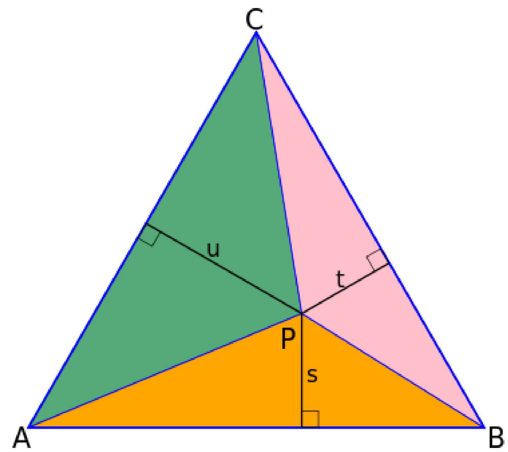
WIKIPEDIA

Viviani's theorem

Viviani's theorem, named after Vincenzo Viviani, states that the sum of the distances from *any* interior point to the sides of an equilateral triangle equals the length of the triangle's altitude.[1]



For any interior point P, the sum of the lengths *s* + *u* + *t* equals the height of the equilateral triangle.

Contents

Proof

This proof depends on the readily-proved proposition that the area of a triangle is half its base times its height—that is, half the product of one side with the altitude from that side.

Let ABC be an equilateral triangle whose height is h and whose side is a.

Let P be any point inside the triangle, and u, s, t the distances of P from the sides. Draw a line from P to each of A, B, and C, forming three triangles PAB, PBC, and PCA.

Now, the areas of these triangles are $\frac{u\cdot a}{2}$, $\frac{s\cdot a}{2}$, and $\frac{t\cdot a}{2}$. They exactly fill the enclosing triangle, so the sum of these areas is equal to the area of the enclosing triangle. So we can write:

$$\frac{u\cdot a}{2}+\frac{s\cdot a}{2}+\frac{t\cdot a}{2}=\frac{h\cdot a}{2}$$

and thus

$$u + s + t = h.$$

Q.E.D.

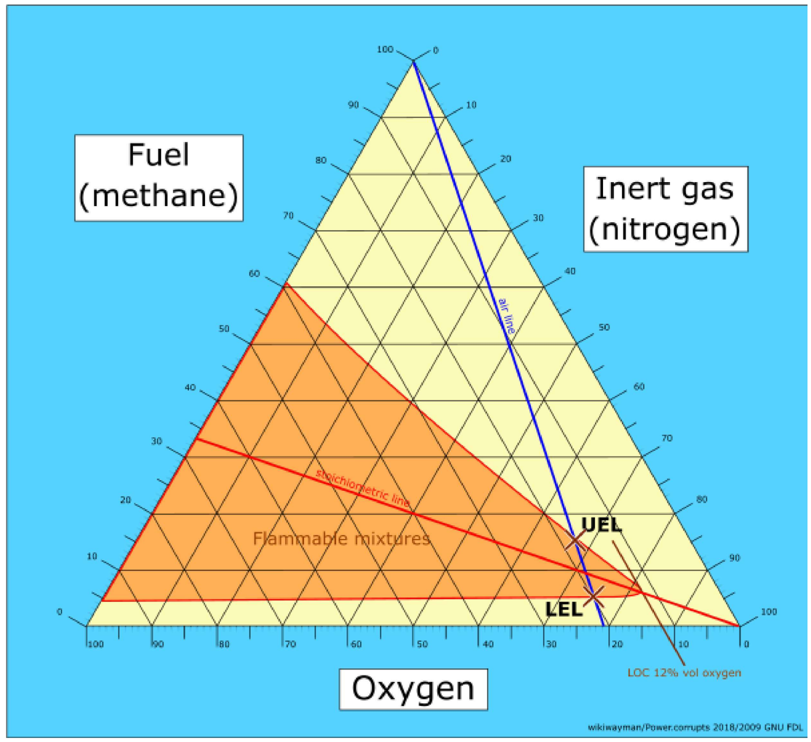
Converse

The converse also holds: If the sum of the distances from an interior point of a triangle to the sides is independent of the location of the point, the triangle is equilateral.[2]

Applications

Viviani's theorem means that lines parallel to the sides of an equilateral triangle give coordinates for making ternary plots, such as flammability diagrams.

More generally, they allow one to give coordinates on a regular simplex in the same way.



Flammability diagram for methane

Extensions

Parallelogram

The sum of the distances from any interior point of a parallelogram to the sides is independent of the location of the point. The converse also holds: If the sum of the distances from a point in the interior of a quadrilateral to the sides is independent of the location of the point, then the quadrilateral is a parallelogram.[2]

The result generalizes to any $2n$-gon with opposite sides parallel. Since the sum of distances between any pair of opposite parallel sides is constant, it follows that the sum of all pairwise sums between the pairs of parallel sides, is also constant. The converse in general is not true, as the result holds for an *equilateral* hexagon, which does not necessarily have opposite sides parallel.

Regular polygon

If a polygon is regular (both equiangular and equilateral), the sum of the distances to the sides from an interior point is independent of the location of the point. Specifically, it equals n times the apothem, where n is the number of sides and the apothem is the distance from the center to a side.[2][3] However, the converse does not hold; the non-square parallelogram is a counterexample.[2]

Equiangular polygon

The sum of the distances from an interior point to the sides of an equiangular polygon does not depend on the location of the point.[1]

Convex polygon

A necessary and sufficient condition for a convex polygon to have a constant sum of distances from any interior point to the sides is that there exist three non-collinear interior points with equal sums of distances.[1]

Regular polyhedron

The sum of the distances from any point in the interior of a regular polyhedron to the sides is independent of the location of the point. However, the converse does not hold, not even for tetrahedra.[2]

Notes

1. Abboud, Elias (2010). "On Viviani's Theorem and its Extensions". *College Mathematics Journal*. **43** (3): 16. arXiv:0903.0753v3 (https://arxiv.org/abs/0903.0753v3). doi:10.4169/074683410X488683 (https://doi.org/10.4169%2F074683410X488683).
2. Chen, Zhibo; Liang, Tian (2006). "The converse of Viviani's theorem". *The College Mathematics Journal*. **37** (5): 390. doi:10.2307/27646392 (https://doi.org/10.2307%2F27646392).
3. Pickover, Clifford A. (2009). *The Math Book*. Stirling. p. 150. ISBN 978-1402788291.

References

- Claudi Alsina, Roger B. Nelsen: *Charming Proofs: A Journey Into Elegant Mathematics*. MAA 2010, ISBN 9780883853481, p. 96 (*excerpt (Google) (https://books.google.com/books?id=mIT5-BN_L0oC&pg=PA96)*, p. 96, at Google Books)
- Weisstein, Eric W. "Viviani's Theorem" (http://mathworld.wolfram.com/VivianisTheorem.html). *MathWorld*.
- Li Zhou, Viviani Polytopes and Fermat Points (https://arxiv.org/abs/1008.1236)

External links

- "Viviani's Theorem: What is it?" (http://www.cut-the-knot.org/Curriculum/Geometry/Viviani.shtml). at Cut the knot.
- Warendorff, Jay. "Viviani's Theorem" (http://demonstrations.wolfram.com/VivianisTheorem/). the Wolfram Demonstrations Project.
- "Some generalizations of Viviani's theorem" (http://frink.machighway.com/~dynamicm/viviani-general.html). at Dynamic Geometry Sketches (http://dynamicmathematicslearning.com/JavaGSPLinks.htm), an interactive dynamic geometry sketch.
- "Clough's Theorem - a variation of Viviani's theorem and some generalizations" (http://frink.machighway.com/~dynamicm/clough.html). at Dynamic Geometry Sketches (http://dynamicmathematicslearning.com/JavaGSPLinks.htm), an interactive dynamic geometry sketch.
- "Some 3D Generalizations of Viviani's theorem" (http://frink.machighway.com/~dynamicm/disphenoid-viviani.html). at Dynamic Geometry Sketches (http://dynamicmathematicslearning.com/JavaGSPLinks.htm), an interactive dynamic geometry sketch.
- Gueron, Shay; Tessler, Ran (2002). "The Fermat-Steiner problem". *Am. Math. Monthly*. **109** (5): 443–451. doi:10.2307/2695644 (https://doi.org/10.2307%2F2695644).
- Samelson, Hans (2003). "Proof without words: Viviani's theorem with vectors". *Math. Mag*. **76** (3): 225. doi:10.2307/3219327 (https://doi.org/10.2307%2F3219327).
- Chen, Zhibo; Liang, Tian (2006). "The converse of Viviani's theorem". *The College Math. J*. **37** (5): 390–391.
- Kawasaki, Ken-Ichiroh; Yagi, Yoshihiro; Yanagawa, Katsuya (2005). "On Viviani's theorem in three dimensions" (https://www.jstor.org/stable/3621243). *Math. Gaz*. **89** (515): 283–287.
- Zhou, Li (2012). "Viviani polytopes and Fermat Points". *Coll. Math. J*. **43** (4): 309–312. doi:10.4169/college.math.j.43.4.309 (https://doi.org/10.4169%2Fcollege.math.j.43.4.309).
- De Villiers, Michael (2013). "3D generalisations of Viviani's theorem". *Math. Gaz*. **97** (540). doi:10.1017/S0025557200000188 (https://doi.org/10.1017%2FS0025557200000188).
- Abboud, Elias (2017). "Loci of points inspired by Viviani's theorem". arXiv:1701.07339 (https://arxiv.org/abs/1701.07339).

- Armstrong, Addie; McQuillan, Dan (2017). "Specialization, generalization, and a new proof of Viviani's theorem". arXiv:1701.01344 (https://arxiv.org/abs/1701.01344).
