Making Change: Analysis of a Greedy Algortithm

Problem: Suppose we want to make change for n cents using pennies (1 cent), nickels (5 cents), dimes (10 cents), and quarters (25 cents), but no other denomination. The goal is to find a combination of coins of these types with total value n that minimizes the number of coins used.

Here is a proposed greedy algorithm for this problem:

> Suppose the amount left to change is m. Add to the solution the largest-denomination coin whose value is no more than m, subtract this coin's value from m, and repeat until $m = 0$.

Prove that this algorithm always yields an optimal solution, i.e. one using the fewest number of coins possible.

1 Small Group Work

For this part, you will divide into small groups. Each group should work through the exercise below collaboratively. *Write down* your solution – you may be asked to present it to another group or to everyone present.

1. Just to show that the problem is not completely trivial, is the algorithm optimal if the only available denominations for making change are 1, 10, and 25 cents? What makes this case behave differently from the problem we posed with four denominations?

2. Now we will begin to write a correctness proof for the (four-denomination) algorithm. Formulate a *greedy choice property* specifically for the algorithm (i.e. translating generic terms like "feasible" and "optimal" into their precise definitions for the problem).

3. Try to formulate an exchange argument to prove this property. *Hint*: you will probably need to consider a number of cases. What is unusual about your argument, and why is it still a proof of the greedy choice property?

4. Formulate an *inductive structure* property specifically for the algorithm, and justify that it is true. How do you derive the subproblem from the original problem?

5. Formulate an *optimal substructure* property for the algorithm.

6. Prove the optimal substructure property. Please write out the whole contradiction argument.

You can use the greedy algorithms handout as a template to help you write the proof.

2 Shared Critique

Your TA will organized this part of the exercise. Be prepared to explain and defend your proofs and to show how the various pieces fit into the inductive structure.