function of the form

$$F(z) = \frac{\Gamma i}{2\pi} \log z + c, \quad z \in \mathbb{C} \setminus \{0\}, \quad c \in \mathbb{C},$$

where the constant  $\Gamma \in \mathbb{R} \setminus \{0\}$  is called the *intensity of the curl*. When we separate into the real and imaginary parts we get

$$\varphi(x, y) = -\frac{\Gamma}{2\pi} \arg z + c_1, \quad \psi(x, y) = \frac{\Gamma}{2\pi} \ln |z| + c_2.$$

In this case the *flow lines* are the circles of centrum at  $z = 0$ , explaining why we call this a model of a curl.

### Example 8.3 Curl sources.

By adding a source and a curl both centred at  $z = 0$ , we get a *curl source* of the complex potential

$$F(z) = \frac{N + i\Gamma}{2\pi} \log z + c, \quad z \in \mathbb{C} \setminus \{0\}.$$

Its real potential is

$$\varphi(x, y) = \frac{N}{2\pi} \ln |z| - \frac{\Gamma}{2\pi} \arg z + c_1,$$

and the corresponding flow function is

$$\psi(x, y) = \frac{\Gamma}{2\pi} \ln |z| + \frac{N}{2\pi} \arg z + c_2.$$

Then, by using *polar coordinates* we get more well-arranged that

$$\psi(x, y) = \frac{1}{2\pi} (\Gamma \cdot \ln r + N\theta) + c_2,$$

so the flow lines in polar coordinates are given by

$$r = C \cdot \exp\left(-\frac{N}{\Gamma} \theta\right).$$

It follows that the flow lines are *logarithmic spirals*. Then a small consideration shows that the *equipotential curves* are also a system of *logarithmic spirals*, which are orthogonal to the system of flow lines.

#### Example 8.4 Dipoles.

For given  $h > 0$  we consider a system consisting of a source at  $z_1 = -\frac{h}{2}$  of strength  $\frac{N}{h}$ , and a drain at  $z_2 = \frac{h}{2}$  of strength  $-\frac{N}{h}$ . The corresponding complex potential is chosen as

$$F_h(z) = \frac{N}{2\pi h} \operatorname{Log}\left(z + \frac{h}{2}\right) - \frac{N}{2\pi h} \operatorname{Log}\left(z - \frac{h}{2}\right).$$

We obtain by taking the limit  $h \rightarrow 0+$  the complex potential of the *dipole*

$$F(z) = \lim_{h \rightarrow 0+} \frac{N}{2\pi} \frac{1}{h} \left\{ \operatorname{Log}\left(z + \frac{h}{2}\right) - \operatorname{Log}\left(z - \frac{h}{2}\right) \right\} = \frac{N}{2\pi} \cdot \frac{1}{z}, \quad z \in \mathbb{C} \setminus \{0\}.$$

When we separate into the real and the imaginary part, we see that the corresponding flow lines are circular arcs of centrum at  $(0, y)$  on the  $y$ -axis and of the radius  $|y| > 0$ . By sketching these we obtain the model of a dipole. We change the orientation of the dipole by also allowing  $N \neq 0$  to be a complex number.