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A Textbook of

Differential Calculus

for

JEE Main & Advanced

The Only *text book* which Starts from fundamentals and gradually builds your *concepts* upto the level required for *Engineering Entrances* and finally will place you among toppers.

Differentiation • Functions
Graphical Transformations
Limits, Continuity & Differentiability
Monotonicity, Maxima & Minima

Amit M Agarwal

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Preface

It is a matter of great pride and honour for me to have received such an overwhelming response to the previous editions of this book from the readers. In a way, this has inspired me to revise this book thoroughly as per the changed pattern of JEE Main & Advanced other Engineering Entrances. I have tried to make the contents more relevant to the needs of students, many topics have been re-written, a lot of new problems of new types have been added in this book now. I have made all possible efforts to remove all the printing errors that had crept in previous editions. The book is now in such a shape that the students would feel at ease while going through the problems which will in turn clear their concepts too.

Some of the Salient Features of the New Edition are:

- Thoroughly revised theory covering latest changes in the subject & the exam pattern.
- Solved Examples of new types has been added.
- Exercise have been enriched with Objective Questions of latest types of questions; Single Option Correct, Multiple Option Correct, Linked Comprehension Based, Matching Type, Assertion-Reason, Single Integer Answer Type.
- All Exercises have been fully solved.

Valuable suggestions from students and teachers are welcome, and these will find due places in the ensuing editions.

Amit M Agarwal

Contents

1

Essential Mathematical Tools

Chapter in a Snapshot

- Basic Definitions
- Closed and Open Intervals
- Modulus or Absolute Value Functions
- Wavy Curve Method
  (Number Line Rule/Sign Scheme for Rational Functions)
- Fundamentals of Quadratic Equations
- Equations and Inequations Containing Modulus

Basic Definitions

(i) **Natural Numbers** The set of numbers {1, 2, 3, 4, ...} are called natural numbers and is **denoted by *N***.

ie, $N = \{1, 2, 3, 4, ...\}$

(ii) **Integers** The set of numbers {......, – 3, – 2, – 1, 0, 1, 2, 3, ...} are called integers and the set is **denoted by *I* or *Z*.**

ie, I (or Z) $= \{....., -4, -3, -2, -1, 0, 1, 2, 3, ...\}$

where we represent;

(a) **Positive integers** by $I^+ = \{1, 2, 3, 4, ...\}$ = natural numbers.

(b) **Negative integers** by $I^- = \{......., -4, -3, -2, -1\}$

(c) **Non-negative integers** $\{0, 1, 2, 3, 4, ...\}$ = whole numbers

(d) **Non-positive integers** $\{......., -3, -2, -1, 0\}$

(iii) **Rational Numbers** All the numbers of the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$ are called rational numbers and their set is **denoted by *Q***.

ie, $Q = \frac{a}{b}$ such that $a, b \in I$ and $b \neq 0$ and HCF of a, b is 1.

Points to Consider

(a) Every integer is a rational number as it can be written as

$$Q = \frac{a}{b} \qquad \text{(where } b = 1\text{)}$$

(b) All recurring decimals are rational numbers.

For example, $Q = \frac{1}{3} = 0.3333$ .....

(iv) **Irrational Numbers** Those values which neither terminate nor could be expressed as recurring decimals are irrational numbers *ie*, they cann't be expressed as $\frac{a}{b}$ form, and are **denoted by $\boldsymbol{Q^c}$** (*ie*, complement of Q).

eg, $\sqrt{2}, 1 + \sqrt{2}, \frac{1}{\sqrt{2}}, \frac{-3}{\sqrt{2}}, \frac{2}{\sqrt{2}}, \sqrt{3}, 1 + \sqrt{3}, \pm \frac{1}{\sqrt{3}}, \pi, ...$ etc.

(v) **Real Numbers** The set which contains both rational and irrational numbers is called the real numbers, and is **denoted by *R***.

ie, $\boldsymbol{R = Q \cup Q^c}$

$$\therefore \quad R = \left\{....., -2, -1, 0, 1, 2, 3, ..., \frac{5}{6}, \frac{3}{4}, \frac{7}{9}, \frac{1}{3}, \frac{1}{7}, \frac{1}{5}, ..., \sqrt{2}, \sqrt{3}, \pi, ...\right\}$$

Point to Consider

As from above definitions;

$$N \subset I \subset Q \subset R,$$

is could be shown that real numbers can be expressed on number line with respect to origin as

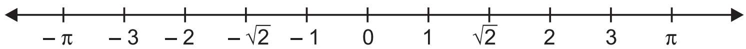


Fig. 1.1

Target Exercise 1.1

Directions (Q. Nos. 1 to 6) : *State whether following statements are True/False:*

1. All the rational numbers are irrational also.

2. All the integers are irrational also.

3. Irrational numbers are real numbers also.

4. Zero is a natural number.

5. Sum of two natural numbers is a rational number.

6. A positive integer is a natural number also.

Directions (Q. Nos. 7 to 10) : *These questions have only one option correct.*

7. Sum of two rational numbers is
(a) rational (b) irrational (c) Both (a) and (b) (d) None of these

8. Sum of two irrational numbers is
(a) rational (b) irrational (c) real (d) None of these

9. Product of two rational numbers is
(a) always rational (b) rational or irrational
(c) always irrational (d) None of these

10. If a is an irrational number which is divisible by b, then the number b
(a) must be rational (b) must be irrational
(c) may be rational or irrational (d) None of these

Closed and Open Intervals

(i) **Open-Open Interval**

If $a < x < b \Rightarrow x \in \,]a, b[$ or $x \in (a, b)$

eg, $1 < x < 2$

$\Rightarrow$ $x \in (1, 2)$ or $x \in \,]1, 2[$ (using above definition)

Illustration 1 Solve $2x + 1 > 3$.

Solution. Here, $2x + 1 > 3$

or $2x > 2$ or $x > 1$

ie, $x \in (1, \infty)$

Point to Consider

At $\pm\infty$ brackets are always open.
This solution can be graphed on a real line as;

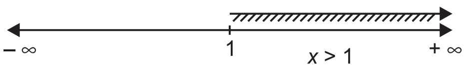


Fig. 1.2

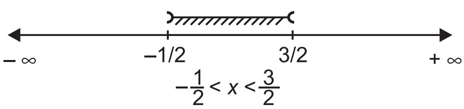
Illustration 2 Solve $-2 < 2x - 1 < 2$.

Solution. Here, $-2 < 2x - 1 < 2$

or $\quad -1 < 2x < 3 \quad$ or $\quad -\dfrac{1}{2} < x < \dfrac{3}{2}$

ie, $\quad x \in \left(-\dfrac{1}{2}, \dfrac{3}{2}\right)$

This solution can be graphed on a real line as;



(ii) **Open-Closed Interval**

If $\quad a < x \leq b$

$\Rightarrow \quad x \in]a, b] \quad$ or $\quad x \in (a, b]$

(iii) **Closed-Open Interval**

If $\quad a \leq x < b \Rightarrow x \in [a, b[$

or $\quad x \in [a, b)$

(iv) **Closed-Closed Interval**

If $\quad a \leq x \leq b \Rightarrow x \in [a, b]$

Illustration 3. Solve the following inequations.

(i) $\dfrac{2x-3}{4} + 9 \geq 3 + \dfrac{4x}{3}$ $\qquad$ (ii) $\dfrac{3(x-2)}{5} \geq \dfrac{5(2-x)}{3}$

Solution. (i) We have, $\dfrac{2x-3}{4} + 9 \geq 3 + \dfrac{4x}{3}$

$$\Rightarrow \quad \frac{2x-3}{4} - \frac{4x}{3} \geq 3 - 9$$

$$\Rightarrow \quad \frac{3(2x-3) - 16x}{12} \geq -6 \quad \Rightarrow \quad \frac{-10x - 9}{12} \geq -6$$

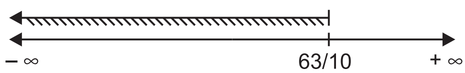
$$\Rightarrow \quad -10x - 9 \geq -72 \quad \Rightarrow \quad -10x \geq -63$$

As we know the inequality sign changes, if multiplied by (–ve).

$$\therefore \quad 10x \leq 63 \quad \text{or} \quad x \leq \frac{63}{10}$$

Hence, the solution set of the given inequation is $\left(-\infty, \frac{63}{10}\right]$.

Graphically it could be shown as



(ii) $\frac{3\,(x-2)}{5} \geq \frac{5\,(2-x)}{3}$

$$\Rightarrow \qquad 3\,(3x-6) \geq 5\,(10-5x)$$

$$\Rightarrow \qquad 9x - 18 \geq 50 - 25x$$

$$\Rightarrow \qquad 34x \geq 68 \quad \Rightarrow \quad x \geq 2$$

Hence, the solution set is $x \in [2, \infty)$ and graphically it could be shown as

Modulus or Absolute Value Function

As we know modulus means numerical value,

ie, $\qquad |3| = |-3| = 3,$

$$|-2| = 2, |-1.3291| = 1.3291 \ldots\ldots \text{etc.}$$

or it is the distance defined with respect to origin, as $|x| = 1$ means distance covered is one unit on right hand side or left hand side of origin shown as;

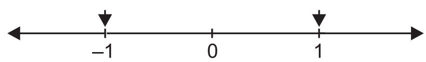


Fig. 1.3

$$\therefore \qquad |x| = 1 \quad \Rightarrow \quad x = \pm 1$$

Again, $|x| < 1$ means distance covered is less than one unit on right hand side or left hand side of origin shown as;

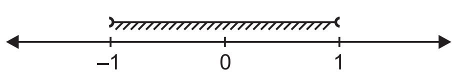


Fig. 1.4

Similarly, $|x| > 1$ means distance covered is more than one unit on right hand side or left hand side of origin shown as;

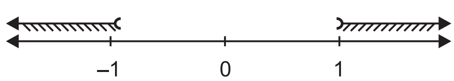


Fig. 1.5

Graphical Representation of Modulus which represents two cases :
As we define,

$$y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

This behaviour is due to two straight lines represented by modulus.
ie, for plotting the graph of the modulus function, we put

x	:	0	1	2	–1	–2
y	:	0	1	2	1	2

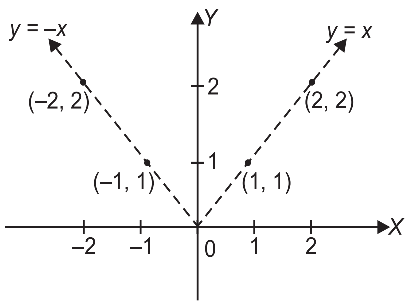


Fig. 1.6

Case I When $x \geq 0$

Equation of the straight line,

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{passing through } (0, 0) \text{ and } (1, 1)$$

ie,
$$\frac{y - 0}{x - 0} = \frac{1 - 0}{1 - 0}$$

$\Rightarrow$ $y = x,$ when $x \geq 0$

Case II When $x \leq 0$

Equation of the straight line,

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{passing through } (-1, 1) \text{ and } (-2, 2)$$

ie,
$$\frac{y - 1}{x + 1} = \frac{2 - 1}{-2 + 1}$$

$$-y + 1 = x + 1$$

or $y = -x$ when $x \leq 0$.

Thus for every modulus function it exhibits two values, which could be shown graphically.

Similarly,
$$y = |x - 1| = \begin{cases} (x - 1), & x \geq 1 \\ -(x - 1), & x \leq 1 \end{cases}$$

Points to Consider

(a) Every modulus function exhibits two values which are represented by +ve and –ve, but it only gives the positive outcomes. So, students shouldn't get confused by +ve or –ve, signs as these signs are in different intervals, but the outcomes are positive.

(b) Modulus function is never negative, thus $|x| \geq 0$ for any real x and $|x| \nless 0$.

Illustration 4 Explain the following :

(i) $|x|=5$ (ii) $|x|=-5$ (iii) $|x|<5$

(iv) $|x|<-5$ (v) $|x|>-5$ (vi) $|x|>5$

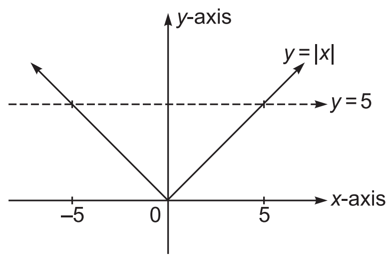
Solution. (i) If $|x|=5$

$\Rightarrow$ $x=\pm 5$, which means, x is at a distance of 5 units from 0, which is certainly 5 and – 5.

Aliter : $|x|=5$.

Here, students are advised to consider two different functions, as

$$y=|x| \text{ and } y=5.$$



which intersect at two points *ie*, $x=5$ and $x=-5$.

$\therefore$ $|x|=5$ posses two solutions $x=5$ and $x=-5$.

(ii) If $|x|=-5$

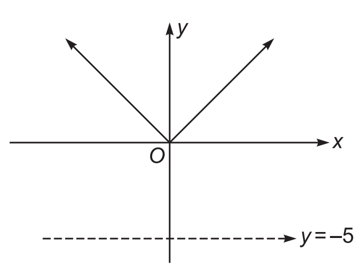
$\Rightarrow$ x has no solution.

As $|x|$ is always positive or zero, it can never be negative.

$\therefore$ RHS > LHS

or given relation has no solution.

Aliter **:** Same as in (i), $y=|x|$ and $y=-5$.



The two graphs do not intersect.

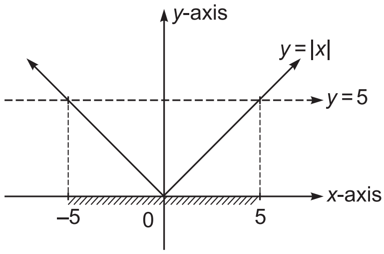
$\therefore$ No solution.

(iii) $|x| < 5$

It means that x is the number, which is at distance less than 5 from 0.

Hence, $\quad -5 < x < 5$

Aliter : $|x| < 5$. From graph, $y = |x|$ and $y = 5$.



We see that, $|x| < 5$, when $-5 < x < 5$

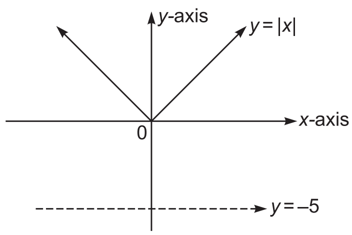
(iv) $|x| < -5$

which shows No solution.

As LHS is non-negative and RHS is negative or $|x| < -5$ does not possess any solution.

Aliter : $|x| < -5$.

From graph of $y = |x|$ and $y = -5$.



We see that, $|x| < -5$ is not possible as $|x| > -5$, for all $x \in R$.

$\therefore \quad |x| < -5 \Rightarrow$ No solution.

(v) $|x| > -5$ We know, here LHS ≥ 0 and RHS < 0

So, $\quad$ LHS > RHS

ie, above statement is true for all real x.

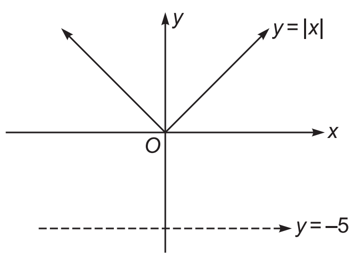
(as we know that non-negative number is always greater than negative).

Aliter : Here, $|x| > -5$

From graph of $y = |x|$ and $y = -5$.

We see that, $|x| > -5$,

for all $x \in$ real number.



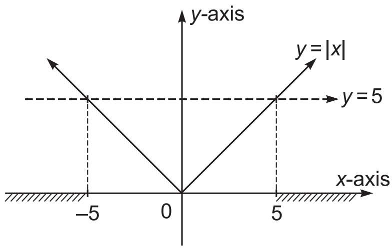
(vi) $|x|>5$

It means that x is the number which is at distance greater than 5 from 0.

Hence, $x<-5$ or $x>5$

Aliter : $|x|>5$.

From graph of $y=|x|$ and $y=5$.



We see that, $x<-5$ or $x>5$.

Generalized Results

(i) For any real number x, we have $x^2 = |x|^2$

(ii) For any real number x, we have

$$\sqrt{x^2} = |x|$$

(iii) If $a > 0$, then

(a) $x^2 \le a^2 \Leftrightarrow |x| \le a \Leftrightarrow -a \le x \le a$
(b) $x^2 < a^2 \Leftrightarrow |x| < a \Leftrightarrow -a < x < a$
(c) $x^2 \ge a^2 \Leftrightarrow |x| \ge a \Leftrightarrow x \le -a$ or $x \ge a$
(d) $x^2 > a^2 \Leftrightarrow |x| > a \Leftrightarrow x < -a$ or $x > a$
(e) $a^2 \le x^2 \le b^2 \Leftrightarrow a \le |x| \le b \Leftrightarrow x \in [-b, -a] \cup [a, b]$
(f) $a^2 < x^2 < b^2 \Leftrightarrow a < |x| < b \Leftrightarrow x \in (-b, -a) \cup (a, b)$

(iv) If $a<0$, then
$|x| \le a \Rightarrow$ No solution.
$|x| \ge a \Rightarrow$ All real numbers solutions.

(v) $|x+y| = |x|+|y|$

$\Leftrightarrow$ $(x \ge 0$ and $y \ge 0)$ or $(x \le 0$ and $y \le 0)$ $\Leftrightarrow$ $xy \ge 0$.

(vi) $|x-y| = |x|-|y|$

$\Leftrightarrow$ $(x \ge 0, y \ge 0$ and $|x| \ge |y|)$ or $(x \le 0, y \le 0$ and $|x| \ge |y|)$

(vii) $|x \pm y| \le |x|+|y|$

(viii) $|x \pm y| \ge \big||x|-|y|\big|$

Illustration 5 Solve for, x where

(i) $f(x)=|x| \ge 0$ (ii) $f(x)=|x|>0$

Solution. (i) $f(x)=|x| \ge 0$

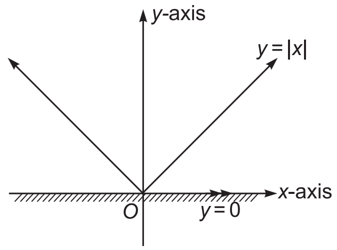
As we know modulus is non-negative quantity.

(*ie*, It is always greater than equal to zero)

$\therefore$ $x \in R$ is the required solution.

Aliter : $f(x) = |x| \geq 0$.

Take two graphs $y = |x|$ and $y = 0$.



From graph $|x| \geq 0$, for all $x \in R$.

(ii) $f(x) = |x| > 0$

Here equal sign is absent, so we have to exclude those value of x for which $|x| = 0$.

$\therefore \quad x \in R$ except $x = 0$ or $x \in R - \{0\}$ is the required solution.

Aliter : $f(x) = |x| > 0$.

Take two graph $y = |x|$ and $y = 0$.

From graph $|x| > 0$

for all $x \in R$ except 0.

$\therefore \qquad x \in R - \{0\}$.

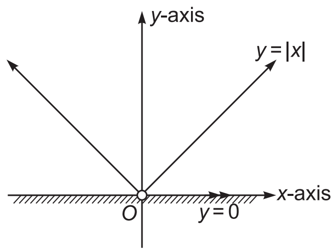


Illustration 6 Solve $|x - 1| \leq 2$.

Solution. $\quad |x - 1| \leq 2$

$\Rightarrow \qquad -2 \leq x - 1 \leq 2$

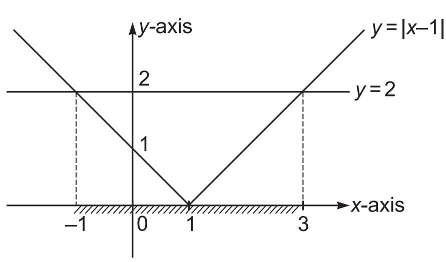
$\Rightarrow \qquad -2 + 1 \leq x \leq 2 + 1$

$\Rightarrow \qquad -1 \leq x \leq 3$

or $\qquad x \in [-1, 3]$

Aliter : For $|x - 1| \leq 2$.

Take two graphs $y = |x - 1|$ and $y = 2$



From graph, $\qquad -1 \leq x \leq 3$

$\therefore \qquad x \in [-1, 3]$

Illustration 7 Solve $|x-3|<5$.

Solution. $|x-3|<5$

$\Rightarrow \quad -5 < x-3 < 5$

$\Rightarrow \quad -5+3 < x < 5+3$

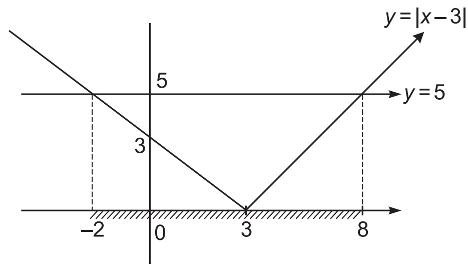
$\Rightarrow \quad -2 < x < 8$

or $\quad x \in]-2, 8[$

or $\quad x \in (-2, 8)$

Aliter : For $|x-3|<5$.

Take two graphs $y=|x-3|$ and $y=5$



From graph, $\quad -2 < x < 8$.

$\therefore \quad x \in (-2, 8)$.

Illustration 8 Solve $1 \le |x-1| \le 3$.

Solution. Here, $1 \le |x-1| \le 3$

$\Rightarrow \quad -3 \le (x-1) \le -1 \quad$ or $\quad 1 \le (x-1) \le 3$

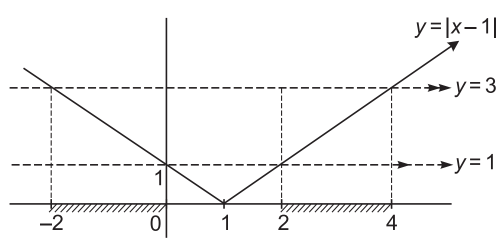
ie., the distance covered is between 1 unit to 3 units.

$\Rightarrow \quad -2 \le x \le 0 \quad$ or $\quad 2 \le x \le 4$

Hence, the solution set of the given inequation is $x \in [-2, 0] \cup [2, 4]$.

Aliter : Here, $1 \le |x-1| \le 3$

Take three graphs $y=1$, $y=|x-1|$ and $y=3$.



From graph, $\quad -2 \le x \le 0 \quad$ or $\quad 2 \le x \le 4$.

$\therefore \quad x \in [-2, 0] \cup [2, 4]$

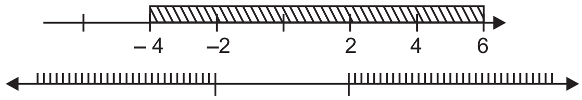
Illustration 9 Solve $|x-1| \le 5, |x| \ge 2$.

Solution. Here, $|x-1| \le 5$ and $|x| \ge 2$

ie, $(-5 \le x-1 \le 5)$ and $(-4 \le x \le 6)$ ...(i)

Similarly, $(x \le -2 \text{ or } x \ge 2)$ and $(x \le -2 \text{ or } x \ge 2)$ ...(ii)

Eqs. (i) and (ii) could be graphically shown as

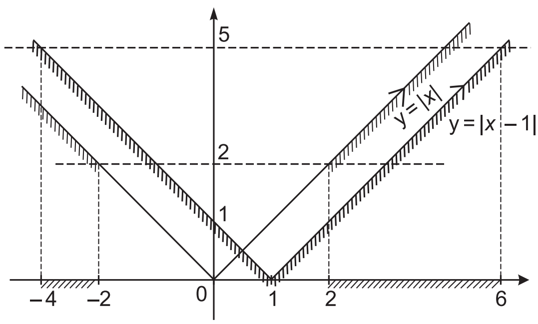


Thus, the shaded portion *ie,* common to both Eqs. (i) and (ii) is the required region.

$\Rightarrow$ $x \in [-4, -2] \cup [2, 6]$

Aliter : Here, $|x-1| \le 5$ and $|x| \ge 2$.

Take $(y = |x-1|,\ y = 5)$ and $(y = |x|,\ y = 2)$ on graph and take common.



From the graph, $-4 \le x \le -2$ or $2 \le x \le 6$

$\therefore$ $x \in [-4, -2] \cup [2, 6]$

Illustration 10 Solve $\left|\dfrac{2}{x-4}\right| > 1,\ x \ne 4$.

Solution. We have, $\left|\dfrac{2}{x-4}\right| > 1$, where $x \ne 4$ ...(i)

$\Rightarrow$ $\dfrac{2}{|x-4|} > 1$ $\left[\because \left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|} \text{ and } |2| = 2\right]$

$\Rightarrow$ $2 > |x-4|$

$\Rightarrow$ $|x-4| < 2$

$\Rightarrow$ $-2 < x-4 < 2$

$\Rightarrow$ $2 < x < 6$

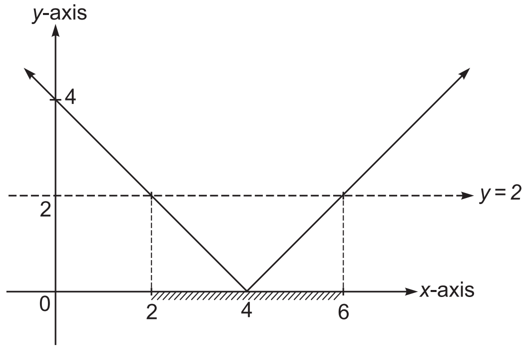
$\therefore$ $x \in (2, 6)$, but $x \ne 4$ [from Eq. (i)]

Hence, the solution set of the given inequation is $x \in (2, 4) \cup (4, 6)$.

Aliter : $\left|\dfrac{2}{x-4}\right| > 1, x \neq 4$

$\Rightarrow \quad |x-4| < 2$

From graph of $y = |x-4|$ and $y = 2$



From graph, $2 < x < 6, x \neq 4$

ie, $\quad x \in (2, 6) - \{4\}$

Point to Consider

Students should always remember that they have to compare the solution set with the initial condition.

Illustration 11 Solve $|x-1| + |x-2| \geq 4$.

Solution. On the LHS of the given inequation, we have two modulus, so we should define each modulus *ie,* by equating it to zero.

ie, $\quad |x-1| = \begin{cases} (x-1), & x \geq 1 \\ -(x-1), & x < 1 \end{cases}$ and $|x-2| = \begin{cases} (x-2), & x \geq 2 \\ -(x-2), & x < 2 \end{cases}$

Thus, it gives three cases :

Case I When $-\infty < x < 1$

ie, $\quad |x-1| + |x-2| \geq 4 \Rightarrow -(x-1) - (x-2) \geq 4$

$\Rightarrow \quad -2x + 3 \geq 4 \Rightarrow -2x \geq 1$

$\Rightarrow \quad x \leq -\dfrac{1}{2}$ ...(i)

But $\quad -\infty < x < 1$

$\therefore$ Solution set is $x \in \left(-\infty, -\dfrac{1}{2}\right]$.

Case II When $1 \leq x \leq 2$

ie, $\quad |x-1| + |x-2| \geq 4 \Rightarrow (x-1) - (x-2) \geq 4$

$\Rightarrow \quad 1 \geq 4$, which is meaningless.

$\therefore$ No solution for $x \in [1, 2]$ ...(ii)

Case III When $x > 2$

ie, $\quad |x-1| + |x-2| \geq 4 \Rightarrow (x-1) + (x-2) \geq 4$

$\Rightarrow \quad 2x - 3 \geq 4 \Rightarrow x \geq 7/2$

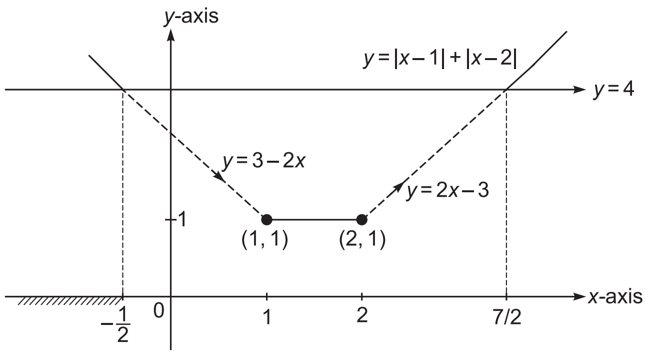
But $\qquad x > 2$

$\therefore \qquad$ Solution set is $\left[\frac{7}{2}, \infty\right)$. $\qquad$ ...(iii)

From Eqs. (i), (ii) and (iii), we get

$$x \in \left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{7}{2}, \infty\right)$$

Aliter : Take two graphs $y = |x - 1| + |x - 2|$ and $y = 4$.



From graph, $\qquad x \le -\frac{1}{2}$ or $x \ge \frac{7}{2}$

$\therefore \qquad x \in \left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{7}{2}, \infty\right)$

Wavy Curve Method
(Number Line Rule/Sign Scheme for Rational Functions)

Method I It is used to solve algebraic inequalities using following steps :

(i) Put only odd power factors in numerator and denominator equal to zero separately.
(*As for polynomial function only numerator* $= 0$, *denominator* $\neq 0$).

(ii) Plot these points on number line in increasing order.

(iii) Now, check the coefficients of x and make them positive.

(iv) Start the number line from right to left taking sign of $f(x)$.

(v) Check your answer so that it should not contain a point for which $f(x)$ doesn't exist.

Method II The method of intervals (or wavy curve) is used for solving inequalities of the form :

$$f(x) = \frac{(x - a_1)^{n_1} (x - a_2)^{n_2} \ldots (x - a_k)^{n_k}}{(x - b_1)^{m_1} (x - b_2)^{m_2} \ldots (x - b_p)^{m_p}} > 0 \qquad (< 0, \le 0 \text{ or } \ge 0)$$

where $n_1, n_2, \ldots\ldots, n_k, m_1, m_2, \ldots\ldots, m_p$ are natural numbers.

a_1, a_2, ..., a_k; b_1, b_2, ..., b_p are any real numbers such that $a_i \neq b_j$, where $i = 1, 2, 3, \ldots, k$ and $j = 1, 2, 3, \ldots, p$

It consists of the following statements :

1. All zeros of the function $f(x)$ contained on the left hand side of the inequality should be marked on the number line with inked (black) circles.
2. All points of discontinuities of the function $f(x)$ contained on the left hand side of the inequality should be marked on the number line with uninked (white) circles.
3. Check the vlaue of $f(x)$ for any real number greater than the rightmost marked number on the number line.
4. From right to left, beginning above the number line (in case of value of $f(x)$ is positive in step (iii) otherwise from below the number line), a wavy curve should be drawn to pass through all the marked points so that when it passes through a simple point, the curve intersects the number line and when passing through a double point, the curve remains located on one side of the number line.
5. The appropriate intervals are chosen in accordance with the sign of inequality (the function $f(x)$ is positive whenever the curve is situated above the number line, it is negative if the curve is found below the number line). Their union just represents the solution of the inequality.

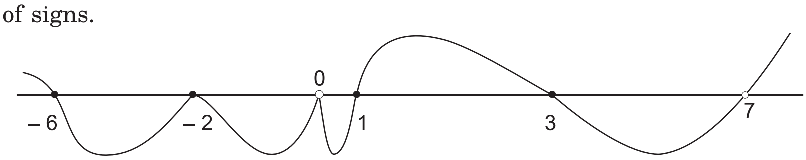
Points to Consider

1. Points of discontinuity will never be include in the answers.
2. If asked to find the intervals where $f(x)$ is non-negative or non-positive, then make the intervals closed, corresponding to the roots of the numerator and let it remain open corresponding to the roots of denominator.

Illustration 12 Let $f(x) = \dfrac{(x-1)^3(x+2)^4(x-3)^5\,(x+6)}{x^2(x-7)^3}$. Solve the following inequalitis :

(i) $f(x) > 0$ (ii) $f(x) \geq 0$ (iii) $f(x) < 0$ (iv) $f(x) \leq 0$

Solution. We mark on the number line zeros of the function: $1, -2, 3$ and -6 (with black circles) and the points of discontinuities 0 and 7 (with white circles). Isolate the double points : -2 and 0 and draw the curve of signs.



From the graph, we get

(i) $x \in (-\infty, -6) \cup (1, 3) \cup (7, \infty)$

(ii) $x \in (-\infty, -6] \cup \{-2\} \cup [1, 3] \cup [7, \infty)$

(iii) $x \in (-6, -2) \cup (-2, 0) \cup (0, 1) \cup (3, 7)$

(iv) $x \in [-6, 0) \cup (0, 1] \cup [3, 7)$

Illustration 13 Let $f(x) = \dfrac{\left(\sin x - \dfrac{1}{2}\right)(\ln x - 1)^2(x-2)(\tan x - \sqrt{3})}{(e^x - e^2)(x-3)^2}$

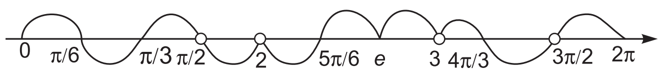
Solve the following inequalities for $x \in [0, 2\pi]$:

(i) $f(x) > 0$ (ii) $f(x) \geq 0$

(iii) $f(x) < 0$ (iv) $f(x) \leq 0$

Solution. Clearly, $x \neq 2, 3, \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ and $f(x) = 0$

for $x = \dfrac{\pi}{6}, \dfrac{\pi}{3}, \dfrac{5\pi}{6}, e, \dfrac{4\pi}{3}$



Now, sign of $f(x)$ will not change around $x = 2, e, 3$.

Then, for $f(x) > 0$

$$\left(\sin x - \frac{1}{2}\right)(\tan x - \sqrt{3}) > 0$$

$$\Rightarrow \quad x \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{\pi}{3}, \frac{\pi}{2}\right) \cup \left(\frac{5\pi}{6}, \frac{4\pi}{3}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right) - \{e, 3\}$$

Hence, solution of

(i) $x \in \left(0, \dfrac{\pi}{6}\right) \cup \left(\dfrac{\pi}{3}, \dfrac{\pi}{2}\right) \cup \left(\dfrac{5\pi}{6}, \dfrac{4\pi}{3}\right) \cup \left(\dfrac{3\pi}{2}, 2\pi\right) - \{e, 3\}$

(ii) $x \in \left(0, \dfrac{\pi}{6}\right] \cup \left[\dfrac{\pi}{3}, \dfrac{\pi}{2}\right) \cup \left[\dfrac{5\pi}{6}, \dfrac{4\pi}{3}\right] \cup \left(\dfrac{3\pi}{2}, 2\pi\right) - \{3\}$

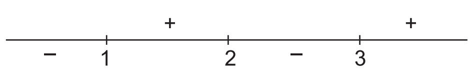
(iii) $x \in \left(\dfrac{\pi}{6}, \dfrac{\pi}{3}\right) \cup \left(\dfrac{\pi}{2}, \dfrac{5\pi}{6}\right) \cup \left(\dfrac{4\pi}{3}, \dfrac{3\pi}{2}\right) - \{2\}$

(iv) $x \in \left[\dfrac{\pi}{6}, \dfrac{\pi}{3}\right] \cup \left(\dfrac{\pi}{2}, \dfrac{5\pi}{6}\right] \cup \left[\dfrac{4\pi}{3}, \dfrac{3\pi}{2}\right) \cup \{e\} - \{2\}$

Illustration 14 Find the interval in which $f(x)$ is positive or negative :
$f(x) = (x-1)(x-2)(x-3)$

Solution. Here, $f(x) = (x-1)(x-2)(x-3)$ has all factors with odd powers, so put them as zero.

ie, $x - 1 = 0$, $x - 2 = 0$, $x - 3 = 0$, we get $x = 1, 2, 3$



Plotting on number line, we get

$f(x) > 0$ when $1 < x < 2$ and $x > 3$

$f(x) < 0$ when $x < 1$ and $2 < x < 3$

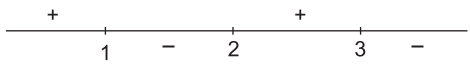
Illustration 15 Solve $f(x) = \dfrac{(x-1)(2-x)}{(x-3)} \geq 0.$

Solution. Here, $f(x) = \dfrac{(x-1)(2-x)}{(x-3)} \geq 0$

or $$f(x) = -\frac{(x-1)(x-2)}{(x-3)}, \text{ which gives}$$

$$x - 3 \neq 0 \quad \text{or} \quad x \neq 3 \qquad \text{...(i)}$$

Using number line rule as shown in the figure,
which shows $f(x) \geq 0$ when $x \leq 1$



or $\quad 2 \leq x < 3$

ie, $\quad x \in (-\infty, 1] \cup [2, 3)$. (as $x \neq 3$)

Illustration 16 Find the values of x for which

$$f(x) = \frac{(2x-1)(x-1)^2(x-2)^3}{(x-4)^4} > 0$$

Solution. $f(x) = \dfrac{(2x-1)(x-1)^2(x-2)^3}{(x-4)^4}$, which gives $x \neq 4$ ...(i)

as denominator $\neq 0$ and $x \neq 1$, as at $x = 1$, $f(x)$ has even powers.

Putting zero to $(2x-1)$ and $(x-2)^3$ as they have odd powers and neglecting $(x-1)^2$ and $(x-4)^4$ on number line as shown in figure.

Which shows $f(x) > 0$ when $x < 1/2$ or $x > 2$, but except for 4 and 1.

$\therefore \quad x \in (-\infty, 1/2) \cup (2, \infty) - \{4\}$

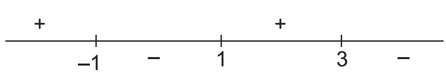
Illustration 17 Find the value of x for which

$$f(x) = \frac{(x-2)^2(1-x)(x-3)^3(x-4)^2}{(x+1)} \leq 0$$

Solution. Here, $f(x) = \dfrac{(x-2)^2(1-x)(x-3)^3(x-4)^2}{(x+1)}$

or $$f(x) = -\frac{(x-2)^2(x-1)(x-3)^3(x-4)^2}{(x+1)}, (x \neq -1) \qquad \text{...(i)}$$

Putting zero to $(x-1)$, $(x-3)^3$, $(x+1)$ as having odd powers and neglecting $(x-2)^2$, $(x-4)^2$, we get, $f(x) \leq 0$ when $-1 < x \leq 1$ [Using Eq. (i) as $x \neq -1$]



or $\quad 3 \leq x < \infty$ or $x \in (-1, 1] \cup [3, \infty)$

Illustration 18 Solve $\dfrac{|x|-1}{|x|-2} \geq 0; \quad x \in R, \quad x \neq \pm 2$

Solution. We have, $\dfrac{|x|-1}{|x|-2} \geq 0$

$$\Rightarrow \qquad \frac{y-1}{y-2} \geq 0; \qquad \text{where } y = |x|$$

$\Rightarrow \quad y \leq 1$ or $y > 2$ using number line rule,

$$\Rightarrow \qquad |x| \leq 1 \qquad \text{or} \qquad |x| > 2$$

$$\Rightarrow \qquad (-1 \leq x \leq 1) \qquad \text{or} \qquad (x < -2 \text{ or } x > 2)$$

$$\Rightarrow \qquad x \in [-1, 1] \cup (-\infty, -2) \cup (2, \infty)$$

Hence, the solution set is

$$x \in (-\infty, -2) \cup [-1, 1] \cup (2, \infty).$$

Illustration 19 Solve $\dfrac{-1}{|x|-2} \geq 1, \quad \text{where } x \in R, x \neq \pm 2$

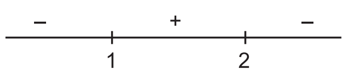
Solution. We have, $\dfrac{-1}{|x|-2} \geq 1$

$$\Rightarrow \qquad \frac{-1}{|x|-2} - 1 \geq 0$$

$$\Rightarrow \qquad \frac{-1-(|x|-2)}{|x|-2} \geq 0$$

$$\Rightarrow \qquad \frac{1-|x|}{|x|-2} \geq 0 \quad \Rightarrow \quad \frac{-(|x|-1)}{(|x|-2)} \geq 0$$

Using number line rule,



$$\Rightarrow \qquad 1 \leq |x| < 2$$

$$\Rightarrow \qquad x \in (-2, -1] \cup [1, 2)$$

$$\{\because \ a \leq |x| < b \Leftrightarrow x \in (-b, -a] \cup [a, b)\}$$

Hence, the solution set is

$$(-2, -1] \cup [1, 2)$$

Illustration 20 Solve $\dfrac{|x+3|+x}{x+2} > 1$

Solution. $\dfrac{|x+3|+x}{x+2} - 1 > 0$

$$\Rightarrow \qquad \frac{|x+3|+x-x-2}{x+2} > 0$$

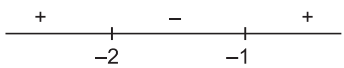
$$\Rightarrow \qquad \frac{|x+3|-2}{x+2} > 0 \qquad \text{...(i)}$$

Now two cases arise :

Case I When $x + 3 \geq 0$. ...(ii)

$$\Rightarrow \quad \frac{x+3-2}{x+2} > 0 \Rightarrow \frac{x+1}{x+2} > 0$$

$\Rightarrow$ $x \in (-\infty, -2) \cup (-1, \infty)$ using number line rule as shown in the figure.



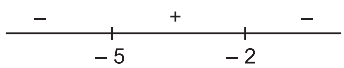
But $\quad x \geq -3$ [from Eq. (ii)]

$\Rightarrow \quad x \in [-3, -2) \cup (-1, \infty)$ ...(a)

Case II When $x + 3 < 0$. ...(iii)

$$\Rightarrow \quad \frac{-(x+3)-2}{x+2} > 0 \Rightarrow \frac{-(x+5)}{(x+2)} > 0$$

$\Rightarrow \quad x \in (-5, -2)$ using number line rule as shown in the figure.



But $\quad x < -3$ [from Eq. (iii)]

$\therefore \quad x \in (-5, -3)$ ...(b)

Thus, from Eqs. (a) and (b), we have

$$x \in [-3, -2) \cup (-1, \infty) \cup (-5, -3)$$

$$\Rightarrow \quad x \in (-5, -2) \cup (-1, \infty)$$

Target Exercise 1.2

1. Solution of the inequality $|x - 1| < 0$ is

(a) $x = 0$ (b) $x = 1$ (c) $x \neq 1$ (d) No solution

2. Solution of inequality $x^2 + x + |x| + 1 \leq 0$ is

(a) (1, 2) (b) (0, 1) (c) No solution (d) None of these

3. Solution of inequality $|x + 3| > |2x - 1|$ is

(a) $\left(-\frac{2}{3}, 4\right)$ (b) $(4, \infty)$ (c) $\left(-\frac{2}{3}, 1\right)$ (d) None of these

4. Solution of inequality $\left|x + \frac{1}{x}\right| < 4$ is

(a) $(2 - \sqrt{3}, 2 + \sqrt{3}) \cup (-2 - \sqrt{3}, -2 + \sqrt{3})$
(b) $R - (2 - \sqrt{3}, 2 + \sqrt{3})$
(c) $R - (-2 - \sqrt{3}, 2 + \sqrt{3})$
(d) None of the above

5. The solution of $|x^2 + 3x| + x^2 - 2 \geq 0$ is

(a) $(-\infty, 1)$ (b) $(0, 1)$

(c) $\left(-\infty, -\frac{2}{3}\right] \cup \left[\frac{1}{2}, \infty\right)$ (d) None of these

6. The solution of $\big||x|-1\big| < |1-x|, x \in R$ is

(a) $(-1, 1)$ (b) $(0, \infty)$
(c) $(-1, \infty)$ (d) None of these

7. The solution of $2^x + 2^{|x|} \geq 2\sqrt{2}$ is

(a) $(-\infty, \log_2(\sqrt{2}+1))$
(b) $(0, \infty)$
(c) $\left(\frac{1}{2}, \log_2(\sqrt{2}-1)\right)$
(d) $(-\infty, \log_2(\sqrt{2}-1)] \cup \left[\frac{1}{2}, \infty\right)$

8. The solution of $|x|^{x^2-x-2} < 1$ is

(a) $(1, 2)$ (b) $(1, \infty)$
(c) $(-\infty, 1)$ (d) None of these

Fundamentals of Quadratic Equations

We know that every quadratic equation represents a parabola (explained in the later part of the book). Thus,

(i) If $ax^2 + bx + c > 0, \forall x \in R \Rightarrow a > 0$, and $D < 0$
(ii) If $ax^2 + bx + c < 0, \forall x \in R \Rightarrow a < 0$, and $D < 0$

Illustration 21 Find a for which $3x^2 + ax + 3 > 0, \forall x \in R$.

Solution. Here, $3x^2 + ax + 3 > 0, \forall x \in R$

$$\Rightarrow D < 0$$
$$\Rightarrow (a)^2 - 4(3)(3) < 0$$
$$\Rightarrow a^2 - 36 < 0$$
$\Rightarrow (a-6)(a+6) < 0$, using number line rule as shown in figure,

which shows $-6 < a < 6$
or $a \in (-6, 6)$

Illustration 22 Find a for which $ax^2 + x - 1 < 0 \; \forall x \in R$?

Solution. Here, $ax^2 + x - 1 < 0, \forall x \in R$

$\Rightarrow a < 0$ and $D < 0$
$\Rightarrow a < 0$ and $1 + 4a < 0$
$\Rightarrow a < 0$ and $a < -1/4$
$\therefore a < -1/4$
or $a \in (-\infty, -1/4)$

Illustration 23 Discuss the following quadratic inequalities :

(i) $x^2 = 4$ (ii) $x^2 = -4$

(iii) $x^2 < 4$ (iv) $x^2 > 4$

(v) $x^2 < -4$ (vi) $x^2 > -4$

Solution. (i) $x^2 = 4$

$\Rightarrow$ $(x + 2)(x - 2) = 0$

$\Rightarrow$ $x = 2, -2$

(ii) $x^2 = -4$

Hence, no solution as LHS ≥ 0 and RHS < 0

(iii) $x^2 < 4$

$\Rightarrow$ $(x - 2)(x + 2) < 0,$

Using number line rule as shown in figure, we get $-2 < x < 2$

(iv) $x^2 > 4$

$\Rightarrow$ $(x - 2)(x + 2) > 0,$

Using number line rule as shown in figure, we get

$x < -2$ or $x > 2$

(v) $x^2 < -4$

$\Rightarrow$ $x^2 + 4 < 0$, which is not possible.

$\therefore$ No solution.

(vi) $x^2 > -4$

$\Rightarrow$ $x^2 + 4 > 0$, which is true for all real number

$\therefore$ $x \in R$

Point to Consider

The sum of several non-negative terms is zero if and only if each term is zero.

Illustration 24 Solve $(x + 1)^2 + (x^2 + 3x + 2)^2 = 0$.

Solution. Here, $(x + 1)^2 + (x^2 + 3x + 2)^2 = 0$ if and only if each term is zero simultaneously,

$(x + 1) = 0$

and $(x^2 + 3x + 2) = 0$

ie, $x = -1$

and $x = -1, -2$

$\therefore$ The common solution is $x = -1$.

Hence, solution of the above equation is $x = -1$.

Illustration 25 Solve $|x+1| + \sqrt{x-1} = 0$.

Solution. Here, $|x+1| + \sqrt{x-1} = 0$, where each term is non-negative.

$\therefore$ $|x+1| = 0$ and $\sqrt{x-1} = 0$ should be zero simultaneously.

ie, $x = -1$ and $x = 1$, which is not possible.

$\therefore$ There is no x for which each term is zero simultaneously.

Hence, there is no solution.

Illustration 26 Solve $|x^2 - 1| + (x-1)^2 + \sqrt{x^2 - 3x + 2} = 0$.

Solution. Here, each of the term is non-negative, thus each must be zero simultaneously.

ie, $(x^2 - 1) = 0, (x-1)^2 = 0$

and $x^2 - 3x + 2 = 0 \Rightarrow x = \pm 1, x = 1$

and $x = 1, 2$

The common solution is $x = 1$.

Therefore, $x = 1$ is the solution of above equation.

Illustration 27 Let $f(x) = x$ and $g(x) = |x|$ be two real-valued functions, $\phi(x)$ be a function satisfying the condition;

$$[\phi(x) - f(x)]^2 + [\phi(x) - g(x)]^2 = 0. \text{ Then, find } \phi(x).$$

Solution. Here, $[\phi(x) - f(x)]^2 + [\phi(x) - g(x)]^2 = 0$ is only possible, if $\phi(x) - f(x) = 0$ and $\phi(x) - g(x) = 0$

$\Rightarrow$ $\phi(x) = f(x) = g(x)$

or $\phi(x) = x = |x|$, which is only possible, if x is non-negative.

Therefore, $\phi(x) = x, \forall x \in [0, \infty)$

Illustration 28 Solve the equation :

$$\sqrt{x^2 + 12y} + \sqrt{y^2 + 12x} = 33, x + y = 23.$$

Solution. Here, $\sqrt{x^2 + 12(23-x)} + \sqrt{(23-x)^2 + 12x} = 33$

$\Rightarrow$ $\sqrt{x^2 - 12x + 276} + \sqrt{x^2 - 34x + 529} = 33$

Let $a = \sqrt{x^2 - 12x + 276}$ and $b = \sqrt{x^2 - 34x + 529}$ ...(i)

$\Rightarrow$ $a + b = 33$ and $a^2 - b^2 = 11(2x - 23)$

$\therefore$ $a - b = \dfrac{2x - 23}{3}$

$\therefore$ $a = \dfrac{x + 38}{3}$ ...(ii)

$\Rightarrow$ $x^2 - 12x + 276 = \left(\dfrac{x+38}{3}\right)^2$ [Using Eqs. (i) and (ii)]

$\Rightarrow$ $8x^2 - 184x + 1040 = 0 \Rightarrow x = 13, 10 \Rightarrow y = 10, 13$

$\therefore$ (13, 10) and (10, 13) are the required solutions.

Illustration 29 Solve $\sqrt{2x-1}+\sqrt{3x-2}=\sqrt{4x-3}+\sqrt{5x-4}$

Solution. Here, $u+v=p+q$ ...(i)

Where $u=\sqrt{2x-1}, v=\sqrt{3x-2}, p=\sqrt{4x-3}, q=\sqrt{5x-4}$

$\therefore$ $$u^2-v^2=1-x$$

$$p^2-q^2=1-x \quad \Rightarrow \quad u^2-v^2=p^2-q^2$$

$\Rightarrow$ $$u-v=p-q \qquad \{\text{using Eq. (i)}, u+v=p+q\} \qquad \text{...(ii)}$$

$\therefore$ From Eqs. (i) and (ii),

$$2u=2p \quad \Rightarrow \quad 2x-1=4x-3$$

or $x=1$, which clearly satisfies

$$\sqrt{2x-1}+\sqrt{3x-2}=\sqrt{4x-3}+\sqrt{5x-4}$$

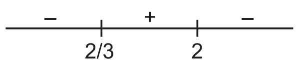
$\therefore$ $x=1$ is the required solution.

Illustration 30 If x, y and z are three real numbers such that $x+y+z=4$ and $x^2+y^2+z^2=6$, then show that each of x, y and z lies in the closed interval $\left[\frac{2}{3}, 2\right]$.

Solution. Here, $x^2+y^2+(4-x-y)^2=6$

$$x^2+(y-4)x+(y^2+5-4y)=0$$

Since, x is real $\Rightarrow D \geq 0$



$\Rightarrow$ $$(y-4)^2-4(y^2-4y+5)\geq 0$$

$\Rightarrow$ $$-3y^2+8y-4\geq 0$$

$\Rightarrow$ $$-(3y-2)(y-2)\geq 0 \quad \Rightarrow \quad \frac{2}{3}\leq y\leq 2$$

Similarly, we can show that x, and $z \in \left[\frac{2}{3}, 2\right]$.

Transformation of Equations

(i) To obtain an equation whose roots are reciprocals of the roots of the given equation, replace x by $\frac{1}{x}$ in the given equation.

(ii) To transform an equation to another equation whose roots are negative of the roots of a given equation, replace x by $-x$.

(iii) To transform an equation to other equation whose roots are square of the roots of a given equation, replace x by $\sqrt{x}$.

(iv) To transform an equation to another equation whose roots are cubes of the roots of a given equation, replace x by $x^{1/3}$.

eg, Form an equation whose roots are cubes of the roots of the equation $ax^3+bx^2+cx+d=0$.

Solution. Replacing x by $x^{1/3}$ in the given equation, we get

$$a(x^{1/3})^3 + b(x^{1/3})^2 + c(x^{1/3}) + d = 0$$

$$\Rightarrow \quad ax + bx^{2/3} + cx^{1/3} + d = 0$$

or $\quad (ax + d)^3 = -(bx^{2/3} + cx^{1/3})^3$

$$\Rightarrow \quad a^3x^3 + 3a^2dx^2 + 3ad^2x + d^3 = -[b^3x^2 + c^3x + 3bcx \cdot (bx^{2/3} + cx^{1/3})]$$

$$\Rightarrow \quad a^3x^3 + 3a^2dx^2 + 3ad^2x + d^3 = -\{b^3x^3 + c^3x - 3bcx(ax + d)\}$$

$$\Rightarrow \quad a^3x^3 + x^2(3a^2d - 3abc + b^3) + x(3ad^2 - 3bcd + c^3) + d^3 = 0$$

This is the required equation.

Illustration 31 If $\{(\alpha + 1)(\beta - 1) + (\beta + 1)(\alpha - 1)\}a + (\alpha - 1)(\beta - 1) = 0$ and $a(\alpha + 1)(\beta + 1) - (\alpha - 1)(\beta - 1) = 0$.

Also, let $A = \left\{\dfrac{\alpha + 1}{\alpha - 1}, \dfrac{\beta + 1}{\beta - 1}\right\}$ and $B = \left\{\dfrac{2\alpha}{\alpha + 1}, \dfrac{2\beta}{\beta + 1}\right\}$. If $A \cap B \neq \phi$, then find all the permissible values of the parameter 'a'.

Solution. Here, $\dfrac{\alpha + 1}{\alpha - 1} + \dfrac{\beta + 1}{\beta - 1} = -\dfrac{1}{a}$ and $\dfrac{(\alpha + 1)(\beta + 1)}{(\alpha - 1)(\beta - 1)} = \dfrac{1}{a}$

Now the quadratic, $x^2 - \left(-\dfrac{1}{a}\right)x + \dfrac{1}{a} = 0$ has roots $\left(\dfrac{\alpha + 1}{\alpha - 1} \text{ and } \dfrac{\beta + 1}{\beta - 1}\right)$

$$\Rightarrow \quad ax^2 + x + 1 = 0 \qquad \text{...(i)}$$

Let $\quad x = \dfrac{2\alpha}{\alpha + 1}$

$$\Rightarrow \quad \alpha = \frac{x}{2 - x} \quad \Rightarrow \quad \frac{\alpha + 1}{\alpha - 1} = \frac{1}{x - 1}$$

Now replacing x by $\dfrac{1}{x - 1}$ in Eq. (i), we get a quadratic whose roots are $\dfrac{2\alpha}{\alpha + 1}$ and $\dfrac{2\beta}{\beta + 1}$

$$\frac{a}{(x - 1)^2} + \frac{1}{(x - 1)} + 1 = 0$$

$$\Rightarrow \quad x^2 - x + a = 0 \qquad \text{...(ii)}$$

Hence, Eqs. (i) and (ii) are the two quadratics whose roots are elements of set 'A' and set 'B', respectively.

$\therefore$ They must have a root in common as $A \cap B \neq \phi$

Case I Both roots are common.

$$\Rightarrow \quad \frac{a}{1} = \frac{1}{-1} = \frac{1}{a} \quad \Rightarrow \quad a = -1$$

and $\quad a^2 = 1 \quad \Rightarrow \quad a = -1$

Case II For a common root,

$$\frac{x^2}{a + 1} = \frac{x}{1 - a^2} = \frac{1}{-(a + 1)} \quad \Rightarrow \quad a = 1 \pm i$$

$$\therefore \quad a \in \{-1, 1 + i, 1 - i\}$$

Illustration 32 Solve $\left|\dfrac{x-1}{3+2x-8x^2}\right| + |1-x| = \dfrac{(x-1)^2}{|3+2x-8x^2|} + 1$

Solution. Here, $\left|\dfrac{x-1}{3+2x-8x^2}\right| + |1-x| = \dfrac{(x-1)^2}{|3+2x-8x^2|} + 1$

Let $a = \dfrac{x-1}{3+2x-8x^2}$; $b = (x-1)$

then $|a| + |b| = |ab| + 1 \Rightarrow |ab| - |a| - |b| + 1 = 0$

$\Rightarrow (|a|-1)(|b|-1) = 0$

$\therefore |a| = 1$ or $|b| = 1$

When $|a| = 1 \Rightarrow \left|\dfrac{x-1}{3+2x-8x^2}\right| = 1$

$$\frac{x-1}{3+2x-8x^2} = 1 \quad \text{or} \quad \frac{x-1}{3+2x-8x^2} = -1$$

$\Rightarrow x - 1 = 3 + 2x - 8x^2$ or $x - 1 = -3 - 2x + 8x^2$

$\Rightarrow 8x^2 - x - 4 = 0$ or $8x^2 - 3x - 2 = 0$

$\Rightarrow x = \dfrac{1 \pm \sqrt{1+128}}{16}$ or $x = \dfrac{3 \pm \sqrt{9+64}}{16}$

$\Rightarrow x = \dfrac{1 \pm \sqrt{129}}{16}$ or $x = \dfrac{3 \pm \sqrt{73}}{16}$ ...(i)

Again, when $|b| = 1, |x-1| = 1$

$\Rightarrow x - 1 = 1$ or $x - 1 = -1$

$\Rightarrow x = 0$ or $x = 2$ ...(ii)

$\therefore$ From Eqs. (i) and (ii) solution set is

$$x \in \left\{0, 2, \frac{1 \pm \sqrt{129}}{16}, \frac{3 \pm \sqrt{73}}{16}\right\}$$

Equations and Inequations Containing Modulus

Illustrations 33 Solve the following equations :

(i) $|x^2 + x - 1| = (2x - 1)$

(ii) $|5x - x^2 - 6| = x^2 - 5x + 6$

(iii) $|x - 1| = |2x - 1|$

(iv) $|x^2 + 3x| > (2 - x^2)$

(v) $|x^2 - 3x - 3| > |x^2 + 7x - 13|$

Solution. As, $|x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$

(i) $\therefore |x^2 + x - 1| = \begin{cases} x^2 + x - 1, & x^2 + x - 1 \ge 0 \\ -(x^2 + x - 1), & x^2 + x - 1 < 0 \end{cases}$

Case I $x^2 + x - 1 \geq 0$

$$\Rightarrow \qquad x^2 + x - 1 = 2x - 1, \; 2x - 1 \geq 0$$

$$\Rightarrow \qquad x^2 - x = 0 \quad \Rightarrow \quad x = 0, 1$$

Caution $|x^2 + x - 1| = 2x - 1$

$$\Rightarrow \qquad 2x - 1 \geq 0$$

$$\Rightarrow \qquad x \geq \frac{1}{2}$$

Neglecting $x = 0 \Rightarrow x = 1$ is the solution.

Case II $x^2 + x - 1 = 1 - 2x$

$$\Rightarrow \qquad x^2 + 3x - 2 = 0$$

$$\Rightarrow \qquad x = \frac{-3 \pm \sqrt{9 + 8}}{2}$$

$$\therefore \qquad x = \frac{-3 \pm \sqrt{17}}{2}$$

Caution $2x - 1 \geq 0$

$$\therefore \qquad x \geq \frac{1}{2}$$

$$\Rightarrow \qquad x = \frac{\sqrt{17} - 3}{2}, \text{ Neglecting } \quad x = \frac{-\sqrt{17} - 3}{2} \text{ as } x \geq \frac{1}{2}.$$

$$\therefore \qquad x = 1 \text{ and } x = \frac{\sqrt{17} - 3}{2} \text{ are required solutions.}$$

(ii) $|5x - x^2 - 6| = x^2 - 5x + 6$

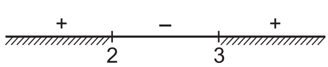
Here, $|f(x)| = -f(x)$

ie, possible only if $f(x) \leq 0$.

$$\therefore \qquad 5x - x^2 - 6 \leq 0$$

$$\text{or} \qquad x^2 - 5x + 6 \geq 0$$

$$\Rightarrow \qquad (x - 2)(x - 3) \geq 0$$



$$\Rightarrow \qquad x \leq 2 \text{ or } x \geq 3 \text{ is the required solution set.}$$

(iii) Here, $|x - 1| = |2x - 1|$

$$\Rightarrow \qquad (x - 1)^2 = (2x - 1)^2$$

$$\Rightarrow \qquad (x - 1)^2 - (2x - 1)^2 = 0$$

$$\Rightarrow \qquad \{(x - 1) - (2x - 1)\}\{(x - 1) + (2x - 1)\} = 0$$

$$\Rightarrow \qquad \{-x\}\{3x - 2\} = 0$$

$$\Rightarrow \qquad x = 0, \frac{2}{3}$$

$$\therefore \qquad x \in \left\{0, \frac{2}{3}\right\}$$

Point to Consider

Whenever both sides are positive, squaring both the sides is the best method *ie*, using $|x|^2 = x^2$.

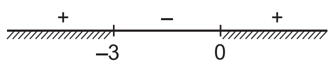
(iv) $|x^2 + 3x| > (2 - x^2)$

Here, $$|x^2 + 3x| = \begin{cases} (x^2 + 3x), & x^2 + 3x \ge 0 \\ -(x^2 + 3x), & x^2 + 3x < 0 \end{cases}$$

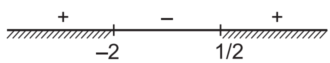
As, squaring is not allowed so we define cases,

Caution Squaring is not allowed, as $(2 - x^2)$ can be positive and negative both.

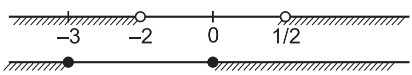
Case I $x^2 + 3x > 2 - x^2$ and $x(x + 3) \ge 0$



$$\Rightarrow \quad 2x^2 + 3x - 2 > 0 \quad \Rightarrow \quad 2x^2 + 4x - x - 2 > 0$$
$$\Rightarrow \quad 2x(x + 2) - 1(x + 2) > 0 \quad \text{and} \quad (x \le -3 \text{ or } x \ge 0)$$
$$\Rightarrow \quad (2x - 1)(x + 2) > 0$$



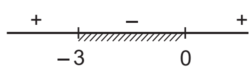
$$\Rightarrow \quad x < -2 \text{ or } x > \frac{1}{2} \text{ and } x \le -3 \text{ or } x \ge 0$$



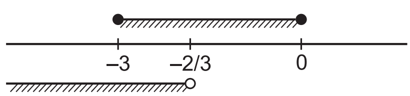
$$\Rightarrow \quad x \le -3 \text{ or } x > \frac{1}{2} \qquad \text{...(i)}$$

Case II $-(x^2 + 3x) > 2 - x^2$

and $\quad x(x + 3) \le 0 \quad \Rightarrow \quad -3x > 2$ and



$$\Rightarrow \quad x < -\frac{2}{3} \text{ and } -3 \le x \le 0$$



$$\Rightarrow \quad -3 \le x \le -\frac{2}{3} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$x \in (-\infty, -3] \cup \left[-3, -\frac{2}{3}\right) \cup \left(\frac{1}{2}, \infty\right)$$

or $$x \in \left(-\infty, -\frac{2}{3}\right) \cup \left(\frac{1}{2}, \infty\right)$$

(v) $|x^2 - 3x - 3| > |x^2 + 7x - 13|$

Here, both the sides are positive. So square both the sides.

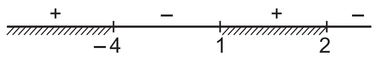
$$(x^2 - 3x - 3)^2 > (x^2 + 7x - 13)^2$$

$$\Rightarrow \quad (x^2 - 3x - 3)^2 - (x^2 + 7x - 13)^2 > 0$$

$$\Rightarrow \quad \{(x^2 - 3x - 3) - (x^2 + 7x - 13)\}$$
$$\{(x^2 - 3x - 3) + (x^2 + 7x - 13)\} > 0$$

$$\Rightarrow \quad (-10x + 10)(2x^2 + 4x - 16) > 0$$

$$\Rightarrow \quad -10(x-1)\cdot 2(x^2 + 2x - 8) > 0 \quad \Rightarrow \quad -20(x-1)(x+4)(x-2) > 0$$



$$\Rightarrow \quad x \in (-\infty, -4) \cup (1, 2)$$

Illustration 34 Let $f(x) = (x^2 - 2|x|)(2|x| - 2) - 9\,\dfrac{(2|x| - 2)}{x^2 - 2|x|}$.

Solve the following inequalites :

(i) $f(x) > 0$ (ii) $f(x) \geq 0$ (iii) $f(x) < 0$ (iv) $f(x) \leq 0$

Solution. We have, $f(x) = (x^2 - 2|x|)(2|x| - 2) - 9\cdot\dfrac{(2|x| - 2)}{x^2 - 2|x|}$

$$= (2|x| - 2)\left\{x^2 - 2|x| - \frac{9}{x^2 - 2|x|}\right\}$$

$$= (2|x| - 2)\left\{\frac{(x^2 - 2|x|)^2 \cdot 9}{x^2 - 2|x|}\right\}$$

$$= \frac{(2|x| - 2)(x^2 - 2|x| + 3)(x^2 - 2|x| - 3)}{x^2 - 2|x|}$$

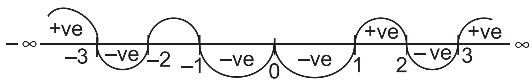
$$= \frac{(2|x| - 2)\{(|x| - 1)^2 + 2\}\{(|x| + 1)(|x| - 3)\}}{x^2 - 2|x|}$$

Taking, $N^r = 0 \quad \Rightarrow \quad |x| = 1, 3$

$\Rightarrow \quad x = \pm 1, \pm 3$ and $D^r = 0$

$\Rightarrow \quad |x|(|x| - 2) = 0 \quad \Rightarrow \quad 0, \pm 2$

We mark these roots on a number line :



From the wavy curve method, we have

(i) $f(x) > 0$

$$\Rightarrow \quad x \in (-\infty, -3) \cup (-2, -1) \cup (1, 2) \cup (3, \infty)$$

(ii) $f(x) \geq 0$

$$\Rightarrow \quad x \in (-\infty, -3] \cup (-2, -1] \cup [1, 2) \cup [3, \infty)$$

(iii) $f(x) < 0$

$$\Rightarrow \quad x \in (-3, -2) \cup (-1, 0) \cup (0, 1) \cup (2, 3)$$

(iv) $f(x) \leq 0$

$$\Rightarrow \quad x \in [-3, -2) \cup [-1, 0) \cup (0, 1] \cup (2, 3]$$

Illustration 35 Solve the inequality $\left|1 - \frac{|x|}{1+|x|}\right| \geq \frac{1}{2}$.

Solution. The domain of admissible values of this inequality consists of all the real numbers. The inequality is equivalent to the collection of two systems :

$$\begin{cases} \left|1 - \dfrac{x}{1+x}\right| \geq \dfrac{1}{2} \\ x \geq 0 \end{cases};\ \begin{cases} \left|1 - \dfrac{-x}{1-x}\right| \geq \dfrac{1}{2} \\ x < 0 \end{cases}$$

We solve the first system :

$$\begin{cases} \left|1 - \dfrac{x}{1+x}\right| \geq \dfrac{1}{2} \\ x \geq 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{1}{1+x} \geq \dfrac{1}{2} \\ x \geq 0 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \dfrac{2-1-x}{1+x} \geq 0 \\ x \geq 0 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} 2 - x \geq 0 \\ x \geq 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} x \leq 0 \\ x \geq 0 \end{cases} \Leftrightarrow 0 \leq x \leq 1$$

Again, we solve the second system. Its first inequality is equivalent to the inequality $\left|\frac{1}{1-x}\right| \geq \frac{1}{2}$.

If $x < 0$, then $1 - x > 0$ and consequently the second system is equivalent to the system

$$\begin{cases} \dfrac{1}{1-x} \geq \dfrac{1}{2} \\ x < 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \dfrac{2-1+x}{1-x} \geq 0 \\ x < 0 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \dfrac{1+x}{1-x} \geq 0 \\ x < 0 \end{cases}$$

$$\Leftrightarrow \quad -1 \leq x < 0$$

Thus, the set of all solutions of the original inequality consists of the numbers belonging to the interval $[-1, 1]$.

Passage

(Illustration Nos. 36 to 38)

Let $f(x) = ax^2 + bx + c$; $a, b, c \in R$

It is given $|f(x)| \leq 1, \forall\, |x| \leq 1$.

Now, answer the following illustrations.

Illustration 36 The possible value of $|a + c|$, if $\frac{8}{3}a^2 + 2b^2$ is maximum, is given by

(a) 1 (b) 0

(c) 2 (d) 3

Illustration 37 The possible value of $|a+b|$, if $\frac{8}{3}a^2+2b^2$ is maximum, is given by

(a) 1 (b) 0

(c) 2 (d) 3

Illustration 38 The possible maximum value of $\frac{8}{3}a^2+2b^2$ is given by

(a) 32 (b) $\frac{32}{3}$

(c) $\frac{2}{3}$ (d) $\frac{16}{3}$

Solution. (For Illustration Nos. 36 to 38)

We know that for $|u|\le 1; |v|\le 1$, then $|u-v|\le 2$

Now, $|f(1)-f(0)|\le 2 \Rightarrow |a+b|\le 2 \Rightarrow (a+b)^2\le 4$ ...(i)

Also, $|f(-1)-f(0)|\le 2 \Rightarrow |a-b|\le 2$

$\Rightarrow$ $(a-b)^2\le 4$ ...(ii)

Now, $4a^2+3b^2=2(a+b)^2+2(a-b)^2-b^2\le 16$

$\Rightarrow$ $(4a^2+3b^2)_{\max}=16$, when $b=0$

$\Rightarrow$ $|a+b|=|a-b||a|=2$

Also, $|f(1)-f(0)|=|(a+c)-c|=|a|=2$

$\Rightarrow$ $|a+c|=|c|=1$

The possible ordered triplets (a, b, c) are $(2, 0, -1)$ or $(-2, 0, 1)$

Also, $\frac{8}{3}a^2+2b^2=\frac{2}{3}(4a^2+3b^2)\le\frac{2}{3}(16)$

Thus, $|a+c|=1$, when $\frac{8a^2}{3}+2b^2$ is maximum.

$|a+b|=1$, when $\frac{8a^2}{3}+2b^2$ is maximum.

and $\left(\frac{8a^2}{3}+2b^2\right)$; the maximum value is $\frac{32}{3}$.

Proficiency in 'Essential Mathematical Tools' (Exercise)

Subjective Questions

Directions (Q. Nos. 1 to 11) : Solve each of the following system of equations :

1. For $a < 0$, determine all solutions of the equation $x^2 - 2a\,|x - a| - 3a^2 = 0$

2. Solve $|x^2 + 4x + 3| + 2x + 5 = 0$.

3. Solve $|x^2 - 3x - 4| = 9 - |x^2 - 1|$.

4. $2^{|x+1|} - 2^x = |2^x - 1| + 1$

5. Find the set of all real 'a' such that $5a^2 - 3a - 2$, $a^2 + a - 2$ and $2a^2 + a - 1$ are the lengths of the sides of a triangle ?

6. Solve $(x + 3)^5 - (x - 1)^5 \geq 244$.

7. Solve $||x - 2| - 1| \geq 3$.

8. Solve $1 \leq \dfrac{3x^2 - 7x + 8}{x^2 + 1} \leq 2$.

9. Let $f(x) = \dfrac{2x}{2x^2 + 5x + 2}$, and $g(x) = \dfrac{1}{x + 1}\cdot$

Find the set of real values of x for which $f(x) > g(x)$.

10. For $x \in R$, $||x||$ is defined as follows; $||x|| = \begin{cases} x + 1, & 0 \leq x < 2 \\ |x - 4|, & 2 \leq x \end{cases}$

Then, solve the equation, $||x||^2 + x = ||x|| + x^2$.

11. Solve the inequality : $|x - 1| + |2 - x| > 3 + x$.

Linked Comprehension Based Questions

Passage I

(Q. Nos. 12 to 14)

Consider the equation $|2x| - |x - 4| = x + 4$.

12. The least integer satisfying the equation, is

(a) -4 (b) 4 (c) 5 (d) -5

13. Total number of prime numbers less than 20 satisfying the equation, is

(a) 3 (b) 4 (c) 5 (d) 6

14. If P = greatest composite number less than 34 satisfying the given equation, then P^{2007} has the digit on its units place as

(a) 8 (b) 1 (c) 7 (d) 0

Passage II

(Q. Nos. 15 to 17)

Consider a number $N = 2\,1\,P\,5\,3\,Q\,4$.

15. The number of ordered pairs (P, Q) so that the number 'N' is divisible by 9, is

(a) 11 (b) 12 (c) 10 (d) 8

16. The number of values of Q so that the number 'N' is divisible by 8, is

(a) 4 (b) 3 (c) 2 (d) 6

17. The number of ordered pairs (P, Q) so that the number 'N' is divisible by 44, is

(a) 2 (b) 3 (c) 4 (d) 5

Passage III

(Q. Nos. 18 to 22)

Consider the nine digit number $n = 7\,3\,\alpha\,4\,9\,6\,1\,\beta\,0$.

18. If p is the number of all possible distinct values of $(\alpha - \beta)$, then p is equal to

(a) 17 (b) 18 (c) 19 (d) 20

19. If q is the number of all possible values of β for which the given number is divisible by 8, then q is equal to

(a) 2 (b) 3 (c) 4 (d) 5

20. The number of ordered pairs (α, β) for which the given number is divisible by 88, is

(a) 1 (b) 2 (c) 3 (d) 4

21. The number of possible values of $(\alpha + \beta)$ for which the given number is divisible by 6, is

(a) 3 (b) 4 (c) 6 (d) 7

22. The number of possible values of β for which $i^N = 1$ (where $i = \sqrt{-1}$), is

(a) 2 (b) 3 (c) 4 (d) 5

Passage IV

(Q. Nos. 23 to 25)

The set of integers can be classified into k classes, according to the remainder obtained when they are divided by k (where k is a fixed natural number). The Classification enables is solving even some more difficult problems of number theory eg,

(i) *even, odd classification is based on whether remainder is* 0 *or* 1 *when divided by* 2.

(ii) *when divided by* 3, *the remainder may by* 0, 2. *Thus, there are three classes.*

23. The remainder obtained, when the square of an integer is divided by 3, is
(a) 0, 1 (b) 1, 2 (c) 0, 2 (d) 0, 1, 2

24. $n^2 + n + 1$ is never divisible by
(a) 2 (b) 3 (c) 111 (d) None of these

25. If n is odd, $n^5 - n$ is divisible by
(a) 16 (b) 15 (c) 240 (d) 720

Answers

Target Exercise 1.1

1. False **2.** False **3.** True **4.** False **5.** True
6. True **7.** (a) **8.** (c) **9.** (a) **10.** (b)

Target Exercise 1.2

1. (d) **2.** (c) **3.** (a) **4.** (a) **5.** (c) **6.** (d) **7.** (d) **8.** (a)

Proficiency in Essential Mathematical Tools Exercise

1. $\{(1-\sqrt{2})a, (-1+\sqrt{6})a\}$ **2.** $\{-4, -1, -1-\sqrt{3}\}$

3. $\{-2, 2\}$ **4.** $\{-2\} \cup [0, \infty)$ **5.** $\left(\dfrac{3+\sqrt{57}}{8} < a < \dfrac{5+\sqrt{17}}{4}\right)$

6. $(-\infty, -2] \cup [0, \infty)$ **7.** $(-\infty, -2] \cup [6, \infty)$ **8.** $[1, 6]$

9. $x \in (-2, -1) \cup \left(-\dfrac{2}{3}, -\dfrac{1}{2}\right)$ **10.** $x = \{0, 2\}$ **11.** $(-\infty, 0) \cup (6, \infty)$ **12.** (a) **13.** (d)

14. (c) **15.** (a) **16.** (b) **17.** (c) **18.** (c) **19.** (a) **20.** (b)
21. (d) **22.** (d) **23.** (a) **24.** (a) **25.** (a, b, c, d)

Solutions

(*Proficiency in 'Essential Mathematical Tools' Exercise*)

1. For $a < 0, |x - a| = \begin{cases} (x-a), & x \geq a \\ (a-x), & x \leq a \end{cases}$.

Case I $x \geq a$

$$x^2 - 2a(x-a) - 3a^2 = 0$$
$$\Rightarrow \quad x^2 - 2ax - a^2 = 0$$
$$\Rightarrow \quad x^2 - 2ax + a^2 = 2a^2$$
$$\Rightarrow \quad (x-a)^2 = 2a^2$$
$$\Rightarrow \quad x = \pm\sqrt{2}a + a \Rightarrow x = a(1+\sqrt{2}), a(1-\sqrt{2})$$

Since, $x \geq a$ and $a < 0$ $\therefore$ Neglecting $\{a(1+\sqrt{2})\}$

$$\Rightarrow \quad x = a(1-\sqrt{2})$$

Case II $x \le a$

$\Rightarrow \quad x^2 + 2a(x - a) - 3a^2 = 0 \quad \Rightarrow \quad x^2 + 2ax = 5a^2$

$\Rightarrow \quad (x + a)^2 = 6a^2 \quad \Rightarrow \quad x = -a \pm \sqrt{6}a$

$\therefore \quad x = a(\sqrt{6} - 1), -a(\sqrt{6} + 1)$

Since, $\quad x \le a$ and $a < 0$

$\therefore$ Neglecting $\{-a(\sqrt{6} + 1)\}$

$\Rightarrow \quad x = a(\sqrt{6} - 1)$

Hence, $x \in \{a(1 - \sqrt{2}), a(\sqrt{6} - 1)\}$.

2. $|x^2 + 4x + 3| + 2x + 5 = 0$

Take two cases, *ie*, $\quad |x^2 + 4x + 3| = \pm(x^2 + 4x + 3)$

Case I $x^2 + 4x + 3 \ge 0$

$\Rightarrow \quad x \notin [-3, -1]$

or $\quad x \in (-\infty, -3) \cup (-1, \infty) \quad \Rightarrow \quad x^2 + 4x + 3 + 2x + 5 = 0$

$\Rightarrow \quad x^2 + 6x + 8 = 0 \quad \Rightarrow \quad x = -2, -4.$

But $\quad x \in [-3, -1]$

$\therefore$ Neglecting $\quad x = -2 \quad \Rightarrow \quad x = -4$ ...(i)

Case II $x^2 + 4x + 3 \le 0 \Rightarrow x \in [-3, -1]$

$\Rightarrow \quad x^2 + 4x + 3 - 2x - 5 = 0 \quad \Rightarrow \quad x^2 + 2x - 2 = 0$

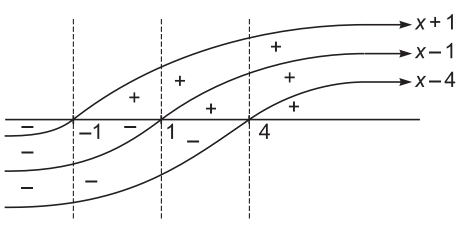
$\Rightarrow \quad x = -1 \pm \sqrt{3}$, as $x \in [-3, -1]$

$\therefore$ Neglecting $\quad x = -1 + \sqrt{3} \quad \Rightarrow \quad x = -1 - \sqrt{3}$ ...(ii)

$\therefore \quad x \in \{-4, -1 - \sqrt{3}\}$.

3. $|x^2 - 3x - 4| + |x^2 - 1| = 9$

$\Rightarrow \quad |x + 1|\{|x - 4| + |x - 1|\} = 9$



Case I $x \ge 4$

$$(x + 1)\{x - 4 + x - 1\} = 9$$

$\Rightarrow \quad (x + 1)(2x - 5) = 9$

$\Rightarrow \quad 2x^2 - 3x - 14 = 0$

$\Rightarrow \quad x = \dfrac{7}{2}, -2$

{Neglecting both as $x \ge 4$}.

Case II $1 < x < 4$

$\Rightarrow \qquad (x+1)\{4-x+x-1\} = 9 \quad \Rightarrow \quad (x+1)\cdot 3 = 9$

$\Rightarrow \qquad x = 2 \in (1, 4)$ ...(i)

Case III $-1 < x < 1$

$$(x+1)\{4-x+1-x\} = 9$$

$\Rightarrow \quad (x+1)(5-2x) = 9 \quad \Rightarrow \quad 2x^2 - 3x + 4 = 0$, having imaginary roots.

$\therefore$ No solution.

Case IV $x \le -1$

$$-(x+1)\{4-x+1-x\} = 9$$

$$(x+1)(2x-5) = 9$$

$\Rightarrow \qquad x = -2, -\dfrac{7}{2}.$

As $x \le -1$,

$\therefore \qquad x = -2$, neglecting $x = -\dfrac{7}{2}$ ...(ii)

Thus, from Eqs. (i) and (ii), $x \in \{-2, 2\}$.

4. $2^{|x+1|} - 2^x = |2^x - 1| + 1$.

Case I $x \ge 0$

$$2^{x+1} - 2^x = 2^x - 1 + 1 \quad \Rightarrow \quad 2^x \cdot 2 = 2 \cdot 2^x$$

ie, true for all $x \ge 0$ ...(i)

Case II $-1 \le x \le 0$

$$2^{+x+1} - 2^x = 1 - 2^x + 1 \quad \Rightarrow \quad 2^{x+1} = 2$$

$\Rightarrow \qquad x = 0$ ...(ii)

Case III $x \le -1$

$\Rightarrow \qquad 2^{-x-1} - 2^x = 1 - 2^x + 1 \quad \Rightarrow \quad 2^{-x-1} = 2$

$\Rightarrow \qquad x = -2$ ...(iii)

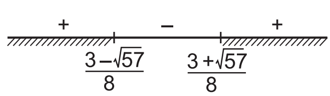
Thus, from Eqs. (i), (ii) and (iii), $x \in \{-2\} \cup [0, \infty)$.

5. If $5a^2 - 3a - 2, a^2 + a - 2, 2a^2 + a - 1$ are lengths of sides of triangle.

Case I $(5a^2 - 3a - 2) + (a^2 + a - 2) > 2a^2 + a - 1$

$\Rightarrow \qquad 6a^2 - 2a - 4 > 2a^2 + a - 1 \quad \Rightarrow \quad 4a^2 - 3a - 3 > 0$

$\Rightarrow \qquad \left\{a - \left(\dfrac{3+\sqrt{57}}{8}\right)\right\}\left\{a - \left(\dfrac{3-\sqrt{57}}{8}\right)\right\} > 0$

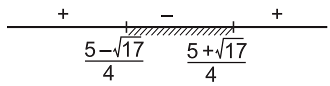


$\Rightarrow \qquad a \in \left(-\infty, \dfrac{3-\sqrt{57}}{8}\right) \cup \left(\dfrac{3+\sqrt{57}}{8}, \infty\right)$ ...(i)

Case II $3a^2 + 2a - 3 > 5a^2 - 3a - 2$

$$\Rightarrow \quad 2a^2 - 5a + 1 < 0$$

$$\Rightarrow \quad \left\{a - \left(\frac{5+\sqrt{17}}{4}\right)\right\}\left\{a - \left(\frac{5-\sqrt{17}}{4}\right)\right\} < 0$$

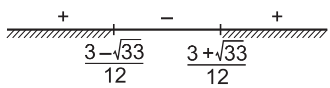


$$\Rightarrow \quad a \in \left(\frac{5-\sqrt{17}}{4}, \frac{5+\sqrt{17}}{4}\right) \qquad \ldots\text{(ii)}$$

Case III $7a^2 - 2a - 3 > a^2 + a - 2$

$$\Rightarrow \quad 6a^2 - 3a - 1 > 0$$

$$\Rightarrow \quad \left\{a - \left(\frac{3+\sqrt{33}}{12}\right)\right\}\left\{a - \left(\frac{3-\sqrt{33}}{12}\right)\right\} > 0$$



$$\Rightarrow \quad a \in \left(-\infty, \frac{3-\sqrt{33}}{12}\right) \cup \left(\frac{3+\sqrt{33}}{12}, \infty\right) \qquad \ldots\text{(iii)}$$

From Eqs. (i), (ii) and (iii),

$$a \in \left(\frac{3-\sqrt{57}}{8}, \frac{5+\sqrt{17}}{4}\right)$$

6. $(x+3)^5 - (x-1)^5 \geq 244$

Let $\quad y = \dfrac{(x+3)+(x-1)}{2}$

$\Rightarrow \quad y = x + 1$

$\therefore \quad (y+2)^5 - (y-2)^5 \geq 244$

$\Rightarrow \quad 2\{{}^5C_1 y^4 \cdot 2 + {}^5C_3 \cdot y^2 \cdot 2^3 + {}^5C_5 \cdot 2^5\} \geq 244$

$\Rightarrow \quad 2\{10y^4 + 80y^2 + 32\} \geq 244$

$\Rightarrow \quad 4\{5y^4 + 40y^2 + 16\} \geq 244$

$\Rightarrow \quad 5y^4 + 40y^2 + 16 \geq 61$

$\Rightarrow \quad y^4 + 8y^2 - 9 \geq 0$

$\Rightarrow \quad (y^2+9)(y^2-1) \geq 0 \quad \Rightarrow \quad y^2 \geq 1$

ie, $\quad y \leq -1 \;$ or $\; y \geq 1$

$\Rightarrow \quad x + 1 \leq -1$ or $x + 1 \geq 1$

$\Rightarrow \quad x \in (-\infty, -2] \cup [0, \infty)$

7. $||x-2|-1| \geq 3$

$\Rightarrow \quad |x-2|-1 \leq -3 \quad$ or $\quad |x-2|-1 \geq 3$

$\Rightarrow \quad |x-2| \leq -2 \quad$ or $\quad |x-2| \geq 4$

$\Rightarrow$ No solution or $\quad x-2 \leq -4 \quad$ or $\quad x-2 \geq 4$

$\Rightarrow \quad x \leq -2 \quad$ or $\quad x \geq 6 \quad \Rightarrow \quad x \in (-\infty, -2] \cup [6, \infty)$

8. $1 \leq \dfrac{3x^2 - 7x + 8}{x^2 + 1} \leq 2$

$\Rightarrow \quad 2x^2 - 7x + 7 \geq 0 \quad$ and $\quad x^2 - 7x + 6 \leq 0$

$\Rightarrow \quad x \in R$ and $x \in [1, 6]$

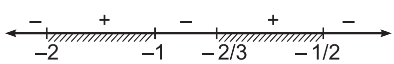
$\therefore \quad x \in [1, 6]$

9. $f(x) > g(x) \Rightarrow \dfrac{2x}{2x^2 + 5x + 2} > \dfrac{1}{x+1}$

$$\Rightarrow \quad \frac{2x}{(2x+1)(x+2)} - \frac{1}{(x+1)} > 0$$

$$\Rightarrow \quad \frac{2x^2 + 2x - (2x^2 + 5x + 2)}{(2x+1)(x+2)(x+1)} > 0$$

$$\Rightarrow \quad \frac{-(3x+2)}{(2x+1)(x+2)(x+1)} > 0$$



$\Rightarrow \quad x \in (-2, -1) \cup (-2/3, -1/2)$

10. $||x||^2 + x = ||x|| + x^2$.

Case I When $0 \leq x < 2$

$\Rightarrow \quad (x+1)^2 + x = (x+1) + x^2 \quad \Rightarrow \quad x^2 + 2x + 1 + x = x^2 + x + 1$

$\Rightarrow \quad 2x = 0$

$\Rightarrow \quad x = 0$ ...(i)

Case II When $x \geq 2$

$\Rightarrow \quad (x-4)^2 + x = |x-4| + x^2$

Now, if $2 \leq x \leq 4$

$\Rightarrow \quad (x^2 - 8x + 16) + x = 4 - x + x^2 \quad \Rightarrow \quad 6x = 12$

$\Rightarrow \quad x = 2$ ...(ii)

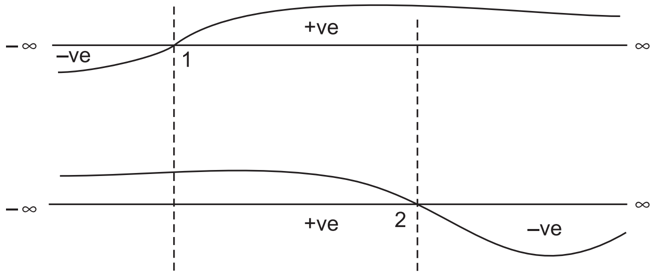
Again, if $\quad x \geq 4$

$\Rightarrow \quad (x-4)^2 + x = x - 4 + x^2 \quad \Rightarrow \quad x^2 - 8x + 16 + x = x^2 + x - 4$

$\Rightarrow \quad 8x = 20 \quad \Rightarrow \quad x = \dfrac{5}{2}, \quad$ but $\quad x \geq 4$.

$\therefore$ Only two solutions $x = 0$ and 2.

11. The points $x = 1$ and $x = 2$ divide the number axis into three intervals as follows :



We solve the inequality on each intervals.

If $x < 1$, then $x - 1 < 0$ and $2 - x > 0$.

$$\therefore \quad |x-1|+|2-x| > 3 - x$$

$$\Rightarrow \quad 1 - x + 2 - x > 3 + x$$

$$\Rightarrow \quad 3 - 2x > 3 + x \quad \Rightarrow \quad x < 0 \qquad \ldots(i)$$

If $1 \le x \le 2$, then $x - 1 \ge 0$ and $2 - x \ge 0$, we have

$$x - 1 + 2 - x > 3 + x$$

$$\Rightarrow \quad 1 > 3 + x \quad \Rightarrow \quad x < -2$$

The system of inequalities obtained has no solution, for $1 \le x \le 2$.

If $x > 2$, then $x - 1 > 0$ and $2 - x < 0$, we have

$$x - 1 + x - 2 > 3 + x$$

$$\Rightarrow \quad 2x - 3 > 3 + x \quad \Rightarrow \quad x > 6$$

Combining the solutions obtained on all parts of the domain of admissible values of the given inequality, we get the solution set $(-\infty, 0) \in (6, \infty)$.

12. Solution of the given equation is $\{-4\} \cup [4, \infty)$.

$\therefore$ Least integer $= -4$.

13. Prime numbers less than 20 satisfying the equation are 5, 7, 11, 13, 17, 19

ie, 6 prime number.

14. $P = 33$

$P^{2007} = (33)^{2007}$ = number has 7 at its units place.

15. Sum of digits $= P + Q + 15$

N is divisible by 9, if

$$P + Q + 15 = 18, 27$$

$$\Rightarrow \quad P + Q = 3 \qquad \ldots(i)$$

$$\text{or} \quad P + Q = 12 \qquad \ldots(ii)$$

$$P = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$$

From Eq. (i), we get

$$\left.\begin{array}{l} P=0, Q=3 \\ P=1, Q=2 \\ P=2, Q=1 \\ P=3, Q=0 \end{array}\right\} \text{Number of ordered pairs is 4.}$$

From Eq. (ii), we get

$$\left.\begin{array}{l} P=3, Q=9 \\ P=4, Q=8 \\ \quad \ldots. \quad \ldots. \\ P=8, Q=4 \\ P=9, Q=3 \end{array}\right\} \text{Number of ordered pairs is 7.}$$

Total number of ordered pairs is 11.

16. N is divisible by 8, if

$$Q = 0, 4, 8$$

Number of values of Q is 3.

17. $S_O = P + 9$

$$S_E = Q + 6$$

$$S_O - S_E = P - Q + 3$$

N is divisible is 11, if

$$P - Q + 3 = 0, 11$$

$$P - Q = -3 \qquad \ldots\text{(i)}$$

or $$P - Q = 8 \qquad \ldots\text{(ii)}$$

N is divisible by 4, if

$$Q = 0, 2, 4, 6, 8$$

From Eq. (i),

$$Q = 0, \quad P = -3 \qquad \text{(Not possible)}$$

$$Q = 2, \quad P = -1 \qquad \text{(Not possible)}$$

$$Q = 4, \quad P = 1$$

$$Q = 6, \quad P = 1$$

$$Q = 8, \quad P = 5$$

$\therefore$ Number of ordered pairs is 3.

From Eq. (ii),

$$Q = 0 \quad P = 8$$

$$Q = 2 \quad P = 10 \qquad \text{(Not possible)}$$

Similarly, $$Q \neq 4, 6, 8$$

$\therefore$ Number of ordered pairs is 1.

$\therefore$ Total number of ordered pairs so that the number N is divisible by 44, is 4.

18. If $\alpha = 0$, then the possible values of $\alpha - \beta$ are

$$\{0, -1, -2, -3, \dots, -9\}$$

If $\alpha = 1$, then the possible values of $\alpha - \beta$ are

$$\{1, 0, -1, -2, -3, \dots, -8\}$$

If $\alpha = 2$, then the possible values of $\alpha - \beta$ are

$$\{2, 1, 0, -1, -2, -3, \dots, -7\} \text{ and so on}$$

If $\alpha = 9$, then the possible values of $\alpha - \beta$ are

$$\{9, 8, 7, 6, 5, 4, 3, 2, 1, 0\}$$

Thus, in all, the values of $\alpha - \beta$ are

$$\{-9, -8, -7, -6, -5, -4, -3, -2, -1, , 0, 1, 2, 3, \dots, 9\}$$

$\therefore$ Total number of values of $\alpha - \beta$ is 19.

$\therefore$ $\quad p = 19$

19. Number N is divisible by 8, if $100 + 10\beta$ is divisible by 8.

$\therefore$ Possible values of β are 2, 6.

$\therefore$ $\quad q = 2$

20. $N = 7\,3\,\alpha\,4\,9\,6\,1\,\beta\,\,0$

N is divisible by 88, if N is divisible by 8 as well as 11.

N is divisible by 8, then $\beta = 2, 6$

For divisibility by 11,

$$S_O = 17 + \alpha \quad S_E = 13 + \beta$$

$$S_O - S_E = (17 + \alpha) - (13 + \beta) = \alpha - \beta + 4$$

$\alpha - \beta + 4$ be the integer lying between –5 and 13.

But, $\alpha - \beta + 4$ is divisible by 11 so $\alpha - \beta + 4 = 0, \; 11$

Case I $\quad \alpha - \beta + 4 = 0$

when $\quad \beta = 2, \alpha = -2 \quad$ (rejected)

when $\quad \beta = 6, \alpha = 2$

Case II $\quad \alpha - \beta + 4 = 11 \quad$ when $\quad \beta = 2, \alpha = 9$

when $\quad \beta = 6, \alpha = 13 \quad$ (rejected)

$\therefore$ $\quad (\alpha, \beta) = (9, 2)(2, 6)$

$\therefore$ Number of ordered pairs are 2.

21. N is divisible by 6, if it is divisible is 2 and 3.

$N = 7\,3\,\alpha\,4\,9\,6\,1\,\beta\,0$ is divisible by 2 for all value of α and β.

$30 \le (30 + \alpha + \beta) \in I \le 48$

But $30 + \alpha + \beta$ is divisible by 3, if

$$30 + \alpha + \beta = 30, 33, 36, 39, 42, 45, 48$$

$$\alpha + \beta = 0, 3, 6, 9, 12, 15, 18$$

Number of possible value of $(\alpha + \beta)$ is equal to 7.

22. $i^N = 1$

$\Rightarrow N$ must be divisible by 4.

$\Rightarrow \qquad \beta = 0, 2, 4, 6, 8$

Number of possible value of β is 5.

23. Let $n = 3k + 4$, $r = 0, 1, 2$

$$n^2 = 9k^2 + 6kr + r^2 = 3m + r^2$$
$$r^2 = 0, 1, 4$$

$\therefore$ Remainder = 0 or 1.

24. (a) $n^2 + n + 1$ is an odd number.

(b) when $n = 1$, $n^2 + n + 1$ is divisible by 3.

(c) $111 = 10^2 + 10 + 1$

$\therefore$ when $n = 10$

$n^2 + n + 1$ is divisible by 111.

25. n = odd

Let $\qquad n = 2k + 1$

$$n^5 - n = n(n^4 - 1) = n(n-1)(n+1)(n^2+1)$$
$$= (2k+1)(2k)(2k+2)(4k^2 + 4k + 2)$$

which is divisible by n, as $(2k+1)$, $(2k)$ and $(2k+2)$ are three consecutive numbers by 16 as 8 (k be taken as common and k and $k+1$ are consecutive numbers) and divisible by 5, as either n in divisible by 5 or $n^4 - 1$ is divisible by 5.

$\therefore$ Options (a), (b), (c) and (d), all are correct.

2

Differentiation

Chapter in a Snapshot

- Definition of *dy/dx*
- Geometrical Meaning of the Derivative
- Differential Coefficients of Standard Functions
- Rules for Differentiation
- Differentiation of Implicit Functions
- Differentiation of Inverse Trigonometric Functions
- Differentiation of a Function in Parametric Form
- Logarithmic Differentiation
- Higher Derivatives of a Function
- Differentiation of a Function Given in the Form of a Determinant
- Derivative of an Inverse Function

Definition of *dy/dx*

Let us consider a function $y = f(x)$ defined in a certain interval. It has a definite value for each value of the argument x in this interval.

Let the argument x receives a certain increment Δx (it is immaterial whether Δx is positive or negative). Obviously, y will also receive a certain increment in Δy.

Since, for the argument x, $y = f(x)$. We have for the argument $(x + \Delta x)$

$$\Rightarrow \qquad y + \Delta y = f(x + \Delta x)$$

$$\therefore \qquad \Delta y = f(x + \Delta x) - y \quad \text{or} \quad \Delta y = f(x + \Delta x) - f(x)$$

Hence, $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$, which is the ratio of increment of the function to the increment of the argument.

Now, as $\Delta y \to 0, \Delta x \to 0$ if $\dfrac{\Delta y}{\Delta x} \to$ finite quantity.

Then, derivative of $f(x)$ exists and is denoted by y' or $f'(x)$ or $\dfrac{d(y)}{dx}$.

Thus, $$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \qquad \text{...(i)}$$

Consequently, the derivative of a given function $y = f(x)$ with respect to the argument x is the limit of ratio $\left(\dfrac{\Delta y}{\Delta x}\right)$, when $\Delta x \to 0$.

Points to Consider

(a) In later chapters, we are going to discuss that differentiation is basically a limit and we know limits come from two sides known as RHL and LHL.

Accordingly, we have two types of derivatives :

The left hand limit of $\left(\dfrac{dy}{dx}\right)$ is called left hand derivative, given by

$$\text{LHD} = f'(x - h) = \lim_{h \to 0} \frac{f(x - h) - f(x)}{-h} = \lim_{h \to 0} \frac{f(x) - f(x - h)}{h}$$

and right hand limit of $\left(\dfrac{dy}{dx}\right)$ is called right hand derivative, given by

$$\text{RHD} = f'(x + h) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

(b) Finding the value of limit given by Eq. (i) in respect of a variety of functions is called finding the derivative by first principles/by delta method/by *ab*-initio/by fundamental definition of Calculus.

(c) If $y = f(x)$, then the symbols $\dfrac{dy}{dx} = Dy = f'(x) = y_1$ or y' have the same meaning.

However, a dot denotes the time derivative.

eg, $$\dot{s} = \frac{ds}{dt}; \dot{\theta} = \frac{d\theta}{dt} \text{ etc.}$$

Geometrical Meaning of the Derivative

Let us consider a function $y = f(x)$ in a rectangular coordinate system. We also consider a point $P(x, y)$ on the curve.

If a point corresponding to an increased value of the argument $x + \Delta x$ is considered, its ordinate value is given by $y + \Delta y = f(x + \Delta x)$.

The point $(x + \Delta x, y + \Delta y)$ is represented by A.

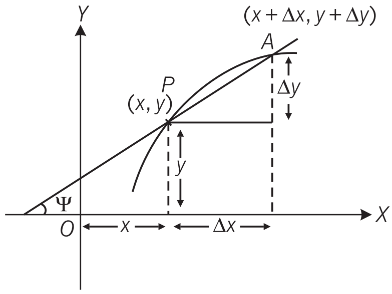


Fig. 2.1

Here, PA is the secant to the curve.

Now, as Δx, $\quad \Delta y \to 0 \quad \Rightarrow \quad PA \to 0$

(*ie*, the distance PA tends to zero or to a single point P)

$$\Rightarrow \qquad \lim_{\Delta x \to 0} \text{(slope of chord } PA) \to \text{(slope of tangent at } P).$$

$$\text{or} \qquad \tan \psi = \lim_{\Delta x \to 0} \tan \theta \quad \Rightarrow \quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \quad \text{or} \quad f'(x)$$

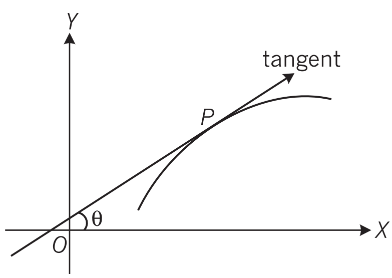


Fig. 2.2

which means that the value of the derivative $f'(x)$ for a given value of x is equal to the tangent of the angle formed by the line tangent to the graph of the function $y = f(x)$ at the point $P(x, y)$ with the positive x-axis.

Point to Consider

The meaning of the term 'rate of change of y w.r.t. x' is that if x is increased by an additional unit the change in y is given by $\frac{dy}{dx}\cdot$ For example, the rate of change of displacement of a particle is defined as its velocity, so if we say that a particle is moving with velocity v km/h, then it means that if time is increased by one hour the displacement changes by v km.

Differential Coefficients of Standard Functions

(i) $\frac{d}{dx}$ (constant) = 0

(ii) $\frac{d}{dx}(x^n) = nx^{n-1}$

(iii) $\frac{d}{dx}(e^x) = e^x$

(iv) $\frac{d}{dx}(a^x) = a^x \log a$

(v) $\frac{d}{dx}(\log_e |x|) = \frac{1}{x}$

(vi) $\frac{d}{dx}(\log_a |x|) = \frac{1}{x \log_e a}$

(vii) $\frac{d}{dx}(\sin x) = \cos x$

(viii) $\frac{d}{dx}(\cos x) = -\sin x$

(ix) $\frac{d}{dx}(\tan x) = \sec^2 x$

(x) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(xi) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(xii) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

(xiii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$ or $|x| < 1$

(xiv) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$, $-1 < x < 1$ or $|x| < 1$

(xv) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, $-\infty < x < \infty$ or $x \in R$

(xvi) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$, $|x| > 1$ or $x \in R - [-1, 1]$

(xvii) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$, $-\infty < x < \infty$

(xviii) $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$, $|x| > 1$ or $x \in R - [-1, 1]$

Rules for Differentiation

Term-by-Term Differentiation

If $y = u(x) \pm v(x) \pm w(x) \pm \ldots\ldots$, then $\frac{dy}{dx} = \frac{du(x)}{dx} \pm \frac{dv(x)}{dx} \pm \frac{dw(x)}{dx} \pm \ldots\ldots$ is known as term-by-term differentiation.

Illustration 1 If $y = x^2 + \sin^{-1} x + \log_e x$, find $\frac{dy}{dx}\cdot$

Solution. $y = x^2 + \sin^{-1} x + \log_e x$

On differentiating, we get $\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin^{-1} x) + \frac{d}{dx}(\log_e x)$

or $$\frac{dy}{dx} = 2(x)^{2-1} + \frac{1}{\sqrt{1-x^2}} + \frac{d}{dx}(\log_e x)$$

$$\frac{dy}{dx} = 2x + \frac{1}{\sqrt{1-x^2}} + \frac{1}{x}$$

Illustration 2 If $y = x^{-1/2} + \log_5 x + \dfrac{\sin x}{\cos x} + 2^x$, then find $\dfrac{dy}{dx}\cdot$

Solution. Here, $y = x^{-1/2} + \log_5 x + \tan x + 2^x$

On differentiating, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x)^{-1/2} + \frac{d}{dx}(\log_5 x) + \frac{d}{dx}\tan x + \frac{d}{dx}(2^x)$$

$$= -\frac{1}{2}(x)^{-1/2-1} + \frac{1}{x\log_e 5} + \sec^2 x + 2^x \log 2$$

$$= -\frac{1}{2}x^{-3/2} + \frac{1}{x\log_e 5} + \sec^2 x + 2^x \log 2$$

Differentiation of a Function Multiplied with a Constant

If $y = k\,f(x)$, then on differentiating, we get $\dfrac{dy}{dx} = k \cdot \dfrac{d}{dx}\, f(x)$.

Illustration 3 If $y = \log x^3 + 3\sin^{-1} x + kx^2$, then find $\dfrac{dy}{dx}\cdot$

Solution. Here, $y = \log x^3 + 3\sin^{-1} x + kx^2$

On differentiating, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\log x^3] + \frac{d}{dx}[3\sin^{-1} x] + \frac{d}{dx}[kx^2]$$

$$= 3\frac{d}{dx}[\log x] + 3\frac{d}{dx}(\sin^{-1} x) + k\frac{d}{dx}(x^2)$$

$$= 3\cdot\frac{1}{x} + 3\cdot\frac{1}{\sqrt{1-x^2}} + k(2x)$$

Product Rule

If $y = u(x)\cdot v(x)$, then $\dfrac{dy}{dx} = \left\{\dfrac{d}{dx}u(x)\right\}\cdot v(x) + u(x)\cdot\left\{\dfrac{d}{dx}v(x)\right\}$, and it is known as product rule.

Illustration 4 If $y = e^x \sin x$, then find $\dfrac{dy}{dx}\cdot$

Solution. $y = e^x \sin x$

On differentiating, we get

$$\frac{dy}{dx} = \left\{\frac{d}{dx}(e^x)\right\}\cdot \sin x + e^x\cdot\left\{\frac{d}{dx}(\sin x)\right\}$$

$$= e^x \cdot \sin x + e^x \cdot \cos x = e^x(\sin x + \cos x)$$

Generalization of the Product Rule Let $f(x)$, $g(x)$, $h(x)$ be three differentiable functions. Then,

$$\frac{d}{dx}\{f(x)\cdot g(x)\cdot h(x)\}=\left(\frac{d}{dx}\{f(x)\}\right)\cdot g(x)h(x)+f(x)\left(\frac{d}{dx}\{g(x)\}\right)\cdot h(x)$$
$$+f(x)\cdot g(x)\cdot\left(\frac{d}{dx}\{h(x)\}\right)$$

Proof We have,

$$\frac{d}{dx}[f(x)\cdot g(x)\cdot h(x)]=\frac{d}{dx}[\{f(x)\cdot g(x)\}\,h(x)]$$
$$=\{f(x)\cdot g(x)\}\frac{d}{dx}\{h(x)\}+h(x)\cdot\frac{d}{dx}\{f(x)\cdot g(x)\}$$
$$=f(x)\cdot g(x)\cdot\left(\frac{d}{dx}\{h(x)\}\right)+h(x)\cdot\left\{f(x)\cdot\frac{d}{dx}\{g(x)\}+g(x)\cdot\frac{d}{dx}\{f(x)\}\right\}$$
$$=f(x)\cdot g(x)\cdot\left(\frac{d}{dx}\{h(x)\}\right)+h(x)\cdot f(x)\left(\frac{d}{dx}\{g(x)\}\right)+h(x)\cdot g(x)\left(\frac{d}{dx}\{f(x)\}\right)$$

Point to Consider

If 3 functions are involved, then remember

$$D\{f(x)\cdot g(x)\cdot h(x)\}=f(x)\cdot g(x)\cdot h'(x)+f(x)\cdot g'(x)\cdot h(x)+f'(x)\cdot g(x)\cdot h(x)$$
$$=\frac{(fg)'h+(gh)'f+(hf)'g}{2}$$

This result can be generalized to product of "n" functions.

Illustration 5 If $y=e^x\tan x+x\cdot\log_e x$, then find $\frac{dy}{dx}$.

Solution. $y=e^x\tan x+x\cdot\log_e x$

On differentiating, we get

$$\frac{dy}{dx}=\frac{d}{dx}(e^x\tan x)+\frac{d}{dx}(x\log x)$$
$$=\left\{\frac{d}{dx}e^x\right\}\cdot\tan x+e^x\cdot\left\{\frac{d}{dx}\tan x\right\}+\left\{\frac{d}{dx}x\right\}\cdot\log x+x\cdot\left\{\frac{d}{dx}\log x\right\}$$
$$=e^x\cdot\tan x+e^x\cdot\sec^2 x+1\cdot\log x+x\cdot\frac{1}{x}$$

Hence, $\frac{dy}{dx}=e^x(\tan x+\sec^2 x)+(\log x+1)$

Illustration 6 Let f, g, h are differentiable functions. If $f(0)=1$, $g(0)=2$, $h(0)=3$ and the derivative of their pairwise product at $x=0$ are $(fg)'(0)=6$; $(gh)'(0)=4$ and $(hf)'(0)=5$, then compute the value of $(fgh)'(0)$.

Solution. We know, $(fgh)'=\frac{(fg)'h+(gh)'f+(hf)'g}{2}$

$$\therefore\qquad (fgh)'(0)=\frac{(fg)'(0)\cdot h(0)+(gh)'(0)\cdot f(0)+(hf)'(0)\cdot g(0)}{2}$$
$$=\frac{(6)\cdot(3)+(4)\cdot(1)+(5)\cdot(2)}{2}=16$$

Quotient Rule

If $y = \dfrac{u(x)}{v(x)}$, then $\dfrac{dy}{dx} = \dfrac{\left\{\dfrac{d}{dx}u(x)\right\}\cdot v(x) - \left\{\dfrac{d}{dx}v(x)\right\}\cdot u(x)}{\{v(x)\}^2}$ is known as the quotient rule of differentiating.

Illustration 7 If $y = \dfrac{e^x - \tan x}{x^n + \cot x}$, then find $\dfrac{dy}{dx}$.

Solution. $y = \dfrac{e^x - \tan x}{x^n + \cot x}$

On differentiating, we get

$$\frac{dy}{dx} = \frac{\left\{\dfrac{d}{dx}(e^x - \tan x)\right\}\cdot(x^n + \cot x) - (e^x - \tan x)\left\{\dfrac{d}{dx}(x^n + \cot x)\right\}}{(x^n + \cot x)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{(e^x - \sec^2 x)(x^n + \cot x) - (e^x - \tan x)(nx^{n-1} - \operatorname{cosec}^2 x)}{(x^n + \cot x)^2}$$

Illustration 8 If $y = \dfrac{\log x}{x} + e^x \sin x + \log_5 x$, then find $\dfrac{dy}{dx}$.

Solution. $y = \dfrac{\log x}{x} + e^x \sin x + \log_5 x$

On differentiating, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\log x}{x}\right) + \frac{d}{dx}(e^x \sin x) + \frac{d}{dx}(\log_5 x)$$

$$= \frac{\left\{\dfrac{d}{dx}(\log x)\right\}\cdot x - \log x\left\{\dfrac{d}{dx}x\right\}}{x^2} + \left\{\frac{d}{dx}e^x\right\}\cdot \sin x + e^x\cdot\left\{\frac{d}{dx}\sin x\right\} + \frac{1}{x\log_e 5}$$

$$= \frac{\dfrac{1}{x}\cdot x - \log x \cdot 1}{x^2} + e^x \sin x + e^x \cdot \cos x + \frac{1}{x\log_e 5}$$

Hence, $\quad \dfrac{dy}{dx} = \left(\dfrac{1 - \log x}{x^2}\right) + e^x(\sin x + \cos x) + \dfrac{1}{x\log_e 5}$

Point to Consider

While applying the quotient rule, think twice and check whether your function could be simplified prior to differentiation. Consider the Illustrations that follow.

Illustration 9 If $y = \dfrac{x^4 + x^2 + 1}{x^2 + x + 1}$ and $\dfrac{dy}{dx} = ax + b$, then find a and b.

Solution. Here, $y = \dfrac{x^4 + x^2 + 1}{x^2 + x + 1}$

$$\Rightarrow \qquad y = \frac{x^4 + 2x^2 + 1 - x^2}{x^2 + x + 1} = \frac{(x^2 + 1)^2 - x^2}{(x^2 + x + 1)}$$

$$\therefore \qquad y = \frac{(x^2 + x + 1)(x^2 - x + 1)}{(x^2 + x + 1)}$$

or $$y = x^2 - x + 1 \quad \Rightarrow \quad \frac{dy}{dx} = 2x - 1 \quad \therefore \quad a = 2,\ b = -1$$

Illustration 10 If $y = \dfrac{\sec x + \tan x - 1}{(-\sec x) + \tan x + 1}$, then find $\left(\dfrac{dy}{dx}\right)_{x = \frac{\pi}{4}}$.

Solution. Here, $y = \dfrac{\sec x + \tan x - 1}{-\sec x + \tan x + 1}$

$$\Rightarrow \qquad y = \frac{(\sec x + \tan x) - (\sec^2 x - \tan^2 x)}{-\sec x + \tan x + 1}$$

$$\Rightarrow \qquad y = \frac{(\sec x + \tan x)\{1 - \sec x + \tan x\}}{\tan x - \sec x + 1}$$

$$\Rightarrow \qquad y = \sec x + \tan x \quad \therefore \quad \frac{dy}{dx} = \sec x \tan x + \sec^2 x$$

$$\left(\frac{dy}{dx}\right)_{\text{at } x = \frac{\pi}{4}} = \sqrt{2} + 2$$

Illustration 11 If $y = \dfrac{\tan^{-1} x - \cot^{-1} x}{\tan^{-1} x + \cot^{-1} x}$, then find $\left(\dfrac{dy}{dx}\right)_{x = -1}$.

Solution. Here, $y = \dfrac{\tan^{-1} x - \cot^{-1} x}{\tan^{-1} x + \cot^{-1} x}$, as $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$

$$\therefore \qquad y = \frac{\tan^{-1} x - (\pi/2 - \tan^{-1} x)}{(\pi/2)}$$

$$\Rightarrow \qquad y = \frac{2}{\pi}\left[2 \tan^{-1} x - \frac{\pi}{2}\right] \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2}{\pi}\left[\frac{2}{1 + x^2}\right]$$

$$\left(\frac{dy}{dx}\right)_{x = -1} = \frac{2}{\pi}$$

Point to Consider

If $f'(x)$ is not defined on $x = c$, then it is wrong to conclude that $f(x)$ is not derivable at $x = c$. In such cases check LHD at $x = c$ and RHD at $x = c$.

eg, $$f(x) = x^{1/3} \cdot \sin x \text{ at } x = 0$$

$$f'(x) = x^{1/3} \cos x + \frac{1}{3x^{2/3}} \cdot \sin x$$

$f'(0)$ seems to be non-defined at $x = 0$. But $f'(0^+)$ and $f'(0^-)$ is 0.

Target Exercise 2.1

Directions (Q. Nos. 1 to 5) : *Differentiate the following functions w.r.t. x* :

1. $e^{x \log a} + e^{a \log x} + e^{a \log a}$

2. $\sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$

3. $\log_3 x + 3 \log_e x + 2 \tan x$

4. $|x| + a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$

5. $\sec^{-1}\left(\dfrac{x+1}{x-1}\right) + \sin^{-1}\left(\dfrac{x-1}{x+1}\right)$

Directions (Q. Nos. 6 to 8) : *Differentiate the following w.r.t. x* :

6. $x^n \log_a xe^x$

7. $\dfrac{2^x \cot x}{\sqrt{x}}$

8. $\dfrac{\sin x - x \cos x}{x \sin x + \cos x}$

9. If $y = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots \infty$, find the value of $\dfrac{dy}{dx}$.

10. Find the values of 'x' for which the rate of change of $\dfrac{x^4}{4} + \dfrac{x^3}{3} - x$ is more than $\dfrac{x^4}{4}$.

Chain Rule

If $y = [u\{v(x)\}]$, then $\dfrac{dy}{dx} = \dfrac{du\{v(x)\}}{d\{v(x)\}} \times \dfrac{d}{dx} v(x)$ is known as chain rule.

or If $y = u[v\{w(x)\}]$, then $\dfrac{dy}{dx} = \dfrac{du[v\{w(x)\}]}{dv\{w(x)\}} \times \dfrac{dv\{w(x)\}}{dw(x)} \times \dfrac{dw(x)}{dx}$

Illustration 12 If $y = \log(\sin x)$, then find $\dfrac{dy}{dx}$.

Solution. $y = \log(\sin x)$

On differentiating, we get $\dfrac{dy}{dx} = \dfrac{d[\log(\sin x)]}{d(\sin x)} \times \dfrac{d}{dx}(\sin x) = \dfrac{1}{\sin x} \times \cos x$

Hence, $\dfrac{dy}{dx} = \cot x$

Illustration 13 If $y = e^{(\tan^{-1} x)^3}$, then find $\dfrac{dy}{dx}$.

Solution. $y = e^{(\tan^{-1} x)^3}$

On differentiating, we get
$$\frac{dy}{dx} = \frac{d\{e^{(\tan^{-1} x)^3}\}}{d\{(\tan^{-1} x)^3\}} \cdot \frac{d\{(\tan^{-1} x)^3\}}{d(\tan^{-1} x)} \cdot \frac{d(\tan^{-1} x)}{dx}$$
$$= e^{(\tan^{-1} x)^3} \cdot 3(\tan^{-1} x)^2 \cdot \frac{1}{1+x^2}$$

Illustration 14 If $y = \log_e (\tan^{-1} \sqrt{1 + x^2})$, then find $\frac{dy}{dx}$.

Solution. $y = \log_e (\tan^{-1} \sqrt{1 + x^2})$

On differentiating, we get

$$\frac{dy}{dx} = \frac{d[\log_e (\tan^{-1} \sqrt{1 + x^2})]}{d(\tan^{-1} \sqrt{1 + x^2})} \cdot \frac{d(\tan^{-1} \sqrt{1 + x^2})}{d(\sqrt{1 + x^2})} \cdot \frac{d(\sqrt{1 + x^2})}{d(1 + x^2)} \cdot \frac{d(1 + x^2)}{dx}$$

$$= \frac{1}{\tan^{-1} \sqrt{1 + x^2}} \cdot \frac{1}{1 + (\sqrt{1 + x^2})^2} \cdot \frac{1}{2\sqrt{1 + x^2}} \cdot 2x$$

$$= \frac{x}{(\tan^{-1} \sqrt{1 + x^2})\{1 + (\sqrt{1 + x^2})^2\} \sqrt{1 + x^2}}$$

$$= \frac{x}{(\tan^{-1} \sqrt{1 + x^2})(2 + x^2)\sqrt{1 + x^2}}$$

Illustration 15 If $y = e^{ax} \cdot \cos(bx + c)$, then find $\frac{dy}{dx}$.

Solution. $y = e^{ax} \cdot \cos(bx + c)$

On differentiating, we get

$$\frac{dy}{dx} = \left\{\frac{d}{dx}(e^{ax})\right\} \cdot \cos(bx + c) + e^{ax} \cdot \left\{\frac{d}{dx}\{\cos(bx + c)\}\right\}$$

$$= \frac{d(e^{ax})}{d(ax)} \cdot \frac{d(ax)}{dx} \cdot \cos(bx + c) + e^{ax} \cdot \frac{d\cos(bx + c)}{d(bx + c)} \cdot \frac{d(bx + c)}{dx}$$

$$= e^{ax} \cdot a \cdot \cos(bx + c) + e^{ax} \cdot \{-\sin(bx + c)\} \cdot b$$

$$= ae^{ax} \cos(bx + c) - be^{ax} \sin(bx + c)$$

Target Exercise 2.2

Directions (Q. Nos. 1 to 20) : *Differentiate the following w.r.t. x :*

1. $(x^2 + x + 1)^4$

2. $\sqrt{x^2 + x + 1}$

3. $\sin^3 x$

4. $\frac{1}{\sqrt{a^2 - x^2}}$

5. $e^{x \sin x}$

6. $\sin^{-1}\left(\frac{a + b\cos x}{b + a\cos x}\right)$, $b > a > 1$

7. e^{e^x}

8. $\sqrt{\log\left\{\sin\left(\frac{x^2}{3} - 1\right)\right\}}$

9. $\log(x + \sqrt{a^2 + x^2})$

10. $\log\left(\frac{a + b\sin x}{a - b\sin x}\right)$

11. $\log\sqrt{\frac{1 + \sin x}{1 - \sin x}}$

12. $\frac{e^x + \log x}{\sin 3x}$

13. $\sin(m\sin^{-1} x)$, $|x| < 1$

14. $a^{(\sin^{-1} x)^2}$, $|x| < 1$

15. $e^{\cos^{-1}(\sqrt{1-x^2})}, |x| < 1$

16. $\dfrac{x \sin^{-1} x}{\sqrt{1-x^2}} + \log \sqrt{1-x^2}, |x| < 1$

17. $\log_{10} x + \log_x 10 + \log_x x + \log_{10} 10$

18. $5^{3-x^2} + (3-x^2)^5$

19. $\dfrac{\sqrt{a^2+x^2} + \sqrt{a^2-x^2}}{\sqrt{a^2+x^2} - \sqrt{a^2-x^2}}$

20. $\sqrt{4 + \sqrt{4 + \sqrt{4 + x^2}}}$

21. The differential coefficient of $f(\log_e x)$ w.r.t. x, where $f(x) = \log_e x$, is

(a) $\dfrac{x}{\log_e x}$ (b) $\dfrac{1}{x} \log_e x$ (c) $\dfrac{1}{x \log_e x}$ (d) None of these

22. If $f(x) = \log_e |x|, x \neq 0$, then $f'(x)$ is equal to

(a) $\dfrac{1}{|x|}$ (b) $\dfrac{1}{x}$ (c) $-\dfrac{1}{x}$ (d) None of these

23. Let $f(x) = \sin x$, $g(x) = x^2$ and $h(x) = \log x$. If $F(x) = h(f(g(x)))$, then $F'(x)$ is

(a) $2x \cot x^2$ (b) $2 \operatorname{cosec}^3 x$ (c) $-2 \operatorname{cosec}^2 x$ (d) None of these

24. If $f(x) = \cos x \cdot \cos 2x \cdot \cos 4x \cdot \cos 8x \cdot \cos 16x$, then $f'\left(\dfrac{\pi}{4}\right)$ is

(a) $\sqrt{2}$ (b) $\dfrac{1}{\sqrt{2}}$

(c) 1 (d) None of these

25. If $y = f\left(\dfrac{3x+4}{5x+6}\right)$ and $f'(x) = \tan x^2$, then $\dfrac{dy}{dx}$ is equal to

(a) $-2 \tan \left(\dfrac{3x+4}{5x+6}\right)^2 \cdot \dfrac{1}{(5x+6)^2}$ (b) $f\left(\dfrac{3 \tan x^2 + 3}{5 \tan x^2 + 6}\right) \tan x^2$

(c) $2x \tan \left(\dfrac{3x+4}{5x+6}\right)$ (d) $\tan x^2$

26. If $y = |\cos x| + |\sin x|$, then $\dfrac{dy}{dx}$ at $x = \dfrac{2\pi}{3}$ is

(a) $\dfrac{1}{2}(\sqrt{3}+1)$ (b) $2(\sqrt{3}-1)$

(c) $\dfrac{1}{2}(\sqrt{3}-1)$ (d) None of these

27. If $f'(x) = \sin x + \sin 4x \cdot \cos x$, then $f'\left(2x^2 + \dfrac{\pi}{2}\right)$ is

(a) $4x \{\cos(2x^2) - \sin 8x^2 \cdot \sin 2x^2\}$
(b) $4x \{\cos(2x^2) + \sin 8x^2 \cdot \cos 2x^2\}$
(c) $\{\cos(2x^2) - \sin 8x^2 \cdot \sin 2x^2\}$
(d) None of the above

28. If $f'(x) = \sqrt{2x^2 - 1}$ and $y = f(x^2)$, then $\dfrac{dy}{dx}$ at $x = 1$, is

(a) 1 (b) -1
(c) -2 (d) 2

Point to Consider

Relation Between dy/dx and dx/dy

Let x and y be two variables connected by a relation of the form $f(x, y) = 0$.

Let Δx be a small change in x and Δy be a small change in y, then

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \quad \text{and} \quad \frac{dx}{dy} = \lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y}$$

Now, $\frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta y} = 1 \quad \Rightarrow \quad \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta y} \right) = 1$

$\Rightarrow \quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \cdot \lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y} = 1 \qquad (\because \Delta x \to 0 \Leftrightarrow \Delta y \to 0)$

$\Rightarrow \quad \frac{dy}{dx} \cdot \frac{dx}{dy} = 1 \qquad$ So, $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

Differentiation of Implicit Functions

In previous rules, we have studied functions in which y was solely expressed in terms of x without complication. (*Called explicit functions*).

But, if the relation between the variables x and y are given by an equation containing both and this equation is not immediately solvable for y, then y is called an implicit function of x.

Implicit functions are given by $\phi(x, y) = 0$.

Short-cut of Differentiation of Implicit Functions
(Only to be applied while solving objective MCQs)

For implicit function put; $\frac{d}{dx}\{ f(x, y) \} = \frac{-\partial f / \partial x}{\partial f / \partial y}$, where $\frac{\partial f}{\partial x}$ is a partial differential of given function w.r.t. 'x' (*ie*, differentiating f w.r.t. x keeping y constant) and $\frac{\partial f}{\partial y}$ means partial differential of given function w.r.t. 'y' (*ie*, differentiating f w.r.t. y keeping x constant).

Illustration 16 If $x^2 + y^2 + xy = 2$, find $\frac{dy}{dx}\cdot$

Solution. $x^2 + y^2 + xy = 2$

Differentiating both sides, we get

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) + \frac{d}{dx}(xy) = \frac{d}{dx}(2)$$

or $\quad 2x + 2y\frac{dy}{dx} + \left\{\frac{dx}{dx}\right\} y + x\left\{\frac{d}{dx} y\right\} = 0$

or $\quad 2x + 2y\frac{dy}{dx} + 1 \cdot y + x \cdot \frac{dy}{dx} = 0$

or $\quad (2y + x)\frac{dy}{dx} = -(2x + y) \quad$ or $\quad \frac{dy}{dx} = -\frac{(2x + y)}{(2y + x)}$

Aliter : Put $f = x^2 + y^2 + xy - 2$

Now, $$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \qquad \text{...(i)}$$

where $$\frac{\partial f}{\partial x} = 2x + 0 + y - 0 \qquad \left(\text{as } \frac{\partial y^2}{\partial x} = 0, \frac{\partial xy}{\partial x} = y\right)$$

and $$\frac{\partial f}{\partial y} = 2y + x \qquad \left(\text{as } \frac{\partial x^2}{\partial y} = 0, \frac{\partial xy}{\partial y} = x\right)$$

Substituting in Eq. (i), we get $\dfrac{dy}{dx} = -\dfrac{(2x + y)}{(2y + x)}$

Illustration 17 If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots \infty}}}$, where $\sin x > 0$, then find $\dfrac{dy}{dx}$.

Solution. We have, $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots \infty}}}$

$\Rightarrow$ $$y = \sqrt{(\sin x) + y} \quad \Rightarrow \quad y^2 = \sin x + y$$

Differentiating both the sides, we get $2y\dfrac{dy}{dx} = \cos x + \dfrac{dy}{dx}$

$\Rightarrow$ $$(2y - 1)\frac{dy}{dx} = \cos x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\cos x}{2y - 1}$$

Illustration 18 If $y = x \cos y + y \cos x$, then find $\dfrac{dy}{dx}$.

Solution. $y = x \cos y + y \cos x$

Differentiating both the sides, we get

$$\frac{dy}{dx} = \left\{\frac{d}{dx} x\right\} \cos y + x\left\{\frac{d}{dx} \cos y\right\} + y\left\{\frac{d}{dx} \cos x\right\} + \left\{\frac{d}{dx} y\right\} \cdot \cos x$$

$$\frac{dy}{dx} = 1 \cdot \cos y + x(-\sin y)\frac{dy}{dx} + y(-\sin x) + \frac{dy}{dx}(\cos x)$$

$$\frac{dy}{dx}(1 + x \sin y - \cos x) = \cos y - y \sin x$$

$$\frac{dy}{dx} = \frac{\cos y - y \sin x}{1 + x \sin y - \cos x}$$

Aliter : $y = x \cos y + y \cos x$

Let $$f = x \cos y + y \cos x - y$$

$\Rightarrow$ $$\frac{\partial f}{\partial x} = \cos y - y \sin x \quad \text{and} \quad \frac{\partial f}{\partial y} = -x \sin y + \cos x - 1$$

$\therefore$ $$\frac{dy}{dx} = \frac{-\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = \frac{-(\cos y - y \sin x)}{(-x \sin y + \cos x - 1)}$$

or $$\frac{dy}{dx} = \frac{\cos y - y \sin x}{1 + x \sin y - \cos x}$$

Illustration 19 If $y\sqrt{1-x^2}+x\sqrt{1-y^2}=1$, when $|x|<1$ and $|y|<1$, then find $\dfrac{dy}{dx}$.

Solution. $y\sqrt{1-x^2}+x\sqrt{1-y^2}=1$

Let $x=\sin\theta$, $y=\sin\phi$, then we get

$$\sin\phi\sqrt{1-\sin^2\theta}+\sin\theta\sqrt{1-\sin^2\phi}=1$$

or $\quad\sin\phi\cos\theta+\sin\theta\cos\phi=1$

or $\quad\sin(\theta+\phi)=1 \quad\text{or}\quad \theta+\phi=\sin^{-1}(1)$

or $\quad\sin^{-1}x+\sin^{-1}y=\sin^{-1}(1)$

Differentiating both the sides, we get $\dfrac{1}{\sqrt{1-x^2}}+\dfrac{1}{\sqrt{1-y^2}}\cdot\dfrac{dy}{dx}=0$

Hence, $\quad\dfrac{dy}{dx}=-\sqrt{\dfrac{1-y^2}{1-x^2}}$

Aliter : Let $\quad f=y\sqrt{1-x^2}+x\sqrt{1-y^2}-1$

$$\Rightarrow\quad \frac{\partial f}{\partial x}=\frac{-2xy}{\sqrt{1-x^2}}+\sqrt{1-y^2}\quad\text{and}\quad\frac{\partial f}{\partial y}=\sqrt{1-x^2}-\frac{2xy}{\sqrt{1-y^2}}$$

$$\therefore\quad \frac{dy}{dx}=\frac{-\left(\dfrac{\partial f}{\partial x}\right)}{\left(\dfrac{\partial f}{\partial y}\right)}=-\frac{[\sqrt{1-x^2}\sqrt{1-y^2}-2xy]/\sqrt{1-x^2}}{[\sqrt{1-x^2}\sqrt{1-y^2}-2xy]/\sqrt{1-y^2}}$$

or $\quad\dfrac{dy}{dx}=-\dfrac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

Illustration 20 If $\sqrt{1-x^6}+\sqrt{1-y^6}=a^3(x^3-y^3)$, then

prove that $\dfrac{dy}{dx}=\dfrac{x^2}{y^2}\sqrt{\dfrac{1-y^6}{1-x^6}}$.

Solution. Here, $\sqrt{1-x^6}+\sqrt{1-y^6}=a^3(x^3-y^3)$

Let $x^3=\sin\theta$, $y^3=\sin\phi$, then we get

$\sqrt{1-\sin^2\theta}+\sqrt{1-\sin^2\phi}=a^3(\sin\theta-\sin\phi)\Rightarrow\cos\theta+\cos\phi=a^3(\sin\theta-\sin\phi)$

$$2\cos\left(\frac{\theta+\phi}{2}\right)\cdot\cos\left(\frac{\theta-\phi}{2}\right)=a^3\left[2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)\right]$$

$$\cos\left(\frac{\theta-\phi}{2}\right)=a^3\sin\left(\frac{\theta-\phi}{2}\right)\ \Rightarrow\ \cot\left(\frac{\theta-\phi}{2}\right)=a^3$$

or $\quad\theta-\phi=2\cot^{-1}(a^3)$ or $\sin^{-1}x^3-\sin^{-1}y^3=2\cot^{-1}(a^3)$

Differentiating both the sides, we get

$$\frac{1}{\sqrt{1-x^6}}\cdot 3x^2-\frac{1}{\sqrt{1-y^6}}\cdot 3y^2\cdot\frac{dy}{dx}=0$$

Hence, $\quad\dfrac{dy}{dx}=\dfrac{x^2\sqrt{1-y^6}}{y^2\sqrt{1-x^6}}$

Point to Consider

In above illustrations it is advised to use $\frac{dy}{dx} = -\left(\frac{\partial f / \partial x}{\partial f / \partial y}\right)$

Target Exercise 2.3

1. If $\log (x^2 + y^2) = 2 \tan^{-1} \left(\frac{y}{x}\right)$, then show that $\frac{dy}{dx} = \frac{x + y}{x - y}$.

2. If $x\sqrt{1 + y} + y\sqrt{1 + x} = 0$, then prove that $\frac{dy}{dx} = \frac{-1}{(x + 1)^2}$.

3. If $\cos^{-1} \left(\frac{x^2 - y^2}{x^2 + y^2}\right) = \tan^{-1} a$, then prove that $\frac{dy}{dx} = \frac{y}{x}$.

4. If $\sin y = x \sin (a + y)$, then prove that $\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$.

5. If $x^2 + y^2 = t - \frac{1}{t}$ and $x^4 + y^4 = t^2 + \frac{1}{t^2}$, then prove that $\frac{dy}{dx} = \frac{1}{x^3 y}$.

Differentiation of Inverse Trigonometric Functions

Some important results on trigonometric and inverse trigonometric functions are given below for reference (as, it sometimes becomes very easy to differentiate a function by using trigonometric transformations).

(i) $\sin 2x = 2 \sin x \cos x$

(ii) $1 + \cos 2x = 2 \cos^2 x$, $1 - \cos 2x = 2 \sin^2 x$

(iii) $\sin 2x = \frac{2 \tan x}{1 + \tan^2 x}$

(iv) $\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$

(v) $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$, where $x \neq (2n + 1) \frac{\pi}{4}$

(vi) $\sin 3x = 3 \sin x - 4 \sin^3 x = 4 \sin (60° - A) \cdot \sin A \cdot \sin (60° + A)$

(vii) $\cos 3x = 4 \cos^3 x - 3 \cos x = 4 \cos (60° - A) \cdot \cos A \cdot \cos (60° + A)$

(viii) $\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x} = \tan (60° - A) \cdot \tan A \cdot \tan (60° + A)$

(ix) $\cos A \cos 2A \cdot \cos 2^2 A \ldots \cos 2^{n-1} A = \frac{\sin 2^n A}{2^n \sin A}$

(x)

S. No.	Function	Domain	Range
1.	$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
2.	$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
3.	$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
4.	$\operatorname{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
5.	$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
6.	$\cot^{-1} x$	R	$(0, \pi)$

(xi)

S. No.	Function	Principal value branch	
1.	$\sin^{-1} x$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$,	where $y = \sin^{-1} x$
2.	$\cos^{-1} x$	$0 \le y \le \pi$,	where $y = \cos^{-1} x$
3.	$\tan^{-1} x$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$,	where $y = \tan^{-1} x$
4.	$\operatorname{cosec}^{-1} x$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$,	where $y = \operatorname{cosec}^{-1} x,\ y \ne 0$
5.	$\sec^{-1} x$	$0 \le y \le \pi$,	where $y = \sec^{-1} x,\ y \ne \frac{\pi}{2}$
6.	$\cot^{-1} x$	$0 < y < \pi$,	where $y = \cot^{-1} x$

Point to Consider

If no branch of an inverse trigonometric function is mentioned, then it means that the principal value branch of the function have to be taken.

Properties of Inverse Trigonometric Functions

Property I

(i) $\sin^{-1}(\sin\theta) = \theta$, for all $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\cos^{-1}(\cos\theta) = \theta$, for all $\theta \in [0, \pi]$

(iii) $\tan^{-1}(\tan\theta) = \theta$, for all $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(iv) $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta$, for all $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \theta \ne 0$

(v) $\sec^{-1}(\sec\theta) = \theta$, for all $\theta \in [0, \pi], \theta \ne \pi/2$

(vi) $\cot^{-1}(\cot\theta) = \theta$, for all $\theta \in (0, \pi)$

Property II

(i) $\sin(\sin^{-1} x) = x$, for all $x \in [-1, 1]$

(ii) $\cos(\cos^{-1} x) = x$, for all $x \in [-1, 1]$

(iii) $\tan(\tan^{-1} x) = x$, for all $x \in R$

(iv) $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(v) $\sec(\sec^{-1} x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\cot(\cot^{-1} x) = x$, for all $x \in R$

Property III

(i) $\sin^{-1}(-x) = -\sin^{-1}(x)$, for all $x \in [-1, 1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$, for all $x \in [-1, 1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x$, for all $x \in R$

(iv) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1} x$, for all $x \in R$

Property IV

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, & \text{for } x > 0 \\ -\pi + \cot^{-1} x, & \text{for } x < 0 \end{cases}$

Property V

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, for all $x \in [-1, 1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, for all $x \in R$

(iii) $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$, for all $x \in (-\infty, -1] \cup [1, \infty)$

Property VI

(i) $$\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } xy < 1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } x < 0, y < 0 \text{ and } xy < 1 \end{cases}$$

(ii) $\tan^{-1} x - \tan^{-1} y = \begin{cases} \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{if } xy > -1 \\ \pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{if } x > 0,\ y < 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{if } x < 0,\ y > 0 \text{ and } xy < -1 \end{cases}$

Remark If $x_1, x_2, x_3, \ldots, x_n \in R$, then

$$\tan^{-1} x_1 + \tan^{-1} x_2 + \ldots + \tan^{-1} x_n = \tan^{-1}\left(\frac{s_1 - s_3 + s_5 - \ldots}{1 - s_2 + s_4 - s_6 + \ldots}\right)$$

where $s_1 = x_1 + x_2 + \ldots + x_n = \Sigma\, x_1$

$s_2 = x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n = \Sigma\, x_1x_2$

$s_3 = \Sigma\, x_1x_2x_3 \ldots$ and so on.

Property VII

(i) $\sin^{-1} x + \sin^{-1} y$

$$= \begin{cases} \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}, & \text{if } -1 \le x, y \le 1 \text{ and } x^2 + y^2 \le 1 \\ & \text{of if } xy < 0 \text{ and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}, & \text{if } 0 < x, y \le 1 \text{ and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}, & \text{if } -1 \le x, y < 0 \text{ and } x^2 + y^2 > 1 \end{cases}$$

(ii) $\sin^{-1} x - \sin^{-1} y$

$$= \begin{cases} \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}, & \text{if } -1 \le x, y \le 1 \text{ and } x^2 + y^2 \le 1 \\ & \text{or if } xy > 0 \text{ and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}, & \text{if } 0 < x \le 1, -1 \le y \le 0 \\ & \text{and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}, & \text{if } -1 \le x < 0, 0 < y \le 1 \\ & \text{and } x^2 + y^2 > 1 \end{cases}$$

Property VIII

(i) $\cos^{-1} x + \cos^{-1} y$

$$= \begin{cases} \cos^{-1}\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1 \le x, y \le 1 \text{ and } x + y \ge 0 \\ 2\pi - \cos^{-1}\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1 \le x, y \le 1 \text{ and } x + y \le 0 \end{cases}$$

(ii) $\cos^{-1} x - \cos^{-1} y$

$$= \begin{cases} \cos^{-1}\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1 \le x, y \le 1 \text{ and } x \le y \\ -\cos^{-1}\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1 \le y \le 0, 0 < x \le 1 \text{ and } x \ge y \end{cases}$$

Property IX

(i) $\sin^{-1} x = \cos^{-1} \sqrt{1-x^2} = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

$$= \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

(ii) $\cos^{-1} x = \sin^{-1} \sqrt{1-x^2} = \tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$

$$= \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

(iii) $\tan^{-1} x = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$

$$= \cot^{-1}\left(\frac{1}{x}\right) = \sec^{-1}(\sqrt{1+x^2})$$

Property X

(i) $$2\sin^{-1} x = \begin{cases} \sin^{-1}(2x\sqrt{1-x^2}), & \text{if } -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ \pi - \sin^{-1}(2x\sqrt{1-x^2}), & \text{if } \dfrac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin^{-1}(2x\sqrt{1-x^2}), & \text{if } -1 \le x \le -\dfrac{1}{\sqrt{2}} \end{cases}$$

(ii) $$3\sin^{-1} x = \begin{cases} \sin^{-1}(3x - 4x^3), & \text{if } -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ \pi - \sin^{-1}(3x - 4x^3), & \text{if } \dfrac{1}{\sqrt{2}} < x \le 1 \\ -\pi - \sin^{-1}(3x - 4x^3), & \text{if } -1 \le x < -\dfrac{1}{\sqrt{2}} \end{cases}$$

Property XI

(i) $$2\cos^{-1} x = \begin{cases} \cos^{-1}(2x^2 - 1), & \text{if } 0 \le x \le 1 \\ 2\pi - \cos^{-1}(2x^2 - 1), & \text{if } -1 \le x \le 0 \end{cases}$$

(ii) $$3\cos^{-1} x = \begin{cases} \cos^{-1}(4x^3 - 3x), & \text{if } \dfrac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}(4x^3 - 3x), & \text{if } -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ 2\pi + \cos^{-1}(4x^3 - 3x), & \text{if } -1 \le x \le -\dfrac{1}{2} \end{cases}$$

Property XII

(i) $$2\tan^{-1}x = \begin{cases} \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{if } -1 < x \le 1 \\ \pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{if } x > 1 \\ -\pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{if } x < -1 \end{cases}$$

(ii) $$3\tan^{-1}x = \begin{cases} \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{if } -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{if } x > \dfrac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{if } x < -\dfrac{1}{\sqrt{3}} \end{cases}$$

Property XIII

(i) $$2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{if } -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{if } x > 1 \\ -\pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{if } x < -1 \end{cases}$$

(ii) $$2\tan^{-1}x = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & \text{if } 0 \le x < \infty \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & \text{if } -\infty < x < 0 \end{cases}$$

Illustration 21 If $y = \tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}$, then find $\dfrac{dy}{dx}$, when $0 < x < \dfrac{\pi}{2}\cdot$

Solution. $$y = \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$$

or $$y = \tan^{-1}\sqrt{\frac{2\sin^2 x/2}{2\cos^2 x/2}} = \tan^{-1}\sqrt{\tan^2 x/2}$$

or $$y = \tan^{-1}(\tan x/2)$$

or $$y = \frac{x}{2}$$

Hence, $$\frac{dy}{dx} = \frac{1}{2}$$

Illustration 22 If $y=\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$, then find $\dfrac{dy}{dx}$ when $-1\le x\le 1$.

Solution. Here, $\quad y=\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$, $\quad$ let $x=\tan\theta$

$$y=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)\Rightarrow y=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$y=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)=\tan^{-1}\left(\frac{2\sin^2\dfrac{\theta}{2}}{2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}\right)$$

$$y=\tan^{-1}\left(\tan\frac{\theta}{2}\right)=\frac{\theta}{2}$$

$$y=\frac{1}{2}\tan^{-1}x \qquad\Rightarrow\qquad \frac{dy}{dx}=\frac{1}{2(1+x^2)}$$

Illustration 23 Sketch the graph for $y=\sin^{-1}(\sin x)$ and hence find $\dfrac{dy}{dx}\cdot$

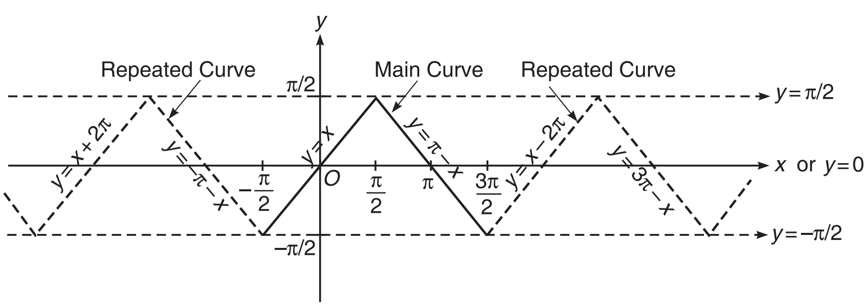
Solution. As, $y=\sin^{-1}(\sin x)$ is periodic with period 2π.

$\therefore$ To draw this graph we should draw the graph for one interval of length 2π and repeat it for entire values of x.

As we know, $\quad \sin^{-1}(\sin x)=\begin{cases} x, & -\dfrac{\pi}{2}\le x\le\dfrac{\pi}{2} \\ (\pi-x), & -\dfrac{\pi}{2}\le\pi-x<\dfrac{\pi}{2} \quad \left(ie,\ \dfrac{\pi}{2}\le x\le\dfrac{3\pi}{2}\right)\end{cases}$

or $\quad \sin^{-1}(\sin x)=\begin{cases} x, & -\dfrac{\pi}{2}\le x\le\dfrac{\pi}{2} \\ \pi-x, & \dfrac{\pi}{2}\le x\le\dfrac{3\pi}{2}\end{cases}$

which is defined for the interval of length 2π, plotted as;



Thus, the graph for $y=\sin^{-1}(\sin x)$, is a straight line up and a straight line down with slopes 1 and -1 respectively lying between $\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$.

Point to Consider

Students are advised to learn the definition of $\sin^{-1}(\sin x)$ as

$$y = \sin^{-1}(\sin x) = \begin{cases} x + 2\pi & , \; -\dfrac{5\pi}{2} \le x \le -\dfrac{3\pi}{2} \\ -\pi - x & , \; -\dfrac{3\pi}{2} \le x \le -\dfrac{\pi}{2} \\ x & , \; -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2} \\ \pi - x & , \; \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2} \\ x - 2\pi & , \; \dfrac{3\pi}{2} \le x \le \dfrac{5\pi}{2} \ldots \text{ and so on} \end{cases}$$

Thus, $$\frac{dy}{dx} = \begin{cases} 1 & , \; \left(2n - \dfrac{1}{2}\right)\pi < x < \left(2n + \dfrac{1}{2}\right)\pi \\ -1 & , \; \left(2n + \dfrac{1}{2}\right)\pi < x < \left(2n + \dfrac{3}{2}\right)\pi \\ \text{does not exist}, & x = \left\{(2n+1)\dfrac{\pi}{2};\, n \in I\right\} \end{cases}$$

Illustration 24 Sketch the graph for $y = \cos^{-1}(\cos x)$, and hence find $\dfrac{dy}{dx}$.

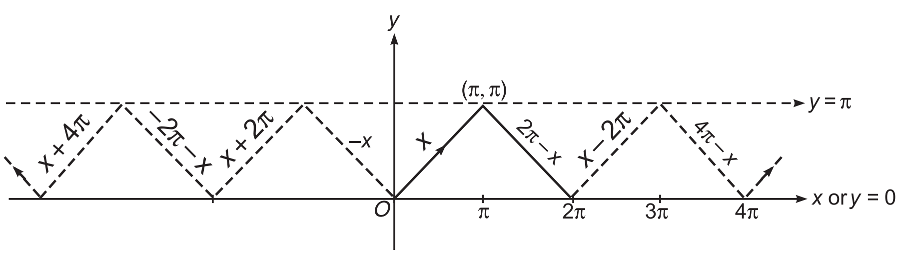
Solution. As, $y = \cos^{-1}(\cos x)$ is periodic with a period 2π.

$\therefore$ To draw this graph we should draw the graph for one interval of length 2π and repeat it for entire values of x of length 2π.

As we know, $$\cos^{-1}(\cos x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & 0 \le 2\pi - x \le \pi \end{cases}$$

or $$\cos^{-1}(\cos x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi. \end{cases}$$

Thus, it has been defined for $0 < x < 2\pi$ that has length 2π. So, its graph could be plotted as



Thus, the curve $y = \cos^{-1}(\cos x)$ and hence

$$\frac{dy}{dx} = \begin{cases} \text{does not exist}, & x = \{n\pi,\, n \in I\} \\ 1, & 2n\pi < x < (2n+1)\pi \\ -1, & (2n+1)\pi < x < (2n+2)\pi \end{cases}$$

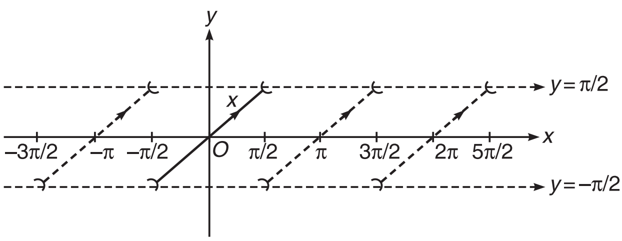
Illustration 25 Sketch the graph for $y = \tan^{-1}(\tan x)$, and hence find $\dfrac{dy}{dx}$.

Solution. As, $y = \tan^{-1}(\tan x)$ is periodic with period π.

$\therefore$ To draw this graph we should draw the graph for one interval of length π and repeat it for entire values of x.

As we know, $$\tan^{-1}(\tan x) = \left\{x, -\frac{\pi}{2} < x < \frac{\pi}{2}\right\}$$

Thus, it has been defined for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ that has length π. So, its graph could be plotted as



Thus, the curve for $y = \tan^{-1}(\tan x)$, where y is not defined for $x \in (2n+1)\dfrac{\pi}{2}$.

and hence $$\frac{dy}{dx} = \begin{cases} 1, & \left(2n - \dfrac{1}{2}\right)\pi < x < \left(2n + \dfrac{1}{2}\right)\pi \\ \text{does not exit}, & x = (2n+1)\dfrac{\pi}{2} \end{cases}$$

Illustration 26 Sketch the graphs for

(i) $y = \sin(\sin^{-1} x)$

(ii) $y = \cos(\cos^{-1} x)$

(iii) $y = \tan(\tan^{-1} x)$

(iv) $y = \operatorname{cosec}(\operatorname{cosec}^{-1} x)$

(v) $y = \sec(\sec^{-1} x)$

(vi) $y = \cot(\cot^{-1} x)$, and hence find $\dfrac{dy}{dx}$

Solution. As we know, all the above mentioned six curves are non-periodic, but have restricted domains and ranges.

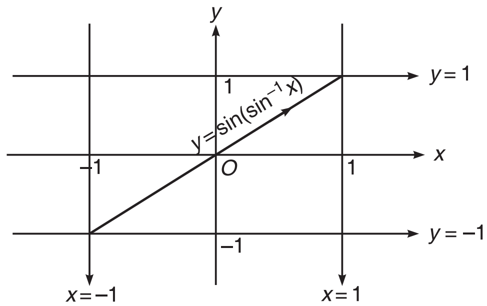
So, we shall first define each curve for its domain and range and then sketch the curves.

(*i*) ***Sketch for the curve y* = sin (sin^{-1} *x*)**

We know, domain $x \in [-1, 1]$ (*ie*, $-1 \le x \le 1$)

and range $y = x \Rightarrow y \in [-1, 1]$

Hence, we should sketch $y = \sin(\sin^{-1} x)$ only when $x \in [-1, 1]$ and $y = x$. So, its graph could be plotted as shown in the figure.



Thus, the graph for $y = \sin(\sin^{-1} x) = x$, $-1 \le x \le 1$.

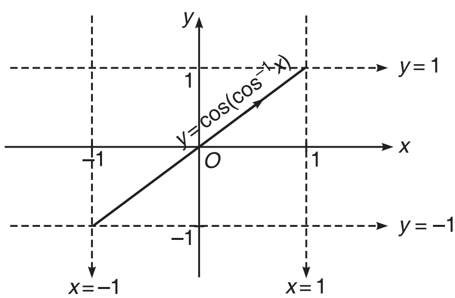
$$\Rightarrow \qquad \frac{dy}{dx} = 1, \quad -1 < x < 1.$$

(*ii*) ***Sketch for the curve y* = cos (cos^{-1} *x*)**

We know, domain $x \in [-1, 1]$ (*ie,* $-1 \le x \le 1$)

and range $y = x \Rightarrow y \in [-1, 1]$

Hence, we should sketch $y = \cos(\cos^{-1} x) = x$ only when $x \in [-1, 1]$. So, its graph could be plotted as shown in the figure.



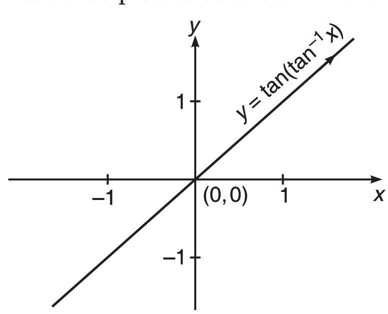
Thus, the graph for $y = \cos(\cos^{-1} x) = x$, $-1 \le x \le 1 \Rightarrow \dfrac{dy}{dx} = 1$, $-1 < x < 1$.

(*iii*) ***Sketch for the curve y* = tan (tan^{-1} *x*)**

We know, domain $x \in R$ (*ie,* $-\infty < x < \infty$) and range $y = x \Rightarrow y \in R$.

Hence, we should sketch $y = \tan(\tan^{-1} x) = x$, $\forall\, x \in R$.

So, its graph could be plotted as shown in the figure.



Thus, the graph for $y = \tan(\tan^{-1} x)$.

(*iv*) ***Sketch for the curve $y = \text{cosec}\,(\text{cosec}^{-1} x)$***

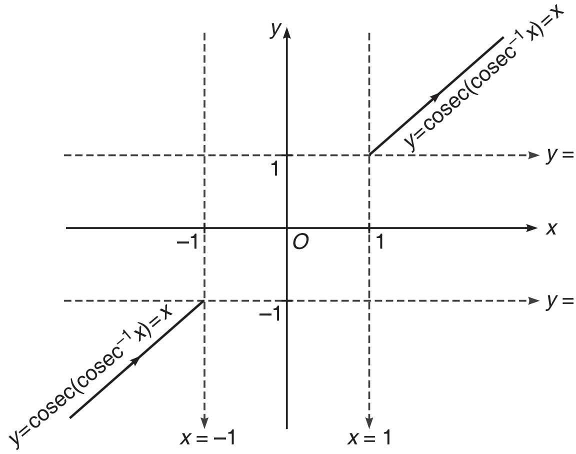
We know, domain $\in R - (-1, 1)$ (*ie*, $-\infty < x \le -1$ or $1 \le x < \infty$)

and range $y = x \Rightarrow y \in R - (-1, 1)$.

Hence, we should sketch

$$y = \text{cosec}\,(\text{cosec}^{-1} x) = x \text{ only when } x \in (-\infty, -1] \cup [1, \infty).$$

So, its graph could be plotted as shown in the figure.



Thus, the graph for $y = \text{cosec}\,(\text{cosec}^{-1} x) = x, |x| \ge 1 \Rightarrow \dfrac{dy}{dx} = 1, |x| > 1$

(*v*) ***Sketch for the curve $y = \sec\,(\sec^{-1} x)$***

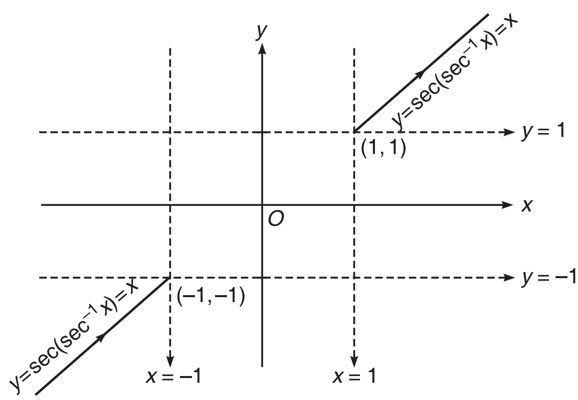
We know, domain $\in R - (-1, 1)$ (*ie*, $-\infty < x \le -1$ or $1 \le x < \infty$)

and range $y = x \Rightarrow y \in R - (-1, 1)$.

Hence, we should sketch $y = \sec\,(\sec^{-1} x) = x$, only when

$$x \in (-\infty, -1] \cup [1, \infty)$$

So, its graph could be plotted as shown in the figure.



Thus, the graph for $y = \sec\,(\sec^{-1} x) = x, |x| \ge 1$

$$\Rightarrow \quad \frac{dy}{dx} = 1, |x| > 1$$

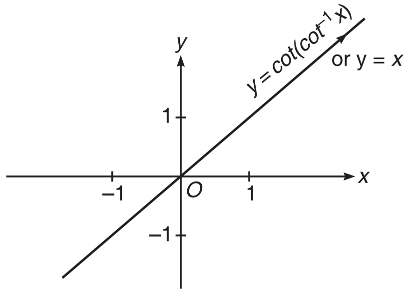
(*vi*) ***Sketch for the curve*** $\boldsymbol{y = \cot(\cot^{-1} x)}$

We know, domain $\in R$ (*ie*, $-\infty < x < \infty$)

and range $y = x \quad \Rightarrow \quad y = R$.

Hence, we should sketch $y = \cot(\cot^{-1} x) = x, \forall\, x \in R$.

Shown as in the figure.



Thus, the graph for $y = \cot(\cot^{-1} x) = x,\ x \in R \Rightarrow \dfrac{dy}{dx} = 1,\ x \in R.$

Illustration 27 Sketch the graph for

(i) $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ (ii) $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

(iii) $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ (iv) $y = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$

(v) $y = \sin^{-1}(3x - 4x^3)$ (vi) $y = \cos^{-1}(4x^3 - 3x)$

and hence find $\dfrac{dy}{dx}$.

Solution. As we know, all the above mentioned six curves are non-periodic, but have restricted domains and ranges.
So, we shall first define each curve for its domain and range, and then sketch these curves.

(*i*) ***Sketch for the curve*** $\boldsymbol{y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)}$

Here, for domain $\left|\dfrac{2x}{1+x^2}\right| \le 1$

$\Rightarrow \quad 2|x| \le 1 + x^2 \qquad \{\because\ 1 + x^2 > 0 \text{ for all } x\}$

$\Rightarrow \quad |x|^2 - 2|x| + 1 \ge 0 \qquad \{\because\ x^2 = |x|^2\}$

$\Rightarrow \quad (|x|^2 - 1)^2 \ge 0 \quad \Rightarrow \quad x \in R$

For range $\quad y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

$\Rightarrow \quad y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] \qquad \left\{\text{as, } y = \sin^{-1}\theta \ \Rightarrow y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]\right\}$

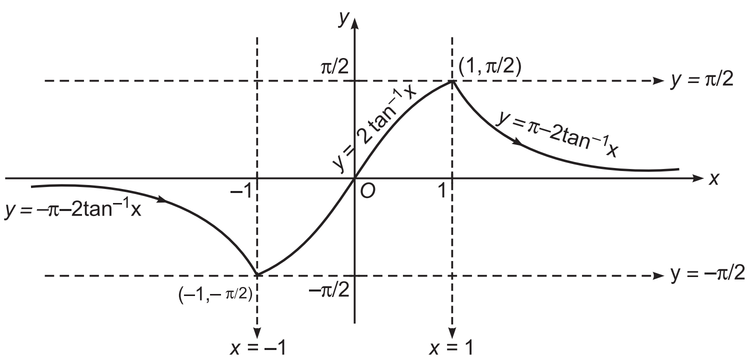
Defining the curve let $x = \tan\theta$

$$\Rightarrow \quad y = \sin^{-1}(\sin 2\theta) = \begin{cases} \pi - 2\theta & , \quad 2\theta > \dfrac{\pi}{2} \\ 2\theta & , \quad -\dfrac{\pi}{2} \le 2\theta \le \dfrac{\pi}{2} \\ -\pi - 2\theta & , \quad 2\theta < -\dfrac{\pi}{2} \end{cases} \qquad \{\text{See Ex. 1}\}$$

$$\text{or} \quad y = \begin{cases} \pi - 2\tan^{-1} x & , \quad \tan^{-1} x > \dfrac{\pi}{4} \\ 2\tan^{-1} x & , \quad -\dfrac{\pi}{4} \le \tan^{-1} x \le \dfrac{\pi}{4} \\ -\pi - 2\tan^{-1} x & , \quad \tan^{-1} x < -\dfrac{\pi}{4} \end{cases} \quad \{\because \ \tan\theta = x \Rightarrow \theta = \tan^{-1} x\}$$

$$\text{or} \quad y = \begin{cases} \pi - 2\tan^{-1} x & , \quad x > 1 \\ 2\tan^{-1} x & , \quad -1 \le x \le 1 \\ -\pi - 2\tan^{-1} x & , \quad x < -1 \end{cases} \qquad \ldots\text{(i)}$$

Thus, $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ is defined for $x \in R$, where $y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, so the graph for Eq. (i) could be shown as in the figure below.



Thus, the graph for $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

$$\text{or } y = \begin{cases} \pi - 2\tan^{-1} x, & x > 1 \\ 2\tan^{-1} x, & -1 \le x \le 1 \\ -\pi - 2\tan^{-1} x, & x < -1 \end{cases} \Rightarrow \frac{dy}{dx} = \begin{cases} \dfrac{-2}{1+x^2}, & |x| > 1 \\ \dfrac{2}{1+x^2}, & |x| < 1 \\ \text{does not exist}, & |x| = 1 \end{cases}$$

(*ii*) ***Sketch for y*** $= \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

Here, for domain $\left|\dfrac{1-x^2}{1+x^2}\right| \le 1$

$\Rightarrow \quad |1 - x^2| \le 1 + x^2 \qquad \{\because \ 1 + x^2 > 0, \ \forall x \in R\}$

which is true for all x; as $1 + x^2 > 1 - x^2$

$\therefore \quad x \in R$

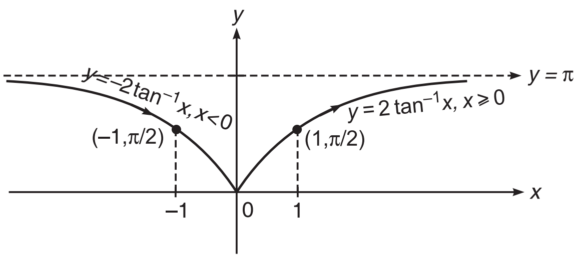
For range $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) \Rightarrow y \in (0, \pi)$

Define the curve Let $x = \tan\theta$

$$\therefore \quad y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta) = \begin{cases} 2\theta, & 2\theta \geq 0 \\ -2\theta, & 2\theta < 0 \end{cases}$$

$$\Rightarrow \quad y = \begin{cases} 2\tan^{-1}x, & \tan^{-1}x \geq 0 \\ -2\tan^{-1}x, & \tan^{-1}x < 0 \end{cases} \qquad (\because \tan\theta = x \Rightarrow \theta = \tan^{-1}x)$$

So, the graph of $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x, & x \geq 0 \\ -2\tan^{-1}x, & x < 0 \end{cases}$ is shown as



Thus, the graph for $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

$$= \begin{cases} 2\tan^{-1}x, & x \geq 0 \\ -2\tan^{-1}x, & x < 0 \end{cases}$$

$$\Rightarrow \quad \frac{dy}{dx} = \begin{cases} \dfrac{2}{1+x^2}, & x > 0 \\ \text{does not exist}, & x = 0 \\ \dfrac{-2}{1+x^2}, & x < 0 \end{cases}$$

(*iii*) ***Sketch for the curve*** $\boldsymbol{y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)}$

Here, for domain $\dfrac{2x}{1-x^2} \in R$ except; $1 - x^2 = 0$

ie, $x \neq \pm 1$ or $x \in R - \{1, -1\}$

For range $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

$$\Rightarrow \quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \left\{ \text{as } y = \tan^{-1}\theta \Rightarrow y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right\}$$

Defining the curve Let $x = \tan\theta$

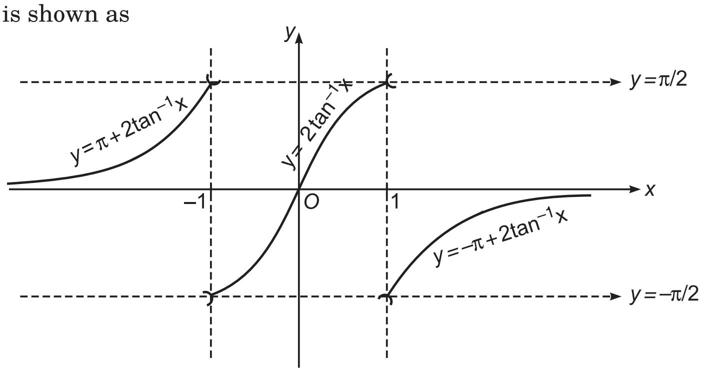
$$\Rightarrow \quad y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = \begin{cases} \pi + 2\theta, & 2\theta < -\frac{\pi}{2} \\ 2\theta, & -\frac{\pi}{2} < 2\theta < \frac{\pi}{2} \\ -\pi + 2\theta, & 2\theta > \frac{\pi}{2} \end{cases}$$

$$= \begin{cases} \pi + 2\tan^{-1}x, & \tan^{-1}x < -\frac{\pi}{4} \\ 2\tan^{-1}x, & -\frac{\pi}{4} < \tan^{-1}x < \frac{\pi}{4} \\ -\pi + 2\tan^{-1}x, & \tan^{-1}x > \frac{\pi}{4} \end{cases} \quad \{\text{as } \tan\theta = x \Rightarrow \theta = \tan^{-1}x\}$$

$$= \begin{cases} \pi + 2\tan^{-1}x, & x < -1 \\ 2\tan^{-1}x, & -1 < x < 1 \\ -\pi + 2\tan^{-1}x, & x > 1 \end{cases}$$

So, the graph of $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \begin{cases} \pi + 2\tan^{-1}x, & x < -1 \\ 2\tan^{-1}x, & -1 < x < 1 \\ -\pi + 2\tan^{-1}x, & x > 1 \end{cases}$

is shown as



Thus, the graph for $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \begin{cases} \pi + 2\tan^{-1}x, & x < -1 \\ 2\tan^{-1}x, & -1 < x < 1 \\ -\pi + 2\tan^{-1}x, & x > 1 \end{cases}$

$$\Rightarrow \quad \frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2}, & x \in R - \{-1, 1\} \\ \text{does not exist}, & x = \{1, -1\} \end{cases}$$

(*iv*) ***Sketch for the curve*** $\boldsymbol{y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)}$

Here, for domain $y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

$\Rightarrow \; x \in R$ except $1 - 3x^2 = 0 \quad \Rightarrow \quad x \neq \pm\frac{1}{\sqrt{3}}$

$\therefore \quad x \in R - \left\{\pm\frac{1}{\sqrt{3}}\right\}$

For range $\quad y = \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right)$

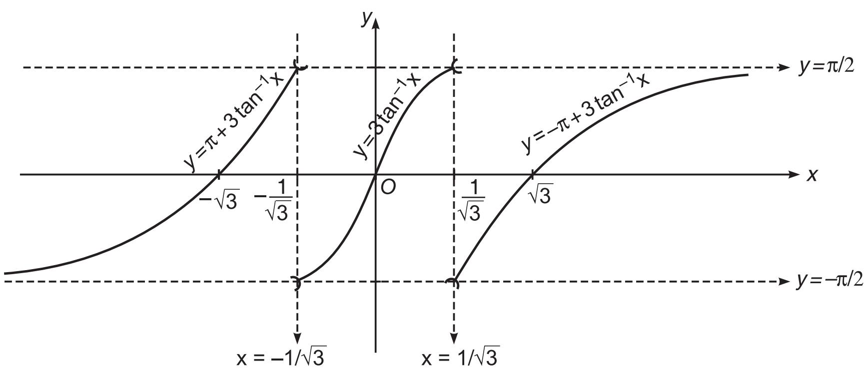
$\Rightarrow \quad y \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \qquad \left(\text{as } y = \tan^{-1}\theta \Rightarrow y \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)\right)$

Defining the curve Let $x = \tan\theta$

$$\Rightarrow \quad y = \tan^{-1}(\tan 3\theta) = \begin{cases} \pi + 3\theta, & 3\theta < -\dfrac{\pi}{2} \\ 3\theta, & -\dfrac{\pi}{2} < 3\theta < \dfrac{\pi}{2} \\ -\pi + 3\theta, & 3\theta > \dfrac{\pi}{2} \end{cases}$$

$$= \begin{cases} \pi + 3\tan^{-1} x, & \tan^{-1} x < -\dfrac{\pi}{6} \\ 3\tan^{-1} x, & -\dfrac{\pi}{6} < \tan^{-1} x < \dfrac{\pi}{6} \\ -\pi + 3\tan^{-1} x, & \tan^{-1} x > \dfrac{\pi}{6} \end{cases} = \begin{cases} \pi + 3\tan^{-1} x, & x < -\dfrac{1}{\sqrt{3}} \\ 3\tan^{-1} x, & -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1} x, & x > \dfrac{1}{\sqrt{3}} \end{cases}$$

So, the graph of $y = \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right) = \begin{cases} \pi + 3\tan^{-1} x, & x < -\dfrac{1}{\sqrt{3}} \\ 3\tan^{-1} x, & -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1} x, & x > \dfrac{1}{\sqrt{3}} \end{cases}$



Thus, the curve for $y = \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right) = \begin{cases} \pi + 3\tan^{-1} x, & x < -\dfrac{1}{\sqrt{3}} \\ 3\tan^{-1} x, & -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1} x, & x > \dfrac{1}{\sqrt{3}} \end{cases}$

$$\frac{dy}{dx} = \begin{cases} \dfrac{3}{1 + x^2}, & x \in R - \left\{\pm \dfrac{1}{\sqrt{3}}\right\} \\ \text{does not exist}, & x = \left\{\pm \dfrac{1}{\sqrt{3}}\right\} \end{cases}$$

(*v*) ***Sketch for the curve*** $\boldsymbol{y = \sin^{-1}(3x - 4x^3)}$

Defining the curve Let $x = \sin\theta$

$$\Rightarrow \qquad y = \sin^{-1}(\sin 3\theta) = \begin{cases} \pi - 3\theta, & \dfrac{\pi}{2} \le 3\theta \le \dfrac{3\pi}{2} \\ 3\theta, & -\dfrac{\pi}{2} \le 3\theta \le \dfrac{\pi}{2} \\ -\pi - 3\theta, & -\dfrac{3\pi}{2} \le 3\theta \le -\dfrac{\pi}{2} \end{cases}$$

$$= \begin{cases} \pi - 3\sin^{-1}x, & \dfrac{\pi}{6} \le \sin^{-1}x \le \dfrac{\pi}{2} \\ 3\sin^{-1}x, & -\dfrac{\pi}{6} \le \sin^{-1}x \le \dfrac{\pi}{6} \\ -\pi - 3\sin^{-1}x, & -\dfrac{\pi}{2} \le \sin^{-1}x \le -\dfrac{\pi}{6} \end{cases}$$

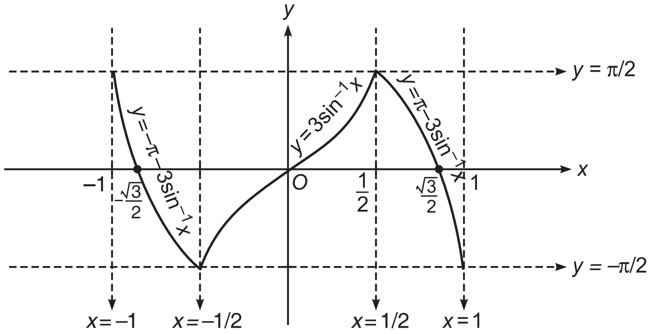
$$\therefore \qquad y = \sin^{-1}(3x - 4x^3) = \begin{cases} \pi - 3\sin^{-1}x, & \dfrac{1}{2} \le x \le 1 \\ 3\sin^{-1}x, & -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ -\pi - 3\sin^{-1}x, & -1 \le x \le -\dfrac{1}{2} \end{cases}$$

For domain, $y = \sin^{-1}(3x - 4x^3) \Rightarrow x \in [-1, 1]$

For range, $y = \sin^{-1}(3x - 4x^3) \Rightarrow y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

So, the graph of $y = \sin^{-1}(3x - 4x^3) = \begin{cases} \pi - 3\sin^{-1}x, & \dfrac{1}{2} \le x \le 1 \\ 3\sin^{-1}x, & -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ -\pi - 3\sin^{-1}x, & -1 \le x \le -\dfrac{1}{2} \end{cases}$

is shown as



Thus, the curve for $y = \sin^{-1}(3x - 4x^3)$

$$= \begin{cases} \pi - 3\sin^{-1}x, & \dfrac{1}{2} \le x \le 1 \\ 3\sin^{-1}x, & -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ -\pi - 3\sin^{-1}x, & -1 \le x \le -\dfrac{1}{2} \end{cases}$$

$$\therefore \qquad \frac{dy}{dx} = \begin{cases} \dfrac{-3}{\sqrt{1-x^2}}, & \dfrac{1}{2} < |x| < 1 \\ \text{does not exist}, & |x| = \left\{\dfrac{1}{2}, 1\right\} \\ \dfrac{3}{\sqrt{1-x^2}}, & |x| < \dfrac{1}{2} \end{cases}$$

(*vi*) *Sketch for the curve* $y = \cos^{-1}(4x^3 - 3x)$

Here, domain $\in [-1, 1]$ range $\in [0, \pi]$

Now, define the curve.

Let $x = \cos\theta$

$$\Rightarrow \qquad y = \cos^{-1}(\cos 3\theta) = \begin{cases} 2\pi - 3\theta, & \pi \le 3\theta \le 2\pi \\ 3\theta, & 0 \le 3\theta \le \pi \\ -2\pi + 3\theta, & -\pi \le 3\theta \le 0 \end{cases}$$

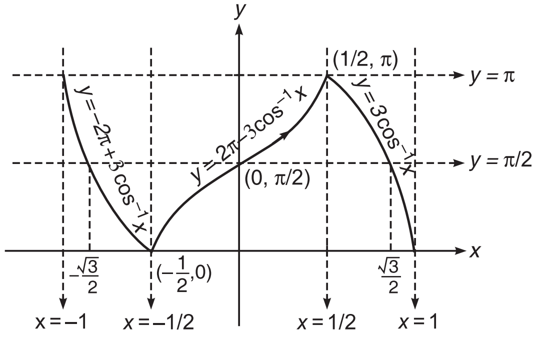
$$= \begin{cases} 2\pi - 3\cos^{-1}x, & \dfrac{\pi}{3} \le \cos^{-1}x \le \dfrac{2\pi}{3} \\ 3\cos^{-1}x, & 0 \le \cos^{-1}x \le \dfrac{\pi}{3} \\ -2\pi + 3\cos^{-1}x, & -\dfrac{\pi}{3} \le \cos^{-1}x \le 0 \end{cases}$$

$$= \begin{cases} 2\pi - 3\cos^{-1}x, & -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ 3\cos^{-1}x, & \dfrac{1}{2} \le x \le 1 \\ -2\pi + 3\cos^{-1}x, & -1 \le x \le -\dfrac{1}{2} \end{cases}$$

{$\because$ If $0 \le \theta \le \dfrac{\pi}{3} \Rightarrow \cos\dfrac{\pi}{3} \le \cos\theta \le \cos 0$ or $\dfrac{1}{2} \le \cos\theta \le 1$. Here, the interval is changed, since $\cos x$ is decreasing in $[0, \pi]$}

So, the graph of

$$y = \cos^{-1}(4x^3 - 3x) = \begin{cases} 2\pi - 3\cos^{-1}x, & -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ 3\cos^{-1}x, & \dfrac{1}{2} \le x \le 1 \\ -2\pi + 3\cos^{-1}x, & -1 \le x \le -\dfrac{1}{2} \end{cases} \quad \text{is shown as}$$



Thus, the curve for $y = \cos^{-1}(4x^3 - 3x)$

$$= \begin{cases} -2\pi + 3\cos^{-1} x, & -1 \le x \le -\frac{1}{2} \\ 2\pi - 3\cos^{-1} x, & -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1} x, & \frac{1}{2} \le x \le 1 \end{cases}$$

$$\frac{dy}{dx} = \begin{cases} -\frac{3}{\sqrt{1-x^2}}, & \frac{1}{2} < |x| < 1 \\ \text{does not exist}, & |x| = \frac{1}{2} \\ \frac{3}{\sqrt{1-x^2}}, & |x| < \frac{1}{2} \end{cases}$$

Target Exercise 2.4

1. If $y = \tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)$, then $\frac{dy}{dx}$ is

(a) $\begin{cases} \frac{1}{2}, & x \in R - \{n\pi\}; \quad n \in I \\ -\frac{1}{2}, & x = \{n\pi\}; \quad n \in I \end{cases}$

(b) $\begin{cases} \frac{1}{2}, & x \in R - \{n\pi\}; \quad n \in I \\ \text{does not exist}, & x = \{n\pi\}; \quad n \in I \end{cases}$

(c) $\begin{cases} -\frac{1}{2}, & x \in R - \{n\pi\}; \quad n \in I \\ \text{does not exist}, & x = \{n\pi\}; \quad n \in I \end{cases}$

(d) None of the above

2. If $y = \cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right)$, then $\frac{dy}{dx}$ is

(a) $\begin{cases} \frac{2}{1+x^2}, & x > 0 \\ -\frac{2}{1+x^2}, & x < 0 \end{cases}$

(b) $\begin{cases} \frac{2}{1+x^2}, & x > 0 \\ \text{does not exist}, & x = 0 \\ \frac{-2}{1+x^2}, & x < 0 \end{cases}$

(c) $\begin{cases} \frac{2}{1+x^2}, & x < 0 \\ \text{does not exist}, & x = 0 \\ \frac{-2}{1+x^2}, & x > 0 \end{cases}$

(d) None of these

3. If $y = \tan^{-1}\left\{\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}$, then $\dfrac{dy}{dx}$ is

(a) $\begin{cases} \dfrac{1}{2}, & \cos\dfrac{x}{2} > \sin\dfrac{x}{2} \\ -\dfrac{1}{2}, & \cos\dfrac{x}{2} < \sin\dfrac{x}{2} \\ \text{does not exist}, & x = \{n\pi\};\ n \in \text{integer} \end{cases}$

(b) $\begin{cases} -\dfrac{1}{2}, & \cos\dfrac{x}{2} > \sin\dfrac{x}{2} \\ \dfrac{1}{2}, & \cos\dfrac{x}{2} < \sin\dfrac{x}{2} \\ \text{does not exist}, & x = \{n\pi\};\ n \in \text{integer} \end{cases}$

(c) $\begin{cases} -\dfrac{1}{2}, & \cos\dfrac{x}{2} \ge \sin\dfrac{x}{2} \\ \dfrac{1}{2}, & \cos\dfrac{x}{2} < \sin\dfrac{x}{2} \end{cases}$

(d) None of the above

4. If $y = \cot^{-1}(\cot x)$, then $\dfrac{dy}{dx}$ is

(a) $1, x \in R$

(b) $1, x \in R - \{n\pi\}$

(c) $\begin{cases} 1, & x \in R - \{n\pi\} \\ \text{does not exist}, & x \in \{n\pi\};\ n \in \text{integer} \end{cases}$

(d) None of these

Differentiation of a Function in Parametric Form

$$\left.\begin{array}{l} \text{let } y = f(t) \\ \text{and } x = g(t) \end{array}\right\} \quad \text{be function of variable } t \qquad \text{...(i)}$$

where t is the parameter, then

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{f'(t)}{g'(t)} \qquad \text{[From Eq. (i)]}$$

Illustration 28 If $x = e^{-t^2}$ and $y = \tan^{-1}(2t+1)$, find $\dfrac{dy}{dx}$.

Solution. Here, $x = e^{-t^2}$

On differentiating both the sides, we get

$$\Rightarrow \qquad \frac{dx}{dt} = e^{-t^2}\cdot(-2t) \quad \text{and} \quad y = \tan^{-1}(2t+1)$$

On differentiating both the sides, we get

$$\Rightarrow \qquad \frac{dy}{dt} = \frac{1}{1+(2t+1)^2}(2)$$

$$\therefore \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2}{1+(4t^2+4t+1)}}{-\frac{2t}{e^{t^2}}}$$

Hence,
$$\frac{dy}{dx} = \frac{-e^{t^2}}{2t(2t^2+2t+1)}$$

Illustration 29 If $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$, then find $\frac{dy}{dx}$.

Solution. Here, $\quad x = a(\theta - \sin\theta)$

On differentiating both the sides, we get

$$\frac{dx}{d\theta} = a(1-\cos\theta)$$

and $\qquad y = a(1-\cos\theta)$

On differentiating both the sides, we get

$$\frac{dy}{d\theta} = a(\sin\theta)$$

So,
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

Logarithmic Differentiation

So far, we have discussed derivatives of the functions of the form $(f(x))^n$, $n^{f(x)}$ and n^n, where $f(x)$ is a function of x and n is a constant. In this section, we will mainly be discussing derivatives of the functions of the form $(f(x))^{g(x)}$, where $f(x)$ and $g(x)$ are functions of x. To find the derivatives of these types of functions, we proceed as follows

Let $y = (f(x))^{g(x)}$, taking logarithm of both the sides, we have

$$\log y = g(x)\log\{f(x)\}$$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = g(x)\cdot\frac{1}{f(x)}\cdot\frac{d\,f(x)}{dx} + \log\{f(x)\}\cdot\frac{d\,g(x)}{dx}$$

$$\therefore \qquad \frac{dy}{dx} = y\left[\frac{g(x)}{f(x)}\cdot\frac{d\,f(x)}{dx} + \log\{f(x)\}\cdot\frac{d\,g(x)}{dx}\right]$$

or
$$\frac{dy}{dx} = (f(x))^{g(x)}\left\{\frac{g(x)}{f(x)}\cdot\frac{d\,f(x)}{dx} + \log\{f(x)\}\cdot\frac{d\,g(x)}{dx}\right\}$$

Logarithmic Differentiation (For Quickly Solving Objective MCQs)

If $y = \{f(x)\}^{\phi(x)}$, then $\dfrac{dy}{dx}$ = d.c. of $\{f(x)\}^{\phi(x)}$ w.r.t. x taking $\phi(x)$ as a constant + d.c. of $\{f(x)\}^{\phi(x)}$ w.r.t. x taking $f(x)$ as a constant.

$$\Rightarrow \quad \frac{dy}{dx} = \phi(x) \cdot \{f(x)\}^{\phi(x)-1} \cdot \frac{df(x)}{dx} + \{f(x)\}^{\phi(x)} \cdot \log f(x) \cdot \frac{d\phi(x)}{dx}$$

or when we have to differentiate the function of the form (Variable)$^{\text{variable}}$, take log on both the sides and then differentiate.

Illustration 30 If $y = x^{\sin x}$, then find $\dfrac{dy}{dx}$.

Solution. $y = x^{\sin x}$, taking log on both the sides, we get

$$\log y = \sin x \log x$$

On differentiating both the sides, we get

$$\frac{1}{y}\frac{dy}{dx} = \sin x \cdot \left(\frac{1}{x}\right) + \log x \cdot (\cos x)$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left[\frac{\sin x}{x} + (\cos x)\log x\right]$$

$$\therefore \quad \frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + (\cos x)\log x\right]$$

Aliter : Here, $y = x^{\sin x}$ could also be differentiated by using definition;

$$\frac{d}{dx}(\text{variable})^{\text{variable}} = \frac{d}{dx}(\text{variable})^{\text{constant}} + \frac{d}{dx}(\text{constant})^{\text{variable}}$$

ie, $$\frac{dy}{dx} = \frac{d}{dx}\underset{\text{variable}}{(x)}^{\overset{\text{constant}}{\sin x}} + \frac{d}{dx}\underset{\text{constant}}{(x)}^{\overset{\text{variable}}{\sin x}}$$

$$= \sin x\,(x)^{\sin x - 1} + (x)^{\sin x} \cdot \log x \cdot \cos x$$

$$= (\sin x)\frac{(x)^{\sin x}}{x} + (x)^{\sin x} \cdot \log x \cdot \cos x$$

$$= x^{\sin x}\left\{\frac{\sin x}{x} + \cos x \cdot \log x\right\}$$

Illustration 31 If $x^y \cdot y^x = 1$, then find $\dfrac{dy}{dx}$.

Solution. Taking log on both the sides, $y \log x + x \log y = \log 1$

Differentiating both the sides, we get

$$y \cdot \frac{d}{dx}(\log x) + \left\{\frac{d}{dx}y\right\} \cdot \log x + \left\{\frac{d}{dx}x\right\} \cdot \log y + x\left\{\frac{d}{dx}\log y\right\} = 0$$

or $$y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx} + 1 \cdot \log y + x \cdot \frac{1}{y} \cdot \frac{dy}{dx} = 0$$

or $$\left[\log x + \frac{x}{y}\right]\frac{dy}{dx} = -\left[\frac{y}{x} + \log y\right]$$

$$\frac{dy}{dx} = -\frac{(y + x\log y)}{(x + y\log x)} \cdot \frac{y}{x}$$

Illustration 32 If $y^{\cot x} + (\tan^{-1} x)^y = 1$, then find $\frac{dy}{dx}$.

Solution. Let $u = y^{\cot x}$ and $v = (\tan^{-1} x)^y$, we get

$$u + v = 1 \qquad \text{...(i)}$$

Taking log on both sides for u and v, we get

$$\log u = \cot x \log y \text{ and } \log v = y \log (\tan^{-1} x)$$

On differentiating both the sides, we get $\frac{1}{u} \cdot \frac{du}{dx} = \cot x \cdot \frac{1}{y} \frac{dy}{dx} + \log y \cdot (-\text{cosec}^2 x)$

$$\frac{du}{dx} = y^{\cot x} \left[\frac{\cot x}{y} \cdot \frac{dy}{dx} - (\text{cosec}^2 x) \log y \right] \qquad \text{...(ii)}$$

Again, differentiating $\log v = y \log (\tan^{-1} x)$, we get

$$\frac{1}{v} \cdot \frac{dv}{dx} = y \cdot \frac{1}{\tan^{-1} x} \cdot \frac{1}{1 + x^2} + \log (\tan^{-1} x) \cdot \frac{dy}{dx}$$

or

$$\frac{dv}{dx} = (\tan^{-1} x)^y \left[\frac{y}{(1 + x^2) \tan^{-1} x} + \log (\tan^{-1} x) \cdot \frac{dy}{dx} \right] \qquad \text{...(iii)}$$

From Eq. (i), we get $\frac{du}{dx} + \frac{dv}{dx} = 0$

$$y^{\cot x} \left\{ \frac{\cot x}{y} \cdot \frac{dy}{dx} - (\text{cosec}^2 x) \log y \right\}$$

$$+ (\tan^{-1} x)^y \left[\frac{y}{(1 + x^2) \tan^{-1} x} + \log (\tan^{-1} x) \cdot \frac{dy}{dx} \right] = 0$$

or

$$\left\{ \frac{y^{\cot x} \cdot \cot x}{y} + (\tan^{-1} x)^y \cdot \log (\tan^{-1} x) \right\} \frac{dy}{dx}$$

$$= \left\{ y^{\cot x} \cdot (\text{cosec}^2 x) \log y - \frac{(\tan^{-1} x)^y \cdot y}{(1 + x^2) \tan^{-1} x} \right\}$$

Hence,

$$\frac{dy}{dx} = \frac{(y^{\cot x} \cdot \text{cosec}^2 x \cdot \log y) - \left\{ \frac{(\tan^{-1} x)^{y-1} \cdot y}{1 + x^2} \right\}}{(y^{\cot x - 1} \cdot \cot x) + \{(\tan^{-1} x)^y \cdot \log (\tan^{-1} x)\}}$$

Derivative of the Product of a Finite Number of Functions

Illustration 33 If $f(x) = (x + 1)(x + 2)(x + 3) \dots (x + n)$, then find $f'(0)$.

Solution. $f(x) = (x + 1)(x + 2)(x + 3) \dots (x + n)$

$$\log f(x) = \log \{(x + 1)(x + 2) \dots (x + n)\}$$

Differentiating both the sides w.r.t. x

$$\frac{1}{f(x)} f'(x) = \frac{1}{x + 1} + \frac{1}{x + 2} + \frac{1}{x + 3} + \dots + \frac{1}{x + n}$$

$\therefore$

$$f'(0) = f(0) \left\{ 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right\}$$

$$= (1 \cdot 2 \cdot 3 \cdot \dots \cdot n) \left\{ 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right\} = (n)! \left\{ 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right\}$$

Illustration 34 If $f(x) = \prod_{n=1}^{100} (x-n)^{n(101-n)}$, then find $\frac{f(101)}{f'(101)}$.

Solution. $f(x) = \prod_{n=1}^{100} (x-n)^{n(101-n)}$

$$\Rightarrow \qquad \log f(x) = \log \left\{ n(101-n) \left\{ \prod_{n=1}^{100} (x-n) \right\} \right\}$$

{Here, Π changes to Σ when taken log}

$$\Rightarrow \qquad \log f(x) = \sum_{n=1}^{100} n(101-n) \log (x-n)$$

Differentiating both the sides, we get

$$\frac{f'(x)}{f(x)} = \sum_{n=1}^{100} n(101-n) \cdot \frac{1}{x-n}$$

$$\therefore \qquad \frac{f'(101)}{f(101)} = \sum_{n=1}^{100} \frac{n(101-n)}{(101-n)} = \sum_{n=1}^{100} n = 5050 \Rightarrow \frac{f(101)}{f'(101)} = \frac{1}{5050}$$

Target Exercise 2.5

Directions (Q. Nos. 1 to 10) : *Differentiate the following functions w.r.t. x :*

1. x^x **2.** $x^{\sqrt{x}}$ **3.** x^{x^x}

4. $(x^x)^x$ **5.** $(x^x)\sqrt{x}$ **6.** $(\cos x)^x$

7. $(\sin x)^{\cos x}$ **8.** $(\sin x)^{\cos^{-1} x}$ **9.** $\cos (x^x)$

10. $\log (x^x + \operatorname{cosec}^2 x)$

11. If $y = (\sin x)^{\tan x} + (\cos x)^{\sec x}$, then find $\frac{dy}{dx}$.

12. If $x^y = e^{x-y}$, then prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

13. If $x^y + y^x = 2$, then find $\frac{dy}{dx}$.

14. If $(\cos x)^y = (\sin y)^x$, then find $\frac{dy}{dx}$.

15. If $x \sin (a+y) + \sin a \cdot \cos (a+y) = 0$. Prove that $\frac{dy}{dx} = \frac{\sin^2 (a+y)}{\sin a}$.

16. If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \dots \infty}}}$, then prove that $\frac{dy}{dx} = \frac{\sin x}{1-2y}$.

17. If $y = (\tan x)^{(\tan x)^{(\tan x)^{\cdots \infty}}}$, then prove that $\frac{dy}{dx} = 2$ at $x = \frac{\pi}{4}$.

18. If $y = e^{x^{e^x}} + x^{e^{e^x}} + e^{x^{x^e}}$, then prove that

$$\frac{dy}{dx} = e^{x^{e^x}} \cdot x^{e^x} \cdot \left\{ \frac{e^x}{x} + e^x \cdot \log x \right\} + x^{e^{e^x}} \cdot e^{e^x} \cdot \left\{ \frac{1}{x} + e^x \cdot \log x \right\}$$
$$+ e^{x^{x^e}} \cdot x^{x^e} \cdot x^{e-1} \{1 + e \log x\}.$$

19. If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$, prove that $\frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$.

20. If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, then prove that $\dfrac{dy}{dx} = \dfrac{-y \log x}{x \log y}$.

21. If $x = \cos t$ and $y = \sin t$, then prove that $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{3}}$, at $t = \dfrac{2\pi}{3}$.

22. If $x = a\left(t + \dfrac{1}{t}\right)$ and $y = a\left(t - \dfrac{1}{t}\right)$, then prove that $\dfrac{dy}{dx} = \dfrac{x}{y}$.

23. If $x = \sin^{-1}\left(\dfrac{2t}{1+t^2}\right)$ and $y = \tan^{-1}\left(\dfrac{2t}{1-t^2}\right)$, then prove that $\dfrac{dy}{dx} = 1$.

24. Differentiate $x^{\sin^{-1} x}$ w.r.t. $\sin^{-1} x$.

25. Differentiate $\sin^{-1}(2ax\sqrt{1-a^2x^2})$ w.r.t. $\sqrt{1-a^2x^2}$.

26. If $y = \tan^{-1}\left(\dfrac{a_1 x - \alpha}{a_1\alpha + x}\right) + \tan^{-1}\left(\dfrac{a_2 - a_1}{1 + a_1 a_2}\right) + \tan^{-1}\left(\dfrac{a_3 - a_2}{1 + a_2 a_3}\right) + \ldots$

$$+ \tan^{-1}\left(\dfrac{a_n - a_{n-1}}{1 + a_n a_{n-1}}\right) - \tan^{-1}(a_n), \text{ then find } \dfrac{dy}{dx}.$$

Higher Derivatives of a Function

If $y = f(x)$, then the derivative of $\dfrac{dy}{dx}$ w.r.t. x is called the second derivative of y w.r.t. x and it is denoted by $\dfrac{d^2y}{dx^2}$.

Also,
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right);\ \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

Point to Consider

If y is a function of x, given parametrically by $y = \psi(t)$, then

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\phi'(t)}{\psi'(t)}\right) = \frac{\dfrac{d}{dx}\cdot\left(\dfrac{\phi'(t)}{\psi'(t)}\right)}{\dfrac{dx}{dt}}$$

Illustration 35 If $x = a(t + \sin t)$ and $y = a(1 - \cos t)$, then find $\dfrac{d^2y}{dx^2}$.

Solution. Here, $x = a(t + \sin t)$ and $y = a(1 - \cos t)$

Differentiating both the sides w.r.t. t, we get

$$\frac{dx}{dt} = a(1 + \cos t) \quad \text{and} \quad \frac{dy}{dt} = a(\sin t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{a \sin t}{a(1 + \cos t)} = \frac{2\sin\frac{t}{2}\cdot\cos\frac{t}{2}}{2\cos^2\frac{t}{2}} = \tan\left(\frac{t}{2}\right)$$

Again, differentiating both the sides, we get

$$\frac{d^2y}{dx^2} = \sec^2\left(\frac{t}{2}\right)\cdot\frac{1}{2}\cdot\frac{dt}{dx}$$

$$= \frac{1}{2}\sec^2\left(\frac{t}{2}\right)\cdot\frac{1}{a\,(1+\cos t)} = \frac{1}{2a}\cdot\frac{\sec^2\left(\frac{t}{2}\right)}{2\left(\cos^2\frac{t}{2}\right)}$$

Hence, $$\frac{d^2y}{dx^2} = \frac{1}{4a}\cdot\sec^4\left(\frac{t}{2}\right)$$

Illustration 36 Show the function $y = f(x)$ defined by the parametric equations $x = e^t\cdot\sin t$, $y = e^t\cdot\cos t$, satisfies the relation $y''(x+y)^2 = 2\,(xy' - y)$.

Solution. Here, $x = e^t\cdot\sin t$

On differentiating w.r.t. t, we get

$$\frac{dx}{dt} = e^t\cdot\cos t + e^t\cdot\sin t = e^t\,(\sin t + \cos t) \quad\text{and}\quad y = e^t\cdot\cos t$$

On differentiating w.r.t. t, we get

$$\frac{dy}{dt} = e^t\cdot(-\sin t) + e^t\cdot(\cos t) = e^t\,(\cos t - \sin t)$$

So, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^t\,(\cos t - \sin t)}{e^t\,(\cos t + \sin t)} = \frac{y-x}{y+x}$$

or $$\frac{dy}{dx} = \frac{y-x}{y+x} \qquad \text{...(i)}$$

Differentiating both the sides w.r.t. x,

$$\frac{d^2y}{dx^2} = \frac{(y+x)\cdot\left\{\frac{dy}{dx} - 1\right\} - (y-x)\left\{\frac{dy}{dx} + 1\right\}}{(y+x)^2}$$

or $$y'' = \frac{(y+x)\cdot(y'-1) - (y-x)(y'+1)}{(y+x)^2}$$

Therefore, $(x+y)^2\cdot y'' = 2(xy' - y)$

Differentiation of a Function Given in the Form of a Determinant

If $y = \begin{vmatrix} u(x) & v(x) & w(x) \\ p(x) & q(x) & r(x) \\ \lambda(x) & \mu(x) & \gamma(x) \end{vmatrix}$, then

$$\frac{dy}{dx} = \begin{vmatrix} u'(x) & v'(x) & w'(x) \\ p(x) & q(x) & r(x) \\ \lambda(x) & \mu(x) & \gamma(x) \end{vmatrix} + \begin{vmatrix} u(x) & v(x) & w(x) \\ p'(x) & q'(x) & r'(x) \\ \lambda(x) & \mu(x) & \gamma(x) \end{vmatrix} + \begin{vmatrix} u(x) & v(x) & w(x) \\ p(x) & q(x) & r(x) \\ \lambda'(x) & \mu'(x) & \gamma'(x) \end{vmatrix}$$

The differentiation can also be done column-wise.

Illustration 37 If $f(x)=\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$, then find $f'(x)$.

Solution. Here, $f(x)=\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$

On differentiating, we get

$$\Rightarrow \quad f'(x)=\begin{vmatrix} \frac{d}{dx}(x) & \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}+\begin{vmatrix} x & x^2 & x^3 \\ \frac{d}{dx}(1) & \frac{d}{dx}(2x) & \frac{d}{dx}(3x^2) \\ 0 & 2 & 6x \end{vmatrix}$$

$$+\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ \frac{d}{dx}(0) & \frac{d}{dx}(2) & \frac{d}{dx}(6x) \end{vmatrix}$$

or $$f'(x)=\begin{vmatrix} 1 & 2x & 3x^2 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}+\begin{vmatrix} x & x^2 & x^3 \\ 0 & 2 & 6x \\ 0 & 2 & 6x \end{vmatrix}+\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix}$$

As we know, if any two rows or columns are equal, then value of the determinant is zero.

$$=0+0+\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix}$$

$\therefore$ $\qquad f'(x)=6\,(2x^2-x^2)$

Therefore, $\quad f'(x)=6\,x^2$

Illustration 38 If $f(x)=\begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$, then find $f'(x)$.

Solution. We have, $f(x)=\begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$

$$\Rightarrow f'(x)=\begin{vmatrix} 1 & 0 & 0 \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}+\begin{vmatrix} x+a^2 & ab & ac \\ 0 & 1 & 0 \\ ac & bc & x+c^2 \end{vmatrix}+\begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ 0 & 0 & 1 \end{vmatrix}$$

$$=\{(x+b^2)(x+c^2)-b^2c^2\}+\{(x+a^2)(x+c^2)-a^2c^2\}$$
$$+\{(x+a^2)(x+b^2)-a^2b^2\}$$

$$=3x^2+2x\,(a^2+b^2+c^2)$$

Illustration 39 Let $f(x)=\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$, where p is constant. Then, find $\dfrac{d^3}{dx^3}[f(x)]$ at $x=0$. **[IIT JEE 1997]**

Solution. Given, $f(x)=\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

On differentiating it, we get $\dfrac{d}{dx}[f(x)]=\begin{vmatrix} \dfrac{d}{dx}(x^3) & \dfrac{d}{dx}(\sin x) & \dfrac{d}{dx}(\cos x) \\ 6 & (-1) & 0 \\ p & p^2 & p^3 \end{vmatrix}$

$$+\begin{vmatrix} x^3 & \sin x & \cos x \\ 0 & 0 & 0 \\ p & p^2 & p^3 \end{vmatrix}+\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

So, $$\frac{d}{dx}[f(x)]=\begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

Again, differentiating it, we get

$$\frac{d^2}{dx^2}[f(x)]=\begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}+\text{ Remaining two determinants as zero}$$

Differentiating it, again at $x=0$

$$\frac{d^3}{dx^3}[f(x)]=\begin{vmatrix} 6 & -\cos x & \sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}+\text{ Remaining two determinants as zero}$$

At $x=0$, $$\frac{d^3}{dx^3}[f(x)]=\begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}=0$$

$\therefore$ $$\left(\frac{d^3}{dx^3}[f(x)]\right)_{\text{at } x=0}=0 \text{ (ie, independent of } p)$$

Illustration 40 If $y=\cos ax$, prove that $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}=0$, where $y_r=\dfrac{d^r}{dx^r}\cdot y$.

Solution. Given, $y=\cos(ax)$

Then, $$y_1=-a\sin(ax)=a\cos\left(\frac{\pi}{2}+ax\right)$$

$$y_2 = -a^2 \cos(ax) = a^2 \cos\left(\frac{2\pi}{2} + ax\right)$$

$$y_3 = +a^3 \sin(ax) = a^3 \cos\left(\frac{3\pi}{2} + ax\right)$$

....

....

$$y_n = a^n \cos\left(\frac{n\pi}{2} + ax\right)$$

$\therefore$ The determinant $\Delta(x) = \begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$

$$\Delta(x) = \begin{vmatrix} \cos ax & a\cos\left(\frac{\pi}{2} + ax\right) & a^2\cos\left(\frac{2\pi}{2} + ax\right) \\ a^3\cos\left(\frac{3\pi}{2} + ax\right) & a^4\cos\left(\frac{4\pi}{2} + ax\right) & a^5\cos\left(\frac{5\pi}{2} + ax\right) \\ a^6\cos\left(\frac{6\pi}{2} + ax\right) & a^7\cos\left(\frac{7\pi}{2} + ax\right) & a^8\cos\left(\frac{8\pi}{2} + ax\right) \end{vmatrix}$$

$$\Delta(x) = a^3 \times a^6 \begin{vmatrix} \cos(ax) & -a\sin(ax) & -a^2\cos(ax) \\ \sin(ax) & a\cos(ax) & -a^2\sin(ax) \\ -\cos(ax) & a\sin(ax) & a^2\cos(ax) \end{vmatrix} \qquad \{R_1 \to -R_3\}$$

$$\Delta(x) = a^9 \times (0)$$

Hence, $\Delta(x) = 0$

Illustration 41 If $f(x) = \begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix}$, then find

(i) constant term (ii) coefficient of x.

Solution. Here,

$$f(x) = \begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix} = A + Bx + Cx^2 + \dots \quad \text{...(i)}$$

Putting $x = 0$, we get

$$f(0) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = A + B(0) + C(0)^2 + \dots$$

$\Rightarrow \qquad A = 0$

Again, differentiating Eq. (i) w.r.t. x, we get

$$f'(x) = \begin{vmatrix} a(1+x)^{a-1} & 2b(1+2x)^{b-1} & 0 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix}$$

$$+\begin{vmatrix}(1+x)^a & (1+2x)^b & 1\\ 0 & a(1+x)^{a-1} & 2b(1+2x)^{b-1}\\ (1+2x)^b & 1 & (1+x)^a\end{vmatrix}$$

$$+\begin{vmatrix}(1+x)^a & (1+2x)^b & 1\\ 1 & (1+x)^a & (1+2x)^b\\ 2b(1+2x)^{b-1} & 0 & a(1+x)^{a-1}\end{vmatrix} = B + 2Cx + \dots$$

$$f'(0) = \begin{vmatrix}a & 2b & 0\\ 1 & 1 & 1\\ 1 & 1 & 1\end{vmatrix} + \begin{vmatrix}1 & 1 & 1\\ 0 & a & 2b\\ 1 & 1 & 1\end{vmatrix} + \begin{vmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 2b & 0 & a\end{vmatrix} = B$$

$\Rightarrow \qquad B = 0 \qquad \dots\text{(ii)}$

$\therefore$ Coefficient of constant term = coefficient of $x = 0$.

Illustration 42 If $a_i, b_i \in N$ for $i = 1, 2, 3$, then coefficient of x in the determinant;

$$\begin{vmatrix}(1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3}\\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3}\\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3}\end{vmatrix}$$

Solution. Here, $\begin{vmatrix}(1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3}\\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3}\\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3}\end{vmatrix} = A + Bx + Cx^2 + \dots$

Differentiating both the sides w.r.t. x, we get

$$\Rightarrow \quad \begin{vmatrix}a_1b_1(1+x)^{a_1b_1-1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3}\\ a_2b_1(1+x)^{a_2b_1-1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3}\\ a_3b_1(1+x)^{a_3b_1-1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3}\end{vmatrix}$$

$$+\begin{vmatrix}(1+x)^{a_1b_1} & a_1b_2(1+x)^{a_1b_2-1} & (1+x)^{a_1b_3}\\ (1+x)^{a_2b_1} & a_2b_2(1+x)^{a_2b_2-1} & (1+x)^{a_2b_3}\\ (1+x)^{a_3b_1} & a_3b_2(1+x)^{a_3b_2-1} & (1+x)^{a_3b_3}\end{vmatrix}$$

$$+\begin{vmatrix}(1+x)^{a_1b_1} & (1+x)^{a_1b_2} & a_1b_3(1+x)^{a_1b_3-1}\\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & a_2b_3(1+x)^{a_2b_3-1}\\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & a_3b_3(1+x)^{a_3b_3-1}\end{vmatrix} = B + 2Cx + \dots$$

Putting $x = 0$,

$$B = \begin{vmatrix}a_1b_1 & 1 & 1\\ a_2b_1 & 1 & 1\\ a_3b_1 & 1 & 1\end{vmatrix} + \begin{vmatrix}1 & a_1b_2 & 1\\ 1 & a_2b_2 & 1\\ 1 & a_3b_2 & 1\end{vmatrix} + \begin{vmatrix}1 & 1 & a_1b_3\\ 1 & 1 & a_2b_3\\ 1 & 1 & a_3b_3\end{vmatrix}$$

$\Rightarrow \qquad B = 0$

Illustration 43 If $y = f\left(\dfrac{2x-1}{x^2+1}\right)$ and $f'(x) = \sin x^2$, then find $\dfrac{dy}{dx}\cdot$

Solution. Here, $\dfrac{dy}{dx} = \dfrac{d}{dx} f\left(\dfrac{2x-1}{x^2+1}\right) = \dfrac{df\left(\dfrac{2x-1}{x^2+1}\right)}{d\left(\dfrac{2x-1}{x^2+1}\right)} \cdot \dfrac{d\left(\dfrac{2x-1}{x^2+1}\right)}{dx}$

$$= f'\left(\frac{2x-1}{x^2+1}\right) \cdot \left\{\frac{(x^2+1)\cdot(2)-(2x-1)\cdot(2x)}{(x^2+1)^2}\right\}$$

$$= \sin\left(\frac{2x-1}{x^2+1}\right)^2 \cdot \frac{2\{x^2+1-2x^2+x\}}{(x^2+1)^2} = \sin\left(\frac{2x-1}{x^2+1}\right)^2 \cdot \frac{2(1+x-x^2)}{(x^2+1)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2(1+x-x^2)}{(x^2+1)^2} \cdot \sin\left(\frac{2x-1}{x^2+1}\right)^2$$

Illustration 44 Let $f(x)$ be a polynomial function of second degree. If $f(1) = f(-1)$ and a_1, a_2, a_3 are in AP, then show that $f'(a_1), f'(a_2), f'(a_3)$ are in AP.

Solution. Let $f(x) = \lambda x^2 + \mu x + v$

Then, $\quad f'(x) = 2\lambda x + \mu$

Also, $\quad f(1) = f(-1) \Rightarrow \lambda + \mu + v = \lambda - \mu + v \Rightarrow \mu = 0 \therefore f'(x) = 2\lambda x$

$\therefore \quad f'(a_1) = 2\lambda a_1, f'(a_2) = 2\lambda a_2$ and $f'(a_3) = 2\lambda a_3$

As, a_1, a_2, a_3 are in AP. $f'(a_1), f'(a_2), f'(a_3)$ are in AP.

Illustration 45 If $5f(x) + 3f\left(\dfrac{1}{x}\right) = x + 2$ and $y = x\,f(x)$, then find $\dfrac{dy}{dx}$ at $x = 1$.

Solution. Here, $\quad 5f(x) + 3f\left(\dfrac{1}{x}\right) = x + 2 \quad$ ...(i)

Put $x = \dfrac{1}{x}$, we get $\quad 5f\left(\dfrac{1}{x}\right) + 3f(x) = \dfrac{1}{x} + 2 \quad$ ...(ii)

Solving Eqs. (i) and (ii), we get

$$16f(x) = 5x - \frac{3}{x} + 4 \quad \text{...(iii)}$$

$$\therefore \quad y = x\,f(x) \quad \Rightarrow \quad y = x \cdot \frac{1}{16}\left\{5x - \frac{3}{x} + 4\right\}$$

$$y = \frac{1}{16}\{5x^2 - 3 + 4x\} \text{ or } \frac{dy}{dx} = \frac{1}{16}\{10x + 4\}$$

Now, $\dfrac{dy}{dx}$ at $x = 1$, $\left(\dfrac{dy}{dx}\right)_{\text{at } x=1} = \dfrac{10+4}{16} = \dfrac{14}{16} = \dfrac{7}{8}$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{\text{at } x=1} = \frac{7}{8}$$

Target Exercise 2.6

1. If $y = x^x$, then prove that $\dfrac{d^2y}{dx^2} - \dfrac{1}{y}\left(\dfrac{dy}{dx}\right)^2 - \dfrac{y}{x} = 0$.

2. If $y = A\cos(\log x) + B\sin(\log x)$, then prove that $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$.

3. If $y = x\log\left(\dfrac{x}{a+bx}\right)$, then prove that $x^3\dfrac{d^2y}{dx^2} = \left(x\dfrac{dy}{dx} - y\right)^2$.

4. If $y = \log(x + \sqrt{x^2 + a^2})$, then prove that $(x^2 + a^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 0$.

5. If $y = (x + \sqrt{x^2 + 1})^m$, then prove that $(x^2 + 1)y_2 + xy_1 - m^2y = 0$.

6. If $x = at^2$, $y = 2at$, then find $\dfrac{d^2y}{dx^2}$.

7. If $x = a\cos^3\theta$, $y = a\sin^3\theta$, then find $\dfrac{d^2y}{dx^2}$.

8. If $x = \tan\left(\dfrac{1}{a}\log y\right)$, then show that $(1 + x^2)\dfrac{d^2y}{dx^2} = (a - 2x)\dfrac{dy}{dx}$.

9. If $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$, then prove that $y^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 0$.

10. If $y = \dfrac{ax+b}{cx+d}$, then prove that $2y_1y_3 = 3(y_2)^2$.

11. If $f(x)$, $g(x)$ and $h(x)$ are three polynomials of degree 2, then prove that

$$\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$$

is a constant polynomial.

12. If f, g, h are differentiable functions of x and $\Delta = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2f)'' & (x^2g)'' & (x^2h)'' \end{vmatrix}$

Prove that $\Delta' = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$

13. If $y = (1 + x)(1 + x^2)(1 + x^4)\dots(1 + x^{2^n})$, find $\dfrac{dy}{dx}$ at $x = 0$.

14. If $x = f(t)$ and $y = \phi(t)$, prove that $\dfrac{d^2y}{dx^2} = \dfrac{f_1\phi_2 - f_2\phi_1}{f_1^3}$, where suffixes denote differentiation w.r.t. 't'.

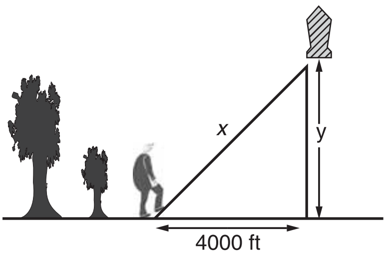
15. If $x = \sin t$, $y = \sin Kt$, then show that $(1 - x^2)y_2 - xy_1 + K^2y = 0$.

16. If $\dfrac{x+b}{2} = a\tan^{-1}(a\log_e y)$, $a > 0$, then prove that $yy'' - yy'\log y = (y')^2$.

17. Find $\dfrac{dy}{dx}$ at $x = -1$, when $(\sin y)^{\sin\frac{\pi x}{2}} + \dfrac{\sqrt{3}}{2}\sec^{-1}(2x) + 2^x\tan\{\log_e(x+2)\} = 0$.

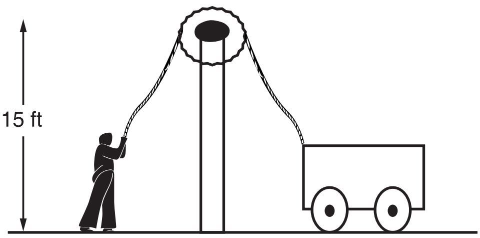
18. Let $f(x)$ be a polynomial function of degree 2 and $f(x) > 0$ for all $x \in R$. If $g(x) = f(x) + f'(x) + f''(x)$, then for any x show that $g(x) > 0$.

19. At a distance of 4000 ft from the launch site, (as shown in figure) a spectator is observing a rocket being launched. If the rocket lifts off vertically and is rising at a speed of 600 ft/s when it is at an altitude of 3000 ft, the distance between the rocket and the spectator is changing at that instant at the rate :



(a) 300 ft/s (b) 360 ft/s (c) 480 ft/s (d) None of these

20. A rope is attached to a pulley mounted on a 15 ft tower. The end of the rope is attached to a heavily loaded cart (as shown in figure).



A worker can pull the rope at a rate of 2 ft/s. The cart is approaching the tower when it is 8 ft from the tower at the rate of

(a) $\frac{9}{4}$ ft/s (b) $\frac{17}{4}$ ft/s

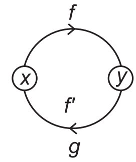
(c) $\frac{13}{4}$ ft/s (d) None of these

Derivative of an Inverse Function

Theorem If the inverse functions f and g are defined by $y = f(x)$ and $x = g(y)$ and if $f'(x)$ exists and $f'(x) \neq 0$, then $g'(y) = \dfrac{1}{f'(x)}$. This result can also be written as, if $\dfrac{dy}{dx}$ exists and $\dfrac{dy}{dx} \neq 0$, then

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \quad \text{or} \quad \frac{dy}{dx} \cdot \frac{dx}{dy} = 1 \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad \left[\frac{dx}{dy} \neq 0\right]$$

Illustration 46 If $y = f(x) = x^3 + x^5$ and g is the inverse of f, then find $g'(2)$ (means dx/dy when $y = 2$)



$y = f(x),\ x = g(y)$

Solution. $\dfrac{dy}{dx} = 3x^2 + 5x^4 \Rightarrow g'(y) = \dfrac{dx}{dy} = \dfrac{1}{3x^2 + 5x^4}$

when $y = 2$, then $2 = x^3 + x^5 \Rightarrow x = 1$

$$\therefore \qquad g'(2) = \left.\frac{dx}{dy}\right]_{\substack{x=1\\y=2}} = \frac{1}{3+5} = \frac{1}{8}$$

Aliter : $(gof)(x) = x$

$$g'[f(x)]f'(x) = 1; \text{ when } f(x) = 2, \text{ then } x = 1$$

$$g'(2) \cdot f'(1) = 1 \qquad [\text{but } f'(1) = 8]$$

$$\therefore \qquad g'(2) = 1/8$$

Illustration 47 Let $f(x) = \exp(x^3 + x^2 + x)$ for any real number x and let g be the inverse function for f. The value of $g'(e)^3$ is

(a) $\dfrac{1}{6e^3}$ (b) $\dfrac{1}{6}$ (c) $\dfrac{1}{34e^{39}}$ (d) 6

Solution. $y = e^{x^3 + x^2 + x}, \dfrac{dy}{dx} = e^{x^3 + x^2 + x} \cdot (3x^2 + 2x + 1)$

$$g'(y) = \frac{dx}{dy} = \frac{1}{e^{x^3 + x^2 + x} \cdot (3x^2 + 2x + 1)}$$

$$y = e^3 \quad \Rightarrow \quad e^3 = e^{x^3 + x^2 + x}$$

$$\Rightarrow \qquad x^3 + x^2 + x - 3 = 0$$

$$\Rightarrow \qquad (x-1)(x^2 + 2x + 3) = 0$$

$$\Rightarrow \qquad x = 1$$

$$\therefore \qquad g'(e^3) = \frac{1}{6e^3}$$

Hence, (a) is the correct answer.

Worked Examples

Type 1 : Subjective Type Questions

Example 1 If $f(x)=\cos\left\{\frac{\pi}{2}[x]-x^3\right\}$, $1<x<2$ and $[x]=$ the greatest integer $\le x$, then find $f'\left(\sqrt[3]{\frac{\pi}{2}}\right)$.

Solution. As we know, $1<\sqrt[3]{\frac{\pi}{2}}<2$

$\therefore$ If $x=\sqrt[3]{\frac{\pi}{2}} \Rightarrow [x]=1$, so $f(x)=\cos\left\{\frac{\pi}{2}-x^3\right\}$

$f(x)=\sin x^3$ $\left[\text{at } x=\sqrt[3]{\frac{\pi}{2}}\in(1,2)\right]$

$\Rightarrow$ $f'(x)=\cos x^3\cdot 3x^2$

$\therefore$ $f'\left(\sqrt[3]{\frac{\pi}{2}}\right)=3\left(\frac{\pi}{2}\right)^{2/3}\cdot\cos\frac{\pi}{2}=0 \Rightarrow f'\left(\sqrt[3]{\frac{\pi}{2}}\right)=0$

Example 2 If $u=f(x^3)$, $v=g(x^2)$, $f'(x)=\cos x$ and $g'(x)=\sin x$, then find $\frac{du}{dv}$.

Solution. Here, $u=f(x^3)$

$\Rightarrow$ $\frac{du}{dx}=f'(x^3)\cdot\frac{d}{dx}(x^3)=\{\cos(x^3)\}\cdot 3x^2=3x^2\cdot\cos x^3$

and $v=g(x^2)$

$\Rightarrow$ $\frac{dv}{dx}=g'(x^2)\cdot\frac{d}{dx}(x^2)=\{\sin x^2\}\cdot(2x)=2x\cdot\sin x^2$

$\therefore$ $\frac{du}{dv}=\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{3x^2\cdot\cos x^3}{2x\cdot\sin x^2} \Rightarrow \frac{du}{dv}=\frac{3}{2}x\cdot\cos x^3\cdot\operatorname{cosec} x^2$

You Must Know !

Identity in *x* If a quadratic equation has more than two roots, then it is an identity in x, that is $a=b=c=0$, for $ax^2+bx+c=0$.

In general any equation is identity, then it has more roots (or infinite roots) than its degree. *eg,* The number of values of the triplet (a, b, c) for which $a\cos 2x+b\sin^2 x+c=0$ is satisfied by all real x.

Here, $a\cos 2x+b\sin^2 x+c=0$, true for all $x\in R$.

ie, $a(1-2\sin^2 x)+b\sin^2 x+c=0, (b-2a)\sin^2 x+(c+a)=0$

It is an identity, if $b-2a=0$ and $a+c=0$.

So, $\frac{a}{1}=\frac{b}{2}=\frac{c}{-1}$

$\therefore$ Number of triplets (a, b, c) are infinite.

Example 3 Find a, b, c and d, where $f(x) = (ax + b)\cos x + (cx + d)\sin x$ and $f'(x) = x\cos x$ is identity in x.

Solution. Here, $f'(x) = x\cos x$

$\Rightarrow \quad a\cos x - (ax + b)\sin x + c\sin x + (cx + d)\cos x \equiv x\cos x$

or $\quad (a + cx + d)\cos x + (-ax - b + c)\sin x \equiv x\cos x + 0\cdot\sin x$

$\Rightarrow \quad a + d + cx = x \quad$ and $\quad -ax - b + c = 0$

which is again identity in 'x'

$\Rightarrow \quad a + d = 0, c = 1, -a = 0, -b + c = 0$

$\Rightarrow \quad a = 0, b = 1, c = 1, d = 0$

Example 4 If $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$ for all $x \in R$. Then, find $f(x)$ independent of $f'(1), f''(2)$ and $f'''(3)$.

Solution. Here, $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$

Put $\quad f'(1) = a, \; f''(2) = b, \; f'''(3) = c \quad$ ...(i)

$\therefore \quad f(x) = x^3 + ax^2 + bx + c$

$\Rightarrow \quad f'(x) = 3x^2 + 2ax + b \quad$ or $\quad f'(1) = 3 + 2a + b \quad$ ...(ii)

$\Rightarrow \quad f''(x) = 6x + 2a \quad$ or $\quad f''(2) = 12 + 2a \quad$ ...(iii)

$\Rightarrow \quad f'''(x) = 6 \quad$ or $\quad f'''(3) = 6 \quad$ ...(iv)

From Eqs. (i) and (iv), $c = 6$

From Eqs. (i), (ii) and (iii), we have $a = -5, \quad b = 2$

$\therefore \quad f(x) = x^3 - 5x^2 + 2x + 6$

Example 5 Let $f(x) = x^2 + xg'(1) + g''(2)$ and $g(x) = f(1)\cdot x^2 + xf'(x) + f''(x)$, then find $f(x)$ and $g(x)$.

Solution. Here, put

$g'(1) = a, \quad g''(2) = b \quad$ ...(i)

Then, $\quad f(x) = x^2 + ax + b, f(1) = 1 + a + b,$

$f'(x) = 2x + a, \quad f''(x) = 2$

$\therefore \quad g(x) = (1 + a + b)x^2 + (2x + a)\cdot x + 2$

$= x^2(3 + a + b) + ax + 2$

$\Rightarrow \quad g'(x) = 2x(3 + a + b) + a$

Hence, $\quad g'(1) = 2(3 + a + b) + a \quad$ ...(ii)

$g''(2) = 4(3 + a + b) \quad$ ...(iii)

From Eqs. (i), (ii) and (iii), we have

$a = 2(3 + a + b) + a \quad$ and $\quad b = 2(3 + a + b)$

ie, $\quad 3 + a + b = 0 \quad$ *ie*, $\quad b + 2a + 6 = 0$

Hence, $\quad b = 0 \quad$ and $\quad a = -3$

So, $\quad f(x) = x^2 - 3x \quad$ and $\quad g(x) = -3x + 2$

Example 6 If $y=\dfrac{\sin x}{1+\dfrac{\cos x}{1+\dfrac{\sin x}{1+\cos x \ldots\ldots\infty}}}$, prove $\dfrac{dy}{dx}=\dfrac{(1+y)\cos x+\sin x}{1+2y+\cos x-\sin x}$.

Solution. Given function is $y=\dfrac{\sin x}{1+\dfrac{\cos x}{1+y}}=\dfrac{(1+y)\sin x}{1+y+\cos x}$

or $$y+y^2+y\cos x=(1+y)\sin x$$

On differentiating both the sides w.r.t. x, we get

$$\frac{dy}{dx}+2y\frac{dy}{dx}+y(-\sin x)+\cos x\cdot\frac{dy}{dx}=(1+y)\cos x+\frac{dy}{dx}\cdot\sin x$$

or $$\frac{dy}{dx}\{1+2y+\cos x-\sin x\}=(1+y)\cos x+y\sin x$$

or $$\frac{dy}{dx}=\frac{(1+y)\cos x+y\sin x}{(1+2y+\cos x-\sin x)}$$

Example 7 If $y=\dfrac{x}{x+\dfrac{\sqrt[3]{x}}{x+\dfrac{\sqrt[3]{x}}{x+\sqrt[3]{x}+\ldots\ldots\ldots\text{infinity}}}}$, find $\dfrac{dy}{dx}$.

Solution.
$$y=\frac{x}{x+\dfrac{\sqrt[3]{x}}{x+\dfrac{\sqrt[3]{x}}{x+\sqrt[3]{x}+\ldots\ldots\ldots\text{infinity}}}}$$

$$y=\frac{x}{x+\dfrac{x}{x+\dfrac{\sqrt[3]{x}}{x+\sqrt[3]{x}+\ldots\ldots\ldots\text{infinity}}}\cdot x^{-2/3}}$$

or $$y=\frac{x}{x+y\cdot x^{-2/3}}$$

$\Rightarrow$ $$y\{x^{5/3}+y\}=x^{5/3}\quad\text{or}\quad x^{5/3}y+y^2=x^{5/3}$$

Differentiating both the sides w.r.t. x,

$$x^{5/3}\cdot\frac{dy}{dx}+\frac{5}{3}\cdot x^{2/3}\cdot y+2y\frac{dy}{dx}=\frac{5}{3}\cdot x^{2/3}$$

or $$(x^{5/3}+2y)\frac{dy}{dx}=\frac{5}{3}x^{2/3}-\frac{5}{3}x^{2/3}y\quad\text{or}\quad\frac{dy}{dx}=\frac{\frac{5}{3}x^{2/3}(1-y)}{(x^{5/3}+2y)}$$

Example 8 If $y=\tan^{-1}\dfrac{1}{x^2+x+1}+\tan^{-1}\dfrac{1}{x^2+3x+3}+\tan^{-1}\dfrac{1}{x^2+5x+7}+\ldots$ to n terms, show that $\dfrac{dy}{dx}=\dfrac{1}{(x+n)^2+1}-\dfrac{1}{x^2+1}$

Solution. Given,

$$y=\tan^{-1}\frac{1}{x^2+x+1}+\tan^{-1}\frac{1}{x^2+3x+3}+\tan^{-1}\frac{1}{x^2+5x+7}+\ldots\text{to } n \text{ terms}$$

$$= \tan^{-1}\left\{\frac{1}{1+x(x+1)}\right\} + \tan^{-1}\left\{\frac{1}{1+(x+1)(x+2)}\right\} + \tan^{-1}\left\{\frac{1}{1+(x+2)(x+3)}\right\}$$

$$+ \ldots\ldots\ldots + \tan^{-1}\left\{\frac{1}{1+\{x+(n-1)\}\{x+n\}}\right\}$$

$$= \tan^{-1}\left\{\frac{(x+1)-x}{1+(x+1)x}\right\} + \tan^{-1}\left\{\frac{(x+2)-(x+1)}{1+(x+2)(x+1)}\right\} + \tan^{-1}\left\{\frac{(x+3)-(x+2)}{1+(x+3)(x+2)}\right\}$$

$$+ \ldots\ldots\ldots + \tan^{-1}\left\{\frac{(x+n)-(x+n-1)}{1+(x+n)(x+n-1)}\right\}$$

$$\therefore \quad y = \{\tan^{-1}(x+1) - \tan^{-1}(x)\} + \{\tan^{-1}(x+2) - \tan^{-1}(x+1)\}$$

$$+ \{\tan^{-1}(x+3) - \tan^{-1}(x+2)\} + \ldots\ldots + \{\tan^{-1}(x+n) - \tan^{-1}(x+n-1)\}$$

$$y = \tan^{-1}(x+n) - \tan^{-1}(x)$$

On differentiating both the sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$$

Example 9 If $f(\theta) = \cos\theta_1 \cdot \cos\theta_2 \cdot \cos\theta_3 \ldots\ldots \cos\theta_n$

Show, $\{\tan\theta_1 + \tan\theta_2 + \tan\theta_3 + \ldots\ldots\ldots + \tan\theta_n\} = -\left\{\frac{f'(\theta)}{f(\theta)}\right\}$,

where $\frac{d\theta_1}{d\theta} = \frac{d\theta_2}{d\theta} = \ldots\ldots\ldots = \frac{d\theta_n}{d\theta} = 1.$

Solution. $f(\theta) = \cos\theta_1 \cdot \cos\theta_2 \cdot \cos\theta_3 \ldots\ldots \cos\theta_n$

Taking log on both the sides,

$$\log f(\theta) = \log(\cos\theta_1) + \log(\cos\theta_2) + \ldots\ldots + \log(\cos\theta_n)$$

On differentiating both the sides w.r.t. θ, we get

$$\frac{1}{f(\theta)} \cdot f'(\theta) = \frac{1}{\cos\theta_1} \cdot (-\sin\theta_1) + \frac{1}{\cos\theta_2} \cdot (-\sin\theta_2) + \ldots + \frac{1}{\cos\theta_n} \cdot (-\sin\theta_n)$$

Hence, $\quad (\tan\theta_1 + \tan\theta_2 + \ldots\ldots\ldots + \tan\theta_n) = -\left\{\frac{f'(\theta)}{f(\theta)}\right\}$

Example 10 Find the sum of the series :

$\sin x + 3\sin 3x + 5\sin 5x + \ldots\ldots + (2k-1)\sin(2k-1)x$ (using calculus)

Solution. Let $S = \cos x + \cos 3x + \cos 5x + \ldots\ldots + \cos(2k-1)x$

Here, the angles are in AP whose first term is x and common difference is $2x$.

$$\therefore \quad S = \frac{\sin\left(\frac{k \cdot 2x}{2}\right)}{\sin\left(\frac{2x}{2}\right)} \cdot \cos\left\{\frac{x}{1} + \frac{(k-1)2x}{2}\right\} = \frac{\sin kx}{\sin x} \cdot \cos kx$$

or $\quad \{\cos x + \cos 3x + \cos 5x + \ldots\ldots + \cos(2k-1)x\} = \frac{\sin 2kx}{2\sin x}$ ...(i)

On differentiating Eq. (i) w.r.t. x, we get

$$\Rightarrow \quad -\{\sin x + 3\sin 3x + 5\sin 5x + \ldots\ldots + (2k-1)\sin(2k-1)x\}$$

$$= \frac{1}{2}\left\{\frac{2k(\cos 2kx)\cdot\sin x - (\sin 2kx)\cdot\cos x}{\sin^2 x}\right\}$$

$$\therefore \quad [\sin x + 3\sin 3x + 5\sin 5x + \ldots + (2k-1)\sin(2k-1)x]$$

$$= -\frac{1}{2\sin^2 x}\left[k\{\sin(2k+1)x - \sin(2k-1)x\} - \frac{1}{2}\{\sin(2k+1)x + \sin(2k-1)x\}\right]$$

$$= \frac{1}{4\sin^2 x}[(2k+1)\sin(2k-1)x - (2k-1)\sin(2k+1)x]$$

Example 11 Find the sum of series $\sum_{r=1}^{n} r\,x^{r-1}$, using calculus.

Solution. Let $\quad S = 1 + x + x^2 + x^3 + x^4 + \ldots\ldots + x^n$

which is a geometric progression.

$$\therefore \quad S = 1 + x + x^2 + x^3 + \ldots\ldots + x^n = \frac{1(1-x^{n+1})}{1-x}$$

On differentiating both the sides, we get

$$0 + 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots + nx^{n-1} = \frac{(1-x)\cdot[-(n+1)x^n] - (1-x^{n+1})\cdot(-1)}{(1-x)^2}$$

$$\therefore \quad \sum_{r=1}^{n} r\,x^{r-1} = \frac{1}{(1-x)^2}\cdot\{1-(n+1)x^n + n\cdot x^{n+1}\}$$

Example 12 Use calculus to find the sum of

$$\left[\frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} + \ldots + \frac{2^n}{x^{2^n}+1}\right]$$

Solution. We know, $(1-x)(1+x) = (1-x^2)$

$$(1-x)(1+x)(1+x^2) = (1-x^4) = (1-x^{2^2})$$

$$(1-x)(1+x)(1+x^2)(1+x^4) = (1-x^8) = (1-x^{2^3})$$

....

.... so on.

$$(1-x)(1+x)(1+x^2)(1+x^4)\ldots\ldots\ldots(1+x^{2^n}) = (1-x^{2^{n+1}}) \quad \ldots\text{(i)}$$

Taking log of Eq. (i) on both the sides, we get

$$\log\{(1-x)(1+x)(1+x^2)(1+x^4)\ldots\ldots\ldots(1+x^{2^n})\} = \log(1-x^{2^{n+1}})$$

or $\quad \log(1-x) + \log(1+x) + \log(1+x^2) + \log(1+x^4) + \ldots\ldots + \log(1+x^{2^n})$

$$= \log(1-x^{2^{n+1}})$$

On differentiating the above equation, we get

$$\frac{1(-1)}{1-x}+\frac{1\cdot 1}{1+x}+\frac{1\,(2x)}{1+x^2}+\frac{1\cdot(4x^3)}{1+x^4}+\ldots..+\frac{1\cdot 2^n\cdot x^{2^n-1}}{1+x^{2^n}}=\frac{-1\cdot 2^{n+1}\cdot x^{2^{n+1}-1}}{1-x^{2^{n+1}}}$$

or $$\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\ldots..+\frac{2^n\cdot x^{2^n-1}}{1+x^{2^n}}=\frac{1}{1-x}-\frac{2^{n+1}\cdot x^{2^{n+1}-1}}{1-x^{2^{n+1}}}$$

Multiplying both the sides by x,

or $$\left[\frac{x}{1+x}+\frac{2x^2}{1+x^2}+\frac{4x^4}{1+x^4}+\ldots..+\frac{2^n\cdot x^{2^n}}{1+x^{2^n}}\right]=\frac{x}{1-x}-\frac{2^{n+1}\cdot x^{2^{n+1}}}{1-x^{2n+1}}$$

or $$\left[\frac{(x+1)-1}{1+x}+\frac{2\,(x^2+1)-2}{1+x^2}+\frac{4\,(x^4+1)-4}{1+x^4}+\ldots\ldots+\frac{2^n\,(x^{2^n}+1)-2^n}{1+x^{2^n}}\right]$$

$$=\frac{x}{1-x}-2^{n+1}\cdot\frac{x^{2^{n+1}}}{1-x^{2^{n+1}}}$$

or $$[(1+2+2^2+\ldots.+2^n)-P\,]=\frac{x}{1-x}-\frac{2^{n+1}\cdot x^{2^{n+1}}}{1-x^{2^{n+1}}}$$

$$\left(\text{where } P=\frac{1}{1+x}+\frac{2}{1+x^2}+\frac{4}{1+x^4}+\ldots\ldots+\frac{2^n}{1+x^{2^n}}\right)$$

Then, $$P=\frac{2^{n+1}-1}{2-1}-\frac{x}{1-x}+\frac{2^{n+1}\cdot x^{2^{n+1}}}{1-x^{2^{n+1}}}$$

$$P=2^{n+1}-1-\frac{x}{1-x}+\frac{2^{n+1}\cdot x^{2^{n+1}}}{1-x^{2^{n+1}}}$$

$$P=2^{n+1}\left[1+\frac{x^{2^{n+1}}}{1-x^{2^{n+1}}}\right]-\left[1+\frac{x}{1-x}\right]$$

$$P=2^{n+1}\left[\frac{1}{1-x^{2^{n+1}}}\right]-\left[\frac{1}{1-x}\right]$$

Hence, $$\left\{\frac{1}{1+x}+\frac{2}{1+x^2}+\frac{4}{1+x^4}+\ldots..+\frac{2^n}{1+x^{2^n}}\right\}=\left\{\frac{2^{n+1}}{1-x^{2^{n+1}}}\right\}-\left\{\frac{1}{1-x}\right\}$$

Example 13 If $f_n(x)=e^{f_{n-1}(x)}$ for all $n\in N$ and $f_0(x)=x$, then find $\frac{d}{dx}\{f_n(x)\}$.

Solution. Here, $$f_n(x)=e^{f_{n-1}(x)} \qquad \text{...(i)}$$

$\Rightarrow$ $$\frac{d}{dx}\{f_n(x)\}=e^{f_{n-1}(x)}\cdot\frac{d}{dx}\{f_{n-1}(x)\}$$

or $$\frac{d}{dx}\{f_n(x)\}=f_n\,(x)\frac{d}{dx}\{f_{n-1}(x)\} \qquad \text{[using Eq.(i)] ...(ii)}$$

From Eq. (ii), $$\frac{d}{dx}\{f_{n-1}(x)\}=f_{n-1}\,(x)\cdot\frac{d}{dx}\{f_{n-2}(x)\} \qquad \text{...(iii)}$$

$\therefore$ From Eqs. (ii) and (iii),

$$\frac{d}{dx}\{f_n(x)\}=f_n\,(x)\cdot f_{n-1}(x)\cdot\frac{d}{dx}\{f_{n-2}(x)\} \qquad \text{...(iv)}$$

Similarly, $$\frac{d}{dx}\{f_{n-2}(x)\}=f_{n-2}(x)\cdot\frac{d}{dx}\{f_{n-3}(x)\}$$

....

....(v)

and so on

$$\frac{d}{dx}\{f_1(x)\}=f_1(x)\frac{d}{dx}\{f_0(x)\}$$

From Eqs. (iv) and (v), we get

$$\frac{d}{dx}\{f_n(x)\}=f_n(x)f_{n-1}(x)\cdot f_{n-2}(x)\ldots\ldots\ldots f_2(x)\cdot f_1(x)\left\{\frac{d}{dx}f_0(x)\right\}$$

$$\frac{d}{dx}\{f_n(x)\}=f_n(x)f_{n-1}(x)\cdot f_{n-2}(x)\ldots\ldots\ldots f_2(x)\cdot f_1(x)\cdot 1$$

$$\left[\because\ f_0(x)=x\ \Rightarrow \frac{d}{dx}\{f_0(x)\}=1\right]$$

Example 14 If $y^3 - y = 2x$, then prove that $\frac{d^2y}{dx^2}=-\frac{24y}{(3y^2-1)^3}$. Hence, show that $\left(x^2-\frac{1}{27}\right)\frac{d^2y}{dx^2}+x\cdot\frac{dy}{dx}=\frac{1}{9}y.$

Solution. $y^3 - y = 2x$ ...(i)

On differentiating both the sides w.r.t. x, we get

$$(3y^2-1)\frac{dy}{dx}=2 \quad\text{or}\quad \frac{dy}{dx}=\frac{2}{3y^2-1} \qquad \text{...(ii)}$$

Again, differentiating w.r.t. x,

$$\frac{d^2y}{dx^2}=\frac{-2(6y)\cdot\frac{dy}{dx}}{(3y^2-1)^2}=\frac{-12y}{(3y^2-1)^2}\cdot\frac{2}{(3y^2-1)} \qquad \text{[from Eq. (ii)]}$$

$$\frac{d^2y}{dx^2}=-\frac{24y}{(3y^2-1)^3} \qquad \text{...(iii)}$$

Now, LHS is $\left(x^2-\frac{1}{27}\right)\frac{d^2y}{dx^2}+x\cdot\frac{dy}{dx}$

$$=\left(\frac{y^2(y^2-1)^2}{4}-\frac{1}{27}\right)\left(\frac{-24y}{(3y^2-1)^3}\right)+\frac{y(y^2-1)}{2}\cdot\frac{2}{(3y^2-1)}$$

[from Eqs. (i), (ii) and (iii)]

$$=\left(\frac{27y^2(y^2-1)^2-4}{108}\right)\frac{-24y}{(3y^2-1)^3}+\frac{y(y^2-1)}{(3y^2-1)}$$

$$=\frac{y}{9}\left[\frac{-54y^2(y^2-1)^2+8}{(3y^2-1)^3}+\frac{9(y^2-1)}{(3y^2-1)}\right]$$

Let $(3y^2-1)=\alpha$

Then, $$=\frac{y}{9}\left[\frac{-2(1+\alpha)(\alpha-2)^2+8}{\alpha^3}+\frac{3(\alpha-1)}{\alpha}\right]$$

$$= \frac{y}{9}\left[\frac{-2\alpha^3 + 6\alpha^2 + 3\alpha^3 - 6\alpha^2}{\alpha^3}\right] = \frac{y}{9} = \text{RHS}$$

Hence, $$\left(x^2 - \frac{1}{27}\right)\frac{d^2y}{dx^2} + x\cdot\frac{dy}{dx} = \frac{y}{9}$$

Example 15 If $2x = y^{1/5} + y^{-1/5}$, then express y as an explicit function of x and prove that $(x^2 - 1)\dfrac{d^2y}{dx^2} + x\cdot\dfrac{dy}{dx} = 25\,y$.

Solution. $$(y^{1/5} + y^{-1/5})^2 = (y^{1/5} - y^{-1/5})^2 + 4$$

Then, $$(y^{1/5} - y^{-1/5})^2 = 4x^2 - 4 \qquad [\text{using } 2x = y^{1/5} + y^{-1/5}]$$

$\therefore$ $$y^{1/5} - y^{-1/5} = 2\sqrt{x^2 - 1} \qquad \text{...(i)}$$

and $$y^{1/5} + y^{-1/5} = 2x \qquad \text{(given)...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2y^{1/5} = 2x + 2\sqrt{x^2 - 1} \quad \therefore \quad y^{1/5} = x + \sqrt{x^2 - 1}$$

or $$y = (x + \sqrt{x^2 - 1})^5 \qquad \text{...(iii)}$$

On differentiating both the sides w.r.t. x, we get

$$\frac{dy}{dx} = 5\,(x + \sqrt{x^2 - 1})^4\cdot\left\{1 + \frac{1\cdot 2x}{2\sqrt{x^2 - 1}}\right\}$$

or $$\frac{dy}{dx} = \frac{5\,(x + \sqrt{x^2 - 1})^5}{\sqrt{x^2 - 1}} \quad \text{or} \quad \frac{dy}{dx} = \frac{5y}{\sqrt{x^2 - 1}} \qquad [\text{using Eq. (iii)}]$$

or $$(x^2 - 1)\left(\frac{dy}{dx}\right)^2 = 25y^2 \qquad \text{...(iv)}$$

Again, differentiating both the sides w.r.t. x, we get

$$2x\left(\frac{dy}{dx}\right)^2 + (x^2 - 1)\cdot 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = 50y\cdot\frac{dy}{dx},$$ dividing by $2\dfrac{dy}{dx}$ on both the sides,

or $$x\cdot\frac{dy}{dx} + (x^2 - 1)\frac{d^2y}{dx^2} = 25y \qquad \left(\because \frac{dy}{dx} \neq 0\right)$$

Example 16 Given, $y = 1 + \dfrac{a_1}{(x - a_1)} + \dfrac{a_2x}{(x - a_1)(x - a_2)} + \dfrac{a_3x^2}{(x - a_1)(x - a_2)(x - a_3)}$ + ... to $(n + 1)$ terms,

show that $$\frac{dy}{dx} = \frac{y}{x}\left[n - \frac{x}{x - a_1} - \frac{x}{x - a_2} - \frac{x}{x - a_3} - \ldots\ldots - \frac{x}{x - a_n}\right]$$

Solution. Here, $y = 1 + \dfrac{a_1}{(x - a_1)} + \dfrac{a_2x}{(x - a_1)(x - a_2)} + \dfrac{a_3x^2}{(x - a_1)(x - a_2)(x - a_3)}$ $+ \ldots.$ to $(n + 1)$ terms

$$y = \frac{x}{(x - a_1)} + \frac{a_2x}{(x - a_1)(x - a_2)} + \frac{a_3x^2}{(x - a_1)(x - a_2)(x - a_3)} + \ldots.\text{ to } (n) \text{ terms}$$

(taking first two terms)

$$y = \frac{x^2}{(x - a_1)(x - a_2)} + \frac{a_3 x^2}{(x - a_1)(x - a_2)(x - a_3)} + \dots \text{ to } (n-1) \text{ terms}$$

(again taking first two terms)

Proceeding in the same way, we get

$$y = \frac{x^n}{(x - a_1)(x - a_2)(x - a_3)\dots\dots(x - a_n)} \qquad \dots\text{(i)}$$

Taking logarithm on both the sides of Eq. (i), we have

$\log y = n \log x - \log (x - a_1) - \log (x - a_2) - \log (x - a_3) - \dots\dots - \log (x - a_n)$

On differentiating both the sides w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{n}{x} - \frac{1}{(x - a_1)} - \frac{1}{(x - a_2)} - \dots - \frac{1}{(x - a_n)}$$

or $$\frac{dy}{dx} = \frac{y}{x}\left[n - \frac{x}{(x - a_1)} - \frac{x}{(x - a_2)} - \dots - \frac{x}{(x - a_n)}\right]$$

Example 17 If for all x, y the function f is defined by;

$$f(x) + f(y) + f(x) \cdot f(y) = 1 \quad \text{and} \quad f(x) > 0.$$

Then, show $f'(x) = 0$, when $f(x)$ is differentiable.

Solution. Here, $f(x) + f(y) + f(x) \cdot f(y) = 1$ ...(i)

Put $x = y = 0$, we get

$$2f(0) + \{f(0)\}^2 = 1 \Rightarrow \{f(0)\}^2 + 2f(0) - 1 = 0$$

$$f(0) = \frac{-2 \pm \sqrt{4+4}}{2} = -1 - \sqrt{2} \quad \text{and} \quad -1 + \sqrt{2}$$

As $f(0) > 0 \Rightarrow f(0) = \sqrt{2} - 1$ [neglecting $f(0) = -1 - \sqrt{2}$ as $f(0)$ is positive]

Again, putting $y = x$ in Eq. (i), $2f(x) + \{f(x)\}^2 = 1$

On differentiating w.r.t. x, $2f'(x) + 2f(x)f'(x) = 0$

$$2f'(x)\{1 + f(x)\} = 0 \Rightarrow f'(x) = 0 \quad \text{because} \quad f(x) > 0$$

Thus, $f'(x) = 0$ when $f(x) > 0$

Example 18 Let f be a twice differentiable such that $f''(x) = -f(x)$ and $f'(x) = g(x)$. If $h(x) = \{f(x)\}^2 + \{g(x)\}^2$, where $h(5) = 11$. Find $h(10)$.

Solution. Given, $h(x) = \{f(x)\}^2 + \{g(x)\}^2$

On differentiating both the sides w.r.t. x, we get

$$h'(x) = 2f(x) \cdot f'(x) + 2g(x) \cdot g'(x) \qquad \dots\text{(i)}$$

Now, $f'(x) = g(x)$. Then, $f''(x) = g'(x) \Rightarrow -f(x) = g'(x)$ ...(ii)

[$\because f''(x) = -f(x)$, given]

From Eqs. (i) and (ii), we get $h'(x) = 2f(x) \cdot g(x) + 2g(x) \cdot \{-f(x)\}$

[using $f'(x) = g(x)$ and $g'(x) = -f(x)$]

$\therefore$ $h'(x) = 0$

So, $h(x)$ must be constant $\left[\text{as } \frac{d}{dx} \text{ constant} = 0\right]$

but $h(5) = 11$. So, $h(x) = 11$ Hence, $h(10) = 11$

Example 19 Let $f(x)$ be a real-valued differentiable function not identically zero such that $f(x+y^{2n+1}) = f(x) + \{f(y)\}^{2n+1}$, $n \in N$ and x, y are any real numbers and $f'(0) \geq 0$. Find the value of $f(5)$ and $f'(10)$.

Solution. Here, $\quad f(x+y^{2n+1}) = f(x) + \{f(y)\}^{2n+1} \quad$ ...(i)

Putting $x = 0$, $y = 0$, we get

$$f(0) = f(0) + \{f(0)\}^{2n+1} \quad \Rightarrow \quad f(0) = 0$$

$$f'(0) \geq 0 \qquad \text{(given)}$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{f(x) - f(0)}{x - 0} \geq 0 \quad \Rightarrow \quad \lim_{h \to 0} \frac{f(x)}{x} \geq 0$$

Now, if $\quad x > 0 \quad \Rightarrow \quad f(x) \geq 0 \quad$ ...(ii)

Putting $x = 0$, $y = 1$ in Eq. (i),

$$f(1) = f(0) + \{f(1)\}^{2n+1} \quad \text{or} \quad f(1)[1 - \{f(1)\}^{2n}] = 0$$

$\therefore \quad f(1) = 0$ or $1 \quad$ [using Eq. (ii)]

Putting $y = 1$ in Eq. (i), for all real x,

$$f(x+1) = f(x) + \{f(1)\}^{2n+1} \qquad \text{...(iii)}$$

Now, two cases arise either $f(1) = 0$ or 1

Case I If $\quad f(1) = 0$

$\Rightarrow \quad f(x+1) = f(x) \quad$ [using Eq. (iii)]

$\Rightarrow \quad f(1) = f(2) = f(3) = \ldots . = 0$

$\therefore \quad f(x)$ is identically zero. (which is not possible)

Case II If $\quad f(1) = 1$

$\Rightarrow \quad f(x+1) = f(x) + 1 \quad$ [using Eq. (iii)]

$$\therefore \quad f(2) = f(1) + 1 = 1 + 1 = 2$$

$$f(3) = f(2) + 1 = 2 + 1 = 3$$

$$f(4) = f(3) + 1 = 3 + 1 = 4$$

$$f(5) = f(4) + 1 = 4 + 1 = 5$$

Proceeding in same way, we get

$$f(x) = x \quad \text{and} \quad f'(x) = 1 \quad \Rightarrow \quad f'(10) = 1$$

Hence, $\quad f(5) = 5 \quad$ and $\quad f'(10) = 1$

Example 20 Let $f\left(\dfrac{x+y}{2}\right) = \dfrac{f(x) + f(y)}{2}$ and $f'(0) = a$ and $f(0) = b$. Find $f''(x)$. (where y is independent of x), when $f(x)$ is differentiable.

Solution. $f\left(\dfrac{x+y}{2}\right) = \dfrac{f(x) + f(y)}{2}$, this holds for any real x, y and y is independent of x. *ie*, $\dfrac{dy}{dx} = 0$

On differentiating w.r.t. x, we get

$$f'\left(\frac{x+y}{2}\right) \cdot \frac{1}{2}\left(1 + \frac{dy}{dx}\right) = \frac{1}{2}\left\{f'(x) + f'(y) \cdot \frac{dy}{dx}\right\}$$

$$\therefore \quad \frac{1}{2} f'\left(\frac{x+y}{2}\right) = \frac{1}{2} f'(x) \qquad \left(\text{as } \frac{dy}{dx} = 0\right)$$

or $$f'\left(\frac{x+y}{2}\right)=f'(x) \qquad \text{...(i)}$$

Taking $x=0$ and $y=x$ in Eq. (i), we get

$$f'\left(\frac{0+x}{2}\right)=f'(0) \Rightarrow f'\left(\frac{x}{2}\right)=a \qquad \text{(given)}$$

which shows $$f\left(\frac{x}{2}\right)=a\left(\frac{x}{2}\right)+c \qquad \text{(using integration)}$$

$\therefore$ $$f(x)=ax+c$$

Let $$x=0$$

$$f(0)=b=c$$

$\therefore$ $$f(x)=ax+b$$

On differentiating $$f'(x)=a \quad \text{and} \quad f''(x)=0$$

Example 21 If $y=\dfrac{(a-x)\sqrt{a-x}-(b-x)\sqrt{x-b}}{\sqrt{a-x}+\sqrt{x-b}}$, then find $\dfrac{dy}{dx}$ whever defined.

Solution. Here, $$y=\frac{(a-x)^{3/2}+(x-b)^{3/2}}{(a-x)^{1/2}+(x-b)^{1/2}}, \text{ use } a^3+b^3=(a+b)(a^2-ab+b^2)$$

$\therefore$ $$y=\frac{\{(a-x)^{1/2}+(x-b)^{1/2}\}\{(a-x)-\sqrt{(a-x)(x-b)}+(x-b)\}}{\{(a-x)^{1/2}+(x-b)^{1/2}\}}$$

$\Rightarrow$ $$y=(a-b)-\sqrt{(ax+bx-ab-x^2)}$$

$\therefore$ $$\frac{dy}{dx}=-\frac{(a+b-2x)}{2\sqrt{(a-x)(x-b)}} \Rightarrow \frac{dy}{dx}=\frac{2x-(a+b)}{2\sqrt{(a-x)(x-b)}}$$

Example 22 If $x^2+y^2=R^2$ $(R>0)$, then $K=\dfrac{y''}{\sqrt{(1+(y')^2)^3}}$. Find K in terms of R.

Solution. Here, $x^2+y^2=R^2$ differentiating w.r.t. x, we get

$\Rightarrow$ $$2x+2yy'=0 \Rightarrow x+yy'=0$$

$\Rightarrow$ $$y'=-\frac{x}{y} \qquad \text{...(i)}$$

Again, differentiating both the sides, $1+yy''+(y')^2=0$

$\therefore$ $$y''=-\frac{(1+(y')^2)}{y} \qquad \text{...(ii)}$$

Now, $$K=\frac{y''}{\sqrt{(1+(y')^2)^3}}=-\frac{(1+(y')^2)}{y\cdot(1+(y')^2)^{3/2}}$$

$$=-\frac{1}{y\sqrt{1+(y')^2}}=-\frac{1}{y\sqrt{1+\dfrac{x^2}{y^2}}} \qquad \text{[from Eq. (i)]}$$

$$=-\frac{1}{\sqrt{x^2+y^2}}=-\frac{1}{R}$$

$\therefore$ $$K=-\frac{1}{R}$$

Example 23 Let $f(x) = x + \sin x$, suppose g denotes the inverse function of f. Then, find the value of $g'\left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right)$.

Solution. Here, $f(x) = x + \sin x$

$$\therefore \quad \frac{dy}{dx} = 1 + \cos x$$

$$g'(y) = \frac{dx}{dy} = \frac{1}{1 + \cos x}$$

where $y = \frac{\pi}{4} + \frac{1}{\sqrt{2}} = x + \sin x \quad \Rightarrow \quad x = \frac{\pi}{4}$

$$\therefore \quad g'\left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right) = \frac{1}{1 + \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2} + 1} = 2 - \sqrt{2}$$

Example 24 Let $e^{f(x)} = \log x$. If $g(x)$ is the inverse of $f(x)$, then find $g'(x)$.

Solution. Let $f(x) = y \quad \Rightarrow \quad x = f^{-1}(y) = g(y)$

$$\Rightarrow \quad x = e^{e^y} \quad \Rightarrow \quad \frac{dx}{dy} = e^{e^y} \cdot e^y = e^{e^y + y} = g'(y)$$

$$\therefore \quad g'(x) = e^{e^x + x}$$

Example 25 The function $f(x) = e^x + x$ being differentiable and one to one, has a differentiable inverse $f^{-1}(x)$, then find $\frac{d}{dx}(f^{-1}(x))$ at the point $f(\log_e 2)$.

Solution. Let $y = e^x + x$

on differentiating w.r.t. y,

$$1 = (e^x + 1)\frac{dx}{dy} \quad \Rightarrow \quad \frac{dx}{dy} = \frac{1}{1 + e^x}$$

$$\therefore \quad \left(\frac{dx}{dy}\right)_{x = \log_e 2} = \frac{1}{1 + e^{\log 2}} = \frac{1}{3}$$

or

$$\left[\frac{d}{dx}(f^{-1}(x))\right]_{x = \log 2} = \frac{1}{3}$$

Example 26 If $f(x) = 4x^3 - 6x^2 \cos 2a + 3x \cdot \sin 2a \cdot \sin 6a + \sqrt{\log (2a - a^2)}$, then show $f'\left(\frac{1}{2}\right) > 0$.

Solution. Here, $f(x) = 4x^3 - 6x^2 \cos 2a + 3x \sin 2a \cdot \sin 6a + \sqrt{\log (2a - a^2)}$ for $f(x)$ to exists.

$$\log (2a - a^2) \geq 0 \Rightarrow (2a - a^2) \geq e^0$$

ie, $\quad 2a - a^2 \geq 1 \quad$ or $\quad a^2 - 2a + 1 \leq 0$

$$\Rightarrow \quad (a - 1)^2 \leq 0,$$

which is only possible, if $(a - 1)^2 = 0$. *ie*, $a = 1$.

$$\therefore \quad f(x) = 4x^3 - 6x^2 \cos 2 + 3x \sin 2 \cdot \sin 6$$

$\Rightarrow \qquad f'(x) = 12x^2 - 12x\cos 2 + 3\sin 2\sin 6$

$\Rightarrow \qquad f'\left(\frac{1}{2}\right) = 3 - 6\cos 2 + 3\sin 2\sin 6 = 3(1 + \sin 2\sin 6) - 6\cos 2$

$\Rightarrow \qquad f'\left(\frac{1}{2}\right) > 0$

Point to Consider

$\cos 2 < 0$ and $1 + \sin 2\sin 6 > 0$

Example 27 Suppose, $f(x) = e^{ax} + e^{bx}$, where $a \neq b$ and $f''(x) - 2f'(x) - 15f(x) = 0$ for all $x \in R$. Then, find ab.

Solution. $f(x) = e^{ax} + e^{bx}$

$\Rightarrow \qquad f'(x) = ae^{ax} + be^{bx}, f''(x) = a^2e^{ax} + b^2e^{bx}$

$\therefore \qquad f''(x) - 2f'(x) - 15f(x) = 0$

$\Rightarrow \qquad \{a^2e^{ax} + b^2e^{bx}\} - 2\{ae^{ax} + be^{bx}\} - 15\{e^{ax} + e^{bx}\} = 0$, for all x.

$\Rightarrow \qquad (a^2 - 2a - 15)e^{ax} + (b^2 - 2b - 15)e^{bx} = 0$, for all x.

$\Rightarrow \qquad a^2 - 2a - 15 = 0$ and $b^2 - 2b - 15 = 0$

$\Rightarrow \qquad (a - 5)(a + 3) = 0$ and $(b - 5)(b + 3) = 0$

$\Rightarrow \qquad a = 5$ or -3 and $b = 5$ or -3

But $\qquad a \neq b$, hence, $a = 5, b = -3$

or $\qquad a = -3, b = 5 \qquad \Rightarrow \qquad ab = -15$

Type 5 : **Linked Comprehension Based Questions**

Passage I
(Q. Nos. 28 to 30)

A non-zero polynomial with real coefficients has the property that $f''(x) \cdot f'(x) = f(x)$. Then,

28. The leading coefficient of $f(x)$, is
(a) 1/6 (b) 1/9 (c) 1/12 (d) 1/18

29. The degree of $f(x)$, is
(a) 2 (b) 3 (c) 4 (d) 5

30. The value of $f'''(x)$, is
(a) 1/18 (b) 1/3 (c) 1/9 (d) 1/6

Solution. (Q. Nos. 28 to 30)

Let $\qquad$ degree of $f(x) = n$

$\therefore \qquad$ degree of $f'(x) = n - 1$, degree of $f''(x) = n - 2$

Since, $\qquad f''(x) \cdot f'(x) = f(x) \qquad \Rightarrow \qquad (n - 1) + (n - 2) = n$

$\Rightarrow \qquad 2n - 3 = n$ or $n = 3$

$\therefore$ Degree of $f(x) = 3$

Hence, $\qquad f(x) = ax^3 + bx^2 + cx + d$

$\qquad f'(x) = 3ax^2 + 2bx + c$

$$f''(x) = 6ax + 2b$$

$$f'''(x) = 6a$$

As, $\qquad f''(x) \cdot f'(x) = f(x)$

$\Rightarrow \qquad (6ax + 2b)(3ax^2 + 2bx + c) = (ax^3 + bx^2 + cx + d)$

On comparing coefficient of x^3, we get

$$18a^2 = a \quad \Rightarrow \quad a = 0, \frac{1}{18} \qquad \text{(but } a \neq 0\text{)}$$

$\therefore \qquad a = \dfrac{1}{18}$

$\therefore$ The leading coefficient of $f(x) = \dfrac{1}{18}$ and $f'''(x) = 6a = \dfrac{1}{3}\cdot$

Ans. 28. (d) **29.** (b) **30.** (b)

Passage II

(Q. Nos. 31 to 32)

The ends A and B of a rod of length $\sqrt{5}$ are sliding along the curve $y = 2x^2$. Let x_A and x_B be the x-coordinate of the ends. Then,

31. The moment when A is at $(0, 0)$ and B is at $(1, 2)$. The derivative $\dfrac{dx_B}{dx_A}$, is

(a) $\dfrac{1}{9}$ (b) $\dfrac{1}{7}$ (c) $\dfrac{1}{5}$ (d) None of these

32. The moment when A is at $(1, 2)$ and B is at $(0, 0)$. The derivative $\dfrac{dx_B}{dx_A}$, is

(a) 16 (b) 8 (c) 9 (d) 2

Solution. (Q. Nos. 31 to 32)

We have, $y = 2x^2$

$$(AB)^2 = (x_B - x_A)^2 + (2x_B^2 - 2x_A^2)^2 = 5$$

$\Rightarrow \qquad (x_B - x_A)^2 + 4(x_B^2 - x_A^2)^2 = 5$

Differentiating w.r.t. x_A and denoting $\dfrac{dx_B}{dx_A} = D$

$$2(x_B - x_A)(D - 1) + 8(x_B^2 - x_A^2) \cdot \{2x_B D - 2x_A\} = 0 \qquad \text{...(i)}$$

Now, when $A(0, 0)$ and $B(1, 2) \quad \Rightarrow \quad x_A = 0, \; x_B = 1$

$\therefore$ Eq. (i), reduces to

$$2(1 - 0)(D - 1) + 8(1 - 0)\{2D - 0\} = 0$$

$\Rightarrow \qquad 2D - 2 + 16D = 0 \quad \Rightarrow \quad D = \dfrac{1}{9} \quad \Rightarrow \quad \dfrac{dx_B}{dx_A} = \dfrac{1}{9}$

Again, when $A(1, 2)$ and $B(0, 0) \quad \Rightarrow \quad x_A = 1, x_B = 0$

$\therefore$ Eq. (i), reduces to

$$2(0 - 1)(D - 1) + 8(0 - 1)\{0 - 2\} = 0$$

$\Rightarrow \qquad -2D + 2 + 16 = 0 \quad \Rightarrow \quad D = 9 \quad \therefore \quad \dfrac{dx_B}{dx_A} = 9$

Ans. 31. (a) **32.** (c)

Proficiency in 'Differentiation'
Exercise 1

Type 1 : Only One Correct Option

1. If $y = at^2 + 2bt + c$ and $t = ax^2 + 2bx + c$, then $\dfrac{d^3y}{dx^3}$ equals to

(a) $24a^2(at+b)$ (b) $24a(ax+b)^2$ (c) $24a(at+b)^2$ (d) $24a^2(ax+b)$

2. Differential coefficient of $\left(x^{\frac{l+m}{m-n}}\right)^{\frac{1}{n-l}} \cdot \left(x^{\frac{m+n}{n-l}}\right)^{\frac{1}{l-m}} \cdot \left(x^{\frac{n+l}{l-m}}\right)^{\frac{1}{m-n}}$ w.r.t. x, is

(a) 1 (b) 0 (c) -1 (d) x^{lmn}

3. If $y = (A + Bx)e^{mx} + (m-1)^{-2}e^x$, then $\dfrac{d^2y}{dx^2} - 2m\dfrac{dy}{dx} + m^2y$ is equal to

(a) e^x (b) e^{mx} (c) e^{-mx} (d) $e^{(1-m)x}$

4. Given, $f(x) = -\dfrac{x^3}{3} + x^2 \sin 1.5\,a - x \sin a \cdot \sin 2a - 5 \text{ arc} \sin (a^2 - 8a + 17)$, then

(a) $f(x)$ is not defined at $x = \sin 8$ (b) $f'(\sin 8) > 0$
(c) $f'(x)$ is not defined at $x = \sin 8$ (d) $f'(\sin 8) < 0$

5. If f and g are differentiable functions such that $g'(a) = 2$ and $g(a) = b$ and if fog is an identity function, then $f'(b)$ has the value equal to

(a) 2/3 (b) 1 (c) 0 (d) 1/2

6. The derivative of the function,

$$f(x) = \cos^{-1}\left\{\frac{1}{\sqrt{13}}(2\cos x - 3\sin x)\right\} + \sin^{-1}\left\{\frac{1}{\sqrt{13}}(2\cos x + 3\sin x)\right\}$$

w.r.t. $\sqrt{1+x^2}$ at $x = \dfrac{3}{4}$ is

(a) $\dfrac{3}{2}$ (b) $\dfrac{5}{2}$ (c) $\dfrac{10}{3}$ (d) 0

7. If $f(x) = \sqrt{x + 2\sqrt{2x-4}} + \sqrt{x - 2\sqrt{2x-4}}$, then the value of $10\,f'(102^+)$, is

(a) -1 (b) 0 (c) 1 (d) does not exist

8. Let $y = \ln (1 + \cos x)^2$, then the value of $\dfrac{d^2y}{dx^2} + \dfrac{2}{e^{y/2}}$ equals to

(a) 0 (b) $\dfrac{2}{1+\cos x}$ (c) $\dfrac{4}{(1+\cos x)}$ (d) $\dfrac{-4}{(1+\cos x)^2}$

9. If $f(x)=\dfrac{a+\sqrt{a^2-x^2}+x}{\sqrt{a^2-x^2}+a-x}$ where $a>0$ and $x<a$, then $f'(0)$ has the value equal to

(a) $\sqrt{a}$ (b) a (c) $\dfrac{1}{\sqrt{a}}$ (d) $\dfrac{1}{a}$

10. Let $u(x)$ and $v(x)$ are differentiable functions such that $\dfrac{u(x)}{v(x)}=7$. If $\dfrac{u'(x)}{v'(x)}=p$ and $\left(\dfrac{u(x)}{v(x)}\right)'=q$, then $\dfrac{p+q}{p-q}$ has the value equal to

(a) 1 (b) 0 (c) 7 (d) –7

11. If $f(x)=|\log_e|x||$, then $f'(x)$ equals to

(a) $\dfrac{1}{|x|}, x\neq 0$ (b) $\dfrac{1}{x}$ for $|x|>1$ and $-\dfrac{1}{x}$ for $|x|<1$

(c) $-\dfrac{1}{x}$ for $|x|>1$ and $\dfrac{1}{x}$ for $|x|<1$ (d) $\dfrac{1}{x}$ for $x>0$ and $-\dfrac{1}{x}$ for $x<0$

12. If $f(x)$ is given by $f(x)=(\cos x+i\sin x)(\cos 3x+i\sin 3x)\ldots$

$\ldots(\cos(2n-1)x+i\sin(2n-1)x)$,

then $f''(x)$ is equal to

(a) $n^3f(x)$ (b) $-n^4f(x)$ (c) $-n^2f(x)$ (d) $n^4f(x)$

13. Let $f(x)=x^n$, n being a non-negative integer. The value of n for which the equality $f'(x+y)=f'(x)+f'(y)$ is valid for all $x, y>0$, is

(a) 0, 1 (b) 1, 2

(c) 2, 4 (d) None of these

14. If $y=\sin^{-1}\left(\dfrac{\sin\alpha\sin x}{1-\cos\alpha\sin x}\right)$, then $y'(0)$ is

(a) 1 (b) $2\tan\alpha$ (c) $\dfrac{1}{2}\tan\alpha$ (d) $\sin\alpha$

15. If $f(x)=\sin\left\{\dfrac{\pi}{3}[x]-x^2\right\}$ for $2<x<3$ and $[x]$ denotes the greatest integer less than or equal to x, then $f'(\sqrt{\pi/3})$ is equal to

(a) $\sqrt{\pi/3}$ (b) $-\sqrt{\pi/3}$ (c) $-\sqrt{\pi}$ (d) None of these

16. The functions $u=e^x\sin x, v=e^x\cos x$ satisfy the equation

(a) $v\dfrac{du}{dx}-u\dfrac{dv}{dx}=u^2+v^2$ (b) $\dfrac{d^2u}{dx^2}=2v$

(c) $\dfrac{d^2v}{dx^2}=-2u$ (d) All of these

17. If $f(x)=\log_x\{\ln(x)\}$, then $f'(x)$ at $x=e$, is

(a) e (b) $-e$ (c) e^2 (d) e^{-1}

18. Let f be a differentiable function satisfying $[f(x)]^n=f(nx)$ for all $x\in R$. Then, $f'(x)f(nx)$

(a) $f(x)$ (b) 0 (c) $f(x)f'(nx)$ (d) None of these

19. If $y = f(x)$ is an odd differentiable function defined on $(-\infty, \infty)$ such that $f'(3) = -2$, then $f'(-3)$ equals

(a) 4 (b) 2 (c) –2 (d) 0

20. If $y = \sqrt{x + \sqrt{y + \sqrt{x + \sqrt{y + \dots \infty}}}}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{y + x}{y^2 - 2x}$ (b) $\frac{y^3 - x}{2y^2 - 2xy - 1}$

(c) $\frac{y^3 + x}{2y^2 - x}$ (d) None of these

21. If $y = (1 + x)(1 + x^2)(1 + x^4)\dots(1 + x^{2^n})$, then $\frac{dy}{dx}$ at $x = 0$ is

(a) 1 (b) –1 (c) 0 (d) None of these

22. If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{4}\right)$ is equal to

(a) $\sqrt{2}$ (b) $-\sqrt{2}$ (c) 0 (d) None of these

23. Let $f(x) = x^2 + xg'(1) + g''(2)$ and $g(x) = x^2 + xf'(2) + f''(3)$. Then,

(a) $f'(1) = 4 + f'(2)$ (b) $g'(2) = 8 + g'(1)$

(c) $g''(2) + f''(3) = 4$ (d) All of these

24. If $f(x) = x^n$, then the value of $f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \frac{f^{(iv)}(1)}{4!} - \dots$ $\dots + \frac{(-1)^n f^n(1)}{n!}$ is

(a) 1 (b) 2^n (c) 2^{n-1} (d) 0

Type 2 : **More than One Correct Options**

25. If $y = x^{(\ln x)^{\ln(\ln x)}}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{y}{x}(\ln x^{\ln x - 1} + 2 \ln x \ln(\ln x))$ (b) $\frac{y}{x}(\ln x)^{\ln(\ln x)}(2 \ln(\ln x) + 1)$

(c) $\frac{y}{x \ln x}((\ln x)^2 + 2 \ln(\ln x))$ (d) $\frac{y}{x} \cdot \frac{\ln y}{\ln x}(2 \ln(\ln x) + 1)$

26. Which of the following functions are not derivable at $x = 0$?

(a) $f(x) = \sin^{-1} 2x\sqrt{1 - x^2}$ (b) $g(x) = \sin^{-1}\left(\frac{2^{x+1}}{1 + 4^x}\right)$

(c) $h(x) = \sin^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$ (d) $k(x) = \sin^{-1}(\cos x)$

27. Let $f(x) = \frac{\sqrt{x - 2\sqrt{x - 1}}}{\sqrt{x - 1} - 1} \cdot x$, then

(a) $f'(10) = 1$ (b) $f'(3/2) = -1$

(c) domain of $f(x)$ is $x \geq 1$ (d) None of these

28. If $2^x + 2^y = 2^{x+y}$, then $\frac{dy}{dx}$ has the value equal to

(a) $-\frac{2^y}{2^x}$ (b) $\frac{1}{1-2^x}$ (c) $1-2^y$ (d) $\frac{2^x(1-2^y)}{2^y(2^x-1)}$

29. For the function $y = f(x) = (x^2 + bx + c)e^x$, which of the following holds?

(a) If $f(x) > 0$ for all real $x \not\Rightarrow f'(x) > 0$ (b) If $f(x) > 0$ for all real $x \Rightarrow f'(x) > 0$
(c) If $f'(x) > 0$ for all real $x \Rightarrow f(x) > 0$ (d) If $f'(x) > 0$ for all real $x \not\Rightarrow f(x) > 0$

30. If $\sqrt{y+x} + \sqrt{y-x} = c$ (where $c \neq 0$), then $\frac{dy}{dx}$ has the value equal to

(a) $\frac{2x}{c^2}$ (b) $\frac{x}{y+\sqrt{y^2-x^2}}$ (c) $\frac{y-\sqrt{y^2-x^2}}{x}$ (d) $\frac{c^2}{2y}$

31. If $y = \tan x \tan 2x \tan 3x$, then $\frac{dy}{dx}$ has the value equal to

(a) $3\sec^2 3x \tan x \tan 2x + \sec^2 x \tan 2x \tan 3x + 2\sec^2 2x \tan 3x \tan x$
(b) $2y(\operatorname{cosec} 2x + 2\operatorname{cosec} 4x + 3\operatorname{cosec} 6x)$
(c) $3\sec^2 3x - 2\sec^2 2x - \sec^2 x$
(d) $\sec^2 x + 2\sec^2 2x + 3\sec^2 3x$

Type 4 : **Linked Comprehension Based Questions**

Passage I

(Q. Nos. 32 to 34)

A curve is represented parametrically by the equations $x = e^t \cos t$ *and* $y = e^t \sin t$, *where* t *is a parameter. Then,*

32. The relation between the parameter 't' and the angle α between the tangent to the given curve and the x-axis is given by, 't' equals to

(a) $\frac{\pi}{2} - \alpha$ (b) $\frac{\pi}{4} + \alpha$ (c) $\alpha - \frac{\pi}{4}$ (d) $\frac{\pi}{4} - \alpha$

33. The value of $\frac{d^2y}{dx^2}$ at the point where $t = 0$ is

(a) 1 (b) 2 (c) –2 (d) 3

34. If $F(t) = \int (x+y)\,dt$, then the value of $F\left(\frac{\pi}{2}\right) - F(0)$ is

(a) 1 (b) –1 (c) $e^{\pi/2}$ (d) 0

Passage II

(Q. Nos. 35 to 37)

Repeated roots *: If equation* $f(x) = 0$, *where* $f(x)$ *is a polynomial function and if it has roots* $\alpha, \alpha, \beta, \ldots$ *or* α *root is repeated root, then* $f(x) = 0$ *is equivalent to* $(x-\alpha)^2 (x-\beta) = 0$, *from which we can conclude that* $f'(x) = 0$

or $$2(x-\alpha)[(x-\beta)\ldots + (x-\alpha)^2 \{(x-\beta)\ldots\}'] = 0$$

or $$(x-\alpha)[2\{(x-\beta)\ldots\}+(x-\alpha)\{(x-\beta)\ldots\}']=0$$

has root α.

Thus, if α occurs twice in equation, then it is common in equations

$$f(x)=0 \quad \text{and} \quad f'(x)=0.$$

Similarly, if α occurs thrice in the equation, then it is common in the equations $f(x)=0$, $f'(x)=0$ and $f''(x)=0$.

35. If $x-c$ is a factor of order m of the polynomial $f(x)$ of degree $n(1<m<n)$, then $x=c$ is a root of the polynomial (where $f^r(x)$ represents rth derivative of $f(x)$ w.r.t. x)

(a) $f^m(x)$ (b) $f^{m-1}(x)$ (c) $f^n(x)$ (d) None of these

36. If $a_1x^3+b_1x^2+c_1x+d_1=0$ and $a_2x^3+b_2x^2+c_2x+d_2=0$ have a pair of repeated common roots, then

$$\begin{vmatrix} 3a_1 & 2b_1 & c_1 \\ 3a_2 & 2b_2 & c_2 \\ a_2b_1-a_1b_2 & c_1a_2-c_2a_1 & d_1a_2-d_2a_1 \end{vmatrix}, \text{ is}$$

(a) 0 (b) $c_1a_2-c_2a_1$ (c) $a_1b_2-a_2b_1$ (d) $d_1a_2-d_2a_1$

37. If α occurs p times and β occurs q times in polynomial equation $f(x)=0$ of degree $n\,(1<p, q<n)$, then which of the following is not true? (where $f^r(x)$ represents rth derivative of $f(x)$ w.r.t. x)

(a) If $p<q<n$, then α and β are two of the roots of the equation $f^{p-1}(x)=0$.
(b) If $q<p<n$, then α and β are two of the roots of the equation $f^{q-1}(x)=0$.
(c) If $p<q<n$, then equations $f(x)=0$ and $f^{q-1}(x)=0$ have exactly one root common.
(d) If $q<p<n$, then equations $f^q(x)=0$ and $f^p(x)=0$ have exactly two roots common.

Passage III

(Q. Nos. 38 to 40)

Equation $x^n-1=0$, $n>1$, $n\in N$, has roots 1, $a_1, a_2, \ldots\ldots a_n$.

38. The value of $(1-a_1)(1-a_2)\ldots(1-a_n)$, is

(a) $\dfrac{n^2}{2}$ (b) n (c) $(-1)^n n$ (d) None of these

39. The value of $\sum\limits_{r=2}^{n}\dfrac{1}{2-a_r}$, is

(a) $\dfrac{2^{n-1}(n-2)+1}{2^n-1}$ (b) $\dfrac{2^n(n-2)+1}{2^n-1}$ (c) $\dfrac{2^{n-1}(n-1)-1}{2^n-1}$ (d) None of these

40. The value of $\sum\limits_{r=2}^{n}\dfrac{1}{1-a_r}$, is

(a) $\dfrac{n}{4}$ (b) $\dfrac{n(n-1)}{2}$ (c) $\dfrac{n-1}{2}$ (d) None of these

Type 5 : Match the Columns

41. Match the entries between following two columns :

Column I	Column II
(A) $y=f(x)$ be given by $x=t^5-5t^3-20t+7$, and $y=4t^3-3t^2-18t+3$, then $-5\times\frac{dy}{dx}$ at $t=1$	(p) 0
(B) $P(x)$ be a polynomial of degree 4 with $P(2)=-1$, $P'(2)=0$, $P''(2)=2$, $P'''(2)=-12$ and $P^{\text{iv}}(2)=24$, then $P''(3)$ is equal to	(q) -2
(C) $y=\frac{1}{x}$, then $\dfrac{\frac{dy}{\sqrt{1+y^4}}}{\frac{dx}{\sqrt{1+x^4}}}$	(r) 2
(D) $f\left(\frac{2x+3y}{5}\right)=\frac{2f(x)+3f(y)}{5}$ and $f'(0)=p$ and $f(0)=q$, then $f''(0)$	(s) -1

42. Match the following :

Column I	Column II
(A) $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then $\frac{dy}{dx}=-\frac{2}{1+x^2}$	(p) for $x<0$
(B) $y=\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$, then $\frac{dy}{dx}=\frac{1}{1+x^2}$	(q) for $x>1$
(C) $y=\lvert e^{\lvert x\rvert}-e\rvert$, then $\frac{dy}{dx}>0$	(r) for $x<-1$
(D) $u=\log\lvert 2x\rvert, v=\lvert\tan^{-1}x\rvert$, then $\frac{du}{dv}>2$	(s) for $-1<x<0$

Type 6 : Integer Answer Type Questions

43. Suppose, $A=\frac{dy}{dx}$ of $x^2+y^2=4$ at $(\sqrt{2},\sqrt{2})$, $B=\frac{dy}{dx}$ of $\sin y+\sin x=\sin x\cdot\sin y$ at (π,π) and $C=\frac{dy}{dx}$ of $2e^{xy}+e^xe^y-e^x-e^y=e^{xy+1}$ at (1, 1), then $(A-B-C)$ has the value equal to ....

44. A function is represented parametrically by the equations $x=\frac{1+t}{t^3}$; $y=\frac{3}{2t^2}+\frac{2}{t}$, then $\frac{dy}{dx}-x\cdot\left(\frac{dy}{dx}\right)^3$ has the absolute value equal to ....

45. Suppose, the function $f(x)-f(2x)$ has the derivative 5 at $x=1$ and derivative 7 at $x=2$. The derivative of the function $f(x)-f(4x)$ at $x=1$, has the value $10+\lambda$, then the value of λ is equal to ....

46. If $x+y=3e^2$, then $D(x^y)$ vanishes when x equals to λe^2, then the value of λ is equal to ....

47. Let $h(x)$ be differentiable for all x and let $f(x)=(kx+e^x)h(x)$ where k is some constant. If $h(0)=5$, $h'(0)=-2$ and $f'(0)=18$, then the value of k is equal to ....

Proficiency in 'Differentiation'
Exercise 2

1. Prove that the derivative of an even function is an odd function and that of an odd function is an even function.

2. If $y^2 = a^2 \cos^2 x + b^2 \sin^2 x$, then prove that $\dfrac{d^2y}{dx^2} + y = \dfrac{a^2b^2}{y^3}$.

3. If $y = a(x + \sqrt{x^2+1})^n + b(x - \sqrt{x^2-1})^{-n}$, then prove that
$(x^2-1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - n^2y = 0$.

4. If $x = \sin\theta$, $y = \cos p\theta$, then prove that $(1-x^2)y_2 - xy_1 + p^2y = 0$, where $y_2 = \dfrac{d^2y}{dx^2}$ and $y_1 = \dfrac{dy}{dx}$.

5. If $x^2 + y^2 + z^2 - 2xyz = 1$, then show that $\dfrac{dx}{\sqrt{1-x^2}} + \dfrac{dy}{\sqrt{1-y^2}} + \dfrac{dz}{\sqrt{1-z^2}} = 0$.

6. If y is a twice differentiable function of x, transform the expression $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y$ by means of the transformation $x = \sin t$ in terms of the independent variable t.

7. Prove that the expression $\dfrac{y'''}{y'} - \dfrac{3}{2}\left(\dfrac{y''}{y}\right)^2 + \dfrac{1}{2}\left(\dfrac{y'}{y}\right)^2$ remains unchanged, if y is replaced by $\dfrac{1}{y^2}$.

8. If $f(x)$ is a real function such that $f(0) = 0$ and $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$, then without using integrating show that $f(x) + f(a) = f(x\sqrt{1-a^2} + a\sqrt{1-x^2})$.

9. Show that the transformation $x = \cos\theta$ reduces the differential equation
$(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 0$ into $\dfrac{d^2y}{d\theta^2} + y = 0$.

10. Show that the transformation $z = \log\tan\dfrac{x}{2}$ reduces the differential equation
$\dfrac{d^2y}{dx^2} + \operatorname{cosec} x \cot x\dfrac{dy}{dx} + 4y\operatorname{cosec}^2 x = 0$ into $\dfrac{d^2y}{dz^2} + 4y = 0$.

Answers

Target Exercise 2.1

1. $a^x \log a + ax^{a-1}$ **2.** $\dfrac{\tan x}{|\tan x|}\sec^2 x$ **3.** $\dfrac{1}{x\log_e 3}+\dfrac{3}{x}+2\sec^2 x$

4. $\dfrac{x}{|x|}+na_0x^{n-1}+(n-1)a_1x^{n-2}+(n-2)a_2x^{n-3}+\ldots+a_{n-i}1$

5. 0 **6.** $e^x x^{n-1}\left\{n\log_a x+\dfrac{1}{\log_e a}+x\log_a x\right\}$

7. $\dfrac{2^x}{\sqrt{x}}\left\{\log 2\cot x-\operatorname{cosec}^2 x-\dfrac{\cot x}{2x}\right\}$ **8.** $\dfrac{x^2}{(x\sin x+\cos x)^2}$

9. y **10.** $x\in(-\infty,-1)\cup(1,\infty)$

Target Exercise 2.2

1. $4(x^2+x+1)^3\cdot(2x+1)$ **2.** $\dfrac{1}{2\sqrt{x^2+x+1}}\cdot(2x+1)$ **3.** $3\sin^2 x\cos x$

4. $\dfrac{x}{(a^2-x^2)^{3/2}}$ **5.** $e^{x\sin x}(x\cos x+\sin x)$ **6.** $-\dfrac{\sqrt{b^2-a^2}}{b+a\cos x}$ **7.** $e^{e^x}\cdot e^x$

8. $\dfrac{x\cot\left(\dfrac{x^2}{3}-1\right)}{3\sqrt{\log\left\{\sin\left(\dfrac{x^2}{3}-1\right)\right\}}}$ **9.** $\dfrac{1}{\sqrt{a^2+x^2}}$ **10.** $\dfrac{2ab\cos x}{a^2-b^2\sin^2 x}$ **11.** $\sec x$

12. $\dfrac{(\sin 3x)\left(e^x+\dfrac{1}{x}\right)-3(e^x+\log x)\cos 3x}{\sin^2 3x}$ **13.** $\dfrac{m}{\sqrt{1-x^2}}\cos(m\sin^{-1}x)$

14. $\dfrac{2\log a\cdot\sin^{-1}x}{\sqrt{1-x^2}}\cdot a^{(\sin^{-1}x)^2}$ **15.** $e^{\cos^{-1}\sqrt{1-x^2}}\cdot\dfrac{1}{\sqrt{1-x^2}}$ **16.** $\dfrac{\sin^{-1}x}{(1-x^2)^{3/2}}$

17. $\dfrac{1}{x\log_e 10}-\dfrac{\log_e 10}{x(\log_e x)^2}$ **18.** $-2x\{5^{3-x^2}\cdot\log_e 5+5(3-x^2)^4\}$

19. $\dfrac{-2a^2}{x^3}\left\{1+\dfrac{a^2}{\sqrt{a^4-x^4}}\right\}$ **20.** $\dfrac{x}{4\sqrt{4+\sqrt{4+\sqrt{4+x^2}}}\,\sqrt{4+\sqrt{4+x^2}}\,\sqrt{4+x^2}}$

21. (c) **22.** (b) **23.** (a) **24.** (a) **25.** (a) **26.** (c) **27.** (a) **28.** (d)

Target Exercise 2.4

1. (b) **2.** (c) **3.** (b) **4.** (c)

Target Exercise 2.5

1. $x^x(1+\log x)$ **2.** $x^{\sqrt{x}}\left(\dfrac{\log x}{2\sqrt{x}}+\dfrac{1}{\sqrt{x}}\right)$ **3.** $x^{x^x}\cdot x^x\cdot\left\{(1+\log x)\cdot\log x+\dfrac{1}{x}\right\}$

4. $x \cdot x^{x^2} \cdot (2 \log x + 1)$ **5.** $x^{x+\frac{1}{2}} \left\{\left(\dfrac{2x+1}{2x}\right) + \log x\right\}$

6. $(\cos x)^x (\log \cos x - x \tan x)$ **7.** $(\sin x)^{\cos x} \left(-\sin x \log \sin x + \dfrac{\cos^2 x}{\sin x}\right)$

8. $(\sin x)^{\cos^{-1} x} \left(\cos^{-1} x \cot x - \dfrac{\log (\sin x)}{\sqrt{1-x^2}}\right)$ **9.** $-x^x (\sin x^x) \cdot \{1 + \log x\}$

10. $\dfrac{1}{x^x + \operatorname{cosec}^2 x} \{x^x(1 + \log x) - 2 \operatorname{cosec}^2 x \cot x\}$

11. $(\sin x)^{\tan x} \cdot \{\sec^2 x (\log \sin x) + 1\} + (\cos x)^{\sec x} \cdot \{\sec x \tan x \cdot \log (\cos x) - \sec x \tan x\}$

13. $-\left(\dfrac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right)$

14. $\dfrac{\log \sin y + y \tan x}{\log \cos x - x \cot y}$ **24.** $x^{\sin^{-1} x} \left\{\log x + \dfrac{\sqrt{1-x^2}}{x} \sin^{-1} x\right\}$ **25.** $-\dfrac{2}{ax}$

26. $\dfrac{\alpha}{\alpha^2 + x^2}$

Target Exercise 2.6

6. $-\dfrac{1}{2at^3}$ **7.** $\dfrac{1}{3a} \sec^4 \theta \operatorname{cosec} \theta$ **13.** 1 **17.** $\dfrac{3}{\pi\sqrt{\pi^2 - 3}}$ **19.** (b) **20.** (b)

Exercise 1

1. (d) **2.** (b) **3.** (a) **4.** (d) **15.** (d) **6.** (c) **7.** (c) **8.** (a) **9.** (d) **10.** (a)
11. (d) **12.** (b) **13.** (b) **14.** (d) **15.** (b) **16.** (d) **17.** (d) **18.** (c) **19.** (c) **20.** (d)
21. (a) **22.** (d) **23.** (d) **24.** (d) **25.** (b, d) **26.** (b, c, d) **27.** (a, b)
28. (a, b, c, d) **29.** (a, c) **30.** (a, b, c) **31.** (a, b, c) **32.** (c) **33.** (b)
34. (c) **35.** (b) **36.** (a) **37.** (d) **38.** (b) **39.** (a) **40.** (c)
41. (A) $\to$ (q), (B) $\to$ (r), (c) $\to$ (s), (D) $\to$ (p)
42. (A) $\to$ (q, r), (B) $\to$ (p, r), (C) $\to$ (q, s), (D) $\to$ (q, r) **43.** (1) **44.** (1) **45.** (9)
46. (1) **47.** (3)

Exercise 2

6. $\dfrac{d^2y}{dt^2} + y$

Solutions

(*Proficiency in 'Differentiation' Exercise* 1)

Type 1 : **Only One Correct Option**

1. Note that in y highest degree of x is 4 and therefore $\dfrac{d^3y}{dx^3}$ is a linear function of x, which is satisfied only in (d).

2. Exponent of $x = \dfrac{1}{(l-m)(m-n)(n-l)}[l^2 - m^2 + m^2 - n^2 + n^2 - l^2] = 0$

$$\Rightarrow \qquad y = x^0 = 1$$

$$\Rightarrow \qquad y = 1 \quad \Rightarrow \quad y' = 0$$

3. $y = (A + Bx)e^{mx} + (m-1)^{-2} \cdot e^x$

$$y \cdot e^{-mx} = (A + Bx) + (m-1)^{-2} \cdot e^{(1-x)x}$$

$$e^{-mx} \cdot y_1 - my + e^{-mx} = B - (m-1)^{-1} \cdot e^{-(m-1)x}$$

$$e^{-mx} \cdot y_2 - y_1 e^{-mx} \cdot m - m[e^{-mx} \cdot y_1 - ye^{-mx} \cdot m] = e^{-(m-1)x}$$

$$e^{-mx} \cdot y_1 - m_2 y_1 e^{-mx} + my \cdot e^{-mx} = e^{-(m-1)x}$$

$$y_2 - 2my_1 + my = e^x$$

4. $f(x) = -\dfrac{x^3}{3} + x^2 \sin 6 - x \sin 4 \cdot \sin 8 \; - 5 \sin^{-1}((a-4)^2 + 1)$

$$f'(x) = -x^2 + 2x \sin 6 - \sin 4 \sin 8 \qquad (a = 4)$$

$$\begin{aligned} f'(\sin 8) &= -\sin^2 8 + 2 \sin 6 \sin 8 - \sin 4 \sin 8 \\ &= \sin 8[-\sin 8 + 2 \sin 6 - \sin 4] \\ &= -\sin 8[\sin 8 + \sin 4 - 2 \sin 6] \\ &= -\sin 8[2 \sin 6 \cos 2 - 2 \sin 6] \\ &= 2 \sin 8 \sin 6[1 - \cos 2] \end{aligned}$$

sin 8	
sin 4	sin 6

5. $f(g(x)) = x; f'(g(x)) \cdot g'(x) = 1; f'(g(a)) \cdot g'(a) = 1; f'(b) \cdot 2 = 1$

$$\Rightarrow \qquad f'(b) = \frac{1}{2}$$

6. Put $\cos \phi = \dfrac{2}{\sqrt{13}}; \sin \phi = \dfrac{3}{\sqrt{13}}; \tan \phi = \dfrac{3}{2}$

$$\begin{aligned} y &= \cos^{-1}\{\cos(x + \phi)\} + \sin^{-1}\{\cos(x - \phi)\} \\ &= \cos^{-1}\{\cos(x + \phi)\} + \frac{\pi}{2} - \cos^{-1}\{\cos(\phi - x)\} \\ &= x + \phi + \frac{\pi}{2} - \phi + x \end{aligned}$$

$$y = 2x + \frac{\pi}{2}; \; z = \sqrt{1 + x^2}$$

Now, compute $\dfrac{dy}{dz}$.

7. $f(x) = \sqrt{x + 2\sqrt{2x-4}} + \sqrt{x - 2\sqrt{2x-4}}$

$\therefore \quad f(x) = \sqrt{(\sqrt{x-2} + \sqrt{2})^2} + \sqrt{(\sqrt{x-2} - \sqrt{2})^2} = |\sqrt{x-2} + \sqrt{2}| + |\sqrt{x-2} - \sqrt{2}|$

For $\sqrt{x-2}$ to exist $x \geq 2$

Also, $\sqrt{x-2} + \sqrt{2} > 0$ (always true)

But $\sqrt{x-2} - \sqrt{2} \geq 0$ only, if $x \geq 4$

< 0 only, if $x < 4$

$\therefore$ Now, $f(x)$ becomes $f(x) = \sqrt{x-2} + \sqrt{2} - \sqrt{x-2} + \sqrt{2}$ for $2 \leq x < 4$

$= \sqrt{x-2} + \sqrt{2} + \sqrt{x-2} - \sqrt{2}$ for $x \geq 4$

$$\therefore \quad f(x) = 2\sqrt{2}, \quad \text{for} \quad 2 \leq x < 4$$

$$= 2\sqrt{x-2}, \text{ for } 4 \leq x < \infty$$

$\because f$ is continuous $[2, 4) \cup [4, \infty)$ (verify)

$$\therefore \quad f'(x) = 0, 2 \leq x < 4$$

$$= \frac{1}{\sqrt{x-2}}, 4 \leq x < \infty$$

$$\therefore \quad f'(102^+) = \frac{1}{\sqrt{102-2}} = \frac{1}{10}$$

$$\therefore \quad 10f'(102^+) = 1$$

8. $y = 2 \ln(1 + \cos x) \Rightarrow \quad y_1 = \dfrac{-2 \sin x}{1 + \cos x}$

$$y_2 = -2\left[\frac{(1+\cos x)\cos x - \sin x(-\sin x)}{(1+\cos x)^2}\right]$$

$$= -2\left[\frac{\cos x + 1}{(1+\cos x)^2}\right] = \frac{-2}{(1+\cos x)}$$

$$\therefore \quad 2e^{-y/2} = 2 \cdot e^{\frac{-\ln(1+\cos x)^2}{2}} = \frac{2}{(1+\cos x)}$$

$$\therefore \quad y_2 + \frac{2}{e^{y/2}} = 0$$

9. $y = \dfrac{(a+x) + \sqrt{a-x} \cdot \sqrt{a+x}}{(a-x) + \sqrt{a-x} \cdot \sqrt{a+x}}$

$$y = \frac{\sqrt{(a+x)}\,(\sqrt{a+x} + \sqrt{a-x})}{\sqrt{a-x}\;(\sqrt{a+x} + \sqrt{a-x})} = \left(\frac{a+x}{a-x}\right)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{a-x}{a+x}}\left(\frac{(a-x)+(a+x)}{(a-x)^2}\right)$$

$$= \frac{1}{2}\frac{\sqrt{a-x}}{\sqrt{a+x}} \times \frac{2a}{(a-x)^2}$$

$$\therefore \quad \left.\frac{dy}{dx}\right|_{x=0} = \frac{1}{a}$$

10. $u(x) = 7v(x)$

$$\Rightarrow \qquad u'(x) = 7v'(x) \quad \Rightarrow \quad p = 7 \qquad \text{(given)}$$

Again, $$\frac{u(x)}{v(x)} = 7 \quad \Rightarrow \quad \left(\frac{u(x)}{v(x)}\right)' = 0$$

$$\Rightarrow \qquad q = 0$$

Now, $$\frac{p+q}{p-q} = \frac{7+0}{7-0} = 1$$

11. For $x > 1$, we have $f(x) = |\log|x|| = \log x$

$$\Rightarrow \qquad f'(x) = \frac{1}{x}$$

For $x < -1$, we have

$$f(x) = |\log|x|| = \log(-x) \quad \Rightarrow \quad f'(x) = \frac{1}{x}$$

For $0 < x < 1$, we have

$$f(x) = |\log|x|| = -\log x \quad \Rightarrow \quad f'(x) = -\frac{1}{x}$$

For $-1 < x < 0$, we have

$$f(x) = -\log(-x) \quad \Rightarrow \quad f'(x) = -\frac{1}{x}$$

Hence, $$f'(x) = \begin{cases} \dfrac{1}{x}, & |x| > 1 \\ -\dfrac{1}{x}, & |x| < 1 \end{cases}$$

12. We have,

$$f(x) = \cos\{x + 3x + \ldots + (2n-1)x\} + i\sin\{x + 3x + 5x + \ldots + (2n-1)x\}$$

$$\Rightarrow \qquad f'(x) = \cos n^2 x + i \sin n^2 x$$

$$\Rightarrow \qquad f'(x) = -n^2(\sin n^2 x) + n^2(i\cos n^2 x)$$

$$\Rightarrow \qquad f''(x) = -n^4 \cos n^2 x - n^4 i \sin n^2 x$$

$$\Rightarrow \qquad f''(x) = -n^4(\cos n^2 x + i \sin n^2 x)$$

$$\Rightarrow \qquad f''(x) = -n^4 f(x)$$

13. We have, $f(x) = x^n$

$$\Rightarrow \qquad f(x+y) = (x+y)^n \quad \Rightarrow \quad f'(x+y) = n(x+y)^{n-1}$$

Also, $$f'(x) = nx^{n-1} \text{ and } f'(y) = ny^{n-1}$$

$$\therefore \qquad f'(x+y) = f'(x) + f'(y)$$

$$\Rightarrow \qquad n(x+y)^{n-1} = n \cdot x^{n-1} + n \cdot y^{n-1}$$

$$\Rightarrow \qquad (x+y)^{n-1} = x^{n-1} + y^{n-1} \qquad \ldots\text{(i)}$$

For $n - 1 > 1$, we find that LHS of Eq. (i) is greater than the RHS. So, we must have $n - 1 \le 1$ *ie*, $n - 1 = 0$ or $n - 1 = 1$.

Hence, $n = 1$ or $n = 2$

14. We have, $y = \sin^{-1}\left(\frac{\sin\alpha \sin x}{1 - \cos\alpha \sin x}\right)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - \frac{\sin^2\alpha \sin^2 x}{(1 - \cos\alpha \sin x)^2}}} \frac{d}{dx}\left(\frac{\sin\alpha \sin x}{1 - \cos\alpha \sin x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1 - \cos\alpha \sin x)}{\sqrt{1 + \sin^2 x - 2\cos\alpha \sin x}} \cdot \frac{\sin\alpha}{\cos\alpha} \frac{d}{dx}\left(1 - \frac{1}{1 - \cos\alpha \sin x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\tan\alpha(1 - \cos\alpha \sin x)}{\sqrt{1 + \sin^2 x - 2\sin x \cos\alpha}} \frac{d}{dx}\left(1 - \frac{1}{\cos\alpha \sin x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\tan\alpha(1 - \cos\alpha \sin x)}{\sqrt{1 + \sin^2 x - 2\sin x \cos\alpha}}\left(\frac{1 - \cos\alpha \cos x}{(1 - \cos\alpha \sin x)^2}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin\alpha \cos x}{(1 - \cos\alpha \sin x)\sqrt{1 + \sin^2 x - 2\cos\alpha \sin x}} \therefore \left(\frac{dy}{dx}\right)_{x=0} = \sin\alpha$$

$$\Rightarrow \quad y'(0) = \sin\alpha$$

15. For $2 < x < 3$, we have $[x] = 2$

$$\therefore \quad f(x) = \sin\left(\frac{2\pi}{3} - x^2\right) \quad \Rightarrow \quad f'(x) = -2x\cos\left(\frac{2\pi}{3} - x^2\right)$$

$$\Rightarrow \quad f'(\sqrt{\pi/3}) = -2\sqrt{\pi/3}\cos\pi/3 = -\sqrt{\pi/3}$$

16. We have, $u = e^x \sin x$

$$\Rightarrow \quad \frac{du}{dx} = e^x \sin x + e^x \cos x = u + v$$

$$v = e^x \cos x$$

$$\Rightarrow \quad \frac{dv}{dx} = e^x \cos x + e^x \sin x = v - u$$

$$\therefore \quad v\frac{du}{dx} - u\frac{dv}{dx} = v(u + v) - u(v - u) = u^2 + v^2$$

$$\frac{d^2u}{dx^2} = \frac{du}{dx} + \frac{dv}{dx} = u + v + v - u = 2v$$

and
$$\frac{d^2v}{dx^2} = \frac{dv}{dx} - \frac{du}{dx} = (v - u) - (v + u) = -2u$$

17. We have, $f(x) = \log_x \{\ln(x)\} = \frac{\ln\{\ln(x)\}}{\ln(x)}$

$$\therefore \quad f'(x) = \frac{\ln(x) \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x} - \ln\{\ln(x)\}\frac{1}{x}}{\{\ln(x)\}^2}$$

$$= \frac{1 - \ln\{\ln(x)\}}{x\{\ln(x)\}^2}$$

$$\Rightarrow \quad f'(e) = \frac{1 - \ln\{\ln(e)\}}{e\{\ln(e)\}^2} = \frac{1 - \ln(1)}{e} = \frac{1}{e}$$

18. We have, $[f(x)]^n = f(nx)$ for all x

$\Rightarrow \quad n[f(x)]^{n-1} f'(x) = nf'(nx)$

$\Rightarrow \quad n[f(x)]^n f'(x) = nf(x)f'(nx)$ [Multiplying both the sides by $f(x)$]

$\Rightarrow \quad nf(nx)f'(x) = nf(x)f'(nx)$ $[\because [f(x)]^n = f(nx)]$

$\Rightarrow \quad f(nx)f'(x) = f(x)f'(nx)$

19. Since, $f(x)$ is an odd differentiable function defined on R. Therefore,

$$f(-x) = -f(x) \text{ for all } x \in R$$

Differentiating both the sides w.r.t. x, we get

$$-f'(-x) = -f'(x) \text{ for all } x \in R$$

$\Rightarrow \quad f'(-x) - f'(x)$ for all $x \in R \quad \Rightarrow f'(-3) = f'(3) = -2$

Aliter : We know that the derivative of a differentiable odd function is an even function. Therefore, $f'(x)$ is an even function.

Hence, $\quad f'(-3) = f'(3) = -2$

20. We have, $y = \sqrt{x + \sqrt{y + \sqrt{x + \sqrt{y + \ldots \infty}}}}$

$\Rightarrow \quad y^2 = x + \sqrt{y + \sqrt{x + \sqrt{y + \ldots \infty}}}$

$\Rightarrow \quad y^2 = x + \sqrt{y + y} \quad \Rightarrow \quad (y^2 - x)^2 = 2y$

Differentiating both the sides w.r.t. x, we get

$$2(y^2 - x)\left(2y\frac{dy}{dx} - 1\right) = 2\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y^2 - x}{y^3 - xy - 1}$$

21. $y = (1 + x)(1 + x^2)(1 + x^4)\ldots(1 + x^{2^n})$

$$\Rightarrow \quad y = \frac{(1 - x)(1 + x)(1 + x^2)(1 + x^4)\ldots(1 + x^{2^n})}{(1 - x)}$$

$$\Rightarrow \quad y = \frac{1 - x^{2^{n+1}}}{1 - x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2^{n+1} x^{2^{n+1}} - 1(1 - x) + (1 - x^{2^{n+1}})}{(1 - x)^2}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=0} = 1$$

22. We have, $f(x) = |\cos x - \sin x|$

$$\Rightarrow \quad f(x) = \begin{cases} \cos x - \sin x, & \text{for } 0 < x \le \frac{\pi}{x} \\ \sin x - \cos x, & \text{for } \frac{\pi}{4} < x < \frac{\pi}{2} \end{cases}$$

Clearly, $\left(\text{LHD at } x = \frac{\pi}{4}\right) = \left\{\frac{d}{dx}(\cos x - \sin x)\right\}_{\text{at } x = \frac{\pi}{4}}$

$$= (-\sin x - \cos x)_{x = \frac{\pi}{4}}$$

$$= -\sqrt{2}$$

$$\text{and}\left(\text{RHD at } x=\frac{\pi}{4}\right)=\left\{\frac{d}{dx}(\sin x-\cos x)\right\}_{\text{at } x=\frac{\pi}{4}}$$

$$=(\cos x+\sin x)_{x=\frac{\pi}{4}}=\sqrt{2}$$

$$\therefore \quad \left(\text{LHD at } x=\frac{\pi}{4}\right)\neq\left(\text{RHD at } x=\frac{\pi}{4}\right)$$

Thus, $f'\left(\frac{\pi}{4}\right)$ doesn't exist.

23. We have, $f(x)=x^2+xg'(1)+g''(2)$

and $g(x)=x^2+xf'(2)+f''(3)$

$\Rightarrow \quad f'(x)=2x+g'(1)$ and $g'(x)=2x+f'(2)$ ...(i)

Putting $x=1$ in Eq (i), we get

$$f'(1)=2+g'(1) \text{ and } g'(1)=2+f'(2) \Rightarrow f'(1)=4+f'(2)$$

Putting $x=2$ in Eq. (i), we get

$$f'(2)=4+g'(1) \text{ and } g'(2)=4+f'(2)$$

$$\Rightarrow \quad g'(2)=4+4+g'(1)=8+g'(1)$$

Differentiating Eq. (i) w.r.t. x, we get

$$f''(x)=2 \text{ and } g''(x)=2 \text{ for all } x$$

$$\Rightarrow \quad f''(3)=2 \text{ and } g''(2)=2 \Rightarrow g''(2)+f''(3)=2+2=4$$

24. We have, $f(x)=x^n$

$$f^r(x)=n(n-1)(n-2)\ldots\{n-(r-1)\}x^{n-r}$$

$$\Rightarrow \quad f^r(x)=\frac{n!}{(n-r)!}x^{n-r} \Rightarrow f^r(1)=\frac{n!}{(n-r)!}$$

$$\therefore \quad f(1)-\frac{f'(1)}{1!}+\frac{f''(1)}{2!}-\frac{f'''(1)}{3!}+\ldots+\frac{(-1)^n f^n(1)}{n!}$$

$$=\sum_{r=0}^{n}(-1)^r\frac{f^r(1)}{r!}, \text{ where } f^0(1)=f(1)$$

$$=\sum_{r=0}^{n}(-1)^r\frac{n!}{(n-r)!\,r!}=\sum_{r=0}^{n}(-1)^r\ {}^nC_r=0$$

Type 2 : **More than One Correct Options**

25. $y=x^{(\ln x)^{\ln(\ln x)}}$, $\ln y=(\ln x)^{\ln(\ln x)}\cdot\ln x$ ...(i)

$$\ln(\ln y)=\ln(\ln x)\cdot\ln(\ln x)+\ln(\ln x)$$

$$\frac{1}{\ln y}\cdot\frac{1}{y}\frac{dy}{dx}=\frac{2\ln(\ln x)}{\ln x}\cdot\frac{1}{x}+\frac{1}{x\ln x}=\frac{2\ln(\ln x)+1}{x\ln x}$$

$$\therefore \quad \frac{dy}{dx}=\frac{y}{x}\cdot\frac{\ln y}{\ln x}(2\ln(\ln x)+1)$$

Substituting the value of $\ln y$ from Eq. (i)

$$\frac{dy}{dx}=\frac{y}{x}(\ln x)^{\ln(\ln x)}(2\ln(\ln x)+1)$$

26. (a) Not derivable at $x = \frac{1}{\sqrt{2}}$, Check with $x = \sin\theta$ or direct differentiation.

(b) $$g'(x) = \frac{1}{\sqrt{1 - \left(\frac{2 \cdot 2^x}{1 + 2^{2x}}\right)^2}} \cdot \frac{(1 + 4^x)(2^{x+1}) \ln 2 - 2^{x+1} \cdot 4^x \cdot \ln 4}{(1 + 4^x)^2}$$

$$= \frac{(1 + 2^{2x})^2}{\sqrt{(1 + 2^{2x})^2 - (2 \cdot 2^x)^2}} = \frac{1}{|1 - 2^{2x}|}$$

$\Rightarrow$ Not derivable at $x = 0$

(c) $$h(x) = \sin^{-1}\left(\frac{1 - x^2}{1 + x^2}\right) = \frac{\pi}{2} - \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$$

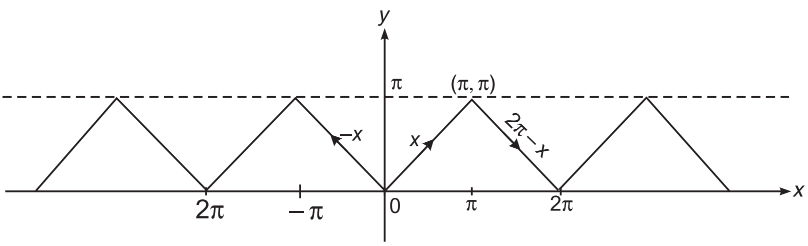
$$\begin{cases} \frac{\pi}{2} - 2\tan^{-1} x, & x \geq 0 \\ \frac{\pi}{2} + 2\tan^{-1} x, & x < 0 \end{cases}$$

$$\therefore \qquad h'(x) = \begin{cases} \frac{-2}{1 + x^2}, & x > 0 \\ \text{not derivable}, & x = 0 \\ \frac{2}{1 + x^2}, & x < 0 \end{cases}$$

$\therefore$ not differentiable at $x = 0$.

(d) $$k(x) = \sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x)$$

and graph for $\cos^{-1}(\cos x)$, is shown as



$\therefore k(x)$ is not differentiable at $x = 0$.

27. $$f(x) = \frac{\sqrt{(\sqrt{x - 1})^2 + 1 - 2\sqrt{x - 1}}}{\sqrt{x - 1} - 1} \cdot x = \frac{|\sqrt{x - 1} - 1|}{\sqrt{x - 1} - 1} \cdot x = \begin{cases} -x, & \text{if } x \in [1, 2) \\ x, & \text{if } x \in (2, \infty) \end{cases}$$

28. $2^x + 2^y = 2^{x+y}$

Differentiating both sides, we get

$$2^x \log 2 + 2^y \log 2 \frac{dy}{dx} = 2^{x+y} \cdot \log 2 \left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \qquad \log 2 \cdot 2^y (2^x - 1) \frac{dy}{dx} = 2^x (1 - 2^y) \log 2$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{2^x (1 - 2^y)}{2^y (2^x - 1)} \qquad \ldots\text{(i)}$$

Also, $2^x = 2^y(1-2^x)$

$\therefore$ $\dfrac{dy}{dx} = (1-2^y)$ ...(ii)

when $2^y = 2^x(2^y - 1)$

$\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{-1}{2^x - 1} = \dfrac{1}{1-2^x}$ ...(iii)

As, $2^x = 2^y(1-2^x)$ and $2^y = 2^x(2^y - 1)$

$\therefore$ $-\dfrac{2^y}{2^x} = \dfrac{2^x(1-2^y)}{2^y(2^x-1)}$ substituting in Eq. (i), we get

$$\frac{dy}{dx} = -\frac{2^y}{2^x}$$

29. $f(x) = (x^2 + bx + c)e^x$

$\therefore$ $f'(x) = \{x^2 + (b+2)x + (b+c)\}e^x$

$f(x) > 0$ iff $D = b^2 - 4c < 0$

Now, $f'(x) > 0$ iff $D' = (b+2)^2 - 4(b+c) = D + 4 < 0$

Thus, for $f'(x) > 0$ $D + 4 < 0$ holds.

$\Rightarrow$ $D < 0$

$\Rightarrow$ $f(x) > 0$

Note that the converse need not be true. *eg,* $b = c = 1$, $f(x) > 0$ but $f'(-1) = 0$

30. Square both the sides, differentiate and rationalize.

31. $y = \tan x \cdot \tan 2x \cdot \tan 3x$ ...(i)

Differentiating both sides, we get

$$\frac{dy}{dx} = 3\sec^2 3x \cdot \tan x \cdot \tan 2x + \sec^2 x \cdot \tan 2x \cdot \tan 3x$$

$$+ 2\sec^2 2x \cdot \tan x \cdot \tan 3x \quad \text{...(ii)}$$

Taking log on both sides of Eq. (i), we get

$$\log y = \log \tan x + \log \tan 2x + \log \tan 3x$$

Differentiating both sides, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2\operatorname{cosec} 2x + 4\operatorname{cosec} 4x + 6\operatorname{cosec} 6x$$

$\therefore$ $\dfrac{dy}{dx} = 2y(\operatorname{cosec} 2x + 2\operatorname{cosec} 4x + 3\operatorname{cosec} 6x)$ ...(iii)

$$\tan(3x) = \tan(2x + x) = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

$\Rightarrow$ $\tan 3x \cdot \tan 2x \cdot \tan x = \tan 3x - \tan 2x - \tan x$

Differentiating both sides, we get

$$\frac{dy}{dx} = 3\sec^2 3x - 2\sec^2 2x - \sec^2 x$$

Type 4 : **Linked Comprehension Based Questions**

Solutions (Q. Nos. 32 to 34)

32. $y = e^t \sin t \Rightarrow \dfrac{dy}{dt} = e^t [\cos t + \sin t]$

$x = e^t \cos t \Rightarrow \dfrac{dx}{dt} = e^t [\cos t - \sin t]$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\cos t + \sin t}{\cos t - \sin t} = \tan \alpha$

$\therefore \quad \tan\left(\dfrac{\pi}{4} + t\right) = \tan \alpha$

$$\left(\frac{\pi}{4} + t\right) = \alpha, \quad t = \alpha - \frac{\pi}{4}$$

33. $\dfrac{d^2y}{dx^2} = \dfrac{\sec^2\left(\dfrac{\pi}{4} + t\right)}{e^t(\cos t - \sin t)}$

$$\left.\frac{d^2y}{dx^2}\right|_{t=0} = 2$$

34. $F(t) = \int e^t(\cos t + \sin t)\, dt = e^t \sin t + C$

$$F\left(\frac{\pi}{2}\right) - F(0) = (e^{\pi/2} + C) - 0 = e^{\pi/2}$$

Solutions (Q. Nos. 35 to 37)

35. From the given information, we have $f(x) = (x - c)^m g(x)$, where $g(x)$ is polynomial of degree $n - m$.

Then, $x = c$ is a common root for the equations $f(x) = 0$, $f^1(x) = 0$, $f^2(x) = 0, \dots,$ $f^{m-1}(x) = 0$ where $f^r(x)$ represents rth derivative of $f(x)$ w.r.t. x.

36. Let $f(x) = a_1x^3 + b_1x^2 + c_1x + d_1 = 0$ has roots α, α, β, then $g(x) = a_2x^3 + b_2x^2 + c_2x + d_2 = 0$ must have roots α, α, γ.

$\Rightarrow \quad a_1\alpha^3 + b_1\alpha^2 + c_1\alpha + d_1 = 0 \quad$ ...(i)

and $\quad a_2\alpha^3 + b_2\alpha^2 + c_2\alpha + d_2 = 0 \quad$ ...(ii)

α is also a root of equations

$$f'(x) = 3a_1x^2 + 2b_1x + c_1 = 0 \text{ and } g'(x) = 3a_2x^2 + 2b_2x + c_2 = 0$$

$\Rightarrow \quad 3a_1\alpha^2 + 2b_1\alpha + c_1 = 0 \quad$ ...(iii)

$3a_2\alpha^2 + 2b_2\alpha + c_2 = 0 \quad$ ...(iv)

Also, from a_1 Eq. (i) $- a_1 \times$ Eq. (ii), we get

$(a_2b_1 - a_1b_2)\alpha^2 + (c_1a_2 - c_2a_1)\alpha + d_1a_2 - d_2a_1 = 0 \quad$ ...(v)

Eliminating α from Eqs. (iii), (iv) and (v), we get

$$\begin{vmatrix} 3a_1 & 2b_1 & c_1 \\ 3a_2 & 2b_2 & c_2 \\ a_2b_1 - a_1b_2 & c_1a_2 - c_2a_1 & d_1a_2 - d_2a_1 \end{vmatrix} = 0$$

37. $f(x) = (x-\alpha)^p \cdot (x-\beta)^q \cdot g(x)$

$$\Rightarrow \quad f(\alpha) = f'(\alpha) = f^2(\alpha) = \ldots. = f^{p-1}(\alpha) = 0 \quad \ldots\text{(i)}$$

$$g(\beta) = g'(\beta) = g^2(\beta) = \ldots. = g^{q-1}(\beta) = 0 \quad \ldots\text{(ii)}$$

$\therefore$ If $\quad p < q < n \Rightarrow \alpha, \beta$ are roots of $f^{p-1}(x) = 0$

If $\quad q < p < n \Rightarrow \alpha, \beta$ are roots of $f^{q-1}(x) = 0$

If $p < q < n$, then $f(x) = 0$ and $f^{q-1}(x) = 0$ has exactly one root common *ie*, $x = \beta$.

Solutions (Q. Nos. 38 to 40)

38. Since, $1, a_1, a_2, \ldots\ldots, a_n$ are roots of $x^n - 1 = 0$, then

$$x^n - 1 = (x-1)(x-a_1)(x-a_2)\ldots(x-a_n) \quad \ldots\text{(i)}$$

$$\Rightarrow \quad \frac{x^n - 1}{x^n - 1} = (x-a_1)(x-a_2)\ldots(x-a_n)$$

$$\Rightarrow \quad \lim_{x\to 1} \frac{x^n - 1}{x^n - 1} = \lim_{x\to 1} [(x-a_1)(x-a_2)\ldots(x-a_n)]$$

$$\Rightarrow \quad (1-a_1)(1-a_2)\ldots(1-a_n) = n$$

39. From Eq. (i), $\log(x^n - 1) = \log(x-1) + \log(x-a_1) + \ldots + \log(x-a_n)$

Differentiating w.r.t. x, we get

$$\frac{nx^{n-1}}{x^n - 1} = \frac{1}{x-1} + \frac{1}{x-a_1} + \frac{1}{x-a_2} + \ldots + \frac{1}{x-a_n} \quad \ldots\text{(ii)}$$

Putting $x = 2$ in Eq. (ii), we get

$$\frac{n2^{n-1}}{x^n - 1} = 1 + \frac{1}{2-a_1} + \frac{1}{2-a_2} + \ldots + \frac{1}{2-a_n}$$

$$\Rightarrow \quad \frac{1}{2-a_1} + \frac{1}{2-a_2} + \ldots + \frac{1}{2-a_n} = \frac{n2^{n-1}}{2^n - 1} = \frac{n2^{n-1} - 2^n + 1}{2^n - 1} = \frac{2^{n-1}(n-2)+1}{2^n - 1}$$

40. From Eq. (ii), $\dfrac{nx^{n-1}}{x^n - 1} - \dfrac{1}{x-1} = \dfrac{1}{x-a_1} + \dfrac{1}{x-a_2} + \ldots + \dfrac{1}{x-a_n}$

$$\Rightarrow \quad \frac{nx^{n-1} - 1(1 + x + x^2 + \ldots + x^{n-1})}{x^n - 1} = \frac{1}{x-a_1} + \frac{1}{x-a_2} + \ldots + \frac{1}{x-a_n}$$

$$\Rightarrow \quad \lim_{x\to 1} \frac{nx^{n-1} - 1(1 + x + x^2 + \ldots + x^{n-1})}{x^n - 1}$$

$$= \lim_{x\to 1} \left(\frac{1}{x-a_1} + \frac{1}{x-a_2} + \ldots + \frac{1}{x-a_n} \right)$$

$$\Rightarrow \quad \lim_{x\to 1} \frac{n(n-1)x^{n-2} - \{1 + 2x + \ldots + (n-1)x^{n-2}\}}{nx^{n-1}}$$

$$= \frac{1}{1-a_1} + \frac{1}{1-a_2} + \ldots + \frac{1}{1-a_n} \quad \text{(Applying L'Hospital's rule on LHS)}$$

$$\Rightarrow \quad \frac{n(n-1) - \{1 + 2 + \ldots + (n-1)\}}{n} = \frac{1}{1-a_1} + \frac{1}{1-a_2} + \ldots + \frac{1}{1-a_n}$$

$$\Rightarrow \quad \frac{1}{1-a_1} + \frac{1}{1-a_2} + \ldots + \frac{1}{1-a_n} = \frac{n-1}{2}$$

Type 5 : **Match the Columns**

41. (A) $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{12t^2-6t-18}{5t^4-15t^2-20}$

$$\Rightarrow \qquad \left[\frac{dy}{dx}\right]_{t=1}=\frac{12-6-18}{5-15-20}=\frac{2}{5} \quad \Rightarrow \quad \left[5\,\frac{dy}{dx}\right]_{t=1}=2$$

(B) Let us take

$$P(x)=a(x-2)^4+b(x-2)^3+c(x-2)^2+d(x-2)+e$$
$$-1=P(2)=e$$
$$0=P'(2)=d$$
$$2=P''(2)=2c \quad \Rightarrow \quad c=1$$
$$-12=P'''(2)=6b \quad \Rightarrow \quad b=-2$$

Thus, $\quad P''(x)=12(x-2)^2-12(x-2)+2$

$$\Rightarrow \qquad P''(3)=12-12(1)+2=2$$

(C) Here, $\sqrt{(1+y^4)}=\sqrt{\left(1+\dfrac{1}{x^4}\right)}=\dfrac{\sqrt{1+x^4}}{x^2} \qquad \left(\because\ y=\dfrac{1}{x}\right)$

$$\Rightarrow \qquad \frac{\sqrt{1+y^4}}{\sqrt{1+x^4}}=\frac{1}{x^2} \qquad \ldots\text{(i)}$$

But $\qquad y=\dfrac{1}{x} \qquad \therefore \qquad \dfrac{dy}{dx}=-\dfrac{1}{x^2} \qquad \ldots\text{(ii)}$

From Eqs. (i) and (ii), we get

$$\frac{\sqrt{1+y^4}}{\sqrt{1+x^4}}=-\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{\sqrt{(1+y^4)}}+\frac{dx}{\sqrt{(1+x^4)}}=0$$

(D) Obviously, $f(x)$ is a linear function.

Also, from $f'(0)=p$ and $f(0)=q$, $f(x)=px+q \quad \Rightarrow \quad f''(0)=0$

42. (A) We know that,

$$2\tan^{-1}x=\begin{cases}\sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{if } -1\le x\le 1\\ \pi-\sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{if } x>1\\ -\pi-\sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{if } x<-1\end{cases}$$

$$\Rightarrow \qquad \frac{dy}{dx}=-\frac{2}{1+x^2},\ \text{if } x<-1 \quad \text{or} \quad x>1$$

(B) $\cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)=\begin{cases}\tan^{-1}x, & x\ge 0\\ -\tan^{-1}x, & x<0\end{cases} \quad \Rightarrow \quad \dfrac{dy}{dx}=-\dfrac{1}{1+x^2},\ \text{if } x<0$

(C) $y = |e^{|x|} - e| = \begin{cases} |e^x - e|, & x \geq 0 \\ |e^{-x} - e|, & x < 0 \end{cases} = \begin{cases} e^x - e, & x \geq 1 \\ e - e^x, & 0 \leq x < 1 \\ e - e^{-x}, & -1 \leq x < 0 \\ e^{-x} - e, & x < -1 \end{cases}$

$$\Rightarrow \qquad \frac{dy}{dx} > 0, \text{ if } x > 1 \text{ or } -1 < x < 0$$

(D) $\dfrac{dy}{dx} = \dfrac{1}{x}$ and $\dfrac{dv}{dx} = \begin{cases} \dfrac{1}{1+x^2}, & x > 0 \\ -\dfrac{1}{1+x^2}, & x < 0 \end{cases}$

$$\Rightarrow \qquad \frac{du}{dv} = \begin{cases} \dfrac{1+x^2}{x}, & x > 0 \\ -\dfrac{1+x^2}{x}, & x < 0 \end{cases}$$

Now, we know that $\dfrac{1+x^2}{x} = x + \dfrac{1}{x} > 2$, if $x > 1$ and < -2, if $x < -1$

$$\Rightarrow \qquad \frac{du}{dv} > 2, \text{ if } x < -1 \text{ or } x > 1$$

Type 6 : **Integer Answer Type Questions**

43. $A : 2x + 2yy' = 0$

$$\Rightarrow \qquad y' = -\frac{x}{y} \quad \therefore \quad y'(\sqrt{2}) = -1$$

$B : \cos y \cdot y' + \cos x = \sin x \cdot \cos y \cdot y' + \sin y \cdot \cos x$

when $\qquad x = y = \pi$

$$-y' - 1 = 0 + 0 \quad \Rightarrow \quad y'(\pi) = -1$$

$C : 2e^{xy}(xy' + y) + e^x e^y y' + e^y e^x - e^x - e^y y'$

$$= e \cdot e^{xy}(xy' + y)$$

At $\qquad x = 1, y = 1$

$$2e(y' + 1) + e^2 y' + e^2 - e - ey' = e^2(y' + 1)$$

$$ey' + e = 0 \quad \Rightarrow \quad y' = -1$$

Hence, $A - B - C = 1$

44. $\dfrac{dx}{dt} = -\dfrac{3}{t^4} - \dfrac{2}{t^3} = -\left(\dfrac{3 + 2t}{t^4}\right)$

$$\frac{dy}{dt} = -\left(\frac{3}{t^3} + \frac{2}{t^2}\right) = -\left(\frac{3 + 2t}{t^3}\right)$$

$$\therefore \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = t$$

$$\Rightarrow \qquad t - \left(\frac{1+t}{t^3}\right) \cdot t^3 = -1$$

45. $y = f(x) - f(2x) \Rightarrow y' = f'(x) - 2f'(2x)$

$$y'(1) = f'(1) - 2f'(2) = 5 \quad \ldots(i)$$

and $$y'(2) = f'(2) - 2f'(4) = 7 \quad \ldots(ii)$$

Now, let $$y = f(x) - f(4x)$$

$$y' = f'(x) - 4f'(4x)$$

$$y'(1) = f'(1) - 4f'(4) \quad \ldots(iii)$$

Substituting the value of $f'(2) = 7 + 2f'(4)$ in Eq. (i),

$$f'(1) - 2[7 + 2f'(4)] = 5$$

$$f'(1) - 4f'(4) = 19$$

Hence, $$19 = 10 + \lambda \quad \Rightarrow \quad \lambda = 9$$

46. $y = 3e^2 - x$

Let $$x^y = x^{3e^2 - x} \Rightarrow f(x) = x^{3e^2 - x}$$

$$\ln(f(x)) = (3e^2 - x)\ln x \Rightarrow \frac{1}{f(x)} \cdot f'(x) = \frac{3e^2 - x}{x} - \ln x$$

$\therefore$ $$f'(x) = 0 \quad \Rightarrow \quad 3e^2 - x = x \ln x$$

$\Rightarrow$ $$3e^2 = x(1 + \ln x) \quad \Rightarrow \quad x = e^2 \quad \text{(By verification)}$$

Hence, $$\lambda = 1$$

47. $f'(x) = (kx + e^x)h'(x) + h(x)(k + e^x)$

$f'(0) = h'(0) + h(0)(k + 1)$

$18 = -2 + 5(k + 1) \quad \Rightarrow k = 3$

(*Proficiency in 'Differentiation' Exercise* 2)

1. Let $f(x)$ be an even function.

Then, $$f(-x) = f(x)$$

$\Rightarrow$ $$\frac{d}{dx}\{f(-x)\} = \frac{d}{dx}\{f(x)\} \quad \Rightarrow \quad f'(-x) \cdot \frac{d}{dx}(-x) = f'(x)$$

$\Rightarrow$ $$-f'(-x) = f'(x) \quad \Rightarrow \quad f'(-x) = -f'(x)$$

$\Rightarrow f'(x)$ is an odd function.

Let $f(x)$ be an odd function.

Then, $f(-x) = -f(x)$

$\Rightarrow$ $$\frac{d}{dx}\{f(-x)\} = -\frac{d}{dx}\{f(x)\} \quad \Rightarrow \quad f'(-x)\frac{d}{dx}(-x) = -f'(x)$$

$\Rightarrow$ $$-f'(-x) = -f'(x) \quad \Rightarrow \quad f'(-x) = f'(x)$$

$\Rightarrow f'(x)$ is an even function.

2. We have, $y^2 = a^2\cos^2 x + b^2\sin^2 x$

$\Rightarrow$ $$2y^2 = a^2(2\cos^2 x) + b^2(2\sin^2 x)$$

$\Rightarrow$ $$2y^2 = a^2(1 + \cos 2x) + b^2(1 - \cos 2x)$$

$\Rightarrow$ $$2y^2 = (a^2 + b^2) + (a^2 - b^2)\cos 2x \quad \ldots(i)$$

Differentiating w.r.t. x, we get

$$4y\frac{dy}{dx} = -2(a^2 - b^2)\sin 2x$$

$$\Rightarrow \qquad 2y\frac{dy}{dx} = -(a^2 - b^2)\sin 2x \qquad \text{...(ii)}$$

From Eq. (i), we get

$$2y^2 - (a^2 + b^2) = (a^2 - b^2)\cos 2x \qquad \text{...(iii)}$$

Squaring Eqs. (i) and (ii) and adding, we get

$$4y^2\left(\frac{dy}{dx}\right)^2 + \{2y^2 - (a^2 + b^2)\}^2 = (a^2 - b^2)^2$$

$$\Rightarrow \qquad 4y^2\left[\left(\frac{dy}{dx}\right)^2 + y^2 - (a^2 + b^2)\right] = (a^2 - b^2)^2 - (a^2 + b^2)^2$$

$$\Rightarrow \qquad 4y^2\left[\left(\frac{dy}{dx}\right)^2 + y^2 - (a^2 + b^2)\right] = -4a^2b^2$$

$$\Rightarrow \qquad \left(\frac{dy}{dx}\right)^2 + y^2 - (a^2 + b^2) = -\frac{a^2b^2}{y^2}$$

Differentiating both the sides w.r.t. x, we get

$$2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} = \frac{2a^2b^2}{y^3}\frac{dy}{dx} \quad \Rightarrow \quad \frac{d^2y}{dx^2} + y = \frac{a^2b^2}{y^3}$$

3. We have, $y = a(x + \sqrt{x^2 + 1})^n + b(x - \sqrt{x^2 - 1})^{-n}$

$$\therefore \quad \frac{dy}{dx} = na\,[x + \sqrt{x^2 - 1}]^{n-1}\left\{1 + \frac{x}{\sqrt{x^2 - 1}}\right\} - bn\,(x - \sqrt{x^2 - 1})^{-n-1}\left\{1 - \frac{x}{\sqrt{x^2 - 1}}\right\}$$

$$\Rightarrow \qquad \frac{dy}{dx}\sqrt{x^2 - 1} = n[a(x + \sqrt{x^2 + 1})^n + b(x - \sqrt{x^2 - 1})^{-n}]$$

$$\Rightarrow \qquad \sqrt{x^2 - 1}\,\frac{dy}{dx} = ny \quad \Rightarrow \quad (x^2 - 1)\left(\frac{dy}{dx}\right)^2 = n^2y^2$$

Differentiating w.r.t. x, we get

$$2(x^2 - 1)\frac{dy}{dx}\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2 = n^2 2y\frac{dy}{dx}$$

$$\Rightarrow \qquad (x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - n^2y = 0$$

4. We have, $x = \sin\theta$ and $y = \cos p\theta$

$$\therefore \qquad \frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{-p\sin p\theta}{\cos\theta} = \frac{-p\sqrt{1 - \cos^2 p\theta}}{\sqrt{1 - \sin^2\theta}}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{-p\sqrt{1 - y^2}}{\sqrt{1 - x^2}} \quad \Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{p^2(1 - y^2)}{(1 - x^2)}$$

$$\Rightarrow \qquad (1 - x^2)\left(\frac{dy}{dx}\right)^2 = p^2(1 - y^2)$$

Differentiating both the sides w.r.t. x, we get

$$(1-x^2)\,2\frac{dy}{dx}\frac{d^2y}{dx^2}-2x\left(\frac{dy}{dx}\right)^2=p^2\left(0-2y\frac{dy}{dx}\right)$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+p^2y=0$$

5. We have,

$$x^2+y^2+z^2-2xyz=1 \qquad \ldots\text{(i)}$$

$$\Rightarrow \quad d(x^2+y^2+z^2-2xyz)=0$$

$$\Rightarrow \quad 2x\,dx+2y\,dy+2z\,dz-2(xy\,dz+yz\,dx+zx\,dy)=0$$

$$\Rightarrow \quad (x-yz)\,dx+(y-zx)\,dy+(z-xy)\,dz=0$$

$$\Rightarrow \quad \frac{dx}{(y-zx)(z-xy)}+\frac{dy}{(x-yz)(z-xy)}+\frac{dz}{(x-yz)(y-zx)}=0 \qquad \ldots\text{(ii)}$$

Now,

$$\begin{aligned}
&(y-zx)^2(z-xy)^2\\
&=(y^2-2xyz+z^2x^2)(z^2-2xyz+x^2y^2)\\
&=(1-x^2-z^2+z^2x^2)(1-x^2-y^2+x^2y^2) && \text{[Using Eq. (i)]}\\
&=(1-x^2)(1-z^2)(1-x^2)(1-y^2)\\
&=(1-x^2)^2(1-y^2)(1-z^2)
\end{aligned}$$

$$\therefore \quad (y-zx)(z-xy)=(1-x^2)\sqrt{(1-y^2)(1-z^2)}$$

Similarly, $(x-yz)(z-xy)=(1-y^2)\sqrt{(1-x^2)(1-z^2)}$

and $(x-yz)(y-zx)=(1-z^2)\sqrt{(1-x^2)(1-y^2)}$

Substituting these values in Eq. (ii), we get

$$\frac{dx}{\sqrt{1-x^2}}+\frac{dy}{\sqrt{1-y^2}}+\frac{dz}{\sqrt{1-z^2}}=0$$

6. We have, $x=\sin t$ or $t=\sin^{-1}x$

Now,
$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{dy}{dt}\times\frac{1}{\sqrt{1-x^2}} \quad\Rightarrow\quad \sqrt{1-x^2}\,\frac{dy}{dx}=\frac{dy}{dt}$$

Differentiating w.r.t. x, we get

$$\sqrt{1-x^2}\,\frac{d^2y}{dx^2}-\frac{x}{\sqrt{1-x^2}}\frac{dy}{dx}=\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$\Rightarrow \quad \sqrt{1-x^2}\,\frac{d^2y}{dx^2}-\frac{x}{\sqrt{1-x^2}}\frac{dy}{dx}=\frac{d^2y}{dt^2}\frac{dt}{dx}$$

$$\Rightarrow \quad \sqrt{1-x^2}\,\frac{d^2y}{dx^2}-\frac{x}{\sqrt{1-x^2}}\frac{dy}{dx}=\frac{d^2y}{dt^2}\times\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=\frac{d^2y}{dt^2}$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=\frac{d^2y}{dt^2}+y \qquad \text{[Adding } y \text{ on both the sides]}$$

7. Let $y_1 = \dfrac{1}{y^2}$, then

$$y_1{'} = \frac{-2}{y^3}\,y' \qquad \ldots(\text{i})$$

$$\Rightarrow \qquad \frac{y_1{'}}{y_1} = -2\,\frac{y'}{y}$$

Differentiating both the sides w.r.t. x, we get

$$\frac{y_1{''}}{y_1} - \frac{y_1'^2}{y_1^2} = -2\left(\frac{y''}{y} - \frac{y'^2}{y^2}\right) \qquad \ldots(\text{ii})$$

Differentiating w.r.t. x, we get

$$\frac{y_1{'''}}{y_1} - \frac{3\,y_1{'}\,y_1{''}}{y_1^2} + \frac{2y_1'^3}{y_1^3} = -2\left(\frac{y'''}{y} - \frac{3\,y'\,y''}{y^2} + \frac{2y^3}{y^3}\right) \qquad \ldots(\text{iii})$$

Using Eqs. (i), (ii) and (iii) can be written as

$$\frac{y_1{'''}}{y_1{'}} - \frac{y_1{'}}{y_1} = \frac{y''}{y'} - \frac{y'}{y} \qquad \ldots(\text{iv})$$

$$\frac{y'''}{y_1{'}} - \frac{3\,y_1{'''}}{y_1} + \frac{2y_1{'}^2}{y_1^2} = \frac{y'''}{y'} - \frac{3\,y''}{y} + \frac{2y'^2}{y^2} \qquad \ldots(\text{v})$$

Subtracting $\dfrac{3}{2}$ times the square of Eq. (iv) from Eq. (v), we get

$$\frac{y_1{'''}}{y_1{'}} - \frac{3}{2}\left(\frac{y_1{''}}{y_1{'}}\right) + \frac{1}{2}\left(\frac{y_1{'}}{y_1}\right)^2 = \frac{y'''}{y} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2 + \frac{1}{2}\left(\frac{y'}{y}\right)^2$$

This shows that the given expression does not change, if y is replaced by $\dfrac{1}{y^2}$.

8. We have, $\quad f'(x) = \dfrac{1}{\sqrt{1-x^2}}$

Therefore, $\quad f'(x\sqrt{1-a^2} + a\sqrt{1-x^2})$

$$= \frac{1}{\sqrt{1-\{x\sqrt{1-a^2} + a\sqrt{1-x^2}\}^2}} = \frac{1}{\sqrt{\{\sqrt{1-a^2}\sqrt{1-x^2} - 9x\}^2}}$$

$$= \frac{1}{\sqrt{(1-a^2)(1-x^2)}} - ax$$

$$\Rightarrow \qquad \frac{\sqrt{(1-a^2)(1-x^2)} - ax}{\sqrt{1-x^2}}\,f'(x\sqrt{1-a^2} + a\sqrt{1-x^2}) = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \qquad \frac{d}{dx}\{f(x\sqrt{1-a^2} + a\sqrt{1-x^2})\} = \frac{d}{dx}\{f(x)\} \qquad \left[\because f'(x) = \frac{1}{\sqrt{1-x^2}}\right]$$

$$\therefore \qquad f(x\sqrt{1-a^2} + a\sqrt{1-x^2}) = f(x) + C$$

Putting $x = 0$, we get

$$f(a) = f(0) + C \quad \Rightarrow \quad C = f(a) \qquad [\because f(0) = 0]$$

$$\therefore \qquad f(x\sqrt{1-a^2} + a\sqrt{1-x^2}) = f(x) + f(a)$$

9. We have, $x = \cos\theta$

$\Rightarrow \quad \dfrac{dx}{d\theta} = -\sin\theta$

Now, $\quad \dfrac{dy}{dx} = \dfrac{dy}{d\theta} \cdot \dfrac{d\theta}{dx}$

$\Rightarrow \quad \dfrac{dy}{dx} = -\text{cosec}\,\theta \cdot \dfrac{dy}{d\theta}$ and $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$

$$= \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx} = \frac{d}{d\theta}\left\{-\text{cosec}\,\theta\,\frac{dy}{d\theta}\right\}\cdot\frac{d\theta}{dx}$$

$$= \left\{\text{cosec}\,\theta\cot\theta\,\frac{dy}{d\theta} - \text{cosec}\,\theta\,\frac{d^2y}{d\theta^2}\right\} \times (-\text{cosec}\,\theta)$$

$$= -\text{cosec}^2\,\theta\cot\theta\,\frac{dy}{d\theta} + \text{cosec}^2\,\theta\,\frac{d^2y}{d\theta^2}$$

Substituting the values of x, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in

$$(1-x^2)\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} + y = 0, \text{ we get}$$

$$(1-\cos^2\theta)\left\{-\text{cosec}^2\,\theta\cot\theta\,\frac{dy}{d\theta} + \text{cosec}^2\,\theta\,\frac{d^2y}{d\theta^2}\right\}$$

$$-\cos\theta\left\{-\text{cosec}\,\theta\,\frac{dy}{d\theta}\right\} + y = 0$$

$\Rightarrow \quad -\cot\theta\,\dfrac{dy}{d\theta} + \dfrac{d^2y}{d\theta^2} + \cot\theta\,\dfrac{dy}{d\theta} + y = 0 \;\Rightarrow\; \dfrac{d^2y}{d\theta^2} + y = 0$

10. We have, $z = \log\tan\dfrac{x}{2}$

$\Rightarrow \quad \dfrac{dz}{dx} = \dfrac{1}{\tan\dfrac{x}{2}} \times \sec^2\dfrac{x}{2} \times \dfrac{1}{2} = \text{cosec}\,x$

Now, $\quad \dfrac{dy}{dx} = \dfrac{dy}{dz}\cdot\dfrac{dz}{dx} \;\Rightarrow\; \dfrac{dy}{dx} = \text{cosec}\,x\,\dfrac{dy}{dx}$

and $\quad \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}\left\{\text{cosec}\,x\,\dfrac{dy}{dx}\right\}$

$$= -\text{cosec}\,x\cot x\,\frac{dy}{dz} + \text{cosec}\,x\,\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$$

$$= -\text{cosec}\,x\cot x\,\frac{dy}{dz} + \text{cosec}^2\,x\,\frac{d^2y}{dz^2}$$

$\therefore \quad \dfrac{d^2y}{dx^2} + \cot x\,\dfrac{dy}{dx} + 4y\,\text{cosec}^2\,x = 0$

$\Rightarrow \quad \left\{-\text{cosec}\,x\cot x\,\dfrac{dy}{dx} + \text{cosec}^2\,x\,\dfrac{d^2y}{dz^2}\right\} + \cot x\,\text{cosec}\,x\,\dfrac{dy}{dx} + 4y\,\text{cosec}^2\,x = 0$

$\Rightarrow \quad \text{cosec}^2\,x\,\dfrac{d^2y}{dz^2} + 4y\,\text{cosec}^2\,x = 0$

$\Rightarrow \quad \dfrac{d^2y}{dz^2} + 4y = 0$

3

Functions

Chapter in a Snapshot

- Functions
- Number of Functions (or Mapping) from *A* to *B*
- Graphical Representation of Functions
- Domain
- Classification of Functions
- Some Special Functions
- Range
- General Definitions
- Odd and Even Functions
- Periodic Functions
- Composite Functions
- Mapping of Functions
- Equal and Identical Functions
- Inverse of a Function
- Graph of the Inverse of an Invertible Function

Functions

Let A and B be two non-empty sets and f be a relation which associates each element of set A with unique element of set B, then f is called a function from A to B.

Here, set A is called the domain of f and B is the co-domain of f.

The set of elements of B, which are the images of the elements of set A is called the range of f.

OR

If for each element in a set A there is assigned, a unique element of a set B, then we call such an assignment a function.

$f : A \to B$. "f is a function of A into B".

The set A is called the domain of the function f and B is called the co-domain of f. If $a \in A$, then the element in B which is assigned to 'a' is called the image of 'a' and denoted by $f(a)$.

eg, Let $A = \{a, b, c, d\}$, $\quad B = \{1, 2, 3, 4, 5\}$

Here, $f(a) = 2$, $\quad f(b) = 3$, $\quad f(c) = 5$, $\quad f(d) = 1$.

given by

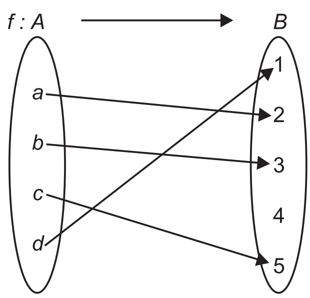


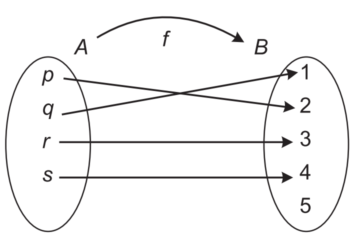
Fig. 3.1

ie, $\quad A \to$ Domain of $f = \{a, b, c, d\}$

$B \to$ Co-domain of $f = \{1, 2, 3, 4, 5\}$

Range of $f = \{1, 2, 3, 5\}$

Illustration 1 In the given figure find the domain, co-domain and range.



Solution. From the given figure, we conclude that

$A \rightarrow$ Domain of $f = \{p, q, r, s\}$

$B \rightarrow$ Co-domain of $f = \{1, 2, 3, 4, 5\}$

Range of $f = \{1, 2, 3, 4\}$

We should always remember "the range is a subset of co-domain".

Number of Functions (or Mapping) from *A* to *B*

Let $A = \{x_1, x_2, x_3, \ldots, x_m\}$ (*ie*, m elements)

and $B = \{y_1, y_2, y_3, \ldots, y_n\}$ (*ie*, n elements)

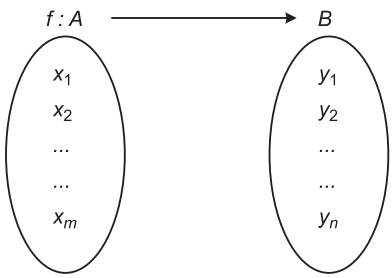


Fig. 3.2

Then, each element in domain x_i's ($i = 1, 2, \ldots, m$) corresponds n images.

ie, x_1 can take n images.

x_2 can take n images.

....

....

x_m can take n images.

Thus, total number of functions from A to B

$\Rightarrow$ $n \times n \times \ldots \times m$ times $= n^m$

ie, (Number of elements in co-domain) $P^{\text{number of elements in domain}}$.

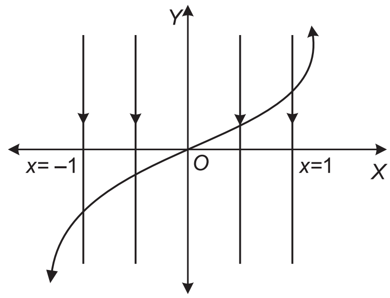
Graphical Representation of Functions

Let f be a mapping with domain D such that $y = f(x)$ should assume single value for each x.

(*ie*, **The straight line drawn parallel to *y*-axis in its domain should cut at only one point**).

Illustration 2 Find whether $f(x) = x^3$ forms a mapping or not.

Solution. $y = f(x) = x^3, \forall x \in R.$



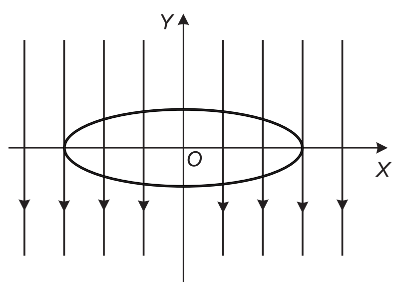
Here, all the straight lines drawn parallel to y-axis cut $y = x^3$ only at one point. Thus, $y = f(x)$ forms a mapping.

Illustration 3 Find whether $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ forms a mapping or not.

Solution. Let us consider an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

ie,
$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$



Here, straight lines drawn parallel to y-axis meet the curve at more than one points. Thus, $f(x) = \pm \frac{b}{a}\sqrt{a^2 - x^2}$ doesn't form a mapping.

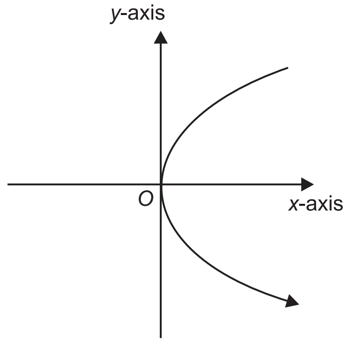
In general, we could say

1. An element of A (*ie*, domain) could not associate with more than one element in B.
2. If graph of a function is plotted and any line parallel to y-axis cuts it at more than one points, then it doesn't form a function.

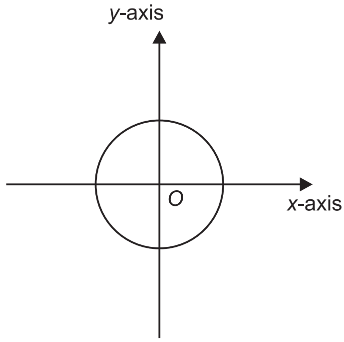
Target Exercise 3.1

1. Which of the following graphs are graphs of a function ?

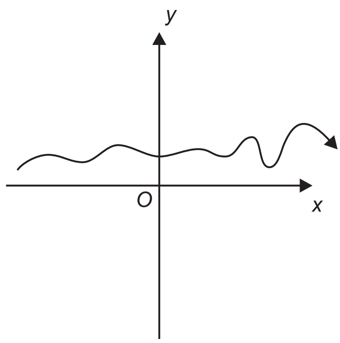
(i)



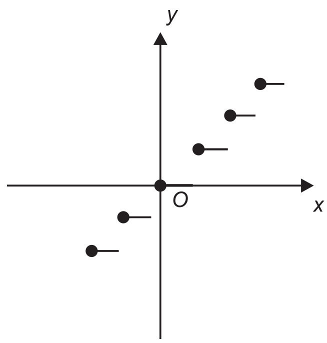
(ii)



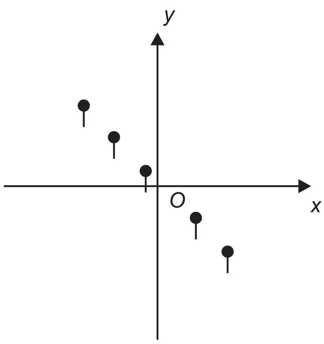
(iii)



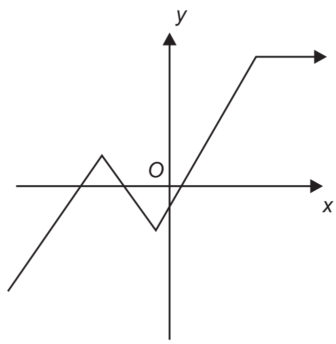
(iv)



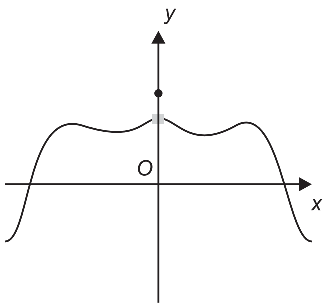
(v)



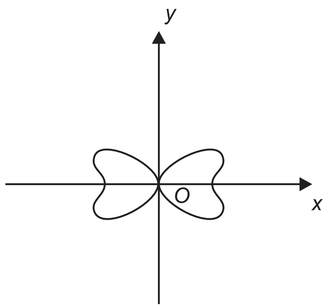
(vi)

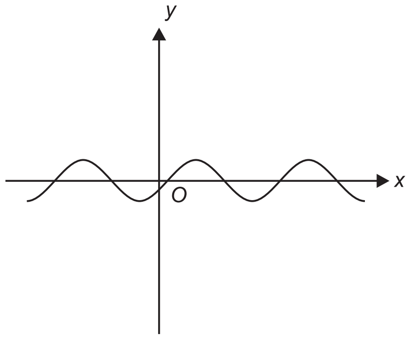


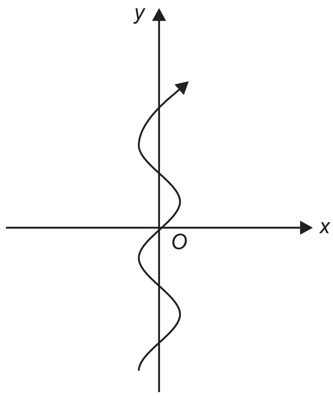
(vii)



(viii)



(ix) 

(x) 

2. For which of the following, y can be a function of x, $(x \in R,\ y \in R)$?

(i) $(x-h)^2 + (y-k)^2 = r^2$ (ii) $y^2 = 4ax$
(iii) $x^4 = y^2$ (iv) $x^6 = y^3$
(v) $3y = (\log x)^2$

Domain

The domain $y = f(x)$ is the set of all real x for which $f(x)$ is defined (real).

Rules for Finding Domain

(i) Expression under even root (*ie*, square root, fourth root etc.) ≥ 0

(ii) Denominator $\neq 0$.

(iii) If domain of $y = f(x)$ and $y = g(x)$ are D_1 and D_2 respectively, then the domain of $f(x) \pm g(x)$ or $f(x) \cdot g(x)$ is $D_1 \cap D_2$.

(iv) While, domain of $\dfrac{f(x)}{g(x)}$ is $D_1 \cap D_2 - \{g(x) = 0\}$.

Point to Consider

For different functions there are different conditions for the domain explained ahead.

For example : $\log_a x$ is defined, if $x, a > 0$ and $a \neq 1$.

Classification of Functions

Constant Function

If the range of a function f consists of only one number, then f is called a constant function. The graph of a constant function is as shown adjacent.

eg, let $f(x) = 3$, where 3 is constant number and thus constant function, it` is all to one correspondence.

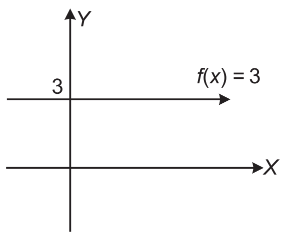


Fig. 3.3

Polynomial Functions or Rational Integral Functions

A function $y = f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$, where $a_0, a_1, a_2, \ldots, a_n$ are real constants and n is a non-negative integer, then $f(x)$ is called a polynomial.

If $a_0 \neq 0$, then n is the degree of polynomial function.

For example :

Expression	Degree
$f(x) = x^{1920} + 5x^{1919} + 6x$	polynomial of degree 1920
$g(x) = x^2 + 3x + 3$	polynomial of degree 2
$h(x) = 7 = 7x^0$	polynomial of degree 0

Points to Consider

(a) A polynomial of degree one with no constant term is called an odd linear function. ie, $f(x) = ax, a \neq 0$.
In case $f(x) = 0$, it is a constant function, degree is not defined.

(b) There are two polynomial functions, satisfying the relation;
$f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$. They are :
(i) $f(x) = 1 + x^n$ and (ii) $f(x) = 1 - x^n$,
where n is a positive integer.
For proof, refer Example 11. in Worked Examples.

(c) A polynomial of odd degree has its range $(-\infty, \infty)$ but a polynomial of even degree has a range which is always subset of R.

Algebraic Functions

A function f is called an algebraic function, if it can be constructed using algebraic operations such as addition, subtraction, multiplication, division and taking roots, starting with polynomials.

eg, $\quad f(x) = \sqrt{1+x}, \; g(x) = \dfrac{x^3 - 16x}{x + \sqrt{x}} + (x-2)\sqrt[3]{x-1}$

Point to Consider

All polynomials are algebraic but not the converse. Functions which are not algebraic are known as transcendental functions.

Rational Functions

It is defined as the ratio of two polynomials.

Let
$$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$$
$$Q(x) = b_0x^m + b_1x^{m-1} + \ldots + b_m.$$

Then, $f(x) = \dfrac{P(x)}{Q(x)}$ is a rational function [provided $Q(x) \neq 0$].

Here, we can say that domain of $f(x)$ is all real numbers except when denominator is zero [*ie*, $Q(x) \neq 0$].

eg, $$f(x)=\frac{x^2-3x+5}{(x-1)(x-2)};\qquad \text{domain} \in R-\{1, 2\}$$

eg, $$f(x)=\frac{1}{2-\cos 3x};\qquad \text{domain} \in \text{real number}$$

as $$2-\cos 3x \neq 0. \qquad (\text{as } -1 \le \cos 3x \le 1)$$

Irrational Functions

In $y = f(x)$, operations of addition, subtraction, multiplication, division and raising to a power with **non-integral** rational exponent are called irrational functions.

eg, $$f(x)=x^{1/2},$$

eg, $$f(x)=\frac{x^2+\sqrt{x}}{\sqrt{1+3x^2}}, \text{ etc.}$$

Identity Function

The function $y = f(x) = x$ for all $x \in R$ is called an identity function on R.

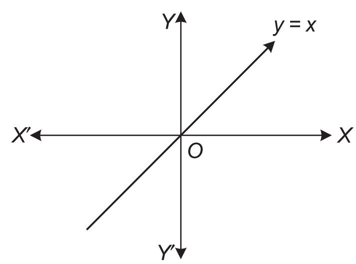


Fig. 3.4

Here, domain of an identity function is all the real number. (As here for all values of x, $f(x)$ exists.)

Thus, domain $\in R$ and range $\in R$.

Exponential Functions

The function $f(x) = a^x$, $a > 0$, $a \neq 1$, $a \in$ constant is said to be an exponential function.

If $0 < a < 1$

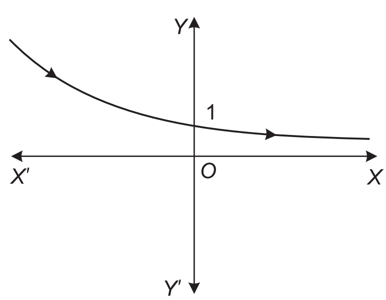


Fig. 3.5

If $a > 1$

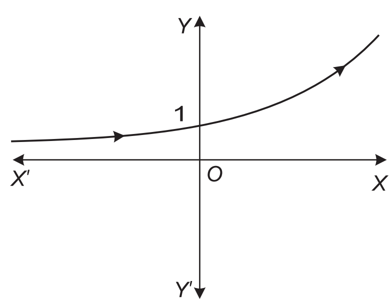


Fig. 3.6

In the given graphs, none of them intersect the x-axis.

Thus, we can say that they exist for all $x \in$ real number and their corresponding values of y are greater than zero.

Thus, Domain $\in$ Real number; Range $\in]\,0, \infty[$.

Logarithmic Functions

The function $f(x) = \log_a x$; $(x, a > 0)$ and $a \neq 1$ is a logarithmic function.

Thus, domain of a logarithmic function is all real positive numbers and their range is the set R of all real numbers.

If $0 < a < 1$

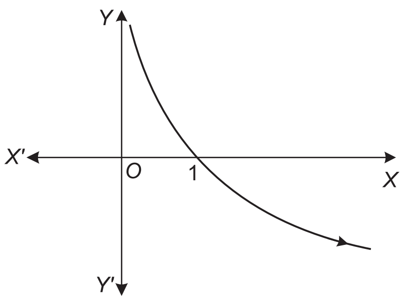


Fig. 3.7

If $a > 1$

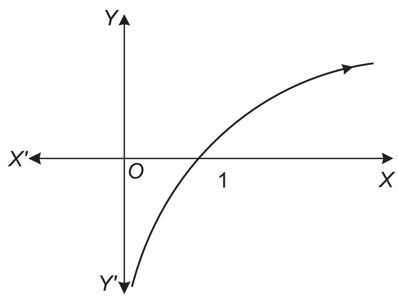


Fig. 3.8

Properties of Logarithmic Functions

1. $\log_e(ab) = \log_e a + \log_e b$ $\quad (a, b > 0)$
2. $\log_e\left(\dfrac{a}{b}\right) = \log_e a - \log_e b$ $\quad (a, b > 0)$
3. $\log_e a^m = m \log_e a$ $\quad (a > 0, m \in R)$
4. $\log_a a = 1$ $\quad (a > 0, a \neq 1)$
5. $\log_{b^m} a = \dfrac{1}{m} \log_b a$ $\quad (a, b > 0, b \neq 1$ and $m \in R - \{0\})$
6. $\log_b a = \dfrac{1}{\log_a b}$ $\quad (a, b > 0)$ and $a, b \neq \{1\}$
7. $\log_b a = \dfrac{\log_m a}{\log_m b}$ $\quad (a, b, m > 0 \neq \{1\})$
8. $a^{\log_a m} = m$ $\quad (a > 0 \neq \{1\}, m > 0)$
9. $a^{\log_c b} = b^{\log_c a}$ $\quad (a, b, c > 0$ and $c \neq 1)$
10. If $\log_m x > \log_m y \Rightarrow \begin{cases} x > y, & \text{if } m > 1 \\ x < y, & \text{if } 0 < m < 1 \end{cases}$ $\quad (m, x, y > 0, m \neq 1)$
11. $\log_m a = b \Rightarrow a = m^b$ $\quad (m, a > 0, m \neq 1, b \in$ real number$)$
12. $\log_m a > b \Rightarrow \begin{cases} a > m^b, & \text{if } m > 1 \\ a < m^b, & \text{if } 0 < m < 1 \end{cases}$
13. $\log_m a < b \Rightarrow \begin{cases} a < m^b, & \text{if } m > 1 \\ a > m^b, & \text{if } 0 < m < 1 \end{cases}$

Illustration 4 Find the domain of a single-valued function $y = f(x)$ given by the equation $10^x + 10^y = 10$.

Solution. Given, $10^x + 10^y = 10$

$\Rightarrow \quad 10^y = 10 - 10^x$

$\Rightarrow \quad y = \log_{10}(10 - 10^x) \qquad$ (as $a^m = b \Rightarrow m = \log_a b$)

Now, $\quad 10^1 - 10^x > 0$

$\Rightarrow \quad 10^1 > 10^x \quad \Rightarrow \quad 1 > x$

Therefore, domain of the single-valued function

$$y = f(x) \quad \text{is} \quad x < 1$$

or $\quad x \in (-\infty, 1)$

Illustration 5 Find the domain of $f(x) = \dfrac{1}{\sqrt{\log_{1/2}(x^2 - 7x + 13)}}$.

Solution. $f(x) = \dfrac{1}{\sqrt{\log_{1/2}(x^2 - 7x + 13)}}$ exists, if

$$\log_{1/2}(x^2 - 7x + 13) > 0$$

$\Rightarrow \quad (x^2 - 7x + 13) < 1 \qquad$ ...(i)

and $\quad x^2 - 7x + 13 > 0 \qquad$ ...(ii)

Considering Eq. (ii), $x^2 - 7x + 13 > 0$, we have

$$\left(x^2 - 7x + \frac{49}{4}\right) + 13 - \frac{49}{4} > 0$$

$\Rightarrow \quad \left(x - \dfrac{7}{2}\right)^2 + \dfrac{3}{4} > 0$

Which is true for all $x \in R$.

As $\quad \left(x - \dfrac{7}{2}\right)^2 \geq 0$ for all x. $\qquad$ ...(a)

Again, taking Eq. (i), $x^2 - 7x + 13 < 1$

$\Rightarrow \quad x^2 - 7x + 12 < 0 \quad \Rightarrow \quad (x - 3)(x - 4) < 0$

$\Rightarrow \quad 3 < x < 4 \qquad$ ...(b)

Combining Eqs. (a) and (b), we have

Hence, domain of $f(x) \in (3, 4)$ or $]3, 4[$.

Illustration 6 Find the domain of function; $f(x) = \dfrac{1}{\log_{10}(1 - x)} + \sqrt{(x + 2)}$.

Solution. $f(x) = \dfrac{1}{\log_{10}(1 - x)} + \sqrt{x + 2}$

(As we know, $\log_a x$ is defined when $x, a > 0$ and $a \neq 1$, also $\log_a 1 = 0$)

Thus, $\log_{10}(1 - x)$ exists when $\quad 1 - x > 0 \qquad$ ...(i)

Also, $\dfrac{1}{\log_{10}(1-x)}$ exists when $\quad 1-x>0$ and $1-x \neq 1$ ...(ii)

$\Rightarrow \quad x<1 \quad$ and $\quad x \neq 0$ ...(iii)

Also, we have $\sqrt{x+2}$ existing when $\quad x+2 \geq 0$

or $\quad x \geq -2$ ...(iv)

Thus, $f(x)=\dfrac{1}{\log_{10}(1-x)}+\sqrt{x+2}$ exists when Eqs. (iii) and (iv) both hold true.

$\Rightarrow \quad -2 \leq x<1$ and $x \neq 0$

$\Rightarrow \quad x \in[-2,0) \cup(0,1)$

Illustration 7 Find the domain of the function,

$$f(x)=\log \left\{\log_{|\sin x|}(x^2-8x+23)-\frac{3}{\log_2|\sin x|}\right\}.$$

Solution. Here, $f(x)$ is defined, if

$$\log_{|\sin x|}(x^2-8x+23)-\frac{3}{\log_2|\sin x|}>0$$

$\Rightarrow \quad \log_{|\sin x|}(x^2-8x+23)-3\log_{|\sin x|}2>0$

$\Rightarrow \quad \log_{|\sin x|}\left(\dfrac{x^2-8x+23}{8}\right)>0$

$\Rightarrow \quad \dfrac{x^2-8x+23}{8}<|\sin x|^0$, and $\quad |\sin x| \neq 0, 1$

$\Rightarrow \quad x^2-8x+23<8$, and $\quad |\sin x| \neq 0, 1$

$\Rightarrow \quad x^2-8x+15<0$, and $\quad \sin x \neq 0, \pm 1$

$\Rightarrow \quad (x-3)(x-5)<0$, and $\quad x \neq n\pi, (2n+1)\dfrac{\pi}{2}$

$\Rightarrow \quad x \in(3,5)$, and $\quad x \neq \pi, \dfrac{3\pi}{2}$

(using number line rule)

$\Rightarrow \quad x \in(3, \pi) \cup\left(\pi, \dfrac{3\pi}{2}\right) \cup\left(\dfrac{3\pi}{2}, 5\right)$

$\therefore \quad$ Domain of $f(x) \in(3, \pi) \cup\left(\pi, \dfrac{3\pi}{2}\right) \cup\left(\dfrac{3\pi}{2}, 5\right)$

Illustration 8 Find domain of $f(x)=\log_{10}(1+x^3)$.

Solution. $f(x)=\log_{10}(1+x^3)$ exists,

if $\quad 1+x^3>0$

$\Rightarrow \quad (1+x)(1-x+x^2)>0$, $\quad$ [where $(1-x+x^2)$ is always positive as $D<0$ and $a>0$]

So, $\quad 1+x>0$

or $\quad x>-1 \quad$ or $\quad x \in]-1, \infty[$

Thus, domain of above function $f(x) \in]-1, \infty[$

Illustration 9 Find domain of $f(x) = \log_{10}\log_{10}(1 + x^3)$.

Solution. $f(x) = \log_{10}\log_{10}(1 + x^3)$ exists

if $\log_{10}(1 + x^3) > 0$

or $1 + x^3 > 10^0$ or $1 + x^3 > 1$

or $x^3 > 0$

or $x \in (0, \infty)$

Thus, domain of above function $f(x)$ exists, if $x > 0$.

Illustration 10 $f(x) = \log_{10}\{\log_{10}\log_{10}\log_{10} x\}$ exists for x.

Solution. $f(x)$ exists, if

$$\{\log_{10}\log_{10}\log_{10} x\} > 0 \quad \text{and} \quad x > 0$$

$$\Rightarrow \log_{10}\log_{10} x > 10^0 \quad \text{and} \quad x > 0$$

$$\Rightarrow \log_{10}\log_{10} x > 1 \quad \text{and} \quad x > 0$$

$$\Rightarrow \log_{10} x > 10^1 \quad \text{and} \quad x > 0$$

$$\Rightarrow x > 10^{10} \quad \text{and} \quad x > 0$$

Thus, $f(x)$ exists, if $x \in (10^{10}, \infty)$.

Therefore, domain of above function $f(x) \in (10^{10}, \infty)$.

Illustration 11 Find the domain for $f(x) = \sqrt{\log_{0.4}\left(\dfrac{x-1}{x+5}\right)}$.

Solution. $f(x) = \sqrt{\log_{0.4}\left(\dfrac{x-1}{x+5}\right)}$ exists, if

$$\log_{0.4}\left(\frac{x-1}{x+5}\right) \geq 0 \quad \text{and} \quad \left(\frac{x-1}{x+5}\right) > 0$$

$$\Rightarrow \frac{x-1}{x+5} \leq (0.4)^0 \quad \text{and} \quad \frac{x-1}{x+5} > 0$$

$$\Rightarrow \frac{x-1}{x+5} \leq 1 \quad \text{and} \quad \frac{x-1}{x+5} > 0$$

$$\Rightarrow \frac{x-1}{x+5} - 1 \leq 0 \quad \text{and} \quad \frac{x-1}{x+5} > 0$$

$$\Rightarrow \frac{-6}{x+5} \leq 0 \quad \text{and} \quad \frac{x-1}{x+5} > 0$$

$$\Rightarrow x+5 > 0 \quad \text{and} \quad \frac{x-1}{x+5} > 0$$

$$\Rightarrow x > -5 \quad \text{and} \quad x - 1 > 0 \qquad (\text{as } x+5 > 0)$$

$$\Rightarrow x > -5 \quad \text{and} \quad x > 1$$

Thus, domain $f(x) \in (1, \infty)$.

Illustration 12 $f(x)=\log_{100x}\left(\dfrac{2\log_{10}x+1}{-x}\right)$ exists, if .... .

Solution. $f(x)$ exists, if $100x>0$ ...(i)

$100x \neq 1$ ...(ii)

$\dfrac{2\log_{10}x+1}{-x}>0$ ...(iii)

$2\log_{10}x+1<0$ (as $x>0$)

$\log_{10}x<-\dfrac{1}{2}$

$x<(10)^{-1/2}$ ... (iv)

Thus, from Eqs. (i), (ii), (iv), we have

$$x \in (0,\ 10^{-2}) \cup (10^{-2},\ 10^{-1/2}).$$

Illustration 13 Find the domain of definition of $f(x)=\dfrac{\log_2(x+3)}{x^2+3x+2}$.

[IIT JEE 2001]

Solution. Here, $f(x)=\dfrac{\log_2(x+3)}{x^2+3x+2}=\dfrac{\log_2(x+3)}{(x+1)(x+2)}$ exists, if

Numerator $x+3>0$

$\Rightarrow$ $x>-3$ ...(i)

and denominator $(x+1)(x+2)\neq 0$

$\Rightarrow$ $x\neq -1,-2$ ...(ii)

Thus, from Eqs. (i) and (ii), we have domain of $f(x)$ as $(-3,\infty)-\{-1,-2\}$.

Trigonometric Functions

Sine Function

$$f(x)=\sin x$$

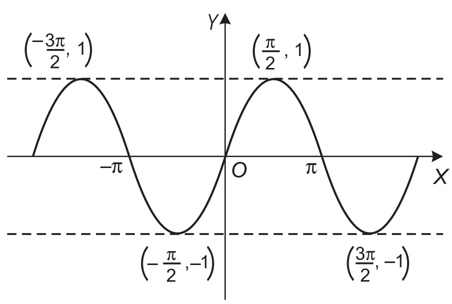


Fig. 3.9

The domain of sine function is R and the range is $[-1, 1]$.

Cosine Function

$$f(x) = \cos x$$

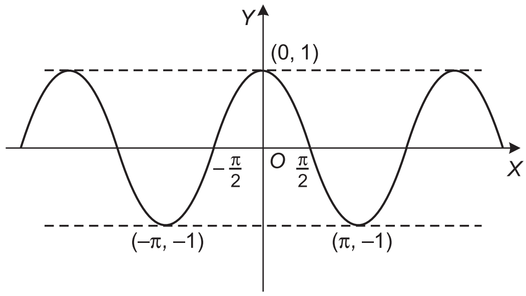


Fig. 3.10

The domain of cosine function is R and the range is $[-1, 1]$.

Tangent Function

$$f(x) = \tan x$$

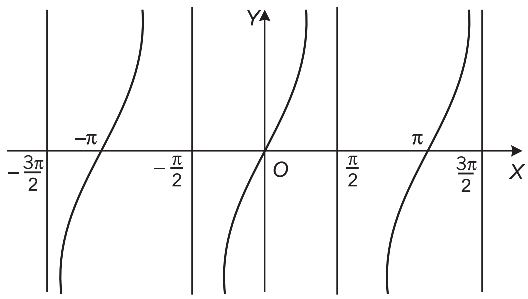


Fig. 3.11

Here, domain of tangent function is $R - \left\{\frac{(2n+1)\pi}{2}, n \in Z\right\}$ and range is R.

Cosecant Function

$$f(x) = \text{cosec } x$$

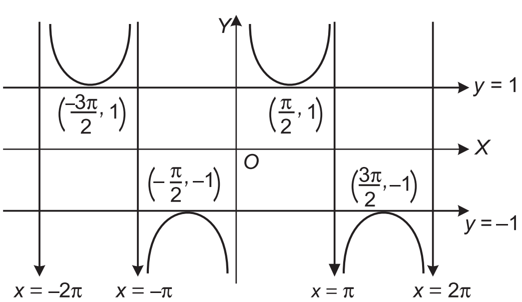


Fig. 3.12

Here, domain $\in R - \{n\pi / n \in Z\}$, and range $\in R - (-1, 1)$.

Secant Function

$f(x) = \sec x$

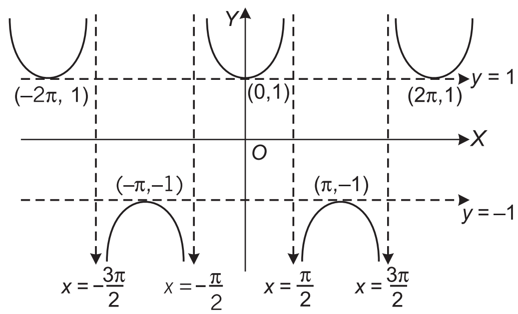


Fig. 3.13

Here, domain $\in R - \{(2n+1)\dfrac{\pi}{2} \mid n \in Z\}$; range $\in R - (-1, 1)$.

Cotangent Function

$f(x) = \cot x$

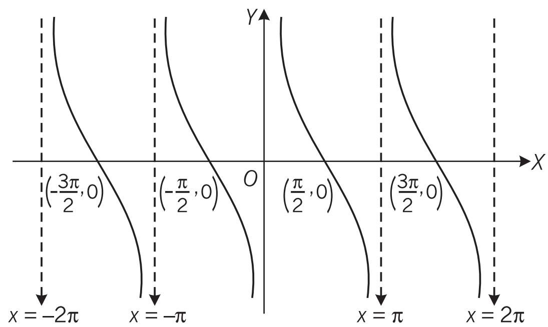


Fig. 3.14

Here, domain $\in R - \{n\pi / n \in Z\}$; range $\in R$.

Inverse of Trigonometric Function

$y = \sin^{-1} x$

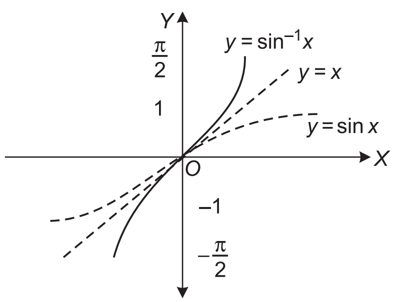


Fig. 3.15

Here, domain $\in [-1, 1]$; range $\in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

$y = \cos^{-1} x$

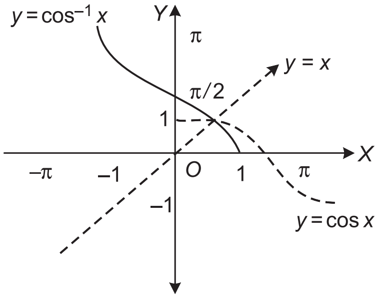


Fig. 3.16

Here, domain $\in [-1, 1]$; range $\in [0, \pi]$.

$y = \tan^{-1} x$

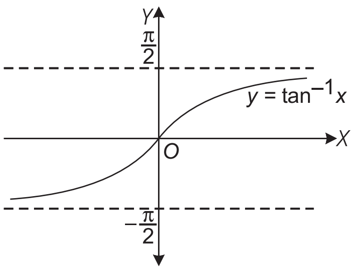


Fig. 3.17

Here, domain $\in R$; range $\in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

$y = \cot^{-1} x$

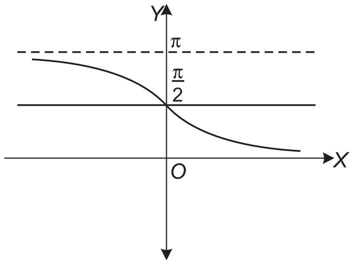


Fig. 3.18

Here, domain $\in R$; range $\in (0, \pi)$.

$\boldsymbol{y = \text{cosec}^{-1}\, x}$

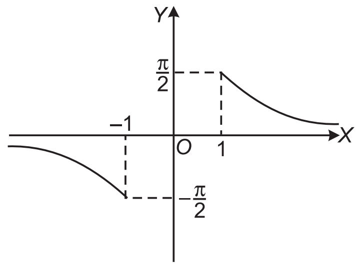


Fig. 3.19

Here, domain $\in R - (-1, 1)$; range $\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

$\boldsymbol{y = \sec^{-1} x}$

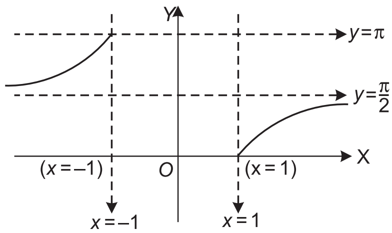


Fig. 3.20

Here, domain $\in R - (-1, 1)$; range $\in [0, \pi] - \{\pi/2\}$.

Illustration 14 Find the domain for $f(x) = \sin^{-1}\left(\frac{x^2}{2}\right)$.

Solution. $f(x) = \sin^{-1}\left(\frac{x^2}{2}\right)$ is defined, if

$$-1 \le \frac{x^2}{2} \le 1 \quad \text{or} \quad -2 \le x^2 \le 2$$

$\Rightarrow$ $0 \le x^2 \le 2$ (as x^2 cannot be negative)

$\Rightarrow$ $-\sqrt{2} \le x \le \sqrt{2}$

Therefore, domain of $f(x) \in [-\sqrt{2}, \sqrt{2}]$

Illustration 15 Find the domain for $y = \sin^{-1}\left(\log_2\left(\frac{x^2}{2}\right)\right)$.

Solution. For y to be defined, $\frac{x^2}{2} > 0$ ...(i)

$$-1 \le \log_2\left(\frac{x^2}{2}\right) \le 1 \qquad \text{...(ii)}$$

From Eq. (i), we have $x \in R - \{0\}$ ...(iii)

From Eq. (ii), we have $2^{-1} \le \frac{x^2}{2} \le 2^1 \Rightarrow 1 \le x^2 \le 4$

$$\Rightarrow \qquad -2 \le x \le -1 \quad \text{or} \quad 1 \le x \le 2 \qquad \text{...(iv)}$$

Thus, from Eqs. (iii) and (iv), we have

$$x \in [-2, -1] \cup [1, 2]$$

Illustration 16 Find domain for $f(x) = \sqrt{\cos(\sin x)}$.

Solution. $f(x) = \sqrt{\cos(\sin x)}$ is defined, if

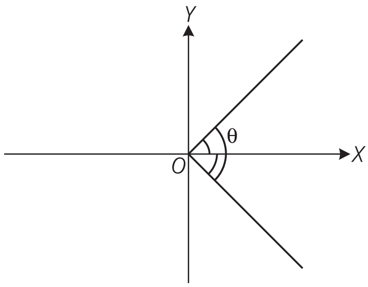
$$\cos(\sin x) \ge 0 \qquad \text{...(i)}$$

As, we know

$$-1 \le \sin x \le 1 \qquad \text{for all } x$$

$$\cos\theta \ge 0$$

(Here, $\theta = \sin x$ lies in the 1st and 4th quadrants)



ie, $\cos(\sin x) \ge 0, \quad$ for all x

ie, $x \in R$

Thus, domain $f(x) \in R$.

Illustration 17 Find domain for $f(x) = \sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$.

Solution. $f(x) = \sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$ is defined, for

$$-1 \le \frac{1+x^2}{2x} \le 1 \quad \text{or} \quad \left|\frac{1+x^2}{2x}\right| \le 1$$

$$\Rightarrow \qquad |1+x^2| \le |2x|, \qquad \text{for all } x$$

$$\Rightarrow \qquad 1+x^2 \le |2x|, \qquad \text{for all } x \qquad (\text{as } 1+x^2 > 0)$$

$$\Rightarrow \qquad x^2 - 2|x| + 1 \le 0$$

$$\Rightarrow \qquad |x|^2 - 2|x| + 1 \le 0 \qquad (\text{as } x^2 = |x|^2)$$

$$\Rightarrow \qquad (|x| - 1)^2 \le 0$$

But $(|x| - 1)^2$ is either always positive or zero.

Thus, $\qquad (|x| - 1)^2 = 0$

$$\Rightarrow \qquad |x| = 1 \quad \Rightarrow \quad x = \pm 1$$

Thus, domain for $f(x)$ is $\{-1, 1\}$.

Illustration 18 Find the domain for $y = \sqrt{\sin^{-1}(\log_2 x)}$.

Solution. y is defined, if $x > 0$ ...(i)

$$-1 \le \log_2 x \le 1 \quad \text{...(ii)}$$

$\Rightarrow$ $$2^{-1} \le x \le 2$$

$\Rightarrow$ $$\frac{1}{2} \le x \le 2$$

and $$\sin^{-1}(\log_2 x) \ge 0 \quad \text{...(iii)}$$

$\Rightarrow$ $$\log_2 x \ge 0$$

$\Rightarrow$ $$x \ge 2^0 \quad \Rightarrow \quad x \ge 1 \quad \text{...(iv)}$$

From Eqs. (i), (ii), (iv), we have

$$1 \le x \le 2$$

Hence, domain is $1 \le x \le 2$ or $x \in [1, 2]$.

Some Special Functions

Absolute Value Function (or Modulus Function)

$$y = |x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$$

It is the numerical value of x.

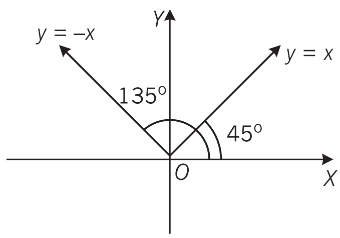


Fig. 3.21

Geometrical Interpretation of Modulus of a Function

Geometrically, $|x|$ represents the distance of the point $P(x, 0)$ or $Q(-x, 0)$ from origin.

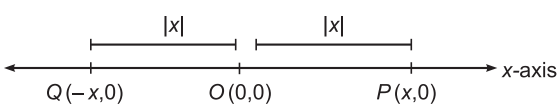


Fig. 3.22

ie, $$d(O, P) = \sqrt{(x-0)^2 + (0-0)^2} = \sqrt{x^2} = |x|$$

$$d(O, Q) = \sqrt{(-x-0)^2 + (0-0)^2} = \sqrt{x^2} = |x|$$

Properties of Modulus

(i) $|x|^2 = x^2$

(ii) $\sqrt{x^2} = |x|$

(iii) $||x|| = |-x| = |x|$

(iv) $|x| = \max\{-x, x\}$

(v) $-|x| = \min\{-x, x\}$

(vi) $\max(a, b) = \dfrac{a+b}{2} + \left|\dfrac{a-b}{2}\right|$

(vii) $\min(a, b) = \dfrac{a+b}{2} - \left|\dfrac{a-b}{2}\right|$

(viii) $|x+y| \le |x| + |y|$

(ix) $|x+y| = |x| + |y|$, iff $xy \ge 0$

(x) $|x-y| = |x| + |y|$, iff $xy \le 0$

(xi) $|x| \le a$, (a is +ve)

$\Rightarrow \quad -a \le x \le a$ or $x \in [-a, a]$

(xii) $|x| \ge a$, (a is +ve)

$\Rightarrow \quad x \le -a$ or $x \ge a$

or $\quad x \in (-\infty, -a] \cup [a, \infty)$

(xiii) $|x| \le a$, (a is –ve)

$\Rightarrow$ No solution or $x \in \phi$.

(xiv) $|x| \ge a$, (a is –ve)

$\Rightarrow \quad x \in$ Real number

or $\quad x \in (-\infty, \infty)$

(xv) $a \le |x| \le b$, where a and b are +ve.

$\Rightarrow \quad -b \le x \le -a$ or $a \le x \le b$

or $\quad x \in [-b, -a] \cup [a, b]$

Illustration 19 Solve for x

(i) $(|x|-1)(|x|-2) \le 0$

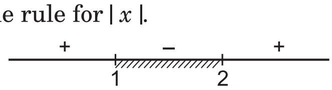
(ii) $\dfrac{|x|-1}{|x|-2} \ge 0$

(iii) $|x-3| + |4-x| = 1$

(iv) $|x| + |x+4| = 4$

Solution. (i) $(|x|-1)(|x|-2) \le 0$

Using, number line rule for $|x|$.

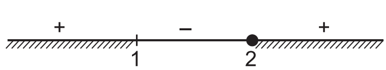


$\Rightarrow \quad | \le |x| \le 2$

ie, $\quad x \in [-2, -1] \cup [1, 2]$

(ii) $\dfrac{|x|-1}{|x|-2} \ge 0$

Using, number line rule for $|x|$.



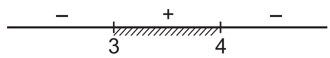
$\Rightarrow \quad |x| \le 1$ or $|x| > 2$

ie, $\quad x \in [-1, 1] \cup (-\infty, -2) \cup (2, \infty)$

(iii) $|x-3|+|4-x|=1$

As, we know, $\quad |x|+|y| = |x+y|$, iff $xy \geq 0$

$\therefore \quad (x-3)(4-x) \geq 0 \quad$ or $\quad -(x-3)(x-4) \geq 0$



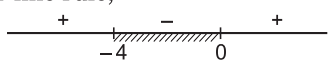
$\Rightarrow \quad x \in [3, 4]$

(iv) $|x|+|x+4|=4$

As, we know, $\quad |x|+|y|=|x-y|$, iff $xy \leq 0$

$\therefore \quad x(x+y) \leq 0$

Using number line rule,



$\Rightarrow \quad x \in [-4, 0]$

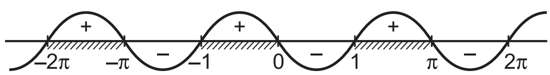
Illustration 20 Solve $|x^2-1+\sin x| = |x^2-1|+|\sin x|$, where $x \in [-2\pi, 2\pi]$.

Solution. Let $f(x)=x^2-1$, $g(x)=\sin x$

$\therefore$ Using $|f(x)+g(x)|=|f(x)|+|g(x)|$, which is true, iff $f(x)\cdot g(x) \geq 0$.

$\therefore \quad (x^2-1)\cdot \sin x \geq 0$ on $[-2\pi, 2\pi]$.

Using number line rule,



$\Rightarrow \quad x \in [-2\pi, -\pi] \cup [-1, 0] \cup [1, \pi]$

Illustration 21 Solve $\left|\dfrac{x}{x-1}\right| + |x| = \dfrac{x^2}{|x-1|}$.

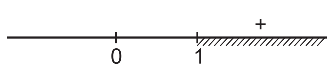
Solution. Let $f(x)=\dfrac{x}{x-1}$ and $g(x)=x$

$\therefore \quad f(x)+g(x)=\dfrac{x}{x-1}+x=\dfrac{x^2}{x-1}$

Using, $|f(x)|+|g(x)|=|f(x)+g(x)|$

ie, $\quad f(x)\cdot g(x) \geq 0$

$\Rightarrow \quad \dfrac{x}{x-1}\cdot x \geq 0 \Rightarrow \dfrac{x^2}{x-1} \geq 0$



$\Rightarrow \quad x \in \{0\} \cup (1, \infty)$

Illustration 22 Find domain for $y = \dfrac{1}{\sqrt{|x| - x}}$.

Solution. y is defined, if

$(|x| - x) > 0 \Rightarrow |x| > x$, which is true for negative x only.

Hence, domain $\in (-\infty, 0)$.

Illustration 23 Find domain for

$$y = \cos^{-1}\left(\frac{1 - 2|x|}{3}\right) + \log_{|x-1|} x.$$

Solution. Here, $\cos^{-1}\left(\dfrac{1 - 2|x|}{3}\right)$ exists, if

$$-1 \le \frac{1 - 2|x|}{3} \le 1$$

$\Rightarrow \quad -3 \le 1 - 2|x| \le 3 \quad \Rightarrow \quad -4 \le -2|x| \le 2$

$\Rightarrow \quad 2 \ge |x| \ge -1 \quad \Rightarrow \quad -2 \le x \le 2$ ...(i)

Also, $\log_{|x-1|} x$ exists, if

$x > 0$ and $|x - 1| > 0$

$x > 0$ and $x \in R - \{1\}$ ...(ii)

Hence, from Eqs. (i) and (ii), we get

$$x \in (0, 1) \cup (1, 2)$$

Illustration 24 The sum of the maximum and minimum values of function $f(x) = \sin^{-1} 2x + \cos^{-1} 2x + \sec^{-1} 2x$ is

(a) π (b) $\dfrac{\pi}{2}$ (c) 2π (d) $\dfrac{3\pi}{2}$

Solution. Here, domain of $f(x) \Rightarrow x \in \left\{-\dfrac{1}{2}, \dfrac{1}{2}\right\}$ only

$\therefore \quad f(x)$ is minimum when $x = \dfrac{1}{2}$, *ie*, $f_{\min}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{2}$

and $f(x)$ is maximum when $x = -\dfrac{1}{2}$, *ie*, $f_{\max}\left(-\dfrac{1}{2}\right) = \dfrac{3\pi}{2}$

$\therefore$ Sum of maximum and minimum value of function is 2π.

Hence, (c) is the correct answer.

Illustration 25 The complete set of values of 'a' for which the function $f(x) = \tan^{-1}(x^2 - 18x + a) > 0 \ \forall \ x \in R$, is

(a) $(81, \infty)$ (b) $[81, \infty)$

(c) $(-\infty, 81)$ (d) $(-\infty, 81]$

Solution. Here, $\tan^{-1}(x^2 - 18x + a) > 0, \forall \ x \in R$

$\Rightarrow \quad x^2 - 18x + a > 0, \forall \ x \in R$

$\Rightarrow \quad (18)^2 - 4a < 0 \quad \Rightarrow \quad a > 81$

$\Rightarrow \quad a \in (81, \infty)$

Hence, (a) is the correct answer.

Illustration 26 The domain of the function

$$f(x) = \sin^{-1} \frac{1}{|x^2 - 1|} + \frac{1}{\sqrt{\sin^2 x + \sin x + 1}} \text{ is}$$

(a) $(-\infty, \infty)$ (b) $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$

(c) $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty) \cup \{0\}$ (d) None of these

Solution. As, $\sin^2 x + \sin x + 1 > 0, \forall x \in R$

$$\therefore \quad \frac{1}{\sqrt{\sin^2 x + \sin x + 1}} \text{ is always exists.}$$

$$\therefore \quad \text{For } \sin^{-1}\left(\frac{1}{|x^2 - 1|}\right) \text{ to exists,}$$

$$0 < \frac{1}{|x^2 - 1|} \le 1 \quad \Rightarrow \quad |x^2 - 1| \ge 1$$

$$\Rightarrow \quad x^2 - 1 \le -1 \quad \text{or} \quad x^2 - 1 \ge 1$$

$$\Rightarrow \quad x^2 \le 0 \quad \text{or} \quad x^2 \ge 2$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad (x \le -\sqrt{2} \text{ or } x \ge \sqrt{2})$$

$$\therefore \quad x \in (-\infty, -\sqrt{2}) \cup [\sqrt{2}, \infty) \cup \{0\}$$

Hence, (c) is the correct answer.

Illustration 27 The domain of $f(x) = \dfrac{\log(\sin^{-1}\sqrt{x^2 + x + 1})}{\log(x^2 - x + 1)}$ is equal to

(a) $(-1, 1)$ (b) $(-1, 0) \cup (0, 1)$

(c) $(-1, 0) \cup \{1\}$ (d) None of these

Solution. Here, $\log(x^2 - x + 1)$ is defined, when

$$x^2 - x + 1 > 0 \quad \text{and} \quad x^2 - x + 1 \ne 1$$

$$\Rightarrow \quad x \in R \quad \text{and} \quad x \ne 0, 1$$

$$\Rightarrow \quad x \in R - \{0, 1\} \qquad \ldots\text{(i)}$$

Again, $\log(\sin^{-1}\sqrt{x^2 + x + 1})$ exists, when

$$0 < x^2 + x + 1 \le 1$$

$$\Rightarrow \quad x^2 + x \le 0 \quad \Rightarrow \quad x \in [-1, 0] \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$x \in [-1, 0)$$

Hence, (d) is the correct answer.

Illustration 28 The domain of $f(x) = \sqrt{\sin^{-1}(3x - 4x^3)} + \sqrt{\cos^{-1} x}$ is equal to

(a) $\left[-1, -\frac{\sqrt{3}}{2}\right] \cup \left[0, \frac{\sqrt{3}}{2}\right]$ (b) $\left[-1, -\frac{1}{2}\right] \cup \left[0, \frac{1}{2}\right]$

(c) $\left[0, \frac{1}{2}\right]$ (d) None of these

Solution. Here, $\sqrt{\cos^{-1} x}$ is defined for $x \in [-1, 1]$ for $\sqrt{\sin^{-1}(3x - 4x^3)}$ for x defined $0 \le 3x - 4x^3 \le 1$

$$\therefore \quad 3x - 4x^3 \ge 0, \quad \text{and} \quad 3x - 4x^3 \le 1$$

$$\Rightarrow \quad x \in \left(-\infty, -\frac{\sqrt{3}}{2}\right] \cup \left[0, \frac{\sqrt{3}}{2}\right], \quad \text{and} \quad x \in [-1, \infty)$$

$$\Rightarrow \quad x \in \left[-1, -\frac{\sqrt{3}}{2}\right] \cup \left[0, \frac{\sqrt{3}}{2}\right]$$

Hence, (a) is the correct answer.

Illustration 29 The domain of the function

$$f(x) = \sqrt[6]{4^x + 8^{2/3(x-2)} - 52 - 2^{2(x-1)}} \text{ is}$$

(a) (0, 1) (b) $[3, \infty)$ (c) [1, 0] (d) None of these

Solution. Here, $f(x)$ exists only, if $4^x + 8^{\frac{2}{3}(x-2)} - 52 - 2^{2(x-1)} \ge 0$.

$$\Rightarrow \quad 2^{2x} + 2^{2(x-2)} - 2^{2(x-1)} \ge 52$$

$$\Rightarrow \quad 2^{2x} \ge 64 \quad \Rightarrow \quad x \ge 3$$

Hence, (b) is the correct answer.

Illustration 30 Domain of the function

$$f(x) = \frac{1}{\sqrt{4x - |x^2 - 10x + 9|}}, \text{ is}$$

(a) $(7 - \sqrt{40}, 7 + \sqrt{40})$ (b) $(0, 7 + \sqrt{40})$

(c) $(7 - \sqrt{40}, \infty)$ (d) None of these

Solution. Here, $f(x) = \dfrac{1}{\sqrt{4x - |x^2 - 10x + 9|}}$ would exists, if

$$4x - |x^2 - 10x + 9| > 0$$

ie, $\quad |x^2 - 10x + 9| < 4x,$

where $\quad |x^2 - 10x + 9| = \begin{cases} x^2 - 10x + 9, & x \le 1 \text{ or } x \ge 9 \\ -(x^2 - 10x + 9), & 1 < x < 9 \end{cases}$

$\Rightarrow$ Two cases.

Case I When $x \le 1$ or $x \ge 9$

$$\therefore \quad x^2 - 10x + 9 < 4x$$

$$\Rightarrow \quad x^2 - 14x + 9 < 0 \quad \Rightarrow \quad (x - 7)^2 < 40$$

$$\Rightarrow \quad x \in (7 - \sqrt{40}, 7 + \sqrt{40}) \qquad (\text{But } x \le 1 \text{ or } x \ge 9)$$

$$\Rightarrow \quad x \in (7 - \sqrt{40}, 1] \cup [9, 7 + \sqrt{40}) \qquad \ldots\text{(i)}$$

Case II When $1 < x < 9$

$$-x^2 + 10x - 9 < 4x \quad \Rightarrow \quad x^2 - 6x + 9 > 0$$

$\Rightarrow \quad (x - 3)^2 > 0$ which is always true except $x = \{3\}$

$$\therefore \quad x \in (1, 9) - \{3\} \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), domain of $f(x) \in (7 - \sqrt{40}, 7 + \sqrt{40}) - \{3\}$

Hence, (d) is the correct answer.

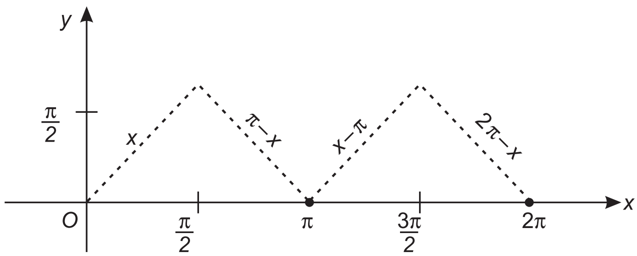
Illustration 31 The domain of the function

$$f(x) = \sqrt{|\sin^{-1}(\sin x)| - \cos^{-1}(\cos x)} \text{ in } [0, 2\pi] \text{ is}$$

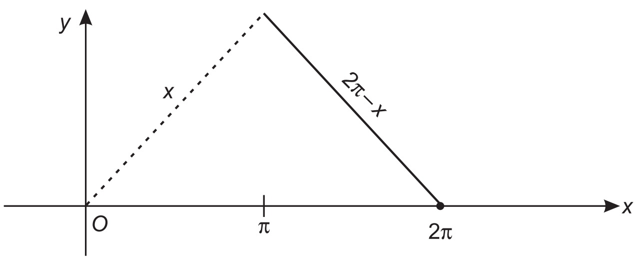
(a) $\left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$ (b) $[\pi, 2\pi]$

(c) $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ (d) $[0, 2\pi] - \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$

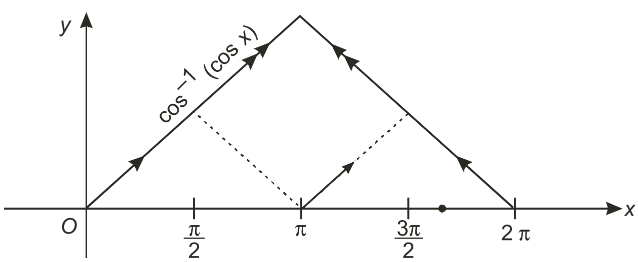
Solution. As, $|\sin^{-1}(\sin x)|$ could be sketched, as



and $\cos^{-1}(\cos x)$ could be sketched, as



$\therefore$ $|\sin^{-1}(\sin x)| > \cos^{-1}(\cos x)$ is not possible. Only equality holds, as



Thus, $|\sin^{-1}(\sin x)| = \cos^{-1}(\cos x)$

when $x \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$

Hence, domain for $f(x) \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$.

Hence, (a) is the correct answer.

Illustration 32 The domain of derivative of the function $f(x) = |\sin^{-1}(2x^2 - 1)|$, is

(a) $(-1, 1)$ (b) $(-1, 1) \sim \left\{0, \pm \frac{1}{\sqrt{2}}\right\}$

(c) $(-1, 1) \sim \{0\}$ (d) $(-1, 1) \sim \left\{\pm \frac{1}{\sqrt{2}}\right\}$

Solution. $y = |\sin^{-1}(2x^2 - 1)|$

$$\therefore \quad \frac{dy}{dx} = \frac{|\sin^{-1}(2x^2-1)|}{\sin^{-1}(2x^2-1)} \cdot \frac{4x}{|2x|\sqrt{1-x^2}}$$

which would exists, if

$$|2x| \neq 0, \ \sin^{-1}(2x^2 - 1) \neq 0 \ \text{ and } \ 1 - x^2 > 0$$

$$\Rightarrow \quad x \neq 0, \ 2x^2 - 1 \neq 0 \ \text{ and } \ |x| < 1$$

$$\Rightarrow \quad x \neq 0, \pm \frac{1}{\sqrt{2}} \ \text{ and } \ |x| < 1$$

$$\Rightarrow \quad x \in (-1, 1) \sim \left\{0, \pm \frac{1}{\sqrt{2}}\right\}$$

Hence, (b) is the correct answer.

Signum Function

$y = \operatorname{sgn}(x)$

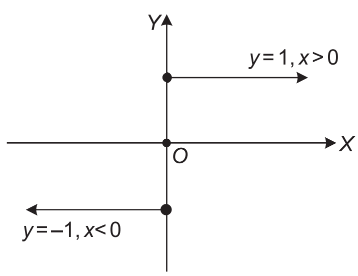


Fig. 3.23

It is defined by

$$y = \operatorname{sgn}(x) = \begin{cases} \dfrac{|x|}{x} \ \text{ or } \ \dfrac{x}{|x|}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}$$

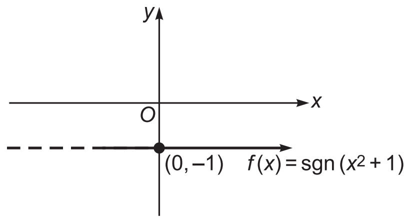
Illustration 33 Sketch the graph of

(i) $f(x) = \operatorname{sgn}(x^2 + 1)$ (ii) $f(x) = \operatorname{sgn}(\log_e x)$

(iii) $f(x) = \operatorname{sgn}(\sin x)$ (iv) $f(x) = \operatorname{sgn}(\cos x)$

Solution. (i) $f(x) = \operatorname{sgn}(x^2 + 1)$

$$= \begin{cases} -1, & \text{if } x^2 + 1 > 0 \\ 0, & \text{if } x^2 + 1 = 0 \\ 1, & \text{if } x^2 + 1 < 0 \end{cases}$$

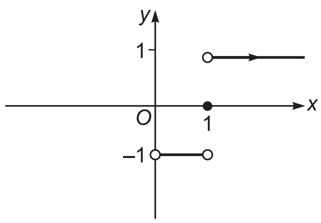


$$f(x) = -1,\ x \in R$$

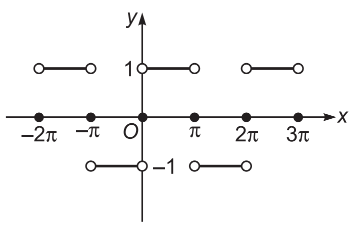
(ii) $f(x) = \text{sgn}\,(\log_e x)$

$$= \begin{cases} 1, & \log_e x > 0 \\ -1, & \log_e x < 0 \\ 0, & \log_e x = 0 \end{cases} \qquad \therefore \qquad f(x) = \begin{cases} 1, & x > 1 \\ -1, & 0 < x < 1 \\ 0, & x = 1 \end{cases}$$

Graph for, $f(x) = \text{sgn}\,(\log_e x)$



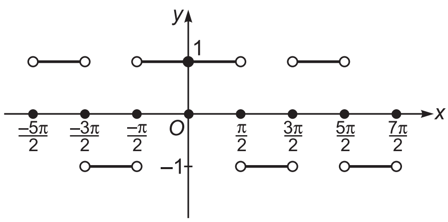
(iii) $f(x) = \text{sgn}\,(\sin x)$



$$f(x) = \begin{cases} 1, & \sin x > 0 \\ -1, & \sin x < 0 \\ 0, & \sin x = 0 \end{cases} = \begin{cases} 1, & 2n\pi < x < (2n+1)\pi \\ -1, & (2n+1)\pi < x < (2n+2)\pi \\ 0, & x = n\pi \end{cases}$$

(iv) $f(x) = \text{sgn}\,(\cos x)$

$$f(x) = \begin{cases} 1, & \cos x > 0 \\ -1, & \cos x < 0 \\ 0, & \cos x = 0 \end{cases} = \begin{cases} 1, & 2n\pi - \pi/2 < x < 2n\pi + \pi/2 \\ -1, & 2n\pi + \pi/2 < x < 2n\pi + 3\pi/2 \\ 0, & x \in (2n+1)\pi/2 \end{cases}$$



Greatest Integer Function or Step up Function

$[x]$ indicates the integral part of x, which is nearest and smaller integer to x. It is also known as floor of x.

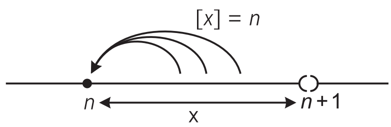


Fig. 3.24

Thus, $[2.3202\,] = 2$, $[0.23] = 0$, $[\,5] = 5$,

$[-8.0725] = -9$, $[-0.6] = -1$

In general $n \le x < n + 1$ ($n \in$ integer)

ie, $[x] = n$

Here, $f(x) = [x]$ could be expressed graphically, as

x	$[x]$
$0 \le x < 1$	0
$1 \le x < 2$	1
$2 \le x < 3$	2
$-1 \le x < 0$	-1
$-2 \le x < -1$	-2

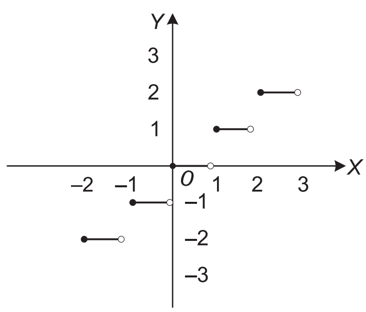


Fig. 3.25

($\bullet$) darkened circle represents value is neglected, ($\circ$) represents value is taken.

Properties of the Greatest Integer Functions

(i) $[x] \le x < [x] + 1$

(ii) $x - 1 < [x] < x$

(iii) $I \le x < I + 1$

$\Rightarrow [x] = I$, where $I \in$ Integer

(iv) $[[x]] = [x]$

(v) $[x] + [-x] = \begin{cases} 0, & \text{if } x \in \text{Integer} \\ -1, & \text{if } x \notin \text{Integer} \end{cases}$

ie, $$[-x] = \begin{cases} -x, & \text{if } x \in \text{Integer} \\ -1-[x], & \text{if } x \notin \text{Integer} \end{cases}$$

(vi) $[x]-[-x] = \begin{cases} 2x, & \text{if } x \in \text{Integer} \\ 2x+1, & \text{if } x \notin \text{Integer} \end{cases}$

(vii) $[x \pm n] = [x] \pm n, n \in \text{Integer}$

(viii) $[x] \geq n \Leftrightarrow x \geq n, n \in \text{Integer}$

(ix) $[x] > n \Leftrightarrow x \geq n+1, n \in \text{Integer}$

(x) $[x] \leq n \Leftrightarrow x < n+1, n \in \text{Integer}$

(xi) $[x] < n \Leftrightarrow x < n, n \in \text{Integer}$

(xii) $[x] = \left[\frac{x}{2}\right] + \left[\frac{x+1}{2}\right]$

(xiii) $\left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \left[\frac{n+4}{8}\right] + \ldots. = n,$

$n \in$ Natural number

(xiv) $[x]+[y] \leq [x+y] \leq [x]+[y]+1$

(xv) $[x] + \left[x+\frac{1}{n}\right] + \left[x+\frac{2}{n}\right] + \ldots + \left[x+\frac{n-1}{n}\right] = [nx], n \in N$

Illustration 34 Find domain for, $f(x) = \cos^{-1}[x]$.

Solution. As, $y = \cos^{-1} x$ exists when $-1 \leq x \leq 1$.

$\therefore f(x) = \cos^{-1}[x]$ exists, when

$$-1 \leq [x] \leq 1$$

$\Rightarrow$ $$-1 \leq x < 2$$

or $$x \in [-1, 2)$$

Illustration 35 Find the value of

$$\left[\frac{3}{4}\right] + \left[\frac{3}{4}+\frac{1}{100}\right] + \left[\frac{3}{4}+\frac{2}{100}\right] + \ldots + \left[\frac{3}{4}+\frac{99}{100}\right].$$

Solution. $$\left[\frac{3}{4}\right] + \left[\frac{3}{4}+\frac{1}{100}\right] + \ldots + \left[\frac{3}{4}+\frac{24}{100}\right] + \left[\frac{3}{4}+\frac{25}{100}\right] + \ldots. + \left[\frac{3}{4}+\frac{99}{100}\right]$$

$$\Rightarrow \underbrace{[0.75]+\ldots+[0.99]}_{\text{25 terms are zero}} + \underbrace{[1.0]+\ldots+[1.74]}_{\text{75 terms are each equal to 1}} \Rightarrow 75$$

Aliter **:** Apply property (xv)

Here, $$x = \frac{3}{4} \text{ and } n = 100$$

$$\therefore \left[\frac{3}{4}\right] + \left[\frac{3}{4}+\frac{1}{100}\right] + \ldots + \left[\frac{3}{4}+\frac{99}{100}\right] = \left[\frac{3}{4} \times 100\right] = 75$$

Illustration 36 Given $y = 2[x] + 3$ and $y = 3[x-2] + 5$, then find the value of $[x+y]$.

Solution. $2[x] + 3 = 3[x-2] + 5$

$\Rightarrow \quad 2[x] + 3 = 3[x] - 6 + 5$

$\Rightarrow \quad [x] = 4 \quad \Rightarrow \quad 4 \le x < 5$

or $\quad x = 4 + f \qquad (f \to \text{fraction})$

$\therefore \quad y = 2[x] + 3 = 11$

Hence, $\quad [x+y] = [4 + f + 11] = [15 + f] = 15$

Illustration 37 Find domain for $f(x) = [\sin x]\cos\left(\dfrac{\pi}{[x-1]}\right)$.

Solution. $[\sin x]$ is always defined. $\cos\left(\dfrac{\pi}{[x-1]}\right)$ is also defined except

when $\quad [x-1] = 0$

$\Rightarrow \quad 0 \le x - 1 < 1$

$\Rightarrow \quad 1 \le x < 2$

Hence, domain $\in R - [1, 2)$.

Illustration 38 Let $[\sqrt{n^2+1}] = [\sqrt{n^2+\lambda}]$, where $n, \lambda \in N$. Show that λ can have $2n$ different values.

Solution. We have, $n^2 + 1 = (n+1)^2 - 2n < (n+1)^2$; $n \in N$

ie, $\quad \sqrt{n^2+1} < n + 1 \quad$ or $\quad n < \sqrt{n^2+1} < n+1$

$\Rightarrow \quad [\sqrt{n^2+1}] = n$

$\therefore \quad [\sqrt{n^2+\lambda}] = n$

$\Rightarrow \quad n < \sqrt{n^2+\lambda} < (n+1)$

or $\quad n^2 < (n^2+\lambda) < (n+1)^2$

$\Rightarrow \quad 0 < \lambda < 2n + 1$

Thus, λ can take $2n$ different values.

Illustration 39 $f(x) = \dfrac{1}{\sqrt{[x]-x}}$, where [ ] denotes the greatest integral function less than or equals to x. Then, find the domain of $f(x)$.

Solution. $f(x) = \dfrac{1}{\sqrt{[x]-x}}$ exists, if

$$[x] - x > 0 \quad \textit{ie,} \quad [x] > x$$

But from definition of greatest integral function, we know,

$[x] \le x \qquad [\text{as } x = [x] + \{x\}]$

Thus, it is not possible that $[x] > x$.

Hence, $\quad$ domain $f(x) = \phi$

Illustration 40 The function $f(x)$ is defined in [0, 1]. Find the domain of $f(\tan x)$.

Solution. Here, $f(x)$ is defined in [0, 1].

$\Rightarrow x \in [0, 1]$ *ie*, the only values of x we can substitute those lie in [0, 1].

For $f(\tan x)$ to be defined, we must have

$$0 \le \tan x \le 1 \qquad \text{(as } x \text{ is replaced by } \tan x\text{)}$$

ie, $$n\pi \le x \le n\pi + \frac{\pi}{4} \qquad \text{(in general)}$$

or $$0 \le x \le \frac{\pi}{4} \qquad \text{(in particular)}$$

Thus, domain for $f(\tan x) \in \left[n\pi, n\pi + \frac{\pi}{4}\right]$.

Illustration 41 If domain for $y = f(x)$ is $[-3, 2]$, find domain of $g(x) = f\{|[x]|\}$.

Solution. Here, $f(x)$ is defined by $[-3, 2]$.

$$\Rightarrow \qquad x \in [-3, 2].$$

(*ie*, the only value of x we can substitute lie between $[-3, 2]$).

For $g(x) = f\{|[x]|\}$ to be defined, we must have

$$-3 \le |[x]| \le 2$$

$$\Rightarrow \qquad 0 \le |[x]| \le 2 \qquad \text{(as } |x| \ge 0 \text{ for all } x\text{)}$$

$$\Rightarrow \qquad -2 \le [x] \le 2 \qquad \text{(as } |x| \le a \Rightarrow -a \le x \le a\text{)}$$

$$\Rightarrow \qquad -2 \le x < 3 \qquad \text{(by definition of greatest integral function)}$$

Hence, domain $g(x) \in [-2, 3[$ or $[-2, 3)$.

Illustration 42 Find the domain of function $f(x) = \dfrac{1}{[|x-1|] + [|7-x|] - 6}$ where [ ] denotes the greatest integral function.

Solution. $f(x)$ is defined when

$$[|x-1|] + [|7-x|] - 6 \ne 0$$

$$\Rightarrow \begin{cases} [1-x] + [7-x] \ne 6, & \text{when } x \le 1 & \quad \ldots\text{(i)} \\ [x-1] + [7-x] \ne 6, & \text{when } 1 \le x \le 7 & \quad \ldots\text{(ii)} \\ [x-1] + [x-7] \ne 6, & \text{when } x \ge 7 & \quad \ldots\text{(iii)} \end{cases}$$

Taking Eq. (i), we have

$$[1-x] + [7-x] \ne 6$$

$$1 + [-x] + 7 + [-x] \ne 6$$

$$\Rightarrow \qquad 2[-x] \ne -2 \quad \Rightarrow \quad [-x] \ne -1$$

$$\Rightarrow \qquad x \notin (0, 1] \qquad \ldots\text{(a)}$$

From Eq. (ii), we have

$$[x-1] + [7-x] \ne 6$$

$$\Rightarrow \quad [x]-1+7+[-x] \neq 6 \Rightarrow [x]+[-x] \neq 0$$

$$\Rightarrow \quad x \notin \text{Integer}$$

$$\Rightarrow \quad x \notin \{1, 2, 3, 4, 5, 6, 7\} \qquad \text{...(b)}$$

From Eq. (iii), we have $[x-1]+[x-7] \neq 6$

$$\Rightarrow \quad [x]-1+[x]-7 \neq 6 \quad \Rightarrow \quad 2[x] \neq 14$$

$$\Rightarrow \quad [x] \neq 7 \quad \Rightarrow \quad x \notin [7, 8) \qquad \text{...(c)}$$

Hence, from Eqs. (a), (b) and (c), we have

Domain $f(x) \in R - \{(0, 1] \cup \{1, 2, 3, 4, 5, 6, 7\} \cup [7, 8)\}$.

Illustration 43 If the function $f(x)=[3.5+b\sin x]$ (where [.] denotes the greatest integer function) is an even function, then complete set of values of 'b' is

(a) $(-0.5, 0.5)$ (b) $[-0.5, 0.5]$

(c) $(0, 1)$ (d) $[-1, 1]$

Solution. $f(x)=[3.5+b\sin x]$ for $f(x)$ to be an even function.

$$3 < 3.5 + b\sin x < 4, \ \forall x \in R$$

$$\Rightarrow \quad -0.5 < b\sin x < 0.5, \ \forall x \in R$$

$$\Rightarrow \quad -0.5 < b < 0.5 \quad \Rightarrow \quad b \in (-0.5, 0.5)$$

Hence, (a) is the correct answer.

Illustration 44 The domain of the function

$$f(x) = \log_3 \log_{1/3}(x^2+10x+25) + \frac{1}{[x]+5}$$

where [.] denotes the greatest integer function) is

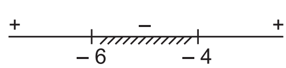
(a) $(-4, -3)$ (b) $(-6, -5)$

(c) $(-6, -4)$ (d) None of these

Solution. Here, $\log_3 \log_{1/3}(x^2+10x+25)$ is defined when

(i) $x^2+10x+25>0$, *ie*, $(x+5)^2>0 \quad \Rightarrow \quad x \neq -5$ ...(i)

(ii) $\log_{1/3}(x^2+10x+25)>0 \quad \Rightarrow \quad x^2+10x+25<1$



$$\text{or} \quad x^2+10x+24<0 \quad \text{or} \quad (x+6)(x+4)<0$$

$$\Rightarrow \quad x \in (-6, -4) \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$x \in (-6, -5) \cup (-5, -4) \qquad \text{...(iii)}$$

Also, Eq. (iii) $[x]+5 \neq 0$

$$\Rightarrow \quad [x] \neq -5$$

$$\Rightarrow \quad x \notin [-5, -4) \qquad \text{...(iv)}$$

From Eqs. (iii) and (iv), domain of $f(x) \in (-6, -5)$.

Hence, (b) is the correct answer.

Illustration 45 Let $[x]$ be the greatest integer less than or equal to x. Then, the equation $\sin x = [1 + \sin x] + [1 - \cos x]$ has

(a) one solution in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (b) one solution in $\left[\frac{\pi}{2}, \pi\right]$

(c) one solution in R (d) no solution in R

Solution. Given that, $[1 + \sin x] + [1 - \cos x] = \sin x$

$\Rightarrow \quad 1 + [\sin x] + 1 + [-\cos x] = \sin x$

$\Rightarrow \quad 2 + [\sin x] + [-\cos x] = \sin x \quad \Rightarrow \quad 2 + [-\cos x] = \{\sin x\}$

Here, LHS is 1, 2, or 3, but RHS $\in [0, 1)$

$\therefore$ No solution.

Hence, (d) is the correct answer.

Fractional Part Function

$y = \{x\}$ It indicates fractional part of x.

In $\quad x = I + f, I = [x] \quad$ and $\quad f = \{x\}$

$\therefore \quad y = \{x\} = x - [x]$

x	$\{x\}$
$0 \le x < 1$	x
$1 \le x < 2$	$x - 1$
$2 \le x < 3$	$x - 2$
$-1 \le x < 0$	$x + 1$
$-2 \le x < -1$	$x + 2$

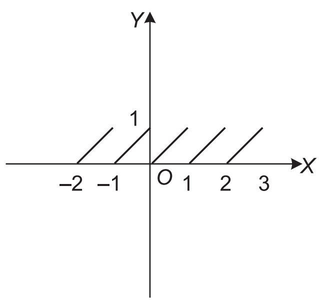


Fig. 3. 26

Properties of Fractional Part of x

(i) $\{x\} = x$, if $0 \le x < 1$. **(ii)** $\{x\} = 0$, if $x \in$ integer.

(iii) $\{-x\} = 1 - \{x\}$, if $x \notin$ integer. **(iv)** $\{x \pm \text{integer}\} = \{x\} \pm$ integer.

Illustration 46 If $\{x\}$ and $[x]$ represent fractional and integral parts of x respectively, then find the value of $[x] + \sum_{r=1}^{2000} \frac{\{x + r\}}{2000}$.

Solution. In $[x] + \sum_{r=1}^{2000} \frac{\{x + r\}}{2000}$, we know that $\{x + r\} = \{x\}$ (as $r \in$ Integer)

$$\Rightarrow \quad [x] + \sum_{r=1}^{2000} \frac{\{x\}}{2000}$$

$$\Rightarrow \quad [x] + \left[\frac{\{x\}}{2000} + \frac{\{x\}}{2000} + \ldots + \text{upto 2000 times}\right]$$

$$\Rightarrow \quad [x] + \frac{2000\{x\}}{2000} \quad \Rightarrow \quad [x] + \{x\} \quad \Rightarrow x$$

Thus, $$[x] + \sum_{r=1}^{2000} \frac{\{x + r\}}{2000} = x$$

Illustration 47 Solve $4\{x\} = x + [x]$.

Solution. We know, $x = [x] + \{x\}$

$$\therefore \quad 4\{x\} = [x] + \{x\} + [x]$$

$$\{x\} = \frac{2[x]}{3} \quad \text{...(i)}$$

But $\quad 0 \le \{x\} < 1$

So, $\quad 0 \le \frac{2[x]}{3} < 1$

$$0 \le [x] < \frac{3}{2}, \text{ therefore } [x] = 0 \text{ or } 1$$

If $\quad [x] = 1, \quad \text{then } \{x\} = \frac{2}{3}$ [From Eq. (i)]

Thus, $\quad x = \frac{5}{3}$

If $\quad [x] = 0, \{x\} = 0 \quad \text{so,} \quad x = 0$

Thus, solutions of $4\{x\} = x + [x]$ are $x \in \left\{0, \frac{5}{3}\right\}$.

Illustration 48 Prove that $[x] + [y] \le [x + y]$, where $x = [x] + \{x\}$ and $y = [y] + \{y\}$ ($[\cdot]$ represents greatest integer function and $\{\cdot\}$ represents fractional part of x.)

Solution. Here, $x + y = [x] + [y] + \{x\} + \{y\}$

$$\therefore \quad [x + y] = [[x] + [y] + \{x\} + \{y\}]$$

$$= [x] + [y] + [\{x\} + \{y\}] \quad (\text{using } [x + I] = [x] + I, I \in \text{Integer})$$

$$\Rightarrow \quad [x + y] \ge [x] + [y]$$

Point to Consider

This could be generalized for n terms as

$$[x_1] + [x_2] + \ldots + [x_n] \le [x_1 + x_2 + \ldots + x_n]$$

Least Integer Function

$y = (x) = \lceil x \rceil, \lceil x \rceil$ or (x) indicates the integral part of x which is the nearest, and greater integer to x.

It is known **as ceiling of *x***.

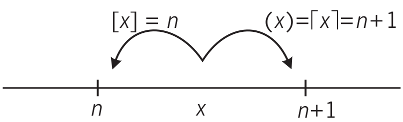


Fig. 3.27

Thus, $\quad [2.3203] = 3, \quad (0.23) = 1$

$$(-8.0725) = -8, (-0.6) = 0$$

In general, $n < x \le n + 1 \quad (n \in \text{integer})$

ie, $\quad (x) = n + 1$

$$[x] = n \qquad (x) = n + 1 = \lceil x \rceil$$

Here, $f(x) = (x) = \lceil x \rceil$ can be expressed graphically as

x	$\lceil x \rceil = (x)$
$-1 < x \le 0$	0
$0 < x \le 1$	1
$1 < x \le 2$	2
$2 < x \le 3$	3
$-2 \le x \le -1$	-1

(•) represents value is taken.
(o) represents value is neglected.

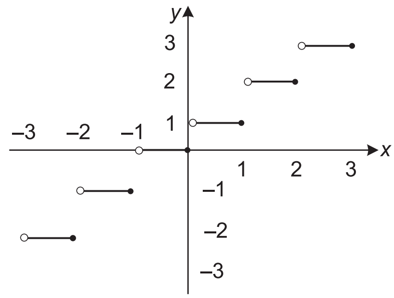


Fig. 3.28

Properties of Least Integer Function

(i) $(x) = x = \lceil x \rceil$ holds, if x is Integer.

(ii) $(x + I) = \lceil x + I \rceil = (x) + I, I \in$ Integer.

(iii) Greatest integer function $[x]$ converts $x = I + f$ into I, while (x) converts to $I + 1$.

(iv) $x = (x) + \{x\} - 1, 0 \le \{x\} < 1$

(v) $(-x) = \begin{cases} -(x), & \text{if } x \in I \\ -(x) - 1, & \text{if } x \notin I \end{cases}$

(vi) $(x) - (-x) = \begin{cases} 2(x), & \text{if } x \in I \\ 2(x) - 1, & \text{if } x \notin I \end{cases}$

(vii) $(x) \ge n \Leftrightarrow x > n - 1, n \in I$

(viii) $(x) > n \Leftrightarrow x > n, n \in I$

(ix) $(x) \le n \Leftrightarrow x \le n, \ n \in I$

(x) $(x) < n \Leftrightarrow x \le n - 1, n \in I$

(xi) $\left(\dfrac{(x)}{n}\right) = \left(\dfrac{x}{n}\right), n \in N$

(xii) $\left(\dfrac{n+1}{2}\right) + \left(\dfrac{n+2}{4}\right) + \left(\dfrac{n+4}{8}\right) + \ldots = 2(n-1), n \in N$

(xiii) $(x) = \begin{cases} [x], & x \in I \\ [x] + 1, & x \notin I \end{cases}$

(xiv) $(x) + \left(x + \dfrac{1}{n}\right) + \left(x + \dfrac{2}{n}\right) + \ldots + \left(x + \dfrac{n-1}{n}\right) = (nx) + n - 1, n \in N$

Illustration 49 Find the solution set of $(x)^2 + (x+1)^2 = 25$, where (x) is the least integer greater than equals to x.

Solution. Let $x = I + f$ where I (integer) and f (fractional part)

Then, $\quad (I + f)^2 + (I + f + 1)^2 = 25 \qquad (ie, 0 < f < 1)$

$\Rightarrow \quad \{I+1\}^2 + \{I+2\}^2 = 25 \quad \Rightarrow \quad I^2 + 2I + 1 + I^2 + 4I + 4 = 25$

$\Rightarrow \quad 2I^2 + 6I + 5 - 25 = 0 \quad \Rightarrow \quad 2I^2 + 6I - 20 = 0$

So, $\quad I = 2, -5$

Thus, $\quad x = 2 + f, -5 + f \quad$ where $0 < f < 1$

$\Rightarrow \quad 2 < 2 + f < 3, \quad -5 < -5 + f < -4$ ...(i)

Again, let $\quad x = I$

$\therefore \quad x^2 + (x+1)^2 = 25$

$\therefore \quad x = 3, -4$ ...(ii)

From Eqs. (i) and (ii), $x\,(-5, -4] \cup [4, \infty]$

Illustration 50 If $[x]$ = the greatest integer less than or equal to x and (x) = the least integer greater than or equal to x and $[x]^2 + (x)^2 > 25$, then x belongs to ...

Solution. Let $x = I + f$ where $I \in$ integer, $f \in$ fractional part (*ie*, $0 \le f < 1$)

$\therefore \quad [x]^2 + (x)^2 > 25$

$\Rightarrow \quad [I+f]^2 + (I+f)^2 > 25 \quad \Rightarrow \quad I^2 + \{I+1\}^2 > 25$

$\Rightarrow \quad I^2 + I^2 + 2I + 1 > 25 \quad \Rightarrow \quad 2I^2 + 2I - 24 > 0$

$\Rightarrow \quad I^2 + I - 12 > 0 \quad \Rightarrow \quad (I+4)(I-3) > 0$

$\therefore \quad I < -4 \quad$ or $\quad I > 3$

Here, $\quad x = I + f$

So, $\quad x < -4 + f \quad$ or $\quad x > 3 + f$

Since, $\quad 0 \le f < 1$

$\therefore \quad x \le -4 \quad$ and $\quad x \ge 4$

Hence, $\quad x \in (-\infty, -4] \cup [4, \infty)$

Illustration 51 Find the number of solutions of $|[x] - 2x| = 4$, where $[x]$ is the greatest integer $\le x$.

Solution. Let $x = I + f$, $I \in$ integer, $f \in$ fractional part (*ie*, $0 \le f < 1$)

$|[x] - 2x| = 4 \quad \Rightarrow \quad |[I+f] - 2(I+f)| = 4$

$\Rightarrow \quad |I - 2I - 2f| = 4 \quad \Rightarrow \quad |I + 2f| = 4,$

which is only possible, if $\left(f = \dfrac{1}{2} \text{ or } 0\right)$

If $\quad f = \dfrac{1}{2} \quad \Rightarrow \quad |I+1| = 4 \quad \Rightarrow \quad I + 1 = \pm 4$

So, $\quad I = 3, -5, \quad$ and $\quad f = \dfrac{1}{2}$

If $\quad f = 0$

Then $\quad |I| = 4$

$I = \pm 4, \quad$ and $\quad f = 0$

Thus, number of solutions are $x = \left\{\pm 4, \dfrac{7}{2}, -\dfrac{9}{2}\right\}$ *ie*, 4 solutions.

Target Exercise 3.2

Directions (Q. Nos. 1 to 30) : *Find the domain of the following :*

1. $f(x)=\sqrt{x^2-5x+6}$

2. $f(x)=\sqrt{\dfrac{2x+1}{x^3-3x^2+2x}}$

3. $f(x)=\sqrt{\log_{1/2}\left(\dfrac{5x-x^2}{4}\right)}$

4. $f(x)=\sin^{-1}\left(\dfrac{2-3[x]}{4}\right)$, which [·] denotes the greatest integer function.

5. $f(x)={}^{16-x}C_{2x-1}+{}^{20-3x}P_{4x-5}$

6. $f(x)=\log(x-[x])$, where [·] denotes the greatest integer function.

7. $f(x)=\sqrt{e^{\cos^{-1}(\log_4 x^2)}}$

8. $f(x)=\operatorname{cosec}^{-1}[1+\sin^2 x]$, where [·] denotes the greatest integer function.

9. $f(x)=\dfrac{1}{\sqrt{[x]^2-[x]-6}}$, where [.] denotes the greatest integer function.

10. $f(x)=\sin|x|+\sin^{-1}(\tan x)+\sin(\sin^{-1}x)$

11. $f(x)=\log_{10}(\sqrt{x-4}+\sqrt{6-x})$

12. $f(x)=\sin^{-1}\left(\dfrac{3-2x}{5}\right)+\sqrt{3-x}$

13. $f(x)=\sqrt{2-|x|}+\sqrt{1+|x|}$

14. $f(x)=\sqrt{x^2-|x|-2}$

15. $f(x)=\log_e|\log_e x|$

16. $f(x)=\sqrt{1-\sqrt{1-\sqrt{1-x^2}}}$

17. $f(x)=\sqrt{\log_{0.3}\left(\dfrac{3x-x^2}{x-1}\right)}$

18. $f(x)=\sqrt{x^{12}-x^9+x^4-x+1}$

19. $f(x)=\log_{10}(1-\log_{10}(x^2-5x+16))$

20. $f(x)=\dfrac{\log_{2x}3}{\cos^{-1}(2x-1)}$

21. $f(x)=\sqrt{\dfrac{\log_{0.3}(x-1)}{x^2-2x-8}}$

22. $f(x)=\log_{10}\log_2\log_{2/\pi}(\tan^{-1}x)^{-1}$

23. $f(x)=\cos^{-1}\sqrt{\log_{[x]}\dfrac{|x|}{x}}$, where [·] denotes the greatest integer function.

24. $f(x) = \sqrt{{}^{x^2 + 4x}C_{2x^2 + 3}}$

25. $f(x) = \sqrt{\dfrac{x-1}{x - 2\{x\}}}$, where {·} denotes the fractional part.

26. $f(x) = \sin^{-1}\left(\dfrac{[x]}{\{x\}}\right)$, where [·] and {·} denote greatest integer and fractional parts.

27. $f(x) = \sqrt{2\{x\}^2 - 3\{x\} + 1}$, $x \in [-1, 1]$, where {·} denotes the fractional part of x.

28. $f(x) = \sin^{-1}[2x^2 - 3]$, where [·] denotes the greatest integer function.

29. $f(x) = \sin^{-1}\left[\log_2\left(\dfrac{x^2}{2}\right)\right]$, where [·] denotes the greatest integer function.

30. $f(x) = \dfrac{1}{[|x-2|] + [|x-10|] - 8}$, where [·] denotes the greatest integer function.

31. If a function is defined, as $g(x) = |\sin x| + \sin x$, $\phi(x) = \sin x + \cos x$, $0 \le x \le \pi$, then find the domain for $f(x) = \sqrt{\log_{\phi(x)} g(x)}$.

32. Solve the equations, $y = \dfrac{1}{3}[\sin x + [\sin x + [\sin x]]]$ and $[y + [y]] = 2\cos x$, where [·] denotes the greatest integer function.

33. If $[x] = \left[\dfrac{x}{2}\right] + \left[\dfrac{x+1}{2}\right]$, where [·] denotes the greatest integer function and n be positive integer, then show that

$$\left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \left[\frac{n+4}{8}\right] + \left[\frac{n+8}{16}\right] + \ldots = n.$$

34. Find the integral solutions to the equation $[x][y] = x + y$. Show that all the non-integral solutions lie on exactly two lines. Determine these lines.

Range

Range of $y = f(x)$ is a collection of all outputs $\{f(x)\}$ corresponding to each real number in the domain.

Rules for Finding the Range

First of all find the domain of $y = f(x)$

(i) If domain $\in$ finite number of points $\Rightarrow$ Range $\in$ set of corresponding $f(x)$ values.

(ii) If domain $\in R$ or R – {some finite points}

Then, express x in terms of y. From this, find y for x to be defined. (*ie*, Find the values of y for which x exists.)

(iii) If domain $\in$ *a* finite interval, then find the least and the greatest values for range, using monotonicity.

Illustration 52 Find the range for $y = \dfrac{x - [x]}{1 - [x] + x}$.

Solution. Here, $y = \dfrac{x - [x]}{1 + x - [x]} = \dfrac{\{x\}}{1 + \{x\}}$

Thus, domain $\in$ real number.

Therefore, from $y = \dfrac{\{x\}}{1 + \{x\}}$

We have, $y + y\{x\} = \{x\} \Rightarrow \{x\} = \dfrac{y}{1 - y}$

Here, $0 \le \{x\} < 1$, so $0 \le \dfrac{y}{1 - y} < 1 \Rightarrow 0 \le y < \dfrac{1}{2}$

Hence, range $= \left[0, \dfrac{1}{2}\right)$

Illustration 53 Find the range for $f(x) = \dfrac{e^x}{1 + [x]}$ when $x \ge 0$.

Solution. Here, $f(x)$ is defined for all $x \ge 0$. Also, $f(x)$ is an increasing function in $[0, \infty)$.

Thus, range $= [f(0), f(\infty))$

Hence, range $= [1, \infty)$

Illustration 54 Find the domain and range of the function

$$y = \log_e(3x^2 - 4x + 5).$$

Solution. y is defined, if $3x^2 - 4x + 5 > 0$

where $D = 16 - 4(3)(5) = -44 < 0$

and coefficient of $x^2 = 3 > 0$

Hence, $(3x^2 - 4x + 5) > 0 \ \forall x \in R$

Thus, domain $\in R$

Now, $y = \log_e(3x^2 - 4x + 5)$

We have, $3x^2 - 4x + 5 = e^y$ or $3x^2 - 4x + (5 - e^y) = 0$

Since, x is real, thus discriminant ≥ 0.

ie, $(-4)^2 - 4(3)(5 - e^y) \ge 0 \Rightarrow 12e^y \ge 44$

So, $e^y \ge \dfrac{11}{3}$

Thus, $y \ge \log\left(\dfrac{11}{3}\right)$

Hence, range is $\left[\log\left(\dfrac{11}{3}\right), \infty\right]$

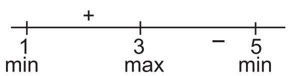
Aliter : Since, log is an increasing function and $3x^2 - 4x + 5$ is minimum at $x = \dfrac{2}{3}$ *ie*, $\log_e(3x^2 - 4x + 5)$ is minimum at $x = \dfrac{2}{3}$ and minimum value of $y = \log\dfrac{11}{3}$ *ie*, $\left(\log\dfrac{11}{3} \le y < \infty\right)$. Thus, range $\left[\log\left(\dfrac{11}{3}\right), \infty\right)$.

Illustration 55 Find the range of $y = \sqrt{x-1} + \sqrt{5-x}$.

Solution. Here, domain $x \geq 1$ and $x \leq 5$ *ie*, $x \in [1, 5]$.

Now, to check monotonicity,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x-1}} - \frac{1}{2\sqrt{5-x}}$$



when $\quad \dfrac{dy}{dx} = 0, \quad$ we get $x = 3$

Thus, $\quad y_{\min}$ at $x = 1$ or $x = 5$

$\quad y_{\min}$ at $x = 1$ is 2; $\quad y_{\min}$ at $x = 5$ is 2.

Thus, minimum value of $y = 2$.

Also, $\quad y_{\max}$ at $x = 3$ is $2\sqrt{2}$.

Hence, $\quad$ range $= [2, 2\sqrt{2}]$

Points to Consider

(a) When y is minimum at two or more points, the smallest value amongst them is taken.

(b) When y is maximum at two or more points, the largest value amongst them is taken.

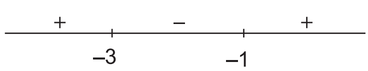
Illustration 56 Find the range of $\log_3\{\log_{1/2}(x^2 + 4x + 4)\}$.

Solution. Firstly, finding domain

$\log_3\{\log_{1/2}(x^2 + 4x + 4)\}$ exists, if

$\log_{1/2}(x^2 + 4x + 4) > 0$

$\Rightarrow \quad x^2 + 4x + 4 < \left(\dfrac{1}{2}\right)^0 \quad$ [using $\log_a x < b \Rightarrow x > a^b$, if $0 < a < 1$]



$\Rightarrow \quad x^2 + 4x + 4 < 1 \quad \Rightarrow \quad x^2 + 4x + 3 < 0$

$\Rightarrow \quad (x+1)(x+3) < 0$

$\Rightarrow \quad -3 < x < -1 \quad$ ...(i)

and $\quad x^2 + 4x + 4 > 0 \Rightarrow (x+2)^2 > 0$,

which is always true except for $x = -2$. $\quad$ ...(ii)

Thus, from Eqs. (i) and (ii), we have

$$x \in (-3, -2) \cup (-2, -1)$$

Therefore, $\quad$ domain $\in (-3, -2) \cup (-2, -1)$

Now, we find out the range.

Since, $\quad 0 < \log_{1/2}(x^2 + 4x + 4) < \infty \; \forall \; x \in$ domain y

$\Rightarrow \quad -\infty < \log_3\{\log_{1/2}(x^2 + 4x + 4)\} < \infty$

Thus, $\quad$ range $(y) \in R$.

Illustration 57 Range of the function $f(x) = (\cos^{-1}|1 - x^2|)$ is

(a) $\left[0, \frac{\pi}{2}\right]$ (b) $\left[0, \frac{\pi}{3}\right]$

(c) $(0, \pi)$ (d) $\left(\frac{\pi}{2}, \pi\right)$

Solution. Here, $f(x) = \cos^{-1}|1 - x^2|$ would exists, only when $|1 - x^2| \le 1$

$\Rightarrow$ $\quad -1 \le 1 - x^2 \le 1 \Rightarrow 2 \ge x^2 \ge 0$

or $\quad x \in [-\sqrt{2}, \sqrt{2}]$

$\therefore$ $\quad 0 \le |1 - x^2| \le 1, \ \forall\, x \in [-\sqrt{2}, \sqrt{2}]$

$\Rightarrow$ $\quad 0 \le \cos^{-1}(1 - x^2) \le \dfrac{\pi}{2}$

Hence, (a) is the correct answer.

Illustration 58 The range of the function $f(x) = \dfrac{1}{|\sin x|} + \dfrac{1}{|\cos x|}$ is

(a) $[2\sqrt{2}, \infty)$ (b) $(\sqrt{2}, 2\sqrt{2})$

(c) $(0, 2\sqrt{2})$ (d) $(2\sqrt{2}, 4)$

Solution. $f(x) = \dfrac{1}{|\sin x|} + \dfrac{1}{|\cos x|}$

Using AM $\ge$ GM, we get

$$\frac{\dfrac{1}{|\sin x|} + \dfrac{1}{|\cos x|}}{2} \ge \left(\frac{1}{|\sin x||\cos x|}\right)^{1/2}$$

$\Rightarrow$ $$\frac{1}{|\sin x|} + \frac{1}{|\cos x|} \ge 2\,(2\,|\operatorname{cosec} 2x\,|)^{1/2}$$

[where $(\operatorname{cosec} 2x) \ge 1$]

$\therefore$ $$\frac{1}{|\sin x|} + \frac{1}{|\cos x|} \ge 2\sqrt{2}$$

$\therefore$ Range of $f(x) \in [2\sqrt{2}, \infty)$.

Hence, (a) is the correct answer.

Illustration 59 If $z = x + iy$ and $x^2 + y^2 = 16$, then the range of $||x| - |y||$ is

(a) [0, 4] (b) [0, 2]

(c) [2, 4] (d) None of these

Solution. Here, $x = 4\cos\theta$, $y = 4\sin\theta$, then

$$|4|\cos\theta| - 4|\sin\theta|| = 4||\cos\theta| - |\sin\theta||$$
$$= 4\sqrt{1 - 2|\cos\theta||\sin\theta|}$$
$$= 4\sqrt{1 - |\sin 2\theta|}$$

$\therefore$ Range $\in [0, 4]$.

Hence, (a) is the correct answer.

Illustration 60 The range of $f(x) = \frac{1}{\pi}(\sin^{-1} x + \tan^{-1} x) + \frac{x+1}{x^2 + 2x + 5}$ is

(a) $\left[-\frac{3}{4}, \frac{1}{5}\right]$ (b) $\left[-\frac{5}{4}, \frac{3}{4}\right]$

(c) $\left[-\frac{3}{4}, \frac{5}{4}\right]$ (d) $\left[-\frac{3}{4}, 1\right]$

Solution. Here, $f(x) = \frac{1}{\pi}(\sin^{-1} x + \tan^{-1} x) + \frac{1}{(x+1) + \frac{4}{(x+1)}}$

$$= g(x) + h(x),$$

where domain of $g(x) \in [-1, 1]$

$\therefore$ Maximum value of $g(x) = g(1) = \frac{3}{4}$

and minimum value of $g(x) = g(-1) = -\frac{3}{4}$

Also, maximum value of $h(x)$ occurs, when $(x+1) + \frac{4}{(x+1)}$ is minimum at $x = 1$.

$\Rightarrow$ Range of $f(x) \in \left[-\frac{3}{4}, 1\right]$

Hence, (d) is the correct answer.

Illustration 61 The range of the function $\sin^2 x - 5 \sin x - 6$ is

(a) $[-10, 0]$ (b) $[-1, 1]$

(c) $[0, \pi]$ (d) $\left[-\frac{49}{4}, 0\right]$

Solution. Here, $f(x) = \sin^2 x - 5 \sin x - 6$

$$= \left(\sin^2 x - 5 \sin x + \frac{25}{4}\right) - 6 - \frac{25}{4}$$

$$= \left(\sin x - \frac{5}{2}\right)^2 - \frac{49}{4} \quad \text{...(i)}$$

where $$\frac{9}{4} \le \left(\sin x - \frac{5}{2}\right)^2 \le \frac{49}{4} \quad \text{...(ii)}$$

$\therefore$ From Eqs. (i) and (ii),

$$-10 \le f(x) \le 0$$

$\Rightarrow$ Range of $f(x) \in [-10, 0]$.

Hence, (a) is the correct answer.

Illustration 62 If $f(x) = [x^2] - [x]^2$, where $[\cdot]$ denotes the greatest integer function and $x \in [0, n]$, $n \in N$, then the number of elements in the range of $f(x)$ is

(a) $(2n + 1)$ (b) $4n - 3$

(c) $3n - 3$ (d) $2n - 1$

Solution. When $x = n-1$, $f(x) = (n-1)^2 - (n-1)^2 = 0$

when $n-1 < x < n$

$$[x] = n-1 \quad \therefore \quad (n-1)^2 \le [x^2] \le n^2 - 1$$

$$\Rightarrow \quad 0 \le [x^2] - [x]^2 \le n^2 - 1 - (n-1)^2$$

$\Rightarrow \quad 0 \le f(x) \le 2n - 2$, since $f(x)$ has to be an integer.

The set of values of $f(x)$ is $\{0, 1, 2, \ldots, 2n-2\}$.

Hence, (d) is the correct answer.

Illustration 63 Range of the function

$$f(x) = \sqrt{|\sin^{-1}|\sin x|| - \cos^{-1}|\cos x|}, \quad \text{is}$$

(a) $\{0\}$ (b) $\left[0, \sqrt{\dfrac{\pi}{2}}\right]$ (c) $[0, \sqrt{\pi}]$ (d) None of these

Solution. As, we know that

$$|\sin^{-1}|\sin x|| = \cos^{-1}|\cos x|, \forall x \in \text{domain}$$

$$\therefore \quad f(x) = \sqrt{|\sin^{-1}|\sin x|| - \cos^{-1}|\cos x|} = 0, \forall x \in \text{domain}$$

$\therefore$ Range of $f(x) \in \{0\}$

Hence, (a) is the correct answer.

Illustration 64 The number of values of y in $[-2\pi, 2\pi]$ satisfying the equation $|\sin 2x| + |\cos 2x| = |\sin y|$ is

(a) 3 (b) 4 (c) 5 (d) 6

Solution. Here, $1 \le |\sin 2x| + |\cos 2x| \le \sqrt{2}$ and $|\sin y| \le 1$.

So, solution is possible only when $|\sin y| = 1$.

$$\Rightarrow \quad \sin y = \pm 1$$

$$\Rightarrow \quad y = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$$

$\therefore$ Number of values of y is 4.

Hence, (b) is the correct answer.

Illustration 65 Let $f(x) = \cot^{-1}(x^2 - 4x + 5)$, then range of $f(x)$ is equal to

(a) $\left(0, \dfrac{\pi}{2}\right)$ (b) $\left(0, \dfrac{\pi}{4}\right]$ (c) $\left[0, \dfrac{\pi}{4}\right)$ (d) None of these

Solution. Here, $x^2 - 4x + 5$

$$\Rightarrow \quad (x-2)^2 + 1 \ge 1$$

$$\therefore \quad 1 \le x - 4x + 5 < \infty$$

$$\Rightarrow \quad 0 < \cot^{-1}(x^2 - 4x + 5) \le \frac{\pi}{4}$$

(using, $x_1 < x_2 \Rightarrow \cot^{-1} x_1 > \cot^{-1} x_2$, since $\cot^{-1} x$ is decreasing)

$$\Rightarrow \quad \text{Range of } f(x) \in \left(0, \frac{\pi}{4}\right]$$

Hence, (b) is the correct answer.

To Find Range for Rational Expressions

The range, for

$$y = \frac{ax^2 + bx + c}{px^2 + qx + r}$$

As, $\quad y = f(x) = \dfrac{ax^2 + bx + c}{px^2 + qx + r}$

Let $\quad A = q^2 - 4pr, B = 4ar + 4pc - 2bq, C = b^2 - 4ac$

Find $\quad \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A} = y_1, y_2$

Let $\quad y_1 < y_2$

$\Rightarrow \quad \begin{cases} \text{Range} \in [y_1, y_2], & \text{if } A < 0 \\ \text{Range} \in R - (y_1, y_2), & \text{if } A > 0 \end{cases}$

Illustration 66 Find the range of $f(x) = \dfrac{x^2 + 14x + 9}{x^2 + 2x + 3}$, where $x \in R$.

Solution. Here, $A = 4 - 12 = -8$, $B = 12 + 36 - 56 = -8$, $C = 160$

$\therefore \quad \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \dfrac{-1 \pm \sqrt{1 + 80}}{-2} = -5, 4$ and $A < 0$

$\therefore \quad \text{Range} \in [-5, 4]$

Illustration 67 For what real values of a does the range of $f(x) = \dfrac{x+1}{a + x^2}$ contains the interval $[0, 1]$?

Solution. Let $y = \dfrac{x+1}{a + x^2}$

$\Rightarrow \quad y = (a + x^2) = x + 1$

$\Rightarrow \quad yx^2 - x + (ay - 1) = 0$ has real roots for every $y \in [0, 1]$.

$\therefore \quad 1 - 4y(ay - 1) \geq 0$

$\Rightarrow \quad 1 - 4ay^2 + 4y \geq 0 \qquad \ldots\text{(i)}$

holds for $\quad 0 \leq y \leq 1$

Case I If $y = 0$

Here, $\quad y = 0$ is assumed at $x = -1$ for any $a \neq -1$.

For $\quad a = -1, y = \dfrac{x+1}{x^2 - 1}$ is undefined for $x = -1$.

$\therefore$ Value $y = 0$ is not assumed.

Case II If $0 < y \leq 1$

Put $\quad z = \dfrac{1}{y}$,

$\therefore \quad 1 \leq z < \infty$

$\therefore$ Eq. (i) holds, if $\quad 0 < y \le 1$, if $\dfrac{1}{y^2} - 4a + \dfrac{4}{y} \ge 0$

ie, $\quad z^2 + 4z - 4a \ge 0$ holds for $1 \le z < \infty$

$\Rightarrow \quad (z+2)^2 - 4 - 4a \ge 0$ holds for $1 \le z < \infty$

If $\quad (z+2)^2 - 4 - 4a \ge 9 - 4 - 4a \ge 0$

$\Rightarrow \quad 5 \ge 4a \qquad$ [as $(u+2)^2 \ge 9$ for $u = 1$]

$\Rightarrow \quad a \le \dfrac{5}{4}, a \ne -1$

$\therefore \quad a \in (-\infty, -1) \cup \left(-1, \dfrac{5}{9}\right)$

Illustration 68 Find the range of a function

$$f(x) = \tan^{-1}\{\log_{5/4}(5x^2 - 8x + 4)\}.$$

Solution. The given function is defined for $5x^2 - 8x + 4 > 0$ which is true $\forall\, x \in R$.

Since, coefficient of $x^2 = 5 > 0$ and $D = 64 - 80 = -16 < 0$

Let $\quad g(x) = 5x^2 - 8x + 4$

Here, $\quad a = 5 > 0$

$\therefore$ Range of $g(x) = \left[-\dfrac{D}{4a}, \infty\right) = \left[-\dfrac{-16}{4 \times 5}, \infty\right) = \left[\dfrac{4}{5}, \infty\right)$

As, $\quad \dfrac{4}{5} \le 5x^2 - 8x + 4 < \infty$

$\therefore \quad \log_{5/4}\left(\dfrac{4}{5}\right) \le \log_{5/4}(5x^2 - 8x + 4) < \log_{5/4}\infty$

$\Rightarrow \quad -1 \le \log_{5/4}(5x^2 - 8x + 4) < \infty$

$\Rightarrow \quad \tan^{-1}(-1) \le \tan^{-1}\{\log_{5/4}(5x^2 - 8x + 4)\} < \tan^{-1}(\infty)$

$\Rightarrow \quad -\dfrac{\pi}{4} \le f(x) < \dfrac{\pi}{2}$

$\therefore \quad R_f = \left[-\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

Illustration 69 Find the range of the function

$$f(x) = \frac{\sin^2 x + \sin x - 1}{\sin^2 x - \sin x + 2}$$

Solution. Let $\quad y = \dfrac{\sin^2 x + \sin x - 1}{\sin^2 x - \sin x + 2}$

Let $\quad t = \sin x \;\Rightarrow\; -1 \le t \le 1$

and $\quad y = \dfrac{t^2 + t - 1}{t^2 - t + 2}$

$\Rightarrow \quad (y-1)t^2 - (y+1)t + (2y+1) = 0,$

since t is real.

$\Rightarrow \quad \dfrac{3 - 2\sqrt{11}}{7} \le y \le \dfrac{3 + 2\sqrt{11}}{7} \qquad \ldots$(i)

Now,

Case I If both roots of Eq. (i) are greater than 1 *ie*, $t_1 > 1$ and $t_2 > 1$.

$$\Rightarrow \quad t_1 + t_2 > 2,$$

$$\text{and} \quad (t_1 - 1)(t_2 - 1) > 0 \Rightarrow \frac{y+1}{y-1} > 2$$

$$\text{and} \quad \frac{2y+1}{y-1} - \frac{y+1}{y-1} + 1 > 0 \Rightarrow 1 < y < 3$$

$$\text{and} \quad y > 1 \text{ or } y < \frac{1}{2}$$

$$\therefore \quad y \in (1, 3) \qquad \ldots\text{(ii)}$$

Case II If $t_1 < -1$ and $t_2 < -1$

$$\Rightarrow \quad t_1 + t_2 < -2 \text{ and } (t_1 + 1)(t_2 + 1) > 0$$

$$\Rightarrow \quad \frac{y+1}{y-1} + 2 < 0 \text{ and } \frac{2y+1}{y-1} + \frac{y+1}{y-1} + 1 > 0$$

$$\Rightarrow \quad \frac{1}{3} < y < 1 \text{ and } y > 1 \text{ or } y < -\frac{1}{4}$$

$$\Rightarrow \quad y \in \phi \qquad \ldots\text{(iii)}$$

Case III If $t_1 < -1$ and $t_2 > 1$

$$\Rightarrow \quad (t_1 + 1)(t_2 + 1) < 0 \text{ and } (t_1 - 1)(t_2 - 1) < 0$$

$$\Rightarrow \quad \frac{2y+1}{y-1} + \frac{y+1}{y-1} + 1 < 0 \text{ and } \frac{2y+1}{y-1} + \frac{y+1}{y-1} + 1 > 0$$

$$\Rightarrow \quad -\frac{1}{4} < y < 1 \quad \text{and} \quad \frac{1}{2} < y < 1 \Rightarrow \frac{1}{2} < y < 1$$

$$\therefore \quad R_f = \left[\frac{3 - 2\sqrt{11}}{7}, \frac{1}{2}\right] \cup \left[3, \frac{3 + 2\sqrt{11}}{7}\right] \cup \{1\}$$

General Definitions

Homogeneous Functions

A function is said to be homogeneous with respect to any set of variables, when each of its terms is of the same degree with respect to those variables.

For example, $5x^2 + 3y^2 - xy$ is homogeneous in x and y. Symbolically, if $f(tx, ty) = t^n \cdot f(x, y)$, then $f(x, y)$ is a homogeneous function of degree n.

Examples of Homogeneous Functions :

$$f(x, y) = \frac{x - y \cos x}{y \sin x + x}$$

is not a homogeneous function, and $f(x, y) = \frac{x}{y} \ln \frac{y}{x} + \frac{y}{x} \ln \frac{x}{y}$;

$\sqrt{x^2 - y^2} + x; x + y \cos \frac{y}{x}$ are homogeneous functions of degree one.

Bounded Functions

A function is said to be bounded, if $|f(x)| \le M$, where M is a finite quantity.

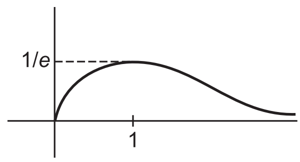
Illustration 70 Which of the following function(s) is(are) bounded on the intervals, as indicated ?

(a) $f(x) = 2^{\frac{1}{x-1}}$ on $(0, 1)$ (b) $g(x) = x \cos \dfrac{1}{x}$ on $(-\infty, \infty)$

(c) $h(x) = xe^{-x}$ on $(0, \infty)$ (d) $l(x) = \arctan 2^x$ on $(-\infty, \infty)$

Solution. (a) $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{h \to 0} 2^{\frac{1}{h-1}} = \dfrac{1}{2}$

$$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} 2^{\frac{1}{-h}} = 0$$



$\Rightarrow$ $f(x) \in (0, 1/2) \Rightarrow$ bounded

(c) $\lim\limits_{h \to 0} x\, e^{-x} = \lim\limits_{h \to 0} he^{-h} = 0$; $\lim\limits_{x \to \infty} xe^{-x} = \lim\limits_{x \to \infty} \dfrac{x}{e^x} = 0$

$\Rightarrow$ Also, $y = \dfrac{x}{e^x}$

$\Rightarrow$ $y' = \dfrac{e^x - xe^x}{e^{2x}}\, e^{x(1-x)} \Rightarrow h(x) = \left(0, \dfrac{1}{e}\right]$

(d) As, $-\infty < x < \infty$

$\Rightarrow$ $0 < 2^x < \infty \Rightarrow 0 < \tan^{-1}(2^x) < \dfrac{\pi}{2}$

$\Rightarrow$ $l(x) \in \left(0, \dfrac{\pi}{2}\right) \Rightarrow$ bounded.

Implicit and Explicit Functions

A function defined by an equation not solved for the dependent variable is called an *implicit function*. For *eg*, the equation $x^3 + y^3 = 1$ defines y as an *implicit function. If* y has been expressed in terms of x alone, then it is called an *explicit function.*

Examples of Implicit and Explicit Function $f(x, y) = 0$

(1) $x\sqrt{1+y} + y\sqrt{1+x} = 0$; explicit $y = -\dfrac{x}{1+x}$ or $y = x$ (rejected)

(2) $y^2 = x$ represents two separate branches *ie*, $y = \sqrt{x}$ and $y = -\sqrt{x}$ shown as

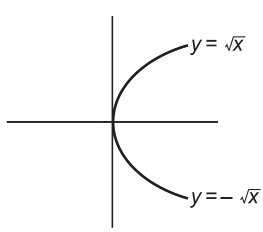


Fig. 3.29

Target Exercise 3.3

Directions (Q. Nos. 1 to 21) **:** *Find the range of the following :*

1. $f(x)=\sqrt{9-x^2}$

2. $f(x)=\dfrac{x}{1+x^2}$

3. $f(x)=\sin x+\cos x+3$

4. $f(x)=|x-1|+|x-2|,\ -1\le x\le 3$

5. $f(x)=\log_3(5+4x-x^2)$

6. $f(x)=\dfrac{x^2-2}{x^2-3}$

7. $f(x)=\dfrac{x^2+2x+3}{x}$

8. $f(x)=|x-1|+|x-2|+|x-3|$

9. $f(x)=\log_{[x-1]}\sin x$, where $[\cdot]$ denotes the greatest integer.

10. $f(x)=\cos^{-1}\sqrt{\log_{[x]}\dfrac{|x|}{x}}$, where $[\cdot]$ denotes the greatest integer.

11. $f(x)=\sqrt{[\sin 2x]-[\cos 2x]}$, where $[\cdot]$ denotes the greatest integer.

12. $f(x)=\sin^{-1}\left[x^2+\dfrac{1}{2}\right]+\cos^{-1}\left[x^2-\dfrac{1}{2}\right]$, where $[\cdot]$ denotes the greatest integer function.

13. $f(x)=\sin^{-1}(\sqrt{x^2+x+1})$

14. $f(x)=\cos^{-1}\left(\dfrac{x^2}{\sqrt{1+x^2}}\right)$

15. $f(x)=\sqrt{\log(\cos(\sin x))}$

16. $f(x)=\dfrac{x-1}{x^2-2x+3}$

17. $f(x)=\dfrac{\sin x}{\sqrt{1+\tan^2 x}}-\dfrac{\cos x}{\sqrt{1+\cot^2 x}}$

18. $f(x)=\dfrac{\tan(\pi[x^2-x])}{1+\sin(\cos x)}$

19. $f(x)=\dfrac{e^x}{[x+1]},\ x\ge 0$

20. $f(x)=[\,|\sin x|+|\cos x|\,]$, where $[\cdot]$ denotes the greatest integer function.

21. $f(x)=\sqrt{-x^2+4x-3}+\sqrt{\sin\dfrac{\pi}{2}\left(\sin\dfrac{\pi}{2}(x-1)\right)}$

22. Find the image of the following sets under the mapping $f(x)=x^4-8x^3+22x^2-24x+10$:

(i) $(-\infty, 1)$

(ii) $[1, 2]$

23. Find the domain and range of $f(x) = \log \left[\cos|x| + \frac{1}{2} \right]$, where $[\cdot]$ denotes the greatest integer function.

24. Find the domain and range of $f(x) = \sin^{-1}(\log[x]) + \log(\sin^{-1}[x])$, where $[\cdot]$ denotes the greatest integer function.

25. Find the domain and range for $f(x) = [\log(\sin^{-1}\sqrt{x^2 + 3x + 2})]$, where $[\cdot]$ denotes the greatest integer function.

Odd and Even Functions

Odd Functions

A function $f(x)$ is said to be an odd function, if $f(-x) = -f(x)$ for all x.

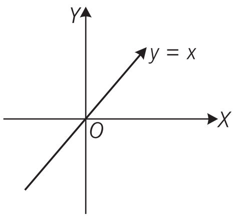


Fig. 3.30

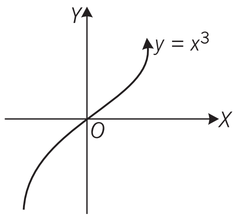


Fig. 3.31

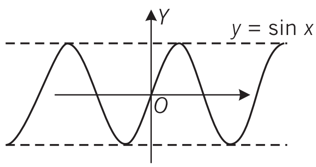


Fig. 3.32

Graph of an odd function is **symmetrical in opposite quadrants** *ie*, the curve in first quadrant is identical to the curve in the third quadrant and the curve in second quadrant is identical to the curve in fourth quadrant. Some graphs which are symmetrical in opposite quadrants (or about origin).

Even Functions

A function $f(x)$ is said to be an even if $f(-x) = f(x)$ for all x.

The graph is always **symmetrical about *y*-axis** *ie*, the graph on left hand side of y-axis is the mirror image of the curve on its right hand side.

Some graphs which are symmetrical about ***y*-axis** are

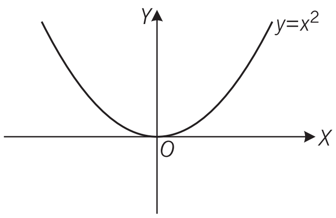


Fig. 3.33

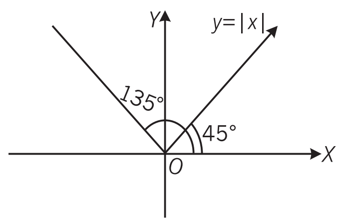


Fig. 3.34

Properties of Odd and Even Functions

(i) Product of two odd functions or two even functions is an even function.

(ii) Product of odd and even function is an odd function.

(iii) Every function $y = f(x)$ can be expressed as the sum of an even and odd function.

(iv) The derivative of an odd function is an even function and derivative of an even function is an odd function.
(v) A function which is even or odd, when squarred becomes an even function.
(vi) The only function which is even and odd both is $f(x) = 0$ *ie*, zero function.

Illustration 71 If f is an even function, find the real values of x satisfying the equation $f(x) = f\left(\dfrac{x+1}{x+2}\right)$. **[IIT JEE 1996, 2001]**

Solution. Since, $f(x)$ is even, so $f(-x) = f(x)$

Thus, $\quad x = \dfrac{x+1}{x+2} \quad$ or $\quad -x = \dfrac{x+1}{x+2}$

$\Rightarrow \quad x^2 + 2x = x + 1 \quad$ or $\quad -x^2 - 2x = x + 1$

$\Rightarrow \quad x^2 + x - 1 = 0 \quad$ or $\quad -x^2 - 3x - 1 = 0$

$\Rightarrow \quad x = \dfrac{-1 \pm \sqrt{5}}{2} \quad$ or $\quad x = \dfrac{-3 \pm \sqrt{5}}{2}$

Thus, $\quad x = \left\{\dfrac{-1+\sqrt{5}}{2}, \dfrac{-1-\sqrt{5}}{2}, \dfrac{-3+\sqrt{5}}{2}, \dfrac{-3-\sqrt{5}}{2}\right\}$

Illustration 72 Find out whether the given function is even, odd or neither even nor odd, where

$$f(x) = \begin{cases} x|x| & , \quad x \le -1 \\ [1+x] + [1-x] & , \quad -1 < x < 1 \\ -x|x| & , \quad x \ge 1 \end{cases}$$

where | | and [ ] represent the modulus and greatest integral functions.

Solution. The given function can be written as

$$f(x) = \begin{cases} -x^2 & , \quad x \le -1 \\ 2 + [x] + [-x] & , \quad -1 < x < 1 \\ -x^2 & , \quad x \ge 1 \end{cases}$$

(Using definition of modulus and the greatest integral functions)

$$\Rightarrow \quad f(x) = \begin{cases} -x^2 & , \quad x \le -1 \\ 2 - 1 + 0 & , \quad -1 < x < 0 \\ 2 & , \quad x = 0 \\ 2 + 0 - 1 & , \quad 0 < x < 1 \\ -x^2 & , \quad x \ge 1 \end{cases}$$

$$f(x) = \begin{cases} -x^2 & , \quad x \le -1 \\ 1 & , \quad -1 < x < 0 \\ 2 & , \quad x = 0 \\ 1 & , \quad 0 < x < 1 \\ -x^2 & , \quad x \ge 1 \end{cases}$$

which is clearly even as, if $f(-x) = f(x)$.

Thus, $f(x)$ is even.

Illustration 73 Find whether the given function is even or odd function, where

$f(x)=\dfrac{x\,(\sin x+\tan x)}{\left[\dfrac{x+\pi}{\pi}\right]-\dfrac{1}{2}}$, when $x \neq n\pi$, where [ ] denotes the greatest integer function.

Solution. $f(x)=\dfrac{x\,(\sin x+\tan x)}{\left[\dfrac{x+\pi}{\pi}\right]-\dfrac{1}{2}}=\dfrac{x\,(\sin x+\tan x)}{\left[\dfrac{x}{\pi}\right]+1-\dfrac{1}{2}}$

$$f(x)=\frac{x\,(\sin x+\tan x)}{\left[\dfrac{x}{\pi}\right]+0.5}$$

$$\Rightarrow \qquad f(-x)=\frac{-x\,(\sin(-x)+\tan(-x))}{\left[-\dfrac{x}{\pi}\right]+0.5}$$

$$\Rightarrow \qquad f(-x)=\begin{cases}\dfrac{x\,(\sin x+\tan x)}{-1-\left[\dfrac{x}{\pi}\right]+0.5}, & x \neq n\pi \\ 0, & x=n\pi\end{cases}$$

Hence, $f(-x)=-\left(\dfrac{x\,(\sin x+\tan x)}{\left[\dfrac{x}{\pi}\right]+0.5}\right)$ and $f(-x)=0$

$$f(-x)=-f(x)$$

Hence, $f(x)$ is an odd function (if $x \neq n\,\pi$).

Target Exercise 3.4

1. Determine whether the following functions are even or odd.

(i) $f(x)=\log\left(x+\sqrt{1+x^2}\right)$

(ii) $f(x)=x\left(\dfrac{a^x+1}{a^x-1}\right)$

(iii) $f(x)=\sin x+\cos x$

(iv) $f(x)=x^2-|x|$

(v) $f(x)=\log\left(\dfrac{1-x}{1+x}\right)$

(vi) $f(x)=\{(\operatorname{sgn} x)^{\operatorname{sgn} x}\}^n$; n is an odd integer.

(vii) $f(x)=\operatorname{sgn}(x)+x^2$

(viii) $f(x+y)+f(x-y)=2f(x)\cdot f(y)$; where $f(0)\neq 0$ and $x, y \in R$.

2. Determine whether function;

$f(x)=(-1)^{[x]}$ is even, odd or neither of two (where $[\cdot]$ denotes the greatest integer function).

3. A function, defined for all real numbers is defined for $x \geq 0$ as follows :

$$f(x) = \begin{cases} x\,|\,x\,|, & 0 \leq x < 1 \\ 2x \quad, & x \geq 1 \end{cases}$$

How, if f defined for $x \leq 0$.

If (i) f is even, (ii) f is odd ?

4. Show that the function, $f(x) = \dfrac{2x\,(\sin x + \tan x)}{2\left[\dfrac{x + 21\pi}{\pi}\right] - 41}$ is symmetric about origin.

5. If $f : [-20, 20] \to R$ defined by $f(x) = \left[\dfrac{x^2}{a}\right] \sin x + \cos x$ is an even function, then find the set of values of 'a'.

(where [·] denotes the greatest integral function.)

Periodic Functions

Definition A function $f(x)$ is said to be periodic function if, there exists a positive real number, such that

$$f(x + T) = f(x), \ \forall \ x \in R.$$

Then, $f(x)$ is periodic with period T, where T is least positive value.

Graphically If the graph repeats at fixed interval, then function is said to be periodic and its period is the width of that interval.

Illustration 74 Prove $\sin x$ is periodic and find its period.

Solution. Let $f(x) = \sin x$ and $T > 0$, then $f(x)$ is periodic, if $f(x + T) = f(x)$.

$\Rightarrow \quad \sin(x + T) = \sin(x), \ \forall \ x \in R \quad \Rightarrow \quad T = 2\pi, 4\pi, 6\pi, \ldots.$

But, period of $f(x)$ is smallest positive real number.

Thus, period of $f(x)$ is 2π.

Aliter : $f(x) = \sin x$ could be expressed graphically as shown in figure.

Here, graph repeats at an interval of 2π.

Thus, $f(x)$ is periodic with period 2π.

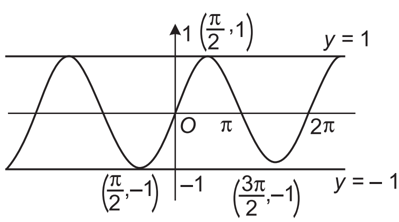


Illustration 75 Prove that $f(x) = x - [x]$ is periodic function. Also, find its period.

Solution. Let $T > 0$.

Then, $\quad f(x + T) = f(x), \forall \ x \in R$

$\Rightarrow \quad (x + T) - [x + T] = x - [x], \ \forall \ x \in R$

$\Rightarrow \quad [x + T] - [x] = T, \ \forall \ x \in R$

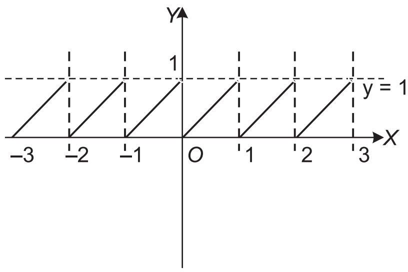
$\Rightarrow \quad T = 1, 2, 3, 4, \ldots \quad$ (since, subtraction of two integers)

The smallest value of T satisfying

$$f(x+T)=f(x) \text{ is } 1.$$

Thus, it is periodic with period 1.

Graphically $\quad f(x)=x-[x]=\{x\}$



Clearly, from the given graph, the function repeats itself at an interval of 1 unit.

Thus, $\quad$ period of $f(x)=1$.

Point to Consider

For those functions whose periods are not deducted by graphs, they can be judged by inspection method.

Illustration 76 Let $f(x)$ be periodic and k be a positive real number such that $f(x+k)+f(x)=0$ for all $x \in R$. Prove that $f(x)$ is a periodic with period $2k$.

Solution. We have,

$$f(x+k)+f(x)=0,\ \forall\ x \in R$$

$\Rightarrow \quad f(x+k)=-f(x),\ \forall\ x \in R$, put $x=x+k$

$\Rightarrow \quad f(x+2k)=-f(x+k),\ \forall\ x \in R \qquad$ [as $f(x+k)=-f(x)$]

$\Rightarrow \quad f(x+2k)=f(x),\ \forall\ x \in R$

which clearly shows that $f(x)$ is periodic with period $2k$.

Some Standard Results on Periodic Functions

Functions	Periods
(i) $\sin^n x$, $\cos^n x$ $\sec^n x$, $\text{cosec}^n x$	π, if n is even. 2π, (if n is odd or fraction).
(ii) $\tan^n x$, $\cot^n x$	π, n is even or odd.
(iii) $\lvert\sin x\rvert$, $\lvert\cos x\rvert$, $\lvert\tan x\rvert$ $\lvert\cot x\rvert$, $\lvert\sec x\rvert$, $\lvert\text{cosec}\, x\rvert$	π
(iv) $x-[x]$	1
(v) Algebraic functions *eg,* $\sqrt{x}$, x^2, x^3+5, ... etc.	Period doesn't exist.
(vi) $f(x)$ = constant	Periodic with no fundamental period.

Illustration 77 Find periods for

(i) $\cos^4 x$ (ii) $\sin^3 x$

(iii) $\cos\sqrt{x}$ (iv) $\sqrt{\cos x}$

Solution. (i) $\cos^4 x$ has a period π as n is even.

(ii) $\sin^3 x$ has period 2π as n is odd.

(iii) $\cos\sqrt{x}$ is not periodic as, for no value of T,

$$f(x+T) = f(x)$$

$$\Rightarrow \quad \cos\sqrt{x+T} = \cos(\sqrt{x})$$

Then, there exists no value for which $f(x+T) = f(x)$.

Hence, $\cos\sqrt{x}$ is not periodic.

(iv) $f(x) = \sqrt{\cos x}$ has the period 2π as n is in fraction.

Aliter : $f(x+T) = f(x)$

$$\Rightarrow \quad \sqrt{\cos(x+T)} = \sqrt{\cos(x)}$$

$$\Rightarrow \quad T = 2\pi, 4\pi, \ldots$$

But, T is the least positive value, hence $f(x)$ is periodic with period 2π.

Properties of Periodic Functions

(i) If $f(x)$ is periodic with period T, then

(a) $c \cdot f(x)$ is periodic with period T,
(b) $f(x+c)$ is periodic with period T,
(c) $f(x) \pm c$ is periodic with period T,
where c is any constant.
We know, $\sin x$ has period 2π.
Then, $f(x) = 5(\sin x) + 4$ *is also periodic with period* 2π.
ie, "If constant is added, subtracted, multiplied or divided in a periodic function, its period remains the same."

(ii) If $f(x)$ is periodic with period T, then

$kf(cx+d)$ has period $\dfrac{T}{|c|}$, *ie*, period is only affected by coefficient of x.

where, $k, c, d \in$ constant.

We know, $f(x) = \left\{7\sin\left(2x + \dfrac{\pi}{9}\right)\right\} - 12$ has the period $\dfrac{2\pi}{|2|} = \pi$; as $\sin x$, is periodic with period 2π.

(iii) If $f_1(x), f_2(x)$ are periodic functions with periods T_1, T_2 respectively, then we have $h(x) = f_1(x) + f_2(x)$ has period, as

$$= \begin{cases} \text{LCM of } \{T_1, T_2\}, \text{ if } h(x) \text{ is not an even function.} \\ \qquad\qquad OR \\ \dfrac{1}{2}\text{ LCM of } \{T_1, T_2\}, \text{ if } f_1(x) \text{ and } f_2(x) \text{ are complementary} \\ \qquad\qquad\qquad\qquad \text{pair-wise comparable functions.} \end{cases}$$

While taking LCM we should always remember,

(a) LCM of $\left(\dfrac{a}{b}, \dfrac{c}{d}, \dfrac{e}{f}\right) = \dfrac{\text{LCM of } (a, c, e)}{\text{HCF of } (b, d, f)}$

eg, $\quad \text{LCM of}\left(\dfrac{2\pi}{3}, \dfrac{\pi}{6}, \dfrac{\pi}{12}\right) = \dfrac{\text{LCM of } (2\pi, \pi, \pi)}{\text{HCF of } (3, 6, 12)} = \dfrac{2\pi}{3}$

$\therefore \quad \text{LCM of}\left(\dfrac{2\pi}{3}, \dfrac{\pi}{6}, \dfrac{\pi}{12}\right) = \dfrac{2\pi}{3}$

(b) LCM of rational with rational is possible.
LCM of irrational with irrational is possible.
But LCM of rational and irrational is not possible.
eg, LCM of $(2\pi, 1, 6\pi)$ is not possible, as $2\pi, 6\pi \in$ irrational and $1 \in$ rational.

Point to Consider

The LCM rule is not applicable, if function reduces to constant, as
eg, $\boldsymbol{f(x) = \sin^2 x + \cos^2 x}$.

Since, period of $\sin^2 x$ and $\cos^2 x$ are π.
$\therefore$ Period of $f(x) = \dfrac{1}{2}$ LCM $\{\pi, \pi\} = \dfrac{\pi}{2}$, which is not correct.

Whereas, $\sin^2 x + \cos^2 x = 1$ is a constant function and period is undetermined.
eg, $\boldsymbol{f(x) = \sec^2 x - \tan^2 x}$.

Period $= \dfrac{1}{2} \cdot$ LCM $\{\pi, \pi\} = \dfrac{\pi}{2}$, which is correct as $f(x)$ is discontinuous at $x = \left\{(2n+1)\dfrac{\pi}{2}\right\}, n \in Z$.

So, we must be particular whether the constant function is continuous or not.

Illustration 78 Find the period, if $f(x) = \sin x + \{x\}$, where $\{x\}$ is fractional part of x.

Solution. Here, $\sin x$ is periodic with period 2π, and $\{x\}$ is periodic with period 1.

Thus, $\quad$ LCM of 2π and 1

$\Rightarrow \quad$ does not exist

Thus, $\quad f(x)$ is not periodic.

Illustration 79 Find period of $f(x) = \tan 3x + \sin\left(\dfrac{x}{3}\right)$.

Solution. Period for $\tan 3x$ is $\left|\dfrac{\pi}{3}\right|$.

$$\text{Period for } \sin\frac{x}{3} \text{ is } \left|2\pi \times \frac{3}{1}\right| = |6\pi|$$

Thus, $\quad$ LCM of $\dfrac{\pi}{3}$ and $\dfrac{6\pi}{1} \Rightarrow \dfrac{6\pi}{1}$

Thus, $f(x)$ is periodic with period 6π.

Illustration 80 Find the period of

$$f(x) = \sin x + \tan\frac{x}{2} + \sin\frac{x}{2^2} + \tan\frac{x}{2^3} + \ldots + \sin\frac{x}{2^{n-1}} + \tan\frac{x}{2^n}.$$

Solution. We have,

Period of $\left(\sin x + \tan\frac{x}{2}\right)$ is 2π,

$\left(\sin\frac{x}{2^2} + \tan\frac{x}{2^3}\right)$ is $2^3\pi$,

....

....

$\left(\sin\frac{x}{2^{n-1}} + \tan\frac{x}{2^n}\right)$ is $2^n\pi$.

Thus, LCM of $\{2\pi, 2^3\pi, \ldots, 2^n\pi\} = 2^n\pi$.

Hence, the period of $f(x)$ is $2^n\pi$.

Illustration 81 Find the period of $f(x) = |\sin x| + |\cos x|$.

Solution. $|\sin x|$ has period π, and $|\cos x|$ has period π.

Here, $f(x)$ is an even function and $\sin x$, $\cos x$ are complementary.

Thus, period of $f(x) = \frac{1}{2}$ {LCM of π and π} $= \frac{\pi}{2}$

Thus, period for $f(x)$ is $\frac{\pi}{2}$.

Illustration 82 Find the period of $f(x) = \sin^4 x + \cos^4 x$.

Solution. $\sin^4 x$ and $\cos^4 x$ both has a period π. (as n is even)

But $f(x)$ is an even function and $\sin x$ and $\cos x$ are complementary.

Hence, $f(x)$ has period $= \frac{1}{2}$ {LCM of π, π} $= \frac{\pi}{2}$

Thus, period of $f(x)$ is $\frac{\pi}{2}$.

Illustration 83 Find the period of $f(x) = \cos(\cos x) + \cos(\sin x)$.

Solution. Here, $\cos(\cos x)$ has period π; as it is even, also $\cos(\sin x)$ has period π; as it is even.

Thus, period of $f(x) = \frac{1}{2}$ {LCM of π and π}

Hence, period of $f(x) = \frac{\pi}{2}$

Illustration 84 Find the period of $f(x) = \cos^{-1}(\cos x)$.

Solution. Here, $f(x) = \cos^{-1}(\cos x)$;

$f(x + T) = f(x)$

$\Rightarrow \quad \cos^{-1}\{\cos(x + T)\} = \cos^{-1}\{\cos(x)\}$

$\Rightarrow \quad T = 2\pi, 4\pi, 6\pi, \ldots, k$

But, T is the least positive value. Hence, $T = 2\pi$ or period is 2π.

Aliter : $f(x) = \cos^{-1}(\cos x)$ has period 2π, since $\cos x$ has period 2π.

(*ie*, In composition of function (fog) or (gof) are periodic, if $g(x)$ and $f(x)$ are periodic, respectively.)

Illustration 85 The period of $f(x) = \cos(|\sin x| - |\cos x|)$ is

(a) π (b) 2π (c) $\frac{\pi}{2}$ (d) None of these

Solution. As, $\cos\theta$ is even and $|\sin x| - |\cos x|$ has the period π.

$\therefore \quad \cos(|\sin x| - |\cos x|)$ has period $\frac{\pi}{2}$.

(*ie*, Half the period of $g(x)$, if $f(x)$ is even in fog).

Hence, (c) is the correct answer.

Illustration 86 Period of the function $f(x) = \sin(\sin(\pi x)) + e^{\{3x\}}$, where $\{.\}$ denotes the fractional part of x is

(a) 1 (b) 2 (c) 3 (d) None of these

Solution. As, $\sin(\pi x)$ has period $= \left|\frac{2\pi}{\pi}\right| = 2$

$\therefore \quad \sin(\sin(\pi x))$ has period 2

and $\quad e^{\{3x\}}$ has period $\frac{1}{3}\cdot$

$\therefore \quad$ Period of $f(x) = \sin(\sin(\pi x)) + e^{\{3x\}}$ is

LCM of $\left\{2, \frac{1}{3}\right\} = 2$

Hence, (b) is the correct answer.

Illustration 87 $\sin ax + \cos ax$ and $|\cos x| + |\sin x|$ are periodic functions of same fundamental period if 'a' equals to

(a) 0 (b) 1 (c) 2 (d) 4

Solution. Fundamental period of $|\sin x| + |\cos x|$ is $\frac{\pi}{2}\cdot$

Fundamental period of $(\sin ax + \cos ax)$ is $\frac{2\pi}{a}\cdot$

$\therefore \quad a = 4$

Hence, (d) is the correct answer.

Illustration 88 Let $f(x) = \sin x + \cos(\sqrt{4 - a^2})\,x$. Then, the integral values of 'a' for which $f(x)$ is a periodic function, are given by

(a) $\{2, -2\}$ (b) $(-2, 2]$

(c) $[-2, 2]$ (d) None of these

Solution. $f(x)$ will be periodic, if $\sqrt{4 - a^2}$ is a rational which is only possible when $(4 - a^2)$ is a perfect square.

$\Rightarrow \quad a = 0, 2, -2$ or $a \in \{-2, 0, 2\}$

Hence, (d) is the correct answer.

Illustration 89 Let $f(x) = \begin{cases} -1 + \sin K_1 \pi x, & x \text{ is rational} \\ 1 + \cos K_2 \pi x, & x \text{ is irrational} \end{cases}$

If $f(x)$ is a periodic function, then

(a) either $K_1, K_2 \in$ rational or $K_1, K_2 \in$ irrational

(b) $K_1, K_2 \in$ rational only

(c) $K_1, K_2 \in$ irrational only

(d) $K_1, K_2 \in$ irrational such that $\dfrac{K_1}{K_2}$ is rational

Solution. Range of $-1 + \sin K_1 \pi x$ is $[-2, 0]$ and range of $1 + \cos K_2\, \pi x$ is $[0, 2]$.

$\Rightarrow$ If $g(x) = -1 + \sin K_1 \pi x$

$\Rightarrow$ $g(x + T_1) = -1 + \sin K_1 \pi x$, where T_1 is a period of $1 + \sin K_1\, \pi x$

$\Rightarrow$ T_1 is rational $\Rightarrow$ period of $f(x)$ is rational.

$\Rightarrow$ K_1 and K_2 are rational.

Hence, (b) is the correct answer.

Illustration 90 If $f(x) = \tan^2 \dfrac{\pi x}{n^2 - 5n + 8} + \cot (n + m)\pi x$; $(n \in N, m \in Q)$, is a periodic function with 2 as its fundamental period, then m can't belong to

(a) $(-\infty, -2) \cup (-1, \infty)$

(b) $(-\infty, -3) \cup (-2, \infty)$

(c) $(-2, -1) \cup (-3, -2)$

(d) $\left(-3, -\dfrac{5}{2}\right) \cup \left(-\dfrac{5}{2}, -2\right)$

Solution. Period is LCM of $n^2 - 5n + 8$ and $\dfrac{1}{n + m}$.

$\Rightarrow$ $n^2 - 5n + 8 = 1$ or $n^2 - 5n + 8 = 2$

$\Rightarrow$ $n^2 - 5n + 7 = 0$ or $n^2 - 5n + 6 = 0$

Since, $n \in N$, $\therefore$ $n = 2, 3$

and $\left(\dfrac{1}{n + m}\right) K_1 = 2$, $(K_1 \in I)$ $\Rightarrow$ $\dfrac{1}{n + m} \not> 1$

$\therefore$ $\dfrac{1}{2 + m} \not> 1$ or $\dfrac{1}{3 + m} \not> 1$

$\Rightarrow$ $m \notin (-2, -1) \cup (-3, -2)$

Hence, (c) is the correct answer.

Illustration 91 Let $f(x)$ be a periodic function with period 3 and $f\left(-\dfrac{2}{3}\right) = 7$ and $g(x) = \int_0^x f(t + n)\, dt$, where $n = 3K, K \in N$. Then, $g'\left(\dfrac{7}{3}\right)$ is equal to

(a) $-\dfrac{2}{3}$

(b) 7

(c) -7

(d) $\dfrac{7}{3}$

Solution. $g'(x) = f(x+n) = f(x+(n-3)+3)$

$$= f(x+n-3)$$
$$= \dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots$$
$$= f(x+3) = f(x) \quad \Rightarrow \quad g'(x) = f(x)$$

$$\Rightarrow \qquad g'\left(\frac{7}{3}\right) = f\left(\frac{7}{3}\right) = f\left(-\frac{2}{3}+3\right) = f\left(-\frac{2}{3}\right) = 7$$

Hence, (b) is the correct answer.

Target Exercise 3.5

1. Find the periods of following functions :

(i) $f(x) = [\sin 3x] + |\cos 6x|$ (ii) $f(x) = \dfrac{1}{2}\left\{\dfrac{|\sin x|}{\cos x} + \dfrac{|\cos x|}{\sin x}\right\}$

(iii) $f(x) = e^{\cos^4 \pi x + x - [x] + \cos^2 \pi x}$ (iv) $f(x) = 3 \sin \dfrac{\pi x}{3} + 4 \cos \dfrac{\pi x}{4}$

(v) $f(x) = \cos 3x + \sin \sqrt{3}\pi x$ (vi) $f(x) = \sin \dfrac{\pi x}{n!} - \cos \dfrac{\pi x}{(n+1)!}$

(vii) $f(x) = x - [x - b]$ (viii) $f(x) = e^{\ln (\sin x)} + \tan^3 x - \operatorname{cosec}(3x - 5)$

2. Find the period of the real-valued function satisfying,

$f(x) + f(x+4) = f(x+2) + f(x+6)$

3. Check whether the function defined by

$f(x+\lambda) = 1 + \sqrt{2f(x) - f^2(x)}, \forall x \in R$ is periodic. If yes, then find its period.

4. Let $f(x+p) = 1 + \{2 - 3f(x) + 3(f(x))^2 - (f(x))^3\}^{1/3}$, $\forall x \in R$. where $p > 0$, then prove $f(x)$ is periodic.

5. Let $f(x)$ be a function such that ;

$f(x-1) + f(x+1) = \sqrt{3}\, f(x), \forall x \in R$. If $f(5) = 100$, then find $\sum\limits_{r=0}^{99} f(5+12r)$.

Composite Functions

Let us consider two functions, $f : X \to Y_1$ and $g : Y_1 \to Y$.

We define function $h : x \to y$

such that, $h(x) = g(f(x)) = (gof)(x)$.

To obtain $h(x)$, we first take f-image of an element $x \in X$ so that $f(x) \in Y_1$, which is the domain of $g(x)$. Then, we take g-image of $f(x)$, *ie*, $g(f(x))$ which would be an element of y.

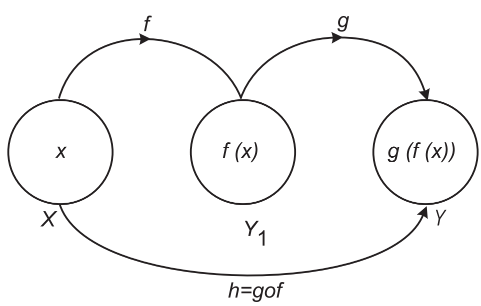


Fig. 3.35

The diagram below shows the steps to be taken :

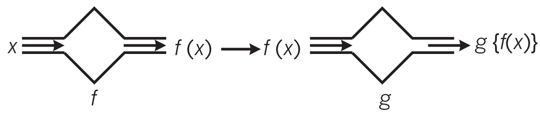


Fig. 3.36

The function h defined in the diagram is called composition of f and g, and is denoted by (gof). It can also expressed clearly as shown

domain $(gof) = \{x : x \in \text{domain}\,(f), f(x) \in \text{domain}\,(g)\}$

Similarly, we can write,

$$(fog)(x) = f\{g(x)\}$$

$= \{x : x \in \text{ domain } (g), g(x) \in \text{ domain } f\}$

In general, $fog \neq gof$.

Points to Consider

(a) It should be noted that gof exists, iff the range of $f \subseteq$ domain of g. Similarly, fog exists; iff, the range of $g \subseteq$ domain of f.

(b) The composite of functions is not commutative *ie*, $gof \neq fog$.

(c) The composite of functions is associative *ie*, if f, g, h are three functions such that $fo(goh)$ and $(fog)oh$ are defined, then $fo(goh) = (fog)oh$.

Associativity $f : (N) \to I_0 \qquad f(x) = 2x$

$$g : I_0 \to Q \qquad g(x) = \frac{1}{x}$$

$$h : Q \to R \qquad h(x) = e^{\frac{1}{x}}$$

$$(hog)of = ho(gof) = e^{2x}$$

(d) The composite of two bijections is a bijection *ie*, if f and g are two bijections such that gof is defined, then gof is also a bijection.

Proof Let $f : A \to B$ and $g : B \to C$ be two bijections. Then, gof exists such that

$gof : A \to C$.

We have to prove that gof is one-one and onto.

One-one Let $a_1, a_2 \in A$ such that $(gof)(a_1) = (gof)(a_2)$, then

$$(gof)(a_1) = (gof)(a_2) \quad \Rightarrow \quad g[f(a_1)] = g[f(a_2)]$$

$\Rightarrow \qquad f(a_1) = f(a_2) \qquad [\because g \text{ is one-one}]$

$\Rightarrow \qquad a_1 = a_2 \qquad [\because f \text{ is one-one}]$

$\therefore$ gof is also one-one function.

Onto Let $c \in C$, then $c \in C$

$\Rightarrow \qquad \exists\, b \in B$ s.t. $g(b) = c \qquad [\because g$ is onto$]$

and $\qquad b \in B \Rightarrow \exists\, a \in A$ s.t. $f(a) = b \qquad [\because f$ is onto$]$

Therefore, we see that

$$c \in C \Rightarrow \exists\, a \in A \text{ s.t. } (gof)(a) = g[f(a)] = g(b) = c$$

ie, Every element of C is the gof image of some element of A. As such gof is an onto function. Hence, gof being one-one and onto is a bijection.

Illustration 92 Let $f: A \to B$ and $g: B \to C$ be functions and $gof : A \to C$. Which of the following statements is true?

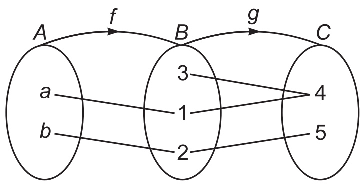
(a) If gof is one-one, then f and g both are one-one.

(b) If gof is one-one, then f is one-one.

(c) If gof is a bijection, then f is one-one and g is onto.

(d) If f and g are both one-one, then gof is one-one.

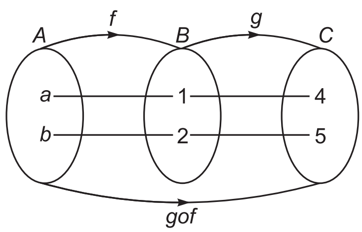
Solution. (a) As shown gof is one-one, but g is many-one.



$\Rightarrow$ (a) is not correct.

(b) If gof is one-one, then f is also one-one,

if f is many-one, then gof cannot be one-one.



(c) and (d) are obviously true.

Hence, (b), (c) and (d) are correct answers.

Illustration 93 If $f : R \to R, f(x) = x^2$ and $g : R \to R;\ g(x) = 2x + 1$. Then, find fog and gof also, show $fog \neq gof$.

Solution. $(gof)(x) = g\{f(x)\} = g\{x^2\}$

$$(gof)(x) = 2x^2 + 1$$

and $(fog)(x) = f\{g(x)\} = f(2x + 1)$

$$(fog)(x) = (2x + 1)^2$$

where $(2x^2 + 1) \neq (2x + 1)^2$. Therefore, $(gof) \neq (fog)$.

Point to Consider

Composition of functions can be expressed graphically. So, students are advised to try these functions graphically.

Illustration 94 Let $g(x)=1+x-[x]$ and $f(x)=\begin{cases}-1, & x<0\\ 0, & x=0\\ 1, & x>0\end{cases}$.

Then, for all x find $f(g(x))$. **[IIT JEE 2001]**

Solution. Here, $g(x)=1+x-[x]$

$\Rightarrow$ $g(x)=1+\{x\}$ [as $x-[x]=\{x\}$]

ie, $g(x)$ is greater than 1.

So, $f(g(x))=1$. Since, $f(x)=1$ for all $x>0$.

Thus, $f(g(x))=1$, for all $x\in R$.

Illustration 95 Let $f(x)=\begin{cases}1+x, & 0\le x\le 2\\ 3-x, & 2<x\le 3\end{cases}$, find $(fof)(x)$.

Solution. $fof(x)=f\{f(x)\}=\begin{cases}f(1+x), & 0\le x\le 2\\ f(3-x), & 2<x\le 3\end{cases}$

$$(fof)(x)=\begin{cases}f(1+x), & 0\le x\le 1\\ f(1+x), & 1\le x\le 2\\ f(3-x), & 2<x\le 3\end{cases}$$

$$(fof)(x)=\begin{cases}1+(1+x), & 0\le x\le 1\\ 3-(1+x), & 1\le x\le 2\\ 1+(3-x), & 2<x\le 3\end{cases}$$

$$(fof)(x)=\begin{cases}2+x, & 0\le x\le 1\\ 2-x, & 1<x\le 2\\ 4-x, & 2<x\le 3\end{cases}$$

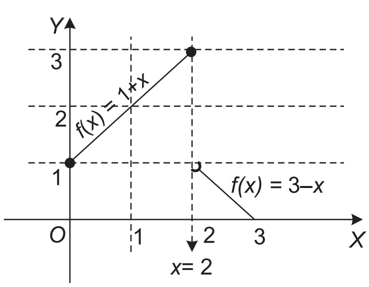
Aliter : $f(x)$ can be expressed graphically as shown in figure below; when

$0\le f(x)<1$; $2<x\le 3$ where $f(x)=3-x$.

$1\le f(x)\le 2$; $1\le x\le 2$ where $f(x)=1+x$.

$2\le f(x)\le 3$; $2\le x\le 3$ where $f(x)=1+x$.

Thus, $(fof)(x)=\begin{cases}1+f(x), & 0\le f(x)\le 2\\ 3-f(x), & 2<f(x)\le 3\end{cases}$



$$(fof)(x)=\begin{cases}1+f(x), & 0\le f(x)\le 1\\ 1+f(x), & 1<f(x)\le 2\\ 3-f(x), & 2<f(x)\le 3\end{cases}$$

$$(fof)(x)=\begin{cases}1+(3-x), & 2<x\le 3\\ 1+(1+x), & 0\le x\le 1\\ 3-(1+x), & 1<x\le 2\end{cases}$$

$$(fof)(x)=\begin{cases}4-x, & 2<x\le 3\\ 2+x, & 0\le x\le 1\\ 2-x, & 1<x\le 2\end{cases}$$

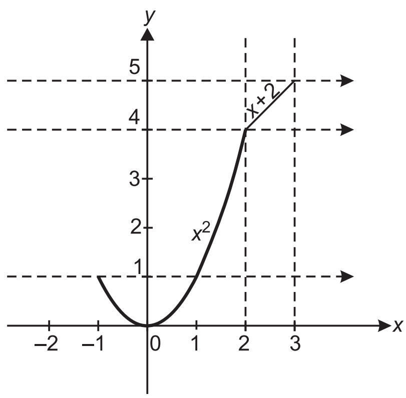
$$(fof)(x)=\begin{cases}2+x, & 0\le x\le 1\\ 2-x, & 1<x\le 2\\ 4-x, & 2<x\le 3\end{cases}$$

Illustration 96 Let $f(x)=\begin{cases}x+1, & x\le 1\\ 2x+1, & 1<x\le 2\end{cases}$

and $g(x)=\begin{cases}x^2, & -1\le x<2\\ x+2, & 2\le x\le 3\end{cases}$. Find (fog).

Solution. $f(g(x))=\begin{cases}g(x)+1, & g(x)\le 1\\ 2g(x)+1, & 1<g(x)\le 2\end{cases}$

Here, $g(x)$ becomes the variable that means we should draw the graph.
It is clear that $g(x)\le 1;\ \forall\ x\in[-1,1]$



and $\quad 1<g(x)\le 2;\ \forall\ x\in(1,\sqrt{2}]$.

$$\Rightarrow \qquad f(g(x))=\begin{cases}x^2+1, & -1\le x<+1\\ 2x^2+1, & 1<x\le\sqrt{2}\end{cases}$$

Properties of Composition of Functions

(i) f is even, g is even $\Rightarrow fog$ is even function.

(ii) f is odd, g is odd $\Rightarrow fog$ is odd function.

(iii) f is even, g is odd $\Rightarrow fog$ is even function.

(iv) f is odd, g is even $\Rightarrow fog$ is even function.

Illustration 97 If $f(x) = 2x + |x|$, $g(x) = \frac{1}{3}(2x - |x|)$ and $h(x) = f(g(x))$, then domain of $\underbrace{\sin^{-1}(h(h(h(h\ldots h(x)\ldots))))}_{n \text{ times}}$ is

(a) $[-1, 1]$ (b) $\left[-1, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right]$

(c) $\left[-1, -\frac{1}{2}\right]$ (d) $\left[\frac{1}{2}, 1\right]$

Solution. Since , $f(x) = \begin{cases} 2x + x, & x \geq 0 \\ 2x - x, & x < 0 \end{cases} = \begin{cases} 3x, & x \geq 0 \\ x, & x < 0 \end{cases}$

and $$g(x) = \frac{1}{3}\begin{cases} 2x - x, & x \geq 0 \\ 2x + x, & x < 0 \end{cases} = \begin{cases} \frac{x}{3}, & x \geq 0 \\ x, & x < 0 \end{cases}$$

$\therefore$ $$f(g(x)) = \begin{cases} 3\left(\frac{x}{3}\right), & x \geq 0 \\ x, & x < 0 \end{cases}$$

$\Rightarrow$ $$f(g(x)) = x, \forall\, x \in R$$

$\therefore$ $$h(x) = x$$

$\Rightarrow$ $$\sin^{-1}(h(h(h\ldots h(x)\ldots))) = \sin^{-1} x$$

$\therefore$ Domain of $\sin^{-1}(h(h(h(h\ldots h(x)..))))$ is $[-1, 1]$.

Hence, (a) is the correct answer.

Illustration 98 A function $f: R \to R$ satisfies

$\sin x \cos y(f(2x + 2y) - f(2x - 2y)) = \cos x \sin y(f(2x + 2y) + f(2x - 2y))$.

If $f'(0) = \frac{1}{2}$, then

(a) $f''(x) = f(x) = 0$ (b) $4f''(x) + f(x) = 0$

(c) $f''(x) + f(x) = 0$ (d) $4f''(x) - f(x) = 0$

Solution. We have, $\dfrac{f(2x + 2y)}{f(2x - 2y)} = \dfrac{\sin(x + y)}{\sin(x - y)}$

$\Rightarrow$ $$\frac{f(\alpha)}{\sin\frac{\alpha}{2}} = \frac{f(\beta)}{\sin\frac{\beta}{2}} = K$$

$\Rightarrow$ $$f(x) = K \sin\frac{x}{2}$$

$\Rightarrow$ $$f'(x) = \frac{K}{2}\cos\frac{x}{2}$$

and $$f''(x) = \frac{-K}{4}\sin\frac{x}{2}$$

$\Rightarrow$ $$4f''(x) + f(x) = 0$$

Hence, (b) is the correct answer.

Target Exercise 3.6

1. Consider the real-valued function satisfying $2f(\sin x)+f(\cos x)=x$. Find the domain and range of $f(x)$.

2. If $f(x)$ is defined in $[-3, 2]$, then find the domain of definition of $f([\,|x|\,])$ and $f([2x+3])$.

3. $f(x)=\begin{cases} x-1, & -1\le x\le 0 \\ x^2, & 0<x\le 1 \end{cases}$, and $g(x)=\sin x$.

Find $h(x)=f(|g(x)|)+|f(g(x))|$.

4. Let $f(x)$ be defined on $[-2, 2]$, and is given by

$$f(x)=\begin{cases} -1, & -2\le x\le 0 \\ x-1, & 0\le x\le 2 \end{cases} \text{ and } g(x)=f(|x|)+|f(x)|, \text{ then find } g(x).$$

5. Let two functions are defined as

$$g(x)=\begin{cases} x^2, & -1\le x<2 \\ x+2, & 2\le x\le 3 \end{cases}, \text{ and } f(x)=\begin{cases} x+1, & x\le 1 \\ 2x+1, & 1<x\le 2 \end{cases}, \text{ then find } gof.$$

Mapping of Functions

As discussed earlier, a function exists only if, "to every element in domain there exists unique image in the co-domain".

ie, To every element of A there exists one and only one element of B.

This is written as $f: A \to B$ and is read as f maps from A to B and this correspondence is denoted by $y=f(x)$.

Point to Consider

From definition, it follows that there may exist some elements in B, which may not have any corresponding element in set A.
But, there should not be any x left (element of A) for which there is no element in set B.

Kinds of Mappings

There are four kinds of mappings defined as

(i) one-one-onto or bijective *ie*, injective and surjective.
(ii) one-one-into or only injective, and not surjective.
(iii) many one-onto or non-injective but surjective.
(iv) many one-into or neither injective nor surjective.

One-one Mapping or Injective or Monomorphic

A function $f: A \to B$ is said to be one-one mapping or injective, if different elements of A have different images in B.

Thus, no two elements of set A can have the same f image.

Verbal Description Let us consider set $A = \{1, 3, 5\}$ and $B = \{3, 7, 11, 15\}$, where $f : A \to B$ and $f(x) = 2x + 1$, then

Here, every element in domain possess distinct images in co-domain.

Thus, $f(x)$ is one-one or injective.

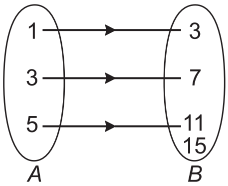


Fig. 3.37

Point to Consider

From above definition, following mappings are not one-one.

(i) $f : A \to B$ (ii) $f : A \to B$

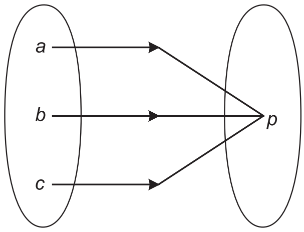


Fig. 3.38

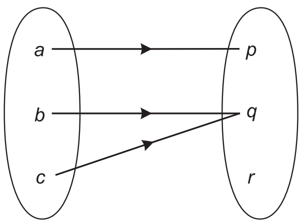


Fig. 3.39

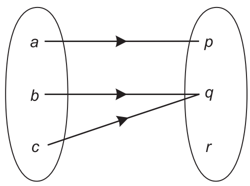


Fig. 3.40

whereas, $f : A \to B$ forms a mapping which is one-one.

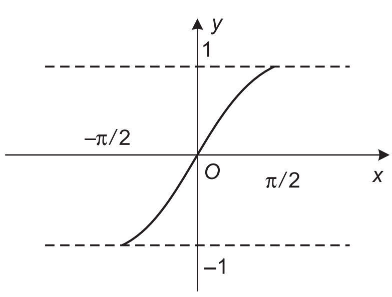
Method to Check One-one Mapping

***Method* 1. Theoretically** If $f(x) = f(y) \Rightarrow x = y$, then $f(x)$ is one-one.

***Method* 2. Graphically** A function is one-one iff no line parallel to x-axis meets the graph of function at more than one point.

Illustration 99 Let $f : [-\pi/2, \pi/2] \to [-1, 1]$ where $f(x) = \sin\ x$. Find whether $f(x)$ is one-one or not.

Solution. Here, $f : [-\pi/2, \pi/2] \to [-1, 1]$ indicates that domain $\in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, and co-domain $\in [-1, 1]$.



Thus, the graph of $f(x) = \sin\ x$ should be plotted in $[-\pi/2, \pi/2]$.

Which is clearly not intersected at more than one point by any straight line parallel to x–axis.

Thus, $f(x)$ is one-one.

***Method* 3. By Calculus** For checking whether $f(x)$ is one-one, find whether function is only increasing or only decreasing in its domain.

If yes, then one-one.

ie, if $f'(x) \geq 0, \ \forall \ x \in$ domain

or if $f'(x) \leq 0, \ \forall \ x \in$ domain, then one-one.

Illustration 100 Let $f:[-\pi/2, \pi/2] \to [-1, 1]$, where $f(x) = \sin x$. Find if one-one or not.

Solution. Here, $f'(x) = \cos x$, which is always positive in $[-\pi/2, \pi/2]$.

$\therefore \quad f'(x) \geq 0, \ \forall \ x \in [-\pi/2, \pi/2]$.

Thus, one-one.

Remark Students are advised to use the graphical or calculus method for finding **one-one**.

Number of One-one Mapping

If A and B are finite sets having m and n elements, then number of one-one function from A to B.

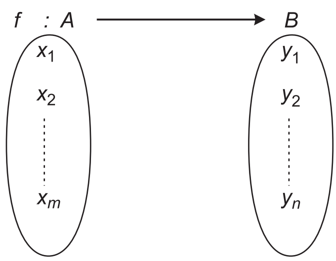


Fig. 3.41

Here, x_1 can take n images,

x_2 can take $(n-1)$ images,

x_3 can take $(n-2)$ images,

....

....

x_m can take $(n-m+1)$ images.

Thus, number of mapping $\Rightarrow n(n-1)(n-2)\ldots(n-m+1)$

$$\Rightarrow \quad \begin{cases} {}^nP_m, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$$

Many-one Mapping

A mapping $f:A \to B$ is said to be many-one function, if two or more elements of set A have the same image in B.

In other words; $f:A \to B$ is a many-one function, if it is not one-one function.

Verbal Description Let $f: A \to B$ and $g: x \to y$ be two functions represented by :

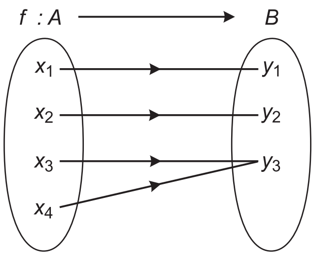


Fig. 3.42

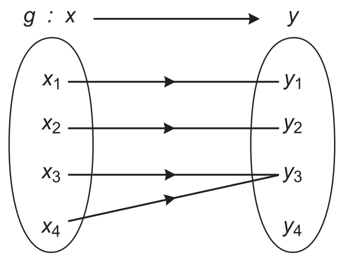


Fig. 3.43

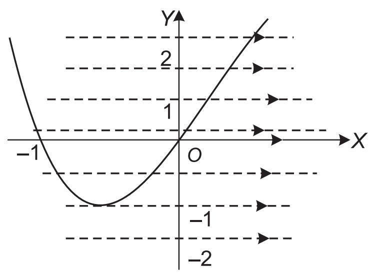
Clearly, f and g both are many-one as there are two elements x_3, x_4 which correspond to the same image.

ie, $f(x_3) = f(x_4) = y_3$. Thus, many-one.

Method to Check Many-one They are same as for one-one because, if mapping is not one-one it is many-one.

Illustration 101 Show $f: R \to R$ defined by $f(x) = x^2 + x$ for all $x \in R$ is many-one.

Solution. By graph



$f(x) = x^2 + x$ can be represented graphically as, shown in figure

where the straight line is parallel to x-axis, which meets the curve at two points. (*ie*, more than one point).

Thus, it is not one-one or many-one.

Aliter : By calculus,

$$f(x) = x^2 + x$$

$$\Rightarrow \quad f'(x) = 2x + 1$$

where $\quad f'(x) > 0, \quad \text{if} \quad x > -1/2$

and $\quad f'(x) < 0, \quad \text{if} \quad x < -1/2$

which shows $f(x)$ is neither increasing nor decreasing *ie*, not monotonic. Hence, many-one.

Onto Function or Surjective

If the function $f: A \to B$ is such that each element of B is the f image of at least one element in A. It is expressed as $f: A \to B$.

Here, range of f = co-domain.

ie, $f(A) = B$

Method to Show onto or surjective Find the range of $y = f(x)$ and show range of $f(x)$ = co-domain of $f(x)$.

> ***Point to Consider***
>
> If range = co-domain, then $f(x)$ is onto. Any polynomial of odd degree has range all real numbers and is onto for $f : R \to R$.

Into Function

A function of $f: A \to B$ is an into function, if there exists an element in B having no pre-image in A.

In other words, $f: A \to B$ is into function, if it is not onto function (mapping).

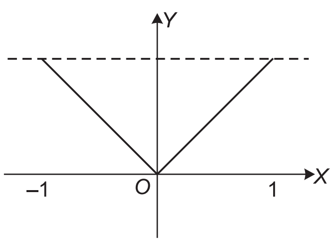
Illustration 102 Let $A = \{x : -1 \le x \le 1\} = B$ be a mapping $f: A \to B$. For each of the following functions from A to B, find whether it is surjective or bijective.

(a) $f(x) = |x|$ (b) $f(x) = x|x|$

(c) $f(x) = x^3$ (d) $f(x) = [x]$

(e) $f(x) = \sin \dfrac{\pi x}{2}$

Solution. (a) $f(x) = |x|$



Which shows many-one, as the straight line is parallel to x-axis and cuts at two points. Here, range for $f(x) \in [0, 1]$.

Which is clearly a subset of co-domain.

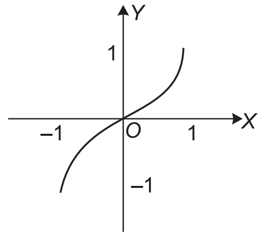
ie, $[0, 1] \subseteq [-1, 1]$

Thus, into.

Hence, function is many-one-into.

$\therefore$ Neither injective nor surjective.

(b) $f(x) = x|x|$



The graph shows that $f(x)$ is one-one, as the straight line parallel to x-axis cuts only at one point.

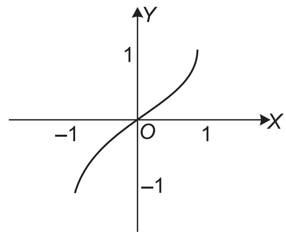
Here, range $f(x) \in [-1, 1]$

Thus, range = co-domain

Hence, onto.

Therefore, $f(x)$ is one-one onto or (bijective).

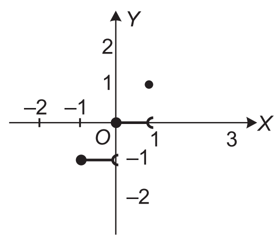
(c) $f(x) = x^3$



Graph shows $f(x)$ is one-one onto (*ie*, bijective).

(as explained in above example)

(d) $f(x) = [x]$



Graph shows that $f(x)$ is many-one, as the straight line parallel to x-axis meets at more than one points.

Here, range $f(x) \in \{-1, 0, 1\}$

which shows into as range $\subseteq$ co-domain.

Hence, many-one-into.

(e) $f(x) = \sin \dfrac{\pi x}{2}$

Graph shows $f(x)$ is one-one and onto as range = co-domain.

Therefore, $f(x)$ is bijective.

Number of Onto Functions If A and B are two sets having m and n elements respectively such that $1 \le n \le m$, then number of onto functions from A to B is

Coefficient of x^m in $m!(e^x - 1)^n$

$\Rightarrow$ Coefficient of x^m in

$$m!\{{}^nC_0 e^{nx} - {}^nC_1 e^{(n-1)x} + \ldots + (-1)^r \, {}^nC_r e^{rx} + \ldots + (-1)^n \, {}^nC_n\}$$

$$\Rightarrow \quad \sum_{r=1}^{n} (-1)^{n-r} \, {}^nC_r \, r^m$$

OR

Consider the set of all possible functions from m to n.

ie, $f : m \to n$

Now, let us define a subset Q such that

$$A_i = \{f \in Q \mid i \notin \text{Range of } f\}, \; i \in n$$

ie, $\bigcup_{i=1}^{n} A_i = \{f \text{ is injective}\}$

To find number of onto functions

$= $ Total number of functions − Number of injective functions

$= n^m - \{\text{Number of injective functions}\}$

$$= n^m - \left| \bigcup_{i=1}^{n} A_i \right| \qquad \ldots\text{(i)}$$

where $\bigcup_{i=1}^{n} A_i = \sum_{i=1}^{n} |A_i| - \sum_{i,j} A_i \cap A_j + \ldots$

$$= n \cdot (n-1)^m - {}^nC_2 (n-2)^m + \ldots + (-1)^{n-1} \cdot {}^nC_{n-1}(1)^m \qquad \ldots\text{(ii)}$$

$\therefore$ Number of onto functions $= n^m - \{n \cdot (n-1)^m - {}^nC_2(n-2)^m + \ldots + (-1)^{n-1} \cdot {}^nC_{n-1}(1)^m\}$

$$= \sum_{r=1}^{n} (-1)^{n-r}\, {}^nC_r \cdot r^m$$

Number of Functions

Let $X = \{x_1, x_2, x_3, \ldots, x_n\}$ (*ie*, n elements)

and $Y = \{y_1, y_2, y_3, \ldots, y_r\}$ (*ie*, r elements)

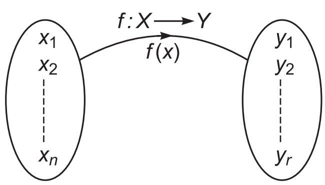


Fig. 3.44

(a) Total number of functions $= r^n$

$= (\text{Number of elements in co-domain})^{\text{number of elements in domain}}$

(b) Total number of one to one functions $= \begin{cases} {}^rC_n \cdot n!, & r \ge n \\ 0, & r < n \end{cases}$

(c) Total number of many-one functions $= \begin{cases} r^n - {}^rC_n \cdot n!, & r \ge n \\ r^n, & r < n \end{cases}$

(d) Total number of constant functions $= r$

(e) Total number of onto functions

$$= \begin{cases} r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - {}^rC_3(r-3)^n + \ldots, & r < n \\ r!, & r = n \\ 0, & r > n \end{cases}$$

(f) Total number of into functions

$$= \begin{cases} {}^rC_1(r-1)^n - {}^rC_2(r-2)^n + {}^rC_3(r-3)^n - \ldots, & r \le n \\ r^n, & r > n \end{cases}$$

Illustration 103 If $X = \{1, 2, 3, 4, 5\}$ and $Y = \{a, b, c, d, e, f\}$ and $f : X \to Y$, then find the total number of

(a) functions
(b) one to one functions
(c) many-one functions
(d) constant functions
(e) onto functions
(f) into functions

Solution. (a) Total number of functions $= 6^5 = 7776$

(b) Total number of one to one functions $= {}^6C_5 \cdot 5! = 6! = 720$
(c) Total number of many-one functions $= 6^5 - 6! = 7056$
(d) Total number of constant functions $= 6$
(e) Total number of onto functions $= 0$ (as $r > n$)
(f) Total number of into functions $= 6^5 = 7776$

Illustration 104 Find the number of surjections from A to B where
$A = \{1, 2, 3, 4\}, B = \{a, b\}$ **[IIT JEE 2000]**

Solution. Number of surjections from A to $B = \sum_{r=1}^{2} (-1)^{2-r}\, {}^2C_r (r)^4$

$$= (-1)^{2-1}\, {}^2C_1 (1)^4 + (-1)^{2-2}\, {}^2C_2 (2)^4 = -2 + 16 = 14$$

Therefore, number of onto mappings from A to $B = 14$.

Aliter : Total number of mapping from A to B is 2^4 of which two functions $f(x) = a$ for all $x \in A$ and $g(x) = b$ for all $x \in A$ are not surjective.

Thus, total number of surjection from A to $B = 2^4 - 2 = 14$

Number of One-one Onto Mappings or Bijections

If A and B are finite sets and $f : A \to B$ is a bijection.

Then, A and B have the same number of elements. If A has n elements, then number of bijection $A \to B$.

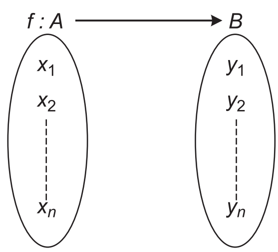


Fig. 3.45

Here, x_1 can take (n) images.
x_2 can take $(n-1)$ images.
x_3 can take $(n-2)$ images.
....
....
x_n can take 1 image.

Thus, total number of mappings are $n(n-1)(n-2)$ .... (2) (1)
Therefore, number of bijections from $A \to B = (n)!$

Equal and Identical Functions

Two functions f and g are said to be equal, if

(i) the domain of f = the domain of g,

(ii) the range of f = the range of g, and

(iii) $f(x) = g(x)$, $\forall x \in$ domain.

Examples of Equal Functions

(i) $f(x) = \ln x^2, g(x) = 2 \ln x$ (NI)

$f(x) = \text{cosec } x, g(x) = \dfrac{1}{\sin x}$ (I)

(ii) $f(x) = \cot(\cot^{-1} x), g(x) = x$ (I)

$f(x) = \tan x, g(x) = \dfrac{1}{\cot x}$ (NI)

$f(x) = \ln e^x, g(x) = e^{\ln x}$ (NI)

$f(x) = \sec x, g(x) = \dfrac{1}{\cos x}$ (I)

(iii) $f(x) = \sin^{-1}(3x - 4x^3), g(x) = 3 \sin^{-1} x$ (NI)

(iv) $f(x) = \sec^{-1} x + \text{cosec}^{-1} x, \; g(x) = \dfrac{\pi}{2}$ (NI)

(v) $f(x) = \cot^2 x \cdot \cos^2 x, \; g(x) = \cot^2 x - \cos^2 x$ (I)

(vi) $f(x) = \text{sgn}(x^2 + 1), g(x) = \sin^2 x + \cos^2 x$ (I)

(vii) $f(x) = \tan^2 x \cdot \sin^2 x, g(x) = \tan^2 x - \sin^2 x$ (I)

(viii) $f(x) = \sec^2 x - \tan^2 x, g(x) = 1$ (NI)

(ix) $f(x) = \log_x e, g(x) = \dfrac{1}{\log_e x}$ (I)

$f(x) = \text{sgn}(\cot^{-1} x), g(x) = \text{sgn}(x^2 - 4x + 5)$ (I)

$f(x) = \log_e x, g(x) = \dfrac{1}{\log_x e}$ (NI)

(x) $f(x) = \tan(\cot^{-1} x), g(x) = \cot(\tan^{-1} x)$ (I)

(xi) $f(x) = \sqrt{x^2 - 1}, g(x) = \sqrt{x - 1} \cdot \sqrt{x + 1}$ (NI)

$f(x) = \sqrt{1 - x^2}, g(x) = \sqrt{1 - x} \cdot \sqrt{1 + x}$ (I)

(xii) $f(x) = \tan x \cdot \cot x, g(x) = \sin x \cdot \text{cosec } x$ (NI)

(xiii) $f(x) = e^{\ln e^x}, g(x) = e^x$ (I)

(xiv) $f(x) = \sqrt{\dfrac{1 - \cos 2x}{2}}, g(x) = \sin x$ (NI)

(xv) $f(x) = \sqrt{x^2}, \; g(x) = (\sqrt{x})^2$ (NI)

(xvi) $f(x) = \log(x + 2) + \log(x - 3)$,

$g(x) = \log(x^2 - x - 6)$ (NI)

(xvii) $f(x) = \dfrac{1}{|x|}, g(x) = \sqrt{x^{-2}}$ (I)

(xviii) $f(x) = x\,|x|, g(x) = x^2 \text{ sgn } x$ (I)

(xix) $f(x) = \lim_{n \to \infty} \dfrac{x^{2n} - 1}{x^{2n} + 1}, g(x) = \text{sgn}\,(|x| - 1)$ (I)

(xx) $f(x) = \sin(\sin^{-1} x), g(x) = \cos(\cos^{-1} x)$ (I)

(xxi) $f(x) = \dfrac{1}{1 + \dfrac{1}{x}}, g(x) = \dfrac{x}{1+x}$ (NI)

(xxii) $f(x) = [\{x\}], g(x) = \{[x]\}$ (I)

[note that $f(x)$ and $g(x)$ are constant functions]

(xxiii) $f(x) = e^{\ln \cot^{-1} x}, g(x) = \cot^{-1} x$ (I)

(xxiv) $f(x) = e^{\ln \sec^{-1} x}, g(x) = \sec^{-1} x$ (NI)

Identical, if $x \in (-\infty, -1] \cup (1, \infty)$

(xxv) $F(x) = (fog)(x), G(x) = (gof)(x)$ where $f(x) = e^x, g(x) = \ln x$ (NI)

Illustration 105 If $f(x) = \log_{x^2} 25$ and $g(x) = \log_x 5$, then $f(x) = g(x)$ holds, now find the interval for x.

Solution. Domain of $f \in R - \{\pm 1, 0\}$

Domain of $g \in (0, \infty) - \{1\}$

For $f(x) = g(x)$,

domain of f = domain of g.

ie, $f(x) = g(x)$, if $x \in (0, \infty) - \{1\}$

Illustration 106 Let $A = \{1, 2\}, B = \{3, 6\}$ and $f : A \to B$ given by $f(x) = x^2 + 2$ and $g : A \to B$ given by $g(x) = 3x$. Find whether they equal or not.

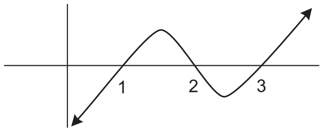
Solution. $f(1) = 3, f(2) = 6$

$g(1) = 3, g(2) = 6$

which shows $f(x)$ and $g(x)$ have same domains and co-domains, thus $f = g$.

Illustration 107 Show that $f : R \to R$ defined by $f(x) = (x-1)(x-2)(x-3)$ is surjective, but not injective.

Solution. Graphically,



$y = f(x) = (x-1)(x-2)(x-3)$,

which is clearly many-one and onto.

Illustration 108 $f(x) = x^3 + 3x^2 + 4x + b \sin x + c \cos x$, $\forall\, x \in R$ is a one-one function, then the value of $b^2 + c^2$ is

(a) ≥ 1 (b) ≥ 2 (c) ≤ 1 (d) None of these

Solution. Here, $f(x) = x^3 + 3x^2 + 4x + b \sin x + c \cos x$

$\Rightarrow$ $f'(x) = 3x^2 + 6x + 4 + b \cos x - c \sin x$

Now, for $f(x)$ to be one-one, the only possibility is

$$f'(x) \geq 0, \ \forall\, x \in R$$

ie, $3x^2 + 6x + 4 + b \cos x - c \sin x \geq 0, \forall\, x \in R$

ie, $3x^2 + 6x + 4 \geq c \sin x - b \cos x, \ \forall\, x \in R$

ie, $3x^2 + 6x + 4 \geq \sqrt{b^2 + c^2}, \ \forall\, x \in R$

ie, $\sqrt{b^2 + c^2} \leq 3(x^2 + 2x + 1) + 1, \forall\, x \in R$

$$\sqrt{b^2 + c^2} \leq 3(x+1)^2 + 1, \forall\, x \in R$$

$$\sqrt{b^2 + c^2} \leq 1, \forall\, x \in R$$

$\Rightarrow$ $b^2 + c^2 \leq 1, \forall\, x \in R$

Hence, (c) is the correct answer.

Illustration 109 Which pair of functions is identical?

(a) $\sin^{-1}(\sin x)$ and $\sin(\sin^{-1} x)$ (b) $\log_e e^x$, $e^{\log_e x}$

(c) $\log_e x^2, 2 \log_e x$ (d) None of these

Solution. Here, (a) $\sin^{-1}(\sin x)$ is defined for $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$,

while $\sin(\sin^{-1} x)$ is defined only for $x \in [-1, 1]$.

(b) $\log_e e^x$ is defined for all x, while $e^{\log_e x}$ is defined for $x > 0$.

(c) $\log_e x^2$ is defined for all $x \in R - \{0\}$, while $2 \log x$ is defined for $x > 0$.

$\therefore$ None is identical.

Hence, (d) is the correct answer.

Illustration 110 If $f : R \to \left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right)$, $f(x) = \sin^{-1}\left(\dfrac{x^2 - a}{x^2 + 1}\right)$ is a onto function, then set of values of 'a' is

(a) $\left\{-\dfrac{1}{2}\right\}$ (b) $\left[-\dfrac{1}{2}, -1\right)$ (c) $(-1, \infty)$ (d) None of these

Solution. Here, $f(x)$ is onto.

$\therefore$ $\dfrac{\pi}{6} \leq \sin^{-1}\left(\dfrac{x^2 - a}{x^2 + 1}\right) < \dfrac{\pi}{2}$

$\Rightarrow$ $\dfrac{1}{2} \leq \dfrac{x^2 - a}{x^2 + 1} < 1$

$\Rightarrow$ $\dfrac{1}{2} \leq 1 - \dfrac{(a+1)}{x^2 + 1} < 1, \forall\, x \in R \quad \Rightarrow \quad a + 1 > 0$

$\Rightarrow$ $a \in (-1, \infty)$

Hence, (c) is the correct answer.

Target Exercise 3.7

1. There are exactly two distinct linear functions, which map $[-1, 1]$ onto $[0, 3]$. Then, find the point of intersection of the two functions.
2. Let f be one-one function with domain $\{x, y, z\}$ and range $\{1, 2, 3\}$. It is given that exactly one of the following statement is true and remaining two are false.
$$f(x) = 1, f(y) \neq 1, f(z) \neq 2. \text{ Determine } f^{-1}(1).$$
3. Let $A = R - \{3\}$, $B = R - \{1\}$ and $f : A \to B$ defined by $f(x) = \dfrac{x-2}{x-3}$. Is 'f' bijective? Give reasons.
4. Let $f : R \to R$ defined by $f(x) = \dfrac{x^2}{1+x^2}$. Prove that f is neither injective nor surjective.
5. If the function $f : R \to A$, given by $f(x) = \dfrac{x^2}{x^2+1}$ is surjection, then find A.
6. If the function $f : R \to A$, given by $f(x) = \dfrac{e^x - e^{-|x|}}{e^x + e^{|x|}}$ is surjection, then find A.
7. Let $f(x) = ax^3 + bx^2 + cx + d \sin x$. Find the condition that $f(x)$ is always one-one function.
8. Let $f : X \to Y$ be a function defined by $f(x) = a \sin\left(x + \dfrac{\pi}{4}\right) + b \cos x + c$. If f is both one-one and onto, then find the sets X and Y.

Inverse of a Function

Let $f : A \to B$ be a one-one and onto function, then there exists a unique function.

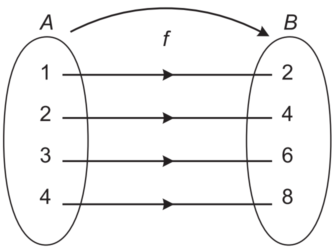


Fig. 3.46

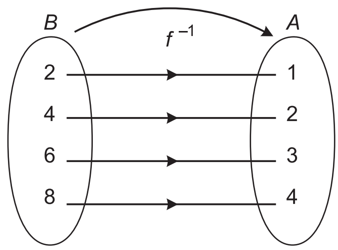


Fig. 3.47

$$g : B \to A \text{ such that } f(x) = y \Leftrightarrow g(y) = x, \ \forall \ x \in A \text{ and } y \in B.$$

Then, g is said to be inverse of f.

Thus, $\quad g = f^{-1} : B \to A = \{(f(x), x) \,|\, (x, f(x)) \in f\}$

Let us consider a one-one function with domain A and range B.

Where $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$ and $f : A \to B$ is given by $f(x) = 2x$, then write f and f^{-1} as a set of ordered pairs.

Here, member $y \in B$ arises from one and only one member $x \in A$.

So, $$f = \{(1, 2)(2, 4)(3, 6)(4, 8)\}$$

and $$f^{-1} = \{(2, 1)(4, 2)(6, 3)(8, 4)\}$$

Point to Consider

In above function,
Domain of $f = \{1, 2, 3, 4\} =$ Range of f^{-1}.
Range of $f = \{2, 4, 6, 8\} =$ Domain of f^{-1}.

Which represents for a function to have its inverse, it must be **one-one onto** or **bijective**.

Illustration 111 If $f(x) = 3x - 5$, then find $f^{-1}(x)$.

Solution. Here, $f(x) = 3x - 5$

which is clearly bijective as it is linear in x.

Thus, let $$f(x) = y$$

$$\Rightarrow \quad y = 3x - 5$$

or $$x = \frac{y+5}{3}$$

or $$f^{-1}(y) = \frac{y+5}{3} \qquad [\text{as } f(x) = y \Rightarrow x = f^{-1}(y)]$$

Therefore, $$f^{-1}(x) = \frac{x+5}{3}$$

Illustration 112 If $f : [1, \infty) \to [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$, then find $f^{-1}(x)$. (assume bijection). **[IIT JEE 2001, 2002]**

Solution. Let $y = f(x)$

$$\therefore \quad y = \frac{x^2+1}{x}$$

$$\Rightarrow \quad x^2 - xy + 1 = 0$$

$$\Rightarrow \quad x = \frac{y \pm \sqrt{y^2-4}}{2}$$

$$\Rightarrow \quad f^{-1}(y) = \frac{y \pm \sqrt{y^2-4}}{2} \qquad [\text{as } f(x) = y \Rightarrow x = f^{-1}(y)]$$

$$\Rightarrow \quad f^{-1}(x) = \frac{x \pm \sqrt{x^2-4}}{2}$$

Since, range of inverse function is $[1, \infty)$, therefore neglecting the negative sign, we have

$$f^{-1}(x) = \frac{x + \sqrt{x^2-4}}{2}$$

Illustration 113 Let $f(x) = x^3 + 3$ be bijective, then find its inverse.

Solution. Let $\quad y = x^3 + 3 \qquad [ie,\ y = f(x)]$

$$\Rightarrow \quad x^3 = y - 3$$
$$\Rightarrow \quad x = (y-3)^{1/3}$$
$$\Rightarrow \quad f^{-1}(y) = (y-3)^{1/3} \qquad [y = f(x) \Rightarrow f^{-1}(y) = x]$$
$$\Rightarrow \quad f^{-1}(x) = (x-3)^{1/3}$$

Thus, $\quad f^{-1}(x) = (x-3)^{1/3}$ when $f(x) = x^3 + 3$ is bijective.

Illustration 114 Find the inverse of the function, (assuming onto).

$$y = \log_a(x + \sqrt{x^2+1}), (a > 1).$$

Solution. The function $y = \log_a(x + \sqrt{x^2+1}), \forall\ x \in R.$

Since, $\quad \sqrt{x^2+1} > |x|$

Thus, it is defined for all x.

Now, $\quad y = \log_a(x + \sqrt{x^2+1})$, which is strictly increasing when $a > 1$.

Thus, one-one also given that $f(x)$ is onto.

Hence, the given function is invertible.

Now, $\quad y = \log_a(x + \sqrt{x^2+1}) \qquad$ [where $y = f(x)$]

$$\Rightarrow \quad a^y = x + \sqrt{x^2+1} \quad \text{and} \quad a^{-y} = \sqrt{x^2+1} - x$$
$$\Rightarrow \quad x = \frac{1}{2}(a^y - a^{-y})$$

Hence, the inverse in the form $y = f^{-1}(x)$ is,

$$y = \frac{1}{2}(a^x - a^{-x})$$

Graph of the Inverse of an Invertible Function

Let (h, k) be a point on the graph of the function f. Then, (k, h) is the corresponding point on the graph of inverse of f ie, g.

The line segment joining the points (h, k) and (k, h) is bisected at right angle by the line $y = x$.

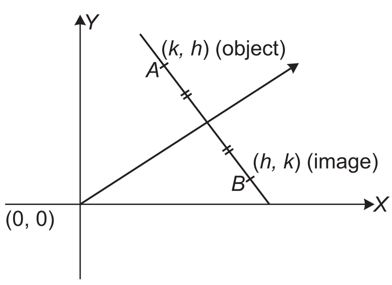


Fig. 3.48

So, that the two points play object-image role in the line $y = x$ as plane mirror.

It follows that the graph of $y=f(x)$ and its inverse written in form $y=g(x)$ are symmetrical about the line $y=x$.

The graphs $y=f(x)$ and $y=f^{-1}(x)$, if they intersect then they meet on the line $y=x$ only.

Hence, the solutions of $f(x)=f^{-1}(x)$ are also the solutions of $f(x)=x$.

Illustration 115 Let $f: R \to R$ be defined by $f(x)=\dfrac{e^x-e^{-x}}{2}$. Is $f(x)$ invertible? If so, find its inverse.

Solution. Let us check for invertibility of $f(x)$:

(a) **One-one** Here, $f'(x)=\dfrac{e^x+e^{-x}}{2}$

$\Rightarrow \quad f'(x)=\dfrac{e^{2x}+1}{2e^x}$, which is strictly increasing as $e^{2x}>0$ for all x.

Thus, it is one-one.

(b) **Onto** Let $y=f(x)$

$\Rightarrow \quad y=\dfrac{e^x-e^{-x}}{2}$, where y is strictly monotonic.

Hence, range of $f(x)=(f(-\infty), f(\infty))$

$\Rightarrow$ Range of $f(x)=(-\infty, \infty)$

So, range of $f(x)=$ co-domain

Hence, $f(x)$ is one-one and onto.

(c) **To find f^{-1}** $\quad y=\dfrac{e^{2x}-1}{2e^x}$

$\Rightarrow \quad e^{2x}-2e^x y-1=0 \quad \Rightarrow \quad e^x=\dfrac{2y\pm\sqrt{4y^2+4}}{2}$

$\Rightarrow \quad x=\log(y\pm\sqrt{y^2+1})$

$\Rightarrow \quad f^{-1}(y)=\log(y\pm\sqrt{y^2+1}) \qquad$ [as $f(x)=y \Rightarrow x=f^{-1}(y)$]

Since, $e^{f^{-1}(x)}$ is always positive.

So, neglecting the negative sign.

Hence, $f^{-1}(x)=\log(x+\sqrt{x^2+1})$

Illustration 116 Let $f:[1/2, \infty) \to [3/4, \infty)$, where $f(x)=x^2-x+1$. Find the inverse of $f(x)$. Hence, solve the equation $x^2-x+1=\dfrac{1}{2}+\sqrt{x-\dfrac{3}{4}}$.

Solution. (a) $f(x)=x^2-x+1$

$\Rightarrow f(x)=\left(x-\dfrac{1}{2}\right)^2+\dfrac{3}{4}$, which is clearly one-one and onto in given domain and co-domain.

(b) Thus, its inverse can be obtained.

Let $\quad f(x)=y$

$\Rightarrow \quad y=\left(x-\dfrac{1}{2}\right)^2+\dfrac{3}{4}$

$$\Rightarrow \qquad x-\frac{1}{2}=\pm\sqrt{y-\frac{3}{4}}$$

$$\Rightarrow \qquad x=\frac{1}{2}\pm\sqrt{y-\frac{3}{4}} \qquad [f(x)=y\Rightarrow x=f^{-1}(y)]$$

$$\Rightarrow \qquad f^{-1}(y)=\frac{1}{2}+\sqrt{y-\frac{3}{4}} \text{ [neglecting – ve sign as always +ve]}$$

$$\Rightarrow \qquad f^{-1}(x)=\frac{1}{2}+\sqrt{x-\frac{3}{4}}$$

(c) **To solve** $x^2-x+1=\frac{1}{2}+\sqrt{x-\frac{3}{4}}$, as $f(x)=f^{-1}(x)$ has only one solution.

ie, $\qquad f(x)=x$

$\Rightarrow \qquad x^2-x+1=x$

$\Rightarrow \qquad x^2-2x+1=0 \;\Rightarrow\; (x-1)^2=0$

$x=1$ is the required solution.

Properties of Inverse Functions

(i) *The inverse of a bijection is unique.*

Proof Let $f:A\to B$ be a bijection. If possible let $g:B\to A$ and $h:B\to A$ be two inverse functions of f. Also, let $a_1, a_2\in A$ and $b\in B$ such that $g(b)=a_1$ and $h(b)=a_2$, then

$$g(b)=a_1\Rightarrow f(a_1)=b$$

$$h(b)=a_2\Rightarrow f(a_2)=b$$

But, since f is one-one, so $f(a_1)=f(a_2)\Rightarrow a_1=a_2\Rightarrow g(b)=h(b), \forall\, b\in B$

(ii) *If $f:A\to B$ is a bijection and $g:B\to A$ is the inverse of f, then $fog=I_B$ and $gof=I_A$ where I_A* and *I_B are identity functions on the sets A* and *B, respectively.*

Note, that the graphs of f and g are the mirror images of each other in the line $y=x$. As shown, in the figure given below a point (x', y') corresponding to $y=x^2 (x\ge 0)$ changes to (y', x') corresponding to $y=+\sqrt{x}$, the changed form of $x=\sqrt{y}$.

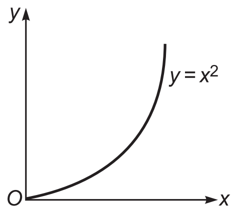


Fig. 3.49

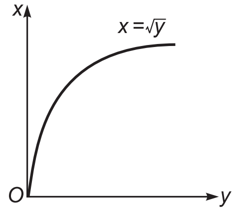


Fig. 3.50

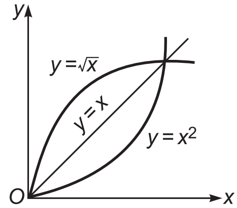


Fig. 3.51

Point to Consider

If $f(x)$ has its own inverse as in $f(x)=\frac{1}{x}$, then $f(x)=f^{-1}(x)$ will have infinite solutions but $f(x)=f^{-1}(x)=x$ will have only one solution.

(iii) *The inverse of a bijection is also a bijection.*

Proof Let $f : A \to B$ be a bijection and $g : B \to A$ be its inverse. We have to show that g is one-one and onto.

One-one Let $g(b_1) = a_1$ and

$$g(b_2) = a_2 : a_1, a_2 \in A \wedge b_1, b_2 \in B$$

Then, $g(b_1) = g(b_2)$

$\Rightarrow \quad a_1 = a_2$

$\Rightarrow \quad f(a_1) = f(a_2)$ $\quad$ [$\because f$ is a bijection]

$\Rightarrow \quad b_1 = b_2$ $\quad$ [$\because g(b_1) = a_1 \Rightarrow b_1 = f(a_1)$
$g(b_2) = a_2 \Rightarrow b_2 = f(a_2)$]

which proves that g is one-one.

Onto Again, if $a \in A$, then

$a \in A \Rightarrow \exists\, b \in B$ s.t. $f(a) = b$ $\quad$ [by definition of f]

$\Rightarrow \exists\, b \in B$ s.t. $g(b) = a$ $\quad$ [$\because f(a) = b \Rightarrow a = g(b)$]

which proves that g is onto.

Hence, g is also a bijection.

(iv) *If f and g are two bijections $f : A \to B$, $g : B \to C$, then the inverse of gof exists and $(gof)^{-1} = f^{-1}og^{-1}$.*

Proof Since, $f : A \to B$ and $g : B \to C$ are two bijections,

$\therefore \quad gof : A \to C$ is also a bijection.

[by theorem, the composite of two bijections is a bijection]

As such gof has an inverse function $(gof)^{-1} : C \to A$. We have to show, that $(gof)^{-1} = f^{-1}og^{-1}$.

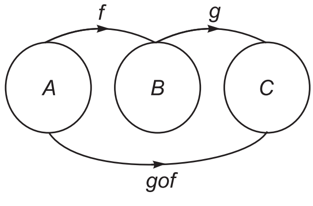


Fig. 3.52

Now, let $\quad a \in A, b \in B, c \in C$ such that

$f(a) = b$ and $g(b) = c$

So, $\quad (gof)(a) = g[f(a)] = g(b) = c$

Now, $\quad f(a) = b \Rightarrow a = f^{-1}(b)$ $\quad$ ...(i)

$g(b) = c \Rightarrow b = g^{-1}(c)$ $\quad$ ...(ii)

$(gof)(a) = c \Rightarrow a = (gof)^{-1}(c)$ $\quad$ ...(iii)

Also, $$(f^{-1}og^{-1})(c) = f^{-1}[g^{-1}(c)] \quad \text{[by definition]}$$
$$= f^{-1}(b) \quad \text{[by Eq. (ii)]}$$
$$= a \quad \text{[by Eq. (i)]}$$
$$= (gof)^{-1}(c) \quad \text{[by Eq. (iii)]}$$

$\therefore$ $$(gof)^{-1} = f^{-1}og^{-1},$$

which proves the theorem.

Illustration 117 Let $g(x)$ be the inverse of $f(x)$ and $f'(x) = \dfrac{1}{1+x^3}$. Then, find $g'(x)$ in terms of $g(x)$.

Solution. We know, if $g(x)$ is inverse of $f(x)$.

$\Rightarrow$ $$g\{f(x)\} = (x)$$

$\Rightarrow$ $$g'\{f(x)\} \cdot f'(x) = 1$$

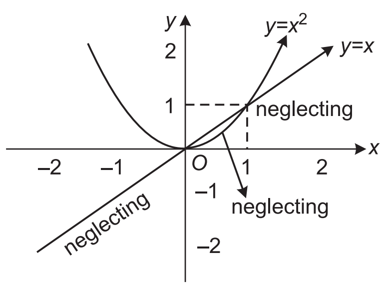
$\Rightarrow$ $$g'\{f(x)\} = \frac{1}{f'(x)} = 1 + x^3$$

$\Rightarrow$ $$g'\{f(g(x))\} = 1 + (g(x))^3$$

$\Rightarrow$ $$g'(x) = 1 + (g(x))^3 \quad [\because f(g(x)) = x]$$

Illustration 118 Let $f(x) = \max\{x, x^2\}$. Then, equivalent definition of $f(x)$.

Solution. **Note :** *These type of questions, where $f(x)$ are either maximum or minimum should be solved graphically for better representation.*



Let $$f_1(x) = x$$

and $$f_2(x) = x^2$$

Now, from graph for $f_1(x) = x$ and $f_2(x) = x^2$.

Here, neglecting the graph, *ie*, below point of intersection. Since, we want to find the maximum of two functions $f_1(x)$ and $f_2(x)$

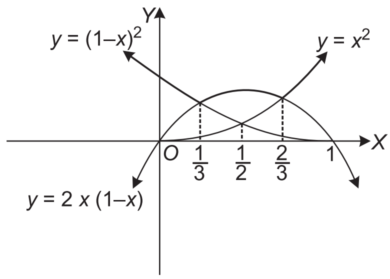
$\therefore$ $$f(x) = \begin{cases} x^2, & x \le 0 \text{ or } x \ge 1 \\ x, & 0 \le x \le 1 \end{cases}$$

Illustration 119 The equivalent definition of

$$f(x) = \max\{x^2, (1-x)^2, 2x(1-x)\}, \text{ where } 0 \le x \le 1.$$

Solution. Here, for maximum, let us consider $f_1(x) = x^2$, $f_2(x) = (1-x)^2$ and $f_3(x) = 2x(1-x)$. Now, taking graph for $f_1(x)$, $f_2(x)$ and $f_3(x)$.

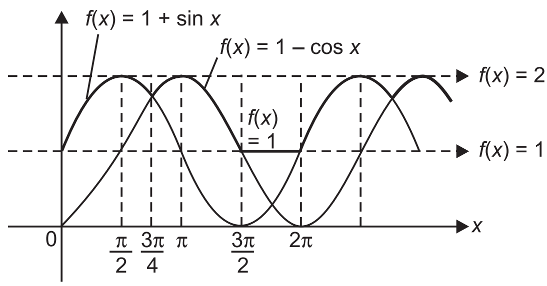
Here, neglecting the graph *ie*, below point of intersection. Since, we want to find the maximum of three functions $f_1(x)$, $f_2(x)$ and $f_3(x)$:



$$\therefore \qquad f(x) = \begin{cases} (1-x)^2, & 0 \le x \le \dfrac{1}{3} \\ 2x(1-x), & \dfrac{1}{3} \le x \le \dfrac{2}{3} \\ x^2, & \dfrac{2}{3} \le x \le 1 \end{cases}$$

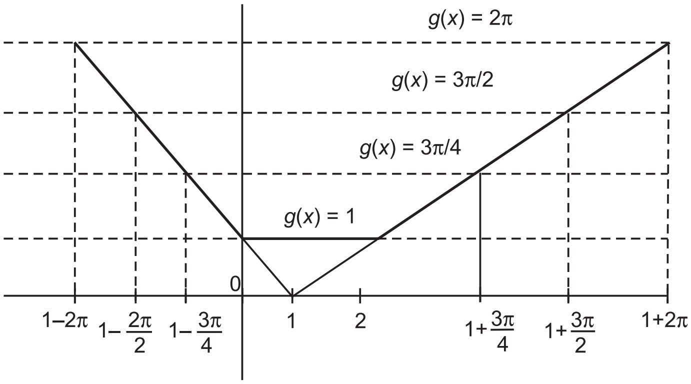
Illustration 120 Let $f(x) = \max\{1 + \sin x, 1 - \cos x, 1\} \;\forall\; x \in [0, 2\pi]$ and $g(x) = \max\{1, |x-1|\} \;\forall\; x \in R$. Determine $f\{g(x)\}$ and $g\{f(x)\}$ in terms of x.

Solution. Here, $f(x) = \max\{1 + \sin x, 1 - \cos x, 1\}$. Graphically it can be shown as



$$f(x) = \begin{cases} 1 + \sin x, & 0 \le x \le \dfrac{3\pi}{4} \\ 1 - \cos x, & \dfrac{3\pi}{4} < x \le \dfrac{3\pi}{2} \\ 1, & \dfrac{3\pi}{2} < x \le 2\pi \end{cases} \qquad \text{(using above graph)}$$

Again, $g(x) = \max\{1, |x-1|\}$, graphically it can be shown as



$$\therefore \qquad g(x) = \begin{cases} 1-x, & x \le 0 \\ 1, & 0 < x \le 2 \\ x-1, & x > 2 \end{cases}$$

Now, $$g\{f(x)\} = \begin{cases} 1-f(x), & f(x) \le 0 \\ 1, & 0 < f(x) \le 2 \\ f(x)-1, & f(x) > 2 \end{cases}$$

Since, $\quad f(x) \in [1, 2], \forall\, x \in [0, 2\pi]$

$\quad g(f(x)) = 1, \forall\, x \in [0, 2\pi]$

Also, $$f\{g(x)\} = \begin{cases} 1+\sin\{g(x)\}, & 0 \le g(x) \le \dfrac{3\pi}{4} \\ 1-\cos\{g(x)\}, & \dfrac{3\pi}{4} < g(x) \le \dfrac{3\pi}{2} \\ 1, & \dfrac{3\pi}{2} < g(x) \le 2\pi \end{cases}$$

$$\Rightarrow \qquad f(g(x)) = \begin{cases} 1, & 1-2\pi \le x < 1-3\pi/2 \\ 1-\cos(1-x), & 1-3\pi/2 \le x < 1-3\pi/4 \\ 1+\sin(1-x), & 1-3\pi/4 \le x \le 0 \\ 1+\sin 1, & 0 < x \le 2 \\ 1+\sin(x-1), & 2 < x \le 1+3\pi/4 \\ 1-\cos(x-1), & 1+3\pi/4 < x \le 1+3\pi/2 \\ 1, & 1+3\pi/2 < x \le 2\pi+1 \end{cases}$$

Illustration 121 If $f : R \to R$ is defined by $f(x) = x^2 + 1$, then find the value of $f^{-1}(17)$ and $f^{-1}(-3)$.

Solution. $f(x) = x^2 + 1$

$f^{-1}(17) \Rightarrow f(x) = 17$

$\Rightarrow \quad x^2 + 1 = 17$

$\Rightarrow \quad x = \pm 4 \quad$ and $\quad f^{-1}(-3)$

$\Rightarrow \quad f(x) = -3$

$\Rightarrow \quad x^2 + 1 = -3$

$\Rightarrow \quad x^2 = -4 \qquad$ (which is not possible)

Hence, $\quad x = \{+4, -4\}$

Illustration 122 If the function f and g be defined as $f(x) = e^x$ and $g(x) = 3x - 2$ where $f : R \to R$ and $g : R \to R$, then find the function fog and gof. Also, find the domain of $(fog)^{-1}$ and $(gof)^{-1}$.

Solution. $(fog)(x) = f\{g(x)\}$

$$\Rightarrow \qquad f\{g(x)\} = f(3x - 2)$$

$$\Rightarrow \qquad f\{g(x)\} = e^{3x - 2} \qquad \text{...(i)}$$

and $\qquad (gof)(x) = g\{f(x)\}$

$$\Rightarrow \qquad g\{f(x)\} = g\{e^x\}$$

$$\Rightarrow \qquad g\{f(x)\} = 3e^x - 2 \qquad \text{...(ii)}$$

For finding $(fog)^{-1}$ and $(gof)^{-1}$,

let $\qquad (fog)(x) = y = e^{3x - 2}$

$$\Rightarrow \qquad 3x - 2 = \log y \quad \Rightarrow \quad x = \frac{\log y + 2}{3}$$

$$\Rightarrow \qquad (fog)^{-1} y = \frac{\log y + 2}{3} \quad \text{and} \quad (fog)^{-1} x = \frac{\log x + 2}{3}$$

and domain of $(fog)^{-1}$ is $x > 0$, *ie*, $x \in (0, \infty)$.

Again, let $\qquad (gof)x = y = 3e^x - 2$

$$\Rightarrow \qquad e^x = \frac{y + 2}{3} \quad \Rightarrow \quad x = \log\left(\frac{y + 2}{3}\right)$$

$$\Rightarrow \qquad (gof)^{-1} y = \log\left(\frac{y + 2}{3}\right) \quad \Rightarrow \quad (gof)^{-1} x = \log\left(\frac{x + 2}{3}\right)$$

and domain of $\qquad (gof)^{-1}$ is $\dfrac{x + 2}{3} > 0$.

Hence, domain of $\quad (gof)^{-1}$ is $x > -2$, *ie*, $x \in (-2, \infty)$.

Illustration 123 If $f(x) = ax + b$ and the equation $f(x) = f^{-1}(x)$ be satisfied by every real value of x, then

(a) $a = 2, b = -1$ (b) $a = -1, b \in R$ (c) $a = 1, b \in R$ (d) $a = 1, b = -1$

Solution. If $f(x) = ax + b$

$$\Rightarrow \qquad f^{-1}(x) = \frac{x}{a} - \frac{b}{a}$$

Since, $\qquad f(x) = f^{-1}(x), \forall\, x \in R$

$$\Rightarrow \qquad \frac{1}{a} = a \quad \text{and} \quad b = -\frac{b}{a}$$

$$\Rightarrow \qquad a = -1 \quad \text{and} \quad b \in R$$

Hence, (b) is the correct answer.

Illustration 124 If $g(x)$ is the inverse of $f(x)$ and $f'(x) = \sin x$, then $g'(x)$ is equal to

(a) $\sin(g(x))$ (b) $\operatorname{cosec}(g(x))$ (c) $\tan(g(x))$ (d) None of these

Solution. Given, $g(x) = f^{-1}(x)$

So, $\qquad x = f(g(x))$

Differentiating w.r.t. 'x', we get $1 = f'(g(x)) \cdot g'(x)$

Therefore, $$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\sin(g(x))}$$

$\therefore$ $$g'(x) = \text{cosec}(g(x))$$

Hence, (b) is the correct answer.

Illustration 125 If A and B are the points of intersection of $y = f(x)$ and $y = f^{-1}(x)$, then

(a) A and B necessarity lie on the line $y = x$

(b) A and B must be coincident

(c) Slope of line AB may be –1

(d) None of the above

Solution. If solution of $f(x) = f^{-1}(x)$ doesn't lie on $y = x$, then they must be of the form (α, β) and (β, α).

$\therefore$ Slope of line AB may be -1.

Hence, (c) is the correct answer.

General Results

If x, y are independent variable, then

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$.

(ii) $f(xy) = f(x) \cdot f(y) \Rightarrow f(x) = x^n$, $n \in R$.

(iii) $f(x + y) = f(x) \cdot f(y) \Rightarrow f(x) = a^{kx}$.

(iv) $f(x + y) = f(x) = f(y) \in f(x) = k$, where k is constant.

(v) $f(x)$ takes rational values for all $x \Rightarrow f(x)$ is a constant function.

(vi) By considering a general nth degree polynomial and writing the expression,

$$f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

$\Rightarrow$ $$f(x) = \pm x^n + 1$$

Target Exercise 3.8

1. Find the inverse of the following functions :

(i) $f(x) = \sin^{-1}\left(\frac{x}{3}\right)$, $x \in [-3, 3]$

(ii) $f(x) = \log_e(x^2 + 3x + 1)$, $x \in [1, 3]$

(iii) $f(x) = 5^{\log_e x}$, $x > 0$

(iv) $f(x) = \log_e(x + \sqrt{x^2 + 1})$

(v) $f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \le x \le 4 \\ 8\sqrt{x}, & x > 4 \end{cases}$

2. If the function $f : [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then find $f^{-1}(x)$.

Functional Equations

We should consider certain examples to make the concept clear.

Illustration 126 For $x \in R$, the function $f(x)$ satisfies $2f(x) + f(1-x) = x^2$. Then, the value of $f(4)$ is equal to

(a) $\frac{13}{3}$ (b) $\frac{43}{3}$

(c) $\frac{23}{3}$ (d) None of these

Solution. If we observe the given equation, then $f(x)$ and $f(1-x)$ are used *ie*, x is to be replaced by $(1-x)$.

$$\therefore \quad 2f(x) + f(1-x) = x^2 \; \} \times 2$$

$$\Rightarrow \quad \underline{\overset{-}{}2f(1-x) \overset{-}{+} f(x) \overset{-}{=} (1-x)^2} \qquad \text{[Replace, } x \to (1-x)\text{]}$$

$$3f(x) = 2x^2 - (1-x)^2$$

$$\Rightarrow \quad f(x) = \frac{1}{3}\{x^2 + 2x - 1\}$$

$$\therefore \quad f(4) = \frac{1}{3}\{16 + 8 - 1\} = \frac{23}{3}$$

Illustration 127 For $x \in R - \{1\}$, the function $f(x)$ satisfies $f(x) + 2f\left(\frac{1}{1-x}\right) = x$. Find $f(2)$.

Solution. Given, $f(x) + 2f\left(\frac{1}{1-x}\right) = x$ ...(i)

As, $x \to \frac{1}{1-x}$

$$\Rightarrow \quad f\left(\frac{1}{1-x}\right) + 2f\left(\frac{x-1}{x}\right) = \frac{1}{1-x} \qquad \text{...(ii)}$$

Again, $x \to \frac{1}{1-x}$

$$\Rightarrow \quad f\left(\frac{x-1}{x}\right) + 2f(x) = \frac{x-1}{x} \qquad \text{...(iii)}$$

From Eq. {(i) – 2(ii)}, we get $f(x) - 4f\left(\frac{x-1}{x}\right) = x - \frac{2}{1-x}$ ...(iv)

From Eq. (iv) + 4 × Eq. (iii), we get $9f(x) = x - \frac{2}{1-x} + \frac{4(x-1)}{x}$

$$\therefore \quad 9f(2) = 2 - \frac{2}{-1} + \frac{4(1)}{2}$$

$$\Rightarrow \quad f(2) = \frac{6}{9} = \frac{2}{3}$$

Illustration 128 Let $f(x)$ and $g(x)$ be functions which take integers as arguments. Let $f(x+y)=f(x)+g(y)+8$ for all integers x and y. Let $f(x)=x$ for all negative integers x and let $g(8)=17$. Find $f(0)$.

Solution. As, $f(x+y)=f(x)+g(y)+8$

Put $\quad x=-8,\ y=8$

$\therefore \quad f(0)=f(-8)+g(8)+8$

$\Rightarrow \quad f(0)=-8+17+8=17 \quad$ [Using, $f(x)=x$]

Illustration 129 Let $f(x)=ax^7+bx^3+cx-5$ where a, b and c are constants. If $f(-7)=7$, then find $f(7)$.

Solution. As, $f(x)=ax^7+bx^3+cx-5$

$\therefore \quad f(-x)=-ax^7-bx^3-cx-5$

$\Rightarrow \quad f(x)+f(-x)=-10$

Put $x=7$, $\quad f(7)+f(-7)=-10$

$\Rightarrow \quad f(7)=-17$

Illustration 130 The function $f:R\to R$ satisfies the condition $mf(x-1)+nf(-x)=2|x|+1$. If $f(-2)=5$ and $f(1)=1$, then find $(m+n)$.

Solution. As, $\quad mf(x-1)+nf(-x)=2|x|+1$

Put $x=2$, $\quad mf(1)+nf(-2)=5$

Put $x=-1$, $\quad mf(-2)+nf(1)=3$

$\therefore \quad m+5n=5$

and $\quad 5m+n=3$

On adding, $\quad 6(m+n)=8$

$\therefore \quad m+n=\dfrac{4}{3}$

Worked Examples

Type 1 : Subjective Type Questions

Example 1 Let a sequence $x_1, x_2, x_3, \ldots$ of complex numbers be defined by $x_1 = 0$, $x_{n+1} = x_n^2 - i$ for $n > 1$ where $i^2 = -1$. Find the distance of x_{2000} from x_{1997} in the complex plane.

Solution. $x_1 = 0$

$$x_2 = 0^2 - i = -i$$
$$x_3 = (-i)^2 - i = -1 - i = -(1+i)$$
$$x_4 = [-(1+i)]^2 - i = 2i - i = i$$
$$x_5 = (i)^2 - i = -1 - i = x_3$$

$\therefore$ $x_6 = x_4$ and hence $x_7 = x_5$ and so on,

$\therefore$ $x_{2n} = i, x_{2n+1} = -(1+i)$

$\therefore$ $x_{2000} = i = (0, 1)$ and $x_{1997} = -1 - i = (-1, -1)$ in the complex plane.

So, the distance between x_{2000} and x_{1997} is $\sqrt{1+4} = \sqrt{5}$.

Example 2 Let $f(x)$ be a polynomial with integral coefficients. Suppose that both $f(1)$ and $f(2)$ are odd. Then, prove that, for any integer n, $f(n) \neq 0$.

Solution. Suppose, $f(n) = 0$ for some integer n.

Then, $(x - n)$ divides $f(x)$.

So, $f(x) = (x - n)\, g(x)$, where $g(x)$ is a polynomial with integral coefficient.

Also, $f(1) = (1 - n)\, g(1)$ and $f(2) = (2 - n)\, g(2)$.

Now, $g(1)$ and $g(2)$ are both integers and one of $(1 - n)$ or $(2 - n)$ is even.

So, one of $f(1)$ or $f(2)$ is even, which is contradiction. So, there exists no integer n, for which $f(n) = 0$.

Example 3 If a, b, c, d, e are positive real numbers, such that $a + b + c + d + e = 8$ and $a^2 + b^2 + c^2 + d^2 + e^2 = 16$, find the range of e.

Solution. As we know,

$$\left(\frac{a+b+c+d}{4}\right)^2 \leq \frac{a^2+b^2+c^2+d^2}{4} \qquad \text{...(i)}$$

(Using Tchebycheff's Inequality)

where $a + b + c + d + e = 8$

and $a^2 + b^2 + c^2 + d^2 + e^2 = 16$

$\therefore$ Eq. (i), reduces to

$$\left(\frac{8-e}{4}\right)^2 \leq \frac{16-e^2}{4}$$

$\Rightarrow$ $64 + e^2 - 16e \leq 4(16 - e^2)$

$\Rightarrow$ $5e^2 - 16e \leq 0$

$\Rightarrow$ $e(5e - 16) \leq 0$ (Using number line rule)

$$\Rightarrow \qquad 0 \le e \le \frac{16}{5}$$

$$\begin{array}{ccccc} + & & - & & + \\ \hline & 0 & & \frac{16}{5} & \end{array}$$

Thus, $\qquad$ range of $e \in \left[0, \frac{16}{5}\right]$

Example 4 Find all positive integers x, y, z satisfying $x^{y^z} \cdot y^{z^x} \cdot z^{x^y} = 5xyz$.

Solution. Here, x, y, z are integers and 5 is prime number.

$$\Rightarrow \qquad \frac{x^{y^z} \cdot y^{z^x} \cdot z^{x^y}}{xyz} = 5$$

$\Rightarrow \qquad x^{y^z - 1} \cdot y^{z^x - 1} \cdot z^{x^y - 1} = 5$ is only possible, if

Case I $\; x^{y^z - 1} = 5, \quad y^{z^x - 1} = 1, \; z^{x^y - 1} = 1$

$$\Rightarrow \qquad x = 5, \; y^z - 1 = 1 \;\; \Rightarrow \;\; y^z = 2$$

$$\Rightarrow \qquad y = 2 \quad \text{and} \quad z = 1$$

$$\therefore \qquad (x, y, z) = (5, 2, 1)$$

Case II $\; x^{y^z - 1} = 1, \; y^{z^x - 1} = 5, \; z^{x^y - 1} = 1$

$$\therefore \qquad (x, y, z) = (1, 5, 2)$$

Case III $\; x^{y^z - 1} = 1, \; y^{z^x - 1} = 1, \; z^{x^y - 1} = 5$

$$\therefore \qquad (x, y, z) = (2, 1, 5)$$

Example 5 Show that for any ΔABC, the following inequality is true: $a^2 + b^2 + c^2 > \sqrt{3} \max \{|a^2 - b^2|, |b^2 - c^2|, |c^2 - a^2|\}$, where a, b, c are the sides of the triangle.

Solution. Let us assume, $a \ge b \ge c$

$\Rightarrow \qquad |c^2 - a^2| = a^2 - c^2$ is maximum of $|a^2 - b^2|, |b^2 - c^2|, |c^2 - a^2|$

$\therefore$ To show, $\qquad a^2 + b^2 + c^2 > \sqrt{3}(a^2 - c^2)$

or $\qquad a^2 + b^2 + c^2 - \sqrt{3}(a^2 - c^2) > 0$

Here, $\quad a^2 + b^2 + c^2 - \sqrt{3}(a^2 - c^2) > a^2 + (a - c)^2 + c^2 - \sqrt{3}(a^2 - c^2)$

$$> a^2 + a^2 + c^2 - 2ac + c^2 - \sqrt{3}a^2 + \sqrt{3}c^2$$

$$> 2a^2 + 2c^2 - \sqrt{3}a^2 + \sqrt{3}c^2 - 2ac$$

$$> (2 - \sqrt{3})a^2 + (2 + \sqrt{3})c^2 - 2ac$$

$$> \frac{2(2 - \sqrt{3})a^2 + 2(2 + \sqrt{3})c^2 - 4ac}{2}$$

$$> \frac{\{(\sqrt{3} - 1)a\}^2 + \{(\sqrt{3} + 1)c\}^2 - 4ac}{2}$$

$$> \frac{1}{2}[(\sqrt{3} - 1)a - (\sqrt{3} + 1)c]^2 \ge 0$$

Thus, $a^2 + b^2 + c^2 > \sqrt{3}(a^2 - c^2)$. Similarly, the other parts.

Example 6 Find the set of all solutions of the equation

$$2^{|y|}-\left|2^{y-1}-1\right|=2^{y-1}+1. \qquad \textbf{[IIT JEE 1997]}$$

Solution. Here, $\quad 2^{|y|}-\left|2^{y-1}-1\right|=2^{y-1}+1$

We know to define modulus, we have three cases as

Case I $y<0$

$$\Rightarrow \quad 2^{-y}+(2^{y-1}-1)=2^{y-1}+1; \qquad \begin{cases}\text{as when } y<0\\ |y|=-y\\ \text{and } \left|2^{y-1}-1\right|=-(2^{y-1}-1)\end{cases}$$

$$\Rightarrow \quad 2^{-y}=2^{1}$$

Hence, $\quad y=-1$, which is true when $y<0$. ...(i)

Case II $0\le y<1$

$$\Rightarrow \quad 2^{y}+(2^{y-1}-1)=2^{y-1}+1; \qquad \begin{cases}\text{as when } 0\le y<1\\ |y|=-y\\ \text{and } \left|2^{y-1}-1\right|=-(2^{y-1}-1)\end{cases}$$

$$\Rightarrow \quad 2^{y}=2$$

$\Rightarrow \quad y=1$, which shows no solution as,

$$0\le y<1 \qquad \text{...(ii)}$$

Case III $y\ge 1$

$$\Rightarrow \quad 2^{y}-(2^{y-1}-1)=2^{y-1}+1 \qquad \begin{cases}\text{as when } y\ge 0\\ |y|=y\\ \text{and } \left|2^{y-1}-1\right|=(2^{y-1}-1)\end{cases}$$

$$\Rightarrow \quad 2^{y}=2^{y-1}+2^{y-1};$$

$$\Rightarrow \quad 2^{y}=2\cdot 2^{y-1}$$

$\Rightarrow \quad 2^{y}=2^{y}$, which is an identity therefore, it is true $\forall\ y\ge 1$ ...(iii)

Hence, from Eqs. (i), (ii) and (iii) the solution of set is $\{y: y\ge 1 \cup\ y=-1\}$.

Example 7 Solve the equation $[x]\{x\}=x$, where [ ] and { } denote the greatest integer function and fractional part, respectively.

Solution. We know that,

$$x=[x]+\{x\} \qquad \text{...(i)}$$

Thus, we have

$$[x]\{x\}=[x]+\{x\}$$

$$\Rightarrow \quad \{x\}=\frac{[x]}{[x]-1} \qquad \text{...(ii)}$$

Here, in Eq. (ii), $\quad [x]\ne 1$

But, if $[x]=1$, then given equation

$$\Rightarrow \quad \{x\}=x$$

which is true only when $\quad x\in[0,1)$ ...(iii)

and $\quad [x]=1 \quad\Rightarrow\quad x\in[1,2)$ ...(iv)

$\therefore$ From Eqs. (iii) and (iv) no value of x,

when $\quad [x]=1$ ...(v)

Now, from Eq. (ii),

$$\{x\} = \frac{[x]}{[x]-1},$$

when $[x] \neq 1$

Again, as we know $\{x\} \in [0, 1)$

$\therefore$ $$0 \leq \frac{[x]}{[x]-1} < 1$$

ie, $\frac{[x]}{[x]-1} < 1$ and $\frac{[x]}{[x]-1} \geq 0$

ie, $\frac{1}{[x]-1} < 0$ and $\{[x] \leq 0$ or $[x] > 1\}$

ie, $[x] < 1$ and $\{[x] \leq 0$ or $[x] > 1\}$

(using number line rule)

ie, $[x] \leq 0$

$\therefore$ $x = [x] + \{x\}$

$\Rightarrow$ $$x = [x] + \frac{[x]}{[x]-1}$$

$$x = \frac{[x]^2}{[x]-1},$$ where x takes values less than 1.

$\therefore$ $$x = \frac{[x]^2}{[x]-1},$$ where $x < 1$

or $$x = \frac{[x]^2}{[x]-1},$$ where $[x]$ is any non-positive integer.

Example 8 Find all possible values of x satisfying

$$\frac{[x]}{[x-2]} - \frac{[x-2]}{[x]} = \frac{8\{x\}+12}{[x-2][x]}$$

(where [ ] denotes the greatest integer function and { } is the fractional part).

Solution. Here, $$\frac{[x]}{[x-2]} - \frac{[x-2]}{[x]} = \frac{8\{x\}+12}{[x-2][x]}$$

$\Rightarrow$ $$\frac{[x]^2 - [x-2]^2}{[x-2][x]} = \frac{8\{x\}+12}{[x][x-2]}$$

$\Rightarrow$ $$\frac{([x]-[x-2])([x]+[x-2])}{[x-2][x]} = \frac{8\{x\}+12}{[x][x-2]}$$

$\Rightarrow$ $([x]-[x-2])([x]+[x-2]) = 8\{x\}+12$

$\Rightarrow$ $([x]-[x]+2)([x]+[x]-2) = 8\{x\}+12$ $\quad (\because [x+I] = [x]+I)$

$\Rightarrow$ $4([x]-1) = 8\{x\}+12$

$\Rightarrow$ $[x]-4 = 2\{x\}$ ...(i)

Now, as we know $0 \leq \{x\} < 1 \Rightarrow 0 \leq 2\{x\} < 2$

$\Rightarrow$ $0 \leq [x]-4 < 2 \Rightarrow 4 \leq [x] < 6$

$\Rightarrow$ $[x] = 4, 5$

$$\left.\begin{array}{l}\text{if } [x]=4 \Rightarrow 2\{x\}=[x]-4 \\ \text{becomes } \{x\}=0\end{array}\right\} \quad \text{...(ii)}$$

$$\left.\begin{array}{l}\text{and if } [x]=5 \Rightarrow 2\{x\}=[x]-4 \\ \text{becomes } \{x\}=\frac{1}{2}\end{array}\right\} \quad \text{...(iii)}$$

Thus, from Eqs. (ii) and (iii), we have

$$x = [x] + \{x\}$$

ie, $x = 4 + 0 = 4$ [Using Eq. (ii)]

and $x = 5 + \frac{1}{2} = \frac{11}{2}$ [Using Eq. (iii)]

$\Rightarrow$ $x \in \left\{4, \frac{11}{2}\right\}$

Example 9 If x and y are real, solve the inequality

$$\log_2 x + \log_x 2 + 2\cos y \le 0.$$

Solution. Here, $\log_2 x + \log_x 2 + 2\cos y \le 0$ holds when $x > 0$ and $x \ne 1$.

Let $\log_2 x = t$

$\Rightarrow$ $t + \frac{1}{t} + 2\cos y \le 0$

ie, $\frac{t^2 + 2t\cos y + 1}{t} \le 0$

Case I When $t > 0$.

$$t^2 + 2t\cos y + 1 \le 0 \text{ for all } t > 0$$

$\Rightarrow$ $(t-1)^2 + 2t(1+\cos y) \le 0$

where LHS ≥ 0, so the above equation will hold only when;

$$t = 1 \quad \text{and} \quad \cos y = -1$$

$\Rightarrow$ $y = (2n+1)\pi$

$\therefore$ $\log_2 x = 1$

and $y = (2n+1)\pi$

$\Rightarrow$ $x = 2$

and $y = (2n+1)\pi$ ...(i)

Case II When $t < 0$.

$\Rightarrow$ $t^2 + 2t\cos y + 1 \ge 0$

$\Rightarrow$ $(t+1)^2 + 2t(\cos y - 1) \ge 0 \Rightarrow (t+1)^2 \ge 0$

and $2t(\cos y - 1) \ge 0$ for all y

$\therefore$ Solution set is $y \in R$ and $t < 0$

$\Rightarrow$ $y \in R$ and $\log_2 x < 0$

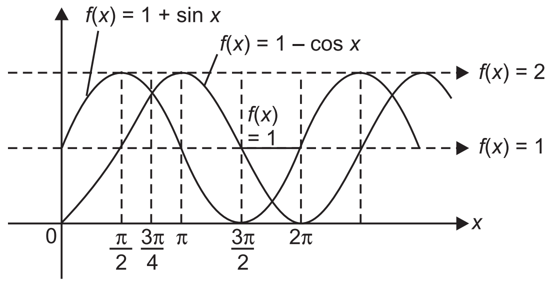
$\Rightarrow$ $y \in R$ and $0 < x < 1$ ...(ii)

Thus, from Eqs. (i) and (ii) solution set is, $0 < x < 1$ and $y \in R$, also when $x = 2$ and $y = (2n+1)\pi$.

Example 10 Let $f(x) = \max\{1 + \sin x, 1 - \cos x, 1\}\ \forall\ x \in [0, 2\pi]$ and $g(x) = \max\{1, |x-1|\}\ \forall\ x \in R$. Determine $f\{g(x)\}$ and $g\{f(x)\}$ in terms of x.

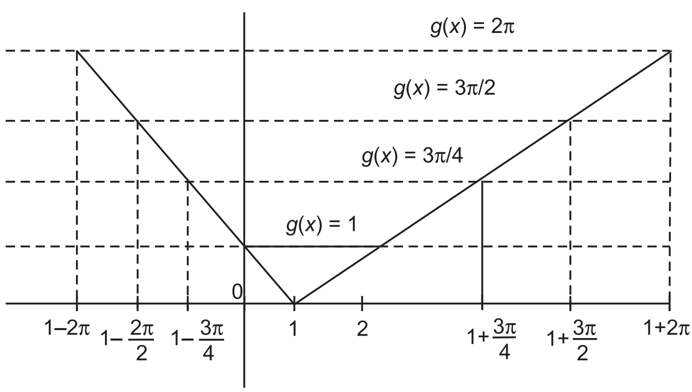
Solution. Here,

$f(x) = \max\{1 + \sin x, 1 - \cos x, 1\}$. Graphically, it can be shown as



$$f(x) = \begin{cases} 1 + \sin x, & 0 \le x \le \dfrac{3\pi}{4} \\ 1 - \cos x, & \dfrac{3\pi}{4} < x \le \dfrac{3\pi}{2} \\ 1, & \dfrac{3\pi}{2} < x \le 2\pi \end{cases} \qquad \text{(using above graph)}$$

Again, $g(x) = \max\{1, |x-1|\}$. Graphically, it can be shown as



$$\therefore \qquad g(x) = \begin{cases} 1 - x, & x \le 0 \\ 1, & 0 < x \le 2 \\ x - 1, & x > 2 \end{cases}$$

Now,
$$g\{f(x)\} = \begin{cases} 1 - f(x), & f(x) \le 0 \\ 1, & 0 < f(x) \le 2 \\ f(x) - 1, & f(x) > 2 \end{cases}$$

Since,
$$f(x) \in [1, 2],\ \forall\ x \in [0, 2\pi]$$
$$g(f(x)) = 1,\ \forall\ x \in [0, 2\pi]$$

Also, $$f\{g(x)\} = \begin{cases} 1+\sin\{g(x)\}, & 0 \le g(x) \le \dfrac{3\pi}{4} \\ 1-\cos\{g(x)\}, & \dfrac{3\pi}{4} < g(x) \le \dfrac{3\pi}{2} \\ 1, & \dfrac{3\pi}{2} < g(x) \le 2\pi \end{cases}$$

$$\Rightarrow \quad f(g(x)) = \begin{cases} 1, & 1-2\pi \le x < 1-3\pi/2 \\ 1-\cos(1-x), & 1-3\pi/2 \le x < 1-3\pi/4 \\ 1+\sin(1-x), & 1-3\pi/4 \le x \le 0 \\ 1+\sin 1, & 0 < x \le 2 \\ 1+\sin(x-1), & 2 < x \le 1+3\pi/4 \\ 1-\cos(x-1), & 1+3\pi/4 < x \le 1+3\pi/2 \\ 1, & 1+3\pi/2 < x \le 2\pi+1 \end{cases}$$

Example 11 If $f(x)$ be a polynomial function satisfying

$$f(x)\cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) \text{ and } f(4) = 65 \text{. Then, find } f(6).$$

Solution. Here, $f(x)\cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$

$$\Rightarrow \quad f(x)\cdot f\left(\frac{1}{x}\right) - f(x) = f\left(\frac{1}{x}\right)$$

$$\Rightarrow \quad f(x) = \frac{f(1/x)}{f(1/x)-1} \qquad \text{...(i)}$$

Also, $$f(x)\cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

$$\Rightarrow \quad f(x)\cdot f\left(\frac{1}{x}\right) - f\left(\frac{1}{x}\right) = f(x)$$

$$\Rightarrow \quad f\left(\frac{1}{x}\right) = \frac{f(x)}{f(x)-1} \qquad \text{...(ii)}$$

On multiplying Eqs. (i) and (ii), we get

$$f(x)\cdot f\left(\frac{1}{x}\right) = \frac{f(1/x)\cdot f(x)}{\{f(1/x)-1\}\{f(x)-1\}}$$

$$\Rightarrow \quad \left(f\left(\frac{1}{x}\right) - 1\right)(f(x)-1) = 1 \qquad \text{...(iii)}$$

Since, $f(x)$ is a polynomial function, so $\{f(x)-1\}$ and $\left\{f\left(\frac{1}{x}\right)-1\right\}$ are reciprocals of each other. Also, x and $\frac{1}{x}$ are reciprocals of each other.

Thus, Eq. (iii) can hold only when

$$f(x) - 1 = \pm x^n, \qquad \text{where } n \in R$$

$$\therefore \quad f(x) = \pm x^n + 1, \qquad \text{but } f(4) = 65$$

$$\Rightarrow \quad \pm 4^n + 1 = 65 \quad \Rightarrow \quad 4^n = 64$$

$$\Rightarrow \quad 4^n = 4^3 \qquad (\because 4^n > 0)$$

$$\Rightarrow \quad n = 3$$

So, $f(x)=x^3+1$

Hence, $f(6)=6^3+1=217$

Aliter : Let $f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n$

Then, $$f(x)\cdot f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)$$

$$\Rightarrow \quad (a_0n^n+a_1x^{n-1}+\ldots+a_n)\left(\frac{a_0}{x^n}+\frac{a_1}{x^{n-1}}+\ldots+a_n\right)$$

$$=(a_0x^n+a_1x^{n-1}+\ldots+a_n)+\left(\frac{a_0}{x^n}+\frac{a_1}{x^{n-1}}+\ldots+a_n\right)$$

On comparing the coefficient of x^n, we have

$$a_0a_n=a_0 \quad\Rightarrow\quad a_n=1 \qquad \text{(as } a_0\neq 0)$$

Comparing the coefficient of x^{n-1}, we have

$$a_0a_{n-1}+a_na_1=a_1$$

$\Rightarrow$ $a_0a_{n-1}+a_1=a_1$ (as $a_n=1$)

$\Rightarrow$ $a_0a_{n-1}=0$

$\Rightarrow$ $a_{n-1}=0$ (as $a_0\neq 0$)

Similarly, $a_{n-1}=a_{n-2}=\ldots\ldots=a_1=0$

and $a_0=\pm 1$

$\therefore$ $f(x)=\pm x^n+1$

$f(4)=\pm 4^n+1$

$\Rightarrow$ $4^n+1=65$ [as $f(4)=65$]

$\Rightarrow$ $4^n=64 \quad\Rightarrow\quad n=3$

So, $f(x)=x^3+1$

Hence, $f(6)=6^3+1=217$

Example 12 If $f(x)$ satisfies the relation, $f(x+y)=f(x)+f(y)$ for all $x,y\in R$ and $f(1)=5$, then find $\sum_{n=1}^{m} f(n)$. Also, prove that $f(x)$ is odd.

Solution. Here, $f(x+y)=f(x)+f(y)$, put $x=r-1$, $y=1$

$$f(r)=f(r-1)+f(1) \qquad \text{...(i)}$$

(Using definition)

$\therefore$ $f(r)=f(r-1)+5$

$\Rightarrow$ $f(r)=\{f(r-2)+5\}+5$

$\Rightarrow$ $f(r)=f(r-2)+2\cdot(5)$

$\Rightarrow$ $f(r)=f(r-3)+3\cdot(5)$

....

....

$\Rightarrow$ $f(r)=f\{r-(r-1)\}+(r-1)\cdot 5$

$\Rightarrow$ $f(r)=f(1)+(r-1)\cdot 5$

$\Rightarrow$ $f(r)=5+(r-1)\cdot 5$

$\Rightarrow$ $f(r)=5r$

$$\therefore \qquad \sum_{n=1}^{m} f(n) = \sum_{n=1}^{m} (5n) = 5[1 + 2 + 3 + \ldots + m]$$

$$= \frac{5m(m+1)}{2}$$

Hence, $$\sum_{n=1}^{m} f(n) = \frac{5m(m+1)}{2} \qquad \ldots\text{(ii)}$$

Now, putting $x = 0$, $y = 0$ in the given function, we have

$$f(0+0) = f(0) + f(0)$$

$$\therefore \qquad f(0) = 0$$

Also, putting $(-x)$ for (y) in the given function,

$$f(x-x) = f(x) + f(-x)$$

$$\Rightarrow \qquad f(0) = f(x) + f(-x)$$

$$\Rightarrow \qquad 0 = f(x) + f(-x)$$

$$\Rightarrow \qquad f(-x) = -f(x) \qquad \ldots\text{(iii)}$$

Thus, $$\sum_{n=1}^{m} f(n) = \frac{5m(m+1)}{2}$$ and $f(x)$ is odd.

Example 13 Let $f(x) = \dfrac{9^x}{9^x + 3}$. Show $f(x) + f(1-x) = 1$, and hence evaluate

$$f\left(\frac{1}{1996}\right) + f\left(\frac{2}{1996}\right) + f\left(\frac{3}{1996}\right) + \ldots + f\left(\frac{1995}{1996}\right).$$

Solution. $$f(x) = \frac{9^x}{9^x + 3} \qquad \ldots\text{(i)}$$

and $$f(1-x) = \frac{9^{1-x}}{9^{1-x} + 3}$$

$$\Rightarrow \qquad f(1-x) = \frac{\frac{9}{9^x}}{\frac{9}{9^x} + 3} = \frac{9}{9 + 3 \cdot 9^x}$$

$$f(1-x) = \frac{9}{3(3 + 9^x)} \qquad \ldots\text{(ii)}$$

Adding Eqs. (i) and (ii), we get

$$f(x) + f(1-x) = \frac{9^x}{9^x + 3} + \frac{9}{3(3 + 9^x)}$$

$$= \frac{3 \cdot 9^x + 9}{3(9^x + 3)} = \frac{3(9^x + 3)}{3(9^x + 3)}$$

$$\therefore \qquad f(x) + f(1-x) = 1 \qquad \ldots\text{(iii)}$$

Now, putting $x = \dfrac{1}{1996}, \dfrac{2}{1996}, \dfrac{3}{1996}, \ldots, \dfrac{998}{1996}$ in Eq. (iii), we get

$$f\left(\frac{1}{1996}\right) + f\left(\frac{1995}{1996}\right) = 1$$

$$\Rightarrow \qquad f\left(\frac{2}{1996}\right) + f\left(\frac{1994}{1996}\right) = 1$$

$$\Rightarrow \qquad f\left(\frac{3}{1996}\right) + f\left(\frac{1993}{1996}\right) = 1$$

....

....

$$\Rightarrow \qquad f\left(\frac{997}{1996}\right) + f\left(\frac{999}{1996}\right) = 1$$

$$\Rightarrow \qquad f\left(\frac{998}{1996}\right) + f\left(\frac{998}{1996}\right) = 1$$

$$\text{or} \qquad f\left(\frac{998}{1996}\right) = \frac{1}{2}$$

Adding all the above expressions, we get

$$f\left(\frac{1}{1996}\right) + f\left(\frac{2}{1996}\right) + \ldots + f\left(\frac{1995}{1996}\right)$$

$$= (1 + 1 + 1 + \ldots + 997) + \frac{1}{2}$$

$$= 997 + \frac{1}{2}$$

$$= 997.5$$

Example 14 If a, b, c, d, e are real numbers, prove that the roots of $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$ cannot all be real, if $2a^2 < 5b$.

Solution. Here,

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0 \text{ has real roots } \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5.$$

$$\Rightarrow \qquad \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = \Sigma\alpha_i = -a$$

$$\sum_{i<j} \alpha_i \alpha_j = b \qquad \ldots\text{(i)}$$

$$(\Sigma\alpha_i)^2 = a^2 \qquad \ldots\text{(ii)}$$

$$\Rightarrow \qquad \Sigma\alpha_i^2 + 2\Big(\sum_{i<j} \alpha_i \alpha_j\Big) = a^2 \qquad \ldots\text{(iii)}$$

$$\Rightarrow \qquad (\Sigma\alpha_i^2) = a^2 - 2b \qquad \text{[from Eqs. (i), (ii) and (iii)]}$$

As we know,

$$\left(\frac{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2}{5}\right) \geq \left(\frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}{5}\right)^2$$

(known as Tchebycheff's inequality)

$$\Rightarrow \qquad 5(a^2 - 2b) \geq a^2$$

$$\text{or} \qquad 4a^2 \geq 10b$$

$$\text{or} \qquad 2a^2 \geq 5b \qquad \text{(for real roots)}$$

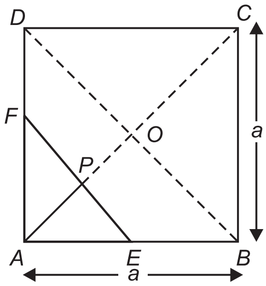
Thus, if $2a^2 < 5b$ cannot all be real roots.

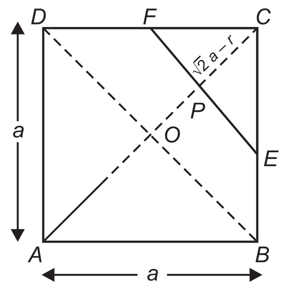
Example 15 *ABCD* is a square of side a. A line parallel to the diagonal *BD* at a distance x from the vertex *A* cuts the two adjacent sides. Express the area of the segment of the square with *A* at a vertex, as a function of x.

Solution. There are two different situations :

(i) when $x = AP \leq OA$, *ie*, $x \leq \frac{a}{\sqrt{2}}$

(ii) when $x = AP \geq OA$ *ie*, $x > \frac{a}{\sqrt{2}}$ but $x \leq \sqrt{2}a$





Case I Area $(\Delta\, AEF) = \frac{1}{2}x \cdot 2x = x^2$

Case II Area $(ABEFDA)$ = Area $(ABCD)$ – Area (ΔCEF)

$$= a^2 - \frac{1}{2}(\sqrt{2}a - x)2(\sqrt{2}a - x)$$

$$\text{Area } (ABEFDA) = 2\sqrt{2}\, ax - x^2 - a^2$$

$$\therefore \qquad f(x) = \begin{cases} x^2 & , \quad 0 \leq x \leq \frac{a}{\sqrt{2}} \\ 2\sqrt{2}\, ax - x^2 - a^2 & , \quad \frac{a}{\sqrt{2}} < x \leq \sqrt{2}a \end{cases}$$

Example 16 A function $R \to R$ is defined by $f(x) = \frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$. Find the integral of values of α for which f is onto. Is the function one-one for $\alpha = 3$? Justify your answer. **[IIT JEE 1996]**

Solution. Since, $f : R \longrightarrow R$ is an onto mapping.

$\therefore$ Range of $f = R$

$\Rightarrow \qquad \frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$ assumes all real values of x.

Let $\qquad y = \frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$

Then, y assumes all real values for real values of x.

$\Rightarrow \qquad \alpha y + 6xy - 8x^2 y = \alpha x^2 + 6x - 8, \ \forall\, y \in R$

$\Rightarrow \qquad x^2(\alpha + 8y) + 6x(1 - y) - (8 + \alpha\, y) = 0, \ \forall\, y \in R$

We know, above equation assumes all real values

$\Rightarrow \qquad D \geq 0$

So, $\qquad 36(1 - y)^2 + 4(\alpha + 8y)\ (8 + \alpha y) \ \geq 0$

$\Rightarrow \quad 4[9(1-2y+y^2)+(8\alpha+\alpha^2 y+64y+8\alpha y^2)] \geq 0$

$\Rightarrow \quad [9-18y+9y^2+8\alpha+\alpha^2 y+64y+8\alpha y^2] \geq 0$

$\Rightarrow \quad [y^2(8\alpha+9)+y(\alpha^2+46)+(8\alpha+9)] \geq 0$

We know, if $\quad ax^2+bx+c>0\ \forall\ x$ and $a>0 \Rightarrow D<0$

So, $\quad (\alpha^2+46)^2-4(8\alpha+9)(8\alpha+9) \leq 0$ and $(8\alpha+9)>0$

$\Rightarrow \quad (\alpha^2+46)^2-[2(8\alpha+9)]^2 \leq 0$ and $\alpha>-9/8$

$\Rightarrow \quad (\alpha^2+46-16\alpha-18)(\alpha^2+46+16\alpha+18) \leq 0$ and $\alpha>-9/8$

$\quad (\alpha^2-16\alpha+28)(\alpha^2+16\alpha+64) \leq 0$ and $\alpha>-9/8$

$\Rightarrow \quad (\alpha-14)(\alpha-2)(\alpha+8)^2 \leq 0 \quad$ and $\quad \alpha>-9/8$

$\alpha \in [2, 14] \cup \{-8\}$ and $\alpha>-9/8$

Thus, $\quad \alpha \in [2, 14]$

Hence, f is onto when $\alpha \in [2, 14]$.

Again, if $\alpha=3$, we have

$$f(x)=\frac{3x^2+6x-8}{3+6x-8x^2}.$$

Clearly, $f(1)=f(-1)$,

which shows $f(x)$ is not one-one.

Example 17 Let $f(x, y)$ be a periodic function satisfying the condition $f(x, y)=f((2x+2y),(2y-2x))\ \forall\ x, y \in R$. Now, define a function g by $g(x)=f(2^x, 0)$. Then, prove that $g(x)$ is a periodic function, and find its period.

Solution. $\quad f(x, y)=f(2x+2y, 2y-2x) \qquad \text{...(i)}$

$\Rightarrow \quad f(2x+2y, 2y-2x)=f\{2(2x+2y)+2(2y-2x),$
$\qquad 2(2y-2x)-2(2x+2y)\} \qquad$ [using Eq. (i)]

$\Rightarrow \quad f(x, y)=f(2x+2y, 2y-2x)=f(8y-8x)$

$\Rightarrow \quad f(x, y)=f(8y-8x) \qquad \text{...(ii)}$

$\Rightarrow \quad f(8y, -8x)=f\{8(-8x), -8(8y)\} \qquad$ [using Eq. (ii)]

$\Rightarrow \quad f(x, y)=f(2x+2y, 2y-2x)=f(8y, -8x)$
$\qquad =f(-64x, -64y)$

$\Rightarrow \quad f(x, y)=f(-64x, -64y) \qquad \text{...(iii)}$

$\Rightarrow \quad f(-64x, -64y)=f(64\times 64x, 64\times 64y)$
$\qquad =f(2^{12}x, 2^{12}y)$

$\Rightarrow \quad f(x, y)=f(2^{12}x, 2^{12}y) \qquad$ [using Eq. (iii)]

$\Rightarrow \quad f(2^x, 0)=f(2^{12}\cdot 2^x, 0)=f(2^{12+x}, 0) \qquad \text{...(iv)}$

Given, $\quad g(x, 0)=f(2^x, 0)$

$\Rightarrow \quad g(x, 0)=f(2^x, 0)=f(2^{12+x}, 0) \qquad$ [using Eq. (iv)]

$\Rightarrow \quad g(x, 0)=g(x+12, 0)$

Hence, $g(x)$ is periodic with period 12.

Example 18 Solve the equation

$$10^{(x+1)(3x+4)} - 2\cdot 10^{(x+1)(x+2)} = 10^{1-x-x^2}.$$

Solution. The given equation is

$$10^{3x^2+7x+4} - 2\cdot 10^{x^2+3x+2} = \frac{10}{10^{x+x^2}}$$

$\Rightarrow$ $\quad 10^{4x^2+8x+4} - 2\cdot 10^{2x^2+4x+2} = 10$

$\Rightarrow$ $\quad 10^{4(x^2+2x+1)} - 2\cdot 10^{2(x^2+2x+1)} = 10$

$\Rightarrow$ $\quad 10^{4(x+1)^2} - 2\cdot 10^{2(x+1)^2} = 10$

$\Rightarrow$ $\quad \{10^{2(x+1)^2}\}^2 - 2\cdot\{10^{2(x+1)^2}\} = 10$ ...(i)

Let $\quad 10^{2(x+1)^2} = y$ ...(ii)

$\Rightarrow$ $\quad y^2 - 2y = 10 \Rightarrow y^2 - 2y - 10 = 0$

$\Rightarrow$ $\quad y = \dfrac{2 \pm \sqrt{4+40}}{2}$

$\Rightarrow$ $\quad y = 1 \pm \sqrt{11}$

$\Rightarrow$ $\quad y = 1 + \sqrt{11}$ (neglecting –ve sign as $y > 0$)

$\Rightarrow$ $\quad 10^{2(x+1)^2} = 1 + \sqrt{11}$

$\Rightarrow$ $\quad 2(x+1)^2 = \log_{10}(1+\sqrt{11})$

$\Rightarrow$ $\quad (x+1)^2 = \dfrac{1}{2}\log_{10}(1+\sqrt{11})$

$\Rightarrow$ $\quad x + 1 = \pm\sqrt{\dfrac{1}{2}\log_{10}(1+\sqrt{11})}$

Hence, $\quad x = -1 \pm \sqrt{\dfrac{1}{2}\log_{10}(1+\sqrt{11})}$

Example 19 Prove that $[x]+[2x]+[4x]+[8x]+[16x]+[32x] = 12345$ has no solution.

Solution. $12345 \le x + 2x + 4x + 8x + 16x + 32x$

or $\quad 12345 \le 63x$

$\Rightarrow$ $\quad x \ge \dfrac{12345}{63} = 195\dfrac{20}{21}$ when $x = 196$

The LHS of the given equation = 12348

$\therefore$ $\quad 195\dfrac{20}{21} \le x < 196$

Now, consider the interval $\left[195\dfrac{31}{32}, 196\right)$

$$[x]+[2x]+[4x]+[8x]+[16x]+[32x]$$

$$= 195 + 391 + 783 + 1567 + 3135 + 6271$$

$$= 12342 < 12345$$

Thus, $\quad x < 195\dfrac{31}{32}$.

The LHS is less than 12342.

$\therefore$ For no value of x, the given equality will hold.

Example 20 Find the real solution of $[x]+[5x]+[10x]+[20x]=36K+35$, $K\in$ integer where [ ] denotes the greatest integral function.

Solution. Let $x=a+r$, where $a\in$ integer and $r\in(0,1)$. Then, there are four cases :

(i) $0\le r<\frac{1}{20}$ (ii) $\frac{1}{20}\le r<\frac{1}{10}$

(iii) $\frac{1}{10}\le r<\frac{1}{5}$ (iv) $\frac{1}{5}\le r<1$

Case I If $0\le r<\frac{1}{20}\cdot$

$\therefore \quad [x]+[5x]+[10x]+[20x]=36K+35$

$\Rightarrow \quad [a+r]+[5a+5r]+[10a+10r]+[20a+20r]$

$=36K+35$

$\Rightarrow \quad a+5a+10a+20a=36K+35$

[by definition of the greatest integer function]

$\Rightarrow \quad 36a=36K+35,$

which is not possible for any value of a and K as they belongs to integers. ...(i)

Case II If $\frac{1}{20}\le r<\frac{1}{10}\cdot$

$\therefore \quad [x]+[5x]+[10x]+[20x]=36K+35$

$\Rightarrow \quad [a+r]+[5a+5r]+[10a+10r]+[20a+20r]$

$=36K+35$

$\Rightarrow \quad a+5a+10a+20a+1=36K+35$

$\Rightarrow \quad 36a+1=36K+35$

$\Rightarrow \quad 36a=36K+34,$

which is again not possible $\forall\ a, K\in$ integer. ...(ii)

Case III If $\frac{1}{10}\le r<\frac{1}{5}\cdot$

$\therefore \quad [x]+[5x]+[10x]+[20x]=36K+35$

$\Rightarrow \quad [a+r]+[5a+5r]+[10a+10r]+[20a+20r]=36K+35$

$\Rightarrow \quad a+5a+10a+1+20a+2=36K+35$

$\Rightarrow \quad 36a+3=36K+35,$

which shows no value of a and K can be given. ...(iii)

Case IV If $\frac{1}{5}\le r<1.$

$\therefore \quad [x]+[5x]+[10x]+[20x]=36K+35$

$\Rightarrow \quad [a+r]+[5a+5r]+[10a+10r]+[20a+20r]=36K+35$

$\Rightarrow \quad 36a+7=36K+35$

Again, no solution is possible. ...(iv)

Hence, from Eqs. (i), (ii), (iii), (iv) there exists no real value which possesses solution.

Example 21 Find the number of elements in the range of

$$f(x)=[x]+[2x]+\left[\frac{2}{3}x\right]+[3x]+[4x]+[5x] \text{ for } 0 \le x < 3$$

and hence when $x \in [0, n]$ where $n \in N$ and [ ] denotes the greatest integer function.

Solution. Given, $f(x)=[x]+[2x]+\left[\frac{2}{3}x\right]+[3x]+[4x]+[5x]$

Since, $[kx]$ changes its value at every integral multiple of $\frac{1}{k}\cdot$

$\Rightarrow$ $[x]$ will change at every integral multiple of 1.

$\Rightarrow$ $[2x]$ will change at every integral multiple of $\frac{1}{2}\cdot$

$\Rightarrow$ $[3x]$ will change at every integral multiple of $\frac{1}{3}\cdot$

$\Rightarrow$ $[4x]$ will change at every integral multiple of $\frac{1}{4}\cdot$

$\Rightarrow$ $[5x]$ will change at every integral multiple of $\frac{1}{5}\cdot$

and $\left[\frac{2}{3}x\right]$ will change at every integral multiple of $\frac{3}{2}\cdot$

They would change all together at every multiple of LCM of

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{3}{2}\right\}=3$$

ie, all will would change at every multiple of 3.

Now, number of total points at which $f(x)$ will change its value in the interval $[0, 3)$ will depend on the total number of different terms in the following cases :

$$[x]=0, 1, 2$$

$$[2x]=0, \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \frac{5}{2}$$

$$[3x]=0, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3}, \frac{7}{3}, \frac{8}{3}$$

$$[4x]=0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \frac{5}{4}, \frac{6}{4}, \frac{7}{4}, \frac{8}{4}, \frac{9}{4}, \frac{10}{4}, \frac{11}{4}$$

$$[5x]=0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5}, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}, \frac{10}{5}, \frac{11}{5}, \frac{12}{5}, \frac{13}{5}, \frac{14}{5}$$

$$\left[\frac{2}{3}x\right]=0, \frac{3}{2}, \frac{6}{2}$$

$\therefore$ $f(x)$ will change its values in the intervals,

$$0 \le x < \frac{1}{5}, \frac{1}{5} \le x < \frac{1}{4}, \frac{1}{4} \le x < \frac{1}{3}, \dots\dots\dots, \frac{14}{5} \le x < 3$$

$\Rightarrow$ Total number of different terms in above equation $=30$.

Hence, number of terms in the range of $f(x)$ for $0 \le x < 3$ is 30.

If $x \in [0, n]$, then

(i) If $n = 3k$, number of terms in the range $=(30k+1)$.

(ii) If $n = 3k + 1$, then number of terms in the range $= 30k + (3 + 4 + 5 + 1) - 2$

$$= 30k + 11.$$

(iii) If $n = 3k + 2$, then number of terms in the range

$$= 30k + (6 + 8 + 10 + 2) - (2 + 2) = 30k + 22.$$

Example 22 Let $f : N \to N$ be a function such that

(i) $x - f(x) = 19\left[\dfrac{x}{19}\right] - 90\left[\dfrac{f(x)}{90}\right] \forall x \in N$, where [ ] denotes the greatest integer function.

(ii) $1900 < f(1990) < 2000$.

Find all the possible values of $f(1990)$.

Solution. Since, $1900 < f(1990) < 2000$

$$\Rightarrow \quad \frac{1900}{90} < \frac{f(1990)}{90} < \frac{2000}{90} \Rightarrow 21.1 < \frac{f(1990)}{90} < 22.2$$

$$\therefore \quad \left[\frac{f(1990)}{90}\right] = 21, 22 \qquad \text{...(i)}$$

Given,
$$x - f(x) = 19\left[\frac{x}{19}\right] - 90\left[\frac{f(x)}{90}\right].$$

Now, there are two cases which arise.

Case I $\quad \left[\dfrac{f(1990)}{90}\right] = 21$

We have,
$$1990 - f(1990) = 19\left[\frac{1990}{19}\right] - 90\left[\frac{f(1990)}{90}\right]$$

$$1990 - f(1990) = 19 \cdot (104) - 90 \cdot (21)$$

$$\Rightarrow \quad f(1990) = 1904 \qquad \text{... (ii)}$$

Case II

$$\left[\frac{f(1990)}{90}\right] = 22$$

We have,
$$1990 - f(1990) = 19\left[\frac{1990}{19}\right] - 90\left[\frac{f(1990)}{90}\right]$$

$$\Rightarrow \quad f(1990) = 1994 \qquad \text{... (iii)}$$

From Eqs. (ii) and (iii), we have $f(1990) = 1904$ or 1994

Example 23 Solve the system of equations; $|x^2 - 2x| + y = 1$, $x^2 + |y| = 1$.

Solution. Here, $|x^2 - 2x| + y = 1$ and $x^2 + |y| = 1$ gives four cases as

Case I $\quad x^2 - 2x \ge 0, \quad$ and $\quad y \ge 0 \qquad$ ...(i)

Equations are $\quad x^2 - 2x + y = 1, \quad$ and $\quad x^2 + y = 1$

$\Rightarrow \quad 1 - 2x = 1 \Rightarrow x = 0, \quad$ and $\quad y = 1 \qquad$ ...(ii)

From Eq. (i), $\quad x^2 - 2x \ge 0 \quad \Rightarrow \quad x \le 0 \quad$ or $\quad x \ge 2, \quad$ and $\quad y \ge 0$

Thus, $x = 0$ and $y = 1$ is the only solution when

$$x^2 - 2x \ge 0, \quad \text{and} \quad y \ge 0$$

Case II $x^2 - 2x \geq 0$, and $y < 0$

$\Rightarrow$ $(x \leq 0$ or $x \geq 2)$ and $(y < 0)$ ...(iii)

$\therefore$ Equations are

$x^2 - 2x + y = 1$ and $x^2 - y = 1$. Adding these equations, we get

$2x^2 - 2x = 2$

or $x = \dfrac{1 \pm \sqrt{5}}{2}$, and $y = \dfrac{1 \pm \sqrt{5}}{2}$ ...(iv)

From Eqs. (iii) and (iv), $x = \dfrac{1 - \sqrt{5}}{2}$, and $y = \dfrac{1 - \sqrt{5}}{2}$

Case III $x^2 - 2x \leq 0$ and $y \geq 0$.

$\Rightarrow$ $(0 \leq x \leq 2)$ and $(y \geq 0)$ ...(v)

Equations are

$-x^2 + 2x + y = 1$ and $x^2 + y = 1$, subtracting, we get

$-2x^2 + 2x = 0 \quad \Rightarrow \quad x = 0, 1$

ie, $x = 0,\ y = 1$ or $x = 1,\ y = 0$ ...(vi)

which satisfy Eq. (v).

Case IV $x^2 - 2x \leq 0$ and $y \leq 0$.

$\Rightarrow$ $(0 \leq x \leq 2)$ and $(y \leq 0)$ ...(vii)

$\therefore$ Equations are $-x^2 + 2x + y = 1$ and $x^2 - y = 1$,

adding, we get $2x = 2 \Rightarrow x = 1$ and $y = 0$ which satisfies Eq. (vii)

Thus, solutions are :

$$(x = 0,\ y = 1)(x = 1,\ y = 0) \text{ and } \left(x = y = \frac{1 - \sqrt{5}}{2}\right).$$

Example 24 Prove that $a^2 + b^2 + c^2 + 2abc < 2$, where a, b, c are the sides of $\Delta\, ABC$ such that $a + b + c = 2$.

Solution. Given, $a + b + c = 2$

or $1 - a + 1 - b + 1 - c = 1$

or $x + y + z = 1$;

where $x = 1 - a,\ y = 1 - b,\ z = 1 - c$.

Since, $a + b > c \Rightarrow 0 < c < 1$

Similarly, $0, < a, b < 1$, hence $0 < x, y, z < 1$.

Now,
$$\begin{aligned} a^2 + b^2 + c^2 + 2abc &= (1 - x)^2 + (1 - y)^2 + (1 - z)^2 + 2(1 - x)(1 - y)(1 - z) \\ &= 3 - 2(x + y + z) + (x^2 + y^2 + z^2) \\ &\quad + 2\{1 - (x + y + z) + (xy + yz + zx) - xyz\} \\ &= 1 + x^2 + y^2 + z^2 - 2xyz + 2(xy + yz + zx) \\ &= 1 + (x + y + z)^2 - 2xyz \\ &= 2 - 2xyz < 2; \text{ as } 0 < x, y, z < 1 \end{aligned}$$

$\therefore$ $a^2 + b^2 + c^2 + 2abc < 2$

Example 25 If the terms of the AP $\sqrt{a-x}, \sqrt{x}, \sqrt{a+x}$, ... are all integers where $a > x > 0$, then find the least composite odd integral value of a.

Solution. Here, $\sqrt{a-x}, \sqrt{x}, \sqrt{a+x}$, ... are in AP.

$\Rightarrow \quad 2\sqrt{x} = \sqrt{a-x} + \sqrt{a+x}$, squaring both sides, we get

$\Rightarrow \quad 4x = 2a + 2\sqrt{a^2 - x^2}$

or $\quad \sqrt{a^2 - x^2} = 2x - a$

Squaring again, we get

$$a^2 - x^2 = (2x - a)^2$$

$\Rightarrow \quad x = 0 \quad \text{or} \quad \dfrac{4a}{5}$

$\therefore$ The terms of the arithmetic progression are

$$\sqrt{\frac{a}{5}}, 2\sqrt{\frac{a}{5}}, 3\sqrt{\frac{a}{5}}, \ldots\ldots. \text{ which are integers.}$$

$\therefore \quad a = 5n^2, n \in N$

for $n = 1$, $a = 5$ which is not composite.

for $n = 2$, $a = 20$ which is composite but not odd.

for $n = 3$, $a = 45$ which is the least composite odd.

Example 26 Show that there exists no polynomial $f(x)$ with integral coefficients which satisfy $f(a) = b, f(b) = c, f(c) = a$, where a, b, c are distinct integers.

Solution. Let $\quad f(x) = a_0 + a_1x + a_2x^2 + \ldots.. + a_nx^n$,

where $a_i \in$ integer $(i = 0, 1, 2, \ldots., n)$

Now, $\quad f(a) = a_0 + a_1a + a_2a^2 + \ldots. + a_na^n = b \quad$ ...(i)

$f(b) = a_0 + a_1b + a_2b^2 + \ldots. + a_nb^n = c \quad$ ...(ii)

$f(c) = a_0 + a_1c + a_2c^2 + \ldots. + a_nc^n = a \quad$ ...(iii)

which given $f(a), f(b), f(c) \in$ integer

$\therefore \quad f(a) - f(b) = (a - b)$. A function in terms of a and b.

$= (a - b) \cdot f_1(a, b) = b - c$, where $f_1(a, b)$ is an integer.

Similarly, $\quad (b - c) f_1(b, c) = c - a$,

and $\quad (c - a) f_1(c, a) = a - b$

Multiplying above expressions, we get

$$f_1(a, b) \cdot f_1(b, c) \cdot f_1(c, a) = 1$$

$\Rightarrow \quad f_1(a, b) = 1, f_1(b, c) = 1, f_1(c, a) = 1$

(as product of integers is 1, if each is one)

$\Rightarrow \quad |a - b| = |b - c| = |c - a|$

$\Rightarrow \quad a - b = b - c$

or $\quad a - b = c - b$

$\Rightarrow \quad a = c$ which is not possible $\quad$ (as a, b, c are distinct.)

Similarly, we get other cases.

Hence, no polynomial exists.

Example 27 If the rational number $\frac{p}{q}$, $q \neq 0$ (where p and q are relatively prime) is a root of the equation, $a_n x^n + a_{n-1} x^{n-1} + \ldots.. + a_1 x + a_0 = 0$, where $a_0, a_1, a_2, \ldots\ldots, a_n$ are integers and $a_n \neq 0$, then show that p is a divisor of a_0 and q that of a_n.

Solution. Given $\frac{p}{q}$ is a root.

$$\Rightarrow \quad a_n \left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \ldots.. + a_1\left(\frac{p}{q}\right) + a_0 = 0$$

$$\Rightarrow \quad a_n p^n + a_{n-1} q p^{n-1} + \ldots.. + a_1 q^{n-1} p + a_0 q^n = 0 \quad \ldots\text{(i)}$$

$$\Rightarrow \quad a_{n-1} p^{n-1} + a_{n-2} p^{n-2} q + \ldots.. + a_1 q^{n-2} p + a_0 q^{n-1} = \frac{-a_n p^n}{q} \quad \ldots\text{(ii)}$$

$\Rightarrow$ Here, $a_0, a_1, \ldots.., a_{n-2}, a_{n-1}$, $p, q \in$ Integers.

$\Rightarrow$ LHS is an integer, so RHS is also an integer.

ie, $-\frac{a_n p^n}{q}$ is an integer, where p and q are relatively prime to each other.

Thus, q must divide a_n.

Again, $\quad a_n p^n + a_{n-1} p^{n-1} q + \ldots.. + a_1 q^{n-1} p = a_0 q^n$

$$\Rightarrow \quad a_n p^{n-1} + a_{n-1} q p^{n-2} + \ldots.. + a_1 q^{n-1} = \frac{a_0 q^n}{p} \quad \ldots\text{(iii)}$$

As from above; $\frac{a_0 q^n}{p} \in$ Integer

$\Rightarrow p$ is divisor of a_0 (as p and q are relatively prime)

thus if the rational number $\frac{p}{q}$ is root of

$$a_n x^n + a_{n-1} x^{n-1} + \ldots.. + a_1 x + a_0 = 0,$$

then p divides a_0 and q divides a_n.

Point to Consider

In the equation, every rational root of

$$x^n + a_{n-1} x^{n-1} + \ldots.. + a_0; \; 0 \leq i \leq n-1$$

where a_i $(i = 0, 1, 2, \ldots.., (n-1))$ is an integer and each of these roots is a divisor of a_0.

Example 28 Let f and g be real-valued functions such that $f(x+y) + f(x-y) = 2f(x) \cdot g(y) \; \forall \; x, y \in R$. Prove that, if $f(x)$ is not identically zero and $|f(x)| \leq 1 \; \forall \; x \in R$, then $|g(y)| \leq 1 \; \forall \; y \in R$.

Solution. Let maximum value of $f(x)$ be M.

$$\Rightarrow \quad \max |f(x)| = M, \text{ where } 0 < M \leq 1 \quad \ldots\text{(i)}$$

(Since, f is not identically zero and $|f(x)| \leq 1 \; \forall \; x \in R$)

Now, $\quad f(x+y) + f(x-y) = 2f(x) \cdot g(y)$

$$\Rightarrow \quad |2f(x)| \cdot |g(y)| = |f(x+y) + f(x-y)|$$

$\Rightarrow \quad 2|f(x)| \cdot |g(y)| \le |f(x+y)| + |f(x-y)| \quad$ (as $|a+b| \le |a| + |b|$)

$\Rightarrow \quad 2|f(x)| \cdot |g(y)| \le M + M \quad$ [Using Eq. (i), *ie,* $\max |f(x)| = M$]

$\Rightarrow \quad |g(y)| \le 1$ for $y \in R$

Thus, if $f(x)$ is not identically zero and $|f(x)| \le 1 \; \forall \; x \in R$

then, $\quad |g(y)| \le 1 \quad$ for $\quad y \in R$

Example 29 If p and q are positive integers, f is a function defined for positive numbers and attains only positive values such that $f(x\,f(y)) = x^p y^q$, then prove that $q = p^2$.

Solution. For $x = \dfrac{1}{f(y)}$, we have

$$f\left(x \cdot \frac{1}{x}\right) = \frac{1}{f(y)^p} \cdot y^q \quad \Rightarrow \quad f(1) = \frac{y^q}{\{f(y)\}^p}$$

$$\Rightarrow \quad f(y) = \frac{y^{q/p}}{\{f(1)\}^{1/p}}$$

For $y = 1$, we have $\quad f(1) = 1$

$\therefore \quad f(y) = y^{q/p} \quad$ or $\quad f(x) = x^{q/p} \quad$ ...(i)

Hence, $\quad f(x \cdot y^{q/p}) = x^p \cdot y^q$

Let $\quad y^{q/p} = z \;\Rightarrow\; y = z^{p/q}$

$\Rightarrow \quad f(x \cdot z) = x^p \cdot z^p$

or $\quad f(x) = x^p \quad$ ...(ii)

From Eqs. (i) and (ii), we have $x^{q/p} = x^p$

$\Rightarrow \quad \dfrac{q}{p} = p \quad$ or $\quad q = p^2$

Example 30 If f is a polynomial with integer coefficients such that there exists four distinct integers a_1, a_2, a_3 and a_4 with $f(a_1) = f(a_2) = f(a_3) = f(a_4) = 1991$, then show that there exists no integer b such that $f(b) = 1993$.

Solution. Let $b \in$ integer, $f(b) = 1993$

Again, let $\quad g(x) = f(x) - 1991$

Now, $\quad g(a_1) = g(a_2) = g(a_3) = g(a_4) = 0$

Thus, $\quad (x - a_1)(x - a_2)(x - a_3)(x - a_4) \times$ integer $= g(x)$

$\therefore \quad g(b) = f(b) - 1991$

$\quad = 1993 - 1991 = 2 \quad$ ...(i)

Also, $\quad g(b) = (b - a_1)(b - a_2)(b - a_3)(b - a_4) \cdot$ integer $\quad$ ...(ii)

From Eqs. (i) and (ii),

$$(b - a_1)(b - a_2)(b - a_3)(b - a_4) \cdot \text{integer} = 2$$

Thus, $\quad (b - a_1)(b - a_2)(b - a_3)(b - a_4)$ are all divisors of 2 and are distinct.

$\therefore \quad (b - a_1)(b - a_2)(b - a_3)(b - a_4)$ are 1, – 1, 2, – 2 in same order.

$\Rightarrow \quad g(b) = 4$ (integer) $\ne 2. \quad$ [From Eq. (i)]

Hence, such 'b' doesn't exist.

Type 2 : **Only One Correct Option**

Example 31 Let $f(x) = \dfrac{a_{2k}x^{2k} + a_{2k-1}\,x^{2k-1} + \ \ldots + a_1x + a_0}{b_{2k}x^{2k} + b_{2k-1}\,x^{2k-1} + \ \ldots + b_1x + b_0}$,

where k is a positive integer, a_i , $b_i \in R$ and $a_{2k} \neq 0$, $b_{2k} \neq 0$ such that $b_{2k}x^{2k} + b_{2k-1}\,x^{2k-1} + \ \ldots + b_1x + b_0 = 0$ has no real roots, then

(a) $f(x)$ must be one to one

(b) $a_{2k}x^{2k} + a_{2k-1}\,x^{2k-1} + \ldots + a_1x + a_0 = 0$ must have real roots

(c) $f(x)$ must be many to one

(d) Nothing can be said about the above options.

Solution. $f(x)$ is continuous $\forall\ x \in R$.

$$\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = \frac{a_{2k}}{b_{2k}}$$

$\Rightarrow f(x)$ is many to one.

Hence, (c) is the correct answer.

Example 32 If $\log_{10}\left(\sin\left(x + \dfrac{\pi}{4}\right)\right) = \dfrac{\log_{10} 6 - 1}{2}$, then the value of $\log_{10}(\sin x) + \log(\cos x)$ is

(a) –1 (b) –2 (c) 2 (d) 1

Solution. $2\log_{10}\left(\dfrac{\sin x + \cos x}{\sqrt{2}}\right) = \log_{10}\left(\dfrac{6}{10}\right)$

$$\Rightarrow \qquad \log_{10}\left(\frac{1 + 2\sin x\cos x}{2}\right) = \log_{10}\left(\frac{6}{10}\right)$$

$$\Rightarrow \qquad \frac{1}{2} + \sin x\cos x = \frac{3}{5} \quad \Rightarrow \quad \sin x\cos x = \frac{1}{10}$$

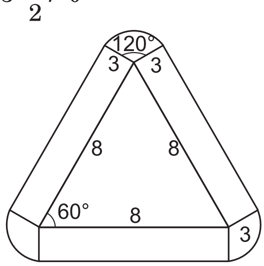
$$\Rightarrow \qquad \log_{10}(\sin x) + \log_{10}(\cos x) = -1$$

Hence, (a) is the correct answer.

Example 33. An equilateral triangle has side length 8. The area of the region containing all points outside the triangle but not more than 3 units from a point on the triangle is

(a) $9(8 + \pi)$ (b) $8(9 + \pi)$ (c) $9\left(8 + \dfrac{\pi}{2}\right)$ (d) $8\left(9 + \dfrac{\pi}{2}\right)$

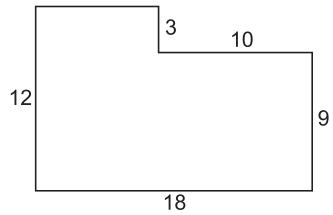
Solution. Area $= 3 \cdot (8 \cdot 3) + 3 \cdot \dfrac{1}{2} r^2\theta$



$$= 72 + \frac{3}{2} \cdot 9 \cdot \frac{2\pi}{3} = 72 + 9\pi = 9(8 + \pi)$$

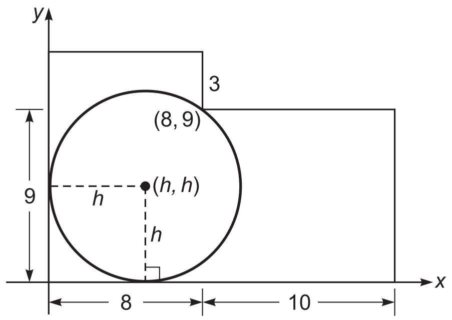
Hence, (a) is the correct answer.

Example 34 The diagram shows the dimensions of the floor of an L-shaped room. (All the angles are right angles). The area of the largest circle that can be drawn on the floor of this room is



(a) 16π (b) 25π (c) $\frac{81\pi}{4}$ (d) $\frac{145\pi}{4}$

Solution.



Here, $(x-h)^2 + (y-h)^2 = h^2$ passes through (8, 9)

$\Rightarrow$ $(8-h)^2 + (9-h)^2 = h^2$

$\Rightarrow$ $h^2 - 34h + 145 = 0 \Rightarrow (h-5)(h-29) = 0$

$\Rightarrow$ $h = 5$, neglecting $h = 29$

$\therefore$ $r = 5$

Area of the largest circle = $\pi(5)^2 = 25\pi$.

Hence, (b) is the correct answer.

Example 35 Suppose that the temperature T at every point (x, y) in the plane cartesian is given by the formula $T = 1 - x^2 + 2y^2$.

The correct statement about the maximum and minimum temperature along the line $x + y = 1$, is

(a) Minimum is –1. There is no maximum.
(b) Maximum is –1. There is no minimum.
(c) Maximum is 0. Minimum is – 1.
(d) There is neither a maximum nor a minimum along the line.

Solution. $T = 1 - x^2 + 2y^2$ where $x + y = 1$

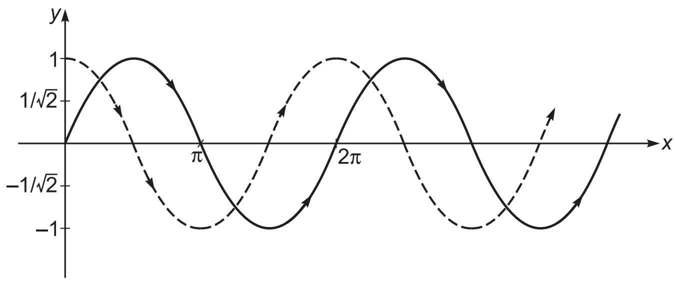
$$T = 1 - x^2 + 2(1-x)^2$$
$$T = 1 - x^2 + 2(1 + x^2 - 2x)$$
$$T = x^2 - 4x + 3 = (x-2)^2 - 1$$
$$T_{\text{max}} = \text{doesn't exist}$$
$$T_{\text{min}} = -1$$

Hence, (a) is the correct answer.

Example 36 The domain of the function $f(x) = \max\{\sin x, \cos x\}$ is $(-\infty, \infty)$. The range of $f(x)$ is

(a) $\left[-\frac{1}{\sqrt{2}}, 1\right]$ (b) $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ (c) $[0, 1]$ (d) $[-1, 1]$

Solution. Here, $f(x) = \max\{\sin x, \cos x\}$



From graph, range of $f(x) \in \left[-\frac{1}{\sqrt{2}}, 1\right]$.

Hence, (a) is the correct answer.

Type 3 : **More than One Correct Options**

Example 37 If the equation $x^2 + 4 + 3\cos(ax + b) = 2x$ has at least one solution where $a, b \in [0, 5]$, then the value of $(a + b)$ equal to

(a) 5π (b) 3π (c) 2π (d) π

Solution. $x^2 - 2x + 4 = -3\cos(ax + b)$

$$(x-1)^2 + 3 = -3\cos(ax + b)$$

For above equation to have at least one solution,

let $\quad f(x) = (x-1)^2 + 3$ and $f(x) = -3\cos(ax + b)$

If $\quad x = 1$, then LHS = 3 and RHS = $-3\cos(a + b)$.

Hence, $\quad \cos(a + b) = -1$

$\therefore \quad a + b = \pi, 3\pi, 5\pi$

But, $\quad 0 \le a + b \le 10 \quad \Rightarrow \quad a + b = \pi$ or 3π

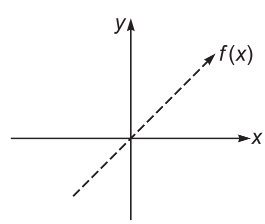
Hence, (b) and (d) are the correct answers.

Example 38 Which of the following functions have the same graph?

(a) $f(x) = \log_e e^x$ (b) $g(x) = |x| \operatorname{sgn} x$

(c) $h(x) = \cot^{-1}(\cot x)$ (d) $k(x) = \lim\limits_{x \to \infty} \dfrac{2|x|}{\pi} \tan^{-1}(nx)$

Solution. (a) $f(x) = \log_e e^x = x \cdot \log_e e = x$ shown as,



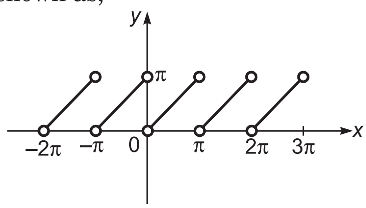
(b) $g(x) = |x| \operatorname{sgn} x$

$$= \begin{cases} |x| \cdot \dfrac{x}{|x|}, & x \ne 0 \\ 0, & x = 0 \end{cases} = \begin{cases} x, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

$\therefore \quad g(x) = x,$

which is same as $f(x)$.

(c) $h(x) = \cot^{-1}(\cot x)$ is shown as,



which is not same as $f(x)$ and $g(x)$.

(d) $k(x) = \lim_{x \to \infty} \frac{2|x|}{\pi} \cdot \tan^{-1}(nx)$

$$= \begin{cases} \frac{2x}{\pi} \cdot \frac{\pi}{2}, & x > 0 \\ -\frac{2x}{\pi} \cdot -\frac{\pi}{2}, & x < 0 \\ 0, & x = 0 \end{cases}$$

$$= \begin{cases} x, & x > 0 \\ x, & x < 0 \\ 0, & x = 0 \end{cases}$$

$\therefore \quad k(x) = x$, for all x.

Hence, (a), (b) and (d) are the correct answers.

Type 4 : **Assertion and Reason**

Directions
(Q. Nos. 39 to 41)

For the following questions, choose the correct answers from the codes (a), (b), (c) and (d) defined as follows :

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I.

(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I.

(c) Statement I is true, Statement II is false.

(d) Statement I is false, Statement II is true.

Example 39 Let $f(\theta) = \frac{\sin^2\theta\cos\theta}{(\sin\theta + \cos\theta)} - \frac{1}{4}\tan\left(\frac{\pi}{4} - \theta\right)$,

$$\forall\, \theta \in R - \left\{n\pi - \frac{\pi}{4}\right\}, n \in I.$$

Then,

Statement I The largest and smallest value of $f(\theta)$ differ by $\frac{1}{\sqrt{2}}$.

Statement II $a\sin x + b\cos x + c \in [c - \sqrt{a^2 + b^2}, c + \sqrt{a^2 + b^2}], \forall\, x \in R$

where $a, b, c \in R$.

Solution. $f(\theta) = \dfrac{4\sin^2\theta\cos\theta - \cos\theta + \sin\theta}{4(\cos\theta + \sin\theta)}$

$$= \frac{2\sin\theta\,((\sin\theta + \cos\theta)^2 - 1) - (\cos\theta + \sin\theta) + 2\sin\theta}{4(\cos\theta + \sin\theta)}$$

$$= \frac{1}{2}\sin\theta\,(\sin\theta + \cos\theta) - \frac{1}{4}$$

$$= \frac{1}{4}(\sin 2\theta - \cos 2\theta) \in \left[-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right]$$

Hence, (a) is the correct answer.

Example 40 Consider two functions $f(x) = 1 + e^{\cot^2 x}$ and $g(x) = \sqrt{2|\sin x| - 1} + \dfrac{1 - \cos 2x}{1 + \sin^4 x}$.

Statement I The solutions of the equation $f(x) = g(x)$ is given by $x = (2n+1)\dfrac{\pi}{2}$, $\forall\, n \in I$.

Statement II If $f(x) \geq k$ and $g(x) \leq k$ (where $k \in R$), then solutions of the equation $f(x) = g(x)$ is the solution corresponding to the equation $f(x) = k$.

Solution. LHS $= 1 + e^{\cot^2 x} \geq 2$

As $\sqrt{2|\sin x| - 1} \leq 1$

and $$\frac{1 - \cos 2x}{1 + \sin^4 x} = \frac{2\sin^2 x}{1 + \sin^4 x} = \frac{2}{\dfrac{1}{\sin^2 x} + \sin^2 x} \leq 1$$

$\therefore$ $$\text{RHS} = \sqrt{2|\sin x| - 1} + \frac{1 - \cos 2x}{1 + \sin^4 x} \leq 2$$

Equation will satisfy, if LHS = RHS $= 2$ which is possible when $\cot^2 x = 0$ and $|\sin x| = 1$.

$\Rightarrow$ $$x = (2n+1)\frac{\pi}{2},\ n \in I$$

$\therefore$ Statement I is correct.

Statement II is not always correct because solution of the equation $f(x) = g(x)$ will be solutions corresponding to $f(x) = g(x) = k$ in the domain of $f(x)$ and $g(x)$ both.

Hence, (c) is the correct answer.

Example 41 **Statement I** The largest exponent of 2 which divides the number $N = 2^{2008} + 10^{2008}$ is 2009.

Statement II $5^n + 1$ is divisible by 2 but not divisible by 4, $\forall\, n \in N$.

Solution. $N = 2^{2008} + 2^{2008} \cdot 5^{2008}$

$$= 2^{2008}[1 + 5^{2008}] = 2^{2008}[1 + (1+4)^{2008}]$$

$$= 2^{2008}[(1+1) + {}^{2008}C_1 \cdot 4 + {}^{2008}C_2 \cdot 4^2 + \ldots + {}^{2008}C_{2008} \cdot 4^{2008}]$$

$$= 2^{2009}[1 + 2 \cdot {}^{2008}C_1 + \ldots] = 2^{2009}\underbrace{[1 + 2m]}_{\text{odd} \Rightarrow \text{not divisible by 2}},\ m \in N$$

Hence, highest exponent of $2 = 2009$.

Hence, (a) is the correct answer.

Type 5 : Linked Comprehension Based Questions

Passage I
(Q. Nos. 42 to 44)

Let $a_m (m = 1, 2, \dots, p)$ be the possible integral values of a for which the graphs of

$$f(x) = ax^2 + 2bx + b \quad \text{and} \quad g(x) = 5x^2 - 3bx - a$$

meet at some points for all real values of b.

Let $$t_r = \prod_{m=1}^{p} (r - a_m),$$

and $$S_n = \sum_{r=1}^{n} t_r, \ n \in N.$$

42. The minimum possible value of a is

(a) $\frac{1}{5}$ (b) $\frac{5}{26}$ (c) $\frac{3}{28}$ (d) $\frac{2}{43}$

43. The sum of values of n for which S_n vanishes is

(a) 8 (b) 9 (c) 10 (d) 15

44. The value of $\sum_{r=5}^{\infty} \frac{1}{t_r}$ is equal to

(a) $\frac{1}{3}$ (b) $\frac{1}{6}$ (c) $\frac{1}{15}$ (d) $\frac{1}{18}$

Solution. (Q. Nos. 42 to 44)

$$ax^2 + 2bx + b = 5x^2 - 3bx - a$$

$$\Rightarrow \quad (a - 5)x^2 + 5bx + (b + a) = 0$$

If $a \neq 5$, then since $x \in R$,

$$D = 25b^2 - 4(b + a)(a - 5) \geq 0, \forall \ b \in R$$

$$\Rightarrow \quad 25b^2 - 4(a - 5)b - 4a(a - 5) \geq 0, \forall \ b \in R$$

$$\therefore \quad 16(a - 5)^2 + 16(25)a(a - 5) \leq 0$$

$$\Rightarrow \quad 16(a - 5)(a - 5 + 25a) \leq 0$$

$$\Rightarrow \quad (a - 5)(26a - 5) \leq 0 \quad \therefore \quad a \in \left[\frac{5}{26}, 5\right)$$

If $a = 5$, $5bx + (b + 5) = 0$ is not satisfied for $b = 0$

$$\therefore \quad a_m \in \{1, 2, 3, 4\}$$

$$t_r = (r - 1)(r - 2)(r - 3)(r - 4)$$

$$S_n = \frac{1}{5} \sum_{r=1}^{n} (r - 4)(r - 3)(r - 2)(r - 1)(r - (r - 5))$$

$$= \frac{1}{5} \sum_{r=1}^{n} [(r - 4)(r - 3)(r - 2)(r - 1)r - (r - 5)(r - 4)(r - 3)(r - 2)(r - 1)]$$

$$= \frac{1}{5} n(n - 1)(n - 2)(n - 3)(n - 4)$$

$$S_n = 0 \Rightarrow n = 1, 2, 3, 4 \qquad (n = 0 \text{ rejected})$$

$$\Sigma \frac{1}{t_r} = \frac{1}{3} \lim_{n \to \infty} \sum_{r=5}^{n} \frac{(r - 1) - (r - 4)}{(r - 4)(r - 3)(r - 2)(r - 1)}$$

$$= \frac{1}{3} \lim_{n \to \infty} \sum_{r=5}^{n} \left(\frac{1}{(r-4)(r-3)(r-2)} - \frac{1}{(r-3)(r-2)(r-1)} \right)$$

$$= \lim_{n \to \infty} \frac{1}{3} \left[\frac{1}{6} - \frac{1}{(n-3)(n-2)(n-1)} \right] = \frac{1}{18}$$

Ans. 42. (b) **43.** (c) **44.** (d)

Passage II

(Q. Nos. 45 to 47)

Consider the two quadratic polynomials :

$$C_a : y = \frac{x^2}{4} - ax + a^2 + a - 2$$

and $$C : y = 2 - \frac{x^2}{4}$$

45. If the origin lies between the zeroes of the polynomial C_a, then the number of integral value(s) of 'a' is

(a) 1 (b) 2 (c) 3 (d) more than 3

46. If 'a' varies, then the equation of the locus of the vertex of C_a is

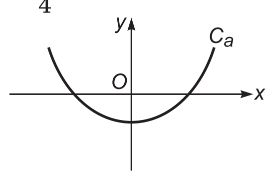
(a) $x - 2y - 4 = 0$ (b) $2x - y - 4 = 0$ (c) $x - 2y + 4 = 0$ (d) $2x + y - 4 = 0$

47. For $a = 3$, if the lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are common tangents to the graph of C_a and C, then the value of $(m_1 + m_2)$ is equal to

(a) -6 (b) -3 (c) $\frac{1}{2}$ (d) None of these

Solution. (Q. Nos. 45 to 47)

$$y = f(x) = \frac{x^2}{4} - ax + a^2 + a - 2$$



45. For zeroes to be on either side of origin $f(x) < 0$

$$a^2 + a - 2 < 0$$

$\Rightarrow$ $(a + 2)(a - 1) < 0 \Rightarrow -2 < a < 1$

$\Rightarrow$ 2 integers *ie*, $\{-1, 0\}$.

Hence, (b) is the correct answer.

46. Vertex of C_a is $(2a, a - 2)$.

Hence, $h = 2a$ and $k = a - 2$

$h = 2(k + 2)$

Locus $x = 2y + 4$

$\Rightarrow$ $x - 2y - 4 = 0$

Hence, (a) is the correct answer.

47. Let $y = mx + c$ is a common tangent to

$$y = \frac{x^2}{4} - 3x + 10 \qquad \text{(for } a = 3)\ldots\text{(i)}$$

and $$y = 2 - \frac{x^2}{4} \qquad \ldots\text{(ii)}$$

where $m = m_1$ or m_2 and $c = c_1$ or c_2

Solving $y = mx + c$ with Eq. (i),

$$mx + c = \frac{x^2}{4} - 3x + 10$$

or $$\frac{x^2}{4} - (m+3)x + 10 - c = 0$$

$$D = 0 \text{ gives } (m+3)^2 = 10 - c$$

$\Rightarrow$ $$c = 10 - (m+3)^2 \qquad \ldots\text{(iii)}$$

Similarly, $$mx + c = 2 - \frac{x^2}{4}$$

$\Rightarrow$ $$\frac{x^2}{4} + mx + c - 2 = 0$$

$$D = 0 \text{ gives } m^2 = c - 2$$

$\Rightarrow$ $$c = 2 + m^2 \qquad \ldots\text{(iv)}$$

From Eqs. (iii) and (iv), we get

$$10 - (m+3)^2 = 2 + m^2 \Rightarrow 2m^2 + 6m + 1 = 0$$

$\Rightarrow$ $$m_1 + m_2 = -\frac{6}{2} = -3$$

Hence, (b) is the correct answer.

Type 6 : **Match the Columns**

Example 48 Match the statements of Column I with values of Column II.

	Column I	Column II	
(A)	Let $f(x)$ be a function on $(-\infty, \infty)$ and $f(x+2) = f(x-2)$. If $f(x) = 0$ has only three real roots in [0, 4] and one of them is 4, then the number of real roots of $f(x) = 0$ in (– 8, 10] is	(p)	4
(B)	Let $r_1, r_2, r_3, \ldots, r_n$ be n positive integers, not necessarily distinct, such that $(x + r_1)(x + r_2)\ldots(x + r_n) = x^n + 56x^{n-1} + \ldots + 2009$ The possible value of n is	(q)	5
(C)	If x and y are positive integers and $2xy = 2009 - 3y$, then the number of ordered pairs of (x, y) is	(r)	8
(D)	If $x, y \in R$, satisfying the equation $\frac{(x-4)^2}{4} + \frac{y^2}{9} = 1$, then the difference between the largest and smallest value of the expression $\frac{x^2}{4} + \frac{y^2}{9}$ is	(s)	9

Solution. (A) $\rightarrow$ (s), (B) $\rightarrow$ (p), (C) $\rightarrow$ (q), (D) $\rightarrow$ (r)

(A) Given, $f(x+2) = f(x-2)$, domain is $R\ x \rightarrow x+2$

$$f(x+4) = f(x)$$

$\Rightarrow f$ is periodic with period 4.

Put $\quad x = 0, f(4) = f(0)$

But $\quad f(4) = 0$

$\Rightarrow \quad f(0) = 0$

$\Rightarrow \quad x = 0$, is also the root.

Therefore, in $[0, 4]$, f has only 3 roots.

So, $f(x)$ is the form of $\sin\left(\frac{\pi x}{2}\right)$ fulfilling all conditions given in the question, hence the 3rd root in $[0, 4]$ is 2 as $f(2) = 0$.

(B) $(r_1 + r_2 + \ldots + r_n) = 56$

and $\quad r_1 r_2 \ldots r_n = 2009 = 7^2 \cdot 41 \quad$ (using theory of equation)

As $\quad 2009 = 7^2 \cdot 49$

Therefore, no value of r_i can be 49 or larger factor of 2009 otherwise their sum is larger than 56 as

$$56 + 41 + 7 + 7 + 1$$

Therefore, $\quad r_1 = 41; r_2 = r_3 = 7$ and $r_n = 1$

Hence, $\quad n = 4$

(C) $2xy = 2009 - 3y$ yields $2xy + 3y = 2009$

or $\quad y = \dfrac{2009}{2x+3}$

So, $2x + 3$ must be the factors of 2009.

Since, $\quad 2009 = 7^2 \cdot 41$

Thus, $2x + 3$ can be one of the factors 7, 49, 41, 287 and 2009.

or $\quad 2x + 3 = 7; 2x + 3 = 49; 2x + 3 = 41; 2x + 3 = 287$

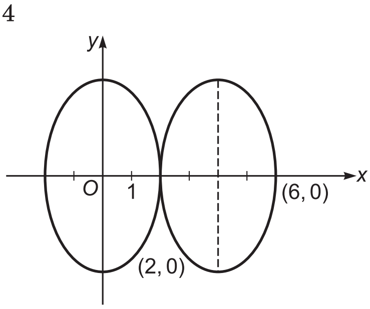
and $2x + 3 = 2009$.

Hence, $\quad x = 2, 23, 19, 142$ and 1003

$\Rightarrow$ Ordered pairs.

(D) Let $x - 4 = 2\cos\theta$

$\Rightarrow \quad x = 2\cos\theta + 4$



$y = 3\sin\theta \Rightarrow y = 3\sin\theta$

Now, $z = \dfrac{x^2}{4} + \dfrac{y^2}{9} = \dfrac{(2\cos\theta + 4)^2}{4} + \sin^2\theta$

$$= \frac{4\cos^2\theta + 16 + 16\cos\theta + 4\sin^2\theta}{4} = \frac{20 + 16\cos\theta}{4}$$

$$z = 5 + 4\cos\theta$$

Hence, $z_{\max} - z_{\min} = (9 - 1) = 8$

Type 7 : **Integer Answer Type Questions**

Example 49 Let $f(x) = \sin^{23} x - \cos^{22} x$ and $g(x) = 1 + \dfrac{1}{2}\tan^{-1}|x|$.

Then, the number of values of x in interval $[-10\pi, 20\pi]$ satisfying the equation $f(x) = \text{sgn}(g(x))$, is $5a$. Then, a is equal to ....

Solution. **(3)** $g(x) = \dfrac{1}{2}\tan^{-1}|x| + 1$

$\Rightarrow$ $\text{sgn}(g(x)) = 1$

$$\sin^{23} x - \cos^{22} x = 1$$

$$\sin^{23} x = 1 + \cos^{22} x$$

which is possible, if $\sin x = 1$ and $\cos x = 0 \Rightarrow \sin x = 1, x = 2n\pi + \dfrac{\pi}{2}$

Hence, $-10\pi \le 2n\pi + \dfrac{\pi}{2} \le 20\pi$

$\Rightarrow$ $-\dfrac{21}{2} \le 2n \le \dfrac{39}{2}$

$\Rightarrow$ $-\dfrac{21}{4} \le n \le \dfrac{39}{4} \Rightarrow -5 \le n \le 9$

Hence, number of values of $x = 15 \Rightarrow a = 3$.

Example 50 Consider the function $g(x)$ defined as

$g(x) \cdot (x^{(2^{2008} - 1)} - 1) = (x + 1)(x^2 + 1)(x^4 + 1) \dots (x^{2^{2007}} + 1) - 1$

the value of $g(2)$ equals to ....

Solution. **(2)** RHS $= \dfrac{(x - 1)(x + 1)(x^2 + 1)(x^4 + 1) \dots (x^{2^{2007}} + 1)}{(x - 1)} - 1$

$$= \frac{(x^2 - 1)(x^2 + 1) \dots (x^{2^{2007}} + 1)}{(x - 1)} - 1$$

$$= \frac{(x^{2^2} - 1)(x^{2^2} + 1) \dots (x^{2^{2007}} + 1)}{(x - 1)} - 1$$

Hence, $g(x) \cdot (x^{(2^{2008} - 1)} - 1) = \dfrac{(x^{2^{2008}} - 1) - (x - 1)}{(x - 1)}$

$$= \frac{x\,(x^{2^{2008} - 1} - 1)}{(x - 1)} \Rightarrow g(x) = \frac{x}{x - 1}$$

$\therefore$ $g(2) = 2$

Proficiency in 'Functions'
Exercise 1

Type 1 : **Only One Correct Option**

1. The sum of the maximum and minimum values of the function $f(x) = \dfrac{1}{1 + (2\cos x - 4\sin x)^2}$ is

(a) $\dfrac{22}{21}$ (b) $\dfrac{21}{20}$
(c) $\dfrac{22}{20}$ (d) $\dfrac{21}{11}$

2. Let $f : X \to Y, f(x) = \sin x + \cos x + 2\sqrt{2}$ is invertible, then $X \to Y$ is/are

(a) $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right] \to [\sqrt{2}, 3\sqrt{2}]$ (b) $\left[-\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right] \to [\sqrt{2}, 3\sqrt{2}]$
(c) $\left[-\dfrac{3\pi}{4}, \dfrac{3\pi}{4}\right] \to [\sqrt{2}, -3\sqrt{2}]$ (d) $\left[-\dfrac{3\pi}{4}, -\dfrac{\pi}{4}\right] \to [\sqrt{2}, 3\sqrt{2}]$

3. The range of values of a so that all the roots of the equation $2x^3 - 3x^2 - 12x + a = 0$ are real and distinct belongs to

(a) (7, 20) (b) (–7, 20) (c) (–20, 7) (d) (–7, 7)

4. If $f(x)$ is continuous such that $|f(x)| \le 1, \forall x \in R$ and $g(x) = \dfrac{e^{f(x)} - e^{|f(x)|}}{e^{f(x)} + e^{|f(x)|}}$, then range of $g(x)$ is

(a) [0, 1] (b) $\left[0, \dfrac{e^2 + 1}{e^2 - 1}\right]$
(c) $\left[0, \dfrac{e^2 - 1}{e^2 + 1}\right]$ (d) $\left[\dfrac{1 - e^2}{1 + e^2}, 0\right]$

5. Let $f(x) = \sqrt{|x| - \{x\}}$ (where {} denotes the fractional part of x and X, Y and its domain and range respectively, then

(a) $f : X \to Y : y = f(x)$ is one-one function
(b) $X \in \left(-\infty, -\dfrac{1}{2}\right] \cup [0, \infty)$ and $Y \in \left[\dfrac{1}{2}, \infty\right)$
(c) $X \in \left(-\infty, -\dfrac{1}{2}\right] \cup [0, \infty)$ and $Y \in [0, \infty)$
(d) None of the above

6. If the graphs of the functions $y = \ln x$ and $y = ax$ intersect at exactly two points, then a must be

(a) $(0, e)$
(b) $\left(\frac{1}{e}, 0\right)$
(c) $\left(0, \frac{1}{e}\right)$
(d) None of these

7. A quadratic polynomial maps from $[-2, 3]$ onto $[0, 3]$ and touches x-axis at $x = 3$, then the polynomial is

(a) $\frac{3}{16}(x^2 - 6x + 16)$
(b) $\frac{3}{25}(x^2 - 6x + 9)$
(c) $\frac{3}{25}(x^2 - 6x + 16)$
(d) $\frac{3}{16}(x^2 - 6x + 9)$

8. The range of the function $y = \sqrt{2\{x\} - \{x\}^2 - \frac{3}{4}}$ is (where $\{\cdot\}$ denotes the fractional part)

(a) $\left[-\frac{1}{4}, \frac{1}{4}\right]$
(b) $\left[0, \frac{1}{2}\right]$
(c) $\left[0, \frac{1}{4}\right]$
(d) $\left[\frac{1}{4}, \frac{1}{2}\right]$

9. Let $f(x)$ be a fourth differentiable function such that $f(2x^2 - 1) = 2xf(x), \forall x \in R$, then $f^{iv}(0)$ is equal to (where $f^{iv}(0)$ represents fourth derivative of $f(x)$ at $x = 0$)

(a) 0
(b) 1
(c) -1
(d) Data insufficient

10. Number of solutions of the equation $[y + [y]] = 2\cos x$ is (where $y = \frac{1}{3}[\sin x + [\sin x + [\sin x]]]$ and $[\cdot]$ denotes the greatest integer function)

(a) 1
(b) 2
(c) 3
(d) None of these

11. If a function satisfies $f(x+1) + f(x-1) = \sqrt{2}f(x)$, then period of $f(x)$ can be

(a) 2 (b) 4 (c) 6 (d) 8

12. If x and α are real, then the inequation $\log_2 x + \log_x 2 + 2\cos\alpha \le 0$

(a) has no solution
(b) has exactly two solutions
(c) is satisfied for any real α and any real x in $(0, 1)$
(d) is satisfied for any real α and any real x in $(1, \infty)$

13. The range of values of 'a' such that $\left(\frac{1}{2}\right)^{|x|} = x^2 - a$ is satisfied for maximum number of values of 'x'

(a) $(-\infty, -1)$ (b) $(-\infty, \infty)$ (c) $(-1, 1)$ (d) $(-1, \infty)$

14. Let $f : R \to R$ be a function defined by $f(x) = \{|\cos x|\}$, where $\{x\}$ represents fractional part of x. Let S be the set containing all real values x lying in the interval $[0, 2\pi]$ for which $f(x) \ne |\cos x|$. Then, number of elements in the set S is

(a) 0 (b) 1 (c) 3 (d) infinite

15. The domain of the function $f(x)=\sqrt{\log_{\sin x+\cos x}(|\cos x|+\cos x)}, 0 \le x \le \pi$ is

(a) $(0, \pi)$ (b) $\left(0, \frac{\pi}{2}\right)$

(c) $\left(0, \frac{\pi}{3}\right)$ (d) None of these

16. If $f(x)=(x^2+2\alpha x+\alpha^2-1)^{1/4}$ has its domain and range such that their union is set of real numbers, then α satisfies

(a) $-1<\alpha<1$ (b) $\alpha \le -1$

(c) $\alpha \ge 1$ (d) $\alpha \le 1$

17. Let $f:(e, \infty) \to R$ be a function defined by $f(x)=\log(\log(\log x))$, the base of the logarithm being e. Then,

(a) f is one-one and onto (b) f is one-one but not onto

(c) f is onto but not one-one (d) the range of f is equal to its domain

18. The expression $x^2-4px+q^2>0$ for all real x and also $r^2+p^2<qr$, then the range of $f(x)=\dfrac{x+r}{x^2+qx+p^2}$ is

(a) $\left[\frac{p}{2r}, \frac{q}{2r}\right]$ (b) $(0, \infty)$

(c) $(-\infty, 0)$ (d) $(-\infty, \infty)$

19. The period of $\sin\dfrac{\pi[x]}{12}+\cos\dfrac{\pi[x]}{4}+\tan\dfrac{\pi[x]}{3}$ where $[x]$ represents the greatest integer less than or equal to x is

(a) 12 (b) 4

(c) 3 (d) 24

20. If $f(2x+3y, 2x-7y)=20x$, then $f(x, y)$ equals to

(a) $7x-3y$ (b) $7x+3y$

(c) $3x-7y$ (d) $x-y$

21. The range of the function $f(x)=\sqrt{x-1}+2\sqrt{3-x}$ is

(a) $[\sqrt{2}, 2\sqrt{2}]$ (b) $[\sqrt{2}, \sqrt{10}]$

(c) $[2\sqrt{2}, \sqrt{10}]$ (d) $[1, 3]$

22. The domain of the function $f(x)=\cos^{-1}(\sec(\cos^{-1}x))+\sin^{-1}(\operatorname{cosec}(\sin^{-1}x))$ is

(a) $x \in R$ (b) $x=1, -1$

(c) $-1 \le x \le 1$ (d) $x \in \phi$

23. Let $f(x)$ be a polynomial one-one function such that

$f(x)f(y)+2=f(x)+f(y)+f(xy), \forall\, x, y \in R-\{0\}, f(1) \ne 1, f'(1)=3.$

Let $g(x)=\dfrac{x}{4}(f(x)+3)-\displaystyle\int_0^x f(x)\,dx$, then

(a) $g(x)=0$ has exactly one root for $x \in (0, 1)$

(b) $g(x)=0$ has exactly two roots for $x \in (0, 1)$

(c) $g(x) \ne 0, \ \forall\, x \in R-\{0\}$

(d) $g(x)=0, \ \forall\, x \in R-\{0\}$

24. Let $f(x)$ be a polynomial with real coefficients such that $f(x)=f'(x)\times f''(x)$. If $f(x)=0$ is satisfied $x=1, 2, 3$ only, then the value of $f'(1)f'(2)f'(3)$ is

(a) positive (b) negative (c) 0 (d) Inadequate data

25. Let $A=\{1, 2, 3, 4, 5\}$ and $f: A \to A$ be an into function such that $f(i) \neq i, \forall\, i \in A$, then number of such functions f are

(a) 1024 (b) 904 (c) 980 (d) None of these

26. If functions $f:\{1, 2, \dots, n\} \to \{1995, 1996\}$ satisfying $f(1)+f(2)+\dots+f(1996)=$ odd integer are formed, then number of such functions can be

(a) 2^n (b) $2^{n/2}$ (c) n^2 (d) 2^{n-1}

27. The range of $y=\sin^3 x - 6\sin^2 x + 11\sin x - 6$ is

(a) $[-24, 2]$ (b) $[-24, 0]$ (c) $[0, 24]$ (d) None of these

28. Let $f(x)=x^2-2x$ and $g(x)=f(f(x)-1)+f(5-f(x))$, then

(a) $g(x)<0, \forall x \in R$ (b) $g(x)<0$ for some $x \in R$

(c) $g(x) \geq 0$ for some $x \in R$ (d) $g(x) \geq 0, \forall x \in R$

29. If $f(x)$ and $g(x)$ are non-periodic functions, then $h(x)=f(g(x))$ is

(a) non-periodic

(b) periodic

(c) may be periodic

(d) always periodic, if domain of $h(x)$ is a proper subset of real numbers

30. If $f(x)$ is a real-valued function discontinuous at all integral points lying in $[0, n]$ and if $(f(x))^2=1\ \forall\ x \in [0, n]$, then number of functions $f(x)$ are

(a) 2^{n+1} (b) 6×3^n (c) $2\times 3^{n-1}$ (d) 3^{n+1}

31. A function f from integers to integers is defined as $f(x)=\begin{cases} n+3, & n \in \text{odd} \\ n/2, & n \in \text{even} \end{cases}$

suppose $k \in$ odd and $f(f(f(k)))=27$, then the sum of digits of k is

(a) 3 (b) 6 (c) 9 (d) 12

32. $f: R \to R, f(x)=\dfrac{\sin(\pi\{x\})}{x^4+3x^2+7}$ where $\{\ \}$ is a fractional function, then

(a) f is injective (b) f is not one-one and non-constant

(c) f is a surjective (d) f is a zero function

33. Let $f: R \to R$ and $g: R \to R$ be two one-one and onto functions, such that they are the mirror images of each other about the line $y=a$. If $h(x)=f(x)+g(x)$, then $h(x)$ is

(a) one-one and onto (b) only one-one and not onto

(c) only onto but not one-one (d) None of these

34. Domain of the function $f(x)$, if $3^x+3^{f(x)}=$ minimum of $\phi(t)$ where $\phi(t)=$ minimum of $\{2t^3-15t^2+36t-25, 2+|\sin t|; 2 \leq t \leq 4\}$ is

(a) $(-\infty, 1)$ (b) $(-\infty, \log_3 e)$

(c) $(0, \log_3 2)$ (d) $(-\infty, \log_3 2)$

Type 2 : **More than One Correct Options**

35. Let $f : R \to R$ be a function defined by $f(x+1) = \dfrac{f(x)-5}{f(x)-3}$ $\forall\, x \in R$. Then, which of the following statements is/are true ?

(a) $f(2008) = f(2004)$ (b) $f(2006) = f(2010)$

(c) $f(2006) = f(2002)$ (d) $f(2006) = f(2018)$

36. Let $f(x) = 1 - x - x^3$. Then, the real values of x satisfying the inequality,

$$1 - f(x) - f^3(x) > f(1-5x), \text{ are}$$

(a) $(-2, 0)$ (b) $(0, 2)$ (c) $(2, \infty)$ (d) $(-\infty, -2)$

37. If a function satisfies

$$(x-y)f(x+y) - (x+y)f(x-y) = 2(x^2y - y^3), \forall\, x, y \in R \text{ and } f(1) = 2, \text{ then}$$

(a) $f(x)$ must be polynomial function (b) $f(3) = 12$

(c) $f(0) = 0$ (d) $f(x)$ may not be differentiable

38. If the fundamental period of function $f(x) = \sin x + \cos(\sqrt{4-a^2})x$ is 4π, then the value of a is/are

(a) $\dfrac{\sqrt{15}}{2}$ (b) $-\dfrac{\sqrt{15}}{2}$

(c) $\dfrac{\sqrt{7}}{2}$ (d) $-\dfrac{\sqrt{7}}{2}$

39. Let $f(x)$ be a real-valued function such that $f(0) = \dfrac{1}{2}$ and

$f(x+y) = f(x)f(a-y) + f(y)f(a-x)\ \forall\, x, y \in R$, then for some real a

(a) $f(x)$ is a periodic function (b) $f(x)$ is a constant function

(c) $f(x) = \dfrac{1}{2}$ (d) $f(x) = \dfrac{\cos x}{2}$

40. If $f(g(x))$ is one-one function, then

(a) $g(x)$ must be one-one (b) $f(x)$ must be one-one

(c) $f(x)$ may not be one-one (d) $g(x)$ may not be one-one

41. Which of the following functions have their range equal to R (the set of real numbers)?

(a) $x \sin x$

(b) $\dfrac{[x]}{\tan 2x} \cdot x \in \left(-\dfrac{\pi}{4} \cdot \dfrac{\pi}{4}\right) - \{0\}$, $[\cdot]$ denotes the greatest integer function

(c) $\dfrac{x}{\sin x}$

(d) $[x] + \sqrt{\{x\}}$, $[\cdot]$ and $\{\cdot\}$ respectively denote the greatest integer and fractional part functions.

42. Which of the following pairs of function are identical?

(a) $f(x) = e^{\ln \sec^{-1} x}$ and $g(x) = \sec^{-1} x$

(b) $f(x) = \tan(\tan^{-1} x)$ and $g(x) = \cot(\cot^{-1} x)$

(c) $f(x) = \text{sgn}(x)$ and $g(x) = \text{sgn}(\text{sgn}(x))$

(d) $f(x) = \cot^2 x \cdot \cos^2 x$ and $g(x) = \cot^2 x - \cos^2 x$

Type 3 : **Assertion and Reason**

Directions

(Q. Nos. 43 to 53)

For the following questions, choose the correct answers from the codes (a), (b), (c) and (d) defined as follows :

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I.

(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I.

(c) Statement I is true, Statement II is false.

(d) Statement I is false, Statement II is true.

43. Statement I The function $f(x) = x \sin x$ and $f'(x) = x \cos x + \sin x$ are both non-periodic.

Statement II The derivative of differentiable function (non-periodic) is non-periodic function.

44. Statement I The maximum value of $\sin \sqrt{2}\, x + \sin ax$ cannot be 2. (where a is positive rational number)

Statement II $\dfrac{\sqrt{2}}{a}$ is irrational.

45. Let $f : R \to R$ be a function such that $f(x) = \dfrac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$.

Statement I $f(x)$ is into function.

Statement II $f(x)$ is many-one function, and the many-one function is not onto.

46. Statement I The range of $f(x) = \sin\left(\dfrac{\pi}{5} + x\right) - \sin\left(\dfrac{\pi}{5} - x\right) - \sin\left(\dfrac{2\pi}{5} + x\right) + \sin\left(\dfrac{2\pi}{5} - x\right)$ is $[-1, 1]$.

Statement II $\cos\dfrac{\pi}{5} - \cos\dfrac{2\pi}{5} = \dfrac{1}{2}$.

47. Statement I The period of $f(x) = 2\cos\dfrac{1}{3}(x - \pi) + 4\sin\dfrac{1}{3}(x - \pi)$ is 3π.

Statement II If T is the period of $f(x)$, then the period of $f(ax + b)$ is $\dfrac{T}{|a|}$.

48. f is a function defined on the interval $[-1, 1]$ such that $f(\sin 2x) = \sin x + \cos x$.

Statement I If $x \in \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$, then $f(\tan^2 x) = \sec x$

Statement II $f(x) = \sqrt{1 + x}, \forall\, x \in [-1, 1]$.

49. Statement I The equation $f(x) = 4x^5 + 20x - 9 = 0$ has only one real root.

Statement II $f'(x) = 20x^4 + 20 = 0$ has no real root.

50. Statement I The range of $\log\left(\dfrac{1}{1 + x^2}\right)$ is $(-\infty, \infty)$.

Statement II When $0 < x \le 1$, $\log x \in (-\infty, 0]$.

51. Let $f : X \to Y$ be a function defined by $f(x) = 2\sin\left(x + \frac{\pi}{4}\right) - \sqrt{2}\cos x + c$.

Statement I For set X, $x \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right]$, $f(x)$ is one-one function.

Statement II $f'(x) \geq 0, x \in \left[0, \frac{\pi}{2}\right]$.

52. Let $f(x) = \sin x$

Statement I f is not a polynomial function.

Statement II nth derivative of $f(x)$, w.r.t. x, is not a zero function for any positive integer n.

53. Statement I The function $f : R \to R$, given $f(x) = \log_a (x + \sqrt{x^2 + 1})$, $a > 0$, $a \neq 1$ is invertible.

Statement II f is many to one and into.

Type 4 : **Linked Comprehension Based Questions**

Passage I

(Q. Nos. 54 to 56)

Let $f : R \to R$ be a continuous function such that $f(x) - 2f\left(\frac{x}{2}\right) + f\left(\frac{x}{4}\right) = x^2$.

54. $f(3)$ is equal to

(a) $f(0)$ (b) $4 + f(0)$ (c) $9 + f(0)$ (d) $16 + f(0)$

55. The equation $f(x) - x - f(0) = 0$ have exactly

(a) no solution (b) one solution

(c) two solutions (d) Infinite solutions

56. $f'(0)$ is equal to

(a) 0 (b) 1 (c) $f(0)$ (d) $-f(0)$

Passage II

(Q. Nos. 57 to 58)

Consider the equation $x + y - [x][y] = 0$, where $[\cdot]$ is the greatest integer function.

57. The number of integral solutions to the equation is

(a) 0 (b) 1

(c) 2 (d) None of these

58. Equation of one of the lines on which the non-integral solution of given equation, lies is

(a) $x + y = -1$ (b) $x + y = 0$

(c) $x + y = 1$ (d) $x + y = 5$

Passage III

(Q. Nos. 59 to 61)

Let $f(x)=\frac{1}{2}\left[f(xy)+f\left(\frac{x}{y}\right)\right]$ *for* $x,\ y\in R^+$ *such that* $f(1)=0; f'(1)=2.$

59. $f(x)-f(y)$ is equal to

(a) $f\left(\frac{y}{x}\right)$ (b) $f\left(\frac{x}{y}\right)$ (c) $f(2x)$ (d) $f(2y)$

60. $f'(3)$ is equal to

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

61. $f(e)$ is equal to

(a) 2 (b) 1 (c) 3 (d) 4

Passage IV

(Q. Nos. 62 to 64)

If $f:R\to R$ *and* $f(x)=g(x)+h(x)$ *where* $g(x)$ *is a polynomial and* $h(x)$ *is a continuous and differentiable bounded function on both sides, then* $f(x)$ *is one-one, we need to differentiate* $f(x)$*. If* $f'(x)$ *changes sign in domain of* f*, then* f*, if many-one else one-one.*

62. $f:R\to R$ and $f(x)=a_1x+a_3x^3+a_5x^5+\ldots+a_{2n+1}x^{2n+1}-\cot^{-1}x$ where $0<a_1<a_3<\ldots<a_{2n+1}$, then the function $f(x)$ is

(a) one-one into (b) many-one onto

(c) one-one onto (d) many-one into

63. $f:R\to R$ and $f(x)=\dfrac{x(x^4+1)(x+1)+x^4+2}{x^2+x+1}$, then $f(x)$ is

(a) one-one into (b) many-one onto

(c) one-one onto (d) many-one into

64. $f:R\to R$ and $f(x)=2ax+\sin 2x$, then the set of values of a for which $f(x)$ is one-one and onto is

(a) $a\in\left(-\frac{1}{2},\frac{1}{2}\right)$ (b) $a\in(-1,1)$

(c) $a\in R-\left(-\frac{1}{2},\frac{1}{2}\right)$ (d) $a\in R-(-1,1)$

Passage V

(Q. Nos. 65 to 67)

Let $g(x)=a_0+a_1x+a_2x^2+a_3x^3$ *and* $f(x)=\sqrt{g(x)}$*,* $f(x)$ *has its non-zero local minimum and maximum values at* -3 *and* 3*, respectively. If* $a_3\in$ *the domain of the function* $h(x)=\sin^{-1}\left(\frac{1+x^2}{2x}\right)$.

65. The value of $a_1 + a_2$ is equal to

(a) 30 (b) -30 (c) 27 (d) -27

66. The value of a_0 is

(a) equal to 50 (b) greater than 54 (c) less than 54 (d) less than 50

67. $f(-10)$ is defined for

(a) $a_0 > 830$ (b) $a_0 < 830$
(c) $a_0 = 830$ (d) None of these

Passage VI

(Q. Nos. 68 to 70)

Let $f:[2, \infty) \to [1, \infty)$ *defined by* $f(x) = 2^{x^4 - 4x^2}$ *and* $g:\left[\frac{\pi}{2}, \pi\right] \to A$ *defined by* $g(x) = \frac{\sin x + 4}{\sin x - 2}$ *be two invertible functions, then*

68. $f^{-1}(x)$ is equal to

(a) $\sqrt{2 + \sqrt{4 - \log_2 x}}$ (b) $\sqrt{2 + \sqrt{4 + \log_2 x}}$

(c) $\sqrt{2 - \sqrt{4 + \log_2 x}}$ (d) None of these

69. The set A is equal to

(a) $[-5, -2]$ (b) $[2, 5]$
(c) $[-5, 2]$ (d) $[-3, -2]$

70. The domain of $f^{-1}g^{-1}(x)$ is

(a) $[-5, \sin 1]$ (b) $\left[-5, \frac{\sin 1}{2 - \sin 1}\right]$

(c) $\left[-5, -\frac{(4 + \sin 1)}{2 - \sin 1}\right]$ (d) $\left[-\frac{(4 + \sin 1)}{2 - \sin 1}, -2\right]$

Passage VII

(Q. Nos. 71 to 73)

$P(x)$ *be polynomial of degree at most 5 which leaves remainders* -1 *and* 1 *upon division by* $(x-1)^3$ *and* $(x+1)^3$ *respectively.*

71. Numbers of real roots of $P(x) = 0$, is

(a) 1 (b) 3 (c) 5 (d) 2

72. The maximum value of $y = P''(x)$ can be obtained at x is equal to

(a) $-\frac{1}{\sqrt{3}}$ (b) 0 (c) $\frac{1}{\sqrt{3}}$ (d) 1

73. The sum of pairwise product of all roots (real and complex) of $P(x) = 0$ is

(a) $-\frac{5}{3}$ (b) $-\frac{10}{3}$ (c) 2 (d) -5

Passage VIII
(Q. Nos. 74 to 76)

Let $f : N \to N$ (N being the set of positive integers) be a function defined by $f(x) =$ the biggest positive integer obtained by reshuffling the digits of x. For example, $f(296) = 962$.

74. f is

(a) one-one and onto
(b) one-one and into
(c) many-one and onto
(d) many-one and into

75. The biggest positive integer which divides $f(n) - n$, for all $n \in N$, is

(a) 3 (b) 9 (c) 18 (d) 27

76. The range of f is

(a) N
(b) set of positive integers whose digits are non-increasing from left to right
(c) set of positive integers whose digits are non-decreasing from left to right
(d) set of positive integers whose digits decrease from left to right

Passage IX
(Q. Nos. 77 to 79)

Let $f(x) = \sin x - x \cos x$, $x \in R$.

77. The least positive value of x, for which $f(x) = 0$, lies in quadrant

(a) I (b) II (c) III (d) IV

78. The set of all values of $x \in (0, 2\pi)$, for which $f(x) > 0$, is

(a) $(0, \pi)$ (b) $(\pi, 2\pi)$ (c) $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ (d) None of these

79. If α is the least positive value for which $\tan \alpha = \alpha$, then the area bounded by $y = f(x)$, x-axis, $x = 0$ and $x = 2\pi$ is

(a) 4
(b) $4(1 - \cos \alpha)$
(c) $4(1 + \cos \alpha)$
(d) None of these

Type 5 : **Match the Columns**

80. Match the statements of Column I with values of Column II.

Column I	Column II
(A) $\sqrt{\sin(\cos x)}$, has domain	(p) $x \in R$
(B) $(\sqrt{\cos(\sin x)})^{-1}$, has domain	(q) $R - \left\{n\pi \pm \frac{\pi}{6}\right\}$
(C) $\tan(\pi \sin x)$, has domain	(r) $x \in \left(n\pi, n\pi + \frac{\pi}{2}\right)$
(D) $\ln(\tan x)$, has domain	(s) $x \in \left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right]$

81. Match the statements of Column I with values of Column II.

Column I	Column II
(A) $\lvert 4\sin x - 1\rvert < \sqrt{5}, x \in [0, \pi]$, then domain is	(p) $\left[0, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right]$
(B) $4\sin^2 x - 8\sin x + 3 \le 0$, $[0, 2\pi]$, then domain is	(q) $\left[\frac{3\pi}{2}, 2\pi\right] \cup \{0\}$
(C) $\lvert \tan x\rvert \le 1$ and $x \in [0, \pi]$, then domain is	(r) $\left[0, \frac{3\pi}{10}\right)$
(D) $\cos x - \sin x \ge 1$ and $[0, 2\pi]$, then domain is	(s) $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$

82. Match the statements of Column I with values of Column II.

Column I	Column II
(A) If $f(x) = x + 1$ when $x < 0 = x^2 - 1$ for $x \ge 0$, then $fof(x)$ for $-1 \le x < 0$ is	(p) $\frac{x-3}{2}$
(B) If $f\left(\frac{2\tan x}{1+\tan^2 x}\right) = \frac{(\cos 2x + 1)(\sec^2 x + 2\tan x)}{2}$, then $f(x)$ is	(q) $x^2 + 2x$
(C) If $f(x + y + 1) = (\sqrt{f(x)} + \sqrt{f(y)})^2$ for all $x, y \in R$ and $f(0) = 1$, then $f(x)$ is	(r) $1 + x$
(D) If $4 < x < 5$ and $f(x) = \left[\frac{x}{4}\right] + 2x + 2$ where $[y]$ is the greatest integer $\le y$, then $f^{-1}(x)$ is	(s) $(x + 1)^2$

83. Match the statements of Column I with values of Column II.

Column I	Column II
(A) $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$ then $f(x)$, is	(p) Defined for all real 'x'
(B) $g(x) = \tan(e^{\{x\}}) + [x + \alpha] - 5 - x$ where $[\cdot]$ denotes the greatest integer less than or equal to x, then $g(x)$, is	(q) Even function
(C) $h(x) = \frac{x}{5^x - 1} + \frac{x}{2} + 5$, then $h(x)$, is	(r) Odd function
(D) $k(x) = 2\sin^2 x - \cos 2\alpha + 4\sin\alpha \cdot \sin x \cos(x + \alpha) + \cos 2(x + \alpha), \alpha \in R$, then $k(x)$, is	(s) Periodic function

Type 6 : **Integer Answer Type Questions**

84. If $a + b = 3 - \cos 4\theta$ and $a - b = 4 \sin 2\theta$, then ab is always less than or equal to ....

85. Let 'n' be the number of elements in the domain set of the function $f(x) = \left| \ln \sqrt{{}^{x^2+4x}C_{2x^2+3}} \right|$, and '$Y$' be the global maximum value of $f(x)$, then $[n + [Y]]$ is .... (where [] = Greatest Integer Function).

86. If $f(x)$ be a function such that $f(x-1) + f(x+1) = \sqrt{3} f(x)$ and $f(5) = 10$, then the sum of digits of the value of $\sum_{r=0}^{19} f(5 + 12r)$ is ..... .

87. If $2f(x) = f(xy) + f\left(\frac{x}{y}\right)$ for all positive values of x and y, $f(1) = 0$ and $f'(1) = 1$, then $f(e)$ is ..... .

88. Let f be a function from the set of positive integers to the set of real number such that

(i) $f(1) = 1$

(ii) $\sum_{r=1}^{n} rf(r) = n(n+1) f(n), \forall\, n \geq 2$

then find the value of $2126\, f(1063)$.

89. If $f(x) = \dfrac{x^4 + x^2 + 1}{x^2 - x + 1}$, then find the value of $f(\omega^n)$

(where 'ω' is the non-real root of the equation $z^3 = 1$ and 'n' is a multiple of 3).

90. If $f^2(x) \cdot f\left(\dfrac{1-x}{1+x}\right) = x^3$, $[x \neq -1, 1$ and $f(x) \neq 0]$, then find $|[f(-2)]|$ (where [] is the greatest integer function).

91. An odd function is symmetric about the vertical line $x = a (a > 0)$ and if $\sum_{r=0}^{\infty} [f(1 + 4r)]^r = 8$, then find the numerical value of $8f(1)$.

92. Let $\dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \ln \sqrt{\dfrac{1+x}{1-x}}$, then find x.

93. The maximum value of $f(x) = \dfrac{3x^2 + 9x + 17}{3x^2 + 9x + 7}$ is $5k + 1$, then the value of k is ..... .

94. The period of the function $f(x)$ which satisfies the relation $f(x) + f(x+4) = f(x+2) + f(x+6)$ is ..... .

Proficiency in 'Functions'
Exercise 2

1. Let x be a real number. $[x]$ denotes the greatest integer function, $\{x\}$ denotes the fractional part and (x) denotes the least integer function, then solve the following :

(i) $(x)^2 = [x]^2 + 2x$ (ii) $[2x] - 2x = [x + 1]$

(iii) $[x^2] + 2[x] = 3x, 0 \le x \le 2$ (iv) $y = 4 - [x]^2$ and $[y] + y = 6$

(v) $[x] + |x - 2| \le 0$ and $-1 \le x \le 3$

2. Let n be a positive integer and define $f(n) = 1! + 2! + 3! + \ldots + n!$. Find polynomials $P(x)$ and $Q(x)$ such that $f(n+2) = Q(n) f(n) + P(n) f(n+1)$ for all $n \ge 1$.

3. If $f(x) = \dfrac{a^x}{a^x + \sqrt{a}}$ $(a > 0)$, evaluate $\displaystyle\sum_{r=1}^{2n-1} 2f\left(\frac{r}{2n}\right)$.

4. Let f and g be real-valued functions such that

$$f(x+y) + f(x-y) = 2f(x) \cdot g(y) \forall x, y \in R.$$

Prove that, if f is not identically zero and $|f(x)| \le 1, \forall x \in R$, then $|g(y)| \le 1, \forall y \in R$.

5. Find the domain of the function $f(x) = \log \left\{ \log_{|\sin x|}(x^2 - 8x + 23) - \dfrac{3}{\log_2 |\sin x|} \right\}$.

6. Let $S(n)$ denote the number of ordered pairs (x, y) satisfying $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{n}$,

where $n > 1$ and $x, y, n \in N$.

(i) Find the value of $S(6)$.

(ii) Show that, if n is prime, then $S(n) = 3$, always.

7. Solve $\dfrac{1}{[x]} + \dfrac{1}{[2x]} = \{x\} + \dfrac{1}{3}$, where [] denotes the greatest integral function and {} denotes fractional part of x.

8. Let $f(x) = x^2 + 3x - 3, x \ge 0$. n points $x_1, x_2, \ldots, x_n$ are so chosen on the x-axis that :

(i) $\displaystyle\frac{1}{n} \sum_{i=1}^{n} f^{-1}(x_i) = f\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right)$

(ii) $\displaystyle\sum_{i=1}^{n} f^{-1}(x_i) = \sum_{i=1}^{n} x_i$, where f^{-1} denotes the inverse of f. Find the AM of $x'_i s$.

9. Let $f(x) = x^2 - 2x$, $x \in R$ and $g(x) = f(f(x) - 1) + f(5 - f(x))$, show that $g(x) \geq 0, \forall x \in R$.

10. If f is a polynomial function satisfying $2 + f(x) \cdot f(y) = f(x) + f(y) + f(xy)$, $\forall\ x, y \in R$ and if $f(2) = 5$, then find $f(f(2))$.

11. $a + b + c = abc$, a, b and $c \in R^+$, then prove that $a + b + c \geq 3\sqrt{3}$.

12. Consider the function $f(x) = \begin{cases} x - [x] - \dfrac{1}{2}, & \text{if } x \notin I \\ 0, & \text{if } x \in I \end{cases}$,

where $[\cdot]$ denotes the greatest integral function and I is the set of integers. Then, find $g(x) = \max\{x^2, f(x), |x|\}; -2 \leq x \leq 2$.

13. If $f(x)$ be continuous function in $[0, 2\pi]$ and $f(0) = f(2\pi)$, then prove that there exists a point $c \in (0, \pi)$ such that $f(c) = f(c + \pi)$.

14. Let $g(t) = |t - 1| - |t| + |t + 1|, \forall\ t \in R$.
Then, find $f(x) = \max\left\{g(t) : -\dfrac{3}{2} \leq t \leq x\right\}, \forall\ x \in \left(\dfrac{-3}{2}, \infty\right)$.

15. Find the integral solution for $n_1 n_2 = 2n_1 - n_2$, where $n_1, n_2 \in$ integer.

Answers

Target Exercise 3.1

1. (i) not a function (ii) not a function (iii) function (iv) function (v) not a function (vi) function (vii) function (viii) not a function (ix) function (x) not a function.

2. (i) not a function (ii) not a function (iii) not a function (iv) is a function (v) not a function

Target Exercise 3.2

1. $(-\infty, 2) \cup [3, \infty)$ **2.** $(-\infty, -1/2] \cup (0, 1)(2, \infty)$ **3.** $(0, 1] \cup [4, 5)$

4. $x \in [0, 3)$ **5.** $x = \{2, 3\}$ **6.** $x \in R - I$ **7.** $x \in [-2, -1/2] \cup [1/2, 2]$

8. $x \in R$ **9.** $x \in (-\infty, -2) \cup (4, \infty)$ **10.** $x \in [-\pi/4, \pi/4]$

11. $x \in [4, 6]$ **12.** $x \in [-1, 3]$ **13.** $x \in [-2, 2]$ **14.** $x \in R - (-2, 2)$

15. $x \in (0, 1) \cup (1, \infty)$ **16.** $x \in [-1, 1]$ **17.** $[1 - \sqrt{2}, 0] \cup [1 + \sqrt{2}, 3)$

18. $(-\infty, \infty)$ **19.** $x \in (2, 3)$ **20.** $x \in \left(0, \dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, 1\right)$ **21.** $x \in (2, 4)$

22. $x \in \phi$ **23.** $x \in (2, \infty)$ **24.** $x \in \{1, 2, 3\}$ **25.** $(-\infty, 0) \cup (0, 1] \cup [2, \infty)$

26. $(0, 1)$ **27.** $\left[-1, \dfrac{-1}{2}\right] \cup \left[0, \dfrac{1}{2}\right] \cup \{1\}$ **28.** $\left[-\sqrt{\dfrac{5}{2}}, -1\right] \cup \left[1, \sqrt{\dfrac{5}{2}}\right]$

29. $(-\sqrt{8} - 1) \cup [1, \sqrt{8})$ **30.** $x \in R - \{1, 2\} \cup \{2, 3, 4, 5, 6, 7, 8, 9, 10\} \cup (10, 11)$

31. $x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right)$ **32.** no solution

34. Integral solutions are (2, 2) and (0, 0), all non-integral solutons lie on exactly two lines $x + y = 0$ and $x + y = 6$.

Target Exercise 3.3

1. Range $\in [0, 3]$ **2.** Range $\in [-1/2, 1/2]$ **3.** Range $\in [3 - \sqrt{2}, 3 + \sqrt{2}]$

4. Range $\in [1, 5]$ **5.** Range $\in (-\infty, \log_3 9)$ **6.** $\left(-\infty, \frac{2}{3}\right] \cup (1, \infty)$

7. $R - (2 - 2\sqrt{3}, 2 + 2\sqrt{3})$ **8.** $[1, 5]$ **9.** $(-\infty, 0]$ **10.** $\left\{\frac{\pi}{2}\right\}$ **11.** $\{0, 1\}$

12. $\{\pi\}$ **13.** $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ **14.** $\left(0, \frac{\pi}{2}\right]$ **15.** $\{0\}$ **17.** $[-1, 1]$ **18.** $\{0\}$

19. $[1, \infty)$ **20.** $\{1\}$ **21.** $[0, 2]$

22. Image of $(-\infty, 1)$ under $f \in (1, \infty)$, Image of $[1, 2]$ under $f \in [1, 2]$

23. Range of $f(x) \in \{0\}$ and domain $\in \cup \left[\left(2n\pi - \frac{\pi}{3}\right), \left(2n\pi + \frac{\pi}{3}\right)\right]$

24. Range of $f(x) \in \left\{\log \frac{\pi}{2}\right\}$ and domain $\in [1, 2)$

25. Range of $f(x) \in$ set of all non-positive integers and domain $\in \left[\frac{-3 - \sqrt{5}}{2}, -2\right) \cup \left(-1, \frac{-3 + \sqrt{5}}{2}\right]$

Target Exercise 3.4

1. (i) odd (ii) even (iii) neither even nor odd (iv) even (v) odd (vi) odd (vii) neither even nor odd (viii) even

2. f is an even function when $x \in$ integer

f is an odd function when $x \notin$ integer

3. (i) $f(x) = \begin{cases} -2x, & x \le -1 \\ -x\,|x|, & -1 < x \le 0 \end{cases}$ (ii) $f(x) = \begin{cases} -2x, & x \le -1 \\ x\,|x|, & -1 < x \le 0 \end{cases}$ **5.** $\alpha \in (400, \infty)$

Target Exercise 3.5

1. (i) $2\pi/3$ (ii) 2π (iii) 1 (iv) 24 (v) does not exist (vi) $2(n + 1)!$ (vii) 1 (viii) 2π

2. Period is 8.

3. $f(x)$ is periodic with period 2λ

4. $f(x)$ is periodic with period $2p$.

5. $f(x)$ is periodic with period 12 and $\sum_{r=0}^{99} f(5 + 12r) = 10000$

Target Exercise 3.6

1. Domain $\in [-1, 1]$ and range $\in \left[\frac{-2\pi}{3}, \frac{\pi}{3}\right]$

2. Domain for $f([\,|x|\,]) \cup (-3, 3)$

Domain for $f([2x + 3]) \in [-3, 0)$

3. $h(x)=\begin{cases}\sin^2 x-\sin x+1, & -1<x<0\\ 2\sin^2 x, & 0<x\le 1\end{cases}$ **4.** $g(x)=\begin{cases}-x, & -2\le x\le 0\\ 0, & 0\le x\le 1\\ 2(x-1), & 1\le x\le 2\end{cases}$

5. $gof=\{(x+1)^2, -2\le x\le 1\}$

Target Exercise 3.7

1. $x=0$ and $y=\dfrac{3}{2}$ **2.** $f^{-1}(1)=y$

3. As monotonic and range = Codomain $\Rightarrow$ Bijective **5.** $A\in[0,1)$ **6.** $A\in\left[0,\dfrac{1}{2}\right)$

7. $b^2<3a(c-|d|)$

8. $X\in\left[-\dfrac{\pi}{2}-\alpha,\dfrac{\pi}{2}-\alpha\right]$ and $Y\in[c-r,c+r]$ where $\alpha=\tan^{-1}\left(\dfrac{a+b\sqrt{2}}{a}\right)$

and $r=\sqrt{a^2+\sqrt{2}ab+b^2}$

Target Exercise 3.8

1. (i) $f^{-1}(x)=3\sin x$ (ii) $f^{-1}(x)=\dfrac{-3+\sqrt{5+4e^x}}{2}$ (iii) $f^{-1}(x)=x^{\log_5 e}, x>0$

(iv) $f^{-1}(x)=\dfrac{1}{2}(e^x-e^{-x})$ (v) $f^{-1}(x)=\begin{cases}x, & x<1\\ \sqrt{x}, & 1\le x\le 16\\ \dfrac{x^2}{64}, & x>16\end{cases}$

2. $f^{-1}(x)=\dfrac{1+\sqrt{1+4\log_2 x}}{2}, x>0$

Exercise 1

1. (a) **2.** (a) **3.** (b) **4.** (d) **5.** (c) **6.** (c) **7.** (b) **8.** (c) **9.** (a) **10.** (d)
11. (d) **12.** (c) **13.** (d) **14.** (c) **15.** (d) **16.** (b) **17.** (a) **18.** (d) **19.** (d) **20.** (b)
21. (b) **22.** (b) **23.** (d) **24.** (c) **25.** (c) **26.** (d) **27.** (b) **28.** (d) **29.** (c) **30.** (c)
31. (b) **32.** (b) **33.** (d) **34.** (d) **35.** (a, b, c) **36.** (a, c) **37.** (a, b, c)
38. (a, b, c, d) **39.** (a, b, c) **40.** (a, c) **41.** (a, d) **42.** (b, c,d)
43. (c) **44.** (a) **45.** (c) **46.** (a) **47.** (d) **48.** (a) **49.** (a) **50.** (d) **51.** (d) **52.** (a)
53. (c) **54.** (d) **55.** (c) **56.** (a) **57.** (c) **58.** (b) **59.** (b) **60.** (b) **61.** (a) **62.** (c)
63. (d) **64.** (d) **65.** (c) **66.** (b) **67.** (d) **68.** (b) **69.** (a) **70.** (c) **71** (a) **72.** (c)
73. (b) **74.** (d) **75.** (b) **76.** (b) **77.** (c) **78.** (d) **79.** (d)
80. (A) $\rightarrow$ (s), (B) $\rightarrow$ (p), (C) $\rightarrow$ (q), (D) $\rightarrow$ (r)
81. (A) $\rightarrow$ (r), (B) $\rightarrow$ (s), (C) $\rightarrow$ (p), (D) $\rightarrow$ (q)
82. (A) $\rightarrow$ (q), (B) $\rightarrow$ (r), (C) $\rightarrow$ (s), (D) $\rightarrow$ (p)
83. (A) $\rightarrow$ (p,q,s), (B) $\rightarrow$ (s), (C) $\rightarrow$ (q), (D) $\rightarrow$ (p,q,r,s) **84.** (1) **85.** (5)
86. (2) **87.** (1) **88.** (2) **89.** (3) **90.** (2) **91.** (7) **92.** (0) **93.** (8) **94.** (8)

Exercise 2

1. (i) 0, $n+\frac{1}{2}$ where $n \in Z$ (ii) $\left\{-1, -\frac{1}{2}\right\}$ (iii) {0, 1}

(iv) {1, – 1, ± 1, + k, where k is any positive proper fraction.} (v) no solution

2. $P(x) = x + 3$ and $Q(x) = -x - 2$ **3.** $(2n - 1)$ **4.** $x \in (3, 5) \cup \left(\pi, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 5\right)$

6. (i) $S(6) = 9$ **7.** Possible solutions are $\frac{29}{12}, \frac{19}{6}, \frac{97}{24}$ **8.** $\frac{1}{n}\sum_{i=1}^{n} x_i = 1$

10. $f(f(2)) = 26$ **12.** $g(x) = \begin{cases} x^2, & -2 \le x \le -1 \\ -x, & -1 \le x \le -1/4 \\ x+\frac{1}{2}, & -\frac{1}{4} \le x \le 0 \\ x, & 0 \le x \le 1 \\ x^2, & 1 \le x \le 2 \end{cases}$

14. $f(x) = \begin{cases} 3/2, & -3/2 \le x \le -1/2 \\ 2+x, & -1/2 \le x \le 0 \\ 2, & 0 \le x \le 2 \\ x, & 2 < x \end{cases}$

15. Integral solution for (n_1, n_2) are $(-3, 3), (-2, 4), (0, 0)$ and $(1, 1)$

Solutions

(*Proficiency in 'Functions' Exercise* 1)

Type 1 : **Only One Correct Option**

1. $-\sqrt{20} \le 2\cos x - 4\sin x \le \sqrt{20}$

$\Rightarrow$ $$0 \le (2\cos x - 4\sin x)^2 \le 20$$

$$\min = \frac{1}{1+20} = \frac{1}{21}; \max = 1$$

$\Rightarrow$ $$M + m = \frac{22}{21}$$

2. $f(x) = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right) + 2\sqrt{2}$

or $$f(x) = \sqrt{2}\cos\left(x+\frac{\pi}{4}\right) + 2\sqrt{2}$$

$\Rightarrow$ $$Y = [\sqrt{2}, 3\sqrt{2}]$$

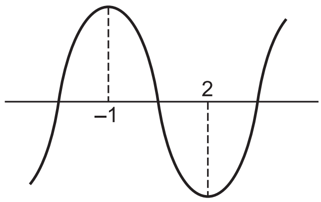
and $$X = \left[-\frac{3\pi}{4}, \frac{\pi}{4}\right]$$

or $$\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$$

3. Let $f(x) = 2x^3 - 3x^2 - 12x + a$, then

$$f'(x) = 6(x^2 - x - 2) = 6(x+1)(x-2)$$

So, the roots of $f'(x) = 0$ are $x = -1, 2$.



Now, $f(x) = 0$ will have all real roots, if $f(-1) > 0$ and $f(2) < 0$.

$\Rightarrow$ $-2 - 3 + 12 + a > 0$

and $16 - 12 - 24 + a < 0$

$\Rightarrow$ $-7 < a < 20$

4. $g(x) = \dfrac{e^{f(x)} - e^{|f(x)|}}{e^{f(x)}, + e^{|f(x)|}}, -1 \le x \le 1$

For, $0 \le x \le 1$,

$g(x) = 0$

$-1 \le f(x) < 0$

$$g(x) = \frac{e^{f(x)} - e^{-f(x)}}{e^{f(x)} + e^{-f(x)}} = \frac{e^{2f(x)} - 1}{e^{2f(x)} + 1}$$

$$= 1 - \frac{2}{e^{2f(x)} + 1}$$

For, $-1 \le f(x) < 0$,

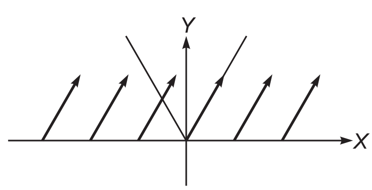
$$g(x) \in \left[\frac{1-e^2}{1+e^2}, 0\right)$$

For, $-1 \le f(x) < 1$,

$$g(x) \in \left[\frac{1-e^2}{1+e^2}, 0\right]$$

5. $f(x) = \sqrt{|x| - \{x\}}$

$$|x| \ge \{x\}$$



$\Rightarrow$ $X \in \left(-\infty, -\dfrac{1}{2}\right] \cup [0, \infty)$

$\Rightarrow$ $Y \in [0, \infty)$ and $f(x)$ is many-one.

6. Given curves are $y = \ln x$ and $y = ax$.

$\Rightarrow \ln x = ax$ has exactly two solutions.

$\Rightarrow \dfrac{\ln x}{x} = a$ has exactly two solutions to find the range of $\dfrac{\ln x}{x}$.

Let $$y = \frac{\ln x}{x}, \; x > 0$$

$$\frac{dy}{dx} = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

y is increasing, if $1 - \ln x > 0$ or $\ln x < 1$

$\Rightarrow$ $$0 < x < e$$

Range of $y \in \left(-\infty, \dfrac{1}{e}\right]$ graph of $y = \dfrac{\ln x}{x}$

For exactly two solutions of $\dfrac{\ln x}{x} = a$

$\Rightarrow$ $$a \in \left(0, \frac{1}{e}\right)$$

7. Let $f(x) = ax^2 + bx + c$ as it touches x-axis at $x = 3$

$\Rightarrow$ $$\frac{-b}{2a} = 3$$

$\Rightarrow$ $$b = -6a \quad \text{...(i)}$$

Also, $$9a + 3b + c = 3 \quad \text{...(ii)}$$

$$4a - 2b + c = 0 \quad \text{...(iii)}$$

$\Rightarrow$ $$a = \frac{3}{25}, \; b = -\frac{18}{25}, \; c = \frac{27}{25}$$

$\Rightarrow$ $$f(x) = \frac{3}{25}(x^2 - 6x + 9)$$

8. $y = \sqrt{2\{x\} - \{x\}^2 - \dfrac{3}{4}}$

$\Rightarrow$ $$2\{x\} - \{x\}^2 - \frac{3}{4} \geq 0$$

$\Rightarrow$ $$\frac{1}{2} \leq \{x\} \leq \frac{3}{2} \;\Rightarrow\; \frac{1}{2} \leq \{x\} < 1$$

$\because$ $$0 \leq \{x\} < 1$$

$\therefore$ $2\{x\} - \{x\}^2 - \dfrac{3}{4}$ is increasing for $\dfrac{1}{2} \leq \{x\} < 1$.

$\Rightarrow$ $$\text{Range} = \left[0, \frac{1}{4}\right]$$

9. Replace x by $-x$

$\Rightarrow$ $$x[f(x) - f(-x)] = 0$$

$\Rightarrow$ $f(x)$ is an odd function.

$\Rightarrow$ $f^{iv}(x)$ is also odd $\Rightarrow f^{iv}(0) = 0$

10. $[y + [y]] = 2\cos x$

$$\Rightarrow \quad [y] = \cos x$$

$$y = \frac{1}{3}[\sin x + [\sin x + [\sin x]]] = [\sin x]$$

$$\Rightarrow \quad [\sin x] = \cos x$$

No. of solutions in $[0, 2\pi]$ is 0.

Hence, total solution is 0.

$\therefore$ Both are periodic with period 2π.

11. By replacing $x = x + 1$ and $x = x - 2$, we get

$$f(x+2) + f(x) = \sqrt{2}f(x+1) \quad \ldots\text{(i)}$$

$$f(x) + f(x-2) = \sqrt{2}f(x-1) \quad \ldots\text{(ii)}$$

Eqs. (i) and (ii), gives

$$f(x+2) + f(x-2) + 2f(x)$$

$$= \sqrt{2}\,[f(x+1) + f(x-1)]$$

$$= \sqrt{2}\sqrt{2}f(x)$$

$$\therefore \quad f(x+2) + f(x-2) = 0$$

On replacing x by $x + 2$, we get

$$f(x+4) + f(x) = 0$$

$$f(x+8) = -f(x+4) = f(x), \forall\, x$$

$\therefore f(x)$ is periodic with period 8.

12. The equation has meaning, if $x > 0, x \neq 1$.

$$\therefore \quad \text{Domain} = (0, 1) \cup (1, \infty)$$

If $\quad x \in (0, 1)$, then $\log_2 x < 0$ and

$$\log_2 x + \log_x 2 = \frac{\log x}{\log 2} + \frac{\log 2}{\log x}$$

$$= \text{sum of a negative number} \leq -2.$$

In this case any α will satisfy, since $2\cos\alpha$ can never be more than 2.

Thus, the inequation is satisfied for any x in (0, 1) and for any α.

If $\quad x \in (1, \infty)$,

then $\quad \log_2 x > 0$

$$\Rightarrow \quad \frac{\log x}{\log 2} = \frac{\log 2}{\log x} > 0$$

The inequation cannot be satisfied unless

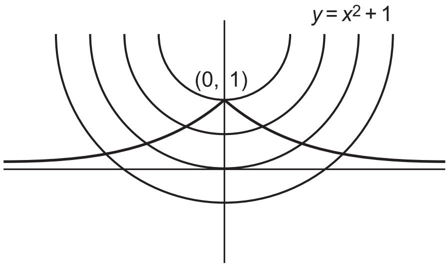
$$\cos\alpha = -1$$

and $\quad x = 2$

ie, $\quad \log_2 x = 1$

Option (d) is wrong, since in the last case there are infinite solution.

13. If we draw the graph of $\left(\frac{1}{2}\right)^{|x|}$ and $x^2 - a$, then the range of value of a will be $(-1, \infty)$.



Maximum possible solution for 'x' is '2'.

14. $f(x) \neq |\cos x|$ is true only when

$$|\cos x| = 1$$

$\Rightarrow$ $\quad x = 0, \pi, 2\pi$

15. $|\cos x| + \cos x > 0 \rightarrow x \in \left[0, \frac{\pi}{2}\right)$

For $\quad x \in \left[0, \frac{\pi}{2}\right)$,

$$1 \le \sin x + \cos x \le \sqrt{2}$$

But $\quad \sin x + \cos x \neq 1 \Rightarrow x \in \left(0, \frac{\pi}{2}\right)$.

Now, for $\quad x \in \left(0, \frac{\pi}{2}\right)$

$$\log_{\sin x + \cos x} (|\cos x| + \cos x) \ge 0$$

$\Rightarrow$ $\quad \cos x \ge \frac{1}{2}$

$\Rightarrow$ $\quad x \in \left(0, \frac{\pi}{3}\right]$

16. $y = ((x + \alpha)^2 - 1)^{1/4}$

$$= [(x + \alpha - 1)][(x + \alpha + 1)]^{1/4}$$

$$(x + \alpha - 1)(x + \alpha + 1) \ge 0$$

$$x \ge 1 - \alpha \text{ and } x \le -1 - \alpha \text{ for } \alpha > 0$$

For $\quad \alpha < 0, x \le 1 - \alpha, x \ge -1 - \alpha$

$$(x + \alpha) \le 1 \text{ and } (x + \alpha) \ge -1$$

For $\quad \alpha \le -1, x \le 0$ and range is $[0, \infty)$.

17. $f(x) = \log(\log(\log x))$

$$\log x > 1 \text{ when } x \in (e, \infty)$$

$\therefore \log(\log x) > 0$ and hence, $\log(\log(\log x))$ is well defined and uniquely.

It is evidently one-one. Since, the range of $\log x = R$, $f(x)$ is one-one and onto.

18. $x^2 - 4px + q^2 > 0, \forall x \in R$

$$\Rightarrow \quad 4p^2 - q^2 < 0 \quad \text{...(i)}$$

$$r^2 + p^2 < qr \quad \text{...(ii)}$$

Let $$y = \frac{x + r}{x^2 + qx + p^2}$$

$$\Rightarrow \quad x^2 y + x(qy - 1) + p^2 y - r = 0 \quad \text{...(iii)}$$

x is real,

$$\Rightarrow \quad (q^2 - 4p^2) y^2 + y(-2q + 4r) + 1 > 0$$

Eq. (i) $\Rightarrow$ Coefficient of y^2 is a positive discriminant.

$$= (4r - 2q)^2 - 4(q^2 - 4p^2)$$

$$= 16(r^2 + p^2 - qr) < 0 \quad \text{[by Eq. (ii)]}$$

Hence, Eq. (iii) is true for all real y or $y \in (-\infty, \infty)$.

19. Since,

$$\sin \frac{\pi [x + 24]}{12} = \sin \frac{\pi}{12}(24 + [x])$$

$$= \sin\left(2\pi + \frac{\pi [x]}{2}\right) = \sin \frac{\pi [x]}{12}$$

The period of $\sin \dfrac{\pi [x]}{12}$ is 24.

Similarly, period of $\cos \dfrac{\pi [x]}{4}$ is 8 and period of $\tan \dfrac{\pi [x]}{3} = 3$.

Hence, the period of the given function = LCM of 24, 8, 3 $= 24$.

20. Let $f(x, y) = ax + by$

Then, $$f(2x + 3y, 2x - 7y)$$

$$= a(2x + 3y) + b(2x - 7y) = 20x \quad \text{(given)}$$

$$\therefore \quad 2a + 2b = 20 \quad \text{and} \quad 3a - 7b = 0$$

$$\therefore \quad a = 7 \text{ and } b = 3$$

$$\therefore \quad f(x, y) = 7x + 3y$$

21. Domain of $f(x)$ is [1, 3] and the function is continuous.

$$f'(x) = \frac{1}{2\sqrt{x - 1}} - \frac{1}{\sqrt{3 - x}} = 0$$

$$\Rightarrow \quad \sqrt{3 - x} = 2\sqrt{x - 1}$$

$$\Rightarrow \quad 3 - x = 4x - 4 \Rightarrow x = \frac{7}{5}$$

$\therefore$ Critical points in [1, 3] are 1, $\dfrac{7}{5}$ and 3.

$$f(1) = 2\sqrt{2}, f(3) = \sqrt{2}$$

and $f\left(\dfrac{7}{5}\right) = \sqrt{\dfrac{2}{5}} + 2\sqrt{\dfrac{8}{5}} = \sqrt{10}$, $2\sqrt{2}$ being $< \sqrt{10}$, the range $= [\sqrt{2}, \sqrt{10}]$.

22. $f(x) = \cos^{-1}(\sec(\cos^{-1} x)) + \sin^{-1}(\text{cosec}(\sin^{-1} x))$

$\Rightarrow \quad -1 \le \sec(\cos^{-1} x) \le 1$

and $\quad -1 \le \text{cosec}(\sin^{-1} x) \le 1$

$\Rightarrow \quad \sec(\cos^{-1} x) = \pm 1$

and $\quad \text{cosec}(\sin^{-1} x) = \pm 1$

$\Rightarrow \quad \cos^{-1} x = 0, \pi$ and $\sin^{-1} x = \dfrac{\pi}{2}, -\dfrac{\pi}{2}$

$\Rightarrow \quad x = \pm 1$ and $x = \pm 1$

$\Rightarrow \quad$ Domain is $x = \pm 1$.

23. Put $x = y = 1 \Rightarrow f(1) = 2$

Put $\quad y = \dfrac{1}{x} \quad \Rightarrow \quad f(x) + f\left(\dfrac{1}{x}\right) = f(x) f\left(\dfrac{1}{x}\right)$

$\Rightarrow \quad f(x) = x^3 + 1$

$\Rightarrow \quad g(x) = 0, \forall x \in R - \{0\}$

24. $f(x) = f'(x) \times f''(x)$ is satisfied by only the polynomial of degree 4.

Since, $f(x) = 0$ satisfies $x = 1, 2, 3$ only. It is clear one of the roots is twice repeated.

$\Rightarrow \quad f'(1) f'(2) f'(3) = 0$

25. Total number of functions for which $f(i) \ne i = 4^5$ and number of onto functions in which $f(i) \ne i = 44$.

$\Rightarrow$ Required number of functions $= 980$

26. We can send $1, 2, \ldots, n-1$ anywhere, and the value of $f(n)$ will then be uniquely determined.

27. Put $\sin x = t$,

$$y = t^3 - 6t^2 + 11t - 6, -1 \le t \le 1$$
$$f(-1) = -24, f(1) = 0$$

28. $g(x) = f(x^2 - 2x - 1) + f(5 - x^2 + 2x)$

$= 2x^4 - 8x^3 - 4x^2 + 24x + 18$

$$g'(x) = 8x^3 - 24x^2 - 8x + 24$$
$$g'(x) = 0 \quad \Rightarrow \quad x = -1, 1, 3$$

We observe that,

$$g(x) \ge \min\{g(-1), g(1), g(3)\} = 0$$

$\therefore \quad g(x) \ge 0, \forall x \in R$

29. Let $f(x) = [x],\ g(x) = \dfrac{e^{-|x|}}{2}$

$\Rightarrow \quad h(x) = \left[\dfrac{e^{-|x|}}{2}\right] = 0, \forall x \in R$

30. There are four possible functions defined in $0 \le x < 1$, of them 2 are continuous and two are discontinuous, now for each of the points $(1, 2, \dots, n-1)$, keep functions fixed from left of the point, so there are 4 possible functions defined in between next two consecutive integral points of them only one is continuous and at last for $x = n$, there is only one possibility of discontinuity of the function. So total number of functions

$$= 2 \times 3^{n-1} \times 1$$

31. $\because$ $k \in$ odd

$$f(x) = k + 3 \qquad \text{(even)}$$

$$f(f(k)) = \frac{k+3}{2}$$

If $\quad \dfrac{k+3}{2} \in \text{odd}$

$\Rightarrow \quad 27 = \dfrac{k+3}{2} + 3$

$\Rightarrow \quad k = 45$ not possible.

Now, let $\quad \dfrac{k+3}{2} \in \text{even}$

$\therefore \quad 27 = f(f(f(k))) = f\left(\dfrac{k+3}{2}\right) = \dfrac{k+3}{4}$

$\therefore \quad k = 105$

Verifying $\quad f(f(f(105))) = f(f(100)) = f(54) = 27$

$\therefore \quad k = 105$

Hence, sum of digits of $k = 1 + 0 + 5 = 6$

32. $f(x) = \dfrac{\sin(\pi\{x\})}{x^4 + 3x^2 + 7}$

Here, $\quad f(1/2) = f(-1/2)$

$\Rightarrow$ Clearly, $f(x)$ is not one-one and also it is dependent on x.

33. Since, $f(x)$ and $g(x)$ are one-one and onto and are also the mirror images of each other which respect to the line $y = a$. It clearly indicates that $h(x) = f(x) + g(x)$ will be a constant function and will always be equal to $2a$.

34. Let $g(t) = 2t^3 - 15t^2 + 36t - 25$

$$g'(t) = 6t^2 - 30t + 36 = 6(t^2 - 5t + 6)$$

$$= 6(t-2)(t-3) = 0 \quad \Rightarrow \quad t = 2, 3$$

For $\quad 2 \le t \le 4,$

$$g(t)_{\min} = g(3) = 2 \times 27 - 15 \times 9 + 36 \times 3 - 25 = 2$$

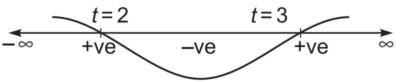
Also, $\quad 2 + |\sin t| \ge 2$

Hence, minimum $\quad \phi(t) = 2$

$\therefore \quad 3^x + 3^{f(x)} = 2$

$\Rightarrow \quad 3^{f'(x)} = 2 - 3^x$

$\because \quad 3^{f'(x)} > 0$



$\Rightarrow \qquad 2-3^x>0 \quad \Rightarrow \quad 3^x<2 \quad \Rightarrow \quad x<\log_3 2$

$\therefore \qquad x \in (-\infty, \log_3 2)$

Type 2 : **More than One Correct Options**

35. $f(x+1)=\dfrac{f(x)-5}{f(x)-3}$ ...(i)

$$f(x)\cdot f(x+1)-3f(x+1)=f(x)-5$$

$$\Rightarrow \qquad f(x)=\frac{3f(x+1)-5}{f(x+1)-1}$$

Replacing x by $(x-1)$, we have

$$f(x-1)=\frac{3f(x)-5}{f(x)-1} \qquad \text{...(ii)}$$

Using Eq. (i), we have $f(x+2)=\dfrac{f(x+1)-5}{f(x+1)-3}$

$$=\frac{\dfrac{f(x)-5}{f(x)-3}-5}{\dfrac{f(x)-5}{f(x)-3}-3}=\frac{2f(x)-5}{f(x)-2} \qquad \text{...(iii)}$$

Using Eq. (ii), we get

$$f(x-2)=\frac{3f(x-1)-5}{f(x-1)-1}$$

$$=\frac{3\left(\dfrac{3f(x)-5}{f(x)-1}\right)-5}{\dfrac{3f(x)-5}{f(x)-1}}=\frac{2f(x)-5}{f(x)-2} \qquad \text{...(iv)}$$

Using Eqs. (iii) and (iv), we have $f(x+2)=f(x-2)$

$\Rightarrow \qquad f(x+4)=f(x)$

$\Rightarrow \qquad f(x)$ is periodic with period 4.

36. $f(x)=1-x-x^3$

Replacing x by $f(x)$, $f(f(x))=1-f(x)-f^3(x)$

Hence, the given equation is

$$f(f(x))>f(1-5x), \qquad f(x)<1-5x$$

$$f(x)=1-x-x^3$$

$$1-x-x^3<1-5x$$

$$x^3-4x>0$$

$$x(x-2)(x+2)>0$$

So, $\qquad x \in (-2, 0) \cup (2, \infty)$

37. $(x-y)f(x+y)-(x+y)f(x-y)$

$$=2y((x-y)(x+y))$$

Let $\quad x-y=u,\ x+y=v$

$$uf(v)-vf(u)=uv(v-u)$$

$$\frac{f(v)}{v}-\frac{f(u)}{u}=v-u$$

$\Rightarrow \quad \left(\frac{f(v)}{v}-v\right)=\left(\frac{f(u)}{u}-u\right)=\text{constant}$

Let $\quad \frac{f(x)}{x}-x=\lambda$

$\Rightarrow \quad f(x)=(\lambda x+x^2)$

$$f(1)=2$$

$$\lambda+1=2 \quad \Rightarrow \quad \lambda=1$$

$$f(x)=x^2+x$$

38. Period of $\sin x=2\pi$ and period of $\cos(\sqrt{4-a^2}x)=\dfrac{2\pi}{\sqrt{4-a^2}}$

$\Rightarrow \quad \text{LCM}\left(2\pi, \frac{2\pi}{\sqrt{4-a^2}}\right)=4\pi \qquad$ (given)

ie, $\quad \sqrt{4-a^2}=\frac{p}{2}$ where $p=1, 3$

Hence, $\quad a^2=\frac{15}{4}, \frac{7}{4}$

$$a=\pm\frac{\sqrt{15}}{2}, \pm\frac{\sqrt{7}}{2}$$

39. $f(x+y)=f(x)f(a-y)+f(y)f(a-x) \qquad \ldots$(i)

Put $\quad x=y=0$, we get $f(a)=\frac{1}{2}$

Let $\quad y=0$

$\Rightarrow \quad f(x)=f(x)f(a)+f(0)\cdot f(a-x)$

$\Rightarrow \quad f(x)=\frac{1}{2}f(x)+\frac{1}{2}f(a-x)$

$\Rightarrow \quad f(x)=f(a-x)$

Put $y=a-x$ in Eq. (i),

$$f(a)=(f(x))^2+(f(a-x))^2$$

$\Rightarrow \quad (f(x))^2=\frac{1}{4}$

$$f(x)=\pm\frac{1}{2} \qquad \left[f(x)\neq-\frac{1}{2}\right]$$

Hence, $\quad f(x)=\frac{1}{2}$

40. Since, $f(g(x))$ is a one-one function.

$\Rightarrow \quad f(g(x_1)) = f(g(x_2))$ whenever $x_1 \neq x_2$

$\Rightarrow \quad g(x_1) = g(x_2)$ whenever $x_1 \neq x_2$

$\Rightarrow \quad g(x)$ must be one-one.

Let $\quad f(x) = y$ is satisfied by $x = x_1$ and x_2.

If $g(x)$ is such that its range has only one of x_1 and x_2, then $f(g(x))$ can be one-one even, if $f(x)$ is many-one.

41. $x \sin x$ is a continuous function value of $|x \sin x|$ can be made as large as we like for sufficiently large values of x.

Therefore, range of $x \sin x = R$

$$\frac{[x]}{\tan 2x} = 0$$

For $\quad x \in \left(0, \frac{\pi}{4}\right)$ as the $[x] = 0$.

For $\quad x \in \left(-\frac{\pi}{4}, 0\right), \frac{[x]}{\tan 2x} > 0.$

Therefore, values of $\frac{[x]}{\tan 2x}$ are never negative.

Thus, range of $\frac{[x]}{\tan 2x} \neq R.$

$$\left|\frac{x}{\sin x}\right| > 1, \text{ whenever defined.}$$

Thus, range of $\frac{x}{\sin x}$ is not R.

$|x| + \sqrt{\{x\}}$ is a continuous function and

$$\lim_{x \to \infty} ([x] + \sqrt{\{x\}}) = \infty, \lim_{x \to -\infty} ([x] - \sqrt{\{x\}}) = -\infty$$

Thus, range of $[x] + [x] = R.$

42. (a) $f(x) = e^{\ln \sec^{-1} x}$

$$g(1) = 0 \text{ but } f(1) \text{ is not defined}$$

Thus, f and g are not identical.

(b) $f(x) = \tan(\tan^{-1} x) = x, \forall\, x \in R$ and

$$g(x) = \cot(\cot^{-1} x) = x, \forall\, x \in R$$

Thus, f and g are identical.

(c) $f(x) = \begin{cases} 1 & , \quad x > 0 \\ 0 & , \quad x = 0 \\ -1 & , \quad x < 0 \end{cases}$

$$g(x) = \text{sgn}(\text{sgn}\, x) = \text{sgn} = \begin{cases} 1 & , \quad x > 0 \\ 0 & , \quad x = 0 \\ -1 & , \quad x < 0 \end{cases}$$

$$g(x) = \begin{cases} 1 & , \quad x > 0 \\ 0 & , \quad x = 0 \\ -1 & , \quad x < 0 \end{cases}$$

Thus, f and g are identical.

(d) $g(x) = \cot^2 x - \cos^2 x$

$$= \frac{\cos^2 x}{\sin^2 x} - \cos^2 x$$

$$= \cot^2 x \cos^2 x = f(x), \forall\, x \neq \pi$$

Thus, f and g are identical.

Type 3 : **Assertion and Reason**

43. If we take example of $x + \sin x$, it is non-periodic whereas its derivative $1 + \cos x$ is periodic.

44. The value of $\sin \sqrt{2}x + \sin ax$ can be equal to 2, if $\sin \sqrt{2}x$ and $\sin ax$ both are equal to one but both are not equal one for any common value of x.

45. Clearly, $f(x)$ is many-one and into function.

46. $f(x) = 2 \cos \dfrac{\pi}{5} \sin x - 2 \cos \dfrac{2\pi}{5} \sin x$

$$= 2 \sin x \left[\cos \frac{\pi}{5} - \cos \frac{2\pi}{5}\right]$$

$$= 2 \sin x \,[\cos 36° - \cos 72°]$$

$$= 2 \sin x \,[\cos 36° - \sin 18°]$$

$$= 2 \sin x \left[\frac{\sqrt{5} + 1}{4} - \left(\frac{\sqrt{5} - 1}{4}\right)\right]$$

$$= \sin x$$

Range is $[-1, 1]$.

47. Period of $2 \cos \dfrac{1}{3}(x - \pi)$ and $4 \sin \dfrac{1}{3}(x - \pi)$ are $\dfrac{2\pi}{\frac{1}{3}}, \dfrac{2\pi}{\frac{1}{3}}$ or $6\pi, 6\pi$.

$\therefore$ Period of their sums $= 6\pi$

48. $(f(\sin 2x))^2 = \sin^2 x + \cos^2 x + 2 \sin x \cos x$

$$= 1 + \sin 2x$$

$\Rightarrow$ $$f(x) = \sqrt{1 + x}, \forall\, x \in [-1, 1]$$

$\Rightarrow$ If $x \in \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$, then

$$f(\tan^2 x) = \sqrt{1 + \tan^2 x} = \sec x$$

49. Fifth degree equation must have at least one real root. If it had two real roots, $f'(x) = 0$ must have one real root.

50. Range of $\frac{1}{1+x^2}$ (0, 1], domain R

$$\therefore \qquad \log\left(\frac{1}{1+x^2}\right) \in (-\infty, 0]$$

51. $f(x) = \sqrt{2}\sin x + \sqrt{2}\cos x - \sqrt{2}\cos x + c$

$$= \sqrt{2}\sin x + c$$

$$\Rightarrow \qquad f(0) = f(\pi) = c$$

Hence, many-one function.

52. If $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, $a_n \neq 0$.

Then, $f^{(n+1)}(x) = 0$, for all x while any derivative of $\sin x$ is never a zero function.

Hence, $\sin x$ is not polynomial function.

53. f is injective, since $x \neq y$ $(x, y \in R)$.

$$\Rightarrow \qquad \log_a \{x + \sqrt{x^2+1}\} \neq \log_a \{y + \sqrt{y^2+1}\}$$

$$\Rightarrow \qquad f(x) \neq f(y)$$

f is onto because $\log_a (x + \sqrt{x^2+1}) = y$

$$\Rightarrow \qquad x = \frac{a^y - a^{-y}}{2}$$

Type 4 : **Linked Comprehension Based Questions**

Solutions (Q. Nos. 54 to 56)

54. $f(x) - 2f\left(\frac{x}{2}\right) + f\left(\frac{x}{4}\right) = x^2$

$$\therefore \qquad f\left(\frac{x}{2}\right) - 2f\left(\frac{x}{4}\right) + f\left(\frac{x}{8}\right) = \left(\frac{x}{2}\right)^2$$

$$f\left(\frac{x}{4}\right) - 2f\left(\frac{x}{8}\right) + f\left(\frac{x}{16}\right) = \left(\frac{x}{4}\right)^2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f\left(\frac{x}{2^n}\right) - 2f\left(\frac{x}{2^{n+1}}\right) + f\left(\frac{x}{2^{n+2}}\right) = \left(\frac{x}{2^n}\right)^2$$

Adding, we get

$$f(x) - f\left(\frac{x}{2}\right) - f\left(\frac{x}{2^{n+1}}\right) + f\left(\frac{x}{2^{n+2}}\right)$$

$$= x^2\left(1 + \frac{1}{2^2} + \ldots + \frac{1}{2^{2n}}\right)$$

As $n \to \infty$, we get $f(x) - f\left(\frac{x}{2}\right) = \frac{4x^2}{3}$

Repeating the same procedure again, we get

$$f(x) - f(0) = \frac{16x^2}{9}$$

$\therefore$ $$f(3) = 16 + f(0)$$

55. $f(x) - f(0) - x = 0$

$\Rightarrow$ $$\frac{16x^2}{9} - x = 0 \quad \Rightarrow \quad x = 0, \frac{9}{16}$$

$\therefore$ Two solutions.

56. $f'(x) = \frac{32x}{9}$

$\therefore$ $f'(0) = 0$

Solutions (Q. Nos. 57 to 58)

57. For integral solution,

$$x + y - [x][y] = 0 \quad \Rightarrow \quad x + y - xy = 0$$

$\Rightarrow$ $$-1 + x + y - xy = -1 \quad \Rightarrow (x - 1)(y - 1) = 1$$

ie, Only possible, if

$$(x - 1 = 1,\ y - 1 = 1) \quad \text{or} \quad (x - 1 = -1,\ y - 1 = -1)$$

$\Rightarrow$ $$x = 2,\ y = 2 \quad \text{or} \quad x = 0,\ y = 0$$

$\therefore$ Solutions are (0, 0) or (2, 2).

58. For non-integral solution,

Let $x = [x] + f_1$

and $Y = [y] + f_2$

$\therefore$ $[x][y] = [x] + f_1 + [y] + f_2$

$([x] - 1)([y] - 1) = f_1 + f_2 + 1$

Now, $0 \le f_1 + f_2 < 2$

$\therefore$ $f_1 + f_2 = 1$

$\Rightarrow$ $([x] - 1)([y] - 1) = 2$

Which is possible for

$\therefore$ $[x] = 3$ and $[y] = 2$

or $[x] = 2$ and $[y] = 3$

or $[x] = -1$ and $[y] = 0$

or $[x] = 0$ and $[y] = -1$

$\therefore$ The $x + y = [x][y]$ becomes $x + y = 6$

or $x + y = 0$

$\therefore$ Non-integer solution lies on $x + y = 6$

or $x + y = 0$

Solutions (Q. Nos. 59 to 61)

59. $f(x)=\frac{1}{2}\left[f(xy)+f\left(\frac{x}{y}\right)\right]$ ...(i)

Interchanging x and y, we get

$$f(y)=\frac{1}{2}\left[f(xy)+f\left(\frac{y}{x}\right)\right] \quad \text{...(ii)}$$

On subtracting,

$$f(x)-f(y)=\frac{1}{2}\left\{f\left(\frac{x}{y}\right)-f\left(\frac{y}{x}\right)\right\} \quad \text{...(iii)}$$

In Eq. (i), put $x=1$, we get

$$f(1)=\frac{1}{2}\left[f(y)+f\left(\frac{1}{y}\right)\right] \quad \text{...(iv)}$$

$$f(1)=0 \Rightarrow f(y)=-f\left(\frac{1}{y}\right)$$

Hence, $\quad f\left(\frac{x}{y}\right)=-f\left(\frac{y}{x}\right)$

$\therefore$ Eq. (iii), becomes $\quad f(x)-f(y)=f\left(\frac{x}{y}\right)$

60. $f'(x)=\lim_{h\to 0}=\frac{f(x+h)-f(x)}{h}$

Using the result in Q. No. 17.

$$\lim_{h\to 0}\frac{f\left(\frac{x+h}{x}\right)}{h}=\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{h}$$

$$=\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}\cdot x}=\frac{1}{x}f'(1)$$

$$=\frac{1}{x}\cdot 2 \quad =\frac{2}{x}$$

$\therefore \quad f'(3)=\frac{2}{3}$

61. From Q. No. 18, $f'(x)=\frac{2}{x}$

$$f(x)=2\log x+C$$

$$f(1)=0\Rightarrow C=0$$

$\therefore \quad f(x)=2\log x$

$\therefore \quad f(e)=2\log e=2$

Solutions (Q. Nos. 62 to 64)

62. $f(x) =$ odd degree polynomial + bounded function

$$\cot^{-1} x \in (0, \pi)$$

Also, $f'(x) > 0, \therefore f(x)$ is one-one,

and Range of $f(x) \in R \therefore$ onto

$\Rightarrow$ $f(x)$ is one-one onto.

63. $f(x) = x^4 + 1 + \dfrac{1}{x^2 + x + 1}$

= even degree polynomial + bounded function

$$\frac{1}{x^2 + x + 1} \in \left(0, \frac{4}{3}\right)$$

$$f'(x) = \frac{4x^3(x^2 + x + 1)^2 - 2x - 1}{(x^2 + x + 1)^2}$$

$\Rightarrow f'(x) = 0$ has at least one root which is repeated odd number of times or it has one root which is not repeated, since numerator of $f'(x)$ is a polynomial of degree 7.

$\Rightarrow$ $f(x) = 0$ has a point of extreme.

$\therefore f(x)$ is many-one into.

64. $f(x) =$ odd degree polynomial + bounded function $\sin 2x \Rightarrow f(x)$ is onto.

$f(x)$ is one-one, if $f'(x) \geq 0$ or $f'(x) \leq 0, \forall x$

$\Rightarrow$ $a \geq 1 \cup a \leq -1$

$\Rightarrow$ $a \in R - (-1, 1)$

Solutions (Q. Nos. 65 to 67)

$D_h = \{-1, 1\}$, as minimum occurs before maxima for $f(x)$.

$\therefore$ $a_3 = -1$

Now, $g(x) = a_0 + a_1 x + a_2 x^2 - x^3$

$$g'(x) = a_1 + 2a_2 x - 3x^2$$

$$= -3(x - 3)(x + 3)$$

$$= -3x^2 + 27$$

$\therefore$ $a_1 = 27, a_2 = 0$

$\therefore$ $a_1 + a_2 = 27$

Also, $g(-3) > 0$ and $g(3) > 0$

$\Rightarrow$ $a_0 > 54$ and $a_0 < -54$

$\therefore$ $a_0 > 54$

Now, $g(x) = a_0 + 27x - x^3$

$$f(x) = \sqrt{a_0 + 27x - x^3}$$

$$f(-10) = \sqrt{a_0 + 270 - 1000}$$

Clearly, $f(-10)$ is defined for $a_0 > 730$.

Solutions (Q. Nos. 68 to 70)

68. $\because ff^{-1}(x) = x$

$$2^{(f^{-1}(x))^4 - 4(f^{-1}(x))^2} = x$$

$\Rightarrow$ $(f^{-1}(x))^4 - 4(f^{-1}(x))^2 - \log_2 x = 0$

$\therefore$ $(f^{-1}(x))^2 = 2 + \sqrt{4 + \log_2 x}$

$\therefore$ Range of $f^{-1}(x)$ is $[2, \infty)$.

$\therefore$ $f^{-1}(x) = \sqrt{2 + \sqrt{4 + \log_2 x}}$

$\therefore$ $f^{-1}(x) > 0$

69. $g(x) = \dfrac{\sin x + 4}{\sin x - 2}$

$\Rightarrow$ $g'(x) = \dfrac{-\cos x}{(\sin x - 2)^2} \geq 0$ $\left(\because x \in \left[\dfrac{\pi}{2}, \pi\right]\right)$

$\Rightarrow g(x)$ is an increasing function, hence one-one function.

$\therefore$ Range is $\left[g\left(\dfrac{\pi}{2}\right), g(\pi)\right]$, and lies in $[-5, -2]$.

70. $x : x$ in domain of $g^{-1}(x)$,

$g^{-1}(x)$ in domain of $f^{-1}(x)$

$g^{-1}(x) = \sin^{-1} \dfrac{2(x+2)}{x-1}, \forall\, x \in [-5, -2]$ ...(i)

$\Rightarrow$ $\sin^{-1} \dfrac{2(x+2)}{x-1} \geq 1$

ie, $1 \leq \sin^{-1} \dfrac{2(x+2)}{x-1} \leq \dfrac{\pi}{2}$

$\Rightarrow$ $\sin 1 \leq \dfrac{2(x+2)}{x-1} \leq 1$

Solving this, we get

$x \leq -\dfrac{(4 - \sin 1)}{2 - \sin 1}$ or $x > 1$ ...(ii)

From Eqs. (i) and (ii), we get

$$x \in \left[-5, -\frac{(4 + \sin 1)}{2 - \sin 1}\right]$$

Solutions (Q. Nos. 71 to 73)

$P(x) + 1 = 0$ has a thrice repeated root at $x = 1$.

$P'(x) = 0$ has a twice repeated root at $x = 1$.

Similarly, $P'(x) = 0$ has a twice repeated root at $x = -1$.

$\Rightarrow P'(x)$ is divisible by $(x - 1)^2(x + 1)^2$

$\because$ $P'(x)$ is degree at most 4

$\Rightarrow$ $P'(x) = \alpha(x - 1)^2(x + 1)^2$

$\therefore \qquad P(x) = \alpha\left(\frac{x^5}{5} - \frac{2}{3}x^3 + x\right) + c$

Now, $\qquad P(1) = -1$ and $P(-1) = 1$

$\Rightarrow \qquad \alpha = -\frac{15}{8}$ and $c = 0$

$\therefore \qquad P(x) = -\frac{3}{8}x^5 + \frac{5}{4}x^3 - \frac{15}{8}x$

$\therefore \qquad P(x) = -\frac{15}{8}\left(\frac{x^5}{5} - \frac{2}{3}x^3 + x\right)$

$$= -\frac{15}{8}x\left(\frac{x^4}{5} - \frac{2x^2}{3} + 1\right)$$

has only one real root $x = 0$, as $\frac{x^4}{5} - \frac{2x^2}{3} + 1$ has imaginary roots.

Also, sum of pairwise product of all roots $= -\frac{10}{3}$.

Also, $\qquad P''(x) = -\frac{15}{2}(x^3 - x)$

Let $\qquad f(x) = -\frac{15}{2}(x^3 - x)$

$$f'(x) = -\frac{15}{2}(3x^2 - 1) = -\frac{15}{2}(\sqrt{3}x - 1)(\sqrt{3}\,x + 1)$$

$-\qquad + \qquad -$

$\Rightarrow$ Maximum at $x = \frac{1}{\sqrt{3}}$.

Solutions (Q. Nos. 74 to 76)

74. As $f(296) = f(926)$, f is many-one.

Further, the numbers whose digits increase from left to right (for example) have no pre-image. Hence, f is into.

75. It is easy to see that the remainder, when a positive integer is divided by 9, is the same as the sum of the digit of the number (until the sum becomes a one digit number). Thus, $f(n)$ and n leave the same remainder, when divided by 9. Hence, 9 divides $f(n) - n$. Further, there is no reason to expect that the number is divisible by 27. The number $f(n) - n$ is not divisible by 18 also, in case $f(n) - n$ is odd.

Hence, 9 is the biggest number.

76. By the definition of f, digits of $f(x)$ are non-increasing from left to right.

Solutions (Q. Nos. 77 to 79)

77. $f(x) = 0 \Rightarrow \tan x = x$ (we can divide by $\cos x$, as $x = \frac{\pi}{2}$ does not satisfy $f(x) = 0$.)

$\Rightarrow x$ lies in the 3rd quadrant.

(can be seen using graphs of $y = \tan x$ and $y = x$.)

78. For $x \in \left(0, \frac{\pi}{2}\right)$, $f(x) > 0$, as here $\tan x > x$. At $x = \frac{\pi}{2}$, obviously $f(x) > 0$.

For $x \in \left(\frac{\pi}{2}, \pi\right]$, $\sin x$ is positive, while $x \cos x$ is negative, and hence, here also $f(x) > 0$.

If α is the least positive value for which $\tan x = x$, then $f(x) > 0$ for $x \in (\pi, \alpha)$.
For $x \in [\alpha, 2\pi)$, $f(x) \le 0$.

Thus, required set is $(0, \alpha)$.

79. Required area

$$= \int_0^\alpha f(x)\,dx - \int_\alpha^{2\pi} f(x)\,dx$$

$$= -(2\cos x + x \sin x)_0^\alpha + (2\cos x + x\sin x)_\alpha^{2\alpha}$$

$$= -(2\cos\alpha + \alpha\sin\alpha) + (2) + (2) - (2\cos\alpha + \alpha\sin\alpha)$$

$$= 4 - 2(2\cos\alpha + \alpha\sin\alpha)$$

$$= 4 - 4\cos\alpha - 2\alpha^2 \cdot \cos\alpha \qquad (\text{as } \sin\alpha = \alpha\cos\alpha)$$

$$= 4 - 2(2 + \alpha^2)\cos\alpha$$

Type 5 : **Match the Columns**

80. (A) $\sqrt{\sin(\cos x)} = \sin(\cos x) \ge 0$

$$2n\pi \le \cos x \le (2n+1)\pi$$

$$0 \le \cos x \le 1$$

$$x \in \left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right]$$

(B) $(\sqrt{\cos(\sin x)})^{-1}$

$$\cos(\sin x) > 0$$

$$-\frac{\pi}{2} < \sin x < \frac{\pi}{2} \quad \Rightarrow x \in R$$

(C) $\tan(\pi \sin x)$

$$\pi \sin x \ne \pm\frac{\pi}{2}$$

$$\sin x \ne \pm\frac{1}{2}$$

$$\therefore \qquad x \ne n\pi \pm \frac{\pi}{6}$$

$$x \in R - \left\{n\pi \pm \frac{\pi}{6}\right\}$$

(D) $\ln(\tan x)$

$$\tan x > 0$$

$$0 < x < \frac{\pi}{2}$$

$$n\pi < x < n\pi + \frac{\pi}{2}$$

81. (A) $|4\sin x - 1| < \sqrt{5} \quad \Rightarrow \quad -\sqrt{5} < 4\sin x - 1 < \sqrt{5}$

$$1 - \sqrt{5} < 4\sin x < 1 + \sqrt{5}$$

$$\frac{1-\sqrt{5}}{4} < \sin x < \frac{1+\sqrt{5}}{4} \quad \Rightarrow \quad -\frac{\pi}{10} < x < \frac{3\pi}{10}$$

But, from the domain $x \in [0, 3\pi/10)$.

(B) $4\sin^2 x - 8\sin x + 3 \le 0, \quad 0 \le x \le 2\pi$

$$(2\sin x - 1)(2\sin x - 3) \le 0 \Rightarrow 2\sin x - 1 \ge 0$$

$$\Rightarrow \quad \sin x \ge \frac{1}{2} \quad \Rightarrow \quad x \in \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$$

(C) $|\tan x| \le 1$

$$x \in [-\pi, \pi], \quad -1 \le \tan x \le 1$$

$$n\pi - \frac{\pi}{4} \le x \le n\pi + \frac{\pi}{4}$$

Put $n = 0$, $\quad -\frac{\pi}{4} \le x \le \frac{\pi}{4}$

Put $n = 1$, $\quad \pi - \frac{\pi}{4} \le x \le \pi + \frac{\pi}{4}$

$$\frac{3\pi}{4} \le x \le \frac{5\pi}{4}$$

But, from domain $\quad \frac{3\pi}{4} \le x \le \pi$

Put $\quad x = -1, \; -\pi - \frac{\pi}{4} \le x \le -\pi + \frac{\pi}{4}$

$$-\frac{5\pi}{4} \le x \le -\frac{3\pi}{4}$$

But, from domain $\quad -\pi \le x \le -\frac{3\pi}{4}$

Finally, $\quad \left[-\pi, -\frac{3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right],$

But, $\quad x \in [0, \pi]$

$\therefore \quad x \in \left[0, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right]$

(D) $\cos x - \sin x \ge 1, 0 \le x \le 2\pi$

$$\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x - \frac{\sin x}{\sqrt{2}}\right) \ge 1$$

$$\left(\cos x \cdot \cos\frac{\pi}{4} - \sin x \cdot \sin\frac{\pi}{4}\right) \ge \frac{1}{\sqrt{2}}$$

$$\cos\left(x + \frac{\pi}{4}\right) \ge \frac{1}{\sqrt{2}}$$

$$2n\pi - \frac{\pi}{4} \le x + \frac{\pi}{4} \le 2n\pi + \frac{\pi}{4}$$

$$2n\pi - \frac{\pi}{2} \le x \le 2n\pi.$$

On substituting suitable values of n, according to domain

$$x \in \left[\frac{3\pi}{2}, 2\pi\right] \cup \{0\}$$

82. (A) When $-1 \le x < 0$

$$f(x) \in [0, 1) \text{ and } f(x) = x + 1$$

$\therefore$ $$f(f(x)) = (x + 1)^2 - 1 = x^2 + 2x$$

(B) $\dfrac{(\cos 2x + 1)(\sec^2 x + 2\tan x)}{2}$

$$= \left(\frac{1 - \tan^2 x}{1 + \tan^2 x} + 1\right)\left(\frac{1 + \tan^2 x + 2\tan x}{2}\right)$$

$$= \frac{1 + \tan^2 x + 2\tan x}{1 + \tan^2 x}$$

ie, $$f\left(\frac{2\tan x}{1 + \tan^2 x}\right) = 1 + \frac{2\tan x}{1 + \tan^2 x}$$

$\therefore$ $$f(x) = 1 + x$$

(C) Put $x = 0,\ y = 0, f(1) = (1 + 1)^2 = 2^2$

Put $x = 0,\ y = 1,\ f(2) = (1 + 2)^2 = 3^2$

By induction, $f(x) = (x + 1)^2$.

(D) When $4 < x < 5, \left[\dfrac{x}{4}\right] = I$

$$f(x) = 2x + 3$$

$\therefore$ $$y = 2x + 3 \Rightarrow x = \frac{y - 3}{2}$$

$$f^{-1}(x) = \frac{x - 3}{2}$$

83. (A) Simplifying the expression, we get $f(x) = \dfrac{5}{4}\cdot$

(B) Period of $\tan(e^{\{x\}}) = 1$, the period $[x + \alpha] - (x + \alpha) + \alpha - 5$ is also 1, hence 1 is the period.

(C) $h(x) - h(-x) = 0$, hence even function.

(D) Simplifying the expression, $k(x) = 0$.

Type 6 : **Integer Answer Type Questions**

84. On adding,

$$a = \frac{3 - \cos 4\theta + 4\sin 2\theta}{2} = (1 + \sin 2\theta)^2$$

On subtracting, $b = (1 - \sin 2\theta)^2$

$\Rightarrow$ $ab = \cos^4 2\theta \le 1$

85. Clearly, $x^2 + 4x \ge 0$

$$2x^2 + 3 \ge 0 \;\Rightarrow\; x^2 + 4x \ge 2x^2 + 3$$

and x is an integer,

$\therefore$ $x \in \{1, 2, 3\}$ $\quad\therefore\; n = 3$

Now, maximum value $\quad {}^{x^2+4x}C_{2x^2+3} = 12$

$\therefore \quad Y = |\ln 12|$

$\therefore \quad [Y] = 2\,(\ln c^2 < \ln 12 < \ln c^3)$

$\therefore \quad [n + [Y]] = [3 + 2] = 5$

86. $f(x-1) + f(x+1) = \sqrt{3} f(x)$

$\Rightarrow \quad f(x) + f(x+2) = \sqrt{3} f(x+1)$

Putting, $\quad x = x + 2,$

$$f(x+1) + f(x+3) = \sqrt{3} f(x+2)$$

$$f(x-1) + 2f(x+2) + f(x+3) = \sqrt{3}[\sqrt{3} f(x+1)]$$

$$f(x-1) + f(x+3) = f(x+1)$$

Putting, $\quad x = x + 2$, again

$$f(x+1) + f(x+5) = f(x+3)$$

$$f(x-1) + f(x+5) = 0$$

$$f(x+5) = -f(x-1)$$

$$f(x) = -f(x+6)$$

$$f(x+12) = f(x)$$

$$\sum_{r=0}^{19} f(5 + 12r) = 20 f(5) = 20 \times 10 = 200$$

Hence, sum of digits $= 2 + 0 + 0 = 2$

87. Put $x = 1$ in $2f(x) = f(xy) + f\left(\dfrac{x}{y}\right)$ $\quad \ldots$(i)

$$2f(1) = f(y) + f\left(\frac{1}{y}\right) \quad \ldots\text{(ii)}$$

$\Rightarrow \quad f(y) = -f\left(\dfrac{1}{y}\right)$

Replacing x by y and y by x in Eq. (i), we get

$$2f(y) = f(yx) + f\left(\frac{y}{x}\right) \quad \ldots\text{(iii)}$$

Eqs. (i) and (iii),

$$2\{f(x) - f(y)\} = f\left(\frac{x}{y}\right) - \left\{-f\left(\frac{x}{y}\right)\right\} = 2f\left(\frac{x}{y}\right) \quad \ldots\text{(iv)}$$

$$f(x) - f(y) = f\left(\frac{x}{y}\right) \quad \ldots\text{(v)}$$

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = f'(1) = 1$$

$$\lim_{h \to 0} \frac{f(1+h)}{h} = 1, \quad \text{as} \quad f(1) = 0$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f\left(1 + \frac{h}{x}\right)}{h} = \frac{1}{x}$$

$$f(x) = \log|x| + c$$

$$f(1) = 0 \quad \Rightarrow \quad c = 0$$

$$f(e) = 1$$

88. $f(1) + 2f(2) + 3f(3) + \ldots + n(f(n)) = n(n+1)f(n)$

$\Rightarrow \quad f(1) + 2(2) + 3(3) + \ldots + (n+1)f(n+1) = (n+1)(n+2)f(n+1)$

$\Rightarrow \quad n(n+1)f(n) = (n+1)^2 f(n+1) \quad \Rightarrow \quad nf(n) = (n+1)f(n+1)$

ie, $\quad 2f(2) = 3f(3) = \ldots = n(f(n))$

ie, $\quad f(n) = \frac{1}{n}$

$$2126\, f(1063) = 2$$

89. Now, $f(x) = \frac{x^4 + x^2 + 1}{x^2 - x + 1}$

$\Rightarrow \quad f(x) = x^2 + x + 1$

Now, $\quad f(\omega^n) = \omega^{2n} + \omega^n + 1 = 3$

$\because \quad \omega^n = 1$ when n is a multiple of 3.

90. $f^2(x) \cdot f\left(\frac{1-x}{1+x}\right) = x^3$ $\quad \ldots$(i)

Replacing x by $\frac{1-x}{1+x}$, we get $\quad f^2\left(\frac{1-x}{1+x}\right) f(x) = \left(\frac{1-x}{1+x}\right)^3$ $\quad \ldots$(ii)

By using Eqs. (i) and (ii), we get $f^3(x) = x^6\left(\frac{1+x}{1-x}\right)^3$

$\Rightarrow \quad f(x) = x^3\left(\frac{1+x}{1-x}\right)$

$$f(-2) = \frac{8}{3}$$

$\Rightarrow \quad [f(-2)] = 2 \quad \Rightarrow \quad |[f(-2)]| = 2$

91. $f(2a - x) = f(x)$

$\Rightarrow \quad f(2a - x) = -f(x)$

$\because \quad f$ is odd $\quad \Rightarrow \quad f(x + 4a) = f(x)$

$\Rightarrow \quad f$ is periodic with period $4a$.

$\Rightarrow \quad f(1 + 4r) = f(1)$

Now, $\quad \sum_{r=0}^{\infty} [f(1)]^r = 8 \quad \Rightarrow \quad \frac{1}{1 - f(1)} = 8$

$\Rightarrow \quad f(1) = \frac{7}{8} \quad \Rightarrow 8f(1) = 7$

92. Consider $f(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}=\dfrac{e^{2x}-1}{e^{2x}+1}=1-\dfrac{2}{e^{2x}+1}$

$$\Rightarrow \quad f'(x)=\frac{4e^{2x}}{(e^{2x}+1)^2}>0,\ \forall\, x\in R$$

$\Rightarrow$ $f(x)$ is an increasing function.

$\Rightarrow$ Domain : R, Range : $(-1, 1)$

For $f: R\to(-1,1)$,

$$f(x)=\frac{e^x-e^{-x}}{e^x+e^{-x}},\ f^{-x}:(-1,1)\to R$$

$$\Rightarrow \quad x=\frac{e^y-e^{-y}}{e^y+e^{-y}}$$

$$\Rightarrow \quad e^y=\sqrt{\frac{1+x}{1-x}} \quad \Rightarrow f^{-1}(x)=\ln\sqrt{\frac{1+x}{1-x}}$$

Hence, given equation is equivalent to $f(x)=f^{-1}(x)$.

$\Leftrightarrow f(x)=x$ (as f is an increasing function)

$$\Rightarrow \quad \ln\sqrt{\frac{1+x}{1-x}}=x \quad \Rightarrow \quad \frac{1+x}{1-x}=e^{2x}$$

Now, draw the graph of $y=\dfrac{1+x}{1-x}$ and $y=e^{2x}$. They intersect each other at $x=0$.

93. Let $y=\dfrac{3x^2+9x+17}{3x^2+9x+7}=1+\dfrac{10}{3x^2+9x-7}$

Now, $3x^2+9x+7=3(x^2+3x)+7$

$$=3\left(x+\frac{3}{2}\right)^2+\frac{1}{4}\ge\frac{1}{4} \text{ for all } x\in R.$$

Maximum value of $\dfrac{10}{3x^2+9x+7}$ is 40.

Maximum value of y is $1+40=41$

$\therefore \quad 5k+1=41$

$\Rightarrow \quad k=8$

94. The period of the function is found as follows :

Given, $f(x)+f(x+4)=f(x+2)+f(x+6)$ ...(i)

$\therefore$ Replacing x by $x+2$, we get

$f(x+2)+f(x+6)=f(x+4)+f(x+8)$ ...(ii)

From Eqs. (i) and (ii), we get

$$f(x)+f(x+4)=f(x+4)+f(x+8)$$

$\Rightarrow \quad f(x)=f(x+8)$

$\Rightarrow$ 8 is the period.

(*Proficiency in 'Functions' Exercise* 2)

2. Given

$$f(n) = 1! + 2! + 3! + \ldots.. + n!$$

$\therefore$ $\quad f(n+2) = Q(n)\, f(n) + P(n) f(n+1)$

$\Rightarrow$ $\quad \{1! + 2! + \ldots. + (n+2)!\} = Q(n)\{1! + 2! + \ldots. + n!\}$
$+P(n)\{1! + 2! + \ldots.+ (n+1)!\}$

$\Rightarrow$ $\quad (1! + 2! + \ldots + n!) + \{(n+1)! + (n+2)!\} = \{Q(n) + P(n)\}$
$\{1! + 2! + 3! + \ldots + n!\} + P(n).(n+1)!$

Equating coefficients of $\{1! + 2! + \ldots.. + n!\}$ and $(n+1)!$ on both sides we get;

$$Q(n) + P(n) = 1 \quad \text{and} \quad P(n) = (n+3)$$

So, $\quad P(n) = n + 3$

or $\quad P(x) = x + 3 \quad$ and $\quad Q(x) = 1 - P(x)$

Hence, $\quad P(x) = x + 3 \quad$ and $\quad Q(x) = -x - 2$

3. Given,

$$f(x) = \frac{a^x}{a^x + \sqrt{a}} \quad \ldots\text{(i)}$$

Now, $\quad f(1-x) = \dfrac{a^{1-x}}{a^{1-x} + \sqrt{a}} = \dfrac{\sqrt{a}}{\sqrt{a} + a^x} \quad \ldots\text{(ii)}$

From Eqs. (i) and (ii); we have $\quad f(x) + f(1-x) = 1 \quad \ldots\text{(iii)}$

$\Rightarrow$ $\quad f\left(\dfrac{r}{2n}\right) + f\left(\dfrac{2n-r}{2n}\right) = 1$

$\Rightarrow$ $\quad \displaystyle\sum_{r=1}^{2n-1} f\left(\frac{r}{2n}\right) + \sum_{r=1}^{2n-1} f\left(\frac{2n-r}{2n}\right) = 2n - 1$

$\Rightarrow$ $\quad \displaystyle\sum_{r=1}^{2n-1} f\left(\frac{r}{2n}\right) + \sum_{t=1}^{2n-1} f\left(\frac{t}{2n}\right) = 2n - 1 \quad$ (putting $2n - r = t$)

Hence, $\quad \displaystyle 2\sum_{r=1}^{2n-1} f\left(\frac{r}{2n}\right) = 2n - 1$

4. Let max $|f(x)| = M$ where $0 < M \le 1$.

(Since, f is not identically zero and $|f(x)| \le 1\ \forall x \in R$)

Now, $\quad f(x+y) + f(x-y) = 2f(x) \cdot g(y)$

$\Rightarrow$ $\quad |2f(x) \cdot g(y)| = |f(x+y) + f(x-y)|$

$\Rightarrow$ $\quad |2f(x)| \cdot |g(y)| \le |f(x+y)| + |f(x-y)| \le M + M$

$\Rightarrow$ $\quad 2|f(x)||g(y)| \le 2M$

$\Rightarrow$ $\quad |g(y)| \le 1 \quad \text{for} \quad y \in R$

5. $f(x)$ is defined if $\left(\log_{|\sin x|}(x^2-8x+23)-\dfrac{3}{\log_2|\sin x|}\right)>0$

$\Rightarrow \qquad \log_{|\sin x|}\left(\dfrac{x^2-8x+23}{8}\right)>0 \qquad \left(\text{as } \dfrac{3}{\log_2|\sin x|}=\dfrac{\log_2 8}{\log_2|\sin x|}=\log_{|\sin x|}8\right)$

This is true, if

$$|\sin x|\neq 0, 1 \quad \text{and} \quad \frac{x^2-8x+23}{8}<1$$

(as $|\sin x|<1 \Rightarrow \log_{|\sin x|} a>0 \Rightarrow a<1$)

Now, $\qquad \dfrac{x^2-8x+23}{8}<1$

$\Rightarrow \qquad x^2-8x+15<0$

$\Rightarrow \qquad x\in(3,5)-\left\{\pi, \dfrac{3\pi}{4}\right\}$

Hence, domain $=(3,\pi)\cup\left(\pi,\dfrac{3\pi}{2}\right)\cup\left(\dfrac{3\pi}{2},5\right)$

6. Given $\qquad nx+ny=xy$

or $\qquad xy-nx-ny+n^2=n^2$

or $\qquad (x-n)(y-n)=n^2$

$\Rightarrow (x-n)$ and $(y-n)$ are two integral factors of n^2. (as $x, y, n\in N$)

Obviously if d is one divisor of n^2, then for each such divisor there will be an ordered pair (x, y).

So, $\qquad S(n)=$ number of divisors of n^2

(i) For $n=6$ we have $d=1, 2, 3, 6, 9, 12, 18, 36.$

Thus, $\qquad S(6)=9$

(ii) If n is prime, then $d=1, n$ and n^2;

hence, $S(n)=3$, when ever n is prime.

8. The condition,

$$\sum_{i=1}^{n} f^{-1}(x_i)=\sum_{i=1}^{n} x_i \text{ can be written as;}$$

$$\frac{1}{n}\sum_{i=1}^{n} f^{-1}(x_i)=\frac{1}{n}\sum_{i=1}^{n} x_i$$

ie, AM of y-coordinates of f^{-1} = AM of x coordinates of f.

The given two conditions hold if and only if $x^2+3x-3=x$ (point where f and f^{-1} meet) .

$\Rightarrow \qquad x^2+2x-3=0$

So, $\qquad x=-3, +1$

But $\qquad x\geq 0 \Rightarrow x=1$ (neglecting $x=-3$)

Hence, we can write $\dfrac{1}{n}\displaystyle\sum_{i=1}^{n} x_i=1$ which is the required result.

9. $f(x)=x^2-2x$ ...(i)

and $g(x)=f(f(x)-1)+f(5-f(x))$ ...(ii)

$\therefore$ $g(x)=f(x^2-2x-1)+f(5-x^2+2x)$

$$=[\{x^2-2x-1\}^2-2\{x^2-2x-1\}]$$
$$+[(5-x^2+2x)^2-2\{5-x^2+2x\}]$$

$\Rightarrow$ $g(x)=2x^4-8x^3-4x^2+24x+18$

To show $g(x)\geq 0$ we find its range; *ie*,

$$g'(x)=8x^3-24x^2-8x+24 \qquad \text{let } g'(x)=0$$

$\Rightarrow$ $x=-1$, 1 and 3

$\Rightarrow$ $g(x)\geq \min\{g(-1), g(1), g(3)\}=0$

Hence, $g(x)\geq 0\ \forall x\in R$

10. Given, $2+f(x)f(y)=f(x)+f(y)+f(xy)$

or $1-f(x)-f(y)+f(x)f(y)=f(xy)-1$

or $(1-f(x))(1-f(y))=f(xy)-1$

The above result holds if and only if,

$$f(x)=1+x^n$$

if $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_0$

Then, consider $(1+f(x))(1-f(y))=f(xy)-1$

Compare constant term on either side, we have

$$1-a_0=a_0-1 \ \Rightarrow \ a_0=1$$

Comparing coefficient of x^ny^n, we get

$a_n^2=a_n \Rightarrow a_n=1$ or otherwise polynomial would not be of n degree.

Comparing coefficient of $x, x^1, \ldots, x^{n-1}$ on either sides, we have

$$a_1=a_2=\ldots=a_{n-1}=0$$

$\Rightarrow$ $a_n=1$ and $f(x)=x^n+1$

Given, $f(2)=5$ *ie*, $2^n+1=5$

$\Rightarrow$ $n=2$

Thus, $f(x)=x^2+1$

$$f(f(2))=f(5)=5^2+1=26$$

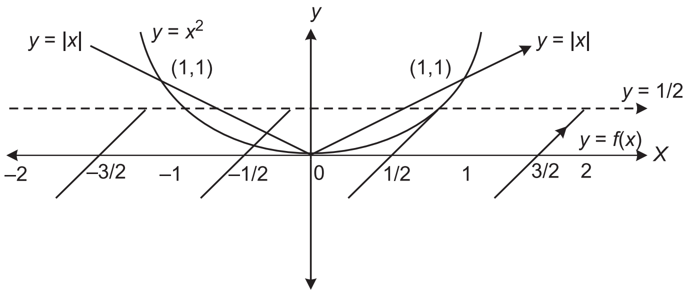
11. Using AM $\geq$ GM, we have $\dfrac{a+b+c}{3}\geq(abc)^{1/3}$

$\Rightarrow$ $\dfrac{a+b+c}{3}\geq(a+b+c)^{1/3}$ (given $a+b+c=abc$)

$\Rightarrow$ $(a+b+c)^{2/3}\geq 3$

$\Rightarrow$ $(a+b+c)\geq 3\sqrt{3}$

12. Clearly, $g(x) = \begin{cases} x^2, -2 \le x \le -1 \\ -x, -1 \le x \le -\frac{1}{4} \\ x + \frac{1}{2}, -\frac{1}{4} \le x < 0 \text{ as graphically if,} \\ x, 0 \le x \le 1 \\ x^2, 1 \le x \le 2 \text{ can be expressed as shown in the following figure} \end{cases}$



13. Let $\qquad g(x) = f(x) - f(x + \pi) \qquad$ ...(i)

at $\qquad x = \pi,\ g(\pi) = f(\pi) - f(2\pi) \qquad$ ...(ii)

at $\qquad x = 0,\ g(0) = f(0) - f(\pi) \qquad$ ....(iii)

Adding Eqs. (ii) and (iii), we have

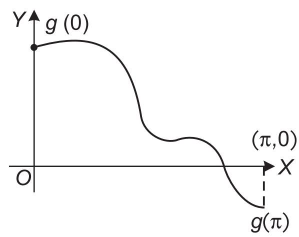
$$g(0) + g(\pi) = f(0) - f(2\pi) = 0$$

$\Rightarrow \qquad g(0) = -g(\pi)$

$\Rightarrow g(0)$ and $g(\pi)$ are of opposite sign.

$\Rightarrow$ There is a point c between 0 and π such that

$$g(c) = 0 \qquad \text{...(iv)}$$



From Eq. (i) putting $x = c$, we have

$$g(c) = f(c) - f(\pi + c) \qquad \text{...(v)}$$

From Eqs. (iv) and (v); we have

$$f(c) - f(\pi + c) = 0$$

Hence, $\qquad f(c) = f(\pi + c)$

15. $n_1 n_2 = 2n_1 - n_2 \qquad$ (given)

$\Rightarrow \qquad n_2(n_1 + 1) = 2n_1 \quad \Rightarrow \quad n_2 = \dfrac{2n_1}{n_1 + 1}$

or $\qquad n_2 = 2 - \dfrac{2}{n_1 + 1}$; $\qquad$ Since, n_1, n_2 is integer.

$\therefore \qquad \dfrac{2}{n_1 + 1} \in$ integer

or $\qquad n_1 + 1 = -2, -1, 1, 2$

or $\qquad n_1 = -3, -2, 0, 1 \quad \Rightarrow \quad n_2 = 3, 4, 0, 1$

$\Rightarrow$ Integral solutions of the form (n_1, n_2) are $(-3, 3), (-2, 4), (0, 0), (1, 1)$.

4

Graphical Transformations

Chapter in a Snapshot

- Graphical Transformations

Graphical Transformations

Students are advised to study this section carefully because it gives very short approach to solve various problems.

When $f(x)$ transforms to $f(x) \pm a$
(where a is a positive constant)

ie, $$f(x) \to f(x) + a$$

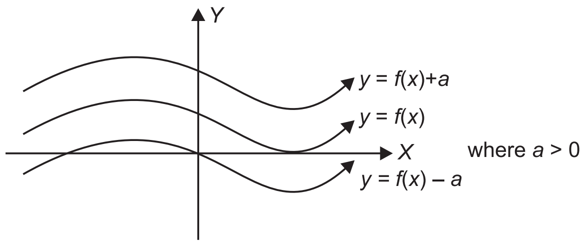


Fig. 4.1

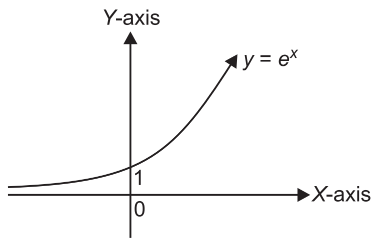
Shift the given graph of $f(x)$ upwards through 'a'.

$$f(x) \to f(x) - a.$$

Shift the given graph of $f(x)$ downwards through 'a'.
Graphically it could be stated as shown in Fig. 4.1.

Illustration 1 Plot $y = e^x$, $y = e^x + 1$, $y = e^x - 1$.

Solution. We know, $y = e^x$ (exponential function) could be plotted as,



$\Rightarrow y = e^x + 1$ is shifted upwards by 1 unit and $y = e^x - 1$ is shifted downwards by 1 unit.

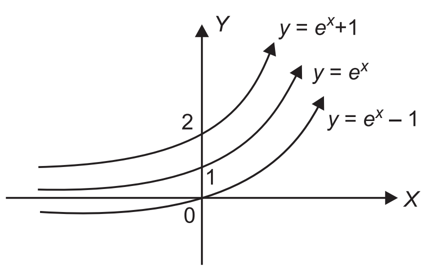
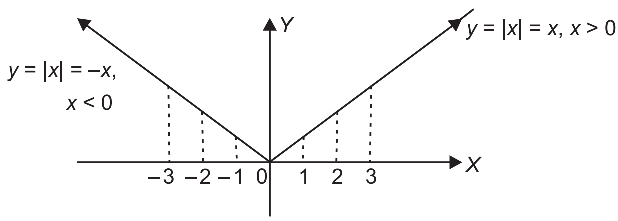
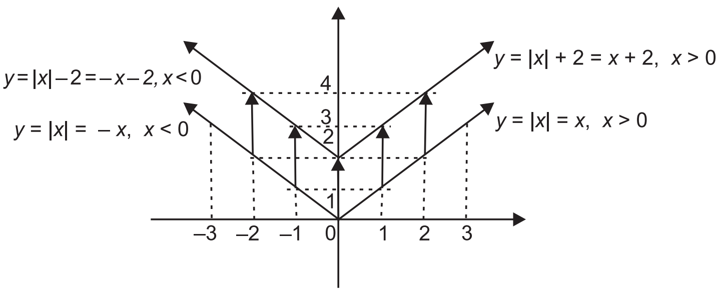


Illustration 2 Plot $y=|x|$, $y=|x|+2$ and $y=|x|-2$.

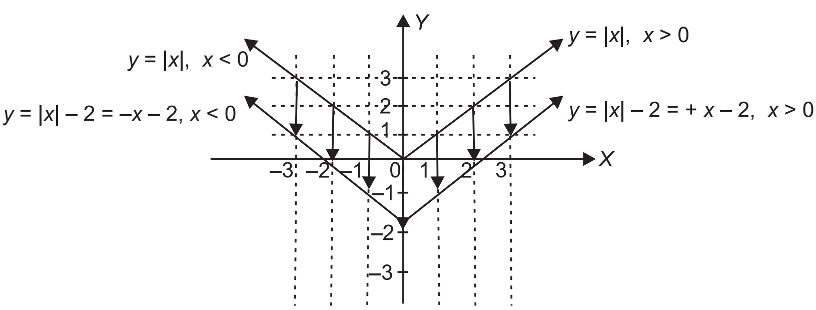
Solution. We know, $y=|x|$ (modulus function) could be plotted as :



$\Rightarrow y=|x|+2$ is shifted upwards by 2 units.



Also, $y=|x|-2$ is shifted downwards by 2 units.



(i) $f(x)$ **transforms to** $f(x-a)$

ie, $f(x) \to f(x-a)$, a is positive.

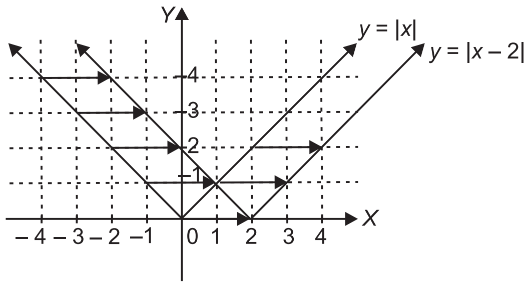
Shift the graph of $f(x)$ through 'a' towards right.

(ii) $f(x) \to f(x+a)$, a **is positive.**

Shift the graph of $f(x)$ through 'a' towards left.

Illustration 3 Plot $y = |x|$ and $y = |x - 2|$.

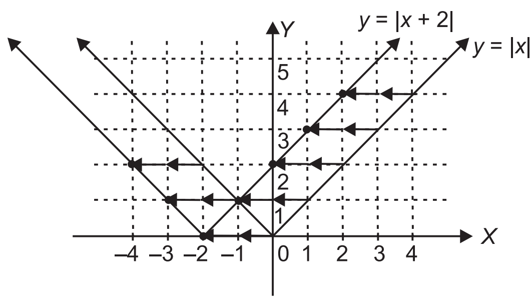
Solution. As discussed $f(x) \to f(x-a)$, shift the graph of $f(x)$ through 'a' towards right.



$\Rightarrow y = |x-2|$ is shifted '2' units towards right.

Illustration 4 Plot $y = |x|$ and $y = |x + 2|$.

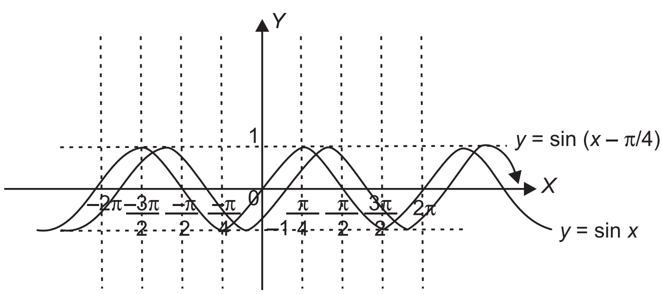
Solution. As discussed $f(x) \to f(x+a)$, shift the graph of $f(x)$ through 'a' units towards left.

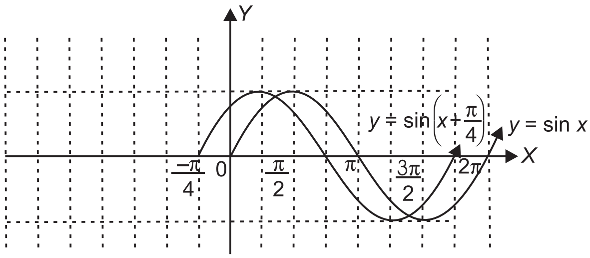


$\Rightarrow y = |x+2|$ is shifted '2' units towards left.

Illustration 5 Plot $y = \sin\left(x + \dfrac{\pi}{4}\right)$ and $y = \sin\left(x - \dfrac{\pi}{4}\right)$.

Solution. We know, $y = \sin x$ could be plotted as,





(i) $\boldsymbol{f(x)}$ **transforms to** $\{af(x)\}$ **where** $\boldsymbol{a > 1}$

ie, $f(x) \to a\ f(x), a > 1$

Stretch the graph of $f(x)$ 'a' times along y-axis.

(ii) $\boldsymbol{f(x) \to \dfrac{1}{a} f(x), a > 1}$

Shrink the graph of $f(x)$ 'a' times along y-axis.

Illustration 6 Plot $y = \sin x$ and $y = 2\sin x$.

Solution. We know, $y = \sin x$ could be plotted as :

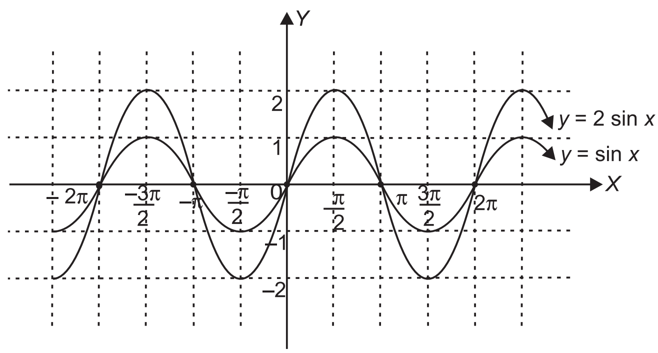
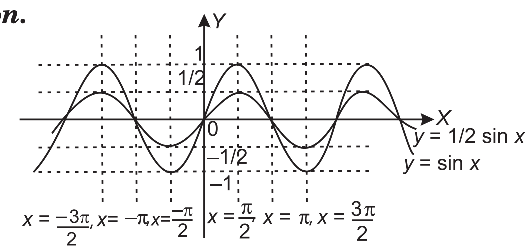


Illustration 7 Plot $y = \sin x$ and $y = \dfrac{1}{2}\sin x$.

Solution.



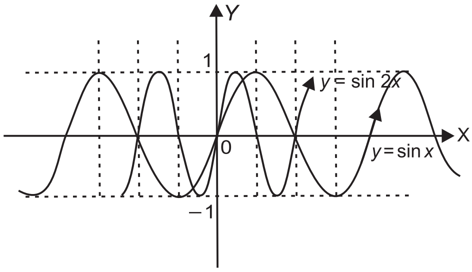
(i) $\boldsymbol{f(x)}$ **transforms to** $\boldsymbol{f(ax), a > 1}$

ie, $f(x) \to f(ax), a > 1$

Shrink the graph of $f(x)$ 'a' times along x-axis.

Illustration 8 Plot $y = \sin x$ and $y = \sin 2x$.

Solution.



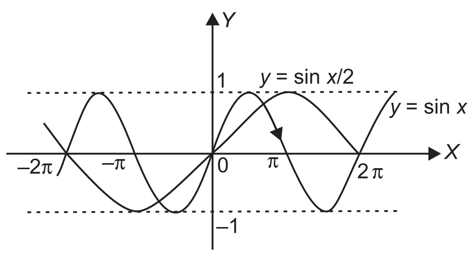
(ii) $\boldsymbol{f(x)}$ **transforms to** $\boldsymbol{f\left(\frac{x}{a}\right), a > 1}$

ie, $$f(x) \to f\left(\frac{x}{a}\right), a > 1$$

Stretch the graph of $f(x)$ 'a' times along x-axis.

Illustration 9 Plot $y = \sin x$ and $y = \sin \frac{x}{2}$.

Solution. We know, $y = \sin x$ could be plotted as :



To draw graph of $y = f(-x)$ when $f(x)$ is given

ie, $$f(x) \to f(-x)$$

To draw $y = f(-x)$, take the image of the curve $y = f(x)$ in the y-axis as plane mirror. Or turn the graph of $f(x)$ by 180° about y-axis.

Illustration 10 Draw the graph of $y = e^{-x}$ when the graph of $y = e^x$ is known.

Solution. As $y = e^x$ is given, then $y = e^{-x}$ is the plane image in y-axis as plane mirror thus, it can be drawn as shown in the figure.

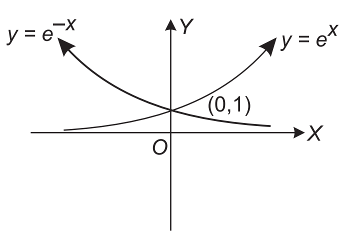
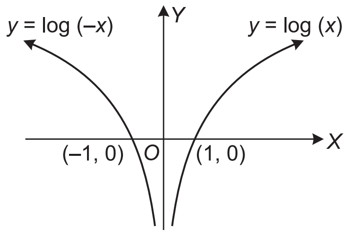


Illustration 11 Draw graph of $y = \log(-x)$ when the graph of $y = \log(x)$ is given.

Solution. As $y = \log(x)$ is given and $y = \log(-x)$ is the image in y-axis as plane mirror. Thus, it can be drawn as shown in figure.



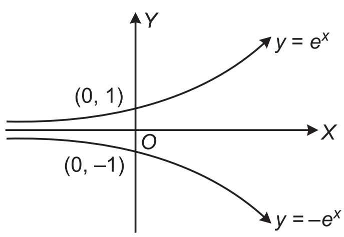
To draw $y = -f(x)$ when $y = f(x)$ is given

ie, $f(x) \to -f(x)$

To draw $y = -f(x)$ take image of $f(x)$ in the x-axis as plane mirror.
Or turn the graph of $f(x)$ by 180° about x-axis.

Illustration 12 Draw the graph of $y = -e^x$ when the graph of $y = e^x$ is known.

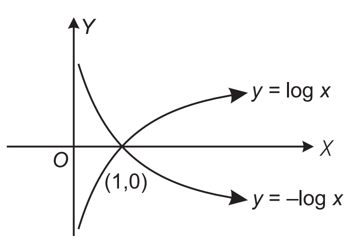
Solution. As $y = e^x$ is given, then $y = -e^x$ is the image of $f(x)$ in the x-axis as plane mirror.



Thus, it can be plotted as shown in figure.

Illustration 13 Draw the graph of $y = -\log(x)$ when the graph of $y = \log x$ is known.

Solution. As $y = \log x$ is given, then $y = -\log x$ is the image of $f(x)$ in the x-axis as plane mirror.



Thus, it can be drawn as shown in figure.

To draw $y = |f(x)|$, when $y = f(x)$ is given

ie, $$|f(x)| = \begin{cases} f(x), & \text{if } f(x) \geq 0 \\ -f(x), & \text{if } f(x) \leq 0 \end{cases}$$

Hence, $y = |f(x)|$ is drawn in two steps :

(a) In the step I, leave the positive part of $f(x)$, (*ie,* the part of $f(x)$ above x-axis) as it is.

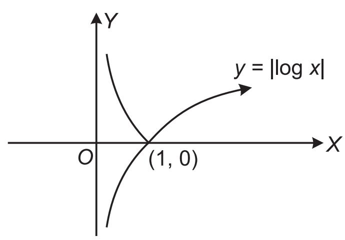
(b) In the step II, take the image of negative part of $f(x)$, (*ie*, the part of $f(x)$ below x-axis) in the x-axis as plane mirror.
Or take the mirror image (in x-axis) of the portion of the graph of $f(x)$ which lies below x-axis.
Or turn the portion of the graph of $f(x)$ lying below x-axis by 180° about x-axis.

Illustration 14 Draw the graph of $y = |\log x|$ when the graph of $y = \log(x)$ is known.

Solution. To draw the graph of $y = |\log x|$, we have to proceed in two steps.

(a) In the first step, leave the positive part of $y = \log x$, *ie*, the part of $y = \log x$ above x-axis as it is.

(b) In the second step, take the image of negative part of $y = \log x$ *ie*, the part below x-axis in the x-axis as plane mirror. Thus, the graph can be drawn as shown.



Point to Consider

The above transformation of graph is very important in discuss explaining the differentiability of $f(x)$.

From above Illustration we could say, $y = \log x$ is differentiable for $(0, \infty)$.

But $y = |\log x|$ is differentiable for $(0, \infty) - \{1\}$ as we have a sharp edge at $x = 1$. (which indicates that it is not differentiable at $x = 1$).

To draw $y = f(|x|)$ when $f(x)$ is given

We know that, $$y = f(x) = \begin{cases} f(x), & \text{if } x \geq 0 \\ f(-x), & \text{if } x \leq 0 \end{cases}$$

Hence, again $f(|x|)$ could be drawn in two steps :

(a) In the step I, leave the graph lying right side of the y-axis, as it is.

(b) In the step II, take the image of $f(x)$ in the y-axis as plane mirror.

The part of $f(x)$ lying left ward of the y-axis (if it exists) is omitted.

Or

On the right of y-axis;

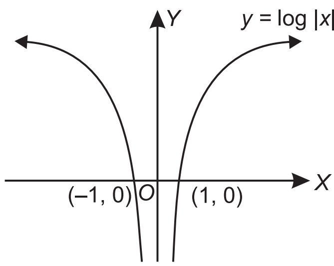
Plot the graph of $f(x)$ as such

On the left of y-axis;

Plot the mirror image of the graph of $f(x)$ (of the portion lying on right of y-axis).

Illustration 15 Draw the graph of $y = \log|x|$ when graph of $y = \log(x)$ is given.

Solution. We know, $y = \log|x|$ could be drawn in two steps



(a) In step I, leave the graph lying right side of y-axis as it is.

(b) In step II, take the image of $f(x)$ in the y-axis as plane mirror.

Thus, it can be plotted as shown in figure.

Point to Consider

$f(|x|)$ is always an even function.

Above mentioned transformations are very useful in explaining continuity and differentiability of functions as,

If $y = \log(x)$, then

(i) $y = \log|x|$ can be plotted as shown in figure below

(ii) $y = |\log|x||$ can be plotted as shown in figure below

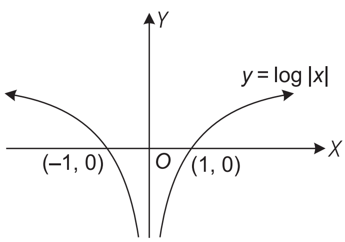


Fig. 4.2

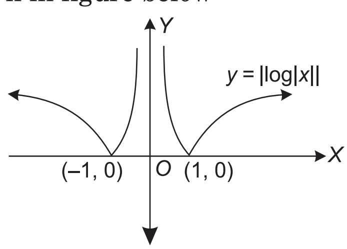


Fig. 4.3

While discussing the case (i), we could say that it is continuous for all $x \in R - \{0\}$ and differentiable for all $x \in R - \{0\}$.

Again, while discussing the case (ii), we could say that it is continuous for all $x \in R - \{0\}$. But differentiable at all $x \in R - \{-1, 0, 1\}$.

(As now it has sharp edges at $\{-1, 1\}$ and is broken at $\{0\}$).

To draw $y = [f(x)]$, when the graph of $y = f(x)$ is given

To draw $y = [f\ (x)]$, where $[\cdot]$ denotes greatest integral function. {*ie*, now y cannot be fraction}.

Here, in order to draw $y = [f\ (x)]$ mark the integer on y-axis. Draw the horizontal lines through integers till they intersect the graph. Draw vertical dotted lines from these intersection points.

Finally, draw horizontal lines parallel to x-axis from any intersection point to the nearest vertical dotted line with blank dot at right end in case $f\ (x)$ increases.

Or

Step 1 Plot $f(x)$.

Step 2 Mark the intervals of unit length with integers as end points on y-axis.

Step 3 Mark the corresponding intervals {with the help of graph of $f(x)$} on x-axis.

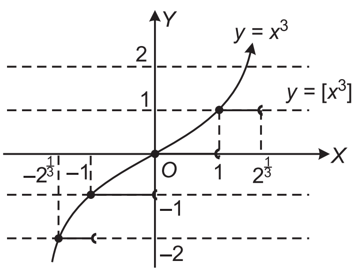
Step 4 Plot the value of $[f(x)]$ for each of the marked intervals.

Illustration 16 Draw the graph of $y = [x^3]$.

Solution. Here, in order to draw the graph of $y = [x^3]$.

(a) Draw horizontal lines through integers till they intersect the graph.

(b) Draw vertical dotted lines from these intersection points.



Finally, draw horizontal lines parallel to x-axis from any intersection point to the nearest vertical dotted line with blank dot at right end.

To draw $y = f([x])$ when the graph of $y = f(x)$ is given

Mark the integers on the x-axis. Draw vertical lines till they intersect the graph of $f\ (x)$. From these intersection points draw horizontal lines (parallel to x-axis) to meet the nearest right vertical line with a blank dot on each nearest right vertical line which can be shown as in the figure.

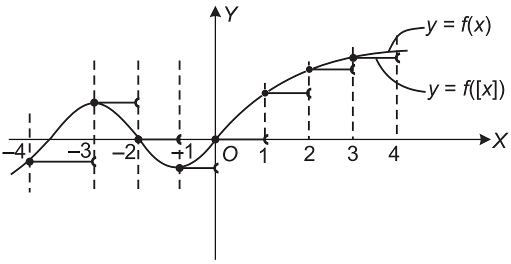


Fig. 4.4

To draw the graph of $|y| = f(x)$ when $y = f(x)$ is given

Clearly, $|y| > 0$, if $f(x) < 0$, graph of $|y| = f(x)$ would not exists.

And if $f(x) \geq 0$, $|y| = f(x)$ would be given as $y = \pm f(x)$.

Hence, the graph of $|y| = f(x)$ exists only in the regions where $f(x)$ is non-negative and will be reflected about x-axis only when $f(x) \geq 0$. Region where $f(x) < 0$ will be neglected.

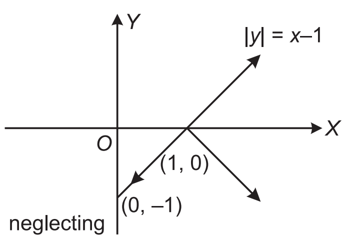
Or

(i) Remove the portion of the graph which lies below x-axis.
($\because$ The equation $|y| = f(x)$ is not satisfied when $f(x)$ is negative.)

(ii) Plot the remaining portion of the graph and also its mirror image in the x-axis. ($\because$ When $f(x) > 0$, then $y = \pm f(x)$)

Illustration 17 Draw graph for $|y| = (x - 1)$.

Solution. We know here, $|y| = (x - 1)$ exists only when $(x - 1)$ is positive.

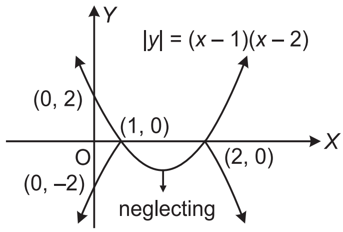


$\therefore$ Neglecting negative values and will be reflected about x-axis.

Thus, it can be drawn as shown.

Illustration 18 Draw the graph for $|y| = (x - 1)(x - 2)$.

Solution. We know, $|y| = (x - 1)(x - 2)$ exists only when $(x - 1)(x - 2) \geq 0$ and neglecting, if $(x - 1)(x - 2) < 0$. Now, $f(x)$ will be reflected about x-axis when $(x - 1)(x - 2) > 0$.



Thus, it can be plotted as shown.

When $f(x)$ and $g(x)$ are two functions and are transformed to their sum

ie, $f(x),\ g(x) \to f(x) + g(x) = h(x)$

There is no direct approach, but we can use the following steps :

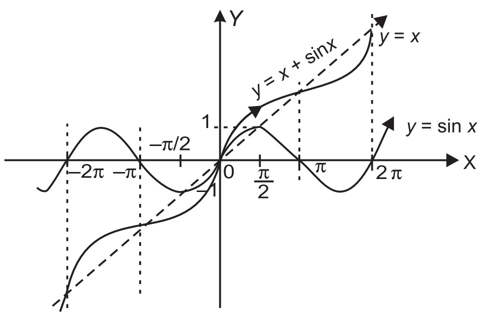
Step 1 Check when $g(x) = 0 \Rightarrow h(x) = f(x)$.

Step 2 When $g(x) > 0$, then $h(x) > f(x)$ *ie,* the graph of $h(x)$ lies above the graph of $f(x)$.

Step 3 When $g(x) < 0$, then $h(x) < f(x)$ *ie,* the graph of $h(x)$ lies below the graph of $f(x)$.

Illustration 19 Plot $y = x + \sin x$.

Solution. Here, $y = f(x) + g(x)$,



where $f(x) = x$ and $g(x) = \sin x$.

Here, $g(x) = 0 \Rightarrow y = x$

Also, when $g(x) > 0 \Rightarrow x + \sin x > x$ and when $g(x) < 0 \Rightarrow x + \sin x < x$

$f(x) \to \dfrac{1}{f(x)} = h(x)$

Here, (i) when $f(x)$ increases, then $h(x)$ decreases.

(ii) when $f(x)$ decreases, then $h(x)$ increases.

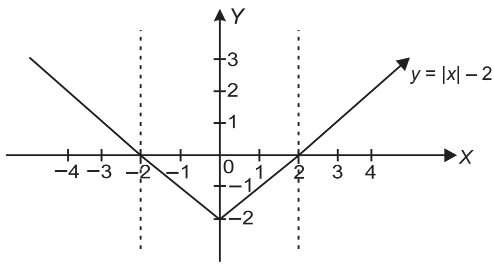
(iii) as $f(x) \to 0, h(x) \to +\infty$

(iv) as $f(x) \to \infty, h(x) \to 0$

(v) when $f(x) = \pm 1$, then $h(x)$ is also equal to ± 1.

Illustration 20 Plot $y = |x| - 2$ and hence $f(x) = \dfrac{1}{|x| - 2}$.

Solution. We know, $y = |x| - 2$ could be plotted as;



Here, (i) $y = |x| - 2$ increases when $x > 0$

$$\Rightarrow \; f(x) = \frac{1}{|x| - 2} \text{ decreases when } x > 0$$

and (ii) $y = |x| - 2$ decreases when $x < 0$

$$\Rightarrow \; f(x) = \frac{1}{|x| - 2} \text{ increases when } x < 0$$

(iii) As, $y \to 0 \;\Rightarrow x = \pm 2$

$\Rightarrow \; f(x) \to \infty$ as $x = \pm 2$

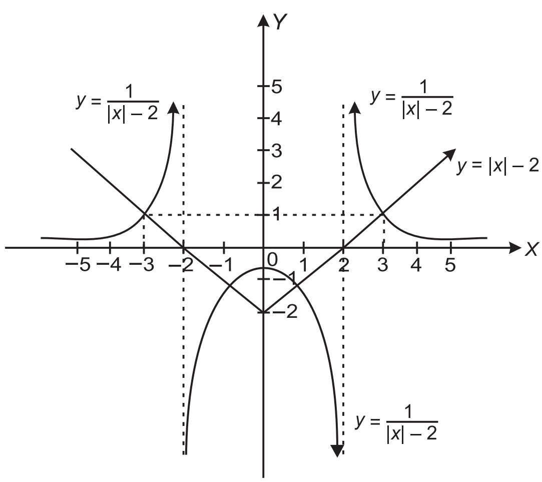
(iv) As, $f(x) \to 0$

ie, $\dfrac{1}{|x| - 2} \to 0$ as $x \to \infty$

and when $x \to \infty \Rightarrow y = |x| - 2 \to \infty$

(v) $y = \pm 1 \Rightarrow f(x) = \pm 1$

$\therefore \; f(x)$ could be plotted as;



To draw the graph for $y = f(x) \cdot \sin x$ when graph of $y = f(x)$ is given

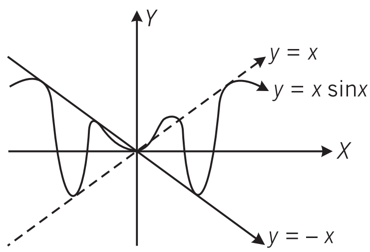
Clearly, $$-f(x) \le f(x) \cdot \sin x \le f(x) \qquad (\text{As } -1 \le \sin x \le 1)$$

Hence, graph for $y = f(x) \cdot \sin x$ would be lying between the graph of $y = f(x)$ and $y = -f(x)$. It amounts to just drawing graph of $\sin x$ in between the graphs of $y = \pm f(x)$.

Illustration 21 Draw graph of $y = x \sin x$.

Solution. Here, $y = x \sin x$

$$\Rightarrow \qquad -x \le y \le x \qquad (\text{As } -1 \le \sin x \le 1)$$



Thus, it can be plotted as shown.

Point to Consider

In case $f(x)$ is even/odd function, then $f(x) \sin x$ would become odd/even function and we have to pay proper attention to the symmetry of $f(x) \sin x$.

Similar treatment can be given to $y = f(x) \cos x$.

When $f(x)$ and $g(x)$ are given, then find

(i) $\boldsymbol{h(x) = \max\{f(x), g(x)\}}$

$$\because \qquad h(x) = \begin{cases} f(x), & \text{when } f(x) > g(x) \\ g(x), & \text{when } f(x) < g(x) \end{cases}$$

$\therefore$ Sketch $f(x)$ when its graph is above the graph of $g(x)$ and sketch $g(x)$ when its graph is above the graph of $f(x)$.

(ii) $\boldsymbol{h(x) = \min\{f(x), g(x)\}}$

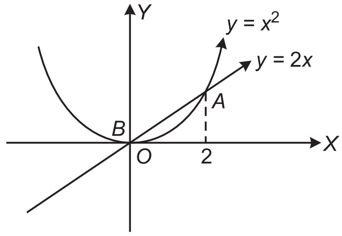
$$\because \qquad h(x) = \begin{cases} f(x), & \text{when } f(x) < g(x) \\ g(x), & \text{when } g(x) < f(x) \end{cases}$$

$\therefore$ Sketch $f(x)$ when its graph is lower and otherwise sketch $g(x)$.

Illustration 22 Draw graph for $y = \max\{2x, x^2\}$ and discuss the continuity and differentiability.

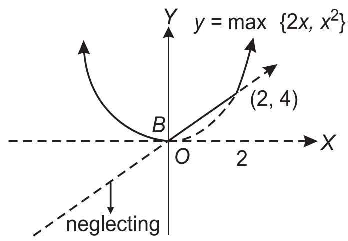
Solution. Here, to draw, $y = \max\{2x, x^2\}$

First, plot $y = 2x$ and $y = x^2$ on graph and put $2x = x^2 \Rightarrow x = 0, 2$ (*ie*, their point of intersection).



Now, since $y = \max\{2x, x^2\}$ we have to neglect the curve below the point of intersections thus, the required graph is, as shown below.

Thus, from the given graph $y = \max\{2x, x^2\}$ we can say $y = \max\{2x, x^2\}$ is continuous for all $x \in R$.



But $y = \max\{2x, x^2\}$ is differentiable for all $x \in R - \{0, 2\}$.

Point to Consider

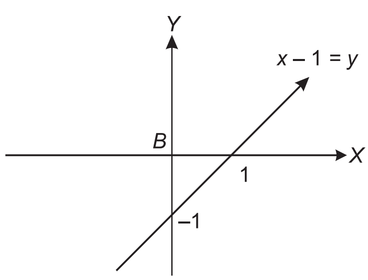
One must remember the formula we can write,

$$\max\{f(x), g(x)\} = \frac{f(x) + g(x)}{2} + \left|\frac{f(x) - g(x)}{2}\right|$$

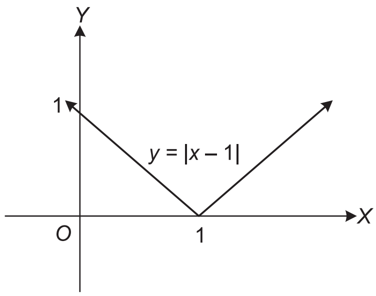
$$\min\{f(x), g(x)\} = \frac{f(x) + g(x)}{2} - \left|\frac{f(x) - g(x)}{2}\right|$$

Illustration 23 Draw the graph for $y = |2 - |x - 1||$.

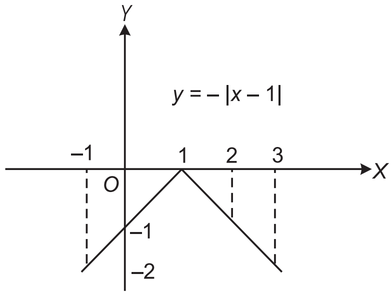
Solution. Here, $y = (x - 1)$ can be plotted as shown below.



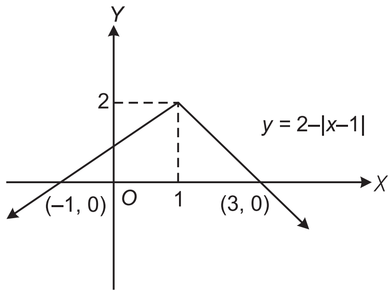
$\Rightarrow$ $y = |x - 1|$ can be plotted as shown below.



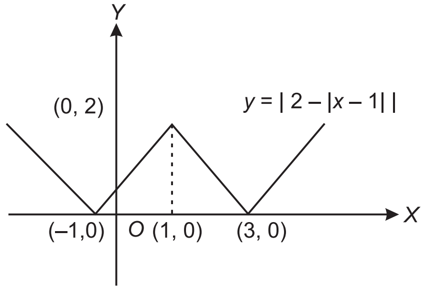
$\Rightarrow$ $y = -|x - 1|$ can be plotted as shown below.



$\Rightarrow$ $y = 2 - |x - 1|$ can be plotted as shown below.



Thus, $y = |2 - |x - 1||$ can be plotted as shown below.



Point to Consider

From above figure we could say $y = |2 - |x - 1||$ is not differentiable at $x = \{-1, 1, 3\}$ as sharp edges at $x = -1, 1, 3$.

Illustration 24 Let $h(x) = \min\{x; x^2\}$ for every real number of x. Then, which one of the following is true? **[IIT JEE 1998]**

(a) h is not continuous for all x

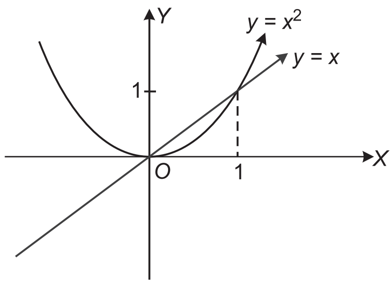
(b) h is differentiable for all x

(c) $h'(x) = 1$, for all x

(d) h is not differentiable at two values of x

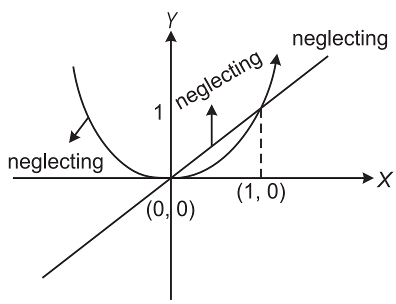
Solution. Here, $h(x) = \min\{x; x^2\}$ can be drawn on graph in two steps.

(a) Draw the graph of $y = x$ and $y = x^2$, also find their point of intersection.



ie, $x = x^2 \Rightarrow x = 0, 1$

(b) To find $h(x) = \min\{x; x^2\}$ neglecting the graph above the point of intersection, we get



Thus, from the given graph,

$$h(x) = \begin{cases} x, & x \le 0 \text{ or } x \ge 1 \\ x^2, & 0 \le x \le 1 \end{cases}$$

which shows $h(x)$ is continuous for all x. But not differentiable at $x = \{0, 1\}$.

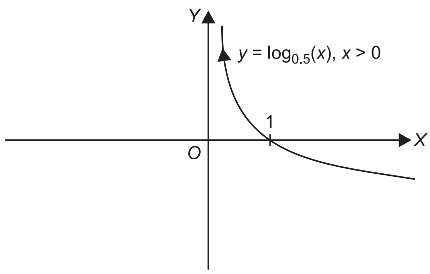
Thus, $h(x)$ is not differentiable at two values of x.

Hence, (d) is the correct answer.

Worked Examples

Example 1 Sketch the graph of $y = \log_{0.5}|x|$.

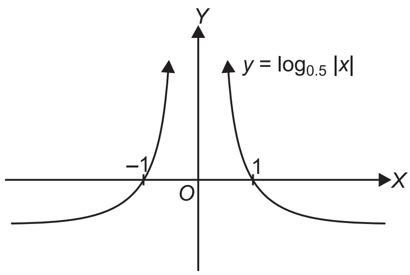
Solution. As we know, $y = \log_{0.5} x$ is a decreasing graph given as;



and $$f(x) \to f(|x|)$$

$\Rightarrow$ Taking images about y-axis for $x > 0$.

$\therefore$ $y = \log_{0.5}|x|$ could be plotted as;



Example 2 Sketch the graph for, $y = \left|\left|\frac{1}{x}\right| - 3\right|$

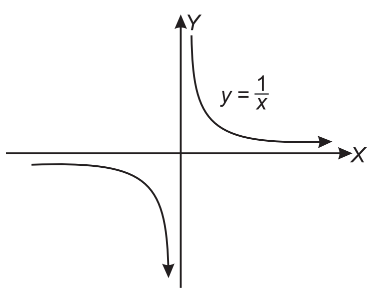
Solution. Here, we follow certain steps to plot

$$y = \left|\left|\frac{1}{x}\right| - 3\right| \text{ as first we plot } \frac{1}{x}$$

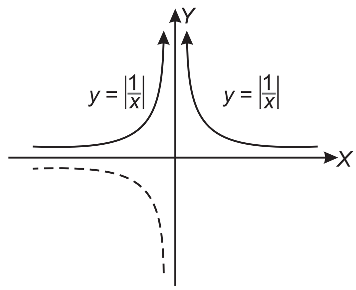
and then successively $\left|\frac{1}{x}\right|$, $\left|\frac{1}{x}\right| - 3$, $\left|\left|\frac{1}{x}\right| - 3\right|$

ie,

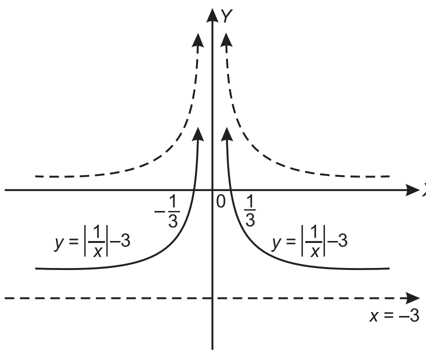
(i)



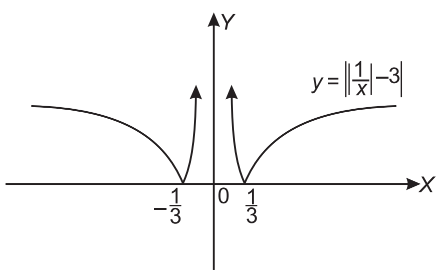
(ii) $\dfrac{1}{x} \to \left|\dfrac{1}{x}\right|$ (*ie,* taking images about x-axis for –ve values of y)



(iii) $\left|\dfrac{1}{x}\right| \to \left|\dfrac{1}{x}\right| - 3$ *ie,* shifting 3 units below



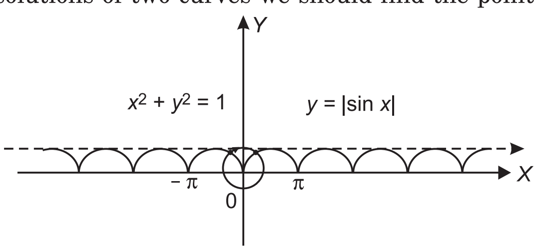
(iv) $\left|\dfrac{1}{x}\right| - 3 \to \left|\left|\dfrac{1}{x}\right| - 3\right|$ (*ie,* taking images about x-axis for –ve values of y)



Example 3 Find the number of solutions of the equations $y = |\sin x|$ and $x^2 + y^2 = 1$.

Solution. To find the number of solutions of two curves we should find the point of intersection of two curves.

As we know, $x^2 + y^2 = 1$ is a circle and $y = |\sin x|$ is the image of –ve values of $y = \sin x$ about x-axis. Thus, we can plot them as;



which shows the two curves intersects at two points.

$\therefore$ Number of solutions is 2.

Example 4 Find the number of solutions of $4\{x\} = x + [x]$.

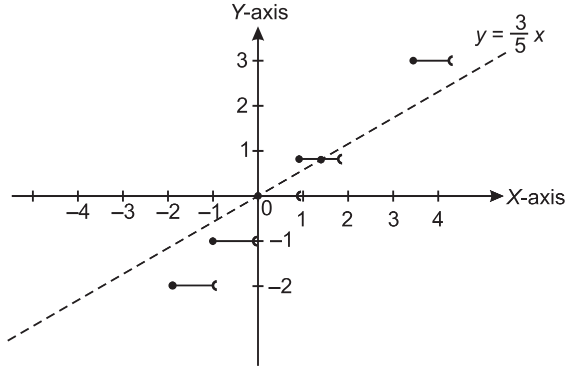
Solution. Here, $4\{x\} = x + [x]$

$$\Rightarrow \quad 4(x - [x]) = x + [x] \quad \Rightarrow \quad 4x = x + 5[x]$$

$$\Rightarrow \quad 3x = 5[x] \quad \Rightarrow \quad [x] = \frac{3}{5}x$$

To find their solution we plot the graph of both $y = [x]$ and $y = \frac{3}{5}x$

ie,



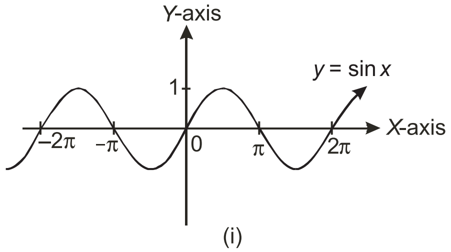
ie, The two graphs intersects when $[x] = 0$ and 1

$$\Rightarrow \quad x = 0 \quad \text{and} \quad x = \frac{5}{3}$$

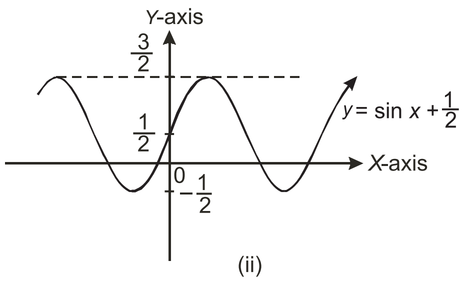
Example 5 Sketch the graph of $\left|\sin x + \frac{1}{2}\right|$.

Solution. We follow certain steps to plot $\left|\sin x + \frac{1}{2}\right|$

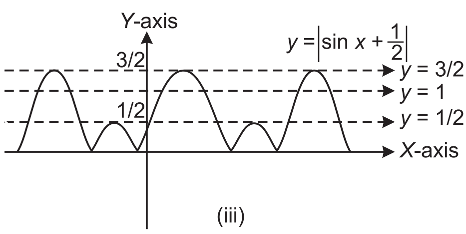
ie,
$$\underset{\text{(i)}}{\sin x} \rightarrow \underset{\text{(ii)}}{\sin x + \frac{1}{2}} \rightarrow \underset{\text{(iii)}}{\left|\sin x + \frac{1}{2}\right|}$$



(i)



(ii)



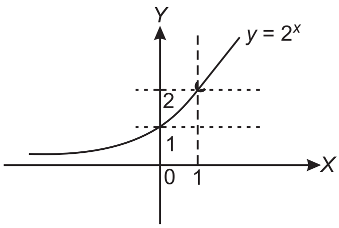
(iii)

Point to Consider

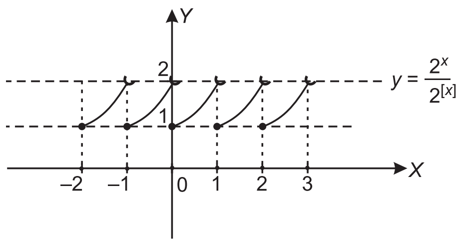
Plotting graph of $f(x-[x])$:
Graph of $f(x-[x])$ can be obtained from the graph of $f(x)$ by the following rule.
"Retain the graph of $f(x)$ for values of x lying between interval [0, 1). Now, it can be repeated for rest of points. New obtained function is periodic with period 1."

Example 6 Sketch the graph of $y = \dfrac{2^x}{2^{[x]}}$.

Solution. As we know, 2^x is exponential function and we want to transform it to $2^{x-[x]}$, it retains, the graph for $x \in [0, 1)$ and repeat for rest points.

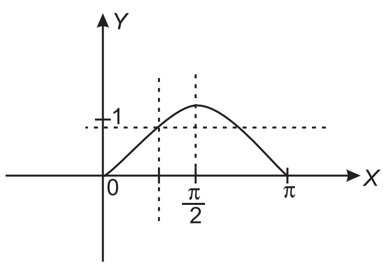


Here, to retain graph between $x \in [0, 1)$, so we get

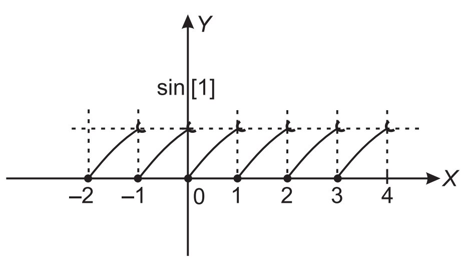


Example 7 Sketch the region for $y = \sin(x-[x])$.

Solution. We know, $y = \sin x$ is the periodic function. So, to plot the graph between $x \in [0, 1)$ and repeat for all values of x.

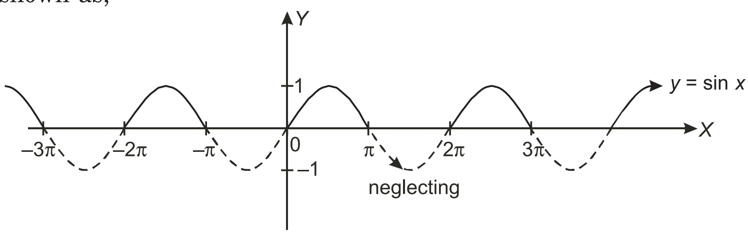


ie, To retain the graph between $x \in [0, 1)$.

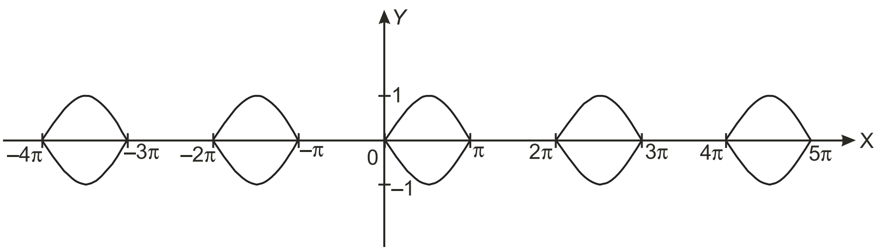


Example 8 Sketch the region for $|y| = \sin x$.

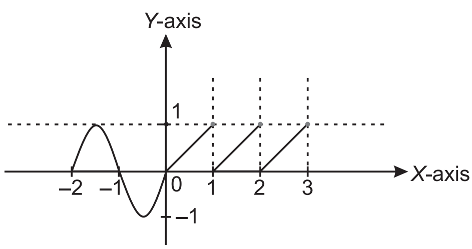
Solution. We know, $y = \sin x$ is a periodic function. So, to plot $|y| = \sin x$ neglect all the points for which y is negative and take the image for +ve values of y about x-axis shown as;



Now, plotting, $|y| = \sin x$.



Example 9 Consider the following function f whose graph is given below.



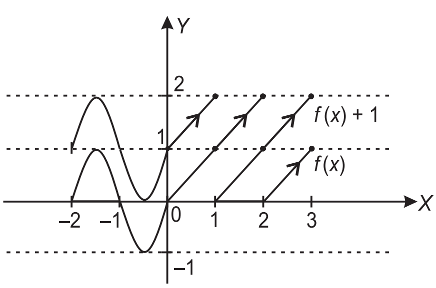
Draw the graph of following functions :

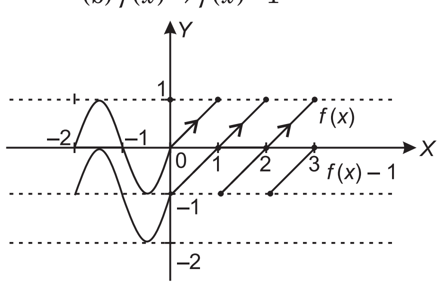
(a) $f(x)+1$ (b) $f(x)-1$ (c) $-f(x)$ (d) $|f(x)|$

(e) $f(-x)$ (f) $f(|x|)$ (g) $2f(x)$ (h) $f(2x)$

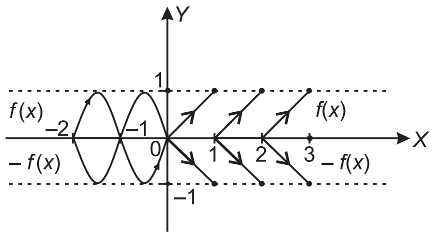
(i) $[f(x)]$ (j) $f(x-[x])$

Solution. (a) $f(x) \to f(x)+1$ (b) $f(x) \to f(x)-1$

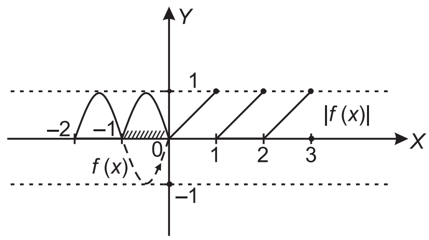




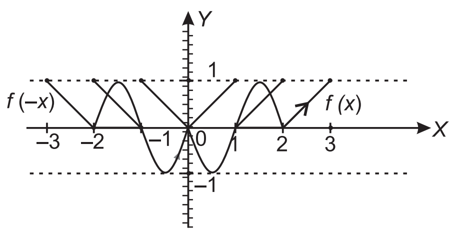
(c) $f(x) \to -f(x)$



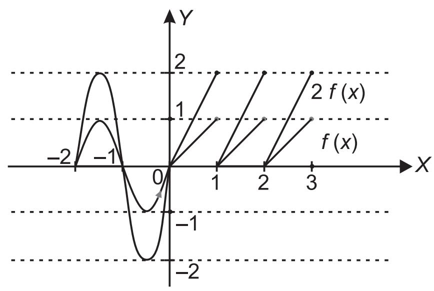
(d) $f(x) \to |f(x)|$, taking image for –ve values of y about x-axis.



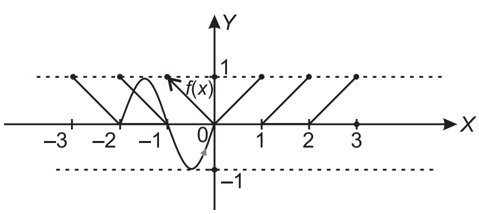
(e) $f(x) \to f(-x)$, taking images of x about y-axis.



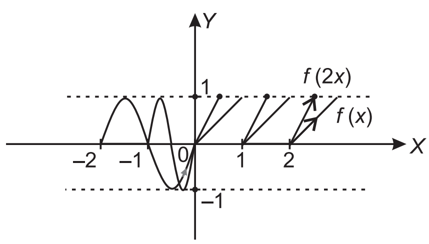
(f) $f(x) \to 2f(x)$



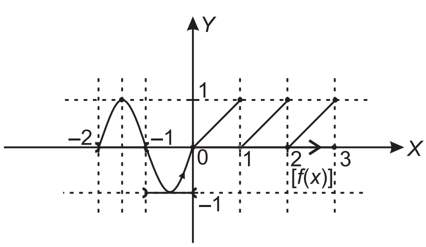
(g) $f(x) \to f(|x|)$, neglecting the graph for –ve values of and taking image for +ve values of x about y-axis.



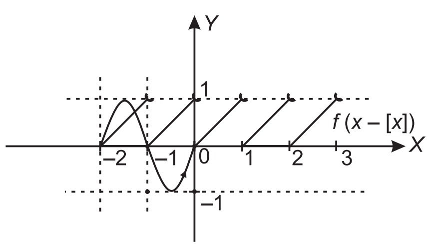
(h) $f(x) \to f(2x)$



(i) $f(x) \to [f(x)]$



(j) $f(x) \to f(x-[x])$



Point to Consider

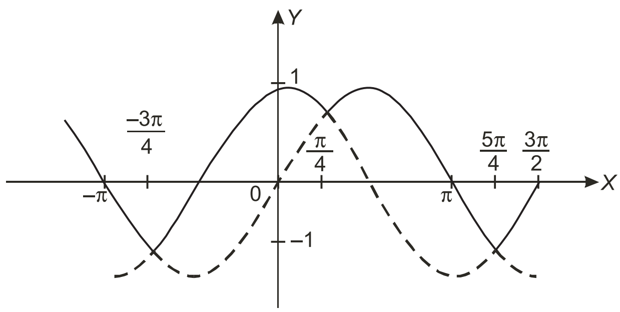
To draw the graph of functions of the form $y = \max\{f(x), g(x)\}$ or $y = \min\{f(x), g(x)\}$.

We first draw the graphs of both the functions $f(x)$ and $g(x)$ and their points of intersection. Then, we find any two consecutive points of intersection. In between these points either $f(x) > g(x)$ or $f(x) < g(x)$ then, in order to $\max\{f(x), g(x)\}$ we take those segments of $f(x)$ for which $f(x) > g(x)$, between any two consecutive points of intersection of $f(x)$ and $g(x)$.

Similarly, in order to $\min f\{(x), g(x)\}$ we take those segments of $f(x)$ for which $f(x) < g(x)$, between any two consecutive points of intersection of $f(x)$ and $g(x)$.

Example 10 Sketch the graph of $y = \max(\sin x, \cos x)$, $\forall\ x \in \left(-\pi, \dfrac{3\pi}{2}\right)$.

Solution. First plot both $y = \sin x$ and $y = \cos x$ by a dotted curve as can be seen from the graph in the interval $\left(-\pi, \dfrac{3\pi}{2}\right)$ and then darken those dotted line for which $f(x) > g(x)$ or $g(x) > f(x)$.

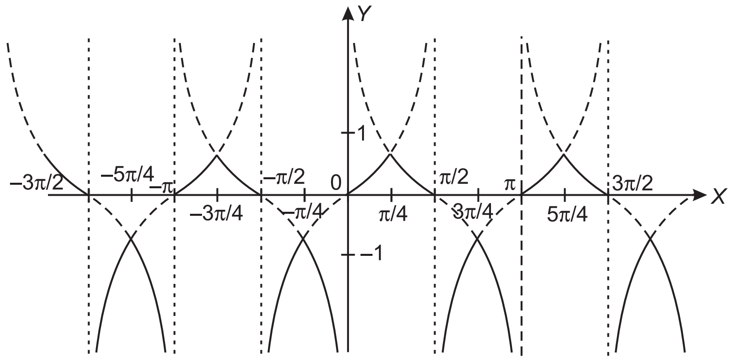


As from the above graph,

$$\max(\sin x, \cos x) = \begin{cases} \sin x, & -\pi < x \le -\dfrac{3\pi}{4} \\ \cos x, & -\dfrac{3\pi}{4} \le x \le \dfrac{\pi}{4} \\ \sin x, & \dfrac{\pi}{4} \le x \le \dfrac{5\pi}{4} \\ \cos x, & \dfrac{5\pi}{4} \le x \le \dfrac{3\pi}{2} \end{cases}$$

Example 11 Sketch the graph for $y = \min\{\tan x, \cot x\}$.

Solution. First plot both $f(x) = \tan x$ and $g(x) = \cot x$ by a dotted curve as can be seen from the graph and then darken those dotted lines for which $f(x) < g(x)$ and $g(x) < f(x)$.

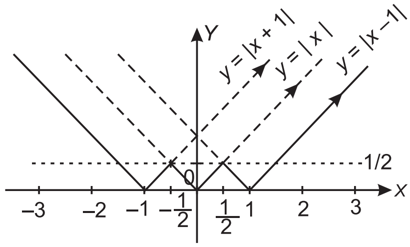


As from the graph we have,

$$\min\{\tan x, \cot x\} = \begin{cases} \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ \tan x, & -\frac{\pi}{2} < x \le -\frac{\pi}{4} \\ \cot x, & -\frac{\pi}{4} \le x < 0 \\ \tan x, & 0 \le x \le \frac{\pi}{4} \\ \cot x, & \frac{\pi}{4} \le x \le \frac{\pi}{2} \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \end{cases}$$

Example 12 Sketch the graph of $y = \min\{|x|, |x-1|, |x+1|\}$.

Solution. First plot the graph for, $y = |x|$, $y = |x-1|$ and $y = |x+1|$ by a dotted curve as can be seen from the graph and then darken those dotted lines for which $|x| < \{|x-1|, |x+1|\}$, $|x-1| < \{|x|, |x+1|\}$ and $|x+1| < \{|x|, |x-1|\}$.



As from the above graph,

$$\min\{|x|, |x-1|, |x+1|\} = \begin{cases} -(x+1), & x \le -1 \\ (x+1), & -1 \le x \le -\frac{1}{2} \\ -(x), & -\frac{1}{2} \le x \le 0 \\ x, & 0 \le x \le \frac{1}{2} \\ -(x-1), & \frac{1}{2} \le x \le 1 \\ (x-1), & 1 \le x \end{cases}$$

Point to Consider

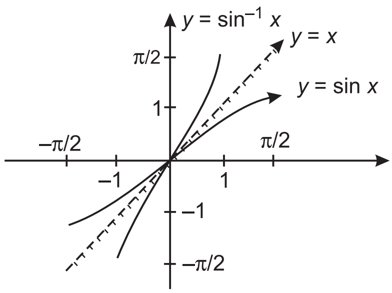
To plot the graph of $f^{-1}(x)$, take reflection of $f(x)$ in $y = x$ line as a mirror.

Example 13 Sketch the graph of $y = \sin^{-1} x$, $\forall\, x \in [-1, 1]$.

Solution. As, $\quad -1 \le x \le 1$

$$\Rightarrow \quad -\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2} \quad \Rightarrow \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

ie, $\sin^{-1} x$ is the reflection of $\sin x$ about $y = x$ (as a mirror) when $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

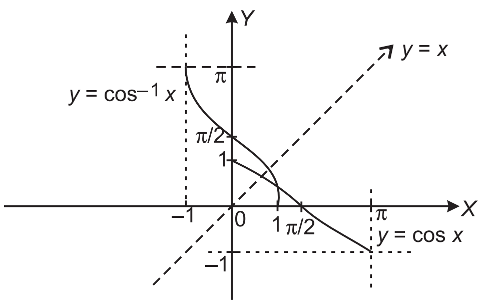


Example 14 Sketch the graph for $y = \cos^{-1} x$, $\forall\, x \in [-1, 1]$.

Solution. As, $\quad -1 \le x \le 1$

$$\Rightarrow \quad 0 \le \cos^{-1} x \le \pi \quad \Rightarrow \quad 0 \le y \le \pi$$

$\therefore$ $\cos^{-1} x$ is reflection of the graph of $\cos x$, $0 \le x \le \pi$ in $y = x$ line as a mirror.

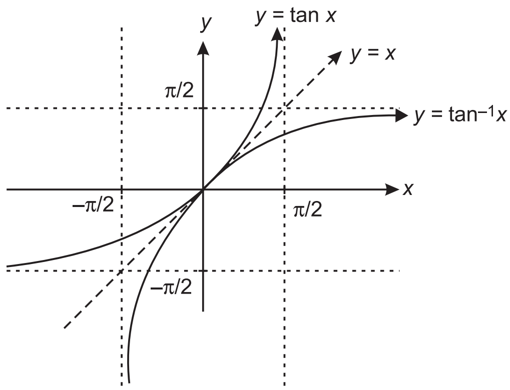


Example 15 Sketch the graph for $y = \tan^{-1} x$, $\forall\, x \in R$.

Solution. As, $x \in R$

$$\Rightarrow \quad \frac{-\pi}{2} < \tan^{-1} x < \frac{\pi}{2} \quad \Rightarrow \quad \frac{-\pi}{2} < y < \frac{\pi}{2}$$

$\therefore$ $\tan^{-1} x$ is the reflection of the graph of $\tan x$, $\frac{-\pi}{2} < x < \frac{\pi}{2}$ in $y = x$ as a mirror.



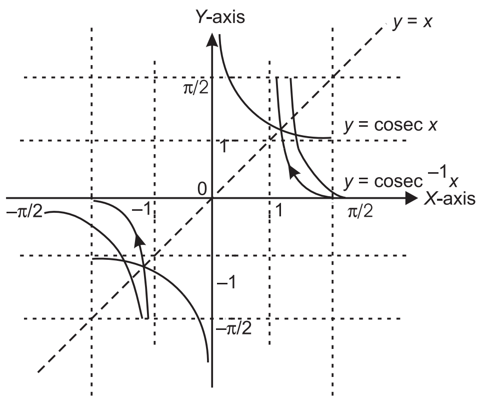
Example 16 Sketch the graph for $y = \operatorname{cosec}^{-1}x,\ \forall\ x \in R-(-1, 1)$.

Solution. As, $x \in R-(-1, 1)$

$$\Rightarrow \qquad x \le -1 \qquad \text{or} \qquad x \ge 1$$

$$\Rightarrow \qquad \frac{-\pi}{2} \le y < 0 \qquad \text{or} \qquad 0 < y \le \frac{\pi}{2}$$

$\therefore$ $\operatorname{cosec}^{-1}x$ is the reflection of the graph of $\operatorname{cosec} x$, $\frac{-\pi}{2} \le x < 0$ and $0 < x \le \frac{\pi}{2}$ in $y = x$ as a mirror.

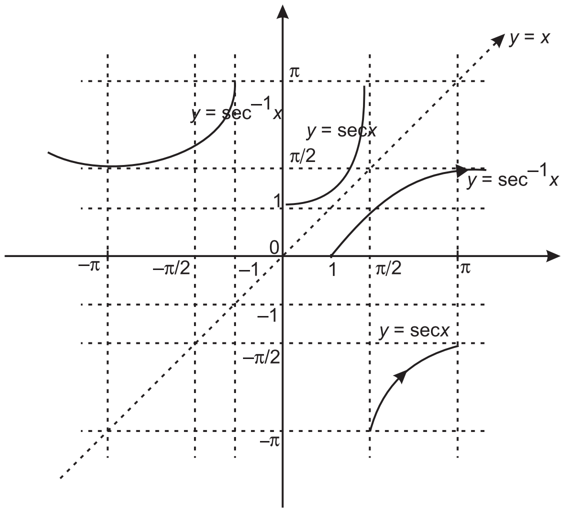


Example 17 Sketch the graph for $y = \sec^{-1} x,\ \forall\ x \in R-(-1, 1)$.

Solution. As, $x \in R-(-1, 1)$

$$\Rightarrow \qquad 0 \le y < \frac{\pi}{2}, \frac{\pi}{2} < y \le \pi$$

$\therefore$ $\sec^{-1} x$ is the reflection of the graph of $\sec x$, $0 \le x < \frac{\pi}{2}$, $\frac{\pi}{2} < x \le \pi$ in $y = x$ as a mirror.

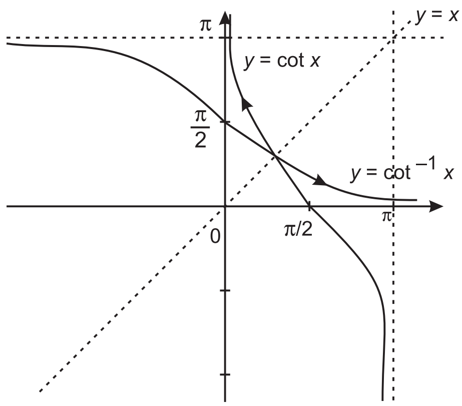


Example 18 Sketch the graph of $y = \cot^{-1} x$, $\forall\, x \in (0, \pi)$.

Solution. As, $\quad x \in (0, \pi)$

$\Rightarrow \qquad -\infty < y < \infty$

$\therefore$ $\cot^{-1} x$ is the reflection of the graph of $\cot x$, $-\infty < x < \infty$ in $y = x$ as a mirror.



Example 19 Sketch the graph for $y = \sin^{-1}(\sin x)$.

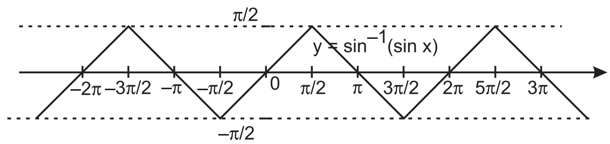
Solution. As, $y = \sin^{-1}(\sin x)$ is periodic with period 2π.

$\therefore$ To draw this graph we should draw the graph for one interval of length 2π and repeat it for entire values of x.

As we know,

$$\sin^{-1}(\sin x) = \begin{cases} (x), & \dfrac{-\pi}{2} \le x \le \dfrac{\pi}{2} \\ (\pi - x), & \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2} \text{ or } \pi - \dfrac{\pi}{2} \ge \pi - x \ge \pi - \dfrac{3\pi}{2} \\ & \text{or } -\dfrac{\pi}{2} \le \pi - x \le \dfrac{\pi}{2} \end{cases}$$

Thus, it has been defined for $\frac{-\pi}{2} \le x \le \frac{3\pi}{2}$, that has length 2π. So, its graph could be plotted as;



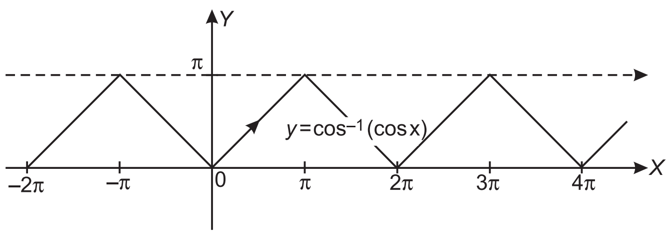
Example 20 Sketch the graph for $y = \cos^{-1}(\cos x)$.

Solution. As, $y = \cos^{-1}(\cos x)$ is periodic with period 2π.

$\therefore$ To draw this graph we should draw the graph for one interval of length 2π and repeat it for entire values of x of length 2π.

As we know, $\cos^{-1}(\cos x) = \begin{cases} x, & 0 \le x \le \pi \\ (2\pi - x), & \pi < x \le 2\pi \text{ or } 0 < 2\pi - x < \pi \end{cases}$

Thus, it has been defined for $0 < x < 2\pi$, that has length 2π. So, its graph could be plotted as;



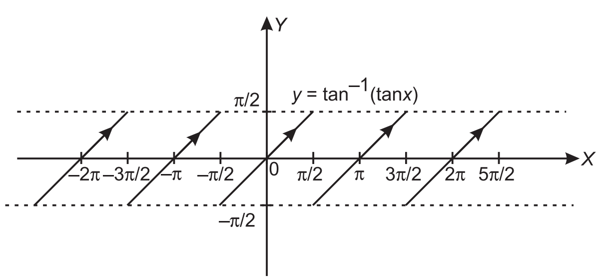
Example 21 Sketch the graph for $y = \tan^{-1}(\tan x)$.

Solution. As, $y = \tan^{-1}(\tan x)$ is periodic with period π.

$\therefore$ To draw this graph we should draw the graph for one interval of length π and repeat for entire values of x.

As we know, $\tan^{-1}(\tan x) = \begin{cases} x, & \frac{-\pi}{2} < x < \frac{\pi}{2} \end{cases}$

Thus, it has been defined for $\frac{-\pi}{2} < x < \frac{\pi}{2}$ that has length π. So, its graph could be plotted as;



Example 22 Find the values of x graphically which satisfy $\left|\frac{x^2}{x-1}\right| \le 1$.

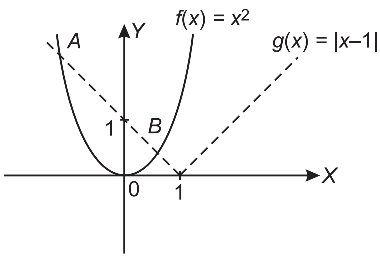
Solution. As, $\left|\frac{x^2}{x-1}\right| \le 1$

$\Rightarrow \quad |x^2| \le |x-1|, \forall\, x \in R-\{1\}$

or $\quad x^2 \le |x-1|, \forall\, x \in R-\{1\}$

Thus, to find the points for which $f(x) = x^2$ is less than or equal to $g(x) = |x-1|$.

Where the two functions $f(x)$ and $g(x)$ could be plotted as;



Thus, from the above graph $f(x) \le g(x)$ when $x \in [A, B]$ where A and B are point of intersection of x^2 and $1-x$.

$\therefore$ Solving $x^2 = 1-x$, we get $x = \frac{-1-\sqrt{5}}{2} = A$

and $\quad x = \frac{-1+\sqrt{5}}{2} = B$

$\therefore \quad \left|\frac{x^2}{x-1}\right| \le 1$ is satisfied, $\forall\, x \in \left[\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right]$

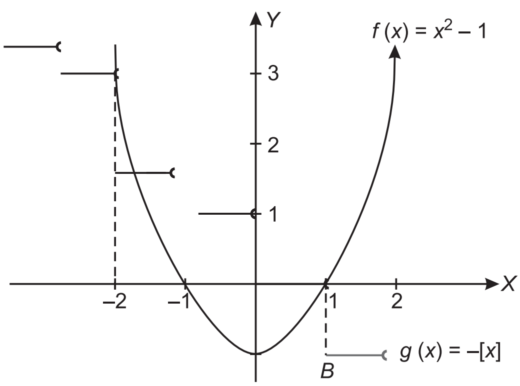
Example 23 Find the values of x graphically satisfying $[x]-1+x^2 \ge 0$; where [.] denotes the greatest integer function.

Solution. As, $\quad [x]-1+x^2 \ge 0$

$\Rightarrow \quad x^2 - 1 \ge -[x]$

Thus, to find the points for which $f(x) = x^2 - 1$ is greater than or equal to $g(x) = -[x]$.

Where the two functions $f(x)$ and $g(x)$ could be plotted as;



Thus, from the above graph $f(x) \ge g(x)$ when $x \in (-\infty, A] \cup [B, \infty)$, where A is point of intersection of x^2-1 and $-[x]$ when $-[x] = +2$.

$\therefore \quad x^2 - 1 = +2$ *ie,* $x = -\sqrt{3} = A$ and $B = 1$

$\therefore \quad [x]-1+x^2 \ge 0$ is satisfied, $\forall\, x \in (-\infty, -\sqrt{3}\,] \cup [1, \infty)$.

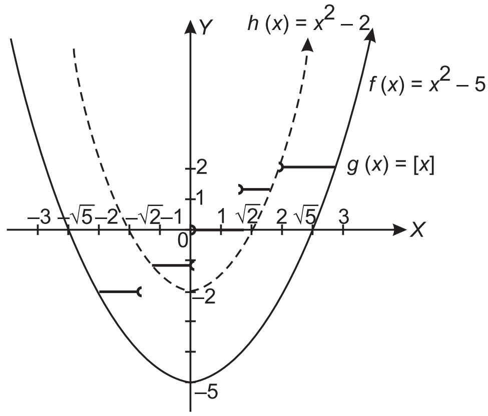
Example 24 Find the values of x graphically which satisfy;

$-1 \le [x] - x^2 + 4 \le 2$, where [.] denotes the greatest integer function.

Solution. As, $-1 \le [x] - x^2 + 4 \le 2 \quad \Rightarrow \quad x^2 - 5 \le [x] \le x^2 - 2$

Thus, to find the points for which $f(x) = x^2 - 5$ is less than or equal to $g(x) = [x]$ and $g(x) = [x]$ is less than or equal to $h(x) = x^2 - 2$.

Where the three functions $f(x)$, $g(x)$ and $h(x)$ could be plotted as;



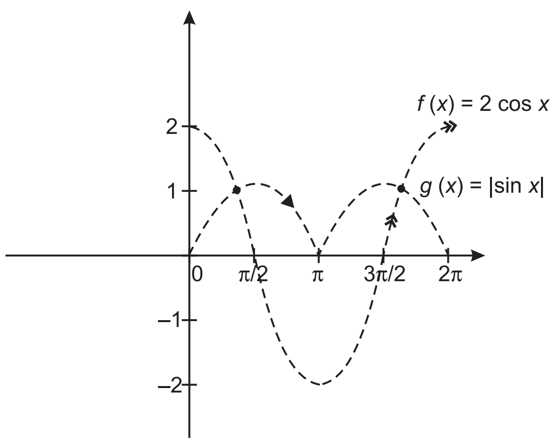
Thus, from the above graph; $x^2 - 5 \le [x] \le x^2 - 2$ when $x \in [A, B] \cup [C, D]$
where A and D is point of intersection of $x^2 - 5 = \pm 2 \Rightarrow x = -\sqrt{3}, \sqrt{7}$ and C is point of intersection of $x^2 - 2 = 1 \Rightarrow x = \sqrt{3}$

$\therefore \qquad -1 \le [x] - x^2 + 4 \le 2$ is satisfied, $x \in [-\sqrt{3}, -1] \cup [\sqrt{3}, \sqrt{7}]$

Example 25 Find the number of solutions of $2\cos x = |\sin x|$ when $x \in [0, 4\pi]$.

Solution. As we know, the graph of both $f(x) = 2\cos x$ and $g(x) = |\sin x|$

$\therefore$ Their point of intersection are number of solutions.



Thus, from the above graph the two functions intersect at two points between $[0, 2\pi]$ and we know that $\cos x$ is periodic with period 2π, so it has same number of solutions for the interval $[2\pi, 4\pi]$.

$\therefore$ Total number of solutions of $2\cos x = |\sin x|$ when $x \in [0, 4\pi]$ is 4.

Example 26 Sketch the curves

(i) $y = \sqrt{x - [x]}$ (ii) $y = [x] + \sqrt{x - [x]}$

(iii) $y = |[x] + \sqrt{x - [x]}|$

(Where $[\cdot]$ denotes the greatest integer function.)

Solution. (i) Here, $0 \le x - [x] < 1$

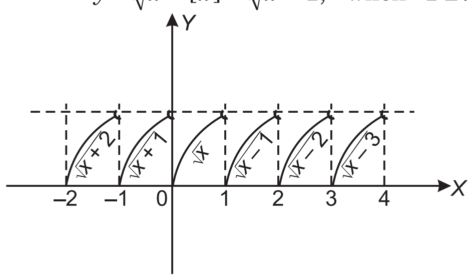
and $x^2 \le x \le \sqrt{x}$

$\therefore$ $x - [x] \le \sqrt{x - [x]}$

$$\Rightarrow \quad y = \sqrt{x - [x]} = \begin{cases} \sqrt{x-1}, & 1 \le x < 2 \\ \sqrt{x}, & 0 \le x < 1 \\ \sqrt{x+1}, & -1 \le x < 0 \text{ and so on.} \end{cases}$$

In general, $y = \sqrt{x - [x]} = \sqrt{x}$, when $0 \le x < 1$

$y = \sqrt{x - [x]} = \sqrt{x - 1}$, when $1 \le x < 2$



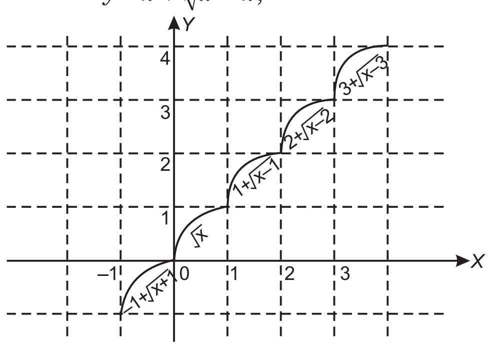
ie, Shifting $\sqrt{x}$ by 1 unit on right side of x-axis. $y = \sqrt{x - [x]} = \sqrt{x - 2}$, when $2 \le x < 3$ Graph of $y = \sqrt{x - [x]}$

ie, Shifting $\sqrt{x - 1}$ by 1 unit on right side of x-axis and so on.

Thus, the graph for $y = \sqrt{x - [x]}$ is shown as in figure.

(ii) Again, $y = [x] + \sqrt{x - [x]}$

$\Rightarrow$ $y = k + \sqrt{x - k}$,

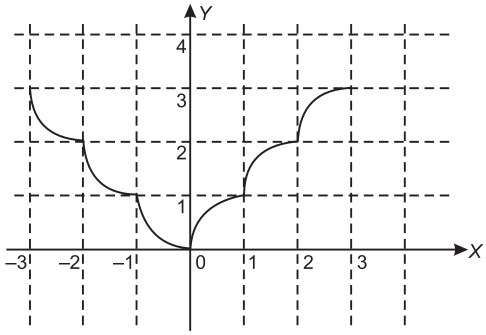


$k \le x < k + 1; k \in$ integer

$$\Rightarrow \quad y = \begin{cases} \sqrt{x}, & 0 \le x < 1 \\ 1 + \sqrt{x-1}, & 1 \le x < 2 \\ 2 + \sqrt{x-2}, & 2 \le x < 3 \text{ and so on.} \end{cases}$$

$\therefore$ Graph for $y = [x] + \sqrt{x - [x]}$

(iii) Graph for $y = |[x] + \sqrt{x - [x]}|$

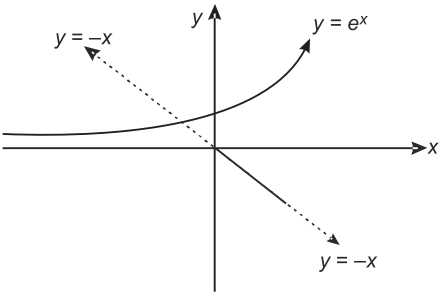


is obtained by reflecting the portion lying below x-axis of the graph of $y = [x] + \sqrt{x - [x]}$.

Example 27 The number of real solutions of the equation $e^x + x = 0$, is

(a) 0 (b) 1 (c) 2 (d) None of these

Solution. It is evident from figure that the curves $y = e^x$ and $y = -x$ intersect exactly at one point. So, the equation $e^x = -x$ or $e^x + x = 0$ has one real solution.



Hence, (b) is the correct answer.

Example 28 The number of real solutions of the equation $\log_a x = |x|, 0 < a < 1$, is

(a) 0 (b) 1 (c) 2 (d) None of these

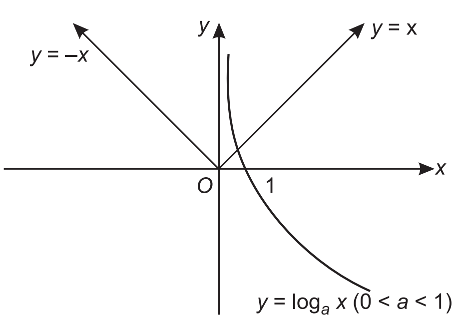
Solution. The number of real solutions of the equation $\log_a x = |x|$ is equal to the number of points of intersection of the curves.

$y = \log_a x$ and $y = |x|$ $(0 < a < 1)$

It is evident from the graph that the two curves intersect at one point only.

$\therefore$ One real solution lying in (0, 1).

Hence, (b) is the correct answer.



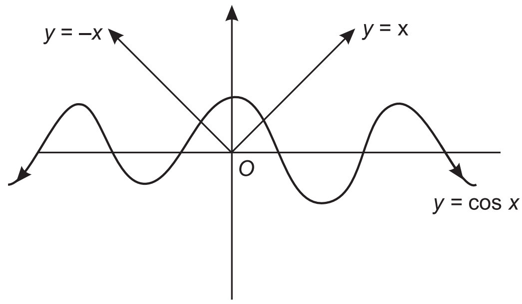
Example 29 The number of solutions of the equation $|x| = \cos x$, is

(a) 0 (b) 1

(c) 2 (d) 3

Solution. The number of real solutions of the equation $|x| = \cos x$ is equal to the number of point of intersection of the curves.

$$y = |x| \quad \text{and} \quad y = \cos x \quad \text{shown as;}$$



It is evident from the graph that the two curves intersect at two points only.

$\therefore$ Two real solutions lying in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Hence, (c) is the correct answer.

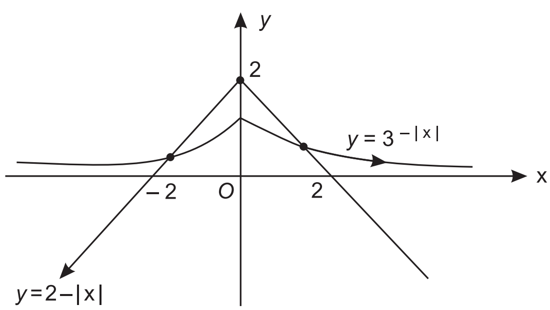
Example 30 How many roots does the following equation possess $3^{|x|}\{|2 - |x||\} = 1$?

(a) 1 (b) 2

(c) 3 (d) 4

Solution. Here, $3^{|x|}\{|2 - |x||\} = 1 \quad \Rightarrow \quad 2 - |x| = 3^{-|x|}$

In order to determine the number of roots, it is sufficient to find the points of intersection of the curves $y = 2 - |x|$ and $y = 3^{-|x|}$, shown as;



We observe the two curves intersect at two points.

$\therefore$ Two real solutions $\in (-2, 2)$.

Hence, (b) is the correct answer.

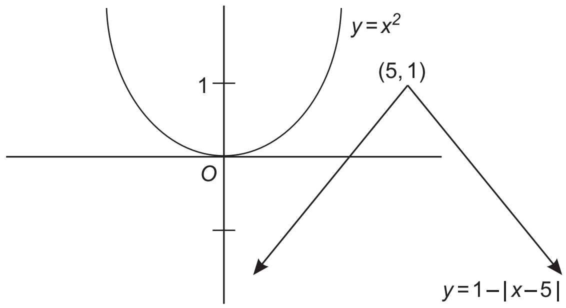
Example 31 The number of real solutions of the equation $x^2 = 1 - |x-5|$ is

(a) 1 (b) 2 (c) 4 (d) None of these

Solution. The number of real solutions of the equation $x^2 = 1 - |x-5|$

is equal to the number of points of intersection of the curves.

$$y = x^2 \quad \text{and} \quad y = 1 - |x-5|\text{, shown as;}$$



It is evident from the graph that the two curves do not intersect.

$\therefore$ No solution.

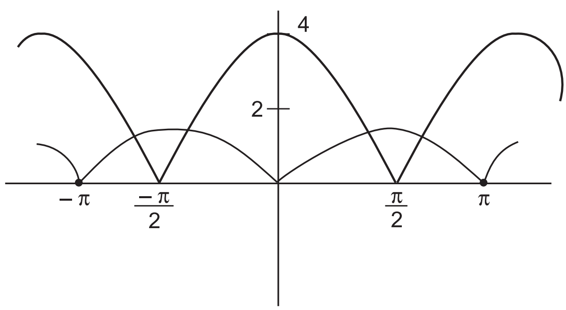
Hence, (d) is the correct answer.

Example 32 Number of solutions for $2^{\sin|x|} = 4^{|\cos x|}$ in $[-\pi, \pi]$ is equal to

(a) 2 (b) 4 (c) 6 (d) 8

Solution. The total number of solutions for the given equation is equal to the number of points of intersection of curves $y = 4^{|\cos x|}$ and $y = 2^{\sin|x|}$.

Clearly, the two curves intersect at four points. So, there are four solutions of the given equation.



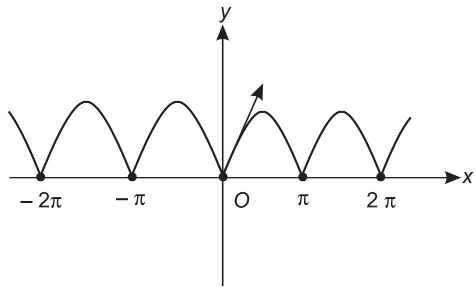
Hence, (b) is the correct answer.

Example 33 Number of roots of $|\sin|x|| = x + |x|$ in $[-2\pi, 2\pi]$ is

(a) 2 (b) 3 (c) 4 (d) 6

Solution. The total number of solutions for the given equation is equal to the number of points of intersection of the curves.

$$y = |\sin|x|| \quad \text{and} \quad y = x + |x| = \begin{cases} 2x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



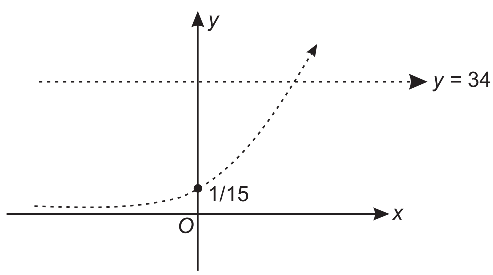
Clearly, the two curves intersect at three points.
∴ There are three solutions.
Hence, (b) is the correct answer.

Example 34 The equation $3^{x-1} + 5^{x-1} = 34$ has

(a) one solution (b) two solutions (c) three solutions (d) four solutions

Solution. The total number of solutions is same as the number of points of intersection of the curves.

$$y = 3^{x-1} + 5^{x-1} \text{ and } y = 34$$

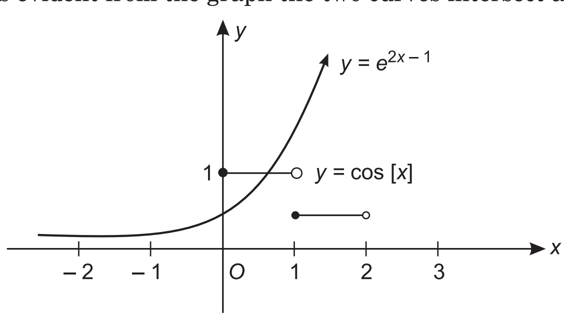


It is evident that these two curves intersect at exactly one point.
∴ Exactly one solution.
Hence, (a) is the correct answer.

Example 35 The number of solutions of the equation $\cos[x] = e^{2x-1}$, $x \in [0, 2\pi]$, where [.] denotes the greatest integer function is

(a) 1 (b) 2 (c) 3 (d) 4

Solution. It is evident from the graph the two curves intersect at only one point.



Hence, (a) is the correct answer.

Proficiency in 'Graphical Transformations'
Exercise 1

1. The number of real solutions of the equation $e^{|x|} - |x| = 0$ is

(a) 0 (b) 1 (c) 2 (d) None of these

2. The number of real solutions of the equation $3^{-|x|} - 2^{|x|} = 0$ is

(a) 0 (b) 1 (c) 2 (d) 3

3. The number of solutions of $3^{|x|} = |2 - |x||$ is

(a) 0 (b) 2 (c) 4 (d) infinite

4. The total number of roots of the equation $|x - x^2 - 1| = |2x - 3 - x^2|$ is

(a) 0 (b) 1 (c) 2 (d) infinitly many

5. The equation $e^x = m(m + 1)$, $m < 0$ has

(a) no real root (b) exactly one real root
(c) two real root (d) None of these

6. The number of integral roots of the equation

(a) 1 (b) 2 (c) 4 (d) 6

7. The number of real solutions of the equation $1 - x = [\cos x]$ is

(a) 1 (b) 2 (c) 3 (d) 4

8. The number of solutions of $[\sin x + \cos x] = 3 + [-\sin x] + [-\cos x]$ in the interval $[0, 2\pi]$ is (where [.] denotes the greatest integer function.)

(a) 0 (b) 4 (c) 3 (d) infinitly many

9. The equation $2^{|x^2 - 12|} = \sqrt{e^{|x| \log 4}}$ has

(a) four real solutions (b) three real solutions
(c) two real solutions (d) no solution

10. The number of real roots of the equation $x^2 + x + 3 + 2 \sin x = 0$, $x \in [-\pi, \pi]$ is

(a) 2 (b) 4 (c) 6 (d) None of these

11. The number of roots of the equation $x^3 + x^2 + 2x + \sin x = 0$ in $[-2\pi, 2\pi]$ is

(a) 1 (b) 2 (c) 3 (d) None of these

12. The number of roots of the equation $1 + 3^{x/2} = 2^x$ is

(a) 0 (b) 1 (c) 2 (d) None of these

13. The equation $x^2 - 2 = [\sin x]$, where [.] denotes the greatest integer function, has

(a) infinity many roots (b) exactly one integer root
(c) exactly one irrational root (d) exactly two roots

Proficiency in 'Graphical Transformations'

Exercise 2

Plot the graph for Q. Nos. 1 to 25

1. $f(x) = \sin(|x| + 1)$

2. $f(x) = \cos(|x| + 1)$

3. $f(x) = |2 - 2^x|$

4. $f(x) = x^2 - 2|x| - 3$

5. $f(x) = |x^2 - 2x - 3|$

6. $f(x) = |\log_2(1 - x)|$

7. $f(x) = \log_2|1 - x|$

8. $f(x) = \log_2(2 - x)^2$

9. $f(x) = |\log_2|x||$

10. $|f(x)| = \log_2 x$

11. $|f(x)| = \log_2(-x)$

12. $f(x) = [|x|]$, where $[\cdot]$ denotes the greatest integer function.

13. $f(x) = [|x - 2|]$, where $[\cdot]$ denotes the greatest integer function.

14. $f(x) = [|x| - 2]$, where $[\cdot]$ denotes the greatest integer function.

15. $f(x) = |\cos|x||$

16. $f(x) = \sin^{-1}(\sin|x|)$

17. $f(x) = \cos(x - [x])$, where $[\cdot]$ denotes the greatest integer function.

18. $f(x) = \tan(|x| - [|x|])$, where $[\cdot]$ denotes the greatest integer function.

19. $f(x) = [x^2]$, where $[\cdot]$ denotes the greatest integer function.

20. $f(x) = \tan^{-1}(\cot x)$

21. $f(x) = \sin^{-1}(\sin x)$

22. $f(x) = [\sin^{-1}(\sin x)]$, where $[\cdot]$ denotes the greatest integer function.

23. $f(x) = [\cos^{-1} x]$, where $[\cdot]$ denotes the greatest integer function.

24. $f(x) = [\sin(|x|-1)]$, where $[\cdot]$ denotes the greatest integer function.

25. $f(x) = \min(x-[x], -x-[-x])$, where $[\cdot]$ denotes the greatest integer function.

26. Find the number of solutions of $2^{\cos x} = |\sin x|$ when $x \in [0, 2\pi]$.

27. Find the solutions of the equation $\dfrac{x^2}{1-|x-2|} = 1$, graphically.

28. Find the number of solutions for; $\tan 4x = \cos x$, when $x \in (0, \pi)$.

29. Find the number of solutions of the equation $[\sin^{-1} x] = x-[x]$, where $[\cdot]$ denotes the greatest integer function.

30. If x and y satisfy the equations $\max(|x+y|, |x-y|) = 1$ and $|y| = x-[x]$, then find the number of ordered pairs (x, y).

31. Find the area enclosed by the curves;
  (i) $\max(|x|, |y|) = 1$
  (ii) $\max(|2x|, |2y|) = 1$
  (iii) $\max(|x+y|, |x-y|) = 1$

32. Find the area enclosed by $|x+y-1| + |2x+y+1| = 1$.

33. Find $f(x)$ when it is given by $f(x) = \max\left\{x^3, x^2, \dfrac{1}{64}\right\}, \forall x \in [0, \infty)$.

34. Find the number of solutions of
(i) $2^{|x|} = \sin x^2$ (ii) $\sin x = x^2 + x + 1$

35. Find the number of solutions of $\sin \pi x = |\log |x||$.

Answers

Exercise 1

1. (a)	**2.** (c)	**3.** (b)	**4.** (a)	**5.** (b)	**6.** (a)	**7.** (b)
8. (d)	**9.** (a)	**10.** (d)	**11.** (d)	**12.** (b)	**13.** (b), (c), (d)	

Exercise 2

26. 4 **27.** no solution **28.** 5 **29.** 1 **30.** 4

31. (i) 4 sq units (ii) 1 sq unit (iii) 2 sq units **32.** 2 sq units

33. $f(x) = \begin{cases} \dfrac{1}{64}, & 0 \le x \le \dfrac{1}{8} \\ x^2, & \dfrac{1}{8} \le x \le 1 \\ x^3, & 1 \le x \end{cases}$ **34.** (i) no solution (ii) no solution

35. six solutions

5

Limits

Chapter in a Snapshot

- Introduction to Limits
- Definition of Limits
- Frequently Used Series Expansions
- Left Hand and Right Hand Limits
- Evaluation of Limits

Introduction to Limits

Let us consider two graphs :

Here, in figure (5.1), if some one asks you what is value of y at $x = 0$. Your answer is $y = 2$.

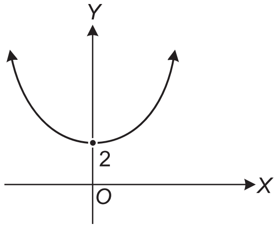


Fig. 5.1

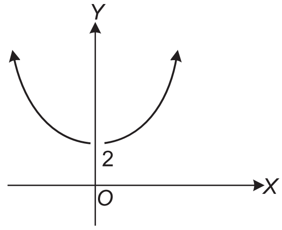


Fig. 5.2

But, if some one asks you what is the position of blank dot in figure (5.2) at $x = 0$, then for defining the coordinate or value of y at $x = 0$, we have to take points on the curve in its vicinity. (*ie,*) We have to take points on the curve situated either on left hand or on right hand side of $x = 0$.

ie, We take a point $x = 0 - h$ on left hand side of $x = 0$ and a point $x = 0 + h$ on right hand side of $x = 0$. Where h is infinitesimal small positive quantity.

As shown in figure (5.3), the abscissae of points on the curve in neighbourhood of blank dot are not exactly equal to 0 rather approximately 0.

$\Rightarrow$ $x \approx 0$ *ie*, $x \neq 0$ rather $x = 0 - h$ for left hand point and $x = 0 + h$ for right hand point.

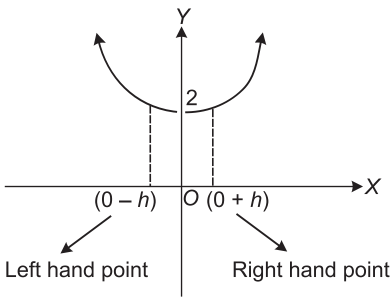


Fig. 5.3

Hence, for $x = 0 \pm h$ we write $x \to 0$ and read it as 'x tends to 0' or 'x approaches to 0'. Now, from the graph, if we ask you what is the position of blank dot in above figure (5.3), as $x \to 0$, your answer would be 2.

Definition of Limits

Let $\lim\limits_{x \to a} f(x) = l$. It would mean that when we approach the point $x = a$ from the values which are just greater than or smaller than $x = a$, $f(x)$ would have a tendency to move closer to the value 'l'.

This is same as saying 'difference between $f(x)$ and l can be made as small as we feel like by suitably choosing x in the neighbourhood of $x = a$'.

Mathematically, we write this, as

$\lim\limits_{x \to a} f(x) = l$, which is equivalent of saying that,

$|f(x) - l| < \varepsilon \; \forall \, x$ whenever $0 < |x - a| < \delta$ and ε and δ sufficiently small +ve numbers.

It is clear from the above discussion that, if we are interested in finding the limit of $f(x)$ at $x = a$, the first thing we have to make sure is that $f(x)$ is well defined in the neighbourhood of $x = a$ and not necessarily at $x = a$ (that means $x = a$ may or may not be in the domain of $f(x)$), because we have to examine its behaviour or tendency in the neighbourhood of $x = a$.

Following possibilities may arise:

(a) **Left tendency is same as it's right tendency** As shown in figure, when we approach $x = a$ from the values which are just less than a, $f(x)$ has a tendency to move towards the value l (**left tendency**).

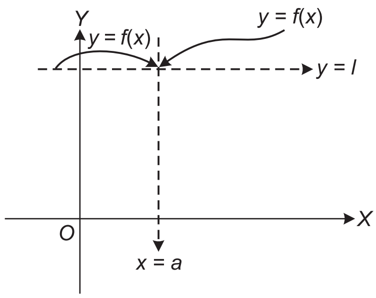


Fig. 5.4

Similarly, when we approach $x = a$ from the values which are just greater than a, $f(x)$ has a tendency to move towards the value l (**right tendency**).

In this case we say $f(x)$ has limit l at $x = a$; *ie*, $\lim\limits_{x \to a} f(x) = l$.

(b) **When the left tendency is not the same as right tendency** Here, left tendency is l_1 and right tendency is l_2, clearly left tendency (l_1) is not same as right tendency (l_2). In this case we say that the limit of $f(x)$ at $x = a$ will not exist.

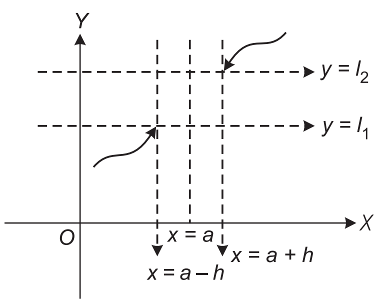


Fig. 5.5

ie, $\lim\limits_{x \to a} f(x) =$ Doesn't exist.

(c) **When the left tendency and/or right tendency is not fixed** As shown in the figure, it is clear that in this case, the function has **erratic** behaviour in the neighbourhood of $x = a$ and it will not be possible to talk about the left and right tendencies of the function in the neighbourhood of $x = a$. In this case we conclude that the limit of $f(x)$ at $x = a$ will not exist.

ie, $\lim_{x \to a} f(x) =$ Doesn't exist.

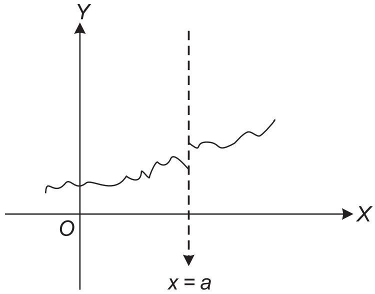


Fig. 5.6

Remarks

(a) Normally students have the perception that limit should be a finite number. But, it is not always so. It is quite possible that $f(x)$ has infinite limit at $x = a$.

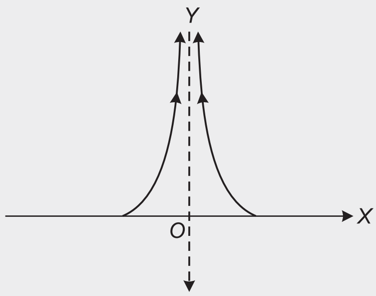


Fig. 5.7

If $\lim_{x \to a} f(x) = \infty$, it would simply mean that function has tendency to assume very large positive values in neighbourhood of $x = a$ (as shown in figure).

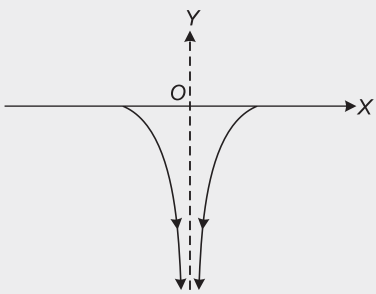


Fig. 5.8

For example $\lim_{x \to 0} \dfrac{1}{|x|} = \infty$

which indicates the left tendency as well as right tendency are the same.

Again, if

$\lim_{x \to a} f(x) = -\infty$, it would simply mean that the function has tendency to assume very large negative values in the neighbourhood of $x = a$.

For example (as shown in figure)

$$\lim_{x \to 0} \frac{-1}{|\sin x|} = -\infty$$

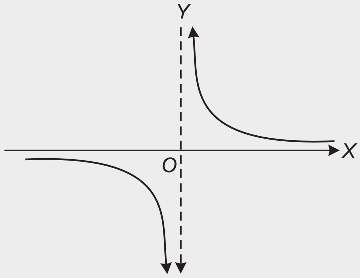


Fig. 5.9

At the end we discuss the case when left tendency is $(-\infty)$ and right tendency is $(+\infty)$ (*ie*, $f(x)$ does not have unique tendency).

Thus, in this case limit does not exist.

For example $\lim_{x \to 0} \frac{1}{x}$ does not exist, since left tendency is $(-\infty)$ and right tendency is $(+\infty)$.

(b) If $f(x)$ is well defined at $x = a$, it doesn't imply that $\lim_{x \to a} f(x) = f(a)$.

Because, it is quite possible that $f(x)$ is well defined at $x = a$ but not in the neighbourhood of $x = a$ or $f(x)$ is well defined in the neighbourhood of $x = a$, but doesn't have a unique tendency.

Frequently Used Series Expansions

1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$

2. $a^x = 1 + \frac{x \cdot \log a}{1!} + \frac{(\log a)^2 x^2}{2!} + \ldots$ where, $a \in R^+$

3. $(1 + x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots.$ $n \in R$ and $|x| < 1$

4. $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$ where, $-1 \le x \le 1$

5. $\frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots\ldots + a^{n-1}$

6. $(1 + x)^{1/x} = e\left(1 - \frac{x}{2} + \frac{11x^2}{24} + \ldots\ldots\right)$

7. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots.$

8. $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots\ldots$

9. $\tan x = x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + \dfrac{17}{315}x^7 + \ldots..$

10. $\sin^{-1} x = x + \dfrac{1^2}{3!}x^3 + \dfrac{1^2 \cdot 3^2}{5!}x^5 + \dfrac{1^2 \cdot 3^2 \cdot 5^2}{7!}x^7 + \ldots.$

11. $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} + \ldots..$

12. $\sec^{-1} x = 1 + \dfrac{x^2}{2!} + \dfrac{5x^4}{4!} + \dfrac{61x^6}{6!} + \ldots..$

13. $(\sin^{-1} x)^2 = \dfrac{2}{2!}x^2 + \dfrac{2 \cdot 2^2}{4!}x^4 + \dfrac{2 \cdot 2^2 \cdot 4^2}{6!}x^6 + \ldots..$

14. $x \cot x = 1 - \dfrac{x^3}{3} + \dfrac{x^4}{45} - \dfrac{2x^6}{945} + \ldots..$

15. $\sec x = 1 + \dfrac{x^2}{2} + \dfrac{5x^4}{24} + \dfrac{61x^6}{720} + \ldots..$

16. $x \operatorname{cosec} x = 1 + \dfrac{x^2}{6} + \dfrac{7x^4}{360} + \dfrac{31x^6}{15120} + \ldots..$

Point to Consider

Using the above formula we can define any other trigonometric expansion. Say, to define $\sin^2 x$. We know, $\sin^2 x = \dfrac{1 - \cos 2x}{2}$.

Infinity (∞) ∞ is a symbol and not a number. It is a symbol for the behaviour of a variable which continuously increases and passes through all limits. Thus, the statement $x = \infty$ is meaningless, we should write $x \to \infty$.

Similarly, $-\infty$ is a symbol for the behaviour of a variable which continuously decreases and passes through all limits. Thus, the statement $x = -\infty$ is meaningless, we should write $x \to -\infty$.

Also, $\dfrac{1}{x} \to 0$, if $x \to +\infty$ and $\dfrac{1}{x} \to 0$, if $x \to -\infty$.

We come across the following concepts :

1. We cannot plot ∞ on paper. Infinity does not obey laws of elementary algebra.

2. $\infty + \infty = \infty$

3. $\infty - \infty$ is indeterminate.

4. $\infty \times \infty = \infty$ is indeterminate.

5. $0 \times \infty$ is indeterminate.

6. $\dfrac{a}{\infty} = 0$, if a is finite.

7. $\dfrac{a}{0}$ is undefined, if $a \neq 0$.

8. $\dfrac{0}{0}, \dfrac{\infty}{\infty}, \infty - \infty, 1^{\infty}, 0^0, \infty^0, \infty \times 0$ are all indeterminate forms.

eg, $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1} = \dfrac{0}{0}$ indeterminate form.

eg, $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = \dfrac{0}{0}$ indeterminate form.

eg, $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = \dfrac{0}{0}$ indeterminate form.

Point to Consider

In all above forms limits are calculated at approximate values.

Left Hand and Right Hand Limits

Let $y = f(x)$ be a given function and $x = a$ is the point under consideration. Left tendency of $f(x)$ at $x = a$ is called its left hand limit and right tendency is called its right hand limit.

Left tendency (left hand limit) is denoted by $f(a - 0)$ or $f(a-)$ and right tendency (right hand limit) is denoted by $f(a + 0)$ or $f(a+)$ and are written as

$$\left.\begin{aligned} f(a - 0) &= \lim_{h \to 0} f(a - h) \\ f(a + 0) &= \lim_{h \to 0} f(a + h) \end{aligned}\right\}, \text{ where } h \text{ is a small positive number.}$$

Thus, for the existence of the limit of $f(x)$ at $x = a$, it is necessary and sufficient that

$f(a - 0) = f(a + 0)$, if these are finite or

$f(a - 0)$ and $f(a + 0)$ both should be either $+\infty$ or $-\infty$.

Illustration 1 Evaluate the right hand limit and left hand limit of the function

$$f(x) = \begin{cases} \dfrac{|x - 4|}{x - 4}, & x \neq 4 \\ 0, & x = 4 \end{cases}$$

Solution. **RHL of $f(x)$ at $x = 4$**

$$= \lim_{x \to 4^+} f(x) = \lim_{h \to 0} f(4 + h) = \lim_{h \to 0} \frac{|4 + h - 4|}{4 + h - 4}$$

$$= \lim_{h \to 0} \frac{|h|}{h} = 1$$

LHL of $f(x)$ at $x = 4$

$$= \lim_{x \to 4^-} f(x) = \lim_{h \to 0} f(4-h) = \lim_{h \to 0} \frac{|4-h-4|}{4-h-4}$$

$$= \lim_{h \to 0} \frac{|h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1$$

Thus, RHL $\neq$ LHL. So, $\lim_{x \to 4} f(x)$ does not exist.

Illustration 2 If $f(x) = \begin{cases} 5x - 4, & 0 < x \le 1 \\ 4x^3 - 3x, & 1 < x < 2 \end{cases}$ show that $\lim_{x \to 1} f(x)$ exists.

Solution. We have,

LHL of $f(x)$ at $x = 1$

$$= \lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h)$$

$$= \lim_{h \to 0} 5(1-h) - 4 = \lim_{h \to 0} 1 - 5h = 1$$

RHL of $f(x)$ at $x = 1$

$$= \lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1+h)$$

$$= \lim_{h \to 0} 4(1+h)^3 - 3(1+h) = 4(1)^3 - 3(1) = 1$$

Thus, RHL = LHL = 1. So, $\lim_{x \to 1} f(x)$ exists and is equal to 1.

Illustration 3 Show $\lim_{x \to 0} \frac{e^{1/x} - 1}{e^{1/x} + 1}$ does not exist.

Solution. Let $f(x) = \frac{e^{1/x} - 1}{e^{1/x} + 1}$. Then,

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1} = \lim_{h \to 0} \frac{(1/e^{1/h} - 1)}{(1/e^{1/h} + 1)} = \frac{0-1}{0+1} = -1$$

$$\left[\text{as } h \to 0 \Rightarrow \frac{1}{h} \to \infty \Rightarrow e^{1/h} \to \infty \Rightarrow 1/e^{1/h} \to 0\right] \quad \text{...(i)}$$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1} = \lim_{h \to 0} \frac{(1 - 1/e^{1/h})}{(1 + 1/e^{1/h})}$$

[Dividing numerator and denominator both by $e^{1/h}$]

$$= \frac{1-0}{1+0} = 1 \quad \text{[Using Eq. (i)]}$$

Clearly, $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

Hence, $\lim_{x \to 0} f(x)$ doesn't exist.

Illustration 4 Evaluate $\lim_{x \to 1} \frac{\sqrt{1 - \cos 2(x-1)}}{(x-1)}$. **[IIT JEE 1998]**

Solution. $\lim_{x \to 1} \frac{\sqrt{1 - \cos 2(x-1)}}{(x-1)} = \lim_{x \to 1} \frac{\sqrt{2 \sin^2 (x-1)}}{(x-1)}$

$$= \lim_{x \to 1} \frac{\sqrt{2}\,|\sin (x-1)|}{(x-1)}$$

$\therefore$ $$\text{LHL} = \lim_{x \to 1^-} \frac{\sqrt{2}\,|\sin (x-1)|}{(x-1)}$$

$$= \lim_{h \to 0} \frac{\sqrt{2}\,|\sin (-h)|}{(-h)} = \lim_{h \to 0} \frac{\sqrt{2} \sin h}{-h} = -\sqrt{2}$$

Again, $$\text{RHL} = \lim_{x \to 1^+} \frac{\sqrt{2}\,|\sin (x-1)|}{(x-1)}$$

$$= \lim_{h \to 0} \frac{\sqrt{2}\,|\sin (h)|}{h} = \lim_{h \to 0} \frac{\sqrt{2} \sin h}{h} = \sqrt{2}$$

Clearly, $$\lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x)$$

Hence, $\lim_{x \to 1} f(x)$ doesn't exist.

Illustration 5 Solve (i) $\lim_{x \to 1} [\sin^{-1} x]$ (ii) $\lim_{x \to 0^+} \left[\frac{\sin x}{x}\right]$ (iii) $\lim_{x \to 0^-} \left[\frac{\sin x}{x}\right]$

(where $[\cdot]$ denotes greatest integer function.)

Solution. (i) Here, $\lim_{x \to 1} [\sin^{-1} x]$

Put $$\sin^{-1} x = t$$

$\therefore$ $$x = \sin t \text{ and } t \to \frac{\pi}{2} \text{ as } x \to 1$$

$\Rightarrow$ $$\lim_{t \to \frac{\pi}{2}} [t] \quad \Rightarrow \quad \left[\frac{\pi}{2}\right] = 1$$

$\therefore$ $$\lim_{x \to 1} [\sin^{-1} x] = 1$$

(ii) Here, $\lim_{x \to 0^+} \left[\frac{\sin x}{x}\right]$

Put $\frac{\sin x}{x} = t \Rightarrow t \to 1^-$ as $x \to 0^+$ $\therefore [t] = 0$ (*ie*, as $x \to 0 + h \Rightarrow t \to 1 - h$)

$\Rightarrow$ $$\lim_{t \to 1^-} [t] \Rightarrow [1 - h] \Rightarrow 0$$

(iii) Here, $\lim_{x \to 0^-} \left[\frac{\sin x}{x}\right]$

Put $\frac{\sin x}{x} = t \Rightarrow t \to 1^-$ as $x \to 0^-$ (*ie*, as $x \to 0 - h \Rightarrow t \to 1 - h \therefore [t] = 0$)

$\therefore$ $$\lim_{t \to 1^-} [t] = [1 - h] = 0$$

Illustration 6 Solve

(i) $\lim_{x \to \infty} [\tan^{-1} x]$ (ii) $\lim_{x \to -\infty} [\tan^{-1} x]$

(where $[\cdot]$ denotes greatest integer function.)

Solution. (i) $\lim_{x \to \infty} [\tan^{-1} x]$

Put $\tan^{-1} x = t$

$\Rightarrow$ $t \to \frac{\pi}{2}$ as $x \to \infty$

$\therefore$ $\lim_{t \to \frac{\pi}{2}} [t] = \left[\frac{\pi}{2}\right] = 1$

(ii) Here, $\lim_{x \to -\infty} [\tan^{-1} x]$

Put $\tan^{-1} x = t \Rightarrow t \to -\frac{\pi}{2}$ as $x \to -\infty$

$\therefore$ $\lim_{t \to -\frac{\pi}{2}} [t] = \left[-\frac{\pi}{2}\right] = [-1.57] = -2$

Illustration 7 Solve

(i) $\lim_{x \to 0^+} \left[\frac{\tan x}{x}\right]$ (ii) $\lim_{x \to 0^-} \left[\frac{\tan x}{x}\right]$

(where $[\cdot]$ denotes greatest integer function.)

Solution. (i) Here, $\lim_{x \to 0^+} \left[\frac{\tan x}{x}\right]$

Put $\frac{\tan x}{x} = t$

$\Rightarrow$ $t \to 1^+$ as $x \to 0^+$ (*ie*, As $x \to 0 + h \Rightarrow t \to 1 + h$)

$\therefore$ $\lim_{t \to 1^+} [t] = [1 + h] = 1$

(ii) Here, $\lim_{x \to 0^-} \left[\frac{\tan x}{x}\right]$

Put $\frac{\tan x}{x} = t$

$\Rightarrow$ $t \to 1^+$ as $x \to 0^-$ (*ie*, As $x \to 0 - h \Rightarrow t \to 1 + h$)

$\therefore$ $\lim_{t \to 1^+} [t] = [1 + h] = 1$

Illustration 8 Solve

(i) $\lim_{x \to 1^-} [\sin(\sin^{-1} x)]$ (ii) $\lim_{x \to \frac{\pi}{2}} [\sin^{-1} (\sin x)]$

(where $[\cdot]$ denotes greatest integer function.)

Solution. (i) Here, $\lim_{x \to 1^-} [\sin(\sin^{-1} x)]$

$\Rightarrow$ $\lim_{x \to 1^-} [x]$ (As $\sin(\sin^{-1} x) = x$, if $-1 \le x \le 1$)

$\Rightarrow$ $[1 - h]$ (As $x \to 1^- \Rightarrow x = 1 - h$)

$\Rightarrow$ 0

Point to Consider

If $\lim_{x \to 1} [\sin(\sin^{-1} x)]$, then it means you have to calculate only left hand limit and not right hand limit as for $x > 1$, $\sin(\sin^{-1} x)$ is not defined.

$\therefore$ $\lim_{x \to 1} [\sin(\sin^{-1} x)] = \lim_{x \to 1^-} [\sin(\sin^{-1} x)]$ and no need to check for $\lim_{x \to 1^+} [\sin(\sin^{-1} x)]$

(ii) $\lim_{x \to \frac{\pi}{2}} [\sin^{-1}(\sin x)]$

$$\Rightarrow \quad \lim_{x \to \frac{\pi}{2}^-} [\sin^{-1}(\sin x)]$$

$$\Rightarrow \quad \lim_{x \to \frac{\pi}{2}^-} [x] \Rightarrow \left[\frac{\pi}{2} - h\right] \Rightarrow 1$$

Illustration 9 Solve

(i) $\lim_{x \to 0} [\cot x]$ (ii) $\lim_{x \to +\infty} [\cot^{-1} x]$

(where $[\cdot]$ denotes greatest integer function.)

Solution. (i) Here, $\lim_{x \to 0} [\cot x]$

Put $\cot x = t$, now as $x \to 0$; $\cot x$ exhibits two values for $x \to 0^+$ and $x \to 0^-$ *ie*, $\cot x \to +\infty$ and $\cot x \to -\infty$ respectively.

$\therefore$ We should apply right hand and left hand limit;

ie, $\lim_{x \to 0^+} [\cot x] = \lim_{t \to +\infty} [t] = \infty$ $\quad (\because \cot x = t \Rightarrow t \to +\infty \text{ as } x \to 0^+)$

and $\lim_{x \to 0^-} [\cot x] = \lim_{t \to -\infty} [t] = -\infty$ $\quad (\because \cot x = t \Rightarrow t \to -\infty \text{ as } x \to 0^-)$

$\therefore$ Limit doesn't exist.

(ii) Here, $\lim_{x \to +\infty} [\cot^{-1} x] \Rightarrow \lim_{t \to 0^+} [t]$

$$(\because \cot^{-1} x = t \Rightarrow t \to 0^+ \text{ as } x \to +\infty)$$

$$\Rightarrow \lim_{h \to 0} [0 + h] = \lim_{h \to 0} 0$$

$$\Rightarrow 0$$

Illustration 10 Solve $\lim_{x \to 0} \left[\sin \frac{|x|}{x}\right]$, where $[.]$ denotes the greatest integer function.

Solution. Here, $\lim_{x \to 0} \left[\sin \frac{|x|}{x}\right]$, since we have greatest integral function we must define function.

Now, RHL (put $x = 0 + h$)

$$\lim_{h \to 0} \left[\frac{\sin |0 + h|}{0 + h}\right]$$

We know, $\dfrac{\sin h}{h} \to 1$ as $h \to 0$ but less than 1.

$\therefore \quad \lim_{h \to 0} 0 = 0 \qquad \left(\because \left[\dfrac{\sin h}{h}\right] = 0 \text{ as } h \to 0\right)$

$\Rightarrow$ RHL = 0

Again, LHL (put $x = 0 - h$) $\lim_{h \to 0}\left[\sin \dfrac{|0-h|}{0-h}\right]$,

we know $\dfrac{\sin h}{-h} \to -1$ as $h \to 0$ but greater than –1.

$\therefore \quad \lim_{h \to 0} -1 = -1 \qquad \left(\because \left[\dfrac{\sin h}{h}\right] = -1 \text{ as } h \to 0\right)$

$\Rightarrow$ LHL = –1

$\therefore$ Limit doesn't exist as RHL = 0 and LHL = –1

Illustration 11 Solve $\lim_{x \to 0}\left[\dfrac{\sin |x|}{|x|}\right]$, where $[\cdot]$ denotes greatest integer function.

Solution. Here, $\lim_{x \to 0}\left[\dfrac{\sin |x|}{|x|}\right]$ ...(i)

As we know, $\dfrac{\sin x}{x} \to 1$ as $x \to 0$

or $\dfrac{\sin |x|}{|x|} \to 1$ as $x \to 0$ from right or left

ie, at $x = 0 + h$ or $x = 0 - h$ and less than one.

$\therefore \quad \left[\dfrac{\sin |x|}{|x|}\right] = 0$ as $\dfrac{\sin |x|}{|x|} < 1$

$\Rightarrow$ From Eq. (i) whether we find RHL or LHL

$$\lim_{x \to 0}\left[\frac{\sin |x|}{|x|}\right] \Rightarrow \lim_{x \to 0} 0 \Rightarrow 0$$

$\therefore \quad \lim_{x \to 0}\left[\dfrac{\sin |x|}{|x|}\right]$, exists and is 0.

Illustration 12 $\lim_{x \to 0}\left[\dfrac{-2x}{\tan x}\right]$, where $[\cdot]$ denotes greatest integer function is

(a) –1 (b) 4 (c) 5 (d) None of these

Solution. We know, when $x \to 0$

$\Rightarrow \quad \dfrac{x}{\tan x} < 1 \quad \Rightarrow \quad \dfrac{-x}{\tan x} > -1 \quad \Rightarrow \quad \dfrac{-2x}{\tan x} > -2$

So, $\quad \lim_{x \to 0}\left[\dfrac{-2x}{\tan x}\right] = -2$

Hence, (d) is the correct answer.

Illustration 13 The value of $\lim\limits_{x \to \pi/2} \dfrac{\left[\dfrac{x}{2}\right]}{\log(\sin x)}$ is equal to

(a) 0 (b) 1

(c) −1 (d) doesn't exist

Solution. As $\lim\limits_{x \to \pi/2} \dfrac{\left[\dfrac{x}{2}\right]}{\sin x}$

$$\therefore \qquad x \to \frac{\pi}{2} \quad \Rightarrow \quad \frac{x}{2} \to \frac{\pi}{4}$$

where,
$$\frac{x}{2} \to \frac{\pi}{4} < 1 \text{ for } x \to \frac{\pi^+}{2} \text{ or } x \to \frac{\pi^-}{2}$$

$$\therefore \qquad \left[\frac{x}{2}\right] = 0, \text{ since } 0 < \left(\frac{x}{2} \to \frac{\pi}{4}\right) < 1$$

$$\Rightarrow \qquad \lim_{x \to \pi/2} \frac{\left[\dfrac{x}{2}\right]}{\sin x} = \lim_{x \to \pi/2} 0 = 0$$

Hence, (a) is the correct answer.

Target Exercise 5.1

1. The value of $\lim\limits_{x \to 1} \{1 - x + [x - 1] + [1 - x]\}$, (where $[\cdot]$ donotes the greatest integral function) is

(a) −1 (b) Doesn't exist

(c) 1 (d) None of these

2. The value of $\lim\limits_{x \to 0} \dfrac{\sin [x]}{[x]}$, (where $[\cdot]$ denotes the greatest integer function) is

(a) 1 (b) sin 1

(c) Doesn't exist (d) None of these

3. The value of $\lim\limits_{x \to 0} \left[\dfrac{|\sin x|}{|x|}\right]$, (where $|\cdot|$, $[\cdot]$ denotes modulus and greatest integer function) is

(a) 0 (b) Doesn't exist

(c) −1 (d) 1

4. The value of $\lim\limits_{x \to 0} \sin^{-1} \{x\}$, (where $\{\cdot\}$ denotes fractional part of x)

(a) 0 (b) $\dfrac{\pi}{2}$

(c) Doesn't exist (d) None of these

5. The value of $\lim\limits_{x \to 0} \left[\dfrac{x^2}{\sin x \tan x}\right]$, (where $[\cdot]$ denotes the greatest integer function)

(a) 0 (b) 1

(c) Doesn't exist (d) None of these

Evaluation of Limits

Now, according to our plan first of all we shall learn the evaluation of limits of different forms and then the existence of limits.

There are eight indeterminate or meaningless forms which are

(i) $\frac{0}{0}$ (ii) $\frac{\infty}{\infty}$ (iii) $\infty - \infty$ (iv) $\infty \times \infty$

(v) $\infty \cdot 0$ (vi) 0^0 (vii) ∞^0 (viii) 1^∞

We shall divide the problems of evaluation of limits in five categories :

(1) **Algebraic Limits**

Limits of algebraic forms are further sub-classified as

(i) $\frac{0}{0}$ form which are based on

(a) Factorisation method

(b) Rationalisation method

(c) Standard formula as $\lim_{x \to a} \left(\frac{x^n - a^n}{x - a} \right) = n \cdot a^{n-1}$,

where *'n'* is a rational number.

(ii) $\infty -$ type algebraic function of ∞ type

(a) $\frac{\infty}{\infty}$ type (b) $\infty - \infty$ type

(2) **Trigonometric Limits**

(3) **Logarithmic Limits**

(4) **Exponential Limits**

(a) Based on series expansion (b) Based on definition of ∞

(5) **Miscellaneous Forms**

Point to Consider

Now, we are familiar with the different indeterminate forms and are in a position to define an important result known as L'Hospital's rule.

L'Hospital's rule This result is applicable to only two indeterminate forms $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty} \right)$.

This result states that, if $\lim_{x \to a} \frac{f(x)}{g(x)}$, reduces to $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Then, **differentiate numerator and denominator till this form is removed**.

ie, $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$ provided the later limit exists.

But, if it again take form $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty} \right)$, then

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)}$ and this process is continued till $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty} \right)$ form is removed.

(1) Algebraic Limits

(i) $\frac{0}{0}$ Form

(a) **Factorisation method** In this method numerators and denominators are factorised. The common factors are cancelled and the rest output is the result.

Illustration 14 Evaluate $\lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1}$.

Solution. **Method I** $\lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1} \left(\frac{0}{0}\text{ form}\right) = \lim_{x \to 1} \frac{(x-1)(x-2)}{(x-1)}$

$$= \lim_{x \to 1} (x - 2) \quad \text{(As } x - 1 \neq 0\text{)}$$

$$= 1 - 2 = -1$$

Method II

$$\lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1} \quad \left(\frac{0}{0}\text{ form}\right)$$

So, applying L'Hospital's rule,

$$\lim_{x \to 1} \frac{2x - 3}{1} = \frac{2 - 3}{1} = -1$$

[*ie*, differentiating, numerator and denominator separately]

Illustration 15 Evaluate $\lim_{x \to 1} \frac{x^3 - x^2 \log x + \log x - 1}{x^2 - 1}$

Solution. $\lim_{x \to 1} \frac{x^3 - x^2 \log x + \log x - 1}{x^2 - 1}$

$$= \lim_{x \to 1} \frac{(x^3 - 1) - (x^2 - 1)\log x}{(x^2 - 1)} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 1} \frac{(x-1)\{x^2 + x + 1 - (x+1)\log x\}}{(x-1)(x+1)}$$

$$= \lim_{x \to 1} \frac{x^2 + x + 1 - (x+1)\log x}{(x+1)}$$

$$= \frac{1^2 + 1 + 1 - (1+1)\log 1}{1 + 1} = \frac{3}{2} \quad \text{(As } \log 1 = 0\text{)}$$

Illustration 16 Evaluate $\lim_{x \to 2} \frac{x^6 - 24x - 16}{x^3 + 2x - 12}$.

Solution. **Method I**

$$\lim_{x \to 2} \frac{x^6 - 24x - 16}{x^3 + 2x - 12} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 2} \frac{(x-2)(x^5 + 2x^4 + 4x^3 + 8x^2 + 16x + 8)}{(x-2)(x^2 + 2x + 6)}$$

$$= \lim_{x \to 2} \frac{x^5 + 2x^4 + 4x^3 + 8x^2 + 16x + 8}{x^2 + 2x + 6}$$

$$= \frac{2^5 + 2(2)^4 + 4(2)^3 + 8(2)^2 + 16(2) + 8}{(2)^2 + 2(2) + 6}$$

$$= \frac{168}{14} = 12$$

Method II (Applying L'Hospital's rule)

If $x = 2$, $\qquad \dfrac{x^6 - 24x - 16}{x^3 + 2x - 12}$

$$\lim_{x \to 2} \frac{x^6 - 24x - 16}{x^3 + 2x - 12} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 2} \frac{6x^5 - 24}{3x^2 + 2} \qquad \text{(Applying L'Hospital's rule)}$$

$$= \frac{6(2)^5 - 24}{3(2)^2 + 2}$$

$$= \frac{168}{14} = 12$$

Illustration 17 Let $f(x)$ be polynomial of degree 4 with roots 1, 2, 3, 4 and leading coefficient 1 and $g(x)$ be the polynomial of degree 4 with roots 1, $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ with leading coefficient 1. Find $\lim\limits_{x \to 1} \dfrac{f(x)}{g(x)}$.

Solution. As, $f(x)$ being polynomial having roots 1, 2, 3, 4 and leading coefficient 1.

$\therefore \qquad f(x) = (x-1)(x-2)(x-3)(x-4)$

Similarly, $\qquad g(x) = (x-1)\left(x - \frac{1}{2}\right)\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)$

$$\therefore \qquad \lim_{x \to 1} \frac{f(x)}{g(x)} = \lim_{x \to 1} \frac{(x-1)(x-2)(x-3)(x-4)}{(x-1)\left(x - \frac{1}{2}\right)\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)}$$

$$= \frac{(-1)(-2)(-3)}{\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)} = -24$$

(b) **Rationalisation method** Rationalisation is followed when we have fractional powers (like $\frac{1}{2}$, $\frac{1}{3}$ etc.) on expressions in numerator or denominator or in both. After rationalisation the terms are factorised which on cancellation gives the result.

Illustration 18 Evaluate $\lim_{h \to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h}$.

Solution. **Method I**

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{(x+h) - (x)}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Method II (L'Hospital's rule)

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$\therefore$ Applying L'Hospital's rule

[differentiating numerator and denominator w.r.t. h]

$$= \lim_{h \to 0} \frac{\dfrac{1}{2\sqrt{x+h}} - 0}{1} = \frac{1}{2\sqrt{x}}$$

(c) **Based on standard formula**

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1} \text{ where '}n\text{' is a rational number.}$$

Proof Let $f(x) = \dfrac{x^n - a^n}{x - a}$

$$= x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-1}$$

$\therefore$ $\lim_{x \to a} f(x) = \lim_{x \to a} (x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-1})$

$$= \underbrace{a^{n-1} + a \cdot a^{n-2} + a^2 \cdot a^{n-3} + \ldots + a^{n-1}}_{n \text{ terms}}$$

$$= a^{n-1} + a^{n-1} + a^{n-1} + \ldots \text{ upto } n \text{ terms}$$

$$= n \cdot a^{n-1}$$

Illustration 19 Evaluate $\lim_{x \to 2} \dfrac{x^3 - 2^3}{x - 2}$.

Solution. $\lim_{x \to 2} \dfrac{x^3 - 2^3}{x - 2}$ $\qquad \left(\dfrac{0}{0} \text{ form}\right)$

$$= 3\,(2)^{3-1} \qquad \left[\text{As } \lim_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}\right]$$

$$= 3\,(2)^2 = 12$$

Illustration 20 Evaluate $\lim_{x \to 1} \frac{\sqrt[3]{x} + \sqrt{x} + x\sqrt{x} - 3}{x^3 - 1}$.

Solution. $\lim_{x \to 1} \frac{(\sqrt[3]{x} - 1) + (\sqrt{x} - 1) + (x^{3/2} - 1)}{(x - 1)(x^2 + x + 1)}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 1} \left\{ \frac{\sqrt[3]{x} - 1}{x - 1} + \frac{(\sqrt{x} - 1)}{x - 1} + \frac{(x^{3/2} - 1)}{x - 1} \right\} \cdot \frac{1}{x^2 + x + 1}$$

$$= \left\{ \left(\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{x - 1} \right) + \left(\lim_{x \to 1} \frac{(\sqrt{x} - 1)}{x - 1} \right) + \left(\lim_{x \to 1} \frac{(x^{3/2} - 1)}{x - 1} \right) \right\} \cdot \lim_{x \to 1} \frac{1}{x^2 + x + 1}$$

(Apply L'Hospital's rule)

$$= \left\{ \frac{1}{3}(1)^{1/3 - 1} + \frac{1}{2}(1)^{1/2 - 1} + \frac{3}{2}(1)^{3/2 - 1} \right\} \cdot \frac{1}{1^2 + 1 + 1}$$

$$= \left\{ \frac{1}{3}(1)^{-2/3} + \frac{1}{2}(1)^{-1/2} + \frac{3}{2}(1)^{1/2} \right\} \cdot \frac{1}{3}$$

$$= \left(\frac{1}{3} + \frac{1}{2} + \frac{3}{2} \right) \cdot \frac{1}{3}$$

$$= \left(\frac{2 + 3 + 9}{6} \right) \cdot \frac{1}{3} = \frac{7}{9}$$

Illustration 21 Evaluate $\lim_{x \to 1} \frac{x^{P+1} - (P + 1)x + P}{(x - 1)^2}$.

Solution. $\lim_{x \to 1} \frac{x^{P+1} - (P + 1)x + P}{(x - 1)^2}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 1} \frac{x^{P+1} - Px - x + P}{(x - 1)^2}$$

$$= \lim_{x \to 1} \frac{x(x^P - 1) - P(x - 1)}{(x - 1)^2}$$

Dividing numerator and denominator by $(x - 1)$, we get

$$\lim_{x \to 1} \frac{x \dfrac{(x^P - 1)}{(x - 1)} - P}{(x - 1)}$$

$$= \lim_{x \to 1} \frac{(x + x^2 + x^3 + \ldots. + x^P) - P}{(x - 1)}$$

$$= \lim_{x \to 1} \frac{(x + x^2 + x^3 + \ldots. + x^P) - (1 + 1 + 1 + \ldots\ldots \text{ upto } P \text{ times})}{(x - 1)}$$

$$= \lim_{x \to 1} \left\{ \frac{(x - 1)}{(x - 1)} + \frac{(x^2 - 1)}{(x - 1)} + \frac{(x^3 - 1)}{(x - 1)} + \ldots. + \frac{(x^P - 1)}{(x - 1)} \right\}$$

$$= 1 + 2(1)^{2-1} + 3(1)^{3-1} + \ldots.. + P(1)^{P-1}$$

$$= 1 + 2 + 3 + \ldots\ldots. + P$$

$$= \frac{P(P + 1)}{2}$$ (Using summation of P terms of AP)

Illustration 22 Let $\Delta(x) = \begin{vmatrix} f(x+\alpha) & f(x+2\alpha) & f(x+3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix}$

for some real values differential function f and constant α. Find $\lim\limits_{x \to 0} \dfrac{\Delta(x)}{x}$.

Solution. Here, $\Delta(0) = 0$

Thus, $$\lim_{x \to 0} \frac{\Delta(x)}{x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$\therefore$ Applying L'Hospital's rule,

$$\lim_{x \to 0} \frac{\Delta(x)}{x} = \lim_{x \to 0} \frac{\Delta'(x)}{1} = \Delta'(0)$$

$\therefore$ $$\lim_{x \to 0} \frac{\Delta(x)}{x} = \Delta'(0) \qquad \text{...(i)}$$

Given, $$\Delta(x) = \begin{vmatrix} f(x+\alpha) & f(x+2\alpha) & f(x+3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix}$$

Using definition of differentiation of determinant

$$\Delta'(x) = \begin{vmatrix} f'(x+\alpha) & f'(x+2\alpha) & f'(x+3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix}$$

$$+ \begin{vmatrix} f(x+\alpha) & f(x+2\alpha) & f(x+3\alpha) \\ 0 & 0 & 0 \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix} + \begin{vmatrix} f(x+\alpha) & f(x+2\alpha) & f(x+3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ 0 & 0 & 0 \end{vmatrix}$$

[As α is constant $\therefore \dfrac{d}{dx}(\alpha) = 0$]

$\therefore$ $$\Delta'(x) = \begin{vmatrix} f'(x+\alpha) & f'(x+2\alpha) & f'(x+3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix}$$

or $$\Delta'(0) = \begin{vmatrix} f'(\alpha) & f'(2\alpha) & f'(3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix} = 0$$

Hence, $$\lim_{x \to 0} \frac{\Delta(x)}{x} = \Delta'(0) = 0$$

Illustration 23 The graph of function $y = f(x)$ has a unique tangent at $(e^a, 0)$ through which the graph passes, then $\lim\limits_{x \to e^a} \dfrac{\log(1 + 7f(x)) - \sin(f(x))}{3f(x)}$ equals to

(a) 1 (b) 2

(c) 7 (d) None of these

Solution. Here, $\lim_{x \to e^a} \dfrac{\log(1+7f(x)) - \sin(f(x))}{3f(x)}$ $\left(\dfrac{0}{0} \text{ form}\right)$

Using L'Hospital's rule

$$= \lim_{x \to e^a} \frac{7f'(x) - \{\cos(f(x)) \cdot f'(x)\}\{1+7f(x)\}}{3f'(x) \cdot \{1+7f(x)\}}$$

$$= \lim_{x \to e^a} \frac{7 - \cos(f(x))\{1+7f(x)\}}{3\{1+7f(x)\}}$$

$$= \frac{7-1}{3} = 2$$

Hence, (b) is the correct answer.

Target Exercise 5.2

1. If $f(a)=2$, $f'(a)=1$, $g(a)=-1$, $g'(a)=-2$, then $\lim_{x \to a} \dfrac{g(x)f(a) - g(a)f(x)}{x-a}$, is

(a) -5 (b) 3
(c) -3 (d) 5

2. The value of $\lim_{\substack{x \to 1 \\ y \to 0}} \dfrac{y^3}{x^3 - y^2 - 1}$ as $(x, y) \to (1, 0)$ along the line $y = x - 1$, is

(a) 1 (b) -1
(c) 0 (d) Doesn't exist

3. The value of $\lim_{x \to 0} \dfrac{x \cos x - \log(1+x)}{x^2}$, is

(a) 1 (b) $\dfrac{1}{4}$
(c) $\dfrac{1}{2}$ (d) None of these

4. The value of $\lim_{x \to 0} \dfrac{e^{x^2} - \cos x}{x^2}$, is

(a) $\dfrac{3}{2}$ (b) $-\dfrac{3}{2}$
(c) $\dfrac{1}{2}$ (d) $-\dfrac{1}{2}$

5. The value of $\lim_{x \to a} \dfrac{\cos x - \cos a}{\cot x - \cot a}$, is

(a) $-\sin^3 a$ (b) $\cos^3 a$
(c) $\sin^3 a$ (d) $\cot a$

(ii) Algebraic Function of ∞ Type

(a) $\dfrac{\infty}{\infty}$ **form** First we should know the limiting values of a^x $(a > 0)$ as $x \to \infty$. See the graphs of these functions.

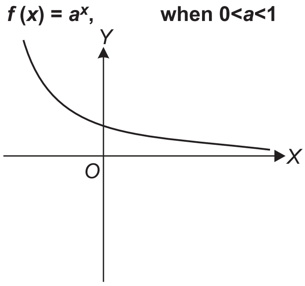


Fig. 5.10

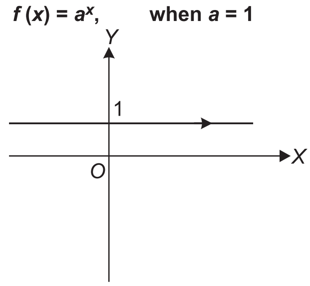


Fig. 5.11

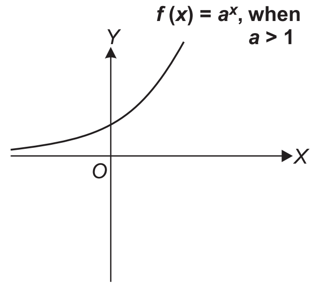


Fig. 5.12

Now, see the graph for a^x when $a > 1$. This graph appears to touch x-axis in the negative side of x-axis and thereafter it increases rapidly.

This's why because $\lim\limits_{x \to -\infty} a^x \to 0$, again you will also find the result, $\lim\limits_{x \to \infty} a^x \to \infty$

Thus, we have $$\lim_{x \to \infty} a^x = \begin{cases} \infty, & \text{if } a > 1 \\ 1, & \text{if } a = 1 \\ 0, & \text{if } 0 \le a < 1 \end{cases}$$

This type of problems are solved by taking the highest power of the terms tending to infinity as common numerator and denominator. That is after they are cancelled and the rest output is the result or (apply L'Hospital's rule).

Illustration 24 Evaluate $\lim\limits_{x \to \infty} \dfrac{x^2 + 5}{x^2 + 4x + 3}$.

Solution. **Method I** $\lim\limits_{x \to \infty} \dfrac{x^2 + 5}{x^2 + 4x + 3}$

Dividing numerator and denominator by x^2

$$= \lim_{x \to \infty} \frac{1 + \frac{5}{x^2}}{1 + \frac{4}{x} + \frac{3}{x^2}} = \frac{1+0}{1+0+0}$$

(because $\dfrac{K}{x} \to 0$, when $x \to \infty$ where K is any constant)

$= 1$

Method II $\lim\limits_{x \to \infty} \dfrac{x^2 + 5}{x^2 + 4x + 3}$ $\left(\dfrac{\infty}{\infty} \text{ form}\right)$

Applying L'Hospital's rule,

$$= \lim_{x \to \infty} \frac{2x}{2x + 4} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

Again, applying L'Hospital's rule,

$$= \lim_{x \to \infty} \frac{2}{2} = 1$$

Illustration 25 Evaluate $\lim_{n\to\infty} \dfrac{(n+2)!+(n+1)!}{(n+2)!-(n+1)!}$.

Solution. $\lim_{n\to\infty} \dfrac{(n+2)(n+1)!+(n+1)!}{(n+2)(n+1)!-(n+1)!}$

$$= \lim_{n\to\infty} \frac{(n+1)![\,n+2+1]}{(n+1)![\,n+2-1]} = \lim_{n\to\infty} \frac{(n+3)}{(n+1)} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{n\to\infty} \frac{1+3/n}{1+1/n} \qquad \left[\text{As } \frac{1}{n}\to 0,\ \text{ as } n\to\infty\right]$$

$$= \frac{1+0}{1+0} = 1$$

(b) $\infty - \infty$ **form** Such problems are simplified (generally rationalised) first, thereafter they generally acquire $\left(\dfrac{\infty}{\infty}\right)$ form.

Illustration 26 Evaluate $\lim_{x\to\infty} (x - \sqrt{x^2+x}\,)$.

Solution. $\lim_{x\to\infty} (x - \sqrt{x^2+x}\,) \qquad (\infty - \infty\ \text{ form})$

$$= \lim_{x\to\infty} \frac{x-\sqrt{x^2+x}}{1} \times \frac{x+\sqrt{x^2+x}}{x+\sqrt{x^2+x}}$$

$$= \lim_{x\to\infty} \frac{x^2-(x^2+x)}{x+\sqrt{x^2+x}} = \lim_{x\to\infty} \frac{-x}{x+\sqrt{x^2+x}}$$

$$= \lim_{x\to\infty} \frac{-x}{x\left\{1+\sqrt{1+\dfrac{1}{x}}\right\}}$$

$$= \lim_{x\to\infty} \frac{-1}{1+\sqrt{1+\dfrac{1}{x}}} = -\frac{1}{2} \qquad \left[\text{As } \frac{1}{x}\to 0, \text{ as } x\to\infty\right]$$

Illustration 27 Evaluate $\lim_{x\to\infty} (\sqrt{x^2+x+1} - \sqrt{x^2+1})$.

Solution. $\lim_{x\to\infty} (\sqrt{x^2+x+1} - \sqrt{x^2+1}) \qquad (\infty - \infty\ \text{ form})$

$$= \lim_{x\to\infty} \frac{(\sqrt{x^2+x+1}-\sqrt{x^2+1})}{1} \times \frac{(\sqrt{x^2+x+1}+\sqrt{x^2+1})}{(\sqrt{x^2+x+1}+\sqrt{x^2+1})}$$

$$= \lim_{x\to\infty} \frac{(x^2+x+1)-(x^2+1)}{\sqrt{x^2+x+1}+\sqrt{x^2+1}}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x+1}+\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\dfrac{1}{x}+\dfrac{1}{x^2}}+\sqrt{1+\dfrac{1}{x^2}}}$$

$$= \frac{1}{1+1} \qquad \left[\text{As } \frac{1}{x}\to 0, \text{ as } x\to\infty\right]$$

$$= \frac{1}{2}$$

An Important Result

If m, n are positive integers and $a_0, b_0 \neq 0$ and non-zero real numbers, then

$$\lim_{x \to \infty} \frac{a_0 x^m + a_1 x^{m-1} + \ldots + a_{m-1}x + a_m}{b_0 x^n + b_1 x^{n-1} + \ldots + b_{n-1}x + b_n} = \begin{cases} 0, & m < n \\ \dfrac{a_0}{b_0}, & m = n \\ \infty, & m > n \quad \text{when } a_0 b_0 > 0 \\ -\infty, & m > n \quad \text{when } a_0 b_0 < 0 \end{cases}$$

Illustration 28 Evaluate $\lim\limits_{x \to \infty} \dfrac{ax^2 + b}{x + 1}$, when $a \geq 0$.

Solution. Here, if $a \neq 0$

$$\lim_{x \to \infty} \frac{ax^2 + b}{x + 1} \quad \text{[As degree of numerator > degree of denominator]}$$

$$= \lim_{x \to \infty} \frac{ax^2 + b}{x + 1} = \infty \qquad \text{(As } a > 0\text{)}$$

Again, if $a = 0$

$$= \lim_{x \to \infty} \frac{0 \cdot x^2 + b}{x + 1}$$

$$= \lim_{x \to \infty} \frac{b}{x + 1}$$

$$= 0 \qquad \text{[As degree of numerator < degree of denominator]}$$

$$\therefore \qquad \lim_{x \to \infty} \frac{ax^2 + b}{x + 1} = \begin{cases} \infty, & a > 0 \\ 0, & a = 0 \end{cases}$$

Illustration 29 If $\lim\limits_{x \to \infty} \left(\dfrac{x^2 + 1}{x + 1} - ax - b \right) = 0$, find the values of a and b.

Solution. We have, $\lim\limits_{x \to \infty} \left(\dfrac{x^2 + 1}{x + 1} - ax - b \right) = 0$

$$= \lim_{x \to \infty} \frac{x^2 + 1 - ax^2 - ax - bx - b}{x + 1} = 0$$

$$= \lim_{x \to \infty} \frac{x^2(1 - a) - x(a + b) + (1 - b)}{x + 1} = 0$$

Since, the limit of above expression is zero.

$\therefore$ Degree of numerator < Degree of denominator

So, numerator must be a constant *ie*, a zero degree polynomial.

$\therefore$ $\quad 1 - a = 0$ and $a + b = 0$

Hence, $\quad a = 1$ and $b = -1$

Illustration 30 If $\lim_{x \to \infty} \left(\frac{x^2 - 1}{x + 1} - ax - b \right) = 2$, then find the values of a and b.

Solution. We have,

$$\lim_{x \to \infty} \left(\frac{x^2 - 1}{x + 1} - ax - b \right) = \lim_{x \to \infty} \frac{x^2 (1 - a) - x (a + b) + (1 + b)}{(x + 1)} = 2$$

Since, limit of above expression is a finite non-zero number.

$\therefore$ Degree of numerator = Degree of denominator

$\Rightarrow \quad 1 - a = 0 \quad \Rightarrow \quad a = 1$

$\therefore$ Putting $a = 1$ in above limit, we get $\lim_{x \to \infty} \frac{-x \ (1 + b) \ + \ (1 + b)}{x \ + \ 1} = 2$

$\Rightarrow \quad -(1 + b) = 2 \quad \Rightarrow \quad b = -3$

Hence, $a = 1$ and $b = -3$

Illustration 31 If $\lim_{x \to \infty} \left(\frac{x^2 + 1}{x + 1} - ax - b \right) = \infty$, then find a and b.

Solution. If $\lim_{x \to \infty} \left(\frac{x^2 + 1}{x + 1} - ax - b \right) = \infty$

$$\Rightarrow \quad \lim_{x \to \infty} \frac{x^2 (1 - a) - x (a + b) + (1 - b)}{x + 1} = \infty$$

The limit of above expression is infinity.

$\therefore$ Degree of numerator > Degree of denominator

$\Rightarrow \quad 1 - a > 0 \quad \Rightarrow \quad a \neq 1$

Hence, $a < 1$ and b can assume any real value.

Illustration 32 Let $S_n = 1 + 2 + 3 + \ldots\ldots + n$

and $$P_n = \frac{S_2}{S_2 - 1} \cdot \frac{S_3}{S_3 - 1} \cdot \frac{S_4}{S_4 - 1} \cdots \frac{S_n}{S_n - 1}$$

where $n \in N \, (n \geq 2)$. Then, find $\lim_{n \to \infty} P_n$.

Solution. As, $S_n = \frac{n (n + 1)}{2}$

$$\therefore \quad S_n - 1 = \frac{n (n + 1)}{2} - 1 = \frac{n^2 + n - 2}{2} = \frac{(n + 2)(n - 1)}{2}$$

$$\therefore \quad \frac{S_n}{S_n - 1} = \left(\frac{n}{n - 1} \right) \left(\frac{n + 1}{n + 2} \right)$$

$$\Rightarrow \quad P_n = \left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n}{n - 1} \right) \left(\frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdots \frac{n + 1}{n + 2} \right)$$

$$= \left(\frac{n}{1} \right) \left(\frac{3}{n + 2} \right)$$

$$\therefore \quad \lim_{n \to \infty} P_n = \lim_{n \to \infty} \frac{3n}{n + 2} = 3$$

Illustration 33 Let $f(x) = \lim_{n \to \infty} \dfrac{1}{\left(\frac{3}{\pi}\tan^{-1} 2x\right)^{2n} + 5}$. Then, the set of values of x for which $f(x) = 0$, is

(a) $|2x| > \sqrt{3}$ (b) $|2x| < \sqrt{3}$

(c) $|2x| \geq \sqrt{3}$ (d) $|2x| \leq \sqrt{3}$

Solution. $f(x) = 0$, if and only if $\left(\frac{3}{\pi}\tan^{-1} 2x\right)^2 > 1$

$\Rightarrow \qquad \tan^{-1} 2x > \frac{\pi}{3}$ or $\tan^{-1} 2x < -\frac{\pi}{3}$

$\Rightarrow \qquad 2x > \sqrt{3}$ or $2x < -\sqrt{3}$

$\Rightarrow \qquad |2x| > \sqrt{3}$

Hence, (a) is the correct answer.

Illustration 34 If $\lim_{n \to \infty} \dfrac{1}{(\sin^{-1} x)^n + 1} = 1$, then x lies in the interval

(a) $(-\sin 1, \sin 1)$ (b) $(-1, 1)$

(c) $(0, 1)$ (d) $(-1, 0)$

Solution. Here, $\lim_{n \to \infty} \dfrac{1}{(\sin^{-1} x)^n + 1} = 1$ is possible only, if

$$-1 < \sin^{-1} x < 1 \quad \Rightarrow \quad x \in (-\sin 1, \sin 1)$$

Hence, (a) is the correct answer.

Illustration 35 The value of $\lim_{x \to \infty} \left(\dfrac{x^2 + 2x - 1}{2x^2 - 3x - 2}\right)^{\frac{2x+1}{2x-1}}$ is equal to

(a) 0 (b) 1

(c) $e^{1/2}$ (d) $\dfrac{1}{2}$

Solution. As, $\lim_{x \to \infty} \left(\dfrac{x^2 + 2x - 1}{2x^2 - 3x - 2}\right)^{\frac{2x+1}{2x-1}}$

$$\left\{\text{where, } \lim_{x \to \infty} \frac{x^2 + 2x - 1}{2x^2 - 3x - 2} = \frac{1}{2} \text{ and } \lim_{x \to \infty} \frac{2x+1}{2x-1} = 1\right\}$$

$$\therefore \qquad \lim_{x \to \infty} \left(\frac{x^2 + 2x - 1}{2x^2 - 3x - 2}\right)^{\frac{2x+1}{2x-1}} = \frac{1}{2}$$

Hence, (d) is the correct answer.

Target Exercise 5.3

1. The value of $\lim\limits_{x \to 0} \left(\frac{1}{x^2} - \cot x \right)$, is

(a) 0 (b) 1

(c) $\frac{1}{4}$ (d) None of these

2. The value of $\lim\limits_{x \to \infty} (\sqrt{a^2x^2 + ax + 1} - \sqrt{a^2x^2 + 1}), (a > 0)$ is

(a) $\frac{1}{2}$

(b) $-\frac{1}{2}$

(c) Doesn't exist

(d) None of the above

3. The value of $\lim\limits_{n \to \infty} \frac{1^3 + 2^3 + 3^3 + \ldots + n^3}{(n^2 + 1)^2}$, is

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$

(c) $\frac{1}{2\sqrt{2}}$ (d) None of these

4. The value of $\lim\limits_{n \to \infty} \frac{1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \ldots + n \cdot 1}{1^2 + 2^2 + \ldots + n^2}$, is

(a) 1 (b) -1

(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{2}$

5. The value of $\lim\limits_{n \to \infty} \frac{a^n + b^n}{a^n - b^n}$, (where $a > b > 1$) is

(a) 1 (b) -1

(c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$

(2) Trigonometrical Limits

To evaluate trigonometric limits the following results are very important.

(i) $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$ **(ii)** $\lim\limits_{x \to 0} \frac{\tan x}{x} = 1$

(iii) $\lim\limits_{x \to 0} \frac{\sin^{-1} x}{x} = 1$ **(iv)** $\lim\limits_{x \to 0} \frac{\tan^{-1} x}{x} = 1$

(v) $\lim\limits_{x \to 0} \frac{\sin x^0}{x} = \frac{\pi}{180°}$ **(vi)** $\lim\limits_{x \to 0} \cos x = 1$

(vii) $\lim\limits_{x \to a} \frac{\sin (x-a)}{x-a} = 1$ **(viii)** $\lim\limits_{x \to a} \frac{\tan (x-a)}{x-a} = 1$

Illustration 36 Evaluate $\lim\limits_{x \to 0} \dfrac{1-\cos x}{x^2}$.

Solution. $\lim\limits_{x \to 0} \dfrac{1-\cos x}{x^2}$ $\qquad\left(\dfrac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{2\sin^2 x/2}{x^2} = \lim_{x \to 0} \frac{2}{4} \frac{\sin^2 x/2}{x^2/4}$$

$$= \lim_{x \to 0} \frac{1}{2}\left(\frac{\sin x/2}{x/2}\right)^2 = \frac{1}{2}(1)^2 = \frac{1}{2}$$

Illustration 37 Solve $\lim\limits_{x \to 0} \dfrac{1-\cos(1-\cos x)}{\sin^4 x}$

Solution. Here, $\lim\limits_{x \to 0} \dfrac{1-\cos(1-\cos x)}{\sin^4 x}$

$$= \lim_{x \to 0} \frac{1-\cos\left(2\sin^2 \frac{x}{2}\right)}{x^4} \Bigg/ \frac{\sin^4 x}{x^4}$$

$$= \lim_{x \to 0} \frac{1-\cos\left(2\sin^2 \frac{x}{2}\right)}{x^4} \Bigg/ \lim_{x \to 0} \frac{\sin^4 x}{x^4}$$

$$= \lim_{x \to 0} \frac{2\sin^2\left(\sin^2 \frac{x}{2}\right)}{x^4} \Bigg/ 1$$

$$= \lim_{x \to 0} 2 \cdot \left(\frac{\sin\left(\sin^2 \frac{x}{2}\right)}{\sin^2 \frac{x}{2}} \cdot \frac{\sin^2 \frac{x}{2}}{4 \cdot \frac{x^2}{4}}\right)^2$$

$$= 2 \times \frac{1}{4^2} = \frac{1}{8}$$

Don't do it

$$\lim_{x \to 0} \frac{1-\cos\left(\frac{1-\cos x}{x^2} \cdot x^2\right)}{x^4} = \lim_{x \to 0} \frac{1-\cos\left(\frac{x^2}{2}\right)}{x^4}, \qquad \left(\text{As } \lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}\right)$$

$$= \lim_{x \to 0} \frac{2\sin^2 \frac{x^2}{4}}{\frac{x^4}{16} \times 16} = \frac{1}{8}$$ is wrong although the answer may be correct.

Illustration 38 Evaluate $\lim\limits_{x \to \infty} 2^{-x} \sin(2^x)$.

Solution. $2^{-x} = \dfrac{1}{2^x}$

We know, as $x \to \infty, 2^x \to \infty$

$\therefore$ The given limit $= 0 \times$ [A finite number between –1 and +1] $= 0$

Hence, $\qquad \lim\limits_{x \to \infty} \dfrac{\sin(2^x)}{(2^x)} = 0$

Illustration 39 Evaluate $\lim_{x \to \infty} e^x \sin(d/e^x)$.

Solution. When $x \to \infty, e^x \to \infty$

But, $\quad$ angle of sine $= \dfrac{d}{e^x} = \dfrac{\text{finite}}{\infty} = 0$

$\therefore \quad$ The given limit $= \lim_{x \to \infty} \dfrac{\sin(d/e^x)}{1/e^x}$

$$= \lim_{x \to \infty} \frac{\sin d/e^x}{d/e^x} \times d = 1 \times d = d$$

Illustration 40 Evaluate $\lim_{x \to \infty} \sqrt{\dfrac{x - \sin x}{x + \cos^2 x}}$.

Solution. $\lim_{x \to \infty} \sqrt{\dfrac{x - \sin x}{x + \cos^2 x}} = \lim_{x \to \infty} \sqrt{\dfrac{x\left(1 - \dfrac{\sin x}{x}\right)}{x\left(1 + \dfrac{\cos^2 x}{x}\right)}}$

$$= \lim_{x \to \infty} \sqrt{\frac{1 - \dfrac{\sin x}{x}}{1 + \dfrac{\cos^2 x}{x}}} = \sqrt{\frac{1-0}{1+0}} = 1$$

Illustration 41 Evaluate $\lim_{x \to y} \dfrac{\sin^2 x - \sin^2 y}{x^2 - y^2}$.

Solution. $\lim_{x \to y} \dfrac{\sin^2 x - \sin^2 y}{x^2 - y^2} = \lim_{x \to y} \dfrac{\sin(x+y)\sin(x-y)}{(x+y)(x-y)}$

$$= \lim_{x \to y} \frac{\sin(x+y)}{(x+y)} \times \lim_{x \to y} \frac{\sin(x-y)}{(x-y)} = \frac{\sin(2y)}{2y} \times 1$$

[As, $x \to y \Rightarrow (x-y) \to 0$, but $\Rightarrow x + y \to 2y$]

$$= \frac{\sin 2y}{2y}$$

Illustration 42 Evaluate $\lim_{x \to 0} \dfrac{x \tan 2x - 2x \tan x}{(1 - \cos 2x)^2}$. **[IIT JEE 1999]**

Solution. $\lim_{x \to 0} \dfrac{x \tan 2x - 2x \tan x}{(1 - \cos 2x)^2}$

$$= \lim_{x \to 0} \frac{x\left[2x + \dfrac{2^3 x^3}{3} + 2\,\dfrac{2^5 x^5}{15} + \ldots\right] - 2x\left[x + \dfrac{x^3}{3} + 2\,\dfrac{x^5}{15} + \ldots\right]}{(2\sin^2 x)^2}$$

$$= \lim_{x \to 0} \frac{x^4\left(\dfrac{8}{3} - \dfrac{2}{3}\right) + x^6\left(\dfrac{64}{15} - \dfrac{4}{15}\right) + \ldots}{4\left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots\right)^4} = \lim_{x \to 0} \frac{2 + 4x^2 + \ldots}{4\left[1 - \dfrac{x^2}{3!} + \ldots\right]}$$

$$= \frac{2}{4} = \frac{1}{2}$$

Illustration 43 Evaluate $\lim_{x \to 0} \dfrac{\sin(\pi \cos^2 x)}{x^2}$. **[IIT JEE 2001]**

Solution. $\lim_{x \to 0} \dfrac{\sin(\pi \cos^2 x)}{x^2}$

$$= \lim_{x \to 0} \frac{\sin\{\pi(1 - \sin^2 x)\}}{x^2} = \lim_{x \to 0} \frac{\sin(\pi - \pi \sin^2 x)}{x^2}$$

$$= \lim_{x \to 0} \frac{\sin(\pi \sin^2 x)}{x^2} = \lim_{x \to 0} \left\{\frac{\sin(\pi \sin^2 x)}{\pi \sin^2 x} \times \frac{\pi}{1} \times \frac{\sin^2 x}{x^2}\right\}$$

$$= \lim_{x \to 0} \frac{\sin(\pi \sin^2 x)}{\pi \sin^2 x} \times \pi \times \lim_{x \to 0} \frac{\sin^2 x}{x^2}$$

$$= 1 \times \pi \times 1 = \pi$$

Illustration 44 Solve $\lim_{x \to \frac{\pi}{4}} \dfrac{\sqrt{1 - \sqrt{\sin 2x}}}{\pi - 4x}$

Solution. Here,

$$\lim_{x \to \frac{\pi}{4}} \frac{\sqrt{1 - \sqrt{\sin 2x}}}{\pi - 4x} = \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{1 - \sqrt{\sin 2x}}}{\pi - 4x} \cdot \frac{\sqrt{1 + \sqrt{\sin 2x}}}{\sqrt{1 + \sqrt{\sin 2x}}}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{1 - \sin 2x}}{\pi - 4x} \cdot \lim_{x \to \frac{\pi}{4}} \frac{1}{\sqrt{1 + \sqrt{\sin 2x}}}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{\sin^2\left(\frac{\pi}{4} - x\right)}}{\pi - 4x} \cdot 1 = \lim_{x \to \frac{\pi}{4}} \frac{\left|\sin\left(\frac{\pi}{4} - x\right)\right|}{4\left(\frac{\pi}{4} - x\right)}$$

which gives RHL at $x = \dfrac{\pi}{4} = -\dfrac{1}{4}$

and LHL at $x = \dfrac{\pi}{4} = \dfrac{1}{4}$

Thus, $\lim_{x \to \frac{\pi}{4}} \dfrac{\sqrt{1 - \sqrt{\sin 2x}}}{\pi - 4x} = \pm \dfrac{1}{4}$

$\therefore$ Limit doesn't exist (as not unique).

(3) Logarithmic Limits

In this section we shall deal with the problems based on expansion of logarithmic series which is given below.

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots \infty$$

where, $-1 \le x \le 1$ and it should be noted that the expansion is true only if the base is e.

To evaluate the logarithmic limit, we use $\lim_{x \to 0} \dfrac{\log(1 + x)}{x} = 1$

Illustration 45 Evaluate $\lim_{x \to a} \dfrac{\log\{1+(x-a)\}}{(x-a)}$.

Solution. Let $x - a = y$, when $x \to a$; $y \to 0$

$\therefore$ The given limit $\Rightarrow \lim_{y \to 0} \dfrac{\log\{1+y\}}{y} = 1$

Illustration 46 Evaluate $\lim_{h \to 0} \dfrac{\log_{10}(1+h)}{h}$.

Solution. $\lim_{h \to 0} \dfrac{\log_{10}(1+h)}{h}$

$$= \lim_{h \to 0} \frac{\log_e(1+h) \times \log_{10} e}{h} = \lim_{h \to 0} \frac{\log_e(1+h)}{h} \times \log_{10} e$$

$$= \log_{10} e \times 1 = \log_{10} e$$

Illustration 47 Evaluate $\lim_{x \to 0} \dfrac{\log(5+x) - \log(5-x)}{x}$.

Solution. $\lim_{x \to 0} \dfrac{\log\left\{5\left(1+\dfrac{x}{5}\right)\right\} - \log\left\{5\left(1-\dfrac{x}{5}\right)\right\}}{x}$

$$= \lim_{x \to 0} \frac{\log 5 + \log\left(1+\dfrac{x}{5}\right) - \log 5 - \log\left(1-\dfrac{x}{5}\right)}{x}$$

$$= \lim_{x \to 0} \frac{\log\left(1+\dfrac{x}{5}\right)}{5\left(\dfrac{x}{5}\right)} - \frac{\log\left(1-\dfrac{x}{5}\right)}{-5\left(-\dfrac{x}{5}\right)} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

Illustration 48 Evaluate $\lim_{h \to 0} \dfrac{\log_e(1+2h) - 2\log_e(1+h)}{h^2}$. **[IIT JEE 1999]**

Solution. $\lim_{h \to 0} \dfrac{\log_e(1+2h) - 2\log_e(1+h)}{h^2}$

$$= \lim_{h \to 0} \frac{\left((2h) - \dfrac{(2h)^2}{2} + \dfrac{(2h)^3}{3} - \ldots.\infty\right) - 2\left(h - \dfrac{h^2}{2} + \dfrac{h^3}{3} - \ldots..\right)}{h^2}$$

$$= \lim_{h \to 0} \frac{-h^2 + 2h^3 - \ldots..}{h^2} = \lim_{h \to 0} \frac{h^2\{-1 + 2h - \ldots..\}}{h^2}$$

$$= \lim_{h \to 0} \{-1 + 2h - \ldots..\} = -1$$

Illustration 49 Solve $\lim_{x \to \infty} \left\{x - x^2 \cdot \log\left(1 + \dfrac{1}{x}\right)\right\}$

Solution. Here, put $x = \dfrac{1}{y}$

$$\Rightarrow \quad \lim_{y \to 0} \left\{\frac{1}{y} - \frac{\log(1+y)}{y^2}\right\}$$

$$\Rightarrow \quad \lim_{y \to 0} \frac{y - \log(1+y)}{y^2}$$

$$\Rightarrow \qquad \lim_{y \to 0} \frac{y - \left\{ y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots \right\}}{y^2}$$

$$\Rightarrow \qquad \lim_{y \to 0} \frac{y^2 \left\{ \frac{1}{2} - \frac{y}{3} + \ldots \right\}}{y^2}$$

$$\Rightarrow \qquad \lim_{y \to 0} \left\{ \frac{1}{2} - \frac{y}{3} + \ldots \right\} = \frac{1}{2}$$

Don't do it

$$\lim_{y \to 0} \left\{ \frac{1}{y} - \frac{\log(1+y)}{y^2} \right\}$$

$$= \lim_{y \to 0} \left\{ \frac{1}{y} - \frac{\log(1+y)}{y} \cdot \frac{1}{y} \right\} \qquad \left[\text{As, } \lim_{y \to 0} \frac{\log(1+y)}{y} = 1 \right]$$

$$= \lim_{y \to 0} \left(\frac{1}{y} - \frac{1}{y} \right) = 0 \text{ is not correct.}$$

(4) **Exponential Limits**

(i) **Based on Series Expansion**

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.. \infty$$

To evaluate the exponential limit, we use the following results.

(a) $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$ **(b)** $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \log_e a$

Illustration 50 Evaluate $\lim\limits_{x \to 0} \dfrac{a^x - b^x}{x}$.

Solution. $\lim\limits_{x \to 0} \dfrac{a^x - b^x}{x} = \lim\limits_{x \to 0} \dfrac{(a^x - 1) - (b^x - 1)}{x}$

$$= \lim_{x \to 0} \frac{a^x - 1}{x} - \lim_{x \to 0} \frac{b^x - 1}{x}$$

$$= \log a - \log b = \log (a/b)$$

Illustration 51 Evaluate $\lim\limits_{x \to 0} \dfrac{(ab)^x - a^x - b^x + 1}{x^2}$.

Solution. $\lim\limits_{x \to 0} \dfrac{(ab)^x - a^x - b^x + 1}{x^2} = \lim\limits_{x \to 0} \dfrac{a^x b^x - a^x - b^x + 1}{x^2}$

$$= \lim_{x \to 0} \frac{a^x (b^x - 1) - (b^x - 1)}{x^2}$$

$$= \lim_{x \to 0} \frac{(a^x - 1)}{x} \times \lim_{x \to 0} \frac{(b^x - 1)}{x}$$

$$= \log a \times \log b$$

Illustration 52 Evaluate $\lim_{x \to 0} \dfrac{e^{\tan x} - e^x}{\tan x - x}$.

Solution. $\lim_{x \to 0} \dfrac{e^{\tan x} - e^x}{\tan x - x} = \lim_{x \to 0} \dfrac{e^x \times e^{(\tan x - x)} - e^x}{(\tan x - x)}$

$$= \lim_{x \to 0} \frac{e^x \{e^{\tan x - x} - 1\}}{(\tan x - x)}$$

$$= e^0 \times 1 \qquad [\text{As } x \to 0, \tan x - x \to 0]$$

$$= 1 \times 1 = 1$$

Illustration 53 Evaluate $\lim_{x \to 0} \dfrac{ae^x - b}{x} = 2$. Find a and b.

Solution. $\lim_{x \to 0} \dfrac{a\left(1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dots \infty\right) - b}{x} = 2$

$$\lim_{x \to 0} \frac{(a - b) + xa + \dfrac{ax^2}{2!} + \dots\dots \infty}{x} = 2$$

Since, limit is finite, $(a - b) = 0 \Rightarrow b = a$

$$\therefore \qquad \lim_{x \to 0} \frac{xa + \dfrac{ax^2}{2!} + \dots \infty}{x} = 2$$

$$\lim_{x \to 0} a + \frac{ax}{2!} + \dots \infty = 2$$

$$\Rightarrow \qquad a = 2 \qquad \therefore \qquad b = 2$$

Illustration 54 Solve $\lim_{x \to 0} \dfrac{a \sin x - bx + cx^2 + x^3}{2x^2 \log(1 + x) - 2x^3 + x^4}$, if it exists and is finite also, then find a, b, c.

Solution. Here, $\lim_{x \to 0} \dfrac{a \sin x - bx + cx^2 + x^3}{2x^2 \log(1 + x) - 2x^3 + x^4}$

$$\Rightarrow \qquad \lim_{x \to 0} \frac{a\left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dots\right) - bx + cx^2 + x^3}{2x^2\left(x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dots\right) - 2x^3 + x^4}$$

$$\Rightarrow \qquad \lim_{x \to 0} \frac{(a - b)x + cx^2 + \left(1 - \dfrac{a}{3!}\right)x^3 + \dfrac{a}{5!}x^5 + \dots}{\dfrac{2}{3}x^5 - \dfrac{1}{2}x^6 + \dots\dots}$$

For finite limit,

$$a - b = 0, c = 0, 1 - \frac{a}{3!} = 0 \qquad (ie, a = b = 6, c = 0)$$

$$= \lim_{x \to 0} \frac{\frac{a}{5!} + \text{higher powers of } x}{\frac{2}{3} - \frac{1}{2}x + \ldots}$$

$$= \frac{a}{5!} \cdot \frac{3}{2} = \frac{a}{80} = \frac{6}{80} = \frac{3}{40}$$

$$\therefore \qquad \lim_{x \to 0} \frac{a \sin x - bx + cx^2 + x^3}{2x^2 \log(1+x) - 2x^3 + x^4} = \frac{3}{40}$$

where $a = 6 = b$, $c = 0$

Illustration 55 Solve $\lim_{x \to \frac{\pi}{2}} \sqrt{\frac{\tan x - \sin\{\tan^{-1}(\tan x)\}}{\tan x + \cos^2(\tan x)}}$

Solution. Here, $\lim_{x \to \frac{\pi}{2}} \sqrt{\frac{\tan x - \sin\{\tan^{-1}(\tan x)\}}{\tan x + \cos^2(\tan x)}}$

where, RHL at $x = \frac{\pi}{2}$

$$\lim_{x \to \frac{\pi}{2}^+} \sqrt{\frac{\tan x - \sin\{\tan^{-1}(\tan x)\}}{\tan x + \cos^2(\tan x)}}$$

$$= \lim_{x \to \frac{\pi}{2}^+} \sqrt{\frac{\tan x - \sin(x - \pi)}{\tan x + \cos^2(\tan x)}} \qquad \left(\because \tan^{-1}(\tan x) = x - \pi, \text{ when } x > \frac{\pi}{2}\right)$$

$$= \lim_{x \to \frac{\pi}{2}^+} \sqrt{\frac{1 + \frac{\sin x}{\tan x}}{1 + \frac{\cos^2(\tan x)}{\tan x}}}$$

$$= \sqrt{\frac{1+0}{1+0}} = 1$$

Again, LHL at $x = \frac{\pi}{2}$

$$\lim_{x \to \frac{\pi}{2}^-} \sqrt{\frac{\tan x - \sin\{\tan^{-1}(\tan x)\}}{\tan x + \cos^2(\tan x)}}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \sqrt{\frac{\tan x - \sin(x)}{\tan x + \cos^2(\tan x)}} \qquad \left(\text{As } \tan^{-1}(\tan x) = x, \text{ when } x < \frac{\pi}{2}\right)$$

$$= \lim_{x \to \frac{\pi}{2}^-} \sqrt{\frac{1 - \frac{\sin x}{\tan x}}{1 + \frac{\cos^2(\tan x)}{\tan x}}} = \sqrt{\frac{1+0}{1+0}} = 1$$

$$\lim_{x \to \frac{\pi}{2}} \sqrt{\frac{\tan x - \sin\{\tan^{-1}(\tan x)\}}{\tan x + \cos^2(\tan x)}} = 1$$

Illustration 56 Evaluate $\lim_{x \to 0} \frac{\tan^{-1} x - \sin^{-1} x}{\sin^3 x}$.

Solution. $\lim_{x \to 0} \frac{\tan^{-1} x - \sin^{-1} x}{\sin^3 x}$

$$= \lim_{x \to 0} \frac{\tan^{-1} x - \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)}{x^3 \left(\frac{\sin^3 x}{x^3}\right)}$$

$$= \lim_{x \to 0} \frac{\tan^{-1} x - \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)}{x^3} \qquad \left(\text{As } \lim_{x \to 0} \frac{\sin x}{x} = 1\right)$$

$$= \lim_{x \to 0} \frac{\tan^{-1}\left(\frac{x - \frac{x}{\sqrt{1-x^2}}}{1 + \frac{x \cdot x}{\sqrt{1-x^2}}}\right)}{x^3} = \lim_{x \to 0} \frac{\tan^{-1}\left[\frac{x\sqrt{1-x^2} - x}{x^2 + \sqrt{1-x^2}}\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{\tan^{-1}\left(\frac{x\sqrt{1-x^2} - x}{x^2 + \sqrt{1-x^2}}\right)}{x^3 \left(\frac{x\sqrt{1-x^2} - x}{x^2 + \sqrt{1-x^2}}\right)} \cdot \frac{x\sqrt{1-x^2} - x}{x^2 + \sqrt{1-x^2}}$$

$$= \lim_{x \to 0} \frac{x(\sqrt{1-x^2} - 1)}{x^3(x^2 + \sqrt{1-x^2})} \qquad \left[\text{As } \lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1\right]$$

$$= \lim_{x \to 0} \frac{x(\sqrt{1-x^2} - 1)}{x^3(x^2 + \sqrt{1-x^2})} \times \frac{\sqrt{1-x^2} + 1}{\sqrt{1-x^2} + 1}$$

$$= \lim_{x \to 0} \frac{-x^3}{x^3(x^2 + \sqrt{1-x^2})} \cdot \left[\frac{1}{\sqrt{1-x^2} + 1}\right]$$

$$= -\frac{1}{2}$$

Illustration 57 If $\lim_{x \to 0} \frac{x^a \sin^b x}{\sin x^c}$, where $a, b, c \in R - \{0\}$, exists and has non-zero value. Then, show $a + b = c$.

Solution. Here, $\lim_{x \to 0} \frac{x^a \sin^b x}{\sin x^c} = \lim_{x \to 0} x^a \cdot \left(\frac{\sin x}{x}\right)^b \cdot \left(\frac{x}{\sin x}\right)^c \cdot x^{b-c}$

$$= \lim_{x \to 0} x^{a+b-c} \cdot \left(\frac{\sin x}{x}\right)^b \cdot \left(\frac{x}{\sin x}\right)^c \qquad \text{...(i)}$$

Eq. (i) has non-zero value if and only if, it is independent of x.

ie, $\quad a + b - c = 0$

or $\quad a + b = c$

Illustration 58 The integer 'n' for which the

$$\lim_{x\to 0}\frac{\cos^2 x-\cos x-e^x\cos x+e^x-\dfrac{x^3}{2}}{x^n}$$ is a finite non-zero number is

(a) 2 (b) 3

(c) 4 (d) None of these

Solution. Given that, $\lim_{x\to 0}\dfrac{\cos^2 x-\cos x-e^x\cos x+e^x-\dfrac{x^3}{2}}{x^n}$

$$\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)-\dfrac{x^3}{2}}{x^n}$$

$$=\lim_{x\to 0}\frac{\left(1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\ldots-1\right)\left[\left(1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\ldots\right)-\left(1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\ldots\right)\right]-\dfrac{x^3}{2}}{x^n}$$

$$=\lim_{x\to 0}\frac{\left(-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\ldots\right)\left(-x-x^2-\dfrac{x^3}{3!}-\dfrac{2x^5}{5!}-\ldots\right)-\dfrac{x^3}{2}}{x^n}$$

$$=\lim_{x\to 0}\frac{\left(\dfrac{x^3}{2}+\dfrac{x^4}{2}+\dfrac{x^5}{12}-\ldots\right)-\dfrac{x^3}{2}}{x^n}$$

$$=\lim_{x\to 0}\frac{\dfrac{x^4}{2}+\dfrac{x^5}{12}-\dfrac{x^5}{24}}{x^n}=\text{a non-zero}$$

If $n=4$

Hence, (c) is the correct answer.

Illustration 59 If $I_1=\lim_{x\to 0}\sqrt{\dfrac{\tan^{-1}x}{x}-\dfrac{\sin^{-1}x}{x}}$

and $$I_2=\lim_{x\to 0}\sqrt{\frac{\sin^{-1}x}{x}-\frac{\tan^{-1}x}{x}},$$

where $|x|<1$, then which of the following statement is true?

(a) Neither I_1 nor I_2 exist

(b) I_1 exists and I_2 doesn't exist

(c) I_1 doesn't exist and I_2 exists

(d) None of the above

Solution. We know, $\dfrac{\tan^{-1}x}{x}<1$ and $\dfrac{\sin^{-1}x}{x}>1, \forall\, x\in R$

$$\therefore \quad \frac{\tan^{-1}x}{x}-\frac{\sin^{-1}x}{x}<0 \text{ and } \frac{\sin^{-1}x}{x}-\frac{\tan^{-1}x}{x}>0$$

$\Rightarrow$ I_1 doesn't exist and I_2 exists.

Hence, (c) is the correct answer.

Target Exercise 5.4

1. The value of $\lim_{x \to 0} |x|^{[\cos x]}$, (where $|\cdot|$ and $[\cdot]$ denotes modulus and greatest integer function respectively) is
   (a) 0
   (b) Doesn't exist
   (c) 1
   (d) None of these

2. The value of $\lim_{x \to a} \dfrac{x \sin a - a \sin x}{x - a}$, is
   (a) $a \sin a - \cos a$
   (b) $\sin a - a \cos a$
   (c) $\cos a + a \sin a$
   (d) $\sin a + a \cos a$

3. The value of $\lim_{x \to 0} \dfrac{e^x - e^{x \cos x}}{x + \sin x}$, is
   (a) 0
   (b) 1
   (c) –1
   (d) None of these

4. The value of $\lim_{x \to y} \dfrac{x^y - y^x}{x^x - y^y}$, is
   (a) $\dfrac{1 - \log x}{1 + \log x}$
   (b) $\dfrac{1 - \log y}{1 + \log y}$
   (c) $\dfrac{\log x - \log y}{\log x + \log y}$
   (d) None of these

5. The value of $\lim_{x \to 0} \dfrac{p^x - q^x}{r^x - s^x}$, is
   (a) $\dfrac{1 - \log p}{1 + \log p}$
   (b) $\dfrac{\log p - \log q}{\log r - \log s}$
   (c) $\dfrac{\log p \cdot \log q}{\log r \cdot \log s}$
   (d) None of these

6. The value of $\lim_{x \to \pi} \dfrac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2}$, is
   (a) $\dfrac{1}{4}$
   (b) $\dfrac{1}{2}$
   (c) 2
   (d) Doesn't exist

7. The value of $\lim_{\theta \to \pi/4} \dfrac{(\sqrt{2} - \cos\theta - \sin\theta)}{(4\theta - \pi)^2}$, is
   (a) $\dfrac{1}{16\sqrt{2}}$
   (b) $\dfrac{1}{16}$
   (c) $\dfrac{1}{8\sqrt{2}}$
   (b) $\dfrac{1}{2\sqrt{2}}$

8. The value of $\lim_{x \to \infty} (x + 2)\tan^{-1}(x + 2) - (x \tan^{-1} x)$, is
   (a) $\dfrac{\pi}{2}$
   (b) Doesn't exist
   (c) $\dfrac{\pi}{4}$
   (d) None of the above

9. The value of $\lim_{x \to \pi/4} \dfrac{(\cos x + \sin x)^3 - 2\sqrt{2}}{1 - \sin 2x}$, is

(a) $\dfrac{\sqrt{3}}{2}$ (b) $-\dfrac{3}{2}$

(c) $\dfrac{1}{2}$ (d) $\dfrac{1}{\sqrt{2}}$

10. $\lim_{x \to 4} \dfrac{(\cos \alpha)^x - (\sin \alpha)^x - \cos 2\alpha}{x - 4}$, $\alpha \in \left(0, \dfrac{\pi}{2}\right)$ is

(a) $\log(\cos \alpha) + (\sin \alpha)^4 \log(\sin \alpha)$

(b) $(\cos^4 \alpha) \log(\cos \alpha) - (\sin \alpha)^4 \log(\sin \alpha)$

(c) $(\cos^4 \alpha) \log(\cos \alpha)$

(d) None of the above

(ii) **Evaluation of Exponential Limits of the Form 1^∞**

To evaluate the exponential form 1^∞, we use the following results.

Result If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, then

$$\lim_{x \to a} \{1 + f(x)\}^{1/g(x)} = e^{\lim_{x \to a} \frac{f(x)}{g(x)}}$$

Or

when $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \infty$

Then, $\lim_{x \to a} \{f(x)\}^{g(x)} = \lim_{x \to a} [1 + f(x) - 1]^{g(x)}$

$$= e^{\lim_{x \to a} (f(x) - 1)\, g(x)}$$

Particular cases

(i) $\lim_{x \to 0} (1 + x)^{1/x} = e$

(ii) $\lim_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x = e$

(iii) $\lim_{x \to 0} (1 + \lambda x)^{1/x} = e^\lambda$

(iv) $\lim_{x \to \infty} \left(1 + \dfrac{\lambda}{x}\right)^x = e^\lambda$

Illustration 60 Evaluate $\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x$.

Solution. $\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x = e^{\lim_{x \to \infty} \frac{2}{x} \cdot x} = e^2$

Illustration 61 Evaluate $\lim_{x \to 1} (\log_3 3x)^{\log_x 3}$.

Solution. $\lim_{x \to 1} (\log_3 3x)^{\log_x 3} = \lim_{x \to 1} (\log_3 3 + \log_3 x)^{\log_x 3}$

$$= \lim_{x \to 1} (1 + \log_3 x)^{1/\log_3 x} \qquad \left[\text{As } \log_b a = \frac{1}{\log_a b}\right]$$

$$= e^{\lim_{x \to 1} \log_3 x \times \frac{1}{\log_3 x}} = e^1$$

Illustration 62 Evaluate $\lim_{x \to a} \left(2 - \frac{a}{x}\right)^{\tan \frac{\pi x}{2a}}$.

Solution. $\lim_{x \to a} \left(2 - \frac{a}{x}\right)^{\tan \frac{\pi x}{2a}} = \lim_{x \to a} \left\{1 + \left(1 - \frac{a}{x}\right)\right\}^{\tan \frac{\pi x}{2a}}$

$$= e^{\lim_{x \to a} \left(1 - \frac{a}{x}\right) \cdot \tan \frac{\pi x}{2a}}$$

$$= e^{\lim_{x \to a} \left(\frac{x - a}{x}\right) \cdot \tan \frac{\pi x}{2a}} = e^{\lim_{x \to a} f(x)} \qquad \ldots(i)$$

Let $x - a = h$, we get

$$e^{\lim_{h \to 0} \left(\frac{h}{a+h}\right) \cdot \tan \frac{\pi}{2a}(a+h)} = e^{\lim_{h \to 0} \frac{h}{a+h} \cdot \tan \left(\frac{\pi}{2} + \frac{\pi h}{2a}\right)}$$

$$= e^{\lim_{h \to 0} \frac{h}{a+h} \cdot \left(-\cot \left(\frac{\pi h}{2a}\right)\right)} = e^{\lim_{h \to 0} \frac{-h}{(a+h)\tan(\pi h/2a)} \cdot \frac{\pi}{2a} \cdot \frac{1}{\pi/2a}}$$

$$= e^{\lim_{h \to 0} \frac{-2a}{\pi(a+h)} \cdot \frac{\pi h/2a}{\tan(\pi h/2a)}} = e^{\frac{-2a}{\pi(a)}} = e^{-2/\pi}$$

Illustration 63 Evaluate $\lim_{x \to \infty} \left(\frac{x+6}{x+1}\right)^{x+4}$.

Solution. As $x \to \infty$, $\lim_{x \to \infty} \frac{x+6}{x+1} = 1$ and $(x+4) \to \infty$

$$\therefore \qquad \lim_{x \to \infty} \left(\frac{x+6}{x+1}\right)^{x+4} = \lim_{x \to \infty} \left[1 + \left(\frac{x+6}{x+1} - 1\right)\right]^{x+4} = \lim_{x \to \infty} \left[1 + \frac{5}{x+1}\right]^{x+4}$$

$$= e^{\lim_{x \to \infty} \left(\frac{5}{x+1}\right) \cdot (x+4)} = e^{\lim_{x \to \infty} 5 \cdot \frac{x+4}{x+1}}$$

$$= e^{5 \cdot (1)} \qquad \left[\text{As } x \to \infty;\ \lim_{x \to \infty} \frac{x+4}{x+1} = 1\right]$$

$$= e^5$$

Illustration 64 Evaluate $\lim\limits_{x\to 0}(1+\tan^2\sqrt{x})^{\frac{1}{2x}}$.

Solution. $\lim\limits_{x\to 0}(1+\tan^2\sqrt{x})^{\frac{1}{2x}} = e^{\lim\limits_{x\to 0}\tan^2\sqrt{x}\cdot\frac{1}{2x}}$

$$= e^{\frac{1}{2}\lim\limits_{x\to 0}\left(\frac{\tan\sqrt{x}}{\sqrt{x}}\right)^2}$$

$$= e^{1/2}$$

Illustration 65 Evaluate $\lim\limits_{x\to 0}\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{\frac{1}{x}}$. **[IIT JEE 1993]**

Solution. $\lim\limits_{x\to 0}\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{\frac{1}{x}} = \lim\limits_{x\to 0}\left\{\frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}\right\}^{\frac{1}{x}}$

$$= \lim_{x\to 0}\left\{\frac{1+\tan x}{1-\tan x}\right\}^{\frac{1}{x}}$$

$$= \lim_{x\to 0}\left\{1+\frac{2\tan x}{1-\tan x}\right\}^{\frac{1}{x}}$$

$$= e^{\lim\limits_{x\to 0}\left(\frac{2\tan x}{1-\tan x}\right)\cdot\frac{1}{x}}$$

$$= e^{\lim\limits_{x\to 0} 2\cdot\frac{\tan x}{x}\cdot\frac{1}{1-\tan x}} = e^2$$

Illustration 66 Evaluate $\lim\limits_{x\to 0}\left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2}$. **[IIT JEE 1996]**

Solution. $\lim\limits_{x\to 0}\left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2} = \lim\limits_{x\to 0}\left(1+\frac{2x^2}{1+3x^2}\right)^{1/x^2}$ $(\because\ 1^\infty \text{ form})$

$$= e^{\lim\limits_{x\to 0}\left(\frac{2x^2}{1+3x^2}\right)\cdot\frac{1}{x^2}}$$

$$= e^{\frac{2}{1+0}} = e^2$$

Illustration 67 Evaluate $\lim\limits_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x$ **[IIT JEE 2000]**

Solution. $\lim\limits_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x = \lim\limits_{x\to\infty}\left(1+\frac{-5}{x+2}\right)^x$

$$= e^{\lim\limits_{x\to\infty}\left(\frac{-5}{x+2}\right)\cdot x} = e^{\lim\limits_{x\to\infty}\frac{-5}{1+\frac{2}{x}}}$$

$$= e^{-5}$$

Illustration 68 Solve $\lim\limits_{n \to \infty} \dfrac{-3n + (-1)^n}{4n - (-1)^n}$.

Solution. Here, $\lim\limits_{n \to \infty} \dfrac{-3n + (-1)^n}{4n - (-1)^n}$, two cases arise, n is even or n is odd.

Case I n is even, say $n = 2k$.

$$\lim_{k \to \infty} \frac{-6k + (-1)^{2k}}{8k - (-1)^{2k}}$$

$$= \lim_{k \to \infty} \frac{-6k + 1}{8k - 1} = \lim_{k \to \infty} \frac{-6 + \frac{1}{k}}{8 - \frac{1}{k}}$$

$$= \frac{-6}{8} = \frac{-3}{4} \qquad \left(\text{As } \frac{1}{k} \to 0, \text{ when } k \to \infty\right)$$

Case II n is odd, say $n = 2k + 1$.

$$\lim_{k \to \infty} \frac{-3(2k + 1) + (-1)^{2k+1}}{4(2k + 1) - (-1)^{2k+1}}$$

$$= \lim_{k \to \infty} \frac{-6k - 3 - 1}{8k + 4 + 1} = \lim_{k \to \infty} \frac{-6 - \frac{4}{k}}{8 + \frac{5}{k}} = \frac{-3}{4}$$

$$\therefore \qquad \lim_{n \to \infty} \frac{-3n + (-1)^n}{4n - (-1)^n} = \frac{-3}{4}$$

Illustration 69 The value of $\lim\limits_{x \to 1} (1 + \sin \pi x)^{\cot \pi x}$ is equal to

(a) e (b) $\dfrac{1}{e}$ (c) -1 (d) e^{π}

Solution. Here, $\lim\limits_{x \to 1} (1 + \sin \pi x)^{\cot \pi x}$ $\qquad$ (1^{∞} form)

$$= e^{\lim\limits_{x \to 1} \sin \pi x \cdot \cot \pi x}$$

$$= e^{\lim\limits_{x \to 1} \cos \pi x}$$

$$= e^{-1}$$

Hence, (b) is the correct answer.

Illustration 70 The value of $\lim\limits_{x \to 0} \left\{\sin^2 \left(\dfrac{\pi}{2 - ax}\right)\right\}^{\sec^2 \frac{\pi}{2 - bx}}$ is equal to

(a) $e^{-a/b}$ (b) e^{-a^2/b^2} (c) $a^{2a/b}$ (d) $e^{4a/b}$

Solution. Here, $\lim\limits_{x \to 0} \left(\sin^2 \left(\dfrac{\pi}{2 - ax}\right)\right)^{\sec^2 \left(\frac{\pi}{2 - bx}\right)}$

$$= \lim_{x \to 0} \left\{1 - \cos^2 \left(\frac{\pi}{2 - ax}\right)\right\}^{\sec^2 \left(\frac{\pi}{2 - bx}\right)} \qquad (1^{\infty} \text{ form})$$

$$= e^{\lim_{x \to 0} -\left\{\cos^2\left(\frac{\pi}{2 - ax}\right) \cdot \frac{1}{\cos^2\left(\frac{\pi}{2 - bx}\right)}\right\}}$$

$$= e^{\lim_{x \to 0} -\left\{\frac{2\sin\left(\frac{\pi}{2 - ax}\right)\cos\left(\frac{\pi}{2 - ax}\right)}{2\sin\left(\frac{\pi}{2 - bx}\right)\cos\left(\frac{\pi}{2 - bx}\right)} \cdot \frac{-\frac{\pi a}{(2 - ax)^2}}{-\frac{\pi b}{(2 - bx)^2}}\right\}}$$

$$= e^{-\lim_{x \to 0} \frac{\sin\left(\frac{2\pi}{2 - ax}\right)}{\sin\left(\frac{2\pi}{2 - bx}\right)} \cdot \frac{a}{b} \cdot \frac{(2 - bx)^2}{(2 - ax)^2}}$$

$$= e^{-\lim_{x \to 0} \frac{a}{b} \cdot \frac{(2 - bx)^3}{(2 - ax)^3}}$$

$$= e^{-\frac{a}{b}}$$

Hence, (a) is the correct answer.

Illustration 71 The value of $\lim_{x \to 7/2} (2x^2 - 9x + 8)^{\cot(2x - 7)}$ is equal to

(a) $e^{5/2}$ (b) $e^{-5/2}$ (c) $e^{7/2}$ (d) $e^{3/2}$

Solution. Here, $\lim_{x \to 7/2} (2x^2 - 9x + 8)^{\cot(2x - 7)}$ (1^∞ form)

$$\Rightarrow \quad \lim_{\pi \to 7/2} \{1 + (2x^2 - 9x + 7)\}^{\cot(2x - 7)}$$

$$= e^{\lim_{x \to 7/2} (2x^2 - 9x + 7) \cdot \cot(2x - 7)}$$

$$= e^{\lim_{x \to 7/2} \frac{4x - 9}{\sec^2(2x - 7) \cdot 2}}$$

$$= e^{5/2}$$

Hence, (a) is the correct answer.

Illustration 72 The value of $\lim_{x \to 1} \left(\tan\left(\frac{\pi}{4} + \log x\right)\right)^{\frac{1}{\log x}}$ is equal to

(a) e (b) e^{-1}

(c) e^2 (d) e^{-2}

Solution. Here, $\lim_{x \to 1} \left(\tan\left(\frac{\pi}{4} + \log x\right)\right)^{\frac{1}{\log x}}$ (1^∞ form)

$$= \lim_{x \to 1} \left(1 - \frac{2\tan(\log x)}{1 - \tan(\log x)}\right)^{\frac{1}{\log x}} = e^{\lim_{x \to 1} \frac{2\tan(\log x)}{\{1 - \tan(\log x)\}} \cdot \frac{1}{\log x}}$$

$$= e^{2\lim_{x \to 1} \frac{\tan(\log x)}{\log x} \cdot \frac{1}{1 - \tan(\log x)}} = e^{2 \cdot (1)} = e^2$$

Hence, (c) is the correct answer.

Illustration 73 The value of $\lim_{x \to 0} \left(\sin \frac{x}{m} + \cos \frac{3x}{m} \right)^{\frac{2m}{x}}$ is equal to

(a) e (b) 1

(c) e^{-1} (d) e^2

Solution. Here, $\lim_{x \to 0} \left(\sin \frac{x}{m} + \cos \frac{3x}{m} \right)^{\frac{2m}{x}}$ (1^∞ form)

$$= \lim_{x \to 0} \left\{ 1 + \left(\sin \frac{x}{m} + \cos \frac{3x}{m} - 1 \right) \right\}^{\frac{2m}{x}}$$

$$= e^{\lim_{x \to 0} \frac{2m}{x} \left(\sin \frac{x}{m} + \cos \frac{3x}{m} - 1 \right)}$$

$$= e^{\lim_{x \to 0} \frac{2m \left(\frac{1}{m} \cos \left(\frac{x}{m} \right) - \frac{3}{m} \sin \frac{3x}{m} \right)}{1}}$$

$$= e^{2m \left(\frac{1}{m} \right)} = e^2$$

Hence, (d) is the correct answer.

Illustration 74 The value of $\lim_{n \to \infty} \left(\frac{a - 1 + \sqrt[n]{b}}{a} \right)^n$ $(a > 0, b > 0)$ is equal to

(a) $\sqrt[a]{b}$ (b) $\sqrt[b]{a}$

(c) $\sqrt{b}$ (d) $\sqrt{a}$

Solution. Here, $\lim_{n \to \infty} \left(\frac{a - 1 + \sqrt[n]{b}}{a} \right)^n$ (1^∞ form)

$$= \lim_{n \to \infty} \left(1 + \frac{\sqrt[n]{b} - 1}{a} \right)^n = e^{\lim_{n \to \infty} \left(\frac{\sqrt[n]{b} - 1}{a} \right) \cdot n}$$

$$= e^{\lim_{n \to \infty} \frac{1}{a} \cdot \frac{b^{1/n} - 1}{1/n}} = e^{\lim_{n \to \infty} \frac{1}{a} \cdot \frac{b^{1/n} \log b \left(\frac{-1}{n^2} \right)}{-1/n^2}}$$

$$= e^{\frac{1}{a} \log_e b} = e^{\log_e b^{1/a}} = b^{1/a}$$

Hence, (a) is the correct answer.

(5) Miscellaneous Forms

(a) **0^0 form** When $\lim_{x \to a} f(x) \neq 1$ but $f(x)$ is positive in the neighbourhood of $x = a$.

In this case, we write $\{f(x)\}^{g(x)} = e^{\log_e \{f(x)\}^{g(x)}}$

$$\Rightarrow \qquad \lim_{x \to a} [f(x)]^{g(x)} = e^{\lim_{x \to a} g(x) \log_e f(x)}$$

Illustration 75 Evaluate $\lim\limits_{x \to 0^+} (\sin x)^x$.

Solution. Let $A = \lim\limits_{x \to 0^+} (\sin x)^x$

$$\Rightarrow \qquad \log A = \lim_{x \to 0^+} x \log (\sin x)$$

$$\log A = \lim_{x \to 0^+} \frac{\log (\sin x)}{1/x}$$

$$= \lim_{x \to 0^+} \frac{\dfrac{1}{\sin x} \cdot \cos x}{-\dfrac{1}{x^2}}$$

$$= -\lim_{x \to 0^+} x^2 \cot x = -\lim_{x \to 0^+} \frac{x^2}{\tan x} = 0$$

$$\Rightarrow \qquad A = 1 \quad \text{or} \quad \lim_{x \to 0^+} (\sin x)^x = 1$$

Illustration 76 Evaluate $\lim\limits_{x \to 0} (\operatorname{cosec} x)^x$.

Solution. Let $A = \lim\limits_{x \to 0} (\operatorname{cosec} x)^x$ $\qquad$ (∞^0 form)

$$\log A = \lim_{x \to 0} x \log (\operatorname{cosec} x) = \lim_{x \to 0} \frac{\log (\operatorname{cosec} x)}{\dfrac{1}{x}} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\dfrac{1}{\operatorname{cosec} x} \cdot (-\operatorname{cosec} x \cot x)}{-\dfrac{1}{x^2}} \qquad \text{(By L'Hospital's rule)}$$

$$= \lim_{x \to 0} \frac{x^2}{\tan x} = 0 \quad \therefore \quad \log A = 0 \text{ or } A = 1$$

$$\Rightarrow \qquad \lim_{x \to 0} (\operatorname{cosec} x)^x = 1$$

Illustration 77 Evaluate $\lim\limits_{x \to 0} e^{\frac{1}{x \log x}}$.

Solution. Let $A = \lim\limits_{x \to 0} e^{\frac{1}{x \log x}}$

$$\log A = \lim_{x \to 0} \frac{1}{x \log x} \cdot \log e = \lim_{x \to 0} \frac{1/x}{\log x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{-1/x^2}{1/x} = -\infty$$

$$\Rightarrow \qquad \log_e A = -\infty$$

$$\Rightarrow \qquad A = e^{-\infty} \quad \text{or} \quad \lim_{x \to 0} e^{\frac{1}{x \log x}} = 0$$

Illustration 78 Evaluate $\lim_{x \to 0} |x|^{\sin x}$.

Solution. $\lim_{x \to 0} |x|^{\sin x} = \lim_{x \to 0} e^{\sin x \log_e |x|} = e^{\lim_{x \to 0} \frac{\log_e |x|}{\operatorname{cosec} x}}$

$$= e^{\lim_{x \to 0} \frac{1/x}{-\operatorname{cosec} x \cot x}} \qquad (ie\text{, Applying L'Hospital's rule})$$

$$= e^{\lim_{x \to 0} -\frac{\sin^2 x}{x \cos x}} = e^{\lim_{x \to 0} -\left(\frac{\sin x}{x}\right)^2 \cdot \left(\frac{x}{\cos x}\right)}$$

$$= e^{-(1)^2 \cdot (0)} = e^0 = 1$$

Illustration 79 Solve $\lim_{x \to 0^+} \log_{\sin x} \sin 2x$.

Solution. Here, $\lim_{x \to 0^+} \log_{\sin x} \sin 2x$

$$= \lim_{x \to 0^+} \frac{\log \sin 2x}{\log \sin x} \qquad \left(\frac{-\infty}{-\infty} \text{ form}\right)$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{\sin 2x} \cdot 2 \cos 2x}{\frac{1}{\sin x} \cdot \cos x} \qquad \text{(Applying L'Hospital's rule)}$$

$$= \lim_{x \to 0^+} \frac{\left(\frac{(2x)}{\sin (2x)}\right) \cos 2x}{\left(\frac{x}{\sin x}\right) \cos x}$$

$$= \lim_{x \to 0^+} \frac{\cos 2x}{\cos x} = 1$$

Illustration 80 Solve $\lim_{x \to 0^+} (\sin x)^{\tan x}$.

Solution. Here, $\lim_{x \to 0^+} (\sin x)^{\tan x}$ (0^0 form)

Let $A = \lim_{x \to 0^+} (\sin x)^{\tan x}$

Taking log on both the sides, we get

$$\log_e A = \lim_{x \to 0^+} \tan x \log (\sin x)$$

$$= \lim_{x \to 0^+} \frac{\log (\sin x)}{\cot x} \qquad \left(\frac{-\infty}{\infty} \text{ form}\right)$$

Applying L'Hospital's rule,

$$= \lim_{x \to 0^+} \frac{\frac{1}{\sin x} \cdot \cos x}{-\operatorname{cosec}^2 x} = \lim_{x \to 0^+} -\sin x \cdot \cos x = 0$$

$\therefore$ $\log_e A = 0$

$\Rightarrow$ $A = e^0 = 1 \quad \Rightarrow A = 1$

Illustration 81 Evaluate $\lim_{n \to \infty} (\pi n)^{2/n}$.

Solution. Here, $A = \lim_{n \to \infty} (\pi n)^{2/n}$ $\quad(\infty^0$ form)

$$\log A = \lim_{n \to \infty} \frac{2 \log (\pi n)}{n} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{n \to \infty} \frac{2 \cdot \dfrac{1}{\pi n} \cdot \pi}{1} \qquad \text{(By L'Hospital's rule)}$$

$$= \lim_{n \to \infty} \frac{2}{n} = 0 \quad \therefore \quad \log_e A = 0 \quad \Rightarrow \quad A = 1$$

Illustration 82 Evaluate $\lim_{n \to \infty} \left(\dfrac{e^n}{\pi}\right)^{1/n}$.

Solution. Here, $A = \lim_{n \to \infty} \left(\dfrac{e^n}{\pi}\right)^{1/n}$ $\quad(\infty^0$ form)

$$\therefore \qquad \log A = \lim_{n \to \infty} \frac{1}{n} \log \left(\frac{e^n}{\pi}\right)$$

$$= \lim_{n \to \infty} \frac{n \log e - \log \pi}{n} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{n \to \infty} \frac{\log e - 0}{1} \qquad \text{(By L'Hospital's rule)}$$

$$\log A = 1 \quad \Rightarrow \quad A = e^1 \text{ or } \lim_{n \to \infty} \left(\frac{e^n}{\pi}\right)^{1/n} = e$$

Illustration 83 Solve $\lim_{x \to 0} \dfrac{(1+x)^{1/x} - e}{x}$.

Solution. Here, $\lim_{x \to 0} \dfrac{(1+x)^{1/x} - e}{x}$

$$= \lim_{x \to 0} \frac{e^{\log (1+x)^{1/x}} - e}{x} = \lim_{x \to 0} \frac{e\left\{e^{\frac{\log (1+x)}{x} - 1} - 1\right\}}{x}$$

$$= \lim_{x \to 0} \frac{e\{e^M - 1\}}{M} \cdot \lim_{x \to 0} \frac{\log(1+x) - x}{x^2}$$

where $\qquad M = \dfrac{\log(1+x)}{x} - 1$

$$= \lim_{x \to 0} \frac{e \cdot \{e^M - 1\}}{M} \cdot \lim_{x \to 0} \frac{\dfrac{1}{1+x} - 1}{2x}$$

$$= e \times 1 \times \lim_{x \to 0} \frac{-x}{2x(1+x)}$$

$$= -\frac{e}{2}$$

Target Exercise 5.5

1. If α and β are roots of $ax^2 + bx + c - 0$, then the value for $\lim_{x \to \alpha} (1 + ax^2 + bx + c)^{2/x - \alpha}$, is

(a) $e^{2a(\alpha - \beta)}$ (b) $e^{a(\alpha - \beta)}$

(c) $e^{\frac{2a}{3}(\alpha - \beta)}$ (d) None of these

2. The value of $\lim_{n \to \infty} ((1.5)^n + [(1 + 0.0001)^{10000}]^n)^{1/n}$, where $[\cdot]$ denotes the greatest integer function is

(a) 1 (b) $\frac{1}{2}$

(c) Doesn't exist (d) 2

3. The value of $\lim_{x \to 0} \frac{e - (1 + x)^{1/x}}{\tan x}$, is

(a) e (b) $\frac{11e}{24}$

(c) $\frac{e}{2}$ (d) None of these

4. The value of $\lim_{x \to 0} \left(\frac{1^x + 2^x + 3^x + \ldots + n^x}{n} \right)^{a/x}$, is

(a) $(n!)^{a/n}$

(b) $n!$

(c) $a^{n!}$

(d) Doesn't exist

5. If $\lim_{x \to 0} (1 + ax + bx^2)^{2/x} = e^3$, then the values of a and b, is

(a) $a = \frac{3}{2}, b \in R$

(b) $a = \frac{1}{2}, b \in R$

(c) $a \in R, b \in R$

(d) None of the above

Use of Standard Theorems/Results

Theorem-1 (Sandwich/Squeeze Play Theorem)

General The Squeeze Principle is used on limit problems where the usual algebraic methods, factorisation or algebraic manipulation etc.) are not effective. However, it requires to "squeeze" our problem in between two other simpler function, whose limits can be easily computed and equal. Use of Squeeze principle requires accurate analysis, indepth algebra skills and careful use of inequalities.

Statement If f, g and h are 3 functions such that $f(x) \le g(x) \le h(x)$ for all x in some interval containing the point $x = c$ and if

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$$

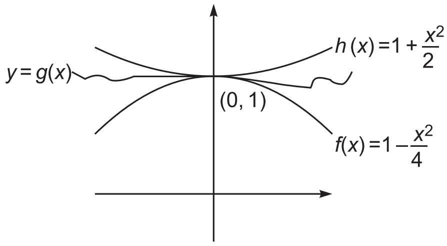


Fig. 5.13

Then, $$\lim_{x\to c} g(x) = L$$

From the figure note that $\lim\limits_{x\to 0} g(x) = 1$

Point to Consider

The quantity c may be a finite number, $+\infty$ or $-\infty$. Similarly, L may also be finite number, $+\infty$ or $-\infty$.

Examples on Sandwich Theorem

Illustration 84 Evaluate $\lim\limits_{x\to 0} x^3 \cos\frac{2}{x}\cdot$

Solution. $-1 \le \cos\frac{2}{x} \le 1$, $-x^3 \le x^3\cos\frac{2}{x} \le x^3$ for $x > 0$

and $$x^3 \le x^3 \cos\frac{2}{x} \le -x^3 \text{ for } x < 0,$$

where $\lim\limits_{x\to 0} x^3\cos\frac{2}{x} = 0$, as in both the cases limit is zero.

Illustration 85 Evaluate $\lim\limits_{x\to\infty} \dfrac{x^2(2+\sin^2 x)}{x+100}$

Solution. $\dfrac{2x^2}{x+100} \le \dfrac{x^2(2+\sin^2 x)}{x+100} \le \dfrac{3x^2}{x+100}$, as $0 \le \sin^2 x \le 1$

$\therefore$ $$\lim_{x\to\infty}\frac{2x^2}{x+100} \le \lim_{x\to 0}\frac{x^2(2+\sin^2 x)}{x+100} \le \lim_{x\to\infty}\frac{3x^2}{x+100}$$

$\Rightarrow$ $$\lim_{x\to\infty}\frac{x^2(2+\sin^2 x)}{x+100} = \infty$$

Illustration 86 Evaluate $\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+2}+\frac{n}{n^2+3}+\ldots+\frac{n}{n^2+n}\right)$.

Solution. Let $f(n)=\frac{n}{n^2+1}+\frac{n}{n^2+2}+\frac{n}{n^2+3}+\ldots+\frac{n}{n^2+n}$

Note that $f(n)$ has n terms which are decreasing.

Suppose, $h(n)=\left(\frac{n}{n^2+1}+\frac{n}{n^2+1}+\frac{n}{n^2+1}+\ldots+\frac{n}{n^2+1}\right)$, n terms

$$h(n)=\frac{n^2}{n^2+1} \qquad \text{[Obviously } f(n)<h(n)]$$

and $$g(n)=\left(\frac{n}{n^2+n}+\frac{n}{n^2+n}+\frac{n}{n^2+n}+\ldots+\frac{n}{n^2+n}\right), n \text{ terms}$$

$$=\frac{n^2}{n^2+n} \qquad \text{[Obviously } g(n)<f(n)]$$

Hence, $g(n)<f(n)<h(n)$

Since, $\lim_{n\to\infty} g(n)=1=\lim_{n\to\infty} h(n)$

Hence, using Sandwich theorem $\lim_{n\to\infty} f(n)=1$

Illustration 87 The value of the $\lim_{x\to 0}\frac{x}{a}\left[\frac{b}{x}\right](a\neq 0)$ (where [·] denotes the greatest integer function) is equal to

(a) a (b) b (c) $\frac{b}{a}$ (d) $1-\frac{b}{a}$

Solution. $\frac{b}{x}-1<\left[\frac{b}{x}\right]\le\frac{b}{x}$

Case I For $\frac{x}{a}>0$

$$\lim_{x\to 0}\left(\frac{b}{x}-1\right)\frac{x}{a}<\left[\frac{b}{x}\right]\frac{x}{a}\le\lim_{x\to 0}\frac{b}{x}\cdot\frac{x}{a},$$

Using Squeeze play theorem $=\frac{b}{a}$

Case II For $\frac{x}{a}<0$

$$\lim_{x\to 0}\left(\frac{b}{x}-1\right)\frac{x}{a}>\left[\frac{b}{x}\right]\frac{x}{a}\ge\lim_{x\to 0}\frac{b}{x}\cdot\frac{x}{a}$$

Using Squeeze play theorem $=\frac{b}{a}$

$$\therefore \qquad \lim_{x\to 0}\frac{x}{a}\left[\frac{b}{x}\right]=\frac{b}{a}$$

Hence, (c) is the correct answer.

Theorem-2 (Limits of Trigonometric Functions)

If x is small and is measured in radians, then

$$\lim_{x\to 0}=\frac{\sin x}{x}=1=\lim_{x\to 0}\frac{x}{\sin x}=\lim_{x\to 0}x\operatorname{cosec}x=\lim_{x\to 0}\frac{\sin^{-1}x}{x}=\lim_{x\to 0}\frac{x}{\sin^{-1}x}$$

Proof Consider a circle with unit radius.

Area of ΔOAP < Area of sector $\widehat{OAP}$ < Area of ΔOAT

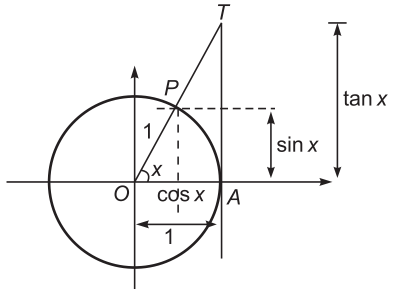


Fig. 5.14

$$\frac{\sin x}{2} < \frac{x}{2} < \frac{\tan x}{2}$$

$$\Rightarrow \qquad 1 < \frac{x}{\sin x} < \frac{1}{\cos x} \qquad (0 < x < \pi/2)$$

$$\Rightarrow \qquad \cos x < \frac{\sin x}{x} < 1$$

Now, using Sandwich theorem $\lim\limits_{x \to 0} \cos x < \lim\limits_{x \to 0^+} \frac{\sin x}{x} < 1$

Obviously, we have $\quad \lim\limits_{x \to 0^+} \frac{\sin x}{x} = 1$

Put $x = -y$, $\quad \lim\limits_{y \to 0^-} \frac{\sin y}{y} = 1$

Hence, $\quad \lim\limits_{x \to 0} \frac{\sin x}{x} = 1 \qquad \ldots$(i)

Point to Consider

The $\lim\limits_{x \to 0} \frac{\sin x}{x}$ always approaches 1 from its left hand *ie*, 0.9999....

$\Rightarrow \lim\limits_{x \to 0} \left[\frac{\sin x}{x}\right] = 0$, where $[\cdot]$ denotes step up function. $\left(\text{Note that} \left[\lim\limits_{x \to 0} \frac{\sin x}{x}\right] = 1\right)$

Using Eq. (i), we can deduce

$$\lim_{x \to 0} \frac{\tan x}{x} = 1 = \lim_{x \to 0} \frac{x}{\tan x} = \lim_{x \to 0} x \cot x$$

$$= \lim_{x \to 0} \frac{\tan^{-1} x}{x} = \lim_{x \to 0} \frac{x}{\tan^{-1} x}$$

Note that the $\lim\limits_{x \to 0} \frac{\tan x}{x}$ approaches 1 from RHS

$$\Rightarrow \qquad \lim_{x \to 0} \left[\frac{\tan x}{x}\right] = 1, \text{ where } [\cdot] \text{ denotes step up function.}$$

Illustration 88 Evaluate $\lim\limits_{n \to \infty} \dfrac{[x]+[2x]+[3x]+\dots+[nx]}{n^2}$,

where $[\cdot]$ denotes the greatest integer function.

Solution. We know that, $x-1<[x]\le x$

$$\Rightarrow \quad 2x-1<[2x]\le 2x$$
$$\Rightarrow \quad 3x-1<[3x]\le 3x$$
$$\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots$$
$$\Rightarrow \quad nx-1<[nx]\le nx$$
$$\therefore \quad (x+2x+3x+\dots+nx)-n<[x]+[2x]+\dots+[nx] \le (x+2x+\dots+nx)$$
$$\Rightarrow \quad \frac{xn(n+1)}{2}-n<\sum_{r=1}^{n}[rx]\le \frac{x\cdot n(n+1)}{2}$$

Thus, $\lim\limits_{n\to\infty}\dfrac{[x]+[2x]+\dots+[nx]}{n^2}$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{x}{2}\left(1+\frac{1}{n}\right)-\frac{1}{n}<\lim_{n\to\infty}\frac{[x]+[2x]+\dots+[nx]}{n^2} \le \lim_{n\to\infty}\frac{x}{2}\left(1+\frac{1}{n}\right)$$
$$\Rightarrow \quad \frac{x}{2}<\lim_{n\to\infty}\frac{[x]+[2x]+\dots+[nx]}{n^2}\le\frac{x}{2}$$
$$\Rightarrow \quad \lim_{n\to\infty}\frac{[x]+[2x]+\dots+[nx]}{n^2}=\frac{x}{2}$$

Aliter : We know that $[x]=x-\{x\}$

$$\sum_{r=1}^{n} rx = [x]+[2x]+\dots+[nx]$$
$$=x-\{x\}+2x-\{2x\}+\dots+nx-\{nx\}$$
$$=(x+2x+3x+\dots+nx)-(\{x\}+\{2x\}+\dots+\{nx\})$$
$$=\frac{xn(n+1)}{2}-(\{x\}+\{2x\}+\dots+\{nx\})$$
$$\therefore \quad \frac{1}{n^2}\sum_{r=1}^{n}[rn]=\frac{x}{2}\left(1+\frac{1}{n}\right)-\frac{\{x\}+\{2x\}+\dots+\{nx\}}{n^2}$$

Since, $0\le\{rx\}<1$

$$\therefore \quad 0\le\sum_{r=1}^{n}[rx]<n \quad \Rightarrow \quad \lim_{n\to\infty}\frac{\sum_{r=1}^{n}[rx]}{n^2}=0$$
$$\therefore \quad \lim_{n\to\infty}\frac{\sum_{r=1}^{n}[rx]}{n^2}=\lim_{n\to\infty}\frac{x}{2}\left(1+\frac{1}{n}\right)-\lim_{n\to\infty}\frac{\sum_{r=1}^{n}\{rx\}}{n^2}$$
$$\lim_{n\to\infty}\frac{\sum_{r=1}^{n}[rx]}{n^2}=\frac{x}{2}$$

(b) **Use of Newton-Leibnitz's formula in evaluating the limits**

Let us consider the definite integral,

$$I(x) = \int_{\phi(x)}^{\psi(x)} f(t)\,dt$$

Newton-Leibnitz's formula states that,

$$\frac{d}{dx}\{I(x)\} = f\{\psi(x)\}\cdot\left\{\frac{d}{dx}\psi(x)\right\} - f\{\phi(x)\}\left\{\frac{d}{dx}\phi(x)\right\}$$

Illustration 89 Evaluate $\lim_{x\to 0}\left(\frac{1}{x^5}\int_0^x e^{-t^2}\,dt - \frac{1}{x^4} + \frac{1}{3x^2}\right)$.

Solution. $\lim_{x\to 0}\dfrac{3\int_0^x e^{-t^2}\,dt - 3x + x^3}{3x^5}$ $\quad\left(\frac{0}{0}\text{ form}\right)$

Applying L'Hospital's rule,

$$= \lim_{x\to 0}\frac{3\frac{d}{dx}\int_0^x e^{-t^2}\,dt - 3 + 3x^2}{15x^4}$$

Applying Newton-Leibnitz's formula to

$$\frac{d}{dx}\int_0^x e^{-t^2}\,dt = e^{-x^2}\cdot\frac{d}{dx}(x) - e^{-0}\frac{d}{dx}(0)$$

$$= e^{-x^2}$$

$$\therefore \qquad \lim_{x\to 0}\frac{3\frac{d}{dx}\int_0^x e^{-t^2}\,dt - 3 + 3x^2}{15x^4}$$

$$= \lim_{x\to 0}\frac{3e^{-x^2} - 3 + 3x^2}{15x^4} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{-3(2x)e^{-x^2} + 6x}{60x^3} \qquad \text{(Again, apply L'Hospital's rule)}$$

$$= \lim_{x\to 0}\frac{-6x(e^{-x^2} - 1)}{60x^3} = \lim_{x\to 0}\frac{-(e^{-x^2} - 1)}{10x^2}$$

$$= \frac{1}{10}\lim_{x\to 0}\left(\frac{e^{-x^2} - 1}{-x^2}\right) = \frac{1}{10}\times 1 = \frac{1}{10}$$

Illustration 90 Evaluate $\lim_{x\to 0}\dfrac{x - \int_0^x \cos t^2\,dt}{x^3 - 6x}$.

Solution. $\lim_{x\to 0}\dfrac{x - \int_0^x \cos t^2\,dt}{x^3 - 6x}$ $\quad\left(\frac{0}{0}\text{ form}\right)$

Applying L'Hospital's rule, we get

$$= \lim_{x\to 0}\frac{1 - \frac{d}{dx}\int_0^x \cos t^2\,dt}{3x^2 - 6}$$

Applying Newton-Leibnitz's rule to

$$\frac{d}{dx}\int_0^x (\cos t^2)\,dt = \cos(x^2)\cdot 1 - 0$$

$$\therefore \quad \lim_{x\to 0} \frac{1 - \frac{d}{dx}\int_0^x \cos t^2\,dt}{3x^2 - 6}$$

$$= \lim_{x\to 0} \frac{1-\cos(x^2)}{3(x^2-2)} = \frac{1-\cos 0}{3(0-2)} = \frac{1-1}{3(-2)} = \frac{0}{-6} = 0$$

Illustration 91 Evaluate $\lim_{x\to 0} \frac{\int_0^{x^2} \cos t^2\,dt}{x\sin x}$. **[IIT JEE 1997]**

Solution. Applying Newton-Leibnitz's rule,

$$\lim_{x\to 0} \frac{\cos(x^2)^2\cdot\{2x\} - 0}{x\cos x + \sin x} = \lim_{x\to 0} \frac{2\cos x^4}{\cos x + \left(\frac{\sin x}{x}\right)} = \frac{2\cos 0}{\cos 0 + 1} = \frac{2}{1+1} = 1$$

(c) **Summation of series using definite integral as the limit**

The expression of the form, $\lim_{n\to\infty} \frac{1}{n} \sum_{r=\phi(x)}^{\psi(x)} f\left(\frac{r}{n}\right) = \int_a^b f(x)\,dx$,

where (i) Σ is replaced by $\int$, (ii) $\frac{r}{n}$ is replaced by x,

(iii) $\frac{1}{n}$ is replaced by dx, (iv) To obtain, $a = \lim_{n\to\infty} \frac{\phi(x)}{n}$

and $b = \lim_{n\to\infty} \frac{\psi(x)}{n}$

The value so obtained is the required sum of the given series.

Illustration 92 Evaluate $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2+r^2}}$. **[IIT JEE 1997]**

Solution. Here, $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2+r^2}}$ dividing numerator and denominator both by n, we get

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r/n}{\sqrt{1+(r/n)^2}}$$

Put $\frac{r}{n} = x;\ \frac{1}{n} = dx;\ \lim_{n\to\infty} \sum_{r=1}^{2n} = \int_a^b$

where $a = \lim_{n\to\infty} \frac{1}{n} = 0$ and $b = \lim_{n\to\infty} \frac{2n}{n} = 2$

$$\therefore \quad \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r/n}{\sqrt{1+(r/n)^2}}$$

$$= \int_0^2 \frac{x}{\sqrt{1+x^2}}\,dx = \left(\sqrt{1+x^2}\right)_0^2 = \sqrt{5} - 1$$

Illustration 93 Evaluate $\lim_{n \to \infty} \left(\frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \dots + \frac{n}{n^2 + n^2} \right)$.

Solution. Let $S = \lim_{n \to \infty} \left(\frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \dots + \frac{n}{n^2 + n^2} \right)$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{n}{n^2 + r^2}$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \frac{n^2}{n^2 + r^2}$$

(dividing numerator and denominator both by n)

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{1 + (r/n)^2}$$

Replace $\frac{r}{n} = x$; $\frac{1}{n} = dx$; $\lim_{n \to \infty} \sum_{r=1}^{n} = \int_0^1$

We get, $\lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{1 + (r/n)^2} = \int_0^1 \frac{1}{1 + x^2}\, dx$

$$= [\tan^{-1} x]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}$$

Illustration 94 The value of $\lim_{n \to \infty} \left(\frac{n!}{n^n} \right)^{1/n}$ is equal to

(a) $\frac{1}{e}$ (b) e (c) e^2 (d) $\frac{1}{e^2}$

Solution. Let $A = \lim_{n \to \infty} \left(\frac{n!}{n^n} \right)^{1/n}$

$\therefore$ $\log A = \lim_{n \to \infty} \frac{1}{n} \log \left(\frac{n}{n} \cdot \frac{(n-1)}{n} \cdot \frac{(n-2)}{n} \dots \frac{3}{n} \cdot \frac{2}{n} \cdot \frac{1}{n} \right)$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{n-1} \log \left(\frac{n-r}{n} \right) = \int_0^1 1 \cdot \log(1-x)\, dx = -1$$

$\Rightarrow$ $\log_e A = -1 \quad \Rightarrow \quad A = e^{-1}$

Hence, (a) is the correct answer.

Illustration 95 The value of $\lim_{n \to \infty} \left(\frac{n!}{n^n} \right)^{\frac{2n^4 + 1}{5n^5 + 1}}$ is equal to

(a) 1 (b) 0 (c) $\left(\frac{1}{e} \right)^{2/5}$ (d) $e^{2/5}$

Solution. $\lim_{n \to \infty} \left(\frac{n!}{n^n} \right)^{\frac{2n^4 + 1}{5n^5 + 1}} = \lim_{n \to \infty} \left\{ \left(\frac{n!}{n^n} \right)^{1/n} \right\}^{\frac{2n^5 + n}{5n^5 + 1}}$

$$= \left(\frac{1}{2} \right)^{2/5} \qquad \left[\text{As } \lim_{n \to \infty} \left(\frac{n!}{n^n} \right)^{1/n} = \frac{1}{e} \text{ and } \lim_{n \to \infty} \frac{2n^5 + n}{5n^5 + 1} = \frac{2}{5} \right]$$

Hence, (c) is the correct answer.

Target Exercise 5.6

1. The value of $\lim_{n \to \infty} \frac{1}{n^3}([1^2x + 1^2] + [2^2x + 2^2] + \dots + [n^2x + n^2])$, is (where $[\cdot]$ denotes the greatest integer function.)

(a) $\frac{x}{3}$ (b) $x + \frac{1}{3}$

(c) $\frac{x}{3} + \frac{1}{3}$ (d) None of these

2. The value of $\lim_{x \to \infty} \left\{ \lim_{n \to \infty} \frac{[1^2 (\sin x)^x] + [2^2 (\sin x)^x] + \dots + [n^2 (\sin x)^x]}{n^3} \right\}$, is (where $[\cdot]$ denotes the greatest integer function.)

(a) $\frac{x}{3} + \frac{\sin x}{3}$ (b) $\frac{x}{3} + (\sin x)^x$

(c) $\frac{1}{3}$ (d) 0

3. The value of $\lim_{x \to 1^+} \frac{\int_1^x |t - 1| \, dt}{\sin (x - 1)}$, is

(a) 0 (b) 1

(c) Doesn't exist (d) None of these

4. The value of $\lim_{n \to \infty} \sum_{K=1}^{n} \log \left(1 + \frac{K}{n}\right)^{1/n}$, is

(a) $\log_e \left(\frac{e}{4}\right)$ (b) $\log_e \left(\frac{4}{e}\right)$

(c) $\log_e 4$ (d) None of these

5. The value of $\lim_{n \to \infty} \left(\frac{1}{na} + \frac{1}{na + 1} + \frac{1}{na + 2} + \dots + \frac{1}{nb} \right)$, is

(a) $\log \left(\frac{a}{b}\right)$ (b) $\log \left(\frac{b}{a}\right)$

(c) $\log (ab)$ (d) None of these

Worked Examples

Type 1 : Subjective Type Questions

Example 1 Solve $\lim\limits_{x \to +\infty} \left(1 + \frac{1}{x^n}\right)^x$, $n > 0$.

Solution. Here, $\lim\limits_{x \to +\infty} \left(1 + \frac{1}{x^n}\right)^x$, $n > 0$

$$= e^{\lim\limits_{x \to \infty} x \cdot \frac{1}{x^n}} = e^{\lim\limits_{x \to \infty} \frac{1}{x^{n-1}}} \qquad \text{...(i)}$$

Now,

if $n = 1$, $\quad \lim\limits_{x \to \infty} \left(1 + \frac{1}{x^n}\right)^x = e^{\lim\limits_{x \to \infty} \frac{1}{x^0}} = e$

if $n > 1$, $\quad \lim\limits_{x \to \infty} \left(1 + \frac{1}{x^n}\right)^x = e^{\lim\limits_{x \to \infty} \frac{1}{x^{n-1}}} = e^{\frac{1}{\infty}} \qquad (\because n - 1 > 0)$

$$= e^0 = 1$$

if $n < 1$, $\quad \lim\limits_{x \to \infty} \left(1 + \frac{1}{x^n}\right)^x = e^{\lim\limits_{x \to \infty} \frac{1}{x^{n-1}}} = e^{\lim\limits_{x \to \infty} \frac{x^{1-n}}{1}} = e^{\infty} \qquad (\because 1 - n > 0)$

$$\therefore \quad \lim_{x \to \infty} \left(1 + \frac{1}{x^n}\right)^x = \begin{cases} e, & n = 1 \\ 1, & n > 1 \\ \infty, & 0 < n < 1 \end{cases}$$

Example 2 Solve $\lim\limits_{n \to \infty} \left[\sum\limits_{r=1}^{n} \frac{1}{2^r}\right]$, where $[\cdot]$ denotes the greatest integer.

Solution. We know, $\sum\limits_{r=1}^{n} \frac{1}{2^r} = \dfrac{\frac{1}{2}\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}}$ (sum of n terms of GP)

$$= 1 - \left(\frac{1}{2}\right)^n$$

which tends to one, as $n \to \infty$ but always remains less than one.

Thus, $\quad \left[\sum\limits_{r=1}^{n} \frac{1}{2^r}\right] = 0$, as $n \to \infty \quad \therefore \quad \lim\limits_{n \to \infty} \left[\sum\limits_{r=1}^{n} \frac{1}{2^r}\right] = 0$

Example 3 Solve

$$\lim_{m \to \infty} \lim_{n \to \infty} \left\{\frac{1 + \sqrt[n]{1^n + 2^n} + \sqrt[n]{2^n + 3^n} + \sqrt[n]{3^n + 4^n} + \ldots + \sqrt[n]{(m-1)^n + m^n}}{m^2}\right\}.$$

Solution. Here,

$$\lim_{m \to \infty} \lim_{n \to \infty} \left\{\frac{1 + \sqrt[n]{1^n + 2^n} + \sqrt[n]{2^n + 3^n} + \sqrt[n]{3^n + 4^n} + \ldots + \sqrt[n]{(m-1)^n + m^n}}{m^2}\right\}$$

$$= \lim_{m \to \infty} \left\{ \lim_{n \to \infty} \frac{1 + 2\sqrt[n]{\left(\frac{1}{2}\right)^n + 1} + 3\sqrt[n]{\left(\frac{2}{3}\right)^n + 1} + \ldots + m\sqrt[n]{\left(\frac{m-1}{m}\right)^n + 1}}{m^2} \right\}$$

$$= \lim_{m \to \infty} \frac{1 + 2 + 3 + 4 + \ldots + m}{m^2}$$

$$\left[\because \left(\frac{1}{2}\right)^n \to 0 \text{ as } n \to \infty; \left(\frac{2}{3}\right)^n \to 0 \text{ as } n \to \infty; \ldots; \left(\frac{m-1}{m}\right)^n \to 0 \text{ as } n \to \infty \right]$$

$$= \lim_{m \to \infty} \frac{m(m+1)}{2m^2}$$

$$= \lim_{m \to \infty} \frac{1}{2}\left(1 + \frac{1}{m}\right) = \frac{1}{2}$$

$$\therefore \quad \lim_{m \to \infty} \lim_{n \to \infty} \left\{ \frac{1 + \sqrt[n]{1^n + 2^n} + \sqrt[n]{2^n + 3^n} + \sqrt[n]{3^n + 4^n} + \ldots + \sqrt[n]{(m-1)^n + m^n}}{m^2} \right\}$$

$$= \frac{1}{2}$$

Example 4 Evaluate $\lim\limits_{n \to \infty} \dfrac{a^n}{n!}$, $a \in R^+$.

Solution. $\lim\limits_{n \to \infty} \dfrac{a^n}{n!}$, $a \in R^+$

Let $\lambda \le a < \lambda + 1$ where $\lambda \in I^+$

$$= \frac{a^n}{n!} = \frac{(a \cdot a \cdot a \cdot a \cdot \ldots\ldots. \cdot a)(a \cdot a \cdot a \cdot \ldots..\cdot a)}{(1 \cdot 2 \cdot 3 \cdot \ldots\ldots\cdot \lambda) \cdot \{(\lambda + 1) \cdot (\lambda + 2) \ldots.. n\}}$$

$$= \frac{a^n}{n!} = \frac{a^\lambda}{\lambda!} \cdot \frac{a}{\lambda + 1} \cdot \frac{a}{\lambda + 2} \cdot \ldots \cdot \frac{a}{n}$$

Here, $$\frac{a}{\lambda + 1} > \frac{a}{\lambda + 2} > \frac{a}{\lambda + 3} > \ldots.. > \frac{a}{n}$$

$$= \frac{a^n}{n!} < \frac{a^\lambda}{\lambda!}\left(\frac{a}{\lambda + 1}\right)^{n - \lambda}$$

Also, $$\frac{a}{\lambda + 1} < 1$$

$\Rightarrow$ $$\lim_{n \to \infty} \left(\frac{a}{\lambda + 1}\right)^{n - \lambda} = 0$$

Using Sandwich theorem, we can say that

$$\lim_{n \to \infty} \frac{a^n}{n!} = 0, a \in R^+$$

Example 5 Evaluate $\lim_{n \to \infty} \left(1 + \frac{1}{a_1}\right)\left(1 + \frac{1}{a_2}\right) \ldots\ldots \left(1 + \frac{1}{a_n}\right)$, where $a_1 = 1$ and $a_n = n(1 + a_{n-1}) \forall\, n \geq 2$.

Solution. $\lim_{n \to \infty} \left(\frac{a_1 + 1}{a_1}\right)\left(\frac{a_2 + 1}{a_2}\right) \ldots\ldots \left(\frac{a_n + 1}{a_n}\right)$

We know, $\quad a_{n-1} + 1 = \frac{a_n}{n} \qquad \ldots\text{(i)}$

$$\lim_{n \to \infty} \left(\frac{a_2}{2}\right)\left(\frac{a_3}{3}\right)\left(\frac{a_4}{4}\right) \ldots\ldots \left(\frac{a_{n+1}}{n+1}\right) \cdot \frac{1}{a_1 \cdot a_2 \cdot \ldots\ldots \cdot a_n}$$

$$= \lim_{n \to \infty} \frac{a_{n+1}}{(n+1)!} = \lim_{n \to \infty} \frac{1 + a_n}{n!} \qquad \text{[using Eq. (i)]}$$

$$= \lim_{n \to \infty} \left(\frac{1}{n!} + \frac{a_n}{n!}\right) = \lim_{n \to \infty} \left(\frac{1}{n!} + \frac{1}{(n-1)!} + \frac{a_{n-1}}{(n-1)!}\right) \qquad \text{[using Eq. (i)]}$$

$$= \lim_{n \to \infty} \left(\frac{1}{n!} + \frac{1}{(n-1)!} + \frac{1}{(n-2)!} + \ldots + \frac{1}{(2)!} + \frac{1}{1!} + \frac{a_1}{1!}\right) \qquad [a_1 = 1; \text{given}]$$

$$= \lim_{n \to \infty} \left(\frac{1}{n!} + \frac{1}{(n-1)!} + \frac{1}{(n-2)!} + \ldots + \frac{1}{(2)!} + \frac{1}{1!} + \frac{1}{1}\right)$$

$$= e \qquad \left[\text{As } e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots\ldots\ldots \infty\right]$$

Example 6 If $f(x + y) = f(x) + f(y)$ for all $x, y \in R$ and $f(1) = 1$, then evaluate $\lim_{x \to 0} \frac{2^{f(\tan x)} - 2^{f(\sin x)}}{x^2 f(\sin x)}$.

Solution. Here, $f(x + y) = f(x) + f(y)$

$$f(2) = f(1) + f(1) = 2$$
$$f(3) = f(1 + 2) = f(1) + f(2) = 3$$
$$f(4) = f(1 + 3) = f(1) + f(3) = 4$$
$$\vdots$$
$$f(x) = x, \text{ for all } x \in R$$

$\therefore \qquad f(\tan x) = \tan x, f(\sin x) = \sin x$

$\Rightarrow \qquad \lim_{x \to 0} \frac{2^{f(\tan x)} - 2^{f(\sin x)}}{x^2 f(\sin x)} = \lim_{x \to 0} \frac{2^{\tan x} - 2^{\sin x}}{x^2 \sin x}$

$$= \lim_{x \to 0} \frac{2^{\sin x} \cdot \{2^{\tan x - \sin x} - 1\}}{\sin x \cdot x^2} \times \frac{\tan x - \sin x}{\tan x - \sin x}$$

$$= \lim_{x \to 0} \frac{\{2^{\tan x - \sin x} - 1\}}{\tan x - \sin x} \times \left\{\frac{\tan x - \sin x}{x^2 \sin x}\right\} \times 2^{\sin x}$$

$$= \lim_{x \to 0} \frac{2^{\tan x - \sin x} - 1}{\tan x - \sin x} \times \frac{1 - \cos x}{x^2 \cos x} \times 2^{\sin x}$$

$$= \lim_{x \to 0} \frac{2^{\tan x - \sin x} - 1}{\tan x - \sin x} \times \frac{2 \sin^2 x/2}{4(x/2)^2} \times 2^{\sin x} \times \frac{1}{\cos x}$$

$$= \log 2 \times \frac{1}{2} \times 1 = \frac{1}{2} \log 2$$

Example 7 Evaluate

$$\lim_{n\to\infty} n^{-n^2}\left[(n+1)\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2^2}\right)\ldots\ldots\left(n+\frac{1}{2^{n-1}}\right)\right]^n.$$

Solution. $\lim_{n\to\infty} n^{-n^2}\left[(n+1)\left(n+\frac{1}{2}\right)\ldots\left(n+\frac{1}{2^{n-1}}\right)\right]^n$

$$= \lim_{n\to\infty}\left[\frac{(n+1)\left(n+\frac{1}{2}\right)\ldots\ldots\left(n+\frac{1}{2^{n-1}}\right)}{n^n}\right]^n$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\left(\frac{n+\frac{1}{2}}{n}\right)^n\ldots\ldots\left(\frac{n+\frac{1}{2^{n-1}}}{n}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot\left(1+\frac{1}{2n}\right)^n\ldots\left(1+\frac{1}{2^{n-1}n}\right)^n \qquad (1^\infty \text{ form})$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot\left(1+\frac{1}{2n}\right)^{\frac{2n}{2}}\ldots\left(1+\frac{1}{2^{n-1}n}\right)^{\frac{2^{n-1}\cdot n}{2^{n-1}}}$$

$$= e^1\cdot e^{1/2}\cdot e^{1/4}\cdot\ldots\ldots\cdot e^{1/2^{n-1}}\cdot\ldots\ldots\infty \qquad \left[\text{Using } \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{an} = e^a\right]$$

$$= e^{(1+1/2+1/4+\ldots\ldots)}$$

$$= e^{\frac{1}{1-\frac{1}{2}}} = e^2$$

Example 8 Find $L = \lim_{x\to\infty}\dfrac{ax^p+bx^{p-2}+c}{dx^q+ex^{q-2}+k}$, where a, b, c, d, e and k are constants and $p>0$, $q>0$.

Solution. Here, $\lim_{x\to\infty}\dfrac{ax^p+bx^{p-2}+c}{dx^q+ex^{q-2}+k}$

$$= \lim_{x\to\infty}\frac{x^p\{a+b/x^2+c/x^p\}}{x^q\{d+e/x^2+k/x^q\}} \qquad \ldots\text{(i)}$$

Now, we have to consider all the we have to consider all the three cases because which one of these p and q is greater, is not given, three cases because which one of these p and q is greater not given.

Case I $p>q$

$$= \lim_{x\to\infty} x^{p-q}\frac{\left\{a+\frac{b}{x^2}+\frac{c}{x^p}\right\}}{\left\{d+\frac{e}{x^2}+\frac{k}{x^q}\right\}} \qquad \left[\begin{array}{l}\text{Since, } p-q>0\\ \Rightarrow x^{p-q}\to\infty \text{ as } x\to\infty\end{array}\right]$$

$$=\infty$$

Case II $p = q$

$$= \lim_{x \to \infty} \frac{\left\{a + \dfrac{b}{x^2} + \dfrac{c}{x^p}\right\}}{\left\{d + \dfrac{e}{x^2} + \dfrac{k}{x^q}\right\}} = \frac{a}{d}$$

Case III $p < q$

$$= \lim_{x \to \infty} \frac{\left\{a + \dfrac{b}{x^2} + \dfrac{c}{x^p}\right\}}{x^{q-p}\left\{d + \dfrac{e}{x^2} + \dfrac{k}{x^q}\right\}} \qquad \left[\begin{array}{l}\text{Since, } \ p - q < 0 \\ \Rightarrow \quad x^{p-q} \to 0 \text{ as } x \to \infty\end{array}\right]$$

$$= 0$$

Hence, $$\lim_{x \to \infty} \frac{ax^p + bx^{p-2} + c}{dx^q + ex^{q-2} + k} = \begin{cases} \infty, & \text{when } p > q \text{ and } ad > 0 \\ -\infty, & \text{when } p > q \text{ and } ad < 0 \\ \dfrac{a}{d}, & \text{when } p = q \\ 0, & \text{when } p < q \end{cases}$$

Example 9 If α and β be the roots of the quadratic equation $ax^2 + bx + c = 0$, then evaluate

$$\lim_{x \to a} \frac{1 - \cos(ax^2 + bx + c)}{(x - \alpha)^2}.$$

Solution. $$\lim_{x \to \alpha} \frac{1 - \cos(ax^2 + bx + c)}{(x - \alpha)^2}$$

$$= \lim_{x \to \alpha} \frac{1 - \cos a(x - \alpha)(x - \beta)}{(x - \alpha)^2}$$

[As α, β are roots of $ax^2 + bx + c$ $\therefore$ $ax^2 + bx + c = a(x - \alpha)(x - \beta)$]

$$= \lim_{x \to \alpha} \frac{2\sin^2 a(x - \alpha)(x - \beta)/2}{(x - \alpha)^2}$$

$$= \lim_{x \to \alpha} \left\{2\left(\frac{\sin a(x - \alpha)(x - \beta)/2}{(x - \alpha)}\right)^2 \cdot \frac{a^2(x - \beta)^2/4}{a^2(x - \beta)^2/4}\right\}$$

$$= \lim_{x \to \alpha} \left\{2\left(\frac{\sin a(x - \alpha)(x - \beta)/2}{a(x - \alpha)(x - \beta)/2}\right)^2 \cdot \frac{a^2(x - \beta)^2}{4}\right\}$$

$$= \frac{2}{4}a^2(\alpha - \beta)^2 = \frac{1}{2}a^2\{(\alpha + \beta)^2 - 4\alpha\beta\}$$

$$= \frac{1}{2}a^2\left\{\left(-\frac{b}{a}\right)^2 - 4\left(\frac{c}{a}\right)\right\} \qquad (\text{As } \alpha, \beta \text{ are roots of } ax^2 + bx + c = 0)$$

$$= \frac{1}{2}a^2\left\{\frac{b^2}{a^2} - \frac{4c}{a}\right\}$$

$$= \frac{\frac{1}{2}a^2(b^2 - 4ac)}{a^2} = \frac{1}{2}(b^2 - 4ac)$$

Example 10 If $f(x)=\begin{cases}\dfrac{\sin[x]}{[x]}, & \text{for } [x]\neq 0\\ 0, & \text{for } [x]=0\end{cases}$

where $[x]$ denotes the greatest integer less than or equal to x, then $\lim\limits_{x\to 0} f(x)$.

Solution. Here, limit can be calculated only after removing greatest integral function (*ie,* $[x]$)

$\therefore$ (LHL at $x=0$)

$$=\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{\sin[x]}{[x]}=\lim_{h\to 0}\frac{\sin[0-h]}{[0-h]}$$

$$=\lim_{h\to 0}\frac{\sin(-1)}{(-1)}=\sin 1 \quad (\text{As } -1\le[0-h]<0 \therefore [0-h]=-1\; ie, [x]\neq 0)$$

Again, (RHL at $x=0$)

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} 0 \qquad \left[\text{As } f(x)=\begin{cases}\dfrac{\sin[x]}{[x]}, & [x]\neq 0\\ 0, & [x]=0\end{cases}\right]$$

$$=0$$

$\therefore$ $\lim\limits_{x\to 0^+} f(x)=0$ Here, $0\le[0+h]<1 \quad \therefore \quad [x]=0 \Rightarrow f(x)=0$

So, $\lim\limits_{x\to 0^-} f(x)\neq \lim\limits_{x\to 0^+} f(x)$

Thus, limit doesn't exist.

Example 11 Evaluate a, b, c and d.

If $\lim\limits_{x\to\infty}\left(\sqrt{x^4+ax^3+3x^2+bx+2}-\sqrt{x^4+2x^3-cx^2+3x-d}\right)=4$

Solution. Here, $\lim\limits_{x\to\infty}\left(\sqrt{x^4+ax^3+3x^2+bx+2}-\sqrt{x^4+2x^3-cx^2+3x-d}\right)=4$

($\infty-\infty$ form)

Rationalising

$$\Rightarrow \quad \lim_{x\to\infty}\frac{(a-2)x^3+(3+c)x^2+(b-3)x+(2+d)}{\sqrt{x^4+ax^3+3x^2+bx+2}+\sqrt{x^4+2x^3-cx^2+3x-d}}$$

Since, limit is finite, the degree of the numerator must be 2. So, $a-2=0$ *ie*, $a=2$.

$$\Rightarrow \quad \lim_{x\to\infty}\frac{(3+c)x^2+(b-3)x+(2+d)}{\sqrt{x^4+ax^3+3x^2+bx+2}+\sqrt{x^4+2x^3-cx^2+3x-d}}$$

On dividing numerator and denominator by x^2. We get,

$$\lim_{x\to\infty}\frac{(3+c)+(b-3)/x+(2+d)/x^2}{\sqrt{1+\dfrac{a}{x}+\dfrac{3}{x^2}+\dfrac{b}{x^3}+\dfrac{2}{x^4}}+\sqrt{1+\dfrac{2}{x}-\dfrac{c}{x^2}+\dfrac{3}{x^3}-\dfrac{d}{x^4}}}=\frac{3+c}{2}$$

Given, $\dfrac{3+c}{2}=4 \quad\Rightarrow\quad c=5 \quad \therefore \; c=5,\; a=2$

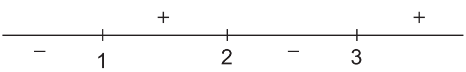
Hence, $a=2$, $c=5$ and b, d are any real numbers.

Example 12 Let $f(x)=\dfrac{|x^3-6x^2+11x-6|}{x^3-6x^2+11x-6}$. Find the set of points a, where $\lim\limits_{x\to a} f(x)$ doesn't exist.

Solution. We write, $f(x)=\dfrac{|(x-1)(x-2)(x-3)|}{(x-1)(x-2)(x-3)}$

$$f(x)=\begin{cases}-1, & x<1\\ 1, & 1<x<2\\ -1, & 2<x<3\\ 1, & x>3\end{cases}\quad \text{(Leaving } x=1,2,3 \text{ as denominator} \neq 0)$$

Using Wavy-curve method, as shown in figure



ie, when $x>3$,

$$|(x-1)(x-2)(x-3)|=(x-1)(x-2)(x-3)$$

when $2<x<3$,

$$|(x-1)(x-2)(x-3)|=-(x-1)(x-2)(x-3)$$

when $1<x<2$,

$$|(x-1)(x-2)(x-3)|=+(x-1)(x-2)(x-3)$$

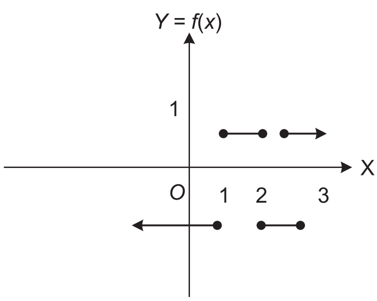
when $x<1$,

$$|(x-1)(x-2)(x-3)|=-(x-1)(x-2)(x-3)$$

Thus, $$f(x)=\begin{cases}-1, & x<1\\ 1, & 1<x<2\\ -1, & 2<x<3\\ 1, & x>3\end{cases}$$

Shows limit exists at all points except at $x=1, 2, 3$.

Graphically, this is shown in given figure.



Which shows limit doesn't exist at $x=1, 2, 3$.

Example 13 Let the r^{th} term, t_r of a series is given by $t_r = \dfrac{r}{1+r^2+r^4}$. Then find the value of $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n} t_r$.

Solution. Here, $t_r = \dfrac{r}{1+r^2+r^4} = \dfrac{1}{2}\left(\dfrac{2r}{1+2r^2+r^4-r^2}\right) = \dfrac{1}{2}\left(\dfrac{2r}{(1+r^2)^2-(r^2)}\right)$

$$= \frac{1}{2}\left(\frac{2r}{(1+r^2+r)(1+r^2-r)}\right) \qquad [\because 2r = (r^2+r+1)-(r^2-r+1)]$$

$$= \frac{1}{2}\left(\frac{(r^2+r+1)-(r^2-r+1)}{(1+r^2+r)(1+r^2-r)}\right) = \frac{1}{2}\left(\frac{1}{1+r^2-r} - \frac{1}{1+r^2+r}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1+r(r-1)} - \frac{1}{1+r(r+1)}\right) \qquad \text{...(i)}$$

Thus,
$$t_r = \frac{1}{2}\left(\frac{1}{1+r(r-1)} - \frac{1}{1+r(r+1)}\right)$$

$$t_1 = \frac{1}{2}\left(\frac{1}{1+0} - \frac{1}{1+2}\right) = \frac{1}{2}\left(1 - \frac{1}{3}\right)$$

$$t_2 = \frac{1}{2}\left(\frac{1}{1+2} - \frac{1}{1+2\cdot 3}\right) = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{7}\right)$$

$$t_3 = \frac{1}{2}\left(\frac{1}{7} - \frac{1}{13}\right) = \frac{1}{2}\left(\frac{1}{7} - \frac{1}{13}\right)$$

....

....

$$t_n = \frac{1}{2}\left(\frac{1}{1+n(n-1)} - \frac{1}{1+n(n+1)}\right)$$

$\therefore$
$$\lim_{n\to\infty}\sum_{r=1}^{n} t_r = \frac{1}{2}\left(1 - \frac{1}{1+n(n+1)}\right) \qquad \text{...(ii)}$$

Hence,
$$\lim_{n\to\infty}\sum_{r=1}^{n} t_r = \lim_{n\to\infty}\frac{1}{2}\left(1 - \frac{1}{1+n(n+1)}\right) = \frac{1}{2}\left(1 - \frac{1}{\infty}\right)$$

$\therefore$
$$\lim_{n\to\infty}\sum_{r=1}^{n} t_r = \frac{1}{2}$$

Example 14 Solve $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\cot^{-1}\left(\dfrac{r^3-r+\frac{1}{r}}{2}\right)$.

Solution. Here, $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\cot^{-1}\left(\dfrac{r^3-r+\frac{1}{r}}{2}\right)$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\frac{2r}{1-r^2+r^4}\right) = \lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\frac{2r}{1+(r^2-r)(r^2+r)}\right)$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \tan^{-1} \left(\frac{(r^2 + r) - (r^2 - r)}{1 + (r^2 + r)(r^2 - r)} \right)$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \{\tan^{-1}(r^2 + r) - \tan^{-1}(r^2 - r)\}$$

$$= \lim_{n \to \infty} [(\tan^{-1} 2 - \tan^{-1} 0) + (\tan^{-1} 6 - \tan^{-1} 2) + (\tan^{-1} 12 - \tan^{-1} 6)$$

$$+ \ldots + \{(\tan^{-1}(n^2 + n) - \tan^{-1}(n^2 - n)\}]$$

$$= \lim_{n \to \infty} \{\tan^{-1}(n^2 + n) - \tan^{-1}(0)\}$$

$$= \tan^{-1}(\infty) - \tan^{-1}(0) = \frac{\pi}{2}$$

$$\therefore \qquad \lim_{n \to \infty} \sum_{r=1}^{n} \cot^{-1} \left(\frac{r^3 - r + \frac{1}{r}}{2} \right) = \frac{\pi}{2}$$

Example 15 Let $(\tan \alpha) x + (\sin \alpha) y = \alpha$ and $(\alpha \operatorname{cosec} \alpha) x + (\cos \alpha) y = 1$ be two variable straight lines, α being the parameter. Let P be the point of intersection of the lines. In the limiting position when $\alpha \to 0$. Then, find the point of intersection of straight lines.

Solution. Here, two straight lines $(\tan \alpha) x + (\sin \alpha) y = \alpha$ and

$(\alpha \operatorname{cosec} \alpha) x + (\cos \alpha) y = 1$ have their point of intersection, as

$$x = \frac{\alpha \cos \alpha - \sin \alpha}{\sin \alpha - \alpha} \text{ and } y = \frac{\alpha - x \tan \alpha}{\sin \alpha}$$

$\therefore$ when $\alpha \to 0$, we obtain the point P.

ie,
$$\lim_{\alpha \to 0} x = \lim_{\alpha \to 0} \frac{\alpha \cos \alpha - \sin \alpha}{\sin \alpha - \alpha} \qquad \left(\frac{0}{0} \text{ form} \right)$$

$$= \lim_{\alpha \to 0} \frac{-\alpha \sin \alpha + \cos \alpha - \cos \alpha}{\cos \alpha - 1} \qquad \text{(Applying L'Hospital's rule)}$$

$$= \lim_{\alpha \to 0} \frac{-\alpha \sin \alpha}{-2 \sin^2 \alpha/2} = \lim_{\alpha \to 0} \frac{\alpha \left(2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \right)}{2 \sin^2 \frac{\alpha}{2}}$$

$$= \lim_{\alpha \to 0} \frac{\alpha}{\tan \frac{\alpha}{2}} = \lim_{\alpha \to 0} \frac{\frac{2\alpha}{2}}{\tan \frac{\alpha}{2}} = 2$$

Again, as
$$\lim_{\alpha \to 0} y = \lim_{\alpha \to 0} \frac{\alpha - x \tan \alpha}{\sin \alpha}$$

$$= \lim_{\alpha \to 0} \left(\frac{\alpha}{\sin \alpha} - \frac{x}{\cos \alpha} \right) = \lim_{\alpha \to 0} \frac{\alpha}{\sin \alpha} - \lim_{\alpha \to 0} \frac{x}{\cos \alpha}$$

$$= 1 - 2 \qquad \left[\text{As } \lim_{\alpha \to 0} x = 2 \right]$$

$$\therefore \qquad \lim_{\alpha \to 0} y = -1$$

Hence, in limiting position, $P(2, -1)$.

Example 16 Let $a = \min\{x^2 + 2x + 3, x \in R\}$ and $b = \lim\limits_{\theta \to 0} \dfrac{1 - \cos\theta}{\theta^2}$. Then, the value of $\sum\limits_{r=0}^{n} a^r \cdot b^{n-r}$.

Solution. Here, $a = \min\{x^2 + 2x + 3, x \in R\}$

ie, $\quad x^2 + 2x + 3 = x^2 + 2x + 1 + 2 = (x+1)^2 + 2$

$\therefore \quad a$ is minimum when $x = -1$ *ie,* $a = 2$

Again, $$b = \lim_{\theta \to 0} \frac{2\sin^2\theta/2}{\theta^2} = \lim_{\theta \to 0} \frac{2}{4} \cdot \frac{\sin^2\theta/2}{\theta^2/4} = \lim_{\theta \to 0} \frac{2}{4} \cdot \frac{\sin^2\theta/2}{(\theta/2)^2} = \frac{1}{2}$$

Hence, $$\sum_{r=0}^{n} a^r\, b^{n-r} = \sum_{r=0}^{n} 2^r \cdot \left(\frac{1}{2}\right)^{n-r}$$

$$\Rightarrow \quad \frac{1}{2^n}\{2^0 + 2^2 + 2^4 + \ldots + 2^{2n}\}$$

$$\frac{1}{2^n}\left\{\frac{1(4^{n+1} - 1)}{4 - 1}\right\} = \frac{4^{n+1} - 1}{2^n \times 3} \quad [\textit{ie}, \text{ sum of } (n+1) \text{ terms of GP}]$$

$$\therefore \quad \sum_{r=0}^{n} a^r\, b^{n-r} = \frac{4^{n+1} - 1}{2^n \times 3}$$

Example 17 Find a polynomial of least degree, such that $\lim\limits_{x \to 0} \left(1 + \dfrac{x^2 + f(x)}{x^2}\right)^{1/x} = e^2$.

Solution. Now, $\lim\limits_{x \to 0} \left(1 + \dfrac{x^2 + f(x)}{x^2}\right)^{1/x}$

$$\Rightarrow \quad \lim_{x \to 0} \left(1 + \frac{x^2 + f(x)}{x^2}\right)^{1/x} = L \quad \text{(say)}$$

exists only when $\lim\limits_{x \to 0} \dfrac{x^2 + f(x)}{x^2} = 0$ [It converts to 1^∞ form]

So, the least degree in $f(x)$ is of degree 2.

ie, $\quad f(x) = a_2x^2 + a_3x^3 + \ldots$

Now, $$L = \lim_{x \to 0} \left(1 + \frac{x^2 + f(x)}{x^2}\right)^{1/x} = e^2$$

$$= e^{\lim\limits_{x \to 0} \left(\frac{x^2 + f(x)}{x^2}\right) \cdot \frac{1}{x}} = e^2 = e^{\lim\limits_{x \to 0} \frac{x^2 + f(x)}{x^3}} = e^2$$

$$\therefore \quad \lim_{x \to 0} \frac{x^2 + f(x)}{x^3} = 2$$

$$\Rightarrow \quad \lim_{x \to 0} \frac{x^2 + a_2x^2 + a_3x^3 + \ldots}{x^3} = 2$$

$\Rightarrow a_2 = -1, a_3 = 2$ and a_4, a_5 are any arbitrary constants. Since, we want polynomial of least degree.

Hence, $\quad f(x) = -x^2 + 2x^3$

Example 18 Find the value of a, b and c such that

$$\lim_{x \to 0} \frac{axe^x - b\log(1+x) + cxe^{-x}}{x^2 \sin x} = 2.$$

Solution. Using the expansion, we have

$$\lim_{x \to 0} \frac{ax\left(1 + x + \frac{x^2}{2!} + \dots\right) - b\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right) + cx\left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right)}{x^2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)} = 2$$

$$\Rightarrow \quad \lim_{x \to 0} \frac{x(a - b + c) - x^2\left(a + \frac{b}{2} - c\right) + x^3\left(\frac{a}{2} - \frac{b}{3} + \frac{c}{2}\right) + \dots}{x^2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)} = 2$$

Now, above limit would exists, if least power in numerator is greater than or equal to least power in denominator.

ie, Coefficient of x and x^2 must be zero and coefficient of x^3 should be 2.

ie, $\quad a - b + c = 0,\ a + \frac{b}{2} - c = 0,\ \frac{a}{2} - \frac{b}{3} + \frac{c}{2} = 2$

On solving, we get $a = 3,\ b = 12,\ c = 9$

Example 19 Evaluate $\lim_{n \to \infty} \cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\dots\dots\cos\left(\frac{x}{2^n}\right)$.

Solution. Here, $\lim_{n \to \infty} \cos\left(\frac{x}{2^n}\right)\cos\left(\frac{x}{2^{n-1}}\right)\dots\dots\cos\left(\frac{x}{8}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{2}\right)$

We know, $\quad \cos A \cos 2A \cos 2^2 A \dots\dots \cos 2^{n-1} A = \frac{\sin 2^n A}{2^n \sin A}$

Thus, $\quad \lim_{n \to \infty} \cos\left(\frac{x}{2^n}\right)\cos\left(\frac{x}{2^{n-1}}\right)\dots\dots\cos\left(\frac{x}{8}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{2}\right)$

$$= \lim_{n \to \infty} \frac{\sin 2^n\left(\frac{x}{2^n}\right)}{2^n \sin\left(\frac{x}{2^n}\right)} = \lim_{n \to \infty} \frac{\sin x}{2^n \sin\left(\frac{x}{2^n}\right)}$$

$$= \sin x \cdot \lim_{n \to \infty} \frac{1}{2^n \sin\left(\frac{x}{2^n}\right)} \cdot \frac{x}{x}$$

$$= \frac{\sin x}{x} \cdot \lim_{n \to \infty} \frac{1}{2^n \sin\left(\frac{x}{2^n}\right)} \cdot x$$

$$= \frac{\sin x}{x} \cdot 1 \qquad \left[n \to \infty,\ \frac{x}{2^n} \to 0 \text{ and } \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$= \frac{\sin x}{x}$$

Example 20 If x is a real number in [0, 1]. Then, the value of $\lim_{m\to\infty}\lim_{n\to\infty}[1+\cos^{2m}(n!\pi x)]$.

Solution. x is a real number in [0, 1], thus we have two cases either,

Case I $x \in Q$ (rational number)

As $x \in Q$, we have $\quad x = 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\ldots\ldots, 1$

which shows $(n!\pi x)$ is integral multiple of π for large values of n

$\therefore \quad \cos(n!\pi x) = \pm 1$

Thus, $\quad \lim_{m\to\infty}\lim_{n\to\infty}[1+\cos^{2m}(n!\pi x)]$

$$= \lim_{m\to\infty}\lim_{n\to\infty}(1+1)$$

$$= 2$$

Case II $x \notin Q$ (irrational number)

As $x \notin Q$, we have $\quad x = \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{5}}, \ldots\ldots\ldots\ldots$

which shows $(n!\pi x)$ is not an integral multiple of π and so, $\cos(n!\pi x)$ will lie between -1 and $+1$.

Thus, $\quad \lim_{m\to\infty}\lim_{n\to\infty}[1+\cos^{2m}(n!\pi x)]$

$\lim_{m\to\infty}\lim_{n\to\infty}[1+(\text{a value between} -1 \text{ and } +1)^{2m}]$

$\lim_{n\to\infty}\{1+0\}$ $\quad$ [As $0 < x < 1 \Rightarrow x^{\infty} = 0$]

Thus, $$\lim_{m\to\infty}\lim_{n\to\infty}[1+\cos^{2m}(n!\pi x)] = \begin{cases} 2, & x \in Q \\ 1, & x \notin Q \end{cases}$$

Example 21 Let $[\cdot]$ represent the greatest integral function less than or equal to x. Then, find the value of $\lim_{x\to 0}\left(\left[\frac{n\sin x}{x}\right]+\left[\frac{n\tan x}{x}\right]\right)$.

Solution. We know, $n \le [x] < n+1 \quad \Rightarrow \quad [x] = n$

Here, $\quad \frac{n\sin x}{x} \to n$, as $x \to 0$ but less than n.

Also, $\quad \frac{n\tan x}{x} \to n$, as $x \to 0$ but more than n.

Thus, $\quad n-1 < \left[\frac{n\sin x}{x}\right] < n$, as $x \to 0 \quad \Rightarrow \quad \left[\frac{n\sin x}{x}\right] = n-1$

Again, $\quad n \le \left[\frac{n\tan x}{x}\right] < n+1$, as $x \to 0 \quad \Rightarrow \quad \left[\frac{n\tan x}{x}\right] = n$

Thus, $$\lim_{x\to 0}\left(\left[\frac{n\sin x}{x}\right]+\left[\frac{n\tan x}{x}\right]\right) = (n-1)+(n) = (2n-1)$$

$\therefore$ $$\lim_{x\to 0}\left(\left[\frac{n\sin x}{x}\right]+\left[\frac{n\tan x}{x}\right]\right) = (2n-1)$$

Example 22 Solve $\lim_{x \to a} [\sqrt{2-x} + \sqrt{1+x}]$, where $a \in \left[0, \frac{1}{2}\right]$ and $[\cdot]$ denote the greatest integer function.

Solution. Here, $\lim_{x \to a} [\sqrt{2-x} + \sqrt{1+x}]$ and we know, we could only apply limit after defining greatest integral function.

Thus, finding range of $[\sqrt{2-x} + \sqrt{1+x}]$ when $x \in \left[0, \frac{1}{2}\right]$.

ie, Let $f(x) = \sqrt{2-x} + \sqrt{1+x}$

For range $f'(x) = \frac{1}{2}\left(-\frac{1}{\sqrt{2-x}} + \frac{1}{\sqrt{1+x}}\right) = \frac{1}{2}\left(\frac{\sqrt{2-x} - \sqrt{1+x}}{\sqrt{2-x} \cdot \sqrt{1+x}}\right)$

$\Rightarrow$ $f'(x)$ will be the +ve for $\sqrt{2-x} > \sqrt{1+x}$.

$\Rightarrow$ $f'(x)$ will be +ve for $2 - x > 1 + x$.

$\Rightarrow$ $f'(x)$ will be +ve for $2x < 1$.

$\Rightarrow$ $x < \frac{1}{2}$

$\therefore$ $f(x)$ will be increasing for $x < \frac{1}{2}$.

$\therefore$ $f(0) = \sqrt{2} + 1$

$f(1/2) = \sqrt{6}$

which shows range of $f(x)$ is $[1 + \sqrt{2}, \sqrt{6}]$ when $x \in \left[0, \frac{1}{2}\right]$

$\therefore$ $[\sqrt{2-x} + \sqrt{1+x}] = 2$

$\Rightarrow$ $\lim_{x \to a} [\sqrt{2-x} + \sqrt{1+x}] = \lim_{x \to a} 2 = 2$

Example 23 Evaluate

$$\lim_{x \to 0^+} \frac{-1 + \sqrt{(\tan x - \sin x) + \sqrt{(\tan x - \sin x) + \sqrt{(\tan x - \sin x) + \ldots \infty}}}}{-1 + \sqrt{x^3 + \sqrt{x^3 + \sqrt{x^3 + \ldots \infty}}}}$$

Solution. Let $y = \sqrt{(\tan x - \sin x) + \sqrt{(\tan x - \sin x) + \sqrt{(\tan x - \sin x) + \ldots \infty}}}$

$\Rightarrow$ $y = \sqrt{(\tan x - \sin x) + y} \quad \Rightarrow \quad y^2 - y - (\tan x - \sin x) = 0$

$\Rightarrow$ $y = \dfrac{1 + \sqrt{1 + (\tan x - \sin x)4}}{2}$ [As $y > 0$] ...(i)

Again, let $Z = \sqrt{x^3 + \sqrt{x^3 + \sqrt{x^3 + \ldots \infty}}}$

$\Rightarrow$ $Z = \sqrt{x^3 + Z} \quad \Rightarrow \quad Z^2 - Z - x^3 = 0$

$\Rightarrow$ $Z = \dfrac{1 + \sqrt{(1 + 4x^3)}}{2}$ [As $Z > 0$] ...(ii)

$\therefore$ $$\lim_{x \to 0^+} \frac{-1 + \sqrt{(\tan x - \sin x) + \sqrt{(\tan x - \sin x) + \ldots \infty}}}{-1 + \sqrt{x^3 + \sqrt{x^3 + \sqrt{x^3 + \ldots \infty}}}}$$

$$= \lim_{x \to 0^+} \frac{-1 + \left(\dfrac{1 + \sqrt{1 + 4(\tan x - \sin x)}}{2}\right)}{-1 + \left(\dfrac{1 + \sqrt{1 + 4x^3}}{2}\right)} \quad \text{[From Eqs. (i) and (ii)]}$$

$$= \lim_{x \to 0^+} \frac{-1 + \sqrt{1 + 4(\tan x - \sin x)}}{-1 + \sqrt{1 + 4x^3}}$$

Rationalising numerator and denominator, we get

$$\lim_{x \to 0^+} \frac{4(\tan x - \sin x)(1 + \sqrt{1 + 4x^3})}{4x^3(1 + \sqrt{1 + 4(\tan x - \sin x)})}$$

$$= \lim_{x \to 0^+} \frac{\left(\dfrac{\sin x}{\cos x} - \dfrac{\sin x}{1}\right)(1 + \sqrt{1 + 4x^3})}{x^3(1 + \sqrt{1 + 4(\tan x - \sin x)})}$$

$$= \lim_{x \to 0^+} \frac{\sin x(1 - \cos x)}{x^3 \cos x} \cdot \frac{(1 + \sqrt{1 + 4x^3})}{(1 + \sqrt{1 + 4(\tan x - \sin x)})}$$

$$= \lim_{x \to 0^+} \frac{\sin x}{x} \cdot \frac{2\sin^2 \dfrac{x}{2}}{\dfrac{4x^2}{4}} \cdot \frac{1}{\cos x} \cdot \frac{(1 + \sqrt{1 + 4x^3})}{(1 + \sqrt{1 + 4(\tan x - \sin x)})}$$

$$= 1 \cdot \frac{1}{2} \cdot 1 \cdot \frac{(1 + 1)}{(1 + 1)} = \frac{1}{2}$$

$$\therefore \quad \lim_{x \to 0^+} \frac{-1 + \sqrt{(\tan x - \sin x) + \sqrt{(\tan x - \sin x) + \sqrt{(\tan x - \sin x) + \ldots \infty}}}}{-1 + \sqrt{x^3 + \sqrt{x^3 + \sqrt{x^3 + \ldots\ldots \infty}}}}$$

$$= \frac{1}{2}$$

Example 24 Evaluate $\lim_{\theta \to 0} \dfrac{\cos^2(1 - \cos^2(1 - \cos^2(1 \ldots \cos^2\theta)) \ldots)}{\sin\left(\dfrac{\pi(\sqrt{\theta + 4} - 2)}{\theta}\right)}$.

Solution. $\lim_{\theta \to 0} \dfrac{\cos^2(1 - \cos^2(1 - \cos^2(1 \ldots \cos^2\theta)) \ldots)}{\sin\left(\dfrac{\pi(\sqrt{\theta + 4} - 2)}{\theta}\right)}$

$$= \lim_{\theta \to 0} \frac{\cos^2(\sin^2(\sin^2 \ldots\ldots (\sin^2\theta)) \ldots)}{\sin\left(\dfrac{\pi(\sqrt{\theta + 4} - 2)}{\theta}\right)}$$

$$= \lim_{\theta \to 0} \frac{\cos^2(\sin^2(\sin^2 \ldots\ldots (\sin^2\theta)) \ldots)}{\sin\left(\pi \lim_{\theta \to 0} \dfrac{\theta}{\theta(\sqrt{\theta + 4} + 2)}\right)}$$

$$= \frac{\cos^2(0)}{\sin\left(\dfrac{\pi}{4}\right)} = \sqrt{2}$$

Example 25 Solve $\lim\limits_{n \to \infty} a_n$ when $a_{n+1} = \sqrt{2 + a_n}$, $n = 1, 2, 3, \ldots$

Solution. Here, $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} a_{n+1}$

So, let $\quad l = \lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} a_{n+1}$

$\therefore \quad l = \lim\limits_{n \to \infty} a_{n+1} = \lim\limits_{n \to \infty} \sqrt{2 + a_n}$

or $\quad l = \sqrt{2 + l} \qquad \left(\because \lim\limits_{n \to \infty} a_n = l \right)$

$\Rightarrow \quad l^2 = 2 + l \qquad \Rightarrow \quad l^2 - l - 2 = 0$

$\Rightarrow \quad (l - 2)(l + 1) = 0 \qquad$ *ie*, $\quad l = 2, -1$

$\therefore \quad \lim\limits_{n \to \infty} a_n = 2$ or $-1 \quad \Rightarrow \quad \lim\limits_{n \to \infty} a_n = 2$

$$\left(\text{Neglecting } \lim_{n \to \infty} a_n = -1, \text{ as } a_n > 0 \right)$$

Example 26 If $a_1 = 1$ and $a_{n+1} = \dfrac{4 + 3a_n}{3 + 2a_n}$, $n \geq 1$, then show $a_{n+2} > a_{n+1}$ and if a_n has a limit l as $n \to \infty$, then evaluate $\lim\limits_{n \to \infty} a_n$.

Solution. Here, $\quad a_1 = 1;$

$$\therefore \quad a_2 = \frac{4 + 3}{3 + 2} = \frac{7}{5} > 1$$

$\therefore \quad a_2 > a_1$

Assuming $a_{n+1} > a_n$,

$$\therefore \quad a_{n+2} - a_{n+1} = \frac{4 + 3a_{n+1}}{3 + 2a_{n+1}} - \frac{4 + 3a_n}{3 + 2a_n}$$

$$= \frac{(4 + 3a_{n+1})(3 + 2a_n) - (4 + 3a_n)(3 + 2a_{n+1})}{(3 + 2a_{n+1})(3 + 2a_n)}$$

$$= \frac{a_{n+1} - a_n}{(3 + 2a_{n+1})(3 + 2a_n)} > 0 \qquad (\because a_{n+1} > a_n)$$

$\therefore \quad a_{n+2} - a_{n+1} > 0$

$\Rightarrow \quad a_{n+2} > a_{n+1}$ whenever $a_{n+1} > a_n$

$\therefore$ The sequence of values a_n is increasing and since $a_1 = 1$, $a_n > 0$, for all n.

Now, let $\quad l = \lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} a_{n+1}$

$$\therefore \quad l = \lim_{n \to \infty} \frac{4 + 3a_n}{3 + 2a_n}$$

$$\Rightarrow \quad l = \frac{4 + 3l}{3 + 2l} \qquad \left(\because \lim_{n \to \infty} a_n = l \right)$$

$\Rightarrow \quad 3l + 2l^2 = 4 + 3l$

$\Rightarrow \quad l^2 = 2$

$\therefore \quad l = \sqrt{2} \qquad$ (Neglecting $l = -\sqrt{2}$, as $l > 0$)

Point to Consider

Before doing the examples, revise introduction of limits, definition of limits and Squeeze theorem.

Example 27 Show $\left(1+\sum_{k=1}^{n}\frac{2}{{}^nC_k}\right)^n \to e^2$, as $n\to\infty$. (For $n\geq 6$)

Solution. Let $a_n = 1+2\sum_{k=1}^{n}\frac{1}{{}^nC_k}$

While, for $n\geq 6$, $a_n = 1+2\cdot\frac{1}{{}^nC_1}+2\cdot\frac{1}{{}^nC_2}+\left(\underbrace{\frac{1}{{}^nC_3}+\dots\dots\dots+\frac{1}{{}^nC_n}}_{(n-2)}\right)$

$$a_n \leq 1+\frac{2}{n}+\frac{4}{n(n-1)}+\frac{(n-2)}{{}^nC_3}$$

$$a_n \leq 1+\frac{2}{n}+\frac{4}{n(n-1)}+\frac{6(n-2)}{n(n-1)(n-2)}$$

$$a_n \leq 1+\frac{2}{n}+\frac{10}{n(n-1)} \qquad \text{...(i)}$$

Also, $$a_n = 1+2\sum_{k=1}^{n}\frac{1}{{}^nC_k} \geq 1+\frac{2}{{}^nC_1} \qquad \text{...(ii)}$$

$\Rightarrow$ $$1+\frac{2}{n}\leq a_n \leq 1+\frac{2}{n}+\frac{10}{n(n-1)}$$

$\Rightarrow$ $$\left(1+\frac{2}{n}\right)^n \leq a_n^n \leq \left(1+\frac{2}{n}+\frac{10}{n(n-1)}\right)^n$$

$\Rightarrow$ $$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n \leq \lim_{n\to\infty} a_n^n \leq \lim_{n\to\infty}\left[1+\frac{2}{n}\left(1+\frac{5}{n-1}\right)\right]^n$$

$\Rightarrow$ $$e^2 \leq \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{{}^nC_k}\right)^n \leq e^2$$

$\therefore$ By Squeeze principle for limits,

$$\lim_{n\to\infty}\left(1+\sum_{k=1}^{n}\frac{2}{{}^nC_k}\right)^n = e^2$$

Example 28 Solve $\lim_{n\to\infty} x_n$, when $x_n^2 = a + x_{n-1}$ and $x_0=\sqrt{a}$.

Solution. Here, $x_0=\sqrt{a}$, $x_1=\sqrt{a+\sqrt{a}}$, which shows $x_1 > x_0$

$\therefore$ Assuming $x_n > x_{n-1}$

$\Rightarrow$ $x_{n-1}^2 - x_{n-1} - a < 0$

$\Rightarrow$ $$\left\{x_{n-1}-\frac{\sqrt{4a+1}+1}{2}\right\}\left\{x_{n-1}+\frac{\sqrt{4a+1}-1}{2}\right\}<0$$

$$\Rightarrow \qquad x_{n-1} < \frac{\sqrt{4a+1}+1}{2}$$

$$\therefore \qquad \lim_{n\to\infty} x_{n-1} = \lim_{n\to\infty} x_n = l$$

$$\Rightarrow \qquad l^2 - l - a = 0$$

$$\Rightarrow \qquad l = \frac{1 \pm \sqrt{1+4a}}{2}, \text{ since } l \geq 0$$

$$\therefore \qquad l = \frac{1+\sqrt{1+4a}}{2}$$

$$\text{or} \qquad \lim_{n\to\infty} x_n < \frac{\sqrt{1+4a}+1}{2}$$

Example 29 Let $a_1, a_2, \ldots, a_n$ be sequence of real numbers with $a_{n+1} = a_n + \sqrt{1+a_n^2}$ and $a_0 = 0$. Prove that $\lim_{n\to\infty}\left(\frac{a_n}{2^{n-1}}\right) = \frac{4}{\pi}$.

Solution. Here, $a_{n+1} = a_n + \sqrt{1+a_n^2}$, where let $a_n = \cot(\alpha_n)$

$$\Rightarrow \qquad a_{n+1} = \cot(\alpha_n) + \operatorname{cosec}(\alpha_n)$$

$$\Rightarrow \qquad a_{n+1} = \frac{\cos(\alpha_n)+1}{\sin(\alpha_n)} = \frac{2\cos^2(\alpha_n/2)}{2\sin(\alpha_n/2)\cos(\alpha_n/2)} = \cot\left(\frac{\alpha_n}{2}\right)$$

Putting $n = 1$,

$$a_1 = \cot(\alpha_1) \text{ and } a_1 = a_0 + \sqrt{1+a_0^2} = 1$$

$$\Rightarrow \qquad \cot(\alpha_1) = 1 \text{ or } \alpha_1 = \frac{\pi}{4}$$

Again,

$$a_2 = \cot\left(\frac{\alpha_1}{2}\right) = \cot\left(\frac{\pi}{8}\right)$$

$$a_3 = \cot\left(\frac{\alpha_2}{2}\right) = \cot\left(\frac{\pi}{4\cdot 2^2}\right)$$

$$a_4 = \cot\left(\frac{\alpha_3}{2}\right) = \cot\left(\frac{\pi}{4\cdot 2^3}\right)$$

....

....

$$a_n = \cot\left(\frac{\pi}{4\cdot 2^{n-1}}\right); \text{ put } \frac{1}{2^{n-1}} = x$$

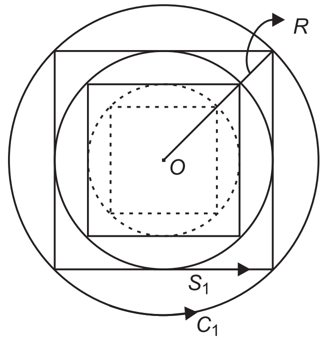
Hence,

$$\lim_{n\to\infty}\left(\frac{a_n}{2^{n-1}}\right) = \lim_{n\to\infty}\frac{\cot\left(\frac{\pi}{4\cdot 2^{n-1}}\right)}{2^{n-1}} = \lim_{x\to 0}\frac{1}{\frac{\tan\left(\frac{\pi}{4}x\right)}{x}}$$

$$\therefore \qquad \lim_{n\to\infty}\left(\frac{a_n}{2^{n-1}}\right) = \frac{4}{\pi}$$

Example 30 A square is inscribed in a circle of radius R, a circle is inscribed in this square, then a square in this circle and so on, n times. Find the limit of the sum of areas of all the squares as $n \to \infty$.

Solution. Let the side of a square, S_1 be 'a' units.



Then, $\quad a\sqrt{2} = 2R$

$\Rightarrow \quad R = \dfrac{a}{\sqrt{2}}$ is radius of circle C_1.

If a_1 be the side of another square, then

$$a_1\sqrt{2} = a \quad \Rightarrow \quad a_1 = \frac{a}{\sqrt{2}}$$

$$a_2\sqrt{2} = a_1 \quad \Rightarrow \quad a_2 = \frac{a_1}{\sqrt{2}} = \frac{a}{2}\cdot$$

....

....

So, sum of areas of all the squares,

$$S_n = a^2 + a_1^2 + a_2^2 + \dots \text{ upto } n \text{ terms}$$

$$= a^2 + \frac{a^2}{2} + \frac{a^2}{4} + \dots \text{ upto } n \text{ terms}$$

$$= a^2\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \text{ upto } n \text{ terms}\right)$$

$$= a^2\left\{\frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}\right\} = 2a^2\left(1 - \frac{1}{2^n}\right)$$

$$\therefore \qquad \lim_{n \to \infty} S_n = \lim_{n \to \infty} 2a^2\left(1 - \frac{1}{2^n}\right)$$

$$= 2\,a^2 = 4\,R^2 \qquad \left(\text{As } n \to \infty \Rightarrow \frac{1}{2^n} \to 0\right)$$

Type 2 : **Only One Correct Option**

Example 31 Let $\lim\limits_{x \to 1} \dfrac{x^a - ax + a - 1}{(x-1)^2} = f(a)$. The value of $f(101)$ equals to

(a) 5050 (b) 5151 (c) 4950 (d) 101

Solution. Put $x = 1 + h$

$$\lim_{h \to 0} \frac{(1+h)^a - a(1+h) + a - 1}{h^2}$$

$$= \lim_{h \to 0} \frac{\left(1 + ah + \frac{a(a-1)}{2!}h^2 + \dots\dots\right) - a - ah + a - 1}{h^2}$$

$$\therefore \qquad f(a) = \frac{a(a-1)}{2}; f(101) = 5050$$

Hence, (a) is the correct answer.

Example 32 $\lim\limits_{x\to 1}\dfrac{nx^{n+1}-(n+1)x^n+1}{(e^x-e)\sin\pi x}$ where $n=100$ is equal to

(a) $\dfrac{5050}{\pi e}$ (b) $\dfrac{100}{\pi e}$ (c) $-\dfrac{5050}{\pi e}$ (d) $-\dfrac{4950}{\pi e}$

Solution. $l=\lim\limits_{x\to 1}\dfrac{nx^n(x-1)-(x^n-1)}{(e^x-e)\sin\pi x}$

Put $x=1+h$, so that as $x\to 1$, $h\to 0$

$$\therefore \quad l=-\lim_{h\to 0}\frac{h\cdot n(1+h)^n-\{(1+h)^n-1\}}{e(e^h-1)\sin\pi h}$$

$$l=-\lim_{x\to 1}\frac{n\cdot h(1+{}^nC_1h+{}^nC_2h^2+{}^nC_3h^3+\ldots)-(1+{}^nC_1h+{}^nC_2h^2+{}^nC_3h^3+\ldots-1)}{\pi e(h^2)\left(\dfrac{e^h-1}{h}\right)\left(\dfrac{\sin\pi h}{\pi h}\right)}$$

$$=-\frac{n^2-{}^nC_2}{\pi e}=-\left[\frac{2n^2-n(n-1)}{2\pi e}\right]=-\frac{n^2+n}{2(\pi e)}=-\frac{n(n+1)}{2(\pi e)}$$

If $n=100 \Rightarrow l=-\left(\dfrac{5050}{\pi e}\right)$

Hence, (c) is the correct answer.

Example 33 If $\lim\limits_{x\to 0}\left(1+x\left(1+\dfrac{f(x)}{kx^2}\right)\right)^{1/x}=e^3$ and $f(4)=64$, then k has value

(a) 1 (b) 2 (c) 4 (d) None of these

Solution. Given, $\lim\limits_{x\to 0}\left(1+x\left(1+\dfrac{f(x)}{kx^2}\right)\right)^{\frac{1}{x}}=e^3$

As, $\lim\limits_{x\to 0}(1+ax)^{1/x}=e^a$

$\therefore \quad 1+\dfrac{f(x)}{kx^2}=3 \quad \therefore \quad f(x)=2kx^2$

Hence, (b) is the correct answer.

Example 34 $\lim\limits_{x\to 0}\dfrac{\left(\sqrt{(1-\cos x)+\sqrt{(1-\cos x)+\sqrt{(1-\cos x)+\ldots\ldots\infty}}}\right)-1}{x^2}$ equals to

(a) 0 (b) $\dfrac{1}{2}$ (c) 1 (d) 2

Solution. Let $y=\sqrt{\alpha+\sqrt{\alpha+\sqrt{\alpha+\ldots\ldots\infty}}}$ where $\alpha=1-\cos x$; as $x\to 0$, $a\to 0$

$$y=\sqrt{\alpha+y};\ y^2=\alpha+y\Rightarrow y^2-y-\alpha=0$$

$$y=\frac{1\pm\sqrt{1+4\alpha}}{2}\quad \text{(Neglecting –ve sign, as } y \text{ can't be –ve)}$$

$$y=\frac{1+\sqrt{1+4\alpha}}{2}$$

Now, $$l = \lim_{\substack{x\to 0\\ \alpha\to 0}} \frac{\left[\dfrac{1+\sqrt{1+4\alpha}}{2} - 1\right]}{\dfrac{x^2}{1-\cos x}\cdot(1-\cos x)} = \lim_{\alpha\to 0} \frac{\sqrt{1+4\alpha}-1}{2\cdot 2\cdot \alpha} \qquad \left(\text{As } \lim_{x\to 0}\frac{x^2}{1-\cos x} = 2\right)$$

$$= \lim_{\alpha\to 0} \frac{(1+4\alpha)-1}{4\alpha(\sqrt{1+4\alpha}+1)} = 2 = \frac{4}{2} \qquad \text{(Rationalising the denominator)}$$

$$= \lim_{\alpha\to 0} \frac{1}{\sqrt{1+4\alpha}+1} = \frac{1}{2}$$

Hence, (b) is the correct answer.

Example 35 Which one of the following statements is true?

(a) If $\lim_{x\to c}\{f(x)\cdot g(x)\}$ and $\lim_{x\to c} f(x)$ exist, then $\lim_{x\to c} g(x)$ exists.

(b) If $\lim_{x\to c}\{f(x)\cdot g(x)\}$ exists, then $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist.

(c) If $\lim_{x\to c}\{f(x)+g(x)\}$ and $\lim_{x\to c} f(x)$ exist, then $\lim_{x\to c} g(x)$ also exists.

(d) If $\lim_{x\to c}\{f(x)+g(x)\}$ exists, then $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ also exist.

Solution. (a) This is false, $f(x)=x$; $g(x)=\operatorname{cosec} x$, now $\lim_{x\to 0}\{f(x)\cdot g(x)\}$ exists $=1$

Also, $\lim_{x\to 0} f(x) = 0$ exists but $\lim_{x\to c} g(x)$ doesn't exist.

(b) This is false. Let f be defined as $f(x) = \begin{cases} 1, & \text{if } x \le 0 \\ 2, & \text{if } x > 0 \end{cases}$. Let $g(x)=0$. Then, $f(x)g(x)=0$ and so $\lim_{x\to 0} f(x)\cdot g(x)$ exists, while $\lim_{x\to 0} f(x)$ doesn't.

(c) This is true. Note that $g=(f+g)-f$. Therefore, by the Limit theorem, $\lim_{x\to c} g(x) = \lim_{x\to c}\{f(x)+g(x)\} - \lim_{x\to c} f(x)$

(d) This is false.

Hence, (c) is the correct answer.

Example 36 If $\lim_{x\to a}[f(x)+g(x)] = 2$ and $\lim_{x\to a}[f(x)-g(x)] = 1$, then $\lim_{x\to a} f(x)g(x)$

(a) need not exist (b) exists and is $\frac{3}{4}$ (c) exists and is $-\frac{3}{4}$ (d) exists and is $\frac{4}{3}$

Solution. Let $f(x)+g(x) = F(x)$

$f(x)-g(x) = G(x)$

Since, $\lim_{x\to a} F(x)$ and $\lim_{x\to a} G(x)$ exist.

Hence, $\lim_{x\to a}\dfrac{F(x)+G(x)}{2}$ and $\lim_{x\to a}\dfrac{F(x)-G(x)}{2}$ must also exist.

$\Rightarrow$ $\lim_{x\to a}\dfrac{F(x)+G(x)}{2} = \dfrac{3}{2}$ and $\lim_{x\to a}\dfrac{F(x)-G(x)}{2} = \dfrac{1}{2}$

$\therefore$ $\lim_{x\to a}\{f(x)\cdot g(x)\} = \dfrac{3}{4}$

Hence, (b) is the correct answer.

Example 37 $\lim\limits_{n\to\infty} \dfrac{1^2 n + 2^2(n-1) + 3^2(n-2) + \ldots. + n^2 \cdot 1}{1^3 + 2^3 + 3^3 + \ldots\ldots + n^3}$ is equal to

(a) $\dfrac{1}{3}$ (b) $\dfrac{2}{3}$ (c) $\dfrac{1}{2}$ (d) $\dfrac{1}{6}$

Solution. $\lim\limits_{n\to\infty} \dfrac{1^2 n + 2^2(n-1) + 3^2(n-2) + \ldots\ldots. + n^2\{n-(n-1)\}}{\Sigma n^3}$

$$\text{Numerator} = n(1^2 + 2^2 + \ldots\ldots + n^2) - \{1\cdot 2^2 + 2\cdot 3^2 + 3\cdot 4^2 + \ldots\ldots + (n-1)n^2\}$$

$$= n\Sigma n^2 - \sum_{r=2}^{n} (r-1)\cdot r^2 = n\Sigma n^2 - \sum_{r=1}^{n} (r^3 - r^2)$$

$$= n\Sigma n^2 - [\Sigma n^3 - \Sigma n^2] = (n+1)\,\Sigma n^2 - \Sigma n^3$$

$$\therefore \qquad l = \lim_{n\to\infty} \frac{(n+1)\,\Sigma n^2 - \Sigma n^3}{\Sigma n^3}$$

$$\lim_{n\to\infty} \frac{(n+1)\,n(n+1)(2n+1)}{6n\,(n+1)\,n\,(n+1)} - 1$$

$$= \frac{4}{3} - 1 = \frac{1}{3}$$

Aliter :

$$l = \frac{1^2 n + 2^2\,(n-1) + \ldots. + n^2\cdot 1}{\underbrace{1^3 + 2^3 + 3^3 + \ldots. + n^3}} + 1 - 1$$

$$l = \frac{1^2(n+1) + 2^2\,(n+1) + \ldots. + n^2\,(n+1)}{\Sigma n^3} - 1$$

$$l = \frac{(n+1)\,\Sigma n^2}{\Sigma n^3} - 1$$

$$\Rightarrow \qquad \lim_{n\to\infty} \frac{(n+1)\cdot n(n+1)(2n+1)}{6\cdot \dfrac{n^2\,(n+1)^2}{4}} - 1 = \frac{4}{3} - 1 = \frac{1}{3}$$

Hence, (a) is the correct answer.

Example 38 The value of $\lim\limits_{x\to\infty} \dfrac{\cot^{-1}(x^{-a}\log_a x)}{\sec^{-1}(a^x \log_x a)}$ $(a > 1)$ is equal to

(a) 1 (b) 0

(c) $\dfrac{\pi}{2}$ (d) Doesn't exist

Solution. $\lim\limits_{x\to\infty} \dfrac{\cot^{-1}\left(\dfrac{\log_a x}{x^a}\right)}{\sec^{-1}\left(\dfrac{a^x}{\log_a x}\right)}$; as $\lim\limits_{x\to\infty} \left(\dfrac{\log_a x}{x^a}\right) \to 0$

and $\left(\dfrac{a^x}{\log_a x}\right) \to \infty$ (Using L' Hospital's rule)

$$\therefore \qquad l = \frac{\frac{\pi}{2}}{\frac{\pi}{2}} = 1$$

Hence, (a) is the correct answer.

Example 39 Let $a = \min[x^2 + 2x + 3, x \in R]$ and $b = \lim_{x \to 0} \dfrac{\sin x \cos x}{e^x - e^{-x}}$. Then, the value of $\sum_{r=0}^{n} a^r b^{n-r}$ is

(a) $\dfrac{2^{n+1}+1}{3 \cdot 2^n}$ (b) $\dfrac{2^{n+1}-1}{3 \cdot 2^n}$ (c) $\dfrac{2^n - 1}{3 \cdot 2^n}$ (d) $\dfrac{4^{n+1}-1}{3 \cdot 2^n}$

Solution. $a = (x+1)^2 + 2 \Rightarrow a = 2$

$$b = \lim_{x \to 0} \frac{\sin 2x}{\dfrac{2(e^{2x}-1) \cdot 2x}{2x}} = \frac{1}{2}$$

Now, $$\sum_{r=0}^{n} a^r b^{n-r} = \Sigma\, 2^r \left(\frac{1}{2}\right)^{n-r} = \frac{1}{2^n} \sum_{r=0}^{n} 2^{2r} = \frac{1}{2^n} \sum_{r=0}^{n} 4^r$$

$$= \frac{1}{2^n}[1 + 4 + 4^2 + \ldots + 4^n] = \frac{1}{2^n}\left[\frac{4^{n+1}-1}{3}\right] = \frac{4^{n+1}-1}{3 \cdot 2^n}$$

Hence, (d) is the correct answer.

Example 40 Suppose that a and $b (b \neq a)$ are real positive numbers, then the value of $\lim_{t \to 0} \left(\dfrac{b^{t+1} - a^{t+1}}{b-a}\right)^{1/t}$ has the value equal to

(a) $\dfrac{a \ln b - b \ln a}{b-a}$ (b) $\dfrac{b \ln b - a \ln a}{b-a}$ (c) $b \ln b - a \ln a$ (d) $\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}}$

Solution. Obviously, limit is of the form 1^∞.

Hence, $$l = e^{\lim_{t \to 0} \frac{1}{t}\left[\frac{b^{t+1}-a^{t+1}}{b-a} - 1\right]} = e^{\lim_{t \to 0}\left(\frac{b^{t+1}-a^{t+1}-b+a}{t(b-a)}\right)}$$

$$= e^{\lim_{t \to 0}\left(\frac{b(b^t-1)-a(a^t-1)}{t(b-a)}\right)} = e^{\frac{b \ln b - a \ln a}{b-a}}$$

$$= e^{\frac{\ln b^b - \ln a^a}{b-a}} = e^{\frac{\ln \frac{b^b}{a^a}}{b-a}} = e^{\ln\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}} = \left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

Hence, (d) is the correct answer.

Example 41 $\lim_{x \to \infty} \dfrac{\cot^{-1}(\sqrt{x+1} - \sqrt{x})}{\sec^{-1}\left\{\left(\dfrac{2x+1}{x-1}\right)^x\right\}}$ is equal to

(a) 1 (b) 0 (c) $\dfrac{\pi}{2}$ (d) non-existent

Solution. As $\lim_{x \to \infty} \sqrt{x+1} - \sqrt{x} = 0 \Rightarrow \cot^{-1}(0) = \dfrac{\pi}{2}$

$$\lim_{x \to \infty}\left(\frac{2x+1}{x-1}\right)^x \to \infty \quad \Rightarrow \sec^{-1}(\infty) = \frac{\pi}{2} \quad \therefore l = 1$$

Hence, (a) is the correct answer.

Example 42 $\lim\limits_{n \to \infty} \cos(\pi\sqrt{n^2 + n})$ (when n is an integer) is equal to

(a) 1 (b) – 1 (c) 0 (d) Doesn't exist

Solution. $l = \lim\limits_{n \to \infty} \pm \cos(n\pi - \pi\sqrt{n^2 + n})$

$$\Rightarrow \quad \lim_{n \to \infty} \pm \cos(\pi(n - \sqrt{n^2 + n})) = \lim_{n \to \infty} \pm \cos\left(\frac{(\pi(+n))}{n + \sqrt{n^2 + n}}\right)$$

$$= \lim_{n \to \infty} \pm \cos\left(\frac{n\pi}{n + n\sqrt{1 + \dfrac{1}{n}}}\right) = \lim_{n \to \infty} \pm \cos\left(\frac{\pi}{1 + \sqrt{1 + \dfrac{1}{n}}}\right) = \cos\frac{\pi}{2} \to 0$$

Aliter : $\pi n\left(1 + \dfrac{1}{n}\right)^{1/2} = n\pi\left(1 + \dfrac{1}{2n} + \dfrac{1}{2}\left(\dfrac{1}{2} - 1\right)\dfrac{1}{2!}\dfrac{1}{n^2} + \ldots.\right)$

$$= \pi\left(n + \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} - 1\right)\frac{1}{2!}\frac{1}{n} + \ldots.\right)$$

As $n \to \infty$; $\quad \dfrac{\pi}{2}\cdot\left(2n + 1 + \left(\dfrac{1}{2} - 1\right)\dfrac{1}{2!}\dfrac{1}{n} + \ldots\right) = (2n + 1)\dfrac{\pi}{2}$

$$\therefore \quad \lim_{n \to \infty} \cos\left((2n + 1)\frac{\pi}{2}\right) = 0$$

Hence, (c) is the correct answer.

Example 43 The value of $\lim\limits_{x \to 0} \dfrac{(\tan(\{x\} - 1))\sin\{x\}}{\{x\}(\{x\} - 1)}$ is, where $\{x\}$ denotes the fractional part function

(a) 1 (b) tan 1 (c) sin 1 (d) Doesn't exist

Solution. $\lim\limits_{x \to 0} \dfrac{(\tan(\{x\} - 1))\sin\{x\}}{\{x\}(\{x\} - 1)}$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0^+} \frac{\tan(h - 1)\cdot\sin h}{h(h - 1)} = \frac{\tan(-1)}{-1} = \tan 1$$

LHL $\quad = \lim\limits_{h \to 0} \dfrac{\tan((1 - h) - 1)\sin(1 - h)}{(1 - h)(1 - h - 1)} = \dfrac{\sin 1}{1} = \sin 1$

Hence, $\quad \lim\limits_{x \to 0} f(x)$ doesn't exist.

Hence, (d) is the correct answer.

Example 44 $\lim\limits_{x \to 0^-} (-\ln(\{x\} + |[x]|))^{\{x\}}$ is

(a) 0 (b) 1 (c) $\ln 2$ (d) $\ln\dfrac{1}{2}$

Solution. $\lim\limits_{x \to 0^-} (-\ln(\{0 - h\} + |[-h]|))^{\{-h\}}$

$$= \lim_{x \to 0^-} (-\ln(1 - h + 1))^{1 - h} = -\ln 2 = \ln\left(\frac{1}{2}\right)$$

Hence, (d) is the correct answer.

Example 45 $\lim\limits_{x \to \infty} \dfrac{2 + 2x + \sin 2x}{(2x + \sin 2x)e^{\sin x}}$ is equal to

(a) 0 (b) 1 (c) – 1 (d) non-existent

Solution. $\lim\limits_{x \to \infty} \dfrac{\dfrac{2}{x} + 2 + \dfrac{\sin 2x}{x}}{\left(2 + \dfrac{\sin 2x}{x}\right)e^{\sin x}}$, as $x \to \infty$

$$l = \lim_{x \to \infty} \frac{2}{2 \cdot e^{\sin x}} = \text{oscillatory between } \frac{1}{e} \text{ and } \frac{1}{e^{-1}} \Rightarrow \text{non-existent}$$

Hence, (d) is the correct answer.

Example 46 The value of $\lim\limits_{x \to 0} (\cos ax)^{\operatorname{cosec}^2 bx}$ is

(a) $e^{\left(-\frac{8b^2}{a^2}\right)}$ (b) $e^{\left(-\frac{8a^2}{b^2}\right)}$ (c) $e^{\left(-\frac{a^2}{2b^2}\right)}$ (d) $e^{\left(-\frac{b^2}{2a^2}\right)}$

Solution. $l = e^{\lim\limits_{x \to \infty} \operatorname{cosec}^2 bx\,(\cos ax - 1)}$

Now, $$-\lim_{x \to 0} \frac{1 - \cos ax}{\sin^2 bx} = -\lim_{x \to 0} \frac{\sin^2 ax}{\sin^2 bx} \cdot \frac{1}{1 + \cos ax} = -\frac{a^2}{2b^2},\ l = e^{-\frac{a^2}{2b^2}}$$

Hence, (c) is the correct answer.

Example 47 $\lim\limits_{n \to \infty} \sum\limits_{r=1}^{n} \dfrac{r}{n^2 + n + r}$ equals to

(a) 0 (b) 1/3 (c) 1/2 (d) 1

Solution. Let $f(n) = \dfrac{1}{n^2 + n + 1} + \dfrac{2}{n^2 + n + 2} + \dots. + \dfrac{n}{n^2 + n + n}$

Consider $$g(n) = \frac{1}{n^2 + n + n} + \frac{2}{n^2 + n + n} + \dots + \frac{n}{n^2 + n + n}$$
$$= \frac{1 + 2 + 3 + \dots + n}{n^2 + 2n} = \frac{n(n + 1)}{2(n^2 + 2n)}$$

$$g(n) < f(n) \qquad \text{...(i)}$$

Similarly, $$h(n) = \frac{1}{n^2 + n + 1} + \frac{2}{n^2 + n + 1} + \dots + \frac{n}{n^2 + n + 1} = \frac{n(n + 1)}{2(n^2 + n + 1)}$$

$\therefore$ $$f(n) < h(n) \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), $g(n) < f(n) < h(n)$

But $\lim\limits_{n \to \infty} g(n) = \lim\limits_{n \to \infty} h(n) = \dfrac{1}{2}$. Hence, using Sandwich theorem

$\therefore$ $$\lim_{n \to \infty} f(n) = \frac{1}{2}$$

Hence, (c) is the correct answer.

Example 48 $\lim\limits_{n \to \infty} \left(\sqrt{n^2 + n + 1} - [\sqrt{n^2 + n + 1}]\right)$ where [] denotes the greatest integer function is

(a) 0 (b) 1/2 (c) 2/3 (d) 1/4

Solution. $n < \sqrt{n^2+n+1} < n+1$

Hence, $[\sqrt{n^2+n+1}] = n$

$\therefore$
$$l = \lim_{n\to\infty} (\sqrt{n^2+n+1} - n) = \lim_{n\to\infty} \frac{n+1}{\sqrt{n^2+n+1}+n} = \frac{1}{2}$$

Hence, (b) is the correct answer.

Example 49 $\lim_{x\to 1} \frac{\sin^2(x^3+x^2+x-3)}{1-\cos(x^2-4x+3)}$ has the value equal to

(a) 18 (b) 9/2 (c) 9 (d) None of these

Solution. $\lim_{x\to 1} \frac{\sin^2(x^3+x^2+x-3)}{(x^3+x^2+x-3)^2} \cdot \frac{(x^3+x^2+x-3)^2}{1-\cos(x^2-4x+3)}$

$$= (1) \lim_{x\to 1} \frac{(x^2-4x+3)^2}{1-\cos(x^2-4x+3)} \cdot \frac{(x^3+x^2+x-3)^2}{(x^2-4x+3)^2}$$

$$= (1)(2) \lim_{x\to 1} \left(\frac{x^3+x^2+x-3}{x^2-4x+3}\right)^2 = 2l^2$$

where $l = \lim_{x\to 1} \frac{3x^2+2x+1}{2x-4}$ (Using L' Hospital's rule)

$$= \frac{6}{-2} = -3 \quad \therefore \quad l = 2(-3)^2 = 18$$

Hence, (a) is the correct answer.

Type 3 : More than One Correct Options

Example 50 $f(x) = \frac{3x^2+ax+a+1}{x^2+x-2}$, then which of the following can be correct?

(a) $\lim_{x\to 1} f(x)$ exists $\Rightarrow a = -2$ (b) $\lim_{x\to -2} f(x)$ exists $\Rightarrow a = 13$

(c) $\lim_{x\to 1} f(x) = 4/3$ (d) $\lim_{x\to -2} f(x) = -1/3$

Solution. $f(x) = \frac{3x^2+ax+a+1}{(x+2)(x-1)}$

As $x \to 1$, Dr. $\to 0$ hence, as $x \to 1$, Nr. $\to 0$

$\therefore$ $3 + 2a + 1 = 0 \quad \Rightarrow \quad a = -2$

As $x \to -2$, Dr. $\to 0$ hence, as $x \to -2$, Nr. $\to 0$

$\therefore$ $12 - 2a + a + 1 = 0 \quad \Rightarrow \quad a = 13$

Now, $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{3x^2-2x-1}{(x+2)(x-1)} = \lim_{x\to 1} \frac{(3x+1)(x-1)}{(x+2)(x-1)} = \frac{4}{3}$

Now, $\lim_{x\to -2} \frac{3x^2+13x+14}{(x+2)(x-1)} = \lim_{x\to -2} \frac{(3x+7)(x+2)}{(x+2)(x-1)} = -\frac{1}{3}$

Hence, (a), (b), (c) and (d) are the correct answer.

Example 51 $\lim_{x \to c} f(x)$ doesn't exist

(a) $f(x) = [[x]] - [2x - 1]$, $c = 3$

(b) $f(x) = [x] - x$, $c = 1$

(c) $f(x) = \{x\}^2 - \{-x\}^2$, $c = 0$

(d) $f(x) = \dfrac{\tan(\text{sgn } x)}{\text{sgn } x}$, $c = 0$

where $[x]$ denotes step up function and $\{x\}$ fractional part function.

Solution. (a) $\lim_{x \to 3} [[x]] - [2x - 1]$

RHL $\lim_{h \to 0} [(3 + h)] - [6 + 2h - 1]$

As $3 < 3 + h < 4 \Rightarrow [3 + h] = 3$ and $5 < 5 + 2h < 6 \Rightarrow [5 + 2h] = 5$

$\Rightarrow$ $\lim_{h \to 0} (3 - 5) = -2$

LHL $d \lim_{h \to 0} [[3 - h]] - [6 - 2h - 1]$

As $2 < 3 - h < 3 \Rightarrow [3 - h] = 2$ and $4 < 5 - 2h < 5 \Rightarrow [5 - 2h] = 4$

$\Rightarrow$ $\lim_{h \to 0} (2 - 4) = -2$

$\therefore$ $\lim_{x \to c} f(x)$ exists, $c = 3$

(b) $\lim_{x \to 1} [x] - x$ $\lim_{x \to 1^+} f(x) = \lim_{h \to 0} [1 + h] - (1 + h) = \lim_{h \to 0} (1 - 1 - h) = 0$

$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} [1 - h] - (1 - h) = \lim_{h \to 0} 0 - (1 - h) = -1$

$\therefore$ $\lim_{x \to 1} f(x)$ doesn't exist.

(c) $\lim_{x \to 0} \{x\}^2 - \{-x\}^2$

RHL $\lim_{h \to 0} \{h\}^2 - \{-h\}^2$

$\Rightarrow$ $\lim_{h \to 0} (h - [h])^2 - ((-h) - [-h])^2 \Rightarrow \lim_{h \to 0} h^2 - (-h + 1)^2 \Rightarrow -1$

LHL $\lim_{h \to 0} \{-h\}^2 - \{h\}^2 \Rightarrow \lim_{h \to 0} ((-h) - [-h])^2 - (h - [h])^2$

$\Rightarrow$ $\lim_{h \to 0} (-h + 1)^2 - (h - 0)^2 \Rightarrow 1$

$\therefore$ $\lim_{x \to c} f(x)$; $c = 0$, doesn't exist.

(d) $\lim_{x \to 0} \dfrac{\tan(\text{sgn } x)}{(\text{sgn } x)} \Rightarrow \lim_{x \to 0^+} f(x) = \lim_{h \to 0} \dfrac{\tan 1}{1} = \tan 1$

$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} \dfrac{\tan(-1)}{-1} = \tan 1 \quad \therefore \quad \lim_{x \to 0} f(x)$ exists.

Hence, (b) and (c) are the correct answers.

Example 52 Let $f(x) = \begin{cases} \dfrac{\tan^2\{x\}}{x^2 - [x]^2} & \text{, for } x > 0 \\ 1 & \text{, for } x = 0 \\ \sqrt{\{x\} \cot \{x\}} & \text{, for } x < 0 \end{cases}$ where $[x]$ is the step up function

and $\{x\}$ is the fractional part function of x, then

(a) $\lim_{x \to 0^+} f(x) = 1$

(b) $\lim_{x \to 0^-} f(x) = 1$

(c) $\cot^{-1} \left(\lim_{x \to 0^-} f(x) \right)^2 = 1$

(d) None of these

Solution. RHL $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{h \to 0} \dfrac{\tan^2 \{h\}}{h^2 - [h]^2} = \lim\limits_{h \to 0} \dfrac{\tan^2 h}{h^2} = 1$

LHL $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{h \to 0} \sqrt{\{-h\} \cot \{-h\}} = \lim\limits_{h \to 0} \sqrt{(1-h) \cot (1-h)} = \sqrt{\cot 1}$

$$\cot^{-1} \left(\lim_{x \to 0^-} f(x) \right)^2 = \cot^{-1} (\sqrt{\cot 1})^2 = 1$$

Hence, (a) and (c) are the correct answers.

Example 53 Given that the derivative $f'(a)$ exists. Indicate which of the following statement(s) is/are always true?

(a) $f'(a) = \lim\limits_{h \to a} \dfrac{f(h) - f(a)}{h - a}$

(b) $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a) - f(a-h)}{h}$

(c) $f'(a) = \lim\limits_{t \to 0} \dfrac{f(a+2t) - f(a)}{t}$

(d) $f'(a) = \lim\limits_{t \to 0} \dfrac{f(a+2t) - f(a+t)}{2t}$

Solution. Here, options (a) and (b) are true by definition.

Option (c) is false, as

$$\lim_{t \to 0} \frac{f(a+2t) - f(a)}{t} = 2f'(a)$$

and
$$\lim_{t \to 0} \frac{f(a+2t) - f(a+t)}{2t} = \frac{1}{2}(2f'(a) - f'(a)) = \frac{1}{2} f'(a)$$

Hence, option (d) is false.

Hence, (a) and (b) are the correct answers.

Type 4 : **Assertion and Reason**

Example 54 Statement I $\lim\limits_{x \to \pi/2} \dfrac{\sin(\cot^2 x)}{(\pi - 2x)^2} = \dfrac{1}{2}$ because

Statement II $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$ and $\lim\limits_{\theta \to 0} \dfrac{\tan\theta}{\theta} = 1$, where θ is measured in radians.

(a) Statement I is true, statement II is true and statement II is correct explanation for statement I.

(b) Statement I is true, statement II is true and statement II is not the correct explanation for statement I.

(c) Statement I is true, statement II is false.

(d) Statement I is false, statement II is true.

Solution. $\lim\limits_{x \to \pi/2} \dfrac{\sin(\cot^2 x)}{\cot^2 x} \cdot \dfrac{\cot^2 x}{(\pi - 2x)^2}$; put $x = \dfrac{\pi}{2} - h$

$$\lim_{h \to 0} \frac{\tan^2 h}{4h^2} = \frac{1}{4}$$

$\Rightarrow$ Statement I is false and statement II is true.

Type 5 : **Linked Comprehension Based Questions**

Passage I

(Q. Nos. 55 to 57)

$$\text{Let } f(x)=\begin{cases} e^{\{x^2\}}-1, & x>1 \\ \dfrac{\sin x-\tan x+\cos x-1}{2x^2+\ln(2+x)+\tan x}, & x<0 \\ 0, & x=0 \end{cases}$$

where {} represents fractional part function. Suppose, lines L_1 and L_2 represent tangent and normal to curve $y=f(x)$ at $x=0$. Consider the family of circles touching both the lines L_1 and L_2.

55. Ratio of radii of two circles belonging to this family cutting each other orthogonally is

(a) $2+\sqrt{3}$ (b) $\sqrt{3}$ (c) $2+\sqrt{2}$ (d) $2-\sqrt{2}$

56. A circle having radius unity is inscribed in the triangle formed by L_1 and L_2 and a tangent to it. Then, the minimum area of the triangle possible is

(a) $3+\sqrt{2}$ (b) $2+\sqrt{3}$ (c) $3+2\sqrt{2}$ (d) $3-2\sqrt{2}$

57. If centres of circles belonging to family having equal radii r are joined, then the area of figure formed is

(a) $2r^2$ (b) $4r^2$ (c) $8r^2$ (d) r^2

Solution. (Q. Nos. 55 to 57)

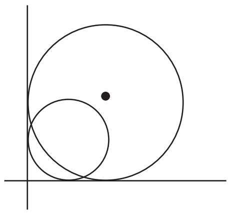
$$\text{LHD} = \lim_{h\to 0} \frac{\dfrac{-\sin h+\tan h+\cos h-1}{2h^2+\ln(2-h)-\tan h}-0}{-h}$$

$$= \lim_{h\to 0} \frac{\dfrac{\sin h}{h}-\dfrac{\tan h}{h}+\dfrac{1-\cos h}{h^2}\times h}{2h^2+\ln(2-h)-\tan h}=0$$

$$f'(0^+)=\text{RHD}=\lim_{h\to 0}\frac{e^{h^2}-1-0}{h}=h\times\frac{e^{h^2}-1}{h^2}=0$$

$$L_1\equiv y=0 \text{ and } L_2\equiv x=0$$

55. $(x-r)^2+(y-r)^2=r^2$ (family of circle)



$$x^2+y^2-2rx-2ry+r^2=0$$

$$2(r_1r_2+r_1r_2)=r_1^2+r_2^2 \quad \text{or} \quad 4r_1r_2=r_1^2+r_2^2$$

$$\left(\frac{r_2}{r_1}\right)^2-4\left(\frac{r_2}{r_1}\right)+1=0$$

$$\frac{r_2}{r_1}=\frac{4\pm\sqrt{12}}{2}=2\pm\sqrt{3}$$

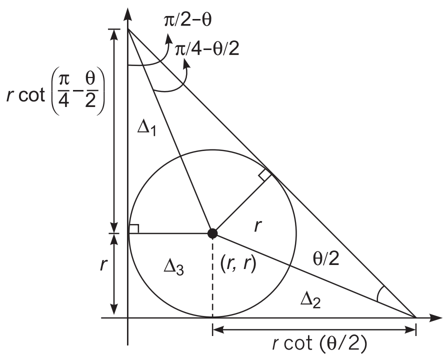
Hence, (a) is the correct answer.

56. $2[\Delta_1 + \Delta_2 + \Delta_3]$

$$\Delta = 2 \times \frac{1}{2}\left(\cot\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + \cot\frac{\theta}{2} + 1\right) \qquad \left[\text{Using } \frac{1}{2}ab\right]$$

$$\Delta = \frac{\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}{\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)} + \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} + 1$$

$$\Delta = 1 + \frac{2\sin\frac{\pi}{4}}{2\sin\frac{\theta}{2}\cdot\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}$$



$$\Delta = 1 + \frac{\sqrt{2}}{\cos\left(\theta - \frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right)}$$

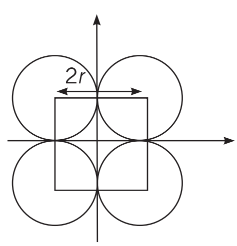
Δ is minimum, if numerator is maximum when $\theta = \frac{\pi}{4}$

$$\Delta_{\min} = 1 + \frac{\sqrt{2}}{1 - \frac{1}{\sqrt{2}}} = 1 + \frac{2}{\sqrt{2} - 1}$$

$$= 1 + 2(\sqrt{2} + 1) = 3 + 2\sqrt{2}$$

Hence, (c) is the correct answer.

57. Area $= (2r)^2 = 4r^2$



Hence, (b) is the correct answer.

Passage II

(Q. Nos. 58 to 60)

Let $f(x)$ is a function continuous for all $x \in R$ except at $x = 0$. Such that $f'(x) < 0\ \forall, x \in (-\infty, 0)$ and $f'(x) > 0,\ \forall\ x \in (0, \infty)$. Let $\lim_{x \to 0^+} f(x) = 2,\ \lim_{x \to 0^-} f(x) = 3$ and $f(0) = 4$

58. The value of λ for which $2\left(\lim_{x \to 0} f(x^3 - x^2)\right) = \lambda\left(\lim_{x \to 0} f(2x^4 - x^5)\right)$ is

(a) $\frac{4}{3}$ (b) 2 (c) 3 (d) 5

59. The values of $\lim_{x \to 0^+} \dfrac{f(-x)\,x^2}{\left\{\dfrac{1 - \cos x}{[f(x)]}\right\}}$ where $[\cdot]$ denotes greatest integer function and $\{\cdot\}$ denote fraction part function

(a) 6 (b) 12 (c) 18 (d) 24

60. $\lim_{x \to 0^-}\left(\left[3f\left(\dfrac{x^3 - \sin^3 x}{x^4}\right)\right] - f\left(\left[\dfrac{\sin x^3}{x}\right]\right)\right)$ where $[\cdot]$ denote greatest integer function

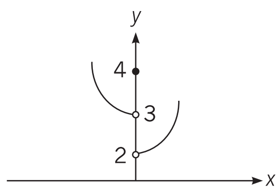
(a) 3 (b) 5 (c) 7 (d) 9

Solution. (Q. Nos. 58 to 60)

58. $x \to 0,\ x^3 - x^2 = x^2(x - 1) \to 0^-$

$$x \to 0,\ 2x^4 - x^5 = x^4(2 - x) \to 0^+$$

$$2(3) = \lambda(2) \Rightarrow \lambda = 3$$



Hence, (c) is the correct answer.

59. $$\lim_{x \to 0^+} \frac{f(-x)x^2}{\left(\dfrac{1 - \cos x}{[f(x)]}\right) - \left[\dfrac{1 - \cos x}{[f(x)]}\right]} = \frac{3x^2}{\dfrac{1 - \cos x}{2} - 0}$$

$$= 6 \times 2 = 12$$

Hence, (b) is the correct answer.

60. $$\lim_{x \to 0^-}\left(\frac{x^3 - \sin^3 x}{x^4}\right) = \left(\frac{x - \sin x}{x^3}\right)\left(\frac{x^2 + \sin^2 x + x\sin x}{x^2}\right)x$$

$$= \frac{1}{6}(3)\,x \to 0^- \Rightarrow f(0^-) = 3 \Rightarrow \lim_{x \to 0^-} \frac{\sin x^3}{x} = \frac{\sin x^3}{x^3}x^2 \to 0^+$$

$$\Rightarrow \left[\frac{\sin x^3}{x}\right] = 0 \Rightarrow f(0) = 4$$

$$\therefore\quad 3f\left(\frac{x^3 - \sin^3 x}{x^4}\right) > 9 \Rightarrow [9^+] - f(0) = 9 - 4 = 5$$

Hence, (b) is the correct answer.

Type 6 : **Match the Columns**

Example 61

	Column I	Column II
(A)	$\lim\limits_{x\to\infty}\left(\dfrac{x}{1+x}\right)^x$ equals to	(p) e^2
(B)	$\lim\limits_{x\to\infty}\left(\sin\dfrac{1}{x}+\cos\dfrac{1}{x}\right)^x$ equals to	(q) $e^{-1/2}$
(C)	$\lim\limits_{x\to 0}(\cos x)^{\cot^2 x}$ equals to	(r) e
(D)	$\lim\limits_{x\to 0}\left(\tan\left(\dfrac{\pi}{4}+x\right)\right)^{1/x}$ equals to	(s) e^{-1}

Solution. (A) → (s); (B) → (r); (C) → (q); (D) → (p)

(A) Put $x=\dfrac{1}{y}$, $\lim\limits_{y\to 0}\left(\dfrac{1}{1+y}\right)^{1/y}=e^{\lim\limits_{y\to 0}\frac{1-y-1}{y(1+y)}}=e^{-1}$

(B) $\lim\limits_{y\to 0}(\sin y+\cos y)^{1/y}=e^{\lim\limits_{y\to 0}\frac{\sin y+\cos y-1}{y}}=e$

(C) $e^{\lim\limits_{x\to 0}\frac{\cos x-1}{\frac{\tan^2 x}{x^2}\cdot x^2}}=e^{-\frac{1}{2}}$

(D) $e^{\lim\limits_{x\to 0}\frac{\tan((\pi/4)+x)-\tan(\pi/4)}{x}}=e^{\lim\limits_{x\to 0}\frac{\tan x\,[1+\tan((\pi/4)+x)\cdot 1]}{x}}=e^2$

Example 62

	Column I	Column II
(A)	$\lim\limits_{x\to\infty}\left(\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}\right)$ equals to	(p) -2
(B)	The value of the $\lim\limits_{x\to 0}\dfrac{\sin 2x-2\tan x}{\ln(1+x^3)}$ is	(q) -1
(C)	$\lim\limits_{x\to 0^+}(\ln\sin^3 x-\ln(x^4+ex^3))$ equals to	(r) 0
(D)	Let $\tan(2\pi\,\lvert\sin\theta\rvert)=\cot(2\pi\,\lvert\cos\theta\rvert)$, where $\theta\in R$ and $f(x)=(\lvert\sin\theta\rvert+\lvert\cos\theta\rvert)^x$. The value of $\lim\limits_{x\to\infty}\left[\dfrac{2}{f(x)}\right]$ equals to (Here, [·] represents greatest integer function)	(s) 1

Solution. (A) → (s); (B) → (p); (C) → (q); (D) → (r)

(A) $\lim\limits_{x\to\infty}\dfrac{(x+\sqrt{x})-(x-\sqrt{x})}{\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}}=\lim\limits_{x\to\infty}\dfrac{2\sqrt{x}}{\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}}=1$

(B) $\lim\limits_{x\to 0}\dfrac{\sin 2x-2\tan x}{x^3}\quad\lim\limits_{x\to 0}\dfrac{x^3}{\log(1+x^3)}$

$$\Rightarrow\quad\lim_{x\to 0}\frac{2\sin x(\cos^2 x-1)}{\cos x\cdot x^3}\cdot 1\Rightarrow\lim_{x\to 0}-\frac{2\tan^3 x}{x^3}=-2$$

(C) $\lim_{x \to 0^+} (\ln \sin^3 x - \ln(x^4 + ex^3)) = \lim_{x \to 0} \ln \left(\dfrac{\sin^3 x}{x^3 (x+e)} \right) = \ln \left(\dfrac{1}{e} \right) = -1$

(D) $\tan(2\pi \,|\sin\theta|) = \tan\left(\dfrac{\pi}{2} - |\cos\theta| 2\pi \right)$

$$2\pi\,|\sin\theta| = n\pi + \frac{\pi}{2} - |\cos\theta|2\pi$$

$$2\pi(|\sin\theta| + |\cos\theta|) = n\pi + \frac{\pi}{2}$$

$$|\sin\theta| + |\cos\theta| = \frac{n}{2} + \frac{1}{4} \qquad \ldots\text{(i)}$$

Since, $\quad 1 \le |\sin\theta| + |\cos\theta| \le \sqrt{2};\; 1 \le \dfrac{n}{2} + \dfrac{1}{4} \le \sqrt{2}$

$$4 \le 2n + 1 \le 4\sqrt{2}$$

$$\frac{3}{2} \le n \le \frac{4\sqrt{2} - 1}{2}$$

Thus, $n = 2$ is only possible value.

Putting in Eq. (i), $|\sin\theta| + |\cos\theta| = \dfrac{5}{4}$

$$g(x) = \lim_{x \to \infty} \left[2\left(\frac{4}{5}\right)^x \right] = 0$$

Example 63

	Column I		Column II
(A)	$\lim_{x \to 0} \dfrac{1 - \cos 2x}{e^{x^2} - e^x + x}$ equals to	(p)	1
(B)	If the value of $\lim_{x \to 0^+} \left(\dfrac{(3/x) + 1}{(3/x) - 1} \right)^{1/x}$ can be expressed in the form of $e^{p/q}$, where p and q are relative prime, then $(p+q)$ is equal to	(q)	2
(C)	$\lim_{x \to 0} \dfrac{\tan^3 x - \tan x^3}{x^5}$ equals to	(r)	4
(D)	$\lim_{x \to 0} \dfrac{x + 2\sin x}{\sqrt{x^2 + 2\sin x + 1} - \sqrt{\sin^2 x - x + 1}}$ equals to	(s)	5

Solution. (A) → (r); (B) → (s); (C) → (p); (D) → (q)

(A) $l = \lim_{x \to 0} \dfrac{e^{x^2} - 1 + x - e^x + 1}{2 \dfrac{\sin^2 x}{x^2} \cdot x^2}$

$$= \frac{1}{2}\left[\lim_{x \to 0} \frac{e^{x^2} - 1}{x^2} + \lim_{x \to 0} \frac{x - e^x + 1}{x^2} \right] = \frac{1}{2}\left[1 - \lim_{x \to 0} \frac{e^x - x - 1}{x^2} \right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{2} \right] = \frac{1}{4} \qquad \Rightarrow \qquad \frac{1}{l} = 4$$

(B) $l = \lim_{x \to 0} \left(\dfrac{3+x}{3-x}\right)^{1/x} = e^{\lim_{x \to 0} \frac{1}{x}\left(\frac{3+x}{3-x} - 1\right)}$

$$= e^{\lim_{x \to 0} \frac{2x}{x(3-x)}} = e^{2/3} \Rightarrow 2 + 3 = 5$$

(C) $\lim_{x \to 0} \dfrac{(\tan^3 x - x^3) - (\tan x^3 - x^3)}{x^5}$

$$= \lim_{x \to 0} \frac{\tan^3 x - x^3}{x^5} - \underbrace{\lim_{x \to 0} \frac{\tan x^3 - x^3}{x^5}}_{\text{zero (by expansion)}}$$

$$= \lim_{x \to 0} \frac{(\tan x - x)}{x^3} \cdot \frac{(\tan^2 x + x \tan x + x^2)}{x^2} = \frac{1}{3} \times 3 = 1$$

(D) Rationalising gives

$$\lim_{x \to 0} \frac{(x + 2\sin x)[\sqrt{(x^2 + 2\sin x + 1)} + \sqrt{\sin^2 x - x + 1}]}{(x^2 + 2\sin x + 1) - (\sin^2 x - x + 1)}$$

$$= 2 \cdot \lim_{x \to 0} \frac{x + \sin 2x}{x^2 - \sin^2 x + 2\sin x + x}$$

$$= 2 \cdot \lim_{x \to 0} \frac{1 + \dfrac{\sin 2x}{x}}{x - \dfrac{\sin^2 x}{x} + 2 + 1} = 2\left(\frac{1+2}{3}\right) = 2$$

Example 64

	Column I		Column II
(A)	$\lim_{n \to \infty} \cos^2(\pi(\sqrt[3]{n^3 + n^2 + 2n} - n))$ where n is an intger, equals to	(p)	$\frac{1}{2}$
(B)	$\lim_{n \to \infty} n \sin(2\pi\sqrt{1+n^2})\,(n \in N)$ equals to	(q)	$\frac{1}{4}$
(C)	$\lim_{n \to \infty} (-1)^n \sin(\pi\sqrt{n^2 + 0.5n + 1})$ $\left(\sin \dfrac{(n+1)\pi}{4n}\right)$ is (where $n \in N$)	(r)	π
(D)	If $\lim_{x \to \infty} \left(\dfrac{x+a}{x-a}\right)^x = e$ where a is some real constant, then the value of a is equal to	(s)	non-existent

Solution. (A) → (q); (B) → (r); (C) → (p); (D) → (p)

(A) $l = \lim_{n \to \infty} \cos^2(\pi(\sqrt[3]{n^3 + n^2 + 2n} - n))$ is

Consider, $\lim_{n \to \infty} [(n^3 + n^2 + 2n)^{1/3} - n] = \lim_{n \to \infty} \left[n\left(1 + \left(\dfrac{1}{n} + \dfrac{2}{n^2}\right)\right)^{1/3} - n\right]$

$$= \lim_{n \to \infty} n\left[\left\{1 + \left(\frac{1}{n} + \frac{2}{n^2}\right)\right\}^{1/3} - 1\right] = \lim_{n \to \infty} n\left[1 + \frac{1}{3}\left(\frac{1}{n} + \frac{2}{n^2}\right) + \ldots - 1\right]$$

$$= \frac{1}{3} + \text{terms containing } \frac{1}{n}, \frac{1}{n^2}, \frac{1}{n^3}, \ldots\ldots.$$

$$= \frac{1}{3}$$

$$\therefore \qquad l = \cos^2\left(\frac{\pi}{3}\right) = \frac{1}{4}$$

(B) $l = \lim\limits_{n \to \infty} n \sin (2\pi \sqrt{1+n^2} - 2n\pi)$

$$= \lim_{n \to \infty} n \sin\left(\frac{2\pi(\sqrt{1+n^2} - n)}{(\sqrt{1+n^2} + n)}(\sqrt{1+n^2} + n)\right)$$

$$= \lim_{n \to \infty}\left(\frac{n \sin\left(\dfrac{2\pi}{\sqrt{1+n^2} + n}\right)}{\left(\dfrac{2\pi}{\sqrt{1+n^2} + n}\right)}\left(\frac{2\pi}{\sqrt{1+n^2} + n}\right)\right)$$

$$= \lim_{n \to \infty} \frac{2n\pi}{n\left(\sqrt{1+\dfrac{1}{n^2}} + 1\right)} = \frac{2\pi}{2} = \pi$$

(C) $\lim\limits_{n \to \infty} (-1)^n (-1)^{n-1} \sin\left(n\pi - \pi\sqrt{n^2 + \dfrac{n}{2} + 1}\right)$

$$= \lim_{n \to \infty} (-1)^{2n-1} \sin \pi\left(\frac{\left(n - \sqrt{n^2 + \dfrac{n}{2} + 1}\right)\left(n + \sqrt{n^2 + \dfrac{n}{2} + 1}\right)}{n + \sqrt{n^2 + \dfrac{n}{2} + 1}}\right)$$

$$= \lim_{n \to \infty} (-1)^{2n-1} \sin \pi\left(\frac{n^2 - n^2 - \dfrac{n}{2} - 1}{n\left(1 + \sqrt{1 + \dfrac{1}{2n} + \dfrac{1}{n^2}}\right)}\right)$$

$$= \lim_{n \to \infty} (-1)^{2n} \sin \pi\left(\frac{\dfrac{1}{2} + \dfrac{1}{n}}{1 + \sqrt{1 + \dfrac{1}{2n} + \dfrac{1}{n^2}}}\right)$$

$$= (1) \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

Also, as $n \to \infty, \sin \dfrac{(n+1)\pi}{4n} \to \dfrac{1}{\sqrt{2}}$

$\therefore$ Final answer is $\dfrac{1}{2}$.

(D) $l = e^{\lim\limits_{x \to \infty} x\left(\frac{x+a}{x-a} - 1\right)} = e^{\lim\limits_{x \to \infty} x\left(\frac{2a}{x-a}\right)} = e^{\lim\limits_{x \to \infty} x\left(\frac{2a}{1-(a/x)}\right)} = e^{2a}$

$$\therefore \qquad e^{2a} = e \Rightarrow a = 1/2$$

Type 7 : **Integer Answer Type Questions**

Example 65 If $\lim_{x \to 0} (x^{-3} \sin 3x + ax^{-2} + b)$ exists and is equal to zero, then the value of $a + 2b$, is ……… .

Solution. **(6)** $\lim_{x \to 0} \dfrac{\sin 3x}{x^3} + \dfrac{a}{x^2} + b = \lim_{x \to 0} \dfrac{\sin 3x + ax + bx^3}{x^3}$

$$= \lim_{x \to 0} \frac{3 \dfrac{\sin 3x}{3x} + a + bx^2}{x^2} \quad \text{for existence of limit } 3 + a = 0 \Rightarrow a = -3$$

$\therefore$
$$l = \lim_{x \to 0} \frac{\sin 3x - 3x + bx^3}{x^3} = 27 \cdot \frac{\sin t - t}{t^3} + b = 0 \qquad (3x = t)$$

$$= -\frac{27}{6} + b = 0 \quad \Rightarrow \quad b = \frac{9}{2}$$

OR

[Use L' Hospital's rule]

Hence, $\quad a + 2b = -3 + 2 \times \dfrac{9}{2} = 6$

Example 66 Let $l = \lim_{x \to \infty} \left(\dfrac{x+1}{x-1} \right)^x$, then $\{l\}$ where { } denotes the fractional part function is $e^2 - \{l\}$ ……… .

Solution. **(7)** $l = \lim_{x \to \infty} \left(1 + \dfrac{2}{x-1} \right)^x = e^{\lim_{x \to \infty} \frac{2x}{x-1}} = e^2$

$\Rightarrow \quad e^2 - \{l\} = 7$

Example 67 For a certain value of c, $\lim_{x \to -\infty} [(x^5 + 7x^4 + 2)^c - x] = \lambda$, is finite and non-zero. Then, the value of $3c + \lambda$ is ……… .

Solution. **(2)** $\lim_{x \to -\infty} \left(x^{5c} \left(1 + \dfrac{7}{x} + \dfrac{2}{x^5} \right)^c - x \right) = \lim_{x \to -\infty} x \left(x^{5c-1} \left(1 + \dfrac{7}{x} + \dfrac{2}{x^5} \right)^c - 1 \right)$

This will be of the form $\infty \times 0$ only, if

$$5c - 1 = 0 \quad \Rightarrow \quad c = \frac{1}{5} \text{ substituting } c = \frac{1}{5}, \lambda \text{ becomes}$$

Thus, $\quad \lambda = \lim_{x \to -\infty} x\, [(1 + x)^{1/5} - 1]$ where $x = \dfrac{7}{x} + \dfrac{2}{x^5}$

$$= \lim_{x \to -\infty} x \left[1 + \frac{x}{5} + \ldots . - 1 \right] = \lim_{x \to -\infty} x \left(\frac{7}{x} + \frac{2}{x^5} \right) \cdot \frac{1}{5} = \frac{7}{5}$$

Hence, $\quad c = \dfrac{1}{5} \quad$ and $\quad \lambda = \dfrac{7}{5}$

$\Rightarrow \quad 3c + \lambda = 2$

Proficiency in 'Limits'
Exercise 1

Type 1 : Only One Correct Option

1. $\lim_{x \to 0} \dfrac{\sin(\pi \cos^2(\tan(\sin x)))}{x^2}$ is equal to

(a) π (b) $\dfrac{\pi}{4}$ (c) $\dfrac{\pi}{2}$ (d) None of these

2. $\lim_{t \to 0} \dfrac{1-(1+t)^t}{\ln(1+t)-t}$ is equal to

(a) $\dfrac{1}{2}$ (b) $-\dfrac{1}{2}$ (c) 2 (d) -2

3. If $I_1 = \lim_{x \to \infty} (\tan^{-1} \pi x - \tan^{-1} x) \cos x$ and $I_2 = \lim_{x \to 0} (\tan^{-1} \pi x - \tan^{-1} x) \cos x$, then (I_1, I_2) is

(a) (0, 0) (b) (0, 1) (c) (1, 0) (d) None of these

4. If $f(x) = 0$ be a quadratic equation such that $f(-\pi) = f(\pi) = 0$ and $f\left(\dfrac{\pi}{2}\right) = -\dfrac{3\pi^2}{4}$, then $\lim_{x \to -\pi} \dfrac{f(x)}{\sin(\sin x)}$ is equal to

(a) 0 (b) π (c) 2π (d) None of these

5. $\lim_{x \to 0} \dfrac{\sqrt[3]{1+\sin^2 x} - \sqrt[4]{1-2\tan x}}{\sin x + \tan^2 x}$ is equal to

(a) -1 (b) 1 (c) $\dfrac{1}{2}$ (d) $-\dfrac{1}{2}$

6. The value of $\lim_{n \to \infty} \dfrac{1}{n^\lambda} [(n+1)^\lambda (n+2)^\lambda \ldots (n+n)^\lambda]^{1/n}$ is equal to

(a) $\left(\dfrac{4}{e}\right)^\lambda$ (b) $\left(\dfrac{1}{e}\right)^\lambda$ (c) $\left(\dfrac{3}{e}\right)^\lambda$ (d) $\left(\dfrac{2}{e}\right)^\lambda$

7. If $x_{n+1} = \sqrt{\dfrac{1+x_n}{2}}$ and $|x_0| < 1, n \geq 0$, then $\lim_{n \to \infty} \left(\dfrac{\sqrt{1-x_0^2}}{x_1 x_2 x_3 \ldots x_{n+1}}\right)$ is equal to

(a) -1 (b) 1 (c) $\cos^{-1}(x_0 + 1)$ (d) $\cos^{-1}(x_0)$

8. For $n \in N$, let $f_n(x) = \tan \dfrac{x}{2} (1+\sec x)(1+\sec 2x)(1+\sec 4x) \ldots (1+\sec 2^n x)$

Then, $\lim_{x \to 0} \dfrac{f_n(x)}{2x}$ is equal to

(a) 0 (b) 2^n (c) 2^{n-1} (d) 2^{n+1}

9. If $f(x)=\begin{cases} 3+|x-k| & , \text{ for } x \le k \\ a^2-2+\dfrac{\sin(x-k)}{x-k} & , \text{ for } x > k \end{cases}$ has minimum at $x=k$, then

(a) $a \in R$ (b) $|a|<2$

(c) $|a|>2$ (d) $1<|a|<2$

10. If $\lim\limits_{n\to\infty} \dfrac{1}{\left(\dfrac{x^2+a}{x^2+1}\right)^n+1}=1$, then condition on a is

(a) $0<a<1$ (b) $-1<a<1$

(c) $-1<a<2$ (d) None of these

11. The vlaue of $\lim\limits_{x\to 0} \dfrac{\sqrt{\sec^2 \dfrac{x}{2}-1}}{x}$ is

(a) $\dfrac{1}{2}$ (b) $-\dfrac{1}{2}$ (c) 1 (d) doesn't exist

12. Let $f(x)$ be a real valued function defined for all $x \ge 1$, satisfying $f(1)=1$ and $f'(x)=\dfrac{1}{x^2+(f(x))^2}$; then $\lim\limits_{x\to\infty} f(x)$

(a) doesn't exist (b) exists and less than $\dfrac{\pi}{4}$

(c) exists and less than $1+\dfrac{\pi}{4}$ (d) exists and equal to 0

13. The quadratic equation whose roots are the minimum value of $\sin^2\theta-\sin\theta+\dfrac{1}{2}$ and $\lim\limits_{x\to\infty} \sqrt{(x+1)(x+2)}-x$ is

(a) $3x^2-7x+3=0$ (b) $8x^2-14x+3=0$

(c) $x^2-7x+3=0$ (d) $2x^2-7x+3=0$

14. Let $[x]$ denotes the greatest integer function. Let $g(x)=\dfrac{\sin\dfrac{\pi}{4}[x]}{[x]}$, then g is such that

(a) it is continuous at $x=\dfrac{3}{2}$ (b) it is continuous at $x=2$

(c) it is not continuous at any point (d) it has its right limit at $x=1$ as $\dfrac{1}{2}$

15. If $\lim\limits_{x\to 0}\dfrac{f(x)}{x^2}=a$ and $\lim\limits_{x\to 0}\dfrac{f(1-\cos x)}{g(x)\sin^2 x}=b$ (where $b\ne 0$), then $\lim\limits_{x\to 0}\dfrac{g(1-\cos 2x)}{x^4}$ is

(a) $\dfrac{4a}{b}$ (b) $\dfrac{a}{4b}$ (c) $\dfrac{a}{b}$ (d) None of these

16. If $x_1=\sqrt{3}$ and $x_{n+1}=\dfrac{x_n}{1+\sqrt{1+x_n^2}}$, $\forall\, n\in N$, then $\lim\limits_{x\to\infty} 2^n x_n$ equal to

(a) $\dfrac{3}{2\pi}$ (b) $\dfrac{2}{3\pi}$ (c) $\dfrac{2\pi}{3}$ (d) $\dfrac{3\pi}{2}$

17. $\lim\limits_{x \to a^-} \dfrac{\sqrt{x-b} - \sqrt{a-b}}{(x^2 - a^2)}$, $(a > b)$ is

(a) $\dfrac{1}{4a}$ (b) $\dfrac{1}{a\sqrt{a-b}}$ (c) $\dfrac{1}{2a\sqrt{a-b}}$ (d) $\dfrac{1}{4a\sqrt{a-b}}$

18. $\lim\limits_{n \to \infty} (\sin^n 1 + \cos^n 1)^n$ is equal to

(a) cot 1 (b) tan 1 (c) cos 1 (d) sin 1

Type 2 : **More than One Correct Options**

19. If $\lim\limits_{x \to \infty} 4x \left(\dfrac{\pi}{4} - \tan^{-1} \dfrac{x+1}{x+2} \right) = y^2 + 4y + 5$, then y can be equal to

(a) 1 (b) -1 (c) -4 (d) -3

20. $\lim\limits_{x \to 0} \dfrac{1 - \cos(x^2)}{x^3 (4^x - 1)}$ is equal to

(a) $\dfrac{1}{2} \ln 2$ (b) $\ln 2$ (c) $\ln 4$ (d) $1 - \dfrac{1}{2} \ln \left(\dfrac{e^2}{4} \right)$

21. If $f(x) = e^{[\cot x]}$ where $[y]$ represents the greatest integer less than or equal to y, then

(a) $\lim\limits_{x \to \frac{\pi^+}{2}} f(x) = 1$ (b) $\lim\limits_{x \to \frac{\pi^+}{2}} f(x) = \dfrac{1}{e}$ (c) $\lim\limits_{x \to \frac{\pi^-}{2}} f(x) = \dfrac{1}{e}$ (d) $\lim\limits_{x \to \frac{\pi^-}{2}} f(x) = 1$

22. $\lim\limits_{x \to 0} \left[m \dfrac{\sin x}{x} \right]$ is equal to (where $m \in I$ and $[\cdot]$ denotes greatest integer function.)

(a) m, if $m \le 0$ (b) $m - 1$, if $m > 0$ (c) $m - 1$, if $m < 0$ (d) m, if $m > 0$

23. If $\lim\limits_{x \to 0} (1 + ax + bx^2)^{2/x} = e^3$, then

(a) $a = 3, b = 0$ (b) $a = \dfrac{3}{2}, b = 1$ (c) $a = \dfrac{3}{2}, b = 4$ (d) $a = 2, b = 3$

Type 3 : **Assertion and Reason**

Directions

(Q. Nos. 24 to 28)

For the following questions, choose the correct answers from the codes (a), (b), (c) and (d) defined as follows :

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I.

(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I.

(c) Statement I is true, Statement II is false.

(d) Statement I is false, Statement II is true.

24. **Statement I** $\lim_{x \to 0} \frac{x}{a}\left(\frac{1}{x}\right)$ does not exist, (where $[\cdot]$ dentoes the greatest integer function)

Statement II $\lim_{x \to 0} \left(\frac{1}{x}\right)$ deos not exist.

25. **Statement I** $\lim_{x \to a} f(x)$ exists $= k$, but $\lim_{x \to k} g(x)$ does not exist. If $\lim_{x \to a} g(f(x))$ exists, then $x = a$ is a point of extremum for $y = f(x)$. If $f(x)$ is non-linear.

Statement II $\lim_{x \to k} g(x)$ does not exist, but $\lim_{x \to a} g(f(x))$ exists, $f(x)$ will approach k when $x \to a$ through only one side.

26. **Statement I** $\lim_{x \to 0} \frac{\sin\left(\pi \sin^2 \frac{x}{2}\right)}{x^2} = \pi$

Statement II $\lim_{x \to 0} \frac{\sin x}{x} = 1$

27. **Statement I** $\lim_{x \to 0} \sec^{-1}\left(\frac{\sin x}{x}\right) = 0$

Statement II $\lim_{x \to 0} \left(\frac{\sin x}{x}\right) = 1$

28. Let $a_n = \underbrace{2.99\ldots9}_{n \text{ times}},\ n \in N$

Statement I $\left[\lim_{n \to \infty} a_n\right] = \lim_{n \to \infty} [a_n]$, $[\cdot]$ denotes the greatest integer function.

Statement II $\lim_{n \to \infty} a_n = 3$

Type 4 : **Linked Comprehension Based Questions**

Passage I

(Q. Nos. 29 to 31)

If $\lim_{x \to a} f(x) = 1$ *and* $\lim_{x \to a} g(x) = \infty$, *then* $\lim_{x \to a} \{f(x)\}^{g(x)} = e^{\lim_{x \to a} (f(x) - 1) \times g(x)}$

29. $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{\frac{\sin x}{x - \sin x}}$ is equal to

(a) $\frac{1}{e}$ (b) $-\frac{1}{e}$ (c) e (d) $-e$

30. $\lim_{x \to 0} \left(\frac{x - 1 + \cos x}{x}\right)^{\frac{1}{x}}$ is equal to

(a) $e^{1/2}$ (b) $e^{-1/2}$ (c) e^1 (d) $\frac{1}{e}$

31. $\lim_{x \to 0} \left(\frac{a^x + b^x + c^x}{3}\right)^{\frac{2}{x}}$ is equal to

(a) $a^{2/3} + b^{2/3} + c^{2/3}$ (b) abc (c) $(abc)^{2/3}$ (d) 1

Passage II

(Q. Nos. 32 to 34)

Let $f(x) = \lim_{n \to \infty} \left(\cos \sqrt{\frac{x}{n}}\right)^n$, $g(x) = \lim_{n \to \infty} (1 + x + x \sqrt[n]{e})^n$.

Now, consider the function $y = h(x)$, *where* $h(x) = \tan^{-1}(g^{-1}f^{-1}(x))$.

32. $\lim_{x \to 0} \frac{\ln(f(x))}{\ln(g(x))}$ is equal to

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 0 (d) 1

33. Domain of the function $y = h(x)$ is

(a) $(0, \infty)$ (b) R (c) $(0, 1)$ (d) $[0, 1]$

34. Range of the function $y = h(x)$ is

(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left(-\frac{\pi}{2}, 0\right)$ (c) R (d) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Type 5 : **Match the Columns**

35. Match the statements of Column I with values of Column II.

Column I	Column II
(A) $\lim_{x \to \frac{\pi}{2}^+} \tan^{-1}(\tan x)$	(p) 0
(B) $\lim_{n \to \infty} \left[\sum_{r=1}^{n} \frac{1}{2^r}\right]$ ([·] denotes the greatest integer function)	(q) Doesn't exist
(C) $\lim_{x \to \infty} \sec^{-1}\left(\frac{x}{x+1}\right)$	(r) $-\frac{\pi}{2}$
(D) $\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{(1 - \sin x)^{2/3}}$	(s) $\frac{\pi}{2}$

36. Match the statements of Column I with values of Column II.

Column I	Column II
(A) $\lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \sqrt{\frac{n+r}{n-r}}$	(p) $\frac{\pi}{2}$
(B) $\lim_{x \to \infty} \left[\frac{1}{\sqrt{n^2 - 1}} + \frac{1}{n^2 - 2^2} + \dots + \frac{1}{\sqrt{n^2 - (n-1)^2}}\right]$	(q) $\frac{\pi}{2} + 1$
(C) $\lim_{x \to \infty} \left(\frac{n!}{n^n}\right)^{1/n}$	(r) π
	(s) $\frac{1}{e}$

Proficiency in 'Limits'
Exercise 2

1. Evaluate the $\lim\limits_{n \to \infty} 0 \cdot 2^{\log_{\sqrt{5}} (1/4 + 1/8 + 1/16 + \ldots\ \text{to}\ n \text{ terms})}$.

2. Let $f(x)$ be twice-differentiable function and $f''(0) = 2$, then evaluate $\lim\limits_{x \to 0} \dfrac{2f(x) - 3f(2x) + f(4x)}{x^2}$.

3. The graph of function $y = f(x)$ has unique tangent at the point $(a, 0)$ through which the graph passes. Then, evaluate $\lim\limits_{x \to a} \dfrac{\log_e \{1 + 6f(x)\}}{3f(x)}$.

4. If $\lim\limits_{x \to 0} \dfrac{x(1 + m \cos x) - n \sin x}{x^3} = 1$. Then, find the value of m and n.

5. Examine whether the following limit exists or not? If exists, find the value of the limit.

(i) $\lim\limits_{x \to 0} \left[\dfrac{\sin x}{x}\right]$

(ii) $\lim\limits_{x \to 0} \left[\dfrac{\sin^{-1} x}{x}\right]$

(iii) $\lim\limits_{x \to 0} \left[\dfrac{\tan x}{x}\right]$ (where $[\cdot]$ denotes greatest integer less than or equal to x.)

6. If $f(x) = \begin{cases} x, & x \le 0 \\ -x, & x > 0 \end{cases}$ and $g(x) = f(x) + |x|$. Then, evaluate $\lim\limits_{x \to 0^+} (\log_{|\sin x|} x)^{g(x)}$.

7. If $f_1(x) = \dfrac{x}{2} + 10$, $\forall\, x \in R$ and defined by $f_n(x) = f_1\{f_{n-1}(x)\}$, $\forall\, n \ge 2$. Then, evaluate $\lim\limits_{n \to \infty} f_n(x)$.

8. Solve $\lim\limits_{x \to 0} (1 + \log^2_{\cos x/2} \cos x)^2$.

9. Evaluate $\lim\limits_{x \to 0} \dfrac{1}{x^3} \displaystyle\int_0^x \dfrac{t^2}{t^4 + 1}\, dt$.

10. Show $\lim\limits_{x \to 0} \dfrac{[x]^2}{x^2}$ does not exist, but $\lim\limits_{x \to 0} \dfrac{[x^2]}{x^2}$ exists.

11. Solve $\lim\limits_{x \to 2^+} \dfrac{\{x\} \sin(x - 2)}{(x - 2)^2}$. (where $\{\cdot\}$ denotes the fractional part of x.)

12. Given a real valued function f such that;

$$f(x)=\begin{cases}\dfrac{\tan^2\{x\}}{x^2-[x]^2}, & \text{for } x>0\\ 1, & \text{for } x=0\\ \sqrt{\{x\}\cot\{x\}}, & \text{for } x<0\end{cases}$$

(where $[\cdot]$ and $\{\cdot\}$ denotes greatest integral and fractional part of x.)

Then, evaluate $\cot^{-1}\left(\lim\limits_{x\to 0^-} f(x)\right)^2$.

13. Evaluate

(i) $\lim\limits_{x\to 0}(\cos x)^{\sin^{-2}x}$

(ii) $\lim\limits_{n\to\infty}\left\{\dfrac{e^{1/n}}{n^2}+\dfrac{2\cdot(e^{1/n})^2}{n^2}+\dfrac{3\cdot(e^{1/n})^3}{n^2}+\ldots+\dfrac{(e^{1/n})^n}{n^2}\right\}$

14. Evaluate the

$$\lim_{x\to 0}\{1^{1/\sin^2 x}+2^{1/\sin^2 x}+\ldots\ldots\ldots+n^{1/\sin^2 x}\}^{\sin^2 x}$$

15. Evaluate the $\lim\limits_{h\to 0}\dfrac{\displaystyle\int_0^{\pi/3+h^4e^{1/h^2}}\cos^3 x\,dx-\int_0^{\pi/3}\cos^3 x\,dx}{h^4\,e^{1/h^2}}$.

16. Evaluate the $\lim\limits_{x\to 3}\left[\dfrac{\sin[x-3]}{[x-3]}\right]$.

(where $[\cdot]$ denotes greatest integral function less than or equal to x.)

17. Evaluate the

$$\lim_{n\to\infty}\frac{[1^2x]+[2^2x]+[3^2x]+\ldots\ldots\ldots+[n^2x]}{n^3}.$$

(where $[\cdot]$ denotes greatest integral function less than or equal to x.)

18. Evaluate the $\lim\limits_{x\to 0^+}\dfrac{\log_{\sin x}\cos x}{\log_{\sin\frac{x}{2}}\cos(x/2)}$

19. If $g(x)=\lim\limits_{m\to\infty}\dfrac{x^m f(x)+h(x)+1}{2x^m+3x+3}$ is continuous at $x=1$ and $g(1)=\lim\limits_{x\to 1}[\log_e(ex)]^{2/\log_e x}$, then find the value of $[2g(1)+2f(1)-h(1)]$.

20. Show that $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\dfrac{{}^nC_k}{n^k\,(k+3)}=e-2$.

21. Evaluate the $\lim\limits_{x\to 0}\left(\dfrac{(1+\{x\})^{1/\{x\}}}{e}\right)^{1/\{x\}}$, if it exists.

(where $\{x\}$ represents the fractional part of x.)

22. If $y_n = \ln\cos\frac{x}{2} + \ln\cos\frac{x}{2^2} + \ldots\ldots + \ln\cos\frac{x}{2^n}$

Then, prove that, $\lim_{n\to\infty}\left(\frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + \ldots\ldots. + \frac{1}{2^n}\tan\frac{x}{2^n}\right) = \frac{1}{x} - \cot x.$

23. Find the polynomial function $f(x)$ for which $\lim_{x\to 0}\left[1 + x + \frac{f(x)}{x}\right]^{1/x} = e^3.$

24. Evaluate $\lim_{n\to\infty}\prod_{r=3}^{n}\frac{r^3-8}{r^3+8}$. (where Π represents product of function.)

25. Evaluate

$$\lim_{x\to\infty}\left(\cfrac{x}{x + \cfrac{\sqrt[3]{x}}{x + \cfrac{\sqrt[3]{x}}{x + \cfrac{\sqrt[3]{x}}{\ldots\ldots\ldots\ldots\ldots\text{Infinity}}}}}\right).$$

26. Let $f(x)$ be a function such that $\lim_{x\to 0}\frac{f(x)}{x} = 1$. Find the values of a and b such that

$$\lim_{x\to 0}\frac{x(1 + a\cos x) - b\sin x}{\{f(x)\}^3} = 1$$

27. Evaluate the $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{[2rx]}{n^2}$. (where $[\cdot]$ denotes greatest integral function less than or equal to x.)

28. Evaluate $\lim_{n\to\infty}\sum_{r=1}^{n}\cot^{-1}(r^2 + 3/4)$.

29. Evaluate

$\lim_{n\to\infty}\{\log_{n-1}(n)\cdot\log_n(n+1)\ldots\ldots\ldots\log_{n^k-1}(n^k)\}$, where $k\in N$.

30. T_1 is an isosceles triangle with circumcircle k. Let T_2 be another an isosceles triangle inscribed in k whose base is one of the equal sides of T_1 and which overlaps the interior of T_1.
Similarly, create isosceles triangles T_3 from T_2, T_4 from T_3 and so on. Do the triangles T_n approaches an equilateral triangle as $n\to\infty$.

Answers

Target Exercise 5.1

1. (a) **2.** (c) **3.** (a) **4.** (c) **5.** (a)

Target Exercise 5.2

1. (c) **2.** (c) **3.** (c) **4.** (a) **5.** (c)

Target Exercise 5.3

1. (d) **2.** (a) **3.** (a) **4.** (d) **5.** (a)

Target Exercise 5.4

1. (c) **2.** (b) **3.** (a) **4.** (b) **5.** (b) **6.** (a) **7.** (a) **8.** (d) **9.** (b) **10.** (b)

Target Exercise 5.5

1. (a) **2.** (d) **3.** (c) **4.** (a) **5.** (a)

Target Exercise 5.6

1. (c) **2.** (c) **3.** (a) **4.** (b) **5.** (b)

Exercise 1

1. (a) **2.** (c) **3.** (a) **4.** (c) **5.** (c) **6.** (a) **7.** (d) **8.** (c) **9.** (c) **10.** (b)
11. (d) **12.** (c) **13.** (b) **14.** (a) **15.** (c) **16.** (c) **17.** (d) **18.** (d) **19.** (b), (d)
20. (b), (d) **21.** (b), (d) **22.** (a), (b) **23.** (b), (c) **24.** (d) **25.** (d)
26. (d) **27.** (d) **28.** (d) **29.** (a) **30.** (b) **31.** (c) **32.** (b) **33.** (c) **34.** (d)
35. (A) $\to$ (r); (B) $\to$ (p); (C) $\to$ (q); (D) $\to$ (q) **36.** (A) $\to$ (q); (B) $\to$ (p); (C) $\to$ (s)

Exercise 2

1. 4 **2.** 6 **3.** 2 **4.** $m = -5/2$, $n = -3/2$ **5.** (i) 0, (ii) 1, (iii) 1 **6.** 1
7. 20 **8.** 289 **9.** $\frac{1}{3}$ **11.** 1 **12.** 1 **13.** 1 **14.** n **15.** 1/8
16. limit doesn't exist **17.** $x/3$ **18.** 4 **19.** 1
21. limit doesn't exist
23. $f(x) = 2x^2 + a_{n-3}x^3 + \ldots + a_0 x^n$; where $a_{n-3}, a_{n-4}, \ldots, a_0 \in R$
24. $\frac{2}{7}$ **25.** 1 **26.** $a = -5/2$, $b = -3/2$ **27.** x **28.** $\tan^{-1} 2$ **29.** k
30. Yes, as $n \to \infty$ T_n approaches to equilateral triangle.

Solutions

(*Proficiency in 'Limits' Exercise* 1)

Type 1 : **Only One Correct Option**

1. $\lim_{x \to 0} \dfrac{\sin \pi (1 - \sin^2 \tan (\sin x))}{x^2}$

$$= \lim_{x \to 0} \frac{\sin \pi \sin^2 \tan (\sin x)}{\pi \sin^2 \tan (\sin x)} \left(\frac{\pi \sin^2 \tan (\sin x)}{\tan^2 (\sin x)} \right) \left(\frac{\tan^2 (\sin x)}{\sin^2 x} \right) \left(\frac{\sin^2 x}{x^2} \right) = \pi.$$

2. $\lim_{t \to 0} \dfrac{1 - (1+t)^t}{\ln(1+t) - t} = \lim_{t \to 0} \dfrac{\dfrac{1 - (1+t)^t}{t^2}}{\dfrac{\ln (1+t) - t}{t^2}} = \lim_{t \to 0} \dfrac{1 - e^{t (\ln (1+t))}}{\dfrac{t^2}{-1/2}} = 2$

3. $I_1 = \lim_{x \to \infty} (\tan^{-1} \pi x - \tan^{-1} x)$

$$= \lim_{x \to \infty} \tan^{-1} \left(\frac{(\pi - 1) x}{1 + \pi x^2} \right) = 0 \quad \Rightarrow \quad I_1 = 0 \text{ and } I_2 = 0$$

4. From given $f(x) = x^2 - \pi^2$,

$$\lim_{x \to -\pi} \frac{x^2 - \pi^2}{\sin (\sin x)} = \lim_{x \to -\pi} \frac{(-\pi + h)^2 - \pi^2}{\sin \sin (-\pi + h)} = \lim_{h \to 0} \frac{-2h\pi + h^2}{-\sin (\sin h)}$$

$$= \lim_{h \to 0} \frac{h - 2\pi}{\dfrac{-\sin (\sin h)}{\sin h} \times \dfrac{\sin h}{h}} = 2\pi$$

5. Using approximations $L = \lim_{x \to 0} \dfrac{\dfrac{\sin^2 x}{3} + \dfrac{2 \tan x}{4}}{\sin x + \tan^2 x} = \lim_{x \to 0} \dfrac{\dfrac{x^2}{3} + \dfrac{2x}{4}}{x + x^2}$

$$= \lim_{x \to 0} \frac{x \left(\dfrac{x}{3} + \dfrac{1}{2} \right)}{x(1+x)} = \frac{1}{2}$$

6. Limit be equal to y, $\log y = \dfrac{1}{n} \lim_{n \to \infty} \left[\log \left(\dfrac{n+1}{n} \right)^\lambda + \log \left(\dfrac{n+2}{n} \right)^\lambda + \dots \right]$

$$= \frac{1}{n} \lim_{n \to \infty} \Sigma \left[\log \left(\frac{n+r}{n} \right)^\lambda \right]$$

$$= \frac{1}{n} \lim_{n \to \infty} \sum_{r=1}^{n} \lambda \log \left(1 + \frac{r}{n} \right) = \lambda \int_0^1 \log (1+x)\, dx$$

$$= \lambda \cdot 2 \left[\log 2 - \frac{1}{2} \right] = \lambda [\log 4 - \log e] = \lambda \log \left(\frac{4}{e} \right)$$

$\Rightarrow$ $$\log y = \log \left(\frac{4}{e} \right)^\lambda$$

7. Let $x_0 = \cos\theta,\ x_1 = \cos\dfrac{\theta}{2},\ x_2 = \cos\dfrac{\theta}{2^2}$

$$= \lim_{n\to\infty}\left(\frac{\sin\theta}{\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\ldots.\cos\frac{\theta}{2^{n+1}}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2^n\cdot\sin\frac{\theta}{2^n}}{\cos\frac{\theta}{2^{n+1}}}\right) = \lim_{n\to\infty}\left(\frac{\sin\frac{\theta}{2^n}}{\frac{\theta}{2^n}}\right)\left(\frac{\theta}{\cos\frac{\theta}{2^{n+1}}}\right)$$

$\Rightarrow$ $$\theta = \cos^{-1}(x_0)$$

8. $f_n(x) = \tan\dfrac{x}{2}\left(\dfrac{1+\cos x}{\cos x}\right)(1+\sec 2x)(1+\sec 4x)\ldots(1+\text{sex } 2^n x)$

$$= \tan x\left(\frac{1+\cos 2x}{\cos 2x}\right)(1+\sec 4x)\ldots(1+\sec 2^n x)$$

$$= \tan 2x\,(1+\sec 2^2 x)\ldots(1+\sec 2^n x)$$

$$= \tan 2^{n-1}x(1+\sec 2^n x) = \tan 2^n x$$

$$= \tan 2^{n-1}x(1+\sec 2^n x) = \tan 2^n x$$

Now, $$\lim_{x\to 0}\frac{f_n(x)}{2x} = \lim_{x\to 0}\frac{\tan 2^n x}{2^n x}\cdot 2^{n-1} = 2^{n-1}$$

9. $\lim\limits_{x\to k} f(x) = 3 + h \Rightarrow f(k) = 3$

$$f(k^-) > f(k) \text{ and } f(k^+) > f(k)$$

$\Rightarrow$ $$a^2 - 2 + 1 > 3$$

$\Rightarrow$ $$|a| > 2$$

10. For finite value

$$-1 < \frac{x^2+a}{x^2+1} < 1$$

$\Rightarrow$ $$-1 < a < 1$$

11. $\lim\limits_{x\to 0^-}\dfrac{\sqrt{\tan^2\frac{x}{2}}}{x} = \lim\limits_{x\to 0^-}\dfrac{|\tan x/2|}{x}$

$\therefore$ $$\lim_{x\to 0^-}\frac{\left|\tan\frac{x}{2}\right|}{x} = \lim_{h\to 0}\frac{\tan\frac{h}{2}}{-\left(\frac{h}{2}\right)2} = -\frac{1}{2}$$

$\therefore$ $$\lim_{x\to 0^+}\frac{\left|\tan\frac{x}{2}\right|}{x} = \lim_{h\to 0}\frac{\tan\frac{h}{2}}{\left(\frac{h}{2}\right)2} = \frac{1}{2}$$

$\therefore$ It doesn't exist.

12. As $f'(x) > 0 \Rightarrow f(x)$ is increasing.

So, for $\quad t > 1; f(t) > 1$

Now, $\quad f'(t) = \dfrac{1}{t^2 + f(t)} < \dfrac{1}{1+t^2}$

$\therefore \quad f(x) = 1 + \int_1^x f'(t)\,dt < 1 + \int_1^x \dfrac{dt}{1+t^2}$

$\Rightarrow \quad \lim_{x \to \infty} f(x) < 1 + \int_1^\infty \dfrac{dt}{1+t^2} \quad \Rightarrow \lim_{x \to \infty} f(x) < 1 + \dfrac{\pi}{4}$

13. $E = \sin^2\theta - \sin\theta + \dfrac{1}{2} = \left(\sin\theta - \dfrac{1}{2}\right)^2 + \dfrac{1}{4}$

$\Rightarrow$ Minimum value is $\dfrac{1}{4}$.

Let $\quad K = \sqrt{(x+1)(x+2)}$, then

$$K - x = \frac{K^2 - x^2}{K + x} = \frac{3x+2}{K+x}$$

$$\lim_{x \to \infty} (K - x) = \lim_{x \to \infty} \frac{3 + \frac{2}{x}}{\frac{K}{x} + 1}$$

$$\lim_{x \to \infty} \frac{K}{x} = 1$$

$\therefore \quad \lim_{x \to \infty} (K - x) = \dfrac{3+0}{1+1} = \dfrac{3}{2}$

$\therefore$ The required equation is $x^2 - \dfrac{7}{4}x + \dfrac{3}{8} = 0$

ie, $\quad 8x^2 - 14x + 3 = 0$

14. $\lim_{x \to \frac{3^-}{2}} g(x) = \dfrac{\sin\frac{\pi}{4}}{1} = \dfrac{1}{\sqrt{2}}$

$$\lim_{x \to \frac{3^+}{2}} g(x) = \frac{\sin\frac{\pi}{4}}{1} = \frac{1}{\sqrt{2}}$$

$\therefore g(x)$ is continuous at $x = \dfrac{3}{2}$

$$\lim_{x \to 2^-} g(x) = \frac{1}{\sqrt{2}}$$

$$\lim_{x \to 2^+} g(x) = \frac{\sin\frac{\pi}{2}}{2} = \frac{1}{2}$$

$g(x)$ is not continuous at $x = 2$

$$\lim_{x \to 1^+} g(x) = \frac{\sin\frac{\pi}{4}}{1} = \frac{1}{\sqrt{2}} \neq \frac{1}{2}$$

15. $$\lim_{x \to 0} \frac{f\left(2\sin^2 \frac{x}{2}\right)}{g(x)\left(2\sin^2 \frac{x}{2}\right)^2} \times \frac{\left(\sin \frac{x}{2}\right)^2}{\left(\sin^2 \frac{x}{2}\right)\left(\cos^2 \frac{x}{2}\right)}$$

$$= \lim_{x \to 0} \frac{a}{g(x)} \times \tan^2 \frac{x}{2} = b$$

$$\lim_{x \to 0} \frac{x^2}{g(x)} = \frac{4b}{a}$$

Thus, $$\lim_{x \to 0} \frac{g(1-\cos 2x)}{x^4} = \frac{a}{b}$$

16. Let $x_n = \tan \theta_n$

Now, $$\tan \theta_{n+1} = x_{n+1} = \frac{x_n}{1+\sqrt{1+x_n^2}}$$

$$= \frac{\tan \theta}{1+\sqrt{1+\tan^2 \theta_n}}$$

$\Rightarrow$ $$\tan \theta_{n+1} = \frac{\tan \theta_n}{1+\sec \theta_n}$$

$$= \frac{\sin \theta_n}{1+\cos \theta_n} = \tan \frac{\theta_n}{2}$$

$\Rightarrow$ $$\theta_{n+1} = \frac{\theta_n}{2} \quad \Rightarrow \quad \theta_n = \frac{\theta_{n-1}}{2}$$

Now, $$\theta_1 = \frac{\pi}{3} \quad \Rightarrow \quad \theta_n = \frac{\pi}{3 \cdot 2^{n-1}}$$

$\Rightarrow$ $$x_n = \tan\left(\frac{2\pi}{3 \cdot 2^n}\right)$$

$\Rightarrow$ $$\lim_{n \to \infty} 2^x x_n = \lim_{n \to \infty} \frac{\tan\left(\frac{2\pi}{3 \cdot 2^n}\right)}{\left(\frac{1}{2^n}\right)} = \frac{2\pi}{3}$$

17. $$L = \lim_{x \to a^-} = \frac{(x-b)-(a-b)}{\sqrt{x-b}+\sqrt{a-b}} \times \frac{1}{(x^2-a^2)}$$

$$= \lim_{x \to a^-} \frac{1}{(x+a)\{\sqrt{x-b}+\sqrt{a-b}\}}$$

$$= \frac{1}{4a\sqrt{a-b}}$$

18. $$\lim_{n \to \infty} (\sin^n 1 + \cos^n 1)^{1/n} = \sin 1 \lim_{n \to \infty} (1+\cot^n 1)^{1/n}$$

$$= \sin 1 \cdot e^{\lim_{n \to \infty} \frac{n}{\tan^n 1}} = \sin 1 \cdot e^{\lim_{n \to \infty} \frac{1}{\tan^n 1 \cdot \ln(\tan 1)}} = \sin 1$$

Type 2 : **More than One Correct Options**

19. $\lim_{x \to \infty} 4x \dfrac{\tan^{-1}\left(\dfrac{1}{2x+3}\right)}{\left(\dfrac{1}{2x+3}\right)} \times \dfrac{1}{2x+3} = 2$

$\Rightarrow \qquad y^2 + 4y + 5 = 2 \Rightarrow y = -1, -3.$

20. $\lim_{x \to 0} \dfrac{2\sin^2\left(\dfrac{x^2}{2}\right)}{2} \cdot \dfrac{\left(\dfrac{x^2}{2}\right)^2}{(4^x - 1)} \cdot \dfrac{x}{x^4} = \dfrac{1}{2} \ln 4 = \ln 2$

Also, $\qquad = 1 - \dfrac{1}{2} \ln\left(\dfrac{e^2}{4}\right)$

21. $f(x) = e^{[\cot x]}$, as $\cot x$ is negative in the II quadrant and $\cot \dfrac{\pi}{2} = 0$

$[\cot x] = -1$

$\therefore \qquad \lim_{x \to \frac{\pi}{2}^+} f(x) = e^{-1} = \dfrac{1}{e}$

As $x \to \dfrac{\pi}{2}^-$, $\cot x$ is positive (being in I quadrant) and hence $[\cot x] = 0$

$\therefore \qquad \lim_{x \to \frac{\pi}{2}^-} f(x) = e^0 = 1$

22. If $m < 0$, then for values of x sufficiently close to 0.

$$1 + \frac{1}{m} < \frac{\sin x}{x} < 1$$

$\therefore \qquad m + 1 > m\dfrac{\sin x}{x} > m \qquad \therefore \qquad \left[m\dfrac{\sin x}{x}\right] = m$

$\therefore \qquad \lim_{x \to 0} \left[m\dfrac{\sin x}{x}\right] = m$

If $m > 0$, then for values of x sufficiently clsoe to 0,we can have

$$1 - \frac{1}{m} < \frac{\sin x}{x} < 1$$

$\therefore \qquad m - 1 < m\dfrac{\sin x}{x} < m$

$\therefore \qquad \lim_{x \to 0}\left[m\dfrac{\sin x}{x}\right] = m - 1$

23. $\lim_{x \to 0} (1 + ax + bx^2)^{2-x} = e^3$

$\Rightarrow \qquad e^{\lim_{x \to \infty} \frac{2}{x}(ax + bx^2)} = e^3$

Since, limit value e^{2a} does not involve b.

$\therefore$ b can have any value.

Thus, $\qquad a = \dfrac{3}{2}, b \in R$

Type 3 : **Assertion and Reason**

24. $\lim_{x \to 0} \frac{x}{a}\left(\frac{b}{x} - \left\{\frac{b}{x}\right\}\right) = \lim_{x \to 0}\left(\frac{b}{a} - \frac{x}{a}\left\{\frac{b}{x}\right\}\right) = \frac{b}{a}$

25. Because maximum and minima is also dependent of $f(a)$.

26. $\lim_{x \to 0} \frac{\sin\left(\pi \sin^2 \frac{x}{2}\right)}{\pi \sin^2 \frac{x}{2}} \times \frac{\pi\left(\sin^2 \frac{x}{2}\right)}{\left(\frac{x}{2}\right)^2 \times 4} = \frac{\pi}{4}$ and not π

Statement II is evidently true.

27. Since, $x > \sin x$ in $\left(0, \frac{\pi}{2}\right)$, $\frac{\sin x}{x} < 1$

$\therefore$ $\sec^{-1}\left(\frac{\sin x}{x}\right)$

is not defined and $\lim_{x \to 0} \sec^{-1}\left(\frac{\sin x}{x}\right)$ does not exist.

28. $\lim_{n \to \infty} 2.\bar{9} = 3$, as there is no real number between $2.\bar{9}$ and 3 and hence $2.\bar{9} = 3$

Hence, $\left[\lim_{n \to \infty} a_n\right] = [3] = 3$, while $[a_n] = 2$, for all $n \in N$ and hence

$$\lim_{n \to \infty} [a_n] = 2 \neq \left[\lim_{n \to \infty} a_n\right]$$

Type 4 : **Linked Comprehension Based Questions**

Solutions (Q. Nos. 29 to 31)

29. $\lim_{x \to 0} \frac{\sin x}{x - \sin x} = \lim_{x \to 0} \frac{\frac{\sin x}{x}}{1 - \frac{\sin x}{x}} = \infty$

Hence, $$L = e^{\lim_{x \to 0}\left(\frac{\sin x}{x} - 1\right)\frac{\sin x}{x - \sin x}} = e^{\lim_{x \to 0} \frac{-\sin x}{x}} = e^{-1} = \frac{1}{e}$$

30. $\lim_{x \to 0} \frac{x - 1 + \cos x}{x} = \lim_{x \to 0} 1 - \frac{2\sin^2 \frac{x}{2}}{x} = 1$

Hence, $$L = e^{\lim_{x \to 0}\left(\frac{x - 1 + \cos x}{x} - 1\right) \times \frac{1}{x}} = e^{\lim_{x \to 0}\left(\frac{-2\sin^2 \frac{x}{2}}{x^2}\right)} = e^{-\frac{2}{4}} = e^{-1/2}$$

31. $L = e^{\lim_{x \to 0}\left(\frac{a^x + b^x + c^x}{3} - 1\right)\frac{2}{x}}$

$$= e^{\lim_{x \to 0}\left(\frac{a^x - 1}{x} + \frac{b^x - 1}{x} + \frac{c^x - 1}{x}\right)\frac{2}{3}}$$

$$= e^{\frac{2}{3}(\log a + \log b + \log c)} = e^{\frac{2}{3}\log abc} = (abc)^{2/3}$$

Solutions (Q. Nos. 32 to 34)

$$f(x)=\lim_{n\to\infty}\left(\cos\sqrt{\frac{x}{n}}\right)^n=\lim_{n\to\infty}\left(1+\left(\cos\sqrt{\frac{x}{n}}-1\right)\right)^n$$

$$=e^{\lim\limits_{n\to\infty}\left(\cos\sqrt{\frac{x}{n}}-1\right)n}=e^{-\lim\limits_{n\to\infty}2\frac{\sin^2\left(\frac{1}{2}\sqrt{\frac{x}{n}}\right)\cdot n}{\left(\frac{1}{2}\sqrt{\frac{x}{n}}\right)^2}\times\left(\frac{1}{2}\sqrt{\frac{x}{n}}\right)^2}$$

$$=e^{-2\lim\limits_{n\to\infty}\frac{\left(\frac{1}{2}\sqrt{\frac{x}{n}}\right)^2}{\frac{1}{n}}}=e^{-2\lim\limits_{n\to\infty}\frac{1}{4}\frac{x/n}{1/n}}=e^{-\frac{x}{2}}$$

$$y=f(x)=e^{-x/2},\ x\ge 0,\ \text{range}=(0,1]$$

$$g(x)=\lim_{n\to\infty}(1-x+x\sqrt[n]{e})^n$$

$$=e^{\lim\limits_{n\to\infty}x\frac{(e^{1/n}-1)}{1/n}}=e^x,\ \forall\ x\in R$$

$$h(x)=\tan^{-1}(g^{-1}(f^{-1}(x)),$$

$$-\frac{x}{2}=\ln y\ x=2\ln\frac{1}{y}\ f^{-1}(x)=2\ln\frac{1}{x}\quad\text{for } 0<x\le 1$$

$$y=g(x)=e^x$$

$$x=\ln y\ g^{-1}(x)=\ln x$$

$\therefore$ $$g^{-1}\left(2\ln\frac{1}{x}\right)=\ln\left(2\ln\left(\frac{1}{x}\right)\right)\qquad\text{for } 0<x<1$$

$\therefore$ $$h(x)=\tan^{-1}\left(\ln\left(\ln\frac{1}{x^2}\right)\right)\qquad\text{for } 0<x<1$$

32. $\lim\limits_{x\to 0^+}\dfrac{\ln f(x)}{\ln g(x)}=\lim\limits_{x\to 0}\dfrac{-x/2}{x}=-\dfrac{1}{2}$

33. Domain of $h(x)$ is $(0, 1)$.

34. $h(x)=\tan^{-1}(\ln(\ln 1/x^2))$ for $0<x<1$

$$1<\frac{1}{x^2}<\infty\quad\Rightarrow\quad 0<\ln\frac{1}{x^2}<\infty$$

$\therefore$ $$-\infty<\ln(\ln(1/x^2))<\infty$$

$\therefore$ Range of $h(x)$ is $(-\pi/2, \pi/2)$.

Type 5 : **Match the Columns**

35. (A) $\lim\limits_{x\to\frac{\pi^+}{2}}\tan^{-1}(\tan x)=-\dfrac{\pi}{2}$, as $\lim\limits_{x\to\frac{\pi^+}{2}}(\tan x)=-\infty$

(B) $\sum\limits_{r=1}^{n}\dfrac{1}{2^r}=\left(1-\dfrac{1}{2^n}\right)<1$, for all $n\in N$.

Thus, $$\left[\sum_{r=1}^{n}\frac{1}{2^r}\right]=0$$

(C) As $x \to \infty$, $\dfrac{x}{x+1} \to 1^-$

and hence $\lim\limits_{x \to \infty} \sec^{-1}\left(\dfrac{x}{x+1}\right)$ doesn't exist.

(D) Put $\dfrac{\pi}{2} - x = \theta$, then the given limit is

$$\lim_{\theta \to 0} \frac{\sin\theta}{(1-\cos\theta)^{2/3}} = \lim_{\theta \to 0} \frac{\sin\theta}{2^{2/3}\cdot\sin^{4/3}\dfrac{\theta}{2}}$$

$$= \lim_{\theta \to 0} \frac{2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}{2^{2/3}\cdot\sin^{4/3}\dfrac{\theta}{2}} = 2^{1/2}\lim_{\theta \to 0}\frac{\cos\dfrac{\theta}{2}}{\sin^{1/3}\dfrac{\theta}{2}},$$

which doesn't exist as for $\lim\limits_{\theta \to 0^+}$ limit is ∞ and for $\lim\limits_{\theta \to 0^+}$ limit is $-\infty$.

36. (A) It can be reduced to

$$\int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx = \int_0^1 \frac{1+x}{\sqrt{1-x^2}}\,dx = [\sin^{-1}x - \sqrt{1-x^2}]_0^1 = \frac{\pi}{2} - (-1) = \frac{\pi}{2} + 1$$

(B) $\lim\limits_{n \to \infty} \dfrac{1}{n}\left[\dfrac{1}{\sqrt{1-\dfrac{1}{n^2}}} + \dfrac{1}{\sqrt{1-\left(\dfrac{2}{n}\right)^2}} + \ldots + \dfrac{1}{\sqrt{1-\left(\dfrac{n-1}{n}\right)^2}}\right]$

$$= \lim_{n \to \infty} \frac{1}{n}\sum_{r=1}^{n-1}\frac{1}{\sqrt{1-\left(\dfrac{r}{n}\right)^2}}$$

Replace $\dfrac{r}{n}$ by x and $\dfrac{1}{n}\,dx$ also when $r = 1$, $x = \dfrac{1}{n} = 0$ when $r = n-1$

$\Rightarrow$ $$\int \frac{1}{\sqrt{1-x^2}}\,dx = (\sin^{-1}x)_0^1 = \sin^{-1}1 - \sin^{-1}0 = \frac{\pi}{2}$$

(C) $A = \lim\limits_{n \to \infty}\left[\dfrac{1}{n}\cdot\dfrac{2}{n}\cdot\dfrac{3}{n}\cdot\dfrac{n}{n}\right]^{1/n}$

$$\log A = \lim_{n \to \infty}\left[\log\frac{1}{n} + \log\frac{1}{n} + \log\frac{3}{n} + \ldots + \log 1\right]$$

$$= \lim_{n \to \infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(\frac{r}{n}\right)$$

Put $\dfrac{r}{n} = x\ \dfrac{1}{n} = dx$ and limits 0 to 1.

$$\log A = \int_0^1 \log x\,dx = (x\log x)_0^1 - \int_0^1 x\,\frac{1}{x}$$

$$= (x\log x - x)_0^1 = -1 \Rightarrow A = e^{-1}$$

(*Proficiency in 'Limits' Exercise* 2)

1. $\lim\limits_{n \to \infty} 0.2^{\log_{\sqrt{5}}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots \text{ to } n \text{ terms}\right)}$

$$\Rightarrow \quad \lim_{n \to \infty} 0.2^{\log_{\sqrt{5}}\left(\frac{\frac{1}{4}\left(1-\frac{1}{2^n}\right)}{\frac{1}{2}}\right)}$$

$$\Rightarrow \quad 0.2^{\log_{\sqrt{5}}\left(\frac{1}{2}\right)} = 4 \qquad \left[\text{as } n \to \infty; \left(\frac{1}{2}\right)^n \to 0\right]$$

2. Given, $f''(0) = 2$ ...(i)

$$\therefore \quad \lim_{x \to 0} \frac{2f(x) - 3f(2x) + f(4x)}{x^2} \qquad \text{(using L'Hospital's rule)}$$

$$\Rightarrow \quad \lim_{x \to 0} \frac{2f'(x) - 6f'(2x) + 4f'(4x)}{2x} \qquad \left[\text{as } \frac{d}{dx} f(x) = f'(x),\ \frac{d}{dx} f(2x) = f'(2x)\cdot 2\right]$$

Again, applying L'Hospital's rule, we get

$$\Rightarrow \quad \lim_{x \to 0} \frac{2f''(x) - 12f''(2x) + 16f''(4x)}{2}$$

$$\Rightarrow \quad f''(0) - 6f''(0) + 8f''(0)$$

$$\Rightarrow \quad 3f''(0) = 6$$

3. Given $f(x)$ has unique tangent at $(a, 0)$

$$\therefore \quad f(a) = 0 \text{ and } f'(x) \text{ exists}$$

$$\Rightarrow \quad \lim_{x \to a} \frac{\log_e\{1 + 6f(x)\}}{3f(x)} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$\Rightarrow \quad \lim_{x \to a} \frac{\left(\frac{1}{1+6f(x)}\right)\cdot 6f'(x)}{3f'(x)} \qquad \text{(using L'Hospital's rule)}$$

$$\Rightarrow \quad \lim_{x \to a} \frac{2}{\{1+6f(x)\}} \Rightarrow \frac{2}{1+6f(a)} \Rightarrow \frac{2}{1+0} = 2 \qquad [\because f(a) = 0]$$

4. $\lim\limits_{x \to 0} \dfrac{x(1 + m\cos x) - n\sin x}{x^3} = 1$ ...(i)

Since, $\lim\limits_{x \to 0} \dfrac{x(1 + m\cos x) - n\sin x}{x^3}$ $\left(\frac{0}{0} \text{ form}\right)$

Applying expansions of $(\sin x)$ and $(\cos x)$, we get

$$\lim_{x \to 0} \frac{x\left\{1 + m\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots\right)\right\} - n\left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\right\}}{x^3}$$

$$\Rightarrow \quad \lim_{x \to 0} \frac{x(1 + m - n) - x^3\left(\frac{1}{2}m - \frac{n}{6}\right) + x^5 \text{ (and higher powers)}}{x^3} \qquad \text{...(ii)}$$

Now, Eq. (ii) to have finite value numerator and denominator must have same power *ie*, coefficient of x must be zero.

ie, $1 + m - n = 0$

$\therefore$ Eq. (ii) reduces to,

$$\lim_{x \to 0} \frac{-x^3\left(\frac{1}{2}m - \frac{n}{6}\right) + x^5 \text{(and higher powers)}}{x^3}$$

$\Rightarrow$ $\frac{n}{6} - \frac{m}{2} = 1$ [using Eq. (i)]

$\therefore$ $1 + m - n = 0$

and $n - 3m = 6$ on solving $m = -\frac{5}{2}$ and $n = -\frac{3}{2}$

5. (i) We know, $\sin x < x$ for $x > 0$ $\therefore$ $\frac{\sin x}{x} < 1$

and $\sin x > x$ for $x < 0$ $\therefore$ $\frac{\sin x}{x} < 1$

$\Rightarrow$ $\left(\frac{\sin x}{x} < 1 \text{ as } x \to 0\right)$, $\therefore$ $\left[\frac{\sin x}{x}\right] = 0$

or $\lim_{x \to 0} \left[\frac{\sin x}{x}\right] = 0$

(ii) We know, $\sin^{-1} x > x$ for $x > 0$ $\Rightarrow$ $\frac{\sin^{-1} x}{x} > 1$

$\therefore$ $\lim_{x \to 0} \frac{\sin^{-1} x}{x} \to 1^+$

and for $x < 0$, $\sin^{-1} x < x$ $\Rightarrow$ $\frac{\sin^{-1} x}{x} > 1$

$\therefore$ $\lim_{x \to 0} \frac{\sin^{-1} x}{x} \to 1^+$

$\therefore$ $\lim_{x \to 0} \left[\frac{\sin^{-1} x}{x}\right] = 1$

(iii) We know, when $x > 0$, $\tan x > x$ and when $x < 0$, $\tan x < x$

$\therefore$ $\frac{\tan x}{x} > 1$ as $x \to 0$

$\Rightarrow$ $\lim_{x \to 0} \left[\frac{\tan x}{x}\right] = 1$

14. We have, $\lim_{x \to 0} \{1^{1/\sin^2 x} + 2^{1/\sin^2 x} + \dots + n^{1/\sin^2 x}\}^{\sin^2 x}$

Put $\frac{1}{\sin^2 x} = t \geq 1$. Such that as $x \to 0$; $t \to \infty$

Then $\lim_{t \to \infty} (1^t + 2^t + \dots + n^t)^{1/t}$ ...(i)

We know, $\frac{1^t + 2^t + \dots + n^t}{n} > \left(\frac{1 + 2 + \dots + n}{n}\right)^t$

$$\Rightarrow \qquad (1^t + 2^t + \ldots. + n^t) > n\left(\frac{n+1}{2}\right)^t$$

$$\therefore \qquad \lim_{t \to \infty} (1^t + 2^t + \ldots. + n^t)^{1/t} > \lim_{t \to \infty} n^{1/t} \left(\frac{n+1}{2}\right)$$

$$\Rightarrow \qquad \lim_{t \to \infty} (1^t + 2^t + \ldots. + n^t)^{1/t} > \left(\frac{n+1}{2}\right)$$

Aliter : $\displaystyle \lim_{x \to 0} n \left\{1 + \left(\frac{1}{n}\right)^{1/\sin^2 x} + \left(\frac{2}{n}\right)^{1/\sin^2 x} + \ldots + \left(\frac{n-1}{n}\right)^{1/\sin^2 x}\right\}^{\sin^2 x}$

$\Rightarrow \quad n\{1 + 0 + 0 + \ldots + 0\}^0$

$\Rightarrow \quad n$

20. $\displaystyle \lim_{n \to \infty} \sum_{k=0}^{n} \frac{{}^nC_k}{n^k(k+3)} = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{{}^nC_k}{n^k} \cdot \int_0^1 x^{k+2}\, dx$

$$= \int_0^1 \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{n^k} \cdot {}^nC_k\, x^{k+2}\, dx$$

$$= \int_0^1 x^2 \left\{\lim_{n \to \infty} \sum_{k=0}^{n} {}^nC_k \left(\frac{x}{n}\right)^k\right\} dx$$

$$= \int_0^1 x^2 \left\{\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n\right\} dx$$

$$= \int_0^1 x^2 \cdot e^x\, dx \qquad \left[\text{using } \lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a\right]$$

$$= (x^2 e^x)_0^1 - \int_0^1 2x \cdot e^x\, dx$$

$$= 1 \cdot e^1 - 2(x \cdot e^x)_0^1 + 2\int_0^1 e^x\, dx = e - 2(e^1) + 2(e^1 - e^0)$$

$$= e - 2e + 2e - 2 = e - 2$$

21. Let $L = \left(\dfrac{(1 + \{x\})^{1/\{x\}}}{e}\right)^{1/\{x\}}$

$$\log L = \frac{1}{\{x\}}\left[\frac{1}{\{x\}} \log(1 + \{x\}) - 1\right] \qquad \ldots\text{(i)}$$

Now, the test for existence of limit in the expression (i).

And hence from Eq. (i)

RHL $\displaystyle \lim_{x \to 0^+} \log L = \lim_{x \to 0^+} \frac{1}{x}\left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots\ldots\right) = -\frac{1}{2}$

$\therefore$ RHL $\displaystyle = \lim_{x \to 0^+} L = \frac{1}{\sqrt{e}}$

$\therefore$ LHL $x \to 0^- \quad \Rightarrow \quad \{x\} = 1 + x$ and hence $\{x\} \to 1^-$

$$\lim_{x \to 0^-} \log_e L = \log 2 - 1 \quad \Rightarrow \quad \lim_{x \to 0^-} L = \frac{2}{e}$$

$\therefore$ Limit doesn't exists, as RHL $\neq$ LHL

22. Let $y_n = \log \cos \frac{x}{2} + \log \cos \frac{x}{2^2} + \ldots. + \log \cos \frac{x}{2^n}$

$$= \log \left(\cos \frac{x}{2} \cdot \cos \frac{x}{2^2} \ \ldots\ldots\ldots \cos \frac{x}{2^n} \right) = \log \left(\frac{\sin x}{2^n \sin \frac{x}{2^n}} \right)$$

Differentiating w.r.t. x, we get

$$\frac{dy_n}{dx} = -\frac{1}{2} \tan \frac{x}{2} - \frac{1}{2^2} \tan \frac{x}{2^2} + \ldots. + \frac{1}{2^n} \tan \frac{x}{2^n} = \frac{d}{dx} \left(\ln \frac{\sin x}{2^n \sin \frac{x}{2^n}} \right) \qquad \ldots\text{(i)}$$

$\therefore$ RHS of Eq. (i)

$$= \frac{2^n \sin \left(\frac{x}{2^n} \right)}{\sin (x)} \cdot \frac{1}{2^n} \left(\frac{\sin \left(\frac{x}{2^n} \right) \cos x - \sin x \cdot \cos \left(\frac{x}{2^n} \right) \cdot \frac{1}{2^n}}{\sin^2 \left(\frac{x}{2^n} \right)} \right)$$

$$= \cot x - \frac{1}{2^n} \cdot \cos \left(\frac{x}{2^n} \right) \cdot \frac{1}{\sin \left(\frac{x}{2^n} \right)}$$

$$= \cot x - \frac{1}{x} \cdot \left(\frac{x}{2^n} \right) \cdot \frac{1}{\sin \left(\frac{x}{2^n} \right)} \cdot \cos \left(\frac{x}{2^n} \right)$$

$$= \cot x - \frac{1}{2^n} \cot \frac{x}{2^n}$$

Taking $\lim\limits_{n \to \infty}$ we have,

$$\lim_{n \to \infty} \left(\frac{1}{2} \tan \frac{x}{2} + \frac{1}{2^2} \tan \frac{x}{2^2} + \ldots.. + \frac{1}{2^n} \tan \frac{x}{2^n} \right)$$

$$= \lim_{n \to \infty} \left\{ \frac{1}{x} \cdot \left(\frac{x}{2^n} \right) \cdot \frac{1}{\sin \left(\frac{x}{2^n} \right)} \cdot \cos \left(\frac{x}{2^n} \right) - \cot x \right\} = \frac{1}{2} - \cot x$$

Aliter : $\tan \frac{x}{2} = \cot \frac{x}{2} - 2 \cot x$

$\therefore$ Series $= \left(\frac{1}{2^n} \cdot \cot \frac{x}{2^n} - \cot x \right)$

23. We have,

$$\lim_{x \to 0} \left(1 + x + \frac{f(x)}{x} \right)^{1/x} = e^3$$

$$\Rightarrow \quad \lim_{x \to 0} \left(1 + \frac{x^2 + f(x)}{x} \right)^{1/x} = e^3, \text{ only if } \lim_{x \to 0} \frac{x^2 + f(x)}{x} = 0 \qquad \ldots\text{(i)}$$

ie, $$e^{\lim\limits_{x \to 0} \frac{x^2 + f(x)}{x} \cdot \frac{1}{x}} = e^3$$

$$\Rightarrow \quad \lim_{x \to 0} \frac{x^2 + f(x)}{x^2} = 3 \qquad \ldots\text{(ii)}$$

As $f(x)$ is polynomial, let $f(x) = a_0x^n + a_1 x^{n-1} + \dots + a_n$

$\therefore$ Eq. (ii) reduces to

$$\lim_{x \to 0} \frac{x^2 + \{a_0x^n + a_1 x^{n-1} + \dots + a_{n-2} x^2 + a_{n-1} x + a_n\}}{x^2} = 3;$$

which shows $a_n = a_{n-1} = 0$ for above limit to exists, thus

$$\lim_{x \to 0} \frac{x^2 (1 + a_{n-2}) + x^3 a_{n-3} + \dots\dots + a_0 x^n}{x^2} = 3$$

$\Rightarrow \quad 1 + a_{n-2} = 3$

$\Rightarrow \quad a_{n-2} = 2$

$\therefore \quad f(x) = 2x^2 + a_{n-3} x^3 + a_{n-4} x^4 + \dots\dots + a_0 x^n;$

where $\quad a_{n-3}, a_{n-4}, \dots\dots, a_0 \in R$

24. $\lim_{n \to \infty} \prod_{r=3}^{n} \left(\frac{r^3 - 8}{r^3 + 8} \right)$

$$= \lim_{n \to \infty} \left(\frac{3^3 - 8}{3^3 + 8} \right) \left(\frac{4^3 - 8}{4^3 + 8} \right) \dots\dots \left(\frac{n^3 - 8}{n^3 + 8} \right)$$

$$= \lim_{n \to \infty} \left(\frac{3-2}{3+2} \cdot \frac{3^2 + 4 + 2(3)}{3^2 + 4 - 2(3)} \right) \cdot \left(\frac{4-2}{4+2} \cdot \frac{4^2 + 4 + 2(4)}{4^2 + 4 - 2(4)} \right) \dots\dots \left(\frac{n-2}{n+2} \cdot \frac{n^2 + 4 + 2n}{n^2 + 4 - 2n} \right)$$

$$= \lim_{n \to \infty} \left\{ \frac{3-2}{3+2} \cdot \frac{4-2}{4+2} \cdot \frac{5-2}{5+2} \dots \frac{n-2}{n+2} \right\} \left\{ \frac{3^2 + 4 + 2(3)}{3^2 + 4 - 2(3)} \cdot \frac{4^2 + 4 + 2(4)}{4^2 + 4 - 2(4)} \cdot \dots \cdot \frac{n^2 + 4 + 2n}{n^2 + 4 - 2n} \right\}$$

$$= \lim_{n \to \infty} \left\{ \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \dots\dots}{5 \cdot 6 \cdot 7 \cdot 8 \dots\dots} \right\} \left\{ \frac{19 \cdot 28 \cdot 39 \cdot 52 \cdot 63 \dots\dots}{7 \cdot 12 \cdot 19 \cdot 28 \cdot 39 \cdot 52 \cdot 63 \dots.} \right\}$$

$$= \lim_{n \to \infty} \frac{(n^2 + 5)(n^2 + 2n + 4)(1 \cdot 2 \cdot 3 \cdot 4)}{(n-1)n(n+1)(n+2)} \left(\frac{1}{7 \cdot 12} \right)$$

$$= \frac{2}{7}$$

25. Let

$$y = \cfrac{x}{x + \cfrac{\sqrt[3]{x}}{x + \cfrac{\sqrt[3]{x}}{x + \sqrt[3]{x} \dots\dots\dots \text{infinity}}}}$$

$$= \cfrac{x}{x + \cfrac{1}{x^{2/3}} \cdot \cfrac{x}{x + \sqrt[3]{x} \dots\dots\dots \text{infinity}}}$$

$$\therefore \quad y = \cfrac{x}{x + \cfrac{1}{x^{2/3}} \cdot (y)}$$

$$\therefore \quad y = \frac{x^{5/3}}{x^{5/3} + y}$$

$$\Rightarrow \quad y^2 + (x^{5/3}) y - x^{5/3} = 0$$

$$\Rightarrow \qquad y = \frac{-(x^{5/3}) \pm \sqrt{x^{10/3} + 4 \cdot x^{5/3}}}{2}$$

$$\Rightarrow \qquad y = \frac{-x^{5/3} + \sqrt{x^{10/3} + 4x^{5/3}}}{2} \qquad (\because\ y > 0)$$

$$\Rightarrow \qquad y = \frac{4x^{5/3}}{2(x^{5/3} + \sqrt{x^{10/3} + 4x^{5/3}})}$$

$$\therefore \qquad \lim_{x \to \infty} y = \lim_{x \to \infty} \frac{2}{\{1 + \sqrt{1 + 4/x^{5/3}}\}}$$

$$= \frac{2}{1+1}$$

$$\lim_{x \to \infty} y = 1$$

$$\Rightarrow \qquad \lim_{x \to \infty} \cfrac{x}{x + \cfrac{3\sqrt{x}}{x + 3\sqrt{x} \ldots\ldots\ldots \text{infinity}}} = 1$$

26. Given, $\lim_{x \to 0} \frac{f(x)}{x} = 1$ ...(i)

$$\therefore \qquad \lim_{x \to 0} \frac{x(1 + a\cos x) - b\sin x}{(f(x))^3} = 1, \text{ applying expansion}$$

$$\Rightarrow \qquad \lim_{x \to 0} \frac{x\left\{1 + a\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\right)\right\} - b\left\{x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots\right\}}{x^3\left(\frac{f(x)}{x}\right)^3} = 1$$

$$\Rightarrow \qquad \lim_{x \to 0} \frac{x(1 + a - b) + x^3\left(-\frac{a}{2!} + \frac{b}{3!}\right) + x^5\left(\frac{9}{4!} - \frac{b}{5!}\right) + \ldots\ldots\ldots}{x^3} = 1 \qquad \text{[using Eq. (i)]}$$

$$\Rightarrow \qquad \lim_{x \to 0} \left\{\left(\frac{1 + a - b}{x^2}\right) + \left(-\frac{a}{2!} + \frac{b}{3!}\right) + x^2\left(\frac{9}{4!} - \frac{b}{5!}\right) + \text{higher powers of } x^2\right\} = 1$$

Now, for limit to exists $(1 + a - b) = 0$; ...(ii)

$$\lim_{x \to 0} \left(-\frac{a}{2!} + \frac{b}{3!}\right) + x^2\left(\frac{9}{4!} - \frac{b}{5!}\right) \ldots\ldots\ldots = 1$$

$$\Rightarrow \qquad \frac{-a}{2} + \frac{b}{6} = 1 \qquad \text{...(iii)}$$

Solving Eqs. (ii) and (iii), we get

$$a = -\frac{5}{2} \quad \text{and} \quad b = -\frac{3}{2}$$

27. $[2rx] = 2rx - \{2rx\}$ ...(i)

So, $$f(x) = \lim_{n \to \infty} \sum_{r=1}^{n} \frac{[2rx]}{n^2} = \lim_{n \to \infty} \sum_{r=1}^{n} \left(\frac{2rx}{n^2} - \frac{\{2rx\}}{n^2}\right) \qquad \text{[using Eq. (i)]}$$

$$\therefore \qquad f(x) = \lim_{n \to \infty} \sum_{r=1}^{n} \frac{2rx}{n^2} - \lim_{n \to \infty} \sum_{r=1}^{n} \frac{\{2rx\}}{n^2} \qquad \text{...(ii)}$$

Let $\quad l_1 = \lim_{n \to \infty} \sum_{r=1}^{n} \frac{2rx}{n^2}$ and $l_2 = \lim_{n \to \infty} \sum_{r=1}^{n} \frac{\{2rx\}}{n^2}$

Now, $\quad l_1 = \lim_{n \to \infty} \frac{2x(1+2+3+\ldots+n)}{n^2}$

$$= \lim_{n \to \infty} \frac{2x \cdot n(n+1)}{2n^2}$$

$$l_1 = \lim_{n \to \infty} x \cdot \left(1 + \frac{1}{n}\right) = x \qquad \ldots\text{(iii)}$$

Again, for second limit, $0 \le \{2rx\} < 1$, for all r

$\Rightarrow \qquad 0 \le \sum_{r=1}^{n} \{2rx\} < n$

$\Rightarrow \qquad \sum_{r=1}^{n} \{2rx\} = \alpha n;$ where $0 \le \alpha < 1$

Thus, $\quad l_2 = \lim_{n \to \infty} \frac{\sum_{r=1}^{n} [2rx]}{n^2} = \lim_{n \to \infty} \frac{\alpha^n}{n^2} = 0 \qquad \ldots\text{(iv)}$

$\therefore$ From Eqs. (ii), (iii) and (iv), we get

$$\lim_{n \to \infty} \sum_{r=1}^{n} \frac{[2rx]}{n^2} = x, \forall \ x \in R$$

28. $\lim_{n \to \infty} \sum_{r=1}^{n} \cot^{-1}\left(r^2 + \frac{3}{4}\right)$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \tan^{-1}\left[\frac{4}{3 + 4r^2}\right]$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \tan^{-1}\left[\frac{4}{4 - 1 + 4r^2}\right]$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \tan^{-1}\left[\frac{1}{1 + (r - 1/2)(r + 1/2)}\right]$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \tan^{-1}\left[\frac{(r + 1/2) - (r - 1/2)}{1 + (r + 1/2)(r - 1/2)}\right] \qquad \left[\text{as, } 1 = \left(r + \frac{1}{2}\right) - \left(r - \frac{1}{2}\right)\right]$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \left\{\tan^{-1}\left(r + \frac{1}{2}\right) - \left(r - \frac{1}{2}\right)\right\}$$

$$= \lim_{n \to \infty} \left\{\tan^{-1}\left(\frac{3}{2}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right\} + \left\{\tan^{-1}\left(\frac{5}{2}\right) - \tan^{-1}\left(\frac{3}{2}\right)\right\} + \ldots$$

$$\ldots + \left\{\tan^{-1}\left(n + \frac{1}{2}\right) - \tan^{-1}\left(n - \frac{1}{2}\right)\right\}$$

$$= \lim_{n \to \infty} \left\{\tan^{-1}\left(\frac{n+1}{2}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right\}$$

$$= \tan^{-1}(\infty) - \tan^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right) \qquad \left[\text{as } \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right]$$

$$= \cot^{-1}\left(\frac{1}{2}\right) = \tan^{-1}(2) \qquad [\tan^{-1} x = \cot^{-1}(1/x)]$$

29. Let $P = \lim_{n \to \infty} \log_{n-1}(n) \cdot \log_n(n+1) \ldots\ldots \log_{n^k - 1}(n^k)$

$$= \lim_{n \to \infty} \left\{ \frac{\log n}{\log(n-1)} \cdot \frac{\log(n+1)}{\log(n)} \cdots\cdots \frac{\log(n^k)}{\log(n^k - 1)} \right\}$$

$$= \lim_{n \to \infty} \frac{\log(n^k)}{\log(n-1)}$$

$$= k \cdot \lim_{n \to \infty} \frac{\log n}{\log(n-1)}$$

$$= k \cdot \lim_{n \to \infty} \frac{1/n}{\dfrac{1}{n-1}} \qquad \text{[applying L'Hospital's rule]}$$

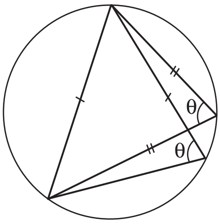
$$= k \cdot \lim_{n \to \infty} \left(1 - \frac{1}{n}\right)$$

$$\Rightarrow \lim_{n \to \infty} \{\log_{n-1}(n) \cdot \log_n(n+1) \cdot \log_{n+1}(n+2) \ldots\ldots \log_{(n^k - 1)} n^k\} = k$$

30. Note that the base angle of T_n is equal to the angle opposite the base of T_{n+1} (as the figure indicates).

Therefore, θ is the base angle for T_n, then the base angle for the next triangle (T_{n+1}) is,

$$\frac{180° - \theta}{2} = 90° - \frac{\theta}{2}$$



Suppose, now that θ is the base angle for T_1.

Then, the base angle for T_2 is $\left(90° - \dfrac{\theta}{2}\right)$.

Again, if base angle for T_2 is $\left(90° - \dfrac{\theta}{2}\right)$.

Then, base angle for T_3 is $90° - \left(\dfrac{90°}{2} - \dfrac{\theta}{4}\right)$.

Proceeding in same way base angle for T_n is

$$90° - \frac{90°}{2} + \frac{90°}{4} - \frac{90°}{8} + \frac{90°}{16} + \ldots\ldots + \frac{(-1)^{n-1}\theta}{2^{n-1}}$$

Here, as $n \to \infty$, above series tends to an infinite geometric progression.

$$\Rightarrow \quad \lim_{n \to \infty} \left(90° - \frac{90°}{2} + \frac{90°}{4} - \frac{90°}{8} + \ldots\ldots + \frac{(-1)^{n-1}\theta}{2^{n-1}}\right) = \frac{90°}{1 + 1/2} = 60°$$

Now, since T_n is isosceles and one of the angle approaches to 60° as $n \to \infty$.

$\therefore$ T_n is equilateral triangle as $n \to \infty$.

6

Continuity and Differentiability

Chapter in a Snapshot

- Introduction to Continuity
- Continuity of a Function
- Graphical View
- Continuity at End Points
- Types of Discontinuity
- Theorems of Continuity
- Continuity of Composite Functions
- A List of Continuous Functions
- Introduction to Differentiability of a Function at a Point
- Differentiability at a Point
- Differentiability in a Set
- Relation Between Continuity and Differentiability
- Some Standard Results on Differentiability
- Theorems of Differentiability

Introduction to Continuity

As, we have discussed in the chapter on Limits, there exist neighbouring points. Graphically, it could be stated as shown in figure.

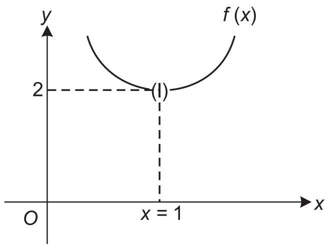


Fig. 6.1

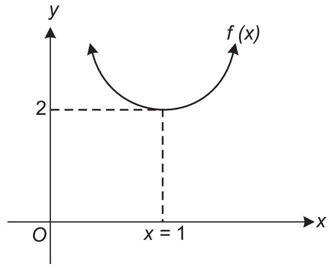


Fig. 6.2

$$\lim_{x \to 1} f(x) = 2$$

But when we say that the function $f(x)$ is continuous at a point $x = a$, we mean that at point $(a, f(a))$ the graph of the function has no holes or gaps. That is, its graph is unbroken at a point $(a, f(a))$.

Graphically, it could be stated as, shown in figure

Here, $\lim_{x \to 1} f(x) = 2$

and $f(1) = 2$

Thus, $\lim_{x \to 1} f(x) = f(1)$ hence, $f(x)$ is continuous.

Point to Consider

Qualitatively, the graph of a function is said to be continuous at $x = a$, if while travelling along the graph of the function and in crossing over the point at $x = a$ either from L to R or from R to L one does not have to lift his pen.

In case one has to lift his pen the graph of the function is said to have a break or discontinuous at $x = a$. Different type of situations, which may come up at $x = a$ along the graph, can be shown as below :

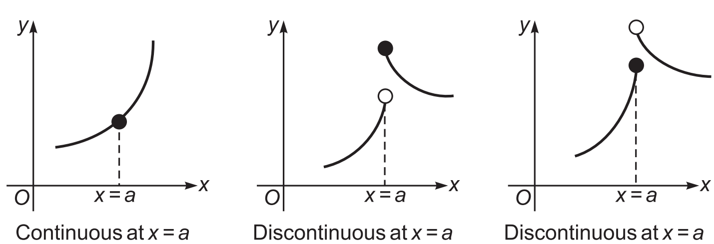


Fig. 6.3

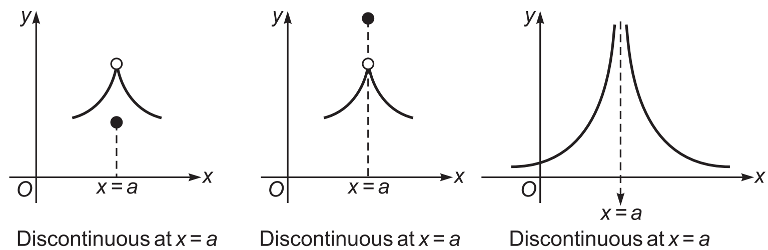


Fig. 6.4

Continuity of a Function

A function $f(x)$ is said to be continuous at $x = a$; where $a \in$ domain of $f(x)$.

If $$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

ie, LHL = RHL = Value of a function at $x = a$

or $$\lim_{x \to a} f(x) = f(a)$$

If $f(x)$ is not continuous at $x = a$, we say that $f(x)$ is discontinuous at $x = a$, $f(x)$ will be discontinuous at $x = a$ in any of the following cases:

(i) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist but are not equal.

(ii) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist and are equal but not equal to $f(a)$.

(iii) $f(a)$ is not defined.

(iv) At least one of the limit doesn't exist.

Graphical View

(i) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exists but are not equal.

Here, $$\lim_{x \to a^-} f(x) = l_1$$
$$\lim_{x \to a^+} f(x) = l_2$$

$\therefore$ $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist but are not equal.

Thus, $f(x)$ is discontinuous at $x = a$.
It doesn't matter whether $f(a)$ exist or not.

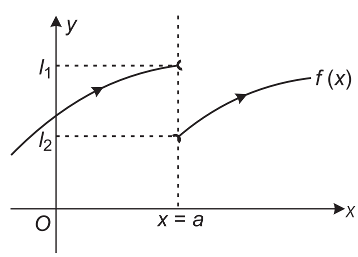


Fig. 6.5

Illustration 1 If $f(x) = \dfrac{|x|}{x}$. Discuss the continuity at $x \to 0$.

Solution. Here, $f(x) = \dfrac{|x|}{x}$

$\therefore$ RHL at $x \to 0$

Let $$x = 0 + h$$

ie, $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{|0+h|}{0+h} = \lim_{h \to 0} \frac{h}{h} = 1 \Rightarrow \lim_{x \to 0^+} f(x) = 1$

Again, LHL at $x \to 0$

Let $x = 0 - h$

ie, $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \frac{|0-h|}{0-h} = \lim_{h \to 0} \frac{h}{-h} = -1 \Rightarrow \lim_{x \to 0^-} f(x) = -1$

$\therefore \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$. Thus, $f(x)$ is discontinuous at $x \to 0$

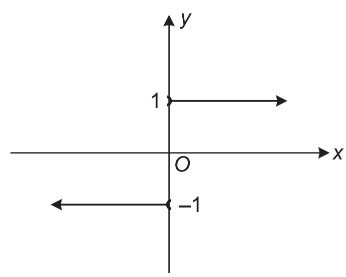
Point to Consider

Here, $f(x)$ is not defined, as $x = 0$, as $f(0) = \frac{0}{0}$ (Indetermined form)

So, we could say directly that the function is discontinuous at $x = 0$

Graphically : Here,

$$f(x) = \frac{|x|}{x} = \begin{cases} \frac{x}{x}, & x > 0 \\ -\frac{x}{x}, & x < 0 \end{cases} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$



and $f(0) = \frac{0}{0}$

(Indetermined form) $\Rightarrow$ not defined.

Which shows, the graph is broken at $x = 0$

Where, $\lim_{x \to 0^-} f(x) = -1$ and $\lim_{x \to 0^+} f(x) = -1$

Thus, $\lim_{x \to 0} f(x) \Rightarrow$ It doesn't exist and hence, function is discontinuous.

(ii) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist and are equal but not equal to $f(a)$.

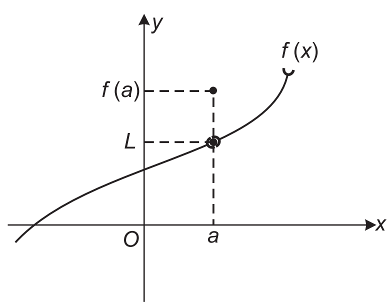


Fig. 6.6

Here, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

$f(a)$ is also defined but $f(a) \neq L$.

So, again limit of $f(x)$ exists at $x = a$.

But, it is not continuous at $x = a$.

Illustration 2 If $f(x)=\begin{cases}2x+3, & \text{when } x<0\\ 0, & \text{when } x=0\\ x^2+3, & \text{when } x>0\end{cases}$. Discuss the continuity.

Solution. Here, $f(x)=\begin{cases}2x+3, & \text{when } x<0\\ 0, & \text{when } x=0\\ x^2+3, & \text{when } x>0\end{cases}$

$\therefore$ RHL at $x=0$, let $x=0+h$

ie, $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{h\to 0} f(0+h)=\lim\limits_{h\to 0}\{(0+h)^2+3\}=3$

$\Rightarrow$ $\lim\limits_{x\to 0^+} f(x)=3$

Again, LHL at $x=0$

Let $x=0-h$

ie, $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{h\to 0} f(0-h)=\lim\limits_{h\to 0}\{2(0-h)+3\}=3$

$\Rightarrow$ $\lim\limits_{x\to 0^-} f(x)=3$

But $f(0)=0$

Therefore, $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^-} f(x)=3\neq f(0)$

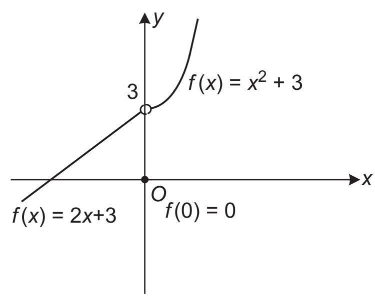
Thus, $f(x)$ is discontinuous at $x\to 0$

Graphically : Here, $\lim\limits_{x\to 0^-} f(x)=3$

$$\lim_{x\to 0^+} f(x)=3$$
$$f(0)=0$$

Thus, $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^+} f(x)=3\neq f(0)$

Hence, $f(x)$ is discontinuous at $x=0$



(iii) $f(a)$ is not defined.

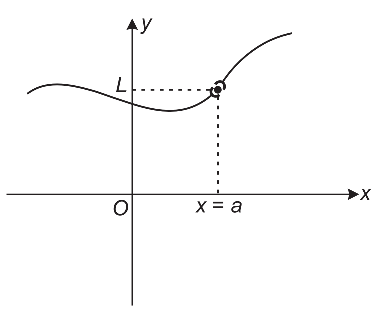


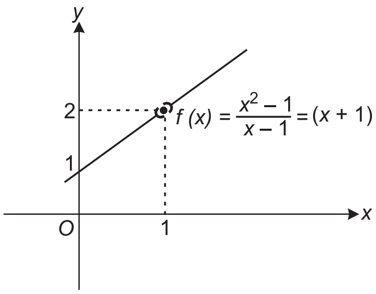
Fig. 6.7

Here, $\lim\limits_{x\to a^+} f(x)=L$ and $\lim\limits_{x\to a^-} f(x)=L$

But, $f(a)$ is not defined. So, $f(x)$ is discontinuous at $x=a$

Illustration 3 If $f(x)=\dfrac{x^2-1}{x-1}$. Discuss the continuity at $x \to 1$.

Solution. Here, $f(x)=\dfrac{x^2-1}{x-1}$



$$\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)}$$

$$=\lim_{x\to 1}(x+1)=2$$

But $f(1)=\dfrac{0}{0}$ (Indetermined form)

$\therefore$ $f(1)$ is not defined at $x=1$

Hence, $f(x)$ is discontinuous at $x=1$

Graphically : Which shows, $\lim\limits_{x\to 1} f(x)=2$ but $f(1)$ is not defined.

So, $f(x)$ is discontinuous at $x=1$

Point to Consider

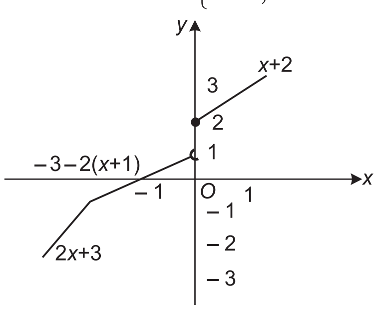
We can also say that while drawing a graph when the pen leaves the paper the function becomes discontinuous at the point where contact with the paper is several.

Illustration 4 Show that the function

$$f(x)=\begin{cases} 2x+3, & -3\le x<-2 \\ x+1, & -2\le x<0 \\ x+2, & 0\le x\le 1 \end{cases}$$

is discontinuous at $x=0$ and continuous at every point in interval $[-3,1]$.

Solution. ***Graphically*** : $f(x)=\begin{cases} 2x+3, & -3\le x<-2 \\ x+1, & -2\le x<0 \\ x+2, & 0\le x\le 1 \end{cases}$ is plotted as shown



Here, if we observe the graph we could conclude that at $x=0$, $\lim\limits_{x\to 0^-} f(x)=1$ and $\lim\limits_{x\to 0^+} f(x)=2$, which shows that the function is discontinuous at $x=0$ and continuous at every other point in $[-3,1]$.

Illustration 5 Examine the function,

$$f(x)=\begin{cases}\dfrac{\cos x}{\pi/2-x}, & x\neq \pi/2\\ 1, & x=\pi/2\end{cases}\qquad \text{for continuity at } x=\pi/2.$$

Solution. We have, $\lim_{x\to\pi/2} f(x)$

$$\Rightarrow \qquad \lim_{x\to\pi/2}\frac{\cos x}{\pi/2-x} \qquad [\text{As } x\neq\pi/2 \text{ but } x\to\pi/2]$$

$$\Rightarrow \qquad \lim_{x\to\pi/2}\frac{\cos x}{\pi/2-x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

Applying L'Hospital's rule, we get

$$\lim_{x\to\pi/2}\frac{-\sin x}{0-1} \quad\Rightarrow\quad \sin\pi/2=1$$

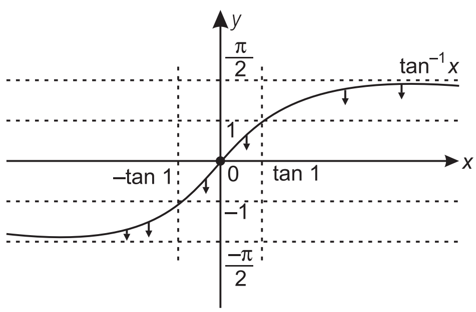
Also, $$f(\pi/2)=1$$

$\therefore$ $$\lim_{x\to\pi/2} f(x)=f(\pi/2)$$

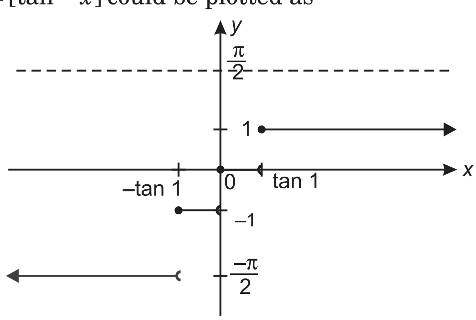
Thus, $f(x)$ is continuous at $x=\pi/2$

Illustration 6 Discuss the continuity of $f(x)=[\tan^{-1}x]$.

Solution. We know, $y=\tan^{-1}x$ could be plotted as



Thus, $f(x)=[\tan^{-1}x]$ could be plotted as



which clearly shows the graph is broken at {– tan 1, 0, tan 1}.

$\therefore$ $f(x)$ is not continuous when $x\in\{-\tan 1, 0, \tan 1\}$.

Illustration 7 Let $y = f(x)$ be defined parametrically as $y = t^2 + t|t|$, $x = 2t - |t|$, $t \in R$. Then, at $x = 0$, find $f(x)$ and discuss continuity.

Solution. As, $\quad y = t^2 + t|t| \quad$ and $\quad x = 2t - |t|$

Thus, when $t \geq 0$

$\Rightarrow x = 2t - t = t, \; y = t^2 + t^2 = 2t^2$

$\therefore \qquad x = t \quad$ and $\quad y = 2t^2$

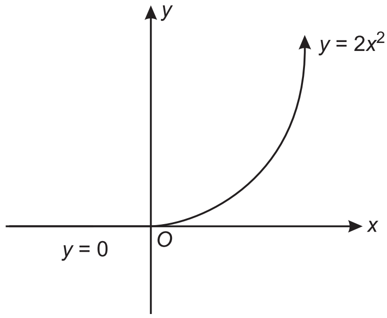
$\Rightarrow \qquad y = 2x^2 \; \forall \; x \geq 0$

Again, when $t < 0$

$\Rightarrow x = 2t + t = 3t \quad$ and $\quad y = t^2 - t^2 = 0$

$\Rightarrow \qquad y = 0 \; \forall \; x < 0$

Hence, $f(x) = \begin{cases} 2x^2, & x \geq 0 \\ 0, & x < 0 \end{cases}$



which is clearly continuous for all x as shown graphically.

Target Exercise 6.1

1. A function is defined as;

$$f(x) = \begin{cases} 0, & \text{where } x \text{ is rational.} \\ 1, & \text{where } x \text{ is irrational.} \end{cases}$$

Then, $f(x)$ is

(a) continuous for all $x \in R$ (b) continuous for all $x \in R - \{0\}$

(c) continuous for all $x \in R - \{0,1\}$ (d) discontinuous for all $x \in R$

2. If $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$ and $g(x) = x(1 - x^2)$, then $f(g(x))$ is continuous for

(a) R (b) $R - \{0\}$

(c) $R - \{0, 1\}$ (d) $R - \{-1, 0, 1\}$

3. If $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$ and $g(x) = x(1 - x^2)$, then $g(f(x))$ is continuous for

(a) R (b) $R - \{0\}$

(c) $R - \{0, 1\}$ (d) $R - \{-1, 0, 1\}$

4. If $f(x) = \begin{cases} |x^2 - 1| - 1, & x \leq 1 \\ |2x - 3| - |x - 2|, & x > 1 \end{cases}$, then $f(x)$ is continuous for

(a) R (b) $R - \{0\}$

(c) $R - \{1\}$ (d) None of these

5. If $f(x) = -1 + |x - 2|, 0 \leq x \leq 4$

$g(x) = 2 - |x|, -1 \leq x \leq 3$

Then, $fog(x)$ is continuous for x belonging to

(a) $[0, 4]$ (b) $[-1, 3]$

(c) $[0, 3]$ (d) $[-1, 2]$

Continuity at End Points

Let a function $y = f(x)$ is defined on $[a, b]$.
Then, the function $f(x)$ is said to be continuous at the left end $x = a$

If $\qquad f(a) = \lim_{x \to a^+} f(x) \qquad$ (Need not check LHL)

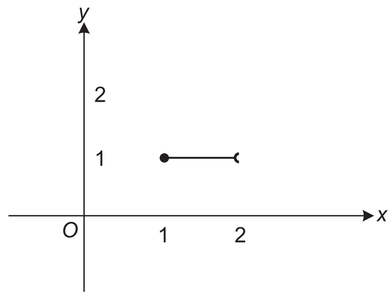
If $f(x)$ is said to be continuous at the right end $x = b$

If $\qquad f(b) = \lim_{x \to b^-} f(x) \qquad$ (Need not check RHL)

Illustration 8 If $f(x) = [x]$, where $[\cdot]$ denotes greatest integral function. Then, check the continuity on $[1, 2]$.

Solution. ***Graphically*** :

$f(x) = [x]$ could be plotted as



Here, in the graph $f(x)$ is continuous at all point where $1 < x < 2$.

To check continuity at $x = 1$ and $x = 2$

(a) **To check continuity at $x = 1$**

Here, $f(1) = 1$ and RHL at $x = 1$

$$\Rightarrow \qquad \lim_{x \to 1^+} f(x) = 1$$

ie, $$f(1) = \lim_{x \to 1^+} f(x)$$

Thus, $f(x)$ is continuous at $x = 1$

(b) **To check continuity at $x = 2$**

Here, $\qquad f(2) = 2$

But LHL at $\qquad x = 2$

$$\Rightarrow \qquad \lim_{x \to 2^-} f(x) = 1$$

which shows $$f(2) \neq \lim_{x \to 2^-} f(x)$$

Thus, $f(x)$ is discontinuous at $x = 2$.

From the above information, it becomes clear that $f(x)$ is continuous at all points on $[1, 2]$ except at $x = 2$.

ie, $f(x)$ is continuous for $x \in [1, 2]$.

Types of Discontinuity

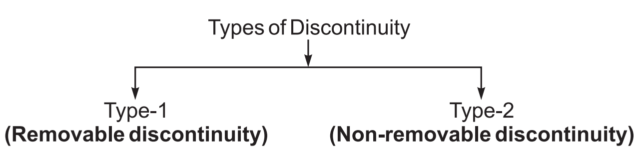


Fig. 6.8

Type 1 : **Removable Discontinuity**

Here, $\lim_{x \to a} f(x)$ necessarily exists, but is either not equal to $f(a)$ or $f(a)$ is not defined.

In this case, therefore, it is possible to redefine the function in such a manner that $\lim_{x \to a} f(x) = f(a)$ and thus making function continuous.

These discontinuities can be further classified as :

(a) Missing point discontinuity

(b) Isolated point discontinuity

Examples of Missing Point Discontinuity

(i) Let $f(x) = \dfrac{(x-1)(9-x^2)}{(x-1)}$, clearly $f(1) \to \dfrac{0}{0}$ form

$\therefore$ $f(x)$ has missing point discontinuity. Shown as,

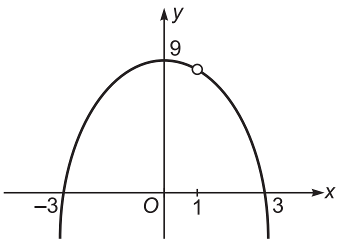


Fig. 6.9

(ii) $f(x) = \dfrac{x^2 - 4}{x - 2}$ at $x = 2$, has missing point discontinuity at $x = 2$.

Shown as,

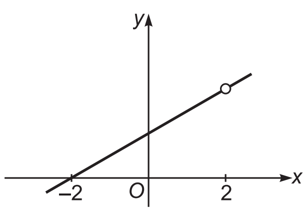


Fig. 6.10

(iii) $f(x) = \dfrac{\sin x}{x}$ at $x = 0$, has missing point discontinuity at $x = 0$.

Shown as,

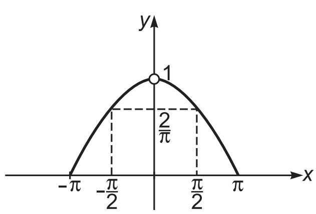


Fig. 6.11

Examples of Isolated Point Discontinuity

(i) Let $f(x) = [x] + [-x]$

$\Rightarrow$ $$f(x) = \begin{cases} 0, \text{ if } x \in I \\ -1, \text{ if } x \notin I \end{cases}$$

where x = Integer, has isolated point discontinuity, can be shown as

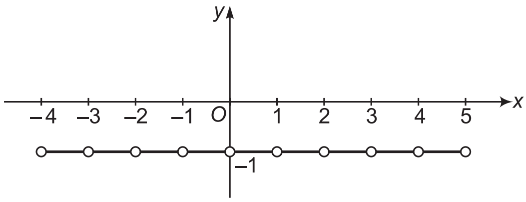


Fig. 6.12

(ii) Let $f(x) = \text{sgn}(\cos 2x - 2\sin x + 3)$

$\Rightarrow$ $$f(x) = \text{sgn}(1 - 2\sin^2 x - 2\sin x + 3)$$
$$= \text{sgn}(2(2 + \sin x)(1 - \sin x))$$
$$= \begin{cases} 0, \text{ if } x = 2n\pi + \dfrac{\pi}{2} \\ 1, \text{ if } x \neq 2n\pi + \dfrac{\pi}{2} \end{cases}$$

$\therefore f(x)$ has an isolated point at $x = 2n\pi + \dfrac{\pi}{2}$

Shown as

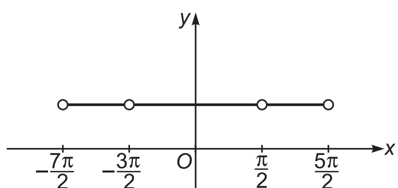


Fig. 6.13

Type 2 : Non-removable Discontinuity

Here, $\lim_{x \to a} f(x)$ doesn't exist and therefore, it is not possible to redefine the function in any manner and make it continuous. Such discontinuities can further be classified into 3 types.

(a) Finite type (Both limits finite and unequal).

(b) Infinite type (At least one of the two limits is infinity).

(c) Oscillatory (Limits oscillate between two finite quantities).

Examples of Finite Type

(i) $\lim\limits_{x \to 0} \tan^{-1}\left(\dfrac{1}{x}\right)$

$$\left.\begin{aligned} &\text{RHL} \quad ie,\ f(0^+) = \frac{\pi}{2} \\ &\text{LHL} \quad ie,\ f(0^-) = -\frac{\pi}{2} \end{aligned}\right\}\text{jump} = \pi$$

(ii) $\lim\limits_{x \to 0} \dfrac{|\sin x|}{x}$

$$\left.\begin{aligned} &\text{RHL} \quad ie,\ f(0^+) = 1 \\ &\text{LHL} \quad ie,\ f(0^-) = -1 \end{aligned}\right\}\text{jump} = 2$$

In this case, the non-negative difference between the two limits is called the jump of discontinuity. A function having a finite number of jumps in a given interval it is called a PIECEWISE CONTINUOUS or SECTIONALLY CONTINUOUS function.

Examples of Infinite Type

(i) $f(x) = \dfrac{x}{1-x}$, at $x = 1$

$$\begin{aligned} &\text{RHL } ie,\ f(1^+) = -\infty \\ &\text{LHL } ie,\ f(1^-) = \infty \end{aligned}$$

(ii) $f(x) = \dfrac{1}{x^2}$, at $x = 0$

$$\begin{aligned} &\text{RHL } ie,\ f(0^+) = \infty \\ &\text{LHL } ie,\ f(0^-) = \infty \end{aligned}$$

Examples of Oscillatory

(i) $f(x) = \sin\left(\dfrac{1}{x}\right)$

$\Rightarrow \lim\limits_{x \to 0} f(x) =$ a value between -1 to 1.

$\therefore$ Limit doesn't exist, as it oscillates between -1 and 1 at $x = 0$.

(ii) $f(x) = \cos\left(\dfrac{1}{x}\right)$

$\Rightarrow \lim\limits_{x \to 0} \cos\left(\dfrac{1}{x}\right) =$ a value between -1 to 1.

$\therefore$ Limit doesn't exist, as it oscillates between -1 to 1 at $x = 0$.

Illustration 9 Examine the function,

$$f(x) = \begin{cases} x - 1, & x < 0 \\ 1/4, & x = 0 \\ x^2 - 1, & x > 0 \end{cases}$$

Discuss the continuity and if discontinuous remove the discontinuity.

Solution. Graphically, $f(x)$ could be plotted as showing

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = -1$, but $f(0) = 1/4$

Thus, $f(x)$ has removable discontinuity and $f(x)$ could be made continuous by taking

$f(0) = -1$

$$\Rightarrow f(x) = \begin{cases} x - 1, & x < 0 \\ -1, & x = 0 \\ x^2 - 1, & x > 0 \end{cases}$$

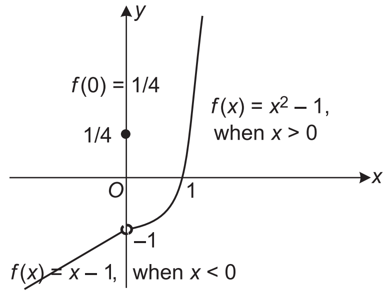


Illustration 10 Show the function, $f(x) = \begin{cases} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$ has non-removable discontinuity at $x = 0$.

Solution. We have, $f(x) = \begin{cases} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$

$\therefore$ RHL at $x = 0$, let $x = 0 + h$

$$\Rightarrow \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} \frac{e^{\frac{1}{0+h}} - 1}{e^{\frac{1}{0+h}} + 1} = \lim_{h \to 0} \frac{e^{\frac{1}{h}} - 1}{e^{\frac{1}{h}} + 1}$$

$$\Rightarrow \lim_{x \to 0^+} f(x) = \lim_{h \to 0} \frac{1 - \dfrac{1}{e^{1/h}}}{1 + \dfrac{1}{e^{1/h}}}$$

$$\Rightarrow \lim_{x \to 0^+} f(x) = \frac{1 - 0}{1 + 0} = 1 \qquad \left[\text{As } h \to 0; \frac{1}{h} \to \infty \Rightarrow e^{1/h} \to \infty; 1/e^{1/h} \to 0\right]$$

$$\therefore \lim_{x \to 0^+} f(x) = 1$$

Again, LHL at $x = 0$, let $x = 0 - h$

$$\Rightarrow \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1} = \frac{0 - 1}{0 + 1} = -1 \quad [\text{As } h \to 0; e^{-1/h} \to 0]$$

$$\lim_{x \to 0^-} f(x) = -1$$

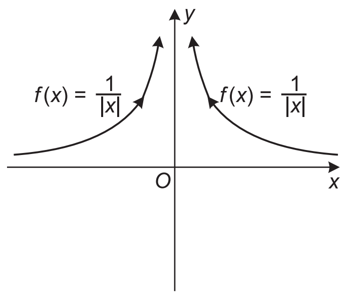
$\Rightarrow \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$. Thus, $f(x)$ has non-removable discontinuity.

Illustration 11 Show $f(x) = \dfrac{1}{|x|}$ has discontinuity of second kind at $x = 0$.

Solution. Here, $f(x) = \dfrac{1}{|x|} = \infty$ which shows function has discontinuity of second kind.

Graphically

Here, the graph is broken at $x = 0$ as $x \to 0 \quad \Rightarrow f(x) \to \infty$



Therefore, $f(x)$ has discontinuity of second kind.

Theorems of Continuity

Theorem 1 Sum, difference, product and quotient of two continuous functions is always a continuous function. However, $h(x) = \dfrac{f(x)}{g(x)}$ is continuous at $x = a$ only if $g(a) \neq 0$.

Theorem 2 If $f(x)$ is continuous and $g(x)$ is discontinuous at $x = a$ then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$.

For Example (i) $f(x) = x$ and $g(x) = \begin{cases} \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

where $f(x)$ is continuous and $g(x)$ is discontinuous at $x = 0$.

But $\qquad \phi(x) = f(x) \cdot g(x) = \begin{cases} x \cdot \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$\Rightarrow \phi(x)$ is continuous at $x = 0$.

(ii) $f(x) = \cos(2x - 1)\dfrac{\pi}{2}$ is continuous at $x = 1$ and $g(x) = [x]$ is discontinuous at $x = 1$.

But $\phi(x) = f(x) \cdot g(x) = [x] \cos\left(\dfrac{2x - 1}{2}\right)\pi$ is continuous at $x = 1$.

Theorem 3 If $f(x)$ and $g(x)$ both are discontinuous at $x = a$ then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$.

For Example $f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$ and $g(x) = \begin{cases} -1, & x \geq 0 \\ 1, & x < 0 \end{cases}$

$\therefore \qquad \phi(x) = f(x) \cdot g(x) = -1, \forall x \in R.$

$\Rightarrow \phi(x)$ is continuous, where as $f(x)$ and $g(x)$ are discontinuous at $x = 0$.

Illustration 12 $f(x)=\begin{cases}\left(\tan\left(\dfrac{\pi}{4}+x\right)\right)^{1/x}, & x \neq 0\\ k, & x=0\end{cases}$. For what value of k, $f(x)$ is continuous at $x=0$?

Solution. Here, $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0}\left\{\tan\left(\dfrac{\pi}{4}+x\right)\right\}^{1/x}$

$$\Rightarrow \quad \lim_{x \to 0} f(x) = \lim_{x \to 0}\left[\frac{1+\tan x}{1-\tan x}\right]^{1/x} \qquad (1^{\infty} \text{ form})$$

$$\Rightarrow \quad \lim_{x \to 0} f(x) = \lim_{x \to 0}\left[1+\left(\frac{1+\tan x}{1-\tan x}-1\right)\right]^{1/x}$$

$$\Rightarrow \quad \lim_{x \to 0} f(x) = e^{\lim\limits_{x \to 0}\left(\frac{2\tan x}{1-\tan x}\right)\cdot\frac{1}{x}}$$

$$\Rightarrow \quad \lim_{x \to 0} f(x) = e^{2\lim\limits_{x \to 0}\frac{\tan x}{x(1-\tan x)}} = e^2$$

Here, $f(x)$ is continuous at $x=0$, when

$$\lim_{x \to 0} f(x) = f(0)$$

$$\Rightarrow \quad k = e^2$$

Illustration 13 A function $f(x)$ is defined by,

$$f(x)=\begin{cases}\dfrac{[x^2]-1}{x^2-1}, & \text{for } x^2 \neq 1\\ 0, & \text{for } x^2=1\end{cases}$$

Discuss the continuity of $f(x)$ at $x=1$.

Solution. We have,

$$f(x)=\begin{cases}\dfrac{[x^2]-1}{x^2-1}, & \text{for } x^2 \neq 1\\ 0, & \text{for } x^2=1\end{cases}$$

$$\Rightarrow \quad f(x)=\begin{cases}\dfrac{-1}{x^2-1}, & \text{for } 0<x^2<1\\ 0, & \text{for } x^2=1\\ \dfrac{1-1}{x^2-1}, & \text{for } 1<x^2<2\end{cases}$$

$$\Rightarrow \quad f(x)=\begin{cases}\dfrac{-1}{x^2-1}, & \text{for } 0<x^2<1\\ 0, & \text{for } x^2=1\\ 0, & \text{for } 1<x^2<2\end{cases}$$

$\therefore$ RHL at $x^2 = 1 \Rightarrow \lim_{x \to 1^+} f(x) = 0$

Also, LHL at $x^2 = 1$

$$\Rightarrow \quad \lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} \frac{-1}{(1-h)^2 - 1} = -\infty$$

$$\therefore \quad \lim_{x \to 1} f(x) \text{ doesn't exist.} \qquad \text{(As RHL} \neq \text{LHL)}$$

Hence, $f(x)$ is not continuous at $x = 1$.

Illustration 14 Discuss the continuity of the function

$$f(x) = \lim_{n \to \infty} \frac{\log(2+x) - x^{2n} \sin x}{1 + x^{2n}} \text{ at } x = 1.$$

Solution. We have, $f(1) = \lim_{n \to \infty} \frac{\log 3 - \sin 1}{2} = \frac{1}{2}(\log 3 - \sin 1)$ ...(i)

We know that, $\lim_{n \to \infty} x^{2n} = \begin{cases} 0, & \text{if } x^2 < 1 \\ \infty, & \text{if } x^2 > 1 \end{cases}$

$\therefore$ For $x^2 < 1$, we have

$$f(x) = \lim_{n \to \infty} \frac{\log(2+x) - x^{2n} \sin x}{1 + x^{2n}} \longrightarrow \log(2+x)$$

Again, for $x^2 > 1$, we have

$$f(x) = \lim_{n \to \infty} \frac{\frac{1}{x^{2n}} \log(2+x) - \sin x}{1 + \frac{1}{x^{2n}}} \longrightarrow -\sin(x)$$

Here, as $x \to 1$

$$\lim_{x \to 1^-} f(x) = \log(3) \text{ and } \lim_{x \to 1^+} f(x) = -\sin(1)$$

So, $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$

Therefore, $f(x)$ is not continuous at $x = 1$.

Illustration 15 Discuss the continuity of the function

$$f(x) = \begin{cases} \dfrac{a^{2[x] + \{x\}} - 1}{2[x] + \{x\}}, & x \neq 0 \\ \log_e a, & x = 0 \end{cases} \qquad \text{(At } x = 0\text{)}$$

where $[\cdot]$ denotes greatest integral part and $\{\cdot\}$ denotes fractional part of x.

Solution. We know, $[x] + \{x\} = x$ for any $x \in R$. Then, the given function can be expressed as follows :

$$f(x) = \begin{cases} \dfrac{a^{[x] + x} - 1}{[x] + x}, & x \neq 0 \\ \log_e a, & x = 0 \end{cases}$$

Now, to check continuity at $x = 0$

RHL at $x = 0$

$$\Rightarrow \quad \lim_{x \to 0^+} f(x) = \lim_{h \to 0} \frac{a^{[0+h]+(0+h)} - 1}{[0+h]+(0+h)} = \lim_{h \to 0} \frac{a^{[h]+h} - 1}{[h]+h}$$

$$= \lim_{h \to 0} \frac{a^h - 1}{h} \qquad [h \to 0^+ \; ie, 0 < h < 1 \Rightarrow [h] = 0]$$

$$= \log a$$

Thus, $\quad$ RHL $= \log a$ $\quad$ ...(i)

Again, LHL at $x = 0$

$$= \lim_{x \to 0^-} f(x) = \lim_{h \to 0} \frac{a^{[0-h]+(0-h)} - 1}{[0-h]+(0-h)}$$

$$= \lim_{h \to 0} \frac{a^{-1-h} - 1}{-1-h} \qquad [\text{As } -1 < 0 - h < 0 \quad \Rightarrow \quad [0-h] = -1]$$

$$= \frac{a^{-1} - 1}{-1} \quad \Rightarrow 1 - \frac{1}{a}$$

$\therefore \quad$ LHL $= \left(1 - \frac{1}{a}\right)$ $\quad$ ...(ii)

From Eqs. (i) and (ii), $f(0) = f(0^+) \neq f(0^-)$

Therefore, $f(x)$ is not continuous at $x = 0$.

Illustration 16 Discuss the continuity of $f(x)$ where $f(x) = \lim_{n \to \infty} \left(\sin \frac{\pi x}{2}\right)^{2n}$.

Solution. Since, $\lim_{n \to \infty} x^{2n} = \begin{cases} 0, & |x| < 1 \\ 1, & |x| = 1 \end{cases}$

$$\therefore \quad f(x) = \lim_{n \to \infty} \left(\sin\left(\frac{\pi x}{2}\right)\right)^{2n} = \begin{cases} 0, & \left|\sin \frac{\pi x}{2}\right| < 1 \\ 1, & \left|\sin \frac{\pi x}{2}\right| = 1 \end{cases}$$

Thus, $f(x)$ is continuous for all x, except for those values of x for which $\left|\sin \frac{\pi x}{2}\right| = 1$ *ie*, x is an odd integer.

$\Rightarrow \quad x = (2n + 1)$ where $n \in I$

Check continuity at $x = (2n + 1)$

$$\text{LHL} = \lim_{x \to 2n+1} f(x) = 0$$

and $\quad f(2n + 1) = 1$

Thus, $\quad$ LHL $\neq f(2n + 1)$

$\Rightarrow \quad f(x)$ is discontinuous at $x = (2n + 1)$ $\quad$ (*ie*, at odd integers)

Hence, $f(x)$ is discontinuous at $x = (2n + 1)$; $n \in$ integer.

Illustration 17 Let $f(x)=\begin{cases}\{1+|\sin x|\}^{a/|\sin x|}, & -\pi/6<x<0\\ b, & x=0\\ e^{\tan 2x/\tan 3x}, & 0<x<\pi/6\end{cases}$

Determine a and b such that $f(x)$ is continuous at $x=0$. **[IIT JEE 1994]**

Solution. Since, f is continuous at $x=0$.

Therefore, RHL = LHL $=f(0)$

RHL at $x=0$

$$\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} e^{\tan 2h/\tan 3h}$$

$$=\lim_{h\to 0} e^{\frac{\tan 2h}{2h}\cdot\frac{3h}{\tan 3h}\cdot\frac{2}{3}}=e^{2/3} \qquad \text{...(i)}$$

Again, LHL at $x=0$

$$\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0}\{1+|\sin(0-h)|\}^{a/|\sin(0-h)|}$$

$$=\lim_{h\to 0}\{1+|\sin h|\}^{\frac{a}{|\sin h|}}=e^{\lim\limits_{h\to 0}|\sin h|\cdot\frac{a}{|\sin h|}}=e^a \qquad \text{...(ii)}$$

and $\qquad f(0)=b \qquad$ ...(iii)

Thus, $\quad e^{2/3}=e^a=b \;\Rightarrow\; a=2/3$ and $b=e^{2/3}$

Illustration 18 Fill in the blanks so that the resulting statement is correct. Let $f(x)=[x]\sin\left(\dfrac{\pi}{[x+1]}\right)$ where $[\cdot]$ denotes greatest integral function.

The domain of f is .... and the points of discontinuity of f in the domain are .... **[IIT JEE 1996]**

Solution. Let $f(x)=[x]\sin\left(\dfrac{\pi}{[x+1]}\right)$

Domain of $f(x)$ is $x\in R$ excluding the points where $[x+1]=0$

($\because$ Denominator can't be zero.)

$\Rightarrow \qquad 0\le x+1<1 \quad\Rightarrow\quad -1\le x<0$

ie, for all $x\in[-1,0)$, denominator is zero. So, domain is $x\in R-[-1,0)$.

$\Rightarrow \qquad$ Domain is $x\in(-\infty,-1)\cup[0,\infty)$

Points of Discontinuity

As greatest integral function is discontinuous at integral points, $f(x)$ is continuous for all non-integer points.

Check the continuity at $x=a$ (where $a\in I$)

$$\text{LHL}=\lim_{h\to 0}[a-h]\sin\left(\frac{\pi}{[a+1-h]}\right)$$

$$\text{LHL}=(a-1)\sin\left(\frac{\pi}{a}\right) \qquad \text{...(i)}$$

$$\text{RHL} = \lim_{h \to 0} [a+h] \sin\left(\frac{\pi}{[a+1+h]}\right)$$

$$\Rightarrow \qquad \text{RHL} = a \sin\left(\frac{\pi}{a+1}\right) \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), LHL $\neq$ RHL

$\Rightarrow$ $f(x)$ is discontinuous at $x = a$. (*ie,* At integral values of x)

So, points of discontinuity are $x \in I \cap D$

(*ie,* Integer lying in the set of domain)

Illustration 19 If $f(x)$ be continuous function for all real values of x and satisfies; $x^2 + \{f(x) - 2\}x + 2\sqrt{3} - 3 - \sqrt{3} \cdot f(x) = 0,\ \forall\, x \in R,$ then find the value of $f(\sqrt{3})$.

Solution. As, $f(x)$ is continuous for all $x \in R$.

Thus, $\displaystyle\lim_{x \to \sqrt{3}} f(x) = f(\sqrt{3})$

where $\displaystyle f(x) = \frac{x^2 - 2x + 2\sqrt{3} - 3}{\sqrt{3} - x},\ x \neq \sqrt{3}$

$$\lim_{x \to \sqrt{3}} f(x) = \lim_{x \to \sqrt{3}} \frac{x^2 - 2x + 2\sqrt{3} - 3}{\sqrt{3} - x}$$

$$= \lim_{x \to \sqrt{3}} \frac{(2 - \sqrt{3} - x)(\sqrt{3} - x)}{(\sqrt{3} - x)} = 2(1 - \sqrt{3})$$

$$\therefore \qquad f(\sqrt{3}) = 2(1 - \sqrt{3})$$

Illustration 20 Let $f(x+y) = f(x) + f(y)$ for all x and y. If the function $f(x)$ is continuous at $x = 0$, show that $f(x)$ is continuous for all x.

Solution. As, the function is continuous at $x = 0$, we have

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

$$\Rightarrow \qquad \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(0+h) = f(0)$$

[Using $f(x+y) = f(x) + f(y)$]

$$\Rightarrow \qquad \lim_{h \to 0} \{f(0) + f(-h)\} = \lim_{h \to 0} \{f(0) + f(h)\} = f(0)$$

$$\Rightarrow \qquad \lim_{h \to 0} f(-h) = \lim_{h \to 0} f(h) = 0 \qquad \ldots\text{(i)}$$

Now, consider some arbitrary point $x = a$

Left Hand Limit

$$= \lim_{h \to 0} f(a-h) = \lim_{h \to 0} f(a) + f(-h) = f(a) + \lim_{h \to 0} f(-h)$$

[Using $f(x+y) = f(x) + f(y)$]

$$\text{LHL} = f(a) + 0 = f(a) \qquad \left[\text{As } \lim_{h \to 0} f(-h) = 0, \text{ using Eq. (i)}\right]$$

Right Hand Limit

$$= \lim_{h \to 0} f(a+h) = \lim_{h \to 0} f(a) + f(h) = f(a) + \lim_{h \to 0} f(h)$$

$$\text{RHL} = f(a) + 0 = f(a) \qquad \left[\text{As } \lim_{h \to 0} f(h) = 0, \text{ using Eq. (i)}\right]$$

at any arbitrary point $(x = a)$

$$\text{LHL} = \text{RHL} = f(a)$$

Therefore, function is continuous for all values of x, if it is continuous at 0.

Illustration 21 Let $f(x)$ be a continuous function defined for $1 \le x \le 3$. If $f(x)$ takes rational values for all x and $f(2) = 10$, then find the value of $f(1.5)$. **[IIT JEE 1997]**

Solution. As, $f(x)$ is continuous in [1, 3], $f(x)$ will attain all values between $f(1)$ and $f(3)$. As, $f(x)$ takes rational values for all x and there are innumerable irrational values between $f(1)$ and $f(3)$ which implies that $f(x)$ can take rational values for all x, if $f(x)$ has a constant rational value at all points between $x = 1$ and $x = 3$.

So, $$f(2) = f(1.5) = 10$$

Continuity of Composite Function

If the function $u = f(x)$ is continuous at the point $x = a$ and the function $y = g(u)$ is continuous at the point $u = f(a)$, then the composite function $y = (gof)(x) = g(f(x))$ is continuous at the point $x = a$

Illustration 22 Discuss the continuity for

$$f(x) = \frac{1-u^2}{2+u^2} \text{ where } u = \tan x.$$

Solution. Here, $u = \tan x$ is discontinuous at $n\pi \pm \frac{\pi}{2}$, $n \in I$

and $$f(x) = \frac{1-u^2}{2+u^2} \text{ is continuous at every } u \in R.$$

Hence, $f(x)$ is continuous on; $x \in R - \left\{n\pi \pm \frac{\pi}{2}, n \in I\right\}$

Also, $$\lim_{x \to n\pi \pm \frac{\pi}{2}} f(x) = \lim_{u^2 \to \infty} \frac{1-u^2}{2+u^2}$$

$$= \lim_{u \to \infty} \frac{\frac{1}{u^2} - 1}{\frac{2}{u^2} + 1} = -1$$

Hence, the points $n\pi \pm \frac{\pi}{2}$, $n \in I$ have removable discontinuity.

ie, If $f(x)$ is defined as

$$f(x) = \begin{cases} \dfrac{1-u^2}{2+u^2}, & x \ne n\pi \pm \dfrac{\pi}{2} \\ -1, & x = n\pi \pm \dfrac{\pi}{2} \text{ and } u = \tan x \end{cases}$$

Illustration 23 Find the points of discontinuity of $y = \dfrac{1}{u^2 + u - 2}$, where $u = \dfrac{1}{x-1}$.

Solution. The function $u = f(x) = \dfrac{1}{x-1}$ is discontinuous at the point $x = 1$. ...(i)

The function $y = g(x) = \dfrac{1}{u^2 + u - 2} = \dfrac{1}{(u+2)(u-1)}$ is discontinuous at $u = -2$ and $u = 1$.

when $u = -2,\ \dfrac{1}{x-1} = u = -2$

$\Rightarrow$ $x - 1 = -\dfrac{1}{2}$ $\Rightarrow$ $x = 1/2$ ...(ii)

when $u = 1,\ \dfrac{1}{x-1} = u = 1$

$\Rightarrow$ $x - 1 = 1$

$\Rightarrow$ $x = 2$ ...(iii)

Hence, the composite function $y = g(f(x))$ is discontinuous at three points $x = \dfrac{1}{2}$, 1 and 2.

A List of Continuous Functions

Function $f(x)$	Interval in which $f(x)$ is continuous
1. constant c	$(-\infty, \infty)$
2. x^n, n is an integer ≥ 0	$(-\infty, \infty)$
3. x^{-n}, n is a positive integer	$(-\infty, \infty) - \{0\}$
4. $\lvert x - a \rvert$	$(-\infty, \infty)$
5. $P(x) = a_0 x^n + a_1 x^{n-1} + \dots\dots + a_n$	$(-\infty, \infty)$
6. $\dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial in x	$(-\infty, \infty) - \{x : q(x) = 0\}$
7. $\sin x$	$(-\infty, \infty)$
8. $\cos x$	$(-\infty, \infty)$
9. $\tan x$	$(-\infty, \infty) - \left\{(2n+1)\dfrac{\pi}{2} : n \in I\right\}$
10. $\cot x$	$(-\infty, \infty) - \{n\pi : n \in I\}$
11. $\sec x$	$(-\infty, \infty) - \{(2n+1)\pi/2 : n \in I\}$
12. $\operatorname{cosec} x$	$(-\infty, \infty) - \{n\pi : n \in I\}$
13. e^x	$(-\infty, \infty)$
14. $\log_e x$	$(0, \infty)$

Illustration 24 Let $f(x)=\begin{cases}1+x\,, & 0\le x\le 2\\ 3-x\,, & 2<x\le 3\end{cases}$

Determine the form of $g(x)=f(f(x))$ and hence find the point of discontinuity of g, if any.

Solution. $g(x)=f(f(x))=\begin{cases}f\,(1+x)\,, & 0\le x\le 2\\ f\,(3-x)\,, & 2<x\le 3\end{cases}$

$$=\begin{cases}f\,(1+x), & 0\le x\le 1\\ f\,(1+x), & 1\le x\le 2\\ f\,(3-x), & 2<x\le 3\end{cases}$$

$$x\in[0,1]\ \Rightarrow\ (1+x)\in[1,2]$$
$$x\in[1,2]\ \Rightarrow\ (1+x)\in[2,3]$$
$$x\in[2,3]\ \Rightarrow\ (3-x)\in[0,1]$$

Hence, $$g\,(x)=\begin{cases}f(1+x), & \text{for } 0\le x\le 1\ \Rightarrow\ 1\le x+1\le 2\\ f(1+x), & \text{for } 1\le x\le 2\ \Rightarrow\ 2\le x+1\le 3\\ f(3-x), & \text{for } 2<x\le 3\ \Rightarrow\ 0\le 3-x\le 1\end{cases}$$

Now, if $(1+x)\in[1,2]$, then

$$f(1+x)=1+(1+x)=2+x \qquad \ldots\text{(i)}$$

[From original definition of $f\,(x)$]

Similarly, if $(1+x)\in(2\,,3)$, then

$$f(1+x)=3-(1+x)=2-x \qquad \ldots\text{(ii)}$$

If $$(3-x)\le(0,1)$$

$$f(3-x)=1+(3-x)=4-x \qquad \ldots\text{(iii)}$$

Using Eqs. (i), (ii) and (iii), we get

$$g\,(x)=\begin{cases}2+x, & 0\le x<1\\ 2-x, & 1<x\le 2\\ 4-x, & 2<x\le 3\end{cases}$$

Here, as $g(x)$ changes the inequality sign at $x=1$ and $x=2$.

Thus, to check continuity at $x=1$ and $x=2$.

Now, we will check the continuity of $g(x)$ at $x=1,2$

At $x=1$

$$\text{LHL}=\lim_{x\to 1^-} g(x)=\lim_{x\to 1^-}(2+x)=3$$
$$\text{RHL}=\lim_{x\to 1^+} g(x)=\lim_{x\to 1^+}(2-x)=1$$

As LHL $\ne$ RHL, $g(x)$ is discontinuous at $x=1$.

At $x=2$

$$\text{LHL}=\lim_{x\to 2^-} g(x)=\lim_{x\to 2^-}(2-x)=0$$
$$\text{RHL}=\lim_{x\to 2^+} g(x)=\lim_{x\to 2^+}(4-x)=2$$

As LHL $\ne$ RHL, $g(x)$ is discontinuous at $x=2$.

Thus, $g\,(x)$ is continuous for all $x\in[0,1)\cup(1,2)$.

Intermediate Value Theorem

If $f(x)$ is continuous on $[a, b]$ and $f(a) \neq f(b)$, then for any value $c \in (f(a), f(b))$, there is at least one number x_0 in (a, b) for which $f(x_0) = c$.

(i)

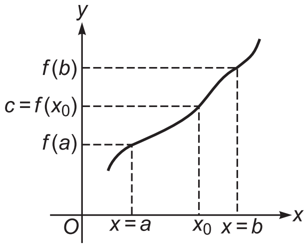


Fig. 6.14

(ii)

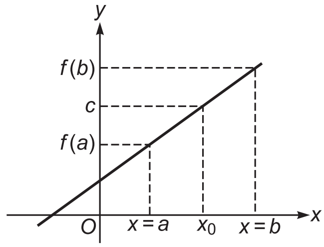


Fig. 6.15

(iii)

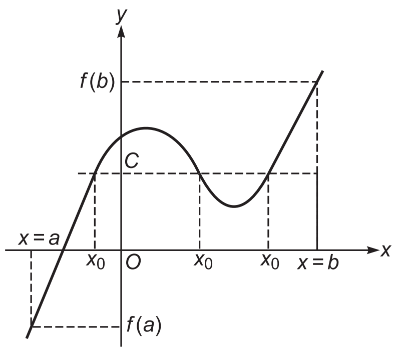


Fig. 6.16

Point to Consider

Continuity through the interval $[a, b]$ is essential for the validity of this theorem.

Illustration 25 Show that the function $f(x) = (x - a)^2 (x - b)^2 + x$ takes the value $\frac{a+b}{2}$ for some value of $x \in [a, b]$.

Solution. Here, $f(a) = a$ and $f(b) = b$

Also, $f(x)$ is continuous in $[a, b]$ and $\frac{a+b}{2} \in [a, b]$.

Therefore, using Intermediate value theorem, there exists some $c \in [a, b]$ such that

$$f(c) = \frac{a+b}{2}$$

Illustration 26 Suppose that $f(x)$ is continuous in $[0, 1]$ and $f(0) = 0, f(1) = 0$. Prove $f(c) = 1 - 2c^2$ for some $c \in (0, 1)$.

Solution. Let $g(x) = f(x) + 2x^2 - 1$ in $[0, 1]$

$$g(0) = 1, g(1) = 1$$

$\therefore$ By Intermediate value theorem, there exists some $c \in (0, 1)$; $g(c) = 0$

$$\Rightarrow \quad f(c) = 1 - 2c^2$$

Point to Consider

Function continuous only at one point and defined everywhere.

(Single Point Continuity)

eg, **(i)** $f(x)=\begin{cases} x, & \text{if } x\in Q \\ 0, & \text{if } x\notin Q \end{cases}$, is continuous only at $x=0$.

(ii) $f(x)=\begin{cases} x, & \text{if } x\in Q \\ -x, & \text{if } x\notin Q \end{cases}$, is continuous only at $x=0$ and defined everywhere.

(iii) $f(x)=\begin{cases} x, & \text{if } x\in Q \\ 1-x, & \text{if } x\notin Q \end{cases}$, is continuous only at $x=\frac{1}{2}$ and defined everywhere.

(iv) $f(x)=\begin{cases} x^2, & \text{if } x\in Q \\ 1, & \text{if } x\notin Q \end{cases}$, is continuous only at $x=1$ or $x=-1$ and defined everywhere.

Target Exercise 6.2

1. Let $f: R\to R$ be any function. Define $g: R\to R$ by $g(x)=|f(x)|$ for $x\in R$. Then, g is

(a) onto, if f is onto (b) one-one, if f is one-one

(c) continuous, if f is continuous (d) None of these

2. Let $f(x)=$ Degree of $(u^{x^2}+u^2+2u+3)$. Then, at $x=\sqrt{2}$, $f(x)$ is

(a) continuous (b) $\lim\limits_{x\to\sqrt{2}} f(x)=-2$ (c) discontinuous (d) None of these

3. Let $f(x)=[\sin x+\cos x]$, $0<x<2\pi$, (where $[\cdot]$ denotes the greatest integer function). Then the number of points of discontinuity of $f(x)$ is

(a) 6 (b) 5 (c) 4 (d) 3

4. If $f(x)=\begin{cases} \dfrac{\sin\{\cos x\}}{x-\pi/2}, & x\neq\dfrac{\pi}{2} \\ 1, & x=\dfrac{\pi}{2} \end{cases}$, where $\{\cdot\}$ denotes the fractional part of x, then $f(x)$ is

(a) continuous at $x=\dfrac{\pi}{2}$ (b) $\lim\limits_{x\to\frac{\pi}{2}} f(x)$ exists, but $f(x)$ is not continuous at $x=\dfrac{\pi}{2}$

(c) $\lim\limits_{x\to\frac{\pi}{2}}$ doesn't exist (d) $\lim\limits_{x\to\frac{\pi}{2}^-} f(x)=1$

5. If α,β $(\alpha<\beta)$ are the points of discontinuity of the function $f(f(f(x)))$, where $f(x)=\dfrac{1}{1-x}$, then the set of values of 'a' for which the points (α,β) and (a,a^2) lie on the same side of the line $x+2y-3=0$, is

(a) $\left(-\dfrac{3}{2},1\right)$ (b) $\left[-\dfrac{3}{2},1\right]$ (c) $[1,\infty)$ (d) $\left(-\infty,-\dfrac{3}{2}\right]$

Introduction to Differentiability of a Function at a Point

We have seen in Continuity that, if a function is continuous at a point $x = a$ (say) then, its graph is an unbroken curve at $x = a$ and there are no holes and jumps in the graph of the function in the neighbourhood of point $x = a$.

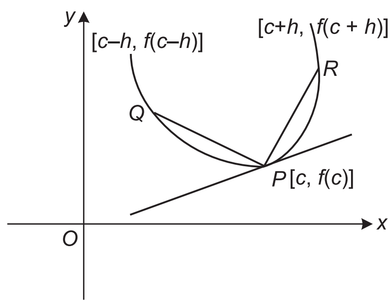


Fig. 6.17

Now a question arises : What do we mean when we say that a function $f(x)$ is differentiable at a point $x = c$?

In the following discussion we shall try to answer this question.

Consider the function $f(x)$, defined on an open interval (a, b), let $P\,(c, f(c))$ be a point on the curve $y = f(x)$ and let $Q\,(c - h, f(c - h))$ and $R(c + h, f(c + h))$ be two neighbouring points on the left hand side and right hand side respectively, of point P as shown in figure.

Then, slope of chord $PQ = \dfrac{f(c-h) - f(c)}{(c-h) - (c)}$

$$= \frac{f(c-h) - f(c)}{-h}$$

and slope of chord $PR = \dfrac{f(c+h) - f(c)}{(c+h) - (c)} = \dfrac{f(c+h) - f(c)}{h}$

We know that the tangent to a curve at a point P (say) is the limiting position of chord PQ when $Q \to P$. Therefore, as $h \to 0$, points Q and R both tend to P from both left hand and right hand sides respectively.

Consequently, chord PQ and PR become tangents at point P.

Thus, $\displaystyle\lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} = \lim_{h \to 0}$ (Slope of chord PQ)

$\displaystyle = \lim_{Q \to P}$ (Slope of chord PQ)

$\Rightarrow$ Slope of the tangent at point P, which is limiting position of the chords drawn on the left hand side of point P.

and $\displaystyle\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0}$ (Slope of chord PR)

$\displaystyle = \lim_{R \to P}$ (Slope of chord PR)

$\Rightarrow$ Slope of the tangent at point P, which is limiting position of the chords drawn on the left hand side of point P.

And $\displaystyle\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0}$ (Slope of chord PR)

$\displaystyle = \lim_{R \to P}$ (Slope of chord PR)

$\Rightarrow$ Slope of the tangent at point P, which is the limiting position of the chords drawn on the right hand side of point P.

Now, $f(x)$ is differentiable at $x = c$

$$\Leftrightarrow \quad \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

$\Leftrightarrow$ There is a unique tangent at point P.

Thus, $f(x)$ is differentiable at point P, if there exists a unique tangent at point P. In other words, $f(x)$ is differentiable at a point P, if the curve does not have P as a corner point.

"*ie*, the function is not differentiable at those points on which function has jumps (or holes) and sharp edges."

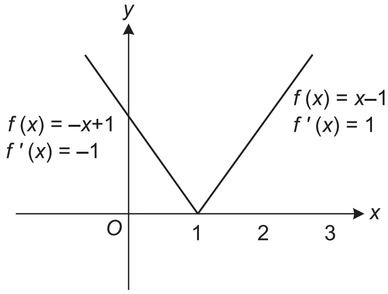


Fig. 6.18

Let us consider the function $f(x) = |x - 1|$ which can be graphically shown.

Which shows, $f(x)$ is not differentiable at $x = 1$. Since, $f(x)$ has sharp edge at $x = 1$.

Mathematically The right hand derivative at $x = 1$ is 1 and left hand derivative at $x = 1$ is -1. Thus, $f(x)$ is not differentiable at $x = 1$.

Differentiability at a Point

Let $f(x)$ be a real valued function defined on an open interval (a, b) where $c \in (a, b)$. Then, $f(x)$ is said to be differentiable or derivable at $x = c$,

$$\text{if} \quad \lim_{x \to c} \frac{f(x) - f(c)}{(x - c)} \text{ exists finitely.}$$

This limit is called the derivative or differential coefficient of the function $f(x)$ at $x = c$ and is denoted by $f'(c)$ or $Df(c)$ or $\frac{d}{dx}(f(x))_{x=c}$.

Thus, $f(x)$ is differentiable at $x = c$

$$\Leftrightarrow \quad \lim_{x \to c} \frac{f(x) - f(c)}{(x - c)} \text{ exists finitely}$$

$$\Leftrightarrow \quad \lim_{x \to c^-} \frac{f(x) - f(c)}{(x - c)} = \lim_{x \to c^+} \frac{f(x) - f(c)}{(x - c)}$$

$$\Leftrightarrow \quad \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

Here, $\lim_{x \to c^-} \dfrac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \dfrac{f(c-h) - f(c)}{-h}$ is called the left hand derivative of $f(x)$ at $x = c$ and is denoted by $f'(c^-)$ or $Lf'(c)$,

while, $\lim_{x \to c^+} \dfrac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$ is called the right hand derivative of $f(x)$ at $x = c$ and is denoted by $f'(c^+)$ or $Rf'(c)$.

Thus, $f(x)$ is differentiable at $x = c$.

$$\Leftrightarrow \qquad Lf'(c) = Rf'(c)$$

If $Lf'(c) \neq Rf'(c)$, we say that $f(x)$ is not differentiable at $x = c$.

Illustration 27 The left hand derivative of $f(x) = [x]\sin(\pi x)$ at $x = k$, k is an integer, is

(a) $(-1)^k (k-1)\pi$ (b) $(-1)^{k-1} (k-1)\pi$

(c) $(-1)^k k\pi$ (d) $(-1)^{k-1} k\pi$

Solution. $f(x) = [x]\sin(\pi x)$

If x is just less than k, $[x] = k - 1$

$\therefore \qquad f(x) = (k-1)\sin(\pi x), \qquad$ when $x < k, \quad \forall\, k \in I$

Now, LHD at $x = k$,

$$= \lim_{x \to k} \frac{(k-1)\sin(\pi x) - k\sin(\pi k)}{x - k}$$

$$= \lim_{x \to k} \frac{(k-1)\sin(\pi x)}{(x - k)} \qquad [\text{As } \sin(\pi k) = 0,\ k \in \text{integer}]$$

$$= \lim_{h \to 0} \frac{(k-1)\sin(\pi(k-h))}{-h} \qquad [\text{Let } x = (k-h)]$$

$$= \lim_{h \to 0} \frac{(k-1)(-1)^{k-1}\sin(h\pi)}{-h}$$

$$= \lim_{h \to 0} (k-1)(-1)^{k-1} \frac{\sin h\pi}{h\pi} \times (-\pi)$$

$$= (k-1)(-1)^k \pi$$

Therefore, (a) is the correct answer.

Illustration 28 Which of the following functions is differentiable at $x = 0$?

[IIT JEE 2001]

(a) $\cos(|x|) + |x|$ (b) $\cos(|x|) - |x|$

(c) $\sin(|x|) + |x|$ (d) $\sin(|x|) - |x|$

Solution. Here, RHD of $\sin(|x|) - |x| = \lim_{h \to 0} \dfrac{(\sin|h| - |h|) - (0 - 0)}{h}$

$$= \lim_{h \to 0} \frac{\sin h}{h} - \frac{h}{h} = 1 - 1 = 0$$

Also, LHD of $\sin(|x|) - |x| = \lim_{h \to 0} \dfrac{(\sin|h| - |h|) - (0 - 0)}{-h}$

$$= \lim_{h \to 0} \frac{\sin h}{-h} + \frac{h}{h} = -1 + 1 = 0$$

$$f(0) = \sin(|0|) - |0| = 0$$

Thus, $\sin(|x|) - |x|$ is differentiable at $x = 0$.

Therefore, (d) is the correct answer. Whereas, (a), (b) and (c) do not satisfy definition of differentiability.

Illustration 29 Let $f(x) = |x-1| + |x+1|$. Discuss the continuity and differentiability of the function.

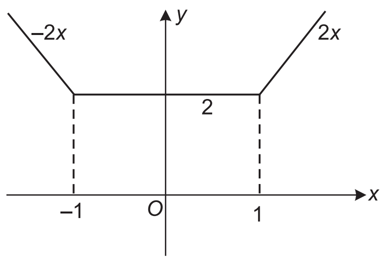
Solution. Here, $f(x) = |x-1| + |x+1|$

$$\Rightarrow \qquad f(x) = \begin{cases} (x-1) + (x+1), & \text{when} \quad x > 1 \\ -(x-1) + (x+1), & \text{when} \quad -1 \le x \le 1 \\ -(x-1) - (x+1), & \text{when} \quad x < -1 \end{cases}$$

$$\Rightarrow \qquad f(x) = \begin{cases} 2x, & \text{when} \quad x > 1 \\ 2, & \text{when} \quad -1 \le x \le 1 \\ -2x, & \text{when} \quad x < -1 \end{cases}$$

Graphically The graph of the function is shown alongside.

From the graph, it is clear that the function is continuous at all real x, also differentiable at all real x except at $x = \pm 1$. Since, it has sharp edges at $x = -1$ and $x = 1$



At $x = 1$ we see that the slope from the right *ie,* RHD $= 2$, while slope from the left *ie,* LHD $= 0$.

Similarly, at $x = -1$ it is clear that RHD = 0, while LHD $= -2$

Aliter : In this method, first of all we differentiate the function and from the derivative equality sign should be removed from doubtful points.

Here, $$f'(x) = \begin{cases} -2, & x < -1 \\ 0, & -1 < x < 1 \\ 2, & x > 1 \end{cases} \qquad \text{(No equality on –1 and +1)}$$

Now, at $x = 1$, $f'(1^+) = 2$, while $f'(1^-) = 0$ and

at $x = -1$, $f'(-1^+) = 0$, while $f'(-1^-) = -2$

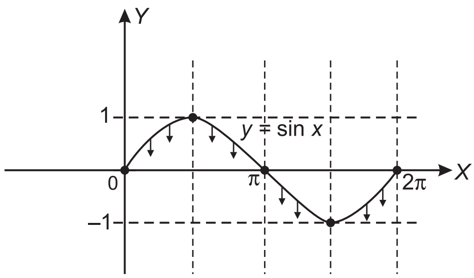
Thus, $f(x)$ is not differentiable at $x = \pm 1$.

Point to Consider

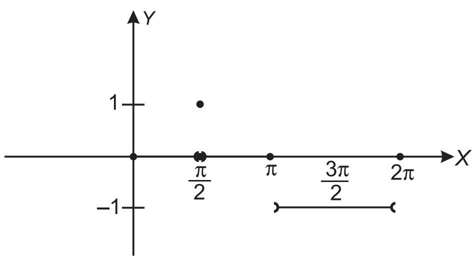
This method is not applicable when function is discontinuous.

Illustration 30 Discuss the continuity and differentiability for $f(x) = [\sin x]$ when $x \in [0, 2\pi]$; where $[\cdot]$ denotes the greatest integer function x.

Solution. In last chapter we have discussed the plotting of curves, so $y = [\sin x]$ could be plotted as



Or



which shows $f(x) = [\sin x]$ is discontinuous for $x = \frac{\pi}{2}, \pi, 2\pi$ when $x \in [0, 2\pi]$ and $f(x)$ is not differentiable at $x = \frac{\pi}{2}, \pi, 2\pi$.

As we know, function is neither differentiable nor continuous at those points for which graph is broken.

Illustration 31 Let $f(x) = \min.\{\tan x, \cot x\}\ \forall\ x \in R$. Find

(i) range of $f(x)$

(ii) period (if periodic)

(iii) points of discontinuity of $f(x)$

(iv) points of non-differentiability of $f(x)$

Solution. We know, $f(x) = \min.\{\tan x, \cot x\}$ can be plotted in two steps.

(i) We should plot the graph of $\tan x$ and $\cot x$

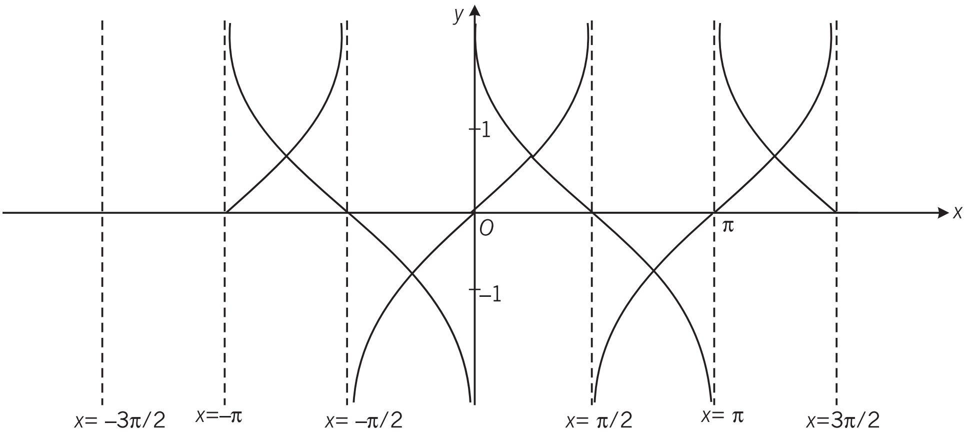
(ii) We should find their point of intersection and neglect the area above their point of intersection.

Here, white lines represent $f(x)$.

It can be seen from the graph that,

(a) range of $f(x) = (-\infty, -1] \cup [0, 1]$

(b) period of $f(x) = \pi$



(c) points of discontinuity are $\pm\pi, \pm\pi/2, 0, \ldots$ which can be put in the form of $n\pi/2,\ n \in I$.

(d) also the points of non-differentiability are $\pm\pi, 3\pi/4, \pm\pi/2,\ 0, \ldots$ which can be put in the form of $n\pi/4,\ n \in I$.

Illustration 32 If $f(x) = \{|x| - |x-1|\}^2$, draw the graph of $f(x)$ and discuss its continuity and differentiability of $f(x)$.

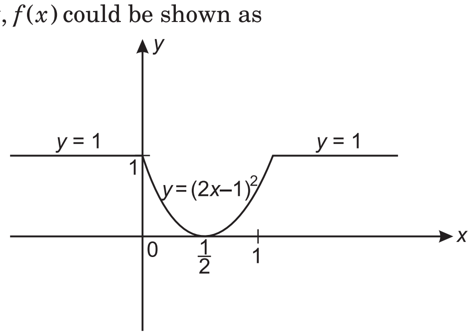
Solution. We know, $|x| - |x-1| = \begin{cases} -x + x - 1, & x < 0 \\ x + x - 1, & 0 \le x < 1 \\ x - (x+1), & 1 \le x \end{cases}$

$$\Rightarrow \qquad |x| - |x-1| = \begin{cases} -1, & x < 0 \\ 2x - 1, & 0 \le x < 1 \\ 1, & 1 \le x \end{cases}$$

$$\therefore \qquad f(x) = \{|x| - |x-1|\}^2$$

$$= \begin{cases} 1, & \text{when } 0 < x \text{ or } x \ge 1 \\ (2x-1)^2, & \text{when } 0 \le x < 1 \end{cases}$$

Graphically, $f(x)$ could be shown as



From given figure, it is clear that $f(x)$ is continuous for $x \in R$; but $f(x)$ is not differentiable at $x = 0, 1$.

$\Rightarrow$ $f(x)$ is continuous for all $x \in R$

$\Rightarrow$ $f(x)$ is differentiable for all $x \in R - \{0, 1\}$.

Illustration 33 If $f(x) = \begin{cases} x - 3, & x < 0 \\ x^2 - 3x + 2, & x \geq 0 \end{cases}$ and let $g(x) = f(|x|) + |f(x)|$.

Discuss the differentiability of $g(x)$.

Solution. $f(|x|) = \begin{cases} |x| - 3, & |x| < 0 \\ |x|^2 - 3|x| + 2, & |x| \geq 0 \end{cases}$

Where, $|x| < 0$ is not possible, thus neglecting, we get

$$f(|x|) = \{ |x|^2 - 3|x| + 2, \quad |x| \geq 0$$

$$f(|x|) = \begin{cases} x^2 + 3x + 2, & x < 0 \\ x^2 - 3x + 2, & x \geq 0 \end{cases} \quad \ldots\text{(i)}$$

Again,
$$|f(x)| = \begin{cases} |x - 3|, & x < 0 \\ |x^2 - 3x + 2|, & x \geq 0 \end{cases}$$

$$|f(x)| = \begin{cases} (3 - x), & x < 0 \\ (x^2 - 3x + 2), & 0 \leq x < 1 \\ -(x^2 - 3x + 2), & 1 \leq x < 2 \\ (x^2 - 3x + 2), & 2 \leq x \end{cases} \quad \ldots\text{(ii)}$$

Now, from Eqs. (i) and (ii), $g(x) = f(|x|) + |f(x)|$

$$g(x) = \begin{cases} x^2 + 2x + 5, & x < 0 \\ 2x^2 - 6x + 4, & 0 \leq x < 1 \\ 0, & 1 \leq x < 2 \\ 2x^2 - 6x + 4, & x \geq 2 \end{cases}$$

and
$$g'(x) = \begin{cases} 2x + 2, & x < 0 \\ 4x - 6, & 0 < x < 1 \\ 0, & 1 < x < 2 \\ 4x - 6, & x > 2 \end{cases}$$

Therefore, $g(x)$ is continuous in $R - \{0\}$ and $g(x)$ is differentiable in $R - \{0, 1, 2\}$.

Illustration 34 Let $f(x) = [n + p \sin x]$, $x \in (0, \pi)$, $n \in Z$ and p is a prime number, where $[\cdot]$ denotes the greatest integer function. Then, find the number of points where $f(x)$ is not differentiable.

Solution. Here, $f(x) = [n + p \sin x]$ is not differentiable at those points where $n + p \sin x$ is an integer.

As, p is a prime number.

$\Rightarrow$ $n + p \sin x$ is an integer of $\sin x = 1, -1, r/p$

ie, $x = \frac{\pi}{2}, -\frac{\pi}{2}, \sin^{-1}\frac{r}{p}, \pi - \sin^{-1}\frac{r}{p}$

where $0 \le r \le p - 1$

But $x \ne -\frac{\pi}{2}, 0$

$\therefore$ Function is not differentiable at $x = \frac{\pi}{2}, \sin^{-1}\frac{r}{p}, \pi - \sin^{-1}\frac{r}{p}$

where $0 < r \le p - 1$

So, the required number of points are

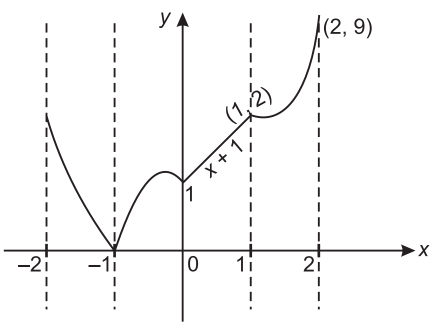
$$= 1 + 2(p-1) = 2p - 1.$$

Illustration 35 If $f(x) = |x+1|\{|x| + |x-1|\}$, then draw the graph of $f(x)$ in the interval $[-2, 2]$ and discuss the continuity and differentiability in $[-2, 2]$.

Solution. Here, $f(x) = |x+1|\{|x| + |x-1|\}$

$$f(x) = \begin{cases} (x+1)(2x-1), & -2 \le x < -1 \\ -(x+1)(2x-1), & -1 \le x < 0 \\ (x+1), & 0 \le x < 1 \\ (x+1)(2x-1), & 1 \le x \le 2 \end{cases}$$

Thus, the graph of $f(x)$ is;



Clearly, continuous for $x \in R$ and has differentiability for

$$x \in R - \{-1, 0, 1\}$$

Illustration 36 Test the continuity and differentiability of the function $f(x) = \left|\left(x + \frac{1}{2}\right)[x]\right|$, by drawing the graph of the function when $-2 \le x \le 2$.

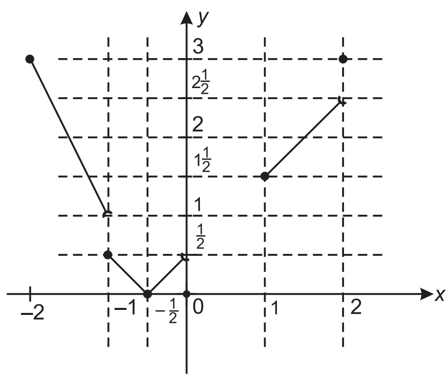
Solution. Here, $f(x) = \left|\left(x + \frac{1}{2}\right)[x]\right|, -2 \le x \le 2$

and we know, $[x] = \begin{cases} -2, & -2 \le x < -1 \\ -1, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$

$$\therefore \qquad f(x) = \begin{cases} \left|\left(x + \dfrac{1}{2}\right)(-2)\right|, & -2 \le x < -1 \\ \left|\left(x + \dfrac{1}{2}\right)(-1)\right|, & -1 \le x < 0 \\ \left|\left(x + \dfrac{1}{2}\right)(-0)\right|, & 0 \le x < 1 \\ \left|\left(x + \dfrac{1}{2}\right)(1)\right|, & 1 \le x < 2 \\ \left|\dfrac{3}{2} \cdot 2\right|, & x = 2 \end{cases}$$

$$\therefore \qquad f(x) = \begin{cases} -(2x+1), & -2 \le x < -1 \\ -\left(x + \dfrac{1}{2}\right), & -1 \le x < -1/2 \\ (x + 1/2), & -\dfrac{1}{2} \le x < 0 \\ 0, & 0 \le x < 1 \\ x + \dfrac{1}{2}, & 1 \le x < 2 \\ 3, & x = 2 \end{cases}$$

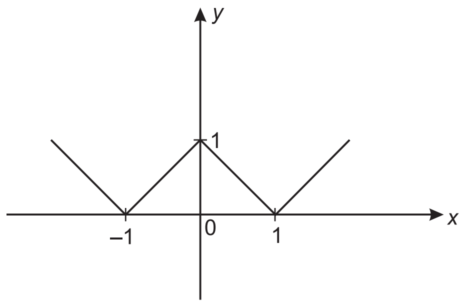
which could be plotted as;



which clearly shows $f(x)$ is not continuous at $x = \{-1, 0, 1, 2\}$ as at these points graph is broken and $f(x)$ is not differentiable at $x = \left\{-1, -\dfrac{1}{2}, 0, 1, 2\right\}$ as at {–1, 0, 1, 2} graph is broken and at $x = -1/2$ we have sharp edge.

Illustration 37 If $f(x) = ||x| - 1|$, then draw the graph of $f(x)$ and $fof(x)$ and also discuss their continuity and differentiability. Also, find derivative of $(fof)^2$ at $x = \dfrac{3}{2}\cdot$

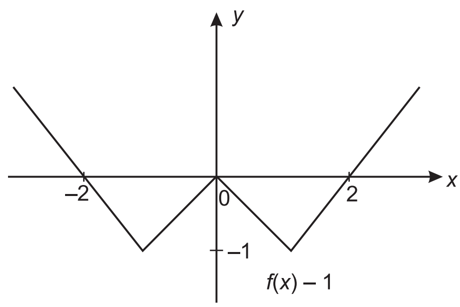
Solution. The graph of $f(x)$ is shown as;



It is clear from the graph that, $f(x)$ is continuous for all x, but $f(x)$ is not differentiable at $x = \{-1, 0, 1\}$.

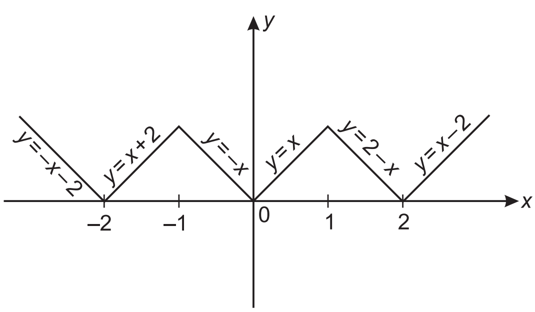
Now, $fof(x) = ||f(x)| - 1| = |f(x) - 1|$ [As $f(x) \geq 0$ for all x]

Now, if $f(x) \to f(x) - 1$, *ie*, shift the graph one unit below the x-axis *ie*, as shown;



Thus, for graph of $fof(x) = |f(x) - 1|$ is taking image of the graph of $f(x) - 1$ below x-axis and leaving the portion above y-axis as it is.

$\therefore$ Graph for $fof(x)$ is shown as;



which is clearly continuous for all $x \in R$, but not differentiable at $x = \{-2, -1, 0, 1, 2\}$.

Which shows,

$$fof(x) = 2 - x, 1 \le x \le 2$$

$\therefore$ $$(fof)^2 = (2 - x)^2, \text{ when } 1 \le x \le 2$$

$\Rightarrow$ $$\frac{d}{dx}(fof)^2 = 2(2 - x)(-1), \text{ when } 1 \le x \le 2$$

$\therefore$ $$\frac{d}{dx}(fof)^2 \text{ (when } x = 3/2)$$

$$= -2(2 - 3/2) = -1$$

$\Rightarrow$ $$\frac{d}{dx}\{(fof)^2\}_{\text{at } x = 3/2} = -1$$

Illustration 38 Draw the graph of the function

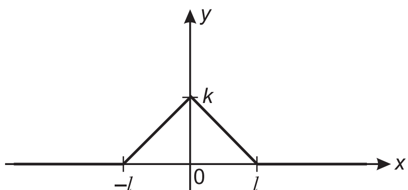
$$g(x) = f(x + l) + f(x - l), \text{ where}$$

$$f(x) = \begin{cases} k\left\{1 - \dfrac{|x|}{l}\right\}, & \text{for } |x| \le l \\ 0, & \text{for } |x| > l \end{cases}$$

Also, discuss the continuity and differentiability of the function $g(x)$.

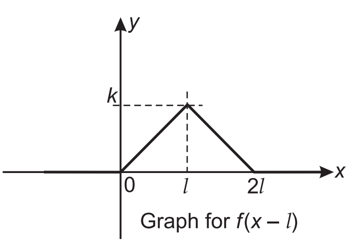
Solution. We have, $$f(x) = \begin{cases} k\left(1 - \dfrac{|x|}{l}\right), & |x| \le l \\ 0, & |x| > l \end{cases}$$

Shown as;

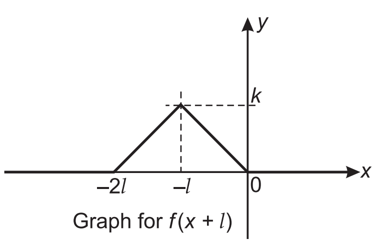


Graph for $f(x)$

Now, to plot $g(x) = f(x + l) + f(x - l)$, we shall first plot $f(x + l)$ and $f(x - l)$. Shown as;

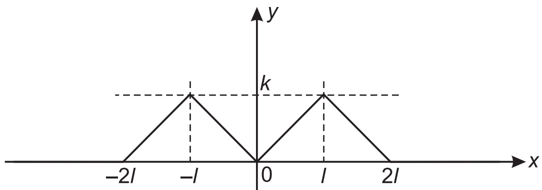


Graph for $f(x - l)$



Graph for $f(x + l)$

Thus, the graph of $g(x) = f(x + l) + f(x - l)$ is obtained by adding $f(x + l)$ and $f(x - l)$ as;



Graph for $g(x) = f(x + l) + f(x - l)$

From the above figure,

$g(x)$ is continuous for all $x \in R$.

And $\quad g(x)$ is differentiable for all $x \in R - \{\pm 2l, \pm l, 0\}$

Illustration 39 Let $f(x) = \begin{cases} \int_0^x \{5 + |1 - t|\}\, dt, & \text{if } x > 2 \\ 5x + 1, & \text{if } x \le 2 \end{cases}$.

Test $f(x)$ for continuity and differentiability for all real x.

Solution. Here, $f(x)$ for $x > 2$.

$$f(x) = \int_0^x \{5 + |1 - t|\}\, dt$$

$$= \int_0^1 (5 + \ 1 - t)\, dt + \int_1^x (5 + \ t - 1)\, dt \qquad \text{(Since } x > 2\text{)}$$

$$= \left(6t - \frac{t^2}{2}\right)_0^1 + \left(4t + \frac{t^2}{2}\right)_1^x.$$

$$= 6 - \frac{1}{2} + 4x + \frac{x^2}{2} - 4 - \frac{1}{2} = 1 + 4x + \frac{x^2}{2}$$

$$\therefore \qquad f(x) = \begin{cases} 1 + 4x + x^2/2, & \text{if } x > 2 \\ 5x + 1, & \text{if } x \le 2 \end{cases}$$

$\therefore$ RHL (at $x = 2$)

$$\lim_{h \to 0} \left\{1 + 4(2 + h) + \frac{(2 + h)^2}{2}\right\} = 1 + 8 + 2 = 11$$

and LHL (at $x = 2$)

$$\lim_{h \to 0} \{5(2 - h) + 1\} = 10 + 1 = 11$$

Thus, $f(x)$ is continuous at $x = 2$. (As $f(2) = 11$)

and RHD (at $x = 2$)

$$Rf'(2) = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{1 + 4(2 + h) + \frac{(2 + h)^2}{2} - 11}{h}$$

$$= \lim_{h \to 0} \frac{11 + 6h + \frac{h^2}{2} - 11}{h} = 6$$

LHD (at $x = 2$)

$$Lf'(2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h \to 0} \frac{5(2-h) + 1 - 11}{-h} = \lim_{h \to 0} \frac{11 - 5h - 11}{-h} = 5$$

$\therefore$ $f(x)$ is not differentiable at $x = 2$.

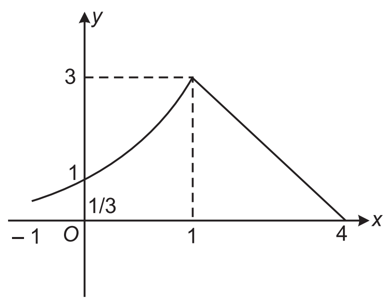
Illustration 40 Draw the graph of the function and discuss the continuity and differentiability at $x = 1$ for,

$$f(x) = \begin{cases} 3^x, & \text{when } -1 \le x \le 1 \\ 4 - x, & \text{when } \quad 1 < x < 4 \end{cases}$$

Solution. Here,

$$f(x) = \begin{cases} 3^x, & \text{when } -1 \le x \le 1 \\ 4 - x, & \text{when } \quad 1 < x < 4 \end{cases}$$

is shown graphically alongside.



From the graph, it is clear that it is continuous for all x in $[-1, 4)$ and not differentiable at $x = 1$.

Because at $x = 1$, LHD > 0,

while RHD < 0

Mathematically,

$$\text{RHD} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{4 - (1+h) - 3}{h} = -1$$

$$\text{LHD} = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{3^{(1-h)} - 3}{-h}$$

$$= \lim_{h \to 0} 3 \left(\frac{3^{-h} - 1}{-h} \right) = 3 \log_e 3$$

Since, LHD $\ne$ RHD, $f(x)$ is not differentiable at $x = 1$.

But $f(x)$ is continuous at $x = 1$, because the derivatives from both the sides are finite and definite.

Aliter **:** The given function is continuous.

Hence, $$f'(x) = \begin{cases} 3^x \log 3, & -1 \le x < 1 \\ -1, & \quad 1 < x < 4 \end{cases}$$

Here, $$f'(1^+) = -1$$

And $$f'(1^-) = \lim_{x \to 1^-} 3^x \log 3 = 3 \log_e 3$$

which shows, $$f'(1^+) \ne f'(1^-)$$

Therefore, $f(x)$ is not differentiable at $x = 1$.

Illustration 41 Match the column I with column II.

Column I	Column II
(i) $\sin(\pi[x])$	(A) differentiable everywhere
(ii) $\sin\{(x-[x])\pi\}$	(B) no where differentiable
	(C) not differentiable at –1 and +1

where $[\cdot]$ denotes greatest integral function. **[IIT JEE 1992]**

Solution. (i) We know, $[x] \in I, \forall\ x \in R$

$\therefore$ $$\sin(\pi[x]) = \sin(I\pi) = 0, \forall\ x \in R$$

By theory, we know that every constant function is differentiable in its domain.

Thus, $\sin(\pi[x])$ is differentiable everywhere.

$\Rightarrow$ (i) $\leftrightarrow$ (A)

(ii) Again, $$f(x) = \sin\{(x-[x])\pi\}$$

Here we know, $$x-[x] = \{x\},$$

then $$\pi(x-[x]) = \pi\{x\}$$

which is not differentiable at integral points.

$\therefore$ $f(x) = \sin\{\pi(x-[x])\}$ is not differentiable at $x \in I$. (integers)

Hence, (ii) $\leftrightarrow$ (C)

Illustration 42 Fill in the blank, in the statement given below let $f(x) = x|x|$. The set of points where $f(x)$ is twice differentiable is ………… .

[IIT JEE 1992]

Solution. The function $f(x) = x|x|$ can be written as,

$$f(x) = \begin{cases} x(x), & \text{if } x \ge 0 \\ x(-x), & \text{if } x < 0 \end{cases}$$

$$\Rightarrow \qquad f(x) = \begin{cases} x^2, & \text{if } x \ge 0 \\ -(x)^2, & \text{if } x < 0 \end{cases}$$

$f(x)$ is not differentiable at $x = 0$ *i.e.*, all $R - \{0\}$

Again, $$f'(x) = \begin{cases} 2x, & \text{if } x > 0 \\ -2x, & \text{if } x < 0 \end{cases}$$

$$f''(x) = \begin{cases} 2, & \text{if } x > 0 \\ -2, & \text{if } x < 0 \end{cases}$$

Hence, $f(x)$ is twice differentiable, $\forall\ x \in R - \{0\}$

Illustration 43 The function $f(x) = (x^2-1)|x^2-3x+2| + \cos(|x|)$ is not differentiable at **[IIT JEE 1992]**

(a) –1 (b) 0 (c) 1 (d) 2

Solution. The function $f(x) = (x^2-1)|x^2-3x+2| + \cos(|x|)$ …(i)

Here, $|x|$ is not differentiable at $x = 0$ but $\cos(|x|) = \begin{cases} \cos(-x), & x < 0 \\ \cos(x), & x \ge 0 \end{cases}$

$$\Rightarrow \qquad \cos(|x|) = \begin{cases} \cos(x), & x \geq 0 \\ \cos(x), & x < 0 \end{cases}$$

$\therefore$ $\cos(|x|)$ is differentiable at $x = 0$ ...(ii)

Again, $|x^2 - 3x + 2| = |(x-1)(x-2)|$

$$= \begin{cases} (x-1)(x-2), & \text{if } \quad x < 1 \\ -(x-1)(x-2), & \text{if } 1 \leq x < 2 \\ (x-1)(x-2), & \text{if } \quad x \geq 2 \end{cases} \qquad \text{...(iii)}$$

So,
$$f(x) = \begin{cases} (x^2-1)(x-1)(x-2) + \cos x, & \text{if } -\infty < x < 1 \\ -(x^2-1)(x-1)(x-2) + \cos x, & \text{if } \quad 1 \leq x < 2 \\ (x^2-1)(x-1)(x-2) + \cos x, & \text{if } \quad 2 \leq x < \infty \end{cases}$$

Now, to check differentiability at $x = 1, 2$ (Using shortcut method)

$$f'(x) = \begin{cases} (x^2-1)(2x-3) + (2x)(x^2-3x+2) - \sin x, & -\infty < x < 1 \\ -(x^2-1)(2x-3) - (2x)(x^2-3x+2) - \sin x, & 1 \leq x < 2 \\ (x^2-1)(2x-3) + (2x)(x^2-3x+2) - \sin x, & 2 \leq x < \infty \end{cases}$$

Thus, for $f'(1)$ we have

$$f'(1) = \begin{cases} -\sin 1, & x < 1 \\ -\sin 1, & x > 1 \end{cases}$$

Thus, $f(x)$ is differentiable at $x = 1$

Also, $$f'(2) = \begin{cases} -3 - \sin 2, & x < 2 \\ 3 - \sin 2, & x > 2 \end{cases}$$

Thus, $f(x)$ is not differentiable at $x = 2$

Hence, (d) is the correct answer.

Illustration 44 If $f(x) = \sum_{r=1}^{n} a_r |x|^r$, where a_i' s are real constants, then $f(x)$ is

(a) continuous at $x = 0$, for all a_i
(b) differentiable at $x = 0$, for all $a_i \in R$
(c) differentiable at $x = 0$, for all $a_{2k+1} = 0$
(d) None of the above

Solution. We know that, $|x|^r$, $r = 0, 1, 2, \ldots$ are all continuous everywhere.

$\therefore$ $f(x) = \sum_{r=1}^{n} a_r |x|^r$ is everywhere continuous.

Since, $|x|, |x|^3, |x|^5, \ldots$ are not differentiable at $x = 0$,
whereas $|x|^2, |x|^4, \ldots$ are everywhere differentiable.

$\therefore$ $f(x) = \sum_{r=1}^{n} a_r |x|^r$ is not differentiable at $x = 0$, if any one of $a_1, a_3, a_5, \ldots$ is non-zero.

Thus, for $f(x)$ to be differentiable at $x = 0$, we must have $a_1 = a_3 = a_5 \ldots = 0$

ie, $a_{2k+1} = 0$

Hence, (a) and (c) are the correct answers.

Illustration 45 The number of points in $(1, 3)$, where $f(x) = a^{[x^2]}$, $a > 1$ is not differentiable is

(a) 0 (b) 3 (c) 5 (d) 7

Solution. Let $g(x) = x^2$. Then, $g(x)$ is an increasing function on $(1, 3)$ such that $g(1) = 1$ and $g(3) = 9$. Clearly, $[g(x)] = [x^2]$ is discontinuous and hence non-differentiable at

$$X = \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7} \text{ and } \sqrt{8}$$

$\therefore$ $f(x)$ is not differentiable at 7 points in (1, 3).

Hence, (d) is the correct answer.

Illustration 46 Let f and g be differentiable functions satisfying $g'(a) = 2$, $g(a) = b$ and $fog = I$ (Identity function).

Then, $f'(b)$ is equal to

(a) 2 (b) $\frac{2}{3}$

(c) $\frac{1}{2}$ (d) None of these

Solution. We have, $fog = I$

$$\Rightarrow \quad f\{g(x)\} = x, \quad \text{for all } x \in R$$

$$\therefore \quad f'\{g(x)\} \cdot g'(x) = 1, \quad \text{for all } x \in R$$

$$\Rightarrow \quad f'(g(a)) = \frac{1}{g'(a)} \Rightarrow f'(b) = \frac{1}{2}$$

Hence, (c) is the correct answer.

Illustration 47 If the function $f(x) = \left[\frac{(x-2)^3}{a}\right] \sin(x-2) + a\cos(x-2)$, (where $[\cdot]$ denotes the greatest integer function) is continuous and differentiable in (4, 6), then

(a) $a \in [8, 64]$ (b) $a \in (0, 8]$

(c) $a \in [64, \infty)$ (d) None of these

Solution. We have, $x \in (4, 6)$

$$\Rightarrow \quad 2 < x - 2 < 4$$

$$\Rightarrow \quad \frac{8}{a} < \frac{(x-2)^3}{a} < \frac{64}{a}, \qquad (a > 0)$$

For $f(x)$ to be continuous and differentiable in (4, 6), $\left[\frac{(x-2)^3}{a}\right]$ must attain a constant value for $x \in (4, 6)$

Clearly, this is possible only when $a \geq 64$

In that case, we have

$f(x) = a\cos(x - 2)$ which is continuous and differentiable

$$\therefore \quad a \in [64, \infty)$$

Hence, (c) is the correct answer.

Target Exercise 6.3

1. Let $f(x)=|x|+|\sin x|, x \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Then, f is

(a) continuous, $\forall\ x \in R-\{0\}$
(b) continuous and differentiable everywhere
(c) nowhere differentiable
(d) differentiable everywhere except at $x=0$

2. If f is a periodic function, then

(a) f' and f'' are also periodic
(b) f' is periodic but f'' is not periodic
(c) f'' is periodic but f' is not periodic
(d) None of these

3. If $f(x)=[\sin^2 x]$, (where $[\cdot]$ denotes the greatest integer function), then

(a) f is everywhere continuous
(b) f is everywhere differentiable
(c) f is a constant function
(d) None of these

4. If $4x+3|y|=5y$, then y as a function of x is

(a) differentiable at $x=0$
(b) continuous for $x \in R$
(c) $\frac{dy}{dx}=\frac{1}{2}$ for all $x>0$
(d) $\frac{dy}{dx}=2$ for all $x>0$

5. Let $f(x)$ be a polynomial of degree one and $f(x)$ be a function defined by

$$f(x)=\begin{cases} g(x) & , \ x \le 0 \\ \dfrac{1+x}{(2+x)^{1/x}}, & x>0 \end{cases}$$

If $f(x)$ is continuous at $x=0$ and $f(-1)=f'(1)$, then $g(x)$ is equal to

(a) $-\frac{1}{9}(1+6\log_e 3)\,x$
(b) $\frac{1}{9}(1+6\log_e 3)\,x$
(c) $-\frac{1}{9}(1-6\log_e 3)\,x$
(d) None of these

Differentiability in a Set

(i) A function $f(x)$ defined on an open interval (a, b) is said to be differentiable or derivable in open interval (a, b), if it is differentiable at each point of (a, b).

(ii) A function $f(x)$ defined on $[a, b]$ is said to be differentiable or derivable at the end points a and b, if it is differentiable from the right at a and from the left at b. In other words $\lim\limits_{x \to a^+} \dfrac{f(x)-f(a)}{x-a}$ and $\lim\limits_{x \to b^-} \dfrac{f(x)-f(b)}{x-b}$ both exist.

"If f is derivable in the open interval (a, b) and also at the end points a and b, then f is said to be derivable in the closed interval $[a, b]$".

For checking differentiability on a closed interval $[a, b]$ we say,

"A function f is said to be a differentiable function, if it is differentiable at every point of its domain."

Illustration 48 Discuss the differentiability of $f(x)=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$.

Solution. We have, $f(x)=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

$$f'(x)=\frac{1}{\sqrt{1-\left(\dfrac{2x}{1+x^2}\right)^2}}\times\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$=\frac{(1+x^2)}{\sqrt{(1+x^2)^2-4x^2}}\times\left[\frac{(1+x^2)(2)-2x(2x)}{(1+x^2)^2}\right]$$

$$=\frac{(1+x^2)}{\sqrt{1+2x^2+x^4-4x^2}}\times\frac{(2+2x^2-4x^2)}{(1+x^2)^2}$$

$$=\frac{(1+x^2)}{\sqrt{1-2x^2+x^4}}\times\frac{(2-2x^2)}{(1+x^2)^2}$$

$$=\frac{(1+x^2)}{\sqrt{(1-x^2)^2}}\times\frac{(2-2x^2)}{(1+x^2)^2}$$

$$=\frac{(1+x^2)}{|1-x^2|}\times\frac{2(1-x^2)}{(1+x^2)^2} \qquad [\text{Since } 1+x^2\neq 0]$$

$$f'(x)=\frac{1}{|(1-x^2)|}\times\frac{2(1-x^2)}{(1+x^2)} \qquad \ldots\text{(i)}$$

Here, in Eq. (i), $f'(x)$ exists only if, $|1-x^2|\neq 0$

$\Rightarrow \qquad 1-x^2\neq 0 \Rightarrow x^2\neq 1 \Rightarrow x\neq \pm 1$

Thus, $f'(x)$ exists only if, $x\in R-\{-1,1\}$

$\therefore\ f(x)$ is differentiable for all $x\in R-\{+1,-1\}$.

Point to Consider

The above Illustration, can also be solved as follows :

$$y=f(x)=\sin^{-1}\left(\frac{2x}{1+x^2}\right),\ \text{let } x=\tan\theta$$

$$\therefore \qquad y=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$y=\sin^{-1}(\sin 2\theta)$$

$$\therefore \qquad y=2\theta \qquad \text{or} \quad y=2\cdot\tan^{-1}x$$

$\dfrac{dy}{dx}=\dfrac{2}{1+x^2}$, which states $f'(x)$ exists for all $x\in R$. "Which is wrong as we have not checked the domain of $f(x)$." So, students are advised to solve these problems carefully, while applying this method.

Illustration 49 Let $[\cdot]$ denotes the greatest integer function and $f(x) = [\tan^2 x]$, then

(a) $\lim_{x \to 0} f(x)$ doesn't exist. (b) $f(x)$ is continuous at $x = 0$

(c) $f(x)$ is not differentiable at $x = 0$ (d) $f'(0) = 1$ **[IIT JEE 1993]**

Solution. Here, $[\cdot]$ denotes the greatest integral function.

Thus, $-45° < x < 45°$

$\Rightarrow$ $\tan(-45°) < \tan x < \tan(45°)$

$\Rightarrow$ $-1 < \tan x < 1 \quad \Rightarrow \quad 0 < \tan^2 x < 1$

Since, $f(x) = [\tan^2 x] = 0$

Therefore, $f(x)$ is zero for all values of x from $(-45°)$ to $(45°)$. Thus, $f(x)$ exists when $x \to 0$ and also it is continuous at $x = 0$, $f(x)$ is differentiable at $x = 0$ and has a value 0 (*ie*, $f(0) = 0$).

Hence, (b) is the correct answer.

Relation Between Continuity and Differentiability

In previous section, we have discussed that if a function is differentiable at a point, then it should be continuous at that point as well and a discontinuous function cannot be differentiable. This fact is proved in the following theorem.

Theorem If a function is differentiable at a point, it is necessarily continuous at that point. But the converse is not necessarily true.

Or

$f(x)$ is differentiable at $x = c \Rightarrow f(x)$ is continuous at $x = c$.

Proof Let a function $f(x)$ be differentiable at $x = c$.

Then, $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists finitely.

Let
$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) \quad \text{...(i)}$$

In order to prove that $f(x)$ is continuous at $x = c$, it is sufficient to show that $\lim_{x \to c} f(x) = f(c)$

Now,
$$\lim_{x \to c} f(x) = \lim_{x \to c} \left[\left(\frac{f(x) - f(c)}{x - c} \right)(x - c) + f(c) \right]$$
$$= \lim_{x \to c} \left[\left\{ \frac{f(x) - f(c)}{x - c} \right\} \cdot (x - c) \right] + f(c)$$
$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} (x - c) + f(c)$$
$$= \lim_{x \to c} f(x) = f'(c) \times 0 + f(c) = \lim_{x \to c} f(x) = f(c)$$

Hence, $f(x)$ is continuous at $x = c$.

Converse The converse of above theorem is not necessarily true *ie*, a function may be continuous at a point but may not be differentiable at that point.

For Example The function $f(x) = |x|$ is continuous at $x = 0$ but it is not differentiable at $x = 0$, as shown by figure.

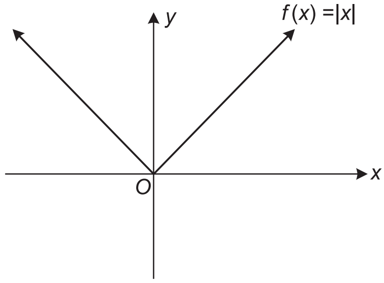


Fig. 6.19

Which shows we have sharp edge at $x = 0$ hence, not differentiable but continuous at $x = 0$.

Illustration 50 Show that $f(x) = \begin{cases} x \sin \dfrac{1}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$ is continuous but not differentiable at $x = 0$.

Solution. (a) **To check continuity at $x = 0$**

Here, $f(0) = 0$

Also, $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} x \sin \dfrac{1}{x}$

$= 0 \times$ (A finite quantity that lies between –1 to +1)

$= 0$ [As $n \to 0$, $\sin \dfrac{1}{x} \to \sin \infty$, which is a finite quantity between –1 to +1]

$\therefore$ $\lim\limits_{x \to 0} f(x) = f(0)$ and hence, $f(x)$ is continuous.

Now,

(b) **To check differentiability at $x = 0$**

(LHD at $x = 0$)

$$= \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{(0-h) - (0)}$$

$$= \lim_{h \to 0} \frac{-h \sin(-1/h)}{(-h)} = -\lim_{h \to 0} \sin\left(\frac{1}{h}\right)$$

(A number which oscillates between –1 and +1)

$\therefore$ (LHD at $x = 0$) doesn't exist.

Similarly, it could be shown that RHD at $x = 0$ doesn't exist.

Hence, $f(x)$ is continuous but not differentiable.

Illustration 51 Let $f(x) = \begin{cases} x \ \exp\left[-\left(\dfrac{1}{|x|} + \dfrac{1}{x}\right)\right], & x \neq 0 \\ 0, & x = 0 \end{cases}$

Test whether (a) $f(x)$ is continuous at $x = 0$

(b) $f(x)$ is differentiable at $x = 0$ **[IIT JEE 1997]**

Solution. Here,

$$f(x) = \begin{cases} x \ \exp\left[-\left(\dfrac{1}{|x|} + \dfrac{1}{x}\right)\right], & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} xe^{-\left\{\frac{1}{x} + \frac{1}{x}\right\}}, & x > 0 \\ xe^{-\left\{\frac{1}{-x} + \frac{1}{x}\right\}}, & x < 0 \\ 0, & x = 0 \end{cases} \qquad \left[\because |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}\right]$$

$$\Rightarrow \quad f(x) = \begin{cases} xe^{-2/x}, & x > 0 \\ x, & x < 0 \\ 0, & x = 0 \end{cases} \qquad \ldots\text{(i)}$$

(a) **To check continuity of $f(x)$ at $x = 0$**

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x = \lim_{h \to 0} (0 - h) = 0$$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} xe^{-2/x}$$

$$= \lim_{h \to 0} \frac{h}{e^{2/h}} = 0, \; f(0) = 0$$

$\therefore$ $f(x)$ is continuous at $x = 0$.

(b) **To check differentiability at $x = 0$**

$$\text{LHD} = Lf'(0) = \lim_{h \to 0} \frac{f(0 - h) - f(0)}{-h}, \; h > 0$$

$$= \lim_{h \to 0} \frac{(-h) - 0}{-h} = 1$$

$$\text{RHD} = Lf'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{he^{\frac{-2}{h}} - 0}{h} = \lim_{h \to 0} e^{\frac{-2}{h}}$$

$$= e^{-\infty} = 0$$

$$\therefore \qquad Lf'(0) \neq Rf'(0)$$

Therefore, $f(x)$ is not differentiable at $x = 0$.

Illustration 52 If $f(x)=(x+[x^3+1])^{x^2+\sin x}$, find the derivative when $x \in (1, 3/2)$ and indicate the points where it does not exist.
(Where $[\cdot]$ denotes the greatest integer function.)

Solution. Here, $\quad f(x)=(x+[x^3+1])^{x^2+\sin x}$

$$f(x)=\begin{cases}(x+2)^{x^2+\sin x}, & 1<x<2^{1/3}\\ (x+3)^{x^2+\sin x}, & 2^{1/3}<x<3^{1/3}\\ (x+4)^{x^2+\sin x}, & 3^{1/3}<x<\dfrac{3}{2}\end{cases}$$

As
$$[x^3+1]=\begin{cases}2, & 1<x<2^{1/3}\\ 3, & 2^{1/3}<x<3^{1/3}\\ 4, & 3^{1/3}<x<\dfrac{3}{2}\end{cases}$$

Which shows $f(x)$ is discontinuous at $x=2^{1/3}$ and $3^{1/3}$ and so not differentiable at $x=2^{1/3}$ and $3^{1/3}$.

Also,

$$f'(x)=\begin{cases}(x+2)^{x^2+\sin x}\left\{(2x+\cos x)\log(x+2)+\dfrac{x^2+\sin x}{x+2}\right\}, & 1<x<2^{1/3}\\ (x+3)^{x^2+\sin x}\left\{(2x+\cos x)\log(x+3)+\dfrac{x^2+\sin x}{x+3}\right\}, & 2^{1/3}<x<3^{1/3}\\ (x+4)^{x^2+\sin x}\left\{(2x+\cos x)\log(x+4)+\dfrac{x^2+\sin x}{x+4}\right\}, & 3^{1/3}<x<\dfrac{3}{2}\end{cases}$$

Illustration 53 Let f be a real function satisfying
$$f(x+y+z)=f(x)f(y)f(z)$$
for all real x, y, z. If $f(2)=4$ and $f'(0)=3$. Then, find $f(0)$ and $f'(2)$.

Solution. Here,

$$f(x+y+z)=f(x)f(y)f(z) \text{ for all } x, y, z \in R \qquad \ldots\text{(i)}$$

Put $\quad x=y=z=0$

$$f(0)=(f(0))^3 \quad \Rightarrow \quad f(0)=0, \pm 1 \qquad \ldots\text{(ii)}$$

Putting $\quad y=z=-1$ in Eq. (i), we get

$$f(x-2)=f(x)\{f(-1)\}^2$$

$\Rightarrow \quad f(0)=f(2)\{f(-1)\}^2$ for all $x \in R$

$\Rightarrow \quad f(0)=4\{f(-1)\}^2$

$\Rightarrow \quad f(0)>0 \qquad \ldots$(iii)

$\therefore \quad$ From Eqs. (ii) and (iii),

$$f(0)=1 \qquad \ldots\text{(iv)}$$

Now, putting $y=2$ and $z=0$ in Eq. (i), we get

$$f(x+2) = f(x)\,f(2)\,f(0)$$
$$f(x+2) = 4f(x)$$
$$f'(x+2) = 4f'(x), \qquad \text{putting } x = 2$$
$$\therefore \qquad f'(4) = 4, f'(2) = 12$$
Thus, $\quad f(0) = 1 \quad \text{and} \quad f'(4) = 12$

Some Standard Results on Differentiability

Functions $f(x)$	Intervals in which $f(x)$ is differentiable
1. Polynomial	$(-\infty\ \infty)$
2. Exponential $(a^x, a > 0)$	$(-\infty\ \infty)$
3. Constant	$(-\infty\ \infty)$
4. Logarithmic	Each point in its domain
5. Trigonometric	Each point in its domain
6. Inverse trigonometric	Each point in its domain

Theorems of Differentiability

Theorem 1 If $f(x)$ and $g(x)$ are both derivable at $x = a$, $f(x) \pm g(x)$, $f(x) \cdot g(x)$ and $\dfrac{f(x)}{g(x)}$ will also be derivable at $x = a$ $\left\{\text{only if } g(a) = 0 \text{ for } \dfrac{f(x)}{g(x)}\right\}$.

Theorem 2 If $f(x)$ is derivable at $x = a$ and $g(x)$ is not differentiable at $x = a$, then $f(x) \pm g(x)$ will not be derivable at $x = a$.

eg, $f(x) = \cos|x|$ is derivable at $x = 0$ and $g(x) = |x|$ is not derivable at $x = 0$.

Then, $\cos|x| + |x|$ is not derivable at $x = 0$.

However, nothing can be said about the product function, as in this case

$f(x) = x$ is derivable at $x = 0$

$g(x) = |x|$ is not derivable at $x = 0$

But, $$f(x) \cdot g(x) = \begin{cases} x^2, & \text{if } x \ge 0 \\ -x^2, & \text{if } x < 0 \end{cases}$$

which is derivable at $x = 0$.

Theorem 3 If both $f(x)$ and $g(x)$ are non-derivable, then nothing can be said about the sum/difference/product function.

eg, $\quad f(x) = \sin|x|$, not derivable at $x = 0$

$g(x) = |x|$, not derivable at $x = 0$

Then, the function

$F(x) = \sin|x| + |x|$, not derivable at $x = 0$

$G(x) = \sin|x| - |x|$, derivable at $x = 0$

Theorem 4 If $f(x)$ is derivable at $x = a$ and $f(a) = 0$ and $g(x)$ is continuous at $x = a$.

Then, the product function

$$F(x) = f(x)\cdot g(x) \text{ will be derivable at } x = a.$$

Proof $F'(a^+) = \lim_{h\to 0} \dfrac{f(a+h)\cdot g(a+h) - 0}{h} = f'(a)\cdot g(a)$

$$F'(a^-) = \lim_{h\to 0} \frac{f(a-h)\cdot g(a-h) - 0}{h} = f'(a)\cdot g(a)$$

$\therefore$ Derivable at $x = a$.

Theorem 5 Derivative of a continuous function need not be a continuous function.

eg, $$f(x) = \begin{cases} x^2\cdot\sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Here, $$f'(0^+) = 0$$
$$f'(0^-) = 0$$

$\therefore$ Derivable at $x = 0$.

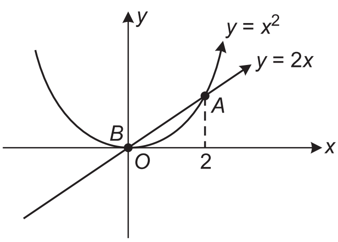
And $$f'(x) = \begin{cases} 2x\cdot\sin\dfrac{1}{x} - x^2\cdot\cos\left(\dfrac{1}{x}\right)\cdot\dfrac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$\Rightarrow$ $f'(x)$ is not continuous at $x = 0$. $\left(\text{As } \lim_{x\to 0} f'(x) \text{ doesn't exist.}\right)$

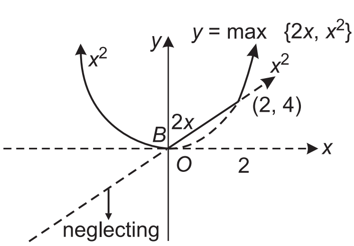
Illustration 54 Draw graph of $y = \max\{2x, x^2\}$ and discuss the continuity and differentiability.

Solution. Here, to draw

$$y = \max\{2x, x^2\}$$



First, plot $y = 2x$ and $y = x^2$ on graph and put $2x = x^2 \Rightarrow x = 0, 2$ (*ie*, Their point of intersection).
Now, since $y = \max\{2x, x^2\}$ we have to neglect the curve below point of intersections thus, the required graph is, as shown.
Thus, from the given graph $y = \max\{2x, x^2\}$ we can say $y = \max\{2x, x^2\}$ is continuous for all $x \in R$.
But $y = \max\{2x, x^2\}$ is differentiable for all $x \in R - \{0, 2\}$



Point to Consider

One must remember the formula which we can write as

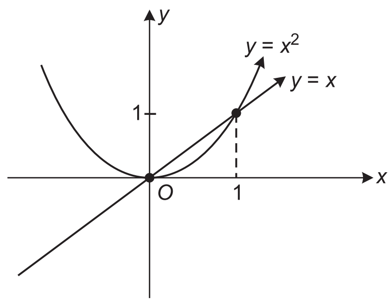
$$\max\{f(x),\ g(x)\} = \frac{f(x)+g(x)}{2} + \left|\frac{f(x)-g(x)}{2}\right|$$

$$\max\{f(x),\ g(x)\} = \frac{f(x)+g(x)}{2} - \left|\frac{f(x)-g(x)}{2}\right|$$

Illustration 55 Let $h(x) = \min\{x, x^2\}$ for every real number of x. Then,

(a) h is not continuous for all x
(b) h is differentiable for all x
(c) $h'(x) = 1$ for all x
(d) h is not differentiable at two values of x **[IIT JEE 1998]**

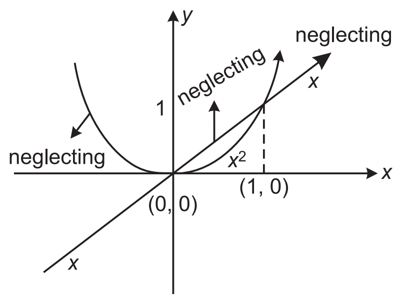
Solution. Here, $h(x) = \min\{x, x^2\}$ can be drawn on graph in two steps.



(a) Draw the graph of $y = x$ and $y = x^2$ also find their point of intersection.
ie, $x = x^2 \quad \Rightarrow \quad x = 0, 1$

(b) To find $h(x) = \min\{x, x^2\}$ neglecting the graph above the point of intersection, we get

Thus, from the given graph, $h(x) = \begin{cases} x\ , & x \le 0 \text{ or } x \ge 1 \\ x^2, & 0 \le x \le 1 \end{cases}$



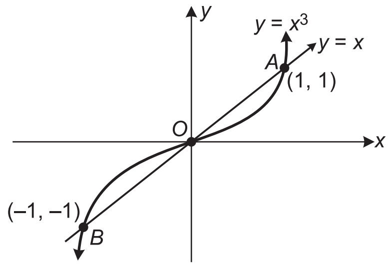
which shows $h(x)$ is continuous for all x. But not differentiable at $x = \{0, 1\}$

Thus, $h(x)$ is not differentiable at two values of x.

Illustration 56 Let $f : R \to R$ be a function defined by $f(x) = \max\{x, x^3\}$. The set of all points where $f(x)$ is not differentiable is **[IIT JEE 2001]**

(a) $\{-1, 1\}$ (b) $\{-1, 0\}$ (c) $\{0, 1\}$ (d) $\{-1, 0, 1\}$

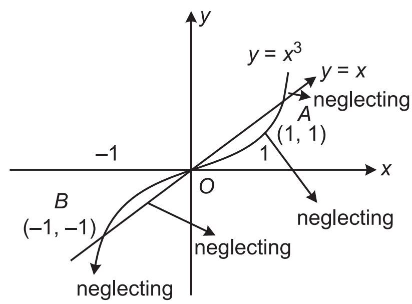
Solution. $f(x) = \max\{x, x^3\}$. Consider the graph separately of $y = x^3$ and $y = x$ and find their point of intersection;



ie, $x^3 = x \Rightarrow x = 0, 1, -1$

Now, to find $f(x) = \max\{x, x^3\}$ neglecting the graph below the point of intersection, we get the required graph of $f(x) = \max\{x, x^3\}$.

Thus, from above graph, $f(x) = \begin{cases} x \ , & \text{if } x \in (-\infty, -1] \cup [0, 1] \\ x^3, & \text{if } x \in [-1, 0] \cup [1, \infty) \end{cases}$



which shows $f(x)$ is not differentiable at 3 points *ie,* $x = \{-1, 0\ 1\}$. (Due to sharp edges)

Hence, (d) is the correct answer.

Illustration 57 Let $f(x)$ be a continuous function $\forall\ x \in R, f(0) = 1$ and $f(x) \neq x$ for any $x \in R$, then show $f(f(x)) > x, \forall\ x \in R_+$

Solution. Let $g(x) = f(x) - x$

So, $g(x)$ is continuous and $g(0) = f(0) - 0$

$\Rightarrow \quad g(0) = 1$

Now, it is given that $g(x) \neq 0$ for any $x \in R$ [As $f(x) \neq x$ for any $x \in R$]

So, $g(x) > 0, \forall\ x \in R_+$

ie, $f(x) > x, \forall\ x \in R_+ \Rightarrow f(f(x)) > f(x) > x, \forall\ x \in R_+$

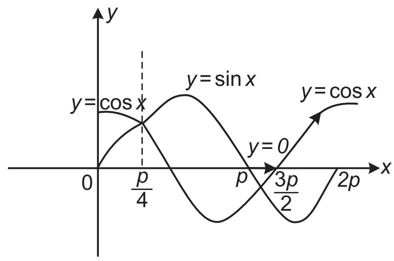
or $f(f(x)) > x, \forall\ x \in R_+$

Illustration 58 Find the number of points of non-differentiability of $f(x) = \max\{\sin x, \cos x, 0\}$ in $(0, 2n\pi)$.

Solution. Here, we know that $\sin x$ and $\cos x$ are periodic with period 2π. Thus, we could sketch the curve as; (In the interval 0 to 2π)

Which shows, $y = \max\{\sin x, \cos x, 0\}$

$$= \begin{cases} \cos x, & 0 < x < \dfrac{\pi}{4} \text{ or } \dfrac{3\pi}{2} < x < 2\pi \\ 0, & \pi < x < \dfrac{3\pi}{2} \\ \sin x, & \dfrac{\pi}{4} < x < \pi \end{cases}$$



Clearly, $y = \max\{\sin x, \cos x, 0\}$ is not differentiable at 3 points when $x = (0, 2\pi)$.

Thus, $y = \max\{\sin x, \cos x, 0\}$ is not differentiable at $3n$ points.

Illustration 59 If $f(x) = x^2 - 2|x|$ and

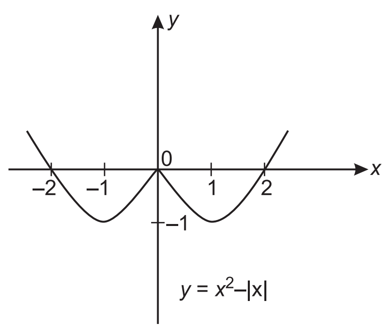
$$g(x) = \begin{cases} \min\{f(t) : -2 \le t \le x, & -2 \le x \le 0\} \\ \max\{f(t) : \ 0 \le t \le x, & 0 \le x \le 3\} \end{cases}$$

(i) Draw the graph of $f(x)$ and discuss its continuity and differentiability.

(ii) Find and draw the graph of $g(x)$. Also, discuss the continuity.

Solution. Here,

(i) $f(x) = \begin{cases} x^2 - 2x, & x \ge 0 \\ x^2 + 2x, & x < 0 \end{cases}$ shown as :



which shows $f(x)$ is continuous for all $x \in R$, but not differentiable for all $x \in R - \{0\}$.

(ii) We know that,

If $f(x)$ is an increasing function on $[a, b]$, then
$\max\{f(t); a \le t \le x, \ a \le x \le b\} = f(x)$

$\min\{f(t); a \le t \le x, \ a \le x \le b\} = f(a)$

If $f(x)$ is decreasing function on $[a, b]$, then

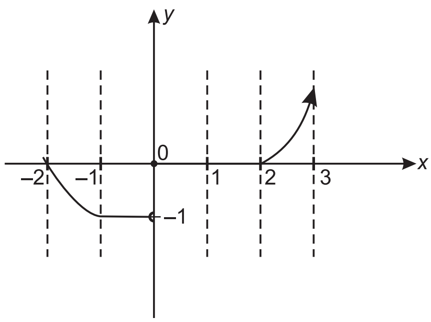
$$\max\{f(t);\ a \le t \le x,\ a \le x \le b\} = f(a)$$
$$\min\{f(t);\ a \le t \le x,\ a \le x \le b\} = f(x)$$

From graph of $f(x)$,

$$g(x) = \begin{cases} f(x), & \text{for } -2 \le x \le -1 \\ -1, & \text{for } -1 \le x < 0 \\ 0, & \text{for } 0 \le x \le 2 \\ f(x), & \text{for } x \ge 2 \end{cases}$$

$\Rightarrow$
$$g(x) = \begin{cases} x^2 + 2x, & \text{for } -2 \le x \le -1 \\ -1, & \text{for } -1 \le x < 0 \\ 0, & \text{for } 0 \le x < 2 \\ x^2 - 2x, & \text{for } x \ge 2 \end{cases}$$

Thus, graph of $g(x)$ is



From above figure, it is clear that $g(x)$ is not continuous at $x = 0$

Illustration 60 Let $f(x) = \phi(x) + \psi(x)$ and $\phi'(a)$, $\psi'(a)$ are finite and definite. Then,

(a) $f(x)$ is continuous at $x = a$

(b) $f(x)$ is differentiable at $x = a$

(c) $f'(x)$ is continuous at $x = a$

(d) $f'(x)$ is differentiable at $x = a$

Solution. We know that the sum of two continuous (differentiable) functions is continuous (differentiable).

$\therefore$ $f(x)$ is continuous and differentiable at $x = a$.

Hence, (a) and (b) are the correct answers.

Illustration 61 If $f(x) = x + \tan x$ and $g(x)$ is the inverse of $f(x)$, then $g'(x)$ is equal to

(a) $\dfrac{1}{1 + (g(x) - x)^2}$ (b) $\dfrac{1}{2 + (g(x) + x)^2}$

(c) $\dfrac{1}{2 + (g(x) - x)^2}$ (d) None of these

Solution. We have, $f(x) = x + \tan x$

$$\Rightarrow \quad f(f^{-1}(x)) = f^{-1}(x) + \tan(f^{-1}(x))$$

$$\Rightarrow \quad x = g(x) + \tan(g(x)) \quad \text{...(i)}$$

$$= g'(x) + \sec^2(g(x))\cdot g'(x) \quad [\because g(x) = f^{-1}(x)]$$

$$\Rightarrow \quad g'(x) = \frac{1}{1 + \sec^2(g(x))}$$

$$\Rightarrow \quad g'(x) = \frac{1}{2 + \tan^2(g(x))}$$

$$\Rightarrow \quad g'(x) = \frac{1}{2 + (x - g(x))^2}$$

Hence, (c) is the correct answer.

Illustration 62 If $f(x)$ is differentiable function and $(f(x)\cdot g(x))$ is differentiable at $x = a$, then

(a) $g(x)$ must be differentiable at $x = a$

(b) If $g(x)$ is discontinuous, then $f(a) = 0$

(c) $f(a) \neq 0$, then $g(x)$ must be differentiable

(d) None of the above

Solution. $$\left[\frac{d}{dx}(f(x)\cdot g(x))\right]_{x=a} = f'(a)\,g(a) + \lim_{h\to 0}\frac{g(a+h) - g(a)}{h}\cdot f(a)$$

If $f(a) \neq 0 \Rightarrow g'(a)$ must exist.

Also, if $g(a)$ is discontinuous, $f(a)$ must be 0 for $f(x)\cdot g(x)$ to be differentiable.

Hence, (b) and (c) are the correct answers.

Target Exercise 6.4

1. If $f(x) = \sin(\pi\,(x - [x]))$, $\forall\, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Then, (where $[\cdot]$ denotes the greatest integer function)

(a) $f(x)$ is discontinuous at $x = \{-1, 0, 1\}$

(b) $f(x)$ is differentiable for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$

(c) $f(x)$ is differentiable for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{-1, 0, 1\}$

(d) None of the above

2. Let $f(x) = \begin{cases} x - 1, & -1 \le x < 0 \\ x^2, & 0 \le x \le 1 \end{cases}$ and $g(x) = \sin x$

further let $h(x) = f(|g(x)|) + |f(g(x))|$. Then,

(a) $h(x)$ is continuous for $x \in [-1, 1]$

(b) $h(x)$ is differentiable for $x \in [-1, 1]$

(c) $h(x)$ is differentiable for $x \in [-1, 1] - \{0\}$

(d) $h(x)$ is differentiable for $x \in (-1, 1) - \{0\}$

3. If $f(x)=\begin{cases}|1-4x^2|, & 0\le x<1\\ [x^2-2x], & 1\le x<2\end{cases}$

(where $[\cdot]$ denotes the greatest integer function). Then,
(a) $f(x)$ is continuous for all $x\in[0,2)$
(b) $f(x)$ is differentiable for all $x\in[0,2)-\{1\}$
(c) $f(x)$ is differentiable for all $x\in[0,2)-\left\{\frac{1}{2},1\right\}$
(d) None of the above

4. Let $f(x)=\int_0^1 |x-t|\,t\,dt$, then
(a) $f(x)$ is continuous but not differentiable for all $x\in R$
(b) $f(x)$ is continuous and differentiable for all $x\in R$
(c) $f(x)$ is continuous for $x\in R-\left\{\frac{1}{2}\right\}$ and $f(x)$ is differentiable for $x\in R-\left\{\frac{1}{4},\frac{1}{2}\right\}$
(d) None of the above

5. Let $f(x)$ be a function such that $f(x+y)=f(x)+f(y)$ for all x and y and $f(x)=(2x^2+3x)\cdot g(x)$ for all x, where $g(x)$ is continuous and $g(0)=3$. Then, $f'(x)$ is equal to
(a) 6 (b) 9 (c) 8 (d) None of these

6. Given a function $g(x)$ which has derivatives $g'(x)$ for every real x and which satisfies the following equation $g(x+y)=e^y g(x)+e^x g(y)$ for all x and y and $g'(0)=2$, then the value of $\{g'(x)-g(x)\}$ is equal to
(a) e^x (b) $\frac{2}{3}e^x$ (c) $\frac{1}{2}e^x$ (d) $2e^x$

7. Let $f:R\to R$ be a function satisfying $f\left(\frac{xy}{2}\right)=\frac{f(x)\cdot f(y)}{2}$, $\forall\, x, y\in R$ and $f(1)=f'(1)\ne 0$ Then, $f(x)+f(1-x)$ is (for all non-zero real values of x)
(a) constant (b) can't be discussed (c) x (d) $\frac{1}{x}$

8. Let $f(x)$ be a derivable function at $x=0$ and $f\left(\frac{x+y}{K}\right)=\frac{f(x)+f(y)}{K}$ $(K\in R, K\ne 0, 2)$. Then, $f(x)$ is
(a) even function
(b) neither even nor odd function
(c) either zero or odd function
(d) either zero or even function

9. Let $f:R-(-\pi,\pi)$ be a differentiable function such that $f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right)$.

If $f(1)=\frac{\pi}{2}$ and $\lim_{x\to 0}\frac{f(x)}{x}=2$. Then, $f(x)$ is equal to
(a) $2\tan^{-1}x$ (b) $\frac{1}{2}\tan^{-1}x$ (c) $\frac{\pi}{2}\tan^{-1}x$ (d) $2\pi\tan^{-1}x$

10. Let $f(x)=\sin x$

$$g(x)=\begin{cases}\max\{f(t), & 0\le x\le\pi \quad \text{for } 0\le x\le\pi\}\\ \dfrac{1-\cos x}{2}, & \text{for } x>\pi\end{cases}$$

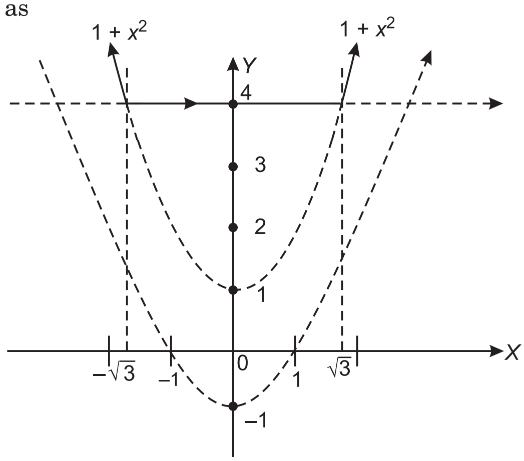
Then, $g(x)$ is
(a) differentiable for all $x\in R$
(b) differentiable for all $x\in R-\{\pi\}$
(c) differentiable for all $x\in(0,\infty)$
(d) differentiable for all $x\in(0,\infty)-\{\pi\}$

Worked Examples

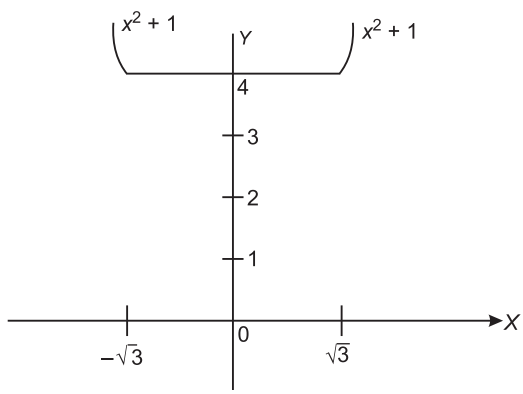
Type 1 : **Subjective Type Questions**

Example 1 Let $f(x) = \text{maximum}\{4, 1 + x^2, x^2 - 1\}, \forall x \in R$. Then, find the total number of points, where $f(x)$ is not differentiable.

Solution. We have discussed in the last chapter for sketching maximum $\{4, 1 + x^2, x^2 - 1\}$ as



Or



Thus, from above graph, we can simply say

$f(x)$ is not differentiable at $x = \pm\sqrt{3}$

And it could be defined as:

$$f(x) = \begin{cases} 4, -\sqrt{3} \le x \le \sqrt{3} \\ x^2 + 1, x \le -\sqrt{3} \text{ or } x \ge \sqrt{3} \end{cases}$$

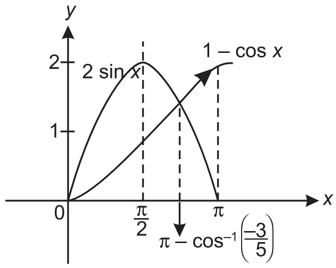
Example 2 Let $f(x) = \text{maximum}\,\{2\sin x, 1-\cos x\}, \forall\, x \in (0, \pi)$, then discuss differentiability of $f(x)$ in $(0, \pi)$.

Solution. We know that $f(x) = \text{maximum}\,\{2\sin x, 1-\cos x\}$ can be plotted as

Thus, $f(x) = \text{maximum}\,\{2\sin x, 1-\cos x\}$ is not differentiable.

When, $\qquad 2\sin x = 1-\cos x$

or $\qquad 4\sin^2 x = (1-\cos x)^2 \quad \text{or} \quad 4(1+\cos x) = (1-\cos x)$



or $\qquad 4 + 4\cos x = 1 - \cos x \quad \text{or} \quad \cos x = -3/5$

$\Rightarrow \qquad x = \cos^{-1}(-3/5)$

$\therefore \qquad f(x)$ is not differentiable at $x = \pi - \cos^{-1}(3/5), \forall\, x \in (0, \pi)$.

Example 3 A function $f(x)$ satisfies the following property: $f(x+y) = f(x)\cdot f(y)$.

Show that the function is continuous for all values of x, if it is continuous at $x = 1$.

Solution. As the function is continuous at $x = 1$, we have

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$\Rightarrow \quad \lim_{h \to 0} f(1-h) = \lim_{h \to 0} f(1+h) = f(1) \qquad [\text{Using } f(x+y) = f(x)\cdot f(y)]$$

$$\Rightarrow \quad \lim_{h \to 0} f(1)\cdot f(h) = f(1)$$

$$\Rightarrow \quad \lim_{h \to 0} f(-h) = \lim_{h \to 0} f(h) = 1 \qquad \text{...(i)}$$

Now, consider some arbitrary points $x = a$.

Left hand limit

$$\Rightarrow \quad \lim_{h \to 0} f(a-h) = \lim_{h \to 0} f(a)\cdot f(-h) = f(a) \lim_{h \to 0} f(-h)$$

LHL $= f(a) \qquad$ (As $\lim_{h \to 0} f(-h) = 1$, using Eq. (i))

Right hand limit

$$\Rightarrow \quad \lim_{h \to 0} f(a+h) = \lim_{h \to 0} f(a)\cdot f(h) = f(a) \lim_{h \to 0} f(h)$$

RHL $= f(a) \qquad$ (As $\lim_{h \to 0} f(h) = 1$, using Eq. (i))

Hence, at any arbitrary point $(x = a)$

$$\text{LHL} = \text{RHL} = f(a)$$

Therefore, function is continuous for all values of x, if it is continuous at 1.

Example 4 Find a and b so that the function

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x, & 0 \le x < \pi/4 \\ 2x\cot x + b, & \pi/4 \le x \le \pi/2 \\ a\cos 2x - b\sin x, & \pi/2 < x \le \pi \end{cases} \quad \text{is continuous for } x \in [0, \pi]$$

Solution. Since, continuous for $x \in [0, \pi]$

$\therefore$ $f(x)$ is continuous at $x = \pi/4$ and $x = \pi/2$ and hence to discuss continuity at

$$x = \pi/4 \quad \text{and} \quad x = \pi/2$$

Now, at $x = \pi/4$

Left hand limit at $x = \pi/4$

$$\Rightarrow \quad \lim_{x \to \frac{\pi}{4}^-} f(x) = \lim_{x \to \frac{\pi}{4}^-} (x + a\sqrt{2}\sin x)$$

$$= \lim_{h \to 0} \{(\pi/4 - h) + a\sqrt{2}\sin(\pi/4 - h)\}$$

$$\therefore \quad \text{LHL} = \lim_{x \to \frac{\pi}{4}^-} f(x) = \frac{\pi}{4} + a \qquad \text{...(i)}$$

Again, Right hand limit at $x = \pi/4$

$$= \lim_{x \to \frac{\pi}{4}^+} f(x) = \lim_{x \to \frac{\pi}{4}^+} (2x\cot x + b) = \lim_{h \to 0} 2\left(\frac{\pi}{4} + h\right)\cot\left(\frac{\pi}{4} + h\right) + b$$

$$\Rightarrow \quad \text{RHL} = \lim_{x \to \frac{\pi}{4}^+} f(x) = \frac{\pi}{2} + b \qquad \text{...(ii)}$$

$$\text{Also} \quad f\left(\frac{\pi}{4}\right) = \frac{\pi}{2} + b \qquad \text{...(iii)}$$

For continuity, these three must be equal

$$\Rightarrow \quad \frac{\pi}{4} + a = \frac{\pi}{2} + b \quad \Rightarrow \quad a - b = \frac{\pi}{4} \qquad \text{...(A)}$$

At $x = \pi/2$

Left hand limit at $x = \pi/2$

$$= \lim_{x \to \frac{\pi}{2}^-} (2x\cot x + b) = \lim_{h \to 0} 2\left(\frac{\pi}{2} - h\right)\cot\left(\frac{\pi}{2} - h\right) + b$$

$$\text{LHL} = b \qquad \text{...(iv)}$$

Right hand limit at $x = \pi/2$

$$= \lim_{x \to \frac{\pi}{2}^+} (a\cos 2x - b\sin x) = \lim_{h \to 0} a\cos 2\left(\frac{\pi}{2} + h\right) - b\sin\left(\frac{\pi}{2} + h\right)$$

$$\text{RHL} = -a - b \qquad \text{...(v)}$$

$$\text{and} \quad f(\pi/2) = 2\frac{\pi}{2}\cot\frac{\pi}{2} + b = b \qquad \text{...(vi)}$$

For continuity, these three must be equal

$$\Rightarrow \quad b = -a - b \quad \Rightarrow \quad a + 2b = 0 \qquad \text{...(B)}$$

Solving Eqs. (A) and (B), we get

$$a = \pi/6, \quad b = -\pi/12$$

Example 5 Discuss the continuity of the function $g(x)=[x]+[-x]$ at integral values of x.

Solution. Here, x can assume two values

(a) integers (b) non-integers

(a) **If x is an integer**

$$[x]=x \text{ and } [-x]=-x \quad \Rightarrow \quad g(x)=x-x=0$$

(b) **If x is not an integer**

Let $x=n+f$ where n is an integer and $f \in (0, 1)$

$\Rightarrow \quad [x]=[n+f]=n$

$\Rightarrow \quad [-x]=[-n-f]=[(-n-1)+(1-f)]=(-n-1)$

$(\because 0<f<1 \Rightarrow (1-f)<1)$

Hence, $\quad g(x)=[x]+[-x]=n+(-n-1)=-1$

So, we get

$$g(x)=\begin{cases} 0, & \text{if } x \text{ is an integer} \\ -1, & \text{if } x \text{ is not an integer} \end{cases}$$

Let us discuss the continuity of $g(x)$ at a point $x=a$ (where $a \in$ integer)

$$\text{LHL}=\lim_{x\to a^-} g(x)=-1 \qquad (\because \text{As } x \to a^-, x \text{ is not an integer.})$$

$$\text{RHL}=\lim_{x\to a^+} g(x)=-1 \qquad (\because \text{As } x \to a^+, x \text{ is not an integer.})$$

But $g(a)=0$ because a is an integer.

Hence, $g(x)$ has a removable discontinuity at integral values of x.

$$\therefore \qquad g(x)=\begin{cases} [x]+[-x], & x \notin \text{integers} \\ -1, & x \in \text{integers} \end{cases}$$

Example 6 Discuss the continuity of $f(x)$ in [0, 2] where

$$f(x)=\begin{cases} [\cos \pi x], & x \le 1 \\ |2x-3|[x-2], & x>1 \end{cases}$$

where $[\cdot]$ denotes the greatest integral function.

Solution. First we shall find the points where $f(x)$ may be discontinuous.

Consider $x \in [0, 1]$

$f(x)=[\cos \pi x]$ is discontinuous where $\cos \pi x \in I$

In [0, 1], $\cos \pi x$ is an integer at $x=0, 1/2, 1$

$\Rightarrow x=0,\ x=\dfrac{1}{2}$ and $x=1$ may be the points at which $f(x)$ may be discontinuous ...(i)

Now, consider $x \in (1, 2]$

$$f(x)=[x-2]\,|2x-3|$$

In $\quad x \in (1, 2),\ [x-2]=-1 \quad$ and $\quad$ for $x=2,\ [x-2]=0$

Also, $\quad |2x-3|=0 \Rightarrow x=3/2$

$\Rightarrow x=3/2$ and 2 may be the points at which $f(x)$ may be discontinuous ...(ii)

Combining Eqs. (i) and (ii), we have

$$x=0, 1/2, 1, 3/2, 2$$

Dividing $f(x)$ by about the 5 critical points, we get

$$f(x)=\begin{cases} 1, \quad x=0 \because \cos(\pi\, 0)=1 \\ 0,\ 0<x\le\frac{1}{2} \because\ 0\le\cos\pi x<1 \Rightarrow [\cos\pi x]=0 \\ -1,\ \frac{1}{2}<x\le 1 \because\ -1\le\cos\pi x<0 \Rightarrow [\cos\pi x]=-1 \\ -(3-2x),\ 1<x\le 3/2 \because\ |2x-3|=3-2x \text{ and } [x-2]=-1 \\ -(2x-3),\ 3/2<x\le 2 \because\ |2x-3|=2x-3 \text{ and } [x-2]=-1 \\ 0,\ x=2 \because\ [x-2]=0 \end{cases}$$

Checking continuity at $x=0$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 0 = 0$$

And $\quad f(0)=1$

As $\quad \text{RHL} \ne f(0)$

Thus, $f(x)$ is discontinuous at $x=0$

Checking continuity at $x=1/2$

$$\text{LHL} = \lim_{x\to \frac{1}{2}^-} f(x) = \lim_{x\to \frac{1}{2}^-} 0 = 0$$

$$\text{RHL} = \lim_{x\to \frac{1}{2}^+} f(x) = \lim_{x\to \frac{1}{2}^+} (-1) = -1$$

As $\quad \text{RHL} \ne \text{LHL}$

Therefore, $f(x)$ is discontinuous at $x=1/2$

Checking continuity at $x=1$

$$\text{LHL} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (-1) = -1$$

$$\text{RHL} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} -(3-2x) = -1$$

And $\quad f(1)=-1$

As $\quad \text{RHL} = \text{LHL} = f(1)$

Therefore, $f(x)$ is continuous at $x=1$

Checking continuity at $x=3/2$

$$\text{LHL} = \lim_{x\to 3/2^-} (2x-3) = 0$$

$$\text{RHL} = \lim_{x\to 3/2^+} (3-2x) = 0$$

And $\quad f(3/2)=0$

As $\quad \text{RHL} = \text{LHL} = f(3/2)$

Hence, $f(x)$ is continuous at $x=3/2$

Checking continuity at $x=2$

$$\text{LHL} = \lim_{x\to 2^-} (3-2x) = -1$$

And $\quad f(2)=0$

As $\quad \text{LHL} \ne f(2)$

Hence, $f(x)$ is discontinuous at $x=2$

Thus, $f(x)$ is continuous when $x\in[0,2]-\{0, 1/2, 2\}$.

Example 7 Let $f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2}, & x<0\\ a, & x=0\\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & x>0\end{cases}$

Determine the value of 'a' if possible, so that the function is continuous at $x=0$.

Solution. As, $f(x)$ is continuous at $x=0$.

$\therefore$ We must have,

RHL (at $x=0$) = LHL (at $x=0$) = $f(0)$

$\Rightarrow$ RHL (at $x=0$)

$$\Rightarrow \quad \lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}$$

Put $x=0+h$

$$\Rightarrow \quad \lim_{h\to 0}\frac{\sqrt{0+h}}{\sqrt{16+\sqrt{0+h}}-4}\times\frac{\sqrt{16+\sqrt{h}}+4}{\sqrt{16+\sqrt{h}}+4}$$

$$\Rightarrow \quad \lim_{h\to 0}\frac{\sqrt{h}\{\sqrt{16+\sqrt{h}}+4\}}{16+\sqrt{h}-16}$$

$$\Rightarrow \quad \lim_{h\to 0}\{\sqrt{16+\sqrt{h}}+4\}=8$$

Also LHL (at $x=0$)

$$\Rightarrow \quad \lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{1-\cos 4x}{x^2}$$

Put $x=0-h$

$$\Rightarrow \quad \lim_{h\to 0}\frac{1-\cos 4(0-h)}{(0-h)^2}=\lim_{h\to 0}\frac{1-\cos 4h}{h^2}$$

$$=\lim_{h\to 0}\frac{2\sin^2 2h}{h^2}=\lim_{h\to 0} 8\left(\frac{\sin 2h}{2h}\right)^2=8$$

And $\quad f(0)=a$

Since, $f(x)$ is continuous at $x=0$

$\Rightarrow \quad f(0)=\text{RHL}=\text{LHL}$

or $\quad f(0)=8$

or $\quad a=8$

Example 8 Let $f(x)$ is defined as follows:

$$f(x)=\begin{cases}(\cos x-\sin x)^{\operatorname{cosec} x}, & \dfrac{-\pi}{2}<x<0\\ a & , x=0\\ \dfrac{e^{1/x}+e^{2/x}+e^{3/x}}{ae^{2/x}+be^{3/x}} & , 0<x<\pi/2\end{cases}$$

If $f(x)$ is continuous at $x=0$, find 'a' and 'b'.

Solution. Here, $f(x)$ is continuous at $x = 0$.

$\Rightarrow$ RHL (at $x = 0$) = LHL (at $x = 0$) = $f(0)$

$\therefore$ RHL (at $x = 0$)

$$\lim_{h \to 0} \frac{e^{1/h} + e^{2/h} + e^{3/h}}{ae^{2/h} + be^{3/h}} \qquad \left(\text{As } \frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{h \to 0} \frac{e^{3/h}\left\{\dfrac{1}{e^{2/h}} + \dfrac{1}{e^{1/h}} + 1\right\}}{e^{3/h}\left\{\dfrac{a}{e^{1/h}} + b\right\}}$$

$$= \frac{1}{b} \qquad \left(\text{As } \lim_{h \to 0} \frac{1}{e^{1/h}} \to 0\right) \quad \text{...(i)}$$

Again, LHL (at $x = 0$)

$$\lim_{h \to 0} (\cos h + \sin h)^{-\operatorname{cosec} h}$$

$$= \lim_{h \to 0} \{1 + (\cos h + \sin h - 1)\}^{\frac{-1}{\sin h}} \qquad [ie, (1)^{\infty} \text{ form}]$$

$$= e^{\lim\limits_{h \to 0} \{\cos h + \sin h - 1\} \cdot \left(-\frac{1}{\sin h}\right)}$$

$$= e^{\lim\limits_{h \to 0} \{-2\sin^2 h/2 + 2\sin h/2 \cos h/2\} \cdot \left\{-\frac{1}{2\sin h/2 \cos h/2}\right\}}$$

$$= e^{\lim\limits_{h \to 0} \frac{\sin h/2 - \cos h/2}{\cos h/2}} = e^{-1} \qquad \text{...(ii)}$$

And $\qquad f(0) = a$

$\therefore \qquad a = e^{-1} = \dfrac{1}{b} \quad$ or $\qquad (a = e^{-1}$ and $b = e)$

Example 9 If $f(x) = \dfrac{\sin 2x + A \sin x + B \cos x}{x^3}$ is continuous at $x = 0$. Find the values of A and B. Also, find $f(0)$.

Solution. As $f(x)$ is continuous at $x = 0$,

$f(0) = \lim\limits_{x \to 0} f(x)$ and both $f(0)$ and $\lim\limits_{x \to 0} f(x)$ are finite.

$$\Rightarrow \qquad f(0) = \lim_{x \to 0} \frac{\sin 2x + A \sin x + B \cos x}{x^3}$$

As denominator $\to 0$ as $x \to 0$,

Numerator should also $\to 0$ as $x \to 0$

Which is possible only if (for $f(0)$ to be finite).

$$\Rightarrow \qquad \sin 2(0) + A \sin (0) + B \cos (0) = 0$$

$$\Rightarrow \qquad B = 0$$

$$\therefore \qquad f(0) = \lim_{x \to 0} \frac{\sin 2x + A \sin x}{x^3}$$

$$\Rightarrow \qquad f(0) = \lim_{x \to 0} \left(\frac{\sin x}{x}\right)\left(\frac{2\cos x + A}{x^2}\right) = \lim_{x \to 0} \left(\frac{2\cos x + A}{x^2}\right)$$

Again, we can see that denominator $\to 0$ as $x \to 0$,

$\therefore$ Numerator should also approach 0 as $x \to 0$ (for $f(0)$ to be finite)

$$\Rightarrow \quad 2 + A = 0$$

$$\Rightarrow \quad A = -2$$

$$\Rightarrow \quad f(0) = \lim_{x \to 0} \left(\frac{2\cos x - 2}{x^2} \right) = \lim_{x \to 0} \left(\frac{-4\sin^2 x/2}{x^2} \right) = \lim_{x \to 0} - \left(\frac{-\sin^2 x/2}{x^2/4} \right) = -1$$

So, we get $\quad A = -2$, $B = 0$ and $f(0) = -1$

Example 10 A function $f : R \to R$ satisfies the equation $f(x + y) = f(x) \cdot f(y)$ for all $x, y \in R$, $f(x) \neq 0$. Suppose that the function is differentiable at $x = 0$ and $f'(0) = 2$. Prove that $f'(x) = 2f(x)$.

Solution. We are given that

$$f(x + y) = f(x) \cdot f(y) \qquad \text{...(i)}$$

And
$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = 2 \qquad \text{...(ii)}$$

Now,
$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x + 0)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{f(x) \cdot f(h) - f(x) \cdot f(0)}{h} \qquad \text{[using Eq. (i)]}$$

$$\Rightarrow \quad f'(x) = f(x) \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$\Rightarrow \quad f'(x) = 2f(x) \qquad \text{[using Eq. (ii)]}$$

Example 11 Let $f\left(\frac{x + y}{2}\right) = \frac{f(x) + f(y)}{2}$ for all real x and y. If $f'(0)$ exists and equals to -1 and $f(0) = 1$, find $f'(x)$. **[IIT JEE 1995]**

Solution. Given equation is $f\left(\frac{x + y}{2}\right) = \frac{f(x) + f(y)}{2}$ ...(i)

Putting $y = 0$ and $f(0) = 1$ in Eq. (i), we have

$$f\left(\frac{x}{2}\right) = \frac{1}{2}[f(x) + 1] \quad \Rightarrow \quad f(x) = 2f\left(\frac{x}{2}\right) - 1 \qquad \text{...(ii)}$$

Now,
$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{f\left(\frac{2x + 2h}{2}\right) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{f(2x) + f(2h)}{2} - f(x)}{h} \qquad \text{[using Eq. (i)]}$$

$$= \lim_{h \to 0} \frac{f(2x) + f(2h) - 2f(x)}{2h}$$

$$= \lim_{h \to 0} \frac{2f(x) - 1 + f(2h) - 2f(x)}{2h} \qquad \text{[using Eq. (ii)]}$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{f(2h) - 1}{2h} = f'(0)$$

Therefore, $\quad f'(x) = -1$

Example 12 (a) Let f be a function such that $f(x+f(y))=f(x)+y\ \forall\ x, y \in R$, then find $f(0)$.
(b) If it is given that there exists a positive real δ, such that $f(h)=h$ for $0<h<\delta$, then find $f'(x)$ and hence $f(x)$.

Solution. (a) Let $x=0$, $y=0$ in $f(x+f(y))=f(x)+y$

$$f(0+f(0))=f(0)+0$$

$\Rightarrow$ $$f(f(0))=f(0)$$

$\Rightarrow$ $$f(a)=a \quad \text{or} \quad f(0)=0$$

(b) Given $$f(h)=h$$

Then, $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad \text{for } 0<h<\delta$$

$\Rightarrow$ $$f'(x)=\lim_{h\to 0}\frac{f(x+f(h))-f(x)}{h} \qquad [\text{Given } f(h)=h]$$

$\Rightarrow$ $$f'(x)=\lim_{h\to 0}\frac{f(x)+h-f(x)}{h}$$

$\Rightarrow$ $$f'(x)=\lim_{h\to 0}\frac{h}{h}$$

$\Rightarrow$ $f'(x)=1$, integrating both sides, we get

$\Rightarrow$ $$f(x)=x+c$$

where $$f(0)=0 \quad \Rightarrow \quad c=0$$

So, $$f(x)=x$$

Thus, $$f'(x)=1 \quad \text{and} \quad f(x)=x$$

Example 13 Let f be an even function and $f'(0)$ exists, then find $f'(0)$.

Solution. Since, f is an even function.

So, $$f(-x)=f(x) \qquad \text{for all } x \qquad \ldots\text{(i)}$$

Also, $f'(0)$ exists

So, $$Rf'(0)=Lf'(0)$$

$\Rightarrow$ $$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$\Rightarrow$ $$\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

$\Rightarrow$ $$\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{-h} \qquad [\text{using Eq. (i) } f(-h)=f(h)]$$

$\Rightarrow$ $$\lim_{h\to 0}\frac{f(h)-f(0)}{h}=-\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$\Rightarrow$ $$2\lim_{h\to 0}\frac{f(h)-f(0)}{h}=0$$

$\Rightarrow$ $$2f'(0)=0 \quad \Rightarrow \quad f'(0)=0$$

Example 14 Let $f(x)=x^n$, n being non-negative integer. Then, find the value of n for which the equality $f'(a+b)=f'(a)+f'(b)$ is valid for all $a, b>0$.

Solution. Since, $f(x)=x^n$, n being non-negative integer.

Then, $$f'(x)=nx^{n-1}$$

$$f'(a)=na^{n-1},\ f'(b)=nb^{n-1},\ f'(a+b)=n(a+b)^{n-1}$$

Now, the equality $f'(a+b)=f'(a)+f'(b)$ holds if

$$n(a+b)^{n-1}=na^{n-1}+nb^{n-1}$$

or $$(a+b)^{n-1}=a^{n-1}+b^{n-1} \quad \text{...(i)}$$

Above statement is true, only if $(n-1)=1 \Rightarrow n=2$

ie, $$(a+b)^{n-1}=a^{n-1}+b^{n-1} \quad (\text{If } n=2)$$

or $$(a+b)^1=a^1+b^1$$

Also, when $n=1, 3, 4, 5, \ldots\ldots\ldots\ldots\ldots$, then LHS > RHS

Again, when $n=0$, $f'(x)=0$ for all x

So, the equality is true for $n=0$ and 2.

Example 15 If $f(x)+f(y)=f\left(\dfrac{x+y}{1-xy}\right)$ for all $x, y \in R$

$(xy \neq 1)$ and $\lim\limits_{x \to 0} \dfrac{f(x)}{x}=2$. Find $f\left(\dfrac{1}{\sqrt{3}}\right)$ and $f'(1)$.

Solution. $$f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right) \quad \text{... (i)}$$

Putting $x=y=0$, we get $f(0)=0$

Putting $y=-x$, we get $f(+x)+f(-x)=f(0)$

$$\Rightarrow \quad f(-x)=-f(x) \quad \text{...(ii)}$$

Also, $$\lim_{x \to 0} \frac{f(x)}{x}=2$$

Now, $$f'(x)=\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \quad \text{...(iii)}$$

$$=\lim_{h \to 0} \frac{f(x+h)+f(-x)}{h} \quad [\text{Using Eq. (ii) } -f(x)=f(-x)]$$

$$f'(x)=\lim_{h \to 0} \frac{f\left(\dfrac{x+h-x}{1-(x+h)(-x)}\right)}{h} \quad [\text{Using Eq. (i)}]$$

$$\Rightarrow \quad f'(x)=\lim_{h \to 0} \left[\frac{f\left(\dfrac{h}{1+x(x+h)}\right)}{h}\right]$$

$$\Rightarrow \quad f'(x)=\lim_{h \to 0} \frac{f\left(\dfrac{h}{1+xh+x^2}\right)}{\left(\dfrac{h}{1+xh+x^2}\right)} \times \left(\frac{1}{1+xh+x^2}\right)$$

$$\Rightarrow \qquad f'(x) = \lim_{h\to 0} \frac{f\left(\dfrac{h}{1+xh+x^2}\right)}{\left(\dfrac{h}{1+xh+x^2}\right)} \times \lim_{h\to 0} \frac{1}{1+xh+x^2} \qquad \left(\text{Using } \lim_{x\to 0} \frac{f(x)}{x} = 2\right)$$

$$\Rightarrow \qquad f'(x) = 2 \times \frac{1}{1+x^2} \quad \Rightarrow \quad f'(x) = \frac{2}{1+x^2}$$

Integrating both the sides,

$$f(x) = 2\tan^{-1}(x) + c$$

Where $\qquad f(0) = 0 \quad \Rightarrow \quad c = 0$

Thus, $\qquad f(x) = 2\tan^{-1} x$

Hence, $\qquad f\left(\dfrac{1}{\sqrt{3}}\right) = 2\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = 2\cdot\dfrac{\pi}{6} = \dfrac{\pi}{3}$

and $\qquad f'(1) = \dfrac{2}{1+1^2} = \dfrac{2}{2} = 1$

Example 16 Let $f : R \to R$ is a function which satisfies condition $f(x + y^3) = f(x) + [f(y)]^3$ for all $x, y \in R$. If $f'(0) \geq 0$. Find $f(10)$.

Solution. Given $f(x + y^3) = f(x) + [f(y)]^3$ ...(i)

And $\qquad f'(0) \geq 0$ ...(ii)

Replacing x, y by 0,

$$f(0) = f(0) + f(0)^3 \qquad \Rightarrow \qquad f(0) = 0 \qquad \text{...(iii)}$$

Also $\qquad f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0} \dfrac{f(h)}{h}$ ...(iv)

Let $\qquad I = f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+(h^{1/3})^3) - f(0)}{(h^{1/3})^3}$

$$= \lim_{h\to 0} \frac{f((h^{1/3}))^3}{(h^{1/3})^3} = \lim_{h\to 0} \left(\frac{f(h^{1/3})}{(h^{1/3})}\right)^3 = I^3$$

$\Rightarrow \qquad I = I^3$

or $\qquad I = 0, 1, -1$ as $f'(0) \geq 0 \qquad \therefore \quad f'(0) = 0, 1$ ...(v)

Thus, $\qquad f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f(x+(h^{1/3})^3) - f(x)}{(h^{1/3})^3}$

$$f'(x) = \lim_{h\to 0} \frac{f(x) + (f(h^{1/3}))^3 - f(x)}{(h^{1/3})^3} \qquad \text{[Using Eq. (i)]}$$

$$\Rightarrow \qquad f'(x) = \lim_{h\to 0} \left(\frac{f(h^{1/3})}{(h^{1/3})}\right)^3 = (f'(0))^3$$

$\Rightarrow \qquad f'(x) = 0, 1 \qquad$ [As $f'(0) = 0, 1$ using Eq. (v)]

Integrating both sides,

$f(x) = c$ or $x + c \qquad$ as $\qquad f(0) = 0$

$\Rightarrow \qquad f(x) = 0 \qquad$ or $\qquad x$

Thus, $\qquad f(10) = 0 \qquad$ or $\qquad 10$

Example 17 Let $f: R \to R$ satisfying $|f(x)| \le x^2, \forall x \in R$, then show $f(x)$ is differentiable at $x=0$.

Solution. Since, $|f(x)| \le x^2, \forall x \in R$

$\therefore$ At $x=0$, $|f(0)| \le 0$

$\Rightarrow \quad f(0)=0 \quad$ ...(i)

$\therefore \quad f'(0) = \lim\limits_{h \to 0} \dfrac{f(h)-f(0)}{h}$

$\lim\limits_{h \to 0} \dfrac{f(h)}{h} \quad [f(0)=0$ from Eq.(i)] ...(ii)

Now, $\left| f\dfrac{(h)}{h} \right| \le |h| \quad \Rightarrow \quad -|h| \le f\dfrac{(h)}{h} \le |h|$

$\Rightarrow \quad \lim\limits_{h \to 0} f\dfrac{(h)}{h} \to 0 \quad$ [Using Cauchy-Squeeze theorem] ...(iii)

$\therefore$ From Eqs. (ii) and (iii), we get $f'(0)=0$

ie, $f(x)$ is differentiable at $x=0$

Example 18 Discuss the continuity and differentiability of the function $y=f(x)$ defined parametrically, $x=2t-|t-1|$ and $y=2t^2+t|t|$

Solution. Here, $x=2t-|t-1|$ and $y=2t^2+t|t|$

Now, when $t<0$

$$x = 2t - \{-(t-1)\} = 3t-1$$

and $\quad y = 2t^2 - t^2 = t^2 \quad \Rightarrow \quad y = \dfrac{1}{9}(x+1)^2$

when $0 \le t < 1$

$$x = 2t - (-(t-1)) = 3t-1$$

and $\quad y = 2t^2 + t^2 = 3t^2 \quad \Rightarrow \quad y = \dfrac{1}{3}(x+1)^2$

when $t \ge 1$

$$x = 2t-(t-1) = t+1$$

and $\quad y = 2t^2 + t^2 = 3t^2 \quad \Rightarrow \quad y = 3(x-1)^2$

Thus,

$$y = f(x) = \begin{cases} \dfrac{1}{9}(x+1)^2, & x < -1 \\ \dfrac{1}{3}(x+1)^2, & -1 \le x < 2 \\ 3(x-1)^2, & x \ge 2 \end{cases}$$

Now, to check continuity at $x=-1$ and 2.

Continuity at $x=-1$;

LHL $\quad \lim\limits_{h \to 0} f(-1-h) = \lim\limits_{h \to 0} \dfrac{1}{9}(-1-h+1)^2 = 0$

RHL $\quad \lim\limits_{h \to 0} f(-1-h) = \lim\limits_{h \to 0} \dfrac{1}{3}(-1+h+1)^2 = 0$

$$f(-1) = 0,$$

$\therefore \quad$ Continuous at $x=-1$.

Now, to check continuity at $x = 2$;

LHL $\quad \lim_{h \to 0} f(2-h) = \lim_{h \to 0} \frac{1}{3}(2-h+1)^2 = 3$

RHL $\quad \lim_{h \to 0} f(2+h) = \lim_{h \to 0} 3(2+h-1)^2 = 3$

$$f(2) = 3$$

Thus, $f(x)$ is continuous at $x = 2$.

Now to check differentiability at $x = -1$ and 2.

Differentiability at $x = -1$;

LHD $\quad Lf'(-1) = \lim_{h \to 0} \dfrac{f(-1-h) - f(-1)}{-h} = \lim_{h \to 0} \dfrac{\frac{1}{9}(-1-h+1)^2 - 0}{-h} = 0$

RHD $\quad Rf'(-1) = \lim_{h \to 0} \dfrac{f(-1+h) - f(-1)}{h} = \lim_{h \to 0} \dfrac{\frac{1}{3}(-1+h+1)^2 - 0}{h} = 0$

Hence, $f(x)$ is differentiable at $x = -1$.

Differentiability at $x = 2$;

LHD $\quad Lf'(2) = \lim_{h \to 0} \dfrac{f(2-h) - f(2)}{-h} = \lim_{h \to 0} \dfrac{\frac{1}{3}(2-h+1)^2 - 3}{-h} = 2$

RHD $\quad Rf'(2) = \lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \dfrac{3(2+h-1)^2 - 3}{h} = 6$

Hence, $f(x)$ is not differentiable at $x = 2$.

$\therefore f(x)$ is continuous for all x and differentiable for all x, except $x = 2$.

Example 19 Let $\quad f(x) = x^4 - 8x^3 + 22x^2 - 24x$ and

$$g(x) = \begin{cases} \min f(x); & x \le t \le x+1, -1 \le x \le 1 \\ x - 10; & x \ge 1 \end{cases}$$

Discuss the continuity and differentiability of $g(x)$ in $[-1, \infty)$.

Solution. Here, $f(x) = x^4 - 8x^3 + 22x^2 - 24x$

$\Rightarrow \qquad f'(x) = 4x^3 - 24x^2 + 44x - 24$

or $\qquad f'(x) = 4(x-1)(x-2)(x-3)$

Which shows $f(x)$ is increasing in $[1, 2] \cup [3, \infty]$ and decreasing in $[-\infty, 1] \cup [2, 3]$.

Thus, minimum $f(x); x \le t \le x+1, -1 \le x \le 1$

$$\Rightarrow \quad \text{minimum } f(x) = \begin{cases} f(x+1), & -1 \le x \le 0 \\ f(1), & 0 < x \le 1 \end{cases}$$

Thus, $\quad g(x) = \begin{cases} f(x+1), & -1 \le x \le 0 \\ f(1), & 0 < x \le 1 \\ x - 10, & x > 1 \end{cases}$

$$= \begin{cases} (x+1)^4 - 8(x+1)^3 + 22(x+1)^2 - 24(x+1), & -1 \le x \le 0 \\ 1 - 8 + 22 - 24, & 0 < x \le 1 \\ x - 10, & x > 1 \end{cases}$$

$$g(x)=\begin{cases} x^4-4x^3+4x^2-9, & -1\le x\le 0 \\ -9, & 0<x\le 1 \\ x-10, & x>1 \end{cases}$$

Also, $$g'(x)=\begin{cases} 4x^3-12x^2+8x, & -1\le x\le 0 \\ 0, & 0<x\le 1 \\ +1, & x>1 \end{cases}$$

which clearly shows $g(x)$ is continuous in $[-1, \infty)$ but not differentiable at $x=1$

Example 20 Let $f(x)=x^3-x^2+x+1$ and

$$g(x)=\begin{cases} \max f(t), & 0\le t\le x \ \text{ for } \ 0\le x\le 1 \\ 3-x, & 1<x\le 2 \end{cases}$$

Discuss the continuity and differentiability of $g(x)$ in $(0, 2)$.

Solution. Here, $f(x)=x^3-x^2+x+1$

$\Rightarrow$ $f'(x)=3x^2-2x+1$ which is strictly increasing in $(0, 2)$.

$\therefore$ $$g(x)=\begin{cases} f(x), & 0\le x\le 1 \\ 3-x, & 1<x\le 2 \end{cases}$$

[As $f(x)$ is increasing, so $f(x)$ is maximum when $0\le t\le x$]

So, $$g(x)=\begin{cases} x^3-x^2+x+1, & 0\le x\le 1 \\ 3-x, & 1<x\le 2 \end{cases}$$

Also, $$g'(x)=\begin{cases} 3x^2-2x+1, & 0\le x\le 1 \\ -1, & 1<x\le 2 \end{cases}$$

Which clearly shows $g(x)$ is continuous for all $x\in[0, 2]$ but $g(x)$ is not differentiable at $x=1$.

Example 21 Let $f(x)=1+4x-x^2, \forall \ x\in R$

$$g(x)=\max\{f(t);\ x\le t\le (x+1);\ 0\le x<3\}$$
$$=\min\{(x+3);\ 3\le x\le 5\}$$

Verify continuity of $g(x)$ for all $x\in[0, 5]$.

Solution. Here, $f(t)=1+4t-t^2$

$$f'(t)=4-2t, \ \text{when } f'(t)=0 \ \Rightarrow \ t=2$$

At $t=2$, $f(t)$ has a maxima.

Since, $g(x)=\max\{f(t)$ for $t\in[x, x+1],\ 0\le x<3\}$

$\therefore$ $$g(x)=\begin{cases} f(x+1), & \text{if } [x, x+1]<2 \\ f(2), & \text{if } x\le 2\le x+1 \\ f(x), & \text{if } 2<[x, x+1] \end{cases}$$

$$\therefore \qquad g(x)=\begin{cases}4+2x-x^2, & \text{if} \quad 0\le x<1\\ 5, & \text{if} \quad 1\le x\le 2\\ 1+4x-x^2, & \text{if} \quad 2<x<3\\ 6, & \text{if} \quad 3\le x\le 5\end{cases}$$

Which is clearly continuous for all $x\in[0,5]$ except $x=3$.

$\therefore$ $g(x)$ is continuous for $x=[0,3)\cup(3,5]$

Example 22 Show that the function defined by

$$f(x)=\begin{cases}x^2\sin 1/x, & x\ne 0\\ 0, & x=0\end{cases}$$

is differentiable for every value of x, but the derivative is not continuous for $x=0$.

Solution. For $x\ne 0$,

$$f'(x)=2x\sin(1/x)+x^2\left(-\frac{1}{x^2}\right)\cos\left(\frac{1}{x}\right)$$

$$f'(x)=2x\sin\frac{1}{x}-\cos\frac{1}{x}$$

For $x=0$, $$f'(x)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{h^2\sin\frac{1}{h}-0}{h}$$

$$\Rightarrow \qquad f'(x)=\lim_{h\to 0} h\sin\frac{1}{h}=0$$

Thus, $$f'(x)=\begin{cases}2x\sin\frac{1}{x}-\cos\frac{1}{x}, & x\ne 0\\ 0, & x=0\end{cases}$$

Now, $f'(x)$ is continuous at $x=0$, if

(i) $\lim\limits_{x\to 0} f'(x)$ exists (ii) $\lim\limits_{x\to 0} f'(x)=f'(0)$

Again, $$\lim_{x\to 0} f'(x)=\lim_{x\to 0}\left(2x\sin\frac{1}{x}-\cos\frac{1}{x}\right), \text{ doesn't exist}$$

Since, $$\lim_{x\to 0}\cos\frac{1}{x} \text{ doesn't exist.}$$

Hence, $f'(x)$ is not continuous at $x=0$.

Example 23 Find the set of points where $x^2|x|$ is thrice differentiable.

Solution. Let $f(x)=x^2|x|$ which could be expressed as

$$f(x)=\begin{cases}-x^3, & x<0\\ 0, & x=0\\ x^3, & x>0\end{cases}$$

This gives, $$f'(x)=\begin{cases}-3x^2, & x<0\\ 0, & x=0\\ 3x^2, & x>0\end{cases}$$

So, $f'(x)$ exists for all real x.

$$f''(x)=\begin{cases}-6x, & x<0\\ 0, & x=0\\ 6x, & x>0\end{cases}$$

So, $f''(x)$ exists for all real x.

$$f'''(x)=\begin{cases}-6, & x<0\\ 0, & x=0\\ 6, & x>0\end{cases}$$

However, $f'''(0)$ doesn't exist, since $f'''(0^-)=-6$ and $f'''(0^+)=6$ which are not equal. Thus, the set of points where $f(x)$ is thrice differentiable is $R-\{0\}$.

Example 24 Find the number of points where $f(x)=[\sin x+\cos x]$ (where $[\cdot]$ denotes greatest integral function), $x\in[0,2\pi]$ is not continuous.

Solution. We know $[\cdot]$ is not continuous at integral points.

Thus, $f(x)=[\sin x+\cos x]$ will be discontinuous at those points, where $\sin x+\cos x$ is an integer, which is the case for

$$x\in\left\{\frac{\pi}{2},\frac{3\pi}{4},\pi,\frac{3\pi}{2},\frac{7\pi}{4}\right\}$$

Thus, the number of points at which $f(x)$ is discontinuous is 5.

Example 25 If $f(x)=\begin{cases}x-[x], & x\notin I\\ 1, & x\in I\end{cases}$

where I is an integer and $[\cdot]$ represents the greatest integer function and

$$g(x)=\lim_{n\to\infty}\frac{\{f(x)\}^{2n}-1}{\{f(x)\}^{2n}+1}, \text{ then}$$

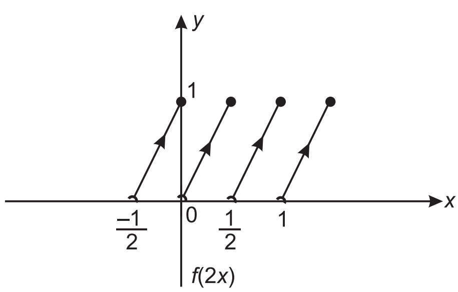
(a) Draw graphs of $f(2x)$, $g(x)$ and $g\{g(x)\}$ and discuss their continuity.

(b) Find the domain and range of these functions.

(c) Are these functions periodic? If yes, find their periods.

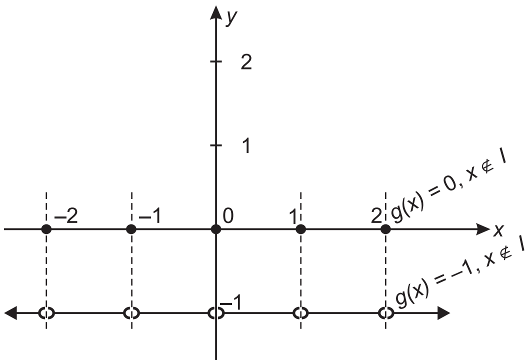
Solution. Here, $f(x)=\begin{cases}x-[x], & x\notin I\\ 1, & x\in I\end{cases}$

(a) Graph of $f(2x)$



As $$f(2x)=\begin{cases}2x-[2x], & 2x\notin I\\ 1, & 2x\in I\end{cases}$$

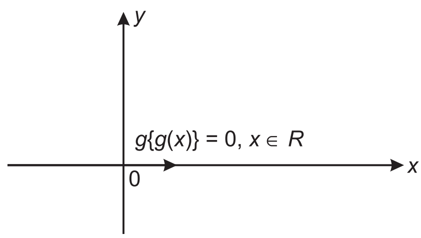
Graph of $g(x)$



Here, $$g(x) = \lim_{n \to \infty} \frac{\{f(x)\}^{2n} - 1}{\{f(x)\}^{2n} + 1}$$

$$g(x) = \begin{cases} 0, x \in I \\ -1, x \notin I, \text{ as } \lim_{n \to \infty} \{f(x)\}^{2n} = 0 \end{cases}$$

Graph of $g\{g(x)\}$



We have,

$$g(x) = \begin{cases} 0, & x \in I \\ -1, & x \notin I \end{cases}$$

$\Rightarrow$ $$g\{g(x)\} = \begin{cases} 0, & g(x) \in I \\ -1, & g(x) \notin I \end{cases}$$

where $g(x) \in \{0, -1\}$

and thus $g(x) \notin I$, should be neglected.

$\Rightarrow$ $g\{g(x)\} = 0, x \in R$ and could be plotted as,

(b) From the above three graphs, we have

Domain of $f(2x) \in R$

and Range of $f(2x) \in (0, 1]$

Domain of $g(x) \in R$

and Range of $g(x) \in \{0, -1\}$

Domain of $g(g(x)) \in R$

and Range of $g(g(x)) \in \{0\}$

(c) Here, $f(2x)$ and $g(x)$ are periodic with period 1/2 and 1 also $g\{g(x)\}$ is constant function.

Example 26 If $g(x) = \lim_{m \to \infty} \dfrac{x^m f(1) + h(x) + 1}{2x^m + 3x + 3}$ is continuous at $x = 1$ and $g(1) = \lim_{x \to 1} \{\log_e(ex)\}^{2/\log_e x}$, then find the value of $2g(1) + 2f(1) - h(1)$. Assume that $f(x)$ and $h(x)$ are continuous at $x = 1$.

Solution. Here, $g(1) = \lim_{x \to 1} \{\log e + \log_e x\}^{2/\log_e x}$

$$= \lim_{x \to 1} \{1 + \log_e x\}^{2/\log_e x} = e^{\lim_{x \to 1} \log_e x \frac{2}{\log_e x}}$$

$$g(1) = e^2 \quad \text{...(i)}$$

Also,
$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} \lim_{m \to \infty} \left\{ \frac{x^m f(1) + h(x) + 1}{2x^m + 3x + 3} \right\}$$

$$= \lim_{m \to \infty} \left\{ \lim_{x \to 1^-} \frac{x^m f(1) + h(x) + 1}{2x^m + 3x + 3} \right\}$$

$$= \frac{h(1) + 1}{3 + 3} \quad \left[\text{As } x < 1 \Rightarrow \lim_{m \to \infty} x^m = 0\right]$$

$$\lim_{x \to 1^-} g(x) = \frac{h(1) + 1}{6} \quad \text{...(ii)}$$

Again,
$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} \lim_{m \to \infty} \left\{ \frac{x^m f(1) + h(x) + 1}{2x^m + 3x + 3} \right\}$$

$$= \lim_{m \to \infty} \left\{ \lim_{x \to 1^+} \frac{x^m f(1) + h(x) + 1}{2x^m + 3x + 3} \right\}$$

$$= \lim_{m \to \infty} \lim_{x \to 1^+} \frac{f(1) + h(x)/x^m + 1/x^m}{2 + 3/x^{m-1} + 3/x^m} = \frac{f(1)}{2}$$

$\therefore$
$$\lim_{x \to 1^+} g(x) = \frac{f(1)}{2} \quad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii)

$$e^2 = \frac{h(1) + 1}{6} = \frac{f(1)}{2} \quad [\text{As } g(x) \text{ is continuous at } x = 1]$$

$\Rightarrow$ $h(1) = 6e^2 - 1$ and $f(1) = 2e^2$

$\therefore$ $2g(1) + 2f(1) - h(1) = 2e^2 + 4e^2 - 6e^2 + 1 = 1$

$\Rightarrow$ $2g(1) + 2f(1) - h(1) = 1$

Example 27 If $f(x) = \begin{cases} \dfrac{\sin [x^2]\pi}{x^2 - 3x - 18} + ax^3 + b, & \text{for } 0 \le x \le 1 \\ 2\cos \pi x + \tan^{-1} x, & \text{for } 1 < x \le 2 \end{cases}$

differentiable function in [0, 2], find a and b. (where $[\ \cdot\ ]$ denotes the greatest integer function.)

Solution. Here; $[x^2] = 0$, for all $0 \le x < 1$

and $[x^2] = 1$, for $x = 1$

$\therefore$ $\sin [x^2]\pi = 0$, for $0 \le x \le 1$

Hence,

$$f(x) = \begin{cases} ax^3 + b, & 0 \le x \le 1 \\ 2\cos \pi x + \tan^{-1} x, & 1 < x \le 2 \end{cases}$$

As $f(x)$ is differentiable in [0, 2] $\Rightarrow$ Continuous and differentiable at $x = 1$

$$\Rightarrow \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$a + b = -2 + \frac{\pi}{4} = a + b$$

$$\Rightarrow \quad a + b = -2 + \frac{\pi}{4} \quad \text{...(i)}$$

Again, since $f(x)$ is differentiable at $x = 1$

(LHD at $x = 1$) = (RHD at $x = 1$)

$$\Rightarrow \quad \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$\Rightarrow \quad \lim_{x \to 1^-} \frac{(ax^3 + b) - (a + b)}{x - 1} = \lim_{x \to 1^+} \frac{(2\cos \pi x + \tan^{-1} x) - (a + b)}{x - 1}$$

$$\Rightarrow \quad 3a = \lim_{x \to 1^+} \frac{-2\pi \sin \pi x + \dfrac{1}{1 + x^2}}{1}$$

$$\Rightarrow \quad 3a = \frac{1}{2} \quad \text{or} \quad a = \frac{1}{6} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$a = \frac{1}{6} \quad \text{and} \quad b = \frac{\pi}{4} - \frac{13}{6}$$

Example 28 Prove that $f(x) = [\tan x] + \sqrt{\tan x - [\tan x]}$. (where $[\cdot]$ denotes greatest integer function) is continuous in $\left[0, \frac{\pi}{2}\right)$.

Solution. Here, $f(x) = [\tan x] + \sqrt{\tan x - [\tan x]}$

or $\quad g(x) = [x] + \sqrt{x - [x]}$, where $x = \tan x \ge 0$

Then, for $a \in N$. We discuss continuity of $f(x)$ as

LHL at $x = a$

$$\lim_{x \to a^-} g(x) = \lim_{h \to 0} [a - h] + \sqrt{a - h - [a - h]}$$

$$= \lim_{h \to 0} (a - 1) + \sqrt{a - h - a + 1}$$

$$= a - 1 + 1$$

$$= a$$

$$\left\{ \begin{array}{l} \text{where ,} \\ a - 1 < a - h < a \\ \therefore [a - h] = a - 1 \end{array} \right.$$

Now, RHL at $x = a$

$$\lim_{x \to a^+} g(x) = \lim_{h \to 0} [a + h] + \sqrt{a + h - [a + h]}$$

$$= \lim_{h \to 0} a + \sqrt{a + h - a}$$

$$= a$$

$$\left\{ \begin{array}{l} \because a < a + h < a + 1 \\ \Rightarrow [a + h] = a \end{array} \right.$$

and $\quad g(a) = [a] + \sqrt{a - [a]} = a \quad \text{as } a \in N$

So, $g(x)$ is continuous $\forall\ a \in N$, now $g(x)$ is clearly continuous in $(a-1, a)$, $\forall\ a \in N$.
Hence, $g(x)$ is continuous in $[0, \infty)$.
Now, let $\phi(x) = \tan x$ which is continuous in $[0, \pi/2]$.
So, $g\{\phi(x)\}$ is continuous in $[0, \pi/2)$.
Hence, $f(x) = [\tan x] + \sqrt{\tan x - [\tan x]}$ is continuous in $[0, \pi/2)$.

Example 29 Discuss the continuity of the function $f(x) = \dfrac{|x+2|}{\tan^{-1}(x+2)}$.

Solution. Clearly, f is continuous except possibility at $x = -2$

RHL $\quad \lim\limits_{x \to -2^+} f(x) = \lim\limits_{x \to -2^+} \dfrac{(x+2)}{\tan^{-1}(x+2)} = 1$

LHL $\quad \lim\limits_{x \to -2^-} f(x) = \lim\limits_{x \to -2^-} \dfrac{-(x+2)}{\tan^{-1}(x+2)} = -1$

So, f is not continuous at $x \in R - \{-2\}$

Example 30 Determine the values of x for which the following functions fails to be continuous or differentiable

$$f(x) = \begin{cases} (1-x) & , x < 1 \\ (1-x)(2-x) & , 1 \le x \le 2 \\ (3-x) & , x > 2 \end{cases}$$

Justify your answer. **[IIT JEE 1995]**

Solution. By the given definition, it is clear that the function f is continuous and differentiable at all points except possibility at $x = 1$ and $x = 2$.

Continuity at $x = 1$

$$\text{LHL} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(1-x) = \lim_{h\to 0}(1-(1-h)) = 0$$

$$\text{RHL} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(1-x)(2-x) = \lim_{h\to 0}\{1-(1+h)\}\{2-(1+h)\} = 0$$

Also, $f(1) = 0$. Hence, LHL = RHL = $f(1)$
Therefore, $f(x)$ is continuous at $x = 1$.
Now, differentiability at $x = 1$

$$Lf'(1) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{h} = \lim_{h\to 0}\frac{1-(1-h)-0}{-h} = -1$$

$$Rf'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{\{1-(1+h)\}\{2-(1+h)\}-0}{h} = -1$$

Since, $\quad L\{f'(1)\} = R\{f'(1)\}$ so, we get $f(x)$ is differentiable at $x = 1$.

Continuity at $x = 2$

$$\text{LHL} = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(1-x)(2-x) = \lim_{h\to 0}(1-(2-h))(2-(2-h)) = 0$$

$$\text{RHL} = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(3-x) = \lim_{h\to 0} 3-(2+h) = 1$$

Since, RHL $\neq$ LHL
Therefore, $f(x)$ is not continuous at $x = 2$.
As such $f(x)$ cannot be differentiable at $x = 2$.
Hence, $f(x)$ is continuous and differentiable at all points except at $x = 2$.

Example 31 Let $g(x)=\int_0^x f(t)\,dt$, where f is such that $1/2 \le f(t) \le 1$ for $t \in [0, 1]$ and $0 \le f(t) \le 1/2$ for $t \in [1,2]$. Then, find the interval in which $g(2)$ lies.
[IIT JEE 2000]

Solution. $g(x)=\int_0^x f(t)\,dt$

$$\Rightarrow \quad g(2)=\int_0^2 f(t)\,dt$$

$$\Rightarrow \quad g(2)=\int_0^1 f(t)\,dt+\int_1^2 f(t)\,dt \quad \text{...(i)}$$

Now, $\quad 1/2 \le f(t) \le 1 \quad \text{for} \quad t \in [0,1]$

We get, $\quad \int_0^1 \frac{1}{2}\,dt \le \int_0^1 f(t)dt \le \int_0^1 1\,dt$

$$\Rightarrow \quad \frac{1}{2} \le \int_0^1 f(t)dt \le 1 \quad \text{...(ii)}$$

Again, $0 \le f(t) \le \frac{1}{2}$, for $t \in [1, 2]$

$$\Rightarrow \quad \int_1^2 0\,dt \le \int_1^2 f(t)dt \le \int_1^2 \frac{1}{2}\,dt$$

$$\Rightarrow \quad 0 \le \int_1^2 f(t)dt \le \frac{1}{2} \quad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get

$$0+\frac{1}{2} \le \int_0^1 f(t)dt + \int_1^2 f(t)dt \le 1+\frac{1}{2}$$

or $\quad \frac{1}{2} \le g(2) \le \frac{3}{2} \quad \Rightarrow \quad g(2) \in \left[\frac{1}{2}, \frac{3}{2}\right]$

Example 32 Let $\alpha \in R$. Prove that a function $f : R \to R$ is differentiable at α, if and only if there is a function $g : R \to R$ which is continuous at $x = \alpha$ and satisfies $f(x)-f(\alpha)=g(x)(x-\alpha)$ for all $x \in R$.
[IIT JEE 2001]

Solution. If $g(x)$ is continuous at $x=\alpha$.

$$\Rightarrow \quad \lim_{x\to\alpha} g(x)=g(\alpha) \quad \text{...(i)}$$

and $\quad f(x)-f(\alpha)=g(x)(x-\alpha),\ \forall\, x \in R$

$$\Rightarrow \quad \frac{f(x)-f(\alpha)}{(x-\alpha)} = g(x)$$

$$\Rightarrow \quad \lim_{x\to\alpha}\frac{f(x)-f(\alpha)}{(x-\alpha)} = \lim_{x\to\alpha} g(x) \Rightarrow f'(\alpha) = g(\alpha)$$

Therefore, $f(x)$ is differentiable at $x=\alpha$ ...(ii)

Conversely, f is differentiable at $x=\alpha$, then $\lim_{x\to\alpha}\frac{f(x)-f(\alpha)}{(x-\alpha)}$ exists finitely.

Let $$g(x) = \begin{cases} \dfrac{f(x)-f(\alpha)}{x-\alpha} &, x \ne \alpha \\ f'(\alpha) &, x = \alpha \end{cases}$$

Clearly, $\lim_{x \to \alpha} g(x) = f'(x)$

Hence, $g(x)$ is continuous at $x = \alpha$

Therefore, $f(x)$ is differentiable at $x = \alpha$

if $g(x)$ is continuous at $x = \alpha$.

Example 33 If $f(x)$ be a continuous function in $[0, 2\pi]$ and $f(0) = f(2\pi)$, then prove that there exists point $c \in (0, \pi)$ such that $f(c) = f(c + \pi)$.

Solution. Let $\quad g(x) = f(x) - f(x + \pi) \quad$ ...(i)

At $\quad x = \pi,\ g(\pi) = f(\pi) - f(2\pi) \quad$ ...(ii)

At $\quad x = 0,\ g(0) = f(0) - f(\pi) \quad$ ...(iii)

Adding Eqs. (ii) and (iii),

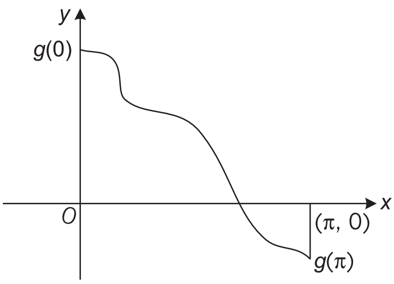
$$g(0) + g(\pi) = f(0) - f(2\pi)$$

$\Rightarrow \quad g(0) + g(\pi) = 0 \quad$ [Given $f(0) = f(2\pi)$]

$\Rightarrow \quad g(0) = - g(\pi)$

$\Rightarrow \quad g(0)$ and $g(\pi)$ are opposite in sign.

$\Rightarrow$ There exists a point c between 0 and π such that $g(c) = 0$ as shown in graph;



From Eq. (i) putting $x = c$

$$g(c) = f(c) - f(c + \pi) = 0$$

Hence, $\quad f(c) = f(c + \pi)$

Example 34 Let $y = f(x)$ be a differentiable function, $\forall\ x \in R$ and satisfies;

$f(x) = x + \int_0^1 x^2\ z\, f(z)\, dz + \int_0^1 x\, z^2\, f(z)\, dz$. Determine the function.

Solution. We have, $\quad f(x) = x + \int_0^1 x^2\ z\, f(z)\, dz + \int_0^1 x\, z^2 f(z)\, dz$

or $\quad f(x) = x + x^2 \int_0^1 z\, f(z)\, dz + x \int_0^1 z^2\, f(z)\, dz$

Let $\quad \lambda_1 = \int_0^1 z\, f(z)\, dz$ and $\lambda_2 = \int_0^1 z^2\, f(z)\, dz$

$\therefore \quad f(x) = x + x^2 \lambda_1 + x\lambda_2 \quad$ ...(i)

Now, $\quad \lambda_1 = \int_0^1 z\, f(z)\, dz$

$$\lambda_1 = \int_0^1 z\,(z + z^2 \lambda_1 + z\lambda_2)\, dz$$

[Using Eq. (i), as $f(z) = z + z^2 \lambda_1 + z\lambda_2$]

$\Rightarrow \quad \lambda_1 = (1 + \lambda_2) \int_0^1 z^2\, dz + \lambda_1 \int_0^1 z^3 dz$

$\Rightarrow \quad \lambda_1 = (1 + \lambda_2) \left(\frac{z^3}{3} \right)_0^1 + \lambda_1 \left(\frac{z^4}{4} \right)_0^1$

$\Rightarrow \quad \lambda_1 = \frac{1 + \lambda_2}{3} + \frac{\lambda_1}{4}$

$\Rightarrow \quad 9\lambda_1 - 4\lambda_2 = 4 \quad$ ...(ii)

Also, $$\lambda_2 = \int_0^1 z^2 f(z)\, dz$$

or $$\lambda_2 = \int_0^1 z^2 \{z + z^2\lambda_1 + z\lambda_2\}\, dz$$

[Using Eq. (i) as $f(z) = z + z^2\lambda_1 + z\lambda_2$]

$\Rightarrow$ $$\lambda_2 = \int_0^1 z^3 (1 + \lambda_2)\, dz + \lambda_1 \int_0^1 z^4\, dz$$

$\Rightarrow$ $$\lambda_2 = \frac{(1 + \lambda_2)}{4} + \frac{\lambda_1}{5}$$

or $$15\lambda_2 - 4\lambda_1 = 5 \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get

$$9\lambda_1 - 4\lambda_2 = 4 \quad \text{and} \quad 15\lambda_2 - 4\lambda_1 = 5$$

$$\lambda_1 = \frac{80}{119}, \lambda_2 = \frac{61}{119} \qquad \text{...(iv)}$$

Thus, Eq. (i) becomes

$$f(x) = x + \frac{80}{119}x^2 + \frac{61}{119}x$$

Hence, $$f(x) = \frac{20x}{119}(4 + 9x)$$

Example 35 Let f be a one-one function such that

$$f(x) \cdot f(y) + 2 = f(x) + f(y) + f(xy), \forall\ x, y \in R - \{0\} \text{ and } f(0) = 1, f'(1) = 2.$$

Prove that $3\left(\int f(x)\, dx\right) - x\,(f(x) + 2)$ is constant.

Solution. We have, $$f(x) \cdot f(y) + 2 = f(x) + f(y) + f(xy) \qquad \text{...(i)}$$

Replacing $x, y \to 1$, we get

$$(f(1))^2 + 2 = 3 f(1) \Rightarrow f^2(1) - 3 f(1) + 2 = 0$$

$\Rightarrow$ $$f(1) = 2, 1$$

But $f(1)$ cannot be equal to one as $f(0) = 1$

$\Rightarrow$ $$f(1) = 2 \qquad \text{...(ii)}$$

Replacing y by $1/x$ in Eq. (i),

$$f(x) \cdot f(1/x) + 2 = f(x) + f(1/x) + f(1)$$

$\Rightarrow$ $$f(x) \cdot f(1/x) + 2 = f(x) + f(1/x) + 2 \qquad \text{[Using Eq. (ii)]}$$

$\Rightarrow$ $$f(x) \cdot f(1/x) = f(x) + f(1/x)$$

$\Rightarrow$ $$f(x) = \frac{f(1/x)}{f(1/x) - 1} \quad \text{and} \quad f(1/x) = \frac{f(x)}{f(x) - 1} \qquad \text{...(iii)}$$

Now, $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x + h) + \dfrac{f(1/x)}{1 - f(1/x)}}{h}$$

$$= \lim_{h \to 0} \frac{f(x + h) - f(x + h) \cdot f(1/x) + f(1/x)}{h\{1 - f(1/x)\}}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x+h) - f(1/x) - f\left(\dfrac{x+h}{x}\right) + 2 + f(1/x)}{h\{1 - f(1/x)\}}$$

[Using Eq. (iii)]

$$= \lim_{h \to 0} \frac{f\left(1 + \dfrac{h}{x}\right) - 2}{h\{f(1/x) - 1\}} = \lim_{h \to 0} \frac{f(1 + h/x) - f(1)}{\dfrac{h}{x} \cdot x\ \{f(1/x) - 1\}}$$

$$= \frac{f'(1)}{x\{f(1/x) - 1\}}$$

$$\left\{\text{From Eq. (iii)}, f(x) \cdot f\left(\frac{1}{x}\right) = \frac{f(x) f\left(\dfrac{1}{x}\right)}{\left\{f\left(\dfrac{1}{x}\right) - 1\right\}\{f(x) - 1\}}\right\}$$

$$f'(x) = \frac{2\{f(x) - 1\}}{x}$$

$\Rightarrow \qquad xf'(x) = 2\{f(x) - 1\}$

Integrating both the sides of the above expression, we get

$$x\, f(x) - \int f(x)\, dx = 2\int f(x)\, dx - 2x + \lambda, \text{ ('}\lambda\text{' is the constant of integral)}$$

$\Rightarrow \qquad 3\int f(x)\, dx = x\{2 + f(x)\} - \lambda$

Hence, $\quad 3\int f(x)\, dx - x\{2 + f(x)\} = \lambda$ (constant)

Example 36 Let $f : R \to R$, such that $f'(0) = 1$ and
$f(x + 2y) = f(x) + f(2y) + e^{x+2y}(x + 2y) - x \cdot e^x - 2y \cdot e^{2y} + 4xy, \forall\ x, y \in R$. Find $f(x)$.

Solution. We have, $f(x + 2y) = f(x) + f(2y) + e^{x+2y}(x + 2y) - x \cdot e^x - 2y \cdot e^{2y} + 4xy$

Replacing $x, y \to 0$, we get

$$f(0) = f(0) + f(0) + 0 - 0 - 0 + 0 \quad \Rightarrow \quad f(0) = 0$$

Replacing $2y \to -x$, we get

$$f(0) = f(x) + f(-x) - x \cdot e^x + xe^{-x} - 2x^2$$

$\Rightarrow \qquad -f(x) = f(-x) - x \cdot e^x + xe^{-x} - 2x^2 \qquad$ ...(i)

Now, $\qquad f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{f(x+h) + f(-x) - xe^x + x \cdot e^{-x} - 2x^2}{h}$

$$\Rightarrow f'(x) = \lim_{h \to 0} \frac{f(h) - e^h \cdot h + (x+h)\, e^{(x+h)} - xe^{-x} + 2(x+h)x - xe^x + xe^{-x} - 2x^2}{h}$$

$\Rightarrow \qquad \lim_{h \to 0} \dfrac{f(h) - e^h \cdot h + x \cdot e^x \cdot e^h + h \cdot e^x \cdot e^h + 2hx - xe^x}{h}$

$\Rightarrow \qquad \lim_{h \to 0} \dfrac{f(h) - f(0)}{h} + \dfrac{h(e^x - 1)}{h} e^h + \dfrac{xe^x(e^h - 1)}{h} + \dfrac{2hx}{h}$

$\Rightarrow \qquad f'(0) + (e^x - 1) + x \cdot e^x + 2x$

$\therefore \qquad f'(x) = 1 + e^x - 1 + xe^x + 2x$

or $\qquad f'(x) = e^x(x + 1) + 2x \qquad$ ...(ii)

Integrating Eq. (ii) both sides,

$$f(x) = \int e^x (x+1)\, dx + 2\int x\, dx$$
$$= (x+1)e^x - \int 1 \cdot e^x dx + 2\frac{x^2}{2} + c$$
$$= (x+1)\, e^x - e^x + x^2 + c$$

$\Rightarrow$ $\qquad f(x) = x^2 + xe^x + c$

But $\qquad f(0) = 0 \quad \Rightarrow \quad c = 0$

So, $\qquad f(x) = x^2 + x \cdot e^x$

Example 37 If $g(x)$ is continuous function in $[0, \infty)$ satisfying $g(1) = 1$. If $\int_0^x 2x \cdot g^2(t)\, dt = \left(\int_0^x 2g\,(x-t)\, dt\right)^2$. Find $g(x)$.

Solution. Here, $\int_0^x g(t)\, dt = \int_0^x g(x-t)\, dt$, the given equation could be written as;

$$x\int_0^x g^2(t)\, dt = 2\left(\int_0^x g(t)\, dt\right)^2 \qquad \text{...(i)}$$

Differentiating both the sides w.r.t 'x', we get

$$\int_0^x g^2(t)\, dt + xg^2(x) = 4g(x)\left\{\int_0^x g(t)\, dt\right\}$$

or $$x\int_0^x g^2(t)\, dt + x^2 g^2(x) = 4xg(x)\int_0^x g(t)\, dt \qquad \text{...(ii)}$$

Using Eq. (i),

$$2\left(\int_0^x g(t)\, dt\right)^2 + x^2 g^2(x) = 4xg\,(x)\int_0^x g(t)\, dt$$

$\Rightarrow$ $$2\left(\int_0^x g(t)\, dt\right)^2 - 4xg(x)\int_0^x g(t)\, dt + x^2 g^2(x) = 0$$

which is quadratic in $\int_0^x g(t)\, dt$

$\therefore$ $\qquad \int_0^x g(t)\, dt = \frac{2 \pm \sqrt{2}}{2} xg(x)$, differentiate w.r.t. '$x$'

$$\frac{g'(x)}{g(x)} = \frac{1 \pm \sqrt{2}}{x} \Rightarrow g(x) = kx^{1 \pm \sqrt{2}}, \text{ since } g(1) = 1 \Rightarrow k = 1$$

$\therefore$ $\qquad g(x) = x^{1 \pm \sqrt{2}}$

which is continuous in $[0, \infty)$

Example 38 Let f is a differentiable function such that

$$f(x) = x^2 + \int_0^x e^{-t} f(x-t)\, dt. \text{ Find } f(x).$$

Solution. We have, $f(x) = x^2 + \int_0^x e^{-t} f(x-t)\, dt$

$\Rightarrow$ $$f(x) = x^2 + \int_0^x e^{-(x-t)} f(x-(x-t))\, dt$$

$$\left[\text{Using } \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$\Rightarrow$ $$f(x) = x^2 + e^{-x}\int_0^x e^t f(t)\, dt \qquad \text{...(i)}$$

Differentiating both the sides, we get

$$f'(x) = 2x + e^{-x}\,[e^x f(x)] - e^{-x}\int_0^x e^t f(t)\,dt \qquad \text{...(ii)}$$

[Using Leibnitz-rule]

$$f(x) + f'(x) = x^2 + 2x + f(x)$$

$$\Rightarrow \qquad f'(x) = x^2 + 2x \qquad \text{...(iii)}$$

Integrating both the sides of Eq. (iii),

$$\Rightarrow \qquad f(x) = \frac{x^3}{3} + x^2 + c$$

But $\qquad f(0) = 0 \quad \Rightarrow \quad c = 0$

Hence, $\qquad f(x) = \dfrac{x^3}{3} + x^2$

Example 39 Let $f : R^+ \to R$ satisfies the functional equation $f(xy) = e^{xy-x-y}\,\{e^y f(x) + e^x f(y)\}, \forall\, x, y \in R^+$. If $f'(1) = e$, determine $f(x)$.

Solution. Given that, $f(xy) = e^{xy-x-y}\,\{e^y f(x) + e^x f(y)\}, \forall\, x, y \in R^+$ ...(i)

Putting $x = y = 1$, we get

$$f(1) = e^{-1}\,\{e^1 f(1) + e^1 f(1)\}$$

$$\Rightarrow \qquad f(1) = 0 \qquad \text{...(ii)}$$

Now, $\qquad f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{f\left\{x\left(1 + \dfrac{h}{x}\right)\right\} - f(x)}{h}$

$$= \lim_{h \to 0} \frac{e^{x\left(1+\frac{h}{x}\right) - x - \left(1+\frac{h}{x}\right)}\left\{e^{1+\frac{h}{x}} f(x) + e^x f\left(1 + \dfrac{h}{x}\right)\right\} - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^h f(x) + e^{h-1-\frac{h}{x}+x} f\left(1 + \dfrac{h}{x}\right) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x)(e^h - 1) + e^{h-1-\frac{h}{x}+x}\left\{f\left(1 + \dfrac{h}{x}\right) - f(1)\right\}}{h} \qquad [\because f(1) = 0]$$

$$= \lim_{h \to 0} \frac{f(x)(e^h - 1)}{h} + \lim_{h \to 0} \frac{e^{h-1-\frac{h}{x}+x}\left\{f\left(1 + \dfrac{h}{x}\right) - f(1)\right\}}{\dfrac{h}{x}\cdot x}$$

$$= f(x) + \frac{e^{x-1}\cdot f'(1)}{x}$$

$$\left[\because \lim_{h \to 0} \frac{f\left(1 + \dfrac{h}{x}\right) - f(1)}{\dfrac{h}{x}} = f'(1) \text{ and } \lim_{h \to 0} \frac{e^h - 1}{h} = 1\right]$$

$$= f(x) + \frac{e^x}{ex}\cdot f'(1) \qquad [\because f'(1) = e]$$

$\therefore \qquad f'(x) = f(x) + \dfrac{e^x}{x}$

$\Rightarrow \qquad \dfrac{e^x}{x} = f'(x) - f(x) \Rightarrow \dfrac{1}{x} = \dfrac{e^x f'(x) - f(x)\cdot e^x}{e^{2x}}$

$$\left[\text{As by quotient rule, we can write } \frac{e^x f'(x) - f(x)\cdot e^x}{(e^x)^2} = \frac{d}{dx}\left\{\frac{f(x)}{e^x}\right\}\right]$$

$\therefore \qquad \dfrac{1}{x} = \dfrac{d}{dx}\left\{\dfrac{f(x)}{e^x}\right\}$

Integrating both the sides w.r.t. 'x', we get

$$\log|x| + c = \frac{f(x)}{e^x}$$

or $\qquad f(x) = e^x\{\log|x| + c\}$

Since, $\qquad f(1) = 0 \quad \Rightarrow \quad c = 0$

Thus, $\qquad f(x) = e^x \log|x|$

Example 40 Let f is a differentiable function such that

$$f'(x) = f(x) + \int_0^2 f(x)\,dx, f(0) = \frac{4-e^2}{3}, \text{ find } f(x).$$

Solution. It is given that

$$f'(x) = f(x) + \int_0^2 f(x)\,dx$$

$\Rightarrow \qquad f'(x) = f(x) + c \qquad \left[c = \int_0^2 f(x)\,dx \text{ (say)}\right]$

$\Rightarrow \qquad \dfrac{f'(x)}{f(x)+c} = 1$

Integrating both the sides of above expression

$\Rightarrow \qquad \log_e\{f(x) + c\} = x$

$\Rightarrow \qquad f(x) + c = ke^x \qquad$ ...(i)

[k being constant of integration]

Since, $\qquad f(0) = \dfrac{4-e^2}{3} \quad \Rightarrow \quad f(0) + c = k$

$\Rightarrow \qquad k = \dfrac{4-e^2}{3} + c \Rightarrow c = k - \left(\dfrac{4-e^2}{3}\right) \qquad$ ...(ii)

From Eqs. (i) and (ii), we get

$$f(x) + k - \left(\frac{4-e^2}{3}\right) = k(e^x)$$

$\Rightarrow \qquad f(x) = k(e^x - 1) + \left(\dfrac{4-e^2}{3}\right) \qquad$ ...(iii)

Now, to find constant of integration k.

Integrating both the sides from 0 to 2, we get

$$\int_0^2 f(x)\,dx = k\int_0^2 (e^x - 1)\,dx + \frac{4-e^2}{3}\int_0^2 dx$$

$$\Rightarrow \quad c = k(e^2 - 2 - 1) + \frac{4-e^2}{3}(2)$$

$$\Rightarrow \quad c = \frac{2}{3}(4 - e^2) + k(e^2 - 3) \quad \text{...(iv)}$$

From Eqs. (ii) and (iv), we get

$$k - \left(\frac{4-e^2}{3}\right) = \frac{2}{3}(4 - e^2) + k(e^2 - 3) \quad \Rightarrow \quad k = 1$$

Putting $k = 1$ in Eq. (iii), we get

$$f(x) = (e^x - 1) + \frac{4 - e^2}{3}$$

Hence, $$f(x) = e^x - \frac{(e^2 - 1)}{3}$$

Example 41 If $f(x) = ax^2 + bx + c$ is such that $|f(0)| \le 1$, $|f(1)| \le 1$ and $|f(-1)| \le 1$, prove that $|f(x)| \le 5/4$, $\forall\, x \in [-1, 1]$.

Solution. We have,

$$f(x) = ax^2 + bx + c$$

$$\Rightarrow \quad f(-1) = a - b + c \quad \text{...(i)}$$

$$\Rightarrow \quad f(0) = c \quad \text{...(ii)}$$

$$\Rightarrow \quad f(1) = a + b + c \quad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$\Rightarrow \quad a = \frac{1}{2}[f(-1) + f(1) - 2f(0)]$$

$$b = \frac{1}{2}[f(1) - f(-1)]$$

and $$c = f(0)$$

$$\Rightarrow \quad f(x) = \frac{1}{2}[f(-1) + f(1) - 2f(0)]x^2 + \frac{1}{2}[f(1) - f(-1)]x + f(0)$$

$$f(x) = \frac{x(x+1)}{2} f(-1) - (x+1)(x-1) f(0) + \frac{(x-1)x}{2} f(1) \quad \text{...(iv)}$$

Since, $|f(-1)|$, $|f(0)|$ and $|f(1)|$ are ≤ 1, we have

$$2|f(x)| \le |x(x+1)| + 2|x^2 - 1| + |x(x-1)| \quad \text{...(v)}$$

In the interval $x \in [-1, 1]$

$$0 \le 1 + x \le 2, \quad 0 \le 1 - x \le 2 \quad \text{and} \quad 0 \le 1 - x^2 \le 2$$

$$\Rightarrow \quad 2|f(x)| \le |x|(1 - x + 1 + x) + 2(1 - x^2)$$

$$2|f(x)| \le 2(|x| + 1 - x^2)$$

Therefore, $$|f(x)| \le \left(|x| - \frac{1}{2}\right)^2 + 5/4 \le 5/4$$

$$\Rightarrow \quad |f(x)| \le \frac{5}{4}, \forall\, x \in [-1, 1]$$

Example 42 Let $f(x)=\begin{cases} x+a, & \text{if } x<0 \\ |x-1|, & \text{if } x\ge 0 \end{cases}$ and $g(x)=\begin{cases} x+1, \text{ if } x<0 \\ (x-1)^2+b, \text{ if } x\ge 0 \end{cases}$

where a and b are non-negative real numbers. Determine the composite function gof. If $(gof)(x)$ is continuous for all real x. Determine the values of a and b. Further for these values of a and b, is gof differentiable at $x=0$. Justify your answer.

[IIT JEE 2002]

Solution. Here, $\quad f(x)=\begin{cases} x+a, & \text{if } x<0 \\ |x-1|, & \text{if } x\ge 0 \end{cases}$

and $\quad g(x)=\begin{cases} x+1, & \text{if } x<0 \\ (x-1)^2+b, & \text{if } x\ge 0 \end{cases}$

$$\therefore \quad gof(x)=g\{f(x)\}=\begin{cases} g(x+a), & x<0 \\ g(|x-1|), & x\ge 0 \end{cases}$$

$$=\begin{cases} x+a+1, & x+a<0 \\ (x+a-1)^2+b, & x+a\ge 0 \\ \{|x-1-1|\}^2+b, & x\ge 0 \end{cases}$$

$$=\begin{cases} x+a+1, & x<-a \\ (x+a-1)^2+b, & 0>x\ge -a \\ x^2+b, & 0\le x<1 \\ (x-2)^2+b, & x\ge 1 \end{cases}$$

$\Rightarrow \quad gof(x)$ is continuous for all real x.

$\therefore$ Continuous at $x=-a, 0, 1$

Since, continuous at $x=-a$

$$\Rightarrow \quad \lim_{x\to a^-} gof(x)=\lim_{x\to a^+} gof(x)=gof(-a)$$

$$\Rightarrow \quad \lim_{x\to -a^-}(x+a+1)=\lim_{x\to -a^+}(x+a-1)^2+b=(-a+a-1)^2+b$$

$$\Rightarrow \quad -a+a+1=(-a+a-1)^2+b=(-a+a-1)^2+b$$

$$\Rightarrow \quad b=0$$

and $gof(x)$ is continuous at $x=0$

$$\Rightarrow \quad \lim_{x\to 0^-} gof(x)=\lim_{x\to 0^+} gof(x)=gof(0)$$

$$\Rightarrow \quad (a-1)^2+b=b \quad \Rightarrow \quad a=1$$

Now, $\quad (\text{LHD at } x=0)=\dfrac{d}{dx}\{(x+a-1)^2+b\}_{\text{at } x=0}$

$$=2(a-1)=0 \ \{\text{As } a=1\}$$

Again, $\quad (\text{RHD at } x=0)=\dfrac{d}{dx}\{(x^2+b)\}_{\text{at } x=0}=0$

$\therefore \quad gof(x)$ is differentiable at $x=0$

Example 43 If a function $f:[-2a, 2a] \to R$ is an odd function such that $f(x) = f(2a - x)$ for $x \in [a, 2a]$ and the left hand derivative at $x = a$ is 0, then find the left hand derivative at $x = -a$. **[IIT JEE 2003]**

Solution. It is given that, (LHD at $x = a$) $= 0$

$$\therefore \quad \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = 0$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{f(a-h) - f(a)}{h} = 0 \quad \text{...(i)}$$

Now, (LHD at $x = -a$)

$$\Rightarrow \quad \lim_{h \to 0} \frac{f(-a-h) - f(-a)}{-h}$$

$$= \lim_{h \to 0} \frac{-f(a+h) + f(a)}{-h} \quad [\text{As } f(-x) = -f(x), \text{ given}]$$

$$= \lim_{h \to 0} \frac{f(a+h) - f(h)}{h}$$

$$= \lim_{h \to 0} \frac{f[2a - (a+h)] - f(h)}{h} \quad [\text{As } f(x) = f(2a - x)]$$

$$= \lim_{h \to 0} \frac{f(a-h) - f(h)}{h} = 0 \quad [\text{Using Eq. (i)}]$$

$$\therefore \quad (\text{LHD at } x = -a) = 0$$

Example 44 Let f be a function such that $f(xy) = f(x) \cdot f(y), \forall\, y \in R$ and $f(1+x) = 1 + x(1 + g(x))$, where $\lim_{x \to 0} g(x) = 0$. Find the value of $\int_1^2 \frac{f(x)}{f'(x)} \cdot \frac{1}{1+x^2}\, dx$.

Solution. We know, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f\left(x\left(1 + \frac{h}{x}\right)\right) - f(x)}{h}$

$$f'(x) = \lim_{h \to 0} \frac{f(x) \cdot f\left(1 + \frac{h}{x}\right) - f(x)}{h} \quad [\text{Given } f(xy) = f(x) \cdot f(y)]$$

$$f'(x) = \lim_{h \to 0} \frac{f(x) \cdot \left\{1 + \frac{h}{x}(1 + g(h/x))\right\} - f(x)}{h}$$

$$[\text{Given } f(1+x) = 1 + x(1 + g(x))]$$

$$f'(x) = \lim_{h \to 0} \frac{f(x) \cdot \left\{1 + \frac{h}{x}(1 + g(h/x)) - 1\right\}}{h}$$

$$f'(x) = \frac{f(x)}{x} \quad [\text{As } \lim_{x \to 0} g(x) = 0] \text{ ...(i)}$$

To find, $$\int_1^2 \frac{f(x)}{f'(x)} \cdot \frac{1}{1+x^2}\, dx = \int_1^2 \frac{x}{1+x^2}\, dx \quad [\text{Using Eq. (i)}]$$

$$= \frac{1}{2}[\log|1 + x^2|]_1^2 = \frac{1}{2}[\log(5/2)]$$

Example 45 Let $\alpha + \beta = 1$, $2\alpha^2 + 2\beta^2 = 1$ and $f(x)$ be a continuous function such that $f(2+x) + f(x) = 2$ for all $x \in [0, 2]$ and $p = \int_0^4 f(x)\,dx - 4$, $q = \dfrac{\alpha}{\beta}$. Then, find the least positive integral value of 'a' for which the equation $ax^2 - bx + c = 0$ has both roots lying between p and q, where $a, b, c \in N$.

Solution. Given, $\alpha + \beta = 1$ ...(i)

$$2\alpha^2 + 2\beta^2 = 1 \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$\alpha = \beta = \frac{1}{2} \quad \Rightarrow \quad q = \frac{\alpha}{\beta} = 1 \quad \text{...(iii)}$$

and given $\quad f(2+x) + f(x) = 2, \forall\, x \in [0, 2]$ ...(iv)

Now,
$$p = \int_0^4 f(x)\,dx - 4$$
$$= \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx - 4$$
$$= \int_0^2 f(x)\,dx + \int_0^2 f(t+2)\,dt - 4$$

[Let $x = t + 2$ for second integration]

$$= \int_0^2 f(x)\,dx + \int_0^2 \{2 - f(x)\}\,dx - 4$$
$$= \int_0^2 f(x)\,dx + 2\int_0^2 dx - \int_0^2 f(x)\,dx - 4 = 0$$

Then, $\quad p = 0, q = 1$

Let the roots of equation $ax^2 - bx + c = 0$ be α and β.

$\therefore \quad f(x) = ax^2 - bx + c = a(x - \alpha)(x - \beta)$ ...(v)

Since, equation $f(x) = 0$ has both roots lying between 0 and 1.

$\therefore \quad f(0) \cdot f(1) > 0$ ...(vi)

But $\quad f(0) \cdot f(1) = c(a - b + c) =$ An Integer ...(vii)

$\therefore$ Least value of $\quad f(0) \cdot f(1) = 1$ ...(viii)

Now, from Eq. (v),

$$f(0) \cdot f(1) = a\,\alpha\beta\, a(1-\alpha)(1-\beta)$$
$$= a^2\,\alpha\beta(1-\alpha)(1-\beta) \quad \text{...(ix)}$$

As we know,

$\alpha(1-\alpha)$ has greatest value $\dfrac{1}{4}$ at $\alpha = \dfrac{1}{2}$ and $\beta(1-\beta)$ has greatest value $\dfrac{1}{4}$ at $\beta = \dfrac{1}{2}$.

But $\alpha \neq \beta$

Thus, from Eq. (viii) greatest value of $f(0) \cdot f(1) < \dfrac{a^2}{16}$ ...(x)

$\therefore$ From Eqs. (viii) and (x), $\quad 1 < \dfrac{a^2}{16} \Rightarrow a^2 - 16 > 0$

$\Rightarrow \quad a < -4 \quad$ or $\quad a > 4 \qquad (\because a \in N)$

$\Rightarrow$ Least value of $a = 5 \qquad$ [As $a \in$ Natural number]

Example 46 Let f be a continuous function defined onto on [0, 1] with range [0, 1]. Show that there exists some 'c' in [0, 1] such that $f(c) = 1 - c$.

Solution. Consider $g(x) = f(x) - 1 + x$

$$g(0) = f(0) - 1 \le 0 \qquad [\text{As } f(0) \le 1]$$

$$g(1) = f(1) \ge 0 \qquad [\text{As } f(1) \ge 0]$$

Hence, $g(0)$ and $g(1)$ are of opposite signs,

hence, $\exists$ at least one $c \in [0, 1]$ such that $g(c) = 0$

$\therefore \qquad g(c) = f(c) - 1 + c = 0, f(c) = 1 - c.$

Example 47 Let $f : [0, 2] \to R$ be continuous and $f(0) = f(2)$. Prove that there exist x_1 and x_2 in (0, 2) such that $x_2 - x_1 = 1$ and $f(x_2) = f(x_1)$.

Solution. Consider continuous function g, as

$$g(x) = f(x + 1) - f(x) \qquad (x_2 = x_1 + 1)$$

Now, $\qquad g(0) = f(1) - f(0) = f(1) - f(2) \qquad$ ...(i)

$\qquad g(1) = f(2) - f(1) = f(2) - f(1) \qquad$ ...(ii)

Hence, $g(0)$ and $g(1)$ are of opposite signs, hence $\exists$ some $c \in (0, 1)$ where $g(c) = 0$

ie, $\qquad f(c + 1) = f(c) \qquad [c + 1 \in (1, 2), \text{ as } c \in (0, 1)]$

Put $\qquad c = x_1; c + 1 = x_2$

$\therefore \qquad f(x_2) = f(x_1)$, where $x_2 - x_1 = 1$

Obviously $\qquad x_1, x_2 \in (0, 2)$.

Example 48 Prove that the function $f(x) = a\sqrt{x - 1} + b\sqrt{2x - 1} - \sqrt{2x^2 - 3x + 1}$ where $a + 2b = 2$ and $a, b \in R$ always has a root in (1, 5), $\forall\, b \in R$.

Solution. Let $b > 0$, then $f(1) = b > 0$

and $\qquad f(5) = 2a + 3b - 6 = 2(a + 2b) - b - 6 = 4 - b - 6 = -(2 + b) < 0$

Hence, by IVT, $\exists$ some $c \in (1, 5)$ such that

$\Rightarrow \qquad f(c) = 0$

If $\qquad b = 0$, then $a = 2$

$$f(x) = 2\sqrt{x - 1} - \sqrt{2x^2 - 3x + 1} = 0$$

$\Rightarrow \qquad 4(x - 1) = 2x^2 - 3x + 1 = (2x - 1)(x - 1)$

$$(x - 1)(2x - 5) = 0 \quad \Rightarrow \quad x = \frac{5}{2}$$

Hence, $\qquad f(x) = 0$, if $x = \frac{5}{2}$, which lies in (1, 5)

If $\qquad b < 0, f(1) = b < 0 \quad$ and

$$\begin{aligned} f(2) &= a + b\sqrt{3} - \sqrt{3} \\ &= (a + 2b) + (\sqrt{3} - 2)b - \sqrt{3} \\ &= (2 - \sqrt{3}) - (2 - \sqrt{3})b \\ &= (2 - \sqrt{3})(1 - b) > 0 \qquad (\text{As } b < 0) \end{aligned}$$

Hence, $f(1)$ and $f(2)$ have opposite signs

$\exists$ some $c \in (1, 2) \subset (1, 5)$ for which $f(c) = 0$.

Type 2 : Only One Correct Option

Example 49 Let $g(x) = \log f(x)$ where $f(x)$ is a twice differentiable positive function on $(0, \infty)$ such that $f(x+1) = xf(x)$. Then, for $N = 1, 2, 3, \ldots, g''\left(N + \frac{1}{2}\right) - g''\left(\frac{1}{2}\right)$ is equal to **[IIT JEE 2008]**

(a) $-4\left\{1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2N-1)^2}\right\}$

(b) $4\left\{1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2N-1)^2}\right\}$

(c) $-4\left\{1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2N+1)^2}\right\}$

(d) $4\left\{1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2N+1)^2}\right\}$

Solution. $f(x) = e^{g(x)} \Rightarrow e^{g(x+1)} = f(x+1) = x f(x) = xe^{g(x)}$

$$g(x+1) = \log x + g(x)$$

ie, $$g(x+1) - g(x) = \log x \quad \ldots\text{(i)}$$

Replacing x by $x - \frac{1}{2}$,

$$g\left(x + \frac{1}{2}\right) - g\left(x - \frac{1}{2}\right) = \log\left(x - \frac{1}{2}\right) = \log(2x-1) - \log 2$$

$\therefore$ $$g''\left(x + \frac{1}{2}\right) - g''\left(x - \frac{1}{2}\right) = \frac{-4}{(2x-1)^2} \quad \ldots\text{(ii)}$$

Substituting, $x = 1, 2, 3, \ldots, N$ in Eq. (ii) and adding

$$g''\left(N + \frac{1}{2}\right) - g''\left(\frac{1}{2}\right) = -4\left\{1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2N-1)^2}\right\}$$

Hence, (a) is the correct answer.

Type 3 : More than One Correct Options

Example 50 Let f be a real-valued function defined on the interval $(0, \infty)$, by $f(x) = \ln x + \int_0^x \sqrt{1 + \sin t}\, dt$. Then, which of the following statement(s) is/are true? **[IIT JEE 2010]**

(a) $f''(x)$ exists for all $x \in (0, \infty)$
(b) $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$ but not differentiable on $(0, \infty)$
(c) There exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (0, \infty)$
(d) There exists $\beta > 0$ such that $|f(x)| + |f'(x)| \le \beta$ from all $x \in (0, \infty)$

Solution. Here, $f'(x) = \frac{1}{x} + \sqrt{1 + \sin x}$, $x > 0$ but $f(x)$ is not differentiable in $(0, \infty)$ as $\sin x$ may be -1 and then $f''(x) = -\frac{1}{x^2} + \frac{\cos x}{\sqrt{1 + \sin x}}$ will not exist.

$\Rightarrow$ $f'(x)$ is continuous for all $x \in (0, \infty)$ but $f'(x)$ is not differentiable on $(0, \infty)$.

$\therefore$ Option (b) is true.

Also, $\quad f'(x) \le 3$, if $x > 1$ and $f(x) > 3$, if $x > e^3$

$\therefore$ Let $\quad \alpha = e^3$

$\Rightarrow$ Option (c) is true.

(d) is not possible, as $f(x) \to \infty$ when $x \to \infty$.

Hence, (b) and (c) are the correct answers.

Example 51 If $f(x)$ be such that $f(x) = \max(|3 - x|, 3 - x^3)$, then

(a) $f(x)$ is continuous $\forall\ x \in R$

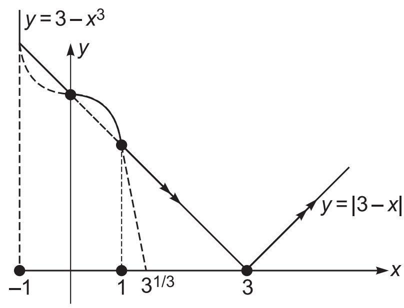
(b) $f(x)$ is differentiable $\forall\ x \in R$

(c) $f(x)$ is non-differentiable at three points only

(d) $f(x)$ is non-differentiable at four points only

Solution. From the graph of $f(x)$,

$f(x)$ is continuous, $\forall\ x \in R$ and $f(x)$ is not differentiable at $x = -1, 0, 1, 3$.



Hence, (a) and (d) are the correct answers.

Example 52 If $\lim_{x \to \infty} f(x^2) = a$ (a is finite number), then which of the following is/are true?

(a) $\lim_{x \to \infty} x^2 f'(x^2) = 0$ (b) $\lim_{x \to \infty} x^2 f'(x^2) = 2a$

(c) $\lim_{x \to \infty} x^4 f''(x^2) = 0$ (d) $\lim_{x \to \infty} x^4 f'''(x^2) = a$

Solution. We have, $\lim_{x \to \infty} f(x^2) = a \Rightarrow \lim_{x \to \infty} \frac{x f(x^2)}{x} = a$

Applying L'Hospital's rule, we have $\lim_{x \to \infty} x^2 f'(x^2) = 0$

Again, applying L'Hospital's rule in

$$\lim_{x \to \infty} \frac{x^3 f'(x^2)}{x} = 0, \text{ we get } \lim_{x \to \infty} x^4 f''(x^2) = 0$$

Hence, (a) and (c) are the correct answers.

Example 53 $f : R \to R$ is one-one, onto and differentiable function and graph of $y = f(x)$ is symmetrical about the point (4, 0), then

(a) $f^{-1}(2010) + f^{-1}(-2010) = 8$

(b) $\int_{-2010}^{2018} f(x)\, dx = 0$

(c) if $f'(-100) > 0$, then roots of $x^2 - f'(10)x - f'(10) = 0$ may be non-real

(d) if $f'(10) = 20$, then $f'(-2) = 20$

Solution. Graph is symmetrical about (4, 0).

$\Rightarrow \qquad f(4+x) = -f(4-x)$

$\Rightarrow \qquad f(x) = -f(8-x) \qquad \ldots(\text{i})$

Now, let $f(x) = 2010$, then $f(8-x) = -2010$

$\Rightarrow \qquad f^{-1}(2010) + f^{-1}(-2010) = 8$

$\Rightarrow$ Option (a) is true.

and $\qquad \int_{-2010}^{4} f(x)dx = -\int_{4}^{2018} f(x)\,dx \quad \Rightarrow$ Option (b) is true.

Also, $\qquad D = (f'(10))^2 + 4f'(10) > 0$

As, $\qquad f'(-100) > 0 \Rightarrow f'(10) \geq 0$

$\Rightarrow \qquad x^2 - f'(10)x - f'(10) = 0$ has real roots.

$\therefore$ Option (c) is false.

As, $\qquad f'(4+x) = f'(4-x)$

$f'(x)$ is symmetric about $x = 4 \Rightarrow f'(10) = f'(-2) = 20$

$\Rightarrow$ Option (d) is true.

Hence, (a), (b) and (d) are the correct answers.

Example 54 $f(x) = \sin^{-1}[e^x] + \sin^{-1}[e^{-x}]$, where $[\cdot]$ greatest integer function, then

(a) domain of $f(x) = (-\ln 2, \ln 2)$

(b) range of $f(x) = \{\pi\}$

(c) $f(x)$ has removable discontinuity at $x = 0$

(d) $f(x) = \cos^{-1} x$ has only solution

Solution. $0 < e^x < 2$ and $0 < e^{-x} < 2$

$\Rightarrow \qquad -\infty < x < \log_e 2 \quad$ and $\quad -\infty < -x < \log_e 2$

$\Rightarrow \qquad (-\infty < x < \log_e 2) \quad$ and $\quad (-\log_e 2 < x < \infty) \Rightarrow -\log_e 2 < x < \log_e 2$

$$\therefore \qquad f(x) = \begin{cases} \pi, & x = 0 \\ \dfrac{\pi}{2}, & x \in (-\log_e 2, 0) \cup (0, \log_e 2) \end{cases}$$

Hence, (a) and (c) are the correct answers.

Type 4 : **Assertion and Reason**

Directions

(Q. Nos. 55 to 56)

For the following questions, choose the correct answer from the codes (a), (b), (c) and (d) defined as follows :

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I.

(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I.

(c) Statement I is true, Statement II is false.

(d) Statement I is false, Statement II is true.

Example 55 **Statement I** The function $f(x) = (3x - 1)|4x^2 - 12x + 5|\cos \pi x$ is differentiable at $x = \frac{1}{2}, \frac{5}{2}$.

Statement II $\cos (2n + 1)\frac{\pi}{2} = 0, \forall\ n \in I$.

Solution. Statement I is correct although $|4x^2 - 12x + 5|$ is non-differentiable at $x = \frac{1}{2}, \frac{5}{2}$ but $\cos \pi x = 0$ those points.

So, $$f'\left(\frac{1}{2}\right) \text{ and } f'\left(\frac{5}{2}\right) \text{ exist.}$$

Hence, (a) is the correct answer.

Example 56 **Statement I** If $f(x)$ is discontinuous at $x = e$ and $\lim\limits_{x \to a} g(x) = e$, then $\lim\limits_{x \to a} f(g(x))$ cannot be equal to $f(\lim\limits_{x \to 0} g(x))$.

Statement II If $f(x)$ is continuous at $x = e$ and $\lim\limits_{x \to a} g(x) = e$, then $\lim\limits_{x \to a} f(g(x)) = f(\lim\limits_{x \to a} g(x))$.

Solution. Statement I is incorrect because, if $\lim\limits_{x \to a^-} g(x)$ and $\lim\limits_{x \to a^+} g(x)$ approach e from the same side of e (say right side). And $\lim\limits_{x \to e^+} f(x) = f(e) \neq \lim\limits_{x \to e^-} f(x)$, then $\lim\limits_{x \to a} f(g(x)) = f(e^+) = f(e)$

Statement II is correct.
Hence, (d) is the correct answer.

Type 6 : Match the Columns

Example 57 Suppose a function $f(x)$ satisfies the following conditions

$$f(x + y) = \frac{f(x) + f(y)}{1 + f(x) f(y)}, \forall\ x, y \text{ and } f'(0) = 1. \text{ Also, } -1 < f(x) < 1, \forall\ x \in R.$$

Match the entries of the following two columns.

Column I	Column II
(A) $f(x)$ is differentiable over the set	(p) $R - (-1, 0, 1)$
(B) $f(x)$ increases in the interval	(q) R
(C) Number of the solutions of $f(x) = 0$ is	(r) 0
(D) The value of the limit $\lim\limits_{x \to \infty} [f(x)]^x$ is	(s) 1

Solution. (A) $\to$ (q), (B) $\to$ (q), (C) $\to$ (s), D $\to$ (s)

Put $$x = y = 0 \Rightarrow f(0) = 0$$

Now, $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{f(x) + f(h)}{1 + f(x) f(h)} - f(x)}{h} = \lim_{h \to 0} \frac{f(h)[1 - \{f(x)\}^2]}{h[1 + f(x) f(h)]}$$

$$= \lim_{h \to 0} \left\{ \frac{f(h) - f(0)}{h - 0} \right\} \left[\frac{1 - \{f(x)\}^2}{1 + f(x) f(h)} \right] = f'(0)[1 - \{(x)\}^2] = 1 - \{f^2(x)\}$$

$\therefore \qquad f'(x) = 1 - \{f^2(x)\}$ ...(i)

Integrating, we get

$$\frac{1}{2} \ln \left[\frac{1 + f(x)}{1 - f(x)} \right] = x + c \text{ or } \frac{1 + f(x)}{1 - f(x)} = ke^{2x}$$

Now, $\qquad f(0) = 0 \Rightarrow k = 1$

$$\therefore \qquad f(x) = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Clearly, $f(x)$ is differentiable for all $x \in R$ and from Eq. (i),
$f'(x) > 0$ for all $x \in R$. Again, $f(x)$ is an odd function, $f(x) = 0 \Rightarrow x = 0$.

Now,
$$\lim_{x \to \infty} [f(x)]^x = \lim_{x \to \infty} \left(\frac{e^x - e^{-x}}{e^x + e^x} \right)^x$$

$$= e^{\lim_{x \to \infty} \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right) x} = e^{-2 \lim_{x \to \infty} \left(\frac{xe^{-x}}{e^x + e^{-x}} \right)}$$

$$= e^{-2 \lim_{x \to \infty} \left(\frac{x}{e^{2x} + 1} \right)} = e^{-2 \lim_{x \to \infty} \left(\frac{1}{2e^{2x}} \right)} = e^0 = 1$$

Example 58 Let $f(x) = \begin{cases} [x], & -2 \le x < 0 \\ |x|, & 0 \le x \le 2 \end{cases}$

(where [·] denotes the greatest integer function) $g(x) = \sec x, x \in R - (2n + 1)\pi / 2$.
Match the following statements in column I with their values in column II in the interval $\left(-\frac{3\pi}{2}, \frac{3\pi}{2} \right)$

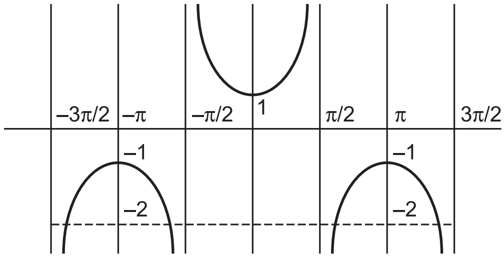
Column I	Column II
(A) Limit of *fog* exists at	(p) -1
(B) Limit of *gof* doesn't exist at	(q) π
(C) Points of discontinuity of *fog* is/are	(r) $\frac{5\pi}{6}$
(D) Points of differentiability of *fog* is/are	(s) $-\pi$

Solution. (A) $\to$ (p, q, s), (B) $\to$ (p), (C) $\to$ (q, s), (D) $\to$ (p, r)

$$f(x) = \begin{cases} [x], & -2 \le x < 0 \\ |x|, & 0 \le x \le 2 \end{cases} \Rightarrow f(x) = \begin{cases} -2, & -2 \le x < -1 \\ -1, & -1 \le x < 0 \\ x, & 0 \le x \le 2 \end{cases}$$

$$g(x) = \sec x, \, x \in R - (2n + 1)\frac{\pi}{2}$$

$$\Rightarrow \qquad fog = \begin{cases} -2, & -2 \le \sec x < -1 \\ -1, & -1 \le \sec x < 0 \\ \sec x, & 0 \le \sec x \le 2 \end{cases}$$



$$\therefore \qquad fog = \begin{cases} -2, & x \in \left[-\frac{4\pi}{3}, -\frac{2\pi}{3}\right] \cup \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right] - \{-\pi, \pi\} \\ -1, & x = -\pi, \pi \\ \sec x, & x \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right] \end{cases}$$

Limit of *fog* exist at $x = -\pi, \pi, -1$ points of discontinuity of *fog* are $-\pi, \pi$ points of differentiability of *fog* are $-1, \frac{5\pi}{6}$.

$$gof = \begin{cases} \sec(-2), & x \in [-2, -1) - \left\{-\frac{\pi}{2}\right\} \\ \sec(-1), & x \in [-1, 0) \\ \sec x, & x \in [0, 2] - \left\{\frac{\pi}{2}\right\} \end{cases}$$

Limit of *gof* doesn't exist at $x = -1$.

Example 59 In the following $[x]$ denotes the greatest integer less than or equal to x.
Match the functions in Column I with the properties Column II. **[IIT JEE 2007]**

Column I	Column II
(A) $x\,|x|$	(p) continuous in $(-1, 1)$
(B) $\sqrt{|x|}$	(q) differentiable in $(-1, 1)$
(C) $x + [x]$	(r) strictly increasing $(-1, 1)$
(D) $|x-1| + |x+1|$	(s) not differentiable at least at one point in $(-1, 1)$

Solution. (A) $\to$ (p, q, r), (B) $\to$ (p, s), (C) $\to$ (r, s), (D) $\to$ (p, q)

(A) $x|x|$ is continuous, differentiable and strictly increasing in $(-1, 1)$

(B) $\sqrt{|x|}$ is continuous in $(-1, 1)$ and not differentiable at $x = 0$

(C) $x + [x]$ is strictly increasing in $(-1, 1)$ and discontinuous at $x = 0$

$\Rightarrow$ Not differentiable at $x = 0$

(D) $|x-1| + |x+1| = 2$ in $(-1, 1)$

$\Rightarrow$ The function is continuous and differentiable in $(-1, 1)$

Example 60 Match the functions in Column I with the properties Column II.

Column I		Column II	
(A)	$g : R \to Q$ (Rational number), $f : R \to Q$ (Rational number); f and g are continuous functions such that $\sqrt{3}\, f(x) + g(x) = 3$, then $(1 - f(x))^3 + (g(x) - 3)^3$ is	(p)	1
(B)	If $f(x)$, $g(x)$ and $h(x)$ are continuous and positive functions such that $f(x) + g(x) + h(x)$ $= \sqrt{f(x)g(x)} + \sqrt{g(x)h(x)} + \sqrt{h(x)f(x)}$, then $f(x) + g(x) - 2h(x)$ is	(q)	0
(C)	$y = f(x)$ satisfies the equation $y^3 - 2y^2(x+1) + 4xy + (x^2 - 1)(y - 2) = 0$, then $y'(1) + y(1)$ would be equal to	(r)	2
(D)	If $y = f(x)$ satisfies $(xf(x))^{90} + (xf(x))^{98} + \ldots$ $\ldots + (xf(x)) + 1 = 0$, then $(1 + f(1))$ is	(s)	3
		(t)	-1

Solution. (A) $\to$ (p), (B) $\to$ (q), (C) $\to$ (p, r, s), (D) $\to$ (q)

(A) On comparing, $f(x) = 0$, $g(x) = 3$

$$\therefore \qquad (1 - f(x))^3 + (g(x) - 3)^3 \Rightarrow 1$$

(B) Here, $f(x) + g(x) + h(x) = \sqrt{f(x) \cdot g(x)} + \sqrt{g(x) \cdot h(x)} + \sqrt{h(x) \cdot f(x)}$

$$\Rightarrow \qquad \frac{1}{2}\{(\sqrt{f(x)} - \sqrt{g(x)})^2 + (\sqrt{g(x)} - \sqrt{h(x)})^2 + (\sqrt{h(x)} - \sqrt{f(x)})^2\} = 0$$

$$\Rightarrow \qquad f(x) = g(x) = h(x)$$

$$\therefore \qquad f(x) + g(x) - 2h(x) = 0$$

(C) $y^3 - 2y^2(x+1) + 4xy + (x^2 - 1)(y - 2) = 0$

Put $y = 2 \Rightarrow 8 - 8(x+1) + 8x \Rightarrow 0$

$\therefore y = 2$ is solution.

Put $\qquad y = (x+1)$

$$\Rightarrow \qquad (x+1)^3 - 2(x+1)^3 + 4x(x+1) + (x^2 - 1)(x - 1)$$

$$\Rightarrow \qquad -(x+1)^3 + 4x(x+1) + (x+1)(x-1)^2$$

$$\Rightarrow \qquad -(x+1)^3 + (x+1)\{4x + (x-1)^2\} \Rightarrow -(x+1)^3 + (x+1)^3 \Rightarrow 0$$

$\therefore$ $y = (x+1)$ is the solution.

Similarly, $y = (x - 1)$ is the solution. $\Rightarrow$ $y = 2, x + 1, x - 1$

$$\therefore \qquad \frac{dy}{dx} = 0, 1$$

$$\Rightarrow \qquad \begin{cases} y'(1) + y(1) = 0 + 2 = 2, \text{ when } y = 2 \\ y'(1) + y(1) = 1 + 2 = 3, \text{ when } y = x + 1 \\ y'(1) + y(1) = 1 + 0 = 1, \text{ when } y = x - 1 \end{cases}$$

(D) $\dfrac{(xf(x))^{100} - 1}{xf(x) - 1} = 0 \Rightarrow xf(x) = -1$

As, $\qquad xf(x) \neq 1 \qquad \therefore \quad xf(x) = -1$

$$\Rightarrow \qquad f(1) = -1 \text{ or } 1 + f(1) = 0$$

Type 7 : **Integer Answer Type Questions**

Example 61 If $f(x)$ is a differentiable function for all $x \in R$ such that $f(x)$ has fundamental period 2. $f(x)=0$ has exactly two solutions in [0, 2], also $f(0) \neq 0$. If minimum number of zeros of $h(x)=f'(x) \cos x - f(x) \sin x$ in (0, 99) is 120 + k, then k is ....

Solution. **(7)** $h(x)=\dfrac{d}{dx}(f(x)\cdot \cos x)$

First find the minimum number of zeros of $(f(x)\cdot \cos x)=0$.

$f(x)=0$ has minimum 98 roots in [0, 99) $\cos x=0$ has 31 roots [0, 99).

Maximum common possible root is only 1.

Hence, minimum number of roots of $f(x)\cos x=0$ is 128.

Thus, $\dfrac{d}{dx}(f(x)\cos x)=0$ has minimum 127 roots.

Example 62 Let $f: R \to R$ be a differentiable function satisfying $f(x)=f(y)f(x-y), \forall x, y \in R$ and $f'(0)=\int_0^4 \{2x\}\, dx$, where $\{\cdot\}$ denotes the fractional part function and $f'(-3)=\alpha e^{\beta}$. Then, $|\alpha+\beta|$ is equal to ....

Solution. **(4)** Given, $f(x)=f(y)f(x-y)$

Replace x by $(x+y)$,

$$f(x+y)=f(y)\cdot f(x)$$

$$\Rightarrow \quad f(x)=e^{kx} \Rightarrow f'(x)=ke^{kx}$$

But, $$f'(0)=\int_0^4 \{2x\}dx=2$$

$$\Rightarrow \quad f'(0)=k=2 \Rightarrow f'(x)=2e^{2x}$$

$$\Rightarrow \quad f'(-3)=2e^{-6} \Rightarrow |\alpha+\beta|=4$$

Example 63 Let $f: R \to R$ is a function satisfying $f(10-x)=f(x)$ and $f(2-x)=f(2+x), \forall x \in R$. If $f(0)=101$. Then, the minimum possible number of values of x satisfying $f(x)=101, x \in [0, 25]$ is ....

Solution. **(9)** Since, $f(10-x)=f(x)=f(4-x)$

$$\Rightarrow \quad f(10-x)=f(4-x)$$

Say, $$4-x=t \Rightarrow f(6+t)=f(t)$$

$\Rightarrow f(x)$ is periodic function with period 6.

So, for $x \in [0, 25]$

$$f(x)=101 \text{ at } x=0, 6, 12, 18, 24$$

Total numbers = 5

Since, $$f(2-x)=f(2+x)$$

$\Rightarrow f(x)$ is symmetric about $x=2$ line.

Due to symmetry in one period length.

$f(x)=101$ has one solution at $x=4$ other than 0 and 6.

Now, $$f(x)=101 \text{ at } x=4, 10, 16, 22$$

Total numbers = 4

Hence, at least minimum possible number of values of $x=9$.

Proficiency in 'Continuity and Differentiability'

Exercise 1

Type 1 : **Only One Correct Option**

1. For $x > 0$, let $h(x) = \begin{cases} \dfrac{1}{q}, & \text{if } x = \dfrac{p}{q} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$ where p and $q > 0$ are relatively prime integers, then which one of the following does not hold good?

(a) $h(x)$ is discontinuous for all x in $(0, \infty)$

(b) $h(x)$ is continuous for each irrational in $(0, \infty)$

(c) $h(x)$ is discontinuous for each rational in $(0, \infty)$

(d) $h(x)$ is not derivable for all x in $(0, \infty)$

2. Let $f(x) = \dfrac{g(x)}{h(x)}$, where g and h are continuous functions on the open interval (a, b). Which of the following statements is true for $a < x < b$?

(a) f is continuous at all x for which $x \neq 0$

(b) f is continuous at all x for which $g(x) = 0$

(c) f is continuous at all x for which $g(x) \neq 0$

(d) f is continuous at all x for which $h(x) \neq 0$

3. $f(x) = \dfrac{2\cos x - \sin 2x}{(\pi - 2x)^2}$, $g(x) = \dfrac{e^{-\cos x} - 1}{8x - 4\pi}$ and $h(x) = f(x)$ for $x < \pi/2$

$= g(x)$ for $x > \pi/2$,

then which of the following holds?

(a) h is continuous at $x = \pi/2$

(b) h has an irremovable discontinuity at $x = \pi/2$

(c) h has a removable discontinuity at $x = \pi/2$

(d) $f\left(\dfrac{\pi^+}{2}\right) = g\left(\dfrac{\pi^-}{2}\right)$

4. If $f(x) = \dfrac{x - e^x + \cos 2x}{x^2}$, $x \neq 0$ is continuous at $x = 0$, then

(a) $f(0) = \dfrac{5}{2}$

(b) $[f(0)] = -2$

(c) $\{f(0)\} = -0.5$

(d) $[f(0)] \cdot \{f(0)\} = -1.5$

(where $[x]$ and $\{x\}$ denotes greatest integer and fractional part function.)

5. Consider the function $f(x)=\begin{cases} x\{x\}+1, & \text{if } 0\le x<1 \\ 2-\{x\}, & \text{if } 1\le x\le 2 \end{cases}$ where $\{x\}$ denotes the fractional part function. Which one of the following statements is NOT correct?

(a) $\lim_{x\to 1} f(x)$ exists

(b) $f(0)\neq f(2)$

(c) $f(x)$ is continuous in $[0, 2]$

(d) Rolles theorem is not applicable to $f(x)$ in $[0, 2]$

6. Let $f(x)=\begin{cases} \dfrac{2^x+2^{3-x}-6}{\sqrt{2^{-x}}-2^{1-x}}, & \text{if } x>2 \\ \dfrac{x^2-4}{x-\sqrt{3x-2}}, & \text{if } x<2 \end{cases}$, then

(a) $f(2)=8 \Rightarrow f$ is continuous at $x=2$

(b) $f(2)=16 \Rightarrow f$ is continuous at $x=2$

(c) $f(2^-)\neq f(2^+) \Rightarrow f$ is discontinuous

(d) f has a removable discontinuity at $x=2$

7. Let $[x]$ denotes the integral part of $x\in R. g(x)=x-[x]$. Let $f(x)$ be any continuous function with $f(0)=f(1)$, then the function $h(x)=f(g(x))$

(a) has finitely many discontinuities (b) is discontinuous at some $x=c$

(c) is continuous on R (d) is a constant function.

8. Let f be a differentiable function on the open interval (a, b). Which of the following statements must be true?

I. f is continuous on the closed interval $[a, b]$

II. f is bounded on the open interval (a, b)

III. If $a<a_1<b_1<b$ and $f(a_1)<0<f(b_1)$, then there exists number c such that $a_1<c<b_1$ and $f(c)=0$

(a) Only I and II (b) Only I and III (c) Only II and III (d) Only III

9. Number of points where the function $f(x)=(x^2-1)|x^2-x-2|+\sin(|x|)$ is not differentiable, is

(a) 0 (b) 1 (c) 2 (d) 3

10. Consider function $f: R-\{-1, 1\}\to R. f(x)=\dfrac{x}{1-|x|}$.

Then, the incorrect statement is

(a) it is continuous at the origin

(b) it is not derivable at the origin

(c) the range of the function is R

(d) f is continuous and derivable in its domain

11. If the functions $f: R\to R$ and $g: R\to R$ are such that $f(x)$ is continuous at $x=\alpha$ and $f(\alpha)=a$ and $g(x)$ is discontinuous at $x=a$ but $g(f(x))$ is continuous at $x=\alpha$, then ($f(x)$ and $g(x)$ are non-constant functions.)

(a) $x=\alpha$ is a extremum of $f(x)$ and $x=a$ is an extremum of $g(x)$

(b) $x=\alpha$ may not be an extremum of $f(x)$ and $x=a$ is an extremum of $g(x)$

(c) $x=\alpha$ is ann extremum of $f(x)$ and $x=a$ may not be an extremum of $g(x)$

(d) None of the above

12. The total number of points of non-differentiability of $f(x) = \min\left[|\sin x|, |\cos x|, \frac{1}{4}\right]$ in $(0, 2\pi)$ is

(a) 8 (b) 9
(c) 10 (d) 11

13. The function $f(x) = [x]^2 - [x^2]$ (where $[y]$ is the greatest integer less than or equal to y) is discontinuous at

(a) all integers (b) all integers except 0 and 1
(c) all integers except 0 (d) all integers except 1

14. The function $f(x) = (x^2 - 1)|x^2 - 6x + 5| + \cos|x|$ is not differentiable at

(a) -1 (b) 0
(c) 1 (d) 5

15. Let $f(x) = \begin{cases} \dfrac{1}{e^{x^2}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

$f'(0)$ is equal to

(a) 0 (b) 1
(c) -1 (d) Doesn't exist

16. Given $f(x) = \dfrac{e^x - \cos 2x - x}{x^2}$, for $x \in R - \{0\}$

$g(x) = f(\{x\})$, for $n < x < n + \frac{1}{2}$

$= f(1 - \{x\})$, for $n + \frac{1}{2} \leq x < n + 1$, $n \in I$ $\left\{\begin{array}{l}\text{where } \{x\} \text{ denotes} \\ \text{fractional part} \\ \text{function}\end{array}\right.$

$= \frac{5}{2}$ otherwise, then $g(x)$ is

(a) discontinuous at all integral values of x only
(b) continuous everywhere except for $x = 0$
(c) discontinuous at $x = n + \frac{1}{2}$, $n \in I$ and at some $x \in I$
(d) continuous everywhere

17. The function $g(x) = \begin{bmatrix} x + b, & x < 0 \\ \cos x, & x \geq 0 \end{bmatrix}$ can be made differentiable at $x = 0$,

(a) if b is equal to zero (b) if b is not equal to zero
(c) if b takes any real value (d) for no value of b

18. The graph of function f contains the point $P(1, 2)$ and $Q(s, r)$. The equation of the secant line through P and Q is $y = \left(\dfrac{s^2 + 2s - 3}{s - 1}\right)x - 1 - s$. The value of $f'(1)$, is

(a) 2 (b) 3
(c) 4 (d) Non existent

19. Consider $f(x)=\left[\dfrac{2(\sin x-\sin^3 x)+|\sin x-\sin^3 x|}{2(\sin x-\sin^3 x)-|\sin x-\sin^3 x|}\right], x\neq\dfrac{\pi}{2}$ for $x\in(0,\pi)\,f(\pi/2)=3$

where [ ] denotes the greatest integer function, then

(a) f is continuous and differentiable at $x=\pi/2$

(b) f is continuous but not differentiable at $x=\pi/2$

(c) f is neither continuous nor differentiable at $x=\pi/2$

(d) None of the above

20. If $f(x+y)=f(x)+f(y)+|x|y+xy^2, \forall x, y\in R$ and $f'(0)=0$, then

(a) f need not be differentiable at every non-zero x

(b) f is differentiable for all $x\in R$

(c) f is twice differentiable at $x=0$

(d) None of the above

21. Let $f(x)=\max\{|x^2-2|x||, |x|\}$ and $g(x)=\min\{|x^2-2|x||, |x|\}$, then

(a) both $f(x)$ and $g(x)$ are non-differentiable at 5 points.

(b) $f(x)$ is not differentiable at 5 points whether $g(x)$ is non-differentiable at 7 points.

(c) number of points of non-differentiability for $f(x)$ and $g(x)$ are 7 and 5 points respectively.

(d) both $f(x)$ and $g(x)$ are non-differentiable at 3 and 5 points respectively

22. Let $g(x)=\begin{cases}3x^2-4\sqrt{x}+1, & \text{for } x<1\\ ax+b, & \text{for } x\geq 1\end{cases}$. If $g(x)$ is the continuous and differentiable for all numbers in its domain, then

(a) $a=b=4$ (b) $a=b=-4$

(c) $a=4$ and $b=-4$ (d) $a=-4$ and $b=4$

23. Let $f(x)$ be continuous and differentiable function for all reals.
$f(x+y)=f(x)-3xy+f(y)$. If $\lim\limits_{h\to 0}\dfrac{f(h)}{h}=7$, then the value of $f'(x)$ is

(a) $-3x$ (b) 7

(c) $-3x+7$ (d) $2f(x)+7$

24. Let $[x]$ be the greatest integer function and $f(x)=\dfrac{\sin\frac{1}{4}\pi[x]}{[x]}$. Then, which one of the following does not hold good?

(a) Not continuous at any point (b) Continuous at 3/2

(c) Discontinuous at 2 (d) Differentiable at 4/3

25. Given, $f(x)=\begin{cases}b([x]^2+[x])+1, & \text{for } x\geq -1\\ \sin(\pi(x+a)), & \text{for } x<-1\end{cases}$ where $[x]$ denotes the integral part of x, then for what values of a, b the function is continuous at $x=-1$?

(a) $a=2n+(3/2);\ b\in R;\ n\in I$

(b) $a=4n+2;\ b\in R;\ n\in I$

(c) $a=4n+(3/2);\ b\in R^+;\ n\in I$

(d) $a=4n+1;\ b\in R^+;\ n\in I$

26. If both $f(x)$ and $g(x)$ are differentiable functions at $x = x_0$, then the function defined as, $h(x) = \text{maximum}\ \{f(x), g(x)\}$

(a) is always differentiable at $x = x_0$
(b) is never differentiable at $x = x_0$
(c) is differentiable at $x = x_0$ when $f(x_0) \neq g(x_0)$
(d) cannot be differentiable at $x = x_0$, if $f(x_0) = g(x_0)$

Type 2 : More than One Correct Options

27. If $f(x) = \begin{cases} \dfrac{x \cdot \ln(\cos x)}{\ln(1+x^2)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then

(a) f is continuous at $x = 0$
(b) f is continuous at $x = 0$ but not differentiable at $x = 0$
(c) f is differentiable at $x = 0$
(d) f is not continuous at $x = 0$

28. Let $[x]$ denotes the greatest integer less than or equal to x. If $f(x) = [x \sin \pi x]$, then $f(x)$ is

(a) continuous at $x = 0$
(b) continuous in $(-1, 0)$
(c) differentiable at $x = 1$
(d) differentiable in $(-1, 1)$

29. The function $f(x) = [\,|x|\,] - |[x]|$, where $[x]$ denotes greatest integer function

(a) is continuous for all positive integers
(b) is discontinuous for all non-positive integers
(c) has finite number of elements in its range
(d) is such that its graph does not lie above the x-axis

30. The function $f(x) = \sqrt{1 - \sqrt{1 - x^2}}$

(a) has its domain $-1 \leq x \leq 1$
(b) has finite one sided derivates at the point $x = 0$
(c) is continuous and differentiable at $x = 0$
(d) is continuous but not differentiable at $x = 0$

31. Consider the function $f(x) = |x^3 + 1|$. Then,

(a) Domain of f $x \in R$
(b) Range of f is R^+
(c) f has no inverse
(d) f is continuous and differentiable for every $x \in R$

32. f is a continuous function in $[a, b]$, g is a continuous function in $[b, c]$. A function $h(x)$ is defined as $h(x) = \begin{cases} f(x) \text{ for } x \in [a, b) \\ g(x) \text{ for } x \in (b, c] \end{cases}$. If $f(b) = g(b)$, then

(a) $h(x)$ has a removable discontinuity at $x = b$
(b) $h(x)$ may or may not be continuous in $[a, c]$
(c) $h(b^-) = g(b^+)$ and $h(b^+) = f(b^-)$
(d) $h(b^+) = g(b^-)$ and $h(b^-) = f(b^+)$

33. Which of the following function(s) has/have the same range?

(a) $f(x)=\dfrac{1}{1+x}$ (b) $f(x)=\dfrac{1}{1+x^2}$ (c) $f(x)=\dfrac{1}{1+\sqrt{x}}$ (d) $f(x)=\dfrac{1}{\sqrt{3-x}}$

34. If $f(x)=\sec 2x+\operatorname{cosec} 2x$, then $f(x)$ is discontinuous at all points in

(a) $\{n\pi,\ n\in N\}$ (b) $\left\{(2n\pm 1)\dfrac{\pi}{4},\ n\in I\right\}$

(c) $\left\{\dfrac{n\pi}{4},\ n\in I\right\}$ (d) $\left\{(2n\pm 1)\dfrac{\pi}{8},\ n\in I\right\}$

35. Let $f(x)=\begin{cases} x^n \sin\left(\dfrac{1}{x^2}\right), & x\neq 0 \\ 0, & x=0 \end{cases}$, $(n\in I)$, then

(a) $\lim\limits_{x\to 0} f(x)$ exists for every $n>1$

(b) f is continuous at $x=0$ for $n>1$

(c) f is differentiable at $x=0$ for every $n>1$

(d) None of the above

36. A function is defined as $f(x)=\begin{cases} e^x, & x\leq 0 \\ |x-1|, & x>0 \end{cases}$, then $f(x)$ is

(a) continuous at $x=0$ (b) continuous at $x=1$

(c) differentiable at $x=0$ (d) differentiable at $x=1$

37. Let $f(x)=\int_{-2}^{x}|t+1|\,dt$, then

(a) $f(x)$ is continuous in $[-1,1]$

(b) $f(x)$ is differentiable in $[-1,1]$

(c) $f'(x)$ is continuous in $[-1,1]$

(d) $f'(x)$ is differentiable in $[-1,1]$

38. A function $f(x)$ satisfies the relation $f(x+y)=f(x)+f(y)+xy(x+y)$, $\forall\ x, y\in R$. If $f'(0)=-1$, then

(a) $f(x)$ is a polynomial function

(b) $f(x)$ is an exponential function

(c) $f(x)$ is twice differentiable for all $x\in R$

(d) $f'(3)=8$

39. If $f(x)=\begin{cases} 3x^2+12x-1, & -1\leq x\leq 2 \\ 37-x, & 2<x\leq 3 \end{cases}$, then

(a) $f(x)$ is increasing on $[-1,2]$

(b) $f(x)$ is continuous on $[-1,3]$

(c) $f'(2)$ doesn't exist

(d) $f(x)$ has the maximum value at $x=2$

40. If $f(x)=0$ for $x<0$ and $f(x)$ is differentiable at $x=0$, then for $x>0$, $f(x)$ may be

(a) x^2 (b) x

(c) $-x$ (d) $-x^{3/2}$

Type 3 : **Assertion and Reason**

Directions

(Q. Nos. 41 to 45)

For the following questions, choose the correct answer from the codes (a), (b), (c) and (d) defined as follows :

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I.

(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I.

(c) Statement I is true, Statement II is false.

(d) Statement I is false, Statement II is true.

41. Let $h(x) = f_1(x) + f_2(x) + f_3(x) + \dots + f_n(x)$, where $f_1(x), f_2(x), f_3(x), \dots, f_n(x)$ are real-valued functions of x.

Statement I $f(x) = |\cos|x|| + \cos^{-1}(\text{sgn } x) + |\ln x|$ is not differentiable at 3 points in $(0, 2\pi)$.

Because

Statement II Exactly one function $f_i(x)$, $i = 1, 2, \dots, n$ not differentiable and the rest of the function differentiable at $x = a$ makes $h(x)$ not differentiable at $x = a$.

42. **Statement I** $f(x) = |x| \sin x$ is differentiable at $x = 0$.

Because

Statement II If $g(x)$ is not differentiable at $x = a$ and $h(x)$ is differentiable at $x = a$, then $g(x) \cdot h(x)$ cannot be differentiable at $x = a$.

43. **Statement I** $f(x) = |\cos x|$ is not derivable at $x = \frac{\pi}{2}$.

Because

Statement II If $g(x)$ is differentiable at $x = a$ and $g(a) = 0$, then $|g(x)|$ is non-derivable at $x = a$.

44. Let $f(x) = x - x^2$ and $g(x) = \{x\}, \forall x \in R$ where { } denotes fractional part function.

Statement I $f(g(x))$ will be continuous, $\forall x \in R$.

Because

Statement II $f(0) = f(1)$ and $g(x)$ is periodic with period 1.

45. Let $f(x) = \begin{cases} -ax^2 - b|x| - c, & -\alpha \le x < 0 \\ ax^2 + b|x| + c, & 0 \le x \le \alpha \end{cases}$ where a, b, c are positive and $\alpha > 0$, then

Statement I The equation $f(x) = 0$ has at least one real root for $x \in [-\alpha, \alpha]$.

Because

Statement II Values of $f(-\alpha)$ and $f(\alpha)$ are opposite in sign.

Type 4 : **Linked Comprehension Based Questions**

Passage I
(Q. Nos. 46 to 48)

Let $y = f(x)$ *be defined in* $[a, b]$, *then*

(i) *Test of continuity at* $x = c, a < c < b$

(ii) *Test of continuity at* $x = a$

(iii) *Test of continuity at* $x = b$

Case I *Test of continuity at* $x = c, a < c < b$

If $y = f(x)$ *be defined at* $x = c$ *and its value* $f(C)$ *be equal to limit of* $f(x)$ *as* $x \to c$ *ie*, $f(c) = \lim_{x \to c} f(x)$

or $$\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x)$$

or $$LHL = f(C) = RHL$$

Then, $y = f(x)$ *is continuous at* $x = c$.

Case II *Test of continuity at* $x = a$

If $$RHL = f(A)$$

Then, $f(x)$ *is said to be continuous at the end point* $x = a$ (*Note*).

Case III *Test of continuity at* $x = b$, *if* $LHL = f(B)$

Then, $f(x)$ *is continuous at right end* $x = b$.

46. $f(x) = \begin{cases} \sin x, & x \le 0 \\ \tan x, & 0 < x < 2\pi \\ \cos x, & 2\pi \le x < 3\pi \\ 3\pi, & x = 3\pi \end{cases}$, then $f(x)$ is discontinuous at

(a) $\frac{\pi}{2}, \frac{3\pi}{2}, 2\pi$ (b) $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 3\pi$

(c) $\frac{\pi}{2}, 2\pi$ (d) None of these

47. Number of points of discontinuity of $[2x^3 - 5]$ in $[1, 2)$ is (where $[\cdot]$ denotes the greatest integral function.)

(a) 14 (b) 13 (c) 10 (d) None of these

48. Max $([x], |x|)$ is discontinuous at,

(a) $x = 0$ (b) ϕ (c) $x = n, n \in I$ (d) None of these

Passage II
(Q. Nos. 49 to 51)

$f(x) = \cos x$ *and* $H_1(x) = \text{Min}\{f(t), 0 \le t < x\}$;

$$0 \le x \le \frac{\pi}{2} = \frac{\pi}{2} - x; \frac{\pi}{2} < x \le \pi$$

$f(x) = \cos x$ *and* $H_2(x) = \text{Max}\{f(t), 0 \le t \le x\}$;

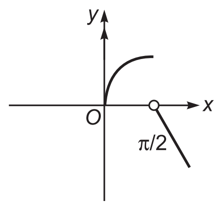
$$0 \le x \le \frac{\pi}{2} = \frac{\pi}{2} - x, \frac{\pi}{2} < x \le \pi$$

$g(x) = \sin x$ *and* $H_3(x) = \text{Min}\{g(t), 0 \le t \le x\};$

$$0 \le x \le \frac{\pi}{2} = \frac{\pi}{2} - x; \frac{\pi}{2} \le x \le \pi$$

$g(x) = \sin x$ *and* $H_4(x) = \text{Max}\{g(t), 0 \le t \le x\};$

$$0 \le x \le \frac{\pi}{2} = \frac{\pi}{2} - x; \frac{\pi}{2} < x \le \pi$$



49. (a) $H_2(x)$ is continuous and derivable in $[0, \pi]$

(b) $H_2(x)$ is continuous but not derivable at $x = \frac{\pi}{2}$

(c) $H_2(x)$ is neither continuous nor derivable at $x = \frac{\pi}{2}$

(d) None of the above

50. (a) $H_3(x)$ is continuous and derivable in $[0, \pi]$

(b) $H_3(x)$ is continuous but not derivable at $x = \frac{\pi}{2}$

(c) $H_3(x)$ is neither continuous nor derivable at $x = \frac{\pi}{2}$

(d) None of the above

51. (a) $H_4(x)$ is continuous and derivable in $[0, \pi]$

(b) $H_4(x)$ is continuous but not derivable at $x = \frac{\pi}{2}$

(c) $H_4(x)$ is neither continuous nor derivable at $x = \frac{\pi}{2}$

(d) None of the above

Type 5 : **Match the Columns**

52. Match the entries of the following two columns.

Column I	Column II
(A) $f(x) = \begin{cases} x+1, & \text{if } x < 0 \\ \cos x, & \text{if } x \ge 0 \end{cases}$ at $x = 0$ is	(p) continuous
(B) For every $x \in R$, the function $g(x) = \dfrac{\sin(\pi[x-\pi])}{1+[x]^2}$ where $[x]$ denotes the greatest integer function is	(q) differentiability
(C) $h(x) = \sqrt{\{x\}^2}$ where $\{x\}$ denotes fractional part function for all $x \in I$, is	(r) discontinuous
(D) $k(x) = \begin{cases} x^{\frac{1}{\ln x}}, & \text{if } x \ne 1 \\ e, & \text{if } x = 1 \end{cases}$ at $x = 1$ is	(s) non-derivable

53. Match the entries of the following two columns.

Column I	Column II
(A) $\lim_{x \to \infty} \left(e^{\sqrt{x^4+1}} - e^{(x^2+1)} \right)$ is	(p) e
(B) For $a > 0$, let $f(x) = \begin{cases} \dfrac{a^x + a^{-x} - 2}{x^2}, & \text{if } x > 0 \\ 3 \ln(a - x) - 2, & \text{if } x \le 0 \end{cases}$	(q) e^2
If f is continuous at $x = 0$, then 'a' equals to	(r) $1/e$
(C) Let $L = \lim_{x \to a} \dfrac{x^x - a^a}{x - a}$ and $M = \lim_{x \to a} \dfrac{x^x - a^x}{x - a} (a > 0)$. If $L = 2M$, then the value of 'a' is equal to	(s) non-existent

Type 6 : **Integer Answer Type Questions**

54. Number of points of discontinuity of $f(x) = \tan^2 x - \sec^2 x$ in $(0, 2\pi)$ is ....

55. Number of point(s) of discontinuity of the function $f(x) = [x^{1/x}], x > 0$, (where [ ] denotes the greatest integral function) is ....

56. Let $f(x) = x + \cos x + 2$ and $g(x)$ be the inverse function of $f(x)$, then $g'(3)$ equals to ....

57. Let $f(x) = x \tan^{-1}(x^2) + x^4$. Let $f^k(x)$ denotes kth derivative of $f(x)$ w.r.t. x, $k \in N$. If $f^{2m}(0) \ne 0$, $m \in N$, then m equals to ....

58. Let $f_1(x)$ and $f_2(x)$ be twice differentiable functions where $F(x) = f_1(x) + f_2(x)$ and $G(x) = f_1(x) - f_2(x), \forall x \in R$, $f_1(0) = 2$ and $f_2(0) = 1$. If $f'_1(x) = f_2(x)$ and $f'_2(x) = f_1(x), \forall x \in R$, then the number of solutions of the equation $(F(x))^2 = \dfrac{9x^4}{G(x)}$ is ....

59. Suppose the function $f(x) - f(2x)$ has the derivative 5 at $x = 1$ and derivative 7 at $x = 2$. The derivative of the function $f(x) - f(4x) - 10x$ at $x = 1$ is equal to ....

60. In a ΔABC, angles A, B, C are in AP.

If $f(x) = \lim_{x \to c} \dfrac{\sqrt{3 - 4 \sin A \sin C}}{|A - C|}$, then $f'(x)$ is equal to ....

Proficiency in 'Continuity and Differentiability'

Exercise 2

1. Find the set of points where $f(x) = \tan 2x$ is discontinuous.

2. Let $f(x) = \begin{cases} (1+3x)^{1/x}, & x \neq 0 \\ (e)^3, & x = 0 \end{cases}$. Discuss the continuity of $f(x)$ at (i) $x = 0$ (ii) $x = 1$

3. Let $f(x) = \begin{cases} ax + 1, & x < 1 \\ 3, & x = 1 \\ bx^2 + 1, & x > 1 \end{cases}$ for what values of a and b, $f(x)$ is continuous at $x = 1$.

4. Let $f(x) = \begin{cases} \dfrac{\sin ax^2}{x^2}, & x \neq 0 \\ \dfrac{3}{4} + \dfrac{1}{4a}, & x = 0 \end{cases}$ for what values of a, $f(x)$ is continuous at $x = 0$.

5. Determine the values of a, b and c for which the function,

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}, & x < 0 \\ \dfrac{(x + bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}, & x > 0, b \neq 0 \\ c, & x = 0 \end{cases}$$ is continuous at $x = 0$.

6. Discuss the continuity of the function $f(x) = [[x]] - [x-1]$. (where [ ] denotes the greatest integral function.)

7. Let $f(x) = \begin{cases} x^2/2, & 0 < x \leq 1 \\ (2x^2 - 3x + 3/2), & 1 < x < 2 \end{cases}$

   Discuss the continuity of f, f' and f'' on $(0, 2)$.

8. Determine the set of all points where the function $f(x) = \dfrac{x}{1 + |x|}$ is differentiable.

9. Examine the continuity or discontinuity of the following :

   (i) $f(x) = [x] + [-x]$ (ii) $g(x) = \lim\limits_{n \to \infty} \dfrac{x^{2n} - 1}{x^{2n} + 1}$

10. Examine the continuity and differentiability at points $x = 1$ and $x = 2$.

    The function f defined by

    $f(x) = \begin{cases} x[x], & 0 \leq x < 2 \\ (x-1)[x], & 2 \leq x \leq 3 \end{cases}$ (where $[\cdot]$ denotes the greatest integral function less than or equal to x.)

11. Let $f(x)=[x]+|1-x|, -1<x\le 3$. Determine the points at which $f(x)$ is not differentiable. (where $[\cdot]$ denotes the greatest integral function)

12. A function f is defined as follows :

$$f(x)=\begin{cases}1 & , \text{ when } -\infty<x<0\\ 1+\sin x & , \text{ when } 0\le x<\pi/2\\ 2+(x-\pi/2)^2 & , \text{ when } \pi/2\le x<+\infty\end{cases}$$

Discuss the continuity and differentiability at

(i) $x=0$ (ii) $x=\pi/2$

13. If $f(x)$ and $g(x)$ has a derivative at $x=a$, then prove that

$$\lim_{x\to a}\frac{g(x)\cdot f(a)-g(a)\cdot f(x)}{\sin(x-a)}=g'(a)f(a)-g(a)f'(a).$$

14. Let $f(x+y)=f(x)\cdot f(y)$ for all x and y. Suppose $f(5)=2$ and $f'(0)=3$, find $f'(5)$.

15. Let f be twice differentiable function, such that $f'(x)=-f(x)$ and $f'(x)=g(x)$, $h(x)=[f(x)]^2+[g(x)]^2$. Find $h(10)$, if $h(5)=11$.

16. Tangent to a graph of a differentiable function $f(x)$ at $x=1$ is parallel to x-axis and $f(1-x)=f(x)$ for all x. If $f'(0)$ exists, find its value.

17. Let R be a set of real numbers and $f:R\to R$ such that for all x and y in R, $|f(x)-f(y)|\le|x-y|^3$. Prove that $f(x)$ is a constant.

18. A function $f:R\to R$ satisfies the equation $f(x+y)=f(x)\cdot f(y)$ for all x, y in R and $f(x)\ne 0$ for any x in R. Let the function be differentiable at $x=0$ and $f'(0)=2$. Show that $f'(x)=2f(x)$ for all x in R. Hence, determine $f(x)$.

19. Suppose the function f satisfies the following two conditions for all $x, y\in R$:

(i) $f(x+y)=f(x)\cdot f(y)$ (ii) $f(x)=1+x\cdot g(x)$ where $\lim\limits_{x\to 0} g(x)=1$

Prove that the derivative $f'(x)$ exists and $f'(x)=f(x)$.

20. Let $f(xy)=f(x)\cdot f(y)$ and f is differentiable at $x=1$ such that $f'(1)=1$ also $f(1)\ne 0$, then show that f is differentiable for all $x\ne 0$. Hence, determine $f(x)$.

21. A function $f:R\to R$, where R is a set of real numbers, satisfying the equation $f\left(\frac{x+y}{3}\right)=\frac{f(x)+f(y)+f(0)}{3}$ for all x, y in R. If the function is differentiable at $x=0$, then show that it is differentiable for all x in R.

22. Let $f(x+y)=f(x)=f(x)+f(y)+2xy-1$ for all real x, y and $f(x)$ be differentiable functions. If $f'(0)=\cos\alpha$, then prove that $f(x)>0$, $\forall x\in R$.

23. If $f(x)$ is a real-valued function not identically equal to zero, such that $f(x+y^n)=f(x)+f(y)^n$; $x, y\in R$ and n is natural number >1 and $f'(0)\ge 0$, then find the values of $f(5)$ and $f'(10)$.

24. Find the area of the region bounded by $y=f(x)$, $y=|g(x)|$ and the lines $x=0, x=2$, where 'f' and 'g' satisfying $f(x+y)=f(x)+f(y)-8xy\ \forall x, y\in R$ and $g(x+y)=g(x)+g(y)+3xy(x+y)\ \forall x, y\in R$ also $f'(0)=8$, $g'(0)=-4$.

25. Let $f:R\to R$ is a real-valued function, $\forall x$ and y in R such that $|f(x)-f(y)|\le|x-y|^3$. Prove that $h(x)=\int f(x)\,dx$ is continuous function of $x, \forall x\in R$.

Answers

Target Exercise 6.1

1. (d) **2.** (d) **3.** (a) **4.** (c) **5.** (d)

Target Exercise 6.2

1. (c) **2.** (a) **3.** (b) **4.** (c) **5.** (a)

Target Exercise 6.3

1. (d) **2.** (a) **3.** (d) **4.** (b,d) **5.** (a)

Target Exercise 6.4

1. (c) **2.** (c) **3.** (c) **4.** (b) **5.** (b) **6.** (d) **7.** (a) **8.** (c) **9.** (a) **10.** (c)

Exercise 1

1. (a) **2.** (d) **3.** (b) **4.** (d) **5.** (c) **6.** (c) **7.** (c) **8.** (d) **9.** (c) **10.** (b)
11. (c) **12.** (d) **13.** (d) **14.** (d) **15.** (a) **16.** (d) **17.** (d) **18.** (c) **19.** (a) **20.** (b)
21. (b) **22.** (c) **23.** (c) **24.** (c) **25.** (a) **26.** (c) **27.** (a, c) **28.** (a, b, d)
29. (a, b, c, d) **30.** (a, b, d) **31.** (a, c) **32.** (a, c)
33. (b, c) **34.** (a, b, c) **35.** (a, b, c) **36.** (a, b)
37. (a, b, c, d) **38.** (a, c, d) **39.** (a, b, d) **40.** (a, d)
41. (a) **42.** (c) **43.** (c) **44.** (a) **45.** (d) **46.** (a) **47.** (b) **48.** (b) **49.** (c) **50.** (b)
51. (c) **52.** (A) $\rightarrow$ (p, s); (B) $\rightarrow$ (p, q); (C) $\rightarrow$ (r, s); (D) $\rightarrow$ (p, q)
53. (A) $\rightarrow$ (s); (B) $\rightarrow$ (p, s); (C) $\rightarrow$ (p)
54. (2) **55.** (1) **56.** (1) **57.** (2) **58.** (2) **59.** (9) **60.** (0)

Exercise 2

1. $(2n+1)\dfrac{\pi}{4} : n$ is any integer **2.** (i) Continuous at $x=0$, (ii) Continuous at $x=1$

3. $a=b=2$ **4.** $a=1, -\dfrac{1}{4}$ **5.** $a=-3/2,\ b\neq 0,\ c=\dfrac{1}{2}$ **6.** Continuous on R

7. $f(x)$ is continuous in $(0,2)$; $f'(x)$ is also continuous on $(0,2)$; $f''(x)$ is not continuous on $(0, 2)$

8. Differentiable in $(-\infty, \infty)$

9. (i) $f(x)$ is discontinuous at $x \in I$, (ii) $g(x)$ is discontinuous at $x=\pm 1$

10. $f(x)$ is discontinuous at $x=1$ but continuous at $x=2$; $f(x)$ is not differentiable at $x=1,2$

11. The function is not differentiable at the points $x=0, 1, 2$ and 3.

12. (i) f is continuous but not differentiable at $x=0$, (ii) f is continuous and differentiable at $x=\pi/2$

14. $f'(5)=6$ **15.** $h(10)=11$ **16.** $f'(0)=0$ **18.** $f(x)=e^{2x}$ **20.** $f(x)=x$

23. $f(x)=x, f(5)=5, f'(10)=1$

Solutions

(*Proficiency in 'Continuity and Differentiability'*)

Exercise 1

Type 1 : **Only One Correct Option**

1. Let $x = \frac{2}{3}$ which is rational.

$$\Rightarrow \qquad h\left(\frac{2}{3}\right) = \frac{1}{3}$$

$$\lim_{t \to 0} h\left(\frac{2}{3} + t\right) = 0 \Rightarrow \text{Discontinuous at } x \in Q$$

Let $\qquad x = \sqrt{2} \notin Q$

$$h(\sqrt{2}) = 0 \text{ consider } \sqrt{2} = 1.4142135624$$

$$h(\sqrt{2}) = h\left(\frac{1.4142135624}{10^{10}}\right) = \frac{1}{10^{10}} \to 0$$

Hence, h is continuous for all irrational.

2. By theorem, if g and h are continuous functions on the open interval (a, b), then g/h is also continuous at all x in the open interval (a, b) where $h(x) \neq 0$.

3. $h(x) = \begin{cases} \dfrac{2\cos x - \sin 2x}{(\pi - 2x)^2}, & x < \dfrac{\pi}{2} \\ \dfrac{e^{-\cos x} - 1}{8x - 4\pi}, & x > \dfrac{\pi}{2} \end{cases}$

LHL at $x = \pi/2$ $\displaystyle\lim_{h \to 0} \frac{2\sin h - \sin 2h}{4h^2} = \lim_{h \to 0} \frac{2\sin h\,(1 - \cos h)}{4h^2} = 0$

RHL $\qquad \displaystyle\lim_{h \to 0} \frac{e^{\sin h} - 1}{((\pi/2) + h) - 4\pi} = \lim_{h \to 0} \frac{e^{\sin h} - 1}{8h} \cdot \frac{\sin h}{\sin h} = \frac{1}{8}$

$\Rightarrow h(x)$ is discontinuous at $x = \pi/2$.

Irremovable discontinuity at $x = \pi/2$.

$$f\left(\frac{\pi^+}{2}\right) = 0 \text{ and } g\left(\frac{\pi^-}{2}\right) = \frac{1}{8} \quad \Rightarrow \quad f\left(\frac{\pi^+}{2}\right) \neq g\left(\frac{\pi^-}{2}\right)$$

4. $\displaystyle\lim_{x \to 0} \frac{x - e^x + 1 - (1 - \cos 2x)}{x^2} = -\frac{1}{2} - 2 = -\frac{5}{2}$, hence for continuity $f(0) = -\frac{5}{2}$

$\therefore \qquad [f(0)] = -3; \{f(0)\} = \left\{-\frac{5}{2}\right\} = \frac{1}{2}$; hence $[f(0)] \cdot \{f(0)\} = -\frac{3}{2} = -1.5$

5. $f(1^+) = f(1^-) = f(1) = 2$

$$f(0) = 1, \qquad f(2) = 2$$

$$f(2^-) = 1, \qquad f(2) = 2$$

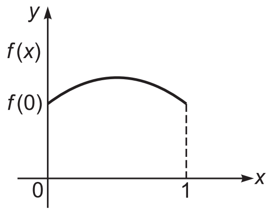
$\Rightarrow f$ is not continuous at $x = 2$.

6. $f(2^+) = 8, f(2^-) = 16$

7. $g(x) = x - [x] = \{x\}$

f is continuous with $f(0) = f(1)$

$$h(x) = f(g(x)) = f(\{x\})$$



Let the graph of f is as shown in the figure satisfying

$$f(0) = f(1)$$

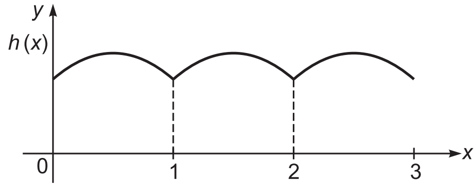
Now, $\quad h(0) = f(\{0\}) = f(0) = f(1)$

$$h(0.2) = f(\{0.2\}) = f(0.2)$$

$$h(1.5) = f(\{1.5\}) = f(0.5) \text{ etc.}$$

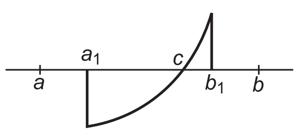
Hence, the graph of $h(x)$ will be a periodic graph as shown

$\Rightarrow h$ is continuous in R.



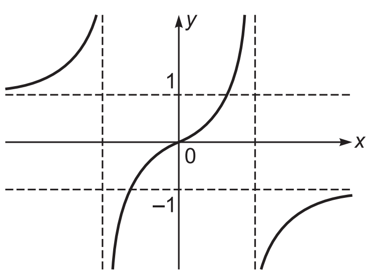
8. Statements I and II are false. The function $f(x) = 1/x, 0 < x < 1$ is a counter example.

Statement III is true. Apply the intermediate value theorem to f on the closed interval $[a_1, b_1]$.



9. Not derivable at $x = 0$ and 2

10. $f(x) = \begin{cases} \dfrac{x}{1-x}, \text{ if } x \ge 0, x \ne 1 \\ \dfrac{x}{1+x}, \text{ if } x < 0, x \ne -1 \end{cases}$



and $\quad f'(x) = \begin{cases} \dfrac{1}{(1-x)^2}, \text{ if } x > 0, x \ne 1 \\ \dfrac{1}{(1+x)^2}, \text{ if } x < 0, x \ne -1 \end{cases}$

11. $g[f(x)]$ is continuous at $x = \alpha$

$\Rightarrow \quad g[f(\alpha)] = g(a)$

Also, $\quad \lim_{x \to \alpha} g(f(x)) = g(a)$

$\Rightarrow \quad g[f(\alpha^-)] = g[f(a^+)] = g(a)$

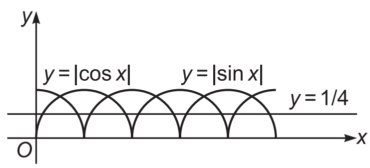
$\Rightarrow g(x)$ takes same limiting values at

$f(\alpha^-), f(\alpha^+)$ and $f(\alpha)$

$\Rightarrow \quad f(\alpha^-) = f(\alpha^+) \Rightarrow h = \alpha$ is an extremum of $f(x)$

and $x = \alpha$ may not be an extremum of $g(x)$.

12. From the graph, number of points of non-differentiability $= 11$



13. Let n be any integer other than 1.

$$\lim_{x \to n^-} f(x) = \lim_{h \to 0} [n-h]^2 - [(n-h)^2]$$

$$= (n-1)^2 - (n^2 - 1) = 2$$

$$\lim_{x \to n^+} f(x) = \lim_{h \to 0} [n+h]^2 - [(n+h)^2] = n^2 - n^2 = 0$$

$\therefore$ LHL $\neq$ RHL unless $n = 1$

Hence, $f(x)$ is discontinuous at all integral values except 1.

14. At $x = 5$, $f'(x) = \lim_{x \to 5} \dfrac{\{(x-1)^2 (x+1)|x-5| + \cos|x|\} - \cos 5}{x-5}$

$$= \lim_{x \to 5} \frac{96|x-5|}{x-5} = +96, \text{ if } x > 5 \text{ and } -96, \text{ if } x < 5$$

Hence, $f'(5)$ doesn't exist.

This ambiguity doesn't occur at other points.

$\therefore f(x)$ is not differentiable at $x = 5$.

15. $f'(0^+) = \lim_{x \to 0^+} \dfrac{f(x) - f(0)}{x}$

$$= \lim_{x \to 0^+} \frac{e^{1/x^2} - 0}{x} = \lim_{x \to 0^+} \frac{-1/x}{e^{1/x^2}}$$

$$= \lim_{x \to 0^+} \frac{-1/x^2}{e^{1/x^2} \cdot \left(-\dfrac{2}{x^3}\right)} = \lim_{x \to 0^+} \frac{x}{2e^{1/x^2}} = 0$$

As, f is even $f'(0^-) = f'(0^+) = 0$. Thus, $f'(0) = 0$

16. $\lim\limits_{h\to 0} g(n+h) = \lim\limits_{h\to 0} \dfrac{e^h - \cos 2h - h}{h^2}$

$$= \lim_{h\to 0} \frac{e^h - h - 1}{h^2} + \lim_{h\to 0} \frac{(1-\cos 2h)}{4h^2}\cdot 4 = \frac{1}{2} + 2 = \frac{5}{2}$$

$$= \lim_{h\to 0} g(n-h) = \frac{e^{1-\{n-h\}} - \cos 2(1-\{n-h\}) - (1-\{n-h\})}{(1-\{n-h\})^2}$$

$$= \lim_{h\to 0} \frac{e^h - \cos 2h - h}{h^2} \ (\{n-h\} = \{-h\} = 1-h) = \frac{5}{2}$$

$g(n) = \dfrac{5}{2}$. Hence, $g(x)$ is continuous at $\forall\ x \in I$.

Hence, $g(x)$ is continuous $\forall\ x \in R$.

17. $g'(0^+) = \lim\limits_{h\to 0} \dfrac{\cos h - 1}{h} = 0$

$g'(0^-) = \lim\limits_{h\to 0} \dfrac{-h+b-1}{-h}$ for existence of limit $b = 1$

Thus, $g'(0^-) = 1$

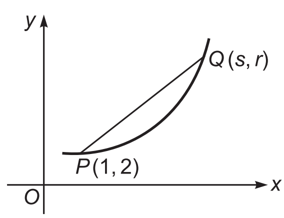
Hence, g cannot be made differentiable for any value of b.

18. **I.** By definition $f'(1)$ is the limit of the slope of the secant line when $s \to 1$.

Thus, $f'(1) = \lim\limits_{s\to 1} \dfrac{s^2+2s-3}{s-1}$

$$= \lim_{s\to 1} \frac{(s-1)(s+3)}{s-1}$$

$$= \lim_{s\to 1} (s+3) = 4$$



II. By substituting $x = s$ into the equation of the secant line and cancelling by $s-1$. Again, we get $y = s^2 + 2s - 1$. This is $f(s)$ and its derivative is $f'(s) = 2s + 2$, so $f'(1) = 4$.

19. In the immediate neighbourhood of $x = \pi/2$, $\sin x > \sin^3 x \Rightarrow |\sin x - \sin^3 x| = \sin x - \sin^3 x$

Hence, for $x \neq \pi/2$

$$f(x) = \left[\frac{2(\sin x - \sin^3 x) + \sin x - \sin^3 x}{2(\sin x - \sin^3 x) - \sin x + \sin^3 x}\right] = \frac{3\sin x - 3\sin^3 x}{\sin x - \sin^3 x} = 3$$

Hence, f is continuous and differentiable at $x = \pi/2$.

20. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f(h) + |x|h + xh^2}{h}$

where $x = h$ and $y = x$

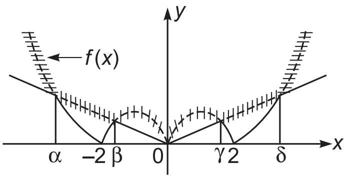
$\because \quad f(0) = 0$

Hence, $\quad f'(x) = \lim\limits_{h\to 0} \left(\dfrac{f(h) - f(0)}{h} + |x| + xh\right)$

$$f'(x) = f'(0) + |x| = |x|$$

21. $f(x)$ is non-differentiable at $x = \alpha, \beta, 0, \gamma, \delta$

and $g(x)$ is non-differentiable at $x = \alpha, \beta, 0, -2, 2$



22. $g(x) = \begin{cases} 3x^2 - 4\sqrt{x} + 1, & \text{for } x < 1 \\ ax + b, & \text{for } x \geq 1 \end{cases}$

For differentiability at $x = 1$, $g'(1^+) = g'(1^-)$

$$a = 6x - \frac{4}{2\sqrt{x}}$$

$\Rightarrow$ $$a = 6 - 2 = 4$$

For continuity at $x = 1$, $g(1^+) = g(1^-)$

$$a + b = 3 - 4 + 1 \Rightarrow a + b = 0$$

$\Rightarrow$ $$b = -4$$

$$a = 4 \text{ and } b = -4$$

23. $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{-3xh + f(h)}{h}$$

$$= \lim_{h \to 0} \left\{-3x + \frac{f(h)}{h}\right\}$$

$$= -3x + \lim_{h \to 0} \frac{f(h)}{h}$$

$$= -3x + 7$$

24. $g(x) = \dfrac{\sin \dfrac{\pi[x]}{4}}{[x]}$

Obviously, continuity at $x = 3/2$

At $x = 2$ $\left.\begin{aligned} f(2^-) &= \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \\ f(2) &= \frac{\sin\frac{\pi}{2}}{2} = \frac{1}{2} \end{aligned}\right\}$

Hence, discontinuous at $x = 2$

25. $f(-1) = b(1 - 1) + 1 = 1$

and $$\lim_{h \to 0} f(-1 + h) = 1$$

$$\lim_{h \to 0} f(-1 - h) = \sin((-1 + h + a)\pi) = -\sin \pi a$$

For continuity $\sin \pi a = -1 = \sin\left(2n\pi + \frac{3\pi}{2}\right)$

$$\Rightarrow \qquad \pi a = 2n\pi + \frac{3\pi}{2} \Rightarrow a = 2n + \frac{3}{2}$$

Hence, $\qquad a = 2n + \frac{3}{2},\ n \in I$ and $b \in R$

26. Consider the graph of $h(x) = \max(x, x^2)$ at $x = 0$ and $x = 1$

For $D : h(x) = \max(x^2, -x^2)$

Type 2 : **More than One Correct Options**

27. $f'(0^+) = \lim_{h \to 0} \dfrac{h \ln(\cos h)}{h \ln(1+h^2)} = \lim_{h \to 0} \dfrac{\ln(\cos h)^{1/h^2}}{\dfrac{\ln(1+h^2)}{h^2}}$

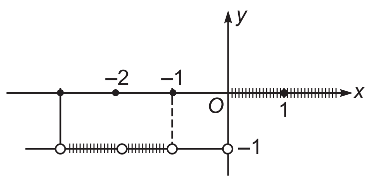
$$= \lim_{h \to 0} \frac{1}{h^2}(\cos h - 1) = -\frac{1}{2};\ \text{similarly } f'(0^-) = -\frac{1}{2}$$

Hence, f is continuous and derivable at $x = 0$.

28. $f(x) = \begin{cases} 0, & 0 < x < 1 \\ 0, & x = 0 \text{ or } 1 \text{ or } -1 \\ 0, & -1 < x < 0 \end{cases} \Rightarrow f(x) = 0$ for all in $[-1, 1]$

29. $[\,|x|\,] - |[x]| = \begin{cases} 0, & x = -1 \\ -1, & -1 < x < 0 \\ 0, & 0 \le x \le 1 \\ 0, & 1 < x \le 2 \end{cases}$

$\Rightarrow$ Range is $\{0, -1\}$. The graph is



30. $f'(0^+) = \dfrac{1}{\sqrt{2}},\ f'(0^-) = -\dfrac{1}{\sqrt{2}};$

$$f(x) = \frac{\sqrt{x^2}}{\sqrt{1+\sqrt{1-x^2}}} = \frac{|x|}{\sqrt{1+\sqrt{1-x^2}}}$$

31. Range is $R^+ \cup \{0\} \Rightarrow$ Option (b) is not correct.

f is not differentiable at $x = -1$

As $f(x) = \begin{cases} x^3 + 1, & \text{if } x \ge -1 \\ -(x^3+1), & \text{if } x < -1 \end{cases} \Rightarrow f'(x) = \begin{cases} 3x^2, & \text{if } x > -1 \\ -3x^2, & \text{if } x < -1 \end{cases}$

$f'(-1^+) = 3, f'(-1^-) = -3 \Rightarrow f$ is not differentiable at $x = -1$ also, since f is not bijective hence, it has no inverse.

32. Given f is continuous in $[a, b]$ ...(i)

$$g \text{ is continuous in } [b, c] \quad \text{...(ii)}$$

$$f(b) = g(b) \quad \text{...(iii)}$$

$$\left.\begin{aligned} h(x) &= f(x) \text{ for } x \in [a, b) \\ &= f(b) = g(b) \text{ for } x = b \\ &= g(x) \text{ for } x \in (b, c] \end{aligned}\right\} \quad \text{...(iv)}$$

$h(x)$ is continuous in $[a, b) \cup (b, c]$ [Using Eqs. (i) and (ii)]

Also, $f(b^-) = f(b),\ g(b^+) = g(b)$ [Using Eqs. (i) and (ii)] ...(iv)

$\therefore$ $h(b^-) = f(b^-) = f(b) = g(b) = g(b^+) = h(b^+)$ [Using Eqs. (iv) and (v)]

Now, verify each alternative. Of course! $g(b^-)$ and $f(b^+)$ are undefined.

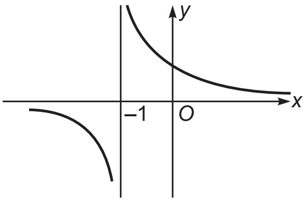
$$h(b^-) = f(b^-) = f(b) = g(b) = g(b^+)$$

and $h(b^+) = g(b^+) = g(b) = f(b) = f(b^-)$

Hence, $h(b^-) = h(b^+) = f(b) = g(b)$

and $h(b)$ is not defined.

33. Domain is $R - \{-1\}$; Range $= R - \{0\}$



Domain is $x \in R$; Range $= (0, 1]$

Domain is $[0, \infty)$; Range $= (0, 1]$

Domain is $(-\infty, 3)$; Range $= (0, \infty)$

34. $f(x) = \sec 2x + \operatorname{cosec} 2x = \dfrac{2(\sin 2x + \cos 2x)}{2\cos 2x \sin 2x} = \dfrac{2(\sin 2x + \cos 2x)}{\sin 4x}$

is discontinuous where $4x = n\pi,\ n \in I$ or $x = \dfrac{n\pi}{4}$

Options (a) and (b) also satisfy the condition since they are subsets of option (c).

35. $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} x^n \cdot \sin\left(\dfrac{1}{x^2}\right) = 0$

If $n > 0$ and hence true for $n > 1$.

Since, $f(0) = 0$, $f(x)$ is continuous at $x = 0$.

Now, $$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^n \sin \dfrac{1}{x^2}}{x}$$

$$= \lim_{x \to 0} x^{n-1} \sin\left(\frac{1}{x^2}\right) = 0, \text{ if } n > 1.$$

Hence, $f(x)$ is differentiable at $x = 0$, if $n > 1$.

36. $f(x) = \begin{cases} e^x \;, & x \le 0 \\ 1 - x \,, & 0 < x < 1 \\ x - 1, & x \ge 1 \end{cases}$

$$\lim_{x \to 0^-} f(x) = 1, \lim_{x \to 0^+} f(x) = 1$$

and $$\lim_{x \to 1^-} f(x) = 0, \lim_{x \to 1^-} f(x) = 0$$

Hence, $f(x)$ is continuous at $x = 0$ and $1\, f'(0^-) = 1$ and $f'(0^+) = -1$.

Hence, $f(x)$ is not differentiable at $x = 1, f'(x) = 6x + 12$.

37. $f(x) = \int_{-2}^{x} |t + 1|\, dt$

$$= -\int_{-2}^{-1} (t + 1)\, dt + \int_{-1}^{x} (t + 1)\, dt$$

$$= \frac{1}{2} + \left(\frac{t^2}{2} + t\right)_{-1}^{x} = \frac{x^2}{2} + x + 1 \text{ for } x \ge -1$$

$f(x)$ is a quadratic polynomial.

$\therefore$ $f(x)$ is continuous as well as differentiable in $[-1, 1]$.

Also, $f'(x)$ is continuous as well as differentiable in $[-1, 1]$.

38. $f(x + y) = f(x) + f(y) + xy\,(x + y)$

$$f(0) = 0$$

$$\therefore \quad \lim_{h \to 0} \frac{f(h)}{h} = -1$$

$$\therefore \quad \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x) + f(h) + xh\,(x + h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(h)}{h} + \lim_{h \to 0} x\,(x + h) = -1 + x^2$$

$$\therefore \quad f'(x) = -1 + x^2$$

$$\therefore \quad f(x) = \frac{x^3}{3} - x + c$$

$\therefore$ $f(x)$ is a polynomial function.

$f(x)$ is twice differentiable for all $x \in R$ and $f'(3) = 3^2 - 1 = 8$.

39. $f'(x) = 6x + 12$

For $f(x)$ is increasing, $f'(x) \ge 0 \;\Rightarrow\; x \ge -2$

Hence, $f(x)$ is increasing in $[-1, 2]$

$$\lim_{x \to 2^-} f(x) = 35, \lim_{x \to 2^-} f(x) = 35 \quad \text{and} \quad \lim_{x \to 2} f(x) = 35$$

Hence, $f(x)$ is continuous on $[-1, 3]\, f'(2^-) = 24$ and $f'(2^+) = -1$, hence $f'(2)$ doesn't exist for maximum, $f(2) = 35$

$$f(-1) = -10, \quad f(3) = 34$$

Hence, $f(x)$ has maximum value at $x = 2$.

40. Since, $f(x) = 0$, $x < 0$ and differentiable at $x = 0 = \text{LHD} = 0$

(Function is x-axis for $x < 0$). If $f(x)$,

(a) $-x^2$, $x > 0$ RHD at $x = 0$

$\Rightarrow \quad f'(0) = 2 \times 0 = 0$ possible

(d) $-x^{3/2}$, $x > 0$ RHD at $x = 0$

$\Rightarrow \quad f'(0) = -3/2\, x^{1/2} = -3/2 \times 0 = 0$ possible

Type 3 : **Assertion and Reason**

41. $y = |\ln x|$ not differentiable at $x = 1$.

$y = |\cos |x||$ is not differentiable at $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

$y = \cos^{-1}(\text{sgn } x) = \cos^{-1}(1) = 0$ differentiable, $\forall\, x \in (0, 2\pi)$.

42. $f'(0^+) = \dfrac{h \sin h - 0}{h} = 0$

$$f'(0^-) = \frac{h \sin(-h) - 0}{-h} = 0$$

$f(x)$ is differentiable at $x = 0$.

eg, $x|x|$ is derivable at $x = 0$.

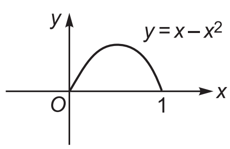
43. Consider $g(x) = x^3$ at $x = 0$, $g(0) = 0$

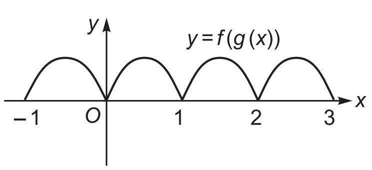
$|g(x)|$ is derivable as $x = 0$.

Actually nothing definite can be said. Also, for $g(x) = x - 1$ with $g(1) = 0$.

Then, $|g(x)|$ is not derivable at $x = 1$.

44.





45. $f(x)$ is discontinuous at $x = 0$ and $f(x) < 0$, $\forall\, x \in [-\alpha, 0)$ and $f(x) > 0$, $\forall\, x \in [0, \alpha]$.

Type 4 : **Linked Comprehension Based Questions**

Solutions (Q. Nos. 46 to 48)

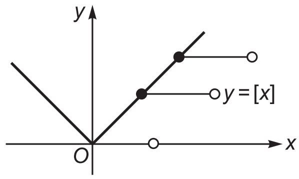
46. $f(x) = \begin{cases} \sin x, & x \le 0 \\ \tan x, & 0 < x < 2\pi \\ \cos x, & 2\pi \le x < 3\pi \\ 3\pi, & x = 3\pi \end{cases}$

$f(x)$ is discontinuous at $\dfrac{\pi}{2}, \dfrac{3\pi}{2}, 2\pi$.

47. $f(x) = [2x^3 - 5]$

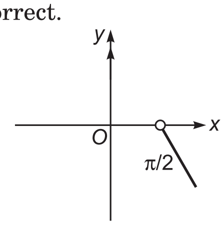
$2x^2 = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15,$ hence number of points of discontinuity $= 13$.

48. Max $([x], |x|)$, hence discontinuity at $x = \phi$.

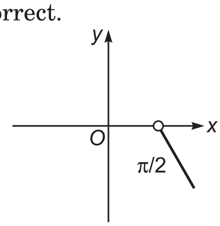


Solutions (Q. Nos. 49 to 51)

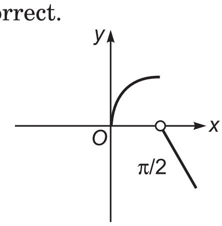
49. From figure option (c) is correct.



50. From figure option (b) is correct.



51. From figure option (c) is correct.



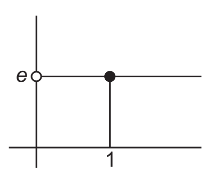
Type 5 : **Match the Columns**

52. (A) $f'(0) = \lim_{h \to 0} \dfrac{\cos h - 0}{h}$ doesn't exist. Obviously, $f(0) = f(0^-) = f(0^+) = 1$

Hence, continuous and not derivable.

(B) $g(x) = 0$ for all x, hence continuous and derivable.

(C) As $0 \le \{f(x)\} < 1$, hence $h(x) = \sqrt{\{x\}^2} = \{x\}$ which is discontinuous, hence non-derivable all $x \in I$



(D) $\lim_{x \to 1} x^{\frac{1}{\ln x}} = \lim_{x \to 1} x^{\log_x e} = e = f(1)$

Hence, $k(x)$ is constant for all $x > 0$, hence continuous and differentiable at $x = 1$.

53. (A) $l = \lim\limits_{x \to \infty} e^{x^2+1}\left[e^{\sqrt{x^4+1}-(x^2+1)} - 1\right]$

Consider $\lim\limits_{x \to \infty} [\sqrt{x^4+1} - (x^2+1)] = \lim\limits_{x \to \infty} \dfrac{x^4+1-(x^4+1+2x^2)}{\sqrt{x^4+1}+(x^2+1)}$

$$= \lim_{x \to \infty} \frac{-2x^2}{x^2(\sqrt{1+(1/x^4)}+1+(1/x^2))} = -1$$

Now, as $x \to \infty$, $\sqrt{x^2+1} - (x^2+1) \to -1$

$$\infty \times \left(\frac{1}{e} - 1\right) \to -\infty \text{ and hence limit doesn't exist.}$$

(B) $f(0^+) = \lim\limits_{h \to 0} \dfrac{a^{2h} - 2a^h + 1}{h^2} = \lim\limits_{h \to 0} \left(\dfrac{a^h - 1}{h}\right)^2 = \ln^2 a$

$$f(0^-) = \lim_{h \to 0} 3 \ln(a+h) - 2 = 3 \ln a - 2 = f(0)$$

For continuous

$$\ln^2 a = 3 \ln a - 2$$

$$\ln^2 a - 3 \ln a + 2 = 0 \Rightarrow (\ln a - 2)(\ln a - 1) = 0$$

$$a = e^2 \text{ or } a = e$$

(C) $L = a^a \ln ae$

$$M = a^a$$

$\therefore$ $\quad a^a \ln ae = 2a^a$

$\therefore$ $\quad \ln ae = 2 \Rightarrow ae = e^2 \Rightarrow a = e$

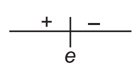
Type 6 : **Integer Answer Type Questions**

54. $\tan^2 x$ is discontinuous at $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

$\sec^2 x$ is discontinuous at $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

$\Rightarrow$ Number of discontinuities $= 2$

55. Let $g(x) = x^{1/x}$, $g'(x)$



$$= x^{1/x} \frac{1 - \ln x}{x^2}$$

$$g_{\max} = e^{1/e} \in (1, 2)$$

$$\lim_{x \to 0} x^{1/x} = 0, \lim_{x \to \infty} x^{1/x} = 1$$

So, $\quad f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 \le x < \infty \end{cases}$

56. $f(x) = x + \cos x + 2, f(0) = 3 \Rightarrow g(3) = 0$

$$g(f(x)) = x \Rightarrow g'(f(x)) \cdot f'(x) = 1, \text{ putting } x = 0$$

$$g'(3) \cdot f'(0) = 1$$

Now, $\quad f'(x) = 1 + \sin x \Rightarrow f'(0) = 1 \quad \Rightarrow \quad g'(3) = 1$

57. Let $g(x) = x \tan^{-1}(x^2)$. It is an odd function.

So, $g^{2m}(0) = 0$. Let $h(x) = x^4$

So, $f(x) = g(x) + h(x) \Rightarrow f^{2m}(0)$

$$= g^{2m}(0) + h^{2m}(0) = h^{2m}(0) \neq 0$$

It happens when $2m = 4 \Rightarrow m = 2$

58. $F(x) = 3e^x$ and $G(x) = e^{-x}$

The equation $9x^4 = F(x)^2\, G(x)$ becomes $x^4 = e^x$

Hence, number of solutions $= 2$

59. $y' = f'(x) - 2f'(2x)$

$$y'(1) = f'(1) - 2f'(2) = 5 \qquad \text{...(i)}$$

and $$y'(2) = f'(2) - 2f'(4) = 7 \qquad \text{...(ii)}$$

Now, let $$y = f(x) - f(4x) - 10$$

$$y' = f'(x) - 4f'(4x) - 10$$

$$y'(1) = f'(1) - 4f'(4) - 10 \qquad \text{...(iii)}$$

Substituting the value of $f'(2) = 7 + 2f'(4)$ in Eq. (i),

$$f'(1) - 2[7 + 2f'(4)] = 5$$

$$f'(1) - 4f'(4) = 19$$

$\Rightarrow$ $$f'(1) - 4f'(4) - 10 = 9$$

60. A, B, C are in AP.

$\therefore$ $$2B = A + C \text{ and } A + B + C = 180^\circ$$

$\therefore$ $$B = 60^\circ \quad \therefore \quad \cos B = \frac{1}{2} = \frac{a^2 + c^2 - b^2}{2ac}$$

$\Rightarrow$ $$a^2 + c^2 = b^2 + ac \qquad \Rightarrow \quad (a - c)^2 = b^2 - ac$$

or $$|\sin A - \sin C| = \sqrt{\sin^2 B - \sin A \sin C}$$

$\Rightarrow$ $$2\cos\left(\frac{A + C}{2}\right)\left|\sin\left(\frac{A - C}{2}\right)\right|$$

$$= \sqrt{\frac{3}{4} - \sin A \sin C}$$

$\Rightarrow$ $$2\left|\sin\left(\frac{A - C}{2}\right)\right| = \sqrt{\frac{3}{4} - \sin A \sin C}$$

$\therefore$ $$\lim_{x \to C} \frac{\sqrt{3 - 4\sin A \sin C}}{|A - C|} = \lim_{x \to C} \frac{2\left|\sin\left(\frac{A - C}{2}\right)\right|}{|A - C|}$$

$$= \lim_{x \to C} \left|\frac{\sin\left(\frac{A - C}{2}\right)}{\left(\frac{A - C}{2}\right)}\right|$$

$$|I| = 1 \Rightarrow f(x) = 1 \;\therefore\; f'(x) = 0$$

(*Proficiency in 'Continuity and Differentiability'*)
Exercise 2

1. We know, tan x is discontinuous at $x = (2n+1)\dfrac{\pi}{2}$

$\therefore$ tan $2x$ is discontinuous at $2x = (2n+1)\dfrac{\pi}{2}$

or $x = (2n+1)\dfrac{\pi}{4}$; n is any integer.

2. Clearly, continuous at $x = 1$

To check continuity at $x = 0$, $f(0) = e^3$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}(1+3x)^{1/x} = e^{\lim\limits_{x\to 0} 3x \times \frac{1}{x}}$$

$$= e^{\lim\limits_{x\to 0}(3)}$$

$$= e^3$$

$\therefore$ $\lim\limits_{x\to 0} f(x) = f(0)$

Thus, continuous at $x = 0$

3. $f(x) = \begin{cases} ax+1, & x<1 \\ 3, & x=1 \\ bx^2+1, & x>1 \end{cases}$ since, continuous at $x = 1$

$\therefore$ RHL (at $x = 1$) = LHL at $(x = 1) = f(1)$

$\Rightarrow$ $a + 1 = b + 1 = 3$

or $a = 2$, $b = 2$

4. $f(x) = \begin{cases} \dfrac{\sin ax^2}{x^2}, & x \neq 0 \\ \dfrac{3}{4} + \dfrac{1}{4a}, & x = 0 \end{cases}$ is continuous at $x = 0$

$\therefore$ $\lim\limits_{x\to 0} f(x) = f(0)$

or $\lim\limits_{x\to 0} \dfrac{a \sin ax^2}{ax^2} = \dfrac{3}{4} + \dfrac{1}{4a}$

$\Rightarrow$ $a = \dfrac{3}{4} + \dfrac{1}{4a}$

or $4a^2 - 3a - 1 = 0$

or $(4a+1)(a-1) = 0$

ie, $a = -1/4, 1$

5. Since, continuous at $x = 0$,

RHL = LHL $= f(0)$

On solving, we get $a = -3/2$, $b \neq 0$, $c = 1/2$

6. $f(x) = [(x)] - [x - 1]$

or $\quad f(x) = [x] - [x] + 1$

or $f(x) = 1$ which is constant function and which is continuous for all real numbers.

7. RHL $\quad$ (at $x = 1$) $= 1/2$

LHL $\quad$ (at $x = 1$) $= 1/2$

$f(1) = 1/2,$ $\quad \therefore \quad f(x)$ is continuous at $x = 1$

$\Rightarrow \quad f(x)$ is continuous on $(0, 2)$ $\quad$ ...(i)

$$f'(x) = \begin{cases} x, & 0 < x \le 1 \\ 4x - 3, & 1 < x < 2 \end{cases}$$

RHL $\quad$ at $(x = 1)$ $\quad$ for $\quad f'(x) = 1$

LHL $\quad$ at $(x = 1)$ $\quad$ for $\quad f'(x) = 1$

$f'(1) = 1, f'(x)$ is continuous at $x = 1$

$\Rightarrow \quad f'(x)$ is continuous on $(0, 2)$ $\quad$ ...(ii)

Also, $$f''(x) = \begin{cases} 1, & 0 < x \le 1 \\ 4, & 1 < x < 2 \end{cases}$$

RHL $\quad$ at $(x = 1)$ $\quad$ for $\quad f''(x) = 4$

LHL $\quad$ at $(x = 1)$ $\quad$ for $\quad f''(x) = 1$

$f''(1) = 1$

$\Rightarrow \quad$ RHL $\ne$ LHL $\quad$ Hence, $\quad f''(x)$ is not continuous at $x = 1$

$\therefore \quad f''(x)$ is continuous at $x \in (0, 2) - \{1\}$

8. $$f(x) = \begin{cases} \dfrac{x}{1 + x}, & x \ge 0 \\ \dfrac{x}{1 - x}, & x < 0 \end{cases}$$

$\therefore$ $$f'(x) = \begin{cases} \dfrac{(1 + x) - x}{(1 + x)^2}, & x \ge 0 \\ \dfrac{(1 - x) + x}{(1 - x)^2}, & x < 0 \end{cases}$$

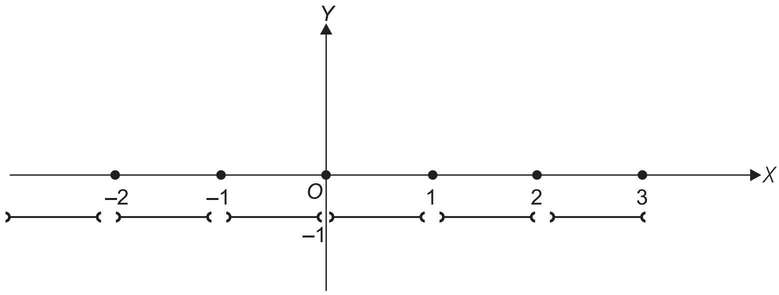
or $$f'(x) = \begin{cases} 1/(1 + x)^2, & x \ge 0 \\ 1/(1 - x)^2, & x < 0 \end{cases}$$ which is differentiable for all x.

Hence, $\quad f(x)$ is differentiable $(-\infty, \infty)$.

9. (i) $$f(x) = [x] + [-x] = \begin{cases} x - x, & x \in \text{integers} \\ [x] - 1 - [x], & x \notin \text{integers} \end{cases}$$

$\therefore$ $$f(x) = \begin{cases} 0, & x \in \text{integers} \\ -1, & x \notin \text{integers} \end{cases}$$

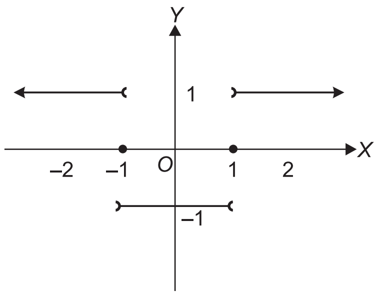
which, shows the graph of $f(x)$ as, shown in figure.



Thus, shows $f(x)$ is discontinuous at $x \in$ integers

(ii) $g(x) = \lim_{n \to \infty} \dfrac{x^{2n} - 1}{x^{2n} + 1} = \begin{cases} -1, & |x| < 1 \\ 0, & |x| = 1 \\ 1, & |x| > 1 \end{cases}$

which can be shown as, in the figure.



Thus, shows $g(x)$ is discontinuous at $x = \pm 1$.

10. $f(x) = \begin{cases} x[x], & 0 \le x < 2 \\ (x-1)[x], & 2 \le x \le 3 \end{cases}$

To check continuity at $x = 1$

RHL (at $x = 1$) $= \lim_{h \to 0} (1+h)[1+h] = 1$

LHL (at $x = 1$) $= \lim_{h \to 0} (1-h)[1-h] = 0$

$\therefore$ discontinuous at $x = 1$

To check continuity at $x = 2$

RHL (at $x = 2$) $= \lim_{h \to 0} (2+h-1)[2+h] = 2$

LHL (at $x = 2$) $= \lim_{h \to 0} (2-h)[2-h] = 2$

$$f(2) = (2-1)[2] = 2$$

$\therefore$ Continuous at $x = 2$

To check differentiability at $x = 2$

RHD (at $x = 2$) $= \lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$

RHD (at $x = 2$) $= \lim_{h \to 0} \dfrac{(2+h-1)[2+h] - 2}{h}$

$$= \lim_{h \to 0} \frac{(1+h)2 - 2}{h} = 2$$

LHD (at $x = 2$) $= \lim_{h \to 0} \dfrac{(2-h)[2-h] - 2}{-h}$

$$\lim_{h \to 0} \frac{2-h-2}{-h} = 1$$

Which shows $f(x)$ is not differentiable at $x = 2$.

Therefore, $f(x)$ is not differentiable as $f(x)$ at $x = 1$ is not continuous.

11. $f(x) = [x] + |1 - x|, -1 < x \le 3$

$$f(x) = \begin{cases} -1 + (1 - x) & , -1 < x < 0 \\ 0 + (1 - x) & , \quad 0 \le x < 1 \\ 1 - (1 - x) & , \quad 1 \le x < 2 \\ 2 - (1 - x) & , \quad 2 \le x < 3 \\ 3 + 2 & , \quad x = 3 \end{cases}$$

$\therefore$
$$f(x) = \begin{cases} -x & , -1 < x < 0 \\ 1 - x & , \quad 0 \le x < 1 \\ x & , \quad 1 \le x < 2 \\ 1 + x & , \quad 2 \le x < 3 \\ 5 & , \quad x = 3 \end{cases}$$

which shows $f(x)$ is not differentiable at $x = 0, 1, 2, 3$.

12. (i) Consider $x = 0$

$$f'(0^+) = \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{(1 + \sin h) - 1}{h} = \lim_{h \to 0} \frac{\sin h}{h} = 1$$

$$f'(0^-) = \lim_{h \to 0^-} \frac{f(-h) - f(0)}{-h} = \lim_{h \to 0} \frac{1 - 1}{-h} = 0.$$

$\therefore$ $\quad f'(0^-) = 0$

$\Rightarrow$ $\quad$ RHD $\ne$ LHD $\Rightarrow$ $f'(0)$ doesn't exist.

$\Rightarrow$ $\quad f$ is not differentiable at $x = 0$.

But $\quad f'(0^+) = 1$ and $f'(0^-) = 0$ both exists.

Hence, $\quad f(x)$ is continuous at $x = 0$.

(ii) Consider $x = \pi/2$

$$f'\left(\frac{\pi^+}{2}\right) = \lim_{h \to 0} \frac{f\left(\frac{\pi}{2} + h\right) - f\left(\frac{\pi}{2}\right)}{h} = \lim_{h \to 0} \frac{2 + h^2 - 2}{h} = 0$$

$\therefore \qquad f'\left(\frac{\pi^+}{2}\right)=0$

$$f'\left(\frac{\pi^-}{2}\right)=\lim_{h\to 0}\frac{f\left(\frac{\pi}{2}-h\right)-f\left(\frac{\pi}{2}\right)}{-h}=\lim_{h\to 0}\frac{1+\sin\left(\frac{\pi}{2}-h\right)-2}{-h}=0$$

$$=\lim_{h\to 0}\frac{1+\cos h-2}{-h}=\lim_{h\to 0}\frac{1-\cos h}{h}$$

$$=\lim_{h\to 0}\frac{2\sin^2\frac{h}{2}}{h\cdot\frac{h}{4}}\cdot\frac{h}{4}=2\cdot(1)\frac{(0)}{4}=0$$

$\therefore \qquad f'\left(\frac{\pi^-}{2}\right)=0$

Hence, f is differentiable at $x=\frac{\pi}{2}\cdot$

Hence, f is continuous at $x=\frac{\pi}{2}\cdot$

13. Since, $f(x)$ and $g(x)$ are differentiable at $x=a$

$\therefore \qquad f'(x)$ and $g'(x)$ exists.

$$\therefore \qquad \lim_{x\to a}\frac{g(x)\cdot f(a)-g(a)\cdot f(x)}{\sin(x-a)} \qquad \left[\text{form }\left(\frac{0}{0}\right)\right]$$

Applying L'Hospital's rule, we get

$$\lim_{x\to a}\frac{g'(x)\cdot f(a)+g(x)\cdot 0-g(a)\cdot f'(x)-0\cdot f(x)}{\cos(x-a)}$$

[as $f(a)$ and $g(a)$ are constant]

Hence, $\qquad g'(a)f(a)-g(a)f'(a)$

14. $f(x+y)=f(x)\cdot f(y)$, $f'(0)=3$ and $f(5)=2$

$$\therefore \qquad f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f(x)\cdot f(h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x)\{f(h)-1\}}{h}$$

$$\Rightarrow \qquad f'(x)=f(x)\cdot\left[\lim_{h\to 0}\frac{f(h)-1}{h}\right]$$

$\Rightarrow \qquad f'(x)=f(x)\cdot f'(0)$

$\Rightarrow \qquad f'(x)=f(x)\cdot(3)$

$\Rightarrow \qquad f'(5)=f(5)\cdot(3)$

$\Rightarrow \qquad f'(5)=(2)\cdot(3)$

Hence, $\qquad f'(5)=6$

$$h(x)=[f(x)]^2+[g(x)]^2$$

15. Differentiating both the sides, we get

$$h'(x) = 2f(x) \cdot f'(x) + 2g(x) \cdot g'(x)$$

$$h'(x) = 2f(x) \cdot g(x) + 2g(x) \cdot f''(x) \qquad [\text{as } g(x) = f'(x), g'(x) = f''(x)]$$

$$h'(x) = 2f(x) \cdot g(x) - 2g(x) \cdot f'(x)$$

$$h'(x) = 2f(x) \cdot g(x) - 2g(x) \cdot f(x) \qquad [f'(x) = f(x)]$$

$$h'(x) = 0$$

$\therefore$ $h(x)$ must be constant function.

Given, $h(5) = 11$

Hence, $h(10) = 11$

16. $\therefore$ $f(x) = f(1 - x)$

Put $x = 0 \Rightarrow f(0) = f(1)$ ...(i)

Now, $f'(1) = 0$

ie, $$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = 0 = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} \qquad \text{...(ii)}$$

$$f(x) = f(1 - x)$$

$\therefore$ $f(1 - h) = f(h)$ $[\because (x = 0 + h)]$

$f(-h) = f(1 + h)$ $\because$ $(x = 0 - h)$ ...(iii)

Let $l = f'(0^+) = f'(0^-)$ [$\because$ $f'(0)$ exists which is given]

$\therefore$ $$l = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{f(1-h) - f(0)}{h}$$

$$= (-1) \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = (-1) \cdot 0 = 0 \quad [\text{using Eqs. (i), (ii), (iii)}]$$

Also,

$$l = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(1+h) - f(0)}{-h}$$

$$= -\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = 0 \qquad [\text{using Eqs. (i), (ii), (iii)}]$$

$\therefore$ $f'(0) = 0$

17. We know

$$\lim_{x \to 0} |f(x)| \to 0 \Leftrightarrow f(x) \to 0 \qquad \text{...(i)}$$

and $|f(x) - f(y)| \le |x - y|^3$ for all real x and y.

Let x be any real number and let y be chosen in the neighbourhood of x, but not equal to x.

$\therefore$ $$\left| \frac{f(x) - f(y)}{x - y} \right| < |x - y|^2$$

Taking $\lim\limits_{y \to x}$ on both sides and using Eq. (i)

$$\lim_{y \to x}\left|\frac{f(x)-f(y)}{x-y}\right| \le \lim_{y \to x}|x-y|^2 \qquad \left[\because \lim_{y \to x}\frac{f(x)-f(y)}{x-y} = f'(x)\right]$$

$\therefore$ $\quad |f'(x)| \le 0$

ie, $\quad |f'(x)| = 0$ $\quad$ [since, absolute value could never be negative]

or $\quad f'(x) = 0$

Hence, $f(x)$ is a constant function.

18. Given that

$$f(x+y) = f(x)\cdot f(y) \qquad \text{for all } x \in R \quad \text{...(i)}$$

Putting $\quad x = y = 0$ in Eq. (i), we get

$$f(0)\{f(0)-1\} = 0$$

$\Rightarrow$ $\quad f(0) = 0 \quad$ or $\quad f(0) = 1$

If $\quad f(0) = 0$, then $f(x) = f(x+0) = f(x)\cdot f(0)$

$\Rightarrow$ $\quad f(x) = 0$ for all $x \in R$

which is not true, so $\quad f(0) = 1$

$\therefore$
$$f'(x) = \lim_{h \to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h \to 0}\frac{f(x)\cdot f(h)-f(x)}{h} \qquad \text{[from Eq. (i)]}$$
$$= \lim_{h \to 0} f(x)\left\{\frac{f(h)-1}{h}\right\}$$
$$= f(x)\cdot f'(0)$$
$$= 2f(x) \qquad [\text{given } f'(0) = 2]$$

or
$$\frac{f'(x)}{f(x)} = 2$$

Integrating both the sides, we get

$$\log_e f(x) = 2x + C$$

at $\quad x = 0, f(0) = 1$

Hence, $\quad \log f(1) = 2(0) + C \Rightarrow \log 1 = 0 + C$

$\Rightarrow$ $\quad C = 0 \quad$ or $\quad \log_e f(x) = 2x + 0$

$\therefore$ $\quad f(x) = e^{2x}$

19. We know that,

$$f'(x) = \lim_{h \to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h \to 0}\frac{f(x)\cdot f(h)-f(x)}{h} \qquad \text{[using (a)]}$$
$$= f(x)\lim_{h \to 0}\frac{f(h)-1}{h}$$
$$= f(x)\lim_{h \to 0}\frac{1+h\cdot g(h)-1}{h} \qquad \text{[using (b)]}$$
$$= f(x)\lim_{h \to 0}\frac{h\cdot g(h)}{h} \qquad (\because h \to 0, h \ne 0)$$

$$= f(x) \cdot \lim_{h \to 0} g(h)$$

$$= f(x) \cdot 1 \qquad \text{[using (b)]}$$

$\therefore \qquad f'(x) = f(x)$

which shows $f'(x)$ exists and $f'(x) = f(x)$.

20. Given, $\quad f(xy) = f(x) \cdot f(y) \quad$ (for all $x \neq 0$)

$\therefore \qquad f(x) = f(x) \cdot f(1) \quad$ for $\quad y = 1$

$\therefore \qquad f(x) = 0 \quad$ or $\quad f(1) = 1$

But $\qquad f(x) \neq 0 \quad \therefore \quad f(1) = 1$

Now, $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f\left(x\left(1 + \frac{h}{x}\right)\right) - f(x)}{h} \qquad (\because x \neq 0)$$

$$f'(x) = \lim_{h \to 0} \frac{f(x) \cdot f\left(1 + \frac{h}{x}\right) - f(x)}{h} = f(x) \cdot \lim_{h \to 0} \frac{f\left(1 + \frac{h}{x}\right) - 1}{x \cdot h / x}$$

$\therefore \qquad f'(x) = \dfrac{f(x)}{x} \cdot f'(1) = \dfrac{f(x)}{x},$

Hence, differentiating for all $x \neq 0$.

To determine $f(x)$ we have, $\dfrac{f'(x)}{f(x)} = \dfrac{1}{x};$

Integrating both the sides

$$\log_e f(x) = \log_e x + C$$

Hence, $\qquad f(x) = x$

21. Since, $f(x)$ is differentiable at $x = 0$.

$$\Rightarrow \qquad \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = p \text{ (say)} \qquad \text{...(i)}$$

Then, $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

or $$f'(x) = \lim_{h \to 0} \frac{f\left\{\frac{3x + 3h}{3}\right\} - f\left(\frac{3x + 3 \cdot 0}{3}\right)}{h}$$

or $$f'(x) = \lim_{h \to 0} \frac{f(3x) + f(3h) + f(0) - f(3x) - f(0) - f(0)}{3h}$$

or $$f'(x) = \lim_{h \to 0} \frac{f(3h) - f(0)}{3h}$$

or $$f'(x) = f'(0) \qquad \left[\text{using } f\left(\frac{x+y}{3}\right) = \frac{f(x) + f(y) + f(0)}{3}\right]$$

or $\qquad f'(x) = p$ (let) $\qquad$ [from Eq. (i)]

$\therefore \qquad f(x) = px + q$

which shows $f(x)$ is differentiable for all x in R.

22. We have,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x) + f(h) + 2hx - 1 - f(x)}{h}$$

[using given definition]

$$= \lim_{h \to 0} \left\{ 2x + \frac{f(h) - 1}{h} \right\}$$

Now, substituting $x = y = 0$ in the given functional relation, we get

$$f(0) = f(0) + f(0) + 0 - 1 \quad \Rightarrow \quad f(0) = 1$$

$\therefore$ $$f'(x) = 2x + \lim_{h \to 0} \frac{f(h) - f(0)}{h} = 2x + f'(0)$$

$\Rightarrow$ $$f'(x) = 2x + \cos \alpha$$

Integrating, $$f(x) = x^2 + x \cos \alpha + C$$

Here, $x = 0$ and $f(0) = 1$

$\therefore$ $1 = C$

$\Rightarrow$ $$f(x) = x^2 + x \cos \alpha + 1$$

It is a quadratic in x with discriminant

$$D = \cos^2 \alpha - 4 < 0$$

and coefficient of $x^2 = 1 > 0$

$\therefore$ $$f(x) > 0 \quad \forall \ x \in R$$

23. Put $x = y = 0 \quad \Rightarrow \quad f(0) = 0$

Now, $$\lim_{y \to 0} \frac{f(x + y^n) - f(x)}{y^n} = \lim_{y \to 0} \frac{(f(y))^n}{y^n}$$

$\Rightarrow$ $$f'(x) = \left(\lim_{y \to 0} \frac{f(y)}{y} \right)^n = \left(\lim_{y \to 0} \frac{f(y + 0) - f(0)}{y} \right)^n$$

$$= (f'(0))^n$$

$\Rightarrow$ $f'(x) = A$ $\qquad$ $[A = f'(0)]$

$\Rightarrow$ $f(x) = Ax + B$ $\qquad$ $[f'(0) = 1 = A$ and $x = 0 \Rightarrow f(0) = 0]$

$\therefore$ $f(x) = x$

Hence, $f(5) = 5$ and $f'(10) = 1$

24. $f(x + y) = f(x) + f(y) - 8xy$ $\qquad$ ...(i)

Replacing $x, y \to 0$, we get $f(0) = 0$ $\qquad$ ...(ii)

Now, $$f'(x) = \lim_{y \to 0} \frac{f(x + y) - f(x)}{y} = \lim_{y \to 0} \frac{f(y) - f(0) - 8xy}{y}$$

$$= \lim_{y \to 0} \frac{f(y)}{y} - 8x$$

[using Eqs. (i) and (ii)]

$$= f'(0) - 8x$$

$\therefore$ $$f'(x) = 8 - 8x$$

$\Rightarrow \quad f(x) = 8x - 4x^2$ [As $f(0) = 0$]

$g(x + y) = g(x) + g(y) + 3xy(x + y)$ ...(iii)

Replacing $x, y \to 0$, we get $g(0) = 0$ ...(iv)

Now, $g'(x) = \lim_{y \to 0} \dfrac{g(x + y) - g(x)}{y} = \lim_{y \to 0} \dfrac{g(y) - g(0) + 3x^2y + 3xy^2}{y}$

[using Eq. (iii)]

$\Rightarrow \quad g'(x) = \lim_{y \to 0} \dfrac{g(y)}{y} + 3x^2 + 3xy = g'(0) + 3x^2$

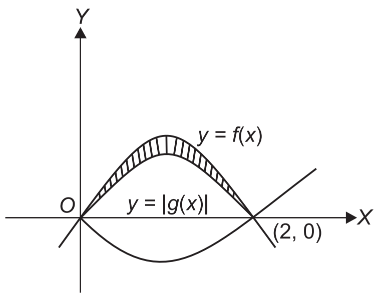
$\therefore \quad g'(x) = -4 + 3x^2 \quad \Rightarrow \quad g(x) = -4x + x^3$

Now, $f(x)$ and $g(x)$ meet, we have $f(x) = g(x)$

or $\quad 8x - 4x^2 = -4x + x^3$

$\Rightarrow \quad x = 0, 2, -6$

Now, $|g(x)| = \begin{cases} x^3 - 4x, & x \in [-2, 0] \cup [2, \infty) \\ 4x - x^3, & x \in (-\infty, -2) \cup (0, 2) \end{cases}$



Area bounded by $y = f(x)$ and $y = |g(x)|$ between $x = 0$ to $x = 2$ is

$$\int_0^2 \{(8x - 4x^2) - (4x - x^3)\}\, dx = \int_0^2 (x^3 - 4x^2 + 4x)\, dx = \frac{4}{3} \text{ sq unit.}$$

7

dy/dx as a Rate Measurer & Tangents, Normals

Chapter in a Snapshot

- Derivative as the Rate of Change
- Velocity and Acceleration in Rectilinear Motion
- Approximation and Differentials
- Slopes of Tangent and Normal
- Equations of Tangent and Normal
- Angle of Intersection of Two Curves
- Length of Tangent, Subtangent, Normal and Subnormal
- Rolle’s Theorem
- Lagrange’s Mean Value Theorem
- Different Graphs of the Cubic

Derivative as the Rate of Change

If a variable quantity y is some function of time t *ie*, $y = f(t)$, then small change in time Δt have a corresponding change Δy in y.

Thus, the average rate of change $= \dfrac{\Delta y}{\Delta t}$

When limit $\Delta t \to 0$ is applied, the rate of change becomes instantaneous and we get the rate of change with respect to at the instant x.

ie, $$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}$$

Hence, it is clear that the rate of change of any variable with respect to some other variable is the derivative of first variable with respect to other variable.

Point to Consider

The differential coefficient of y with respect to x *ie*, $\dfrac{dy}{dx}$ is nothing but the rate of increase of y relative to x.

Illustration 1 If the radius of a circle is increasing at a uniform rate of 2 cm/s, find the rate of increase of area of circle, at the instant when the radius is 20 cm.

Solution. Given, $\dfrac{dr}{dt} = 2$ cm/s (where r radius and t time)

Now, area of circle is given by $A = \pi r^2$

Differentiating it with respect to time t, we get

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dA}{dt} = 2\pi \cdot 20 \cdot 2 \text{ cm}^2/\text{s}$$

$$\Rightarrow \quad \frac{dA}{dt} = 80\pi \text{ cm}^2/\text{s}$$

Thus, the rate of change of area of circle with respect to time is 80π cm^2/s.

Illustration 2 On the curve $x^3 = 12y$, find the interval at which the abscissa changes at a faster rate than the ordinate?

Solution. Given, $x^3 = 12y$, differentiating it with respect to y

$$3x^2 \frac{dx}{dy} = 12$$

$$\therefore \quad \frac{dx}{dy} = \frac{12}{3x^2}$$

In the interval, at which the abscissa changes at a faster rate than the ordinate, we must have

$$\Rightarrow \qquad \left|\frac{dx}{dy}\right| > 1 \quad \text{or} \quad \left|\frac{12}{3x^2}\right| > 1$$

$$\text{or} \qquad \frac{4}{x^2} > 1 \quad \Rightarrow \quad \frac{4 - x^2}{x^2} > 0, \text{ when } x \neq 0$$

Using number line rule, $4 - x^2 > 0$ or $x^2 - 4 < 0$

$$(x - 2)(x + 2) < 0 \Rightarrow -2 < x < 2, \neq \{0\}$$

Thus, $x \in (-2, 2) - \{0\}$ is the required interval between which abscissa changes at a faster rate than ordinate.

Velocity and Acceleration in Rectilinear Motion

Students who are studying Physics, are quite familiar with the definition of velocity and acceleration.

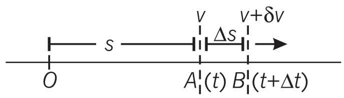


Fig. 7.1

The velocity of a moving particle is defined as the rate of change of its displacement with respect to time and the acceleration is defined as the rate of change of its velocity with respect to time.

Let a particle A move rectilinearly as shown in figure.

Let s be the displacement from a fixed point O along the path at time t; s is considered to be positive on right of the point O and negative on the left of it.

Also, Δs is positive when s increases *ie*, when the particle moves towards right.

Thus, if Δs be the increment in s in time Δt. The average velocity in this interval is

$$\frac{\Delta s}{\Delta t}$$

and the instantaneous velocity *ie*, velocity at time t is

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

If the velocity varies, then there is change of velocity Δv in time Δt.

Hence, the acceleration at time

$$t = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

Illustration 3 If the displacement of a particle is given by $s=\left(\frac{1}{2}t^2+4\sqrt{t}\right)$ m. Find the velocity and acceleration at $t=4$ s.

Solution. We know that the displacement of particle is given by, $s=\left(\frac{1}{2}t^2+4\sqrt{t}\right)$ m

$\therefore$ Velocity is given by, $v=\frac{ds}{dt}=\left(t+\frac{2}{\sqrt{t}}\right)$ m/s ...(i)

and acceleration is given by, $a=\frac{d^2s}{dt^2}=\frac{dv}{dt}=\left(1-\frac{1}{t^{3/2}}\right)$ m/s ...(ii)

$\therefore$ Velocity when $t=4 \Rightarrow v=\left(4+\frac{2}{\sqrt{4}}\right)$ m/s $\Rightarrow v=5$ m/s

Also, acceleration when $t=4 \Rightarrow a=\left(1-\frac{1}{4^{3/2}}\right)$ m/s^2

$$\Rightarrow \qquad a=\left(1-\frac{1}{8}\right)\text{ m/s}^2 \Rightarrow a=\frac{7}{8}\text{ m/s}^2$$

Illustration 4 If $s=\frac{1}{2}t^3-6t$, find the acceleration at time when the velocity tends to zero.

Solution. We know that the displacement s is given by, $s=\frac{1}{2}t^3-6t$

Thus, velocity $\quad v=\frac{ds}{dt}=\left(\frac{3t^2}{2}-6\right)$ unit/s ...(i)

and acceleration, $\quad a=\frac{dv}{dt}=\frac{d^2s}{dt^2}=(3t)$ unit/s^2 ...(ii)

To find acceleration when velocity $\to 0$

$$\Rightarrow \qquad \frac{3t^2}{2}-6=0 \Rightarrow t^2=4 \Rightarrow t=2\text{ s}$$

Thus, acceleration when velocity tends to zero,

$$\Rightarrow \qquad a=3t\text{ unit/s}^2=6\text{ unit/s}^2$$

Illustration 5 If r be the radius, S the surface area and V the volume of a spherical bubble, prove that

(i) $\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$ (ii) $\frac{dV}{dS}\propto r$

Solution. (i) Since, $V=\frac{4}{3}\pi r^3$

$$\therefore \qquad \frac{dV}{dt}=\frac{4}{3}\pi\cdot 3r^2\frac{dr}{dt}=4\pi r^2\frac{dr}{dt} \qquad ...(i)$$

(ii) We know, $S=4\pi r^2$

$$\Rightarrow \qquad \frac{dS}{dt}=8\pi r\frac{dr}{dt} \qquad ...(ii)$$

Thus, $\quad \frac{dV}{dS}=\frac{dV/dt}{dS/dt}=\frac{1}{2}\cdot r$ [from Eqs. (i) and (ii)]

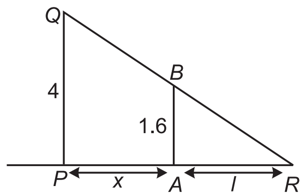
$$\Rightarrow \qquad \frac{dV}{dS}=\frac{1}{2}r \quad \text{or} \quad \frac{dV}{dS}\propto r$$

Illustration 6 A man who is 1.6 m tall walks away from a lamp which is 4 m above ground at the rate of 30 m/min. How fast is the man's shadow lengthening?

Solution. Let $PQ = 4$ m be the height of pole and $AB = 1.6$ m be height of man.

Let the end of shadow is R and it is at a distance of l from A when the man is at a distance x from PQ at some instance.

Since, $\Delta\, PQR$ and $\Delta\, ABR$ are similar, we have



$$\frac{PQ}{AB} = \frac{PR}{AR}$$

$$\Rightarrow \quad \frac{4}{1.6} = \frac{x + l}{l} \quad \Rightarrow \quad 2x = 3l$$

$$\Rightarrow \quad 2\frac{dx}{dt} = 3\frac{dl}{dt} \qquad \left[\text{given } \frac{dx}{dt} = 30 \text{ m/min}\right]$$

$$\Rightarrow \quad \frac{dl}{dt} = \frac{2}{3} \cdot 30 \text{ m/min } = 20 \text{ m/min lengthening}$$

Illustration 7 If x and y are the sides of two squares such that $y = x - x^2$, find the rate of change of the area of the second square with respect to the first square.

Solution. Given, x and y are sides of two squares, thus the area of two squares are x^2 and y^2.

We have to obtain $\quad \dfrac{d(y^2)}{d(x^2)} = \dfrac{2y\dfrac{dy}{dx}}{2x} = \dfrac{y}{x} \cdot \dfrac{dy}{dx} \qquad$ ...(i)

where the given curve is,

$$y = x - x^2 \quad \Rightarrow \quad \frac{dy}{dx} = 1 - 2x \qquad \text{...(ii)}$$

Thus, $\quad \dfrac{d(y^2)}{d(x^2)} = \dfrac{y}{x}(1 - 2x) \qquad$ [from Eqs. (i) and (ii)]

or $\quad \dfrac{d(y^2)}{d(x^2)} = \dfrac{(x - x^2)(1 - 2x)}{x}$

$$\Rightarrow \quad \frac{d(y^2)}{d(x^2)} = (2x^2 - 3x + 1)$$

The rate of change of the area of second square with respect to first square is $(2x^2 - 3x + 1)$.

Approximation and Differentials

From the figure, it is clear that if Δx and Δy are sufficiently small quantities, then

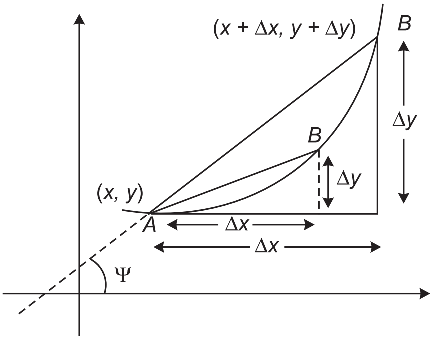


Fig. 7.2

$$\frac{\Delta y}{\Delta x} = \tan \psi \cong \frac{dy}{dx} = f'(x)$$

Hence, approximate change in the value of y, called its differential, is given by

$$\Delta y = f'(x) \cdot \Delta x \qquad \ldots(\text{i})$$

Illustration 8 Use differential to approximate $\sqrt{101}$.

Solution. Let $f(x) = y = \sqrt{x}$, $y + \Delta y = \sqrt{x + \Delta x}$

$\therefore$ $$\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = f'(x) \cdot \Delta x \qquad [\text{using Eq. (i)}]$$

Put $$x = 100, \Delta x = 1$$

$$\sqrt{101} = 10 + \frac{1}{20} \times 1 = 10 + 0.05 = 10.05$$

Illustration 9 In an acute ΔABC, if sides a, b be constants and the base angles A and B vary, show that $\dfrac{dA}{\sqrt{a^2 - b^2 \sin^2 A}} = \dfrac{dB}{\sqrt{b^2 - a^2 \sin^2 B}}$

Solution. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

or $$b \sin A = a \sin B \quad \Rightarrow \quad b \cos A \, dA = a \cos B \, dB$$

$$\frac{dA}{a \cos B} = \frac{dB}{b \cos A} \quad \Rightarrow \quad \frac{dA}{a\sqrt{1 - \sin^2 B}} = \frac{dB}{b\sqrt{1 - \sin^2 A}}$$

$$\frac{dA}{a\sqrt{1 - \dfrac{b^2 \sin^2 A}{a^2}}} = \frac{dB}{b\sqrt{1 - \dfrac{a^2 \sin^2 B}{b^2}}}$$

$\Rightarrow$ $$\frac{dA}{\sqrt{a^2 - b^2 \sin^2 A}} = \frac{dB}{\sqrt{b^2 - a^2 \sin^2 B}}$$

Slopes of Tangent and Normal

Slope of Tangent Let $y = f(x)$ be a continuous curve and let $P(x_1, y_1)$ be a point on it.

Then, $\left(\dfrac{dy}{dx}\right)_{(x_1,\, y_1)}$ is the slope of tangent to the curve $y = f(x)$ at a point P.

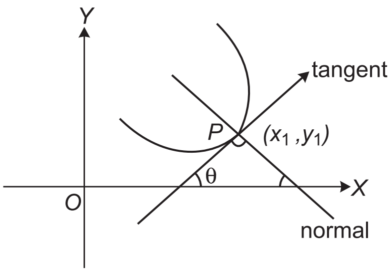


Fig. 7.3

$$\Rightarrow \qquad \left(\frac{dy}{dx}\right)_P = \tan\theta = \text{ slope of tangent at } P.$$

Where θ is the angle which the tangent at $P(x_1, y_1)$ forms with the positive direction of x-axis as shown in the figure.

Points to Consider

(i) If tangent is parallel to x-axis $\theta = 0° \quad \Rightarrow \quad \tan\theta = 0$

$$\therefore \qquad \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = 0$$

(ii) If tangent is perpendicular to x-axis (or parallel to y-axis), then

$$\theta = 90° \qquad \Rightarrow \qquad \tan\theta = \infty \quad \text{or} \quad \cot\theta = 0$$

$$\therefore \qquad \left(\frac{dx}{dy}\right)_{(x_1,\, y_1)} = 0$$

Slope of Normal We know that the normal to the curve at $P(x_1, y_1)$ is a line perpendicular to tangent at $P(x_1, y_1)$ and passes through P.

$$\therefore \quad \text{Slope of the normal at } P = -\frac{1}{\text{Slope of the tangent at } P}$$

$$\Rightarrow \quad \text{Slope of normal at } P(x_1, y_1) = -\frac{1}{\left(\dfrac{dy}{dx}\right)_{(x_1,\, y_1)}}$$

or slope of normal at $P(x_1, y_1) = -\left(\dfrac{dx}{dy}\right)_{(x_1,\, y_1)}$

Points to Consider

(i) If normal is parallel to x-axis

$$\Rightarrow \qquad -\left(\frac{dx}{dy}\right)_{(x_1,\, y_1)} = 0 \text{ or } \left(\frac{dx}{dy}\right)_{(x_1,\, y_1)} = 0$$

(ii) If normal is perpendicular to x-axis (or parallel to y-axis)

$$\Rightarrow \qquad -\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = 0$$

Illustration 10 In which of the following cases the function $f(x)$ has a vertical tangent at $x = 0$?

(i) $f(x) = x^{1/3}$ (ii) $f(x) = \text{sgn } x$ (iii) $f(x) = x^{2/3}$

(iv) $f(x) = \sqrt{|x|}$ (v) $f(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}$

Solution. **Vertical Tangent**

Concept $y = f(x)$ has a vertical tangent at the points $x = x_0$, if

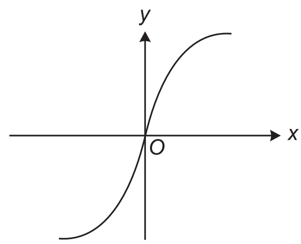
$\lim\limits_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h} = \infty$ or $-\infty$ but not both, for example the functions

$f(x) = x^{1/3}$ and $f(x) = \text{sgn } x$ both have a vertical tangent at $x = 0$

but $f(x) = x^{2/3}, f(x) = \sqrt{|x|}$ and $f(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}$ have no vertical tangent.

Explanation

(i) $f(x) = x^{1/3}$



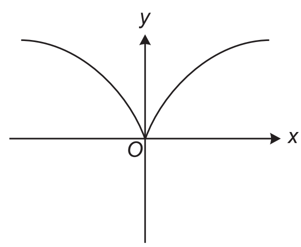
$$\left.\begin{aligned} f'(0^+) &= \lim_{h \to 0} \frac{h^{1/3}}{h} = \frac{1}{h^{2/3}} \to \infty \\ f'(0^-) &= \lim_{h \to 0} \frac{(-h)^{1/3}}{-h} = \frac{1}{h^{2/3}} \to \infty \end{aligned}\right\}$$

$\Rightarrow x = 0$ is a vertical tangent.

(ii) $f(x) = \text{sgn } x = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$

$$\left.\begin{aligned} f'(0^+) &= \lim_{h \to 0} \frac{1 - 0}{h} \to \infty \\ f'(0^-) &= \lim_{h \to 0} \frac{-1}{-h} \to \infty \end{aligned}\right\} \Rightarrow x = 0 \text{ is a vertical tangent.}$$

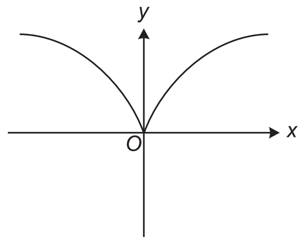
(iii) $f(x) = x^{2/3}$



$$\left.\begin{aligned} f'(0^+) &= \lim_{h \to 0} \frac{h^{2/3}}{h} \to \infty \\ f'(0^-) &= \lim_{h \to 0} \frac{(-h)^{2/3}}{-h} = -\frac{1}{h^{2/3}} \to -\infty \end{aligned}\right\}$$

no vertical tangent at $x = 0$.

(iv) $f(x)=\sqrt{|x|}=\begin{bmatrix}\sqrt{x}, \text{ if } x\geq 0\\ \sqrt{-x}, \text{ if } x<0\end{bmatrix}$



$$\left.\begin{aligned} f'(0^+)&=\lim_{h\to 0}\frac{\sqrt{h}-0}{h}\to\infty\\ f'(0^-)&=\lim_{h\to 0}\frac{\sqrt{h}}{-h}=\to-\infty\end{aligned}\right\}$$

no vertical tangent at $x=0$.

(v) $f(x)=\begin{cases}0, \text{ if } x<0\\ 1, \text{ if } x\geq 0\end{cases}$

$$\left.\begin{aligned} f'(0^+)&=\lim_{h\to 0}\frac{1-1}{h}=0\\ f'(0^-)&=\lim_{h\to 0}\frac{0-1}{h}=-\infty\end{aligned}\right\}$$

no vertical tangent at $x=0$.

Illustration 11 Find the slopes of the tangent and normal to the curve

$$x^3+3xy+y^3=2 \text{ at } (1, 1).$$

Solution. The equation of the curve is, $x^3+3xy+y^3=2$

Differentiating it w.r.t. x, we get

$$3x^2+3x\frac{dy}{dx}+3y+3y^2\frac{dy}{dx}=0$$

$$\Rightarrow \qquad \frac{dy}{dx}=-\frac{(3x^2+3y)}{(3x+3y^2)}$$

$$\Rightarrow \qquad \frac{dy}{dx}=-\frac{(x^2+y)}{(x+y^2)}$$

$$\Rightarrow \qquad \left(\frac{dy}{dx}\right)_{(1,1)}=-\left(\frac{2}{2}\right)=-1$$

$\therefore$ Slope of tangent at $(1,1)=\left(\frac{dy}{dx}\right)_{(1,1)}=-1$

and slope of normal at $(1,1)=-\dfrac{1}{\left(\frac{dy}{dx}\right)_{(x_1,\,y_1)}}=\dfrac{-1}{-1}=1$

Illustration 12 Find the point on the curve $y=x^3-3x$ at which tangent is parallel to x-axis.

Solution. Let the point at which tangent is parallel to x-axis be $P(x_1, y_1)$.

Then, it must lie on curve *ie*, $y_1=x_1^3-3x_1$ ...(i)

Differentiating it w.r.t. x, we get

$$\frac{dy}{dx}=3x^2-3$$

$$\Rightarrow \qquad \left(\frac{dy}{dx}\right)_{(x_1,\,y_1)}=3x_1^2-3$$

Since, the tangent is parallel to x-axis,

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$$

$$\Rightarrow \quad 3x_1^2 - 3 = 0$$

$$\Rightarrow \quad x_1 = \pm 1 \quad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$y_1 = x_1^3 - 3x_1$$

when $x_1 = 1$ when $x_1 = -1$

$y_1 = 1 - 3 = -2$ $y_1 = -1 + 3 = 2$

$\therefore$ Points at which tangent is parallel to x-axis are $(1, -2)$ and $(-1, 2)$.

Illustration 13 Find the point on the curve $y = x^3 - 2x^2 - x$ at which the tangent line is parallel to the line $y = 3x - 2$.

Solution. Let $P(x_1, y_1)$ be the required point. The given curve can be written as

$$y_1 = x_1^3 - 2x_1^2 - x_1 \quad \text{...(i)}$$

Differentiating the curve $y = x^3 - 2x^2 - x$ w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 - 4x - 1$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3x_1^2 - 4x_1 - 1$$

Since, tangent at (x_1, y_1) is parallel to the line $y = 3x - 2$.

$\therefore$ Slope of the tangent at $P(x_1, y_1)$ = Slope of the line $y = 3x - 2$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3$$

$$\Rightarrow \quad 3x_1^2 - 4x_1 - 1 = 3$$

$$\Rightarrow \quad 3x_1^2 - 4x_1 - 4 = 0$$

$$\Rightarrow \quad (x_1 - 2)(3x_1 + 2) = 0$$

$$\Rightarrow \quad x_1 = 2, -2/3 \quad \text{...(ii)}$$

From Eqs. (i) and (ii),

when $x_1 = 2$ when $x_1 = -2/3$

$y_1 = x_1^3 - 2x_1^2 - x_1$ $y_1 = x_1^3 - 2x_1^2 - x_1$

$\Rightarrow$ $y_1 = 8 - 8 - 2$ $\Rightarrow$ $y_1 = \frac{-8}{27} - \frac{8}{9} + \frac{2}{3}$

$\Rightarrow$ $y_1 = -2$ $\Rightarrow$ $y_1 = \frac{-14}{27}$

Thus, the point at which tangent is parallel to $y = 3x - 2$ are $(2, -2)$ and $\left(-\frac{2}{3}, -\frac{14}{27}\right)$.

Equations of Tangent and Normal

We know that, the equation of line passing through (x_1, y_1) and having slope $\tan\theta$ (or m) is $(y - y_1) = \tan\theta\,(x - x_1)$ or $(y - y_1) = m\,(x - x_1)$.

Equation of Tangent

We know that the slope of tangent at $P(x_1, y_1) = m = \tan\theta = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

Thus, we can write equation of tangent as

$$\frac{y - y_1}{x - x_1} = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} \quad \textit{ie}, \text{ equation of tangent.}$$

Since, the normal passes through $P(x_1, y_1)$ at $P(x_1, y_1)$ and has slope $-\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$.

$\therefore$ Equation of normal at $P(x_1, y_1)$ to the curve is,

$$\frac{y - y_1}{x - x_1} = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$$

or
$$\frac{y - y_1}{x - x_1} = -\left(\frac{dx}{dy}\right)_{(x_1, y_1)} \quad \textit{ie}, \text{ equation of normal.}$$

Illustration 14 Find the equation of tangent and normal to the curve $2y = 3 - x^2$ at (1, 1).

Solution. The equation of given curve is,

$$2y = 3 - x^2 \qquad \text{...(i)}$$

Differentiating Eq. (i) w.r.t. x, we get

$$2\left(\frac{dy}{dx}\right) = -2x \quad \Rightarrow \quad \frac{dy}{dx} = -x$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,1)} = -1 \qquad \text{...(ii)}$$

So, the equation of tangent at (1, 1) is,

$$\frac{y - 1}{x - 1} = \left(\frac{dy}{dx}\right)_{(1,1)} = -1$$

$$y - 1 = -x + 1$$

$$y + x = 2 \qquad \text{(required equation of tangent)}$$

and the equation of the normal at (1, 1) is,

$$\frac{y - 1}{x - 1} = -\frac{1}{(dy/dx)_{(1,1)}} = 1$$

$$\therefore \quad y - x = 0 \qquad \text{(required equation of normal)}$$

Illustration 15 Find the equation of tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

Solution. The equation of given curve is,

$$y^2 = 4ax \quad \text{...(i)}$$

Differentiating Eq. (i) w.r.t. x, we get $2y\dfrac{dy}{dx} = 4a$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(at^2,\, 2at)} = \frac{4a}{4at} = \frac{1}{t} \quad \text{...(ii)}$$

So, the equation of tangent at $(at^2, 2at)$ is,

$$\frac{y-2at}{x-at^2} = \left(\frac{dy}{dx}\right)_{(at^2,\, 2at)} = \frac{1}{t} \quad \text{[using Eq. (ii)]}$$

$$\Rightarrow \quad (y-2at)t = x - at^2$$

$$\Rightarrow \quad yt - 2at^2 = x - at^2$$

$$\Rightarrow \quad yt = x + at^2 \quad \text{(required equation of tangent)}$$

and the equation of normal at $(at^2, 2at)$ is,

$$\frac{y-2at}{x-at^2} = -\frac{1}{\left(\frac{dy}{dx}\right)_{(at^2,\, 2at)}} = -t \quad \text{[using Eq. (ii)]}$$

$$\Rightarrow \quad y - 2at = -xt + at^3$$

$$\Rightarrow \quad y + xt = 2at + at^3 \quad \text{(required equation of normal)}$$

Illustration 16 Find the point on the curve $y - e^{xy} + x = 0$ at which we have vertical tangent.

Solution. The equation of the given curve is,

$$y - e^{xy} + x = 0 \quad \text{...(i)}$$

Differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} - e^{xy}\left\{1\cdot y + x\cdot\frac{dy}{dx}\right\} + 1 = 0$$

$$\Rightarrow \quad \frac{dy}{dx}(1 - xe^{xy}) = -1 + y\cdot e^{xy}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1 + y\cdot e^{xy}}{1 - xe^{xy}} \quad \text{...(ii)}$$

Let the point be (x_1, y_1) on the curve at which we have vertical tangent (*ie*, parallel to y-axis). Then,

$$\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = \infty \text{ or } \left(\frac{dx}{dy}\right)_{(x_1,\, y_1)} = 0$$

$$\Rightarrow \quad \frac{1 - x_1e^{x_1y_1}}{-1 + y_1e^{x_1y_1}} = 0 \Rightarrow 1 - x_1e^{x_1y_1} = 0$$

$$\Rightarrow \quad x_1e^{x_1y_1} = 1$$

Which is possible only if $x_1 = 1$ and $y_1 = 0$.

Thus, the required point is (1, 0).

Point to Consider

For standard curves students are advised to use direct method of finding equation of tangent on the curve of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

We must make replacement in the above equation as :

x^2 to xx_1. $\quad$ $2x$ to $x + x_1$.

y^2 to yy_1. $\quad$ $2y$ to $y + y_1$.

xy to $\dfrac{xy_1 + x_1y}{2}$.

Therefore, equation of tangent is,

$$\Rightarrow \quad axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Illustration 17 Find equation of tangent to the curve $2y = x^2 + 3$ at (x_1, y_1).

Solution. We can replace $2y$ to $y + y_1$ and x^2 to xx_1

ie, equation of tangent is,

$$(y + y_1) = xx_1 + 3$$

Illustration 18 Find equation of tangent to the curve $y^2 = 4ax$ at $(at^2, 2at)$.

Solution. We know that the equation of curve $y^2 = 4ax$ is a standard curve, thus replacing y^2 to yy_1 and $2x$ to $x + x_1$.

We get, the equation of tangent as,

$$yy_1 = 2a(x + x_1)$$

where $\quad (x_1, y_1) = (at^2, 2at)$

$$\Rightarrow \quad y(2at) = 2a(x + at^2)$$

$$\Rightarrow \quad yt = x + at^2$$

Illustration 19 Find the equation of tangent, at point $P(x_1, y_1)$, to the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution. We know that the equation of curve $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is a standard curve equation thus, equation of tangent is obtained by replacing x^2 to xx_1 and y^2 to yy_1

ie, $\quad \dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$

Points to Consider

1. If a curve passes through the origin, then the equation of the tangent at the origin can be directly written by equating the lowest degree terms appearing in the equation of the curve to zero.

eg, In

(i) $x^2 + y^2 + 2gx + 2fy = 0$

equation of tangent is $gx + fy = 0$

(ii) $x^3 + y^3 - 3x^2y + 3xy^2 + x^2 - y^2 = 0$

equation of tangents at origin is $x^2 - y^2 = 0$

(iii) Equation of tangents to $x^3 + y^2 - 3xy = 0$ is $xy = 0$

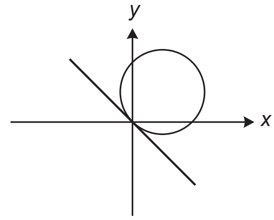


Fig. 7.4

2. If the curve is $x^4 + y^4 = x^2 + y^2$ then the equation of the tangent would be $x^2 + y^2 = 0$ which would indicate that the origin is an isolated point on the graph.

Proof Let the equation of the curve be

$$a_1x + b_1y + a_2x^2 + b_2xy + c_2y^2 = 0 \quad \ldots\text{(i)}$$

Tangent $y - 0 = \lim\limits_{\substack{x_1 \to 0 \\ y_1 \to 0}} \dfrac{y_1}{x_1}(x - 0)$

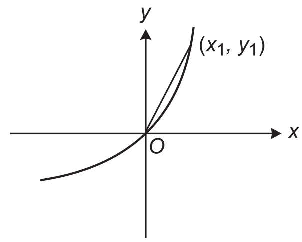


Fig. 7.5

Now, Eq. (i) becomes

$$a_1 + b_1\frac{y_1}{x_1} + a_2x_1 + b_2\frac{y_1}{x_1}\cdot x_1 + c_2\frac{y_1}{x_1}\cdot y_1 = 0 \quad \ldots\text{(ii)}$$

at $\quad x_1 \to 0 \quad$ and $\quad y_1 \to 0, \dfrac{y_1}{x_1} \to m$

From Eq. (ii), $a_1 + b_1m = 0$ $\qquad \left[\because \quad m = -\dfrac{a_1}{b_1}\right]$

Hence, tangent is

$$y = -\frac{a_1}{b_1}x \quad \Rightarrow \quad a_1x + b_1y = 0$$

3. Same line could be the tangent as well as normal to a given curve at a given point.

eg, In $x^3 + y^3 - 3xy = 0$ (Folium of descartes)

The line pair $xy = 0$ is both the tangent as well as normal at $x = 0$.

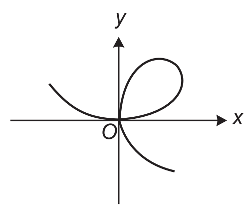


Fig. 7.6

4. Some common parametric coordinates on a curve

(i) for $x^{2/3} + y^{2/3} = a^{2/3}$ take parametric coordinate $x = a\cos^3\theta$ and $y = a\sin^3\theta$.

(ii) for $\sqrt{x} + \sqrt{y} = \sqrt{a}$ take $x = a\cos^4\theta$ and $y = a\sin^4\theta$.

(iii) $\dfrac{x^n}{a^n} + \dfrac{y^n}{b^n} = 1$ take $x = a(\sin\theta)^{2/n}$ and $y = b(\sin\theta)^{2/n}$.

(iv) for $c^2(x^2 + y^2) = x^2y^2$ take $x = c\sec\theta$ and $y = c\,\mathrm{cosec}\,\theta$.

(v) for $y^2 = x^3$, take $x = t^2$ and $y = t^3$.

Illustration 20 Find the sum of the intercepts on the axes of coordinates by any tangent to the curve $\sqrt{x}+\sqrt{y}=2$.

Solution. Here, $\sqrt{x}+\sqrt{y}=2$

whose parametric coordinates are given by,

$$\sqrt{x}=2\cos^2\theta \quad \text{and} \quad \sqrt{y}=2\sin^2\theta$$

ie, $$x=4\cos^4\theta \quad \text{and} \quad y=4\sin^4\theta$$

$\therefore$ $$\frac{dy}{dx}=\frac{4\times 4\sin^3\theta\cdot\cos\theta}{4\times 4\cos^3\theta(-\sin\theta)}=-\tan^2\theta$$

$\Rightarrow$ Equation of tangent, $\dfrac{y-4\sin^4\theta}{x-4\cos^4\theta}=-\tan^2\theta$

$\therefore$ x-intercept,

$\Rightarrow$ $$\left|4\cos^4\theta+\frac{4\sin^4\theta}{\tan^2\theta}\right|=4\cos^2\theta$$

and y-intercept,

$\Rightarrow$ $$\left|4\sin^4\theta+\frac{4\sin^4\theta}{\tan^2\theta}\right|=4\sin^2\theta$$

Hence, the sum of intercepts made on the axes of coordinates is,

$$4\cos^2\theta+4\sin^2\theta=4$$

Illustration 21 The tangent represented by the graph of the function $y=f(x)$ at the point with abscissa $x=1$ form an angle $\pi/6$, at the point $x=2$ an angle of $\pi/3$ and at the point $x=3$ an angle of $\pi/4$. Then, find the value of,

$$\int_1^3 f'(x)f''(x)\,dx+\int_2^3 f''(x)\,dx.$$

Solution. Given, at $x=1$, $\dfrac{dy}{dx}=\tan\pi/6=1/\sqrt{3}$

or at $\quad x=1 \Rightarrow f'(1)=\tan\pi/6=1/\sqrt{3}$

also at $\quad x=2 \Rightarrow f'(2)=\tan\pi/3=\sqrt{3}$

and at $\quad x=3 \Rightarrow f'(3)=\tan\pi/4=1$

Then, $$\int_1^3 f'(x)f''(x)\,dx+\int_2^3 f''(x)\,dx$$

$\Rightarrow$ $$\int_{f'(1)}^{f'(3)} t\,dt+(f'(x))_2^3 \qquad \text{Let} \quad f'(x)=t,\ f''(x)\,dx=dt$$

$\Rightarrow$ $$\frac{1}{2}(t^2)_{f'(1)}^{f'(3)}+\{f'(3)-f'(2)\}$$

$\Rightarrow$ $$\frac{1}{2}\{(f'(3))^2-(f'(1))^2\}+\{f'(3)-f'(2)\}$$

$\Rightarrow$ $$\frac{1}{2}\left\{(1)^2-\left(\frac{1}{\sqrt{3}}\right)^2\right\}+\{1-\sqrt{3}\}$$

$\Rightarrow$ $$\frac{1}{2}\left(1-\frac{1}{3}\right)+(1-\sqrt{3})$$

$\Rightarrow$ $$\frac{4}{3}-\sqrt{3} \quad \Rightarrow \quad \frac{4-3\sqrt{3}}{3}$$

Illustration 22 Three normals are drawn from the point $(c, 0)$ to the curve $y^2 = x$, show that c must be greater than 1/2. One normal is always the x-axis. Find c for which the other two normals are perpendicular to each other.

Solution. The slope of the normal to the curve $y^2 = 4ax$ is,

$$y = mx - 2am - am^3 \qquad \text{...(i)}$$

For the curve given $y^2 = x$, we have

$$4a = 1 \quad \Rightarrow \quad a = 1/4$$

$\therefore$ Equation of normal is, $y = mx - \dfrac{m}{2} - \dfrac{m^3}{4}$

The equation passes through $(c, 0)$, then

$$\therefore \qquad 0 = mc - \frac{1}{2}m - \frac{1}{4}m^3$$

$$ie, \qquad m\left(c - \frac{1}{2} - \frac{m^2}{4}\right) = 0$$

$$\therefore \qquad m = 0, \qquad \frac{m^2}{4} + \frac{1}{2} - c = 0$$

For $m = 0$, the normal is $y = 0$ which is the x-axis.

The other two values of m are given by,

$$m = \pm 2\sqrt{(c - 1/2)}$$

$\therefore$ For m to be real, $c \geq 1/2$

If $c = 1/2$, then $m = 0$ which is already considered

So, $\qquad c > 1/2$

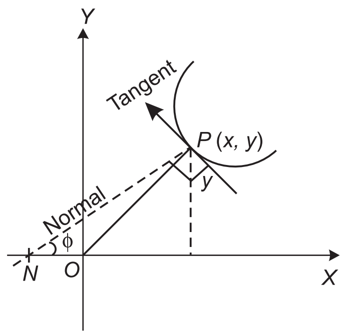
Now, for the other two normals to be perpendicular to each other, we must have

$$\left(2\cdot\sqrt{c - \frac{1}{2}}\right)\left(-2\cdot\sqrt{c - \frac{1}{2}}\right) = -1$$

$$\therefore \qquad c - \frac{1}{2} = \frac{1}{4} \quad \Rightarrow \quad c = \frac{3}{4}$$

Illustration 23 Find the equation for family of curves for which the length of normal is equal to the radius vector.

Solution. Let $P(x, y)$ be the point on the curve.



$$OP = \text{radius vector} = \sqrt{x^2 + y^2}$$

$$PN = \text{length of normal}$$

Now, $\tan\phi = -\dfrac{1}{\left(\dfrac{dy}{dx}\right)} \Rightarrow PN = \dfrac{y}{\sin\phi}$

It is given $OP = PN$

$$\Rightarrow \quad \sqrt{x^2 + y^2} = y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow \quad x^2 + y^2 = y^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]$$

$$\Rightarrow \quad x^2 = y^2\left(\frac{dy}{dx}\right)^2 \Rightarrow \frac{dy}{dx} = \pm\frac{x}{y}$$

or $y\,dy = \pm\, x\,dx$

Integrating both the sides,

$y^2 = \pm\, x^2 + c$ is required family of curves.

Illustration 24 Find the condition that the line $x\cos\alpha + y\sin\alpha = P$ may touch the curve $\left(\dfrac{x}{a}\right)^m + \left(\dfrac{y}{b}\right)^m = 1.$

Solution. Given, equation $\left(\dfrac{x}{a}\right)^m + \left(\dfrac{y}{b}\right)^m = 1$

Differentiating the equation of curve w.r.t. x, we get

$$m\left(\frac{x}{a}\right)^{m-1} \cdot \frac{1}{a} + m\left(\frac{y}{b}\right)^{m-1} \cdot \frac{1}{b} \cdot \frac{dy}{dx} = 0$$

On simplifying, we get $\dfrac{dy}{dx} = \dfrac{-b^m x^{m-1}}{a^m y^{m-1}}$

Hence, at any point $P(x_1, y_1)$ on the curve.

Slope of tangent $= \left(\dfrac{dy}{dx}\right)_{(x_1,\, y_1)} = \dfrac{-\,b^m x_1^{m-1}}{a^m y_1^{m-1}}$

$\therefore$ Equation of tangent at P is

$$y - y_1 = \frac{-\,b^m x_1^{m-1}}{a^m y_1^{m-1}}(x - x_1)$$

$$\Rightarrow \quad \frac{y y_1^{m-1}}{b^m} - \frac{y_1^m}{b^m} = -\frac{x x_1^{m-1}}{a^m} + \frac{x_1^m}{a^m}$$

ie, $$\frac{x}{a}\left(\frac{x_1}{a}\right)^{m-1} + \frac{y}{b}\left(\frac{y_1}{b}\right)^{m-1} = \left(\frac{x_1}{a}\right)^m + \left(\frac{y_1}{b}\right)^m = 1$$

(since P lies on the curve at any point)

Hence, the equation of tangent at $P(x_1, y_1)$ on the curve is,

$$\frac{x}{a}\left(\frac{x_1}{a}\right)^{m-1} + \frac{y}{b}\left(\frac{y_1}{b}\right)^{m-1} = 1 \qquad \text{...(i)}$$

and $$x\cos\alpha + y\sin\alpha = P \qquad \text{...(ii)}$$

If Eq. (ii) is the equation of tangent, then coefficients of Eqs. (i) and (ii) must be proportional for point (x_1, y_1).

ie, $$\frac{\cos\alpha}{\frac{1}{a}\left(\frac{x_1}{a}\right)^{m-1}} = \frac{\sin\alpha}{\frac{1}{b}\left(\frac{y_1}{b}\right)^{m-1}} = \frac{P}{1}$$

This gives $$\frac{x_1}{a} = \left(\frac{a\cos\alpha}{P}\right)^{\frac{1}{m-1}}, \frac{y_1}{b} = \left(\frac{b\sin\alpha}{P}\right)^{\frac{1}{m-1}}$$

Since, point $P(x_1, y_1)$ lies on the curve,

$$\left(\frac{x_1}{a}\right)^m + \left(\frac{y_1}{b}\right)^m = 1$$

ie, $$\left(\frac{a\cos\alpha}{P}\right)^{\frac{m}{m-1}} + \left(\frac{b\sin\alpha}{P}\right)^{\frac{m}{m-1}} = 1$$

ie, $$(a\cos\alpha)^{\frac{m}{m-1}} + (b\sin\alpha)^{\frac{m}{m-1}} = P^{\frac{m}{m-1}},$$

which is the required condition.

Illustration 25 If tangent and normal to the curve $y = 2\sin x + \sin 2x$ are drawn at $P\left(x = \frac{\pi}{3}\right)$, then area of the quadrilateral formed by the tangent, the normal at P and the coordinate axes is

(a) $\frac{\pi}{3}$ (b) 3π

(c) $\frac{\pi\sqrt{3}}{2}$ (d) None of these

Solution. Here, $\frac{dy}{dx} = 0$ at $\left(x = \frac{\pi}{3}, y = \frac{3\sqrt{3}}{2}\right)$

$\Rightarrow$ Tangent at $x = \frac{\pi}{3}$ is parallel to x-axis.

$\Rightarrow$ Equation of tangent is, $y = \frac{3\sqrt{3}}{2}$

Also, equation of normal is, $x = \frac{\pi}{3}$

$\therefore$ Area of quadrilateral $= \frac{\pi}{3} \cdot \frac{3\sqrt{3}}{2} = \frac{\pi\sqrt{3}}{2}$ sq unit

Hence, (c) is the correct answer.

Illustration 26 The maximum value of the sum of the intercepts made by any tangent to the curve $(a\sin^2\theta, 2a\sin\theta)$ with the axes is

(a) $2a$ (b) $a/4$ (c) $a/2$ (d) a

Solution. The curve is $(a\sin^2\theta, 2a\sin\theta) \Rightarrow x = a\sin^2\theta,\ y = 2a\sin\theta$

$\Rightarrow$ The equation of any tangent to the curve is,

$$\frac{y - 2a\sin\theta}{x - a\sin^2\theta} = \frac{1}{\sin\theta} \Rightarrow y\sin\theta = x + a\sin^2\theta$$

$$\Rightarrow \quad \frac{x}{-a\sin^2\theta} + \frac{y}{a\sin\theta} = 1, \text{ is the equation of the tangent}$$

$$\Rightarrow \quad \text{Sum of intercepts} = a\,(\sin^2\theta + \sin\theta) = a\left\{\left(\sin\theta + \frac{1}{2}\right)^2 - \frac{1}{4}\right\}$$

which is maximum, when $\sin\theta = 1$

ie, $(\text{Sum of intercept})_{\max} = 2a$

Hence, (a) is the correct answer.

Illustration 27 If $g(x)$ is a curve which is obtained by the reflection of $f(x) = \dfrac{e^x - e^{-x}}{2}$ by the line $y = x$, then

(a) $g(x)$ has more than one tangent parallel to x-axis

(b) $g(x)$ has more than one tangent parallel to y-axis

(c) $y = -x$ is a tangent to $g(x)$ at (0, 0)

(d) $g(x)$ has no extremum

Solution. As $g(x)$ is a curve which is obtained by the reflection of $f(x) = \dfrac{e^x - e^{-x}}{2}$ on $y = x$

$$\Rightarrow \quad g(x) \text{ is inverse of } f(x)$$

$$\therefore \quad g(x) = \log\left(x + \sqrt{1+x^2}\right) = f^{-1}(x)$$

$$\Rightarrow \quad g'(x) = \frac{1}{x + \sqrt{1+x^2}} \cdot \left(1 + \frac{2x}{2\sqrt{1+x^2}}\right)$$

$$= \frac{1}{\sqrt{1+x^2}} \neq 0, \forall x \in R$$

$\Rightarrow g(x)$ has no tangent parallel to x-axis also $g'(x)$ is always defined, $\forall x \in R$.

$\Rightarrow g(x)$ has no tangent parallel to y-axis since, $g'(x) > 0 \Rightarrow g(x)$ doesn't have any extremum.

Hence, (d) is the correct answer.

Target Exercise 7.1

1. If the line $ax + by + c = 0$ is normal to the $xy + 5 = 0$, then a and b have

(a) same sign (b) opposite sign
(c) cannot be discussed (d) None of these

2. The equation of tangent drawn to the curve $y^2 - 2x^3 - 4y + 8 = 0$ from the point (1, 2) is given by

(a) $y - 2(1 \pm \sqrt{2}) = \pm 2\sqrt{3}\,(x - 2)$ (b) $y - 2(1 \pm \sqrt{3}) = \pm 2\sqrt{2}\,(x - 2)$
(c) $y - 2(1 \pm \sqrt{3}) = \pm 2\sqrt{3}\,(x - 2)$ (d) None of these

3. The equation of the tangents to the curve $(1 + x^2)y = 1$ at the points of its intersection with the curve $(x + 1)y = 1$, is given by

(a) $x + y = 1,\ y = 1$ (b) $x + 2y = 2,\ y = 1$
(c) $x - y = 1,\ y = 1$ (d) None of these

4. The tangent lines for the curve $y=\int_0^x 2|t|\,dt$ which are parallel to the bisector of the first coordinate angle, is given by

(a) $y=x+\frac{3}{4},\ y=x-\frac{1}{4}$ (b) $y=-x+\frac{1}{4},\ y=-x+\frac{3}{4}$

(c) $x+y=2,\ x-y=1$ (d) None of these

5. The equation of normal to $x+y=x^y$, where it intersects x-axis, is given by

(a) $x+y=1$ (b) $x-y-1=0$

(c) $x-y+1=0$ (d) None of these

6. The equation of normal at any point θ to the curve $x=a\cos\theta+a\theta\sin\theta,\ y=a\sin\theta-a\theta\cos\theta$ is always at a distance of

(a) $2a$ unit from origin (b) a unit from origin

(c) $\frac{1}{2}a$ unit from origin (d) None of these

7. If the tangent at (x_0, y_0) to the curve $x^3+y^3=a^3$ meets the curve again at (x_1, y_1), then $\frac{x_1}{x_0}+\frac{y_1}{y_0}$ is equal to

(a) a (b) $2a$

(c) 1 (d) None of these

8. The area bounded by the axes of reference and the normal to $y=\log_e x$ at $(1, 0)$, is

(a) 1 sq unit (b) 2 sq unit

(c) $\frac{1}{2}$ sq unit (d) None of these

9. If $\frac{x}{a}+\frac{y}{b}=2$ touches the curve $\frac{x^n}{a^n}+\frac{y^n}{b^n}=2$ at the point (α, β), then

(a) $\alpha=a^2, \beta=b^2$ (b) $\alpha=a, \beta=b$

(c) $\alpha=-2a, \beta=4b$ (d) $\alpha=3a, \beta=-2b$

10. The equation of tangents to the curve $y=\cos(x+y)$, $-2\pi\le x\le 2\pi$ that are parallel to the line $x+2y=0$, is

(a) $x+2y=\frac{\pi}{2}$ and $x+2y=-\frac{3\pi}{2}$ (b) $x+2y=\frac{\pi}{2}$ and $x+2y=\frac{3\pi}{2}$

(c) $x+2y=0$ and $x+2y=\pi$ (d) None of these

Angle of Intersection of Two Curves

The angle of intersection of two curves is defined as the angle between the tangents to the two curves at their point of intersection.

Let C_1 and C_2 be two curves having equations

$y=f(x)$ and $y=g(x)$ respectively.

Let PT_1 and PT_2 be tangents to the curves C_1 and C_2 at their point of intersection.

Let θ be the angle between the two tangents PT_1 and PT_2 and θ_1 and θ_2 are the angles made by tangents with the positive direction of x-axis in anti-clockwise sense.

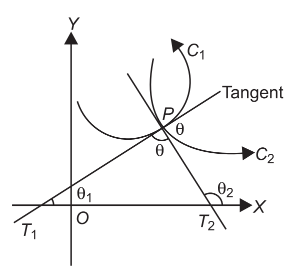


Fig. 7.7

Then, $$m_1 = \tan\theta_1 = \left(\frac{dy}{dx}\right)_{C_1}$$

$$m_2 = \tan\theta_2 = \left(\frac{dy}{dx}\right)_{C_2}$$

From the figure it follows,

$$\theta = \theta_2 - \theta_1$$

$$\Rightarrow \quad \tan\theta = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$$

$$\Rightarrow \quad \tan\theta = \left|\frac{\left(\frac{dy}{dx}\right)_{C_1} - \left(\frac{dy}{dx}\right)_{C_2}}{1 + \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2}}\right|$$

Angle of intersection of these curves is defined as acute angle between the tangents.

Orthogonal Curves If the angle of intersection of two curves is a right angle, the two curves are said to be orthogonal and curves are called orthogonal curves.

$\therefore$ If the curves are orthogonal $\theta = \pi/2$

$$\Rightarrow \quad 1 + \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2} = 0$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2} = -1$$

Illustration 28 Find the angle of intersection of the curves $y = x^2$ and $y = 4 - x^2$.

Solution. For the intersection points of the given curves,

$$x^2 = 4 - x^2 \quad \Rightarrow \quad x = \pm\sqrt{2}$$

Now, at $x = \sqrt{2}$, $\frac{dy}{dx}$ for first curve

$$\Rightarrow \quad \left(\frac{dy}{dx}\right) = 2x \quad \text{or} \quad \left(\frac{dy}{dx}\right) = 2\sqrt{2}$$

while at $x = \sqrt{2}$, $\frac{dy}{dx}$ for second curve,

$$\Rightarrow \quad \frac{dy}{dx} = -2x \quad \text{or} \quad \left(\frac{dy}{dx}\right) = -2\sqrt{2}$$

Hence, if θ_1 be the acute angle of intersection of the curves, then

$$\tan\theta_1 = \left|\frac{2\sqrt{2} - (-2\sqrt{2})}{1 + 2\sqrt{2}\,(-2\sqrt{2})}\right| = \left|\frac{4\sqrt{2}}{-7}\right|$$

$$\therefore \quad \theta_1 = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right) \qquad \text{...(i)}$$

Again, at $x = -\sqrt{2}$,

$$\left(\frac{dy}{dx}\right) \text{ for first curve} = -2\sqrt{2}$$

and $$\left(\frac{dy}{dx}\right) \text{ for second curve} = 2\sqrt{2}$$

Hence, if θ_2 be the acute angle of intersection of the curves, then

$$\tan\theta_2 = \left|\frac{-2\sqrt{2}-2\sqrt{2}}{1+(-2\sqrt{2})(2\sqrt{2})}\right|$$

$$\therefore \qquad \theta_2 = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right) \qquad \text{...(ii)}$$

From Eqs. (i) and (ii) the two acute angles are equal.

Illustration 29 Find the acute angle between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersection when $x > 0$.

Solution. For the intersection of the given curves

$$|x^2 - 1| = |x^2 - 3| \quad \Rightarrow \quad (x^2-1)^2 = (x^2-3)^2$$

$$\Rightarrow \quad (x^2-1)^2 - (x^2-3)^2 = 0$$

$$\Rightarrow \quad [(x^2-1)-(x^2-3)][(x^2-1)+(x^2-3)] = 0$$

$$\Rightarrow \quad 2[2x^2 - 4] = 0 \quad \Rightarrow \quad 2x^2 = 4 \Rightarrow x = \pm\sqrt{2}$$

neglecting $x = -\sqrt{2}$ as $x > 0$

We have, point of intersection as $x = \sqrt{2}$

Here, $y = |x^2 - 1| = (x^2 - 1)$ in the neighbouring of $x = \sqrt{2}$

and $y = -(x^2 - 3)$ in the neighbouring of $x = \sqrt{2}$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{C_1} = 2x = 2\sqrt{2}$$

and $$\left(\frac{dy}{dx}\right)_{C_2} = -2x = -2\sqrt{2}$$

Hence, if θ is angle between them,

$$\Rightarrow \quad \tan\theta = \left|\frac{2\sqrt{2}-(-2\sqrt{2})}{1+2\sqrt{2}(-2\sqrt{2})}\right| = \left|\frac{4\sqrt{2}}{-7}\right| = \left(\frac{4\sqrt{2}}{7}\right)$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$$

Illustration 30 Find the angle of intersection of curves, $y = [|\sin x| + |\cos x|]$ and $x^2 + y^2 = 5$ where $[\cdot]$ denotes the greatest integral function.

Solution. We know that,

$$1 \le |\sin x| + |\cos x| \le \sqrt{2}$$

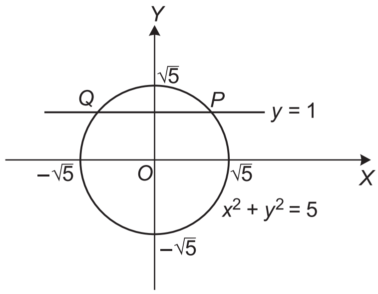
$$\therefore \quad y = [|\sin x| + |\cos x|] = 1$$

Let P and Q be the points of intersection of given curves.

Clearly, the given curves meet at points where $y = 1$ so, we get $x^2 + 1 = 5$

$$x = \pm 2$$

Now, $\qquad P(2, 1)$ and $Q(-2, 1)$



Now, $\qquad x^2 + y^2 = 5$

Differentiating the above equation w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = -\frac{x}{y}$$

$$\left(\frac{dy}{dx}\right)_{(2,\,1)} = -2 \quad \text{and} \quad \left(\frac{dy}{dx}\right)_{(-2,\,1)} = 2$$

Clearly, the slope of line $y = 1$ is zero and the slope of the tangents at P and Q are (-2) and (2) respectively.

Thus, the angle of intersection is $\tan^{-1}(2)$.

Length of Tangent, Subtangent, Normal and Subnormal

Length of Tangent The length of the segment PT of the tangent between the point of tangent and x-axis is called the length of tangent.

Subtangent The projection of the segment PT along x-axis (ST) is called the subtangent.

Length of Normal The length of segment PN *ie*, the portion of the normal intercepted between the point on the curve and x-axis is called the length of normal.

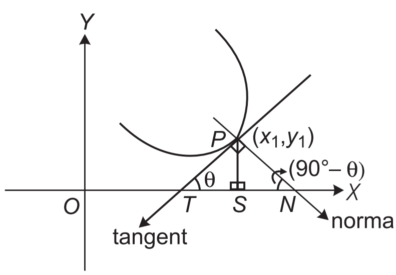


Fig. 7.8

Subnormal The projection of the segment PN along x-axis (SN) is called the subnormal. From the figure,

If PT makes an angle θ with x-axis, then

$$\tan\theta = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

Then, we can say

$$\frac{ST}{PS} = \cot\theta$$

or $$\text{Subtangent} = ST = \left| y_1 \left(\frac{dx}{dy}\right)_{(x_1, y_1)} \right|$$

$$\frac{SN}{PS} = \cot(90^\circ - \theta)$$

$\Rightarrow$ $$\text{Subnormal} = SN = PS\tan\theta$$

$\Rightarrow$ $$\text{Subnormal} = SN = \left| y_1 \left(\frac{dy}{dx}\right)_{(x_1, y_1)} \right|$$

$$\text{Length of tangent} = PT = \left| \sqrt{y_1^2 + y_1^2 \left(\frac{dx}{dy}\right)^2_{(x_1, y_1)}} \right|$$

$\Rightarrow$ $$PT = \left| y_1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2_{(x_1, y_1)}} \right|$$

$$\text{Length of normal} = PN = \left| \sqrt{y_1^2 + y_1^2 \left(\frac{dy}{dx}\right)^2_{(x_1, y_1)}} \right|$$

$\Rightarrow$ $$PN = \left| y_1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2_{(x_1, y_1)}} \right|$$

Illustration 31 Show that for the curve $y = be^{x/a}$, the subtangent is of constant length and the subnormal varies as the square of ordinate.

Solution. The given curve is $y = be^{x/a}$

Let us consider a point (x_1, y_1) on the curve.

Then, $$y_1 = be^{x_1/a} \qquad \text{...(i)}$$

Differentiating the curve $y = be^{x/a}$ w.r.t. x, we get

$$\frac{dy}{dx} = be^{x/a} \cdot \frac{1}{a}$$

$\therefore$ $$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{b}{a} e^{x_1/a} \qquad \text{...(ii)}$$

Thus, the length of subtangent $= \left| y_1 \left(\frac{dx}{dy}\right)_{(x_1, y_1)} \right|$

$$= \left| y_1 \cdot \frac{a}{be^{x_1/a}} \right| = \left| be^{x_1/a} \cdot \frac{a}{be^{x_1/a}} \right|$$

$$= a \text{ (constant)} \qquad \text{[using Eqs. (i) and (ii)]}$$

$\Rightarrow$ Subtangent is of constant length a.

Again, length of subnormal $= \left| y_1 \left(\frac{dy}{dx} \right)_{(x_1, y_1)} \right|$

$$= \left| be^{x_1/a} \cdot \frac{be^{x_1/a}}{a} \right|$$

$$= \frac{1}{a} (be^{x_1/a})^2 = \frac{1}{a} y^2 \qquad \text{[using Eq. (i)]}$$

Therefore, subnormal varies as the square of ordinate.

Illustration 32 Find the lengths of tangent, subtangent, normal and subnormal to $y^2 = 4ax$ at $(at^2, 2at)$.

Solution. We have, the given curve,

$$y^2 = 4ax \qquad \text{...(i)}$$

Differentiating Eq. (i) both the sides w.r.t. x, we get

$$2y \frac{dy}{dx} = 4a$$

$$\left[\frac{dy}{dx} \right]_{(at^2, 2at)} = \frac{4a}{4at} = \frac{1}{t} \qquad \text{...(ii)}$$

Now, the length of tangent at $(at^2, 2at)$ is

$$= y_1 \sqrt{1 + \left(\frac{dx}{dy} \right)^2_{(x_1, y_1)}}$$

$$= 2at \sqrt{1 + t^2} \qquad \text{[using Eq. (ii)]}$$

$\therefore$ Length of normal at $(at^2, 2at)$ is

$$= y_1 \sqrt{1 + \left(\frac{dy}{dx} \right)^2_{(x_1, y_1)}}$$

$$= 2at \sqrt{1 + 1/t^2}$$

$$= 2a \sqrt{t^2 + 1}$$

$\therefore$ Length of subtangent

$$\frac{y_1}{\left[\frac{dy}{dx} \right]_{(x_1, y_1)}} = \frac{2at}{1/t} = 2at^2$$

Length of subnormal

$$y_1 \left[\frac{dy}{dx} \right]_{(x_1, y_1)} = 2at \cdot \frac{1}{t} = 2a$$

Illustration 33 Find the equation of tangent and normal, the length of subtangent and subnormal of the circle $x^2 + y^2 = a^2$ at the point (x_1, y_1).

Solution. The equation of given curve is,

$$x^2 + y^2 = a^2 \quad \text{...(i)}$$

Differentiating Eq. (i) w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{x_1}{y_1} \quad \text{...(ii)}$$

Thus, the equation of tangent is,

$$y - y_1 = \left[\frac{dy}{dx}\right]_{(x_1, y_1)} (x - x_1)$$

$$\Rightarrow \quad y - y_1 = -\frac{x_1}{y_1}(x - x_1) \quad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad yy_1 - y_1^2 = -xx_1 + x_1^2$$

$$\Rightarrow \quad xx_1 + yy_1 = x_1^2 + y_1^2 \quad \Rightarrow \quad xx_1 + yy_1 = a^2$$

[using Eq. (i) as (x_1, y_1) lies on $x^2 + y^2 = a^2 \Rightarrow x_1^2 + y_1^2 = a^2$]

While the equation of normal is,

$$y - y_1 = +\frac{y_1}{x_1}(x - x_1)$$

$$\Rightarrow \quad x_1 y - x_1 y_1 = xy_1 - x_1 y_1$$

$$\Rightarrow \quad xy_1 - x_1 y = 0$$

The length of subtangent = $\left| y_1 \cdot \left(\frac{dx}{dy}\right)_{(x_1, y_1)} \right|$

$$\Rightarrow \quad = \left| y_1 \cdot \left(\frac{-y_1}{x_1}\right) \right| \quad \text{[using Eq. (ii)]}$$

The length of subtangent = $\left| \frac{y_1^2}{x_1} \right|$

While the length of subnormal = $\left| y_1 \cdot \left(\frac{dy}{dx}\right)_{(x_1, y_1)} \right|$

$$\Rightarrow \quad \left| y_1\left(-\frac{x_1}{y_1}\right) \right| = |x_1|$$

Illustration 34 If the relation between subnormal SN and subtangent ST at any point S on the curve, $by^2 = (x + a)^3$ is $p(SN) = q(ST)^2$, then find the value of $\frac{p}{q}\cdot$

Solution. Here, $\quad by^2 = (x + a)^3$

Differentiating both the sides, we get $2\,by\frac{dy}{dx} = 3\,(x + a)^2 \cdot 1$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3}{2}\frac{(x + a)^2}{by}$$

$\therefore$ Length of subnormal $\Rightarrow SN = y\dfrac{dy}{dx} = \dfrac{3}{2}\cdot\dfrac{(x+a)^2}{b}$ ...(i)

and length of subtangent $\Rightarrow ST = y\dfrac{dx}{dy} = \dfrac{2\,by^2}{3\,(x+a)^2}$ ...(ii)

$$\therefore \quad \frac{p}{q} = \frac{(ST)^2}{(SN)} \qquad \text{(given)}$$

$$\Rightarrow \quad \frac{p}{q} = \frac{(2\,by^2)^2\cdot 2b}{\{3\,(x+a)^2\}^2\cdot 3\,(x+a)^2} \qquad \text{[using, Eqs. (i) and (ii)]}$$

$$= \frac{8b}{27}\cdot\frac{\{(x+a)^3\}^2}{(x+a)^6} \qquad [\text{using, } by^2 = (x+a)^3]$$

$$= \frac{8b}{27}$$

$$\therefore \quad \frac{p}{q} = \frac{8b}{27}$$

Illustration 35 If the length of subnormal is equal to length of subtangent at any point $(3, 4)$ on the curve $y = f(x)$ and the tangent at (3, 4) to $y = f(x)$ meets the coordinate axes at A and B, then maximum area of the ΔOAB where O is origin, is

(a) $\dfrac{45}{2}$ (b) $\dfrac{49}{2}$

(c) $\dfrac{25}{2}$ (d) $\dfrac{81}{2}$

Solution. Length of subnormal = length of subtangent

$$\Rightarrow \quad \frac{dy}{dx} = \pm 1$$

If $\dfrac{dy}{dx} = 1$, equation of tangent is

$$y - 4 = x - 3$$

$$\Rightarrow \quad y - x = 1$$

Area of $\Delta OAB = \dfrac{1}{2}\times 1\times 1 = \dfrac{1}{2}$ ...(i)

If $\dfrac{dy}{dx} = -1$, equation of tangent is

$$y - 4 = -x + 3$$

$$\Rightarrow \quad x + y = 7$$

Area of $\Delta OAB = \dfrac{1}{2}\times 7\times 7 = \dfrac{49}{2}$ ...(ii)

$\therefore$ Maximum area $= \dfrac{49}{2}$

Hence, (b) is the correct answer.

Target Exercise 7.2

1. The angle of intersection of $y = a^x$ and $y = b^x$, is given by

(a) $\tan\theta = \left|\dfrac{\log\,(ab)}{1-\log(ab)}\right|$

(b) $\tan\theta = \left|\dfrac{\log(a/b)}{1+\log a\log b}\right|$

(c) $\tan\theta = \left|\dfrac{\log(a/b)}{1-\log(a/b)}\right|$

(d) None of these

2. The angle between the curves $x^2 + 4y^2 = 32$ and $x^2 - y^2 = 12$, is

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{4}$ (c) $\dfrac{\pi}{6}$ (d) $\dfrac{\pi}{2}$

3. If $ax^2 + by^2 = 1$ cuts $a'x^2 + b'y^2 = 1$ orthogonally, then

(a) $\dfrac{1}{a} - \dfrac{1}{a'} = \dfrac{1}{b} - \dfrac{1}{b'}$

(b) $\dfrac{1}{a} + \dfrac{1}{a'} = \dfrac{1}{b} + \dfrac{1}{b'}$

(c) $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{1}{a'} + \dfrac{1}{b'}$

(d) None of these

4. The length of subtangent to the curve, $y = e^{x/a}$, is

(a) $2a$ (b) a (c) $a/2$ (d) $a/4$

5. The length of normal to the curve $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$, at $\theta = \dfrac{\pi}{2}$ is

(a) $2a$ (b) a (c) $\sqrt{2}\,a$ (d) $2\sqrt{2}a$

Rolle's Theorem

The theorem was named after the French mathematician Michel Rolle (1652–1719) who first gave it in his book *Methode pour resoudre lesegalites* (1691).

The theorem states :

Let f be a real-valued function defined on the closed interval $[a, b]$ such that,

(i) $f(x)$ is continuous in the closed interval $[a, b]$

(ii) $f(x)$ is differentiable in the open interval $]a, b[$ and

(iii) $f(a) = f(b)$

Then, there is at least one value c of x in open interval $]a, b[$ for which $f'(c) = 0$.

Analytical Proof

Now, Rolle's theorem is valid for a function such that

(i) $f(x)$ is continuous in the closed interval $[a, b]$

(ii) $f(x)$ is differentiable in open interval $]a, b[$ and

(iii) $f(a) = f(b)$

So, generally two cases arise in such circumstances.

Case I $f(x)$ is constant in the interval $[a, b]$, then $f'(x) = 0$ for all $x \in [a, b]$.

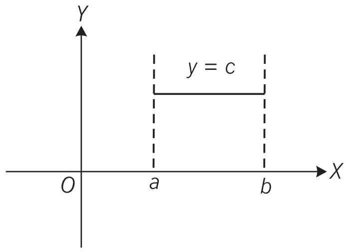


Fig. 7.9

Hence, Rolle's theorem follows and we can say, $f'(c) = 0$, where $a < c < b$
Case II $f(x)$ is not constant in the interval $[a, b]$ and since $f(a) = f(b)$.

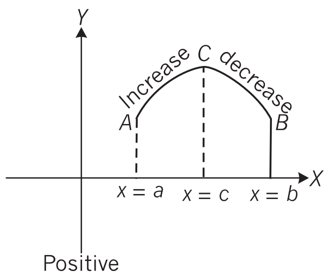


Fig. 7.10

The function should either increase or decrease when x assumes values slightly greater than a.

Now, let $f(x)$ increases for $x > a$

Since, $f(a) = f(b)$, hence the function must cease to increase at some value $x = c$ and decrease upto $x = b$.

Clearly, at $x = c$ function has maximum value.

Now, let h be a small positive quantity, then from definition of maximum value of the function,

$$f(c+h) - f(c) < 0$$

and $$f(c-h) - f(c) < 0$$

$\therefore$ $$\frac{f(c+h) - f(c)}{h} < 0$$

and $$\frac{f(c-h) - f(c)}{-h} > 0$$

So, $$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \le 0$$

and $$\lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} \ge 0 \qquad \text{...(i)}$$

But, if $$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \ne \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}$$

Then, Rolle's theorem cannot be applied because in such case,

$$\text{RHD at } x = c \neq \text{LHD at } x = c.$$

Hence, $f(x)$ is not differentiable at $x = c$, which contradicts the condition of Rolle's theorem.

$\therefore$ Only one possible solution arises, when

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} = 0$$

which implies that, $f'(c) = 0$ where $a < c < b$

Hence, Rolle's theorem is proved.

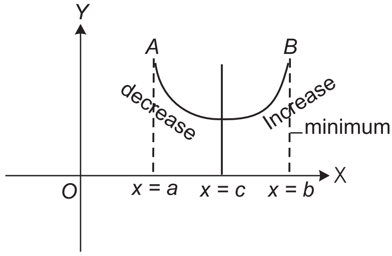


Fig. 7.11

Similarly, the case where $f(x)$ decreases in the interval $a < x < c$ and then increases in the interval $c < x < b$, $f'(c) = 0$, but when $x = c$, the minimum value of $f(x)$ exists in the interval $[a, b]$.

Geometrical Proof

Consider the portion AB of the curve $y = f(x)$, lying between $x = a$ and $x = b$, such that

(i) it goes continuously from A to B

(ii) it has tangent at every point between A and B, and

(iii) ordinate of A = ordinate of B

From the figure, it is clear that $f(x)$ increases in the interval AC_1, which implies that $f'(x) > 0$ in this region and decreases in the interval C_1B which implies $f'(x) < 0$ in this region. Now, since there is unique tangent to be drawn on the curve lying in between A and B and since each of them has a unique slope *ie*, unique value of $f'(x)$.

$\therefore$ Due to continuity and differentiability of the function $f(x)$ in the region A to B, there is a point $x = c$ where $f'(c) = 0$ should be zero.

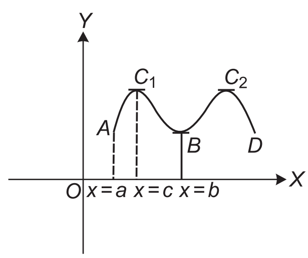


Fig. 7.12

Hence, $f'(c) = 0$ where $a < c < b$

Thus, Rolle's theorem is proved.

Similarly, the other parts of the figure given above can be explained establishing Rolle's theorem throughout.

Points to Consider

Generally two types of problems are formulated on Rolle's theorem

(i) To check the applicability of Rolle's theorem to a given function on a given interval.

(ii) To verify Rolle's theorem for a given function in a given interval.

In both the types of problems we first check whether $f(x)$ satisfies the conditions of Rolle's theorem or not.

The following results are very useful in doing so.

(i) A polynomial function is everywhere continuous and differentiable.

(ii) The exponential function, sine and cosine functions are everywhere continuous and differentiable.

(iii) Logarithmic functions is continuous and differentiable in its domain.

(iv) $\tan x$ is not continuous and differentiable at $x = \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots\ldots\ldots$

(v) $|x|$ is not differentiable at $x = 0$.

(vi) If $f'(x)$ tends to $\pm\infty$ as $x \to k$, then $f(x)$ is not differentiable at $x = k$.

For example, If $f(x) = (2x-1)^{1/2}$, then $f'(x) = \dfrac{1}{\sqrt{2x-1}}$ is such that as

$$x \to \left(\frac{1}{2}\right)^{+} \quad \Rightarrow \quad f'(x) \to \infty$$

So, $f(x)$ is not differentiable at $x = 1/2$.

Illustration 36 If $ax^2 + bx + c = 0$, $a, b, c \in R$. Find the condition that this equation would have at least one root in (0, 1).

Solution. Let $f'(x) = ax^2 + bx + c$

Integrating both sides,

$$\Rightarrow \quad f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + d$$

$$\Rightarrow \quad f(0) = d$$

$$\text{and} \quad f(1) = \frac{a}{3} + \frac{b}{2} + c + d$$

Since, Rolle's theorem is applicable

$$\Rightarrow \quad f(0) = f(1)$$

$$\Rightarrow \quad d = \frac{a}{3} + \frac{b}{2} + c + d$$

$$\Rightarrow \quad 2a + 3b + 6c = 0$$

Here, the required condition is $2a + 3b + 6c = 0$

Illustration 37 Verify Rolle's theorem for the function $f(x) = x^3 - 3x^2 + 2x$ in the interval [0, 2].

Solution. Here, we observe that

(a) $f(x)$ is polynomial and since polynomials are always continuous, $f(x)$ is continuous in the interval [0, 2]

(b) $f'(x) = 3x^2 - 6x + 2$ clearly exists for all $x \in (0, 2)$. So, $f(x)$ is differentiable for all $x \in (0, 2)$ and

(c) $f(0) = 0, f(2) = 2^3 - 3 \cdot (2)^2 + 2(2) = 0$

$$\therefore \qquad f(0) = f(2)$$

Thus, all the conditions of Rolle's theorem are satisfied.

So, there must exists some $c \in (0, 2)$ such that $f'(c) = 0$

$$\Rightarrow \qquad f'(c) = 3c^2 - 6c + 2 = 0$$

$$\Rightarrow \qquad c = 1 \pm \frac{1}{\sqrt{3}}$$

Where $c = 1 \pm \dfrac{1}{\sqrt{3}} \in (0, 2)$, thus Rolle's theorem is verified.

Illustration 38 If $f(x)$ and $g(x)$ are continuous functions in $[a, b]$ and they are differentiable in (a, b), then prove that there exists $c \in (a, b)$ such that

$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b - a) \begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$$

Solution. Consider a function, $\phi(x) = f(a)\, g(x) - f(x)\, g(a)$ for all $x \in [a, b]$.

As $f(x)$ is continuous and differentiable on (a, b).

$$\Rightarrow \qquad \phi'(c) = \frac{\phi(b) - \phi(a)}{b - a} \qquad \text{...(i)}$$

Now, $\qquad \phi(x) = f(a)\, g(x) - f(x)\, g(a)$

$$\Rightarrow \qquad \phi'(x) = f(a)\, g'(x) - f'(x)\, g(a)$$

$$\Rightarrow \qquad \phi'(c) = f(a)\, g'(c) - f'(c)\, g(a)$$

$$\Rightarrow \qquad \phi'(c) = \begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$$

Also, $\qquad \phi(b) = f(a)\, g(b) - f(b)\, g(a) = \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix}$

and $\qquad \phi(a) = f(a)\, g(a) - f(a)\, g(a) = 0$

$\therefore$ Eq. (i) reduces to;

$$\begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix} = \frac{1}{(b - a)} \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix}$$

or $$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b - a) \begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$$

Illustration 39 If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b), prove that there is at least one $c \in (a, b)$, such that $\dfrac{f'(c)}{3c^2} = \dfrac{f(b) - f(a)}{b^3 - a^3}$.

Solution. Let us consider a function, $h(x) = f(x) - f(a) + A(x^3 - a^3)$

where A is obtained from the relation $h(b) = 0$.

So that, $\quad 0 = h(b) = f(b) - f(a) + A(b^3 - a^3) \quad$ ...(i)

Also, $\quad h(a) = 0 \quad$ ...(ii)

Since, (1) $h(x)$ is continuous in $[a, b]$

(2) $h(x)$ is differentiable in (a, b)

and (3) $h(a) = 0 = h(b)$

Hence, all the three conditions of Rolle's theorem are fulfilled.

Then, there must exists a 'c' $\in (a, b)$ such that $f'(c) = 0$.

$$\Rightarrow \quad f'(c) + A(3c^2) = 0$$

$$\text{or} \quad f'(c) = 3c^2 \frac{f(b) - f(a)}{b^3 - a^3} \quad \text{[using Eq. (i)]}$$

$$\Rightarrow \quad \frac{f'(c)}{3c^2} = \frac{f(b) - f(a)}{b^3 - a^3}$$

Illustration 40 Use Rolle's theorem to find the condition for the polynomial equation $f(x) = 0$ to have a repeated real roots. Hence, or otherwise prove that the equation;

$1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dots + \dfrac{x^n}{n!} = 0$, cannot have repeated roots.

Solution. By Rolle's theorem, we can say that between any two roots of a polynomial there is always a root of its derivative. Thus, if α is a repeated root of a polynomial $f(x)$, then there must be a root of $f'(x)$ in the interval.

$$\Rightarrow \quad f'(\alpha) = 0$$

ie, $\quad f(\alpha) = f'(\alpha) = 0$, for α to be a repeated root.

Let $\quad \phi(x) = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dots + \dfrac{x^n}{n!}$ has a repeated root α.

$$\Rightarrow \quad \phi(\alpha) = 0 \text{ and } \quad \phi'(\alpha) = 0$$

$$\Rightarrow \quad 1 + \frac{\alpha}{1!} + \frac{\alpha^2}{2!} + \dots + \frac{\alpha^n}{n!} = 0$$

$$\text{and} \quad 1 + \alpha + \frac{\alpha^2}{2!} + \dots + \frac{\alpha^{n-1}}{(n-1)!} = 0$$

Solving which, we get

$$\frac{\alpha^n}{n!} = 0$$

or $\alpha = 0$, thus 0 is the repeated root of $\phi(x) = 0$.

But, 0 doesn't satisfy $\phi(x)$. $\therefore$ There is no repeated root of $\phi(x) = 0$.

Illustration 41 If $f''(x) < 0$ for all $x \in (a, b)$, then $f'(x) = 0$

(a) exactly once in (a, b) (b) at most once in (a, b)

(c) at least once in (a, b) (d) None of these

Solution. Let x_1, x_2 be two distinct points in (a, b) such that $f'(x_1) = f'(x_2) = 0$.

Then, by Rolle's theorem there exists a point $c \in (a, b)$ such that $f''(c) = 0$. This contradicts the given condition that $f''(x) < 0$. For all $x \in (a, b)$.

Hence, our supposition is wrong. Consequently, there can be at most one point in (a, b) at which $f'(x)$ is zero.

Hence, (a) is the correct answer.

Illustration 42 If $f(x) = x^{\alpha} \log x$ and $f(0) = 0$, then the value of 'α' for which Rolle's theorem can be applied in [0, 1] is **[IIT JEE 2004]**

(a) – 2 (b) – 1 (c) 0 (d) 1/2

Solution. Clearly, $f(x)$ is continuous and differentiable on (0, 1) for $\alpha > 0$.

Also, $f(0) = 0 = f(1)$

For $f(x)$ to be continuous at $x = 0$, we must have

$$\lim_{x \to 0^+} f(x) = f(0), \quad ie, \quad \lim_{x \to 0^+} x^{\alpha} \log x = 0$$

Now, $$\lim_{x \to 0^+} x^{\alpha} \log x = \lim_{x \to 0^+} \frac{\log x}{\frac{1}{x^{\alpha}}} = \lim_{x \to 0} \frac{\frac{1}{x}}{\frac{-\alpha}{x^{\alpha+1}}} = 0 \qquad (\text{as } \alpha > 0)$$

So, $f(x)$ is continuous at $x = 0$ for $\alpha > 0$.

$\therefore$ Rolle's theorem can be applied on $f(x)$ in [0, 1] for all $\alpha > 0$.

Hence, (d) is the correct answer.

Illustration 43 If a, b, c be non-zero real numbers such that

$$\int_0^1 (1 + \cos^8 x)(ax^2 + bx + c)\, dx = \int_0^2 (1 + \cos^8 x)(ax^2 + bx + c)\, dx = 0.$$

Then, the equation $ax^2 + bx + c = 0$ will have

(a) one root between 0 and 1 and another between 1 and 2

(b) both the roots between 0 and 1

(c) both the roots between 1 and 2

(d) None of the above

Solution. Consider the function $\phi(x)$ given by

$$\phi(x) = \int_0^x (1 + \cos^8 t)(at^2 + bt + c)\, dt$$

$$\Rightarrow \quad \phi'(x) = (1 + \cos^8 x)(ax^2 + bx + c) \qquad \ldots\text{(i)}$$

We observe that $\phi(0) = 0$

$$\phi(1) = \int_0^1 (1 + \cos^8 t)(at^2 + bt + c)\, dt = 0 \qquad \text{(given)}$$

and $\phi(2) = \int_0^2 (1 + \cos^8 t)(at^2 + bt + c)\, dt = 0$ (given)

$\therefore$ 0, 1 and 2 are the roots of $\phi(x)$.

By Rolle's theorem $\phi'(x) = 0$ will have at least one real root between 0 and 1 and at least one real root between 1 and 2.

Hence, (a) is the correct answer.

Illustration 44 Let $n \in N$. If the value of c prescribed in Rolle's theorem for the function $f(x) = 2x(x-3)^n$ on [0, 3] is 3/4, then n is equal to

(a) 1 (b) 3 (c) 5 (d) 7

Solution. $f(0) = f(3)$; $f(x) = 2x\,(x-3)^n$

Hence, $f'(c) = 0$, $f'(x) = 2\,[(x-3)^n + nx\,(x-3)^{n-1}]$

$$f'(c) = 2\,[(c-3)^n + nc\,(c-3)^{n-1}] = 0$$

or $2\,(c-3)^{n-1}\,[c - 3 + nc] = 0$; but $c = 3/4$

$\therefore$ $(n+1)\dfrac{3}{4} = 3 \quad \Rightarrow \quad n = 3$

Hence, (b) is the correct answer.

Illustration 45 Let $f(x)$ and $g(x)$ be differentiable functions such that $f'(x)g(x) \neq f(x)g'(x)$ for any real x. Show that between any two real solutions of $f(x) = 0$, there is at least one real solution of $g(x) = 0$.

Solution. Let a, b be the solutions of $f(x) = 0$

Suppose $g(x)$ is not equal to zero for any x belonging to $[a, b]$.

Now, consider $h(x) = f(x)/g(x)$

Since, $g(x)$ is not equal to zero.

$h(x)$ is differentiable and continuous in $[a, b]$.

$h(a) = h(b) = 0$ [as $f(a) = 0$ and $f(b) = 0$ but $g(a)$ or $g(b) \neq 0$]

Applying Rolle's theorem for $h(x)$ in $[a, b]$

$h'(c) = 0$ for some c belonging to (a, b)

$$f(x)g'(x) = f'(x)g(x)$$

This gives the contradiction. **Hence proved**.

Illustration 46 Let $P(x)$ be a polynomial with real coefficients. Let $a, b \in R$, $a < b$, be two consecutive roots of $P(x)$. Show that there exists 'c' such that $a \le c \le b$ and $P'(c) + 100\,P(c) = 0$.

Solution. Consider $f(x) = e^{100x} \cdot P(x)$

Now, $f(a) = f(b) = 0$ [as $P(a) = P(b) = 0$]

Also, as $P(x)$ is polynomial $\Rightarrow$ $f(x)$ is continuous and differentiable in $[a, b]$.

$\Rightarrow$ Rolle's theorem can be applied here

$\Rightarrow$ $\exists\, c \in (a, b)$ such that $f'(c) = 0$

Now, $f'(x) = e^{100x}\,(P'(x) + 100 \cdot P(x))$

$\Rightarrow$ $e^{100c}\,(P'(c) + 100 \cdot P(c)) = 0$, from Eq. (i)

$\Rightarrow$ $(P'(c) + 100 \cdot P(c)) = 0$ (as $[e^{100c} \neq 0]$) **Hence proved**.

Illustration 47 Consider the function $f(x) = \begin{cases} x \sin \dfrac{\pi}{x}, & \text{for } x > 0 \\ 0, & \text{for } x = 0 \end{cases}$, then the number of points in (0, 1), where the derivative $f'(x)$ tends to zero is

(a) 0 (b) 1 (c) 2 (d) infinite

Solution. $f(x)$ tends to zero at points, where $\sin \dfrac{\pi}{x} = 0$

ie, $\dfrac{\pi}{x} = k\pi,\ k = 1, 2, 3, 4, \ldots$

Hence, $x = \dfrac{1}{k}\cdot$ Also, $f'(x) = \sin \dfrac{\pi}{x} - \dfrac{\pi}{x} \cos \dfrac{\pi}{x}$, if $x \neq 0$

Since, the function has a derivative at any interior point of the interval (0, 1) is also continuous in [0, 1] and $f(0) = f(1)$ hence Rolle's theorem is applicable to any one of the intervals $\left[\dfrac{1}{2}, 1\right], \left[\dfrac{1}{3}, \dfrac{1}{2}\right], \ldots, \left[\dfrac{1}{k+1}, \dfrac{1}{k}\right]$.

Hence, $\exists$ some c in each of these interval where $f'(c) = 0 \Rightarrow$ Infinite points.

Hence, (d) is the correct answer.

Illustration 48 Let f be a continuous function on $[a, b]$. If $F(x) = \left(\int_a^x f(t)dt - \int_x^b f(t)dt\right)(2x - (a + b))$, then there exist some $c \in (a, b)$ such that

(a) $\int_a^c f(t)dt = \int_c^b f(t)dt$

(b) $\int_a^c f(t)dt - \int_c^b f(t)dt = f(c)(a + b - 2c)$

(c) $\int_a^c f(t)dt - \int_c^b f(t)dt = f(c)[2c - (a + b)]$

(d) $\int_a^c f(t)dt + \int_c^b f(t)dt = f(c)[2c - (a + b)]$

Solution. Given, $F(x) = \left(\int_a^x f(t)dt - \int_x^b f(t)dt\right)[2x - (a + b)]$ ...(i)

As f is continuous, hence $F(x)$ is also continuous. Also, put $x = a$

$$F(a) = \left(-\int_a^b f(t)dt\right)(a - b) = (b - a)\int_a^b f(t)dt$$

and put $x = b$

$$f(b) = \left(\int_a^b f(t)dt\right)(b - a)$$

Hence, $F(a) = F(b)$

Hence, Rolle's theorem is applicable to $F(x)$.

$\therefore \exists$ some $c \in (a, b)$ such that $F'(c) = 0$

Now, $$F'(x) = 2\left(\int_a^x f(t)dt - \int_x^b f(t)dt\right) + [2x - (a + b)][f(x) + f(x)] = 0$$

$\therefore$ $$F'(c) = \left(\int_a^c f(t)dt - \int_c^b f(t)dt\right) = f(c)[(a + b) - 2c]$$

Hence, (b) is the correct answer.

Lagrange's Mean Value Theorem

First Form

If a function $f(x)$,

(i) is continuous in the closed interval $[a, b]$ and

(ii) is differentiable in the open interval $]a, b[$

Then, there is at least one value $c \in (a, b)$, such that;

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof Consider the function,

$$\phi(x) = f(x) - \frac{f(b) - f(a)}{b - a}x$$

Since, $f(x)$ is continuous in $[a, b]$.

$\therefore$ $\phi(x)$ is also continuous in $[a, b]$.

Since, $f'(x)$ exists in (a, b), hence $\phi'(x)$ also exists in (a, b) and

$$\phi'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \qquad \text{...(i)}$$

Clearly, $\phi(x)$ satisfies all the conditions of Rolle's theorem.

$\therefore$ There is at least one value of c of x between a and b such that $\phi'(c) = 0$ substituting $x = c$ in Eq. (i), we get

$$f'(c) = \frac{f(b) - f(a)}{b - a} \text{ which proves the theorem.}$$

Second Form

If we write $b = a + h$, then since $a < c < b$, $c = a + \theta h$ where $0 < \theta < 1$

Thus, the mean value theorem can be stated as follows:

If (i) $f(x)$ is continuous in closed interval $[a, a + h]$

(ii) $f'(x)$ exists in the open interval $]a, a + h[$, then there exists at least one number θ $(0 < \theta < 1)$

Such that $f(a + h) = f(a) + hf'(a + \theta h)$

Geometrical Interpretation of Lagrange's Theorem

Let A, B be the points on the curve $y = f(x)$ corresponding to $x = a$ and $x = b$, so that $A = [a, f(a)]$ and $B = [b, f(b)]$

$$\therefore \quad \text{Slope of chord } AB = \frac{f(b) - f(a)}{b - a}$$

The slope of the chord $AB = f'(c)$, the slope of the tangent to the curve at $x = c$.

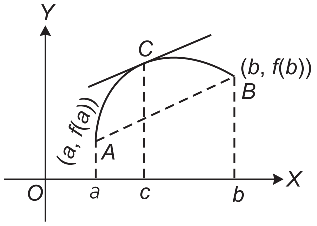


Fig. 7.13

Hence, the Lagrange's mean value theorem asserts that if a curve AB has a tangent at each of its points then there is a point c on this curve in between A and B, the tangent at which is parallel to the chord AB.

Illustration 49 Find c of the Lagrange's mean value theorem for which $f(x) = \sqrt{25 - x^2}$ in [1, 5].

Solution. It is clear that $f(x)$ has a definite and unique value of each $x \in [1, 5]$.

Thus, every point in the interval [1, 5] the value of $f(x)$ is equal to the limit of $f(x)$.

So, $f(x)$ is continuous in the interval [1, 5].

Also, $f'(x) = \dfrac{-x}{\sqrt{25 - x^2}}$, which clearly exists for all x in open interval (1, 5)

Hence, $f(x)$ is differentiable in (1, 5).

So, there must be a value $c \in (1, 5)$ such that,

$$f'(c) = \frac{f(5) - f(1)}{5 - 1} = \frac{0 - \sqrt{24}}{4} = \frac{-\sqrt{6}}{2}$$

But
$$f'(c) = \frac{-c}{\sqrt{25 - c^2}}$$

$\therefore$
$$\frac{-c}{\sqrt{25 - c^2}} = \frac{-\sqrt{6}}{2} \Rightarrow 4c^2 = 6(25 - c^2) \Rightarrow c = \pm\sqrt{15}$$

Clearly, $c = \sqrt{15} \in (1, 5)$ such that Lagrange's theorem is satisfied.

Illustration 50 Let $f(x)$ and $g(x)$ be differentiable for $0 \le x \le 2$ such that $f(0) = 2$, $g(0) = 1$ and $f(2) = 8$. Let there exists a real number c in [0, 2] such that $f'(c) = 3g'(c)$, then the value of $g(2)$ must be

(a) 2 (b) 3 (c) 4 (d) 5

Solution. As $f(x)$ and $g(x)$ are continuous and differentiable in [0, 2], then there exists at least one value 'c' such that

$$\frac{f'(c)}{g'(c)} = \frac{f(2) - f(0)}{g(2) - g(0)} \Rightarrow \frac{8 - 2}{g(2) - 1} = 3$$

$$\Rightarrow \quad g(2) - 1 = 2 \Rightarrow g(2) = 3$$

Hence, (b) is the correct answer.

Illustration 51 If $f(x) = \log_e x$ and $g(x) = x^2$ and $c \in (4, 5)$, then $c \log\left(\frac{4^{25}}{5^{16}}\right)$ is equal to

(a) $c \log_e 5 - 8$ (b) $2(c^2 \log_e 4 - 8)$

(c) $2(c^2 \log_e 5 - 8)$ (d) $c \log_e 4 - 8$

Solution. Let $\phi(x) = x^2 (\log 4) - 16 \log x$ which is continuous [4, 5] and differentiable on (4, 5).

$\Rightarrow$ By Lagrange's theorem $\frac{\phi(5) - \phi(4)}{5 - 4} = \phi'(c),\ c \in (4, 5)$

$$\therefore \qquad \phi(5) - \phi(4) = \log\left(\frac{4^{25}}{5^{16}}\right)$$

$$\text{Also,} \qquad \phi'(c) = \frac{2}{c}(c^2 \log 4 - 8)$$

$$\therefore \qquad \phi'(c) = \frac{\phi(5) - \phi(4)}{5 - 4}$$

$$\Rightarrow \qquad \frac{2}{c}(c^2 \log_e 4 - 8) = \log\left(\frac{4^{25}}{5^{16}}\right)$$

$$\text{or} \qquad c \log\left(\frac{4^{25}}{5^{16}}\right) = 2(c^2 \log 4 - 8)$$

Hence, (b) is the correct answer.

Illustration 52 If $0 < a < b < \frac{\pi}{2}$ and $f(a, b) = \frac{\tan b - \tan a}{b - a}$, then

(a) $f(a, b) \geq 2$ (b) $f(a, b) > 1$

(c) $f(a, b) \leq 1$ (d) None of these

Solution. Consider the function $f(x) = \tan x$, defined on $[a, b]$ such that $a, b \in \left(0, \frac{\pi}{2}\right)$.

Applying Lagrange's mean value theorem, we have

$$f'(c) = \frac{f(b) - f(a)}{b - a} \text{ for some } c \in (a, b)$$

$$\Rightarrow \qquad \sec^2 C = \frac{\tan b - \tan a}{b - a}$$

$$\Rightarrow \qquad f(a, b) = \sec^2 C$$

$$\Rightarrow \qquad f(a, b) > 1 \qquad [\because \sec^2 C > 1 \text{ as } C \in (0, \pi/2)]$$

Hence, (b) is the correct answer.

Illustration 53 In [0, 1] Lagrange's mean value theorem is not applicable to **[IIT JEE 2003]**

(a) $f(x) = \begin{cases} \frac{1}{2} - x, & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2, & x \geq \frac{1}{2} \end{cases}$ (b) $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

(c) $f(x) = x|x|$ (d) $f(x) = |x|$

Solution. For the function $f(x)$ given in option (a), we have (LHD at $x = 1/2) = -1$ and, (RHD at $x = 1/2) = 0$.

So, it is not differentiable at $x = 1/2 \in (0, 1)$.

$\therefore$ Lagrange's mean value theorem is not applicable.

Hence, (a) is the correct answer.

Illustration 54 Let $f(x)$ satisfy the requirements of Lagrange's mean value theorem in [0, 2]. If $f(0) = 0$ and $|f'(x)| \le 1/2$ for all $x \in [0, 2]$, then

(a) $f(x) \le 2$

(b) $|f(x)| \le 2x$

(c) $|f(x)| \le 1$

(d) $f(x) = 3$, for at least one $x \in [0, 2]$

Solution. Let $x \in (0, 2)$. Since, $f(x)$ satisfies the requirements of Lagrange's mean value theorem in [0, 2]. So, it also satisfies in $[0, x]$. Consequently, there exists $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

$$\Rightarrow \qquad f'(c) = \frac{f(x)}{x}$$

$$\Rightarrow \qquad \left|\frac{f(x)}{x}\right| = |f'(c)| \le 1/2 \qquad (\because\ |f'(x)| \le 1/2)$$

$$\Rightarrow \qquad |f(x)| \le \frac{x}{2} \ \Rightarrow\ |f(x)| \le 1 \qquad (\because\ x \in [0, 2],\ \therefore |x| \le 2)$$

Hence, (c) is the correct answer.

Illustration 55 Let $f : [2, 7] \to [0, \infty)$ be a continuous and differentiable function. Then, the value of $(f(7) - f(2)) \dfrac{(f(7))^2 + (f(2))^2 + f(2) \cdot f(7)}{3}$, is (where $c \in (2, 7)$)

(a) $3f^2(c)f'(c)$

(b) $5f^2(c) \cdot f(c)$

(c) $5f^2(c) \cdot f'(c)$

(d) None of these

Solution. Let $g(x) = f^3(x)$

$$\Rightarrow \qquad g'(x) = 3f^2(x) \cdot f'(x)$$

$$\because \qquad f : [2, 7] \to [0, \infty)$$

$$\Rightarrow \qquad g : [2, 7] \to [0, \infty)$$

Using Lagrange's mean value theorem on $g(x)$, we get

$$g'(c) = \frac{g(7) - g(2)}{5},\ c \in (2, 7)$$

$$\Rightarrow \qquad f^2(c) f'(c) = \frac{f^3(7) - f^3(2)}{5}$$

$$5f^2(c) \cdot f'(c) = \frac{(f(7) - f(2))(f^2(7) + f^2(2) + f(5) \cdot f(2))}{3}$$

Hence, (c) is the correct answer.

Different Graphs of the Cubic

$$y = ax^3 + bx^2 + cx + d$$

1. One real and two imaginary roots. (always monotonic) $\forall\ x \in R$

Condition $f'(x) \geq 0$ or $f'(x) \leq 0$ together with either $f'(x) = 0$ has no root (*ie*, $D < 0$) or $f'(x) = 0$ has a root $x = \alpha$, then $f(\alpha) = 0$.

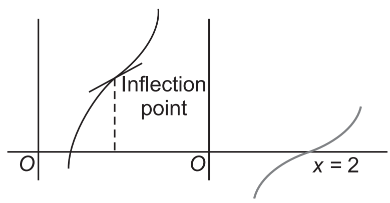


Fig. 7.14

(i) either $f'(x) = 0$ has no real root or

(ii) if $f'(x) = 0$ has a root $x = \alpha$, then $f(\alpha) = 0$

eg, $y = x^3 - 2x^2 + 5x + 4$ $\quad$ $y = (x-2)^3$

$y' = 3x^2 - 4x + 5 \quad (D < 0)$ $\quad$ $y' = 3(x-2)^2 = 0 \Rightarrow x = 2$, also $f(2) = 0$ gives $x = 2$, $y(2) = 0$

Point to Consider

In this case, if $f'(x) = 0$ has a root $x = \alpha$ and $f(\alpha) = 0$ this would mean $f(x) = 0$ has repeated roots, which is dealt with separately.

2. Exactly one root and non-monotonic.

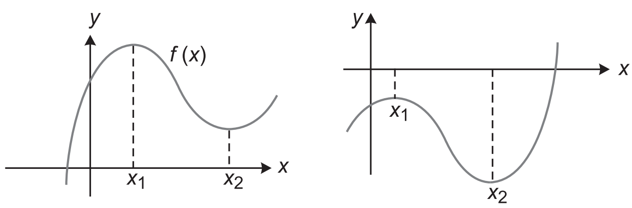


Fig. 7.15

$$f(x_1) \cdot f(x_2) > 0$$

where x_1 and x_2 are the roots of $f'(x) = 0$

3. Three roots < two coincident / one different

$$f(x_1) \cdot f(x_2) = 0$$

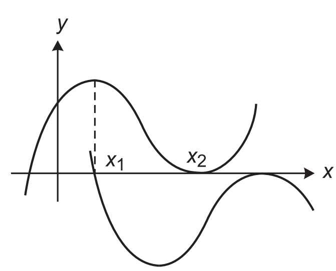


Fig. 7.16

4. All three distinct real roots

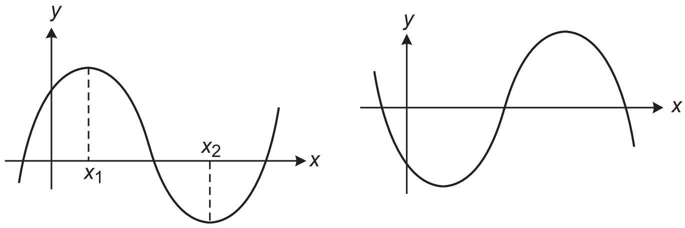


Fig. 7.17

$$f(x_1) \cdot f(x_2) < 0$$

where x_1 and x_2 are the roots of $f'(x) = 0$

5. All the three roots are coincident

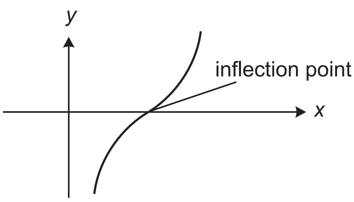


Fig. 7.18

$$f'(x) \geq 0 \quad \text{or} \quad f'(x) \leq 0 \quad \text{and} \quad f(\alpha) = 0$$

where α is a root of $f'(x) = 0$

eg, $\qquad y = (x-1)^3$

Points to Consider

(i) Graph of every cubic polynomial must have exactly one point of *inflection*.

(ii) In case (4),if $f(a), f(b), f(c)$ and $f(d)$ alternatively change sign.

Illustration 56 If the cubic $y = x^3 + px + q$ has 3 distinct real roots, then prove that $4p^3 + 27q^2 < 0$.

Solution. $f'(x) = 3x^2 + 0x + p$

$$x_1 + x_2 = 0 \quad \text{and} \quad x_1 x_2 = \frac{p}{3}$$

$$\Rightarrow \quad (x_1^3 + px_1 + q)(x_2^3 + px_2 + q) < 0$$

$$\Rightarrow \quad x_1^3 \cdot x_2^3 + px_1^3 x_2 + qx_1^3 + p^2 x_1 x_2 + px_1 x_2^3 + qx_2^3 + pqx_1 + q^2 + pqx_2 < 0$$

$$\Rightarrow \quad (x_1 x_2)^3 + px_1 x_2 (x_1^2 + x_2^2) + q(x_1^3 + x_2^3) + pq(x_1 + x_2) + p^2 x_1 x_2 + q^2 < 0$$

$$\Rightarrow \quad (x_1 x_2)^3 + px_1 x_2 \{(x_1 + x_2)^2 - 2x_1 x_2\} + q\{(x_1 + x_2)^3 - 3(x_1 x_2)(x_1 + x_2)\} + pq(x_1 + x_2) + p^2 x_1 x_2 + q^2 < 0$$

$$\Rightarrow \quad \frac{p^3}{27} + \frac{p^2}{3}\left\{-\frac{2p}{3}\right\} + p^2\left\{\frac{p}{3}\right\} + q^2 < 0$$

$$\Rightarrow \quad 4p^3 + 27q^2 < 0$$

Illustration 57 Find all the possible values of the parameter a so that $x^3 - 3x + a = 0$ has three real and distinct roots.

Solution. Let $f(x) = x^3 - 3x + a \quad \Rightarrow f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$

Now, $\qquad f(1) = a - 2, f(-1) = a + 2$

We know that the roots would be real and distinct, if

$$f(1) f(-1) < 0 \quad \Rightarrow \quad (a-2)(a+2) < 0$$

$\Rightarrow \qquad -2 < a < 2 \qquad$ (using wavy curve method)

Thus, the given equation would have real and distinct roots, if $a \in (-2, 2)$.

Illustration 58 The equation $\sin x + x\cos x = 0$ has at least one root in the interval

(a) $\left(-\frac{\pi}{2}, 0\right)$ (b) $(0, \pi)$ (c) $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ (d) None of these

Solution. Consider the function given by,

$$f(x) = \int (\sin x + x\cos x)\, dx = x \sin x$$

we observe that $\qquad f(0) = f(\pi) = 0$

$\therefore$ 0 and π are two roots of $f(x) = 0$.

Consequently, $f'(x) = 0$ *ie*, $\sin x + x\cos x = 0$ has at least one root in $(0, \pi)$.

Hence, (b) is the correct answer.

Illustration 59 Let $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex$, where $a, b, c, d, e \in R$ and $f(x) = 0$ has a positive root α. Then,

(a) $f'(x) = 0$ has root α_1 such that $0 < \alpha_1 < \alpha$

(b) $f''(x) = 0$ has at least one real root

(c) $f'(x) = 0$ has at least two real roots

(d) All of the above

Solution. It is given that α is a positive root of $f(x)$ and by inspection, we have $f(0) = 0$

$\therefore \qquad x = 0$ and $x = \alpha$ are roots of $f(x) = 0$.

By Rolle's theorem, $f'(x) = 0$ has a root α_1 between 0 and α *ie,* $0 < \alpha_1 < \alpha$

$\therefore$ (a) is correct.

Clearly, $f'(x) = 0$ is a fourth degree equation in x and imaginary roots always occurs in pairs.

Since, $x = \alpha_1$ is a root of $f'(x) = 0$.

$\therefore$ $f'(x) = 0$ will have another real root, α_2 (say)

Now, α_1 and α_2 are real roots of $f'(x) = 0$.

$\therefore$ By Rolle's theorem $f''(x) = 0$ will have a real root between α_1 and α_2.

$\therefore$ (b) is correct.

We have seen that $x = 0, x = \alpha$ are two real roots of $f(x) = 0$. As $f(x) = 0$ is fifth degree equation, it will have at least three real roots. Consequently by Rolle's theorem $f'(x) = 0$ will have at least two real roots.

$\therefore$ (c) is correct.

Hence, (d) is the correct answer.

Illustration 60 Between any two real roots of the equation $e^x \sin x - 1 = 0$, the equation $e^x \cos x + 1 = 0$ has

(a) at least one root

(b) at most one root

(c) exactly one root

(d) no root

Solution. Let $f(x) = e^{-x} - \sin x$ and α and β be two roots of the equation $e^x \sin x - 1 = 0$. Such that $\alpha < \beta$. Then,

$$e^{\alpha} \sin \alpha = 1 \quad \text{and} \quad e^{\beta} \sin \beta = 1$$

$$\Rightarrow \quad e^{-\alpha} - \sin \alpha = 0 \quad \text{and} \quad e^{-\beta} - \sin \beta = 0 \quad \ldots\text{(i)}$$

Clearly, $f(x)$ is continuous on $[\alpha, \beta]$ and differentiable on (α, β).

Also, $f(\alpha) = f(\beta) = 0$ [using Eq. (i)]

$\therefore$ By Rolle's theorem there exists $c \in (\alpha, \beta)$ such that $f'(c) = 0$

$\Rightarrow \quad - e^{-c} - \cos c = 0$

$\Rightarrow \quad e^c \cos c + 1 = 0$

$\Rightarrow \quad x = c$ is root of $e^x \cos x + 1 = 0$; where $c \in (\alpha, \beta)$

Hence, (a) is the correct answer.

Illustration 61 $f(x)$ is a polynomial of degree 4 with real coefficients such that $f(x) = 0$ is satisfied by $x = 1, 2, 3$ only, then $f'(1) \cdot f'(2) \cdot f'(3)$ is equal to

(a) 0 (b) 2

(c) –1 (d) None of these

Solution. $f(x) = 0$ has roots 1, 2, 3 only

$\Rightarrow$ Any one of 1, 2 or 3 is a repeated root of $f(x) = 0$

$\Rightarrow$ $f'(1)$ or $f'(2)$ or $f'(3)$ any one of them must be zero.

$\Rightarrow$ $f'(1) \cdot f'(2) \cdot f'(3) = 0$

Hence, (a) is the correct answer.

Illustration 62 If $f(x)$ is a polynomial of degree 5 with real coefficients such that $f(|x|) = 0$ has 8 real roots, then $f(x) = 0$ has

(a) 4 real roots

(b) 5 real roots

(c) 3 real roots

(d) nothing can be said

Solution. Given that $f(|x|) = 0$ has 8 real roots.

$\Rightarrow$ $f(x) = 0$ has 4 positive roots.

Since, $f(x)$ is a polynomial of degree 5, $f(x)$ cannot have even number of real roots.

$\Rightarrow f(x)$ has all the five roots real, in which four positive and one root is negative.

Hence, (b) is the correct answer.

Illustration 63 If the function $f(x) = |x^2 + a|x| + b|$ has exactly three points of non-differentiability, then which of the following can be true?

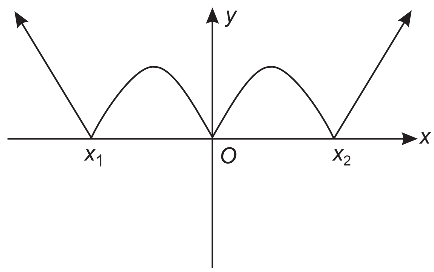
(a) $b = 0, a < 0$

(b) $b < 0, a \in R$

(c) $b > 0, a \in R$

(d) All of the above

Solution. Here, $f(x) = |x^2 + a|x| + b|$ has three points of non-differentiability.



$\therefore$ $f(x)$ is non-differentiable at $x = 0, x_1, x_2$

$\Rightarrow$ $x^2 + ax + b = 0$ has one root zero and other positive root.

$\Rightarrow$ $b = 0$ and $a < 0$, is one of the case.

Hence, (a) is the correct answer.

Illustration 64 If the equation $e^{||x|-2|+b} = 2$ has four solutions, then b lies in

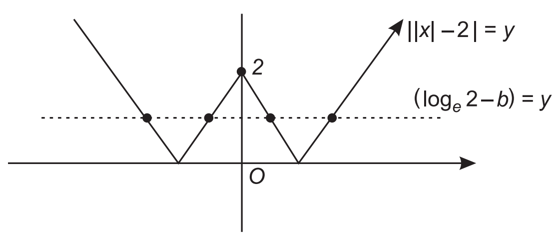
(a) (log 2, – log 2)

(b) (log 2 – 2, log 2)

(c) (– 2, log 2)

(d) (0, log 2)

Solution. Here, $e^{||x|-2|+b} = 2$



$\Rightarrow$ $||x|-2| + b = \log_e 2 \quad \Rightarrow ||x|-2| = \log_e 2 - b$

Clearly, above equation has 4 real roots, if

$$0 < (\log_e 2 - b) < 2$$

$\Rightarrow$ $$-\log_e 2 < -b < 2 - \log_e 2$$

or $$b \in (\log_e 2 - 2, \log_e 2)$$

Hence, (b) is the correct answer.

Illustration 65 If the function $f(x) = x^3 - 9x^2 + 24x + c$ has three real and distinct roots α, β and γ, then the value of $[\alpha]+[\beta]+[\gamma]$ is

(a) 5, 6

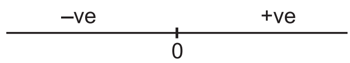
(b) 6, 7

(c) 7, 8

(d) None of the above

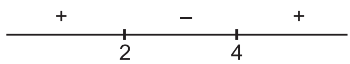
Solution. Take $y = x^3 - 9x^2 + 24x = x\,(x^2 - 9x + 24)$

$\therefore \quad y = x\,(x^2 - 9x + 24) = x\,\{(x-3)^2 + 15\}$



$$\Rightarrow \quad \frac{dy}{dx} = 3x^2 - 18x + 24 = 3\,(x^2 - 6x + 8)$$

$$= 3\,(x-2)(x-4)$$



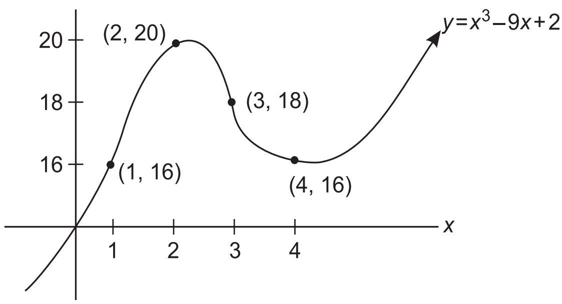
for three real roots of

$f(x) = x^3 - 9x^2 + 24x + c$, c must lie in the interval $(-20, -16)$.

Now, if $\quad c \in (-20, -18)$

$\alpha \in (1, 2), \beta \in (2, 3),$

$\gamma \in (4, 5)$



$\Rightarrow \quad [\alpha]+[\beta]+[\gamma] = 7$

If $\quad c \in (-18, -16)$

$\Rightarrow \quad \alpha \in (1, 2), \beta \in (3, 4), \gamma \in (4, 5)$

$\Rightarrow \quad [\alpha]+[\beta]+[\gamma] = 8$

Hence, (c) is the correct answer.

Worked Examples

Type 1 : Subjective Type Questions

Example 1 If the line $x\cos\alpha + y\sin\alpha = P$ is the normal to the curve $(x+a)y = c^2$ then show,

$$\alpha \in \left(2n\pi + \frac{\pi}{2}, (2n+1)\pi\right) \cup \left(2n\pi + \frac{3\pi}{2}, (2n+2)\pi\right)$$

Solution. Here, $y = \dfrac{c^2}{x+a}$

$$\Rightarrow \qquad \frac{dy}{dx} = -\frac{c^2}{(x+a)^2}$$

Slope of normal is $\Rightarrow \quad \dfrac{(x+a)^2}{c^2} > 0$ (for all x)

$\therefore$ $x\cos\alpha + y\sin\alpha = P$ is normal

if, $\quad -\dfrac{\cos\alpha}{\sin\alpha} > 0$

or $\quad \cot\alpha < 0$

ie, α lies in II or IV quadrant.

So, $$\alpha \in \left(2n\pi + \frac{\pi}{2}, (2n+1)\pi\right) \cup \left(2n\pi + \frac{3\pi}{2}, (2n+2)\pi\right)$$

Example 2 Find the total number of parallel tangents of $f_1(x) = x^2 - x + 1$ and $f_2(x) = x^3 - x^2 - 2x + 1$.

Solution. Here,

$$f_1(x) = x^2 - x + 1 \quad \text{and} \quad f_2(x) = x^3 - x^2 - 2x + 1$$

$$\Rightarrow \quad f_1'(x_1) = 2x_1 - 1 \quad \text{and} \quad f_2'(x_2) = 3x_2^2 - 2x_2 - 2$$

Let the tangents drawn to the curves $y = f_1(x)$ and $y = f_2(x)$ at $(x_1, f_1(x_1))$ and $(x_2, f_2(x_2))$ are parallel,

$$\Rightarrow \quad 2x_1 - 1 = 3x_2^2 - 2x_2 - 2$$

or $$2x_1 = (3x_2^2 - 2x_2 - 1)$$

which is possible for infinite numbers of ordered pairs.

$\therefore$ It will have infinite number of parallel tangents.

Example 3 Find the point on the curve $3x^2 - 4y^2 = 72$ which is nearest to the line $3x + 2y + 1 = 0$.

Solution. Slope of the given line $3x + 2y + 1 = 0$ is $(-3/2)$.

Let us locate the point on the curve at which the tangent is parallel to the given line.

Differentiating the curve both sides with respect to x, we get

$$6x - 8y\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = \frac{3x_1}{4y_1} = \frac{-3}{2} \qquad \text{[since parallel to } 3x + 2y = 1]$$

Also, the point (x_1, y_1) lies on, $3x^2 - 4y^2 = 72$

$$3x_1^2 - 4y_1^2 = 72$$

$$\Rightarrow \qquad 3\,\frac{x_1^2}{y_1^2} - 4 = \frac{72}{y_1^2}$$

$$\Rightarrow \qquad 3(4) - 4 = \frac{72}{y_1^2} \qquad \left[\text{as } \frac{x_1}{y_1} = -2\right]$$

$$\Rightarrow \qquad y_1^2 = 9 \qquad \Rightarrow \quad y_1 = \pm 3$$

Required points are $(-6, 3)$ and $(6, -3)$.

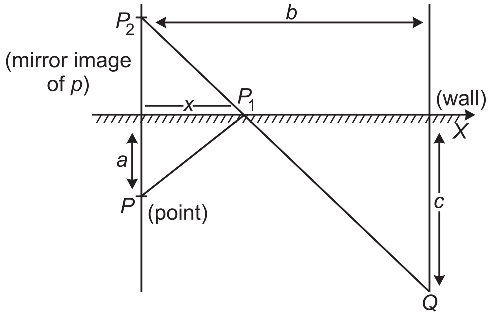
Distance of $(-6, 3)$ from the given line $= \dfrac{|-18 + 6 + 1|}{\sqrt{13}} = \dfrac{11}{\sqrt{13}}$

and distance of $(6, 3)$ from the given line $= \dfrac{|18 - 6 + 1|}{\sqrt{13}} = \dfrac{13}{\sqrt{13}} = \sqrt{13}$

Thus, $(-6, 3)$ is the required point.

Example 4 Consider a man standing at point P at a distance a unit from the wall. This man has to reach at a point Q (which is on the same side of the wall). Distance between P and Q, measured along the wall is b units, while the distance of Q from the wall is c units. The man has to touch the wall before arriving at point Q. Locate the point on the wall that the man should touch before arriving at Q, so that the distance covered by him is the least?

Solution. Let P be the required point on the wall such that distance of P_1 and P (measured along the wall) is x.



Clearly, the man should follow a path that a ray of light would have followed if the ray were reflected from mirror (in this case wall). Let P_2 be the mirror image of P.

Clearly, $P_2 \equiv (0, -a)\, Q \equiv (b, c)$

Equation of P_1Q is : $\quad y + a = \dfrac{c + a}{b - 0}(x)$

$$\Rightarrow \qquad y = \left(\frac{c + a}{b}\right)x - a$$

$$\text{If} \qquad y = 0 \qquad \Rightarrow \qquad x = \frac{ab}{a + c}$$

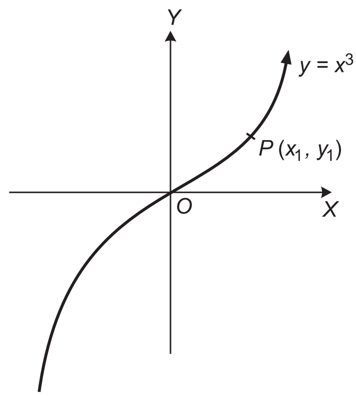
Point P would be at a distance $\dfrac{ab}{a + c}$ units from P (measured along the wall).

Example 5 Tangent at a point P_1 (other than (0, 0)) on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3 and so on, show that the abscissae of $P_1, P_2, P_3, \ldots\ldots, P_n$, form a GP. Also, find the ratio of $\dfrac{\text{area}(\Delta P_1P_2P_3)}{\text{area}(\Delta P_2P_3P_4)}$.

Solution. Let $P_1(x_1, y_1)$ be a point on the curve

$$y = x^3 \quad \text{...(i)}$$

$$y_1 = x_1^3 \quad \text{...(ii)}$$



Now, $\dfrac{dy}{dx} = 3x^2$

$\therefore$ Slope of the tangent at $P_1 = m_1 = 3x_1^2$

$\therefore$ Equation of the tangent at $P_1(x_1, y_1)$ is

$$y - x_1^3 = 3x_1^2(x - x_1)$$

ie, $y = 3x_1^2x - 2x_1^3$ ...(iii)

Solving Eqs. (i) and (iii), we get

$$x^3 - 3x_1^2x + 2x_1^3 = 0$$

ie, $(x - x_1)\cdot(x^2 + xx_1 - 2x_1^2) = 0$

ie, $(x - x_1)(x - x_1)(x + 2x_1) = 0$

$\therefore$ $x = x_1$ (neglecting) or $x = -2x_1$

$\therefore$ $x_2 = -2x_1, \quad y_2 = x_2^3 = -8x_1^3$

$\therefore$ $P_2(x_2, y_2) = (-2x_1, -8x_1^3)$

Now, we find P_3, the point where the curve meets again at P_3.

Slope of the tangent at $P_2 = \left(\dfrac{dy}{dx}\right)_{(x_2,\, y_2)}$

$$= 3x_2^2 = 3\cdot 4x_1^2 = 12x_1^2$$

$\therefore$ Equation of tangent at P_2 is,

$$y - x_2^3 = 3x_2^2(x - x_2) \quad \text{...(iv)}$$

To get, $P_3 = (x_3, y_3)$, solve Eqs. (i) and (iv)

$\therefore$ $P_3 = (x_3, y_3) = (-2x_2, -8x_2^3) = (4x_1, 64x_1^3)$ and so on.

$\therefore$ Abscissa of $P_1, P_2, P_3, \ldots$ are given by $x_1, -2x_1, 4x_1, -8x_1, \ldots$, which is GP with common ratio $= -2$.

Now,
$$\text{area}(\Delta P_1P_2P_3) = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$
$$\text{area}(\Delta P_1P_2P_3) = \frac{1}{2}\begin{vmatrix} x_1 & x_1^3 & 1 \\ -2x_1 & -8x_1^3 & 1 \\ 4x_1 & 64x_1^3 & 1 \end{vmatrix}$$
$$\text{area}(\Delta P_1P_2P_3) = \frac{x_1^4}{2}\begin{vmatrix} 1 & 1 & 1 \\ -2 & -8 & 1 \\ 4 & 64 & 1 \end{vmatrix}$$

Similarly,
$$\text{area}(\Delta P_2P_3P_4) = \frac{1}{2}\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$$
$$\text{area}(\Delta P_2P_3P_4) = \frac{1}{2}\begin{vmatrix} -2x_1 & -8x_1^3 & 1 \\ 4x_1 & 64x_1^3 & 1 \\ -8x_1 & 512x_1^3 & 1 \end{vmatrix}$$
$$\text{area}(\Delta P_2P_3P_4) = 8x_1^4\begin{vmatrix} 1 & 1 & 1 \\ -2 & -8 & 1 \\ 4 & 64 & 1 \end{vmatrix}$$

$\therefore$
$$\frac{\text{area}(\Delta P_1P_2P_3)}{\text{area}(\Delta P_2P_3P_4)} = \frac{1}{16}$$

Example 6 Determine all polynomial $P(x)$ with rational coefficient so that for all x with $|x| \le 1$; $P(x) = P\left(\dfrac{-x+\sqrt{3-3x^2}}{2}\right)$.

Solution. Here, $P(x) = P\left(\dfrac{-x+\sqrt{3-3x^2}}{2}\right)$ for $|x| \le 1$ ...(i)

Let $x = 0$;

$\Rightarrow$
$$P(0) = P(\sqrt{3}/2)$$

Which shows, $P(x) - P(0)$ is divisible by $x\left(x - \dfrac{\sqrt{3}}{2}\right)$.

Since, $P(x) - P(0)$ has rational coefficients and $\dfrac{\sqrt{3}}{2}$ is one of the roots,

$\therefore$ $-\dfrac{\sqrt{3}}{2}$ is also a root of $P(x) - P(0)$.

Thus, $x\left(x - \dfrac{\sqrt{3}}{2}\right)\left(x + \dfrac{\sqrt{3}}{2}\right) = x^3 - \dfrac{3}{4}x = \dfrac{4x^3 - 3x}{4}$ is factor of $P(x) - P(0)$.

$\therefore$ $P(x) = P(0) + (3x - 4x^3)\,P_1(x)$ for $|x| \le 1$ ...(ii) as $$P(x) = P\left(\frac{-x+\sqrt{3-3x^2}}{2}\right)$$

$$\Rightarrow \quad 3\left(\frac{-x+\sqrt{3-3x^2}}{2}\right) - 4\left(\frac{-x+\sqrt{3-3x^2}}{2}\right) = 3x - 4x^3$$

$$\therefore \quad P_1(x) = P_1\left(\frac{-x+\sqrt{3-3x^2}}{2}\right)$$

$$\therefore \quad P_1(x) = P_1(0) + (3x-4x^3)\,P_2(x) \qquad \text{[using Eq. (ii)]}$$

$$\Rightarrow \quad P\,(x) = P(0) + (3x-4x^3)\,(P_1(0)) + (3x-4x^3)P_2(x))$$
$$= P(0) + (3x-4x^3)\,P_1(0) + (3x-4x^3)^2\,P_2(x)$$

Thus, in general,

$$P\,(x) = a_0 + a_1(3x-4x^3) + a_2\,(3x-4x^3)^2 + \ldots + (3x-4x^3)^k \cdot k\,(x)$$

where $k\,(x)$ is a polynomial with rational coefficient.

Example 7 Let $f(x) = (x-a)(x-b)(x-c)$, $a < b < c$. Show that $f'(x) = 0$ has two roots one belonging to (a, b) and other belonging to (b, c).

Solution. Here, $f(x)$ being a polynomial, is continuous and differentiable for all real values of x.

We also have, $f(a) = f(b) = f(c)$

If we apply Rolle's theorem to $f(x)$ in $[a, b]$ and $[b, c]$ we would observe that $f'(x) = 0$ would have at least one root in (a, b) and at least one root in (b, c).

But $f'(x) = 0$ is a polynomial of degree two, hence $f'(x) = 0$ cannot have more than two roots.

It implies that exactly one root of $f'(x) = 0$ would lie in (a, b) and exactly one root of $f'(x) = 0$ would be in (b, c).

Example 8 Prove that if $2a_0^2 < 15a$, all roots of

$$x^5 - a_0x^4 + 3ax^3 + bx^2 + cx + d = 0$$

cannot be real. It is given that $a_0, a, b, c, d \in R$.

Solution. Let $$f(x) = x^5 - a_0x^4 + 3ax^3 + bx^2 + cx + d$$

$$f'(x) = 5x^4 - 4a_0x^3 + 9ax^2 + 2bx + c$$

$$f''(x) = 20x^3 - 12a_0x^2 + 18ax + 2b$$

$$f'''(x) = 60x^2 - 24a_0x + 18a$$

or $$f'''(x) = 6(10x^2 - 4a_0x + 3a)$$

Now, discriminant $= 16a_0^2 - 4 \cdot 10 \cdot 3a$

$$\Rightarrow \quad D = 8\,(2a_0^2 - 15a) < 0 \qquad [\text{as } 2a_0^2 - 15a < 0 \text{ given}]$$

Hence, the roots of $f''(x) = 0$ cannot be real.

And therefore, all the roots of $f(x) = 0$ will not be real.

Example 9 If at each point of the curve $y = x^3 - ax^2 + x + 1$ the tangents is inclined at an acute angle with the positive direction of the x-axis, a lies in the interval.

Solution. As, $y = x^3 - ax^2 + x + 1$

and the tangent is inclined at an acute angle with the positive direction of x-axis,

$$\Rightarrow \quad \frac{dy}{dx} \geq 0$$

$$\therefore \quad 3x^2 - 2ax + 1 \geq 0, \qquad \text{for all } x \in R$$

(and we know, $ax^2 + bx + c \geq 0$ for all $x \in R \Rightarrow a > 0$ and $D \leq 0$)

$$\Rightarrow \quad (2a)^2 - 4(3)(1) \leq 0$$

$$\Rightarrow \quad 4(a^2 - 3) \leq 0$$

$$\Rightarrow \quad (a - \sqrt{3})(a + \sqrt{3}) \leq 0$$

$$\therefore \quad -\sqrt{3} \leq a \leq \sqrt{3}$$

Example 10 Show that if $P(x)$ is a polynomial of odd degree greater than 1, then through any point P in the plane, there will be at least one tangent line to the curve $y = P(x)$. Is this true if $P(x)$ is a curve of even degree?

Solution. If $y = P(x)$, where $P(x)$ is a polynomial of odd degree $d > 1$, the point $P(a, b)$ is on some tangent to the curve if and only if,

$$(y - b) = P'(x)(x - a)$$

or $\quad \{P(x) - b\} = P'(x)\{(x - a)\}$ has a real solution.

But, $xP'(x) - P(x) - aP'(x) + b$ has a degree d with leading coefficient $(d - 1)$ times the leading coefficient of $P(x)$ and by Intermediate value theorem it has root (real) *ie*,. There is a real number x_0 for which tangent to $y = P(x)$ at $(x_0, P(x_0))$ passes through $P(a, b)$.

For even degree it may not be true, (consider $y = x^2$).

Example 11 Show that there is no cubic curve for which the tangent lines at two distinct points coincide.

Solution. Suppose $y \equiv ax^3 + bx^2 + cx + d = 0 \ (a \neq 0)$ be a cubic curve.

We assume that (x_1, y_1) and (x_2, y_2), $(x_1 < x_2)$ are two distinct points on the curve at which tangents coincide.

Then, by Mean value theorem there exists x_3 $(x_1 < x_3 < x_2)$ such that

$$\frac{y_2 - y_1}{x_2 - x_1} = y'(x_3)$$

Since, tangent x_1, x_2, x_3 are solutions of equation

$$3ax^2 + 2bx + c = M$$

But, it is a quadratic and thus cannot have more than two roots. Therefore, no such cubic is possible.

Example 12 A line is drawn from a point $P(x, y)$ on curve $y = f(x)$ making an angle with the x-axis which is supplementary to the one made by the tangent to the curve at $P(x, y)$. The line meets the x-axis at A. Another line perpendicular to the first, is drawn from $P(x, y)$ meeting the y-axis at B. If $OA = OB$, where O is the origin, find the curve which passes through (1, 1).

Solution. The equation of the line passing through $P(x, y)$ making an angle with the x-axis which is supplementary to the angle made by the tangent at $P(x, y)$ is,

$$Y - y = -\frac{dy}{dx}(X - x) \qquad \text{...(i)}$$

When it meets x-axis, $Y = 0$

$$\Rightarrow \qquad X = x + y\frac{dx}{dy}$$

$$\Rightarrow \qquad OA = x + y\frac{dx}{dy} \qquad \text{...(ii)}$$

The line passing through $P(x, y)$ and perpendicular to (i) is

$$Y - y = \frac{dx}{dy}(X - x)$$

When it meets y-axis, $X = 0$

$$\Rightarrow \qquad Y = y - x\frac{dx}{dy}$$

$$\Rightarrow \qquad OB = y - x\frac{dx}{dy} \qquad \text{...(iii)}$$

Since, $OA = OB$

$$\Rightarrow \qquad x + y\frac{dx}{dy} = y - x\frac{dx}{dy} \quad \text{or} \quad (y - x) = (y + x)\frac{dx}{dy}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{y + x}{y - x}$$

Putting $y = vx$,

$$v + x\frac{dv}{dx} = \frac{1 + v}{v - 1}$$

$$\Rightarrow \qquad \frac{xdv}{dx} = \frac{1 + 2v - v^2}{v - 1}$$

$$\Rightarrow \qquad \frac{(1 - v)\,dv}{1 + 2v - v^2} = -\frac{dx}{x}$$

Integrating both the sides, we get

$$\log(1 + 2v - v^2) = -\log(x^2) + c_1$$

$$\text{or} \qquad \log(x^2) + \log\left(1 + \frac{2y}{x} - \frac{y^2}{x^2}\right) = c_1$$

$$\text{or} \qquad x^2 + 2xy - y^2 = c \qquad (c = e^{c_1})$$

Since, the curve passes through (1, 1), $c = 2$

Hence, the required curve is, $x^2 + 2xy - y^2 = 2$.

Example 13 Let $y = f(x)$ be a curve passing through $(1, \sqrt{3})$ such that tangent at any point P on the curve lying in the first quadrant has positive slope and the tangent and the normal at the point P intersect the x-axis at A and B respectively, so that the mid-point of AB is the origin. Find the differential equation of the curve and hence determine $f(x)$.

Solution. Let $P\{x, f(x)\}$ be a point lying on the curve in first quadrant.

Equation of normal and tangent at P are, $\{Y - f(x)\} = -\dfrac{1}{f'(x)}(X - x)$

and $\{Y - f(x)\} = -f'(x)(X - x)$, respectively.

$\Rightarrow \quad A = \left(x - \dfrac{f(x)}{f'(x)}, 0\right)$ *ie*, x-intercept and

$B = (x + f(x)f'(x), 0)$ *ie*, x-intercept

Since, mid-point of the segment AB is the origin,

$$\Rightarrow \quad 2x - \frac{f(x)}{f'(x)} + f(x)f'(x) = 0$$

$$\Rightarrow \quad f(x)\cdot\{f'(x)\}^2 + 2x\cdot\{f'(x)\} - f(x) = 0$$

$$\Rightarrow \quad f'(x) = \frac{-2x \pm \sqrt{4x^2 + 4f^2(x)}}{2f(x)}$$

$$= \frac{-x + \sqrt{x^2 + f^2(x)}}{f(x)}$$

[negative sign is being neglected as $f'(x) > 0$]

Thus we have, $\quad x + f(x)\cdot f'(x) = \sqrt{x^2 + f^2(x)}$

$$\Rightarrow \quad \frac{d}{2\,dx}(x^2 + f^2(x)) = \sqrt{x^2 + f^2(x)} \qquad \left[\text{as } \frac{d}{dx}\{x^2 + f^2(x)\} = 2\{x + f(x)\cdot f'(x)\}\right]$$

$$\Rightarrow \quad \frac{d\{x^2 + f^2(x)\}}{2\sqrt{x^2 + f^2(x)}} = dx,$$

Integrating both the sides, we get

$$\sqrt{x^2 + f^2(x)} = x + \lambda$$

As it passes through $(1, \sqrt{3})$.

$$\Rightarrow \quad \sqrt{1+3} = 1 + \lambda \quad \Rightarrow \quad \lambda = 1$$

Thus, the curve is, $\quad x^2 + f^2(x) = (x + 1)^2$

or $\quad y^2 = 1 + 2x$, is the required curve [where $y = f(x)$].

Example 14 The tangent at a point P to the rectangular hyperbola $xy = c^2$ meets the lines $x - y = 0$, $x + y = 0$ at Q and R, Δ is the area of the $\Delta\, OQR$, where O is the origin. The normal at P meets the x-axis at M and y-axis at N. If Δ_2 is the area of the ΔOMN, show that Δ_2 varies inversely as the square of Δ_1.

Solution. Tangent at $P(x_1, y_1)$ is $xy_1 + yx_1 = 2c^2$

The point of intersection of this line with $x - y = 0$ is given by,

$$x(x_1 + y_1) = 2c^2, \quad ie, \quad x = 2c^2/(x_1 + y_1)$$

$$\therefore \quad Q\left(\frac{2c^2}{x_1 + y_1}, \frac{2c^2}{x_1 + y_1}\right)$$

The point of intersection of the tangent with $x + y = 0$ is,

$$x(y_1 - x_1) = 2c^2, \qquad x = \frac{2c^2}{y_1 - x_1}$$

$$\therefore \qquad R \text{ is } \left(\frac{2c^2}{y_1 - x_1}, \frac{2c^2}{x_1 - y_1}\right)$$

$$\therefore \qquad \Delta_1 = \frac{1}{2}\{x_1 y_2 - x_2 y_1\}$$

$$= \frac{1}{2}\{-x_1 x_2 - x_2 x_1\} = -x_1 x_2$$

$$\Delta_1 = \frac{4c^4}{x_1^2 - y_1^2}$$

The normal at (x_1, y_1) is $y - y_1 = \dfrac{x_1}{y_1}(x - x_1)$

Intersection with $y = 0$ is $x - x_1 = -\dfrac{y_1^2}{x_1}$

$$\Rightarrow \qquad x = \frac{x_1^2 - y_1^2}{x_1}$$

Let $x = 0$, $\qquad y - y_1 = -\dfrac{x_1^2}{y_1} \qquad \Rightarrow \qquad y = \dfrac{y_1^2 - x_1^2}{y_1}$

$$\Delta_2 = \frac{1}{2}\frac{(x_1^2 - y_1^2)}{x_1} \cdot \frac{(y_1^2 - x_1^2)}{y_1}$$

$$= -\frac{1}{2}\frac{(x_1^2 - y_1^2)^2}{x_1 y_1} = -\frac{1}{2}\frac{(x_1^2 - y_1^2)^2}{c^2}$$

$$\therefore \qquad \Delta_1^2\,\Delta_2 = \frac{16c^8}{(x_1^2 - y_1^2)^2}\left(-\frac{1}{2}\right)\frac{(x_1^2 - y_1^2)^2}{c^2}$$

$$= -8c^6 = \text{constant}$$

$$\therefore \qquad \Delta_2 \propto \frac{1}{\Delta_1^2}$$

or Δ_2 varies inversely as the square of Δ_1.

Example 15 If the function of $f : [0, 4] \to R$ is differentiable, then show that,

(i) $(f(4))^2 - (f(0))^2 = 8 f'(a) f(b)$ for $a, b \in (0, 4)$

(ii) $\int_0^4 f(t)\,dt = 2\{\alpha f(\alpha^2) + \beta f(\beta^2)\}$ for some $0 < \alpha, \beta < 2$ **[IIT JEE 2003]**

Solution. (i) Since, f is differentiable $\Rightarrow f$ is also continuous.

Thus, by Lagrange's mean value theorem, $a \in (0, 4)$ such that

$$f'(a) = \frac{f(4) - f(0)}{4 - 0} = \frac{f(4) - f(0)}{4} \qquad \text{...(i)}$$

Also, by Intermediate mean value theorem there exists $b \in (0, 4)$ such that

$$f(b) = \frac{f(4) + f(0)}{2} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$f'(a)f(b) = \frac{f(4)-f(0)}{4} \times \frac{f(4)+f(0)}{2}$$

$$f'(a)f(b) = \frac{(f(4))^2-(f(0))^2}{8}$$

$$\Rightarrow \quad (f(4))^2-(f(0))^2 = 8f'(a)f(b) \text{ for some } a, b \in (0, 4)$$

(ii) Putting $t = z^2$, we have

$$\int_0^4 f(t)\,dt = \int_0^2 2\,z\,f(z^2)\,dz$$

Consider the function of $\phi(t)$ given by,

$$\phi(t) = \int_0^t 2\,z\,f(z^2)\,dz;\ t \in [0, 2]$$

Clearly, $\phi(t)$ being an integral function of a continuous function, is continuous and differentiable on [0, 2].

$\therefore$ By Lagrange's mean value theorem there exists $c \in (0, 2)$ such that

$$\frac{\phi(2)-\phi(0)}{2-0} = \phi'(c)$$

$$\Rightarrow \quad \frac{\int_0^2 2z\,f(z^2)\,dz - \int_0^0 2z\,f(z^2)\,dz}{2-0} = 2cf(c^2)$$

[using $\phi'(t) = 2t\,f(t^2)$]

$$\Rightarrow \quad \int_0^2 2z\,f(z^2)\,dz = 4cf(c^2) \qquad \text{...(i)}$$

Also, by Intermediate mean value theorem for $c \in (0, 2)$ there exists $\alpha, \beta \in (0, 2)$ such that

$$\frac{\phi'(\alpha)+\phi'(\beta)}{2} = \phi'(c), \quad \text{where } 0 < \alpha < c < \beta < 2$$

$$\Rightarrow \quad 2\alpha f(\alpha^2) + 2\beta\,f(\beta^2) = 2\{2c\,f(c^2)\} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$\int_0^2 2z\,f(z^2)\,dz = 2\alpha\,f(\alpha^2) + 2\beta\,f(\beta^2) \quad \text{for all } 0 < \alpha, \beta < 2$$

$$\Rightarrow \quad \int_0^4 f(t)\,dt = 2(\alpha\,f(\alpha^2) + \beta\,f(\beta^2)) \quad \text{for all } 0 < \alpha, \beta < 2$$

Type 2 : **Only One Correct Option**

Example 16 The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c-1, e^{c-1})$ and $(c+1, e^{c+1})$ **[IIT JEE 2007]**

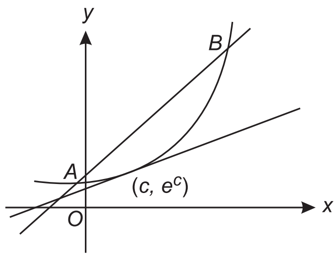
(a) on the left of $x = c$

(b) on the right of $x = c$

(c) at no point

(d) at all points

Solution. Slope of the line joining the points $(c-1, e^{c-1})$ and $(c+1, e^{c+1})$ is equal to $\dfrac{e^{c+1}-e^{c-1}}{2} > e^c$



$\Rightarrow$ Tangent to the curve $y = e^x$ will intersect the given line to the left of the line $x = c$.

Aliter : The equation of the tangent to the curve $y = e^x$ at (c, e^c) is

$$y - e^c = e^c (x - c) \quad \text{...(i)}$$

Equation of the line joining the given points is

$$y - e^{c-1} = \frac{e^c (e - e^{-1})}{2} [x - (c-1)] \quad \text{...(ii)}$$

Eliminating y from Eqs. (i) and (ii), we get

$$[x - (c-1)][2 - (e - e^1)] = 2e^{-1}$$

or

$$x - c = \frac{e + e^{-1} - 2}{2 - (e - e^{-1})} < 0 \quad \Rightarrow \quad x < c$$

Hence, (a) is the correct answer.

Type 3 : More than One Correct Options

Example 17 The coordinate of the point(s) on the graph of the function, $f(x) = \dfrac{x^3}{3} - \dfrac{5x^2}{2} + 7x - 4$ where the tangent drawn cuts off intercepts from the coordinate axes which are equal in magnitude but opposite in sign, is

(a) $\left(2, \dfrac{8}{3}\right)$ (b) $\left(3, \dfrac{7}{2}\right)$ (c) $\left(1, \dfrac{5}{6}\right)$ (d) None of these

Solution. Since, intercepts are equal in magnitude but opposite in sign

$\Rightarrow$ $$\left[\frac{dy}{dx}\right]_P = 1$$

Now, $$\frac{dy}{dx} = x^2 = 5x + 7 = 1$$

$\Rightarrow$ $$x^2 - 5x + 6 = 0$$

$\Rightarrow$ $$x = 2 \text{ or } 3$$

Hence, (a) and (b) are the correct answers.

Type 4 : Assertion and Reason

Directions
(Q. Nos. 18 to 21)

For the following questions, choose the correct answers from the option (a), (b), (c) and (d) defined as follows :

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I.

(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I.

(c) Statement I is true, Statement II is false.

(d) Statement I is false, Statement II is true.

Example 18 **Statement I** The tangent at $x = 1$ to the curve $y = x^3 - x^2 - x + 2$ again meets the curve at $x = -2$.

Statement II When a equation of a tangent solved with the curve, repeated roots are obtained at the point of tendency.

Solution. When $x = 1$, $y = 1$, $y' = 3x^2 - 2x - 1$

$$\Rightarrow \qquad y'\big|_{x=1} = 0 \text{ equation of tangent is } y = 1.$$

Solving with the curve $x^3 - x^2 - x + 2 = 1$

$$\Rightarrow \qquad x^3 - x^2 - x + 1 = 0 \quad \Rightarrow \quad x = -1, 1$$

$\therefore$ The tangent meets the curve again at $x = -1$.

$\therefore$ Statement I is false and Statement II is true.

Hence, (d) is the correct answer.

Example 19 **Statement I** The ratio of length of tangent to length of normal is inversely proportional to the ordinate of the point of tendency at the curve $y^2 = 4ax$.

Statement II Length of normal and tangent to a curve $y = f(x)$ is $\left| y\sqrt{1 + m^2} \right|$ and $\left| \dfrac{y\sqrt{1 + m^2}}{m} \right|$, Where $m = \dfrac{dy}{dx}$.

Solution. Let the slope of the tangent be denoted by tan ψ

length of tangent = y cosec ψ

length of normal = y sec ψ

$$\therefore \frac{\text{length of tangent}}{\text{length of normal}} = \cot \psi \propto \frac{1}{y}$$

$\therefore$ Statement I is true.

$$\text{Length of normal} = y \sec \psi = \left| y\sqrt{1 + m^2} \right|$$

$$\text{Length of tangent} = y \operatorname{cosec} \psi = \left| \frac{\gamma\sqrt{1 + m^2}}{m} \right|$$

$\therefore$ Statement II is true and explains Statement I.

Hence, (a) is the correct answer.

Example 20 Statement I Tangent drawn at the point (0, 1) to the curve $y = x^3 - 3x + 1$ meets the curve thrice at one point only.

Statement II Tangent drawn at the point $(1, -1)$ to the curve $y = x^3 - 3x + 1$ meets the curve at one point only.

Solution. $\dfrac{dy}{dx} = 3x^2 - 3$

Statement I $\therefore$ $\left.\dfrac{dy}{dx}\right|_{\text{at }(0, 1)} = -3$

$\therefore$ Equation of the tangent is $y - 1 = -3(x - 0)$ *ie*, $y = -3y + 1$

$$-3x + 1 = x^3 - 3x + 1 \Rightarrow x = 0$$

$\therefore$ The tangent meets the curve at one point only.

$\therefore$ Statement is true.

Statement II $\left.\dfrac{dy}{dx}\right|_{\text{at }(1, 1)} = 0$

$\therefore$ Equation of the tangent is $y + 1 = 0(x - 1)$

ie, $y = -1$

$$-1 = x^3 - 3x + 1 \Rightarrow x^3 - 3x + 2 = 0$$

$\Rightarrow$ $(x - 1)(x^2 + x - 2) = 0$

The tangent meets the curve at two points.

$\therefore$ Statement II is false.

Hence, (c) is the correct answer.

Example 21 Statement I f is a differentiable function such that $\lim\limits_{x \to \infty} f(x) = a$ where a is a finite real number, then $\lim\limits_{x \to \infty} x f'(x)$ if exists, then it is equal to zero.

Statement II Applying Langrange's Mean Value Theorem

$f(N) - f(n) = (N - n) f'(\varepsilon_0)$ Where $\varepsilon_0 \in (n, N)$.

Solution. Clearly, Statement I is true, Reason is also true, and Statement II is a correct explanation for Statement I.

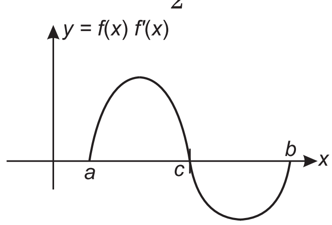
Hence, (a) is the correct answer.

Type 5 : Linked Comprehension Based Questions

Passage I

(Q. Nos. 22 to 24)

$f(x)$ is continuous and differentiable function. Given, $f(x)$ assumes values of the form $\pm\sqrt{I}$ where I denotes set of whole numbers whenever $x = a$ or b; otherwise $f(x)$ assumes real values. Also, $f(c) = -\dfrac{3}{2}$ and $|f(a)| \le |f(b)|$.



22. The number of rational values that $f(a)+f(b)+f(c)$ can assume is/are

(a) 4 (b) 2 (c) 3 (d) 5

23. The number of values that $(f(a))^2+(f(b))^2+(f(c))^2$ can assume is

(a) 4 (b) 2
(c) 3 (d) 7

24. The possible number of triplets $(f(a), f(b), f(c))$ is/are

(a) 4 (b) 5
(c) 7 (d) 6

Solution. (Q. Nos. 22 to 24)

$$f(x)f'(x) \geq 0 \quad \text{in } [a, c] \text{ and } \leq 0 \text{ in } [c, b]$$

$$\because \quad f(c) = -\sqrt{2} \quad \text{and } f(x)f'(x) \neq 0 \text{ in } (a, c) \cup (c, b)$$

$$\Rightarrow \quad f(a), f(b) \leq 0.$$

$f(a)$	$f(b)$	$f(c)$
0	0	$-\frac{3}{2}$
0	-1	$-\frac{3}{2}$
0	$-\sqrt{2}$	$-\frac{3}{2}$
-1	-1	$-\frac{3}{2}$
-1	$-\sqrt{2}$	$-\frac{3}{2}$
$-\sqrt{2}$	$-\sqrt{2}$	$-\frac{3}{2}$

Ans. 22. (c) **23.** (a) **24.** (d)

Passage II

(Q. Nos. 25 to 29)

Let $f(x)=x^3+ax^2+bx+c$ be the given cubic polynomial and $f(x)=0$ be the corresponding cubic equation, where $a, b, c \in R$.
Now, $f'(x)=3x^2+2ax+b$
Let $D=4a^2-12b=4(a^2-3b)$ be the discriminant of the equation $f'(x)=0$. Then,

25. If $D=4a^2-12b=4(a^2-3b)$, then the discriminant of the equation $f'(x)=0$. Then,

(a) $f(x)$ has all real roots
(b) $f(x)$ has one real and two imaginary roots
(c) $f(x)$ has repeated roots
(d) None of the above

26. If $D = 4(a^2 - 3b) > 0$ and $f(x_1) \cdot f(x_2) > 0$ where x_1, x_2 are the roots of $f(x)$, then

(a) $f(x)$ has all real and distinct roots
(b) $f(x)$ has three real roots but one of the roots would be repeated
(c) $f(x)$ would have just one real root
(d) None of the above

27. If $D = 4(a^2 - 3b) > 0$ and $f(x_1) \cdot f(x_2) < 0$ where x_1, x_2 are the roots of $f(x)$, then

(a) $f(x)$ has all real and distinct roots
(b) $f(x)$ has three real roots but one of the roots would be repeated
(c) $f(x)$ would have just one real root
(d) None of the above

28. If $D = 4(a^2 - 3b) > 0$ and $f(x_1) \cdot f(x_2) = 0$ where x_1, x_2 are the roots of $f(x)$, then

(a) $f(x)$ has all real and distinct roots
(b) $f(x)$ has three real roots but one of the roots would be repeated
(c) $f(x)$ would have just one real root
(d) $f(x)$ has three real roots but all are same

29. If $D = 4(a^2 - 3b) = 0$, then

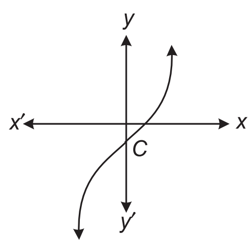
(a) $f(x)$ has all real and distinct roots
(b) $f(x)$ has three real roots but one of the roots would be repeated
(c) $f(x)$ would have just one real root
(d) None of the above

Solution. (Q. Nos. 25 to 29)

Case I If $D < 0 \quad \Rightarrow \quad f'(x) > 0, \forall\, x \in R.$

That means $f(x)$ would be an increasing function of x.

Also, $\quad \lim\limits_{x \to \infty} f(-x) = -\infty \; \lim\limits_{x \to \infty} f(x) = \infty$, thus



The graph of $f(x)$ would look like as shown in this figure.

It is clear that the graph of $f(x)$ would intersect the x-axis only once.

That means we would have just one real root, (say x_0).

Case II Clearly, $x_0 > 0$, if $c < 0$ and $x_0 < 0$ if $c > 0$.

If $D > 0$, $f'(x) = 0$ would have two real roots (say x_1 and x_2, let $x_1 < x_2$).

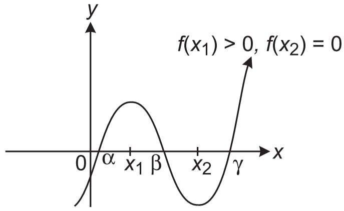
$$\Rightarrow \qquad f'(x) = 3\,(x - x_1)(x - x_2)$$

Clearly, $$f'(x) < 0,\; x \in (x_1, x_2)$$

$$f'(x) > 0,\; x \in (-\infty, x_1)\;\; (x_2, \infty)$$

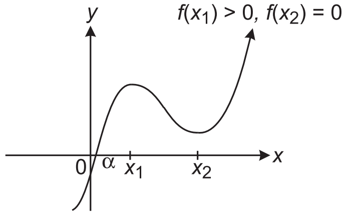
That means $f(x)$ would increase in $(-\infty, x_1)$ and (x_2, ∞) and would decrease in (x_1, x_2). Hence, $x = x_1$ would be a point of local maxima and $x = x_2$ would be a point of local minima. Thus, the graph of $y = f(x)$ could have these five possibilities.

25. Here, $f(x) = 0$ with three distinct roots $x = \alpha, \beta, \gamma$



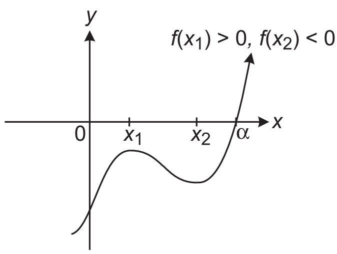
Hence, (b) is the correct answer.

26. Here, $f(x) = 0$ with one real root $x = \alpha$ and other two imaginary roots.



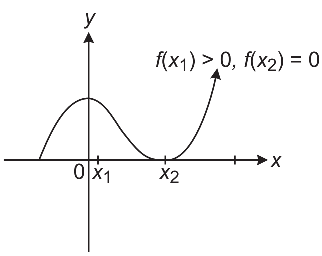
Hence, (c) is the correct answer.

27. Here, $f(x) = 0$ with one real root $x = \alpha$ and other two imaginary roots.



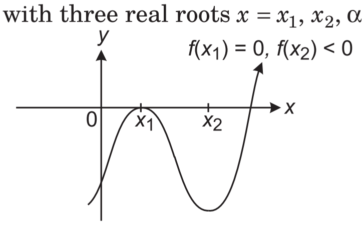
Hence, (a) is the correct answer.

28. Here, $f(x) = 0$ with three roots $x = \alpha, x_2$ (x_2 being repeated root).



Hence, (b) is the correct answer.

29. Here, $f(x) = 0$ with three real roots $x = x_1, x_2, \alpha$ (x_1 being repeated root).



Hence, (d) is the correct answer.

Thus, the following results are obtained from the above graphs :

(a) $f(x_1) \cdot f(x_2) > 0$, $f(x) = 0$ would have just one real root.

(b) $f(x_1) \cdot f(x_2) < 0$, $f(x) = 0$ would have three real and distinct roots.

(c) $f(x_1) \cdot f(x_2) = 0$, $f(x) = 0$ would have three real roots about one of the roots would be repeated.

Case III If $D = 0, f'(x) = 3(x - x_1)^2$ where x_1 is root of $f'(x) = 0$

$\Rightarrow$ $f(x) = (x - x_1)^3 + k$

$\therefore$ $f(x) = 0$ has three real roots, if $k = 0$

$f(x) = 0$ have one real root, if $k \neq 0$.

Passage III
(Q. Nos. 30 to 32)

If $y = f(x)$ is a curve and if there exists two points $A(x_1, f(x_1))$ and $B(x_2, f(x_2))$ on it such that $f'(x_1) = -\dfrac{1}{f'(x_2)} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$, then the tangent at x_1 is normal at x_2 for that curve. Now, answer the following questions

30. Number of such lines on the curve $y = \sin x$ is

(a) 1 (b) 0 (c) 2 (d) Infinite

Solution. $f(x) = y = \sin x$

$$f'(x) = \frac{dy}{dx} = \cos x$$

$\therefore$ $$\cos x_1 = -\frac{1}{\cos x_2} = \frac{\sin x_2 - \sin x_1}{x_2 - x_1}$$

ie, $\cos x_1 \cos x_2 = -1 \quad \therefore \quad \sin x_1 = \sin x_2 = 0$

$\therefore$ There is no solution.

Hence, (b) is the correct answer.

31. Number of such lines on the curve $y = |\ln x|$ is

(a) 1 (b) 2 (c) 0 (d) Infinite

Solution. $f(x) = y = |\ln x|$

$\therefore$ $$f'(x_1) = -\frac{1}{f'(x_2)} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$\Rightarrow$ $$\frac{\ln x_1}{x_1 |\ln x_1|} = -\frac{x_2 |\ln x_2|}{\ln x_2} = \frac{|\ln x_2| - |\ln x_1|}{x_2 - x_1} \qquad \ldots\text{(i)}$$

$\Rightarrow$ $\ln x_1 \cdot \ln x_2 < 0$

Let $0 < x_1 < 1$, then $1 < x_2$ and $x_1 \cdot x_2 = 1$

From Eq. (i), we get

$$-\frac{1}{x^1} = -x_2 = \frac{\ln x_2 + \ln x_1}{x_2 - x_1} = \frac{\ln x_1 x_2}{x_2 - x_1} = 0 \quad \text{which is not possible}$$

$\therefore$ There is no solution.

Hence, (c) is the correct answer.

32. Number of such lines on the curve $y^2 = x^3$ is

(a) 1 (b) 2

(c) 3 (d) 0

Solution. $y^2 = x^3$

$\therefore$ $y = \sqrt{x^3}$ or $-\sqrt{x^3}$

$$2y\frac{dy}{dx} = 3x^2$$

$$\frac{3x_1^2}{2y_1} = -\frac{2y_2}{3x_2^2}$$

$\Rightarrow$ $\frac{9}{4}x_1^2\,x_2^2 = -\,y_1y_2$

$\therefore$ $y_1y_2 < 0$

Let $y_1 = \sqrt{x_1^3}$ and $y_2 = -\sqrt{x_2^3}$

Thus, $\dfrac{3x_1^2}{2\sqrt{x_1^3}} = -\dfrac{2\sqrt{x_2^3}}{3x_2^2} = -\dfrac{-\sqrt{x^3}}{x}\;\dfrac{-\sqrt{x_1^3}}{x_1}$

$\Rightarrow$ $\dfrac{3\sqrt{x_1}}{2} = \dfrac{2}{3\sqrt{x_2}} = \dfrac{\sqrt{x_2^3} + \sqrt{x_1^3}}{x_1 - x_2}$

$\Rightarrow$ $\sqrt{x_1\,x_2} = \dfrac{4}{9}$

$\Rightarrow$ $3x_1\sqrt{x_1} - 3\sqrt{x_1}\,x_2 = 2\sqrt{x_2^3} + 2\sqrt{x_1^3}$

$\Rightarrow$ $3(\sqrt{x_1})^3 - \dfrac{3 \times 16}{31\sqrt{x_1}} = 2\cdot\dfrac{64}{729\sqrt{x_1^3}} + 2\sqrt{x_1^3}$

$\Rightarrow$ $3x_1^3 - \dfrac{16}{27}x_1 = \dfrac{128}{729} + 2x_1^2$

$\Rightarrow$ $x_1^3 - \dfrac{16}{27}x_1 = \dfrac{128}{729}$

$\Rightarrow$ $739x_1^3 - 432x_1 - 12 = 0$

Consider, $h(t) = 729x^3 - 432t - 128$

$$h'(t) = 3 \times 729t^2 - 432 = 0$$

Gives $t = \pm\dfrac{4}{9}$

$$h\left(-\frac{4}{9}\right) = 0$$

$\therefore$ There are two distinct solutions of

$$729x_1^3 - 432x_1 - 128 = 0.$$

Hence, (b) is the correct answer.

Type 7 : Integer Answer Type Questions

Example 33 Let f be a real-valued differentiable function on R (the set of all real numbers) such that $f(1)=1$. If the y-intercept of the tangent at any point $P(x, y)$ on the curve $y=f(x)$ is equal to the cube of the abscissa of P, then the value of $f(-3)$ is equal to ……… . **[IIT JEE 2010]**

Solution. **(9)** The equation of the tangent at (x, y) to the given curve $y=f(x)$ is

$$Y-y=\frac{dy}{dx}(X-x)$$

$$Y-\text{intercept}=y-x\frac{dy}{dx}$$

According to the question

$$x^3=y-x\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}-\frac{y}{x}=-x^2$$

which is linear is x

$$\text{IF}=e^{\int\frac{-1}{x}dx}=\frac{1}{x}$$

Required solution is $y\frac{1}{x}=\int -x^2\cdot\frac{1}{x}dx$

$$\frac{y}{x}=\frac{-x^2}{2}+C$$

$$y=\frac{-x^3}{2}+Cx \quad \text{at} \quad x=1, y=1$$

$$\therefore \quad 1=\frac{-1}{2}+C \quad \Rightarrow \quad C=\frac{3}{2}$$

Now, $$f(-3)=\frac{27}{2}+\frac{3}{2}(-3)$$

$$=\frac{27-9}{2}=9$$

Example 34 Suppose that $f(0)=-3$ and $f'(x)\le 5$ for all values of x. Then the largest value which $f(2)$ can assume is ………… .

Solution. **(7)** Using LMVT in [0, 2]

$$\frac{f(2)-f(0)}{2-0}=f'(c) \qquad \text{where } c\in(0,2)$$

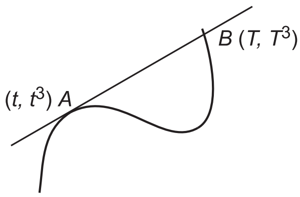
$$\frac{f(2)+3}{2}\le 5$$

$$f(2)\le 7$$

Example 35 Let C be the curve $y=x^3$ (where x assumes all real values). The tangent at A meets the curve again at B. If the gradient at B is K times the gradient at A, then K is equal to ………… .

Solution. **(4)** $\frac{dy}{dx}=3x^2=3t^2$ at 'A'

$$\therefore \qquad 3t^2 = \frac{T^3 - t^3}{T - t} = T^2 + Tt + t^2$$



$$T^2 + Tt - 2t^2 = 0$$

$$(T - t)(T + 2t) = 0$$

$$\Rightarrow \qquad T = t \quad \text{or} \quad T = -2t$$

$$(T = t \text{ is not possible})$$

Now, $$m_A = \frac{t^3}{t} = t^2; \;\; m_B = T^2$$

$$\frac{m_B}{m_A} = \frac{T^2}{t^2} = \frac{4t^2}{t^2} \qquad \text{(using } T = -2t\text{)}$$

$$m_B = 4$$

Example 36 Consider the two graphs $y = 2x$ and $x^2 - xy + 2y^2 = 28$. The absolute value of the tangent of the angle between the two curves at the points where they meet, is ....

Solution. **(2)** $y = 2x$, $x^2 - xy + 2y^2 = 28$

Solving the point of intersection are (2, 4) and $(-2, -4)$

For 1st curve, $$\frac{dy}{dx} = 2 = m_1 \qquad \text{...(i)}$$

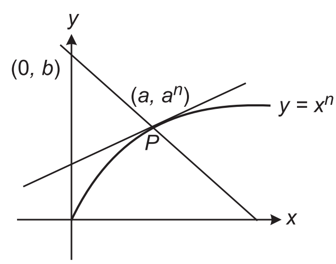
For 2nd curve, $$\frac{dy}{dx} = \frac{y - 2x}{4y - x} = 0 = m_2 \qquad \text{...(ii)}$$

$$\therefore \qquad \tan\theta = 2$$

Example 37 At the point $P(a, a^n)$ on the graph of $y = x^n$ $(n \in N)$ in the first quadrant a normal is drawn. The normal intersects the y-axis at the point $(0, b)$. If $\lim\limits_{a \to 0} b = \frac{1}{2}$, then n equals to ....

Solution. **(2)** $y = x^n$

$$\frac{dy}{dx} = n\,x^{n-1} = na^{n-1}$$



Slope of normal $$= -\frac{1}{na^{n-1}}$$

Equation of normal $$y - a^n = -\frac{1}{na^{n-1}}(x - a)$$

Put $x = 0$ to get y-intercept

$$y = a^n + \frac{1}{na^{n-2}}$$

Hence, $$b = a^n + \frac{1}{na^{n-2}}$$

$$\lim_{a \to 0} b = \begin{cases} 0, & \text{if} \quad n < 2 \\ \frac{1}{2}, & \text{if} \quad n = 2 \\ \infty, & \text{if} \quad n > 2 \end{cases}$$

$$\Rightarrow \qquad \lim_{a \to 0} b = \frac{1}{2}, \text{ if } n = 2$$

Proficiency in '*dy*/*dx* as a Rate Measurer & Tangents, Normals'

Exercise 1

Type 1 : **Only One Correct Option**

1. If $3(a+2c)=4(b+3d)\neq 0$, then the equation $ax^3+bx^2+cx+d=0$ will have

(a) no real solution
(b) at least one real root in $(-1, 0)$
(c) at least one real root in $(0, 1)$
(d) None of these

2. Let $f(x)$ be a differentiable function in the interval $(0, 2)$, then the value of $\int_0^2 f(x)\,dx$

(a) $f(c)$ where $c\in(0,2)$
(b) $2f(c)$ where $c\in(0,2)$
(c) $f'(c)$ where $c\in(0,2)$
(d) None of these

3. Let $f(x)$ be a fourth differentiable function such that $f(2x^2-1)=2xf(x),\ \forall\ x\in R$, then $f^{\text{iv}}(0)$ is equal to (where $f^{\text{iv}}(0)$ represents fourth derivative of $f(x)$ at $x=0$)

(a) 0
(b) 1
(c) -1
(d) data insufficient

4. The curve $x+y-\ln(x+y)=2x+5$ has a vertical tangent at the point (α,β). Then, $\alpha+\beta$ is equal to

(a) -1
(b) 1
(c) 2
(d) -2

5. Let $y=f(x)$, $f:R\to R$ be on odd differentiable function such that $f'''(x)>0$ and $g(\alpha,\beta)=\sin^8\alpha+\cos^8\beta+2-4\sin^2\alpha\cos^2\beta$.
If $f''(g(\alpha,\beta))=0$, then $\sin^2\alpha+\sin^2\beta$ is equal to

(a) 0
(b) 1
(c) 2
(d) 3

6. A polynomial of 6th degree $f(x)$ satisfies $f(x)=f(2-x),\ \forall\ x\in R$, if $f(x)=0$ has 4 distinct and two equal roots, then sum of the roots of $f(x)=0$ is

(a) 4
(b) 5
(c) 6
(d) 7

7. Let a curve $y=f(x)$, $f(x)\geq 0,\ \forall\ x\in R$ has property that for every point P on the curve length of subnormal is equal to abscissa of P. If $f(1)=3$, then $f(4)$ is equal to

(a) $-2\sqrt{6}$
(b) $2\sqrt{6}$
(c) $3\sqrt{5}$
(d) None of these

8. If a variable tangent to the curve $x^2 y = c^3$ makes intercepts, a, b on x and y-axes respectively, then the value of $a^2 b$ is

(a) $27\, c^3$ (b) $\frac{4}{27} c^3$ (c) $\frac{27}{4} c^3$ (d) $\frac{4}{9} c^3$

9. Let $f(x) = \begin{vmatrix} 1 & 1 & 1 \\ 3 - x & 5 - 3x^2 & 3x^3 - 1 \\ 2x^2 - 1 & 3x^5 - 1 & 7x^8 - 1 \end{vmatrix}$. Then, the equation $f(x) = 0$ has

(a) no real root
(b) at most one real root
(c) at least two real roots
(d) exactly one real root in (0, 1) and no other real root

10. The graphs $y = 2x^3 - 4x + 2$ and $y = x^3 + 2x - 1$ intersect at exactly 3 distinct points. The slope of the line passing through two of these points

(a) is equal to 4 (b) is equal to 6
(c) is equal to 8 (d) is not unique

11. In which of the following functions Rolle's theorem is applicable?

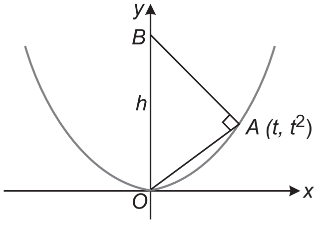
(a) $f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$ on [0, 1]

(b) $f(x) = \begin{cases} \dfrac{\sin x}{x}, & -\pi \le x < 0 \\ 0 , & x = 0 \end{cases}$ on $[-\pi, 0]$

(c) $f(x) = \dfrac{x^2 - x - 6}{x - 1}$ on $[-2, 3]$

(d) $f(x) = \begin{cases} \dfrac{x^3 - 2x^2 - 5x + 6}{x - 1}, & \text{if } x \ne 1 \\ -6 & , \text{if } x = 1 \end{cases}$ on $[-2, 3]$

12. The figure shows a right triangle with its hypotenuse OB along the y-axis and its vertex A on the parabola $y = x^2$. Let h represent the length of the hypotenuse which depends on the x-coordinate of the point A. The value of $\lim_{x \to 0} (h)$ is equal to



(a) 0 (b) 1/2 (c) 1 (d) 2

13. Number of positive integral value(s) of 'a' for which the curve $y = a^x$ intersects the line $y = x$ is

(a) 0 (b) 1
(c) 2 (d) more than 2

14. Given, $f(x)=4-\left(\frac{1}{2}-x\right)^{2/3}$, $g(x)=\begin{cases}\frac{\tan[x]}{x}, & x\neq 0\\ 1, & x=0\end{cases}$

$h(x)=\{x\}$, $k(x)=5^{\log_2(x+3)}$, then in [0, 1], Lagrange's mean value theorem is not applicable to

(a) f, g, h (b) h, k (c) f, g (d) g, h, k

where $[x]$ and $\{x\}$ denote the greatest integer and fraction part function.

15. If the function $f(x)=x^4+bx^2+8x+1$ has a horizontal tangent and a point of inflection for the same value of x, then the value of b is equal to

(a) -1 (b) 1 (c) 6 (d) -6

16. Coffee is coming out from a conical filter, with height and diameter both are 15 cm into a cylindrical coffee pot with a diameter 15 cm. The rate at which coffee comes out from the filter into the pot is 100 cu cm/min. The rate in cm/min at which the level in the pot is rising at the instance when the coffee in the pot is 10 cm, is

(a) $\frac{9}{16\pi}$ (b) $\frac{25}{9\pi}$ (c) $\frac{5}{3\pi}$ (d) $\frac{16}{9\pi}$

17. A horse runs along a circle with a speed of 20 km/h. A lantern is at the centre of the circle. A fence is there along the tangent to the circle at the point at which the horse starts. The speed with which the shadow of the horse moves along the fence at the moment when it covers 1/8 of the circle in km/h is

(a) 20 (b) 40 (c) 30 (d) 60

18. Water runs into an inverted conical tent at the rate of 20 cu ft/min and leaks out at the rate of 5 cu ft/min. The height of the water is three times the radius of the water's surface. The radius of the water surface is increasing when the radius is 5 ft, is

(a) $\frac{1}{5\pi}$ ft/min (b) $\frac{1}{10\pi}$ ft/min

(c) $\frac{1}{15\pi}$ ft/min (d) None of these

19. Let $f(x)=x^3-3x^2+2x$. If the equation $f(x)=k$ has exactly one positive and one negative solution, then the value of k equals to

(a) $-\frac{2\sqrt{3}}{9}$ (b) $-\frac{2}{9}$ (c) $\frac{2}{3\sqrt{3}}$ (d) $\frac{1}{3\sqrt{3}}$

20. The x-intercept of the tangent at any arbitrary point of the curve $\frac{a}{x^2}+\frac{b}{y^2}=1$ is proportional to

(a) square of the abscissa of the point of tangency
(b) square root of the abscissa of the point of tangency
(c) cube of the abscissa of the point of tangency
(d) cube root of the abscissa of the point of tangency

21. If $f(x)$ is continuous and differentiable over $[-2, 5]$ and $-4\leq f'(x)\leq 3$ for all x in $(-2, 5)$, then the greatest possible value of $f(5)-f(-2)$ is

(a) 7 (b) 9 (c) 15 (d) 21

22. A curve is represented parametrically by the equations $x = t + e^{at}$ and $y = -t + e^{at}$ when $t \in R$ and $a > 0$. If the curve touches the axis of x at the point A, then the coordinates of the point A are

(a) $(1, 0)$ (b) $(1/e, 0)$ (c) $(e, 0)$ (d) $(2e, 0)$

23. At any two points of the curve represented parametrically by $x = a\,(2\cos t - \cos 2t)$, $y = a\,(2\sin t - \sin 2t)$, the tangents are parallel to the axis of x corresponding to the values of the parameter t differing from each other by

(a) $2\pi/3$ (b) $3\pi/4$ (c) $\pi/2$ (d) $\pi/3$

24. Let $F(x) = \int_{\sin x}^{\cos x} e^{(1 + \arcsin t)^2}\, dt$ on $\left[0, \frac{\pi}{2}\right]$, then

(a) $F''(c) = 0$ for all $c \in \left(0, \frac{\pi}{2}\right)$ (b) $F''(c) = 0$ for some $c \in \left(0, \frac{\pi}{2}\right)$

(c) $F'(c) = 0$ for some $c \in \left(0, \frac{\pi}{2}\right)$ (d) $F(c) \neq 0$ for all $c \in \left(0, \frac{\pi}{2}\right)$

25. Given $f'(1) = 1$ and $\frac{d}{dx}(f(2x)) = f'(x), \forall\, x > 0$. If $f'(x)$ is differentiable, then there exists a number $c \in (2, 4)$ such that $f''(c)$ is equal to

(a) $-1/4$ (b) $-1/8$ (c) $1/4$ (d) $1/8$

26. Let $f(x)$ and $g(x)$ are two functions which are defined and differentiable for all $x \geq x_0$. If $f(x_0) = g(x_0)$ and $f'(x) > g'(x)$ for all $x > x_0$, then

(a) $f(x) < g(x)$ for some $x > x_0$ (b) $f(x) = g(x)$ for some $x > x_0$

(c) $f(x) > g(x)$ only for some $x > x_0$ (d) $f(x) > g(x)$ for all $x > x_0$

27. The range of values of m for which the line $y = mx$ and the curve $y = \frac{x}{x^2 + 1}$ enclose a region, is

(a) $(-1, 1)$ (b) $(0, 1)$ (c) $[0, 1]$ (d) $(1, \infty)$

28. For a steamer the consumption of petrol (per hour) varies as the cube of its speed (in km). If the speed of the water current is steady at C km/h, then the most economical speed of the steamer going against the current will be

(a) $1.25\,C$ (b) $1.5\,C$ (c) $1.75\,C$ (d) $2\,C$

29. Let S be a square with sides of length x. If we approximate the change in size of the area of S by $h \cdot \left.\frac{dA}{dx}\right|_{x = x_0}$, when the sides are changed from x_0 to $x_0 + h$, then the absolute value of the error in our approximation, is

(a) h^2 (b) $2hx_0$ (c) x_0^2 (d) h

30. Consider $f(x) = \int_1^x \left(t + \frac{1}{t}\right) dt$ and $g(x) = f'(x)$ for $x \in \left[\frac{1}{2}, 3\right]$. If P is a point on the curve $y = g(x)$ such that the tangent to this curve at P is parallel to a chord joining the points $\left(\frac{1}{2}, g\left(\frac{1}{2}\right)\right)$ and $(3, g(3))$ of the curve, then the coordinates of the point P

(a) can't be found out (b) $\left(\frac{7}{4}, \frac{65}{28}\right)$ (c) $(1, 2)$ (d) $\left(\sqrt{\frac{3}{2}}, \frac{5}{\sqrt{6}}\right)$

Type 2 : More than One Correct Options

31. If f is an odd continuous function in $[-1, 1]$ and differentiable in $(-1, 1)$, then

(a) $f'(A) = f(1)$ for some $A \in (-1, 0)$
(b) $f'(B) = f(1)$ for some $B \in (0, 1)$
(c) $n(f(A))^{n-1} f'(A) = (f(1))^n$ for some $A \in (-1, 0)$, $n \in N$
(d) $n(f(B))^{n-1} f'(B) = (f(1))^n$ for some $B \in (0, 1)$, $n \in N$

32. The parabola $y = x^2 + px + q$ intercepts the straight line $y = 2x - 3$ at a point with abscissa 1. If the distance between the vertex of the parabola and the x-axis is least, then

(a) $p = 0$ and $q = -2$
(b) $p = -2$ and $q = 0$
(c) least distance between the parabola and x-axis is 2
(d) least distance between the parabola and x-axis is 1

33. The abscissa of the point on the curve $\sqrt{xy} = a + x$, the tangent at which cuts off equal intersects from the coordinate axes is $(a > 0)$

(a) $\frac{a}{\sqrt{2}}$ (b) $-\frac{a}{\sqrt{2}}$ (c) $a\sqrt{2}$ (d) $-a\sqrt{2}$

34. If the side of a triangle vary slightly in such a way that its circumradius remains constant, then $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C}$ is equal to

(a) $6R$ (b) $2R$ (c) 0 (d) $2R(dA + dB + dC)$

35. Let $f(x)$ satisfy the requirements of Lagrange's mean value theorem in [0, 1], $f(0) = 0$ and $f'(x) \le 1 - x$, $\forall\, x \in (0, 1)$, then

(a) $f(x) \ge x$ (b) $|f(x)| \ge 1$ (c) $f(x) \le x(1 - x)$ (d) $f(x) \le 1/4$

36. For function $f(x) = \frac{\ln x}{x}$, which of the following statements are true?

(a) $f(x)$ has horizontal tangent at $x = e$
(b) $f(x)$ cuts the x-axis only at one point
(c) $f(x)$ is many-one function
(d) $f(x)$ has one vertical tangent

37. Equation of a line which is tangent to both the curves $y = x^2 + 1$ and $y = -x^2$ is

(a) $y = \sqrt{2}\,x + \frac{1}{2}$ (b) $y = \sqrt{2}\,x - \frac{1}{2}$ (c) $y = -\sqrt{2}\,x + \frac{1}{2}$ (d) $y = -\sqrt{2}\,x - \frac{1}{2}$

Type 3 : Assertion and Reason

Directions
(Q. Nos. 38 to 44)

For the following questions, choose the correct answer from the options (a), (b), (c) and (d) defined as follows :

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I.
(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I.
(c) Statement I is true, Statement II is false.
(d) Statement I is false, Statement II is true.

38. **Statement I** If $g(x)$ is a differentiable function $g(1) \neq 0$, $g(-1) \neq 0$ and Rolle's theorem is not applicable to $f(x) = \dfrac{x^2 - 1}{g(x)}$ in $[-1, 1]$, then $g(x)$ has at least one root in $(-1, 1)$.

Statement II If $f(a) = f(b)$, then Rolle's theorem is applicable for $x \in (a, b)$.

39. **Statement I** Shortest distance between $|x| + |y| = 2$ and $x^2 + y^2 = 16$ is $4 - \sqrt{2}$.

Statement II Shortest distance between the two smooth curves lies along the common normal.

40. **Statement I** If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the n real roots of a polynomial equation of nth degree with real coefficients such that sum of the roots taken $r(1 \leq r \leq n)$ at a time is positive, then all the roots are positive.

Statement II The number of times sign of coefficients change while going left to right of a polynomial equation is the number of maximum positive roots.

41. **Statement I** Tangents at two distinct points of a cubic polynomial cannot coincide.

Statement II If $P(x)$ is a polynomial of degree n $(n \geq 2)$, then $P'(x) + k$ cannot hold for n or more distinct values of x.

42. **Statement I** $f(x) = \begin{cases} \dfrac{1}{2} - x, & x < \dfrac{1}{2} \\ \left(\dfrac{1}{2} - x\right)^2, & x \geq \dfrac{1}{2} \end{cases}$. Mean value theorem is applicable in the interval $[0, 1]$.

Statement II For application of mean value theorem, $f(x)$ must be continuous in $[0, 1]$ and differentiable in $(0, 1)$.

43. Let $f(x) = \ln(2 + x) - \dfrac{2x + 2}{x + 3}$.

Statement I The function $f(x) = 0$ has a unique solution in the domain of $f(x)$.

Statement II If $f(x)$ is continuous in $[a, b]$ and is strictly monotonic in (a, b), then f has a unique root in (a, b).

44. Consider the polynomial function

$$f(x) = \frac{x^7}{7} - \frac{x^6}{6} + \frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x$$

Statement I The equation $f(x) = 0$ cannot have two or more roots.

Statement II Rolle's theorem is not applicable for $y = f(x)$ on any interval $[a, b]$, where $a, b \in R$.

Type 4 : **Linked Comprehension Based Questions**

Passage I

(Q. Nos. 45 to 47)

We say an equation $f(x) = g(x)$ is consistent, if the curves $y = f(x)$ and $y = g(x)$ touch or intersect at at least one point. If the curves $y = f(x)$ and $y = g(x)$ do not intersect or touch, then the equation $f(x) = g(x)$ is said to be inconsistent ie, has no solution.

45. The equation $\cos x + \cos^{-1} x = \sin x + \sin^{-1} x$ is

(a) consistent and has infinite number of solutions
(b) consistent and has finite number of solutions
(c) inconsistent
(d) None of the above

46. The equation $\sin x = x^2 + x + 1$ is

(a) consistent and has infinite number of solutions
(b) consistent and has finite (many) number of solutions
(c) inconsistent
(d) consistent and has unique solution

47. Among the following equations which is consistent in $(0, \pi / 2)$?

(a) $\sin x + x^2 = 0$ (b) $\cos x = x$ (c) $\tan x = x$ (d) All of these

Passage II

(Q. Nos. 48 to 50)

To find the point of contact $P \equiv (x_1, y_1)$ of a tangent to the graph of $y = f(x)$ passing through origin O, we equate the slope of tangent to $y = f(x)$ at P to the slope of OP. Hence, we solve the equation $f'(x_1) = \dfrac{f(x_1)}{x_1}$ to get x_1 and y_1.

48. The equation $|\ln mx| = px$ where m is a positive constant has a single root for

(a) $0 < p < \dfrac{m}{e}$ (b) $p < \dfrac{e}{m}$ (c) $0 < p < \dfrac{e}{m}$ (d) $p > \dfrac{m}{e}$

49. The equation $|\ln mx| = px$ where m is a positive constant has exactly two roots for

(a) $p = \dfrac{m}{e}$ (b) $p = \dfrac{e}{m}$
(c) $0 < p \le \dfrac{e}{m}$ (d) $0 < p \le \dfrac{m}{e}$

50. The equation $|\ln mx| = px$ where m is a positive constant has exactly three roots for

(a) $p < \dfrac{m}{e}$ (b) $0 < p < \dfrac{m}{e}$
(c) $0 < p < \dfrac{e}{m}$ (d) $p < \dfrac{e}{m}$

Type 5 : **Match the Columns**

51. Match the statements of Column I with values of Column II.

Column I	Column II
(A) The equation $x \log x = 3 - x$ has at least one root in	(p) (0, 1)
(B) If $27a + 9b + 3c + d = 0$, then the equation $4ax^3 + 3bx^2 + 2cx + d = 0$ has at least one root in	(q) (1, 3)
(C) If $c = \sqrt{3}$ and $f(x) = x + \frac{1}{x}$, then interval of x in which LMVT is applicable for $f(x)$ is	(r) (0, 3)
(D) If $c = \frac{1}{2}$ and $f(x) = 2x - x^2$, then interval of x in which LMVT is applicable for $f(x)$ is	(s) $(-1, 1)$

52. Match the statements of Column I with values of Column II.

Column I	Column II
(A) A circular plate is expanded by heat from radius 5 cm to 5.06 cm. Approximate increase in area is	(p) 4
(B) If an edge of a cube increases by 1%, then percentage increase in volume is	(q) $0.6\,\pi$
(C) If the rate of decrease of $\frac{x^2}{2} - 2x + 5$ is twice the rate of decrease of x, then x is equal to (rate of decrease is non-zero)	(r) 3
(D) Rate of increase in area of equilateral triangle of side 15 cm, when each side is increasing at the rate of 0.1 cm/s, is	(s) $\frac{3\sqrt{3}}{4}$

Type 6 : **Integer Answer Type Questions**

53. Consider the function $f(x) = 8x^2 - 7x + 5$ on the interval $[-6, 6]$. The value of c that satisfies the conclusion of the mean value theorem, is …… .

54. Suppose that f is differentiable for all x and that $f'(x) \le 2$ for all x. If $f(1) = 2$ and $f(4) = 8$, then $f(2)$ has the value equal to ……… .

Proficiency in '*dy*/*dx* as a Rate Measurer & Tangents, Normals'

Exercise 2

1. A particle moves along the curve, $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as the x-coordinate.

2. The curve $y = ax^3 + bx^2 + cx + 5$ touches the x-axis at the point $(-2, 0)$ and intercepts y-axis at the point Q where its gradient is 3. Find a, b and c.

3. Find the equation of tangents at the origin to the curve $y^2 = x^2(1 + x)$.

4. If tangents are drawn from origin to the curve $y = \sin x$. Prove their point of contact lie on $x^2y^2 = x^2 - y^2$.

5. Show that the curve $x = 1 - 3t^2$, $y = t - 3t^3$ is symmetrical about x-axis and has no real points for $x > 1$. If the tangent at the point 't' is inclined at an angle θ to the x-axis, prove that $3t = \tan\theta + \sec\theta$. If the tangent at $P(-2, 2)$ meets the curve again at Q, prove that tangent at P and Q are mutually perpendicular.

6. On the graph of function $f(x) = \dfrac{3}{\sqrt{2}} x \log_e x$, $x \in (e^{-1.5}, \infty)$, find the point P such that segment of tangent at point P between y-axis and point P is of least length.

7. Show that a tangent to an ellipse whose segment intercepted by the axes is the shortest, is divided at the point of tangency into two parts respectively, is equal to the semi-axes of the ellipse.

8. Tangents are drawn from the origin to the curve $y = \sin x$. Prove that points of contact lie on $y^2 = \dfrac{x^2}{1 + x^2}$.

9. If f is a continuous function with $\int_0^x f(t)\,dt \to \infty$ as $|x| \to \infty$, then show that every line $y = mx$ intersects the curve $y^2 + \int_0^x f(t)\,dt = 2$.

10. A man is standing on a straight bridge over a river and another man on a boat on the river just below the man on the bridge. If the first man starts walking at the uniform speed of 4 m/min and the boat moves perpendicularly towards the bridge at the speed of 5 m/min, then at what rate are they separating after 4 min, if the height of the bridge above the boat is 3 m.

11. Find the equation of the straight line which is a tangent at one point and normal at another point to the curve $y = 8t^3 - 1$, $x = 4t^2 + 3$.

12. Let a curve $y = f(x)$ pass through (1, 1) at any point P on the curve, tangent and normal to the curve are drawn to intersect the x-axis at Q and R respectively. If $QR = 2$, find the equation of all such possible curves.

13. If the tangent at (a, b) to the curve $x^3 + y^3 = c^3$ meet the curve again at (a_1, b_1), then prove that $\frac{a_1}{a} + \frac{b_1}{b} + 1 = 0$.

14. Show that the angle between the tangents at any point P and the line joining P to the origin 'O' is the same at all points of the curve $\log(x^2 + y^2) = c \tan^{-1}(y/x)$, where c is constant.

15. If the equation of two curves is $y^2 = 4ax$ and $x^2 = 4ay$

(i) Find the angle of intersection of two curves.

(ii) Find the equation of common tangents to these curves.

16. A straight line intersects the three concentric circles at A, B, C. If the distance of the line from the centre of the circles is 'P', prove that the area of the triangle formed by the tangents to the circle at A, B, C is $\left(\frac{1}{2P} \cdot AB \cdot BC \cdot CA\right)$.

17. Find the equation of all possible curves such that length of intercept made by any tangent on x-axis is equal to the square of x-coordinate of the point of tangency. Given that the curve passes through $(2, 1)$.

18. The tangent to the curve $y = x - x^3$ at a point P meets the curve again at Q. Prove that one point of trisection of PQ lies on the y-axis. Find the locus of the other points of trisection.

19. Determine all the curves for which the ratio of the length of the segment intercepted by any tangent on the y-axis to the length of the radius vector is constant.

20. If t be a real number satisfying $2t^3 - 9t^2 + 30 - a = 0$, then the values of the parameter a for which the equation $x + \frac{1}{x} = t$ gives six real and distinct values of x.

Answers

Target Exercise 7.1

1. (a)	**2.** (c)	**3.** (b)	**4.** (a)	**5.** (b)
6. (b)	**7.** (d)	**8.** (c)	**9.** (b)	**10.** (a)

Target Exercise 7.2

1. (b)	**2.** (d)	**3.** (a)	**4.** (b)	**5.** (c)

Exercise 1

1. (b)	**2.** (b)	**3.** (a)	**4.** (b)	**5.** (b)	**6.** (c)	**7.** (b)	**8.** (c)	**9.** (c)	**10.** (c)
11. (d)	**12.** (c)	**13.** (b)	**14.** (a)	**15.** (d)	**16.** (d)	**17.** (b)	**18.** (a)	**19.** (a)	**20.** (c)
21. (d)	**22.** (d)	**23.** (a)	**24.** (b)	**25.** (b)	**26.** (d)	**27.** (b)	**28.** (b)	**29.** (a)	**30.** (d)
31. (a, b, d)		**32.** (b, d)		**33.** (a, b)		**34.** (c, d)		**35.** (c, d)	
36. (a, b, c)		**37.** (a, c)		**38.** (c)	**39.** (d)	**40.** (a)	**41.** (a)	**42.** (d)	**43.** (c)
44. (a)	**45.** (b)	**46.** (c)	**47.** (b)	**48.** (d)	**49.** (a)	**50.** (b)			

51. (A) $\to$ (q); (B) $\to$ (r); (C) $\to$ (q); (D) $\to$ (p)

52. (A) $\to$ (q); (B) $\to$ (r); (C) $\to$ (p); (D) $\to$ (s)

53. (0) **54.** (4)

Exercise 2

1. (4, 11) and $\left(-4, -\frac{31}{3}\right)$ **2.** $a=-1/2,\ b=-3/4,\ c=3$ **3.** $y=\pm x$

6. $P=(e^{-4/3}, 2\sqrt{2}\,e^{-4/3})$ **10.** $\frac{164}{\sqrt{665}}$ m/min

11. $27(y+1)\mp 16\sqrt{2}=\pm\sqrt{2}(27x-105)$

12. $y=x\pm\left[\log\left|\frac{1-\sqrt{1-y^2}}{y}\right|+\sqrt{(1-y^2)}\right]$ **15.** (i) $\tan^{-1}(3/4)$ (ii) $x+y+a=0$

17. $y=\frac{x}{2(x-1)}$ and $y=\frac{3x}{2(1+x)}$ **18.** $y=x-5x^3$

19. $(y+\sqrt{x^2+y^2})\,x^{k-1}=C_1$ **20.** No real value of 'a' exists

Solutions

(*Proficiency in 'dy/dx as a Rate Measurer & Tangents, Normals' Exercise* 1)

Type 1 : **Only One Correct Option**

1. Let $f(x)=\frac{ax^4}{4}+\frac{bx^3}{3}+\frac{cx^2}{2}+dx$,

which is continuous and differentiable.

$$f(0)=0,\ f(-1)=\frac{a}{4}-\frac{b}{3}+\frac{c}{2}-d$$
$$=\frac{1}{4}(a+2c)-\frac{1}{3}(b+3d)=0$$

So, according to Rolle's theorem, there exist at least one root of $f'(x)=0$ in $(-1, 0)$.

2. Let us consider $g(t) = \int_0^t f(x)\, dx$

Applying LMVT in (0, 2)

$\Rightarrow$ $$\frac{g(2) - g(0)}{2 - 0} = g'(c), \text{ where } c \in (0, 2)$$

$\Rightarrow$ $$\int_0^2 f(x)\, dx = 2f(c), \text{ where } c \in (0, 2).$$

3. Replace x by $-x$

$\Rightarrow$ $$x[f(x) - f(-x)] = 0 \Rightarrow f(x) \text{ is an odd function}$$

$\Rightarrow$ $$f^{\text{iv}}(x) \text{ is also odd} \Rightarrow f^{\text{iv}}(0) = 0$$

4. Given, $x + y - \ln(x + y) = 2x + 5$

$\Rightarrow$ $$1 + \frac{dy}{dx} - \frac{1}{x + y}\left(1 + \frac{dy}{dx}\right) = 2 \Rightarrow \frac{dy}{dx} = \frac{x + y + 1}{x + y - 1}$$

$\Rightarrow$ $$\left.\frac{dy}{dx}\right|_{(\alpha, \beta)} = \frac{\alpha + \beta}{\alpha + \beta - 1} = \infty, \text{ when } \alpha + \beta = 1$$

5. $f''(x)$ is odd function

$\Rightarrow$ $$g(\alpha, \beta) = 0$$

$\Rightarrow$ $$(\sin^4 \alpha - 1)^2 + (\cos^4 \beta - 1)^2 + 2(\sin^2 \alpha - \cos^2 \beta)^2 = 0$$

$\Rightarrow$ $$\sin^2 \alpha + \sin^2 \beta = 1$$

6. Let α be the root of $f(x) = 0$

$\Rightarrow$ $$f(\alpha) = f(2 - \alpha) = 0$$

$f(x)$ has 4 distinct and two equal roots

$\therefore$ Sum of the roots $= 6$

7. Given, $y\dfrac{dy}{dx} = x$

$$y\, dy = x\, dx, \quad y^2 = x^2 + c$$

$$f(1) = 3 \Rightarrow 9 = 1 + c \Rightarrow c = 8$$

$\Rightarrow$ $$y^2 = x^2 + 8 \Rightarrow f(x) = \sqrt{x^2 + 8}$$

$\Rightarrow$ $$f(4) = \sqrt{16 + 8} = 2\sqrt{6}$$

8. $x^2 y = c^3$

$$x^2 \frac{dy}{dx} + 2xy = 0 \Rightarrow \frac{dy}{dx} = -\frac{2y}{x}$$

Equation of tangent at (x, y), $Y - y = -\dfrac{2y}{x}(X - x)$

$$Y = 0, \text{ gives, } X = \frac{3x}{2} = a \quad \text{and} \quad X = 0, \text{ gives, } Y = 3y = b$$

Now, $$a^2 b = \frac{9x^2}{4} \cdot 3y = \frac{27}{4} x^2 y = \frac{27}{4} c^3$$

9. $f(0) = f\ (1) = 0$ (obviously) and $f(x)$ is a polynomial of degree 10.

Therefore, by LMVT, we must have at least one root in (0, 1).

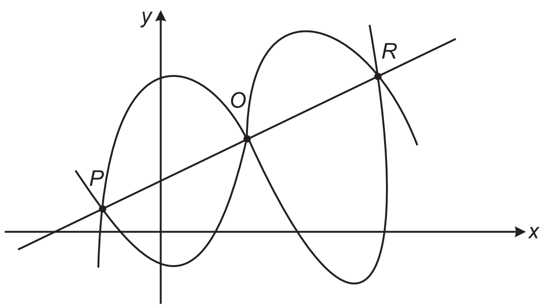
Since, the degree of $f(x)$ is even.

Hence, it has at least two real roots.

10. Let (x_1, y_1) and (x_2, y_2) are two of these points. Given, $y = x^3 + 2x - 1$ and $y = 2x^3 - 4x + 2$

$$\therefore \qquad y_1 = 2x_1^3 - 4x_1 + 2 \qquad \text{...(i)}$$

$$\text{and} \qquad 2y_1 = 2x_1^3 + 4x_1 - 2 \qquad \text{...(ii)}$$



Subtracting Eq. (i) from Eq. (ii),

$$y_1 = 8x_1 - 4 \qquad \text{...(iii)}$$

$$\text{Similarly,} \qquad y_2 = 8x_2 - 4 \qquad \text{...(iv)}$$

$$y_2 - y_1 = 8(x_2 - x_1)$$

$$\frac{y_2 - y_1}{x_2 - x_1} = 8$$

11. (a) Discontinuous at $x = 1 \quad \Rightarrow \quad$ not applicable.

(b) $f(x)$ is not continuous at $x = 0$, hence (b) is incorrect.

(c) Discontinuity at $x = 1 \quad \Rightarrow \quad$ not applicable.

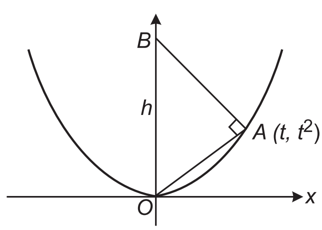
(d) Note that $x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 - x - 6)$. Hence, $f(x) = x^2 - x - 6$, if $x \neq 1$ and $f(1) = -6$

$\Rightarrow f$ is continuous at $x = 1$. So, $f(x) = x^2 - x - 6$ throughout the interval $[-2, 3]$.

Also, note that $f(-2) = f(3) = 0$. Hence, Rolle's theorem applies $f'(x) = 2x - 1$.

Setting $f'(x) = 0$, we obtain $x = 1/2$ which lies between -2 and 3.

12. Let $A = (t, t^2)$, $m_{OA} = t$, $m_{AB} = -\dfrac{1}{t}$



Equation of AB, $\quad y - t^2 = -\dfrac{1}{t}(x - t^2)$

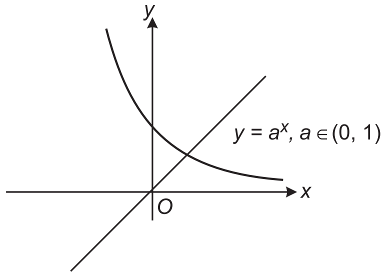
Put $\qquad x = 0$

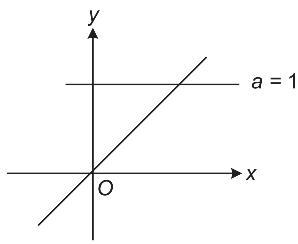
$$h = t^2 + 1$$

(as $x \to 0$, then $t \to 0$)

Now, $\qquad \lim\limits_{t \to 0}(h) = \lim\limits_{t \to 0}(1 + t^2) = 1$

13. For $0 < a \leq 1$ the line always intersects $y = a^x$ for $a > 1$ say $a = e$, consider $f(x) = e^x - x, f'(x) = e^x - 1$





$f'(x) > 0$ for $x > 0$ and $f'(x) < 0$ for $x < 0$

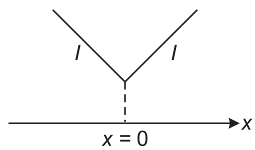
$\therefore f(x)$ is increasing $(\uparrow)$ for $x > 0$

and decreasing $(\downarrow)$ for $x < 0$

$y = e^x$ always lies above $y = x$

ie, $e^x - x \geq 1$ for $a > 1$

Hence, never intersects $= a = (0, 1]$



14. f is not differentiable at $x = \dfrac{1}{2}$

g is not continuous in [0, 1] at $x = 0$ and 1

h is not continuous in [0, 1] at $x = 1$

$$k(x) = (x + 3)^{\ln_2 5} = (x + 3)^p \text{ where } 2 < p < 3$$

15. $f'(x) = 0$ and $f''(x) = 0$ for the same $x = x_1$ (say)

Now, $$f'(x) = 4x^3 + 2bx + 8$$

$$f'(x_1) = 2[2x_1^3 + bx_1 + 4] = 0 \quad \ldots\text{(i)}$$

$$f''(x_1) = 2[6x_1^2 + b] = 0 \quad \ldots\text{(ii)}$$

From Eq. (ii), $$b = -6x_1^2$$

Substituting this value of b in Eq. (i)

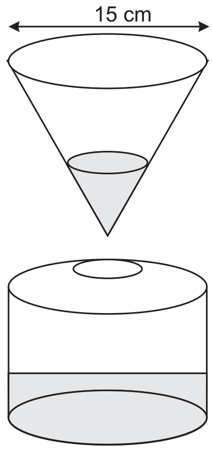
$$2x_1^3 + (-6x_1^3) + 4 = 0$$

$\Rightarrow$ $$4x_1^3 = 4$$

$\Rightarrow$ $x_1 = 1$. Hence, $b = -6$

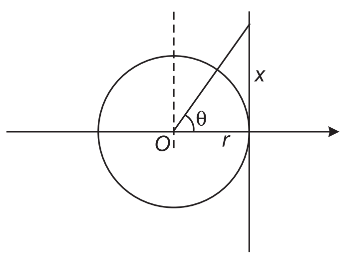
16. For a cylindrical pot $V = \pi r^2 h$

$$\frac{dV}{dt} = \pi\left[r^2\frac{dh}{dt} + h\cdot 2r\frac{dr}{dt}\right] \qquad \left(r = \text{constant}, \frac{dr}{dt} = 0\right)$$



Hence,
$$100 = \pi r^2\frac{dh}{dt}$$
$$100 = \pi\cdot\frac{225}{4}\cdot\frac{dh}{dt} \qquad \left(r = \frac{15}{2}\text{ cm}\right)$$
$$\frac{dh}{dt} = \frac{400}{225\pi} = \frac{16}{9\pi}\text{ cm/min}$$

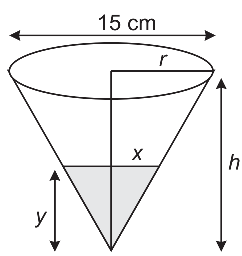
17. $\tan\theta = x/r \quad \Rightarrow \quad x = r\tan\theta$



$\Rightarrow$ $\quad dx/dt = r\sec^2\theta\,(d\theta/dt) = r\,\omega\sec^2\theta = v\sec^2\theta$

where $\quad \theta = \pi/8,\ dx/dt = v\sec^2(\pi/4) = 2v = 40\text{ km/h},\ \theta = 45°$

18. $\dfrac{dV}{dt} = 15,\ h = 3r$



$V = \dfrac{1}{3}\pi x^2 y,\ \dfrac{dx}{dt} = ?$ when $x = 5$

$\dfrac{x}{y} = \dfrac{r}{h} = \dfrac{1}{3}$

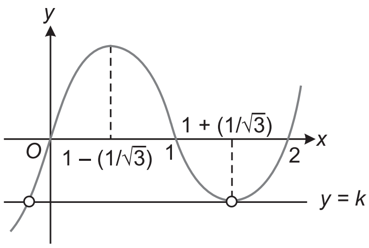
$V = \dfrac{2}{3}\pi r^2 3x = \pi x^3$

$$\frac{dV}{dt} = 3\pi x^2\frac{dx}{dt} \quad \Rightarrow \quad 15 = 3\pi\cdot 25\frac{dx}{dt} \quad \Rightarrow \quad \frac{dx}{dt} = \frac{1}{5\pi}$$

19. $f(x) = x(x^2 - 3x + 2)$

$$f(x) = x(x-2)(x-1)$$

Graph of $y = f(x)$ is shown as



Now, $f(x) = k$ to have exactly one positive and negative solution

We have, $\quad k = f\left(1 + \frac{1}{\sqrt{3}}\right)$

$1 - \frac{1}{\sqrt{3}}$ and $1 + \frac{1}{\sqrt{3}}$ are the roots of $f'(x) = 0$

$$\therefore \quad k = \underbrace{\left(1 + \frac{1}{\sqrt{3}}\right)}_{x}\underbrace{\left(\frac{1}{\sqrt{3}} - 1\right)}_{x-2}\underbrace{\left(\frac{1}{\sqrt{3}}\right)}_{x-1} = \left(\frac{1}{3} - 1\right)\left(\frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$$

20. $\frac{a}{x^2} + \frac{b}{y^2} = 1$

$$\Rightarrow \quad ay^2 + bx^2 = x^2y^2 \qquad \text{...(i)}$$

$$-\frac{2a}{x^3} - \frac{2b}{y^3}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{ay^3}{bx^3}$$

Equation of tangent $\quad Y - y = -\frac{ay^3}{bx^3}(X - x)$

for x-intercept, put $\quad Y = 0 \quad \therefore \quad X = \frac{bx^3}{ay^2} + x$

$$X = x\left[\frac{bx^2 + ay^2}{ay^2}\right] = x\left[\frac{x^2y^2}{ay^2}\right] = \frac{x^3}{a}$$

$\Rightarrow$ x-intercept is proportional to cube of abscissa.

21. Using LMVT in $[-2, 5]$

$$-4 \le \frac{f(5) - f(-2)}{7} \le 3$$

$$-28 \le f(5) - f(-2) \le 21$$

22. $x = t + e^{at}, \quad y = -t + e^{at}$

$$\frac{dx}{dt} = 1 + ae^{at}, \quad \frac{dy}{dt} = -1 + ae^{at}, \quad \frac{dy}{dx} = \frac{-1 + ae^{at}}{1 + ae^{at}}$$

At the point A, $y = 0$ and $\frac{dy}{dx} = 0$ for some $t = t_1$

$$\therefore \quad ae^{at_1} = 1 \qquad \text{...(i)}$$

$$\text{Also,} \quad 0 = -t_1 + e^{at_1} \quad \therefore \quad e^{at_1} = t_1 \qquad \text{...(ii)}$$

Putting this value in Eq. (i), we get

$$at_1 = 1 \quad \Rightarrow \quad t_1 = \frac{1}{a}, \text{ now from Eq. (i) } ae = 1 \quad \Rightarrow \quad a = \frac{1}{e}$$

Hence, $\quad x_A = t_1 + e^{at_1} = e + e = 2e \quad \Rightarrow \quad A \equiv (2e, 0)$

23. $\dfrac{dy}{dx} = \dfrac{\cos 2t - \cos t}{\sin 2t - \sin t} = 0 \quad \Rightarrow \quad \cos 2t = \cos t \quad \Rightarrow \quad \cos 2t = \cos(2\pi - t) \quad \Rightarrow \quad t = 2\pi/3$

24. $F'(x) = e^{(1 + \sin^{-1}(\cos x))^2} \cdot (-\sin x) - e^{(1+x)^2} \cdot \cos x$

$$\left.\begin{aligned} F'(0) &= 0 - e = -e \\ F'\left(\frac{\pi}{2}\right) &= -e - 0 = -e \end{aligned}\right]$$

Hence, Rolle's theorem is applicable for the function $F'(x)$.

$\Rightarrow$ There lies c in $\left(0, \dfrac{\pi}{2}\right)$ for which $F''(c) = 0$ as

Rolle's theorem is applicable for $F'(x)$ in $\left[0, \dfrac{\pi}{2}\right]$,

also $F(0) = \int_0^1 f(t)\, dt$ and $F\left(\dfrac{\pi}{2}\right) = \int_1^0 f(t)\, dt$.

Hence, $F(0)$ and $F\left(\dfrac{\pi}{2}\right)$ have opposite signs.

$\Rightarrow \quad F(c) = 0$ for some $c \in \left(0, \dfrac{\pi}{2}\right)$

25. $f'(1) = 1, 2 \cdot f'(2x) = f'(x)$

Put $\quad x = 1, f'(2) = \dfrac{f'(1)}{2} = \dfrac{1}{2}$

and $\quad f'(4) = \dfrac{1}{2} f'(2) = \dfrac{1}{4}$

Applying LMVT for $y = f'(x)$ is $[2, 4]$

$$f''(c) = \frac{f'(4) - f'(2)}{2} = \frac{\frac{1}{4} - \frac{1}{2}}{2} = -\frac{1}{8}$$

26. Consider $\phi(x) = f(x) - g(x) \Rightarrow \phi'(x) = f'(x) - g'(x) > 0$

$\phi(x)$ is also continuous and derivable in $[x_0, x]$.

Using LMVT for $\phi(x)$ in $[x_0, x]$

$$\phi'(x) = \frac{\phi(x) - \phi(x_0)}{x - x_0}$$

Since, $\phi'(x) = f'(x) - g'(x)$ are $f'(x) - g'(x) > 0$ for all $x > x_0$

$\therefore \quad \phi'(x) > 0$

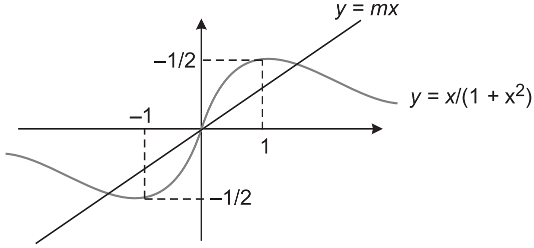
Hence, $\quad \phi(x) - \phi(x_0) > 0$

$\Rightarrow \quad \phi(x) > \phi(x_0)$

$\Rightarrow \quad \{\phi(x_0) = f(x_0) - g(x_0) = 0\}$

$\Rightarrow \quad f(x) - g(x) > 0$

27. Solving, $mx = \dfrac{x}{x^2+1} \Rightarrow x^2 + 1 = \dfrac{1}{m}$ or $x = 0$



$$x^2 = \frac{1}{m} - 1 > 0 \text{ for a region}$$

$$\frac{m-1}{m} < 0 \Rightarrow m \in (0, 1)$$

Point to Consider

For $m = 0$ or 1 the line does not enclose a region.

28. Time of journey $= \dfrac{d}{V - C}$ when V is the speed in the still water.

$\therefore$ Petrol burnt per hour $= kV^3$

$\therefore$ Fuel's cost $= \dfrac{kV^3 d}{V - C} = kd\ \dfrac{V^3}{V - C} = f(v)$ now proceed.

29. $A = x^2$, $\dfrac{dA}{dx} = 2x$. So, $\left(\left.\dfrac{dA}{dx}\right]_{x = x_0} \times h\right) = 2x_0 h$

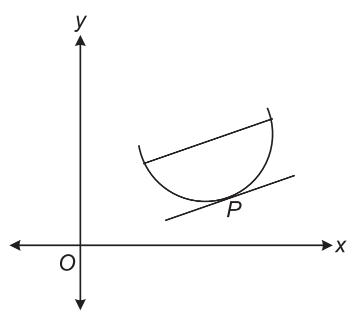
The exact change in the area of S when x is changed from x_0 to $x_0 + h$ is

$$(x_0 + h)^2 - x_0^2 = x_0^2 + 2x_0 h + h^2 - x_0^2 = 2x_0 h + h^2$$

The difference between the exact change and the approximate change, is

$$2x_0 h + h^2 - 2x_0 h = h^2$$

30. $f(x) = \int_1^x \left(t + \dfrac{1}{t}\right) dt \Rightarrow f'(x) = x + \dfrac{1}{x}$



$\therefore$ $g(x) = x + \dfrac{1}{x}$

for $x \in \left[\dfrac{1}{2}, 3\right]$

$$g\left(\frac{1}{2}\right) = 2 + \frac{1}{2} = \frac{5}{2},\ g(3) = 3 + \frac{1}{3} = \frac{10}{3}$$

Let $P \equiv (c, g(c)),\ c \in \left[\dfrac{1}{2}, 3\right]$

By LMVT, $g'(c) = \dfrac{g(3) - g\left(\frac{1}{2}\right)}{3 - \frac{1}{2}}$ $\therefore$ $1 - \dfrac{1}{c^2} = \dfrac{\frac{10}{3} - \frac{5}{2}}{3 - \frac{1}{2}}$

$\Rightarrow$ $c^2 = \dfrac{3}{2} \Rightarrow c = \sqrt{\dfrac{3}{2}}$

$\therefore$ $g(c) = \sqrt{\dfrac{3}{2}} + \dfrac{1}{\sqrt{\frac{3}{2}}} = \dfrac{5}{\sqrt{6}}$ $\therefore$ $P \equiv \left(\sqrt{\dfrac{3}{2}}, \dfrac{5}{\sqrt{6}}\right)$

Type 2 : More than One Correct Options

31. $f(-1) = -f(1), f(0) = 0$. For (a) and (b) apply LMVT for the function $f(x)$ in $(-1, 0)$ and $(0, 1)$ respectively. For (d) apply LMVT for $(f(x))^n$ in $(0, 1)$.

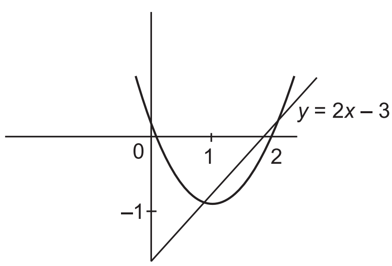
32. When $x = 1$, $y = -1$ (from the line)

Thus, it must lie on the parabola $y^2 = x^2 + px + q$

$\Rightarrow$ $-1 = 1 + p + q \Rightarrow p + q = -2$

$\therefore$ Now, distance of the vertex of the parabola from the x-axis is

$$d = f\left(-\frac{p}{2}\right) = \frac{p^2}{4} - \frac{p^2}{2} + q = q - \frac{p^2}{4}$$



Substituting $q = -2 - p$, here

$$f(p) = -2 - p - \frac{p^2}{4}$$

Hence, $f'(p) = -1 - \dfrac{p}{2} = 0 \Rightarrow p = -2$

Hence, $q = 0$

Note the least distance of the vertex from x-axis is 1.

33. $xy = a^2 + x^2 + 2ax$

$\Rightarrow$ $y = \dfrac{a^2}{x} + x + 2a$

$\Rightarrow$ $\dfrac{dy}{dx} = -\dfrac{a^2}{x^2} + 1 = -1$

$\Rightarrow$ $2x^2 = a^2 \Rightarrow x = \pm \dfrac{a}{\sqrt{2}}$

34. Given, $\dfrac{a}{\sin A} + \dfrac{b}{\sin B} + \dfrac{c}{\sin C} = 2R$ (say)

$\therefore$ $da = 2R \cos A \, dA$, $db = 2R \cos B \, dB$,

$dc = 2R \cos C \, dC$

$$\therefore \quad \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R\,(dA + dB + dC) \quad \text{...(i)}$$

Also, $A + B + C = \pi$

So, $dA + dB + dC = 0$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0$$

35. $\dfrac{f(x) - f(0)}{x - 0} = f'(e) \le 1 - x$ for some

$$c \in (0, 1) \quad \Rightarrow \quad f(x) \le x\,(1 - x) \le 1/4$$

36. $f(x) = \dfrac{\ln x}{x}$ ...(i)

$\therefore$ Domain is R^+.

$$\because \quad f'(x) = \frac{x \cdot \dfrac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

(a) For horizontal tangent $f'(x) = 0$

$\Rightarrow$ $\ln x = 1 \quad \Rightarrow \quad x = e$ (True)

(b) If Eq. (i) intersects the x-axis

$\Rightarrow$ $\dfrac{\ln x}{x} = 0 \quad \Rightarrow \quad x = 1$ (True)

(c) $f'(x)$ is positive, if $x \in (0, e)$ and $f'(x)$ negative, if $x \in (e, \infty)$

$\therefore$ $f(x)$ is not monotonic.

$\therefore$ $f(x)$ is many-one.

(d) For vertical tangent $f'(x) = \infty$

$\Rightarrow$ $\dfrac{1 - \ln x}{x^2} = \infty \quad \Rightarrow \quad \dfrac{x^2}{1 - \ln x} = 0$

$\Rightarrow$ $x = 0$ which is not in the domain of $f(x)$ (False)

37. Let the tangent line be $y = ax + b$

The equation for its intersection with the upper parabola is

$$x^2 + 1 = ax + b$$

$$x^2 - ax + (1 - b) = 0$$

This has a double root when $a^2 - 4\,(1 - b) = 0$

or $a^2 + 4b = 4$

For the lower parabola $ax + b = -x^2$

$$x^2 + ax + b = 0$$

This has a double root when $a^2 - 4b = 0$

Subtract these two equations to get $8b = 4$ or $b = \dfrac{1}{4}$

Add them to get $2a^2 = 4$ or $a = \pm\sqrt{2}$

The tangent lines are $y = \sqrt{2}\,x + \dfrac{1}{2}$ and $y = -\sqrt{2}\,x + \dfrac{1}{2}$

Aliter : Let the common tangent is $y = mx + c$

Solving with first curve, we get

$$x^2 + 1 = mx + c$$
$$x^2 - mx + 1 - c = 0$$
$$D = 0, \quad m^2 = 4(1 - c) \qquad \text{...(A)}$$

Solving with second curve

$$mx + c = x^2$$
$$x^2 + mx + c = 0$$
$$D = 0, \; m^2 = 4c \qquad \text{...(B)}$$

From Eqs. (A) and (B), we get $8c = 4$, $c = \dfrac{1}{2}$, $m = \pm\sqrt{2}$

The tangent lines are $y = \sqrt{2}\,x + \dfrac{1}{2}$ and $y = -\sqrt{2}\,x + \dfrac{1}{2}$.

Type 3 : **Assertion and Reason**

38. Statement I as $f(-1) = f(1)$ and Rolle's theorem is not applicable, then it implies that $f(x)$ is either discontinuous or $f'(x)$ does not exist at at least one point in $(-1, 1)$.

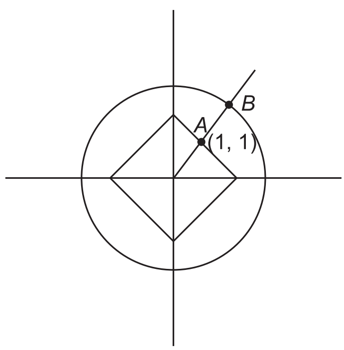
$\Rightarrow$ $g(x) = 0$ at at least one value of x in $(-1, 1)$.

Statement II is false. Consider the example in Statement I.

39. Common normal is $y = x$

Solving, $x + y = 2 \Rightarrow (1, 1)$

and $x^2 + y^2 = 16 \Rightarrow (2\sqrt{2}, 2\sqrt{2})$



The distance between is $AB = 4 - \sqrt{2}$.

But as the curves are not smooth, check at slope points. The coordinates in 1st quadrant are (2, 0) and (4, 0) and here distance = 2

$\therefore$ $4 - \sqrt{2}$ is not the shortest.

40. If $P(x)=0$ is a polynomial equation, then $P(-x)=0$ has no positive root.

$\Rightarrow \quad P(x)=0$ cannot have negative roots.

41. Let $A(a, P(a)), B(b, P(b))$, then slope of $AB = P'(a) = P'(b)$ from LMVT $\exists\, c \in (a, b)$ where $P'(c) =$ slope of AB.

42. $f(x)=\begin{cases} \dfrac{1}{2}-x, & x<\dfrac{1}{2}, \\ \left(\dfrac{1}{2}-x\right)^2, & x \geq \dfrac{1}{2} \end{cases} \quad f'(x)=\begin{cases} -1, & x<\dfrac{1}{2} \\ 2\left(\dfrac{1}{2}-x\right)(-1), & x \geq \dfrac{1}{2} \end{cases}$

Left hand derivative at $x = 1/2$ is (-1) and right hand derivative at $x = 1/2$ is 0, so the function is not differentiable at $x = 1/2$.

43. $f(x) = \ln(2+x) - \dfrac{2x+2}{x+3}$ is continuous in $(-2, \infty)$.

$$f'(x) = \frac{1}{x+2} - \frac{4}{(x-3)^2} = \frac{(x+3)^2 - 4(x+2)}{(x+2)(x+3)^2}$$

$$= \frac{x^2+2x+1}{(x+2)(x+3)^2} = \frac{(x+1)^2}{(x+2)(x+3)^2} > 0 \qquad [f'(x)=0 \text{ at } x=-1]$$

$\Rightarrow \quad f$ is increasing in $(-2, \infty)$.

Also, $\quad \lim_{x \to -2^+} f(x) \to -\infty$ and $\lim_{x \to \infty} f(x) \to \infty \quad \Rightarrow$ unique root.

44. Let $f(x)=0$ has two roots say $x = r_1$ and $x = r_2$ where $r_1 r_2 \in [a, b]$.

$$\Rightarrow \quad f(r_1) = f(r_2)$$

Hence, $\exists$ there must exist some $c \in (r_1, r_2)$, where $f'(c) = 0$

But $\quad f'(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$ for $x \geq 1$,

$\quad f'(x) = (x^6 - x^5) + (x^4 - x^3) + (x^2 - x) + 1 > 0$ for $x \leq 1$,

$\quad f'(x) = (1-x) + (x^2 - x^3) + (x^4 - x^5) + x^6 > 0$

Hence, $f'(x) > 0$ for all x.

$\therefore \quad$ Rolle's theorem fails.

$\Rightarrow \quad f(x)=0$ cannot have two or more roots.

Type 4 : **Linked Comprehension Based Questions**

Solutions (Q. Nos. 45 to 47)

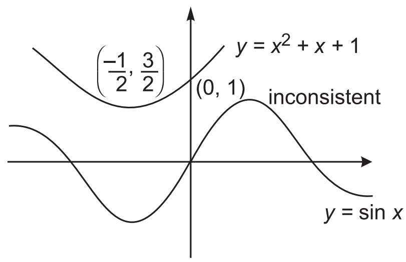
45. Let $f(x) = \cos x - \sin x + \cos^{-1} x - \sin^{-1} x, x \in [-1, 1]$

$$\because \quad f(-1) f(1) < 0$$

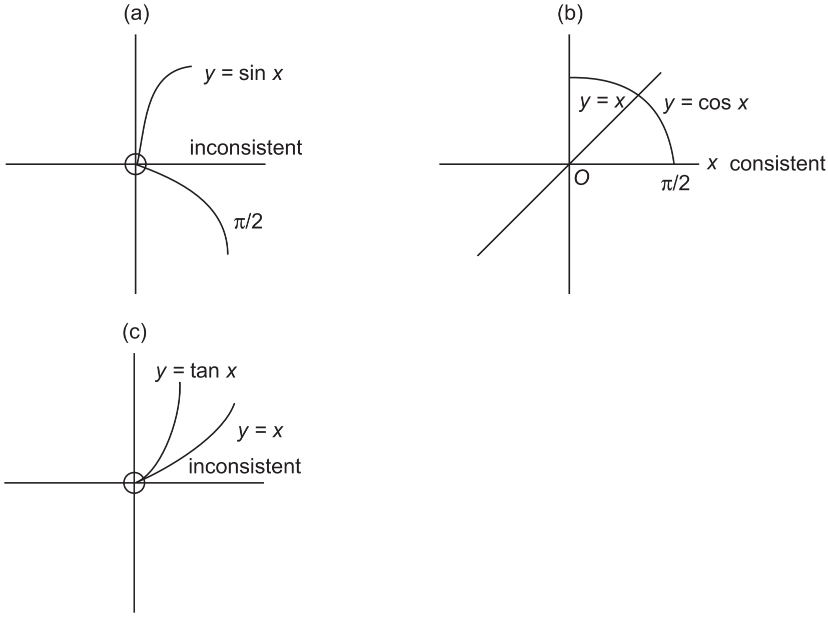
$\Rightarrow \quad \exists$ at least $c \in (-1, 1)$ such that $f(c) = 0$

$\Rightarrow \quad$ The curves $y = \cos x + \cos^{-1} x$ and $y = \sin x + \sin^{-1} x$ intersect each other at at least one point.

46. $\sin x = x^2 + x + 1 = \left(x + \dfrac{1}{2}\right)^2 + \dfrac{3}{4}$



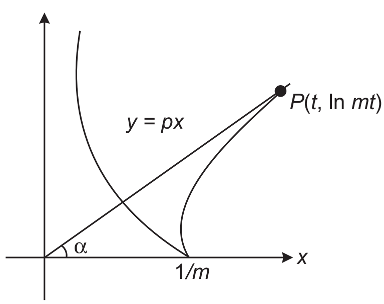
47.



Solutions (Q. Nos. 48 to 50)

Slope of tangent at P = slope of OP

$$\Rightarrow \qquad \frac{1}{t} = \frac{\ln mt}{t} \quad \Rightarrow \quad t = \frac{e}{m}$$



$$\Rightarrow \qquad P \equiv \left(\frac{e}{m}, 1\right) \quad \Rightarrow \quad \tan\alpha = p = \frac{m}{e}$$

Type 5 : Match the Columns

51. (A) $f'(x) = \log x - \dfrac{3}{x} + 1$

$\Rightarrow \quad f(x) = (x-3)\log x + c \quad \therefore \quad f(1) = f(3)$

(B) $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$\therefore \quad f(x) = ax^4 + bx^3 + cx^2 + dx + e$

$\because \quad f(0) = f(3) \Rightarrow 27a + 9b + 3c + d = 0$

(C) $\dfrac{f(b) - f(a)}{b - a} = f'(\sqrt{3}) = \dfrac{2}{3}$

$\Rightarrow \quad \dfrac{ab - 1}{ab} = \dfrac{2}{3}$

(D) $\dfrac{f(b) - f(a)}{b - a} = f'\left(\dfrac{1}{2}\right) \Rightarrow a + b = 1$

52. (A) $r = 5$ cm, $\delta r = 0.06$

$$A = \pi r^2 = 10\pi \times 0.06, \qquad \delta A = 2\pi r\,\delta r = 0.6\,\pi$$

(B) $v = x^3$, $\delta v = 3x^2\,\delta x$

$$\frac{\delta v}{v} \times 100 = 3\,\frac{\delta x}{x} \times 100 = 3 \times 1 = 3$$

(C) $(x-2)\dfrac{dx}{dt} = 2\,\dfrac{dx}{dt} \quad x = 4$

(D) $A = \dfrac{\sqrt{3}}{4}x^2$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2} \times \frac{dx}{dt} = \frac{\sqrt{3}}{2} \cdot 15 \cdot \frac{1}{10} = \frac{3\sqrt{3}}{4}$$

Type 6 : Integer Answer Type Questions

53. $f'(c) = 16c - 7 = \dfrac{f(6) - f(-6)}{12}$

$$= \frac{(8\cdot 36 - 7\cdot 6 + 5) - (8\cdot 36 + 7\cdot 6 + 5)}{12} = -\frac{2\cdot 7\cdot 6}{12} = -7$$

$$16c = 0 \Rightarrow c = 0$$

54. Using LMVT for f is [1, 2],

$$\forall\, c \in (1, 2) \qquad \frac{f(2) - f(1)}{2 - 1} = f'(c) \le 2$$

$$f(2) - f(1) \le 2 \quad \Rightarrow \quad f(2) \le 4 \qquad \text{...(i)}$$

Again, using LMVT in [2, 4],

$$\forall\, d \in (2, 4) \qquad \frac{f(4) - f(2)}{4 - 2} = f'(d) \le 2$$

$\therefore \quad f(4) - f(2) \le 4, \quad 8 - f(2) \le 4, \quad 4 \le f(2)$

$\Rightarrow \quad f(2) \ge 4 \qquad$ ...(ii)

From Eqs. (i) and (ii), we get $f(2) = 4$

(*Proficiency in 'dy/dx as a Rate Measurer & Tangents, Normals' Exercise* 2)

5. Since, $t \to -t$, does not change the value of x, then the curve is symmetrical about x-axis.

For $x > 1$

$$1 - 3t^2 > 1 \quad \Rightarrow \quad -3t^2 > 0$$

So, there is no real points for $x > 1$.

Now, $$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$\frac{dy}{dx} = \frac{1 - 9t^2}{-6t} = \tan\theta \qquad \text{(given)} \quad \text{...(i)}$$

$$\sec^2\theta = 1 + \tan^2\theta$$

$$= 1 + \frac{(1 + 9t^2)^2}{-36t^2} \qquad \text{[from Eq. (i)]}$$

$$= \frac{(1 + 9t^2)^2}{36t^2}$$

$\Rightarrow$ $$\sec\theta = \frac{1 + 9t^2}{6t} \qquad \text{...(ii)}$$

Adding Eqs. (i) and (ii)

$$\tan\theta + \sec\theta = \frac{1 - 9t^2}{-6t} + \frac{1 + 9t^2}{6t} = 3t$$

$\because$ $P(-2, 2)$; $x = -2$, $y = 2$

$\Rightarrow$ $1 - 3t^2 = -2$, $t - 3t^3 = 2$ after solving $t = -1$

$\therefore$ $$\left(\frac{dy}{dx}\right)_{t=-1} = \frac{1 - 9}{6} = \frac{-4}{3}$$

Tangent at $(-2, 2)$

$$Y - 2 = \frac{-4}{3}(X + 2)$$

$\Rightarrow$ $$t - 3t^3 - 2 = \frac{-4}{3}(1 - 3t^2 + 2)$$

$\Rightarrow$ $$(t + 1)^2 (3t - 2) = 0$$

$\therefore$ Tangent at $t = -1$ meets the curve again at $t = 2/3$, $Q = (-1/3, -2/9)$

$\therefore$ $$\left(\frac{dy}{dx}\right)_{(t = 2/3)} = \frac{1 - 9(4/9)}{-6(2/3)} = \frac{3}{4}$$

Hence, $$\left(\frac{dy}{dx}\right)_{(t=-1)} \left(\frac{dy}{dx}\right)_{(t = 2/3)} = -1$$

Hence, the tangents at P and Q are at right angle.

6. The slope of the tangent to the curve is $\dfrac{dy}{dx} = \dfrac{3}{\sqrt{2}}(1 + \log x)$

Equation of tangent at $\left(x, \dfrac{3}{\sqrt{2}} x \log x\right)$ is;

$$Y - \frac{3}{\sqrt{2}} x \log x = \frac{3}{\sqrt{2}}(1 + \log x)(X - x)$$

The lines intersects the y-axis (*ie*, $x = 0$) at the point $\left(0, -\dfrac{3}{\sqrt{2}} x\right)$.

The square of distance between this point and $P\left(x, \dfrac{3x}{\sqrt{2}} \log x\right)$ is

$$S = x^2 + \left(\frac{3}{\sqrt{2}} x \log x + \frac{3}{\sqrt{2}} x\right)^2 = x^2\left[\frac{9}{2}(1 + \log x)^2 + 1\right]$$

$$\Rightarrow \quad \frac{dS}{dx} = 2x\left(1 + \frac{9}{2}(1 + \log x)^2\right) + 9x\,[1 + \log x]$$

$$= x\,[2 + 9(1 + \log x)^2] + 9x\,[1 + \log x]$$

For minimum, $\dfrac{dS}{dx}$ must be zero.

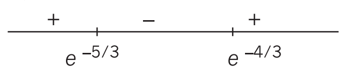
Excluding $x = 0$ as it lies outside the given interval, then we get

$$9(1 + \log x)^2 + 9(1 + \log x) + 2 = 0$$

$$\Rightarrow \quad 1 + \log x = \frac{-9 \pm \sqrt{81 - 72}}{18} = -\frac{2}{3}, -\frac{1}{3}$$

$$\Rightarrow \quad \log x = -\frac{5}{3}, -\frac{4}{3} \quad \Rightarrow \quad x = e^{-5/3}, e^{-4/3}$$

Using number line rule,



Clearly, $x = e^{-4/3}$ is point of minima.

Hence, S is minimum at $x = e^{-4/3}$ and the required point P is

$$(e^{-4/3}, -2\sqrt{2}\, e^{-4/3})$$

7. Equation of ellipse is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Equation of tangent at $(a\cos\theta, b\sin\theta)$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

Intercept on the x-axis $= (a\sec\theta)$

Intercept on the y-axis $= (b\,\mathrm{cosec}\,\theta)$

Length of intercept of the tangent by the axes $= \sqrt{a^2\sec^2\theta + b^2\,\mathrm{cosec}^2\theta}$

Let $\quad l = a^2\sec^2\theta + b^2\,\mathrm{cosec}^2\theta$

$$\frac{dl}{d\theta} = 2a^2\sec^2\theta\tan\theta - 2b^2\,\mathrm{cosec}^2\theta\cot\theta$$

$$a^2\sin^4\theta = b^2\cos^4\theta$$

$$\Rightarrow \qquad \frac{a}{b} = \cot^2\theta$$

$$\Rightarrow \qquad \sin^2\theta = \frac{b}{a+b}, \ \cos^2\theta = \frac{a}{a-b}$$

Distance between $(a \sec\theta, 0)$ and point of tangency $(a\cos\theta, b\sin\theta)$ is

$$= \sqrt{a^2(\sec\theta - \cos\theta)^2 + b^2\sin^2\theta} = \sqrt{a^2\cos^2\theta(\sec^2\theta - 1)^2 + b^2\sin^2\theta}$$

$$= \sqrt{a^2\cos^2\left(\frac{a+b}{a} - 1\right)^2 + b^2\sin^2\theta} = \sqrt{a^2\cos^2\theta\,\frac{b^2}{a^2} + b^2\sin^2\theta}$$

$$= b$$

Similarly, distance between $(0, b\operatorname{cosec}\theta) = a$

8. Let (x_1, y_1) be a point of contact of tangents from the origin (0, 0).

Here, $y = \sin x$

$$\therefore \qquad \frac{dy}{dx} = \cos x; \qquad \therefore \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = (\cos x_1)$$

$\therefore$ Equation of tangent at (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x - x_1)$$

or $\qquad y - y_1 = (\cos x_1)(x - x_1)$

It passes through (0, 0).

So, $\qquad -y_1 = (\cos x_1)(-x_1) \qquad$ ...(i)

Also, (x_1, y_1) is on the curve.

So, $\qquad y_1 = \sin x_1 \qquad$ ...(ii)

Squaring and adding Eqs. (i) and (ii), we get

$$\left(\frac{y_1}{x_1}\right)^2 + y_1^2 = \cos^2 x_1 + \sin^2 x_1 = 1$$

or $\qquad y_1^2 + x_1^2 y_1^2 = x_1^2 \qquad$ *ie*, $\quad (x_1^2 + 1)y_1^2 = x_1^2$

$\therefore$ The point of contact (x_1, y_1) lies on the curve, $y^2 = \dfrac{x^2}{x^2+1}$.

9. We must show that for a given $m \in R$ these exists $x \in R$ such that,

$$m^2x^2 + \int_0^x f(t)\,dt = 2$$

Let $\qquad f(x) = \int_0^x [2m^2t + f(t)]\,dt - 2, \quad x \in R$

Since, $f(x)$ is continuous at $2m^2t^2$ is continuous,

$$\int_0^x [2m^2t + f(t)]\,dt \text{ continuous on } R,$$

$\therefore \quad f$ is continuous on R, also

$$f(0) = \int_0^0 [2m^2t + f(t)]\,dt - 2 = 0 - 2 = -2$$

and $\quad f(x) = \int_0^x [2m^2 t + f(t)]\, dt - 2$

where; $\quad f(x) = m^2 x^2 + \int_0^x f(t)\, dt - 2 \to \infty \qquad \left(\because \int_0^x f(t)\, dt \to \infty\right)$

As $\quad |x| \to \infty$

Thus, these exists some $a \in R$ such that;

$$f(x) > 1 \qquad \text{for } |x| > a$$

Note that f is a continuous on $[0, a+1]$ and $f(0) f(a+1) < 0$. By the intermediate value theorem of continuous functions, we have that there exists some $b \in (0, a+1)$ such that $f(b) = 0$, *ie*, there exists a real β which satisfies the equation

$$m^2 x^2 + \int_0^x f(t)\, dt = 2$$

10. As shown, in the beginning the man is at A on the bridge and the boat is at B on the river after t minutes, the man is at C and the boat it at D.

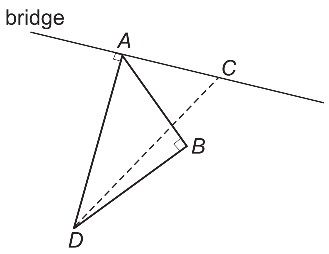
From the question, $\quad AB \perp AC$.

$\therefore$ AC is perpendicular to the plane of AB and BD.

$\therefore \quad AC \perp AD$

Also, $AC = 4t\,(\text{m})$ and $BD = 5t$ m and $AB = 3$ m

$\therefore$ From the right angled ΔABD,



$$AD = \sqrt{AB^2 + BD^2} = \sqrt{3^2 + (5t)^2} \text{ m}$$

and from the right angled ΔDAC,

$$DC = \sqrt{AD^2 + AC^2}$$

If $\quad DC = Z$ m,

$$Z = \sqrt{3^2 + (5t)\ + (4t)^2} = \sqrt{9 + 41t^2}$$

$$\therefore \quad \frac{dZ}{dt} = \frac{1}{2\sqrt{9 + 41t^2}} \cdot (41)(2t), \text{ at } t = 4 \text{ min}$$

$$\left(\frac{dZ}{dt}\right)_{t=4} = \frac{164}{\sqrt{665}} \text{ m / min}$$

11. Tangent at any point $P\ t_1$. *ie*, $(4t_1^2 + 3,\ 8t_1^3 - 1)$ be normal to the curve at $Q\ \ t_2$. *ie*, $(4t_2^2 + 3,\ 8t_2^3 - 1)$

The equation of the tangent at t_1 is

$$y-(8t_1^3-1)=\left(\frac{dy}{dx}\right)_{t_1}\cdot\{x-(4t_1^2+3)\}$$

or $$y-(8t_1^3-1)=\left(\frac{dy/dt}{dx/dt}\right)_{t_1}\cdot\{x-(4t_1^2+3)\}$$

or $$y-(8t_1^3-1)=\frac{24t_1^2}{8t_1}\cdot\{x-(4t_1^2+3)\}$$

or $$y-(8t_1^3-1)=3t_1\ \{x-(4t_1^2+3)\} \qquad \text{...(i)}$$

Clearly, slope of tangent at $t_1=$ slope of tangent at t_2.

$\therefore$ $$\left(\frac{dy}{dx}\right)_{t_1}=\frac{-1}{\left(\frac{dy}{dx}\right)_{t_2}}\quad ie,\ 3t_1=\frac{-1}{3t_2} \qquad \text{...(ii)}$$

and $$(8t_2^3-1)-(8t_1^3-1)=3t_1\ \{(4t_2^2+3)-(4t_1^2+3)\}$$

$\Rightarrow$ $$2t_2^2=t_1t_2+t_1^2$$

$\Rightarrow$ $$2\cdot\left(\frac{-1}{9t_1}\right)^2=-\frac{1}{9}+t_1^2 \qquad \text{[using Eq. (ii)]}$$

$\Rightarrow$ $$2=-9t_1^2+81t_1^4$$

$\therefore$ $$81t_1^4-9t_1^2-2=0$$

$$t_1=\pm\frac{\sqrt{2}}{3}$$

Putting in Eq. (i), the equation is

$$27(y+1)\mp 16\sqrt{2}=\pm\sqrt{2}\,(27x-105)$$

12. Equation of tangent at (x, y)

$$Y-y=\frac{dy}{dx}(X-x)$$

$\therefore$ $$Q=\left(x-y\frac{dx}{dy},0\right)$$

Equation of normal at (x, y)

$$Y-y=-\frac{dy}{dx}(X-x)$$

$\therefore$ $$R=\left(x+y\frac{dy}{dx},0\right)$$

Given, $$QR=2$$

$$y\frac{dy}{dx}+y\frac{dx}{dy}=2$$

$\Rightarrow$ $$y\left(\frac{dy}{dx}\right)^2-2\left(\frac{dy}{dx}\right)+y=0$$

or $$\frac{dy}{dx}=\frac{2\pm\sqrt{4-4y^2}}{2y}=\frac{1\pm\sqrt{1-y^2}}{y}$$

$\Rightarrow$ $$\frac{y\,dy}{1\pm\sqrt{1-y^2}}=dx$$

or $$\frac{1 \mp \sqrt{1-y^2}}{y}\,dy = dx$$

$\Rightarrow$ $$dy \mp \frac{\sqrt{1-y^2}}{y}\,dy = dx$$

Integrating both the sides, we get

$\Rightarrow$ $$y \mp \left(\log \left| \frac{1-\sqrt{1-y^2}}{y} \right| + \sqrt{1-y^2} \right) = x + c$$

The curve passes through $(1, 1)$, so $c = 0$

Hence, the possible curves

$$y - x = \pm \left(\log \left| \frac{1-\sqrt{1-y^2}}{y} \right| + \sqrt{1-y^2} \right)$$

13. Slope of the tangent at $(a, b) = -\dfrac{a^2}{b^2}$. The tangent cuts the curve again at (a_1, b_1).

$\therefore$ Slope of tangent $= \dfrac{b_1 - b}{a_1 - a} \Rightarrow -\dfrac{a^2}{b^2} = \dfrac{b_1 - b}{a_1 - a}$ ...(i)

Also, $a^3 + b^3 = c^3$

and $a_1^3 + b_1^3 = c_1^3$

$a^3 - a_1^3 = b_1^3 - b^3$

$\Rightarrow$ $(a - a_1)(a^2 + a_1^2 + a_1 a_2) = (b - b_1)(b^2 + b_1 b + b_1^2)$

$\Rightarrow$ $-\dfrac{(b - b_1)}{(a_1 - a)} = \dfrac{a^2 + a_1^2 + aa_1}{b^2 + b_1^2 + bb_1}$

$\Rightarrow$ $\dfrac{a^2}{b^2} = \dfrac{a^2 + a_1^2 + aa_1}{b^2 + b_1^2 + bb_1}$ [from Eq. (i)]

$\Rightarrow$ $a^2b^2 + a^2b_1^2 + a^2bb_1 = a^2b^2 + a_1^2b^2 + ab^2a_1$

$\Rightarrow$ $a^2b_1^2 - a_1^2b^2 = ab(a_1b - ab_1)$

$\Rightarrow$ $ab_1 + a_1b = -ab$

or $\dfrac{a_1}{a} + \dfrac{b_1}{b} + 1 = 0$

14. Let the point $P(x, y)$ be on the curve,

$$\log(x^2 + y^2) = c \tan^{-1}\left(\frac{y}{x}\right)$$

Differentiating both the sides w.r.t. 'x', we get

$$\frac{2x + 2yy'}{(x^2 + y^2)} = \frac{c(xy' - y)}{(x^2 + y^2)}$$

$\Rightarrow$ $$y' = \frac{2x + cy}{cx - 2y} = m_1 \text{ (say)}$$

Slope of $OP = \dfrac{y}{x} = m_2$ (say)

Let the angle between the tangent at P and OP be θ.

$\therefore \qquad \tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\dfrac{\dfrac{2x+cy}{cx-cy} - \dfrac{y}{x}}{1 + \dfrac{2xy+cy^2}{x(cx-2y)}}\right| = \dfrac{2}{c}$

$\therefore \qquad \theta = \tan^{-1}\left(\dfrac{2}{c}\right) = \text{constant.}$

Hence, the angle between the tangents at any point P and the line joining P to the origin O is the same.

15. (i) The given curves are

$$y^2 = 4ax \qquad \text{...(i)}$$

$$x^2 = 4ay \qquad \text{...(ii)}$$

Point of intersection of Eqs. (i) and (ii) are (0, 0) and $(4a, 4a)$.

From Eq. (i), $\qquad \dfrac{dy}{dx} = \dfrac{2a}{y} = m_1$ (say)

From Eq. (ii), $\qquad \dfrac{dy}{dx} = \dfrac{x}{2a} = m_2$ (say)

Let the angle of intersection of two curves is θ, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\dfrac{2a}{y} - \dfrac{x}{2a}}{1 + \dfrac{x}{y}}\right| = \left|\frac{4a^2 - xy}{2a(x+y)}\right|$$

$\therefore \qquad (\tan\theta)_{(0,0)} = \infty \quad \text{or} \quad \theta = 90^\circ$

and $\qquad (\tan\theta)_{(4a,\,4a)} = \left|\dfrac{\dfrac{1}{2} - 2}{1+1}\right| = \left|-\dfrac{3}{4}\right| = \dfrac{3}{4}$

Hence, $\qquad \theta = \tan^{-1}\left(\dfrac{3}{4}\right)$

(ii) The given curves are $y^2 = 4ax \qquad \text{...(i)}$

$$x^2 = 4ay \qquad \text{...(ii)}$$

Tangents of Eq. (i) in terms of slope is,

$$y = mx + \frac{a}{m} \qquad \text{...(iii)}$$

Eliminating y from Eqs. (ii) and (iii), we have

$$x^2 = 4a\left(mx + \frac{a}{m}\right)$$

$$x^2 - 4amx - \frac{4a^2}{m} = 0 \qquad \text{...(iv)}$$

Now, Eq. (iii) is also tangent of Eq. (ii).

[$\because$ Eqs. (i) and (ii) have common tangent]

$\Rightarrow$ From Eq. (iv) $\qquad B^2 - 4AC = 0$

$$16a^2m^2 - 4\left(-\frac{4a^2}{m^2}\right) = 0$$

$\Rightarrow$ $$m^3 = -1$$

or $$m = -1$$

From Eq. (iii) common tangent is

$$y = -x - a$$

or $$y + x + a = 0$$

Hence, the common tangent is $x + y + a = 0$.

16. Let the coordinate system be chosen that the given straight line is $x = p$ and the equations of the circles are $x^2 + y^2 = a^2, x^2 + y^2 = b^2, x^2 + y^2 = c^2$.

The line $x = p$ cuts these circles at A, B, C respectively.

The coordinates of these points are $A(p, \sqrt{a^2 - p^2}), B(p, \sqrt{b^2 - p^2}),$

$C(p, \sqrt{c^2 - p^2})$.

Equations of the tangents at these points are

$$px + \sqrt{a^2 - p^2}\,y = a^2,\ px + \sqrt{b^2 - p^2}\,y = b^2,\ px + \sqrt{c^2 - p^2}\,y = c^2$$

These tangents intersect at

$$\left[\frac{p^2 - \sqrt{a^2 - p^2}\sqrt{b^2 - p^2}}{p}, \sqrt{a^2 - p^2} + \sqrt{b^2 - p^2}\right],$$

$$\left[\frac{p^2 - \sqrt{b^2 - p^2}\sqrt{c^2 - p^2}}{p}, \sqrt{b^2 - p^2} + \sqrt{c^2 - p^2}\right],$$

$$\left[\frac{p^2 - \sqrt{c^2 - p^2}\sqrt{c^2 - p^2}}{p}, \sqrt{c^2 - p^2} + \sqrt{a^2 - p^2}\right]$$

Area of Δ formed by the tangents at A, B, C is

$$\Delta = \frac{1}{2}\begin{vmatrix} \frac{p^2 - \sqrt{a^2 - p^2}\sqrt{b^2 - p^2}}{p} & \sqrt{a^2 - p^2} + \sqrt{b^2 - p^2} & 1 \\ \frac{p^2 - \sqrt{b^2 - p^2}\sqrt{c^2 - p^2}}{p} & \sqrt{b^2 - p^2} + \sqrt{c^2 - p^2} & 1 \\ \frac{p^2 - \sqrt{c^2 - p^2}\sqrt{c^2 - p^2}}{p} & \sqrt{c^2 - p^2} + \sqrt{a^2 - p^2} & 1 \end{vmatrix}$$

$\Rightarrow$ $$= \frac{(\sqrt{a^2 - p^2} - \sqrt{c^2 - p^2})(\sqrt{c^2 - p^2} - \sqrt{b^2 - p^2})(\sqrt{b^2 - p^2} - \sqrt{a^2 - p^2})}{2p}$$

$$= \frac{CA \cdot BC \cdot AB}{2p}$$

17. Let the curve be $y = f(x)$ and tangent drawn at $P(x, y)$ meets the x-axis at T.

We have, $OT = x^2$

Equation of tangent at $P(x, y)$;

$$Y - y = f'(x)(X - x)$$

$\Rightarrow \qquad T \equiv \left(\left|x - \dfrac{y}{f'(x)}, 0\right|\right) \quad \Rightarrow \quad \left|x - \dfrac{y}{f'(x)}\right| = x^2$

$\Rightarrow \qquad x - \dfrac{f(x)}{f'(x)} = \pm x^2$

$\Rightarrow \qquad \dfrac{x f'(x) - f(x)}{x^2} = \pm f'(x)$

$\Rightarrow \qquad \dfrac{d}{dx}\left(\dfrac{f(x)}{x}\right) = \pm f'(x)$

Integrating both the sides, we get

$$\frac{f(x)}{x} = \pm f(x) + c$$

Now, the curve passes through (2, 1).

$\Rightarrow \qquad \dfrac{1}{2} = \pm 1 + c \quad \Rightarrow \quad c = -\dfrac{1}{2}, \dfrac{3}{2}$

$\Rightarrow \qquad f(x) = \dfrac{x}{2(x-1)} \quad \text{or} \quad f(x) = \dfrac{3x}{2(1+x)}$

Hence, possible curves are $y = \dfrac{x}{2(x-1)}$ and $y = \dfrac{3x}{2(1+x)}$.

18. For $y = x - x^3$, $\dfrac{dy}{dx} = 1 - 3x^2$

Therefore, the equation of the tangent at the point $P(x_1, y_1)$ is

$$y - y_1 = (1 - 3x_1^2)(x - x_1)$$

It meets the curve again at $Q(x_2, y_2)$.

Hence, $\qquad x_2 - x_2^3 - (x_1 - x_1^3) = (1 - 3x_1^2)(x_2 - x_1)$

$\Rightarrow \qquad (x_2 - x_1)[1 - (x_2^2 + x_1x_2 + x_1^2)] = (x_2 - x_1)(1 - 3x_1^2)$

$\Rightarrow \qquad 1 - x_2^2 - x_1x_2 - x_1^2 = 1 - 3x_1^2$

$\Rightarrow \qquad x_2^2 + x_1x_2 - 2x_1^2 = 0$

or $\qquad \left(x_2 + \dfrac{x_1}{2}\right)^2 = \dfrac{9x_1^2}{4}$, so that $x_2 + \dfrac{x_1}{2} = \pm \dfrac{3x_1}{2}$

Since, $\qquad x_1 \neq x_2$, we have $x_2 = -2x_1$

$\Rightarrow \qquad Q$ is $(-2x_1, -2x_1 + 8x_1^3)$

If $L_1(\alpha, \beta)$ is the point of trisection of PQ, then $\alpha = \dfrac{2x_1 - 2x_1}{3} = 0$. Hence, L_1 lies on the y-axis. If $L_2(h, k)$ is the other point of trisection, then $h = \dfrac{x_1 - 4x_1}{3} = -x_1$ and $k = \dfrac{y_1 - 4x_1 + 16x_1^3}{3}$

ie, $\qquad k = \dfrac{x_1 - x_1^3 - 4x_1 + 16x_1^3}{3} = -x_1 + 5x_1^3$

$$k = h - 5h^3$$

$\therefore \qquad$ Locus of (h, k) is $y = x - 5x^3$.

19. Let the curve be $y = f(x)$

The equation of the tangent at any point (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

Its intercept on the y-axis is given by $(X = 0)$

$$Y = y - x\frac{dy}{dx} = k\sqrt{x^2 + y^2}$$

So that $x\dfrac{dy}{dx} - y + k\sqrt{x^2 + y^2} = 0$ is the differential equation governing the curve.

This can be written as

$$\frac{dy}{dx} = \frac{y - k\sqrt{x^2 + y^2}}{x}$$

Let $y = vx$, so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

The differential equation becomes

$$v + x\frac{dy}{dx} = v - k\sqrt{1 + v^2}$$

$$\Rightarrow \quad \frac{dv}{\sqrt{1 + v^2}} + k\frac{dx}{x} = 0$$

Let $\quad v = \tan\theta \quad \Rightarrow \quad dv = \sec^2\theta\, d\theta$

Hence, $\quad \sec^2\theta\, d\theta + k\dfrac{dx}{x} = 0$

$$\Rightarrow \quad \log|\sec\theta + \tan\theta| + k\log|x| = c \qquad \text{[on integrating]}$$

$$\Rightarrow \quad \log|V + \sqrt{1 + V^2}| + k\log|x| = c$$

$$\Rightarrow \quad \log\left|\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}\right| + k\log|x| = c$$

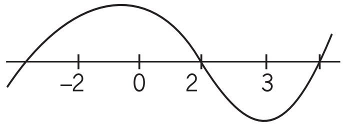
$$\Rightarrow \quad \log|y + \sqrt{1 + y^2}| + (k - 1)\log|x| = c$$

$$\Rightarrow \quad (y + \sqrt{x^2 + y^2})\, x^{k-1} = c_1$$

20. We have, $2t^3 - 9t^2 + 30 - a = 0$

Any real root t_0 of this equation gives two real and distinct values of x if $|t_0| > 2$.

Thus, we need to find the condition for the equation in t to have three real and distinct roots none of which lies in [–2, 2].



Let $\quad f(t) = 2t^3 - 9t^2 + 30 - a$

$$f'(t) = 6t^2 - 8t = 0 \quad \Rightarrow \quad t = 0, 3$$

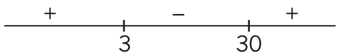
So, the equation $f(t) = 0$ has three real and distinct roots, if $f(0) \cdot f(3) < 0$

$\Rightarrow \quad (30 - a)(54 - 81 + 30 - a) < 0$

$\Rightarrow \quad (30 - a)(3 - a) < 0$

$\Rightarrow \quad (a - 3)(a - 30) < 0$, using number line rule *ie*,

$\Rightarrow \quad a \in (3, 30)$ ...(i)



Also, none of the roots lie in $[-2, 2]$

if $\quad f(-2) > 0 \quad$ and $\quad f(2) > 0$

$\Rightarrow \quad (-16 - 36 + 30 - a) > 0 \quad$ and $\quad (16 - 36 + 30 - a) > 0$

$\Rightarrow \quad a + 22 < 0 \quad$ and $\quad a - 10 < 0$

$\Rightarrow \quad a < -22$ and $a < 10$

$\Rightarrow \quad a < -22$ ...(ii)

From Eqs. (i) and (ii) no real value of a exists.

8

Monotonicity, Maxima and Minima

Chapter in a Snapshot

- Increasing Functions
- Decreasing Functions
- Properties of Monotonic Functions
- Critical Points
- Comparison of Functions Using Calculus
- Introduction to Maxima and Minima
- Method of Finding Extrema of Continuous Functions
- Concept of Global Maximum/Minimum
- Maxima and Minima in Discontinuous Functions
- Some Geometrical Results

From the word monotonic we deduce the monotonous behaviour *ie,* in the sense of ascending (increasing) or descending (decreasing).

Obviously there are two types of monotonic functions:

(I) **Increasing functions** It can further be studied under two sub-heads:

(a) Strictly increasing functions.

(b) Increasing or non-decreasing functions.

(II) **Decreasing functions** It can further be studied under two sub-categories:

(a) Strictly decreasing functions.

(b) Decreasing or non-increasing functions.

Following the above plan, we have:

Increasing Functions

(a) **Strictly Increasing Functions**

A function $f(x)$ is known as strictly increasing function in its domain, if $x_1 < x_2$.

$$\Rightarrow \qquad f(x_1) < f(x_2)$$

ie, for the smaller input we have smaller output and for higher value of input we have higher output.

Graphically it can be expressed as, shown in figure.

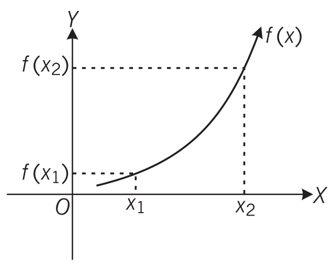


Fig. 8.1

Here, $\qquad x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

Thus, the function is strictly increasing.

In the graph,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{+\text{ve}}{+\text{ve}} \qquad \left\{ \begin{array}{l} \text{as} \quad x_1 < x_2 \\ \Rightarrow \quad f(x_1) < f(x_2) \\ \text{Thus, } f(x) < f(x+h) \end{array} \right.$$

ie, $\qquad f'(x) > 0$

Point to Consider

It means that the value of $f(x)$ will keep on increasing with an increase in the value of x or $f'(x) > 0$, $\forall\ x \in$ domain.

Classification of Strictly Increasing Functions

Increasing functions can be classified as

(i) **Concave up when**

$f'(x) > 0$ and $f''(x) > 0, \forall x \in$ domain

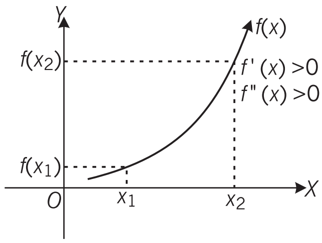


Fig. 8.2

(ii) **When $f'(x) > 0$ and $f''(x) = 0, \forall x \in$ domain**

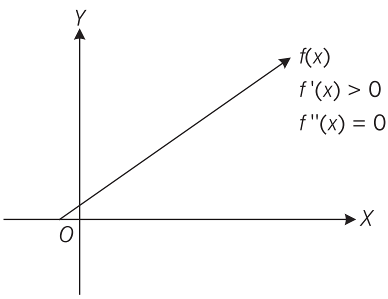


Fig. 8.3

(iii) **Concave down when $f'(x) > 0$ and $f''(x) < 0, \forall x \in$ domain**

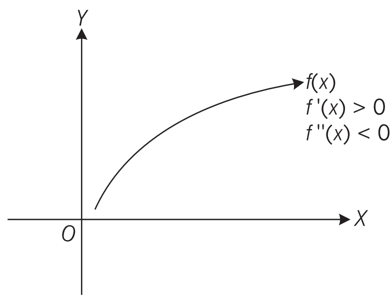


Fig. 8.4

(b) **Only Increasing or Non-decreasing Functions**

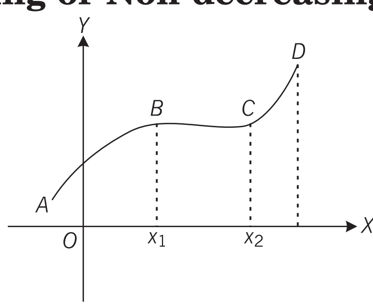


Fig. 8.5

A function $f(x)$ is said to be non-decreasing, if for $x_1 < x_2$.

$$\Rightarrow \qquad f(x_1) \le f(x_2)$$

As shown in the graph,

for AB and CD portion $x_1 < x_2$

$$\Rightarrow \qquad f(x_1) < f(x_2)$$

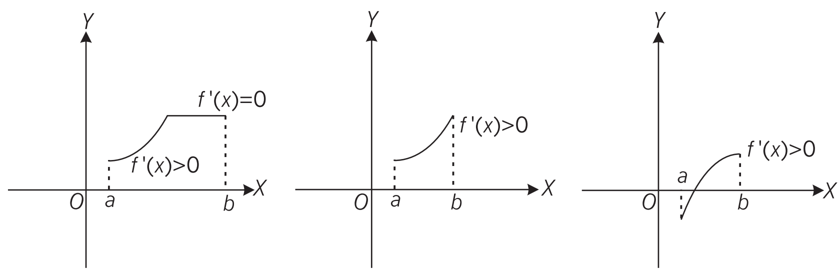
and for $\qquad BC, x_1 < x_2 \quad \Rightarrow \quad f(x_1) = f(x_2).$

Hence, as a whole we can say that for non-decreasing functions

$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) < f(x_2).$$

Obviously, for this $f'(x) \ge 0$, where equality holds for horizontal path of the graph *ie,* in the interval of BC.

Point to Consider



(i) Increasing ($f'(x) \ge 0$)

(ii) Strictly increasing ($f'(x) > 0$)

(iii) Strictly increasing ($f'(x) > 0$)

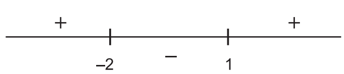
Fig. 8.6

In above figure (i) function is only increasing or it is a non-decreasing function and figures (ii), (iii) show strictly increasing functions on $[a, b]$.

Illustration 1 Find the interval in which $f(x) = 2x^3 + 3x^2 - 12x + 1$ is increasing.

Solution. Given, $f(x) = 2x^3 + 3x^2 - 12x + 1$

Differentiating both the sides, we have



$$f'(x) = 6x^2 + 6x - 12$$

$$\Rightarrow \qquad f'(x) = 6(x^2 + x - 2)$$

$$\Rightarrow \qquad f'(x) = 6(x + 2)(x - 1)$$

Using number line rule

Hence, $\qquad f'(x) \ge 0$

when $\quad x \in (-\infty, -2] \cup [1, \infty)$

$$\Rightarrow \qquad f(x) \text{ is increasing when } x \in (-\infty, -2] \cup [1, \infty)$$

Decreasing Functions

(a) Strictly Decreasing Functions

A function $f(x)$ is known as strictly decreasing function in its domain, if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

ie, for the smaller input we have higher output and for higher value of input we have smaller output.

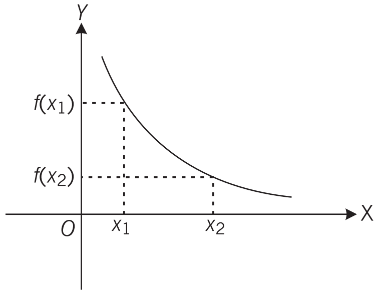


Fig. 8.7

Graphically, it can be expressed as shown in the figure.

Here, $x_1 < x_2$

$\Rightarrow$ $f(x_1) > f(x_2)$ thus, strictly decreasing.

In above graph, $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

As $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Thus, $f(x+h) < f(x)$

$\Rightarrow$ $f'(x) = \lim_{h \to 0} \dfrac{-\text{ve}}{+\text{ve}}$

ie, $f'(x) < 0$

Point to Consider

It means that the curve of $f(x)$ will keep on decreasing with an increase in the value of x or $f'(x) < 0, \forall x \in$ domain.

Classification of Strictly Decreasing Functions

Decreasing functions can be classified as

(i) **Concave up** when $f'(x) < 0$ and $f''(x) > 0, \forall x \in$ domain.

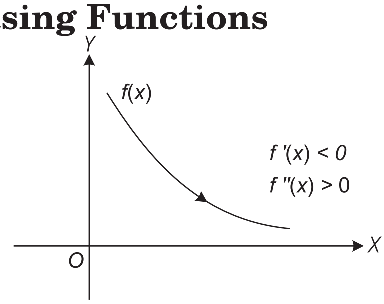


Fig. 8.8

(ii) When $f'(x) < 0$ and $f''(x) = 0, \forall x \in$ domain.

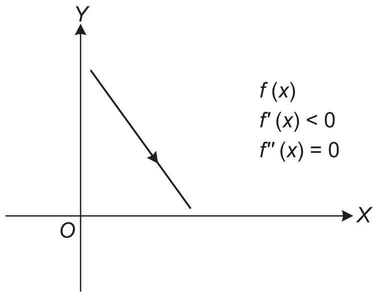


Fig. 8.9

(iii) **Concave down** when $f'(x) < 0$ and $f''(x) < 0, \forall x \in$ domain.

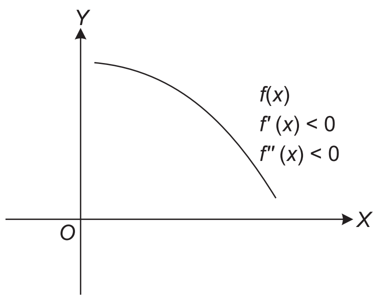


Fig. 8.10

(b) Only Decreasing or Non-increasing Functions

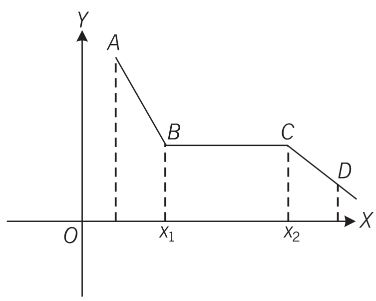


Fig. 8.11

A function $f(x)$ is said to be non-increasing, if for $x_1 < x_2$.

$\Rightarrow$ $$f(x_1) \geq f(x_2)$$

As shown in graph,

for AB and CD portion $x_1 < x_2$ $\Rightarrow$ $f(x_1) > f(x_2)$

and for BC, $x_1 < x_2$ $\Rightarrow$ $f(x_1) = f(x_2)$

Hence, on the whole we can say that for non-increasing functions $x_1 < x_2$

$\Rightarrow$ $$f(x_1) \geq f(x_2)$$

Obviously, for this $f'(x) \leq 0$ where equality holds for horizontal path of the graph *ie,* in the interval of BC.

Points to Consider

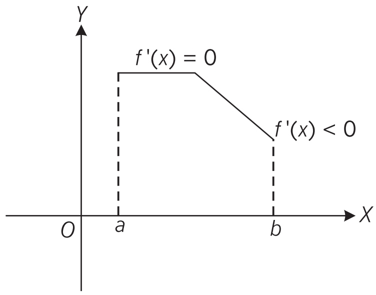


Fig. 8.12

(i) Only decreasing or non-increasing

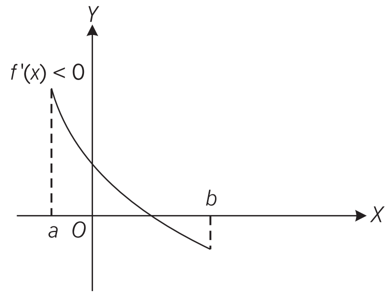


Fig. 8.13

(ii) Strictly decreasing

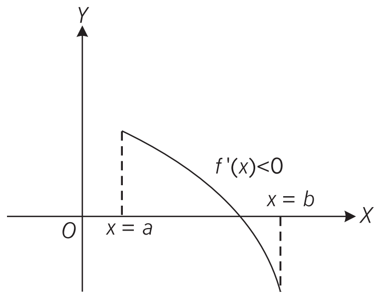


Fig. 8.14

(iii) Strictly decreasing

In the above figure (i) is only decreasing or non-increasing and figures (ii), (iii) are strictly decreasing on $[a, b]$.

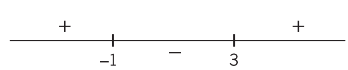
Illustration 2 Find the interval in which $f(x) = x^3 - 3x^2 - 9x + 20$ is strictly increasing or strictly decreasing.

Solution. Given, $f(x) = x^3 - 3x^2 - 9x + 20$

$$f'(x) = 3x^2 - 6x - 9$$

$$\Rightarrow \quad f'(x) = 3(x^2 - 2x - 3)$$

$$\Rightarrow \quad f'(x) = 3(x-3)(x+1)$$



Using number line method, as shown in figure

$\Rightarrow \quad f'(x) > 0,$

for $\quad x \in (-\infty, -1) \cup (3, \infty)$

$\quad f'(x) < 0,$

for $\quad x \in (-1, 3)$

Thus, $f(x)$ is strictly increasing for $x \in (-\infty, -1) \cup (3, \infty)$ and strictly decreasing for $x \in (-1, 3)$.

Illustration 3 Show that the function $f(x) = x^2$ is a strictly increasing function on $(0, \infty)$.

Solution. Given, $f(x) = x^2 \quad \Rightarrow \quad f'(x) = 2x$

which is clearly increasing for all $x > 0 \Rightarrow f'(x) > 0$ for $x \in (0, \infty)$.

Thus, $f(x)$ is strictly increasing for $x \in (0, \infty)$.

Illustration 4 Find the interval of increase or decrease of the

$$f(x) = \int_{-1}^{x} (t^2 + 2t)(t^2 - 1)\, dt$$

Solution. Given, $f(x) = \int_{-1}^{x} (t^2 + 2t)(t^2 - 1)\, dt$

On differentiating both the sides, we have

$$f'(x) = (x^2 + 2x)(x^2 - 1)\left\{\frac{d}{dx}(x)\right\} - (1+2)(1-1)\left\{\frac{d}{dx}(-1)\right\} \text{ [using Leibnitz-rule]}$$

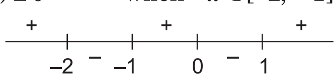
$$\Rightarrow \qquad f'(x) = (x^2 + 2x)(x^2 - 1)$$

$$\Rightarrow \qquad f'(x) = x(x+2)(x+1)(x-1)$$

Using number line rule as shown in figure,

Clearly, $\quad f'(x) \ge 0 \quad$ when $\quad x \in (-\infty, -2] \cup [-1, 0] \cup [1, \infty)$ and

$\qquad\qquad f'(x) \le 0 \quad$ when $\quad x \in [-2, -1] \cup [0, 1]$.



Hence, $f(x)$ is increasing, when $x \in (-\infty, -2] \cup [-1, 0] \cup [1, \infty)$ and $f(x)$ is decreasing, when $x \in [-2, -1] \cup [0, 1]$.

Point to Consider

In above Illustration

Leibnitz-rule is stated as :

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(t)\, dt\right] = f(\psi(x))\left\{\frac{d}{dx}\psi(x)\right\} - f(\phi(x))\left\{\frac{d}{dx}\phi(x)\right\}$$

Illustration 5 If $f(x) = \dfrac{x}{\sin x}$ and $g(x) = \dfrac{x}{\tan x}$, where $0 < x \le 1$, then in this interval $f(x)$ and $g(x)$ are increasing or decreasing?

Solution. Here, $\quad f(x) = \dfrac{x}{\sin x}$

$$\Rightarrow \qquad f'(x) = \frac{\sin x \cdot 1 - x \cdot \cos x}{\sin^2 x} \qquad \text{...(i)}$$

where $\sin^2 x$ is always +ve when $0 < x \le 1$.

But to check numerator we again let, $h(x) = \sin x - x\cos x$

$\Rightarrow \qquad h'(x) = \cos x - 1 \cdot \cos x + x\sin x = x\sin x$, which is +ve for $0 < x \le 1$

$\therefore \qquad h'(x) > 0 \quad \Rightarrow \quad h(x)$ is increasing, when $0 < x \le 1$

$\Rightarrow \qquad h(0) < h(x) \quad \Rightarrow \quad 0 < \sin x - x\cos x$

$\therefore \qquad$ In Eq. (i) $\qquad (\sin x - x\cos x) > 0$,

$\Rightarrow \qquad f'(x) > 0$, when $0 < x \le 1$

Hence, $f(x)$ is increasing, when $0 < x \le 1$

Hence, again, $g(x) = \dfrac{x}{\tan x}$ (given)

$$\Rightarrow \quad g'(x) = \frac{\tan x \cdot 1 - x \cdot \sec^2 x}{\tan^2 x} \qquad \text{...(ii)}$$

where $\tan^2 x > 0$

for $\tan x - x\sec^2 x$, we let $\phi(x) = \tan x - x\sec^2 x$

$\Rightarrow$ $\phi'(x) = \sec^2 x - \sec^2 x - x(2\sec x)\cdot(\sec x \tan x)$

$\phi'(x) = -2x\sec^2 x \tan x$

As $\phi'(x) < 0, \forall 0 < x \le 1$

$\therefore$ $\phi(x)$ is decreasing when $0 < x \le 1$

$\Rightarrow$ $\phi(0) > \phi(x)$

or $0 < \tan x - x\sec^2 x$

$\therefore$ In Eq. (ii) $(\tan x - x\sec^2 x) < 0$

$\Rightarrow$ $g'(x) < 0$, when $0 < x \le 1$

$\therefore$ $g(x)$ is decreasing when $0 < x \le 1$

Illustration 6 The function $f(x) = \sin^4 x + \cos^4 x$ increasing, if

(a) $0 < x < \pi/8$ (b) $\pi/4 < x < 3\pi/8$ **[IIT JEE 1999]**

(c) $3\pi/8 < x < 5\pi/8$ (d) $5\pi/8 < x < 3\pi/4$

Solution. Here, $f(x) = \sin^4 x + \cos^4 x$

$\Rightarrow$ $f'(x) = 4\sin^3 x \cdot \cos x + 4\cos^3 x(-\sin x)$

$f'(x) = 4\sin x \cos x(\sin^2 x - \cos^2 x)$

$f'(x) = 2(\sin 2x)(-\cos 2x) \Rightarrow f'(x) = -\sin 4x$

Now, $f'(x) \ge 0$, if $\sin 4x \le 0 \Rightarrow \pi \le 4x \le 2\pi \Rightarrow \pi/4 \le x \le \pi/2$

Here, (b) is only subset of $\left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right]$.

Hence, (b) is the correct answer.

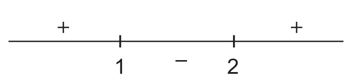
Illustration 7 Let $f(x) = \int_0^x e^t(t-1)(t-2)\,dt$. Then, f decreases in the interval **[IIT JEE 2000]**

(a) $(-\infty, -2)$ (b) $(-2, -1)$ (c) $(1, 2)$ (d) $(2, \infty)$

Solution. Here, $f(x) = \int_0^x e^t(t-1)(t-2)\,dt$

$f'(x) = e^x(x-1)(x-2)$, (as e^x is always +ve) (using Leibnitz-rule)

Using number line rule for $f'(x)$, we get



$f'(x) \le 0$ when $1 < x < 2$

$\therefore$ f decreases when $1 \le x \le 2$

Hence, (c) is the correct answer.

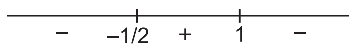
Illustration 8 If $f(x) = x \cdot e^{x(1-x)}$, then $f(x)$ is **[IIT JEE 2000]**

(a) increasing on $\left[-\frac{1}{2}, 1\right]$ (b) decreasing on R

(c) increasing on R (d) decreasing on $\left[-\frac{1}{2}, 1\right]$

Solution. Here, $f'(x) = x \cdot e^{x(1-x)} \cdot (1-2x) + 1 \cdot e^{x(1-x)}$

$$f'(x) = e^{x(1-x)}[x - 2x^2 + 1]$$
$$f'(x) = -e^{x(1-x)}(x-1)(2x+1)$$



Using number line rule for $f'(x)$ we get, $f'(x) \geq 0$, when $x \in \left[-\frac{1}{2}, 1\right]$ as shown in figure.

Hence, (a) is the correct answer.

Illustration 9 Find the interval for which $f(x) = x - \sin x$ is increasing or decreasing.

Solution. Given, $f(x) = x - \sin x$,

Differentiating both the sides w.r.t. x, we have

$$f'(x) = 1 - \cos x$$

we know, $-1 \leq \cos x \leq 1$ or $\cos x \leq 1 \Rightarrow 1 - \cos x \geq 0$

Therefore, $f'(x) \geq 0, \forall x \in R$

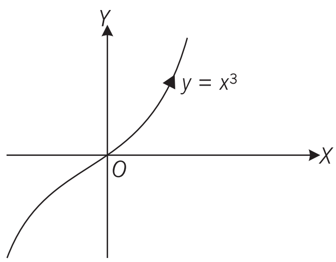
which shows $f(x)$ is increasing for the entire number scale,

ie, all real number.

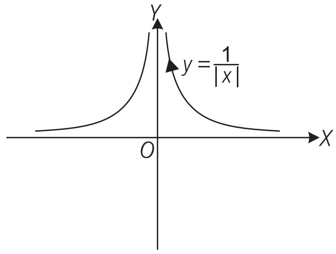
Illustration 10 Discuss the nature of following functions, graphically.

(i) $f(x) = x^3$ (ii) $f(x) = \frac{1}{|x|}$ (iii) $f(x) = e^x$ (iv) $f(x) = [x]$

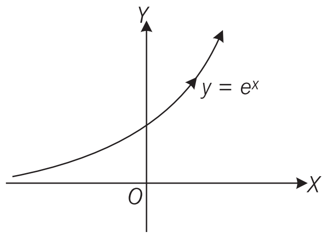
Solution. (i) $f(x) = x^3$ can be graphically plotted as shown in the following figure, which shows $f(x) = x^3$ is strictly increasing in R.



(ii) $f(x) = \frac{1}{|x|}$ can be graphically plotted as shown in the following figure, which shows $f(x) = \frac{1}{|x|}$ is strictly increasing in $]-\infty, 0[$ and strictly decreasing in $]0, \infty[$.



(iii) $f(x) = e^x$ can be graphically plotted as shown in the following figure, which shows $f(x) = e^x$ is strictly increasing in R.



(iv) $f(x) = [x]$ can be plotted as shown in the following figure, which shows $f(x) = [x]$ is increasing but not strictly increasing *ie,* non-decreasing in R.

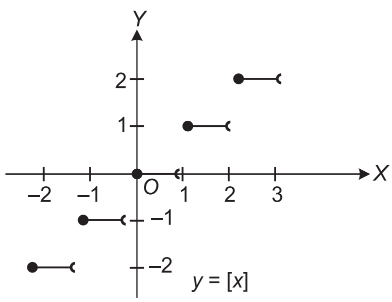


Illustration 11 If $H(x_0) = 0$ for some $x = x_0$ and $\frac{d}{dx} H(x) > 2cx\, H(x)$ for all $x \geq x_0$, where $c > 0$, then prove that $H(x)$ cannot be zero for any $x > x_0$.

Solution. Given that, $\frac{d}{dx} H(x) > 2\, cx\, H(x)$

$$\Rightarrow \quad \frac{d}{dx} H(x) - 2\, cx\, H(x) > 0$$

$$\Rightarrow \quad \frac{d}{dx}\{H(x)\}e^{-cx^2} - 2\, cxe^{-cx^2} \cdot H(x) > 0$$

$$\Rightarrow \quad \left\{\frac{d}{dx} H(x)\right\} e^{-cx^2} + H(x)\left\{\frac{d}{dx} e^{-cx^2}\right\} > 0 \Rightarrow \left\{\frac{d}{dx} H(x) \cdot e^{-cx^2}\right\} > 0$$

$\therefore$ $\quad H(x)\, e^{-cx^2}$ is an increasing function.

But, $\quad H(x_0) = 0$ and e^{-cx^2} is always positive.

$\Rightarrow$ $\quad H(x_0) > 0$ for all $x > x_0$

$\Rightarrow$ $\quad H(x)$ cannot be zero for any $x > x_0$.

Properties of Monotonic Functions

(a) If $f(x)$ is strictly increasing function on $[a, b]$

$$\Rightarrow \quad \begin{cases} f^{-1}(x) \text{ exists} \\ f^{-1}(x) \text{ is also strictly increasing on [a, b]} \end{cases}$$

Illustration 12 Let $f(x) = 3x + 5$, then show $f(x)$ is strictly increasing and $f^{-1}(x)$ exists and is strictly increasing for $x \in R$.

Solution. Here, $f(x) = 3x + 5$

$f'(x) = 3 > 0$, which is strictly increasing for $x \in R$.

Thus, finding $f^{-1}(x)$, let $f(x) = y$, $\quad y = 3x + 5 \quad$ or $\quad x = \dfrac{y-5}{3}$

or $$f^{-1}(y) = \frac{y-5}{3} \qquad [\text{as } f(x) = y \Rightarrow x = f^{-1}(y)]$$

$$f^{-1}(x) = \frac{x-5}{3}$$

which shows $f^{-1}(x) = \dfrac{x-5}{3}$ exists for all $x \in R$ and is strictly increasing as $\dfrac{d}{dx}(f^{-1}(x)) = \dfrac{1}{3} > 0$ for all $x \in R$.

(b) If $f(x)$ and $g(x)$ are two continuous and differentiable functions and $fog\,(x)$ and $gof\,(x)$ exists, then

$$\Rightarrow \quad \begin{cases} f'(x) > 0,\ g'(x) > 0 \Rightarrow (fog)'(x) > 0 \text{ and } (gof)'(x) > 0 \\ f'(x) > 0,\ g'(x) < 0 \Rightarrow (fog)'(x) < 0 \text{ and } (gof)'(x) < 0 \\ f'(x) < 0,\ g'(x) > 0 \Rightarrow (fog)'(x) < 0 \text{ and } (gof)'(x) < 0 \\ f'(x) < 0,\ g'(x) < 0 \Rightarrow (fog)'(x) > 0 \text{ and } (gof)'(x) > 0 \end{cases}$$

ie, from above definition,

(i) If $f(x)$ and $g(x)$ are both strictly increasing or strictly decreasing $\Rightarrow (fog)(x)$ and $(gof)(x)$ both are strictly increasing.

(ii) If amongst the two functions $f(x)$ and $g(x)$ one is strictly increasing and other is strictly decreasing.
$\Rightarrow (fog)(x)$ and $(gof)(x)$ both are strictly decreasing.

Point to Consider

Students are advised to learn it as the simple product of signs,

$f'(x)$	$g'(x)$	$(fog)'(x)$ or $(gof)'(x)$
+	+	+
+	–	–
–	+	–
–	–	+

where, (+) indicates strictly increasing and (–) indicates strictly decreasing.

Illustration 13 Let $\phi(x) = \sin(\cos x)$, then check of whether it is increasing or decreasing in $[0, \pi/2]$.

Solution. Given, $\phi(x) = \sin(\cos x)$

$$\phi'(x) = \cos(\cos x) \cdot (-\sin x)$$
$$\phi'(x) = -\cos(\cos x) \cdot \sin x$$

Therefore, it is clearly decreasing for $x \in [0, \pi/2]$ as $\phi'(x) \leq 0$.

Aliter **:** Here, $f(x) = \sin x$ and $g(x) = \cos x$ are increasing and decreasing in $[0, \pi/2]$.

$\Rightarrow (fog)(x) = \phi(x) = \sin(\cos x)$ is decreasing using above mentioned property (ii).

Illustration 14 Let $\phi(x) = \cos(\cos x)$, then check of whether it is increasing or decreasing in $[0, \pi/2]$.

Solution. Given, $\phi(x) = \cos(\cos x)$

$$\phi'(x) = -\sin(\cos x) \cdot (-\sin x)$$
$$\phi'(x) = \sin x \sin(\cos x),$$

Therefore, it is clearly increasing for $x \in [0, \pi/2]$ as $\phi'(x) \geq 0$.

Aliter Here, $f(x) = \cos x$ and $g(x) = \cos x$ are decreasing in $[0, \pi/2]$.

$\Rightarrow (fog)(x) = \cos(\cos x) = \phi(x)$ is increasing using above mentioned property (ii).

Illustration 15 Let $f(x) = \begin{cases} xe^{ax}; & x \leq 0 \\ x + ax^2 - x^3; & x > 0 \end{cases}$ where a is positive constant. Find the interval in which $f'(x)$ is increasing. **[IIT JEE 1996]**

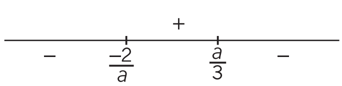
Solution. Given, $f(x) = \begin{cases} xe^{ax}; & x \leq 0 \\ x + ax^2 - x^3; & x > 0 \end{cases}$

Differentiating both the sides, we have

$$f'(x) = \begin{cases} axe^{ax} + e^{ax}; & x \leq 0 \\ 1 + 2ax - 3x^2; & x > 0 \end{cases}$$

Again, differentiating both sides, we have

$$f''(x) = \begin{cases} 2ae^{ax} + a^2xe^{ax}; & x \leq 0 \\ 2a - 6x; & x > 0 \end{cases}$$



Now, $f''(x) = 0$, then in the interval $x \leq 0$ the root is $x = -\dfrac{2}{a}$

and in interval when $x > 0$ root is $x = \dfrac{a}{3}$

Using sign scheme or number line rule, as shown in figure.

Hence, $f'(x)$ decreases on $\left(-\infty, -\dfrac{2}{a}\right) \cup \left(\dfrac{a}{3}, \infty\right)$ and increases on $\left(-\dfrac{2}{a}, \dfrac{a}{3}\right)$.

Illustration 16 If $a < 0$ and $f(x) = e^{ax} + e^{-ax}$ is monotonically decreasing. Find the interval to which x belongs.

Solution. Given, $a < 0$ and ...(i)

$$f(x) = e^{ax} + e^{-ax} \text{ is decreasing}$$

$$\Rightarrow \quad f'(x) < 0 \quad \Rightarrow \quad ae^{ax} - ae^{-ax} < 0$$

$$\Rightarrow \quad a\left(\frac{e^{2ax} - 1}{e^{ax}}\right) < 0 \qquad \text{...(ii)}$$

As from Eq. (i) $a < 0$

$$\Rightarrow \quad (e^{2ax} - 1) > 0 \quad \Rightarrow \quad e^{2ax} > 1$$

$$\Rightarrow \quad 2ax > 0 \quad \Rightarrow \quad ax > 0$$

$$\Rightarrow \quad x < 0 \text{ (as } a < 0)$$

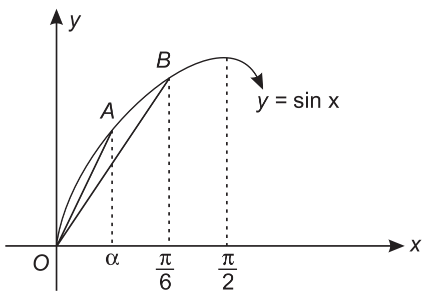
Thus, $f(x)$ is monotonically decreasing, if $x < 0$.

Illustration 17 If $0 < \alpha < \frac{\pi}{6}$, then the value of $(\alpha \operatorname{cosec} \alpha)$ is

(a) less than $\frac{\pi}{3}$ (b) more than $\frac{\pi}{3}$

(c) less than $\frac{\pi}{6}$ (d) more than $\frac{\pi}{6}$

Solution. As we know, $y = \sin x$ could be plotted as



From the figure, we can say

Slope of $OA >$ Slope of OB

where $A = (\alpha, \sin \alpha)$ and $B\left(\frac{\pi}{6}, \sin\frac{\pi}{6}\right)$

$$\Rightarrow \quad \frac{\sin \alpha - 0}{\alpha - 0} > \frac{\sin\frac{\pi}{6} - 0}{\frac{\pi}{6} - 0}$$

$$\Rightarrow \quad \frac{\sin \alpha}{\alpha} > \frac{3}{\pi} \quad \text{or} \quad \frac{\alpha}{\sin \alpha} < \frac{\pi}{3}$$

$$\Rightarrow \quad \alpha \operatorname{cosec} \alpha < \frac{\pi}{3}$$

Hence, (a) is the correct answer.

Illustration 18 If $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, d are real numbers and $3b^2 < c^2$, is an increasing cubic function and $g(x) = af'(x) + bf''(x) + c^2$, then

(a) $\int_a^x g(t)\, dt$ is a decreasing function

(b) $\int_a^x g(t)\, dt$ is an increasing function

(c) $\int_a^x g(t)\, dt$ is a neither increasing nor a decreasing function

(d) None of the above

Solution. $f'(x) = 3ax^2 + 2bx + c > 0$ [since $f(x)$ is increasing]

$\Rightarrow$ $a > 0$ and $b^2 - 3ac < 0$

$\Rightarrow$ $a > 0$ and $b^2 < 3ac$

Also, $g(x) = af'(x) + bf''(x) + c^2$

$g(x) = 3a^2x^2 + 2abx + ac + 6abx + 2b^2 + c^2$

$g(x) = 3a^2x^2 + 8abx + (2b^2 + c^2 + ac)$

where $D = 64a^2b^2 - 4 \cdot 3a^2 \cdot (2b^2 + c^2 + ac)$

$= 4a^2(16b^2 - 6b^2 - 3c^2 - 3ac)$

$= 4a^2(10b^2 - 3c^2 - 3ac) < 4a^2(10b^2 - 3c^2 - b^2)$

(as $3ac > b^2 \Rightarrow -3ac < -b^2$)

$= 4a^2(9b^2 - 3c^2)$

$= 12a^2(3b^2 - c^2)$ (given $3b^2 < c^2$)

$\therefore$ $D < 0$

$\Rightarrow$ $g(x) < 0, \forall\, x \in R$

$\therefore$ $\int_0^x g(t)\, dt$ is an increasing function.

Hence, (b) is the correct answer.

Illustration 19 If $f : R \to R$, $f(x)$ is a differentiable bijective function, then which of the following is true?

(a) $(f(x) - x) f''(x) < 0, \forall\, x \in R$

(b) $(f(x) - x) f''(x) > 0, \forall\, x \in R$

(c) If $(f(x) - x) f''(x) > 0$, then $f(x) = f^{-1}(x)$ has no solution

(d) If $(f(x) - x) f''(x) > 0$, then $f(x) = f^{-1}(x)$ has at least one real solution

Solution. As, $(f(x) - x) f''(x) < 0, \forall\, x \in R$

$\Rightarrow$ $(f(x) - x > 0$ and $f''(x) < 0)$ or $(f(x) - x < 0$ and $f''(x) > 0)$

Can't be true as $f(x) - x > 0$ and $f'(x)$ are decreasing. Then, $f(x)$ has to intersect the line $y = x$.

Similarly, $f(x) - x < 0$ and $f'(x)$ is increasing, is not possible.

Also, $f(x) - x \neq 0 \Rightarrow f(x) = f^{-1}(x)$ has no solution.

Hence, (c) is the correct answer.

Illustration 20 If $f(x)$ and $g(x)$ are two positive and increasing functions, then

(a) $(f(x))^{g(x)}$ is always increasing

(b) if $(f(x))^{g(x)}$ is decreasing, then $f(x) < 1$

(c) if $(f(x))^{g(x)}$ is increasing, then $f(x) > 1$

(d) if $f(x) > 1$, then $(f(x))^{g(x)}$ is increasing

Solution. Let $h(x) = (f(x))^{g(x)}$

$$\Rightarrow \qquad \log (h(x)) = g(x) \{\log f(x)\}$$

$$\Rightarrow \qquad \frac{1}{h(x)} \cdot h'(x) = \frac{g(x)}{f(x)} \cdot f'(x) + \{\log f(x)\}\, g'(x)$$

$\therefore$ $h(x)$ is decreasing, if $\log (f(x)) < 0 \Rightarrow f(x) < 1$

and $h(x)$ is increasing, if $\log (f(x)) > 0 \Rightarrow f(x) > 1$

Hence, (b) and (d) are the correct answers.

Illustration 21 If the function $y = \sin (f(x))$ is monotonic for all values of x (where $f(x)$ is continuous), then the maximum value of the difference between the maximum and the minimum value of $f(x)$, is

(a) π (b) 2π (c) $\frac{\pi}{2}$ (d) None of these

Solution. As, $y = \sin (f(x))$ is monotonic for

$$f(x) \in \left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right] \quad \text{or} \quad \left[2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right]$$

$\therefore$ The maximum value of difference is π.

Hence, (a) is the correct answer.

Illustration 22 If $f''(x) > 0$ and $f'(1) = 0$ such that $g(x) = f(\cot^2 x + 2\cot x + 2)$, where $0 < x < \pi$, then the interval in which $g(x)$ is decreasing is

(a) $(0, \pi)$ (b) $\left(\frac{\pi}{2}, \pi\right)$ (c) $\left(\frac{3\pi}{4}, \pi\right)$ (d) $\left(0, \frac{3\pi}{4}\right)$

Solution. Here, $g(x) = f(\cot^2 x + 2\cot x + 2)$

$$\Rightarrow \qquad g'(x) = f'(\cot^2 x + 2\cot x + 2) \cdot \{-2\cot x \operatorname{cosec}^2 x - 2\operatorname{cosec}^2 x\}$$

for $g(x)$ to be decreasing, $g'(x) < 0$

$$\Rightarrow \qquad f'\{(\cot x + 1)^2 + 1\} \cdot (-2\operatorname{cosec}^2 x)(\cot x + 1) < 0$$

$$\Rightarrow \qquad f'\{(\cot x + 1)^2 + 1\} \cdot (\cot x + 1) > 0 \qquad \ldots\text{(i)}$$

$$\left[\text{as } f''(x) > 0 \Rightarrow f'(x) \text{ is increasing, then } f'\{(\cot x + 1)^2 + 1\} > f'(1) = 0, \ \forall\ x \in \left(0, \frac{3\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \pi\right)\right]$$

Thus, Eq. (i) holds, if $\cot x + 1 > 0$

$$\Rightarrow \qquad \cot x > -1, \forall\ x \in \left(0, \frac{3\pi}{4}\right)$$

Hence, (d) is the correct answer.

Target Exercise 8.1

1. The interval in which $f(x) = \cot^{-1} x + x$ increases, is

(a) R (b) $(0, \infty)$ (c) $R - \{n\pi\}$ (d) None of these

2. The interval in which $f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3$ increase or decrease in $(0, \pi)$

(a) decreases on $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and increases on $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{2\pi}{3}, \pi\right)$

(b) decreases on $\left(\frac{\pi}{2}, \pi\right)$ and increases on $\left(0, \frac{\pi}{2}\right)$

(c) decreases on $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{2\pi}{3}, \pi\right)$ and increases on $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$

(d) decreases on $\left(0, \frac{\pi}{2}\right)$ and increases on $\left(\frac{\pi}{2}, \pi\right)$

3. The interval in which $f(x) = \int_0^x \{(t+1)(e^t - 1)(t-2)(t+4)\}\, dt$ increases and decreases

(a) increases on $(-\infty, -4) \cup (-1, 0) \cup (2, \infty)$ and decreases on $(-4, -1) \cup (0, 2)$
(b) increases on $(-\infty, -4) \cup (-1, 2)$ and decreases on $(-4, -1) \cup (2, \infty)$
(c) increases on $(-\infty, -4) \cup (2, \infty)$ and decreases on $(-4, 2)$
(d) increases on $(-4, -1) \cup (0, 2)$ and decreases on $(-\infty, -4) \cup (-1, 0) \cup (2, \infty)$

4. Let $f(x) = \int_0^x \left|\log_2(\log_3(\log_4(\cos t + a)))\right| dt$, be increasing for all real values of x, then 'a' belongs to

(a) $(0, \infty)$ (b) R (c) $(-\infty, 5)$ (d) $[5, \infty)$

5. The interval of monotonicity of the function $f(x) = x^2 - \log_e |x|$, when $(x \neq 0)$ is

(a) increasing for all $x > 0$ and decreasing for all $x < 0$

(b) increasing when $x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(0, \frac{1}{\sqrt{2}}\right)$ and decreasing when $x \in \left(-\frac{1}{\sqrt{2}}, 0\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$

(c) increasing when $x \in \left(-\frac{1}{\sqrt{2}}, 0\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$ and decreasing when $x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(0, \frac{1}{\sqrt{2}}\right)$

(d) increasing when $x \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) - \{0\}$ and decreasing when $x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$

6. The interval of monotonicity of the function $f(x) = \dfrac{x}{\log_e x}$, is

(a) increases when $x \in (e, \infty)$ and decreases when $x \in (0, e)$
(b) increases when $x \in (e, \infty)$ and decreases when $x \in (0, e) - \{1\}$
(c) increases when $x \in (0, e)$ and decreases when $x \in (e, \infty)$
(d) increases when $x \in (0, e) - \{1\}$ and decreases when $x \in (e, \infty)$

7. The function $f(x)=\cos\left(\frac{\pi}{x}\right)$ is

(a) increasing when $x \in \left(\frac{1}{2K+1}, \frac{1}{2K}\right)$; $K \in$ +ve integer and decreasing when $x \in \left(\frac{1}{2K+2}, \frac{1}{2K+1}\right)$; $K \in$ +ve integer

(b) increasing when $x \in \left(\frac{1}{2K+2}, \frac{1}{2K+1}\right)$ and decreasing when $x \in \left(\frac{1}{2K+1}, \frac{1}{2K}\right)$ $K \in$ +ve integer

(c) increasing when $x \in (2K, 2K+1)$ and decreasing when $x \in (2K+1, 2K+2)$, $K \in$ +ve integer

(d) increasing when $x \in (2K+1, 2K+2)$ and decreasing when $x \in (2K, 2K+1)$, $K \in$ +ve integer

8. The interval in which $f(x)=\sin(\log_e x)-\cos(\log_e x)$ is strictly increasing, when

(a) $x \in [e^{2n\pi+\pi/4}, e^{2n\pi+5\pi/4}]$
(b) $x \in [e^{2n\pi-\pi/4}, e^{2n\pi+3\pi/4}]$
(c) $x \in [e^{2n\pi-\pi/4}, e^{2n\pi+5\pi/4}]$
(d) $x \in [e^{2n\pi\pm\pi/4}, e^{2n\pi\pm3\pi/4}]$

9. Let $f(x)=x^3+ax^2+bx+5\sin^2 x$ be an increasing function on the set R. Then,

(a) $a^2-3b+15>0$
(b) $a^2-3b+5<0$
(c) $a^2-3b+15<0$
(d) $a^2-3b+5>0$

10. Let $g(x)=f(x)+f(1-x)$ and $f''(x)>0, \forall\, x \in (0, 1)$. Then, $g(x)$ is

(a) increasing on $\left(0, \frac{1}{2}\right)$ and decreasing on $\left(\frac{1}{2}, 1\right)$

(b) increasing on $\left(\frac{1}{2}, 1\right)$ and decreasing on $\left(0, \frac{1}{2}\right)$

(c) increasing on (0, 1)

(d) decreasing on (0, 1)

Critical Points

It is a collection of points for which,

(i) $f(x)$ does not exists.

(ii) $f'(x)$ does not exists.

(iii) $f'(x)=0$

All the values of x obtained from above conditions are said to be the critical points.

It should be noted that critical points are the interior points of an interval.

Illustration 23 Find the critical points for $f(x)=(x-2)^{2/3}(2x+1)$.

Solution. Given, $f(x)=(x-2)^{2/3}(2x+1)$

$$\Rightarrow \qquad f'(x)=\frac{2}{3}(x-2)^{-1/3}(2x+1)+(x-2)^{2/3}\cdot 2$$

$$\text{or} \qquad f'(x)=2\left[\frac{(2x+1)}{3(x-2)^{1/3}}+\frac{(x-2)^{2/3}}{1}\right]$$

Clearly, $f'(x)$ is not defined at $x=2$, so $x=2$ is a critical point.

Another critical point is given by,

$$f'(x)=0$$

$$\text{ie,} \qquad 2\left[\frac{(2x+1)+3\,(x-2)}{(x-2)^{1/3}}\right]=0$$

$$\Rightarrow \qquad 5x-5=0$$

$$\Rightarrow \qquad x=1$$

Hence, $x=1$ and $x=2$ are two critical points of $f(x)$.

Illustration 24 Find all the values of a for which the function $f(x)=(a^2-3a+2)\cos\left(\frac{x}{2}\right)+(a-1)\,x$, possess critical points.

Solution. Given, $f(x)=(a^2-3a+2)\cos\frac{x}{2}+(a-1)\,x$

$$\Rightarrow \qquad f'(x)=-\frac{1}{2}(a-1)(a-2)\sin\left(\frac{x}{2}\right)+(a-1)$$

$$\Rightarrow \qquad f'(x)=(a-1)\left[1-\frac{1}{2}(a-2)\sin\left(\frac{x}{2}\right)\right]$$

If $f(x)$ possess critical points, then

$$f'(x)=0$$

$$\Rightarrow \qquad (a-1)\left[1-\left(\frac{a-2}{2}\right)\sin\frac{x}{2}\right]=0$$

$$\Rightarrow \qquad a=1 \quad \text{and} \quad 1-\left(\frac{a-2}{2}\right)\sin\frac{x}{2}=0,$$

$$\text{but at} \qquad a=1$$

$$\Rightarrow \qquad f(x)=0$$

$$\text{Thus,} \qquad \sin\left(\frac{x}{2}\right)=\frac{2}{a-2}$$

$$\left|\frac{2}{a-2}\right|\le 1$$

$$|a-2|\ge 2$$

$$a-2\ge 2 \qquad \text{or} \qquad a-2\le -2$$

$$a\ge 4 \qquad \text{or} \qquad a\le 0$$

$$a\in(-\infty,0]\cup[4,\infty)$$

$$\text{Therefore,} \qquad a\in(-\infty,0]\cup[4,\infty)$$

Comparison of Functions Using Calculus

If we want to compare $f(x)$ and $g(x)$ consider a function $\phi(x) = f(x) - g(x)$ or $\phi(x) = g(x) - f(x)$ and check whether $\phi(x)$ is increasing or decreasing in given domain of $f(x)$ and $g(x)$.

Illustration 25 Using calculus, find the order relation between x and $\tan^{-1} x$ when $x \in [0, \infty)$.

Solution. Let $f(x) = x - \tan^{-1}(x)$

$$\Rightarrow \quad f'(x) = 1 - \frac{1}{1+x^2}$$

$$\Rightarrow \quad f'(x) = \frac{x^2}{1+x^2} \geq 0, \forall x \in [0, \infty)$$

Thus, $f(x)$ is a increasing function.

(As, we know $x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$ for increasing functions)

$\therefore \quad x \geq 0, \ \forall x \in [0, \infty)$

$\Rightarrow \quad f(x) \geq f(0), \ \forall x \in [0, \infty)$

$\Rightarrow \quad x - \tan^{-1} x \geq 0 - \tan^{-1}(0), \forall x \in [0, \infty)$

$\Rightarrow \quad x \geq \tan^{-1} x, \forall x \in [0, \infty)$

Thus, the above is order relation between x and $\tan^{-1} x$.

Illustration 26 Using calculus, find the order relation between x and $\tan^{-1} x$ when $x \in (-\infty, 0]$.

Solution. Let $f(x) = (x) - \tan^{-1}(x)$

$$\Rightarrow \quad f'(x) = 1 - \frac{1}{1+x^2}$$

$$\Rightarrow \quad f'(x) = \frac{x^2}{1+x^2} \geq 0, \forall x \in (-\infty, 0]$$

Thus, $f(x)$ is increasing function.

(As, we know $x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$ for increasing functions).

$\therefore \quad x \leq 0, \forall x \in (-\infty, 0]$

$\Rightarrow \quad f(x) \leq f(0), \forall x \in (-\infty, 0]$

$\Rightarrow \quad x - \tan^{-1} x \leq 0, \forall x \in (-\infty, 0]$

$\Rightarrow \quad x \leq \tan^{-1} x, \forall x \in (-\infty, 0]$

Thus, the above is order relation between x and $\tan^{-1} x$.

Illustration 27 The set of all values of x for which $\log(1+x) \leq x$.

Solution. Let $f(x) = \log(1+x) - x$

where $f(x)$ is defined for $x > -1$.

Now, $\quad f(x) = \log(1+x) - x$

$$\Rightarrow \quad f'(x) = \frac{1}{1+x} - 1 = \frac{-x}{1+x}$$

Now, if $\quad x \geq 0 \quad \Rightarrow \quad f'(x) \leq 0 \qquad$ ...(i)

Again, if $\quad -1 < x \le 0 \quad \Rightarrow \quad f'(x) \ge 0$ ...(ii)

Taking Eq. (i) $\quad f(x)$ is decreasing when $x \ge 0$,

$\therefore \quad f(x) \le f(0) \quad \Rightarrow \quad \log(1+x) \le x$

Taking Eq. (ii) $\quad f(x)$ is increasing, when $-1 < x \le 0$

$\therefore \quad f(x) \le f(0) \quad \Rightarrow \quad \log(1+x) \le x$

Thus, the set of values of x for which $\log(1+x) \le x \Rightarrow x \in (-1, \infty)$. *ie,* in its domain.

Illustration 28 For all $x \in (0, 1)$ **[IIT JEE 2000]**

(a) $e^x < 1 + x$ (b) $\log_e(1+x) < x$ (c) $\sin x > x$ (d) $\log_e x > x$

Solution. (a) Let $f(x) = e^x - 1 - x$

$\Rightarrow \quad f'(x) = e^x - 1 > 0, \forall x \in (0, 1)$

So, $\quad f(x)$ is increasing, when $0 < x < 1$

$\Rightarrow \quad f(x) > f(0) \quad$ or $\quad e^x - 1 - x > 0$

$\Rightarrow \quad e^x > 1 + x$

Hence, (a) is false.

(b) Let $g(x) = \log(1+x) - x$

$$\Rightarrow \quad g'(x) = \frac{1}{1+x} - 1 = \frac{-x}{1+x} < 0, \ \forall x \in (0, 1)$$

So, $\quad g(x)$ is decreasing, when $0 < x < 1$

$\Rightarrow \quad g(0) > g(x) \ \Rightarrow \ \log(1+x) < x$

Hence, (b) is correct.

(c) Let $\quad h(x) = \sin x - x$

$\Rightarrow \quad h'(x) = \cos x - 1 < 0, \ \forall x \in (0, 1)$

So, $h(x)$ is decreasing, when $0 < x < 1 \ \Rightarrow \ h(x) < h(0)$

$\Rightarrow \quad \sin x < x$

Hence, (c) is false.

(d) Let $g(x) = \log x - x \quad \Rightarrow \quad g'(x) = \dfrac{1}{x} - 1$

$\therefore \quad g'(x) > 0, \forall x \in (0, 1)$

or $\quad g(x) < g(1) \quad \Rightarrow \quad \log x - x < -1$

$\Rightarrow \quad x - 1 > \log x \quad$ or $\quad x > \log x$

Hence, (d) is false.

Hence, (b) is the correct answer.

Illustration 29 Prove that $\left(\tan^{-1}\dfrac{1}{e}\right)^2 + \dfrac{2e}{\sqrt{e^2+1}} < (\tan^{-1} e)^2 + \dfrac{2}{\sqrt{e^2+1}}$.

Solution. Let us consider a function $f(x)$,

$$ie, \quad f(x) = (\tan^{-1} x)^2 + \frac{2}{\sqrt{x^2+1}} \text{ for all } x \in R$$

$$\therefore \quad f'(x) = \frac{2\tan^{-1} x}{1+x^2} - \frac{2x}{(x^2+1)^{3/2}}$$

$$= \frac{2}{1+x^2}\left[\tan^{-1} x - \frac{x}{\sqrt{x^2+1}}\right]$$

$$f'(x) = \frac{2}{1+x^2}\, g(x), \text{ where } g(x) = \tan^{-1} x - \frac{x}{\sqrt{x^2+1}} \qquad \text{...(i)}$$

$$\therefore \qquad g'(x) = \frac{1}{1+x^2} - \frac{1}{\sqrt{x^2+1}} + \frac{x^2}{(x^2+1)^{3/2}}$$

$$= \frac{1}{1+x^2} - \frac{1}{(x^2+1)^{3/2}}$$

$$= \frac{1}{1+x^2}\left(1 - \frac{1}{\sqrt{(x^2+1)}}\right) > 0 \text{ for all } x \in R$$

$\Rightarrow \qquad g(x)$ is increasing for $x \in R$

$\Rightarrow \qquad g(x) > g(0)$ for all $x \in R$

$$\Rightarrow \qquad \tan^{-1} x - \frac{x}{\sqrt{x^2+1}} > 0 \qquad \text{...(ii)}$$

$\therefore$ From Eqs. (i) and (ii),

$f'(x) > 0$ for all $x > 0$

$\therefore \qquad f(x)$ is increasing for all $x > 0$

$\Rightarrow \qquad f(1/e) < f(e)$

$$\Rightarrow \qquad \left(\tan^{-1}\frac{1}{e}\right)^2 + \frac{2e}{\sqrt{e^2+1}} < (\tan^{-1} e)^2 + \frac{2}{\sqrt{e^2+1}}$$

Illustration 30 The set of all values of 'b' for which the function $f(x) = (b^2 - 3b + 2)(\cos^2 x - \sin^2 x) + (b-1)x + \sin 2$ does not possess stationary points is

(a) $[1, \infty)$ (b) $(0, 1) \cup (1, 4)$

(c) $\left(\frac{3}{2}, \frac{5}{2}\right)$ (d) None of these

Solution. Here, $f'(x) = (b^2 - 3b + 2)\cdot(-2\sin 2x) + (b-1)$

As, $f(x)$ does not possess stationary points

$\Rightarrow \qquad f'(x) \neq 0$

$\Rightarrow \qquad (b-1)(b-2)(-2\sin 2x) + (b-1) \neq 0$ for any $x \in R$

$\Rightarrow \qquad (b-1)\{1 - 2(b-2)\sin 2x\} \neq 0$

$$\Rightarrow \qquad \left|\frac{1}{2(b-2)}\right| > 1 \text{ and } b \neq 1$$

$$\Rightarrow \qquad -\frac{1}{2} < b - 2 < \frac{1}{2} \text{ and } b \neq 1$$

$$\Rightarrow \qquad b \in \left(\frac{3}{2}, \frac{5}{2}\right)$$

Hence, (c) is the correct answer.

Target Exercise 8.2

1. The number of critical points of $f(x) = \max\{\sin x, \cos x\}, \forall\, x \in (-2\pi, 2\pi)$.

(a) 5 (b) 6 (c) 7 (d) 8

2. Which of the following is true?

(a) $1 + x \log(x + \sqrt{x^2+1}) \geq \sqrt{1+x^2}$ for all $x \geq 0$

(b) $1 + x \log(x + \sqrt{x^2+1}) > \sqrt{1+x^2}$ for all $x \geq 0$

(c) $1 + x \log(x + \sqrt{1+x^2}) \leq \sqrt{1+x^2}$ for all $x \geq 0$

(d) $1 + x \log(x + \sqrt{x^2+1}) < \sqrt{1+x^2}$ for all $x \geq 0$

3. Which of the following holds true for $x \in \left[0, \dfrac{\pi}{2}\right)$?

(a) $x - \dfrac{x^3}{6} \leq \sin x \leq x$ (b) $x - \dfrac{x^3}{6} \geq \sin x \geq x$

(c) $2\sin x + \tan x \geq 3x$ (d) $2\sin x + \tan x \leq 3x$

4. Let $f(x) = x\tan x, \forall\, 0 < x < \dfrac{\pi}{2}$. Then

(a) $\dfrac{\tan x_1}{\tan x_2} > \dfrac{x_2}{x_1}$, where $0 < x_1 < x_2 < \dfrac{\pi}{2}$ (b) $\dfrac{\tan x_1}{\tan x_2} > \dfrac{x_1}{x_2}$, where $0 < x_1 < x_2 < \dfrac{\pi}{2}$

(c) $\dfrac{\tan x_2}{\tan x_1} > \dfrac{x_1}{x_2}$, where $0 < x_1 < x_2 < \dfrac{\pi}{2}$ (d) None of these

5. Let $f(x) = \dfrac{\sin x}{x}$, where $0 < x < \dfrac{\pi}{2}$. Then

(a) $\sin^2 x < x\sin(\sin x)$ (b) $\sin^2 x > x\sin(\sin x)$

(c) $\sin^2 x > 1 + x\sin(\sin x)$ (d) None of these

Introduction to Maxima and Minima

By the maximum/minimum value of function $f(x)$ we mean local or regional maximum/minimum and not the greatest/least value attainable by the function. It is also possible in a function that local maximum at one point is smaller than local minimum at another point. Sometimes we use the word extrema for maxima and minima.

Definition A function $f(x)$ is said to have a maximum at $x = a$, if $f(a)$ is greatest of all values in the suitably small neighbourhood of a, where $x = a$ is an interior point in the domain of $f(x)$. Analytically this means $f(a) \geq f(a+h)$ and $f(a) \geq f(a-h)$, where $h \geq 0$ (very small quantity).

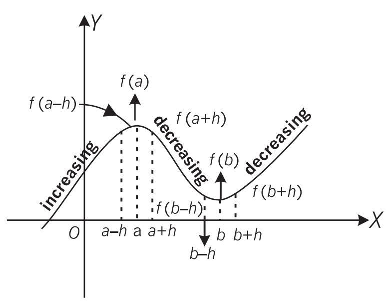


Fig. 8.15

Similarly, a function $y = f(x)$ is said to have a minimum at $x = b$. If $f(b)$ is smallest of all values in the suitably small neighbourhood of b, where $x = b$ is an interior point in the domain of $f(x)$. Analytically,

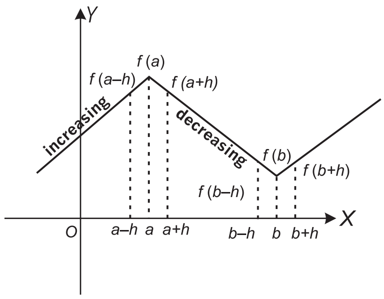


Fig. 8.16

$f(b) \leq f(b+h)$ and $f(b) \leq f(b-h)$ where $h \geq 0$ (very small quantity)

Method of Finding Extrema of Continuous Functions

(a) First Derivative Test

The following test applies to a continuous function in order to get the extrema.

As we know the function attains maximum, when it has assumed its maximum value and attains minimum, when it has assumed its minimum value which could be shown as :

(1) At a critical point $x = x_0$

(i) When $f(x)$ attains maximum at ($x = a$)

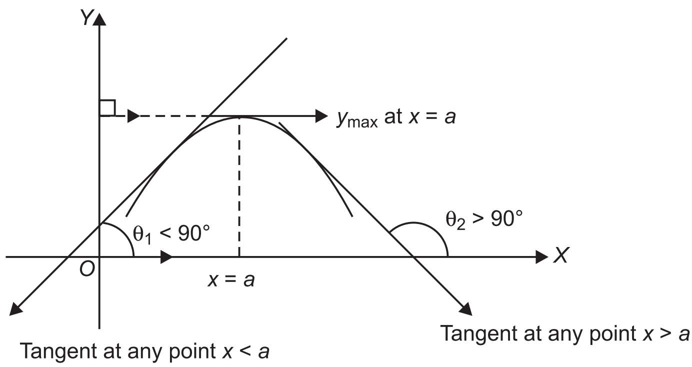


Fig. 8.17

ie, from above graph,

$$\begin{cases} \text{for } x < a, \theta_1 < 90° \Rightarrow \tan\theta_1 > 0 \text{ or increasing for } x < a \\ \text{for } x = a, \tan\theta = 0 \quad \text{ or neither increasing nor decreasing for } x = a \\ \text{for } x > a, \theta_2 > 90° \Rightarrow \tan\theta_2 < 0 \text{ or decreasing for } x > a \end{cases}$$

Thus, we can say,

$f(x)$ is maximum at some point $(x = a)$.

$$\Rightarrow \quad \begin{cases} f(x) \text{ is increasing for } x < a \\ f(x) \text{ is decreasing for } x > a \end{cases}$$

(ii) **When $f(x)$ attains minimum at $(x = a)$**

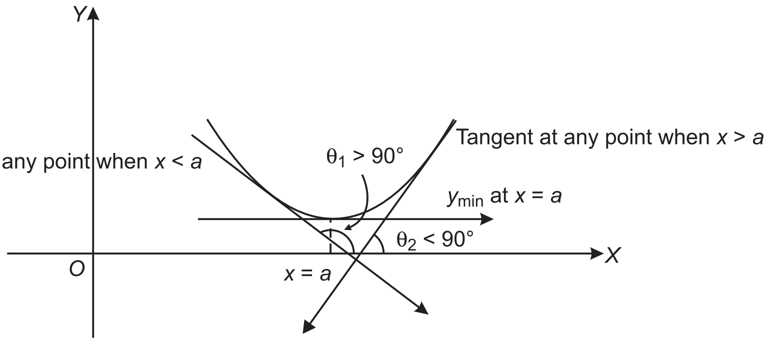


Fig. 8.18

ie, from the above graph,

$$\begin{cases} \text{for } x < a, \theta_1 > 90^\circ \Rightarrow \tan\theta_1 < 0 \text{ or decreasing when } x < a \\ \text{for } x = a, \tan\theta = 0 \text{ or neither increasing nor decreasing for } x = a \\ \text{for } x > a, \theta_2 < 90^\circ \Rightarrow \tan\theta_2 > 0 \text{ or increasing when } x > a \end{cases}$$

Thus, we can say,

$f(x)$ is minimum at some points '$x = a$',

$$\Rightarrow \quad \begin{cases} f(x) \text{ is decreasing for } x < a \\ f(x) \text{ is increasing for } x > a \end{cases}$$

Here, some of the Illustrations are given to make it more clear.

Illustration 31 If $f'(x)$ changes from positive to negative at x_0 while moving from left to right,

ie, $f'(x) > 0, \quad x < x_0$

$f'(x) < 0, \quad x > x_0$, then $f(x)$ has local maximum value at $x = x_0$.

Solution.

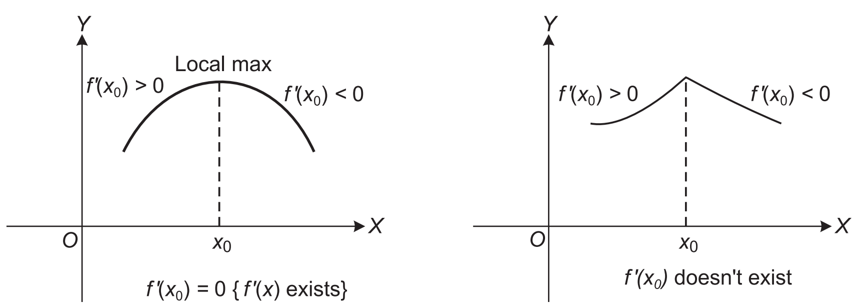
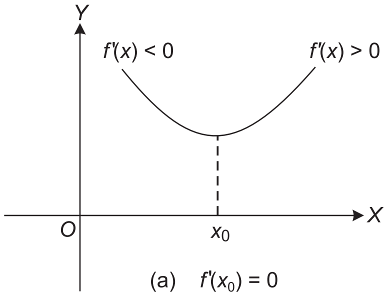


Illustration 32 If $f'(x)$ changes from negative to positive at x_0 while moving from left to right,

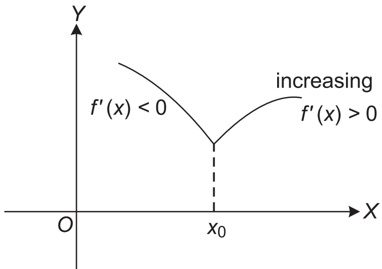
ie, $$f'(x) < 0,\ x < x_0$$
$$f'(x) > 0,\ x > x_0,$$

then $f(x)$ has local minimum value at $x = x_0$

Solution.



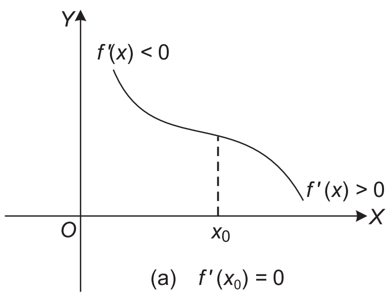
(a) $f'(x_0) = 0$



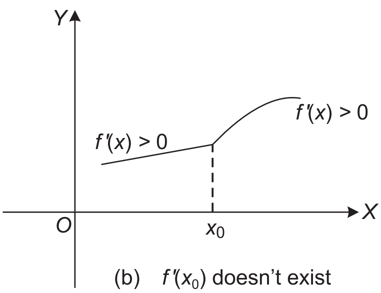
(b) $f'(x_0)$ doesn't exist

Illustration 33 If sign of $f'(x)$ doesn't change at x_0, then $f(x)$ has neither a maximum nor a minimum at x_0.

Solution.



(a) $f'(x_0) = 0$



(b) $f'(x_0)$ doesn't exist

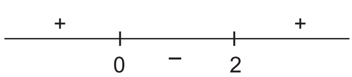
Illustration 34 Let $f(x) = x^3 - 3x^2 + 6$ find the point at which $f(x)$ assumes local maximum and local minimum.

Solution. Given,

$$f(x) = x^3 - 3x^2 + 6$$
$$\Rightarrow \quad f'(x) = 3x^2 - 6x = 3x(x-2)$$

using number line rule,

$f'(x)$ changes sign from +ve to –ve at $x = 0$ and $f'(x)$ changes sign from –ve to +ve at $x = 2$.



Therefore, at $x = 0$, we have local maxima and at $x = 2$, we have local minima.

Illustration 35 Let $f(x) = x^3$ find the point at which $f(x)$ assumes local maximum and local minimum.

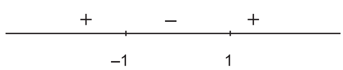
Solution. Given, $f(x) = x^3 \quad \Rightarrow \quad f'(x) = 3x^2$

Here, $f'(x) \geq 0$ for all x, it does not change sign from –ve to +ve or +ve to –ve.

Thus, at $x = 0$, we have neither maximum nor minimum.

Illustration 36 Let $f(x) = x + \dfrac{1}{x}$, $x \neq 0$. Discuss the maximum and minimum value of $f(x)$.

Solution. Here, $f'(x) = 1 - \dfrac{1}{x^2}$



$$f'(x) = \frac{x^2 - 1}{x^2} = \frac{(x-1)(x+1)}{x^2}$$

Using number line rule, we have maximum at $x = -1$ and minimum at $x = 1$

$\therefore$ At $x = -1$, we have local maximum $\Rightarrow f(x)_{\text{max}} = -2$

and at $x = 1$, we have local minimum $\Rightarrow f(x)_{\text{min}} = 2$

(2) At a left end point *a* and right end point *b* in [*a*, *b*]

Let $f(x)$ be defined on $[a, b]$.

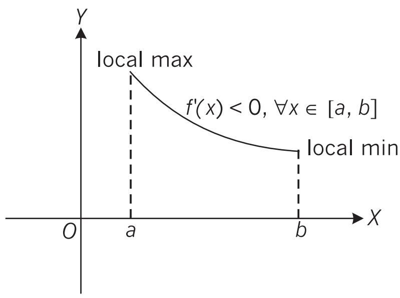


Fig. 8.19

If $f'(x) < 0$ for $x > a$, then $f(x)$ has local maximum at $x = a$ and local minimum at $x = b$.

Again, if

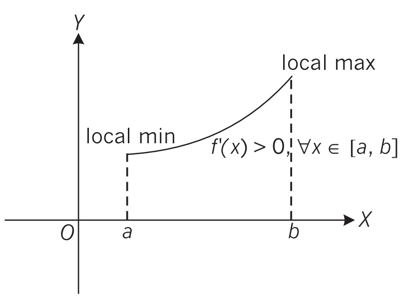


Fig. 8.20

$f'(x) > 0$ for $x > a$, then $f(x)$ has local minimum at $x = a$ and local maximum at $x = b$.

Points to Consider

(i) If a function is strictly increasing in $[a, b]$, then

$$\begin{cases} f(a) \text{ is local minimum} \\ f(b) \text{ is local maximum} \end{cases}$$

(ii) If a function is strictly decreasing in $[a, b]$, then

$$\begin{cases} f(a) \text{ is local maximum} \\ f(b) \text{ is local minimum} \end{cases}$$

Illustration 37 Find the local maximum and local minimum of $f(x) = x^3 + 3x$ in $[-2, 4]$.

Solution. Given, $f(x) = x^3 + 3x$

$\Rightarrow f'(x) = 3x^2 + 3$ which is strictly increasing for all $x \in R$ and thus, increasing for $[-2, 4]$.

Hence, local minimum is $f(-2) = (-2)^3 + 3\,(-2) = -14$

and local maximum is $f(4) = (4)^3 + 3\,(4) = 76$

Illustration 38 If $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3, \end{cases}$, then **[IIT JEE 1993]**

(a) $f(x)$ is increasing on $[-1, 2]$

(b) $f(x)$ is continuous on $[-1, 3]$

(c) $f'(x)$ does not exist at $x = 2$

(d) $f(x)$ has the maximum value at $x = 2$

Solution. Given, $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3 \end{cases}$

$$\Rightarrow \qquad f'(x) = \begin{cases} 6x + 12, & -1 \le x \le 2 \\ -1, & 2 < x \le 3 \end{cases}$$

(a) Which shows $f'(x) > 0$ for $x \in [-1, 2]$

So, $f(x)$ is increasing on $[-1, 2]$.

Hence, (a) is correct.

(b) For continuity of $f(x)$. (check at $x = 2$)

RHL $= 35$, LHL $= 35$ and $f(2) = 35$

So, (b) is correct.

(c) As discussed in previous chapter,

$$R\,f'(2) = -1 \qquad \text{and} \qquad L\,f'(2) = 24$$

So, not differentiable at $x = 2$

Hence, (c) is correct.

(d) We know $f(x)$ is increasing on $[-1, 2]$ and decreasing on $[2, 3]$.

Thus, maximum at $x = 2$

Hence, (d) is correct.

$\therefore$ Hence, (a), (b), (c) and (d) all are correct answers.

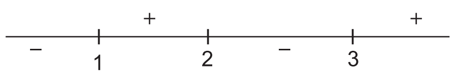
Illustration 39 The function $f(x) = \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3(t-3)^5 dt$ has a local maximum at x equals to **[IIT JEE 1999]**

(a) 0 (b) 1 (c) 2 (d) 3

Solution. Given, $f(x) = \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3(t-3)^5 dt$

$$f'(x) = x(e^x - 1)(x-1)(x-2)^3(x-3)^5 \qquad \text{(using Leibnitz rule)}$$

Using number line rule for $f'(x)$, we get given figure which shows local maxima at $x = 2$ as $f'(x)$ changes from (+ve) to (–ve) and local minima at $x = 1$ and $x = 3$ as $f'(x)$ changes from (–ve) to (+ve).



$\therefore$ Local minima at $x = 1, 3$ and local maxima at $x = 2$.

Hence, (c) is the correct answer.

Illustration 40 Find the set of critical points of the function

$$f(x) = x - \log x + \int_2^x \left(\frac{1}{z} - 2 - 2\cos 4z\right) dz$$

Solution. Here, $\quad f(x) = x - \log x + \int_2^x \left(\frac{1}{z} - 2 - 2\cos 4z\right) dz$

$\Rightarrow \quad f'(x) = 1 - \frac{1}{x} + \left(\frac{1}{x} - 2 - 2\cos 4x\right)(1) - 0 = -1 - 2\cos 4x$

Put $\quad f'(x) = 0,$

$\Rightarrow \quad \cos 4x = -\frac{1}{2} \quad$ or $\quad \cos 4x = \cos\left(\pi - \frac{\pi}{3}\right)$

or $\quad 4x = 2n\pi + \frac{2\pi}{3}, n \in$ integer $\quad \Rightarrow \quad x = \frac{n\pi}{2} \pm \frac{\pi}{6}, n \in I$

But $\log x$ is defined for $x > 0$

$\therefore$ For $n = 0$, $x = \pm\frac{\pi}{6}$ (neglecting $x = -\pi/6$) $\quad \therefore \quad x = \pi/6$

$\therefore$ Set of critical points $= \left\{\frac{\pi}{6}, \frac{n\pi}{2} \pm \frac{\pi}{6}\right\}$ where $n \in N$.

Illustration 41 Let $f(x) = \sin x - x$ on $[0, \pi/2]$, find local maximum and local minimum.

Solution. Given, $\quad f(x) = \sin x - x$

$\quad f'(x) = \cos x - 1, \ \forall x \in [0, \pi/2]$

$\quad (\cos x - 1) \le 0, \forall x \in [0, \pi/2]$, as $\cos x \le 1$

$\therefore \quad f'(x) < 0, \forall x \in [0, \pi/2]$

$\therefore \quad f(x)$ is decreasing for $x \in [0, \pi/2]$.

Hence, maximum value of $f(x)$ is at $x = 0$

ie, $\quad f_{\max}(0) = \sin 0 - 0 = 0$

and minimum value of $f(x)$ is at $x = \pi/2$

ie, $\quad f_{\min}\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} - \frac{\pi}{2} = 1 - \frac{\pi}{2}$

(b) Second Derivative Test

First we find the root of $f'(x)=0$. Suppose $x=a$ is one of the roots of $f'(x)=0$.

Now, find $f''(x)$ at $x=a$

(i) If $f''(a)=$ negative, then $f(x)$ is maximum at $x=a$

(ii) If $f''(a)=$ positive, then $f(x)$ is minimum at $x=a$

(iii) If $f''(a)=$ zero

Then, we find $f'''(x)$ at $x=a$

If $f'''(a)\neq 0$, then $f(x)$ has neither maximum nor minimum (inflexion point). at $x=a$.

But, if $f'''(a)=0$, then find $f^{iv}(a)$

If $f^{iv}(a)=$ positive, then $f(x)$ is minimum at $x=a$

If $f^{iv}(a)=$ negative, then $f(x)$ is maximum at $x=a$

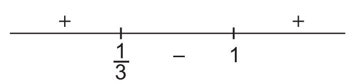
and so on, process is repeated till point is discussed.

Point to Consider

It must be remembered that this method is not applicable to those critical points where $f'(x)$ remains undefined.

Illustration 42 Let $f(x)=x(x-1)^2$, find the point at which $f(x)$ assumes maximum and minimum.

Solution. Given, $f(x)=x(x-1)^2$



$$f'(x)=2x(x-1)+(x-1)^2$$
$$f'(x)=(x-1)[2x+x-1]$$
$$f'(x)=(x-1)(3x-1)$$

Using number line rule for $f'(x)$, we have given figure which shows $f'(x)$ changes sign from +ve to –ve at $x=1/3$. Hence, at $x=1/3$, we have maximum and $f'(x)$ changes sign from – ve to +ve at $x=1$.

Hence, minimum at $x=1$

Aliter We have, $f'(x)=(x-1)(3x-1)$ $f''(x)=6x-4$

Let $f'(x)=0 \Rightarrow x=1, 1/3$ (critical points)

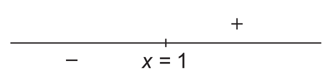
$\therefore$ $f''(1)=2>0$

ie, minimum at $x=1$ and $f''(1/3)=-2<0$

ie, maximum at $x=1/3$

Illustration 43 Let $f(x)=(x-1)^4$ discuss the point at which $f(x)$ assumes maximum or minimum value.

Solution. Given, $f(x)=(x-1)^4$



$$f'(x)=4(x-1)^3$$

Using number line rule.

Shows $f'(x)$ changes sign from –ve to +ve.

Therefore, $f(x)$ assumes minimum at $x=1$.

Aliter : Given, $f(x) = (x-1)^4 \Rightarrow f'(x) = 4(x-1)^3$

Let $f'(x) = 0 \Rightarrow x = 1$

Now, $f''(x) = 12(x-1)^2$ which is zero at $x = 1$

ie, $f''(1) = 0$

Thus, finding $f'''(x) = 24(x-1)$ which is again zero at $x = 1$

ie, $f'''(1) = 0$

Again, finding $f^{iv}(x) = 24$ which is positive at $x = 1$,

ie, $f^{iv}(1) > 0$

Therefore, minimum at $x = 1$.

Concept of Global Maximum/Minimum

Let $y = f(x)$ be a given function with domain D.

Let $[a, b] \leq D$, then global maximum/minimum of $f(x)$ in $[a, b]$ is basically the greatest/least value of $f(x)$ in $[a, b]$.

Global maxima/minima in $[a, b]$ would always occur at critical points of $f(x)$ with in $[a, b]$ or at the end points of the interval.

Global Maximum/Minimum in $[a, b]$

In order to find the global maximum and minimum of $f(x)$ in $[a, b]$, find out all critical points of $f(x)$ in $[a, b]$ (*ie*, all points at which $f'(x) = 0$).

Let $c_1, c_2, c_3, \ldots, c_n$ be the points at which $f'(x) = 0$
and let $f(c_1), f(c_2), \ldots, f(c_n)$ be the values of the function at these points.
Then, $M_1 \rightarrow$ Global maxima or the greatest value.
and $M_2 \rightarrow$ Global minima or the least value.

where, $M_1 = \max\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$

and $M_2 = \min\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$

Then, M_1 is the greatest value or global maxima in $[a, b]$ and M_2 is the least value or global minima in $[a, b]$.

Illustration 44 Let $f(x) = 2x^3 - 9x^2 + 12x + 6$. Discuss the global maxima and minima of $f(x)$ in [0, 2].

Solution. Given, $f(x) = 2x^3 - 9x^2 + 12x + 6 \Rightarrow f'(x) = 6x^2 - 18x + 12$

$\Rightarrow f'(x) = 6(x^2 - 3x + 2)$

$\Rightarrow f'(x) = 6(x-1)(x-2)$

Let $f'(x) = 0 \quad \therefore \quad x = 1, 2$ (say c_1 and c_2)

Then, for global maximum or global minimum.

We have, $f(0) = 6, \ f(1) = 11, \ f(2) = 10$,

$\therefore$ Global maximum $\Rightarrow M_1 = \max\{6, 10, 11\} = 11$

and global minimum $\Rightarrow M_2 = \min\{6, 10, 11\} = 6$

$\therefore f(1) = 11$ global maximum and $f(0) = 6$ global minimum.

Global Maximum/Minimum in (a, b)

Method for obtaining the greatest and least values of $f(x)$ in (a, b) is almost the same as the method used for obtaining the greatest and least values in $[a, b]$.

However a caution may be exercised

let $M_1 = \max\{f(c_1), f(c_2), \dots, f(c_n)\}$

and $M_2 = \min\{f(c_1), f(c_2), \dots, f(c_n)\}$

Now, M_1 and M_2 are global maximum and global minimum respectively.

But, if $\lim_{x \to a^+} f(x) > M_1$ or $\lim_{x \to b^-} f(x) < M_2$

$\Rightarrow f(x)$ would not possess global maximum or global minimum in (a, b).

This means that the limiting values at the end points are greater than M_1 or less than M_2, then global maximum or global minimum does not exist in (a, b).

Illustration 45 Let $f(x) = 2x^3 - 9x^2 + 12x + 6$. Discuss the global maxima and global minima of $f(x)$ in (1, 3).

Solution. $f(x) = 2x^3 - 9x^2 + 12x + 6$

$\Rightarrow$ $f'(x) = 6x^2 - 18x + 12 \Rightarrow f'(x) = 6(x-1)(x-2)$

Let $f'(x) = 0 \Rightarrow x = 1, 2$

$\therefore$ $f(1) = 11$ and $f(2) = 10$ ...(i)

Let us consider the open interval (1, 3).

Clearly, $x = 2$ is the only point in (1, 3).

And $f(2) = 10$ [from Eq. (i)]

Now, $\lim_{x \to 1^+} f(x) = 11$ and $\lim_{x \to 3^-} f(x) = 15$

Thus, $x = 2$ is the point of global minima in (1, 3) and global maxima does not exist in (1, 3).

Summary Graph

Point to Consider

Based on the above discussion we can summarize things in a single graph as given below.

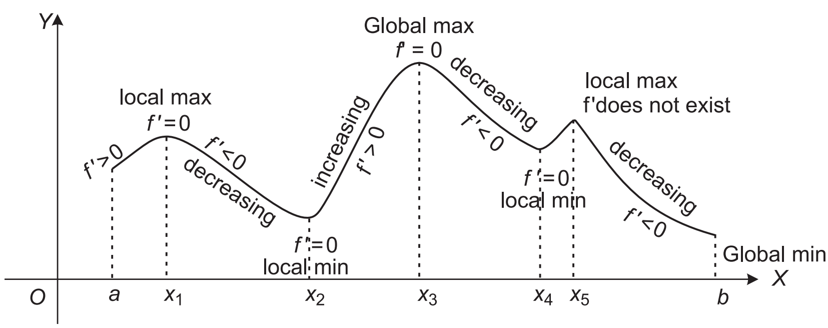


Fig. 8.21

Maxima and Minima in Discontinuous Functions

In discontinuous functions we don't apply any of the methods discussed in previous pages but we observe certain conditions and their graphs which would give you more clear picture.

(1) Minimum of Discontinuous Functions

The following four cases arise for minimum at $x = a$

(i) From this figure,

$f(a) < f(a + h)$
$f(a) < f(a - h)$

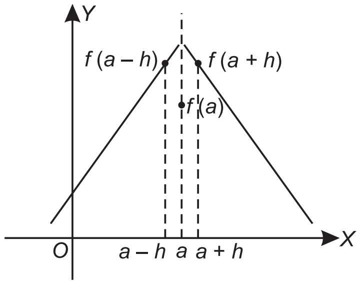


Fig. 8.22

(ii) From this figure,

$f(a) < f(a + h)$
$f(a) \le f(a - h)$

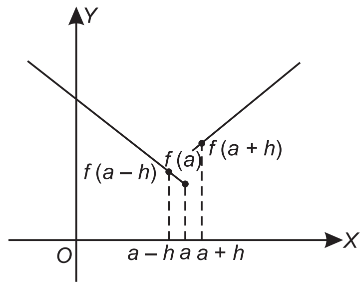


Fig. 8.23

(iii) From this figure,

$f(a) < f(a + h)$
$f(a) < f(a - h)$

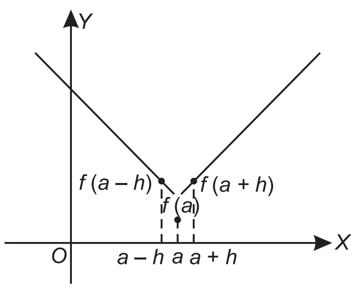


Fig. 8.24

(iv) From this figure,

$f(a) \le f(a + h)$
$f(a) < f(a - h)$

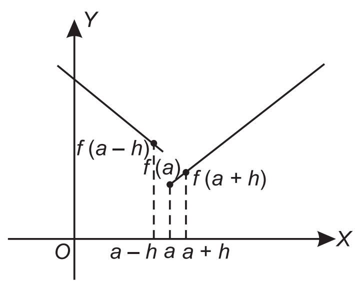


Fig. 8.25

from all the four above mentioned cases for minimum of discontinuous functions, we have

$$f(a) \le f(a + h)$$

and $$f(a) \le f(a - h)$$

Illustration 46 Discuss the maxima and minima of $f(x)=\{x\}$, (where $\{.\}$ denotes the fractional part of x for $x = 6$.

Solution. As we have, discussed for discontinuous functions, maximum and minimum at $x = a$ is attained when,

$$\begin{cases} f(a) \geq f(a+h) \text{ and } f(a) \geq f(a-h) \Rightarrow f(x) \text{ is maximum at } x = a \\ f(a) \leq f(a+h) \text{ and } f(a) \leq f(a-h) \Rightarrow f(x) \text{ is minimum at } x = a \end{cases}$$

Here, $f(x)=\{x\}$ is discontinuous function at $x=6$

where, $f(6)=0$

$f(6+h)>0$ and $f(6-h)>0$

So, $f(6)<f(6+h)$ and $f(6)<f(6-h)$

$\Rightarrow$ $f(x)$ is minimum at $x=6$.

$\therefore$ $f(x)$ for $x=6$ attains local minima.

Illustration 47 Let $f(x)=\begin{cases} |x-2|+a^2-9a-9, & \text{if } x<2 \\ 2x-3, & \text{if } x \geq 2 \end{cases}$.

Then, find the value of 'a' for which $f(x)$ has local minimum at $x=2$.

Solution. We have, $f(x)=\begin{cases} |x-2|+a^2-9a-9, & \text{if } x<2 \\ 2x-3, & \text{if } x \geq 2 \end{cases}$

$f(x)$ has local minima at $x=2$.

Since, $f(x)=2x-3$ for $x \geq 2$ (is strictly increasing)

$\therefore$ $\lim_{x \to 2^-} f(x) \geq f(2)$

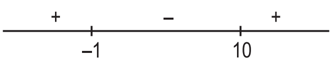
or $\lim_{h \to 0} f(2-h) \geq f(2)$ $\quad [\because f(2)=2\times 2-3=1]$

$\lim_{h \to 0} \{|2-h-2|+a^2-9a-9\} \geq 1$

$a^2-9a-10 \geq 0$

$(a+1)(a-10) \geq 0$

$a \leq -1$ or $a \geq 10$



(2) Maximum of Discontinuous Functions

The following four cases arise for maximum at $x = a$.

(i) From this figure,

$f(a) > f(a+h)$

$f(a) > f(a-h)$

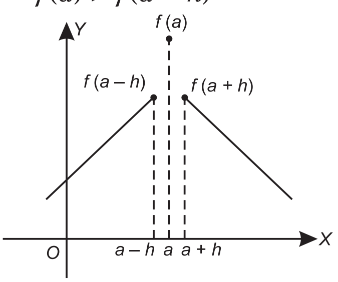


Fig. 8.26

(ii) From this figure,

$f(a) > f(a+h)$

$f(a) > f(a-h)$

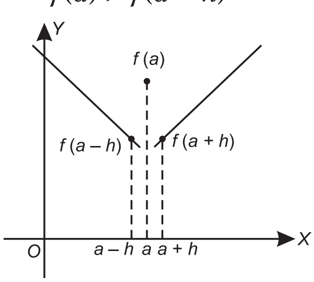


Fig. 8.27

(iii) From this figure,
$f(a) > f(a+h)$
$f(a) \geq f(a-h)$

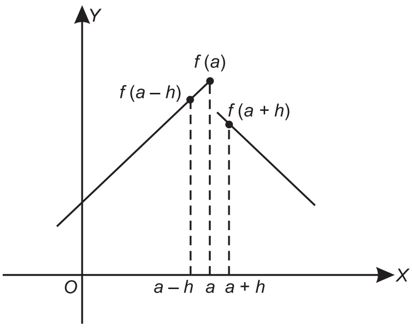


Fig. 8.28

(iv) From this figure,
$f(a) \geq f(a+h)$
$f(a) > f(a-h)$

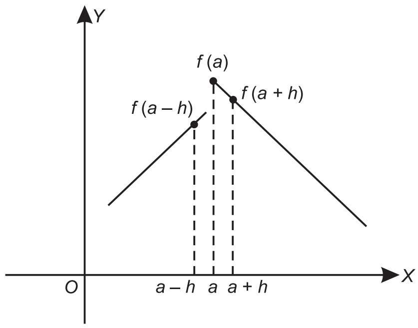
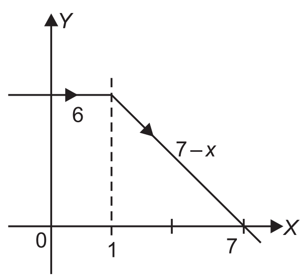


Fig. 8.29

From all the above four mentioned cases for maximum of discontinuous functions, we have

$$f(a) \geq f(a+h),\ f(a) \geq f(a-h)$$

Illustration 48 Let $f(x) = \begin{cases} 6, & x \leq 1 \\ 7-x, & x > 1 \end{cases}$, then for $f(x)$ at $x = 1$ discuss maxima and minima.



Solution. Here, $f(x) = \begin{cases} 6, & x \leq 1 \\ 7-x, & x > 1 \end{cases}$

$\Rightarrow \qquad f(1) = 6$

$\Rightarrow \qquad f(1-h) = 6$

and $\qquad f(1+h) = 7-(1+h) = 6-h < 6$

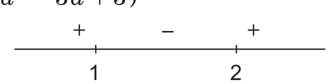
Thus, at $x = 1$ is a point of maxima.

Illustration 49 Find the values of 'a' for which,
$f(x) = \begin{cases} 4x - x^3 + \log(a^2 - 3a + 3), & 0 \leq x < 3 \\ x - 18, & x \geq 3 \end{cases}$ has a local maxima at $x = 3$.

Solution. Since, function attains maxima at $x = 3$

$\Rightarrow \qquad f(3) \geq f(3-0) \quad \Rightarrow \quad -15 \geq 12 - 27 + \log(a^2 - 3a + 3)$

$\Rightarrow \qquad \log(a^2 - 3a + 3) \leq 0$



where for 'log' to exists,

$a^2 - 3a + 3 > 0$ and $\log(a^2 - 3a + 3) \leq 0$

$\Rightarrow \qquad 0 < a^2 - 3a + 3 \leq 1 \quad ie, \quad (a-2)(a-1) \leq 0$

ie, $\qquad 1 \leq a \leq 2$

$\therefore$ $f(x)$ attains local maxima at $x = 3$, when $a \in [1, 2]$.

(3) Neither Maximum Nor Minimum for Discontinuous Functions

The following cases arise for neither maximum nor minimum at $x = a$

(i) $f(a) < f(a-h)$ and $f(a) \geq f(a+h)$

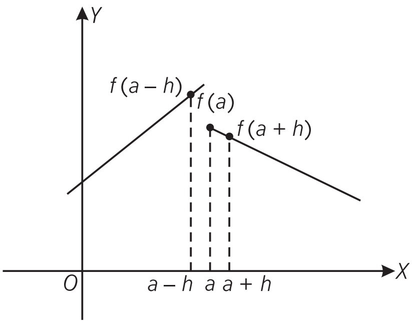


Fig. 8.30

(ii) $f(a) \leq f(a+h)$

$f(a) > f(a-h)$

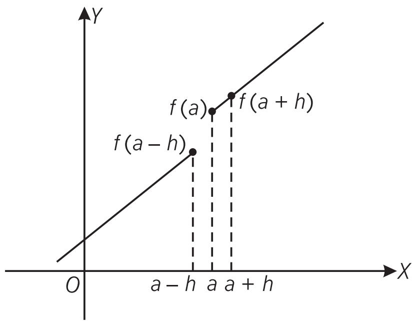


Fig. 8.31

(iii) $f(a) < f(a+h)$

$f(a) \geq f(a-h)$

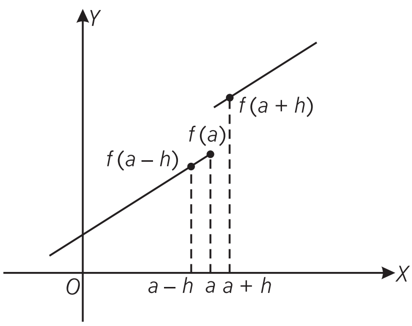


Fig. 8.32

(iv) $f(a) \leq f(a-h)$

$f(a) > f(a+h)$

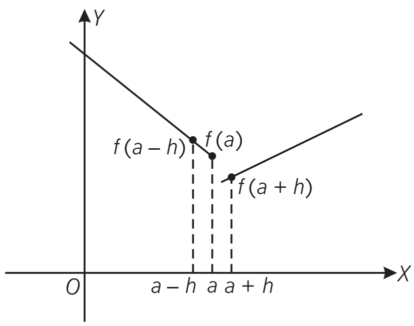


Fig. 8.33

(v) $f(a) < f(a-h)$

$f(a) > f(a+h)$

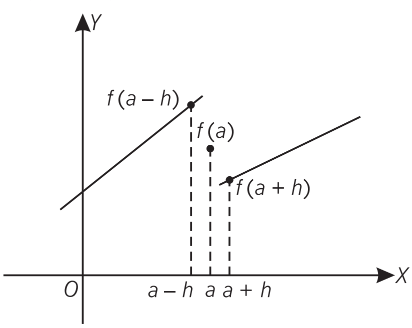


Fig. 8.34

(vi) $f(a) < f(a-h)$

$f(a) \geq f(a+h)$

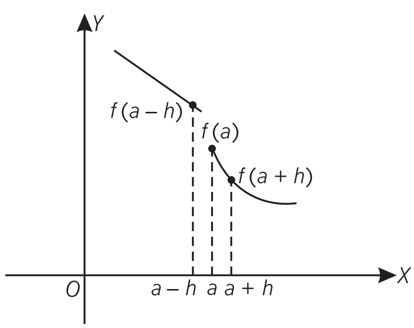


Fig. 8.35

In all above cases no extremum exist.

Some Geometrical Results

	In Usual Notations	Results
(1)	Area of equilateral	$\Delta = \frac{\sqrt{3}}{4}(\text{side})^2$
	and its perimeter	$= 3\,(\text{side})$
(2)	Area of square	$= (\text{side})^2$
	Perimeter	$= 4\,(\text{side})$
(3)	Area of rectangle	$= l \times b$
	Perimeter	$= 2\,(l + b)$
(4)	Area of trapezium	$= \frac{1}{2}$ (sum of parallel sides) $\times$ (distance between them)
(5)	Area of circle	$= \pi r^2$
	Perimeter of circle	$= 2\pi r$
(6)	Volume of sphere	$= \frac{4}{3}\pi r^3$
	Surface area of sphere	$= 4\pi r^2$
(7)	Volume of cone	$= \frac{1}{3}\pi r^2 h$
	Curved surface area of cone	$= \pi r l$
	Total surface area of cone	$= \pi r\,(r + l)$
(8)	Volume of cylinder	$= \pi r^2 h$
	Curved surface area	$= 2\pi r h$
	Total surface area	$= 2\pi r\,(h + r)$
(9)	Volume of cuboid	$= l \times b \times h$
	Surface area of cuboid	$= 2\,(lb + bh + hl)$
	Area of four walls	$= 2\,(l + b)\,h$
(10)	Volume of cube	$= l^3$
	Surface area of cube	$= 6l^2$
	Area of four walls of cube	$= 4l^2$

Illustration 50 A cubic $f(x)$ tends to zero at $x = -2$ and has relative maximum/ minimum at $x = -1$ and $x = \frac{1}{3}$. If $\int_{-1}^{1} f(x)\,dx = \frac{14}{3}$. Find the cubic $f(x)$. **[IIT JEE 1992]**

Solution. $f(x)$ is a cubic polynomial. Therefore, $f'(x)$ is a quadratic polynomial and $f(x)$ has relative maximum and minimum at $x = \frac{1}{3}$ and $x = -1$ respectively.

$$\therefore \qquad -1 \text{ and } \frac{1}{3} \text{ are roots of } f'(x) = 0$$

$$\Rightarrow \qquad f'(x) = a\,(x+1)\left(x - \frac{1}{3}\right) = a\left(x^2 + \frac{2}{3}x - \frac{1}{3}\right)$$

Now, integrating w.r.t. x, we get

$$f(x) = a\left[\frac{x^3}{3} + \frac{x^2}{3} - \frac{x}{3}\right] + c, \; c \text{ is constant of integration.}$$

Again, $\quad f(-2) = 0 \quad$ (given)

$\therefore \quad f(-2) = a\left(\frac{-8}{3} + \frac{4}{3} + \frac{2}{3}\right) + c = 0$

$\Rightarrow \quad \frac{-2a}{3} + c = 0 \quad$ or $\quad c = \frac{2a}{3}$

$\Rightarrow \quad f(x) = a\left(\frac{x^3}{3} + \frac{x^2}{3} - \frac{x}{3}\right) + \frac{2a}{3}$

$\Rightarrow \quad f(x) = a\left(\frac{x^3}{3} + \frac{x^2}{3} - \frac{x}{3} + \frac{2}{3}\right)$

Again, $\quad \int_{-1}^{1} f(x)\, dx = \frac{14}{3} \quad$ (given)

$\Rightarrow \quad \frac{a}{3}\int_{-1}^{1} (x^3 + x^2 - x + 2)\, dx = \frac{14}{3}$

Because, $\quad y = x^3$ and $y = -x$ are odd functions.

So, $\quad \int_{-1}^{1} x^3 dx = \int_{-1}^{1} x\, dx = 0$

$\therefore \quad \frac{a}{3}\left[\frac{2x^3}{3} + 4x\right]_0^1 = \frac{14}{3}$

$\Rightarrow \quad a = 3$

$\therefore \quad f(x) = x^3 + x^2 - x + 2$

Illustration 51 Find the maximum and minimum value of

$$f(x) = \frac{40}{3x^4 + 8x^3 - 18x^2 + 60}.$$

Solution. Given, $f(x) = \dfrac{40}{3x^4 + 8x^3 - 18x^2 + 60}$ is maximum or minimum

according by $\quad 3x^4 + 8x^3 - 18x^2 + 60$ is minimum or maximum.

Then, $\quad f_{\text{max}}$, if $3x^4 + 8x^3 - 18x^2 + 60$ is minimum

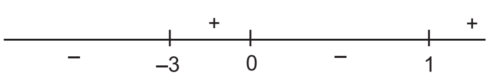
$\quad f_{\text{min}}$, if $3x^4 + 8x^3 - 18x^2 + 60$ is maximum

Let $\quad g(x) = 3x^4 + 8x^3 - 18x^2 + 60$

$\Rightarrow \quad g'(x) = 12x\,(x^2 + 2x - 3)$

$\Rightarrow \quad g'(x) = 12x\,(x + 3)(x - 1)$

Using number line rule,



which indicates $g'(x)$ changes sign from –ve to +ve at $x = -3, 1$.

$\therefore$ Local minimum at $x = -3, 1$

and local maximum at $x = 0$ [as changes from +ve to –ve]

$\Rightarrow$ $g(x)$ is maximum at $x = 0$

ie, $g_{\max}(0) = 60$

and for $g(x)$ to be minimum, $g_{\min}(-3) = 243 - 216 - 162 + 60 = -75$

$$g_{\min}(1) = 3 + 8 - 18 + 60 = 53$$

Substituting these values in Eq. (i), we get

$$f_{\max}(x) \text{ when } g_{\min} \text{ } ie, \quad f(x) = \frac{40}{-75} \text{ and } \frac{40}{53}$$

$$\text{Maximum value} = \frac{40}{53}, \frac{-8}{15}$$

$$f_{\min}(x) \text{ when } g_{\max} \text{ } ie, f(x) = \frac{40}{60} = \frac{2}{3}$$

$$\text{Minimum value} = \frac{2}{3}$$

Illustration 52 Use the function $f(x) = x^{1/x}$, $x > 0$ to determine the bigger of the two numbers e^{π} and π^{e}.

Solution. $f(x) = x^{1/x}$, $x > 0$

Let $y = f(x) = x^{1/x}$

Taking log of both the sides, we have $\log y = \dfrac{1}{x} \log x$

Differentiating both the sides, we have

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x^2} + \log x \left(-\frac{1}{x^2}\right)$$

or $$\frac{dy}{dx} = y\left[\frac{1 - \log x}{x^2}\right]$$

$\therefore$ $$f'(x) = \frac{x^{1/x}}{x^2}[1 - \log x]$$

Let $f'(x) = 0$,

$\Rightarrow$ $\log x = 1$ or $x = e$

Again, $$f''(x) = \frac{x^{1/x}}{x^2}\left[0 - \frac{1}{x}\right] + (1 - \log x)\frac{d}{dx}\left(\frac{x^{1/x}}{x^2}\right)$$

$\therefore$ $$f''(e) = \frac{e^{1/e}}{e^2}\left(-\frac{1}{e}\right) + 0 \quad \Rightarrow \quad f''(e) < 0$$

$\therefore$ 'f' has a maximum at $x = e$

But $x = e$ is the only extreme value.

$\therefore$ f has the greatest value at $x = e$

$\Rightarrow$ $f(e) > f(\pi)$ for all $x > 0$

$\Rightarrow$ $(e)^{\frac{1}{e}} > (\pi)^{\frac{1}{\pi}}$

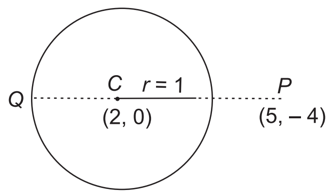
$\Rightarrow$ $e^{\pi} > \pi^{e}$

Illustration 53 The maximum value of $(\sqrt{-3+4x-x^2}+4)^2+(x-5)^2$. (where $1 \le x \le 3$) is

(a) 34 (b) 36 (c) 32 (d) 20

Solution. Here, $(\sqrt{-3+4x-x^2}+4)^2+(x-5)^2$, represents the square of the distance between circle $y=\sqrt{-3+4x-x^2}$ and point $(5, -4)$.

ie, Maximum distance between $x^2+y^2-4x+3=0$ and $(5, -4)$, squared



$$\Rightarrow \quad PQ^2 = (PC + \text{radius})^2 = (\sqrt{(5-2)^2+(4-0)^2}+1)^2 = 6^2 = 36$$

Hence, (b) is the correct answer.

Illustration 54 If $a > b > 0$ and $f(\theta) = \dfrac{(a^2-b^2)\cos\theta}{a - b\sin\theta}$, then the maximum value of $f(\theta)$, is

(a) $2\sqrt{a^2+b^2}$ (b) $\sqrt{a^2+b^2}$ (c) $\sqrt{a^2-b^2}$ (d) $\sqrt{b^2-a^2}$

Solution. Here, $f(\theta) = \dfrac{(a^2-b^2)\cos\theta}{a-b\sin\theta} = \dfrac{(a^2-b^2)}{a\sec\theta - b\tan\theta}$

or $\quad f(\theta) = \dfrac{a^2-b^2}{h(\theta)}$, where $h(\theta) = a\sec\theta - b\tan\theta$

$\therefore$ $f(\theta)$ is maximum and minimum as $h(\theta)$ is minimum and maximum respectively.

$\Rightarrow \quad h(\theta) = a\sec\theta - b\tan\theta \quad \Rightarrow \quad h'(\theta) = \sec\theta\,(a\tan\theta - b\sec\theta)$

For maximum and minimum put $h'(\theta) = 0$

$\Rightarrow \quad \sin\theta = \dfrac{b}{a} \qquad (\sec\theta \ne 0) \ldots\text{(i)}$

Also, $\quad h''(\theta) = a\sec^3\theta + a\sec\theta\cdot\tan^2\theta - 2b\sec^2\theta\tan\theta$

or $\quad h''(\theta) = \dfrac{a + a\sin^2\theta - 2b\sin\theta}{\cos^3\theta}$

$$(h''(\theta))_{\text{when }\sin\theta = \frac{b}{a}} = \frac{a + a\cdot\dfrac{b^2}{a^2} - 2b\cdot\dfrac{b}{a}}{\left(1-\dfrac{b^2}{a^2}\right)^{3/2}} = \frac{a^2-b^2}{a\left(\dfrac{a^2-b^2}{a^2}\right)^{3/2}} > 0 \quad (\text{as } a > b)$$

$\Rightarrow h(\theta)$ is minimum when $\sin\theta = \dfrac{b}{a}$

$\therefore \quad f(\theta)$ is maximum when $\sin\theta = \dfrac{b}{a} \Rightarrow f_{\max}(\theta) = \sqrt{a^2-b^2}$

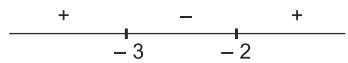
Hence, (c) is the correct answer.

Illustration 55 The values of 'K' for which the point of minimum of the function $f(x) = 1 + K^2x - x^3$ satisfy the inequality $\dfrac{(x^2 + x + 2)}{(x^2 + 5x + 6)} < 0$, belongs to

(a) $(-3\sqrt{3}, \infty)$ (b) $(-3\sqrt{3}, -2\sqrt{3}) \cup (0, \infty)$

(c) $(-3\sqrt{3}, -2\sqrt{3}) \cup (2\sqrt{3}, 3\sqrt{3})$ (d) $(0, \infty)$

Solution. Here, $\dfrac{x^2 + x + 2}{x^2 + 5x + 6} < 0$



$$\Rightarrow \quad \frac{\left(x + \frac{1}{2}\right)^2 + \frac{7}{4}}{(x + 2)(x + 3)} < 0$$

where $\left(x + \dfrac{1}{2}\right)^2 + \dfrac{7}{4}$ is always positive.

$\therefore$ Using number line rule for $(x + 2)(x + 3)$ as shown above, we get :

$$\frac{x^2 + x + 2}{x^2 + 5x + 6} < 0 \quad \Rightarrow \quad -3 < x < -2 \qquad \text{...(i)}$$

Now, consider $\quad f(x) = 1 + K^2x - x^3$

$$f'(x) = K^2 - 3x^2$$

$$f''(x) = -6x$$

For maximum/minimum let $f'(x) = 0$,

$$\Rightarrow \quad x = \pm \frac{|K|}{\sqrt{3}}$$

Let $\quad x_1 = \dfrac{|K|}{\sqrt{3}}$ and $x_2 = -\dfrac{|K|}{\sqrt{3}}$

$\therefore \quad f''(x_1) < 0$ and $f''(x_2) > 0$

$\therefore$ $f(x)$ is maximum at $x = x_1$ and $f(x)$ is minimum at $x = x_2$.

$\therefore \quad -3 < x_2 < -2$ [from Eq. (i)]

$$\Rightarrow \quad -3 < \frac{-|K|}{\sqrt{3}} < -2 \quad \Rightarrow \quad 3\sqrt{3} > |K| > 2\sqrt{3}$$

$$\Rightarrow \quad K \in (-3\sqrt{3}, -2\sqrt{3}) \cup (2\sqrt{3}, 3\sqrt{3})$$

Hence, (c) is the correct answer.

Illustration 56 The values of a and b for which all the extrema of the function, $f(x) = a^2x^3 - 0.5\,ax^2 - 2x - b$, is positive and the minimum is at the point $x_0 = \dfrac{1}{3}$, are

(a) when $a = -2 \Rightarrow b < -\dfrac{11}{27}$ and when $a = 3 \Rightarrow b < -\dfrac{1}{2}$

(b) when $a = 3 \Rightarrow b < -\dfrac{11}{27}$ and when $a = 2 \Rightarrow b < -\dfrac{1}{2}$

(c) when $a = -2 \Rightarrow b < -\dfrac{1}{2}$ and when $a = 3 \Rightarrow b < -\dfrac{11}{27}$

(d) None of the above

Solution. For extrema, $f'(x)=0$

$\Rightarrow \quad 3a^2x^2 - ax - 2 = 0$ at $x = \frac{1}{3}$ $\quad \left(\text{as at } x = \frac{1}{3} \text{ function is minimum}\right)$

$$\therefore \quad 3a^2\left(\frac{1}{3}\right)^2 - a\left(\frac{1}{3}\right) - 2 = 0$$

$$\Rightarrow \quad \frac{a^2}{3} - \frac{a}{3} - 2 = 0$$

$$\Rightarrow \quad a^2 - a - 6 = 0 \quad \text{or} \quad a = -2, 3$$

So, their arise two cases as :

Case I At $a=3$, if function attains minimum and is positive.

$$\therefore \quad 9\left(\frac{1}{3}\right)^3 - (0.5)(3)\left(\frac{1}{3}\right)^2 - 2\left(\frac{1}{3}\right) - b > 0$$

$$\left[\because \text{ minimum at } x = \frac{1}{3} \text{ and } a = 3 \;\Rightarrow\; f\left(\frac{1}{3}\right) > 0 \text{ when } a = 3\right]$$

$$\Rightarrow \quad b < \frac{1}{3} - \frac{1.5}{9} - \frac{2}{3}$$

$$\Rightarrow \quad b < -\frac{1}{2}$$

Case II At $a=-2$, if function attains minimum and is positive.

$$\therefore \quad (-2)^2\left(\frac{1}{3}\right)^3 - (0.5)(-2)\left(\frac{1}{3}\right)^2 - 2\left(\frac{1}{3}\right) - b > 0$$

$$\left[\text{since minimum at } x = \frac{1}{3}, \text{ when } a = -2 \Rightarrow f\left(\frac{1}{3}\right) > 0 \text{ when } a = -2\right]$$

$$\Rightarrow \quad b < \frac{4}{27} + \frac{1}{9} - \frac{2}{3}$$

$$\text{or} \quad b < -\frac{11}{27}$$

$\therefore$ When $a=3 \Rightarrow b < -\frac{1}{2}$ and when $a=-2 \Rightarrow b < -\frac{11}{27}$

Hence, (a) is the correct answer.

Illustration 57 If $f''(x) + f'(x) + f^2(x) = x^2$ be the differential equation of a curve and let P be the point of maxima, then number of tangents which can be drawn from P to $x^2 - y^2 = a^2$ is/are

(a) 2 (b) 1 (c) 0 (d) either 1 or 2

Solution. At point of maxima $f'(x)=0$ and $f''(x)<0$

$$\Rightarrow \quad f''(x) = x^2 - f^2(x) \le 0$$

(Since, the curve is $x^2 - y^2 = a^2$ and $x^2 - f^2(x) \le 0 \therefore x_1^2 - y_1^2 < a^2 \Rightarrow$ point lies outside the hyperbola)

$\Rightarrow \quad$ Point $P(x, f(x))$ lies outside $x^2 - y^2 = a^2$

$\therefore$ Two normals can be drawn.

Hence, (a) is the correct answer.

Illustration 58 A solid cylinder of height H has a conical portion of same height and radius 1/3rd of height removed from it. Rain water is accumulating in the at a with rate equal to π times the instantaneous radius of the water surface inside the hole, the time after which hole will filled with water is

(a) $\frac{H^2}{3}$ (b) H^2

(c) $\frac{H^2}{6}$ (d) $\frac{H^2}{4}$

Solution. Here, $r = \frac{H}{3}, \frac{x}{r} = \frac{y}{H}$

$$\Rightarrow \quad 3x = y \Rightarrow \frac{dv}{dt} = \pi x$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{3}\pi x^2 y\right) = \pi x \quad \Rightarrow 3\int_0^r x\,dx = \int_0^t dt$$

$$\Rightarrow \quad 3\,\frac{r^2}{2} = t \Rightarrow \frac{3}{2}\cdot\frac{H^2}{9} = t \Rightarrow t = \frac{H^2}{6}$$

Hence, (c) is the correct answer.

Target Exercise 8.3

1. The point at which the slope of the tangent of the function $f(x) = e^x \cdot \cos x$ attains minima, when $x \in [0, 2\pi]$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{3\pi}{4}$ (d) π

2. If λ, μ be real numbers such that, $x^3 - \lambda x^2 + \mu x - 6 = 0$ has its roots real and positive, then the minimum value of μ, is

(a) $3\,(6)^{1/3}$ (b) $3\,(6)^{2/3}$

(c) $(6)^{1/3}$ (d) $(6)^{2/3}$

3. The points for which the function $f(x) = \int_1^x \{2\,(t-1)\,(t-2)^3 + 3\,(t-1)^2 (t-2)^2\}\,dt$ attains maxima and minima, is

(a) maximum when $x = \frac{7}{5}$ and minimum when $x = 1$

(b) maximum when $x = 1$ and minimum when $x = 0$

(c) maximum when $x = 1$ and minimum when $x = 2$

(d) maximum when $x = 1$ and minimum when $x = \frac{7}{5}$

4. The maximum value of $f(x) = \sin x + \cos 2x$ when $\in [0, 2\pi]$ is

(a) 0 (b) -2

(c) $\frac{9}{8}$ (d) None of these

5. The point on the curve $y = \dfrac{x}{1+x^2}$, where the tangent to the curve has the greatest slope, is at

(a) $x = 0$
(b) $x = \sqrt{3}$
(c) $x = -\sqrt{3}$
(d) None of these

6. Let $f(x) = 4\tan x - \tan^2 x + \tan^3 x, x \neq n\pi + \dfrac{\pi}{2}$ has

(a) only one point of local maxima
(b) only one point of local minima
(c) neither local minima nor maxima
(d) None of the above

7. The global maxima and global minima of $f(x) = \dfrac{e^x + e^{-x}}{2}$, when $x \in [-\log_e 2, \log_e 7]$ are

(a) $\dfrac{25}{7}, 1$
(b) $1, \dfrac{25}{7}$
(c) $\dfrac{5}{4}, \dfrac{25}{7}$
(d) $\dfrac{25}{7}, \dfrac{5}{4}$

8. The least and the greatest value of $f(x, y) = x^2 + y^2 - xy$, where x and y are connected by the relation $x^2 + 4y^2 = 4$, are

(a) $\dfrac{5-\sqrt{13}}{2}, \dfrac{5+\sqrt{13}}{2}$

(b) $\dfrac{5-\sqrt{13}}{4}, \dfrac{5+\sqrt{13}}{4}$

(c) $\dfrac{1}{4}, -\dfrac{1}{4}$

(d) $-\dfrac{2}{3}, \dfrac{2}{3}$

9. The value of 'a' for which the function $f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 2$ has a positive point of maxima, is

(a) $(-\infty, -3) \cup \left(3, \dfrac{29}{7}\right)$
(b) $(-3, 3)$
(c) $(3, \infty)$
(d) $\left(-\infty, \dfrac{29}{7}\right) - \{\pm 3\}$

10. Let $f(x) = \begin{cases} -x^3 + \dfrac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)}, & 0 \le x < 1 \\ 2x - 3, & 1 \le x \le 3 \end{cases}$.

Then, the values of 'b' for which $f(x)$ has the smallest value at $x = 1$, belongs to

(a) $(-2, -1) \cup (2, \infty)$
(b) $(-2, -1) \cup [1, \infty)$
(c) $(1, 2)$
(d) None of these

Worked Examples

Type 1 : **Subjective Type Questions**

Example 1 Let $g'(x) > 0$ and $f'(x) < 0, \forall x \in R$, then show

(i) $g(f(x+1)) < g(f(x-1))$ (ii) $f(g(x+1)) < f(g(x-1))$

Solution. Here, $g'(x) > 0$ and $f'(x) < 0, \forall x \in R$

ie, $g(x)$ is increasing [or if $x_1 > x_2 \Rightarrow g(x_1) > g(x_2)$]

and $f(x)$ is decreasing [or if $x_1 > x_2 \Rightarrow f(x_1) < f(x_2), \forall x \in R$]

$\therefore$ $f(x+1) < f(x-1)$ and $g(x+1) > g(x-1)$ [as $(x+1) > (x-1)$] ...(i)

Case I As, $g(x)$ is increasing (so greater input gives greater output)

$\Rightarrow$ $g(f(x-1)) > g(f(x+1))$ [using Eq. (i)]

or $g(f(x+1)) < g(f(x-1))$

Case II $f(x)$ is decreasing (so greater input gives smaller output)

$\Rightarrow$ $f(g(x+1)) < f(g(x-1))$ [using Eq. (i)]

Example 2 Let $f'(\sin x) < 0$ and $f''(\sin x) > 0, \forall x \in \left(0, \frac{\pi}{2}\right)$ and $g(x) = f(\sin x) + f(\cos x)$, then find the interval in which $g(x)$ is increasing and decreasing.

Solution. Here, $f'(\sin x) < 0$ and $f''(\sin x) > 0, \forall x \in \left(0, \frac{\pi}{2}\right)$ ...(i)

and $g(x) = f(\sin x) + f(\cos x)$

$\Rightarrow$ $g'(x) = f'(\sin x) \cdot \cos x + f'(\cos x)(-\sin x)$

$\Rightarrow$ $g''(x) = \{-f'(\sin x) \cdot \sin x + f''(\sin x) \cdot \cos^2 x\}$

$- \{f'(\cos x) \cdot \cos x - f''(\cos x) \cdot \sin^2 x\}$ ...(ii)

As, $f'(\sin x) < 0, f''(\sin x) > 0,$

$\sin x > 0, \cos x > 0, \forall x \in \left(0, \frac{\pi}{2}\right)$ ...(iii)

$\therefore$ From Eqs. (ii) and (iii), we can say

$$g''(x) = \underbrace{\{-f'(\sin x) \cdot \cos x\}}_{+\text{ ve}} + \underbrace{\{f''(\sin x) \cdot \cos^2 x\}}_{+\text{ ve}} + \underbrace{\{f''(\cos x) \cdot \sin^2 x\}}_{+\text{ ve}}$$

$$+ \underbrace{\{-f'(\cos x) \cdot \cos x\}}_{+\text{ ve}}$$

$\Rightarrow$ $g''(x) > 0, \forall x \in \left(0, \frac{\pi}{2}\right)$ ..(iv)

$\Rightarrow$ $g'(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$ ...(v)

Now, putting $g'(x) = 0$, $g'(x) = f'(\sin x) \cdot \cos x - f'(\cos x) \cdot \sin x = 0$

$\Rightarrow \qquad x = \dfrac{\pi}{4}$

and $\qquad g'(x) > 0$, when $x \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

$g'(x) < 0$, when $x \in \left(0, \dfrac{\pi}{4}\right)$

$\therefore$ $g(x)$ is increasing when $x \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

$g(x)$ is decreasing when $x \in \left(0, \dfrac{\pi}{4}\right)$

Example 3 Let $f:[0, \infty) \to [0, \infty)$ and $g:[0, \infty) \to [0, \infty)$ be non-increasing and non-decreasing function and $h(x) = g(f(x))$. If f and g are differentiable functions, $h(x) = g(f(x))$ and if f and g are differentiable for all points in their respective domains and $h(0) = 0$. Then, show $h(x)$ is always identically zero.

Solution. Here, $\qquad h(x) = g(f(x))$, since, $g(x) \in [0, \infty)$

$h(x) \geq 0, \forall\, x \in$ domain

Also, $\qquad h'(x) = g'(f(x)) \cdot f'(x) \leq 0$ as $g'(x) \geq 0$

and $\qquad h(x) \leq 0, \forall\, x \in$ domain as $h(0) = 0$

Hence, $\qquad h(x) = 0, \forall\, x \in$ domain

Example 4 Find the value of n, for which $f(x) = (x^2 - 4)^n (x^2 - x + 1)$, $n \in N$ assumes a local minima at $x = 2$.

Solution. Here, $f(x) = (x^2 - 4)^n (x^2 - x + 1)$ assumes local minima at $x = 2$

$\Rightarrow \qquad f(2) < f(2 - h)$ and $f(2) < f(2 + h)$, where $h > 0$

where $\qquad f(2) = 0$

$\Rightarrow \qquad f(2 - h) > 0$ and $f(2 + h) > 0, \forall\, h > 0$

$\Rightarrow \qquad (-h)^n (4 - h)^n \cdot \{h^2 - 3h + 1\} > 0$

and $\qquad h^n (4 + h)^n (h^2 + 5h + 1) > 0$

ie, $\qquad (-h)^n > 0 \qquad [\because (4 - h) > 0, h^2 - 3h + 1 > 0, 4 + h > 0,$
$h^2 + 5h + 1 > 0, \forall\, h > 0]$

$\Rightarrow n \in$ even number.

Example 5 The interval to which b may belong so that the function,

$$f(x) = \left(1 - \frac{\sqrt{21 - 4b - b^2}}{b + 1}\right) x^3 + 5x + \sqrt{16}, \text{ increases for all } x.$$

Solution. If $f(x) = \left(1 - \dfrac{\sqrt{21 - 4b - b^2}}{b + 1}\right) x^3 + 5x + \sqrt{16}$, increases we must have

$f'(x) > 0, \forall\, x \in$ real number.

Then, $$f'(x) = \left(1 - \frac{\sqrt{21 - 4b - b^2}}{b+1}\right) 3x^2 + 5 > 0, \forall x \in R$$

(as we know $ax^2 + bx + c > 0, \forall x \in R$, we must have $a > 0$ and $D < 0$)

$\therefore$ $$1 - \frac{\sqrt{21 - 4b - b^2}}{b+1} > 0 \text{ and } (0)^2 - 4 \times 3 \left(1 - \frac{\sqrt{21 - 4b - b^2}}{b+1}\right) 5 < 0$$

$\Rightarrow$ $$1 - \frac{\sqrt{21 - 4b - b^2}}{b+1} > 0$$

The above inequality holds, when, (i) $b + 1 < 0$ and (ii) $21 - 4b - b^2 > 0$

$\therefore$ $b < -1$ and $b^2 + 4b - 21 < 0$

$\Rightarrow$ $b < -1$ and $(b + 7)(b - 3) < 0$

$\Rightarrow$ $b < -1$ and $-7 < b < 3$ (using number line rule)

$\therefore$ $b \in (-7, -1)$ ...(i)

Again, when $b + 1 > 0, f(x)$ will be increasing for all x, if

$$21 - 4b - b^2 > 0 \quad \text{and} \quad 1 > \frac{\sqrt{21 - 4b - b^2}}{b+1}$$

or $b^2 + 4b - 21 < 0$ and $(b + 1)^2 > (21 - 4b - b^2)$ (as $b + 1 > 0$)

or $(b + 7)(b - 3) < 0$ and $b^2 + 3b - 10 > 0$

$\Rightarrow$ $(-7 < b < 3)$ and ($b < -5$ or $b > 2$)

$\Rightarrow$ $2 < b < 3$ ...(iii)

From Eqs. (i) and (ii), we have concluded that.

$$b \in (-7, -1) \cup (2, 3)$$

Example 6 Find the set of all values of 'a' for which

$f(x) = \left(\frac{\sqrt{a+4}}{1-a} - 1\right) x^5 - 3x + \log 5$ monotonically decreases for all x.

Solution. Given, $f(x) = \left(\frac{\sqrt{a+4}}{1-a} - 1\right) x^5 - 3x + \log 5$ decreases for all x.

Then, $$f'(x) = \left(\frac{\sqrt{a+4}}{1-a} - 1\right) 5x^4 - 3 < 0, \forall x \in R$$

ie, $$5\left(\frac{\sqrt{a+4}}{1-a} - 1\right) x^4 - 3 < 0, \forall x \in R$$

$\Rightarrow$ $$\left(\frac{\sqrt{a+4}}{1-a} - 1\right) < 0, \forall x \in R \quad \text{...(i)}$$

(as $ax^2 + bx + c < 0 \Rightarrow a < 0$ and $D < 0$)

Now, two cases arise :

Case I If $1 - a < 0 \Rightarrow a > 1$,

then $\sqrt{a+4} > (1 - a)$ and $a + 4 > 0$

which is always true as LHS > 0 and RHS < 0

$\therefore$ Above inequality is true for all $a > 1$ ...(ii)

Case II If $1-a>0 \qquad \Rightarrow \qquad a<1$

$$\Rightarrow \qquad \sqrt{a+4}<1-a \text{ and } a>-4$$

$$\Rightarrow \qquad a^2-3a-3>0 \text{ and } a>-4$$

$$\Rightarrow \qquad a<\frac{3-\sqrt{21}}{2} \text{ or } a>\frac{3+\sqrt{21}}{2} \text{ and } a>-4$$

$$\Rightarrow \qquad -4<a<\frac{3-\sqrt{21}}{2} \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii), we conclude

$$a \in \left(-4, \frac{3-\sqrt{21}}{2}\right) \cup (1, \infty)$$

Example 7 Let $a+b=4$, where $a<2$ and let $g(x)$ be a differentiable function. If $\frac{dg}{dx}>0$, $\forall\, x$ prove that $\int_0^b g(x)\,dx+\int_0^a g(x)\,dx$ increases as $(b-a)$ increases.

[IIT JEE 1997]

Solution. Let $(b-a)=t$ and since $a+b=4$, we have $a=\frac{4-t}{2}$ and $b=\frac{t+4}{2}$ ...(i)

where $t>0$ (as $a<2$ and $b>2$)

Let $\qquad \int_0^a g(x)\,dx+\int_0^b g(x)\,dx=\phi(t)$

$$\therefore \qquad \phi(t)=\int_0^{\frac{4-t}{2}} g(x)\,dx+\int_0^{\frac{4+t}{2}} g(x)\,dx$$

$$\therefore \qquad \phi'(t)=g\left(\frac{4-t}{2}\right)\cdot\left(\frac{-1}{2}\right)+g\left(\frac{4+t}{2}\right)\left(\frac{1}{2}\right)$$

(using Leibnitz rule)

$$\Rightarrow \qquad \phi'(t)=\frac{1}{2}\left[g\left(\frac{4+t}{2}\right)-g\left(\frac{4-t}{2}\right)\right] \qquad \text{...(ii)}$$

Since, $g(x)$ is increasing and we know

If $\qquad x_1>x_2$

$\Rightarrow \qquad g(x_1)>g(x_2)$

Here, $\qquad \frac{4+t}{2}>\frac{4-t}{2}$ and $g(x)$ is increasing.

$$\therefore \qquad g\left(\frac{4+t}{2}\right)>g\left(\frac{4-t}{2}\right) \qquad \text{...(iii)}$$

$$\therefore \qquad \phi'(t)=\frac{1}{2}\left[g\left(\frac{4+t}{2}\right)-g\left(\frac{4-t}{2}\right)\right]>0 \qquad \text{[using Eq. (iii)]}$$

$$\Rightarrow \qquad \phi'(t)>0$$

Hence, $\phi(t)$ increases as t increases.

or $\qquad \int_0^a g(x)\,dx+\int_0^b g(x)\,dx$ is increasing as $(b-a)$ increases.

Example 8 Let $g(x)=2f\left(\frac{x}{2}\right)+f(2-x)$ and $f''(x)<0,\ \forall\, x\in(0,2)$. Find the intervals of increase and decrease of $g(x)$.

Solution. We have, $$g(x)=2f\left(\frac{x}{2}\right)+f(2-x)$$

$$\Rightarrow \quad g'(x)=2f'\left(\frac{x}{2}\right)\cdot\left(\frac{1}{2}\right)+f'(2-x)\cdot(-1)$$

$$\Rightarrow \quad g'(x)=f'\left(\frac{x}{2}\right)-f'(2-x) \qquad \text{...(i)}$$

We are given that $f''(x)<0,\ \forall\, x\in(0,2)$

It means that $f'(x)$ would be decreasing on (0, 2).

Now, two cases arise

Case I $\frac{x}{2}>(2-x)$ and $f'(x)$ is decreasing,

$$\Rightarrow \quad f'\left(\frac{x}{2}\right)<f'(2-x),\ \forall\, x>\frac{4}{3}$$

or $$g'(x)=f'\left(\frac{x}{2}\right)-f'(2-x)<0,\ \forall\, \frac{4}{3}<x<2$$

$\therefore$ $g(x)$ is decreasing in $\left(\frac{4}{3},2\right)$. ...(ii)

Case II $\frac{x}{2}<(2-x)$ and $f'(x)$ is decreasing,

$$\Rightarrow \quad f\left(\frac{x}{2}\right)>f(2-x),\ \forall\, x<\frac{4}{3}$$

or $$g'(x)=f\left(\frac{x}{2}\right)-f(2-x)>0,\ \forall\, 0<x<\frac{4}{3}$$

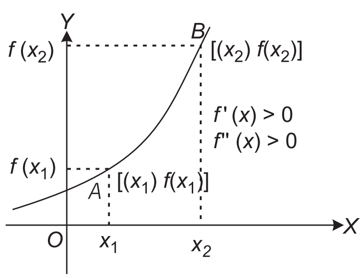
$\therefore$ $g(x)$ is increasing in $\left(0,\frac{4}{3}\right)$. ...(iii)

From Eqs. (ii) and (iii), we conclude :

$g(x)$ is increasing in $\left(0,\frac{4}{3}\right)$ and decreasing in $\left(\frac{4}{3},2\right)$.

Example 9 Let $f'(x)>0$ and $f''(x)>0$ where $x_1<x_2$. Then, show $f\left(\frac{x_1+x_2}{2}\right)<\frac{f(x_1)+f(x_2)}{2}$.

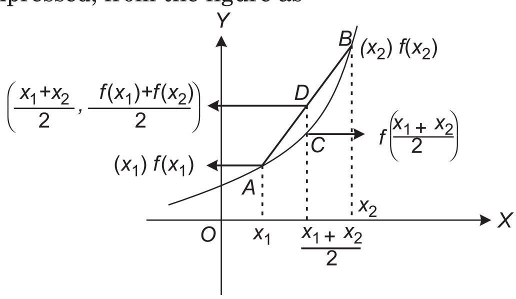
Solution. As we have discussed in theory, if $f'(x)>0$ and $f''(x)>0$, then graphically it can be expressed as shown in the following figure:



We know, $$x_1 < \frac{x_1 + x_2}{2} < x_2$$

and $\dfrac{f(x_1) + f(x_2)}{2}$ is mid-point of the chord joining A and B.

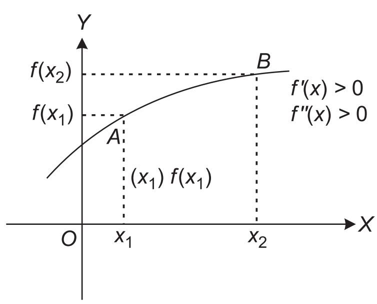
Thus, it can be expressed, from the figure as



$$f\left(\frac{x_1 + x_2}{2}\right) < \frac{f(x_1) + f(x_2)}{2}$$

Example 10 Let $f'(x) > 0$ and $f''(x) < 0$ where $x_1 < x_2$. Then, show $f\left(\dfrac{x_1 + x_2}{2}\right) > \dfrac{f(x_1) + f(x_2)}{2}$.

Solution. As we know, if $f'(x) > 0$ and $f''(x) < 0$. Then, it could be expressed graphically as shown in the following figure :



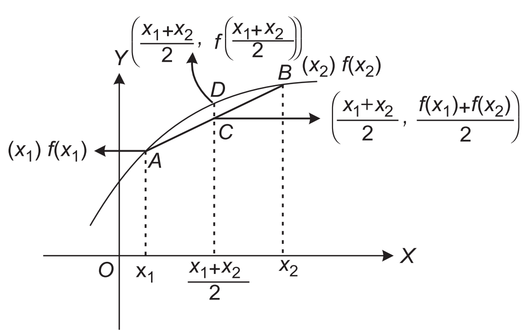
We know, $$x_1 < \frac{x_1 + x_2}{2} < x_2$$

and $\dfrac{f(x_1) + f(x_2)}{2}$ is the mid-point of chord joining A and B.

Thus, it can be expressed as shown in figure

From the adjacent figure,

$$f\left(\frac{x_1 + x_2}{2}\right) > \frac{f(x_1) + f(x_2)}{2}.$$



Example 11 If $f(x)$ is monotonically increasing function for all $x \in R$, such that $f''(x) > 0$ and $f^{-1}(x)$ exists, then prove that

$$\frac{f^{-1}(x_1) + f^{-1}(x_2) + f^{-1}(x_3)}{3} < f^{-1}\left(\frac{x_1 + x_2 + x_3}{3}\right)$$

Solution. Let $g(x) = f^{-1}(x)$

Since, g is the inverse of f

$\Rightarrow$ $\quad fog(x) = gof(x) = x \Rightarrow g'(x) = \dfrac{1}{f'(g(x))}$

$\Rightarrow$ $\quad g'(x) > 0$ [as $f(x)$ is increasing] ...(i)

$\Rightarrow$ $\quad g(x)$ is increasing for all $x \in R$.

$\Rightarrow$ $\quad f^{-1}(x)$ is increasing for all $x \in R$.

Again, $\quad g'(x) = \dfrac{1}{f'(g(x))}$

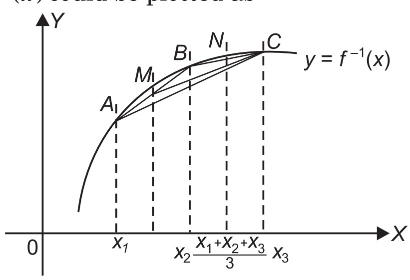
$\Rightarrow$ $\quad g''(x) = -\dfrac{1}{(f'(g(x)))^2} f''(g(x))\, g'(x)$, for all $x \in R$

$\Rightarrow$ $\quad g''(x) < 0, \begin{cases} \text{as } g'(x) > 0, & \text{from relation (i)} \\ f''(x) > 0, & \text{given} \end{cases}$

$\Rightarrow$ $\quad g'(x)$ is decreasing for all $x \in R$.

$\Rightarrow$ $\quad f^{-1}(x)$ is increasing and $\dfrac{d}{dx}\{f^{-1}(x)\}$ is decreasing.

Thus, the graph for $f^{-1}(x)$ could be plotted as



In above figure, we have taken three points A, B, C as; $A(x_1, f^{-1}(x_1))$, $B(x_2, f^{-1}(x_2))$, $C(x_3, f^{-1}(x_3))$.

Also, M is the mid-point of AB as $\left(\dfrac{x_1 + x_2}{2}, \dfrac{f^{-1}(x_1) + f^{-1}(x_2)}{2}\right)$

and L as the centroid of ΔABC,

ie, $\quad L = \left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{f^{-1}(x_1) + f^{-1}(x_2) + f^{-1}(x_3)}{3}\right)$

Correspondingly a point N lies on the curve;

$$N = \left(\frac{x_1 + x_2 + x_3}{3}, f^{-1}\left(\frac{x_1 + x_2 + x_3}{3}\right)\right)$$

Also, from above figure it is clear that ordinate of $N >$ ordinate of L

$\Rightarrow$ $\quad f^{-1}\left(\dfrac{x_1 + x_2 + x_3}{3}\right) > \dfrac{f^{-1}(x_1) + f^{-1}(x_2) + f^{-1}(x_3)}{3}$

Example 12 Find the points on the curve $ax^2 + 2bxy + ay^2 = c$, $0 < a < b < c$, whose distance from the origin is minimum.

Solution. Let $P(x, y)$ be a point on the curve $ax^2 + 2bxy + ay^2 = c$ whose distance from the origin is r.

$\therefore \quad x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$

As $r\cos\theta$ and $r\sin\theta$ lies on $ax^2 + 2bxy + ay^2 = c$,

$\Rightarrow \quad ar^2\cos^2\theta + 2br^2\sin\theta\cos\theta + ar^2\sin^2\theta = c$

$\Rightarrow \quad (a + b\sin 2\theta)r^2 = c$

$$\Rightarrow \quad r^2 = \frac{c}{a + b\sin 2\theta} \qquad \text{...(i)}$$

From Eq. (i) r is minimum when $(a + b\sin 2\theta)$ is maximum.

ie, $\sin 2\theta$ is maximum.

ie, $\quad 2\theta = \frac{\pi}{2}$ or $\frac{5\pi}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{4}, \frac{5\pi}{4}$

For θ, maximum value of $(a + b\sin 2\theta) = a + b$

$$\therefore \quad r_{\text{min}} = \sqrt{\frac{c}{a+b}}$$

Also, when $\quad \theta = \frac{\pi}{4}, P\left(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right) = \left(\sqrt{\frac{c}{2(a+b)}}, \sqrt{\frac{c}{2(a+b)}}\right)$

Again, when $\quad \theta = \frac{5\pi}{4}, P\left(\frac{-r}{\sqrt{2}}, \frac{-r}{\sqrt{2}}\right) = \left(-\sqrt{\frac{c}{2(a+b)}}, -\sqrt{\frac{c}{2(a+b)}}\right)$

Thus, the required points are $\pm\left(\sqrt{\frac{c}{2(a+b)}}, \sqrt{\frac{c}{2(a+b)}}\right)$.

Example 13 Determine the points of maxima and minima of the function, $f(x) = \frac{1}{8}\log x - bx + x^2$, $x > 0$ when $b \geq 0$ is a constant. **[IIT JEE 1996]**

Solution. Here, $f(x) = \frac{1}{8}\log x - bx + x^2$ is defined and continuous for all $x > 0$.

Then, $\quad f'(x) = \frac{1}{8x} - b + 2x \quad$ or $\quad f'(x) = \frac{16x^2 - 8bx + 1}{8x}$

For extrema let $\quad f'(x) = 0$

$\Rightarrow \quad 16x^2 - 8bx + 1 = 0$

So, $\quad x = \frac{8b \pm \sqrt{64(b^2 - 1)}}{2 \times 16} \quad$ or $\quad x = \frac{b \pm \sqrt{b^2 - 1}}{4}$

Obviously the roots are real, if $b^2 - 1 \geq 0$

$\Rightarrow \quad b > 1 \qquad$ (as $b > 0$)

Hence, when $b > 1$, then using number line rule for $f'(x)$ as shown in given figure.

We know $f'(x)$ changes sign from +ve to –ve at

$$x = \frac{b - \sqrt{b^2 - 1}}{4}$$

$$\therefore \quad f(x)_{\text{max}} \text{ at } x = \frac{b - \sqrt{b^2 - 1}}{4}$$

and $f'(x)$ changes sign from –ve to +ve at

$$x = \frac{b + \sqrt{b^2 - 1}}{4}$$

$$\begin{array}{ccccc} + & | & - & | & + \\ & \frac{b - \sqrt{b^2-1}}{4} & & \frac{b + \sqrt{b^2-1}}{4} & \end{array}$$

$\therefore$ $\quad f(x)_{\min}$ at $x = \dfrac{b + \sqrt{b^2 - 1}}{4}$

Also, if $b = 1$ $\quad f'(x) = \dfrac{16x^2 - 8x + 1}{x} = \dfrac{(4x - 1)^2}{x}$ no change in sign.

$\therefore$ Neither maximum nor minimum, if $b = 1$

Thus,

$$f(x) \Rightarrow \begin{cases} f(x)_{\max} \text{ when } x = \dfrac{b - \sqrt{b^2 - 1}}{4} \text{ and } b > 1 \\ f(x)_{\min} \text{ when } x = \dfrac{b + \sqrt{b^2 - 1}}{4} \text{ and } b > 1 \\ f(x) \text{ neither maximum nor minimum when } b = 1 \end{cases}$$

Example 14 Let $f(x) = \sin^3 x + \lambda \sin^2 x$ where $-\pi/2 < x < \pi/2$. Find the intervals in which λ should lie in order that $f(x)$ has exactly one minimum.

Solution. $\quad f(x) = \sin^3 x + \lambda \sin^2 x$

$\therefore$ $\quad f'(x) = 3 \sin^2 x (\cos x) + \lambda \cdot 2 \sin x (\cos x)$

$$= \sin x \cos x (3 \sin x + 2\lambda)$$

For extremum, let $\quad f'(x) = 0$

$\therefore$ $\quad \sin x = 0, \quad \cos x = 0, \quad \sin x = -2\lambda/3$

Since, $\quad -\pi/2 < x < \pi/2 \quad \therefore \quad \cos x \neq 0$

$\Rightarrow$ $\quad \sin x = 0 \quad \Rightarrow \quad x = 0$

and $\quad \sin x = \dfrac{-2\lambda}{3} \quad \Rightarrow \quad x = \sin^{-1}\left(\dfrac{-2\lambda}{3}\right) \quad$ ...(i)

One of these from Eq. (i) will give maximum and one minimum, provided

$$-1 < \sin x = \frac{-2\lambda}{3} < 1$$

ie, $\quad -1 < \dfrac{-2\lambda}{3} < 1$

$\Rightarrow$ $\quad -3 < -2\lambda < 3$

$\Rightarrow$ $\quad -3 < 2\lambda < 3$

ie, $\quad -3/2 < \lambda < 3/2$

But, if $\lambda = 0$, then $\sin x = 0$ has only one solution.

$\therefore$ $\quad \lambda \in (-3/2, 3/2) - \{0\}$

$\Rightarrow$ $\quad \lambda \in (-3/2, 0) \cup (0, 3/2)$

For this value of λ there are two distinct solutions.

Since, $f(x)$ is continuous, these solutions give one maximum and one minimum because for a continuous function, between two maxima there must lie one minima and *vice-versa*.

Example 15 Which normal to the curve $y = x^2$ forms the shortest chord?

[IIT JEE 1992]

Solution. Let (t, t^2) be any point on the parabola $y = x^2$

Now, $$\frac{dy}{dx} = 2x \Rightarrow \left(\frac{dy}{dx}\right)_{(t,\, t^2)} = 2t, \text{ which is the slope of tangent.}$$

So, the slope of the normal to $y = x^2$ at (t, t^2) is $\left(-\frac{1}{2t}\right)$.

$\therefore$ The equation of the normal to $y = x^2$ at (t, t^2) is

$$y - t^2 = \left(-\frac{1}{2t}\right)(x - t) \qquad \text{...(i)}$$

Suppose Eq. (i) meets the curve again at $B(t_1, t_1^2)$, then

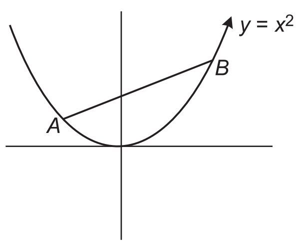
$$t_1^2 - t^2 = -\frac{1}{2t}(t_1 - t)$$

$$\Rightarrow \quad t_1 + t = -\frac{1}{2t}$$

$$\Rightarrow \quad t_1 = -t - \frac{1}{2t} \qquad \text{...(ii)}$$

Let L be the length of the chord AB (as normal)

$$L = AB^2 = (t - t_1)^2 + (t^2 - t_1^2)^2$$
$$= (t - t_1)^2 [1 + (t + t_1)^2]$$
$$= \left(t + t + \frac{1}{2t}\right)^2 \left[1 + \left(t - t - \frac{1}{2t}\right)^2\right] \qquad \text{[using Eq. (ii)]}$$
$$= \left(2t + \frac{1}{2t}\right)^2 \left(1 + \frac{1}{4t^2}\right)$$



$$L = 4t^2 \left(1 + \frac{1}{4t^2}\right)^3$$

$$\Rightarrow \quad \frac{dL}{dt} = 8t\left(1 + \frac{1}{4t^2}\right)^2 + 12t^2\left(1 + \frac{1}{4t^2}\right)^2 \cdot \left(-\frac{2}{4t^3}\right)$$

$$\Rightarrow \quad \frac{dL}{dt} = 2\left(1 + \frac{1}{4t^2}\right)^2 \left[4t\left(1 + \frac{1}{4t^2}\right) - \frac{3}{t}\right]$$

$$\Rightarrow \quad \frac{dL}{dt} = 2\left(1 + \frac{1}{4t^2}\right)^2 \left(4t - \frac{2}{t}\right) = 4\left(1 + \frac{1}{4t^2}\right)^2 \left(2t - \frac{1}{t}\right)$$

For extremum, let $\dfrac{dL}{dt}=0$

$\Rightarrow \qquad t=\pm\dfrac{1}{\sqrt{2}}$

Again, $$\frac{d^2L}{dt^2}=8\left(1+\frac{1}{4t^2}\right)\left(-\frac{1}{2t^2}\right)\left(2t-\frac{1}{t}\right)+4\left(1+\frac{1}{4t^2}\right)^2\left(2+\frac{1}{t^2}\right)$$

$\Rightarrow \qquad \left(\dfrac{d^2L}{dt^2}\right)_{t=\pm\frac{1}{\sqrt{2}}}>0 \qquad \therefore$ Minimum when $t=\pm\dfrac{1}{\sqrt{2}}$

Thus, points are $A\left(\pm\dfrac{1}{\sqrt{2}},\dfrac{1}{2}\right)$ and $B(\mp\sqrt{2},2)$.

$\Rightarrow$ Equation of normal AB is $\sqrt{2}x+2y-2=0$ and $\sqrt{2}x-2y+2=0$.

Example 16 Find the minimum value of $f(x)=\big||x+2|-2|x-2|\big|+|x|$

Solution. Here, $f(x)=\big||x+2|-2|x-2|\big|+|x|$, which gives rise to four cases as :

Case I $\boldsymbol{x<-2}$

$$\begin{aligned} f(x)&=|-(x+2)+2(x-2)|-x\\ &=|-x-2+2x-4|-x\\ &=|x-6|-x=-(x-6)-x \end{aligned}$$

$\Rightarrow \qquad f(x)=-2x+6 \qquad$ ...(i)

Case II $\boldsymbol{-2\le x<0}$

$$\begin{aligned} f(x)&=|(x+2)+2(x-2)|-x\\ &=|3x-2|-x=-(3x-2)-x \end{aligned}$$

$\Rightarrow \qquad f(x)=-4x+2 \qquad$ ...(ii)

Case III $\boldsymbol{0\le x<2}$

$$\begin{aligned} f(x)&=|x+2+2(x-2)|+x\\ &=|3x-2|+x\\ &=\begin{cases}-(3x-2)+x, & \text{for } 0\le x<\dfrac{2}{3}\\[2mm] (3x-2)+x, & \text{for } \dfrac{2}{3}\le x<2\end{cases} \end{aligned}$$

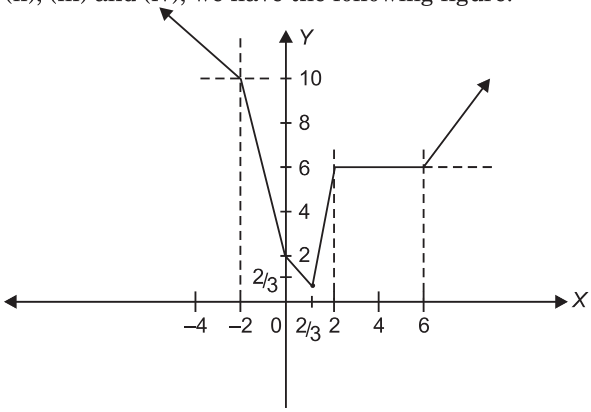
$\Rightarrow \qquad f(x)=\begin{cases}-2x+2, & 0\le x<2/3\\ 4x-2, & 2/3\le x<2\end{cases} \qquad$ ...(iii)

Case IV $\boldsymbol{x\ge 2}$

$$\begin{aligned} f(x)&=|x+2-2(x-2)|+x=|-x+6|+x=|x-6|+x\\ &=\begin{cases}-(x-6)+x, & 2\le x<6\\ (x-6)+x, & x\ge 6\end{cases} \end{aligned}$$

$f(x)=\begin{cases}6, & 2\le x<6\\ 2x-6, & x\ge 6\end{cases} \qquad$ ...(iv)

From Eqs. (i), (ii), (iii) and (iv), we have the following figure.



From the graph, minimum value of $f(x)=\dfrac{2}{3}$.

Example 17 Let $f(x)=\sin^{-1}\left(\dfrac{2\,\phi(x)}{1+\phi^2(x)}\right)$, then find the interval in which $f(x)$ is increasing or decreasing.

Solution. Here, $f(x)=\sin^{-1}\left(\dfrac{2\,\phi(x)}{1+\phi^2(x)}\right)$

Case I $|\phi(x)|<1$

Let $\qquad \phi(x)=\tan\theta$

$$\therefore \qquad f(x)=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\sin 2\theta)=2\theta$$

$$\Rightarrow \qquad f(x)=2\tan^{-1}\{\phi(x)\}$$

$$\Rightarrow \qquad f'(x)=\frac{2}{1+\{\phi(x)^2\}}\cdot\phi'(x)$$

where $\phi'(x)>0\Rightarrow f'(x)>0\Rightarrow f(x)$ is increasing

[since $\phi(x)=\tan\theta$ which is increasing]

Case II When $|\phi(x)|>1$ or $\left|\dfrac{1}{\phi(x)}\right|<1$

Now, put $\qquad \dfrac{1}{\phi(x)}=\tan\theta$

$$\therefore \qquad f(x)=\sin^{-1}\left(\frac{2\cdot\dfrac{1}{\phi(x)}}{1+\left(\dfrac{1}{\phi(x)}\right)^2}\right)$$

$$\Rightarrow \qquad f(x)=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$f(x)=\sin^{-1}(\sin 2\theta)=2\theta=2\tan^{-1}\left(\frac{1}{\phi(x)}\right)$$

$$=2\cot^{-1}(\phi(x))$$

$\Rightarrow \qquad f'(x) = -2 \cdot \dfrac{1}{1 + (\phi(x))^2} \cdot \phi'(x)$

where $\qquad \phi'(x) > 0$

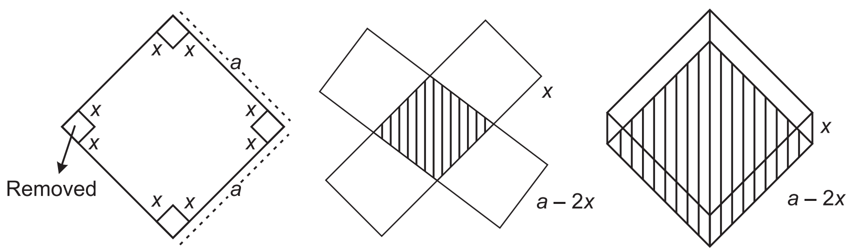
$\Rightarrow \qquad f'(x) < 0$ for all $|\phi(x)| > 1$

Hence, $f(x)$ is increasing when, $|\phi(x)| < 1$

and $f(x)$ is decreasing when, $|\phi(x)| > 1$.

Example 18 A box of maximum volume with top open is to be made by cutting out four equal squares from four corners of a square tin sheet of side length a feet, and then folding up the flaps. Find the side of the square cut off.

Solution. Volume of the box is, $V = (a - 2x)^2 \cdot x$ *ie*, squares of side x are cut out, then we will get a box with a square base of side $(a - 2x)$ and height x.



$\therefore \qquad \dfrac{dV}{dx} = (a - 2x)^2 + x \cdot 2(a - 2x) \cdot (-2)$

$\qquad \dfrac{dV}{dx} = (a - 2x)(a - 6x)$

For V to be extremum, $\qquad \dfrac{dV}{dx} = 0$

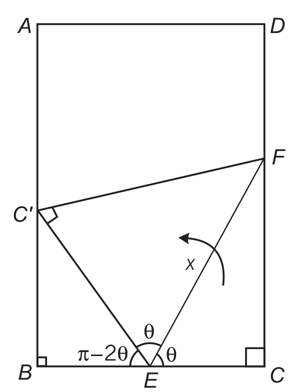
$\Rightarrow \qquad x = a/2, a/6$

But when $x = a/2$; $V = 0$ (minimum) and we know minimum and maximum occurs simultaneously in a continuous function.

Hence, V is maximum when $x = a/6$.

Example 19 One corner of a long rectangular sheet of paper of width 1 unit is folded over so as to reach the opposite edge of the sheet. Find the minimum length of the crease.

Solution. Let $ABCD$ be the rectangular sheet whose corner C is folded over along EF so as to reach the edge AB at C'.



Let $\qquad EF = x$

$\qquad \angle FEC = \theta = \angle FEC'$

$\therefore \qquad EC = x \cos\theta = EC'$

From $\Delta BEC'$, we have $BE = C'E \cos(\pi - 2\theta)$

$\Rightarrow \qquad BE = -x \cos\theta \cdot \cos 2\theta$

$\therefore \qquad BC = BE + EC$

$\qquad 1 = -x \cos\theta \cos 2\theta + x \cos\theta$

$$\Rightarrow \qquad x = \frac{1}{\cos\theta\,(1 - \cos 2\theta)} \qquad \text{...(i)}$$

x to be minimum, Z has to be maximum.

ie, $$Z = \frac{1}{x} = \cos\theta\,(1 - \cos 2\theta) \qquad \text{...(ii)}$$

Differentiating Eq. (ii) w.r.t. θ, we get

$$\frac{dZ}{d\theta} = \cos\theta\,(+2\sin 2\theta) - \sin\theta\,(1 - \cos 2\theta)$$

and $$\frac{d^2Z}{d\theta^2} = \cos\theta\,(4\cos 4\theta) - 2\sin 2\theta \cdot \sin\theta - \sin\theta\,(2\sin 2\theta) - \cos\theta\,(1 - \cos 2\theta)$$

For maximum/minimum,

$$\frac{dZ}{d\theta} = 0 \quad \Rightarrow \quad 2\sin\theta\,(2 - 3\sin^2\theta) = 0$$

$\therefore$ $$\sin\theta = +\sqrt{2/3} \qquad (\because \quad \sin\theta \neq 0)$$

When $$\sin\theta = \sqrt{\frac{2}{3}} \quad \Rightarrow \quad \frac{d^2Z}{d\theta^2} = -\frac{5}{5\sqrt{3}} - \frac{16}{3\sqrt{3}} - \frac{1}{\sqrt{3}} = -\frac{8}{\sqrt{3}} < 0$$

Hence, Z is maximum.

$\Rightarrow$ $$x = \frac{1}{Z} \text{ is minimum} \qquad \text{[from Eq. (i)]}$$

$\therefore$ x is minimum.

$\Rightarrow$ $$\min x \quad \Rightarrow \quad \frac{1}{Z} = \frac{1}{(1/\sqrt{3})\left(1 + \frac{1}{3}\right)} = \frac{3\sqrt{3}}{4} \text{ unit}$$

Example 20 Find the volume of the greatest right circular cone that can be described by the revolution about a side of a right angled triangle of hypotenuse 1 feet.

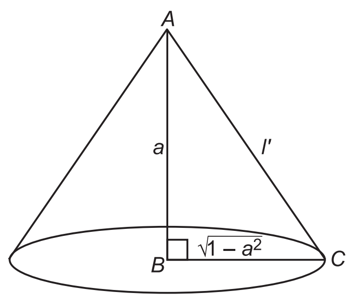
Solution. Let ABC be right angled triangle .

Let the cone be revolved about AB.

$$AC = 1 \text{ feet} \qquad \text{(given)}$$

Let $AB = a$ = height of cone

$\therefore$ $$BC = \sqrt{1 - a^2} = \text{radius of cone}$$



$\therefore$ Volume of cone $= \frac{1}{3}\pi\,(1 - a^2)a$

$$V = \frac{1}{3}\pi\,(a - a^3)$$

$\Rightarrow$ $$\frac{dV}{da} = \frac{1}{3}\pi\,(1 - 3a^2)$$

and $$\frac{d^2V}{da^2} = \frac{1}{3}\pi\,(-6a) < 0$$

$\therefore$ Maximum volume when $\frac{dV}{da} = 0$

ie, when $a = \frac{1}{\sqrt{3}}$

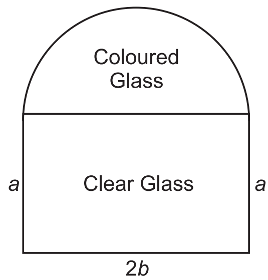
Putting $a = \frac{1}{\sqrt{3}}$, we get $\qquad V_{\max} = \frac{2\sqrt{3}}{27}\pi$ cu ft

Example 21 A window of fixed perimeter (including the base of the arc) is in the form of a rectangle surmounted by a semi-circle. The semi-circular portion is fitted with coloured glass while the rectangular portion is fitted with clear glass. The clear glass transmits three times as much light per square metre as the coloured glass does. What show be the ratio of the sides of the rectangle so that the window transmits the maximum light?

Solution. Let $2b$ be the diameter of the circular portion and a be the lengths of the other side of the rectangle.

$$\text{Total perimeter} = 2a + 4b + \pi b = K \quad \text{(say)} \quad \text{...(A)}$$

Now, let the light transmission rate (per square metre) of the coloured glass be L and Q be the total amount of transmitted light.



Then, $$Q = 2ab \cdot (3L) + \frac{1}{2}\pi b^2 \cdot (L)$$

$$Q = \frac{L}{2}[\pi b^2 + 12ab]$$

$$Q = \frac{L}{2}[\pi b^2 + 6\,b(K - 4\,b - \pi b)]$$

$$Q = \frac{L}{2}[6Kb - 24b^2 - 5\pi b^2]$$

$\therefore$ $$\frac{dQ}{db} = \frac{L}{2}[6K - 48b - 10\pi b] = 0$$

$$b = \frac{6K}{48 + 10\pi} \quad \text{...(B)}$$

and $$\frac{d^2Q}{db^2} = \frac{L}{2}[-48 - 10\pi] < 0$$

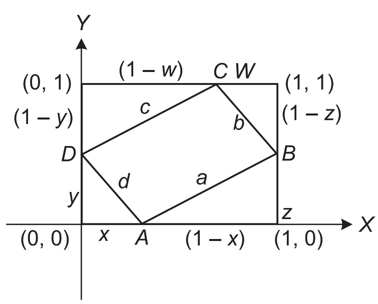
Thus, Q is maximum and from Eqs. (A) and (B)

$$(48 + 10\pi)\,b = 6K = 6\{2a + 4b + \pi b\}$$

Thus, $$\text{the ratio} = \frac{2b}{a} = \frac{6}{6 + \pi}$$

Example 22 Let S be a square of unit area. Consider any quadrilateral which has one vertex on each side of S. If a, b, c and d denote the lengths of the sides of the quadrilateral, prove that $2 \le a^2 + b^2 + c^2 + d^2 \le 4$ **[IIT JEE 1997]**

Solution. Let S be the square of unit area and $ABCD$ be the quadrilateral of sides a, b, c and d.



Here, $$a^2 = (1 - x^2) + z^2,\ \ b^2 = w^2 + (1 - z)^2$$

$$c^2 = (1 - w^2) + (1 - y)^2,\ d^2 = x^2 + y^2$$

Adding all the above, $a^2 + b^2 + c^2 + d^2$

$$= \{x^2 + (1-x)^2\} + \{y^2 + (1-y)^2\} + \{z^2 + (1-z)^2\} + \{w^2 + (1-w)^2\}$$

where $0 \le x, y, z, w \le 1$

Let us consider a function,

$$f(x) = x^2 + (1-x)^2, 0 \le x \le 1$$

Then, $f'(x) = 2x - 2(1-x)$

Let $f'(x) = 0$ for maximum/minimum.

$\Rightarrow \quad 4x - 2 = 0$

$\Rightarrow \quad x = 1/2$

Again, $f''(x) = 4 > 0$ when $x = 1/2$

$\therefore$ $f(x)$ is minimum at $x = 1/2$ and maximum at $x = 1$

$\Rightarrow \quad 2 \le a^2 + b^2 + c^2 + d^2 \le 4$

Example 23 Show that a triangle of maximum area that can be inscribed in a circle of radius a is an equilateral triangle.

Solution. Let BC be one of the sides of the triangle and the third vertex A should be in a position that the altitude AD is maximum (for area of the triangle to be maximum).

For that the ΔABC must be symmetric about AD.

ie, D should be the mid-point of BC.

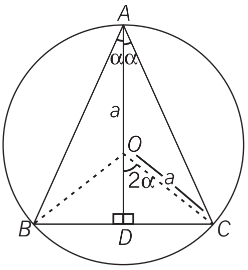
Let $\angle A = 2\alpha$

$\therefore \quad \angle BOD = \angle COD = 2\alpha$

Thus, $AD = AO + OD = a + a\cos 2\alpha$

and $CD = a \sin 2\alpha$

Hence, area of $\Delta ABC = A = \frac{1}{2} \cdot AD \cdot BC$



Area, $A = \frac{1}{2} \cdot a(1 + \cos 2\alpha) \cdot a \sin 2\alpha$

$$A = \frac{a^2}{2}\left(\sin 2\alpha + \frac{1}{2}\sin 4\alpha\right)$$

Differentiating w.r.t. α, we get

$\therefore \quad \frac{dA}{d\alpha} = \frac{a^2}{2}[2\cos 2\alpha + 2\cos 4\alpha] = 0$

$$2a^2 \cos 3\alpha \cdot \cos \alpha = 0$$

$\Rightarrow$ Either $\cos 3\alpha = 0$ or $\cos \alpha = 0$

$\Rightarrow \quad \alpha = \pi/6, \pi/2$

But $\alpha = \pi/2$ is not possible.

Now, $\frac{d^2A}{d\alpha^2} = 2a^2[-\sin 2\alpha - 4\sin 4\alpha] = -\text{ve at } \alpha = \pi/6$

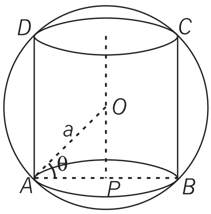
$\therefore$ A is maximum when $\alpha = \pi/6$.

Also, $\angle A = \pi/3$ and triangle is isosceles.

Hence, ΔABC must be equilateral.

Example 24 Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a is $2a/\sqrt{3}$.

Solution. Let h be the height and r be the radius of the cylinder. Let O be the centre of the sphere, as shown in figure.



From the figure,

$$OA^2 = OP^2 + PA^2$$

ie, $$a^2 = \frac{h^2}{4} + r^2 \quad \text{...(i)}$$

Now, volume of the cylinder, $V = \pi r^2 h = \pi h \left(a^2 - \frac{h^2}{4}\right)$ [using Eq. (i)]

Differentiating w.r.t. h, we have

$$\therefore \quad \frac{dV}{dh} = \pi \left[a^2 - \frac{3h^2}{4}\right] = 0 \text{ for extremum} \Rightarrow h = \frac{2a}{\sqrt{3}}$$

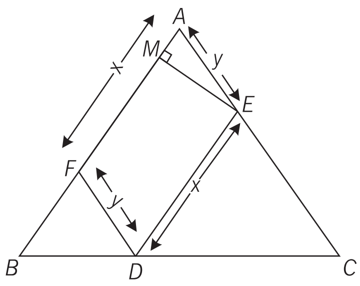
Also, $$\frac{d^2V}{dh^2} = -\frac{3\pi}{2} h < 0 \text{ at } h = \frac{2a}{\sqrt{3}}$$

Thus, volume is maximum when $h = \frac{2a}{\sqrt{3}}$

Example 25 Let $A(p^2, p)$, $B(q^2, -q)$ and $C(r^2, -r)$ be the vertices of the ΔABC. A parallelogram $AFDE$ is drawn with vertices D, E, F on the line segments BC, CA and AB respectively. Show that maximum area of such a parallelogram is $\frac{1}{4}(p-q)(q-r)(r-p)$.

Solution. Let $AF = x = DE$ and $AE = y = DF$

Since, $\Delta' s$ CAB and CED are similar.



We have, $$\frac{CE}{CA} = \frac{DE}{AB} \quad \text{(as shown in figure)}$$

$$\Rightarrow \quad \frac{b-y}{b} = \frac{x}{c} \quad \text{...(i)}$$

(here, $BC = a$, $AC = b$ and $AB = c$)

Now, area of parallelogram,

$$S = AF \cdot EM = xy \sin A$$

$$S = x \cdot b \left(1 - \frac{x}{c}\right) \sin A \quad \text{[from Eq. (i)] ...(ii)}$$

Differentiating w.r.t. x, we have

$$\frac{dS}{dx} = \frac{b}{c}(c - 2x)\sin A \qquad \text{(where } \sin A \text{ is constant)}$$

For extremum, $\dfrac{dS}{dx} = 0 \quad \Rightarrow \quad x = \dfrac{c}{2}$

Also, $\dfrac{d^2S}{dx^2} = -\dfrac{2b}{c} < 0$ at $x = \dfrac{c}{2}$

Hence, S is maximum when $x = \dfrac{c}{2}$

Now,

$$S_{\max} = \frac{1}{4} bc \sin A \qquad \text{[from Eq. (ii)]}$$

$$S_{\max} = \frac{1}{2}\left(\frac{1}{2} bc \sin A\right)$$

$$S_{\max} = \frac{1}{2}(\text{area of } \Delta ABC)$$

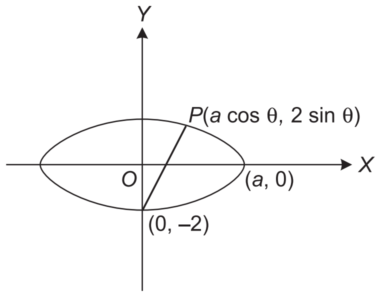
$$S_{\max} = \frac{1}{4}\begin{vmatrix} p^2 & -p & 1 \\ q^2 & -q & 1 \\ r^2 & -r & 1 \end{vmatrix}$$

$$S_{\max} = \frac{1}{4}(p - q)(q - r)(r - p)$$

Example 26 Find the point on the curve $4x^2 + a^2y^2 = 4a^2$; $4 < a^2 < 8$ that is farthest from the point $(0, -2)$.

Solution. The equation of given curve can be expressed as shown in the figure

$$\frac{x^2}{a^2} + \frac{y^2}{4} = 1, \text{ where } 4 < a^2 < 8 \text{ which is equation of ellipse.}$$



Hence, let us consider a point $P\,(a \cos \theta, 2 \sin \theta)$ on the ellipse.

Let the distance of $P\,(a \cos \theta, 2 \sin \theta)$ from $(0, -2)$ is L.

Then, $L^2 = (a \cos \theta - 0)^2 + (2 \sin \theta + 2)^2$

Differentiating w.r.t. θ, we have

$$\therefore \quad \frac{d\,(L^2)}{d\theta} = a^2 \cdot 2 \cos \theta\,(-\sin \theta) + 4 \cdot 2\,(\sin \theta + 1) \cdot \cos \theta$$

$$= \cos \theta\,[-2a^2 \sin \theta + 8 \sin \theta + 8] = 0$$

$\Rightarrow$ Either $\cos \theta = 0$ or $(8 - 2a^2) \sin \theta + 8 = 0$

ie, $\theta = \pi/2$ or $\sin \theta = \dfrac{4}{a^2 - 4}$

Since, $\quad a^2 < 8 \quad \Rightarrow \quad a^2 - 4 < 4$

$\Rightarrow \quad \dfrac{4}{a^2 - 4} > 1$

$\Rightarrow \quad \sin\theta > 1$ which is not possible.

Further, $\dfrac{d^2(L^2)}{d\theta^2} = \cos\theta\,[-2a^2\cos\theta + 8\cos\theta] + (-\sin\theta)\,[-2a^2\sin\theta + 8\sin\theta + 8]$

At $\quad \theta = \pi/2, \dfrac{d^2(L^2)}{d\theta^2} = 0 - [16 - 2a^2] = 2\,(a^2 - 8) < 0 \qquad$ (as $a^2 < 8$)

Hence, L is maximum at $\theta = \pi/2$ and the farthest point is (0, 2).

Example 27 LL' be latusrectum of the parabola $y^2 = 4ax$ and PP' is a double ordinate between the vertex and the latusrectum. Show that the area of trapezium $PP'L'L$ is maximum when the distance of PP' from the vertex is $\dfrac{a}{9}$.

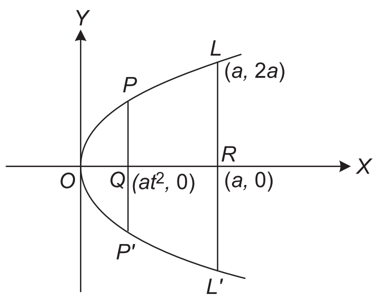
Solution. Let the double ordinate PP' is drawn at a distance $x = at^2$ from the origin. (vertex).

Thus, the coordinate of P is $(at^2, 2at)$ as shown in figure.

Hence, the area of trapezium is,

$$A = \frac{1}{2}(PP' + LL')\cdot QR$$

$$= \frac{1}{2}(4at + 4a)\cdot(a - at^2)$$



$$A = 2a^2\,(t + 1)(1 - t^2)$$

$\therefore \quad \dfrac{dA}{dt} = 2a^2\,(-3t^2 - 2t + 1) = 0 \qquad$ (for extremum)

$\Rightarrow \quad t = -1, 1/3$

Also, $\quad \dfrac{d^2A}{dt^2} = 2a^2\,(-6t - 2)$

Thus, A is maximum when $t = 1/3$ as $\dfrac{d^2A}{dt^2} < 0$

Hence, $x = at^2 = \dfrac{a}{9}$ is the point at which area of trapezium is maximum.

Example 28 The circle $x^2 + y^2 = 1$ intersects the x-axis at P and Q. Another circle with centre at Q and variable radius intersects the first circle at R, above the x-axis and the line segment PQ at S. Find the maximum area of the ΔQSR.

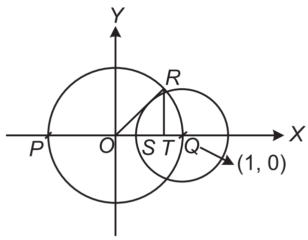
[IIT JEE 1994]

Solution. The centre of the circle

$$x^2 + y^2 = 1 \quad \text{...(i)}$$

is (0, 0) and radius $OP = 1 = OQ$

so, coordinates of Q are $(1, 0)$.



Let the radius of the variable circle be r.

Hence, its equation is

$$(x-1)^2 + (y)^2 = r^2 \quad \text{...(ii)}$$

Subtracting Eq. (ii) from Eq. (i), we get

$$2x - 1 = 1 - r^2$$

$$x = 1 - \frac{r^2}{2} = OT \quad \text{...(iii)}$$

Now, $$RT = \sqrt{OR^2 - OT^2} = \sqrt{1 - \left(1 - \frac{r^2}{2}\right)^2} \quad \text{..(iv)}$$

Now, the area of $\Delta\, QSR$ is, $A = \frac{1}{2} \cdot QS \cdot RT$

$$\therefore \quad A^2 = \frac{1}{4}(QS^2)\cdot(RT^2)$$

$$A^2 = \frac{1}{4} r^2 \left(r^2 - \frac{r^4}{4}\right) \quad \text{[using Eqs. (ii) and (iv)]}$$

$$A^2 = \frac{1}{16}(4r^4 - r^6)$$

Thus, $$\frac{d\,(A^2)}{dr} = \frac{1}{16}(16r^3 - 6\,r^5) = 0 \quad \text{(for extremum)}$$

$$\Rightarrow \quad r = 2\sqrt{\frac{2}{3}}$$

Also, $$\frac{d^2(A^2)}{dr^2} = \frac{1}{16}(48r^2 - 30r^4) = -\frac{16}{3} < 0 \quad \text{where } r = 2\sqrt{\frac{2}{3}}$$

Hence, area is maximum at $r = 2\sqrt{\frac{2}{3}}$ and $A_{\max} = \frac{4}{3\sqrt{3}}$ sq unit.

Example 29 John has x children from his first wife. Mary has $(x + 1)$ children from her first husband. They marry and have children of their own. The whole family has 24 children. Assuming that the two children of same parents do not fight, then find the maximum possible number of fights that can take place.

Solution. Since, the whole family has 24 children, those of John and Mary are

$$24 - x - (x + 1)$$

ie, $$(23 - 2x)$$

Now, F = Total number of fights.

= (number of fights when a John's child fights a Mary's child) + (number of fights when a John child fights a John-Mary's child) + (number of fights when a Mary's child fights a John-Mary's child)

$$= x(x+1) + x(23-2x) + (x+1)(23-2x) = 23 + 45x - 3x^2$$

For maximum, $\dfrac{dF}{dx} = 0$

$\Rightarrow$ $45 - 6x = 0$ or $x = 7.5$

$\therefore f(x)$ is minimum when $x = 7.5$

But in this case fractional value is not possible.

The nearest integral values are $x = 7$ and $x = 8$.

In either case the total number of fights $= 23 + 45 \times 7 - 3(7)^2 = 191$

Example 30 In the graph of the function $y = \dfrac{3}{\sqrt{2}} x \log x$, where $x \in [e^{-1.5}, \infty[$ find the point $P(x, y)$ such that the segment of the tangent to the graph of the function at the point, intercepted between the point P and y-axis, is shortest.

Solution. The point $P(x, y)$ $\Rightarrow$ $P\left(x, \dfrac{3}{\sqrt{2}} x \log_e x\right)$

Differentiating w.r.t. x, the given function, we have

$$\frac{dy}{dx} = \frac{3}{\sqrt{2}}\left(x \cdot \frac{1}{x} + \log_e x\right) = \frac{3}{\sqrt{2}}(1 + \log_e x)$$

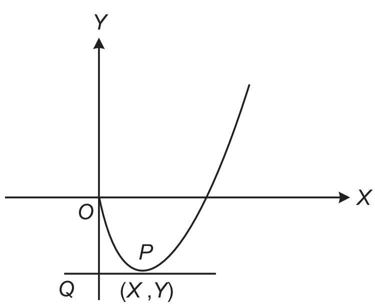
Hence, the equation of tangent to the curve at the point $P\left(x, \dfrac{3}{\sqrt{2}} x \log_e x\right)$ is

$$Y - \frac{3}{\sqrt{2}} x \log_e x = \frac{3}{\sqrt{2}}(1 + \log_e x)(X - x)$$

When it cuts Y-axis, $X = 0$

Thus, $y = \dfrac{3}{\sqrt{2}} x \log_e x - \dfrac{3}{\sqrt{2}}(1 + \log_e x)\, x$

So, $y = -\dfrac{3}{\sqrt{2}} x$



Hence, the given tangent intersects axis at

$$Q\left(0, -\frac{3}{\sqrt{2}} x\right)$$

Now, $PQ^2 = x^2 + \left(y + \dfrac{3}{\sqrt{2}} x\right)^2$

$$= x^2 + \left(\frac{3}{\sqrt{2}} x(1 + \log_e x)\right)^2$$

$$PQ^2 = x^2\left[1 + \frac{9}{2}(1 + \log_e x)^2\right]$$

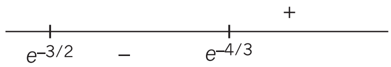
Differentiating w.r.t. x, we have

$$\therefore \quad \frac{d(PQ^2)}{dx} = x^2\left[\frac{9}{2} \cdot 2(1 + \log_e x) \cdot \frac{1}{x}\right] + \left[1 + \frac{9}{2}(1 + \log_e x)^2\right] 2x$$

$$= x[9(1 + \log_e x) + 2 + 9(1 + \log_e x)^2]$$

For extremum $\dfrac{d(PQ^2)}{dx} = 0$

Since, $x \neq 0, 9(1 + \log x)^2 + 9(1 + \log x) + 2 = 0$



$\Rightarrow \quad (1 + \log x) = -1/3, -2/3$

$\Rightarrow \quad \log x = -4/3, -5/3$

$\Rightarrow \quad x = e^{-4/3}, e^{-5/3}$

$x = e^{-5/3}$ lie outside the interval $(e^{-1.5}, \infty)$.

The sign scheme for $\dfrac{d(PQ^2)}{dx}$ is shown in figure which shows that PQ^2 is minimum.

Therefore, PQ is minimum when $x = e^{-4/3}$.

Example 31 A despatch rider is in open country at a distance of 6 km from the nearest point P of a straight road. He wishes to proceed as quickly as possible to a point Q on the road 20 km from P. If his maximum speed across country is 40 km/h, 50 km/h on road, then at what distance from P, he should touch the road.

Solution. Let A be the initial position of rider.

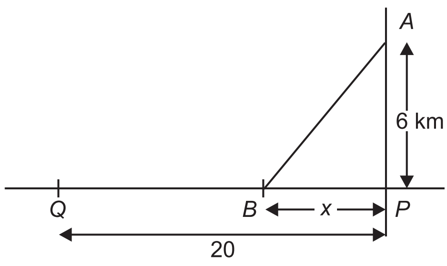
Let $\quad PB = x$

$\therefore \quad QB = 20 - x$

and $\quad AB = \sqrt{AP^2 + BP^2} = \sqrt{6^2 + x^2}$ km

$\therefore$ Total time T, $\quad T = \dfrac{\sqrt{x^2 + 36}}{40} + \dfrac{20 - x}{50}$

$$\frac{dT}{dx} = \frac{1}{40} \cdot \frac{(2x)}{2\sqrt{x^2 + 36}} - \frac{1}{50}$$



For maximum and minimum value, we must have

$$\frac{dT}{dx} = 0$$

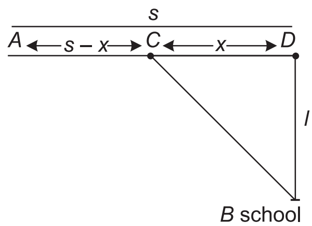
$\Rightarrow \quad \dfrac{x}{40\sqrt{x^2 + 36}} = \dfrac{1}{50}$ or $\dfrac{5x}{4} = \sqrt{x^2 + 36}$

or $\quad \dfrac{25x^2}{16} - x^2 = 36 \Rightarrow x^2 = \dfrac{36 \times 16}{9}$

$\Rightarrow \quad x = \dfrac{6 \times 4}{3} = 8$ km

Example 32 From point A located on a highway a boy has to get his bus to his school B located in the field at a distance l from the highway in the least possible time. At what distance from D should the bus leave the highway when the bus moves n times slower in the field than on the highway?

Solution. Let $AD = s$ and $CD = x$ where C is a point where the bus leaves the highway.



Also, let the speed of the bus is 'v' on the highway.

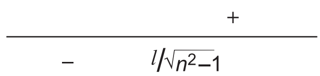
$\therefore$ Total time taken,

$$t = \frac{AC}{v} + \frac{BC}{(v/n)}$$

$$\Rightarrow \quad t = \frac{s-x}{v} + \frac{n\sqrt{l^2+x^2}}{v}$$

Differentiating w.r.t. x, we have

$$\therefore \quad \frac{dt}{dx} = \frac{1}{v}\left[-1 + \frac{n}{2\sqrt{l^2+x^2}}(+2x)\right] = 0 \qquad \text{(for extremum)}$$



$$\Rightarrow \quad n^2x^2 = l^2 + x^2$$

$$\Rightarrow \quad x = \frac{l}{\sqrt{n^2-1}}$$

Thus, 't' is minimum when $x = \dfrac{l}{\sqrt{n^2-1}}$, as shown in figure.

Example 33 Two men are walking on a path $x^3 + y^3 = a^3$ when the first man arrives at a point (x_1, y_1), he finds the second man in the direction of his own instantaneous motion. If the coordinates of the second man are (x_2, y_2), then show that

$$\frac{x_2}{x_1} + \frac{y_2}{y_1} + 1 = 0.$$

Solution. Since, (x_1, y_1) and (x_2, y_2) lies on the curve.

$$\therefore \quad x_1^3 + y_1^3 = a^3 \qquad \text{...(i)}$$

$$\text{and} \quad x_2^3 + y_2^3 = a^3 \qquad \text{...(ii)}$$

Subtracting Eqs. (i) and (ii), we get

$$(x_2^3 - x_1^3) + (y_2^3 - y_1^3) = 0$$

$$\text{or} \quad x_2^3 - x_1^3 = -(y_2^3 - y_1^3) \qquad \text{...(iii)}$$

Now, differentiating both sides of $x^3 + y^3 = a^3$ w.r.t. x, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 0$$

Slope of tangent at $(x_1, y_1) \Rightarrow -\dfrac{x_1{}^2}{y_1{}^2}$

$\therefore$ The equation of tangent at (x_1, y_1)

$$y - y_1 = -\frac{x_1^2}{y_1^2}(x - x_1)$$

It passes through (x_2, y_2).

$$\therefore \quad y_2 - y_1 = \frac{-x_1^2}{y_1^2}(x_2 - x_1)$$

$$\text{or} \quad x_1^2(x_2 - x_1) = -y_1^2(y_2 - y_1) \qquad \text{...(iv)}$$

Dividing Eqs. (iii) and (iv), we get

$$\frac{x_2^3 - x_1^3}{x_1^2(x_2 - x_1)} = \frac{-(y_2^3 - y_1^3)}{-y_1^2(y_2 - y_1)}$$

$$\Rightarrow \qquad \frac{x_2^2 + x_1^2 + x_1x_2}{x_1^2} = \frac{y_2^2 + y_1^2 + y_1y_2}{y_1^2}$$

$$\Rightarrow \qquad \left(\frac{x_2}{x_1}\right)^2 + \left(\frac{x_2}{x_1}\right) + 1 = \left(\frac{y_2}{y_1}\right)^2 + \left(\frac{y_2}{y_1}\right) + 1$$

$$\Rightarrow \qquad \left(\frac{x_2}{x_1}\right)^2 - \left(\frac{y_2}{y_1}\right)^2 = \left(\frac{y_2}{y_1}\right) - \left(\frac{x_2}{x_1}\right)$$

$$\Rightarrow \qquad \left(\frac{x_2}{x_1} - \frac{y_2}{y_1}\right) \cdot \left(\frac{x_2}{x_1} + \frac{y_2}{y_1}\right) + \left(\frac{x_2}{x_1} - \frac{y_2}{y_1}\right) = 0$$

$$\Rightarrow \qquad \left(\frac{x_2}{x_1} - \frac{y_2}{y_1}\right) \cdot \left(\frac{x_2}{x_1} + \frac{y_2}{y_1} + 1\right) = 0$$

ie, either $\dfrac{x_2}{x_1} = \dfrac{y_2}{y_1}$ or $\dfrac{x_2}{x_1} + \dfrac{y_2}{y_1} + 1 = 0$

But, $\dfrac{x_2}{x_1} \neq \dfrac{y_2}{y_1}$

$\therefore \qquad \dfrac{x_2}{x_1} + \dfrac{y_2}{y_1} + 1 = 0$

Example 34 In still water a boat moves with a velocity which is K times less than velocity the river has current. At what angle to the stream direction must the boat move to minimize drifting.

Solution. Let the flow velocity of river is u and the velocity of boat in still water is v.

Thus, $\qquad v = u/K$

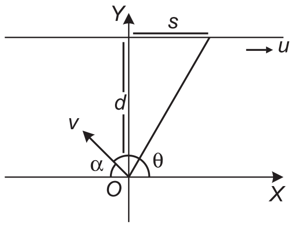
Also, let the boat moves at an angle θ with direction of stream.

Now, the velocity of boat in the river is vector resultant of the velocity of boat and velocity of following water or water current, which can be written as,

$$\bar{v}_B = (u - v\cos\alpha)\,\hat{\mathbf{i}} + (u\sin\alpha)\,\hat{\mathbf{j}} = (u + v\cos\theta)\,\hat{\mathbf{i}} + (u\sin\theta)\,\hat{\mathbf{j}}$$

Hence, the time taken to cross the river

$$= \frac{d}{u\sin\theta} \qquad (d = \text{width of river})$$



Thus, the drift $s = (u + v\cos\theta) \cdot d$

$$\Rightarrow \qquad s = d\,(\operatorname{cosec}\theta + \frac{v}{u}\cot\theta)$$

$$\Rightarrow \qquad \frac{ds}{d\theta} = d\,(-\operatorname{cosec}\theta\cot\theta - \frac{v}{u}\operatorname{cosec}^2\theta) = 0$$

$$\Rightarrow \qquad -\frac{v}{u}\operatorname{cosec}^2\theta = \operatorname{cosec}\theta\cot\theta$$

$$\Rightarrow \qquad \operatorname{cosec}\theta = -1/K \qquad \Rightarrow \qquad \theta = \cos^{-1}(-1/K)$$

Example 35 Consider a square with vertices at (1, 1) (–1, 1) (–1, –1) and (1, –1). Let S be the region consisting of all points inside the square which are nearer to the origin than to any edge. Sketch the region S and find its area. **[IIT JEE 1995]**

Solution. Let the square $ABCD$ and the equations of the sides of the square are as follows be :

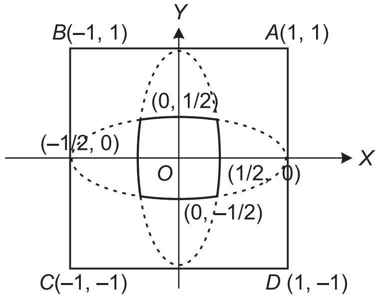
$$AB : y = 1, \quad BC : x = -1, \quad CD : y = -1, \quad DA : x = 1$$

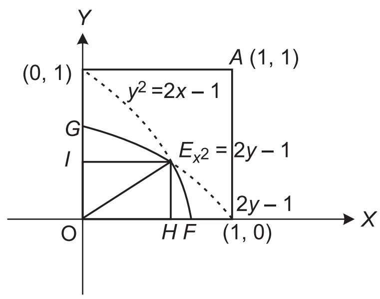
Let the region be S and (x, y) is any point inside it. Then, according to given conditions,

$$\sqrt{x^2 + y^2} < |1 - x|, |1 + x|, |1 - y|, |1 + y|$$

$$\Rightarrow \quad (x^2 + y^2) < x^2 - 2x + 1, x^2 + 2x + 1, y^2 - 2y + 1, y^2 + 2y + 1$$

$$\Rightarrow \quad y^2 < 1 - 2x, y^2 < 1 + 2x, x^2 < 1 - 2y \text{ and } x^2 < 2y + 1$$





Now, in $y^2 = 1 - 2x$ and $y^2 = 2x + 1$, the first equation represents a parabola with vertex $(1/2, 0)$ and second equation represents a parabola with vertex $(-1/2, 0)$ and in $x^2 = 1 - 2y$ and $x^2 = 1 + 2y$, the first equation represents parabola with vertex at $(0, 1/2)$ and second equation represents a parabola with vertex at $(0, -1/2)$.

So, the region S is the region lying inside the four parabolas.

$$y^2 = 1 - 2x, y^2 = 1 + 2x, x^2 = 1 - 2y, x^2 = 1 + 2y$$

Now, S is symmetrical in all four quadrants.

$$\therefore \quad S = 4 \times \text{area lying in first quadrant.}$$

Now, $\quad y^2 = 1 - 2x$ and $x^2 = 1 - 2y$ intersect on $y = x$

The point of intersection is $E(\sqrt{2} - 1, \sqrt{2} - 1)$.

$\therefore$ Area of region $OEFO$ = area of $\Delta\, OEH$ + area of region $HEFH$

$$= \frac{1}{2}(\sqrt{2} - 1)^2 + \int_{\sqrt{2}-1}^{1/2} \sqrt{1 - 2x}\, dx$$

$$= \frac{1}{2}(2 + 1 - 2\sqrt{2}) + \frac{2}{3}[(1 - 2x)^{3/2}]_{\sqrt{2}-1}^{1/2}$$

$$= \frac{1}{2}(3 - 2\sqrt{2}) + \frac{1}{3}(3 - 2\sqrt{2})^{3/2} = \frac{1}{2}(3 - 2\sqrt{2}) + \frac{1}{3}(\sqrt{2} - 1)^3$$

$$= \frac{1}{2}(3 - 2\sqrt{2}) + \frac{1}{3}(5\sqrt{2} - 7) = \frac{1}{6}[4\sqrt{2} - 5]$$

Similarly, area of region $OEGO = \frac{1}{6}(4\sqrt{2} - 5)$

So, area of S lying in first quadrant $= \frac{2}{6}(4\sqrt{2} - 5)$

Hence, $\quad S = \frac{4}{3}(4\sqrt{2} - 5)$

Example 36 If $P(1)=0$ and $\frac{d}{dx}\{P(x)\} > P(x)$ for all $x \geq 1$, then prove that $P(x) > 0$ for all $x > 1$. **[IIT JEE 2003]**

Solution. Here, $\frac{d}{dx}\{P(x)\} > P(x)$, for all $x \geq 1$

or $\quad \frac{d}{dx}\{P(x)\}e^{-x} > P(x)e^{-x}$, for all $x \geq 1$

$\Rightarrow \quad \frac{d}{dx}\{P(x)\}e^{-x} - P(x)e^{-x} > 0$, for all $x \geq 1$

$\Rightarrow \quad \frac{d}{dx}\{e^{-x}P(x)\} > 0,$

for all $\quad x \geq 1$

$\Rightarrow \quad P(x)e^{-x}$ is an increasing function for all $x \geq 1$

$\Rightarrow \quad P(x)e^{-x} > P(1)e^{-1}$, for all $x > 1$

$\Rightarrow \quad P(x)e^{-x} > 0$, for all $x > 1$ [as $P(1) = 0$, given]

Thus, $\quad P(x) > 0$, for all $x > 1$ [as $e^{-x} > 0$, for all x]

Example 37 Using the relation $2(1-\cos x) < x^2$, $x \neq 0$ or otherwise prove that $\sin(\tan x) \geq x$ for all $x \in \left[0, \frac{\pi}{4}\right]$. **[IIT JEE 2003]**

Solution. Let $f(x) = \sin(\tan x) - x$

Then, $\quad f'(x) = \cos(\tan x)\cdot\sec^2 x - 1$

$\Rightarrow \quad f'(x) = \cos(\tan x)\{1 + \tan^2 x\} - 1$

$\Rightarrow \quad f'(x) = \tan^2 x \cos(\tan x) - \{1 - \cos(\tan x)\}$

[using $2(1-\cos x) < x^2$]

$$f'(x) > \tan^2 x \cos(\tan x) - \frac{1}{2}\tan^2 x$$

$\Rightarrow \quad f'(x) > \frac{1}{2}\tan^2 x\,\{2\cos(\tan x) - 1\}$

[again, using $2(1-\cos x) < x^2$]

$$f'(x) > \frac{1}{2}\tan^2 x\,(1 - \tan^2 x)$$

$\Rightarrow \quad f'(x) \geq 0, \forall\; x \in \left[0, \frac{\pi}{4}\right]$

$\Rightarrow \quad f(x)$ is increasing function for all $x \in \left[0, \frac{\pi}{4}\right]$

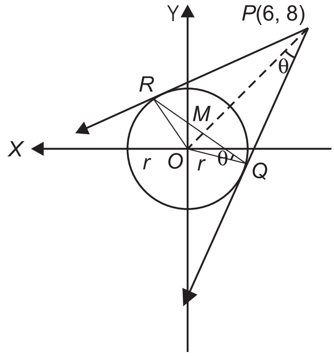
$\therefore \quad f(x) \geq f(0)$, for all $x \in \left[0, \frac{\pi}{4}\right]$

$\Rightarrow \quad \sin(\tan x) - x > \sin(\tan 0) - 0$

$\Rightarrow \quad \sin(\tan x) \geq x$, for all $x \in \left[0, \frac{\pi}{4}\right]$

Example 38 For the circle $x^2 + y^2 = r^2$, find the value of r for which the area enclosed by the tangents drawn from the point $P(6, 8)$ to the circle and the chord of contact is maximum. **[IIT JEE 2003]**

Solution. Here, $x^2 + y^2 = r^2$ and tangents from $P(6, 8)$ are shown as :



From above figure, in $\Delta\, OMQ$, we have

$$\cos\theta = \frac{MQ}{OQ} \quad \text{and} \quad \sin\theta = \frac{OM}{OQ}$$

$\therefore \qquad MQ = r\cos\theta \quad$ and $\quad OM = r\sin\theta$

$\therefore \qquad QR = 2r\cos\theta$

$$PM = OP - OM = 10 - r\sin\theta$$

$\therefore \qquad \text{Area of } \Delta\, PQR = \frac{1}{2}(2r\cos\theta)(10 - r\sin\theta)$

$\therefore \qquad f(\theta) = r\cos\theta\,(10 - r\sin\theta), \qquad \left[\text{using } \frac{OQ}{OP} = \sin\theta \Rightarrow r = 10\sin\theta\right]$

$\Rightarrow \qquad f(\theta) = 100\sin\theta\cos\theta\,(1 - \sin^2\theta)$

$$f(\theta) = 100\sin\theta\cos^3\theta \qquad \ldots\text{(i)}$$

$\therefore \qquad f'(\theta) = 100(\cos^4\theta - 3\cos^2\theta\sin^2\theta)$

$$f''(\theta) = 100(-10\cos^3\theta\sin\theta + 6\sin^3\theta\cos\theta)$$

Put $\qquad f'(\theta) = 0$

$\Rightarrow \qquad \tan^2\theta = 1/3 \quad \text{or} \quad \theta = \pi/6$

$\therefore \qquad f''(\pi/6) = 100\left(\frac{-15\sqrt{3}}{8} + \frac{\sqrt{3}}{8}\right) < 0$

$\therefore$ Area is maximum when $\theta = \frac{\pi}{6}$ and hence, $r = 10\sin\frac{\pi}{6} = 5$

Example 39 If x is increasing at the rate of 2 cm/s at the instant when $x = 3$ cm and $y = 1$ cm, at what rate y must be changing in order that the quantity $(2xy - 3x^2y)$ shall be neither increasing nor decreasing.

If $S = \{(a, b) \in R \times R : x = a,\ y = b,\ 2xy - 3x^2y = \text{constant} \Rightarrow \frac{dy}{dx} > 0\}$

$S' = \{(x, y) \in A \times B : -1 \le A \le 1 \text{ and } -1 \le B \le 1\}$, then find the area $S \cap S'$.

Solution. Here, $x = 3$ cm, $y = 1$ cm and $\dfrac{dx}{dt} = 2$ cm/s

Let $f = 2xy - 3x^2 y$

$$\Rightarrow \quad \frac{df}{dt} = 2y\frac{dx}{dt} + 2x\frac{dy}{dt} - 3x^2\frac{dy}{dt} - 6xy\frac{dx}{dt} \quad \text{...(i)}$$

$\because f$ is neither increasing nor decreasing.

$$\therefore \quad \left(\frac{df}{dt}\right)_{(3,\,1)} = 0$$

$$\Rightarrow \quad (2 \times 3 - 27)\frac{dy}{dt} + (2 - 18)2 = 0 \quad \Rightarrow dy/dt = -32/21$$

Now, for $S, f =$ constant

$$\Rightarrow \quad 0 = (2x - 3x^2)\frac{dy}{dt} + (2y - 6xy)\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{6xy - 2y}{2x - 3x^2} \quad \Rightarrow \quad \frac{dy}{dx} > 0$$

$$\Rightarrow \quad \frac{6xy - 2y}{2x - 3x^2} > 0 \quad \Rightarrow \quad \frac{y(3x - 1)}{x(3x - 2)} < 0,$$

Thus, there arise two cases :

Case I $y(3x - 1) > 0$ and $x(3x - 2) < 0$

$$\Rightarrow \quad (y > 0 \text{ and } x > 1/3) \text{ or } (y < 0 \text{ and } x < 1/3) \text{ and } \left(0 < x < \frac{2}{3}\right)$$

$$\Rightarrow \quad \left(y > 0 \text{ and } \frac{1}{3} < x < \frac{2}{3}\right) \text{ or } \left(y < 0 \text{ and } 0 < x < \frac{1}{3}\right)$$

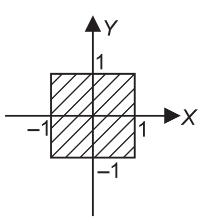
Case II $y(3x - 1) < 0$ and $x(3x - 2) > 0$

$$\Rightarrow \quad (y > 0 \text{ and } x < 1/3) \text{ or } (y < 0 \text{ and } x > 1/3) \text{ and } (x < 0 \text{ or } x > 2/3)$$

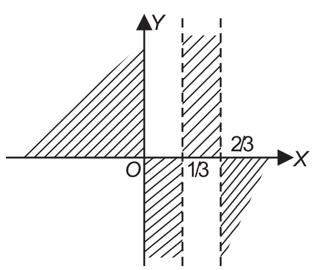
$$\Rightarrow \quad (y > 0 \text{ and } x < 0) \text{ or } (y < 0 \text{ and } x > 2/3)$$

Thus, S is shaded portion shown in figure (i).

Also, S' represents shown in figure (ii).

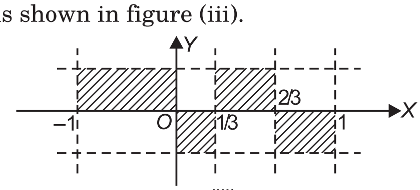


(i)



(ii)

Thus, area for $S \cap S'$ is shown in figure (iii).



(iii)

Thus, the area of $S \cap S' = 2$

Example 40 In how many parts an integer $N \geq 5$ should be dissected so that the product of the parts is maximized?

Solution. Let $x_1 + x_2 + x_3 + \ldots + x_n = N$

$\therefore$ To maximize the value of $x_1 x_2 x_3 \ldots\ldots\ldots x_n$

Using AM > GM

$$\frac{x_1 + x_2 + \ldots + x_n}{n} > (x_1 x_2 \ldots x_n)^{1/n}$$

$$\Rightarrow \qquad x_1 x_2 \ldots x_n \leq \left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)^n$$

which shows maximum value of $x_1 x_2 \ldots x_n$ is obtained when $x_1 = x_2 = x_3 = \ldots = x_n$

Now, function to be maximized is $\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)^n$ which is discrete function of n. In order to arrive at some possible neighbourhood first we make it continuous. Thus by changing the variable n to x.

We write $\qquad f(x) = \left(\frac{N}{x}\right)^x$

for maxima $\qquad f'(x) = 0$

ie, $\qquad \left(\frac{N}{x}\right)^x \left\{\log \frac{N}{x} - 1\right\} = 0 \quad \Rightarrow \quad \frac{N}{x} = e$

or $\qquad x = \frac{N}{e}$, now as $x \in$ integer

$\Rightarrow$ The nearest integer is $\left[\frac{N}{e}\right]$ or $\left[\frac{N}{e}\right] + 1$.

Type 2 : **Only One Correct Option**

Example 41 Least natural number a for which $x + ax^{-2} > 2, \forall\, x \in (0, \infty)$ is

(a) 1 (b) 2 (c) 5 (d) None of these

Solution. Here, $x + ax^{-2} > 2$

$\Rightarrow \qquad x^3 - 2x^2 + a > 0$

Let $\qquad f(x) = x^3 - 2x^2 + a$

Since, $\qquad f(x) > 0, \forall\, x \in (0, \infty), f(0) > 0$

and $\qquad \min f(x) > 0$

$\Rightarrow \qquad a > 0$ and for minimum $f(x)$

$\qquad f'(x) = 3x^2 - 4x$

$\Rightarrow \qquad x = 0, \frac{4}{3}$

$\qquad f\left(\frac{4}{3}\right) > 0 \quad \Rightarrow \quad a > \frac{32}{27}$

Hence, (b) is the correct answer.

Example 42 If $k\sin^2 x + \frac{1}{k}\operatorname{cosec}^2 x = 2$, $x \in \left(0, \frac{\pi}{2}\right)$, then $\cos^2 x + 5\sin x\cos x + 6\sin^2 x$ is equal to

(a) $\frac{k^2+5k+6}{k^2}$ (b) $\frac{k^2-5k+6}{k^2}$ (c) 6 (d) None of these

Solution. Given, $k\sin^2 x + \frac{1}{k\sin^2 x} = 2$

$$\Rightarrow \quad \left(\sqrt{k}\sin x - \frac{1}{\sqrt{k}\sin x}\right)^2 = 0$$

$$\Rightarrow \quad \sin^2 x = \frac{1}{k}$$

So,
$$\cos^2 x + 5\sin x\cos x + 6\sin^2 x$$
$$= \frac{k-1}{k} + \frac{5\sqrt{k-1}}{k} + \frac{6}{k} = \frac{k+6+5\sqrt{k-1}}{k}$$

Hence, (d) is the correct answer.

Example 43 The least value of the expression $x^2 + 4y^2 + 3z^2 - 2x - 12y - 6z + 14$ is

(a) 0 (b) 1 (c) no least value (d) None of these

Solution. Let $f(x, y, z) = x^2 + 4y^2 + 3z^2 - 2x - 12y - 6z + 14$

$$= (x-1)^2 + (2y-3)^2 + 3(z-1)^2 + 1$$

For least value of $f(x, y, z)$

$$x - 1 = 0,\ 2y - 3 = 0 \text{ and } z - 1 = 0$$

$$\therefore \quad x = 1,\quad y = \frac{3}{2},\quad z = 1$$

Hence, least value of $f(x, y, z)$ is $f\left(1, \frac{3}{2}, 1\right) = 1$

Hence, (b) is the correct answer.

Example 44 On the interval $\left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$, the least value of the function $f(x) = \int_{5\pi/4}^{x} (3\sin t + 4\cos t)\,dt$ is

(a) $\frac{3}{2} + \frac{1}{\sqrt{2}} - 2\sqrt{3}$ (b) $\frac{3}{2} - \frac{1}{\sqrt{2}} + 2\sqrt{3}$ (c) $\frac{3}{2} - \frac{1}{\sqrt{2}} - 2\sqrt{3}$ (d) None of these

Solution. $f(x) = \int_{5\pi/4}^{x} (3\sin t + 4\cos t)\,dt$, $x \in \left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$.

$$f'(x) = 3\sin x + 4\cos x$$

$f'(x) < 0$ as $\sin x, \cos x$ are negative for $x \in \left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$.

$$\Rightarrow \quad f(x)|_{\min} = f\left(\frac{4\pi}{3}\right)$$
$$= \int_{5\pi/4}^{4\pi/3} (3\sin t + 4\cos t)\,dt = \frac{3}{2} + \frac{1}{\sqrt{2}} - 2\sqrt{3}$$

Hence, (a) is the correct answer.

Example 45 For any real θ, the maximum value of $\cos^2(\cos\theta) + \sin^2(\sin\theta)$ is

(a) 1 (b) $1 + \sin^2 1$ (c) $1 + \cos^2 1$ (d) does not exist

Solution. The maximum value of $\cos^2(\cos\theta)$ is 1 and that of $\sin^2(\sin\theta)$ is $\sin^2 1$, both exists for $\theta = \frac{\pi}{2}\cdot$ Hence, maximum value is $1 + \sin^2 1$.

Hence, (b) is the correct answer.

Example 46 If $\sin\theta + \cos\theta = 1$, then the minimum value of $(1 + \operatorname{cosec}\theta)(1 + \sec\theta)$ is

(a) 3 (b) 4 (c) 6 (d) 9

Solution. AM ≥ GM

$$\frac{\sin\theta + \cos\theta}{2} \geq \sqrt{\sin\theta\cos\theta} \quad \Rightarrow \quad \sin\theta\cos\theta \leq \frac{1}{4}$$

Now, let $\sin\theta = x, \cos\theta = y$

Let $(1 + \operatorname{cosec}\theta)(1 + \sec\theta) \geq p$

$$\Rightarrow \left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) \geq p \quad \Rightarrow \left(\frac{1+x}{x}\right)\left(\frac{1+y}{y}\right) \geq p$$

$$\Rightarrow xy + x + y + 1 \geq pxy \quad \Rightarrow x + y + 1 \geq (p-1)xy$$

$$\Rightarrow 2 \geq (p-1)xy \qquad \text{(since } x + y = 1\text{)}$$

$$\Rightarrow xy \leq \frac{2}{p-1}$$

But, $\dfrac{2}{p-1} = \dfrac{1}{4}$, $p - 1 = 8 \quad \Rightarrow \quad p = 9$

Hence, (d) is the correct answer.

Example 47 If composite function $f_1(f_2(f_3(....(f_n(x)))))$ n times, is an increasing function and if r of f_i's are decreasing function while rest are increasing, then maxium value of $r(n-r)$ is

(a) $\dfrac{n^2-1}{4}$ when n is an even number (b) $\dfrac{n^2}{4}$ when n is an odd number

(c) $\dfrac{n^2-1}{4}$ when n is an odd number (d) None of these

Solution. r must be an even integer because two decreasing are required to make it increasing function.

Let $y = r(n-r)$,

when n is odd

$$r = \frac{n-1}{2} \text{ or } \frac{n+1}{2} \text{ for maximum value of } y$$

when n is even

$$r = \frac{n}{2} \text{ for maximum value of } y$$

$\therefore$ Maximum $(y) = \dfrac{n^2-1}{4}$ when n is odd and $\dfrac{n^2}{4}$ when n is even.

Hence, (c) is the correct answer.

Example 48 The coordinates of the point on the curve $x^3 = y(x-a)^2$, $a > 0$, where the ordinate is minimum

(a) $(2a, 8a)$ (b) $\left(-2a, \frac{-8a}{9}\right)$

(c) $\left(3a, \frac{27a}{4}\right)$ (d) $\left(-3a, \frac{-27a}{16}\right)$

Solution. The ordinates of any point on the curve is given by $y = \dfrac{x^3}{(x-a)^2}$

$$\frac{dy}{dx} = \frac{x^2(x-3a)}{(x-a)^3}$$

Now, $$\frac{dy}{dx} = 0 \quad \Rightarrow \quad x = 0 \text{ or } x = 3a$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=0} = 0 \text{ and } \left[\frac{d^2y}{dx^2}\right]_{x=3a} = \frac{72a^5}{(2a)^6} > 0$$

Hence, y is minimum at $x = 3a$ and is equal to $\dfrac{27a}{4}$.

Hence, (c) is the correct answer.

Example 49 If $a, b \in R$ distinct numbers satisfying $|a-1| + |b-1| = |a| + |b| = |a+1| + |b+1|$, then the minimum value of $|a-b|$ is

(a) 3 (b) 0

(c) 1 (d) 2

Solution. Let $a < b$ and $f(x) = |x-a| + |x-b|$, $\forall\, x \in R$

So, $f(x)$ is decreasing in $(-\infty, a]$ constant in $[a, b]$ and increasing in $[b, \infty)$,we have

$$f(0) = f(1) = f(-1)$$

$\Rightarrow$ $$\{-1, 0, 1\} \in [a, b]$$

$\therefore$ $$|a-b|_{\min} = 2$$

Hence, (d) is the correct answer.

Example 50 If $x^2 + y^2 + z^2 = 1$ for $x, y, z \in R$, then the maximum value of $x^3 + y^3 + z^3 - 3xyz$ is

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) 3

Solution. Let $t = xy + yz + zx$, so $-\dfrac{1}{2} \le t \le 1$

$\therefore$
$$\begin{aligned} x^3 + y^3 + z^3 - 3xyz &= (x+y+z)(x^2+y^2+z^2-xy-yz-zx) \\ &= \sqrt{(1+2t)}\,(1-t) \end{aligned}$$

$\therefore$ Let $$f(t) = (1+2t)(1-t)^2$$

Clearly, $$t_{\max} = f(0) = 1$$

Hence, (b) is the correct answer.

Example 51 Let $f(x) = x^4 + ax^3 + 3x^2 + bx + 1$, $a, b \in R$. If $f(x) \geq 0, \forall\, x \in R$, then the maximum value of $a^2 + b^2$ is equal to

(a) 10 (b) 12 (c) 16 (d) None of these

Solution. $f(x) \leq 0$

$$\Rightarrow \qquad \left(x^2 + \frac{a}{2}x\right)^2 + \left(3 - \frac{a^2 + b^2}{4}\right)x^2 + \left(\frac{b}{2}x + 1\right)^2 \geq 0$$

which holds only when $3 - \dfrac{a^2 + b^2}{4} \geq 0$

$$\Rightarrow \qquad a^2 + b^2 \leq 12$$

Hence, (b) is the correct answer.

Type 3 : More than One Correct Options

Example 52 Let $f(x) = \sin x + ax + b$. Then, $f(x) = 0$ has

(a) only one real root which is positive, if $a > 1$, $b < 0$

(b) only one real root which is negative, if $a > 1$, $b > 0$

(c) only one real root which is negative, if $a < -1$, $b < 0$

(d) None of the above

Solution. $f'(x) = -\cos x + a$, if $a > 1$, then $f(x)$ is entirely increasing. So, $f(x) = 0$ has only one real root, which is positive, if $f(x) < 0$ and negative. If $f(0) > 0$ similarly, when $a < -1$. Then $f(x)$ entirely decreasing. Therefore, $f(x)$ has only one real root which is positive, if $f(0) < 0$ and positive, if $f(0) > 0$.
Hence, (a), (b) and (c) are the correct answers.

Example 53 If $a > 0, b > 0, c > 0$ and $a + b + c = abc$, then at least one of the number a, b, c exceeds

(a) $\dfrac{3}{2}$ (b) $\dfrac{17}{10}$

(c) 2 (d) $\dfrac{13}{10}$

Solution. We may suppose $a \geq b \geq c$

$$abc = a + b + c \geq 3c$$

So, $\qquad ab \geq 3$, $a \geq b$ and $a \geq \sqrt{3} > \dfrac{17}{10}$

Aliter $\qquad (a + b + c)^3 \geq 27\, abc \geq 27\,(a + b + c)$

$\therefore \qquad a + b + c \geq 3\sqrt{3}$

$\Rightarrow$ At least one of then $\geq \sqrt{3}$

Hence, (a), (b) and (d) are the correct answers.

Example 54 Let $f(x, y) = x^2 + 2xy + 3y^2 - 6x - 2y$, where $x, y \in R$, then

(a) $f(x, y) \geq -11$ (b) $f(x, y) \geq -10$

(c) $f(x, y) > -11$ (d) $f(x, y) > -12$

Solution. Let $z = x^2 + 2xy + 3y^2 - 6x - 2y$

$\Rightarrow \quad x^2 + 2xy + 3y^2 - 6x - 2y - z = 0$ as $x \in R$

$\therefore \quad D \ge 0$

$\Rightarrow \quad 4(y-3)^2 - 4(3y^2 - 2y - z) \ge 0$

$\Rightarrow \quad y^2 + 9 - 6y - 3y^2 + 2y + z \ge 0 \quad \Rightarrow \quad -2y^2 - 4y + 9 + z \ge 0$

$\Rightarrow \quad z \ge 2(y^2 + 2y + 1) - 11 = 2(y+1)^2 - 11 \quad \Rightarrow \quad z \ge -11$

Hence, (a), (c) and (d) are the correct answers.

Example 55 Let $g(x) = f(\tan x) + f(\cot x), \forall\, x \in \left(\frac{\pi}{2}, \pi\right)$. If $f''(x) < 0, \forall\, x \in \left(\frac{\pi}{2}, \pi\right)$, then

(a) $g(x)$ is increasing in $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$

(b) $g(x)$ has local minimum at $x = \frac{3\pi}{4}$

(c) $g(x)$ is decreasing in $\left(\frac{3\pi}{4}, \pi\right)$

(d) $g(x)$ has local maximum at $x = \frac{3\pi}{4}$

Solution. $g'(x) = t'(\tan x)\sec x - f'(\cot x)\operatorname{cosec}^2 x$

For increasing $g(x) > 0$

$f'(\tan x) > f'(\cot x)$

$[\because f''(x) < 0$ and $\tan x < \cot x, \forall\, x \left(\frac{\pi}{4}, \frac{\pi}{2}\right)]$

Also, $\quad \sec^2 x > \operatorname{cosec}^2 x - x = \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$

$g'(x) > 0 \Rightarrow g(x)$ is increasing in $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$

Similarly, $\quad g(x)$ is decreasing $\left(\frac{3\pi}{4}, \pi\right)$

Also, $g(x)$ has local maximum at $x = \frac{3\pi}{4}$.

Hence, (a), (c) and (d) are the correct answers.

Example 56 The function $f(x) = \int_0^x \sqrt{1 - t^4}\, dt$ is such that

(a) it is defined on the interval $[-1, 1]$

(b) it is an increasing function

(c) it is an odd function

(d) the point (0, 0) is the point of inflection

Solution. $f'(x) = \sqrt{1 - x^4} > 0$ in $(-1, 1) \Rightarrow f$ is increasing

Now, $\quad f(x) + f(-x) = \int_0^x \sqrt{1 - t^4}\, dt + \int_0^{-x} \sqrt{1 - t^4}\, dt$

$\Rightarrow \quad \int_0^x \sqrt{1 - t^4}\, dt + \left(-\int_0^y \sqrt{1 - y^4}\, dy\right) (t = -y)$

$= 0$

$\Rightarrow \quad f(x)$ is odd

Again, $\quad f''(x) = \dfrac{-4x^3}{2\sqrt{1 - x^4}}$ which vanished at $x = 0$

and changes sign $\Rightarrow (0, 0)$ is inflection since f is well defined in $[-1, 1]$.

Hence, (a), (b), (c) and (d) are the correct answers.

Example 57 The function $\dfrac{\sin(x+a)}{\sin(x+b)}$ has no maxima or minima, if

(a) $b - a = n\pi,\ n \in I$
(b) $b - a = (2n+1)\pi,\ n \in I$
(c) $b - a = 2n\pi,\ n \in I$
(d) None of these

Solution. $f(x) = \dfrac{\sin(x+a)}{\sin(x+b)}$

$$f(x) = \frac{\sin(x+b) \times \cos(x+a) - \sin(x+a)\cos(x+b)}{\sin^2(x+b)} = \frac{\sin(b-a)}{\sin^2(x+b)}$$

If $\quad \sin(b-a) = a$, then $f'(x) = 0 \Rightarrow f(x)$ will be constant.

ie, $\quad b = a = n\pi$ or $b - a = (2n+1)\pi$ or $b - a = 2n\pi$, then $f(x)$ has no minima.

Hence, (a), (b) and (c) are the correct answers.

Example 58 Let $F(x) = 1 + f(x) + (f(x))^2 + (f(x))^3$, where $f(x)$ is an increasing differentiable function and $F(x) = 0$ has a positive root, then

(a) $F(x)$ is an increasing function
(b) $F(0) \le 0$
(c) $f(0) \le -1$
(d) $F'(0) > 0$

Solution. $F'(x) = (1 + 2f(x) + 3(f(x))^2) f'(x) > 0$, so $F(x)$ is increasing.

So, $\quad F(0) < 0$

$\Rightarrow \quad (1 + f(0))(1 + f(0))^2) < 0 \Rightarrow f(0) < -1$

Hence, (a), (b), (c) and (d) are the correct answers.

Example 59 The extremum values of the function $f(x) = \dfrac{1}{\sin x + 4} - \dfrac{1}{\cos x - 4}$, where $x \in R$ is

(a) $\dfrac{4}{8-\sqrt{2}}$
(b) $\dfrac{2\sqrt{2}}{8-\sqrt{2}}$
(c) $\dfrac{2\sqrt{2}}{4\sqrt{2}+1}$
(d) $\dfrac{4\sqrt{2}}{8+\sqrt{2}}$

Solution. $f'(x) = 0$

$\Rightarrow \quad (\sin x + \cos x)$ (non-zero quantity) $= 0 \Rightarrow \tan x = -1$

$\Rightarrow \quad x = \dfrac{3\pi}{4}$ or $\dfrac{7\pi}{4}$

Global Minimum $= x = 2n\pi + \left(\dfrac{3\pi}{4}\right)$

Global Maximum $= x = 2n\pi + \left(\dfrac{7\pi}{4}\right)$

$$M = \frac{4}{8-\sqrt{2}},\ m = \frac{4}{8+\sqrt{2}}$$

Hence, (a) and (c) are the correct answers.

Example 60 The function $f(x) = x^{1/3}(x-1)$

(a) has 2 inflection oints

(b) is strictly increasing for $x > \dfrac{1}{4}$ and strictly decreasing for $x < \dfrac{1}{4}$

(c) is concavpe down in $\left(-\dfrac{1}{2}, 0\right)$

(d) area enclosed by the curve lying in the fourth quadrant is $\dfrac{9}{28}$

Solution. $y = x^{1/3}(x-1)$

$$\frac{dy}{dx} = \frac{4}{3}x^{1/3} - \frac{1}{3}\cdot\frac{1}{x^{2/3}} = \frac{1}{3x^{2/3}}[4x-1]$$

$x^{2/3}$ is always positive and $x = \frac{1}{4}$ the curves has a local minima.

Hence, f is increasing for $x > \frac{1}{4}$

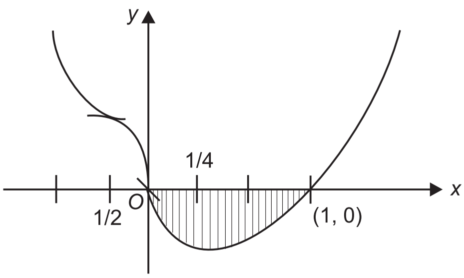
and f is decreasing for $x < \frac{1}{4}$

Now, $f'(x) = \frac{4}{3}x^{1/3} - \frac{1}{3}\cdot x^{-2/3}$ (non-existent at $x = 0$, vertical tangent)

$$f''(x) = \frac{4}{9}\cdot\frac{1}{x^{2/3}} + \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{x^{5/3}} = \frac{2}{9x^{2/3}}\left[2 + \frac{1}{x}\right] = \frac{2}{9x^{2/3}}\left[\frac{2x+1}{x}\right]$$

$\therefore$ $f''(x) = 0$ at $x = -\frac{1}{2}$ (inflection point)

Graph of $f(x)$ is as



$$A = \int_0^1 (x^{4/3} - x^{1/3})\,dx = \left[\frac{3}{7}x^{3/7} - \frac{3}{4}x^{4/3}\right]_0^1$$

$$= \left|\frac{3}{7} - \frac{3}{4}\right| = 3\left|\frac{4-7}{28}\right| = \frac{9}{28}$$

Hence, (a), (b), (c) and (d) are the correct answers.

Example 61 Assume that inverse of the function f is denoted by g, then which of the following statement hold good?

(a) If f is increasing, then g is also increasing
(b) If f is decreasing, then g is increasing
(c) The function f is injective
(d) The function g is onto

Solution. If f and g are inverse, then $(fog)(x) = x$

$$f'[g(x)]g'(x) = 1$$

If f is increasing $\Rightarrow f' > 0 \Rightarrow$ Sign of g' is also positive.

Therefore, option (a) is correct.

If f is decreasing $\Rightarrow f' < 0 \Rightarrow$ Sign of g' is negative.

Therefore, option (b) is false.

Since, f has an inverse.

$\Rightarrow$ f is bijective $\Rightarrow$ f is injective

Therefore, option (c) is correct.

Inverse of a bijectibe mapping is bijective.

$\Rightarrow \qquad g$ is also bijective $\Rightarrow g$ is onto

Therefore, option (d) is correct.

Hence, (a), (c) and (d) are the correct answers.

Type 4 : **Assertion and Reason**

Directions

(Q. Nos. 62 to 68)

For the following questions, choose the correct answers from the codes (a), (b), (c) and (d) defined as follows :

(a) Statement I is true, Statement II is also true, Statement II is the correct explanation of Statement I.

(b) Statement I is true, Statement II is also true, Statement II is not the correct explanation of Statement I.

(c) Statement I is true, Statement II is false.

(d) Statement I is false, Statement II is true.

Example 62 Statement I Among all the rectangles of the given perimeter, the square has the largest area. Also, among all the rectangles of given area, the square has the least perimeter.

Statement II For $x > 0$, $y > 0$, if $x + y =$ constant, then xy will be maximum for $y = x$ and if $xy =$ constant, then $x + y$ will be minimum for $y = x$.

Solution. Statement II $x + y = k$

$$xy = x(k - x) = f(x), f'(x) = k - 2x = 0$$
$$x = \frac{k}{2}, y = \frac{k}{2} \quad \Rightarrow \quad y = x$$
$$x + y = x + \frac{k}{x} = f(x), f'(x) = 1 - \frac{k}{x^2}$$
$$x = \sqrt{k}, y = \sqrt{k}$$

So, Statement II is true and it explains Statement I.

Hence, (a) is the correct answer.

Example 63 Statement I The function $f(x) = (x^3 + 3x - 4)(x^2 + 4x - 5)$ has local extremum at $x = 1$.

Statement II $f(x)$ is continuous and differentiable and $f'(1) = 0$

Solution. Statement I is correct because $f(1)^- > f(1) < f(1)^+$

Statement II is correct as $f(x)$ has a repeated root at $x = 1$.

Statement II is not the correct explanation of Statement I as $f'(c) = 0$ doesn't imply that f has an extrema at $x = c$.

Hence, (b) is the correct answer.

Example 64 Statement I If $f(x)$ is increasing function with upward concavity, then concavity of $f^{-1}(x)$ is also upwards.
Statement II If $f(x)$ is decreasing function with upwards concavity, then concavity of $f^{-1}(x)$ is also upwards.
Solution. Let $g(x)$ be the inverse function of $f(x)$. Then, $f(g(x)) = x$

$$\therefore \qquad f'(g(x)) \cdot g'(x) = 1$$

$$ie, \qquad g'(x) = \frac{1}{f'(g(x))}$$

$$\therefore \qquad g'' = -\frac{1}{(f'(g(x)))^2} \cdot f''(g(x)) \cdot g'(x)$$

In Statement I $f''(g(x)) > 0$ and $g'(x) > 0 \Rightarrow g''(x) < 0$
$\Rightarrow$ Concavity of $f^{-1}(x)$ is downwards.
$\therefore$ Statement I is false.
In Statement II $f''(g(x)) > 0$ and $g'(x) < 0 \Rightarrow g''(x) > 0$
$\Rightarrow$ Concavity of $f^{-1}(x)$ is upwards.
$\therefore$ Statement II is true.
Hence, (d) is the correct answer.

Example 65 Statement I The minimum distance of the fixed point $(0, y_0)$, where $0 \le y_0 \le \frac{1}{2}$, from the curve $y = x^2$ is y_0.
Statement II Maxima and minima of a function is always a root of the equation $f'(x) = 0$.
Solution. Let the point on the parabola be (t, t^2)

Let d be the distance between (t, t^2) and $(0, y_0)$,

$$\text{then} \qquad d^2 = t^2 + (t^2 - y_0)^2 = t^4 + (1 - 2y_0)\,t^2 + y_0^2$$
$$= z^2 + (1 - 2y_0)\,z + y_0^2,\ z \ge 0$$

Its vertex is at $x = y_0 - \frac{1}{2} < 0$

$\therefore$ The minimum value of d^2 is at $z = 0$ *ie*, $t^2 = 0$

$$\therefore \qquad d = y_0$$

$\therefore$ Statement I is true. Statement II is false because extremum can occur at a point where $f'(x)$ does not exist.
Hence, (c) is the correct answer.

Example 66 Let $f : R \to R$ is differentiable and strictly increasing function throughout its domain.

Statement I If $|f(x)|$ is also strictly increasing function, then $f(x) = 0$ has no real roots.

Statement II At ∞ of $-\infty$, $f(x)$ may approach to 0, but cannot be equal to zero.

Solution. Suppose $f(x) = 0$ has a real root say $x = a$, then $f(x) < 0$ for $x < a$. Thus, $|f(x)|$ becomes strictly decreasing on $(-\infty, a)$, which is a contradiction.
Hence, (a) is the correct answer.

Example 67 Statement I $f(x) = x + \cos x$ is strictly increasing.

Statement II If $f(x)$ is strictly increasing, then $f'(x)$ may tend to zero at some finite number of points.

Solution. $f(x) = x + \cos x$

$\therefore$ $f'(x) = 1 - \sin x > 0, \ \forall x \in R,$

except at $x = 2n\pi + \dfrac{\pi}{2}$ and $f'(x) = 0$ at $x = 2n\pi + \dfrac{\pi}{2}$

$\therefore f(x)$ is strictly increasing.

Statement II is true but does not explain Statement I.

$\therefore$ Statement II gives $f'(x)$ may tend to zero. at finite number of points but in Statement I $f'(x)$ tend to zero at infinite number of points.

Hence, (b) is the correct answer.

Example 68 Statement I The largest term in the sequence

$a_n = \dfrac{n^2}{n^3 + 200}, n \in N$ is $\dfrac{(400)^{2/3}}{600}$

Statement II $f(x) = \dfrac{x^2}{x^3 + 200}$, $x > 0$, then at $x = (400)^{1/3}$, $f(x)$ is maximum.

Solution. Statement II $f(x) = \dfrac{x^2}{x^3 + 200}$

$$f'(x) = \frac{(x^3 + 200)\,2x - 3x^2 x^2}{(x^3 + 200)^2} = \frac{-x^4 + 400x}{(x^3 + 200)^2}$$

$$x \to 0^+ \ f(x) = 0^+$$

$$x = 400^{1/3} \ f(x) = \frac{400^{2/3}}{600}$$

$$x \to \infty \ f(x) \to 0$$

So, Statement II is true.

But Statement I is false as $x \in N$.

Hence, (d) is the correct answer.

Type 5 : **Linked Comprehension Based Questions**

Passage I

(Q. Nos. 69 to 71)

We are given the curves $y = \int_{-\infty}^{x} f(t)\,dt$ *through the point* $\left(0, \dfrac{1}{2}\right)$ *and* $y = f(x)$*, where* $f(x) > 0$ *and* $f(x)$ *is differentiable,* $\forall x \in R$ *through* $(0, 1)$*. If tangents drawn to both the curves at the points with equal abscissae intersect on the same point on the x-axis, then*

69. Number of solutions $f(x) = 2ex$ is equal to

(a) 0 (b) 1 (c) 2 (d) None of these

70. $\lim_{x \to \infty} (f(x))^{f(-x)}$ is

(a) 3 (b) 6 (c) 1 (d) None of these

71. The function $f(x)$ is

(a) increasing for all x (b) non-monotonic
(c) decreasing for all x (d) None of these

Solution. (Q. Nos. 69 to 71)

We have, the equations of the tangents to the curves

$y = \int_{-\infty}^{x} f(t)\,dt$ and $y = f(x)$ at arbitrary points on them are

$$Y - \int_{-\infty}^{x} f(t)\,dt = f(x)(X - x) \quad \text{...(i)}$$

and $$Y - f(x) = f'(x)(X - X) \quad \text{...(ii)}$$

As Eqs. (i) and (ii), intersect at the same point on x-axis.

On putting $Y = 0$ and equating x-coordinates, we have

$$x - \frac{f(x)}{f'(x)} = x - \frac{\int_{-\infty}^{x} f(t)\,dt}{f(x)} \Rightarrow \frac{f(x)}{\int_{-\infty}^{x} f(t)\,dt} = \frac{f'(x)}{f(x)}$$

$$\Rightarrow \int_{-\infty}^{x} f(t) = cf(x) \quad \text{...(iii)}$$

As, $$f(0) = 1 \Rightarrow \int_{-\infty}^{0} f(t)\,dt = \frac{1}{2} = c \times 1 \Rightarrow c = \frac{1}{2}$$

$\Rightarrow \int_{-\infty}^{x} f(t)\,dt = \frac{1}{2} f(x)$, differentiating both the sides and on integrating and using boundary condition, we get $f(x) = e^{2x}$, $y = 2ex$ is tangent to $y = e^{2x}$

$\Rightarrow$ Number of solutions $= 1$

Clearly, $f(x)$ is increasing for all x.

$$\lim_{x \to \infty} (e^{2x})^{e^{-2x}} = 1 \quad (\infty^0 \text{ form})$$

Ans. 69. (b) **70.** (c) **71.** (a)

Passage II

(Q. Nos. 72 to 74)

Let x_1, x_2, x_3, x_4 be the roots (real or complex, of the equation $x^4 + ax^3 + bx^2 + cx + d = 0$. If $x_1 + x_2 = x_3 + x_4$ and $a, b, c, d \in R$, then

72. If $a = 2$, then the value of $b - c$ is

(a) -1 (b) 1 (c) -2 (d) 2

73. If $b < 0$, then how many different values of a, we may have

(a) 3 (b) 2
(c) 1 (d) 0

74. If $b + c = 1$ and $a \neq -2$, then for real values of a, $c \in$

(a) $\left(-\infty, \frac{1}{4}\right)$ (b) $(-\infty, 3)$

(c) $(-\infty, 1)$ (d) $(-\infty, 4)$

Solution. (Q. Nos. 72 to 74)

Let $x^4 + ax^3 + bx^2 + cx + d$

$= (x - x_1)(x - x_2)(x - x_3)(x - x_4)$

Let $(x - x_1)(x - x_2) = x^2 + px + q$

and $(x - x_3)(x - x_4) = x^2 + px + r$

$\therefore$ $q = x_1x_2$ and $r = x_3x_4$

$\therefore$ $x^4 + ax^3 + bx^2 + cx + d$

$= x^4 + 2px^3 + (p^2 + q + r)x^2 + p(q + r)x + qr$

$\therefore$ $a = 2p,\ b = p^2 + q + r,\ c = p(q + r),\ d = qr$

Clearly, $a^3 - 4ab + 8c = 0$ ...(i)

72. If $a = 2 \Rightarrow b - c = 1$

Hence, (b) is the correct answer.

73. Investigating the nature of the cubic equation of a.

Let $f(a) = a^3 - 4ab + 8c$

$f'(a) = 3a^2 - 4b$

if $b < 0 \quad \Rightarrow \quad f'(a) > 0$

$\therefore$ The equation $a^3 - 4ab + 8c = 0$ hence, has only one real root.

Hence, (c) is the correct answer.

74. Substituting $c = 1 - b$ in Eq. (1), we have

$(a + 2)[(a - 1)^2 + 3 - 4b] = 0$

$\Rightarrow$ $4b - 3 > 0$

$\Rightarrow$ $b > \frac{3}{4} \quad \Rightarrow \quad c < \frac{1}{4}\cdot$

Hence, (c) is the correct answer.

Passage III

(Q. Nos. 75 to 77)

Consider a ΔOAB formed by the point $O(0, 0)$, $A(2, 0)$, $B(1, \sqrt{3})$. $P(x, y)$ is an arbitrary interior point of triangle moving in such a way that $d(P, OA) + d(P, AB) + d(P, OB) = \sqrt{3}$,
where $d(P, OA)$, $d(P, AB)$, $d(P, OB)$ represent the distance of P from the sides OA, AB and OB respectively.

75. Area of region representing all possible position of point P is equal to

(a) $2\sqrt{3}$ (b) $\sqrt{6}$

(c) $\sqrt{3}$ (d) None of these

Solution. ΔOAB is clearly equilateral

$$\Delta OAB = \Delta OPA + \Delta APB + \Delta OPB = \frac{\sqrt{3}}{4} \times 4 = \frac{1}{2} \cdot 2$$

$$(d(P, OA) + (P, AB) + (P, OB))$$

$$d(P, OA) + d(P, AB) + d(P, OB) = \frac{4}{\sqrt{3}}$$

Hence, (c) is the correct answer.

76. If the point P moves in such a way that $d(P, OA) \le \min(d(P, OB), d(P, AB))$, then area of region representing all possible position of point P is equal to

(a) $\sqrt{3}$ (b) $\sqrt{6}$

(c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{6}}$

Solution. We must have, $d(P, OA) \le d(P, OB)$ as well as $d(P, OA) \le d(P, AB)$, then P lies either on or below the angle bisector of $\angle BOA$ and $\angle BAO$ area

$$= \frac{1}{3} \Delta OAB = \frac{1}{3} \cdot$$

Hence, (c) is the correct asnwer.

77. If the point P moves in such a way that $d(P, OA) \le \min(d(P, OB), d(P, AB))$, then area of region representing all possible position of point P is equal to

(a) $\sqrt{3}$ (b) $\sqrt{6}$

(c) $1/\sqrt{3}$ (d) $\frac{1}{\sqrt{6}}$

Solution. We must have $d(P, QA) \ge d(P, OB)$ as well as $d(P, OA) \ge d(P, AB)$, then P must be above bisector of $\angle BOA$ and $\angle BAO$.

Area of triangle $= \frac{1}{3} \Delta OAB = \frac{1}{\sqrt{3}}$

Hence, (c) is the correct answer.

Passage IV

(Q. Nos. 78 to 80)

Let $f(x) = ax^2 + bx + c$, $a, b, c \in R$. It is given $|f(x)| \le 1$, $|x| \le 1$

78. The possible value of $|a + c|$, if $\frac{8}{3}a^2 + 2b^2$ is maximum, is given by

(a) 1 (b) 0 (c) 2 (d) 3

79. The possible value of $|a + b|$, if $\frac{8}{3}a^2 + 2b^2$ is maximum, is given by

(a) 1 (b) 0 (c) 2 (d) 3

80. The possible maximum value of $\frac{8}{3}a^2 + 2b^2$ is given by

(a) 32 (b) $\frac{32}{3}$ (c) $\frac{2}{3}$ (d) $\frac{16}{3}$

Solution. (Q. Nos. 78 to 80)

We know that for $|u| \le 1$; $|v| \le 1$, then $|u - v| \le 2$

Now, $|f(1) - f(0)| \le 2 \quad \Rightarrow \quad |a + b| \le 2$

$\Rightarrow \quad (a + b)^2 \le 4$ ...(i)

Also, $|f(-1) - f(0)| \le 2 \quad \Rightarrow \quad |a - b| \le 2$

$\Rightarrow \quad (a - b)^2 \le 4$ ...(ii)

Now, $4a^2 + 3b^2 = 2(a + b)^2 + 2(a - b)^2 - b^2 \le 16$

$\Rightarrow \quad (4a^2 + 3b^2)_{max} = 16$ when $b = 0$

$\Rightarrow \quad |a + b| = |a - b| = |a| = 2$

Also, $|f(1) - f(0)| = |(a + c) - c| = |a| = 2 \quad \Rightarrow \quad |a + c| = |c| = 1$

The possible ordered triplet (a, b, c) are $(2, 0, -1)$ or $(-2, 0, 1)$.

Also, $\dfrac{8}{3}a^2 + 2b^2 = \dfrac{2}{3}(4a^2 + 3b^2) \le \dfrac{2}{3} + 16$

Ans. 78. (a) **79.** (b) **80.** (b)

Passage V

(Q. Nos. 81 to 85)

The absolute maximum and minimum values of functions can be found by their monotonic and asymptotic behaviour provided they exist. We may agree that finite limiting values may be regarded as absolute maximum or minimum. For instance the absolute maximum value of $\dfrac{1}{1 + x^2}$ *is unity. It is attained at* $x = 0$ *while absolute minimum value of the same function is zero which is a limiting value of* $(x \to \infty$ *or* $x \to -\infty)$

81. The function $x^4 - 4x + 1$ will have

(a) absolute maximum value
(b) absolute minimum value
(c) both absolute maximum and minimum values
(d) None of the above

Solution. Since, $f'(x) = 4x^3 - 4 = 0 \quad \Rightarrow \quad x = 1$

$\Rightarrow$ There is only one extrema which is minima.

$\Rightarrow$ 1 is a point of absolute minima.

Hence, (b) is the correct answer.

82. The absolute minimum value of the function $\dfrac{x - 2}{\sqrt{x^2 + 1}}$

(a) -1 (b) $\dfrac{1}{2}$ (c) $-\sqrt{5}$ (d) None of these

Solution. Since, $f'(x) = \dfrac{1 + 2x}{(x^2 + 1)^{3/2}}$

$\Rightarrow f(x)$ is increasing in $\left(-\dfrac{1}{2}, \infty\right)$ and decreasing in $\left(-\infty, -\dfrac{1}{2}\right)$

$\Rightarrow$ Absolute minimum occurs at $x = -\dfrac{1}{2}$, we have $= -\sqrt{5}$

Hence, (c) is the correct answer.

83. The absolute minimum and maximum values of the function

$$\frac{x^2 - x + 1}{x^2 + x + 1} \text{ is}$$

(a) 1 and 3 (b) $\frac{1}{2}$ and 3

(c) $\frac{1}{3}$ and 3 (d) None of these

Solution. $yx^2 + yx + y = x^2 - x + 1$

$$x^2(1-y) - x(1+y) + (1-y) = 0 \quad \because D \geq 0$$

$$\Rightarrow \quad (1+y)^2 - 4(1-y)^2 \geq 0 \quad \Rightarrow \quad -(3y-1)(y-3) \geq 0$$

$$\Rightarrow \quad \frac{1}{3} \leq y \leq 3$$

Hence, (c) is the correct answer.

84. The function $f(x) = \dfrac{4}{x-1} - \dfrac{9}{x+1}$ will

(a) have absolute maximum value $-\frac{1}{2}$

(b) have absolute minimum value $-\frac{25}{2}$

(c) not lie between $-\frac{25}{2}$ and $-\frac{1}{2}$

(d) always be negative

Solution. Should be the correct choice (we can prove by using monotonically that f cannot lie between $-\frac{25}{2}$ and $-\frac{1}{2}\cdot$ This is an example of a function whose maximum (local) value is smaller than minimum value).

Hence, (c) is the correct asnwer.

85. Which of the following functions will have absolute minimum value ?

(a) $\cot(\sin x)$ (b) $\tan(\log x)$

(c) $x^{2005} - x^{1947} + 1$ (d) $x^{2006} + x^{1947} + 1$

Solution. As even degree polynomial will have absolute minimum essentially. Hence, (d) is the correct answer.

Passage VI

(Q. Nos. 86 to 88)

Let $f(x) = \left(1 + \dfrac{1}{x}\right)^x \ (x > 0)$ *and* $g(x) = \begin{cases} x \ln(1 + (1/x)), & \text{if } 0 < x \leq 1 \\ 0, & \text{if } x = 0 \end{cases}$

86. $\lim\limits_{x \to 0^+} g(x)$

(a) is equal to 0 (b) is equal to 1

(c) is equal to e (d) is non-existent

Solution. $\lim_{x \to 0^+} x \ln\left(1 + \frac{1}{x}\right) = \lim_{x \to 0^+} \frac{\ln\left(\frac{x+1}{x}\right)}{\frac{1}{x}} \left(\frac{\infty}{\infty}\right)$

Using L'Hospital's Rule

$$l = \lim_{x \to 0} -\left(\frac{1}{x+1} - \frac{1}{x}\right)x^2 = \lim_{x \to 0}\left(\frac{1}{x} - \frac{1}{x+1}\right)\cdot x^2$$

$$= \lim_{x \to 0} \frac{1}{x(x+1)}\cdot x^2$$

$$= \lim_{x \to 0} \frac{x}{(x+1)} = 0$$

Hence, (a) is the correct answer.

87. The function f

(a) has a maxima but not minima

(b) has a minima but not maxima

(c) has both of maxima and minima

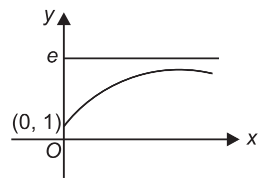
(d) is a monotonic

Solution. $\lim_{x \to 0} f(x) = 1$ (can be verified)

$\lim_{x \to \infty} f(x) = e$

Also, f is increasing for all $x > 0$

$\Rightarrow$ (d) (can be verified)



Hence, (d) is the correct answer.

88. $\lim_{n \to \infty} \left\{f\left(\frac{1}{n}\right)\cdot f\left(\frac{2}{n}\right)\cdot f\left(\frac{3}{n}\right)\ldots . f\left(\frac{n}{n}\right)\right\}^{1/n}$ equals to

(a) $\sqrt{2}\,e$ (b) $\sqrt{2e}$ (c) $2\sqrt{e}$ (d) $\sqrt{e}$

Solution. $l = \left(\prod_{k=1}^{n}\left(1 + \frac{n}{k}\right)^{k/n}\right)^{1/n}$ [given $f(x) = (1 + 1/x)^x$ and $f(k/n) = \left(1 + \frac{n}{k}\right)^{k/n}$]

Taking log,

$$\ln l = \lim_{n \to \infty} \frac{1}{n}\cdot \sum_{k=1}^{n} \ln\left(1 + \frac{n}{k}\right)^{k/n} = \lim_{n \to \infty} \frac{1}{n}\cdot \sum_{k=1}^{n} \frac{k}{n} \ln\left(1 + \frac{1}{k/n}\right) dx$$

$$= \int_0^1 \underbrace{x}_{\text{II}} \underbrace{\ln\left(1 + \frac{1}{x}\right)}_{\text{I}} dx = \ln\left(1 + \frac{1}{x}\right)\cdot \frac{x^2}{2}\Bigg]_0^1 + \int_0^1 \left(\frac{1}{x} - \frac{1}{x+1}\right)\cdot \frac{x^2}{2}\, dx$$

$$= \left(\frac{1}{2}\ln 2 - 0\right) + \frac{1}{2}\int_0^1 \frac{x+1-1}{x+1}\, dx = \frac{1}{2}\ln 2 + \frac{1}{2}[x - \ln(x+1)]_0^1$$

$$= \frac{1}{2}\ln 2 + \frac{1}{2}[(1 - \ln 2) - 0] = \frac{1}{2}$$

$$l = \sqrt{e}$$

Hence, (d) is the correct answer.

Passage VII

(Q. Nos. 89 to 91)

Consider the cubic $f(x) = 8x^3 + 4ax^2 + 2bx + a$ where $a, b \in R$.

89. For $a = 1$, if $y = f(x)$ is strictly increasing, $\forall\, x \in R$, then maximum range of the values of b is

(a) $\left(-\infty, \frac{1}{3}\right]$ (b) $\left(\frac{1}{3}, \infty\right)$ (c) $\left[\frac{1}{3}, \infty\right)$ (d) $(-\infty, \infty)$

Solution. $a = 1$

$$f(x) = 8x^3 + 4x^2 + 2bx + 1$$
$$f'(x) = 24x^2 + 8x + 2b = 2(12x^2 + 4x + b)$$

For increasing function $f'(x) \geq 0, \forall\, x \in R$

$\therefore$ $$D \leq 0 \quad \Rightarrow \quad 16 - 48b \leq 0 \quad \Rightarrow \quad b \geq \frac{1}{3}$$

Hence, (c) is the correct answer.

90. For $b = 1$, if $y = f(x)$ is non-monotonic, then the sum of the integral values of $a \in [1, 100]$, is

(a) 4950 (b) 5049 (c) 5050 (d) 5047

Solution. If $b = 1$

$$f(x) = 8x^3 + 4ax^2 + 2x + a$$
$$f'(x) = 24x^2 + 8ax + 2$$

or $$2(12x^2 + 4ax + 1)$$

For non-monotonic $f'(x) = 0$ must have distinct roots hence,

$$D > 0 \quad ie, \quad 16a^2 - 48 > 0 \quad \Rightarrow a^2 > 3,$$

$\therefore$ $$a > \sqrt{3} \text{ or } a < -\sqrt{3}$$

$\therefore$ $$a \in 2, 3, 4, \ldots,$$

$$\text{Sum} = 5050 - 1 = 5049$$

Hence, (b) is the correct answer.

91. If the sum of the base 2 logarithms of the roots of the cubic $f(x) = 0$ is 5, then the value of a is

(a) -64 (b) -8

(c) -128 (d) -256

Solution. If x_1, x_2 and x_3 are the roots, then $\log_2 x_1 + \log_2 x_2 + \log_2 x_3 = 5$

$$\log_2 (x_1 x_2 x_3) = 5$$
$$x_1 x_2 x_3 = 32$$
$$-\frac{a}{8} = 32 \Rightarrow a = -256$$

Hence, (d) is the correct answer.

Type 6 : **Match the Columns**

Example 92 Match the statements of Column I with values of Column II.

	Column I	Column II
(A)	Number of the values of x lying in $\left(0, \frac{\pi}{2}\right)$ at which $f(x) = \ln(\sin x)$ is not monotonic is	(p) 0
(B)	If the greatest interval of decrease of the function $f(x) = x^3 - 3x + 2$ is $[a, b]$, then $a + b$ equals	(q) 2
(C)	Let $f(x) = \frac{x^2 + 2}{[x]}$, $1 \le x \le 3$ where [.] greatest integer function, then least value of $f(x)$ is	(r) -3
(D)	Set of all possible values of a such that $f(x) = e^{2x} - (a + 1)e^x + 2x$ is MI for all $x \in R$ is $(-\infty, a]$, then a equals	(s) 3

Solution. (A) → (p), (B) → (p), (C) → (s), (D) → (s)

(A) $f(x) = \ln(\sin x)$

$$f'(x) = \frac{\cos x}{\sin x} > 0$$

$\therefore$ Required number of values of x is 0.

(B) $f'(x) = 3x^2 - 3 \le 0$, if $-1 \le x \le 1$

$\therefore$ $$a = -1, b = 1 \quad \therefore \quad a + b = 0$$

(C) $$f(x) = \begin{cases} x^2 + 2, & 1 \le x < 2 \\ \dfrac{x^2 + 2}{2}, & 2 \le x < 3 \\ \dfrac{x^2 + 2}{3}, & x = 3 \end{cases}$$

$$f'(x) = \begin{cases} 2x, & 1 < x < 2 \\ x, & 2 < x < 3 \end{cases}$$

$\therefore$ Least value of $f(x)$ in [1, 2] is 3

Least value of $f(x)$ in [2, 3] is 3

$$f(3) = \frac{11}{3}$$

$\therefore$ Least value of $f(x)$ is 3.

(D) $f(x) = e^{2x} - (a + 1)e^x + 2x$

$$f'(x) = 2e^{2x} - (a + 1)e^x + 2$$

Now, $$2e^{2x} - (a + 1)e^x + 2 \ge 0 \text{ for all } x \in R$$

ie, $$2\left(e^x + \frac{1}{e^x}\right) - (a + 1) \ge 0 \text{ for all } x \in R$$

ie, $$4 - (a + 1) \ge 0$$

ie, $$a \le 3 \quad \therefore \quad a = 3$$

Example 93 Match the statements of Column I with values of Column II.

	Column I		Column II
(A)	The dimensions of the rectangle of perimeter 36 cm, which sweeps out the largest volume when revolved about one of its sides, are	(p)	6
(B)	Let $A(-1, 2)$ and $B(2, 3)$ be two fixed points, A point P lying on $y = x$ such that perimeter of ΔPAB is minimum, then sum of the abscissa and ordinate of point P, is	(q)	12
(C)	If x_1 and x_2 are abscissae of two points on the curve $f(x) = x - x^2$ in the interval [0, 1], then maximum value of expression $(x_1 + x_2) - (x_1^2 + x_2^2)$ is	(r)	4
(D)	The number of non-zero integral values of a for which the function $f(x) = x^4 + ax^3 + \dfrac{3x^2}{2} + 1$ is concave upward along the entire real line is	(s)	$\dfrac{1}{2}$

Solution. (A) → (p, q), (B) → (r), (C) → (s), (D) → (r)

(A) Perimeter of the rectangle is 36 cm

If one side is x, then the other side $= 18 - x$

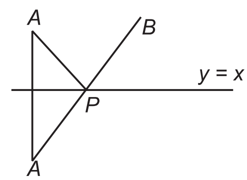
If the rectangle is revolved around the side x, then

volume swept out $V = \pi x (18 - x)^2$

$$\frac{dV}{dx} = \pi [(18 - x)^2 - 2x (18 - x)] = \pi (18 - x)(18 - x - 2x)$$

$\therefore$ $x = 6$ and $y = 12$

(B) $A(-1, 2)$, $B(2, 3)$ and P in a point on $y = x$ perimeter of ΔPAB is minimum when $PA + PB$ is minimum image of $A(-1, 2)$ in the line $y = x$ is $A'(-1, 2)$ in the line $y = x$ is $A'(2, -1)$. Equation of $A'B$ is $x = 2$ hence, P is (2, 2).



(C) Let (x_1, y_1) and (x_2, y_2) are two points.

$\therefore$ $y_1 + y_2 = (x_1 + x_2) - (x_1^2 + x_2^2)$

Now, $y = x - x^2 = -\left(x - \dfrac{1}{2}\right)^2 + \dfrac{1}{4}$

$\therefore$ $(y_1 + y_2)_{max} = 2 \times \dfrac{1}{4} = \dfrac{1}{2}$

(D) $f''(x) = 12x^2 + 6ax + 3 \geq 0, \forall x \in R \Rightarrow a \in [-2, 2]$

$\Rightarrow$ Number of non-zero integer values of a is 4.

Type 7 : **Integer Answer Type Questions**

Example 94 The function $S(x) = \int_0^x \sin\left(\dfrac{\pi t^2}{2}\right) dt$ has two critical points in the interval [1, 2.4]. One of the critical points is a local minimum and the other is a local maximum. The local minimum occurs at x equals ……… .

Solution. **(2)** $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt,\ S'(x) = \sin\left(\frac{\pi x^2}{2}\right) = 0$

$$\frac{\pi x^2}{2} = n\pi \quad \Rightarrow \quad x^2 = 2n \qquad (1 \le x^2 \le 5.76 \text{ as is given})$$

hence, $n = 1$ or 2

$$x = \sqrt{2} \text{ or } x = 2,\ S''(x) = \cos\left(\frac{\pi x^2}{2}\right) \cdot \pi x$$

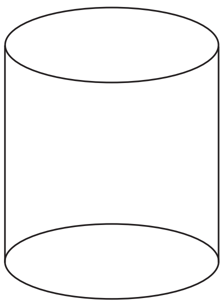
$$S''(\sqrt{2}) < 0 \text{ and } S''(2) > 0 \Rightarrow \text{ minimum at } x = 2$$

Example 95 The radius of a right circular cylinder increases at a constant rate. Its altitude is a linear function of the radius and increases three times as fast as radius. When the radius is 1 cm the altitude is 6 cm. When the radius is 6 cm, the volume is increasing at the rate of 1 cu cm/s. When the radius is 36 cm, the volume is increasing at a rate of n cu cm/s. The value of $n/11$ is equal to ....

Solution. **(3)** $\frac{dr}{dt} = c$ and $h = ar + b$

Also, $\frac{dh}{dt} = 3\frac{dr}{dt}$ (given)

$\therefore$ $a\frac{dr}{dt} = 3\frac{dr}{dt} \quad \Rightarrow \quad a = 3$



Hence, $h = 3r + b$

when $r = 1, h = 6 \quad \Rightarrow \quad 6 = 3 + b \quad \Rightarrow \quad b = 3$

$\therefore$ $h = 3(r + 1)$

$$V = \pi r^2 h = 3\pi r^2 (r + 1) = 3\pi (r^3 + r^2)$$

$$\frac{dV}{dt} = 3\pi (3r^2 + 2r)\frac{dr}{dt}$$

where $r = 6, \frac{dV}{dt} = 1$ cc/s

$\therefore$ $1 = 3\pi (108 + 12)\frac{dr}{dt} \quad \Rightarrow \quad 360\,\pi\frac{dr}{dt} = 1$

Again when $r = 23, \frac{dV}{dt} = n$

$$n = 3\pi ((3.36)^2 + 2.36)\frac{dr}{dt}$$

$$n = 3\pi \cdot 36\,(110) \cdot \frac{1}{360\,\pi} \quad \Rightarrow \quad n = 33 \quad \Rightarrow \quad \frac{n}{11} = \frac{33}{11} = 3$$

Example 96 The set of all points where $f(x)$ is increasing is $(a, b) \cup (c, \infty)$, then find $[a + b + c]$ {where [.] denotes GIF}. Given that

$$f(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2), \forall\, x \in R \text{ and } f''(x) > 0, \forall\, x \in R.$$

Solution. **(0)** $f(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)$

$$f'(x) = 2f'\left(\frac{x^2}{2}\right) \cdot x - 2x\, f'(6 - x^2)$$

$$f'(x) = 2x\left(f'\left(\frac{x^2}{2}\right) - f'(6 - x^2)\right)$$

$$f'\left(\frac{x^2}{2}\right) > f'(6-x)^2, \text{ if } \frac{x^2}{2} > 6 - x^2 \qquad [\because f'(x) \text{ is increasing}]$$

$$\frac{x^2}{2} > 6 - x^2 \quad \Rightarrow \quad x^2 > 4$$

$\Rightarrow$ $$f'\left(\frac{x^2}{2}\right) - f'(6-x^2) > 0 \text{ when } x < -2 \text{ or } x > 2 \;\Rightarrow\; f'(x) > 0$$

When $x \ (-2, 0) \cup (2, \infty)$

$\therefore$ $a + b + c = 0 \quad \Rightarrow \quad [a + b + c] = 0$

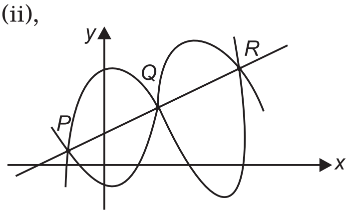
Example 97 The graphs $y = 2x^3 - 4x + 2$ and $y = x^3 + 2x - 1$ intersect at exactly 3 distinct points. The slope of the line passing through two of these points is equal to......... .

Solution. **(8)** Let (x_1, y_1) and (x_2, y_2) are two of these points. Given $y = x^3 + 2x - 1$ and $y = 2x^3 - 4x + 2$

$\therefore$ $$y_1 = 2x_1^3 - 4x_1 + 2 \qquad \text{...(i)}$$

and $$2y_1 = 2x_1^3 + 4x_1 - 2 \qquad \text{...(ii)}$$

Subtracting Eq. (i) in Eq. (ii),



$$y_1 = 8x_1 - 4 \qquad \text{...(iii)}$$

Similarly, $$y_2 = 8x_2 - 4 \qquad \text{...(iv)}$$

$$y_2 - y_1 = 8(x_2 - x_1), \ \frac{y_2 - y_1}{x_2 - x_1} = 8$$

Example 98 The length of the shortest path that begins at the point (2, 5), touches the x-axis and then ends at a point on the circle $x^2 + y^2 + 12x - 20y + 120 = 0$, is......... .

Solution. **(13)** Circles with centre $(-6, 10)$ and radius $= \sqrt{36 + 100 - 120} = 4$

Now, let $(a, 0)$ be a point on the x-axis. If y is the distance from A to P and P to M

$$y = \sqrt{(a-2)^2 + 25} + \sqrt{(a+6)^2 + 100} - 4$$

$$\frac{dy}{dx} = \frac{2(a-2)}{2\sqrt{(a-2)^2 + 25}} + \frac{2(a+6)}{2\sqrt{(a+6)^2 + 100}}$$

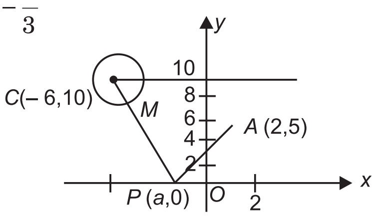
$\frac{dy}{da}$ can be zero only if $a - 2 > 0$ and $a + 6 < 0$ not possible or $a - 2 < 0$ and $a + 6 > 0$, hence $a \in (-6, 2)$

Solving $\frac{dy}{da} = 0$, gives $a = 10$ (rejected) or $a = -\frac{2}{3}$

Hence, $$y_{\min} = \sqrt{\frac{64}{9} + 25} + \sqrt{\frac{256}{9} + 100} - 4$$

$$= \frac{17}{3} + \frac{\sqrt{1156}}{3} - 4 = \frac{17}{3} + \frac{34}{3} - 4$$

$$= 17 - 4 = 13$$



Proficiency in 'Monotonicity, Maxima and Minima'
Exercise 1

Type 1 : Only One Correct Option

1. If $f : [1, 10] \to [1, 10]$ is a non-decreasing function and $g : [1, 10] \to [1, 10]$ is a non-increasing function. Let $h(x) = f(g(x))$ with $h(1) = 1$. Then, $h(2)$

(a) lies in (1, 2)
(b) is more than two
(c) is equal to on
(d) is not defined

2. P is a variable point on the curve $y = f(x)$ and A is a fixed point in the plane not lying on the curve. If PA^2 is minimum, then the angle between PA and the tangent at P is

(a) $\frac{\pi}{4}$
(b) $\frac{\pi}{3}$
(c) $\frac{\pi}{2}$
(d) None of these

3. Let $f x) = \begin{cases} 1 + \sin x, & x < 0 \\ x^2 - x + 1, & x \ge 0 \end{cases}$. Then,

(a) f has a local maximum at $x = 0$
(b) f has a local minimum at $x = 0$
(c) f is increasing everywhere
(d) f is decreasing everywhere

4. If m and n are positive integers and $f(x) = \int_1^x (t - a)^{2n} (t - b)^{2m+1} dt,\ a \ne b$, then

(a) $x = b$ is a point of local minimum
(b) $x = b$ is a point of local maximum
(c) $x = a$ is a point of local minimum
(d) $x = a$ is a point of local maximum

5. Let $f(x) = x^{n+1} + a \cdot x^n$, where be a is a positive real number. Then, $x = 0$ is a point of

(a) local minimum for any integer n
(b) local maximum for any integer n
(c) local minimum if n is an even integer
(d) local minimum if n is an odd integer

6. If f is twice differentiable such that $f''(x) = -f(x)$; $f'(x) = g(x)$, $h'(x) = [f(x)]^2 + [g(x)]^2$ and $h(0) = 2$, $h(1) = 4$, then the equation $y = h(x)$ represents

(a) a straight line with slope (-2)
(b) a straight line with y-intercept 1
(c) a straight line with x-intercept 2
(d) None of these

7. If $f(x)=\begin{cases}2x^2+\dfrac{2}{x^2}, & 0<|x|\le 2\\ 3, & x>2\end{cases}$, then

(a) $x=1,\ -1$ are the points of global minima
(b) $x=1,\ -1$ are the points of local minima
(c) $x=0$ is the points of local minima
(d) None of the above

8. Given a function $y=x^x$, $x>0$ and $0<x<1$. The values of x for which the function attain values exceeding the values of its inverse are
(a) $0<x<1$ (b) $1<x<\infty$
(c) $0<x<2$ (d) None of these

9. $f(x)=\max(\sin x,\ \cos x)$, $\forall\, x\in R$. Then, number of critical points $\in(-2\pi,\ 2\pi)$ is/are
(a) 5 (b) 4 (c) 7 (d) None of these

10. $\sin x+\cos x=y^2-y+a$ has no value of x for any y, if a belongs to
(a) $(0,\ \sqrt{3})$ (b) $(-\sqrt{3},\ 0)$ (c) $(-\infty,\ -\sqrt{3})$ (d) $(\sqrt{3},\ \infty)$

11. If $f: R\to R$ is the function defined by $f(x)=\dfrac{e^{x^2}-e^{-x^2}}{e^{x^2}+e^{-x^2}}$, then
(a) $f(x)$ is an increasing function (b) $f(x)$ is a decreasing function
(c) $f(x)$ is a onto (d) None of these

12. Let $f(x)$ be a quadratic expression possible for all real x. If $g(x)=f(x)-f'(x)+f''(x)$, then for any real x
(a) $g(x)>0$ (b) $g(x)\le 0$ (c) $g(x)\ge 0$ (d) $g(x)<0$

13. $f(x)=\min\{1,\ \cos x,\ 1-\sin x\}$, $-\pi\le x\le\pi$, then
(a) $f(x)$ is differentiable at 0
(b) $f(x)$ is differentiable at $\dfrac{\pi}{2}$
(c) $f(x)$ has local maxima at 0
(d) None of the above

14. P is a variable point on the curve $y=f(x)$ and A is fixed point in the plane not lying on the curve. If PA^2 is a maximum or minimum, then the angle between PA and the tangent at P is
(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{3}$ (c) $\dfrac{\pi}{2}$ (d) None of these

15. $f(x)=\begin{cases}2-|x^2+5x+6|, & x\ne -2\\ a^2+1, & x=-2\end{cases}$, then the range of a, so that $f(x)$ has maxima at $x=-2$ is
(a) $|a|\ge 1$ (b) $|a|<1$ (c) $a>1$ (d) $a<1$

16. Maximum number of real solution for the equation $ax^n+x^2+bx+c=0$, where $a,\ b,\ c\in R$ and n is an even positive number is
(a) 2 (b) 3 (c) 4 (d) infinite

17. Maximum area of rectangle whose two sides are $x = x_0$, $x = \pi - x_0$ and which is inscribed in a region bounded by $y = \sin x$ and x-axis is obtained when $x_0 \in$

(a) $\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$ (b) $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$

(c) $\left(0, \frac{\pi}{6}\right)$ (d) None of these

18. $f(x) = -1 + kx + k$ neither touches nor intercepts the curve $f(x) = \ln x$, then minimum value of $k \in$

(a) $\left(\frac{1}{e}, \frac{1}{\sqrt{e}}\right)$ (b) (e, e^2)

(c) $\left(\frac{1}{\sqrt{e}}, e\right)$ (d) None of these

19. Let $f(x)$ be a polynomial with real coefficients satisfies $f(x) = f'(x) \times f'''(x)$. If $f(x) = 0$ satisfies $x = 1, 2, 3$ only, then the value of $f'(1) \times f'(2) \times f'(3)$ is equal to

(a) positive (b) negative

(c) 0 (d) inadequate data

20. A curve whose concavity is directly proportional to the logarithm of its x-coordinates at any of the curve, is given by

(a) $c_1 \cdot x^2(2\log x - 3) + c_2 x + c_3$ (b) $c_1 x^2(2\log x + 3) + c_2 x + c_3$

(c) $c_1 x^2(2\log x) + c_2$ (d) None of these

21. $f(x) = 4\tan x - \tan^2 x + \tan^3 x, \forall x \neq np + \frac{\pi}{2}, \forall\ n \in I$, then

(a) $f(x)$ is increasing for all $x \in R$ (b) $f(x)$ is decreasing for all $x \in R$

(c) $f(x)$ is increasing in its domain (d) None of these

22. If $f(x) = \begin{cases} 3 + |x - k|, & \text{for } x \le k \\ a^2 - 2 + \dfrac{\sin(x-k)}{x-k}, & \text{for } x > k \end{cases}$ has minimum at $x = k$, then

(a) $a \in R$ (b) $|a| < 2$ (c) $|a| > 2$ (d) $1 < |a| < 2$

23. Let f be a linear function with properties $f(1) \le f(2)$, $f(3) \ge f(4)$ and $f(5) = 5$, then which of the following is true

(a) $f(0) < 0$ (b) $f(0) = 0$

(c) $f(1) < f(0) < f(-1)$ (d) $f(0) = 5$

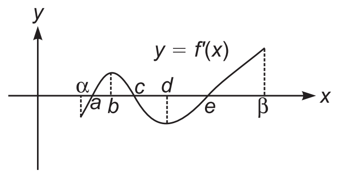
24. If $P(x)$ is polynomial satisfying $P(x^2) = x^2 P(x)$ and $P(0) = -2$, $P'(3/2) = 0$ and $P(1) = 0$. The maximum value of $P(x)$ is

(a) $-\frac{1}{3}$ (b) $-\frac{1}{4}$ (c) $-\frac{1}{2}$ (d) None of these

25. If the curve $x^2 = -4(y - a)$ does not intersect the curve $y = [x^2 - x + 1]$ (where [.] denotes the greatest integer function) in $\left[0, \frac{1+\sqrt{5}}{2}\right]$, then

(a) $\frac{1}{3} < a < 1$ (b) $-1 < a < 1$ (c) $\frac{1}{4} < a < 1$ (d) None of these

26. Analyze the following graph of $f'(x)$



which is incorrect about $f(x)$ for $\alpha < x < \beta$?

(a) only three extreme points
(b) two inflexion points
(c) $f'''(x) > 0$ for $d < x < e$
(d) $x = e$ is the point of local maxima

27. Let $f(x) = x^2 - 2x$ and $g(x) = f(f(x) - 1) + f(5 - f(x))$, then

(a) $g(x) < 0, \forall x \in R$
(b) $g(x) < 0$ for some $x \in R$
(c) $g(x) \geq 0$ for some $x \in R$
(d) $g(x) \geq 0, \forall x \in R$

28. Let $f : N \to N$ be such that $f(n+1) > f(f(n))$ for all $n \in N$, then

(a) $f(x) = n^2 - n + 1$
(b) $f(x) = n - 1$
(c) $f(x) = n^2 + 1$
(d) None of these

29. The equation $|2ax - 3| + |ax + 1| + |5 - ax| = \frac{1}{2}$ possesses

(a) infinite number of real solutions for some $a \in R$
(b) finitely many real solutions for some $a \in R$
(c) no real solutions for some $a \in R$
(d) no real solutions for all $a \in R$

Type 2 : More than One Correct Options

30. Which of the following is/are true?

(You may use $f(x) = \frac{\ln(\ln x)}{\ln x}$)

(a) $(\ln 2.1)^{\ln 2.2} > (\ln 2.2)^{\ln 2.1}$
(b) $(\ln 4)^{\ln 5} > (\ln 5)^{\ln 4}$
(c) $(\ln 30)^{\ln 31} > (\ln 31)^{\ln 30}$
(d) $(\ln 28)^{30} < (\ln 30)^{\ln 28}$

31. If $\lim_{x \to a} f(x) = \lim_{x \to a} [f(x)]$ ([·] denotes the greatest integer function) and $f(x)$ is non-constant continuous function, then

(a) $\lim_{x \to a} f(x)$ is an integer
(b) $\lim_{x \to a} f(x)$ is non-integer
(c) $f(x)$ has local maximum at $x = a$
(d) $f(x)$ has local minimum at $x = a$

32. Let S be the set of real values of parameter λ for which the equation $f(x) = 2x^3 - 3(2 + \lambda)x^2 + 12\lambda x$ has exactly one local maximum and exactly one local minimum. Then, S is a subset of

(a) $(-4, \infty)$
(b) $(-3, 3)$
(c) $(3, \infty)$
(d) $(-\infty, 0)$

33. $h(x) = 3f\left(\frac{x^2}{3}\right) + f(3 - x^2)$, $\forall\ x \in (-3, 4)$ where $f''(x) > 0$, $\forall\ x \in (-3, 4)$, then $h(x)$ is

(a) increasing in $\left(\frac{3}{2}, 4\right)$
(b) increasing in $\left(-\frac{3}{2}, 0\right)$
(c) decreasing in $\left(-3, -\frac{3}{2}\right)$
(d) decreasing in $\left(0, \frac{3}{2}\right)$

34. Let $f(x) = \ln(2x - x^2) + \sin\frac{\pi x}{2}$. Then,

(a) graph of f is symmetrical about the line $x = 1$
(b) graph of f is symmetrical about the line $x = 2$
(c) maximum value of f is 1
(d) minimum value of f does not exist

35. $f(x) = \tan^{-1}(\sin x + \cos x)$, then $f(x)$ is increasing in

(a) $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$
(b) $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$
(c) $\left(\frac{5\pi}{4}, \frac{3\pi}{2}\right)$
(d) $\left(-2\pi, -\frac{7\pi}{4}\right)$

36. If maximum and minimum values of the determinant

$$\begin{vmatrix} 1+\sin^2 x & \cos^2 x & \sin 2x \\ \sin^2 x & 1+\cos^2 x & \sin 2x \\ \sin^2 x & \cos^2 x & 1+\sin 2x \end{vmatrix}$$

are α and β, then

(a) $\alpha + \beta^{99} = 4$
(b) $\alpha^3 - \beta^{17} = 26$
(c) $(\alpha^{2n} - \beta^{2n})$ is always an even integer for $n \in N$
(d) a triangle can be drawn having it's sides as α, β and $\alpha - \beta$

37. Let $f(x) = \begin{cases} x^2 + 4x, & -3 \le x \le 0 \\ -\sin x, & 0 < x \le \frac{\pi}{2} \\ -\cos x - 1, & \frac{\pi}{2} < x \le \pi \end{cases}$. Then,

(a) $x = -2$ is the point of global minima
(b) $x = \pi$ is the point of global maxima
(c) $f(x)$ is non-differentiable at $x = \frac{\pi}{2}$
(d) $f(x)$ is discontinuous at $x = 0$

38. Let $f(x) = ab\sin x + b\sqrt{1-a^2}\cos x + c$, where $|a| < 1$, $b > 0$, then

(a) maximum value of $f(x)$ is b, if $c = 0$
(b) difference of maximum and minimum value of $f(x)$ is $2b$
(c) $f(x) = c$, if $x = -\cos^{-1} a$
(d) $f(x) = c$, if $x = \cos^{-1} a$

39. If $f(x) = \int_{x^m}^{x^n} \frac{dt}{\ln t}$, $x > 0$ and $n > m$, then

(a) $f'(x) = \frac{x^{m-1}(x-1)}{\ln x}$ (b) $f(x)$ is decreasing for $x > 1$

(c) $f(x)$ is increasing in $(0, 1)$ (d) $f(x)$ is increasing for $x > 1$

40. $f(x) = \sqrt{x-1} + \sqrt{2-x}$ and $g(x) = x^2 + bx + c$ are two given functions such that $f(x)$ and $g(x)$ attain their maximum and minimum values respectively for same value of x, then

(a) $f(x)_{\max} = \frac{1}{2}$ (b) $f(x)_{\max} = \frac{3}{2}$

(c) $b = 3$ (d) $b = -3$

41. If $f(x) = a^{\{a^{|x|} \operatorname{sgn} x\}}$; $g(x) = a^{[a^{|x|} \operatorname{sgn} x]}$ for $a > 0$, $a \neq 1$ and $x \in R$, where $\{\}$ and $[\cdot]$ denote the fractional part and integral part functions respectively, then which of the following statements can hold good for the function $h(x)$, where

$(\ln a)\, h(x) = (\ln f(x) + \ln g(x))$

(a) h is even and increasing (b) h is odd and decreasing

(c) h is even and decreasing (d) h is odd and increasing

42. For the function $f(x) = \ln(1 - \ln x)$ which of the following do not hold good?

(a) increasing in $(0, 1)$ and decreasing in $(1, e)$

(b) decreasing in $(0, 1)$ and increasing in $(1, e)$

(c) $x = 1$ is the critical number for $f(x)$

(d) f has two asymptotes

43. The function $f(x) = \begin{bmatrix} x + 2, & \text{if } x < -1 \\ x^2, & \text{if } -1 \le x < 1 \\ (x-2)^2, & \text{if } x \ge 1 \end{bmatrix}$

(a) is continuous for all $x \in R$

(b) is continuous but not differentiable, $\forall\, x \in R$

(c) is such that $f'(x)$ change its sign exactly twice

(d) has two local maxima and two local minima

44. A function f is defined by $f(x) = \int_0^{\pi} \cos t \cos(x - t)\, dt$, $0 \le x \le 2\pi$, then which of the following hold(s) good?

(a) $f(x)$ is continuous but not differentiable in $(0, 2\pi)$

(b) Maximum value of f is π

(c) There exists atleast one $c \in (0, 2\pi)$ it $f'(c) = 0$

(d) Minimum vlaue of f is $-\frac{\pi}{2}$

45. Let $f(x) = \frac{x-1}{x^2}$, then which of the following is correct?

(a) $f(x)$ has minima but no maxima.

(b) $f(x)$ increases in the interval $(0, 2)$ and decreases in the interval $(-\infty, 0) \cup (2, \infty)$

(c) $f(x)$ can come down in $(-\infty, 0) \cup (0, 3)$

(d) $x = 3$ is the point of inflection

Type 3 : **Assertion and Reason**

Directions

(Q. Nos. 46 to 54)

For the following questions, choose the correct answers from the codes (a), (b), (c) and (d) defined as follows:

(a) Statement I is true, Statement II is also true, Statement II is the correct explanation of Statement I.

(b) Statement I is true, Statement II is also true, Statement II is not the correct explanation of Statement I.

(c) Statement I is true, Statement II is false.

(d) Statement I is false, Statement II is true.

46. Statement I The equation $3x^2 + 4ax + b = 0$ has atleast one root in (0, 1), if $3 + 4a = 0$.

Statement II $f(x) = 3x^2 + 4x + b$ is continuous and differentiable in (0, 1).

47. Statement I For the function $f(x) = \begin{cases} 15 - x, & x < 2 \\ 2x - 3, & x \geq 2 \end{cases}$ $x = 2$ has neither a maximum nor a minimum point.

Statement II $f'(x)$ does not exist at $x = 2$.

48. Statement I $\phi(x) = \int_0^x (3\sin t + 4\cos t)\,dt$, $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ $\phi(x)-$ attains its maximum value at $x = \frac{\pi}{3}$

Statement II $\phi(x) = \int_0^x (3\sin t + 4\cos t)\,dt$, $\phi(x)$ is increasing function in $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$.

49. Let $f(x)$ be a twice differentiable function in $[a, b]$, given that $f(x)$ and $f''(x)$ has same sign in $[a, b]$.

Statement I $f'(x) = 0$ has at the most one real root in $[a, b]$.

Statement II An increasing function can intersect the x-axis at the most once.

50. Let $u = \sqrt{c+1} - \sqrt{c}$ and $v = \sqrt{c} - \sqrt{c-1}$, $c > 1$ and let $f(x) = \ln(1 + x)$, $\forall\, x \in (-1, \infty)$.

Statement I $f(u) > f(v)$, $\forall\, c > 1$ because

Statement II $f(x)$ is increasing function, hence for $u > v$, $f(u) > f(v)$.

51. Let $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 1$, $f\left(\frac{3\pi}{2}\right) = -1$ be a continuous and twice differentiable function.

Statement I $|f''(x)| \leq 1$ for atleast one $x \in \left(0, \frac{3\pi}{2}\right)$ because

Statement II According to Rolle's theorem, if $y = g(x)$ is continuous and differentiable $\forall\, x \in [a, b]$ and $g(a) = g(b)$, then there exists atleast one c such that $g'(c) = 0$.

52. Statement I For any $\Delta\, ABC$ $\sin\left(\frac{A+B+C}{3}\right) \geq \frac{\sin A + \sin B + \sin C}{3}$

Statement II $y = \sin x$ is concave downward for $x \in (0, \pi]$.

53. **Statement I** If $f(x)=[x](\sin x+\cos x-1)$

(where $[\cdot]$ denotes the greatest integer function), then $f'(x)=[x](\cos x-\sin x)$ for any $x \in$ integer.

Statement II $f'(x)$ does not exist for any $x \in$ integer.

54. $f(x)$ is a polynomial of degree 3 passing through origin having local extrema at $x=\pm a$.

Statement I Ratio of areas in which $f(x)$ cuts the circle $x^2+y^2=36$ is 1 : 1.

Statement II Both $y=f(x)$ and the circle are symmetric about origin.

Type 4 : **Linked Comprehension Based Questions**

Passage I

(Q. Nos. 55 to 57)

Let $f(x)=\dfrac{1}{1+x^2}$, *let m be the slope, a be the x-intercept and b be the y-intercept of a tangent to* $y=f(x)$.

55. Abscissa of the point of contact of the tangent for which m is greatest, is

(a) $\dfrac{1}{\sqrt{3}}$ (b) 1 (c) -1 (d) $-\dfrac{1}{\sqrt{3}}$

56. Value of b for the tangent drawn to the curve $y=f(x)$ whose slope is greatest, is

(a) $\dfrac{9}{8}$ (b) $\dfrac{3}{8}$ (c) $\dfrac{1}{8}$ (d) $\dfrac{5}{8}$

57. Value of a for the tangent drawn to the curve $y=f(x)$ whose slope is greatest, is

(a) $-\sqrt{3}$ (b) 1 (c) -1 (d) $\sqrt{3}$

Passage II

(Q. Nos. 58 to 60)

Consider the function $f(x)=\max x^2,(1-x)^2,\ 2x(1-x),\ x\in[0,1]$

58. The interval in which $f(x)$ is increasing is

(a) $\left(\dfrac{1}{3},\dfrac{2}{3}\right)$ (b) $\left(\dfrac{1}{3},\dfrac{1}{2}\right)$ (c) $\left(\dfrac{1}{3},\dfrac{1}{2}\right)\cup\left(\dfrac{1}{2},\dfrac{2}{3}\right)$ (d) $\left(\dfrac{1}{3},\dfrac{1}{2}\right)\cup\left(\dfrac{2}{3},1\right)$

59. Let RMVT is applicable for $f(x)$ on (a,b), then $a+b+c$ is (where c is the point such that $f'(c)=0$)

(a) $\dfrac{2}{3}$ (b) $\dfrac{1}{3}$ (c) $\dfrac{1}{2}$ (d) $\dfrac{3}{2}$

60. The interval in which $f(x)$ is decreasing is

(a) $\left(\dfrac{1}{3},\dfrac{2}{3}\right)$ (b) $\left(\dfrac{1}{3},\dfrac{1}{2}\right)$ (c) $\left(0,\dfrac{1}{3}\right)\cup\left(\dfrac{1}{2},\dfrac{2}{3}\right)$ (d) $\left(\dfrac{1}{3},\dfrac{1}{2}\right)\cup\left(\dfrac{2}{3},1\right)$

Passage III

(Q. Nos. 61 to 63)

$f(x)$, $g(x)$, $h(x)$ all are continuous and differentiable functions in $[a, b]$ also $a < c < b$ and $f(a) = g(a) = h(a)$. Point of intersection of the tangent at $x = c$ with chord joining $x = a$ and $x = b$ is on the left of c in $y = f(x)$ and on the right in $y = h(x)$. And tangent at $x = c$ is parallel to the chord in case of $y = g(x)$.
Now, answer the following questions.

61. If $f'(x) > g'(x) > h'(x)$, then

(a) $f(b) < g(b) < h(b)$ (b) $f(b) > g(b) > h(b)$
(c) $f(b) \le g(b) \le h(b)$ (d) $f(b) \ge g(b) \ge h(b)$

62. If $f(b) = g(b) = h(b)$, then

(a) $f'(c) = g'(c) = h'(c)$ (b) $f'(c) > g'(c) > h'(c)$
(c) $f'(c) < g'(c) < h'(c)$ (d) None of these

63. If $c = \dfrac{a+b}{2}$ for each b, then

(a) $g(x) = Ax^2 + Bx + C$ (b) $g(x) = \log x$
(c) $g(x) = \sin x$ (d) $g(x) = e^x$

Passage IV

(Q. Nos. 64 to 66)

In the non decreasing sequence of odd integers $(a_1, a_2, a_3, \dots) = \{1, 3, 3, 3, 5, 5, 5, 5, 5, \dots\}$ each positive odd integer k appears k times. It is a fact that there are integers b, c and d such that for all positive integer n, $a_n = b\,[\sqrt{n + c}] + d$ (where $[.]$ denotes greatest integer function).

64. The possible value of $b + c + d$ is
(a) 0 (b) 1 (c) 2 (d) 4

65. The possible value of $\dfrac{b - 2d}{8}$ is

(a) 0 (b) 1 (c) 2 (d) 4

66. The possible value of $\dfrac{c + d}{2b}$ is

(a) 0 (b) 1 (c) 2 (d) 4

Passage V

(Q. Nos. 67 to 69)

Let $g(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ and $f(x) = \sqrt{g(x)}$, $f(x)$ have its non-zero local minimum and maximum values at -3 and 3 respectively. If $a_3 \in$ the domain of the function $h(x) = \sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$

67. The value of $a_1 + a_2$ is equal to
(a) 30 (b) – 30 (c) 27 (d) – 27

68. The value of a_0 is

(a) equal to 50
(b) greater than 54
(c) less than 54
(d) less than 50

69. $f(-10)$ is defined for

(a) $a_0 > 830$
(b) $a_0 < 830$
(c) $a_0 = 830$
(d) None of these

Passage VI

(Q. Nos. 70 to 74)

If $f : D \to R$, $f(x) = \dfrac{x^2 + bx + c}{x^2 + b_1x + c_1}$, where α, β are the roots of the equation $x^2 + bx + c = 0$ and α_1, β_1 are the roots of $x^2 + b_1x + c_1 = 0$. Now answer the following questions for $f(x)$. A combination of graphical and analytical approach may be helpful in solving these problems. (If α_1 and β_1 are real, then $f(x)$ has vertical asymptote at $x = \alpha_1, \beta_1$). Then

70. If $\alpha_1 < \alpha < \beta_1 < \beta$, then

(a) $f(x)$ is increasing in (α_1, β_1)
(b) $f(x)$ is decreasing in (α, β)
(c) $f(x)$ is decreasing in (β_1, β)
(d) $f(x)$ is decreasing in $(-\infty, \alpha)$

71. If $\alpha_1 < \beta_1 < \alpha < \beta$, then

(a) $f(x)$ has a maxima in $[\alpha_1, \beta_1]$ and a minima is $[\alpha, \beta]$
(b) $f(x)$ has a minima in (α_1, β_1) and a maxima in (α, β)
(c) $f'(x) > 0$ where ever defined
(d) $f'(x) < 0$ where ever defined

72. If the equations $x^2 + bx + c = 0$ and $x^2 + b_1x + c_1 = 0$ do not have real roots, then

(a) $f'(x) = 0$ has real and distinct roots
(b) $f'(x) = 0$ has real and equal roots
(c) $f'(x) = 0$ has imaginary roots
(d) nothing can be said

73. In the above problem $\lim\limits_{x \to \infty} [f(x)] \cdot \lim\limits_{x \to \infty} [f(x)]$ (where [.] denotes the greatest integer function) is equal to

(a) 1
(b) 0
(c) – 1
(d) does not exists

74. In the last problem, if $b > b_1$, then

(a) x-coordinate of point of minima is greater than the x-coordinate of point of maxima
(b) x-coordinate of point of minima is less than x-coordinate of point of maxima
(c) it also depends upon c and c_1
(d) nothing can be said.

Passage VII
(Q. Nos. 75 to 77)

Consider the function $f(x)=\dfrac{x^2}{x^2-1}$

75. The interval in which f is increasing is

(a) $(-1, 1)$
(b) $(-\infty, -1)\cup(-1, 0)$
(c) $(-\infty, \infty)-\{-1, 1\}$
(d) $(0, 1)\cup(1, \infty)$

76. If f is defined from $R-\{-1, 1\}\to R$, then f is

(a) injective but not surjective
(b) surjective but not inective
(c) injective as well as surjective
(d) neither injective nor surjective

77. f has

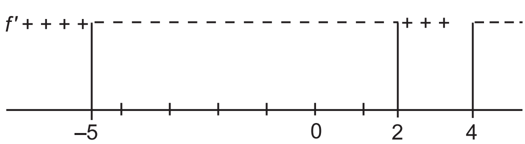
(a) local maxima but not local minima
(b) local minima but not local maxima
(c) both local maxima and local minima
(d) neither local maxima nor local minima

Passage VIII
(Q. Nos. 78 to 80)

Suppose you do not know the function $f(x)$, however some information about $f(x)$ is listed below.

Read the following carefully before attempting the questions

(i) *$f(x)$ is continuous and defined for all real numbers*
(ii) $f'(-5)=0$, $f'(2)$ *is not defined and* $f'(4)=0$
(iii) $(-5, 12)$ *is a point which lies on the graph of* $f(x)$
(iv) $f''(2)$ *is undefined, but* $f''(x)$ *is negative everywhere else*
(v) *The signs of* $f'(x)$ *is given below*



78. On the possible graph of $y=f(x)$, we have

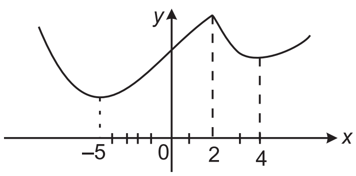
(a) $x=-5$ is a point of relative minima
(b) $x=2$ is a point of relative maxima
(c) $x=4$ is a point of relative minima
(d) graph of $y=f(x)$ must have a geometricaly sharp corner

79. From the possible graph of $y=f(x)$, we can say that

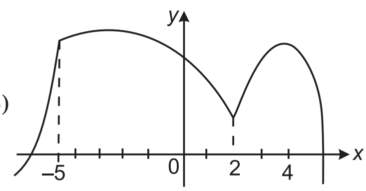
(a) there is exactly one point of inflection on the curve.
(b) $f(x)$ increases on $-5<x<2$ and $x>4$ and decreases on $-\infty<x<-5$ and $2<x<4$
(c) the curve is always concave down
(d) curve always concave up

80. Possible graph of $y = f(x)$ is

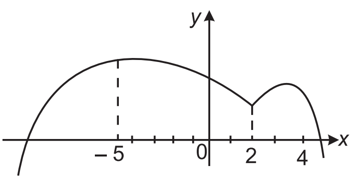
(a)



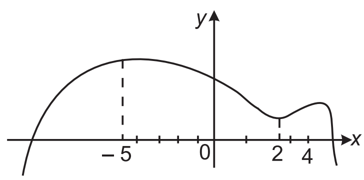
(b)



(c)



(d)



Type 5 : **Match the Columns**

81. Match the following :

	Column I	**Column II**
(A)	The maximum value attained by $y = 10 - \|x - 10\|,\ -1 \le x \le 3$ is	(p) 3
(B)	If $P(t^2, 2t),\ t \in [0, 2]$ is an arbitrary point on parabola $y^2 = 4x$, Q is foot of perpendicular from focus S on the tangent at P, then maximum area of ΔPQS is	(q) $\frac{1}{3}$
(C)	If $a + b = 1,\ a,\ b > 0$, then minimum value of $\sqrt{\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)}$ is	(r) 5
(D)	For real values of x, the greatest and least value of expression $\frac{x+2}{2x^2 + 3x + 6}$ is	(s) $-\frac{1}{13}$

82. Match the entries of the following two columns.

	Column I	**Column II**
(A)	The least value of the function $f(x) = 2 \cdot 3^{3x} - 3^{2x} \cdot 4 + 2 \cdot 3^x$ in [– 1, 1] is	(p) 5
(B)	The minimum value of the polynomial $f(x) = (x-1)x(x+1)$ is	(q) – 1
(C)	The value of the polynomial $\int_{-1}^{3} (\|x-2\| - [x])\,dx$ (where [·] denotes the greatest integer function) is	(r) 3
(D)	If period of the function $f(x) = \sin 36x \tan 42x$ is p, then $\frac{18p}{\pi}$ equals	(s) 0

83. Match the following :

	Column I	Column II
(A)	$f(x)=\int_0^{2x^2}(t-2)(t+1)^3(t-3)^2\,dt$ is/has in (– 1, 1)	(p) local maxima
(B)	$f(x)=\begin{cases}\sin\left(\dfrac{\pi x}{4}\right), & x\le 2\\ 9-4x, & x>2\end{cases}$ is/has in (0, 2)	(q) local minima
(C)	$f(x)=\{2x\}$ denotes fractional part of x is/has in (0, 1)	(r) continuous
(D)	$f(x)=\begin{cases}\lvert\lvert x-2\rvert-2\rvert, & x<2\\ [x], & x\ge 2\end{cases}$ (where [·] denotes the greatest integer function), then $f(x)$ is/has in (– 1, 4)	(s) non-differentiable

84. Match the statements of Column I with values of Column II.

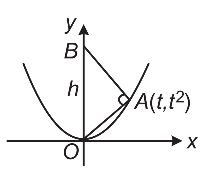
	Column I	Column II
(A)	$f(x)=\cos\pi x+10x+3x^2+x^3,\quad -2\le x\le 3$. The absolute minimum value of $f(x)$ is	(p) 3/4
(B)	If $x\in[-1,1]$, then the minimum value of $f(x)=x^2+x+1$ is	(q) 2
(C)	Let $f(x)=\dfrac{4}{3}x^3-4x,\ 0\le x\le 2$. Then, the global minimum value of the function is	(r) – 15
(D)	Let $f(x)=6-12x+9x^2-2x^3,\ 1\le x\le 4$. Then, the absolute maximum value of $f(x)$ in the interval is	(s) – 8/3

Type 6 : Integer Answer Type Questions

85. A particular substance is being cooled by a streem of cold air (temperature of the air is constant and is 5°C) where rate of cooling is directly proportional to square of difference of temperature of the substance and the air. If the substance is cooled from 40°C to 30°C in 15 min and temperature after 1 hour is T°C, then find the value of $[T]/2$, where [.] represents the greatest integer function.

86. The minimum value of $\dfrac{\tan\left(x+\dfrac{\pi}{6}\right)}{\tan x}$ is ....

87. The figure shows a right triangle with its hypotenuse OB along the y-axis and its vertex A on the parabola $y=x^2$. Let h represents the length of the hypotenuse which depends on the x-coordinate of the point A. The value of $\lim\limits_{x\to 0}(h)$ equals to ....

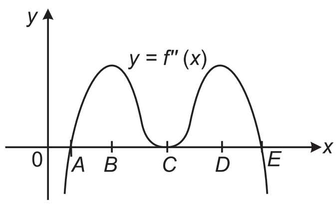


88. Number of positive integral values of a for which the curve $y = a^x$ intersects the line $y = x$ is ....

89. The least value of a for which the equation, $\dfrac{4}{\sin x} + \dfrac{1}{1 - \sin x} = a$ has atleast one solution on the interval $(0, \pi/2)$ is ....

90. Let $f(x) = \begin{bmatrix} x^{3/5} & \text{if } x \le 1 \\ -(x-2)^3 & \text{if } x > 1 \end{bmatrix}$, then the number of critical points on the graph of the function are ....

91. The graph of $y = f''(x)$ for a function f is shown. Number of points of inflection for $y = f(x)$ is ....



92. Number of critical points of the function, $f(x) = \dfrac{2}{3}\sqrt{x^3} - \dfrac{x}{2} + \int_1^x \left(\dfrac{1}{2} + \dfrac{1}{2}2t - \sqrt{t}\right)dt$ which lie in the interval $[-2\pi, 2\pi]$ is ....

93. Let $f(x)$ and $g(x)$ be two continuous functions defined from $R \to R$, such that $f(x_1) > f(x_2)$ and $g(x_1) < g(x_2), \forall\, x_2 > x_2$, then solution set of $f(g(\alpha^2 - 2\alpha)) > f(g(3\alpha - 4))$, then the least value of a for which $f(g(\alpha^2 - 2\alpha)) > f(g(3\alpha - 4))$ is ....

94. If the function $f(x) = \dfrac{t + 3x - x^2}{x - 4}$, where t is a parameter, has a minimum and a maximum, then the greatest value of t is ....

Proficiency in 'Monotonicity, Maxima and Minima'
Exercise 2

1. Find the maximum slope of the curve,
$$y = -x^3 + 3x^2 + 2x - 27.$$

2. Find the coordinates of the point on the parabola $y = x^2 + 7x + 2$ which is closest to the straight line $y = 3x - 3$.

3. Find the interval of increasing and decreasing of the function $y^2 = \dfrac{x^3}{1-x}$.

4. Let $f(x) = x^3 - x^2 + x + 1$

and $$g(x) = \max \begin{cases} f(t) : 0 \le t \le x; & 0 \le x \le 1 \\ 3 - x & ; 1 < x \le 2 \end{cases}$$

Discuss the continuity and differentiability of $g(x)$ in the interval (0, 2).

5. If $ax^2 + \dfrac{b}{x} \ge c$ for all positive x where $a > 0$, and $b > 0$ show that $27ab^2 \ge 4c^3$.

6. Show that $\left(a - \dfrac{1}{a} - x\right)(4 - 3x^2)$, where a is a positive constant, has one maximum and one minimum. Find these value and show that the difference between them is $\dfrac{4}{9}\left(a + \dfrac{1}{a}\right)^3$. What is the least value of this difference for various value of a.

7. A chord of length R divides a circular area of radius R into two regions. Find the sides of the rectangle with the largest area that can be inscribed in the smaller region with one side along the given chord .

8. The function $\sqrt{ax^3 + bx^2 + cx + d}$ has its non-zero local minimum and maximum values at -2 and 2 respectively. If 'a' is root of $x^2 - x - 6 = 0$, find the possible values of a, b, c and d.

9. Find the area of the region of the point P satisfying maximum $\{PA + PB, PB + PC\} < 2$, where $A(1/2, 0)$, $B(3/2, 0)$ and $C = (5/2, 0)$.

10. Two squares are inscribed in a circle of diameter $\sqrt{2}$ units. Prove that the area of the common region of the squares is at least $2(\sqrt{2} - 1)$.

11. Discuss the monotonicity of $g(x)$, where $g(x) = 2f\left(\dfrac{x^2}{2}\right) + f(6 - x^2)\ \forall\ x \in R$. It is given that $f''(x) > 0\ \forall\ x \in R$.

Also find the points of maxima and minima of $g(x)$.

12. Find the greatest value of $f(x)=\dfrac{1}{2ax-x^2-5a^2}$ in $[-3,5]$ depending upon the parameter a .

13. Find the maximum and minimum areas of the triangle formed by x-axis, tangent and normal at a point on the parabola $y=x^2+1\,(1\le x\le 3)$.

14. Consider the circle, $x^2+y^2=9$. Let P be any point lying on the positive x-axis. Tangents are drawn from this point to the given circle, meeting the y-axis at P_1 and P_2 respectively. Find the coordinates of a point 'P' so that the area of the $\Delta\, P, P_1, P_2$ is minimum .

15. Consider the curve, $5x^2-8xy+5y^2=4$. Locate the points on the curve whose distance from the origin is maximum or minimum .

16. Find the range of parameter b, for which the function $f(x)$ is entirely increasing or decreasing for all values of x where,

$$f(x)=\int_0^x (bt^2+b+\cos t)\,dt.$$

17. A box is made with square base and open top. The area of the material used is 192 sq cm. Find the dimensions of the box, if its volume is maximum.

18. A closed rectangular box with a square base is to be made to hold 1000 cu feet. The cost of the material per sq foot for the bottom is 15 paise, for the top 25 paise and for the sides 20 paise. The labour charge for making the box is ₹ 3/-. Find the dimensions of the box when the cost is minimum.

19. A running track of 440 yards is to be laid out encircling a football field the shape of which is rectangle with a semi-circle at each end. If the area of the rectangular position is to be maximum find the lengths of its sides.

20. Show that a right circular conical tent of volume V will require the least amount of canvas to make it if it's height is $\sqrt{2}$ times the radius of its base.

21. Two corridors of a width a and b are right angles. Show that the length of the longest pipe that can be passed round the corner horizontally is $(a^{2/3}+b^{2/3})^{3/2}$.

22. A point P is given on the circumference of a circle of radius r. Chord QR is parallel to the tangent at P. Determine the maximum possible area of the $\Delta\, PQR$.

23. Two cars are travelling along two roads which cross each other at right angles at A. One car is travelling towards A at a speed of 21 miles per hour while the other is travelling towards it at a speed of 28 miles per hour. If initially their distance from A are 1500 feet and 2100 feet respectively, prove that the least distance between them is 60 feet.

24. Two roads OA and OB intersect at an angle of 60°. A car driver approaches O from A, where $AO=800$ m at a uniform speed of 20 m/s. Simultaneously, a runner starts running from O towards B at a uniform speed of 5 m/s. Find the time when the car and the runner are closest.

25. A swimmer S is in the sea at a closest distance d km from the A on a straight shore. The house of the swimmer is on his shore at a distance L km from A. He can swim at a speed of u km h^{-1} and walk at a speed of v km h^{-1} $(v>u)$. At what point on the shore should he land so that he reaches his house in the shortest possible time.

Answers

Target Exercise 8.1

1. (a)	**2.** (c)	**3.** (a)	**4.** (d)	**5.** (c)	**6.** (b)	**7.** (a)
8. (b)	**9.** (c)	**10.** (b)				

Target Exercise 8.2

1. (c)	**2.** (a)	**3.** (a)	**4.** (c)	**5.** (a)

Target Exercise 8.3

1. (d)	**2.** (b)	**3.** (d)	**4.** (c)	**5.** (a)	**6.** (c)	**7.** (a)
8. (a)	**9.** (a)	**10.** (b)				

Exercise 1

1. (c)	**2.** (c)	**3.** (a)	**4.** (a)	**5.** (c)	**6.** (d)	**7.** (b)
8. (a)	**9.** (a)	**10.** (d)	**11.** (d)	**12.** (a)		**13.** (c)
14. (c)	**15.** (a)	**16.** (d)	**17.** (b)	**18.** (a)		**19.** (c)
20. (a)	**21.** (c)	**22.** (c)	**23.** (d)	**24.** (b)		**25.** (c)
26. (d)	**27.** (d)	**28.** (a)	**29.** (d)	**30.** (b,c)		**31.** (a,d)
32. (c,d)	**33.** (a,b,c,d)	**34.** (a,c,d)	**35.** (a,b,c,d)	**36.** (a,b,c)		**37.** (a,b,c)
38. (a,b,c)	**39.** (c,d)	**40.** (b,d)	**41.** (b,d)	**42.** (a,b,c)		**43.** (a,b,d)
44. (c,d)	**45.** (b,c,d)	**46.** (d)	**47.** (d)	**48.** (a)		**49.** (a)
50. (d)	**51.** (a)	**52.** (b)	**53.** (b)	**54.** (a)		**55.** (d)
56. (a)	**57.** (a)	**58.** (d)	**59.** (d)	**60.** (c)		**61.** (b)
62. (c)	**63.** (a)	**64.** (c)	**65.** (a)	**66.** (a)		**67.** (c)
68. (b)	**69.** (a)	**70.** (a)	**71.** (a)	**72.** (a)		**73.** (b)
74. (b)	**75.** (b)	**76.** (d)	**77.** (a)	**78.** (d)		**79.** (c)

80. (c) **81.** (A) $\to$ (p), (B) $\to$ (r), (C) $\to$ (p), (D) $\to$ (q, s) **86.** (3)

82. (A) $\to$ (s), (B) $\to$ (q), (C) $\to$ (r), (D) $\to$ (r)

83. (A) $\to$ (p, r), (B) $\to$ (p, r, s), (C) $\to$ (q, s), (D) $\to$ (p, s)

84. (A) $\to$ (r), (B) $\to$ (p), (C) $\to$ (s), (D) $\to$ (q) **85.** (9) **86.** (3) **87.** (1)

88. (1) **89.** (9) **90.** (3) **91.** (2) **92.** (4) **93.** (2) **94.** (3)

Exercise 2

1. Maximum slope $= 5$ at $(1, -23)$

2. Hence the point $(-2, -8)$ on the parabola is closest to the given line

3. The function is increasing when $x < 3/2$, *ie*, $0 \le x < 1$; as domain $f(x)$ is $0 \le x < 1$

4. $f(x)$ is continuous on (0, 2) and differentiable for $x \in (0, 2) - \{1\}$

6. $a \in (4, 5)$

7. Sides are $2R\sqrt{1-\left(\frac{\sqrt{3}+\sqrt{35}}{8}\right)^2}$ and $R\left[\frac{\sqrt{35}-3\sqrt{3}}{8}\right]$

8. $a=-2,\ b=0, c=24$ and $d>32$

9. Maximum area $=\sqrt{3}\left[\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right]$ sq units

11. $g(x)$ is decreasing in $(-\infty,-2)\cup(0,2)$ and increasing in $(-2,0)\cup(2,\infty)$ also $x=-2,2$ are points of local minima and $x=0$ is the point of local maxima.

12. $f(x)_{\max}=f(5)=\frac{-1}{5(a^2-2a+5)}$ for $a\le 1$, $f(x)_{\max}=f(-3)=\frac{-1}{(5a^2+6a+9)}$ for $a>1$

13. Minimum area of $\Delta=A(1)=5$ sq units and maximum area of

$\Delta=A(3)=\frac{925}{3}$ sq units

14. The required point is $P(3\sqrt{2},0)$

15. Maximum when $(x,y)\in(\pm\sqrt{2},\pm\sqrt{2})$ and minimum when

$(x,y)\in\left(\pm\frac{\sqrt{2}}{3},\pm\frac{\sqrt{2}}{3}\right)$

16. Increasing when $b\ge 1$ and decreasing $b\in(-\infty,-1)$, range $\in R-(-1,1)$.

17. Maximum volume when dimensions are 8, 8, 4

18. Minimum when $x=h=10$

19. Lengths of the sides are 110 yard and diameter is 70 yard for maximum area

22. Maximum area $=\frac{3\sqrt{3}r^2}{4}$ sq units

23. Least distance $=60$ feet

24. Distance between them is minimum at $t=\frac{240}{7}$ s.

25. Minimum at $x=\frac{ud}{\sqrt{v^2-u^2}}$

Solutions

(*Proficiency in 'Monotonicity, Maxima and Minima' Exercise* 1)

Type 1 : **Only One Correct Option**

1. Since, f is non-decreasing and g is non-increasing, so h is a non-increasing function.

Now, $h(1) = 1$

$\Rightarrow$ $h(x)$ is a constant function

$\Rightarrow$ $h(2) = 1$

2. Obviously AP is perpendicular on the tangent drawn to the curve.

3. f is continuous at 0 and $f'(0-) > 0$ and $f'(0+) < 0$. Thus, f has a local maximum at 0.

4. $f'(x) = (x-a)^{2n}(x-b)^{2m+1}$.

Obviously, $f'(a-)f'(a+) > 0$

while $f'(b-) < 0$ and $f'(b+) > 0$

Hence, $x = b$ is a point of local minima.

5. $f'(x) = x^{n-1}[(n+1)x + a]$.

If n is even then, $f'(0-) < 0$ and $f'(0+) > 0$

Hence, 0 is a point of minimum when n is even.

6. $h'(x) = [f(x)]^2 + [g(x)]^2$

$$h''(x) = 2f(x)f'(x) + 2g(x)\,g'(x)$$

$$= 2f(x)\,g(x) + 2g(x)\cdot f''(x)$$

$$2f(x)\cdot g(x) - 2g(x)\cdot f(x) = 0$$

$\therefore$ $h'(x) = \text{constant} = k$ and $h(x) = kx + 1$

Since, $h(0) = 2$ and $h(1) = 4$

$\therefore$ $k_1 = 2$ and $k + k_1 = 4$ or $k = 2$

$\therefore$ $h(x) = y = 2x + 2$

7. $f'(x) = 4x - \dfrac{4}{x^3} = \dfrac{4}{x^3}(x^4 - 1)$

$$= \frac{4}{x^3}(x-1)(x+1)(x^2+1)$$

$\Rightarrow$ $f'(x) = 0$ at $x = 1$ and -1

at $x = \pm 1,\ f(x) = 4$

Since, $f(x) = 3,\ \forall\ x > 2$

$\Rightarrow$ $x = 1$ and -1 are points of local minima.

8. Since, $y_{\min} = \frac{1}{e^{1/e}} > \frac{1}{e}$

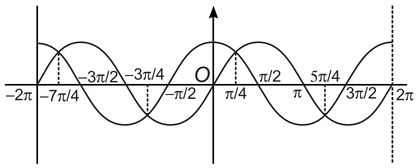
Hence, graph always lie above the line $y = x$.

Hence, $0 < x < 1$ is correct answer.

9. Clearly, $f(x)$ is not differentiable at $-\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$

and $f'(x) = 0$ at $x = -\frac{3\pi}{2}, 0, \frac{\pi}{2}$

$\Rightarrow$ Total number of critical points = 7



10. $y^2 - y + a = \left(y - \frac{1}{2}\right)^2 + a - \frac{1}{4}$

Since, $-\sqrt{2} \le \sin x + \cos x \le \sqrt{2}$, given equation will have no real value of x for any y, if $a - \frac{1}{4} > \sqrt{2}$

ie, $a \in \left(\sqrt{2} + \frac{1}{4}, \infty\right) \Rightarrow a \in (\sqrt{3}, \infty)$ (as $\sqrt{2} + \frac{1}{4} < \sqrt{3}$)

11. $f(x) = 1 - \frac{2e^{-x^2}}{e^{x^2} + e^{-x^2}} = 1 - \frac{2}{e^{2x^2} + 1}$

As $x \to +\infty, \ f(x) \to 1$

As $x \to -\infty, \ f(x) \to 1$

$\Rightarrow f(x)$ is increasing as well as decreasing in some intervals. Since, the range of $f(x)$ is $[0, 1)$ which does not coincide with the codomain R and hence f is not an onto function.

12. Let $f(x) = ax^2 + bx + c > 0, \ \forall \ x \in R$

$\Rightarrow$ $b^2 - 4ac < 0$ and $a > 0$ ... (i)

Now, $g(x) = f(x) - f'(x) + f''(x) = (ax^2 + bx + c) - (2ax + b) + 2a$

$= ax^2 + (b - 2a)x + (2a - b + c)$

$\Rightarrow$ $\Delta = (b - 2a)^2 - 4a(2a - b + c)$

$= (b^2 - 4ac) - 4a^2 < 0$ [Using Eq. (i)]

$\Rightarrow$ $g(x) > 0, \ \forall \ x \in R$

13. We have, $f(x) = \min\{1, \cos x, 1 - \sin x\}$

$$f(x) = \begin{cases} \cos x, & -\frac{\pi}{2} \le x \le 0 \\ 1 - \sin x, & 0 < x \le \frac{\pi}{2} \\ \cos x, & \frac{\pi}{2} < x \le \pi \end{cases}$$

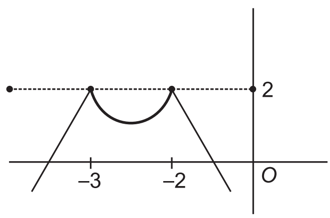
$$f'(x)=\begin{cases}-\sin x, & -\dfrac{\pi}{2}\le x\le 0\\[2mm] -\cos x, & 0<x\le\dfrac{\pi}{2}\\[2mm] -\sin x, & \dfrac{\pi}{2}<x\le\pi\end{cases}$$

Therefore, $f'(0)=0$

Hence, $f(x)$ has local maximum at 0 and $f(x)$ is not differentiable at $x=0$.

14. Maximum/minimum distance between two curves always takes place along the common normal.

15. $f(x)$ will have maxima only, if $a^2+1\ge 2 \Rightarrow |a|\ge 1$



16. $ax^n=-x^2-bx-c$

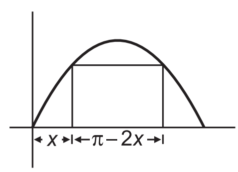
For $a=-1$ and $b=c=0$ and $n=2$, it will have infinite solutions.

17. $A=\text{Area}=\sin x(\pi-2x)$

$$\frac{dA}{dx}=(\pi-2x)\cos x-2\sin x=0 \quad\Rightarrow\ \tan x=\frac{\pi}{2}-x$$

Let $\quad f(x)=\tan x+x-\dfrac{\pi}{2}$

$f\left(\dfrac{\pi}{6}\right)$ is negative, $f\left(\dfrac{\pi}{4}\right)$ is positive

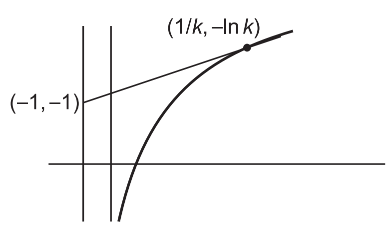


So, one root lies between $\left(\dfrac{\pi}{6},\dfrac{\pi}{4}\right)$.

18. $f(x)+1=k(x+1)$ always passes through $(-1,-1)$. Clearly, its maximum slope can go upto ∞. For minimum slope this line should touch $y=\ln x$.

$\dfrac{dy}{dx}=\dfrac{1}{x}=k$. So, $\left(\dfrac{1}{k},-\ln k\right)$ is point of tangency.

Now, $\quad \dfrac{-\ln k+1}{\dfrac{1}{k}+1}=k$



$\Rightarrow \quad -\ln k+1=k+1\Rightarrow -\ln k=k$

Let $\quad f(x)=k+\ln k$

$f\left(\dfrac{1}{e}\right)=\dfrac{1}{e}-1 \qquad$ (negative)

$f\left(\dfrac{1}{\sqrt{e}}\right)=\dfrac{1}{\sqrt{e}}-\dfrac{1}{2} \qquad$ (positive)

So, one root must lie between $\dfrac{1}{e}$ and $\dfrac{1}{\sqrt{e}}$.

19. $f(x)=f'(x)\times f'''(x)$ is satisfied only by the polynomial of degree 4.

Since, $f(x)=0$ satisfies $x=1,\ 2,\ 3$ only. It is clear that one of the roots is repeated twice.

$$\Rightarrow \qquad f'(1)f'(2)f'(3)=0$$

20. $\dfrac{d^2y}{dx^2}=k\log x \Rightarrow \dfrac{dy}{dx}=k(x\log x-x)+A$

$$\Rightarrow \qquad y=k\left[\frac{1}{2}x^2\log x-\int\frac{1}{2}x^2\cdot\frac{1}{x}-\frac{x^2}{2}\,dx\right]+Ax+B$$

$$\Rightarrow \qquad y=\frac{k}{4}\{2x^2\log x-x^2-2x^2\}+Ax+B$$

$$\Rightarrow \qquad y=c_1(2\log x-3)x^2+c_2x+c_3$$

21. $f'(x)=3\sec^2\left[\left(\tan x-\dfrac{1}{3}\right)^2+\dfrac{11}{9}\right]>0$ for all x in its domain.

22. $\lim\limits_{x\to k} f(x)=3+h \quad \Rightarrow f(k)=3$

$$f(k^-)>f(k) \text{ and } f(k^+)>f(k)$$

$$\Rightarrow \qquad a^2-2+1>3$$

$$|a|>2$$

23. Let $f(x)=mx+b$

$\because \qquad f(1)\le f(2) \ \Rightarrow\ m\ge 0$, similarly

$\qquad f(3)\ge f(4) \ \Rightarrow\ m\le 0\Rightarrow m=0$

$\therefore \qquad f(0)=f(5)=5$

24. Let $P(x)$ is of degree n

$\Rightarrow \qquad 2n=n+2 \quad\Rightarrow\quad n=2$

$\Rightarrow \qquad P(x)=ax^2+bx+c$ form

$\Rightarrow \qquad P(x)=-x^2+3x-2$

Hence, maximum $P(x)=-\dfrac{1}{4}$

25. In [0, 1]

$$[x^2-x+1]=0$$

$$\ln\left[1,\frac{1+\sqrt{5}}{2}\right]$$

$$[x^2-x+1]=1$$

$$x^2=-4(y-a)$$

put $\qquad y=0$

$$x^2=4(a-1) \quad\Rightarrow\quad 4a>1$$

put $\qquad y=1$

$$x^2=4(a-1)\Rightarrow a<1$$

$$\Rightarrow \qquad \frac{1}{4}<a<1.$$

26. Clearly, $f'(a)=0, f'(c)=0, f'(e)=0, x=b$ and two points of inflexion

$$f'''(x)>0, d<x<e$$

$x=e$ is point of local minima.

27. $g(x)=f(x^2-2x-1)+f(5-x^2+2x)$

$$=2x^4-8x^3-4x^2+24x+18$$

$$g'(x)=8x^3-24x^2-8x+24$$

$$g'(x)=0 \quad \Rightarrow \quad x=-1, 1, 3$$

We observe that $g(x)\geq \min\{g(-1), g(1), g(3)\}=0$

$\therefore \qquad g(x)\geq 0, \forall\, x\in R$

28. Using the fact that every set of natural numbers have the smallest element.

$\therefore f(f(1))$ is the smallest element in $\{f(f(1)), f(2), f(f(2)), \ldots\}$

Same argument implies that $f(1)=1$

Repeating the argument for $f:\{n\geq 2\}\to\{n\geq 2\}$, we get $f(2)=2$

$\therefore \qquad$ Clearly, $f(x)=x$

29. $\dfrac{1}{2}=|2ax-3|+|ax+1|+|5-ax|\geq|2ax-3-(ax+1)+5-ax|=1$

which is impossible.

Type 2 : **More than One Correct Options**

30. Let $f(x)=\dfrac{\ln(\ln x)}{\ln x}$

$$f'(x)=\frac{1-\ln(\ln x)}{\ln x}<0,\ \forall\ x>e^e$$

So, $f(x)$ is increasing in $(1, e^e)$ and decreasing in (e^e, ∞).

Therefore, $\qquad x>y$

$\Rightarrow \qquad (\ln x)^{\ln y}<(\ln y)^{\ln x},\ \forall\ x, y\in(e^e, \infty)$ and $x<y$

$\Rightarrow \qquad (\ln x)^{\ln y}<(\ln y)^{\ln x},\ \forall\ x, y\in(1, e^e)$

31. $\lim\limits_{x\to a}[f(x)]$ can exist only when $f(x)$ either increases

or decreases at both sides of the point $x=a$.

Since, $\qquad \lim\limits_{x\to a} f(x)=\lim\limits_{x\to a}[f(x)]$

So, this can occur only when $\lim\limits_{x\to a} f(x)$ is an integer.

32. $f(x)=2x^3-3(2+\lambda)x^2+12\lambda x$

$\Rightarrow \qquad f'(x)=6x^2-6(2+\lambda)x+12\lambda$

$\Rightarrow \qquad f'(x)=0 \ \Rightarrow\ x=2, \lambda$

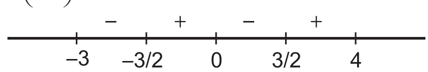
If $f(x)$ has exactly one local maximum and exactly one local minimum, then $\lambda\neq 2$.

33. $h(x) = 3f\left(\frac{x^2}{3}\right) + f(3 - x^2)$

$\Rightarrow \qquad h'(x) = 2x\left(f'\left(\frac{x^2}{3}\right) - f'(3 - x^2)\right)$

$\Rightarrow \qquad f'\left(\frac{x^2}{3}\right) > f'(3 - x^2),\ \forall\, x$ such that $\frac{x^2}{3} > 3 - x^2 \quad \Rightarrow x^2 > \frac{9}{4}$

$\Rightarrow \qquad f'\left(\frac{x^2}{3}\right) < f'(3 - x^2),\ \forall\ x$ such that $x^2 < \frac{9}{4}$



Sign of $h'(x)$

$\Rightarrow \qquad h(x)$ increases in $\left(-\frac{3}{2},\ 0\right) \cup \left(\frac{3}{2},\ 4\right)$

and $\qquad h(x)$ decreases in $\left(-3,\ -\frac{3}{2}\right) \cup \left(0,\ \frac{3}{2}\right)$

34. Obviously, $f(1 + \alpha) = f(1 - \alpha),\ \forall\ \alpha \in (0,\ 1)$. Domain of f is $(0,\ 2)$.

Hence, option (a) is true.

Maxima of $\ln(2x - x^2)$ as well as that of $\sin\left(\frac{\pi x}{2}\right)$ occur at $x = 1$

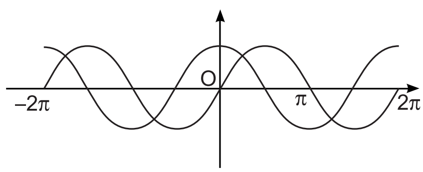
Hence, option (c) is true.

As $x \to 0+$ or $x \to 2-$, $f(x) \to -\infty$.

Hence, option (d) is true.

35. $f'(x) = \dfrac{\cos x - \sin x}{1 + (\sin x + \cos x)^2} \geq 0 \ \Rightarrow \cos x \geq \sin x$

$$\Rightarrow \qquad x \in \left[-2\pi,\ -\frac{7\pi}{4}\right] \cup \left[-\frac{3\pi}{4},\ \frac{\pi}{4}\right] \cup \left[\frac{5\pi}{4},\ 2\pi\right]$$



36. Applying $C_1 \to C_1 + C_2$, we get $\begin{vmatrix} 2 & \cos^2 x & \sin 2x \\ 2 & 1 + \cos^2 x & \sin 2x \\ 1 & \cos^2 x & 1 + \sin 2x \end{vmatrix}$

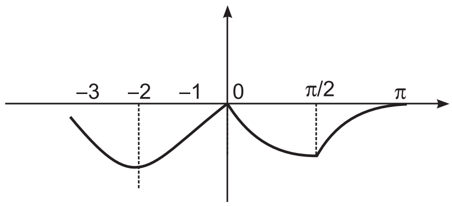
Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\begin{vmatrix} 2 & \cos^2 x & \sin 2x \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 2 + \sin 2x$$

Since, the maximum value of $\sin 2x$ is 1 and min value of $\sin 2x$ is (– 1).

Therefore, $\alpha = 3, \beta = 1$.

37.



From above figure clearly options (a), (b) and (c) are correct.

38. $f(x)=\sqrt{a^2b^2+b^2-b^2a^2}\sin(x+\alpha)+c$

$$= b\sin(x+\alpha)+c, \text{ where } \tan\alpha=\frac{\sqrt{1-a^2}}{a}$$

$$(f(x))_{\max}-(f(x))_{\min}=2b$$

Also, at $x=-\cos^{-1}a,\ f(x)=c.$

39. $f'(x)=\dfrac{1\cdot x^{n-1}}{(\ln x^n)}-\dfrac{1\cdot mx^{m-1}}{(\ln x^m)},$

Clearly, (c) and (d) are the answers.

40. Maximum occurs at $x=\dfrac{3}{2}$ and minimum occurs at $x=-\dfrac{b}{2}\cdot$

41. $h(x)=\dfrac{\ln(f(x)\cdot g(x))}{\ln a}=\dfrac{\ln a^{|x|\{a^{|x|}\cdot\operatorname{sgn}x\}+[a^{|x|}\cdot\operatorname{sgn}x]}}{\ln a}$

$$=\{a^{|x|}\cdot\operatorname{sgn}x\}+[a^{|x|}\cdot\operatorname{sgn}x]=a^{|x|}\operatorname{sgn}x \qquad [\because \{y\}+[y]=y]$$

$$=\begin{cases} a^x & \text{for } x>0 \\ 0 & \text{for } x=0 \\ -a^{-x} & \text{for } x<0 \end{cases}$$

$\Rightarrow$ $h(x)$ is an odd function.

42. $f(x)=\ln(1-\ln x)$

Domain $(0, e)$

$$f'(x)=-\frac{1}{(1-\ln x)}\cdot\frac{1}{x}<0$$

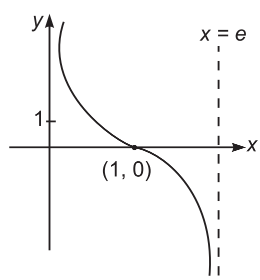
$\Rightarrow$ Decreasing, $\forall\ x$ in its domain

$\Rightarrow$ (a) and (b) are incorrect.

$f'(1)=-1\Rightarrow$ (c) is also incorrect.

Also, $f(1)=0,\ \lim\limits_{x\to e^{-1}}\to-\infty,\ \lim\limits_{x\to 0^+}f(x)\to\infty$

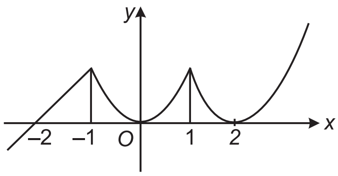
$$f''(x)=\frac{-\ln x}{x^2(1-\ln x)^2}$$



$f''(1)=0$ which is a point of inflection graph is as shown y-axis and $x=e$ are two asymptotes.

43. f is obvious continuous, $\forall\, x \in R$ and not derivable at -1 and 1.

$f'(x)$ changes sign 4 times at $-1, 0, 1, 2$



Local maxima at 1 and -1

Local minima at $x = 0$ and 2.

44. $f(x) = \int_0^{\pi} \cos t \cos(x - t)\, dt$ ...(i)

$$= \int_0^{\pi} -\cos t \cdot \cos(x - \pi + t)\, dt$$

$$f(x) = \int_0^{\pi} -\cos t \cdot \cos(x + t)\, dt \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2f(x) = \int_0^{\pi} \cos t\,(2\cos x \cdot \cos t)\, dt$$

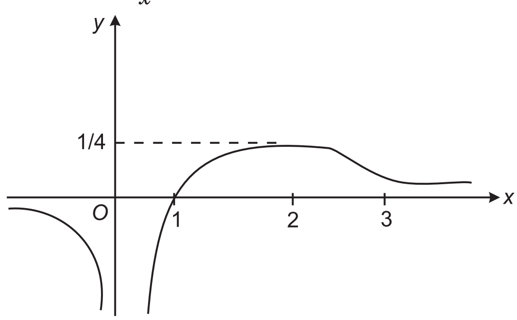
$$2f(x) = 2\ \cos x \int_0^{\pi/2} \cos^2 t\, dt$$

$$f(x) = \frac{\pi \cos x}{2} \text{ Now, verify.}$$

Only (a) and (b) are correct.

Aliter **:** Convert the integer and into sum of two cosine functions.

45. $f'(x) = \dfrac{2 - x}{x^3}$ and $f''(x) = \dfrac{x - 3}{x^4}$ Now, interpret



Type 3 : Assertion and Reason

46. If $b < 0$, then $f(0) = b < 0, f(1) = b < 0$

$\therefore$ 0, 1 lie between the roots, Statement I is false.

47. $x = 2$ is a point of local minima.

48. $\phi(x) = 3\sin x + 4\cos x > 0$

$\phi(x)$ is increasing in $\left[\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$.

$\therefore f(x)$ attain maximum value at $x = \dfrac{\pi}{3}$.

49. Let $g(x) = f(x) \cdot f'(x) \Rightarrow g'(x) > 0$ in $[a, b]$.

50. Let $g(x) = \sqrt{x} - \sqrt{x-1}, x > 1$

$$\Rightarrow \qquad g'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x-1}} - \frac{1}{2}\left(\frac{\sqrt{x-1}-\sqrt{x}}{\sqrt{x(x-1)}}\right)$$

$$= \frac{-1}{2\sqrt{x}\sqrt{x-1}(\sqrt{x}+\sqrt{x-1})} < 0, \forall\, x > 1$$

Hence, $g(x)$ is a decreasing function

$\Rightarrow \qquad c + 1 > c$

$g(c+1) < f(c) \;\Rightarrow\; f(u) < f(v)$.

51. Let $h(x) = f(x) - \sin x$

or $\qquad x \in \left[0, \frac{\pi}{2}\right] \;\Rightarrow\; h(0) = h\left(\frac{\pi}{2}\right) = 0$

According to Rolle's theorem for atleast one $c \in \left(0, \frac{\pi}{2}\right)$.

$h'(c) = f'(c) - \sin c = 0$

for $\quad x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], h\left(\frac{\pi}{2}\right) = h\left(\frac{5\pi}{2}\right) = 0$

According to Rolle's theorem for atleast one $d \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

$h'(d) = f'(d) - \cos d = 0$

for $x \in [c, d]$, $h'(c) = h'(d) = 0$

According to Rolle's theorem for atleast one $x \in (c, d)$.

$h'(x) = f''(x) + \sin x = 0$

$\Rightarrow \quad |f''(x)| \le 1$ for atleast one $x \in \left(0, \frac{3\pi}{2}\right)$.

52. Statements I and II both are true but Statement II does not explain Statement I.

53. Statements II is true as $f'(a)^+ = f'(a)^-, \forall\; a \in I$.

Statement I is true and is obtained from differentiable rule.

54. Statement II is correct as $y = f(x)$ is odd and hence Statement I is correct.

Type 4 : **Linked Comprehension Based Questions**

Solutions (Q. Nos. 55 to 57)

Here, we have $\quad f'(x) = \dfrac{-2x}{(1+x^2)^2}$ and $f''(x) = \dfrac{6x^2 - 2}{(1+x^2)^3}$

$\therefore \qquad f'(x)$ is maximum at $x = -\dfrac{1}{\sqrt{3}}$

If m is greatest, then $m = \dfrac{3\sqrt{3}}{8}$

y-coordinate of the point of contact is $\frac{3}{4}$.

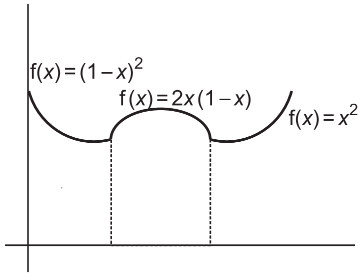
$\therefore$ Equation of the tangent is $y - \frac{3}{4} = \frac{3\sqrt{3}}{8}\left(x + \frac{1}{\sqrt{3}}\right)$

$\therefore \quad a = -\sqrt{3}$ and $b = \frac{9}{8}$

Solutions (Q. Nos. 58 to 60)

We draw the graphs of $f_1(x) = x^2$, $f_2(x) = (1-x)^2$ and $f_3(x) = 2x(1-x)$

Here, $f(x)$ is redefined as



$$f(x) = \begin{cases} (1-x)^2, & 0 \le x < \frac{1}{3} \\ 2x(1-x), & \frac{1}{3} \le x \le \frac{2}{3} \\ x^2, & \frac{2}{3} < x \le 1 \end{cases}$$

Interval of increase of $f(x)$ is $\left(\frac{1}{3}, \frac{1}{2}\right) \cup \left(\frac{2}{3}, 1\right)$.

Interval of decrease of $f(x)$ is $\left(0, \frac{1}{3}\right) \cup \left(\frac{1}{2}, \frac{2}{3}\right)$.

Clearly, Rolle's theorem is applicable on $\left[\frac{1}{3}, \frac{2}{3}\right]$, where $f(x) = 2x(1-x)$.

$\Rightarrow \quad f'(c) = 2 - 4c = 0 \quad \Rightarrow \quad c = \frac{1}{2}$

$\Rightarrow \quad a + b + c = \frac{1}{3} + \frac{2}{3} + \frac{1}{2} = \frac{3}{2}$

Solutions (Q. Nos. 61 to 63)

According to paragraph $\frac{f(b)-f(a)}{b-a} > f'(c)$, $\frac{g(b)-g(a)}{b-a} = g'(c)$

and $\quad \frac{h(b)-h(a)}{b-a} < h'(c)$

As, $\quad f'(x) > g'(x) > h'(x)$

$\Rightarrow \quad \frac{f(b)-f(a)}{b-a} > \frac{g(b)-g(a)}{b-a} > \frac{h(b)-h(a)}{b-a}$

63. If $g(x)=Ax^2+Bx+C$

$$\Rightarrow \quad \frac{g(b)-g(a)}{b-a}=\frac{A(b^2-a^2)+B(b-a)}{b-a}$$

$$\Rightarrow \quad 2A\frac{(b+a)}{2}+B=g'\left(\frac{b+a}{2}\right)$$

Solutions (Q. Nos. 64 to 66)

$a_n > a_{n-1}$ iff $n+c$ is a perfect square, since $a_2 > a_1$ and $a_5 > a_4$

$\Rightarrow$ $2+c$ and $5+c$ are perfect squares

$\Rightarrow$ $c=-1$ ...(i)

Now, $a_2=3=b[\sqrt{2+c}]+d$

$\therefore$ $b+d=2$ ...(ii)

Also, $a_{10}>a_9$

$\Rightarrow$ $a_{10}=7=3b+d$ ...(iii)

On solving Eqs. (ii) and (iii), $b=2,\ d=1$

Solutions (Q. Nos 67 to 69)

$D_h=\{-1, 1\}$, as minimum occurs before maxima

$\therefore$ $a_3=-1$

Now,
$$g(x)=a_0+a_1x+a_2x^2-x^3$$
$$g'(x)=a_1+2a_2x-3x^2$$
$$=-3(x-3)(x+3)=-3x^2=27$$

$\therefore$ $a_1=27,\ a_2=0$ $\quad\therefore\ \alpha_1+\alpha_2=27$

Also, $g(-3)>0$ and $g(3)>0$

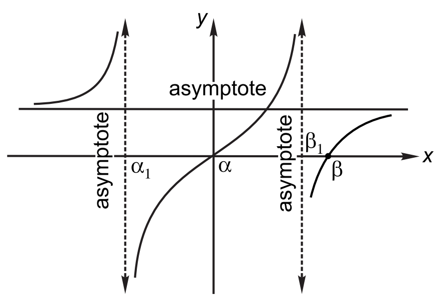
$\Rightarrow$ $a_0>54$ and $a_0<-54$ $\quad\therefore\ a_0>54$

Now,
$$g(x)=a_0+27x-x^3$$
$$f(x)=\sqrt{a_0+27x-x^3}$$
$$f(-10)=\sqrt{a_0+270-1000}$$

Clearly, $f(-10)$ is defined for $a_0>830$.

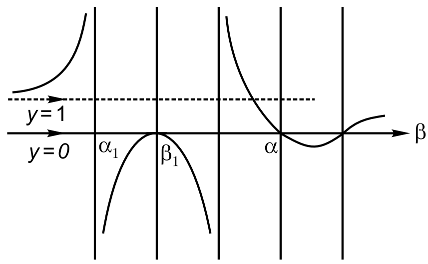
Solutions (Q. Nos. 70 to 74)

70. Graph of $f(x)$ is shown.



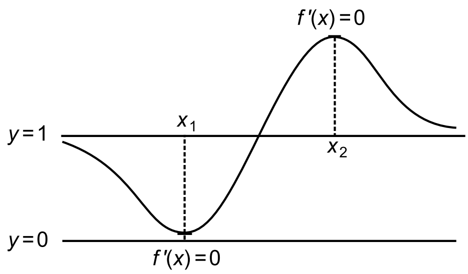
Clearly, $f(x)$ is increasing in $(\alpha_1,\ \beta_1)$.

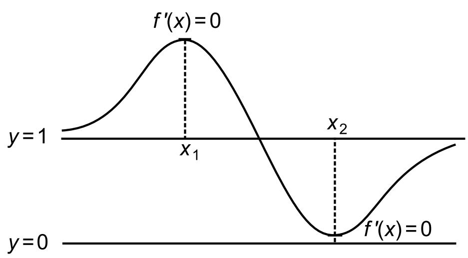
71. Clearly, $f(x)$ has a maximum in $[\alpha_1, \beta_1]$ and a minima in $[\alpha, \beta]$, shown as.



72. $f(x)$ has one of the two graphs.

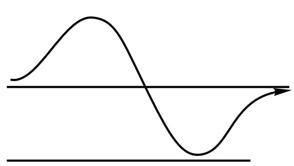
$\Rightarrow$ $f'(x) = 0$ has real and distinct roots.





73. Clearly, $\lim_{x \to \infty} [f(x)] = 0$ and $\lim_{x \to -\infty} [f(x)] = 1$

$\therefore$ $\lim_{x \to \infty} [f(x)] \lim_{x \to -\infty} [f(x)] = 0$



74. $$f(x) = \frac{x^2 + bx + c}{x^2 + b_1 x + c_1} = 1 + \frac{(b - b_1)x + (c - c_1)}{x^2 + b_1 x + c_1}$$

$$= 1 + \frac{\dfrac{(b - b_1)}{x} + \dfrac{(c - c_1)}{x^2}}{1 + \dfrac{b_1}{x} + \dfrac{c_1}{x^2}}$$

For $b > b_1$,

$$\lim_{x \to \infty} f(x) \to 1^+$$

$\Rightarrow$ Point of maxima is greater than point of minima.

Solutions (Q. Nos. 75 to 77)

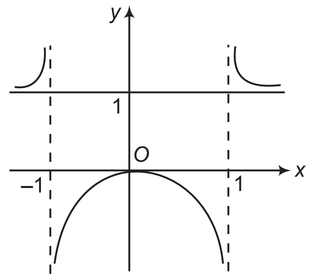
$$y = \frac{x^2}{x^2 - 1}, \text{ not defined at } x = \pm 1$$

$$= 1 + \frac{1}{x^2 - 1}; \; y' = -\frac{2x}{(x^2 - 1)^2}$$

$$\frac{dy}{dx} = 0 \quad \Rightarrow \quad x = 0 \qquad \text{(point of maxima)}$$

as $\qquad x \to 1^+, y \to \infty, \quad x \to 1^-, y \to -\infty$

Similarly, $\qquad x \to -1^+, y \to -\infty, \; x > -1^-, y \to \infty$

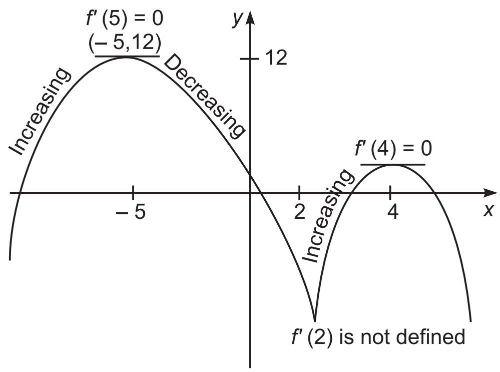


The graph of $y = \dfrac{x^2}{x^2 - 1}$ is as shown

verify all alternatives from the graph.

Solutions (Q. Nos. 78 to 80)

From given statements (i) to (v), one of the graph of $f(x)$ can be plotted as

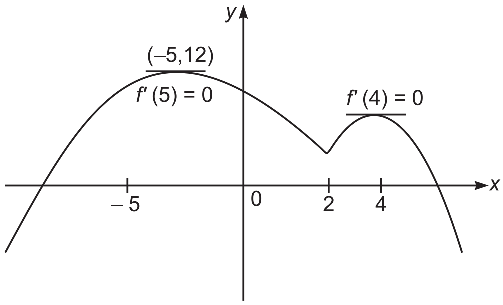


78. Since, $f'(2)$ is not defined and continuous for $x \in R$

$\Rightarrow y = f(x)$ must have a geometrically sharp corner at $x = 2$.

79. At $x = -5$, $f'(x)$ changes from + ve to – ve and $x = 4$, $f'(x)$ change sign for + ve to – ve, hence maxima at $x = -5$ and 4. f is continuous and $f'(x)$ is not defined, hence $x = 2$ must be geometrical sharp corner.

80. Clearly, one of the graph of $f(x)$ is



Type 5 : **Match the Columns**

81. (A) $\tan^2 x$ and $\sec^2 x$ are discontinuous at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$

$\therefore$ Number of discontinuities $= 2$

(B) Since, $f(x) = \sin^{-1} x + \tan^{-1} x + \cot^{-1} x = \sin^{-1} \frac{\pi}{2}$

$\therefore f(x)$ is differentiable in $(-1, 1)$.

$\Rightarrow$ Number of points of non-differentiability $= 0$

(C) $y = [\sin x]$ is discontinuous at $x = \frac{\pi}{2}$ and π

(D) $y = |(x-1)|^3 + |(x-2)^5| + |x-3|$

is non-differentiable at $x = 3$ only.

82. (A) $3^x = t \Rightarrow Q(t) = 2t^3 - 4t^2 + 2t$ in $t \in \left[\frac{1}{3}, 3\right]$

$\Rightarrow \quad Q'(t) < 0$ in $\left(\frac{1}{3}, 1\right)$ and $Q'(t) > 0$ in $(1, 3)$

$\Rightarrow \quad f(x)_{\min} = Q(1) = 0$

(B) Take $x^2 + x = t \Rightarrow Q(t) = t(t-2) \Rightarrow Q(t)_{\min} = -1$

(C) $\int_{-1}^{3} |x-2|\, dx = 5$ and $\int_{-1}^{3} [x]\, dx = 2 \Rightarrow I = 3$

(D) Period of $\sin 36x = \frac{\pi}{18}$

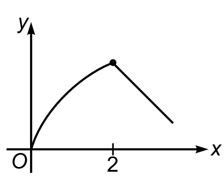
Period of $\tan 42x = \frac{\pi}{42}$

$\Rightarrow \quad p = \frac{\pi}{6}$

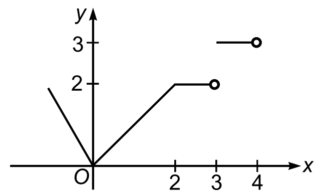
83. (A) $f'(x) = 8x(x-1)(x+1)$ (positive factor)

$\Rightarrow$ Local maxima at $x = 0$

(B) Clearly, we see (p), (r), (s)



(C) $f\left(\frac{1}{2}\right)^- > f\left(\frac{1}{2}\right) < f\left(\frac{1}{2}\right)^+$



Also, $f(x)$ is non-differentiable at $x = \frac{1}{2}$

$\Rightarrow$ Clearly, we see (p), (s).

(D) Clearly, we see (q), (s).

84. (A) $f'(x) = -\pi \sin \pi x + 10 + 6x + 3x^2$

$$= 3(x+1)^2 + 7 - \pi \sin \pi x > 0 \text{ for all } x$$

$\therefore$ $f(x)$ is minimum in $-2 \le x \le 3$. So, the absolute minimum $= f(-2) = -15$

(B) $f'(x) = 2x + 1$. Therefore, for $-1 \le x < -\frac{1}{2}$, we get

$f'(x) < 0$ and for $-\frac{1}{2} < x \le 1$, we get $f'(x) > 0$

$\therefore$ $f(x)$ is minimum decrease in $f\left[-1, -\frac{1}{2}\right]$.

and minimum decrease in $\left(-\frac{1}{2}, 1\right]$.

$$\therefore \min f(x) = f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) + 1 = \frac{3}{4}$$

(C) $f'(x) = 4(x^2 - 1)$. So, for $0 \le x < 1$, we get

$f'(x) < 0$, *ie*, $f(x)$ is monotonically decreasing and for $1 < x \le 2$, we get $f'(x) > 0$, $f(x)$ is monotonically increasing.

$$\therefore \quad \min f(x) = f(1) = \frac{4}{3} - 4 = -\frac{8}{3}.$$

(D) $f'(x) = -12 + 18x - 6x^2$

$$= -6(x^2 - 3x + 2) = -6(x-1)(x-2)$$

$\therefore$ $\quad f'(x) > 0$, if $1 < x < 2$

and $\quad f'(x) < 0$, if $2 < x \le 4$

$\therefore$ $f(x)$ is monotonically increasing in $1 < x < 2$ and monotonically decreasing in $2 < x \le 4$.

$\therefore$ Absolute maximum

= The greatest among $\{f(1), f(2)\}$

= The greatest among $\{1, 2\} = 2$

Type 6 : **Integer Answer Type Questions**

85. $\frac{dT}{dt} = k(T-5)^2$

$$\frac{dT}{(T-5)^2} = k\,dt$$

$$\therefore \quad \frac{(T-5)^{-1}}{-1} = kt + c$$

$$\therefore \qquad -(40-5)^{-1} = c$$

ie, $$c = -\frac{1}{35}$$

$$\therefore \qquad -\frac{1}{T-5} = kt - \frac{1}{35}$$

After 15 min

$$-\frac{1}{30-5} = 15k - \frac{1}{35}$$

$$\therefore \qquad k = \frac{1}{15}\left(\frac{1}{35} - \frac{1}{25}\right) = \frac{-10}{15 \times 35 \times 25} = \frac{-2}{75 \times 35}$$

$\therefore$ Temperature after 60 min is given by

$$-\frac{1}{T-5} = 60\left(\frac{-2}{75 \times 35}\right) - \frac{1}{35}$$

$$\frac{1}{T-5} = \frac{120+75}{75 \times 35} + 5 = \frac{195}{75 \times 35}$$

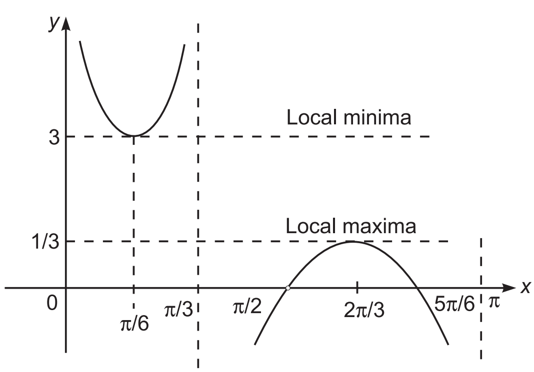
$$\therefore \qquad T = \frac{75 \times 35}{195} + 5 = \frac{5 \times 35}{13} + 5 = \frac{175}{13} + 5 \qquad \therefore \quad [T] = 18$$

$$\therefore \qquad [T]/2 = 18/2 = 9$$

86. $f(x)$ has a period equal to π and can take values $(-\infty, \infty) \Rightarrow 3$ is the local minimum value.

$$y = \frac{2\sin\left(x+\frac{\pi}{6}\right)\cos x}{2\sin x \cos\left(x+\frac{\pi}{6}\right)} = \frac{\sin\left(2x+\frac{\pi}{6}\right)+\sin\frac{\pi}{6}}{\sin\left(2x+\frac{\pi}{6}\right)-\sin\frac{\pi}{6}}$$

$$= 1 + \frac{1}{\sin\left(2x+\frac{\pi}{6}\right)-\sin\frac{\pi}{6}}$$



y is minimum, if $2x + \frac{\pi}{6} = \frac{\pi}{2}$

$$\Rightarrow \qquad x = \frac{\pi}{6}$$

$$\Rightarrow \qquad y_{\min} = 1 + 2 = 3$$

87. Let $A=(t, t^2)$, $m_{OA}=t$, $m_{AB}=-\frac{1}{t}$

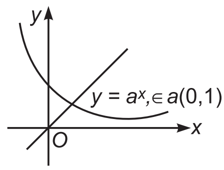
Equation of AB, $y-t^2=-\frac{1}{t}(x-t^2)$

Put $\qquad x=0, h=t^2+1$ (as $x \to 0$, then $t \to 0$)

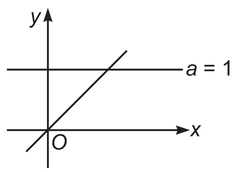
Now, $\qquad \lim_{t \to 0}(h)=\lim_{t \to 0}(1+t^2)=1$

88. For $0<a \le 1$ the line always intercepts $y=a^x$ for $a>1$ say $a=e$ consider $f(x)=e^x-x$

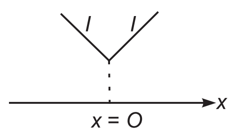
$$f'(x)=e^x-1$$



$f'(x)>0$ for $x>0$ and $f'(x)<0$ for $x<0$



$\therefore f(x)$ is increasing $(\uparrow)$ for $x>0$ and decreasing $(\downarrow)$ for $x<0$

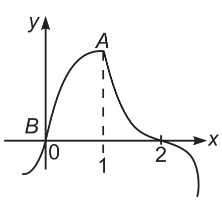


$y=e^x$ always lies above $y=x$ *ie*, $e^x-x \ge 1$ for $a>1$, hence, never intercepts $=a=(0,1]$

89. $\frac{dy}{dx}=-\frac{4\cos x}{\sin^2 x}+\frac{\cos x}{(1-\sin x)^2}=0$ gives $\sin x=\frac{2}{3}$

note that $f(x) \to \infty$ as $x \to 0^+$ or $x \to \frac{\pi^-}{2}$ and between two maxima, we have a minima.

90.



A, B, C are the 3 critical points of $y=f(x)$.

91. Only at A and E only at $D f'(x)=0$ but does not change sign.

92. Note that f is defined for $x > 0$

$$f'(x) = \frac{1}{2}\cos 2x = 0 \quad \Rightarrow \quad x = n\pi \pm \frac{\pi}{4}$$

93. Obviously f is increasing and g is decreasing in (x_1, x_2) hence,
$f(g(\alpha^2 - 2\alpha)) > f(g(3\alpha - 4))$ as f is increasing

$\Rightarrow$ $\quad g(\alpha^2 - 2\alpha) > g(3\alpha - 4)$

$\therefore$ $\quad \alpha^2 - 2\alpha < 3\alpha - 4$ as g is decreasing.

$$\alpha^2 - 5\alpha + 4 < 0$$

$$(\alpha - 1)(\alpha - 4) < 0 \quad \Rightarrow \quad \alpha \in (1, 4)$$

94. $f(x) = \dfrac{t + 3x - x^2}{x - 4}$, $f'(x) = \dfrac{(x-4)(3-2x) - (t + 3x - x^2)}{(x-4)^2}$

For maximum or minimum, $f'(x) = 0$

$$-2x^2 + 11x - 12 - t - 3x + x^2 = 0$$

$$-x^2 + 8x - (12 + t) = 0$$

For one M and m,

$$D > 0$$

$$64 - 4(12 + t) > 0$$

$$16 - 12 - t > 0 \quad \Rightarrow \quad 4 > t \text{ or } t < 4$$

Hence, the greatest value of t is 3.

(Proficiency in 'Monotonicity, Maxima and Minima' Exercise 2)

1. $y = -x^3 + 3x^2 + 2x - 27$

$\therefore$ Slope $\quad m = \dfrac{dy}{dx} = -3x^2 + 6x + 2$

$$\frac{dm}{dx} = -6x + 6$$

$$\frac{d^2m}{dx^2} = -6 < 0$$

$\therefore$ $\quad \dfrac{d^2m}{dx^2} = 0$

$\Rightarrow$ $\quad x = 1$ gives a maximum.

When $\quad x = 1, y = -23$ and $m = 5$

$\therefore$ Maximum slope $= 5$, at the point $(1, -23)$

2. For any point (x, y) on the parabola, its distance from the $y = 3x - 3$

ie, $\quad y - 3x + 3 = 0$ is given by;

$$f(x) = \frac{|y - 3x + 3|}{\sqrt{10}}$$

But $\quad y = x^2 + 7x + 2$, since (x, y) lies on the parabola.

Hence, distance $\quad f(x) = \dfrac{|x^2 + 7x + 2 - 3x + 3|}{\sqrt{10}} = \dfrac{|x^2 + 4x + 5|}{\sqrt{10}}$

$$\therefore \qquad f(x) = \frac{|(x+2)^2 + 1|}{\sqrt{10}}$$

$$\therefore \qquad f(x) = \frac{(x+2)^2 + 1}{\sqrt{10}} \qquad \{\because (x+2)^2 + 1 > 0\}$$

$$\therefore \qquad f'(x) = \frac{2(x+2)}{\sqrt{10}}$$

For either a maximum or minimum f, $f'(x) = 0$, if it exists.

$f'(x) = 0$ if $x = -2$, $f''(x) = +\text{ve} \quad \Rightarrow \quad f$ has minimum at $x = -2$.

Hence, the point $(-2, -8)$ on the parabola is closest to the given line.

3. Given; $y^2 = \dfrac{x^3}{1-x}$ ...(i)

$$\Rightarrow \qquad y = +\sqrt{\frac{x^3}{1-x}} \qquad \text{[Since } y^2 > 0, \textit{ ie}, \text{ single valued function.]}$$

$$\frac{dy}{dx} = \left(\frac{3}{2} - x\right)\frac{x^2}{(1-x)^2} \cdot \frac{1}{y} \qquad \text{[differentiating Eq. (i)]}$$

Now function is increasing for $\dfrac{dy}{dx} > 0$

$$\Rightarrow \qquad \left(\frac{3}{2} - x\right)\frac{x^2}{(1-x)^2} \cdot \frac{1}{y} > 0$$

$$\Rightarrow \qquad \frac{3}{2} - x > 0 \qquad \left[\because \frac{x^2}{(1-x)^2} \cdot \frac{1}{y} > 0\right]$$

$$\Rightarrow \qquad x < \frac{3}{2} \qquad \text{...(ii)}$$

Again,

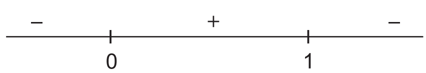
function is decreasing for $\dfrac{dy}{dx} < 0$

when $\qquad x > \dfrac{3}{2}$ ...(iii)

But domain of given function shows,

$$\frac{x^3}{1-x} \geq 0 \quad \textit{ie}, \text{ using number line rule;}$$

$$x \in [0, 1) \qquad \text{...(iv)}$$



Thus from (ii), (iii) and (iv) $f(x)$ could not be decreasing but only increasing when $0 \leq x < 1$.

4. If $f(x)$ is increasing for $a \leq x \leq b$; then $f(a) \leq f(x) \leq f(b)$ maximum $f(x) = f(b)$ and minimum $f(x) = f(a)$; vice-versa for $f(x)$ decreasing for $a \leq x \leq b$

$$f(t) = t^3 - t^2 + t + 1$$

$$f'(t) = 3t^2 - 2t + 1$$

Since this is not factorable we try the method of completion of square.

$$f'(t) = 3\left(t^2 - \frac{2t}{3}\right) + 1$$

$$= 3\left(t^2 - \frac{2}{3}t + \frac{1}{9}\right) - \frac{3}{9} + 1$$

$$f'(t) = 3\left(t - \frac{1}{3}\right)^2 + \frac{2}{3}$$

$\therefore$ $f'(t) > 0$

$\therefore$ f is increasing.

$\therefore$ $\max\{f(t) : 0 \le t \le x\} = f(x)$ *ie*, $x^3 - x^2 + x + 1$

$\therefore$ $$g(x) = \begin{cases} x^3 - x^2 + x + 1, & 0 \le x \le 1 \\ 3 - x, & 1 < x \le 2 \end{cases}$$

Here $g'(1^+) = -1$ and $g'(1^-) = 2$

$\therefore$ $g'(1^+) \ne g'(1^-)$

$\therefore$ $g(x)$ is not differentiable at $x = 1$.

Since, $g'(1^-)$, $g'(1^+)$ exists and are finite, $g(x)$ is continuous at $x = 1$.

At any other point of $(0, 2)$ $g(x)$ is continuous and differentiable.

5. Let $f(x) = ax^2 + \dfrac{b}{x} - c$

$$f'(x) = 2ax - \frac{b}{x^2} = \frac{2ax^3 - b}{x^2}$$

$$f''(x) = 2a + \frac{2b}{x^3} > 0 \qquad (\because\ a, b, x \text{ are all positive})$$

$\therefore$ $$f'(x) = 0 \Rightarrow x = \left(\frac{b}{2a}\right)^{1/3} > 0 \qquad (\because\ a > 0, b > 0)$$

$$f''(x) = +\text{ve} \Rightarrow f(x) \text{ is minimum at } x = \left(\frac{b}{2a}\right)^{1/3}$$

and $$f\left(\left(\frac{b}{2a}\right)^{1/3}\right) = \left(\frac{2a}{b}\right)^{1/3} \cdot \frac{3b}{2} - c \ge 0 \qquad \text{(given)}$$

$\Rightarrow$ $$\left(\frac{2a}{b}\right)^{1/3} \cdot \frac{3b}{2} \ge c$$

Cubing : $$\frac{2a}{b} \cdot \frac{27b^3}{8} \ge c^3 \quad \text{or} \quad 27ab^2 \ge 4c^3$$

6. Let $f(x) = \left(a - \dfrac{1}{a} - x\right)(4 - 3x^2)$

$\therefore$ $$f'(x) = 9x^2 - 6\left(a - \frac{1}{a}\right)x - 4$$

This is quadratic in x and hence $f'(x) = 0$ gives one value-which is maximum and another value which is minimum.

The two roots are;

$$x = \frac{6\left(a - \frac{1}{a}\right) \pm \sqrt{36\left(a - \frac{1}{a}\right)^2 + 144}}{18} = \frac{2a}{3} \quad \text{or} \quad \frac{-2}{3a}$$

The difference between the values of $f\ (x)$ at $x = \dfrac{-2}{3a}$ and $x = \dfrac{2a}{3}$ is

$$= \left(a - \frac{1}{a} + \frac{2}{3\,a}\right)\left(4 - \frac{4}{3a^2}\right) - \left(a - \frac{1}{a} - \frac{2}{3a}\right)\left(4 - \frac{4\,a^2}{3}\right)$$

$$= 4\left(a - \frac{1}{a}\right) - \frac{4}{3a^2}\left(\alpha - \frac{1}{\alpha}\right) + \frac{8}{3a} - \frac{8}{9a^3} - 4\left(a - \frac{1}{a}\right)$$

$$+ \frac{4\,a^2}{3}\left(a - \frac{1}{a}\right) + \frac{8\,a}{3} - \frac{8\,a^3}{9}$$

$$\Rightarrow \quad \frac{8}{3}\left(a + \frac{1}{a}\right) + \frac{4}{3}\left(a - \frac{1}{a}\right)\left(a^2 - \frac{1}{a^2}\right) - \frac{8}{9}\left(a^3 + \frac{1}{a^3}\right)$$

$$\Rightarrow \quad \frac{1}{9}\left(a + \frac{1}{a}\right)\left[24 + 12\left(a - \frac{1}{a}\right)^2 - 8\left(a^2 + \frac{1}{a^2} - 1\right)\right]$$

$$\Rightarrow \quad \frac{4}{9}\left(a + \frac{1}{a}\right)^3$$

This difference is least when $\alpha = 1$, and the least difference value $= \dfrac{32}{9}$

7. Let $ABCD$ be a rectangle inscribed in the segment $PQCDP$. Let E be the mid-point of DC.

Join OE and OC. Let $\angle ECO = \theta$

$$\Rightarrow \quad OE = R\cos\theta \quad \text{and } OF = \sqrt{R^2 - \frac{R^2}{4}} = \frac{R\sqrt{3}}{2}$$

$$\Rightarrow \quad EF = OE - OF$$

$$= R\cos\theta - \frac{\sqrt{3}}{2}\,R$$

$$\Rightarrow \quad EC = R\sin\theta \quad \Rightarrow \quad CD = 2R\sin\theta$$

Area of $\quad ABCD = A\,(\theta) = FE \times DC = 2R\sin\theta\left(R\cos\theta - \dfrac{\sqrt{3}}{2}\,R\right)$

$$\therefore \quad A'(\theta) = 2R\cos\theta\left(R\cos\theta - \frac{\sqrt{3}}{2}\,R\right) - 2R\sin\theta\,(R\sin\theta)$$

Let $A'(0) = 0$ for maximum or minimum.

$$\Rightarrow \quad \cos^2\theta - \frac{\sqrt{3}}{2}\cos\theta - \sin^2\theta = 0$$

$$\Rightarrow \quad 4\cos^2\theta - \sqrt{3}\cos\theta - 2 = 0$$

$$\Rightarrow \quad \cos\theta = \frac{\sqrt{3} \pm \sqrt{35}}{8}$$

Since θ is acute, neglecting negative sign, we have $\cos\theta = \dfrac{\sqrt{3} + \sqrt{35}}{8}$.

Also $\quad f''(\theta) < 0$ where $\cos\theta = \dfrac{\sqrt{3} + \sqrt{35}}{8}$

So, A is maximum for this value of θ.

Now, $$AB = 2R\sin\theta = 2R\sqrt{1-\left(\frac{\sqrt{3}+\sqrt{35}}{8}\right)^2}$$

and $$BC = R\cos\theta - R\frac{\sqrt{3}}{2}$$
$$= R\left[\frac{\sqrt{3}+\sqrt{35}}{8}-\frac{\sqrt{3}}{2}\right] = R\left[\frac{\sqrt{35}-3\sqrt{3}}{8}\right]$$

8. Since minimum occurs before maximum, $a < 0$

Since a is a root of $x^2 - x - 6 = 0$, $a = -2$

Let $g(x) = ax^3 + bx^2 + cx + d$

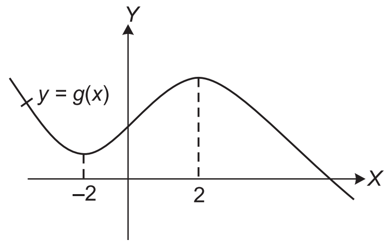
or $g(x) = -2x^3 + bx^2 + cx + d$

$\Rightarrow$ $g'(x) = -6x^2 + 2bx + c = -6(x+2)(x-2)$

$\Rightarrow$ $b = 0$ and $c = 24$

Since minimum and maximum values are non-zero.

$g(-2)$ and $g(2)$ are positive.



Now, $g(x) = -2x^3 + 24x + d$

$g(-2) > 0 \quad \Rightarrow \quad 16 - 48 + d > 0$

$\Rightarrow$ $d > 32$

$g(2) > 0 \quad \Rightarrow \quad -16 + 48 + d > 0$

$\Rightarrow$ $d > -32$

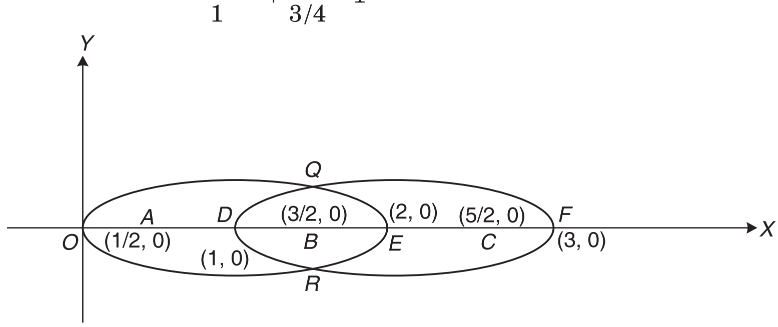
Thus, $a = -2, b = 0, c = 24, d > 32$

9. According to given condition $PA + PB < 2$ as well as $PB + PC < 2$. If $PA + PB < 2$, then P lies inside the ellipse having foci at A and B and major axis of length 2.

The equation of ellipse is $\frac{(x-1)^2}{1} + \frac{y^2}{3/4} = 1$ ...(i)

The equation of ellipse satisfying $PB + PC < 2$ is,

$$\frac{(x-2)^2}{1} + \frac{y^2}{3/4} = 1 \quad \text{...(ii)}$$



The required region is $DQERD$.

The x-coordinate of $Q = 3/2$

From Eq. (ii) $\dfrac{4y^2}{3} = 1 - (x-2)^2$

$$y = \frac{\sqrt{3}}{2}\sqrt{1-(x-2)^2}$$

Required area $= 4\int_1^{3/2} y\,dx$

$$= 2\sqrt{3}\int_1^{3/2}\sqrt{1-(x-2)^2}\,dx$$

$$= 2\sqrt{3}\left[\frac{x-2}{2}\sqrt{1-(x-2)^2} + \frac{1}{2}\sin^{-1}(x-2)\right]_1^{3/2}$$

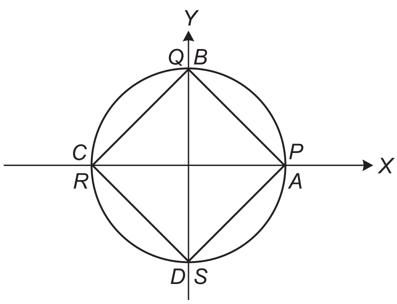
$$= \sqrt{3}\left[(x-2)\sqrt{4x-x^2-3} + \frac{1}{2}\sin^{-1}(x-2)\right]_1^{3/2}$$

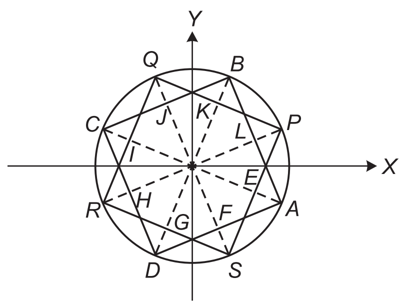
$$= \sqrt{3}\left[-\frac{1}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\pi}{6} - \left(0 - \frac{\pi}{2}\right)\right]$$

$$= \sqrt{3}\left[\frac{\pi}{2} - \frac{\pi}{6} - \frac{\sqrt{3}}{4}\right]$$

$$= \sqrt{3}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right] \text{ sq units}$$

10. Since the radius of circle is $\dfrac{1}{\sqrt{2}}$, side of the square $= 2\left(\dfrac{1}{\sqrt{2}}\right)\cos 45° = 1$ unit.





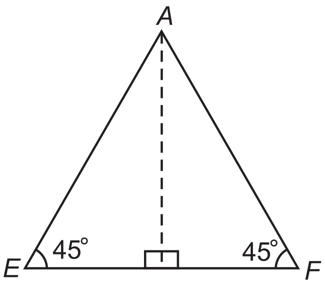
Let us assume that initially both squares $\square PQRS$ and $\square ABCD$ are kept upon each other.

Now, $\square PQRS$ is rotated anti-clockwise by an angle $\theta/2$ and $\square ABCD$ is rotated clockwise by an angle $\theta/2$.

Due to symmetry in figure it is clear that $\Delta\, LPE \cong \Delta\, EAF$. Since all corners are symmetrically place, all the triangle thus formed will be identical (*ie*, $\Delta\, LPE$, $\Delta\, EAF$, $\Delta\, FSG$, $\Delta\, GDH$, $\Delta\, HRI$, $\Delta\, ICJ$, $\Delta\, JQK$ and $\Delta\, KBL$ will be identical).

$\Rightarrow$ Area of *EFGHIJKLE* will be least when area of $\Delta\, EAF$ will be maximum. Since $AE = PE$ and $AF = FS$, $AE + EF + FA = 1 \Rightarrow$ perimeter of $\Delta\, EAF = 1$ unit; which is constant.

Since $\Delta\, EAF$ is a right angle triangle having constant perimeter, area of $\Delta\, EAF$ will be maximum when it becomes isosceles right angled triangled.



Let $\quad AE = x = AF$

$\Rightarrow \quad EM = x \cos 45° = \dfrac{x}{\sqrt{2}}$

$\Rightarrow \quad x + x + \dfrac{2x}{\sqrt{2}} = 1 \quad \Rightarrow \quad x = \dfrac{1}{2+\sqrt{2}} = \dfrac{2-\sqrt{2}}{2}$

$\Rightarrow$ Area of $\Delta EAF = \dfrac{1}{2} AE \cdot AF$

$$= \frac{1}{2}\left(\frac{2-\sqrt{2}}{2}\right)^2 = \frac{3-2\sqrt{2}}{4}$$

Hence, least area of $EFGHIJKLE =$ (Area of square) $- 4$ (Area of Δ EAF)

$$= 1 - 4\left(\frac{3-2\sqrt{2}}{4}\right) = 1 - 3 + 2\sqrt{2} = 2(\sqrt{2}-1)$$

11. $g(x) = 2f\left(\dfrac{x^2}{2}\right) + f(6 - x^2)$

$$g'(x) = f'\left(\frac{x^2}{2}\right) \cdot 2x + f'(6-x^2) \cdot (-2x)$$

$$\Rightarrow \quad g'(x) = 2x\left[f'\left(\frac{x^2}{2}\right) - f'(6-x^2)\right]$$

We have been given that $f''(x) > 0 \ \Rightarrow f'(x)$ is increasing for all real values of x.

Let $\quad \dfrac{x^2}{2} > 6 - x^2 \quad \Rightarrow \quad x^2 > 4 \quad \Rightarrow \quad x \in (-\infty, -2) \cup (2, \infty)$

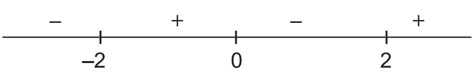
$\Rightarrow \quad f'\left(\dfrac{x^2}{2}\right) > f'(6-x^2) \qquad \forall\, x \in (-\infty, -2) \cup (2, \infty) \qquad$ ...(i)

Also, $\quad \dfrac{x^2}{2} < 6 - x^2 \quad \Rightarrow \quad x \in (-2, 2)$

$\Rightarrow \quad f'\left(\dfrac{x^2}{2}\right) < f'(6-x^2) \qquad \forall\ x \in (-2, 2) \qquad$ ...(ii)

Let us check the sign of $g'(x)$.

From the given sign scheme we get,



$g'(x) < 0, \qquad \forall\ x \in (-\infty, -2) \cup (0, 2)$

$g'(x) > 0, \qquad \forall\ x \in (-2, 0) \cup (2, \infty)$

$g'(x) = 0, \qquad \forall\ x = -2, 0, 2$

Thus, $g(x)$ is monotonically decreasing in $(-\infty, -2) \cup (0, 2)$ and monotonically increasing in $(-2, 0) \cup (2, \infty)$.

Clearly $x = 2, -2$ are point of local minima of $g(x)$ whereas, $x = 0$ is the point of local maxima of $g(x)$.

12. $f(x) = \dfrac{1}{2ax - x^2 - 5a^2}$

or $\quad f(x) = \dfrac{1}{-[(x-a)^2 + 4a^2]}$

$\Rightarrow \quad f'(x) = \dfrac{2(x-a)}{[(x-a)^2 + 4a^2]^2}$

$\Rightarrow \quad f'(x) > 0, \forall\ x > a$ and $f'(x) < 0, \forall\ x < a$

$\Rightarrow$ $x = a$ is the point of minima of the function $y = f(x)$.

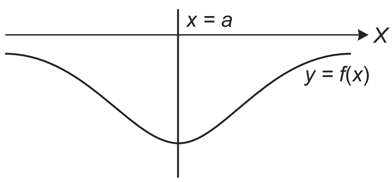
The given figure shows the rough sketch of $y = f(x)$.

Clearly if $a > 5$, greatest value of $y = f(x)$ in $[-3, 5]$ occurs at $x = -3$

Also if, $a < -3$, greatest value of $y = f(x)$ in $[-3, 5]$ occurs at $x = 5$.

Now, let us consider the case when $a \in [-3, 5]$

Clearly in this case greatest value of $f(x)$ occurs either at $x = -3$ or $x = 5$, depending upon the order relation between $f(-3)$ and $f(5)$



Let us assume $f(-3) \le f(5)$
(**Note :** $f(-3)$ and $f(5)$ are both negative)

$\Rightarrow \quad \dfrac{1}{-(5a^2 + 6a + 9)} \le \dfrac{1}{-(a^2 - 2a + 5) \cdot 5}$

$\Rightarrow \quad 5(a^2 - 2a + 5) \ge 5a^2 + 6a + 9$

$\Rightarrow \quad a \le 1$

Hence for $\quad a \in [-3, 1],\ f_{\max}(x) = f(5)$

and for $\quad a \in (1, 5],\ f_{\max}(x) = f(-3)$

Putting the three cases together, we get the following results;

for $a \le 1$, $\quad f_{\max}(x) = f(5) = \dfrac{1}{-5(a^2 - 2a + 5)}$

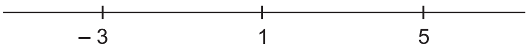
for $a > 1$, $\quad f_{\max}(x) = f(-3) = \dfrac{1}{-(5a^2 + 6a + 9)}$

Aliter : $\quad f(x) = -\dfrac{1}{(x-a)^2 + 4a^2}$

Obviously $f(x)$ is maximum when $(x-a)^2 + 4a^2$ is maximum *ie*, $|x - a|$ is maximum.

Taking mid-point of the interval $[-3, 5]$, namely $x = 1$, it is clear that;

If $a \le 1$, then $|x - a|$ is maximum if $x = 5$



and if $a > 1$, then $|x - a|$ is maximum if $x = -3$

$$\max f(x) = \begin{cases} f(5), & \text{if } a \le 1 \\ f(-3), & \text{if } a > 1 \end{cases}$$

13. $y = x^2 + 1$

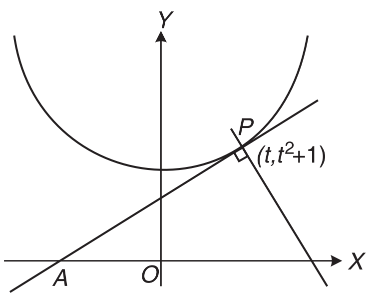
Any point in parabola is $(t, t^2 + 1)$

$$\left(\frac{dy}{dx}\right)_{(t,\, t^2+1)} = 2t$$

Equation of tangent at P, $y - (t^2 + 1) = 2t(x - t)$

at $\quad A,\ y = 0$, So $x = t - \left(\dfrac{t^2 + 1}{2t}\right)$

Hence, $\quad A = \left(t - \dfrac{t^2 + 1}{2t}, 0\right)$



Equation of normal at $P : y - (t^2 + 1) = -\dfrac{1}{2t}(x - t)$

At $\quad B,\ y = 0$, so $x = t + 2t(t^2 + 1)$

Hence, $\quad B \equiv (t + 2t(t^2 + 1), 0)$

Now, $\quad AB = 2t(t^2 + 1) + \dfrac{t^2 + 1}{2t}$

Area $\quad (\Delta APB) = A(t) = \dfrac{1}{2}(t^2 + 1)\left[2t(t^2 + 1) + \dfrac{t^2 + 1}{2t}\right] = \dfrac{(t^2 + 1)^2(4t^2 + 1)}{2t}$

$$A'(t) = \frac{(t^2 + 1)(20t^4 + 7t^2 - 1)}{4t^2}$$

$\Rightarrow \quad A'(t) > 0$

Hence, $A(t)$ is increasing function for $1 \le t \le 3$.

$\therefore$ Minimum area of $\Delta APB = A(1) = 5$ sq units.

And, maximum area of $\Delta APB = A(3) = \dfrac{925}{3}$ sq units.

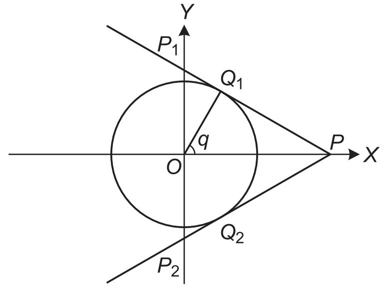
14. Let Q_1 and Q_2 be the points of contact.

Let $OP = h$ and $\angle QOP = \theta$

Clearly, $OQ_1 = 3 \quad \Rightarrow \quad OP = h = 3 \sec\theta$

And $OP_1 = 3 \operatorname{cosec}\theta$

Now, area of $\Delta P\, P_1 P_2$



$$\Delta = \frac{1}{2}\cdot(P_1\ P_2)\cdot(OP) = (OP_1\ )\cdot(OP)$$

$$= 3 \operatorname{cosec}\theta \cdot 3 \sec\theta$$

$$\Delta = \frac{18}{\sin 2\theta}$$

Clearly, triangle is minimum of $\sin 2\theta = 1$

$\Rightarrow \qquad \theta = \pi/4$

$\Rightarrow \qquad h = 3 \sec\left(\frac{\pi}{4}\right) = 3\sqrt{2}$

Hence, the required point is $P\,(3\sqrt{2}, 0)$.

15. Let $P\,(r, \theta)$ be any point on the curve where r and θ are the polar coordinate of the point P. Clearly r is the distance of P from the origin and θ is the angle that the line OP would make with positive x-axis.

$P \equiv (r\cos\theta, r\sin\theta)$ lies on the curve

$\Rightarrow \qquad 5r^2\cos^2\theta - 8r^2\sin\theta\cos\theta + 5r^2\sin^2\theta = 4$

$\Rightarrow \qquad 5r^2 - 4r^2\sin 2\theta = 4$

$\Rightarrow \qquad r^2 = \dfrac{4}{(5 - 4\sin 2\theta)}$

$\Rightarrow \qquad r^2_{\max} = 4 \quad \text{and} \quad r^2_{\min} = \dfrac{4}{9}$

$\Rightarrow \qquad r_{\max} = 2 \quad \text{and} \quad r_{\min} = \dfrac{2}{3}$

Clearly r is maximum, if $\sin 2\theta = 1$

$\Rightarrow \qquad \theta = \pi/4$

r is minimum, if $\sin 2\theta = -1$

$\Rightarrow \qquad \theta = 3\pi/4$

$\Rightarrow$ Required points are $\left(\pm 2\cos\dfrac{\pi}{4}, \pm 2\sin\dfrac{\pi}{4}\right)$ and $\left(\pm\dfrac{2}{3}\cos\dfrac{3\pi}{4}, \pm\dfrac{2}{3}\sin\dfrac{3\pi}{4}\right)$

or $(\pm\sqrt{2}, \pm\sqrt{2})$ and $\left(\pm\dfrac{\sqrt{2}}{3}, \pm\dfrac{\sqrt{2}}{3}\right)$, points to be taken either with upper signs or with lower signs together.

16. $f(x) = \int_0^x (bt^2 + b + \cos t)\,dt$

$\Rightarrow \qquad f'(x) = bx^2 + b + \cos x$

***Case* (a)** $\qquad f'(x) \geq 0, \qquad \forall\ x \in R$

$\Rightarrow \qquad bx^2 + b + \cos x \geq 0, \qquad \forall\ x \in R$

$\Rightarrow \quad bx^2 + b - 1 \geq 0, \quad \forall\ x \in R$

$\Rightarrow \quad b > 0$ and $(-4b(b-1)) \leq 0$

$\Rightarrow \quad b \geq 1 \quad$ ...(i)

***Case* (b)** $\quad f'(x) \leq 0, \quad \forall\ x \in R$

$\Rightarrow \quad bx^2 + b + 1 \leq 0, \quad \forall\ x \in R$

$\Rightarrow \quad b < 0$ and $(0 - 4b(b+1)) \leq 0$

$\Rightarrow \quad b + 1 \leq 0 \quad \Rightarrow \quad b \leq -1 \quad$ ...(ii)

Combining (i) and (ii), we get $b \in (-\infty, -1] \cup [1, \infty)$.

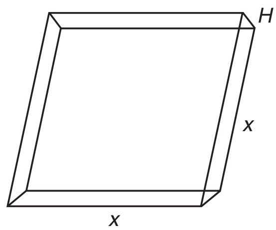
17. Let the length of each base side $= x$, height $= H$

$\therefore$ Area of material used = area of base + area of 4 vertical sides

$\Rightarrow \quad x^2 + 4xH = 192 \quad$ (given)

$$\therefore \quad H = \frac{192 - x^2}{4x} \quad \text{...(i)}$$

Hence, the volume $= f(x) = x^2 H$



$$= x^2 \left[\frac{192 - x^2}{4x}\right] \quad \text{[using Eq. (i)]}$$

$$\therefore \quad f(x) = \frac{192x - x^3}{4}$$

$$f'(x) = \frac{1}{4}(192 - 3x^2) = \frac{3}{4}(64 - x^2) = \frac{3}{4}(8 - x)(8 + x)$$

For either maximum or minimum,

$f'(x) = 0$ if it exists

$f'(x) = 0$

$\Rightarrow \quad x = 8, -8$

But $x \neq -8$, we consider $x = 8$ only

$$f''(x) = \frac{1}{4}(0 - 6x)$$

$$f''(8) = \frac{-48}{4} = (-12) < 0 \quad \Rightarrow \quad f \text{ has a maximum at } x = 8$$

So that maximum volume is attained for $x = 8$

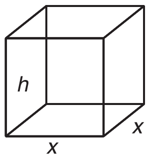
$$\therefore \quad H = \frac{192 - x^2}{4x} = \frac{192 - 64}{32} = 4$$

Hence for maximum volume the dimensions are 8, 8, 4.

18. Let the side of the base be x and height h.

The volume $= x^2 h = 1000$ (given)

$\therefore \quad h = \dfrac{1000}{x^2}$ ...(i)



Area of base $= x^2$, area of top $= x^2$

and total area of sides $= 4xh$

$\therefore$ Cost for making the box,

$$f(x) = \text{cost for the (base + top + side) + Labour}$$
$$= 15x^2 + 25x^2 + 20\cdot(4xh) + 300$$

$\therefore \quad f(x) = 40x^2 + \dfrac{80000}{x} + 300$ [using Eq. (i)]

$$f'(x) = 80x - \frac{80000}{x^2}$$

For either maximum or a minimum f, $f'(x) = 0$;

$\Rightarrow \quad x = 10$

Also $\quad f''(x) = 80 + \dfrac{160000}{x^3} > 0 \quad \because x > 0$

$\therefore \quad f$ is minimum at $x = 10$

when $\quad x = 10, h = \dfrac{1000}{x^2} = 10$

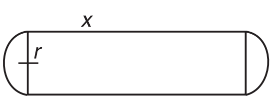
$\therefore \quad x = h = 10$

19. Let length $= x$ and radius of circle $= r$

Then, length of track = perimeter of the field.

$$= 2x + \pi r + \pi r$$
$$= 2x + 2\pi r = 440 \quad \text{(given)}$$

$\therefore \quad 2r = \dfrac{440 - 2x}{\pi}$ ...(i)



Area of rectangular portion;

$$f(x) = 2rx = \left(\frac{440 - 2x}{\pi}\right)\cdot x$$

$$f(x) = \frac{2}{\pi}(220x - x^2)$$

$\therefore \quad f'(x) = \dfrac{2}{\pi}(220 - 2x) = \dfrac{4}{\pi}(110 - x)$

For either a maximum or a minimum f, $f'(x) = 0$, if it exists.

$$f'(x) = 0, \text{ if } x = 110$$

Also $\quad f'(110 - h) = \dfrac{4}{\pi}[110 - (110 - h)] = \dfrac{4h}{\pi} > 0$, where $h > 0$ and sufficiently small. $\quad f'(110 + h) = \dfrac{4}{\pi}(-h) < 0 \quad \Rightarrow \quad f$ is maximum at $x = 110$

$\therefore$ Area is maximum for $x = 110$

Hence, $\quad 2r = \dfrac{440 - 2x}{\pi} = \dfrac{440 - 220}{22/7} = 70$

Thus, the lengths of the sides are 110 yards, 70 yards for maximum area.

20. Let x, h, l be the radius of the base, vertical height and slant height of the cone respectively.

Given, volume $= V = \frac{1}{3}\pi x^2 b$

$\therefore \quad h = \frac{3V}{\pi x^2}$ ...(i)

Surface area $= \pi x l = \pi x\sqrt{x^2 + h^2} = \pi x\sqrt{x^2 + \frac{9V^2}{\pi^2 x^4}}$

The value of x which makes surface area minimum will also make its square minimum and conversely.

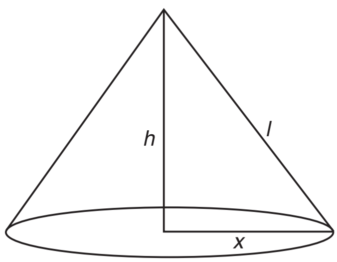
Hence we consider,

$$f(x) = (\text{Surface area})^2 = \pi^2 x^2\left(x^2 + \frac{9V^2}{\pi^2 x^4}\right) = \pi^2 x^4 + \frac{9V^2}{x^2}$$

$$\therefore \quad f'(x) = 4\pi^2 x^3 - \frac{18V^2}{x^3} = \frac{4\pi^2}{x^3}\left(x^6 - \frac{9V^2}{2\pi^2}\right)$$

$$f''(x) = 4\pi^2\left\{3x^2 + \frac{9V^2}{2\pi^2}\cdot\frac{3}{x^4}\right\}$$

For either a maximum or a minimum f, $f'(x) = 0$, if it exists.



$$f'(x) = 0 \text{ if } x^6 = \frac{9V^2}{2\pi^2}, f''(x) = +\text{ve}$$

$\Rightarrow \quad f$ has minimum when $x^6 = \frac{9V^2}{2\pi^2}$

Since, $\quad h = \frac{3V}{\pi x^2}$

$\Rightarrow \quad 9V^2 = h^2\pi^2 x^4$

$\therefore \quad x^6 = \frac{9V^2}{2\pi^2}$

$\Rightarrow \quad x^6 = \frac{h^2\pi^2 x^4}{2\pi^2}$

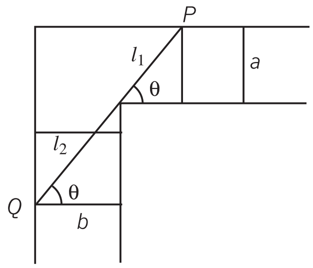
$\Rightarrow \quad h^2 = 2x^2$

$\Rightarrow \quad h = \sqrt{2}x$

Thus, for minimum surface area, we have height $= \sqrt{2}$ (radius of base)

21. Consider a line PQ of length $l_1 + l_2$ as shown in figure :

from the figure,



$$l_1 = \frac{a}{\sin\theta},\ l_2 = \frac{b}{\cos\theta}$$

$$\therefore \qquad f(\theta) = l_1 + l_2 = \frac{a}{\sin\theta} + \frac{b}{\cos\theta}$$

$$f'(\theta) = \frac{-a\cos\theta}{\sin^2\theta} + \frac{b\sin\theta}{\cos^2\theta} = \frac{b\sin^3\theta - a\cos^3\theta}{\sin^2\theta\cos^2\theta}$$

For maximum or minimum of f, $f'(\theta) = 0$ if it exists.

$$f'(\theta) = 0 \text{ if } b\sin^3\theta = a\cos^3\theta \ \ ie, \tan^3\theta = \frac{a}{b}$$

ie,
$$\tan\theta = \left(\frac{a}{b}\right)^{1/3}$$

$$\tan\theta < \left(\frac{a}{b}\right)^{1/3}$$

$$\Rightarrow \qquad \frac{\sin^3\theta}{\cos^3\theta} < \frac{a}{b}$$

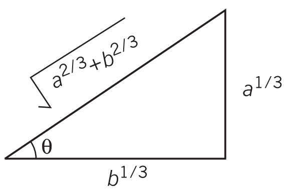
$$\Rightarrow \qquad b\sin^3\theta < a\cos^3\theta$$

$$\Rightarrow \qquad f'(\theta) = \frac{b\sin^3\theta - a\cos^3\theta}{\sin^2\theta\cos^2\theta} < 0$$

Similarly,
$$\tan\theta > \left(\frac{a}{b}\right)^{1/3} \quad \Rightarrow \quad f'(\theta) > 0$$

Thus, for all positions of line PQ, its minimum length is attained when $\tan\theta = \left(\frac{a}{b}\right)^{1/3}$. Since, the pipe must pass through all these positions PQ, the longest pipe that can be passed through must not exceed the minimum of PQ.

For $\tan\theta = \left(\frac{a}{b}\right)^{1/3}$, we have,



$$\sin\theta = \frac{a^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}},$$

$$\cos\theta = \frac{b^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}},$$

$$\therefore \qquad l_1 + l_2 = \frac{a}{\sin\theta} + \frac{b}{\cos\theta} = \frac{a\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}} + \frac{b\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}}$$

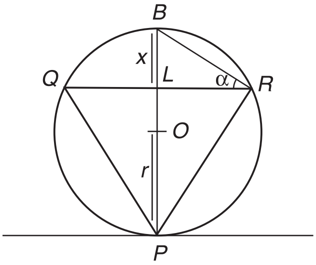
Hence, the longest pipe that can be carried round the band is,

$$l_1 + l_2 = \sqrt{a^{2/3}+b^{2/3}}\left[\frac{a}{a^{1/3}} + \frac{b}{b^{1/3}}\right] = (a^{2/3}+b^{2/3})^{3/2}$$

22. Let $BL = x$ and $PQ = r$

$$\therefore \qquad PL = 2r - x \text{ and } QR = 2LR$$

From $\Delta\, BLR$ and $\Delta\, PLR$



$$\frac{x}{LR} = \frac{LR}{PL} \quad\Rightarrow\quad \frac{x}{LR} = \frac{LR}{2r-x}$$

$$\Rightarrow \qquad LR^2 = (2r-x)\,x$$

$$\therefore \qquad QR^2 = 4x\,(2r-x)$$

Area of $\quad \Delta\, PQR = \dfrac{1}{2}\, QR \cdot PL$

$$(\text{Area})^2 = \frac{1}{4}(QR)^2 \cdot (PL)^2 = S \text{ (say)}$$

$$\therefore \qquad S = \frac{1}{4}\cdot 4x(2r-x)\cdot(2r-x)^2 = x\,(2r-x)^3$$

and $\quad \dfrac{dS}{dx} = (2r-x)^3 - 3x\,(2r-x)^2$

$$\therefore \qquad \frac{dS}{dx} = 0 \quad\Rightarrow\quad x = \frac{r}{2}$$

Now, $\quad \dfrac{d^2S}{dx^2} = 12\,(2r-x)(x-r) = -9r^2 < 0.$ For $x = \dfrac{r}{2}$

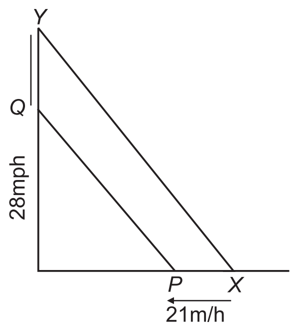
$\therefore$ Area is maximum when $x = \dfrac{r}{2}$ and $QR = \sqrt{3}r$, $PL = \dfrac{3}{2}r$

$\therefore$ Maximum area $= \dfrac{1}{2}\sqrt{3}\, r \cdot \dfrac{3}{2} r = \dfrac{3\sqrt{3}}{4} r^2$ sq units

23. Let two cars be at P and Q at time 't'

It the 1st car travels x ft,

then the 2nd car travels $\left(\dfrac{28}{21}x\right)$ ft in the same time.



$$\therefore \quad PQ^2 = d^2 = (1500-x)^2 + \left(2100 - \frac{4}{3}x\right)^2 = f(x)$$

$$\therefore \quad f'(x) = 2(1500-x)(-1) + 2\left(2100 - \frac{4x}{3}\right)^2 = (-4/3)$$

Let $\qquad f'(x) = 0$

$$\Rightarrow \qquad x = 1584$$

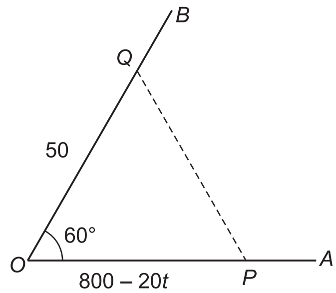
$\therefore$ The least distance between two cars

$$\Rightarrow \qquad d = \sqrt{(1500-1548)^2 + (2100-2064)^2}$$

$$d = \sqrt{(48)^2 + (36)^2} = 60 \text{ ft}$$

24. Let the positions of the car and runner be at P and Q respectively, after t seconds.

Then, $AP = 20$ ft and $OQ = 5\,t$



Hence, $OP = (800 - 20\,t)$ and by cosine rule applied to $\Delta\, OPQ$;

$$PQ^2 = OP^2 + OQ^2 - 2OP \cdot OQ \cos 60^\circ$$
$$= (800 - 20\,t)^2 + 25\,t^2 - 2 \cdot (800 - 20\,t)\, 5\,t \cdot \frac{1}{2}$$
$$= (800 - 20\,t)^2 + 25\,t^2 - 5(800\,t - 20\,t^2)$$

[**Note :** The values of t for which PQ is minimum will also make PQ^2 minimum and conversely.]

$\therefore \quad f(t) = PQ^2 = (800 - 20\,t)^2 + 25\,t^2 - 5\,(800\,t - 20\,t^2)$

$\therefore \quad f'(t) = 2\,(800 - 20\,t)(-20) + 50\,t - 5\,(800 - 40\,t)$

$$= 1050\,t - 36000$$

$$f'(t) = 1050\left(t - \frac{240}{7}\right)$$

for either a maximum or a minimum f, $f'(t) = 0$,

if it exists

$\therefore \quad f'(t) = 1050\left(t - \dfrac{240}{7}\right) = 0$

$\Rightarrow \quad t = \dfrac{240}{7}$ and $f''(t) = +\text{ve}$

$\Rightarrow \quad f$ has a minimum at $t = \dfrac{240}{7}$

Thus, the distance between car and runner is minimum at $t = \dfrac{240}{7}$ s.

25. Let the house of the swimmer be at B.

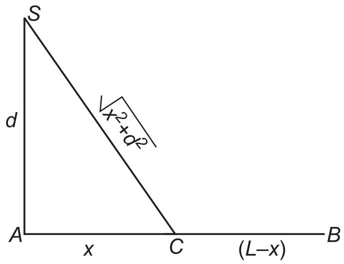
$\therefore \quad d\,(AB) = L$ km

Let the swimmer land at C, on the shore and let $d\,(AC) = x$ km

$\therefore \quad d\,(SC) = \sqrt{x^2 + d^2}$

and $\quad d\,(CB) = (L - x)$

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$



Time from S to B

$$= \text{time from } S \text{ to } C + \text{time from } C \text{ to } B.$$

$\therefore$ $$T = \frac{\sqrt{x^2 + d^2}}{u} + \frac{L - x}{v}$$

Hence, we take $f(x) = \frac{1}{u}\sqrt{x^2 + d^2} + \frac{L}{v} - \frac{x}{v}$

$\Rightarrow$ $$f'(x) = \frac{1}{u} \cdot \frac{1 \cdot 2x}{2\sqrt{x^2 + d^2}} + 0 - \frac{1}{v}$$

for either a maximum or minimum f, $f'(x) = 0$

$\Rightarrow$ $$f'(x) = \frac{x}{u\sqrt{x^2 + d^2}} - \frac{1}{v} = 0$$

ie, $$v^2x^2 = u^2(x^2 + d^2)$$

ie, if $$x^2(v^2 - u^2) = u^2d^2$$

ie, if $$x^2 = \frac{u^2d^2}{(v^2 - u^2)}$$

$\therefore$ $$f'(x) = 0 \text{ of } x = \pm \frac{ud}{\sqrt{v^2 - u^2}} \, (v > u)$$

But $$x \neq \frac{-ud}{\sqrt{v^2 - u^2}}$$

$\therefore$ We consider $$x = \frac{ud}{\sqrt{v^2 - u^2}}$$

$$f''(x) = \frac{1}{u}\left[\frac{\sqrt{x^2 + d^2} \cdot (1) - x \cdot \frac{2x}{2\sqrt{x^2 + d^2}}}{(\sqrt{x^2 + d^2})^2}\right] = \frac{1}{u}\left[\frac{d^2}{\sqrt{x^2 + d^2} \cdot (x^2 + d^2)}\right]$$

$$f''(x) = \frac{d^2}{u(x^2 + d^2)^{3/2}} > 0 \;\; \forall \, x$$

$\therefore$ f has minimum at $x = \frac{ud}{\sqrt{v^2 - u^2}}$

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