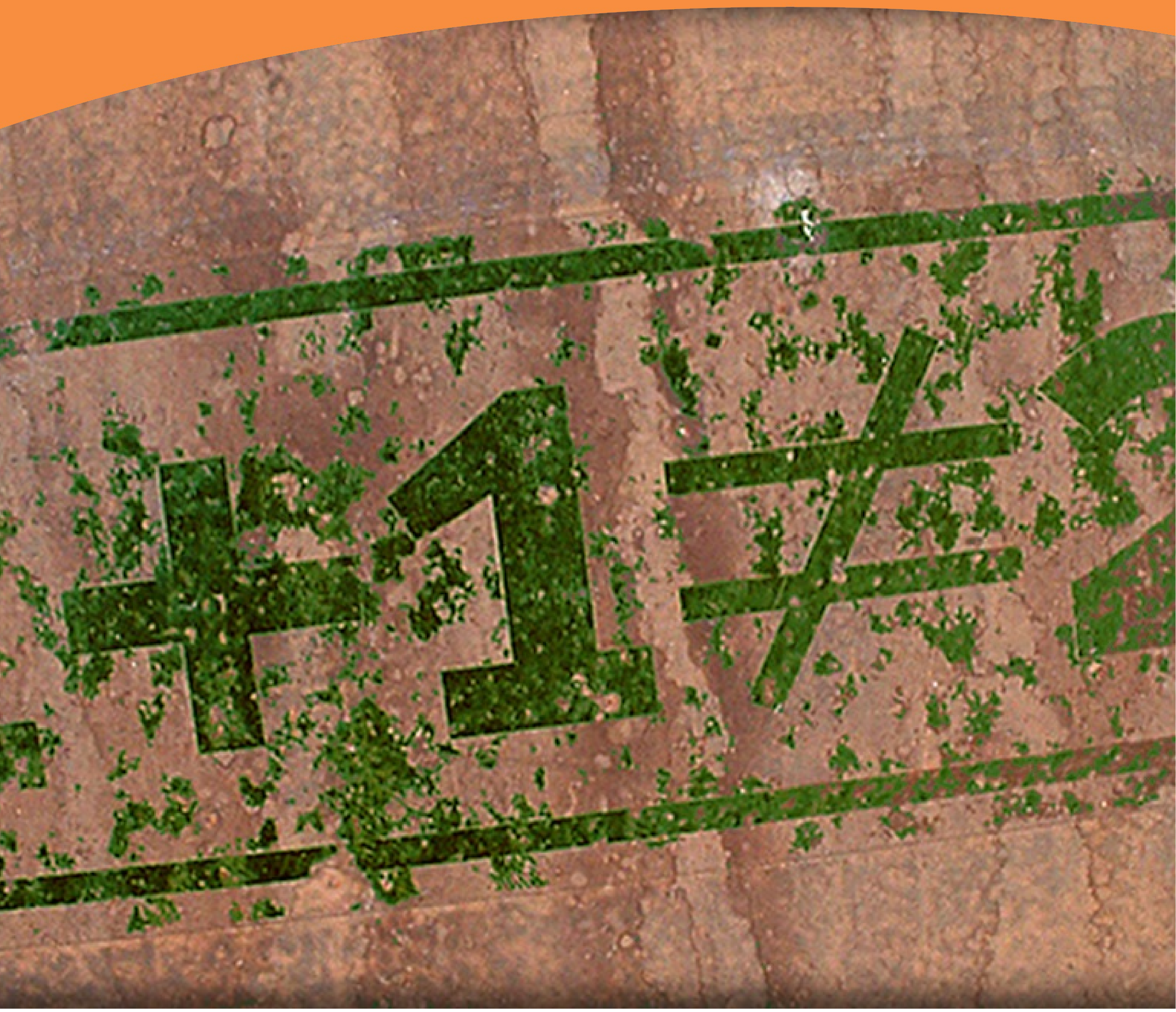
Complex Functions Examples c-8

Some Classical Transforms

Leif Mejlbro



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Complex Functions Examples c-8 – Some Classical Transforms

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Contents

Introduction

This is the eighth book containing examples from the *Theory of Complex Functions*. In this volume we show how we can apply the calculations of residues in connection with some classical transforms like the *Laplace transform*, the *Mellin transform*, the $\mathfrak{z}$-*transform* and the *Fourier transform*. I have further supplied with some examples from the *Theory of Linear Difference Equations* and from the *Theory of Distributions*, also called *generalized functions*.

Even if I have tried to be careful about this text, it is impossible to avoid errors, in particular in the first edition. It is my hope that the reader will show some understanding of my situation.

Leif Mejlbro
22nd June 2008

1 Some theoretical background

1.1 The Laplace transform

In the elementary Calculus one introduces the class $\mathcal{E}$ of *piecewise continuous and exponentially bounded functions* $f : [0,+\infty[\to \mathbb{C}$ as the class of such functions, for which there exist constants $A > 0$ and $B \in \mathbb{R}$, such that

$$|f(t)| \le A\, e^{Bt} \qquad \text{for every } t \in [0,+\infty[.$$

If f is exponentially bounded, we put

$$\varrho(f) := \inf\left\{B \in \mathbb{R} \mid \text{there exists an } A > 0, \text{ such that } |f(t)| \le A\, e^{Bt} \text{ for every } t \ge 0\right\}.$$

This class of functions is sufficient for most of the applications in practice. On the other hand, it is easy to extend the theory to the larger class of functions with mathematically better properties, defined in the following way:

Definition 1.1 *A function* $f : [0,+\infty[\to \mathbb{C}^* := \mathbb{C} \cup \{\infty\}$ *belongs to the class of functions* $\mathcal{F}$, *if there exists a constant* $\sigma \in \mathbb{R}$, *such that*

$$\int_0^{+\infty} |f(t)|\, e^{-\sigma t}\, dt < +\infty.$$

When $f \in \mathcal{F}$, *we define the* radius of convergence *by*

$$\sigma(f) := \inf\left\{\sigma \in \mathbb{R} \;\middle|\; \int_0^{+\infty} |f(t)|\, e^{-\sigma t}\, dt < +\infty\right\}.$$

It is easy to prove that $\mathcal{E} \subset \mathcal{F}$, and that the function $f(x) = 1/\sqrt{x}$ for $x > 0$, and $f(0) = 0$ lies in $\mathcal{F}$, and not in $\mathcal{E}$, so $\mathcal{F}$ is indeed an extension of the class $\mathcal{E}$.

Definition 1.2 *We define the* Laplace transformed $\mathcal{L}\{f\}$ *of a function* $f \in \mathcal{F}$ *as the complex function*

$$\mathcal{L}\{f\}(z) = \int_0^{+\infty} e^{-zt} f(t)\, dt,$$

where z belongs to the set of complex numbers, for which the improper integral on the right hand side is convergent.

Remark 1.1 One usually denotes the complex variable by s. However, in order to underline the connection with the *Theory of Complex Functions* we here write z instead. $\Diamond$

The purpose of these definitions is that we have the following theorem:

Theorem 1.1 *Assume that $f \in \mathcal{F}$. Then the integral representation of $\mathcal{L}\{f\}(z)$ is convergent for* $\text{Re}\, z > \sigma(f)$ *and divergent for* $\text{Re}\, z < \sigma(f)$.
The function $\mathcal{L}\{f\}$ is analytic in (at least) the open half plane $\text{Re}\, z > \sigma(f)$, *and its derivative is obtained in this set by differentiating below the sign of integral*

$$\frac{d}{dz}\, \mathcal{L}\{f\}(z) = -\int_0^{+\infty} t\, e^{-zt} f(t)\, dt.$$

If furthermore, $f \in \mathcal{E}$, then $\sigma(f) \leq \varrho(f)$.

We shall here assume the well-known rules of calculations of the Laplace transform. What is new here is that we in some cases are able to compute the *inverse Laplace transform* of an analytic function by a *residuum formula*, which will reduce the practical computation considerably.

First we perform a small natural extension. If $\mathcal{L}\{f\}(z)$, which is defined for $\text{Re}\, z > \sigma(f)$, can be extended analytically to a function $F(z)$ in a bigger domain Ω, then $F(z)$ is also called a Laplace transformed of $f(t)$, even if $F(z)$ does not have a representation as a convergent integral in all its points of definition. Then the following theorem makes sense:

Theorem 1.2 Complex inversion formula for the Laplace transform by a residuum formula. *Assume that $F(z)$ is analytic in a set of the form $\mathbb{C} \setminus \{z_1, \dots, <_n\}$. If there exist positive constants M, R and $a > 0$, such that we have the estimate*

$$|F(z)| \leq \frac{M}{|z|^a} \qquad \text{for } |z| \geq R,$$

then $F(z)$ has an inverse Laplace transformed function $f(t)$, given by

$$f(t) = \mathcal{L}^{-1}\{F\}(t) = \sum_{j=1}^{n} \text{res}\left(e^{zt}\, F(z); z_j\right), \qquad \text{for } t > 0.$$

Conversely, this constructed function $f \in \mathcal{F}$ satisfies

$$\mathcal{F}\{f\}(z) = \int_0^{+\infty} e^{-zt} f(t)\, dt = F(z) \qquad \text{for } \text{Re}\, z > \sigma(f),$$

where $\sigma(f) = \max_{j=1,\dots,n} \text{Re}\, z_j$.

Remark 1.2 This theorem is particular useful in e.g. the *Theory of Cybernetics* and in the *Theory of Circuits*, where a typical task is to find the inverse Laplace transformed of a rational function with a zero of at least first order in ∞. Also, this residuum formula may be an alternative to the usual use of tables. $\Diamond$

A particular simple example of a residuum formula is given by:

Theorem 1.3 HEAVISIDE'S EXPANSION THEOREM. *Assume that $P(z)$ and $Q(z)$ are two polynomials, where the degree of the polynomial of the denominator $Q(z)$ is strictly larger than the degree of the polynomial of the numerator $P(z)$. If $Q(z)$ only has simple zeros $z_1, \dots, z_n$, then the inverse Laplace transformed of $F(z) = \dfrac{P(z)}{Q(z)}$ is given by*

$$f(t) = \sum_{j=1}^{n} \frac{P\left(z_j\right)}{Q'\left(z_j\right)} \exp\left(z_j t\right), \qquad \text{for } t > 0,$$

and $\sigma(f) = \max_{j=1,\dots,n} \operatorname{Re} z_j$.

1.2 The Mellin transform

The Mellin transform is closely connected with the Laplace transform. Assuming that the integrals are convergent, we define the *Mellin transform* of a function f by

$$\mathcal{M}\{f\}(a) := \int_0^{+\infty} f(x)\, x^a \, \frac{dx}{x} = \int_{-\infty}^{+\infty} f\left(e^{-t}\right) e^{-at} \, dt,$$

where the latter integral is the *two-sided Laplace transformed* of the function $g(t) := f\left(e^{-t}\right)$, generated at the point a. We may therefore also here expect a residuum formula:

Theorem 1.4 *Assume that f is analytic in the set $\mathbb{C} \setminus \{z_1, \dots, x_n\}$, where none of the numbers z_j, $j = 1, \dots, n$, is a real and positive number, $z_j \notin \mathbb{R}_+$.*
If there exist real constants $\alpha < \beta$ and C, R_0, $r_0 > 0$, such that the following estimates hold

$$|z^\alpha f(z)| \le C \qquad \text{for } |z| \le r_0 \text{ and } z \ne z_j,\ j = 1, \dots, n,$$

$$\left|z^\beta f(z)\right| \le C \qquad \text{for } |z| \ge R_0 \text{ and } z \ne z_j,\ j = 1, \dots, n,$$

then the Mellin transformed is convergent for every $a \in]\alpha, \beta[\setminus \mathbb{Z}$, and the value is given by

$$\mathcal{M}\{f\}(a) := \int_0^{+\infty} f(z)\, x^a \, \frac{dx}{x} = -\frac{\pi \exp(-\pi i a)}{\sin \pi a} \sum_{z_j \ne 0} \operatorname{res}\left(f(z) z^{a-1}; z_j\right),$$

where we define

$$z^a := \exp\left(a \operatorname{Log}_0 z\right) \qquad \text{for } z \in \mathbb{C} \setminus \left(\mathbb{R}_+ \cup \{0\}\right),$$

and

$$\operatorname{Log}_0 z := \ln |z| + i \operatorname{Arg}_0 z, \qquad \operatorname{Arg}_0 z \in]0, 2\pi[, \qquad z \notin \mathbb{R}_+ \cup \{0\}.$$

1.3 The $\mathfrak{z}$-transform

Definition 1.3 *Given a continuous function $f(t)$ defined for $t \geq 0$. Assume that*

$$R = \limsup_{n \to +\infty} \sqrt[n]{|f(nT)|} < +\infty.$$

The $\mathfrak{z}$-transformed $\mathfrak{z}_T\{f\}(z)$ of f with the sample interval T is defined as the analytic function (a Laurent series)

$$\mathfrak{z}_T\{f\}(z) := \sum_{n=0}^{+\infty} f(nT)\, z^{-n} \qquad \text{for } |z| > R.$$

Let $H(t)$ denote the *Heaviside function*, defined by

$$H(t) = \begin{cases} 1 & \text{for } t \geq 0, \\ 0 & \text{for } t < 0. \end{cases}$$

Then the $\mathfrak{z}$-transformed of H is the important

$$\mathfrak{z}_T\{H\}(z) = \frac{z}{z-1}, \qquad |z| > 1,$$

which is independent of the sample interval T.

Definition 1.4 *Assume that $(a_n)_{n \geq 0}$ is a sequence, for which*

$$R = \limsup_{n \to +\infty} \sqrt[n]{|a_n|} < +\infty.$$

We defined the $\mathfrak{z}$-transformed of the sequence as the following analytic function which is defined outside a disc (again a Laurent series),

$$\mathfrak{z}\left\{a_n\right\}(z) := \sum_{n=0}^{+\infty} a_n\, z^{-n}, \qquad |z| > R.$$

One may consider the $\mathfrak{z}$-transform as a discrete Laplace transform, and we have quite similar rules of computations for the two transforms. These are not given her. Instead we mention that we have a simple residuum formula for the inverse $\mathfrak{z}$-transformed of some analytic functions:

Theorem 1.5 *Assume that $F(z)$ is analytic in $\mathbb{C} \setminus \{z_1, \dots, z_n\}$. Then $F(z)$ has an inverse $\mathfrak{z}$-transformed. If the sample interval is $T > 0$, then this inverse $\mathfrak{z}$-transformed is given by*

$$f(nT) = \sum_{k=0}^{n} \operatorname{res}\left(F(z)\, z^{n-1}; z_k\right) \qquad \text{for } n \in \mathbb{N}.$$

1.4 Linear difference equations of second order and of constant coefficients

The following theorem is similar to a theorem for ordinary linear differential equations of second order with constant coefficients,

Theorem 1.6 *Let* (x_n) *denote any particular solution of the linear and inhomogeneous difference equation of second order and of constant coefficients,*

$$x_{n+2} + c_1\, x_{n+1} + c_0\, x_n = a_n, \qquad n \in \mathbb{N}_0.$$

The complete solution of this equation is obtained by adding to (x_n) *the complete solution* (y_n) *of the corresponding homogeneous equation*

$$y_{n+2} + c_1\, y_{n+1} + c_0\, y_n = 0, \qquad n \in \mathbb{N}_0.$$

It should not be surprising that we have

Theorem 1.7 *Assume that $\lambda^2 + c_1\lambda + c_0 = (\lambda-\alpha)(\lambda-\beta)$, where $\alpha \neq \beta$. The complete solution of the homogeneous difference equation*

$$y_{n+2} + c_1\, y_{n+1} + c_0\, y_n = 0, \qquad n \in \mathbb{N}_0,$$

is given by

$$y_n = A \cdot \alpha^n + B \cdot \beta^n, \qquad n \in \mathbb{N}_0,$$

where A and B are arbitrary constants.
If instead $\alpha = \beta$, then the complete solution is given by

$$y_n = A \cdot \alpha^n + B \cdot n\, \alpha^n, \qquad n \in \mathbb{N}_0,$$

where A and B are arbitrary constants.

We can find a particular solution by using the $\mathfrak{z}$-transform.

Theorem 1.8 *Let $(a_n)_{n\geq 0}$ be a complex sequence with the $\mathfrak{z}$-transformed $A(z)$. For given initial conditions x_0, $x_1 \in \mathbb{C}$ the uniquely determined solution of the difference equation*

$$x_{n+2} + c_1\, x_{n+1} + c_0\, x_n = a_n, \qquad n \in \mathbb{N}_0,$$

is given by the sequence $(x_n)_{n\geq 0}$, which is the inverse $\mathfrak{z}$-transformed of

$$X(z) = \frac{A(z) + x_0\, z^2 + (c_1\, x_0 + x_1)\, z}{z^2 + c_1\, z + c_0}.$$

2 The Laplace transform

Example 2.1 *Prove that*

$$
\begin{array}{lll}
(a) & \mathcal{L}\{1\}(z)=\dfrac{1}{z}, & Re(z)>0,\\
(b) & \mathcal{L}\left\{t^n\right\}=\dfrac{n!}{z^{n+1}}, & Re(z)>0,\\
(c) & \mathcal{L}\left\{e^{a\,t}\right\}=\dfrac{1}{z-a}, & Re(z)>Re(a),\\
(d) & \mathcal{L}\{\sin at\}=\dfrac{a}{z^2+a^2}, & Re(z)>|Im(z)|,\\
(e) & \mathcal{L}\{\cos at\}(z)=\dfrac{z}{z^2+a^2}, & Re(z)>|Im(z)|,\\
(f) & \mathcal{L}\{\sinh at\}(z)=\dfrac{a}{z^2-a^2}, & Re(z)>|Re(a)|,\\
(g) & \mathcal{L}\{\cosh at\}(z)=\dfrac{z}{z^2-a^2}, & Re(z)>|Re(a)|.
\end{array}
$$

We shall use the definition

$$\mathcal{L}\{f\}(z)=\int_0^{+\infty} e^{-zt}\,f(t)\,dt$$

of the Laplace transform of f.

(a) If $\mathrm{Re}(z)>0$, then

$$\mathcal{L}\{1\}(z)=\int_0^{+\infty} e^{-zt}\,dt=\lim_{R\to+\infty}\left[-\frac{1}{z}\,e^{-zt}\right]_0^R=\frac{1}{z}-\frac{1}{z}\lim_{R\to+\infty}e^{-R\,x-i\,R\,y}=\frac{1}{z},$$

because $\left|e^{-z\,R}\right|=e^{-R\cdot z}\to 0$ for $R\to+\infty$, når $x>0$.

(b) The integral is convergent for $\mathrm{Re}(z)>0$. Assuming this, it follows by a partial integration and a recursion,

$$
\begin{aligned}
\mathcal{L}\left\{t^n\right\}(z) &= \int_0^{+\infty} t^n e^{-zt}\,dt=\left[-\frac{1}{z}\,t^n\,e^{-zt}\right]_0^{+\infty}+\frac{n}{z}\int_0^{+\infty} t^{n-1}\,e^{-zt}\,dt\\
&= \frac{n}{z}\,\mathcal{L}\left\{t^{n-1}\right\}(z)=\cdots=\frac{n!}{z^n}\,\mathcal{L}\{1\}(z)=\frac{n!}{z^{n+1}}.
\end{aligned}
$$

(c) If $\mathrm{Re}(z)>\mathrm{Re}(a)$, then

$$\mathcal{L}\left\{e^{at}\right\}(z)=\int_0^{+\infty} e^{at}e^{-zt}\,dt=\int_0^{+\infty} e^{-(z-a)t}\,dt=\mathcal{L}\{1\}(z-a)=\frac{1}{z-a}.$$

(d) If $\mathrm{Re}(z)>|\mathrm{Im}(a)|$, then it follows from Euler's formulæ and **(c)**,

$$
\begin{aligned}
\mathcal{L}\{\sin at\} &= \frac{1}{2i}\,\mathcal{L}\left\{e^{iat}\right\}(z)-\frac{1}{2i}\,\mathcal{L}\left\{e^{-iat}\right\}(z)\\
&= \frac{1}{2}\left\{\frac{1}{z-ia}+\frac{1}{z+ia}\right\}=\frac{1}{2}\cdot\frac{2z}{z^2+a^2}=\frac{z}{z^2+a^2}.
\end{aligned}
$$

(e) In the same way we get for $\mathrm{Re}(z) > |\mathrm{Im}(a)|$ that

$$\begin{aligned}\mathcal{L}\{\cos at\}(z) &= \frac{1}{2}\,\mathcal{L}\left\{e^{iat}\right\}(z) + \frac{1}{2}\,\mathcal{L}\left\{e^{-iat}\right\}(z)\\ &= \frac{1}{2}\left\{\frac{1}{z-ia}+\frac{1}{z+ia}\right\} = \frac{1}{2}\cdot\frac{2z}{z^2+a^2} = \frac{z}{z^2+a^2}.\end{aligned}$$

(f) In the same way we get for $\mathrm{Re}(z) > |\mathrm{Re}(z)|$ that

$$\begin{aligned}\mathcal{L}\{\sinh at\}(z) &= \frac{1}{2}\,\mathcal{L}\left\{e^{at}\right\}(z) - \frac{1}{2}\,\mathcal{L}\left\{e^{-at}\right\}(z)\\ &= \frac{1}{2}\left\{\frac{1}{z-a}-\frac{1}{z+a}\right\} = \frac{1}{2}\cdot\frac{2a}{z^2-a^2} = \frac{a}{z^2-a^2}.\end{aligned}$$

(g) In the same way we get for $\mathrm{Re}(z) > |\mathrm{Re}(a)|$ that

$$\begin{aligned}\mathcal{L}\{\cosh at\}(z) &= \frac{1}{2}\,\mathcal{L}\left\{e^{at}\right\}(z) + \frac{1}{2}\,\mathcal{L}\left\{e^{-at}\right\}(z)\\ &= \frac{1}{2}\left\{\frac{1}{z-a}+\frac{1}{z+a}\right\} = \frac{1}{2}\cdot\frac{2z}{z^2-a^2} = \frac{z}{z^2-a^2}.\end{aligned}$$

Remark 2.1 We note that the assumptions of convergence of the integrals are necessary in all cases. $\Diamond$

Example 2.2 *Find the Laplace transform of* $e^t\cos t$ *and of* $e^{2t}\cos 2t$.

If $\mathrm{Re}(z) > 1$, then

$$\begin{aligned}\mathcal{L}\left\{e^t\cos t\right\}(z) &= \int_0^{+\infty} e^t\cos t e^{-zt}\,dt = \int_0^{+\infty}\cot t\cdot e^{-(z-1)t}dt\\ &= \mathcal{L}\{\cos t\}(z-1) = \frac{z-1}{(z-1)^2+1}.\end{aligned}$$

If $\mathrm{Re}(z) > 2$, then

$$\begin{aligned}\mathcal{L}\left\{e^{2t}\cos 2t\right\}(z) &= \int_0^{+\infty} e^{2t}\cos 2t\, e^{-zt}\,dt = \int_0^{+\infty}\cos 2t\cdot e^{-(z-2)t}\,dt\\ &= \mathcal{L}\{\cos 2t\}(z-2) = \frac{z-2}{(z-2)^2+2^2}.\end{aligned}$$

Alternatively it follows by the rule of similarity, where $k=\dfrac{1}{2}$,

$$\begin{aligned}\mathcal{L}\left\{e^{2r}\cos 2t\right\}(z) &= \mathcal{L}\left\{\exp\left(\frac{t}{\frac{1}{2}}\right)\cos\left(\frac{t}{\frac{1}{2}}\right)\right\}(z) = \frac{1}{2}\,\mathcal{L}\left\{e^t\cos t\right\}\left(\frac{1}{2}\,z\right)\\ &= \frac{1}{2}\cdot\frac{\frac{z}{2}-1}{\left(\frac{z}{2}-1\right)^2+1} = \frac{z-2}{(z-2)^2+4}.\end{aligned}$$

Example 2.3 *Find the Laplace transform of the function*

$$f(t)=\begin{cases}1 & \text{for } t\in[0,1],\\[2ex] 0 & \text{ellers.}\end{cases}$$

By the definition,

$$\mathcal{L}\{f\}(z)=\int_0^{+\infty} f(t)\,e^{-zt}\,dt=\int_0^a e^{-zt}\,dt=\begin{cases}\dfrac{1}{z}\left(1-e^{-az}\right), & z\neq 0,\\[2ex] a, & z=0.\end{cases}$$

Example 2.4 *Which of the following functions has a Laplace transform?*

$$(a)\quad \frac{1}{1+t},\qquad (b)\quad \exp\left(t^2-1\right),\qquad (c)\quad \cos\left(t^2\right).$$

(a) It follows from

$$0<\frac{1}{1+t}\le t \qquad \text{for } t\ge 0,$$

and from the continuity of the function, that the Laplace transform exists.

Remark 2.2 If we put

$$f(z) = \left\{\frac{1}{1+t}\right\}(z) = \int_0^{+\infty} \frac{e^{-zt}}{1+t}\,dt, \qquad \mathrm{Re}(z) < 0,$$

then we get by differentiating with respect to the parameter z under the sign of integration,

$$f'(z) = \int_0^{+\infty} \frac{-t\,e^{-zt}}{1+t}\,dt,$$

hence

$$f'(z) - f(z) = \int_0^{+\infty} \frac{-t-1}{1+t}\,e^{-zt}\,dt = -\frac{1}{z}.$$

This will give us the structure of the Laplace transform,

$$f(z) = C\,e^z - e^z \int \frac{e^{-z}}{z}\,dz,$$

where we still shall find the constant C. The integral is of a type, which cannot be expressed by elementary functions, so the example shows that even if we can prove that the Laplace transform exists, it is not always possible to find an exact expression for it. $\Diamond$

(b) We have for every $\sigma \in \mathbb{R}$,

$$\int_0^{+\infty} \exp\left(t^2 - 1\right)\,e^{-\sigma t}\,dt = +\infty,$$

so the Laplace transform is not defined.

(c) It follows from

$$\left|\cos\left(t^2\right)\right| \le 1 \qquad \text{for every } t,$$

and from the continuity of the function that the Laplace transform exists. We cannot either in this case express the Laplace transform as an elementary function.

Example 2.5 *Given $f(t) = \min\{t,1\}$ for $t \ge 0$. Sketch the graph of f. Then find $\mathcal{L}\{f\}(z)$ and $\sigma(f)$.*

By the definition,

$$\begin{aligned}\mathcal{L}\{f\}(z) &= \int_0^1 t\,e^{-zt}\,dt + \int_1^{+\infty} e^{-zt}\,dt = \left[-\frac{1}{z}\,t\,e^{-zt}\right]_0^1 + \frac{1}{z}\int_0^1 e^{-zt}\,dt + \int_0^{+\infty} e^{-z(t+1)}\,dt \\ &= -\frac{1}{z}\,e^{-z} - \frac{1}{z^2}\,\left[e^{-zt}\right]_0^1 + e^{-z}\cdot\frac{1}{z} = \frac{1}{z^2}\,\left(1 - e^{-z}\right),\end{aligned}$$

where the improper integral is convergent, if and only if $\mathrm{Re}(z) > 0$, so $\sigma(f) = 0$.

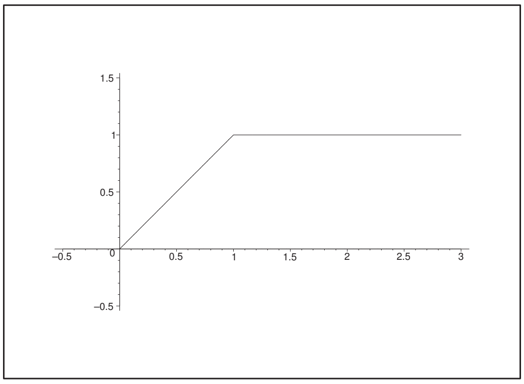

Figure 1: The graph of $f(t) = \min\{t, 1\}$ for $t \geq 0$.

Example 2.6 *Find the Laplace transform of*

$$(a)\ \ (\sin t - \cos t)^2, \qquad (b)\ \ \cosh^2 4t, \qquad (c)\ \ \left(5e^{2t} - 3\right)^2.$$

The function belongs in all three cases to the class $\mathbb{E}$. The rest follows by some simple computations.

(a) We get

$$(\sin t - \cos t)^2 = \sin^2 t + \cos^2 t - 2\,\sin t\,\cos t = 1 - \sin 2t.$$

Hence

$$\mathcal{L}\left\{(\sin t - \cos t)^2\right\}(z) = \mathcal{L}\{1 - \sin 2t\}(z) = \frac{1}{z} - \frac{2}{z^2+4} = \frac{z^2 - 2z + 4}{z\,(z^2+4)}.$$

(b) Now,

$$\cosh^2 4t = \frac{1}{2}\,\cosh 8t + \frac{1}{2},$$

so we get in a similar way for $\mathrm{Re}(z) > 8$ that

$$\mathcal{L}\left\{\cosh^2 4t\right\}(z) = \frac{1}{2}\,\mathcal{L}\{1 + \cosh 8t\}(z) = \frac{1}{2}\left\{\frac{1}{z} + \frac{z}{z^2 - 64}\right\} = \frac{z^2 - 32}{z\,(z^2 - 64)}.$$

(c) In this case we have for $\mathrm{Re}(z) > 4$,

$$\begin{aligned}\mathcal{L}\left\{\left(5\,e^{2t} - 3\right)^2\right\}(z) &= \mathcal{L}\left\{25\,e^{4t} - 30\,e^{2t} + 9\right\}(z) = \frac{25}{z-4} - \frac{30}{z-2} + \frac{9}{z} \\ &= \frac{4z^2 + 16z + 72}{z(z-4)(z-2)}.\end{aligned}$$

Example 2.7 *Find the Laplace transform of*

$$f(t)=\begin{cases}\sin t & \textit{for } t\in[0,\pi],\\[2mm] 0 & \textit{ellers.}\end{cases}$$

If $z \neq \pm i$, then it follows from the definition,

$$\begin{aligned}\mathcal{L}\{f\}(z) &= \int_0^{\pi}\sin t\cdot e^{-zt}\,dt=\frac{1}{2i}\int_0^{\pi}e^{-(z-i)t}\,dt-\frac{1}{2i}\int_0^{\pi}e^{-(z+i)t}\,dt=\frac{1}{2i}\left[\frac{e^{-(z-i)t}}{z+i}-\frac{e^{-(z+i)t}}{z-i}\right]_0^{\pi}\\ &= \frac{1}{2i}\left(e^{-z\pi}-1\right)\cdot\left\{\frac{1}{z-i}-\frac{1}{z+i}\right\}=\frac{e^{-z\pi}-1}{z^2+1}.\end{aligned}$$

We have removable singularities at $z_0=\pm i$, so

$$\mathcal{L}\{f\}\left(z_0\right)=\lim_{z\to z_0}\frac{-\pi\,e^{-z\pi}}{2z}=\frac{\pi}{2z_0},$$

and the result above is supplied with

$$\mathcal{L}\{f\}(i)=\frac{\pi}{2i}=-\frac{\pi i}{2}\qquad\text{og}\qquad\mathcal{L}\{f\}(-i)=\frac{\pi i}{2}.$$

Example 2.8 *Find the Laplace transform of*

$$(a)\;\; t^2\cos^2 t,\qquad (b)\;\; \left(t^2-3t+2\right)\sin 3t.$$

(a) Since

$$t^2\cos^2 t=\frac{1}{2}\,t^2\,(1+\cos 2t),$$

we get for $\text{Re}(z)>0$,

$$\begin{aligned}&\mathcal{L}\left\{t^2\cos^2 t\right\}(z)=\frac{1}{2}\,\mathcal{L}\left\{t^2\right\}(z)+\frac{1}{2}\,\mathcal{L}\left\{t^2\cos 2t\right\}(z)=\frac{1}{2}\cdot\frac{2!}{z^3}+\frac{1}{2}\,(-1)^2\,\frac{d^2}{dz^2}\,\mathcal{L}\{\cos 2t\}(z)\\ &= \frac{1}{z^3}+\frac{1}{2}\,\frac{d^2}{dz^2}\left\{\frac{z}{z^2+4}\right)=\frac{1}{z^3}+\frac{1}{2}\,\frac{d}{dz}\left(\frac{1}{z^2+4}-\frac{2z^2}{\left(z^2+4\right)^2}\right)\\ &= \frac{1}{z^3}+\frac{1}{2}\left\{-\frac{2z}{\left(z^2+4\right)^2}-\frac{4z}{\left(z^2+4\right)^2}+\frac{2\cdot 2z^2\cdot 2z}{\left(z^2+4\right)^3}\right\}\\ &= \frac{1}{z^3}+\frac{4z^3}{\left(z^2+4\right)^3}-\frac{3z}{\left(z^2+4\right)^2}=\frac{1}{z^3}+\frac{4z^3-3z\left(z^2+4\right)}{\left(z^2+4\right)^3}=\frac{1}{z^3}+\frac{z^3-12z}{\left(z^2+4\right)^3}.\end{aligned}$$

(b) Analogously,

$$
\begin{aligned}
\mathcal{L}\left\{\left(t^2-3t+2\right)\sin 3t\right\}(z) &= \mathcal{L}\left\{t^2\sin 3t\right\}(z)+3\,\mathcal{L}\{-t\,\sin 3t\}(z)+2\,\mathcal{L}\{\sin 3t\}(z)\\
&= (-1)^2\,\frac{d^2}{dz^2}\,\mathcal{L}\{\sin 3t\}(z)+3\,\frac{d}{dz}\,\mathcal{L}\{\sin 3t\}(z)+2\,\mathcal{L}\{\sin 3t\}(z)\\
&= \frac{d^2}{dz^2}\left\{\frac{3}{z^2+9}\right\}+3\,\frac{d}{dz}\left\{\frac{3}{z^2+9}\right\}+2\,\frac{3}{z^2+9}=\frac{d}{dz}\left\{-\frac{6z}{\left(z^2+9\right)^2}\right\}-\frac{18z}{\left(z^2+9\right)^2}+\frac{6}{z^2+9}\\
&= -\frac{6}{\left(z^2+9\right)^2}+\frac{12z\cdot 2z}{\left(z^2+9\right)^3}-\frac{18z}{\left(z^2+9\right)^2}+\frac{6}{z^2+9}\\
&= \frac{6}{\left(z^2+9\right)^3}\left\{-\left(z^2+9\right)+4z^2-3z\left(z^2+9\right)+\left(z^2+9\right)^2\right\}\\
&= \frac{6}{\left(z^2+9\right)^3}\left\{z^4-3z^3+21z^2-27z+72\right\}.
\end{aligned}
$$

Example 2.9 *Find the Laplace transform of*

$$f(t) = \begin{cases} \cos t, & t \in [0, \pi[, \\ \\ \sin t, & t \in [\pi, +\infty[. \end{cases}$$

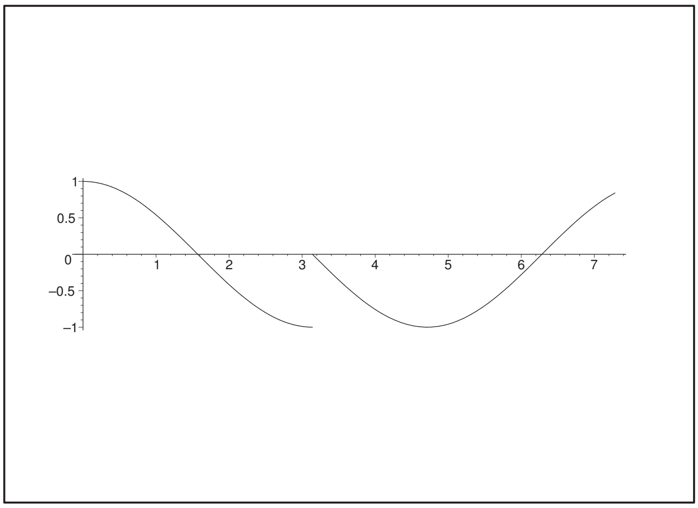

Figure 2: The graph of $f(t)$.

We have in this case $\sigma(f) = 0$. Thus for $\text{Re}(z) > 0$,

$$\begin{aligned}
\mathcal{L}\{f\}(z) &= \int_0^{\pi} \cos t \cdot e^{-zt}\, dt + \int_{\pi}^{+\infty} \sin t \cdot e^{-zt}\, dt \\
&= \frac{1}{2} \int_0^{\pi} \left\{ e^{-(z-i)t} + e^{-(z+i)t} \right\} dt + \int_0^{+\infty} \sin(u+\pi)\, e^{-z(\pi+u)}\, du \\
&= \frac{1}{2} \left[-\frac{e^{-zt+it}}{z-i} - \frac{e^{-zt-it}}{z+i} \right]_0^{\pi} - e^{-\pi z} \int_0^{+\infty} \sin t \cdot e^{-zt}\, dt \\
&= \frac{1}{2} \left\{ \frac{1}{z-i} \left(e^{-z\pi}+1\right) + \frac{1}{z+i} \left(e^{-z\pi}+1\right) \right\} - e^{-\pi z}\, \mathcal{L}\{\sin t\}(z) \\
&= \frac{z}{z^2+1} \left(e^{-z\pi}+1\right) - e^{-z\pi} \cdot \frac{1}{z^2+1} = \frac{z}{z^2+1} + \frac{z-1}{z^2+1} \cdot e^{-z\pi}.
\end{aligned}$$

Example 2.10 *Give an example of a non-periodic function f, the Laplace transform of which has the same form as in the Rule of Periodicity.*

A simple example is $f(t) := 1 + [t]$, where $[t]$ indicates the entire part of $t \in \mathbb{R}$, i.e. the largest integer $[t] \in \mathbb{Z}$ satisfying $[t] \leq t$. It is obvious that $f(t)$ is not periodic.

The Laplace transform is

$$\begin{aligned}\mathcal{L}\{f\}(z) &= \int_0^1 e^{-zt}dt + 2\int_1^2 e^{-zt}dt + \cdots + n\int_{n-1}^n e^{-zt}dt + \cdots \\ &= \sum_{n=1}^{+\infty}\int_{n-1}^n n\, e^{-zt}dt = \sum_{n=1}^{+\infty} n\left[-\frac{1}{z}\,e^{-zt}\right]_{t=n-1}^n = \frac{1}{z}\sum_{n=1}^{+\infty} n\left\{e^{-(n-1)z} - e^{-nz}\right\} \\ &= \frac{e^z-1}{z}\sum_{n=1}^{+\infty} n\, e^{-nz} = \frac{e^z-1}{z}\sum_{n=1}^{+\infty} n\left\{e^{-z}\right\}^n.\end{aligned}$$

The series is of course convergent for $\mathrm{Re}\, z > 0$. If we put $w = e^{-z}$, then $|w| < 1$, and

$$\frac{1}{1-w} = \sum_{n=0}^{+\infty} w^n \qquad \text{where} \qquad \frac{1}{(1-w)^2} = \frac{d}{dw}\,\frac{1}{1-w} = \sum_{n=1}^{+\infty} n\, w^{n-1},$$

and thus

$$\frac{w}{(1-w)^2} = \sum_{n=1}^{+\infty} n\, w^n.$$

We get by insertion for $\mathrm{Re}\, z > 0$ and $w = e^{-z}$ that

$$\mathcal{L}\{f\}(z) = \frac{e^z-1}{z}\,\frac{w}{(1-w)^2} = \frac{e^z-1}{z}\cdot\frac{e^{-z}}{\left(1-e^{-z}\right)^2} = \frac{1}{1-e^{-z}}\cdot\frac{1}{z}.$$

Hence, one must not be misled by the formal form of the Laplace transform to believe that the function is periodic.

On the other hand, one should here also mention that e.g. $\sin t$ is periodic and yet its Laplace transform

$$\mathcal{L}\{\sin t\}(z) = \frac{1}{z^2+1}, \qquad \mathrm{Re}\, z > 0,$$

does not at all have the same formal form as given by the Rule of Periodicity.

Example 2.11 *Find the Laplace transform of*

$$f(t) = \begin{cases} \cos\left(t - \dfrac{2\pi}{3}\right), & t > \dfrac{2\pi}{3}, \\[2ex] 0, & t \le \dfrac{2\pi}{3}. \end{cases}$$

It follows from the definition and the change of variable $u = t - \dfrac{2\pi}{3}$ that

$$\begin{aligned}\mathcal{L}\{f\}(z) &= \int_{\frac{2\pi}{2}}^{+\infty}\cos\left(t-\frac{2\pi}{3}\right)e^{-zt}\,dt = \int_0^{+\infty}\cos u\cdot\exp\left(-z\left(u+\frac{2\pi}{3}\right)\right)du \\ &= \exp\left(-z\cdot\frac{2\pi}{3}\right)\int_0^{+\infty}\cos t\cdot e^{-zt}\,dt = \exp\left(-z\cdot\frac{2\pi}{3}\right)\mathcal{L}\{\cos t\}(z) \\ &= \exp\left(-z\cdot\frac{2\pi}{3}\right)\cdot\frac{z}{z^2+1} \qquad \text{for } \mathrm{Re}(z) > 0.\end{aligned}$$

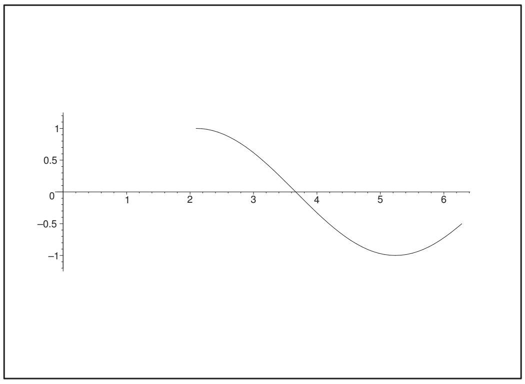

Figure 3: The graph of $f(t)$.

Example 2.12 *Find the Laplace transforms of*

$$(a)\quad \sin^3 t, \qquad (b)\quad e^{-t}\sin^2 t, \qquad (c)\quad \left(1+t\,e^{-t}\right)^3.$$

(a) First we see that

$$\begin{aligned}\sin^3 t &= \left\{\frac{e^{it}-e^{-it}}{2i}\right\}^3 = \frac{1}{8i^3}\left\{e^{3it}-3\,e^{it}+3\,e^{-it}-e^{-3it}\right\}\\ &= \frac{1}{4}\left\{\frac{3}{2i}\left(e^{it}-e^{-it}\right)-\frac{1}{2i}\left(e^{3it}-e^{-3it}\right)\right\} = \frac{3}{4}\sin t-\frac{1}{4}\sin 3t,\end{aligned}$$

hence we get for $\mathrm{Re}(z)>0$ that

$$\mathcal{L}\left\{\sin^3 t\right\}(z) = \frac{3}{4}\,\mathcal{L}\{\sin t\}(z)-\frac{1}{4}\,\mathcal{L}\{\sin 3t\}(z) = \frac{3}{4}\cdot\frac{1}{z^2+1}-\frac{1}{4}\cdot\frac{3}{z^2+9}.$$

(b) Using that

$$\sin^2 t = \frac{1}{2}\,(1-\cos 2t),$$

we get

$$\begin{aligned}\mathcal{L}\left\{e^{-t}\sin^2 t\right\}(z) &= \frac{1}{2}\,\mathcal{L}\left\{1\cdot e^{-t}\right\}(z)-\frac{1}{2}\,\mathcal{L}\left\{e^{-t}\cos 2t\right\}(z) = \frac{1}{2}\cdot\frac{1}{z+1}\frac{1}{2}\cdot\frac{z+1}{(z+1)^2+4}\\ &= \frac{2}{(z+1)\left\{(z+1)^2+4\right\}} \qquad \text{for } \mathrm{Re}(z)>-1.\end{aligned}$$

(c) From

$$\left(1+t\,e^{-t}\right)^3 = 1+3t\,e^{-t}+3t^2e^{-2t}+t^3e^{-3t},$$

follows that

$$
\begin{aligned}
&\mathcal{L}\left\{\left(1+t\,e^{-t}\right)^3\right\}(z)=\mathcal{L}\{1\}(z)+3\,\mathcal{L}\left\{t\,e^{-t}\right\}(z)+3\,\mathcal{L}\left\{t^2e^{-2t}\right\}(z)+\mathcal{L}\left\{t^3e^{3t}\right\}(z)\\
&=\frac{1}{z}+3\,\mathcal{L}\{t\}(z+1)+3\,\mathcal{L}\left\{t^2\right\}(z+2)+\mathcal{L}\left\{t^3\right\}(z+3)\\
&=\frac{1}{z}+\frac{3}{(z+1)^2}+\frac{6}{(z+2)^3}+\frac{6}{(z+3)^4}\qquad\text{for }\operatorname{Re}(z)>0.
\end{aligned}
$$

Example 2.13 *Find by using the Laplace transform the values of*

$$
(a)\quad \int_0^{+\infty} t\,e^{-3t}\sin t\,dt,\qquad (b)\quad \int_0^{+\infty} t^3e^{-t}\sin t\,dt.
$$

(a) We shall only give an interpretation of the integrals. We see that

$$
\begin{aligned}
&\int_0^{+\infty} t\,e^{-3t}\sin t\,dt=\mathcal{L}\{t\,\sin t\}(3)=\lim_{z\to 3}\left\{-\frac{d}{dz}\,\mathcal{L}\{\sin t\}(z)\right\}\\
&=\lim_{z\to 3}\left\{-\frac{d}{dz}\left(\frac{1}{z^2+1}\right)\right\}=\lim_{z\to 3}\frac{2z}{\left(z^2+1\right)^2}=\frac{6}{100}=\frac{3}{50}.
\end{aligned}
$$

(b) We get in the same way

$$\int_0^{+\infty} t^3 e^{-t}\sin t\,dt = \mathcal{L}\left\{t^3\sin t\right\}(1) = \lim_{z\to 1}\left(-\frac{d^3}{dz^3}\,\mathcal{L}\{\sin t\}(z)\right) = \lim_{z\to 1}\left\{-\frac{d^3}{dz^3}\left(\frac{1}{z^2+1}\right)\right\}$$

$$= \lim_{z\to 1}\left\{\frac{d^2}{dz^2}\left(\frac{2z}{\left(z^2+1\right)^2}\right)\right\} = \lim_{z\to 1}\frac{d}{dz}\left\{\frac{2}{\left(z^2+1\right)^2} - \frac{8z^2}{\left(z^2+1\right)^3}\right\}$$

$$= \lim_{z\to 1}\left\{-\frac{8z}{\left(z^2+1\right)^3} - \frac{16z}{\left(z^2+1\right)^3} + \frac{3\cdot 8z^2\cdot 2z}{\left(z^2+1\right)^4}\right\} = -\frac{8}{2^3} - \frac{16}{2^3} + \frac{48}{2^4} = 0.$$

Alternatively,

$$\begin{aligned}
\int_0^{+\infty} t^3 e^{-t}\sin t\,dt &= \frac{1}{2i}\int_0^{+\infty} t^3 e^{-(1-i)t}\,dt - \frac{1}{2i}\int_0^{+\infty} t^3 e^{-(1+i)t}\,dt \\
&= \frac{1}{2i}\,\mathcal{L}\left\{t^3\right\}(1-i) - \frac{1}{2i}\,\mathcal{L}\left\{t^3\right\}(1+i) \\
&= \frac{3!}{2i}\left\{\frac{1}{(1-i)^4} - \frac{1}{(1+i)^4}\right\} = \frac{3}{i}\left\{\frac{1}{-4} - \frac{1}{-4}\right\} = 0.
\end{aligned}$$

Example 2.14 *Find the values of the following integrals*

$$(a)\quad \int_0^{+\infty} t\,e^{-2t}\cos t\,dt, \qquad (b)\quad \int_0^{+\infty} \frac{e^{-t}-e^{-3t}}{t}\,dt.$$

(a) We have

$$\begin{aligned}
\int_0^{+\infty} t\,e^{-2t}\cos t\,dt &= \mathcal{L}\{t\,\cos t\}(2) = -\frac{d}{dz}\,\mathcal{L}\{\cos t\}(t) = -\lim_{z\to 2}\frac{d}{dz}\left(\frac{z}{z^2+1}\right) \\
&= -\lim_{z\to 2}\left\{\frac{1}{z^2+1} - \frac{2z^2}{\left(z^2+1\right)^2}\right\} = -\left\{\frac{1}{5} - \frac{8}{25}\right\} = \frac{3}{25}.
\end{aligned}$$

(b) First note that the integrand lies in $\mathbb{F}$. Then we get by the Rule of Division by t (because $0 > -1$, -3),

$$\begin{aligned}
\int_0^{+\infty} \frac{e^{-t}-e^{-3t}}{t}\,dt &= \mathcal{L}\left\{\frac{e^{-t}-e^{-3t}}{t}\right\}(0) = \int_0^{+\infty}\left(\mathcal{L}\left\{e^{-t}\right\}(x) - \mathcal{L}\left\{e^{-3t}\right\}(x)\right)\,dx \\
&= \int_0^{+\infty}\left\{\frac{1}{x+1} - \frac{1}{x+3}\right\}dx = \lim_{R\to+\infty}\left[\ln\left(\frac{x+1}{x+3}\right)\right]_0^R = \ln 3.
\end{aligned}$$

Example 2.15 *Prove that*

$$\mathcal{L}\left\{\frac{\cos at-\cos bt}{t}\right\}(z)=\frac{1}{2}\,\mathrm{Log}\left(\frac{z^2+b^2}{z^2+a^2}\right),\qquad a,\,b\in\mathbb{R},$$

and find $\sigma(f)$.

Since $t=0$ is a removable singularity, we have

$$\frac{\cos at-\cos bt}{t}\in\mathbb{F}.$$

Obviously, $\sigma(f)\le 0$, and due to the rules of magnitudes it is almost clear that $\sigma(f)\ge 0$, hence summing up,

$$\sigma(f)=0.$$

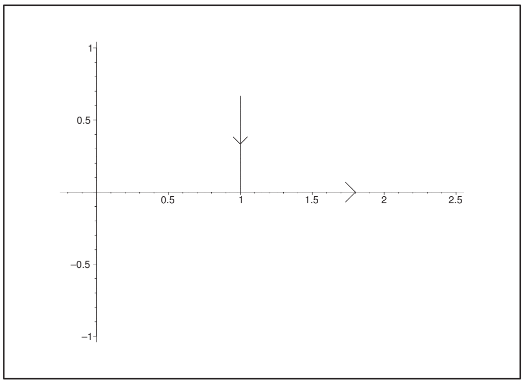

Figure 4: The path of integration Γ_z.

Finally, it follows from the Rule of Division by t that if $\mathrm{Re}(z)>0$ and the path of integration is the curve Γ_z, which consists of the line segment from z to $x=\mathrm{Re}(z)$, and then the line segment from x to $+\infty$ along the positive real axis that

$$\begin{aligned}\mathcal{L}\left\{\frac{\cos at-\cos bt}{t}\right\}(z) &= \int_z^{+\infty}\mathcal{L}\{\cos at-\cos bt\}(\zeta)\,d\zeta=\int_z^{+\infty}\left\{\frac{z}{z^2+a^2}-\frac{z}{z^2+b^2}\right\}dz\\ &= \frac{1}{2}\,\mathrm{Log}\left(\frac{z^2+b^2}{z^2+a^2}\right).\end{aligned}$$

Example 2.16 *Prove that*

$$\mathcal{L}\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}(z) = \operatorname{Log}\left(\frac{z+b}{z+a}\right) \qquad \textit{for } a,\, b \in \mathbb{R},$$

and find $\sigma(f)$.
Then compute the improper integral

$$\int_0^{+\infty} \frac{e^{-3t}-e^{-6t}}{t}\,dt.$$

We shall apply the Rule of Division by t. The function

$$\frac{e^{-at}-e^{-bt}}{t}$$

has a removable singularity at $t=0$ with the value $b-a$.

Furthermore, if $\operatorname{Re}(z) > \max\{-a,\, -b\}$, then the integral is convergent, while it is divergent, if $\operatorname{Re}(z) < \max\{-a, -b\}$, so

$$\sigma(f) = \max\{-a,-b\} = -\min\{a,b\}.$$

If $\operatorname{Re}(z) > -\min\{a,b\}$, then

$$\begin{aligned}\mathcal{L}\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}(z) &= \int_z^{+\infty} \left(\mathcal{L}\left\{e^{-at}\right\}(\zeta) - \mathcal{L}\left\{e^{-bt}\right\}(\zeta)\right)\,d\zeta \\ &= \int_z^{+\infty} \left\{\frac{1}{z+a} - \frac{1}{z+b}\right\} dz = \operatorname{Log}\left(\frac{z+b}{z+a}\right).\end{aligned}$$

Now, $0 > -\min\{3,6\} = -3$, so we conclude that

$$\int_0^{+\infty} \frac{e^{-3t}-e^{-6t}}{t}\,dt = \mathcal{L}\left\{\frac{e^{-3t-e^{-6t}}}{t}\right\}(0) = \operatorname{Log}\left(\frac{0+6}{0+3}\right) = \ln 2.$$

Example 2.17 *Find*

$$\mathcal{L}\left\{\frac{\sinh t}{t}\right\}(x) \qquad \textit{og} \qquad \mathcal{L}\left\{\frac{\cosh t - 1}{t}\right\}(x)$$

for $x > 1$.

We conclude from

$$\lim_{t\to 0} \frac{\sinh t}{t} = 1,$$

that

$$\frac{\sinh t}{t} \in \mathbb{F},$$

and that $\sigma(f) = 1$.

Then we get for $x > 1$ by the Rule of Division by t that

$$\begin{aligned}\mathcal{L}\left\{\frac{\sinh t}{t}\right\}(x) &= \int_x^{+\infty} \mathcal{L}\{\sinh t\}(\xi)\, d\xi = \int_x^{+\infty} \frac{1}{\xi^2-1}\, d\xi = \frac{1}{2}\int_x^{+\infty}\left\{\frac{1}{\xi} - \frac{1}{\xi+1}\right\} d\xi \\ &= \left[\frac{1}{2}\ln\left(\frac{\xi-1}{\xi+1}\right)\right]_x^{+\infty} = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right).\end{aligned}$$

It follows from

$$\lim_{t\to 0}\frac{\cosh t - 1}{t} = 0,$$

that

$$\frac{\cosh t - 1}{t} \in \mathbb{F},$$

and $\sigma(f) = 1$. Finally, if $x > 1$, then we get by the Rule of Division by t that

$$\begin{aligned}\mathcal{L}\left\{\frac{\cosh t - 1}{t}\right\}(x) &= \int_x^{+\infty} \left(\mathcal{L}\{\cosh t\}(\xi) - \mathcal{L}\{1\}(\xi)\right) d\xi = \int_x^{+\infty}\left(\frac{\xi}{\xi^2-1} - \frac{1}{\xi}\right) d\xi \\ &= \left[\frac{1}{2}\ln\left(\xi^2-1\right) - \ln \xi\right]_x^{+\infty} = \frac{1}{2}\ln\left(\frac{x^2}{x^2-1}\right).\end{aligned}$$

Example 2.18 *Compute*

$$(a)\quad \int_0^{+\infty} \frac{e^{-t}\sin t}{t}\,dt, \qquad (b)\quad \int_0^{+\infty} \frac{e^{-\sqrt{2}\,t}\sinh t\cdot \sin t}{t}\,dt.$$

We apply again the Rule of Division by t.

(a) Since $\dfrac{\sin t}{t} \in \mathbb{F}$, where $\sigma(f) = 0$, we get

$$\begin{aligned}\int_0^{+\infty} \frac{e^{-t}\sin t}{t}\,dt &= \mathcal{L}\left\{\frac{\sin t}{t}\right\}(1) = \int_1^{+\infty} \mathcal{L}\{\sin t\}(\xi)\,d\xi \\ &= \int_1^{+\infty} \frac{1}{1+\xi^2}\,d\xi = [\text{Arctan}\ \xi]_1^{+\infty} = \frac{\pi}{4}.\end{aligned}$$

(b) Since

$$e^{-\sqrt{2}\,t}\sinh t = \frac{1}{2}\,e^{-(\sqrt{2}-1)t} - \frac{1}{2}\,e^{-(\sqrt{2}+1)t},$$

we get as in **(a)**,

$$\begin{aligned}&\int_0^{+\infty} \frac{e^{-\sqrt{2}\,t}\sinh t\cdot\sin t}{t}\,dt = \frac{1}{2}\,\mathcal{L}\left\{\frac{\sin t}{t}\right\}(\sqrt{2}-1) - \frac{1}{2}\,\mathcal{L}\left\{\frac{\sin t}{t}\right\}(\sqrt{2}+1) \\ &= \frac{1}{2}\,[\text{Arctan}\ \xi]_{\sqrt{2}-1}^{+\infty} - \frac{1}{2}\,[\text{Arctan}\ \xi]_{\sqrt{2}+1}^{+\infty} = \frac{1}{2}\,[\text{Arctan}\ \xi]_{\sqrt{2}-1}^{\sqrt{2}+1} \\ &= \frac{1}{2}\,\{\text{Arctan}(\sqrt{2}+1) - \text{Arctan}(\sqrt{2}-1)\}.\end{aligned}$$

Now

$$\tan(\text{Arctan}(\sqrt{2}+1) - \text{Arctan}(\sqrt{2}-1)) = \frac{(\sqrt{2}+1)-(\sqrt{2}-1)}{1+(\sqrt{2}+1)(\sqrt{2}-1)} = 1,$$

so

$$\text{Arctan}(\sqrt{2}+1) - \text{Arctan}(\sqrt{2}-1) = \text{Arctan}\ 1 = \frac{\pi}{4},$$

hence by insertion,

$$\int_0^{+\infty} \frac{e^{-\sqrt{2}\,t}\sinh t\cdot t}{t}\,dt = \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8}.$$

Example 2.19 *Prove that*

$$\int_0^{+\infty} e^{-t}\left\{\int_0^t \frac{\sin u}{u}\,du\right\} dt = \frac{\pi}{4}.$$

We shall only apply the Rule of Integration and the Rule of Division by t,

$$\begin{aligned}\int_0^{+\infty} e^{-t}\left\{\int_0^t \frac{\sin u}{u}\,du\right\} dt &= \mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\,du\right\}(1) = \frac{1}{1}\,\mathcal{L}\left\{\frac{\sin t}{t}\right\}(1) = \int_1^{+\infty} \mathcal{L}\{\sin t\}(\xi)\,d\xi \\ &= \int_1^{+\infty} \frac{d\xi}{\xi^2+1} = [\text{Arctan}\;\xi]_1^{+\infty} = \frac{\pi}{4}.\end{aligned}$$

Example 2.20 *Prove that*

$$\int_0^{+\infty} e^{-t}\left\{\int_0^t \frac{1-e^{-u}}{u}\,du\right\} dt = \ln 2.$$

The integral $\int_0^t \frac{1-e^{-u}}{u}\,du$ exists, and we get by the Rule of Integration and the Rule of Division by t,

$$\begin{aligned}&\int_0^{+\infty} e^{-t}\left\{\int_0^t \frac{1-e^{-u}}{u}\,du\right\} dt = \mathcal{L}\left\{\int_0^t \frac{1-e^{-u}}{u}\,du\right\}(1) = \frac{1}{1}\,\mathcal{L}\left\{\frac{1-e^{-t}}{t}\right\}(1) \\ &\quad = \int_1^{+\infty}\left(\mathcal{L}\{1\}(\xi) - \mathcal{L}\left\{e^{-t}\right\}(\xi)\right) d\xi = \int_1^{+\infty}\left\{\frac{1}{\xi} - \frac{1}{\xi+1}\right\} d\xi = \left[\ln\left(\frac{\xi}{\xi+1}\right)\right]_1^{+\infty} = \ln 2.\end{aligned}$$

Example 2.21 *Find* $\mathcal{L}\{|\sin t|\}(z)$ *and* $\mathcal{L}\{\max(0,\sin t)\}(z)$.

Here we shall apply the Rule of Periodicity.

Since $|\sin t|$ is periodic of period π, we get for $\text{Re}(z) > 0$ that

$$\begin{aligned}\mathcal{L}\{|\sin t|\}(z) &= \frac{1}{1-e^{-\pi z}}\int_0^{\pi} e^{-zt}|\sin t|\,dt = \frac{e^{\pi z}}{e^{\pi z}-1}\int_0^{\pi} e^{-zt}\sin t\,dt \\ &= \frac{1}{2i}\cdot\frac{e^{\pi z}}{e^{\pi z}-1}\left\{\int_0^{\pi} e^{-(z-i)t}\,dt - \int_0^{\pi} e^{-(z+i)t}\,dt\right\} = \frac{1}{2i}\cdot\frac{e^{\pi z}}{e^{\pi z}-1}\left[-\frac{e^{-zt+it}}{z-i} + \frac{e^{-zt-it}}{z+i}\right]_0^{\pi} \\ &= \frac{1}{2i}\cdot\frac{e^{\pi z}}{e^{\pi z}-1}\left\{-\frac{1}{z-i}\left(-e^{-\pi z}-1\right) + \frac{1}{z+i}\left(-e^{-\pi z}-1\right)\right\} \\ &= \frac{1}{2i}\cdot\frac{e^{\pi z}}{e^{\pi z}-1}\left(1+e^{-\pi z}\right)\cdot\left\{\frac{1}{z-i} - \frac{1}{z+i}\right\} = \frac{e^{\pi z}+1}{e^{\pi z}-1}\cdot\frac{1}{z^2+1}.\end{aligned}$$

Analogously we may find $\mathcal{L}\{\max(0,\sin t)\}(z)$ by the Rule of Periodicity, because the period is 2π. However, it is easier to notice that

$$\max(0,\sin t) = \frac{1}{2}\sin t + \frac{1}{2}|\sin t|,$$

so if we use the result above, then

$$\begin{aligned}\mathcal{L}\{\max(0,\sin t)\}(z) &= \frac{1}{2}\,\mathcal{L}\{\sin t\}(z)+\frac{1}{2}\,\mathcal{L}\{|\sin t|\}(z)=\frac{1}{2}\cdot\frac{1}{z^2+1}+\frac{1}{2}\cdot\frac{\pi z+1}{e^{\pi z}-1}\cdot\frac{1}{z^2+1}\\ &= \frac{1}{2}\cdot\frac{1}{z^2+1}\cdot\frac{e^{\pi z}-1+e^{\pi z}+1}{e^{\pi z}-1}=\frac{e^{\pi z}}{e^{\pi z}-1}\cdot\frac{1}{z^2+1}.\end{aligned}$$

Example 2.22 *Find the inverse Laplace transform of*

$$(a)\ \frac{3z-12}{z^2+8},\qquad (b)\ \frac{2z+1}{z(z+1)},\qquad (c)\ \frac{z}{(z+1)^5}.$$

We have in all three cases given rational functions with a zero at ∞, so the inverse Laplace transform exists and is given by a residuum formula.
ALTERNATIVELY one may use a decomposition.

(a) *Inspection and rules of calculation.* Since

$$\begin{aligned}\frac{3z-12}{z^2+8} &= 3\cdot\frac{z}{z^2+(\sqrt{8})^2}-\frac{12}{\sqrt{8}}\cdot\frac{\sqrt{8}}{z^2+(\sqrt{8})^2}\\ &= 3\,\mathcal{L}\{\cos(2\sqrt{2}\,t)\}(z)-3\sqrt{2}\,\mathcal{L}\{\sin(2\sqrt{2}\,t)\}(z),\end{aligned}$$

the inverse Laplace transform is given by

$$f(t)=\mathcal{L}^{\circ-1}\left\{\frac{3z-12}{z^2+8}\right\}(t)=3\,\cos(2\sqrt{2}\,t)-3\sqrt{2}\,\sin(2\sqrt{2}\,t).$$

Residuum formula. The singularities are $z=\pm i\,2\sqrt{2}$. They are both simple poles, hence by the residuum formula and RULE IA,

$$\begin{aligned}f(t) &= \operatorname{res}\left(\frac{(3z-12)e^{zt}}{(z-2i\sqrt{2})(z+2i\sqrt{2})}\,;\,2i\sqrt{2}\right)+\operatorname{res}\left(\frac{(3z-12)e^{zt}}{(z-2i\sqrt{2})(z+2i\sqrt{2})}\,;\,-2i\sqrt{2}\right)\\ &= \frac{-12+6i\sqrt{2}}{i4\sqrt{2}}\,e^{i\,2\sqrt{t}}+\frac{-12-6i\sqrt{2}}{-i4\sqrt{2}}\,e^{-i\,2\sqrt{t}}\\ &= \frac{3}{2}\left\{e^{i2\sqrt{2}\,t}+e^{-2\sqrt{2}\,t}\right\}-\frac{12}{2\sqrt{2}}\cdot\frac{1}{2i}\left\{e^{i2\sqrt{2}\,t}-e^{-i2\sqrt{2}\,t}\right\}\\ &= 3\,\cos(2\sqrt{2}\,t)-3\sqrt{2}\,\sin(2\sqrt{2}\,t).\end{aligned}$$

(b) *Decomposition and rules of calculation.* Since

$$\frac{2z+1}{z(z+1)}=\frac{1}{z}+\frac{1}{z+1}=\mathcal{L}\{1\}(z)+\mathcal{L}\left\{e^{-t}\right\}(z)=\mathcal{L}\left\{1+e^{-t}\right\}(z),$$

the inverse Laplace transform is given by

$$f(t)=\mathcal{L}^{\circ-1}\left\{\frac{2z+1}{z(z+1)}\right\}(t)=1+e^{-t}.$$

Residuum formula. The two singularities $z=0$ and $z=-1$ are both simple pole. Hence, by the residuum formula,

$$\begin{aligned} f(t) &= \operatorname{res}\left(\frac{2z+1}{z(z+1)}\cdot e^{zt}\,;\,0\right) + \operatorname{res}\left(\frac{2z+1}{z(z+1)}\cdot e^{zt}\,;\,-1\right) = \frac{0+1}{0+1}\cdot e^{0} + \frac{-2+1}{-1}\cdot e^{-t} \\ &= 1+e^{-t}. \end{aligned}$$

(c) *Decomposition and rules of calculation.* It follows from

$$\begin{aligned} \frac{z}{(z+1)^5} &= \frac{z+1-1}{(z+1)^5} = \frac{1}{(z+1)^4} - \frac{1}{(z+1)^5} = \frac{1}{3!}\,\frac{3!}{(z+1)^4} - \frac{1}{4!}\,\frac{4!}{(z+1)^5} \\ &= \frac{1}{6}\,\mathcal{L}\left\{t^3\right\}(z+1) - \frac{1}{24}\,\mathcal{L}\left\{t^4\right\}(z+1) = \mathcal{L}\left\{\frac{1}{6}\,e^{-t}\,t^3 - \frac{1}{24}\,e^{-t}\,t^4\right\})z), \end{aligned}$$

that the inverse Laplace transform is given by

$$f(t) = \mathcal{L}^{\circ -1}\left\{\frac{z}{(z+1)^5}\right\} = \frac{1}{6}\,t^3e^{-t} - \frac{1}{24}\,t^4e^{-t} = \frac{1}{24}\,(t-t)t^3e^{-t}.$$

Residuum formula. The only singularity $z=-1$ is a five-tuple pole, so

$$\begin{aligned} f(t) &= \operatorname{res}\left(\frac{z}{(z+1)^5}\,e^{zt}\,;\,-1\right) = \frac{1}{4!}\lim_{z\to -1}\frac{d^4}{dz^4}\left(z\,e^{zt}\right) = \frac{1}{24}\lim_{z\to -1}\frac{d^3}{dz^3}\left(t\,z\,e^{zt}+e^{zt}\right) \\ &= \frac{1}{24}\lim_{z\to -1}\frac{d^2}{dz^2}\left(t^2 z\,e^{zt}+2t\,e^{zt}\right) = \frac{1}{24}\lim_{z\to -1}\frac{d}{dz}\left(t^3 z\,e^{zt}+3t^2 e^{zt}\right) \\ &= \frac{1}{24}\lim_{z\to -1}\left(t^4 z\,e^{zt}+4t^3 e^{zt}\right) = -\frac{1}{24}\,t^4 e^{-t}+\frac{1}{6}\,t^3 e^{-t}. \end{aligned}$$

Example 2.23 *Find the inverse Laplace transform of*

$$(a)\quad \frac{z}{(z+1)(z+2)}, \qquad (b)\quad \frac{1}{(z+1)^3}, \qquad (c)\quad \frac{3z-14}{z^2-4z+8}.$$

We have in all three cases given a rational function with a zero at ∞, so the inverse Laplace transform exists and is given by a residuum formula. Alternatively we may decompose and then use a table. We shall demonstrate both methods here.

(a) *Decomposition and rules of calculations.* It follows from

$$\frac{z}{(z+1)(z+2)} = \frac{-1}{z+1}+\frac{2}{z+2} = \mathcal{L}\left\{2\,e^{-2t}-e^{-t}\right\}(z),$$

that the inverse Laplace transform is given by

$$f(t) = \mathcal{L}^{\circ -1}\left\{\frac{z}{(z+1)(z+2)}\right\} = 2\,e^{-2t}-e^{-t}.$$

Residuum formula. Since $z=-1$ and $z=-2$ are simple poles, we get by the residuum formula and Rule Ia that

$$\begin{aligned} f(t) &= \operatorname{res}\left(\frac{z\,e^{zt}}{(z+1)(z+2)}\,;\,-1\right)+\operatorname{res}\left(\frac{z\,e^{zt}}{(z+1)(z+2)}\,;\,-2\right) = \frac{-1}{1}\,e^{-t}+\frac{-2}{-1}\,e^{-2t} \\ &= 2\,e^{-2t}-e^{-t}. \end{aligned}$$

(b) *Rules of calculation and use of tables.* Since

$$\frac{1}{(z+1)^3} = \frac{1}{2}\cdot\frac{2!}{(z+1)^3} = \frac{1}{2}\,\mathcal{L}\left\{t^2\right\}(z+1) = \mathcal{L}\left\{\frac{1}{2}\,t^2 e^{-t}\right\}(z),$$

the inverse Laplace transform is given by

$$f(t) = \mathcal{L}^{\circ -1}\left\{\frac{1}{(z+1)^3}\right\}(t) = \frac{1}{2}\,t^2 e^{-t}.$$

Residuum formula. Here $z=-1$ is a triple pole, so by the residuum formula and Rule I,

$$f(t) = \operatorname{res}\left(\frac{e^{zt}}{(z+1)^3}\,;\,-1\right) = \frac{1}{2!}\lim_{z\to -1}\frac{d^2}{dz^2}\,e^{zt} = \frac{1}{2}\,t^2 e^{-t}.$$

(c) *Rules of calculation.* (Here a little quibbling). We conclude from

$$\begin{aligned}\frac{3z-14}{z^2-4z+8} &= \frac{3(z-2)-8}{(z-2)^2+2^2} = 3\cdot\frac{z-2}{(z-2)^2+2^2} - 4\cdot\frac{2}{(z-2)^2+2^2} \\ &= 3\,\mathcal{L}\{\cos 2t\}(z-2) - 4\,\mathcal{L}\{\sin 2t\}(z-2) = \mathcal{L}\left\{3\,e^{2t}\cos 2t - 4\,e^{2t}\,\sin 2t\right\}(z),\end{aligned}$$

that

$$f(t) = \mathcal{L}^{\circ -1}\left\{\frac{3z-14}{z^2-4z+8}\right\}(t) = 3\,e^{2t}\cos 2t - 4\,e^{2t}\sin 2t.$$

Residuum formula. The roots of the denominator $Q(z) = z^2 - 4z + 8$ are $z = 2\pi 2i$. Since $Q'(z) = 2z - 4$, we get from the residuum formula and Heaviside's expansion theorem that

$$\begin{aligned} f(t) &= \operatorname{res}\left(\frac{3z-14}{z^2-4z+8}\,e^{zt}\,;\,2+2i\right) + \operatorname{res}\left(\frac{3z-14}{z^2-4z+8}\,e^{zt}\,;\,2-2i\right) \\ &= \frac{3(2+2i)-14}{2(2+2i)-4}\,e^{(2+2i)t} + \frac{3(2-2i)-14}{2(2-2i)-4}\,e^{(2-2i)t} = \frac{-8+6i}{4i}\,e^{2t}e^{2it} + \frac{-8-6i}{-4i}\,e^{2t}e^{-2it} \\ &= e^{2t}\cdot\frac{1}{2}\left\{(3+4i)e^{2it} + (3-4i)e^{-2it}\right\} = e^{2t}\,(3\,\cos 2t - 4\,\sin 2t).\end{aligned}$$

Example 2.24 *Find the inverse Laplace transform of*

$$(a)\quad \frac{1}{\left(z^2+1\right)^2}, \qquad (b)\quad \frac{1}{z^4-1}, \qquad (c)\quad \frac{z^2}{z^3-1}.$$

We have in all three cases a rational function with a zero at ∞, so the inverse Laplace transform exists and is given by a residuum formula. We shall treat the examples in various alternative ways, so one can compare the different methods.

(a) *Decomposition and rules of calculations.* We conclude from

$$\begin{aligned}\frac{1}{\left(z^2+1\right)^2} &= \left\{\frac{1}{(z-i)(z+i)}\right\}^2 = \left\{\frac{1}{2i}\cdot\frac{1}{z-i} - \frac{1}{2i}\cdot\frac{1}{z+i}\right\}^2 \\ &= -\frac{1}{4}\left\{\frac{1}{z-i} - \frac{1}{z+i}\right\}^2 = -\frac{1}{4}\left\{\frac{1}{(z-i)^2} + \frac{1}{(z+i)^2} - \frac{2}{z^2+1}\right\} \\ &= -\frac{1}{4}\,\mathcal{L}\{t\}(z-i) - \frac{1}{4}\,\mathcal{L}t(z+i) + \frac{1}{2}\,\mathcal{L}\{\sin t\}(z) \\ &= \mathcal{L}\left\{\frac{1}{2}\,\sin t - \frac{1}{4}\,t\,e^{it} - \frac{1}{4}\,t\,e^{-it}\right\}(z) = \mathcal{L}\left\{\frac{1}{2}\,\sin t - \frac{1}{2}\,t\,\cos t\right\}/z).\end{aligned}$$

that

$$f(t) = \mathcal{L}^{\circ -1}\left(\frac{1}{\left(z^2+1\right)^2}\right\}(t) = \frac{1}{2}\,\sin t - \frac{1}{2}\,t\,\cos t.$$

Rule of Convolution. We conclude from

$$\mathcal{L}\{f\}(z) = \frac{1}{\left(z^2+1\right)^2} = \frac{1}{z^2+1}\cdot\frac{1}{z^2+1} = \mathcal{L}\{\sin t\}(z)\cdot\mathcal{L}\{\sin t\}(z) = \mathcal{L}\{(\sin\star\sin)(t)\}(z).$$

that

$$\begin{aligned}
f(t) &= \mathcal{L}^{\circ -1}\left(\frac{1}{(z^2+1)^2}\right\}(t) = \int_0^t \sin(t-\tau)\sin\tau\,d\tau = \frac{1}{2}\int_0^t \{\cos(t-2\tau)-\cos t\}\,d\tau \\
&= \frac{1}{2}\left[-\frac{1}{2}\sin(t-2\tau)-\tau\,\cos t\right]_{\tau=0}^{t} = \frac{1}{2}\left\{-\frac{1}{2}\sin(-t)-t\,\cos t+\frac{1}{2}\sin t+0\right\} \\
&= \frac{1}{2}\sin t - \frac{1}{2}t\,\cos t.
\end{aligned}$$

Residuum formula. The two singularities $z = \pm i$ are both double poles, so by the residuum formula and RULE I,

$$\begin{aligned}
f(t) &= \operatorname{res}\left(\frac{e^{zt}}{(z^2+1)^2}\,;\,i\right)+\operatorname{res}\left(\frac{e^{zt}}{(z^2+1)^2}\,;\,-i\right) = \lim_{z\to i}\frac{d}{dz}\left\{\frac{e^{zt}}{(z+i)^2}\right\}+\lim_{z\to -i}\frac{d}{dz}\left\{\frac{e^{zt}}{(z-i)^2}\right\} \\
&= \lim_{z\to i}\left\{\frac{t\,e^{zt}}{(z+i)^2}-2\,\frac{e^{zt}}{(z+i)^3}\right\}+\lim_{z\to -i}\left\{\frac{t\,e^{zt}}{(z-i)^2}-2\,\frac{e^{zt}}{(z-i)^3}\right\} \\
&= \frac{t\,e^{it}}{(2i)^2}-2\,\frac{e^{it}}{(2i)^3}+\frac{t\,e^{-it}}{(-2i)^2}-2\,\frac{e^{-it}}{(-2i)^3} = -\frac{1}{4}\left(e^{it}+e^{-it}\right)+\frac{2}{4}\cdot\frac{1}{2i}\left(e^{it}-e^{-it}\right) \\
&= -\frac{1}{2}t\,\cos t+\frac{1}{2}\sin t.
\end{aligned}$$

(b) *Decomposition.* We conclude from

$$\frac{1}{z^4-1} = \frac{1}{z^2+1}\cdot\frac{1}{z^2-1} = \frac{1}{2}\cdot\frac{1}{z^2-1}-\frac{1}{2}\cdot\frac{1}{z^2+1} = \mathcal{L}\left\{\frac{1}{2}\sinh t-\frac{1}{2}\sin t\right\}(z),$$

that

$$f(t) = \mathcal{L}^{\circ -1}\left\{\frac{1}{z^4-1}\right\} = \frac{1}{2}\sinh t-\frac{1}{2}\sin t.$$

Residuum formula. The denominator $Q(z) = z^4-1$ has the four simple zeros 1, i, -1, $-i$, and since $Q'(z) = 4z^3$, it follows by Heaviside's expansion theorem that

$$\begin{aligned}
f(t) &= \sum_{n=0}^{3}\operatorname{res}\left(\frac{e^{zt}}{z^4-1}\,;\,i^n\right) = \sum_{n=0}^{3}\frac{\exp\left(i^n t\right)}{4\left(i^n\right)^3} = \frac{1}{4}\sum_{n=0}^{3} i^n\exp\left(i^n t\right) \\
&= \frac{1}{4}\left\{1\cdot e^t+i\cdot e^{it}-1\cdot e^{-t}-i\cdot e^{-it}\right\} \\
&= \frac{1}{2}\left\{\frac{1}{2}e^t-\frac{1}{2}e^{-t}\right\}-\frac{1}{2}\left\{\frac{1}{2i}e^{it}-\frac{1}{2i}e^{-it}\right\} = \frac{1}{2}\sinh t-\frac{1}{2}\sin t.
\end{aligned}$$

(c) *Decomposition.* It follows from

$$\frac{z^2}{z^3-1} = \frac{z^3}{(z-1)\left(z^2+z+1\right)} = \frac{1}{3}\cdot\frac{1}{z-1}+\frac{Az+B}{z^2+z+1},$$

that

$$Az+B = \frac{1}{z-1}\left\{z^2-\frac{1}{3}\left(z^2+z+1\right)\right\} = \frac{1}{3}\cdot\frac{1}{z-1}\left\{2z^2-z-1\right\} = \frac{1}{3}(2z+1),$$

hence by insertion,

$$
\begin{aligned}
\frac{z^2}{z^3-1} &= \frac{1}{3}\,\frac{1}{z-1}+\frac{1}{3}\,\frac{2z+1}{z^2+z+1}=\frac{1}{3}\,\mathcal{L}\left\{e^t\right\}(z)+\frac{2}{3}\,\frac{z+\frac{1}{2}}{\left(z+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \\
&= \frac{1}{3}\,\mathcal{L}\left\{\,e^t\right\}(z)+\frac{2}{3}\,\mathcal{L}\left\{\exp\left(-\frac{1}{2}\,t\right)\cos\left(\frac{\sqrt{3}}{2}\,t\right)\right\}(z).
\end{aligned}
$$

We conclude that

$$
f(t)=\mathcal{L}^{\circ -1}\left\{\frac{z^2}{z^3-1}\right\}=\frac{1}{3}\,e^t+\frac{2}{3}\,\exp\left(-\frac{1}{2}\,t\right)\cos\left(\frac{\sqrt{3}}{2}\,t\right).
$$

Residuum formula. The denominator $Q(z)=z^3-1$ has the simple zeros

$$
1, \qquad -\frac{1}{2}\pm i\,\frac{\sqrt{3}}{2},
$$

thus we get from the residuum formula for the inverse Laplace transform and from Heaviside's expansion theorem,

$$\begin{aligned}
f(t) &= \operatorname{res}\left(\frac{z^2e^{zt}}{z^3-1}\,;\,1\right)+\operatorname{res}\left(\frac{z^2e^{zt}}{z^3-1}\,;\,-\frac{1}{2}+i\,\frac{\sqrt{3}}{2}\right)+\operatorname{res}\left(\frac{z^2e^{zt}}{z^3-1}\,;\,-\frac{1}{2}-i\,\frac{\sqrt{3}}{2}\right)\\
&= \frac{1}{3}\,e^t+\frac{1}{3}\exp\left(\left(-\frac{1}{2}+i\,\frac{\sqrt{3}}{2}\right)t\right)+\frac{1}{3}\exp\left(\left(-\frac{1}{2}-i\,\frac{\sqrt{3}}{2}\right)t\right)\\
&= \frac{1}{3}\,e^t+\frac{2}{3}\,\exp\left(-\frac{1}{2}\,t\right)\cdot 12\left\{\exp\left(i\,\frac{\sqrt{3}}{2}\,t\right)+\exp\left(-i\,\frac{\sqrt{3}}{2}\,t\right)\right\}\\
&= \frac{1}{3}\,e^t+\frac{2}{3}\,\exp\left(-\frac{1}{2}\,t\right)\cos\left(\frac{\sqrt{3}}{2}\,t\right).
\end{aligned}$$

Example 2.25 *Find the inverse Laplace transform of*

$$(a)\ \frac{1}{z(z+3)^3},\qquad (b)\ \frac{1}{(z+1)(z-2)^2},\qquad (c)\ \frac{z}{(z+1)^3(z-1)^2},\qquad (d)\ \frac{1}{z(z+3)^2}.$$

We have in all four cases a rational function with a zero at ∞, so we may apply the residuum formula. We shall try various methods so the reader can compare them.

(a) *Decomposition.* The structure is

$$\frac{1}{z(z+3)^3}=\frac{1}{27}\cdot\frac{1}{z}-\frac{1}{3}\cdot\frac{1}{(z+3)^3}+\frac{Az+B}{(z+3)^2},$$

thus

$$\begin{aligned}
\frac{Az+B}{(z+3)^2} &= \frac{1}{z(z+3)}-\frac{1}{27}\cdot\frac{1}{z}+\frac{1}{3}\cdot\frac{1}{(z+3)^3}=\frac{1}{27z(z+3)^3}\,\left\{27-(z+3)^3+9z\right\}\\
&= \frac{1}{27z(z+3)^3}\,\left\{27-z^3-9z^2-27z-27+9z\right\}=-\frac{1}{27z(z+3)^3}\,\left\{z^3+9z^2+18z\right\}\\
&= -\frac{1}{27(z+3)^3z}\,z(z+3)(z+6)=-\frac{1}{27}\cdot\frac{z+6}{(z+3)^2}=-\frac{1}{27}\cdot\frac{1}{z+3}-\frac{1}{9}\cdot\frac{1}{(z+3)^2}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{z(z+3)^3} &= \frac{1}{27}\cdot\frac{1}{z}-\frac{1}{3}\cdot\frac{1}{(z+3)^3}-\frac{1}{9}\cdot\frac{1}{(z+3)^2}-\frac{1}{27}\cdot\frac{1}{z+3}\\
&= \mathcal{L}\left\{\frac{1}{27}\right\}(z)-\mathcal{L}\left\{\frac{1}{6}\,e^{-3t}t^2\right\}(z)-\mathcal{L}\left\{\frac{1}{9}\,e^{-3t}t\right\}(z)-\frac{1}{27}\,\mathcal{L}\left\{e^{-3t}\right\}(z)\\
&= \mathcal{L}\left\{\frac{1}{27}-e^{-3t}\left(\frac{1}{6}\,t^2+\frac{1}{9}\,t+\frac{1}{27}\right)\right\}(z).
\end{aligned}$$

Then it follows that

$$f(t)=\mathcal{L}^{\circ-1}\left\{\frac{1}{z(z+3)^3}\right\}(t)=\frac{1}{27}-e^{-3t}\left(\frac{1}{6}\,t^2+\frac{1}{9}\,t+\frac{1}{27}\right).$$

Residuum formula. Here we get

$$\begin{aligned}
f(t) &= \operatorname{res}\left(\frac{e^{zt}}{z(z+3)^3};0\right)+\operatorname{res}\left(\frac{e^{zt}}{z(z+3)^3};-3\right)=\frac{1}{27}+\lim_{z\to-3}\frac{1}{2!}\frac{d^2}{dz^2}\left\{\frac{e^{zt}}{z}\right\}\\
&= \frac{1}{27}+\frac{1}{2}\lim_{z\to-3}\frac{d}{dz}\left\{t\cdot\frac{e^{zt}}{z}-\frac{e^{zt}}{z^2}\right\}=\frac{1}{27}+\frac{1}{2}\lim_{z\to-3}\left\{t^2\cdot\frac{e^{zt}}{z}-2t\cdot\frac{e^{zt}}{z^2}+2\cdot\frac{e^{zt}}{z^3}\right\}\\
&= \frac{1}{27}+\frac{1}{2}\left\{\frac{t^2e^{-3t}}{-3}-\frac{2t\,e^{-3t}}{9}+\frac{2e^{-3t}}{-27}\right\}=\frac{1}{27}-\frac{1}{6}\,t^2e^{-3t}-\frac{1}{9}\,t\,e^{-3t}-\frac{1}{27}\,e^{-3t}.
\end{aligned}$$

(b) *Decomposition.* The structure is

$$\frac{1}{(z+1)(z-2)^2}=\frac{1}{(-3)^2}\cdot\frac{1}{z+1}+\frac{Az+B}{(z-2)^2}=\frac{1}{9}\cdot\frac{1}{z+1}+\frac{Az+B}{(z-2)^2},$$

hence by a rearrangement,

$$\begin{aligned}
\frac{Az+B}{(z-2)^2} &= \frac{1}{(z+1)(z-2)^2}-\frac{1}{9}\,\frac{1}{z+1}=\frac{1}{9}\cdot\frac{1}{(z+1)(z-2)^2}\left\{9-(z-2)^2\right\}\\
&= \frac{1}{9}\cdot\frac{1}{(z+1)(z-2)^2}\left\{-z^2+4z+5\right\}=\frac{1}{9}\,\frac{1}{(z+1)(z-2)^2}\,(z-1)(-z+5)\\
&= \frac{1}{9}\cdot\frac{1}{(z-2)^2}\,(-z+5)=\frac{1}{9}\cdot\frac{-2+2+3}{(z-2)^2}=\frac{1}{3}\cdot\frac{1}{(z-2)^2}-\frac{1}{9}\cdot\frac{1}{z-2},
\end{aligned}$$

giving

$$\begin{aligned}
\frac{1}{(z+1)(z-2)^2} &= \frac{1}{9}\cdot\frac{1}{z+1}+\frac{1}{3}\cdot\frac{1}{(z-2)^2}-\frac{1}{9}\cdot\frac{1}{z-2}\\
&= \mathcal{L}\left\{\frac{1}{9}\,e^{-t}\right\}(z)+\mathcal{L}\left\{\frac{1}{3}\,t\,e^{2t}\right\}(z)-\mathcal{L}\left\{\frac{1}{9}\,e^{2t}\right\}(z)\\
&= \mathcal{L}\left\{\frac{1}{9}\,e^{-t}+\frac{1}{3}\,t\,e^{2t}-\frac{1}{9}\,e^{2t}\right\}(z).
\end{aligned}$$

We therefore conclude that

$$f(t)=\mathcal{L}^{\circ-1}\left\{\frac{1}{(z+1)(z-2)^2}\right\}(t)=\frac{1}{9}\,e^{-t}+\frac{1}{3}\,t\,e^{2t}-\frac{1}{9}\,e^{2t}.$$

Residuum formula. Here we get

$$\begin{aligned}
f(t) &= \operatorname{res}\left(\frac{e^{zt}}{(z+1)(z-2)^2};-1\right)+\operatorname{res}\left(\frac{e^{zt}}{(z+1)(z-2)^2};2\right)=\frac{e^{-t}}{(-3)^2}+\lim_{z\to2}\frac{d}{dz}\left\{\frac{e^{zt}}{z+1}\right\}\\
&= \frac{1}{9}\,e^{-t}+\lim_{z\to2}\left\{t\cdot\frac{e^{zt}}{z+1}-\frac{e^{zt}}{(z+1)^2}\right\}=\frac{1}{9}\,e^{-t}+\frac{1}{3}\,t\,e^{2t}-\frac{1}{9}\,e^{2t}.
\end{aligned}$$

(c) *Decomposition.* Using only the standard procedure this may give us an extremely tedious calculation. However, if we use some small tricks, this method may be quite reasonable:

$$
\begin{aligned}
\frac{z}{(z+1)^3(z-1)^2} &= \frac{1}{4}\cdot\frac{(z+1)^2-(z-1)^2}{(z+1)^3(z-1)^2} = \frac{1}{4}\cdot\frac{1}{(z+1)(z-1)^2} - \frac{1}{4}\cdot\frac{1}{(z+1)^3}\\
&= \frac{1}{8}\cdot\frac{(z+1)-(z-1)}{(z+1)(z-1)^2} - \frac{1}{4}\mathcal{L}\left\{\frac{1}{2!}t^2\,e^{-t}\right\}(z)\\
&= \frac{1}{8}\cdot\frac{1}{(z-1)^2} - \frac{1}{8}\cdot\frac{1}{(z+1)(z-1)} - \frac{1}{8}\mathcal{L}\left\{t^2e^{-t}\right\}(z)\\
&= \frac{1}{8}\mathcal{L}\left\{\frac{1}{1!}te^{+t}\right\}(z) - \frac{1}{16}\,\frac{(z+1)-(z-1)}{(z+1)(z-1)} - \frac{1}{8}\mathcal{L}\left\{t^2e^{-t}\right\}(z)\\
&= \frac{1}{16}\mathcal{L}\left\{e^{-t}\right\}(z) - \frac{1}{16}\mathcal{L}\left\{e^{t}\right\}(z) + \frac{1}{8}\mathcal{L}\left\{t\,e^{t}\right\}(z) - \frac{1}{8}\mathcal{L}\left\{t^2e^{-t}\right\}(z)\\
&= \mathcal{L}\left\{\frac{1}{16}e^{-t} - \frac{1}{16}e^{t} + \frac{1}{8}t\,e^{t} - \frac{1}{8}t^2e^{-t}\right\}(z).
\end{aligned}
$$

Then by the inverse Laplace transform,

$$
f(t) = \mathcal{L}^{\circ-1}\left\{\frac{z}{(z+1)^3(z-1)^2}\right\}(t) = \frac{1}{16}\,e^{-t}\left(1-2t^1\right) + \frac{1}{16}\,(2t-1)e^{t}.
$$

Residuum formula. This will also give some difficult computations:

$$\begin{aligned}
f(t) &= \operatorname{res}\left(\frac{z\,e^{zt}}{(z+1)^3(z-1)^2}\,;\,-1\right)+\operatorname{res}\left(\frac{z\,e^{zt}}{(z+1)^3(z-1)^2}\,;\,1\right)\\
&= \frac{1}{2}\lim_{z\to-1}\frac{d^2}{dz^2}\left\{\frac{z\,e^{zt}}{(z-1)^2}\right\}+\lim_{z\to1}\frac{d}{dz}\left\{\frac{z\,e^{zt}}{(z+1)^3}\right\}\\
&= \frac{1}{2}\lim_{z\to-1}\frac{d}{dz}\left\{t\cdot\frac{z\,e^{zt}}{(z-1)^2}+\frac{e^{zt}}{(z-1)^2}-\frac{2z\,e^{zt}}{(z-1)^3}\right\}\\
&\qquad+\lim_{z\to1}\left\{t\,\frac{z\,e^{zt}}{(z+1)^3}+\frac{e^{zt}}{(z+1)^3}-\frac{3z\,e^{zt}}{(z+1)^4}\right\}\\
&= \frac{1}{2}\lim_{z\to-1}\left\{t^2\,\frac{z\,e^{zt}}{(z-1)^2}+t\,\frac{e^{zt}}{(z-1)^2}-2t\,\frac{z\,e^{zt}}{(z-1)^3}+t\,\frac{e^{zt}}{(z-1)^2}\right.\\
&\qquad\left.-2\,\frac{e^{zt}}{(z-1)^3}-2t\,\frac{z\,e^{zt}}{(z-1)^3}-2\,\frac{e^{zt}}{(z-1)^3}+6\,\frac{z\,e^{zt}}{(z-1)^4}\right\}+\left\{t\cdot\frac{e^t}{2^3}+\frac{e^t}{2^3}-\frac{3\,e^t}{2^4}\right\}\\
&= \frac{1}{2}\,e^{-t}\left\{t^2\cdot\frac{-1}{(-2)^2}+t\cdot\frac{1}{(-2)^2}-2t\cdot\frac{(-1)}{(-2)^3}+t\cdot\frac{1}{(-2)^2}\right.\\
&\qquad\left.-\frac{2}{(-2)^3}-2t\cdot\frac{-1}{(-2)^3}-2\cdot\frac{1}{(-2)^3}+6\cdot\frac{(-1)}{(-2)^4}\right\}+\frac{1}{16}\,e^t\,\{2t+2-3\}\\
&= \frac{1}{2}\,e^{-t}\left\{-\frac{1}{4}\,t^2+\frac{1}{4}\,t-\frac{1}{4}\,t+\frac{1}{4}-\frac{1}{4}\,t+\frac{1}{4}-\frac{3}{8}\right\}+\frac{1}{16}\,e^t\,\{2t-1\}\\
&= \frac{1}{16}e^{-t}\,\{1-2t^2\}+\frac{1}{16}\,e^t\,\{2t-1\}.
\end{aligned}$$

(d) *Residuum formula.* We find

$$\begin{aligned}
f(t) &= \operatorname{res}\left(\frac{e^{zt}}{z(z+3)^2}\,;\,0\right)+\operatorname{res}\left(\frac{e^{zt}}{z(z+3)^2}\,;\,-3\right)=\frac{1}{9}+\frac{1}{1!}\lim_{z\to-3}\frac{d}{dz}\left\{\frac{e^{zt}}{z}\right\}\\
&= \frac{1}{9}+\lim_{z\to-3}\left\{t\cdot\frac{e^{zt}}{z}-\frac{e^{zt}}{z^2}\right\}=\frac{1}{9}-\frac{1}{3}\,t\,e^{-3t}-\frac{1}{9}\,e^{-3t}.
\end{aligned}$$

Example 2.26 *Find the inverse Laplace transform of*

$$(a)\ \ \frac{1}{z^3+1},\qquad (b)\ \ \frac{1}{z^4+4},\qquad (c)\ \ \frac{1}{z^3\,(z^2+1)}.$$

It is possible in all three cases to apply the residuum formula. Furthermore, since a decomposition is rather difficult in all three cases, we shall at least for the former two tasks only use the *residuum formula.*

(a) The roots of the denominator $A(z)=z^3+1$ are

$$-1\qquad\text{and}\qquad\frac{1}{2}\pm i\,\frac{\sqrt{3}}{2},$$

and we notice that if z_0 is one of these, then

$$\frac{1}{Q'(z_0)}=\frac{1}{3z_0^2}=\frac{z_0}{3z_0^3}=-\frac{1}{3}\,z_0.$$

Hence, we get by Heaviside's expansion theorem,

$$
\begin{aligned}
f(t) &= \operatorname{res}\left(\frac{e^{zt}}{z^3+1}\,;\,-1\right) + \operatorname{res}\left(\frac{e^{zt}}{z^3+1}\,;\,\frac{1}{2}+i\,\frac{\sqrt{3}}{2}\right) + \operatorname{res}\left(\frac{e^{zt}}{z^3+1}\,;\,\frac{1}{2}-i\,\frac{\sqrt{3}}{2}\right) \\
&= -\frac{1}{3}\left\{-e^{-t}+\left(\frac{1}{2}+i\,\frac{\sqrt{3}}{2}\right)\exp\left(\frac{1}{2}\,t+i\,\frac{\sqrt{3}}{2}\,t\right)\right\}+\left(\frac{1}{2}-i\,\frac{\sqrt{3}}{2}\right)\exp\left(\frac{1}{2}\,t-i\,\frac{\sqrt{3}}{2}\,t\right) \\
&= \frac{1}{3}\,e^{-t}-\frac{1}{3}\,e^{\frac{1}{2}\,t}\left\{\frac{1}{2}\,e^{i\,\frac{\sqrt{3}}{2}\,t}+\frac{1}{2}\,e^{-i\,\frac{\sqrt{3}}{2}\,t}-\sqrt{3}\cdot\frac{1}{2i}\left(e^{i\,\frac{\sqrt{3}}{2}\,t}-e^{-i\,\frac{\sqrt{3}}{2}\,t}\right)\right\} \\
&= \frac{1}{3}\,e^{-t}-\frac{1}{3}\,e^{\frac{1}{2}\,t}\left\{\cos\left(\frac{\sqrt{3}}{2}\,t\right)-\sqrt{3}\cdot\sin\left(\frac{\sqrt{3}}{2}\,t\right)\right\} \\
&= \frac{1}{3}\,e^{-t}+\frac{2}{3}\,e^{\frac{1}{2}\,t}\sin\left(\frac{\sqrt{3}}{2}\,t-\frac{\pi}{6}\right).
\end{aligned}
$$

(b) The roots of the denominator $Q(z)=z^4+4$ are $z=\pm 1\pm i$, and if z_0 is one of these, then

$$
\frac{1}{Q'(z_0)}=\frac{1}{4z_0^3}=\frac{z_0}{4z_0^4}=-\frac{1}{16}\,z_0.
$$

All poles are simple, so by Heaviside's expansion theorem,

$$
\begin{aligned}
f(t) &= \operatorname{res}\left(\frac{e^{zt}}{z^4+4}\,;\,1+i\right)+\operatorname{res}\left(\frac{e^{zt}}{z^4+4}\,;\,1-i\right) \\
&\qquad +\operatorname{res}\left(\frac{e^{zt}}{z^4+4}\,;\,-1+i\right)+\operatorname{res}\left(\frac{e^{zt}}{z^4+4}\,;\,-1-i\right) \\
&= -\frac{1}{16}\left\{(1+i)e^{t+it}+(1-i)e^{t-it}+(-1+i)e^{-t+it}+(-1-i)e^{-t-it}\right\} \\
&= -\frac{1}{16}\,e^{t}\left\{e^{it}+e^{-it}+i\left(e^{it}-e^{-it}\right)\right\}-\frac{1}{16}\,e^{-t}\left\{-e^{it}-e^{-it}+\left(e^{it}-e^{-it}\right)\right\} \\
&= -\frac{1}{16}\,e^{t}\left\{2\cos t-2\sin t\right\}-\frac{1}{16}\,e^{-t}\left\{-2\cos t-2\sin t\right\} \\
&= \frac{1}{8}\,e^{t}\left\{\sin t-\cos t\right\}+\frac{1}{8}\,e^{-t}\left\{\sin t+\cos t\right\} \\
&= \frac{1}{4}\,\cosh t\cdot\sin t-\frac{1}{4}\,\sinh t\cdot\cos t.
\end{aligned}
$$

(c) Here it is actually possible to use the *Rule of Convolution*, because

$$
\mathcal{L}\{f\}=\frac{1}{z^3\left(z^2+1\right)}=\frac{1}{2}\,\mathcal{L}\left\{t^2\right\}\cdot\mathcal{L}\{\sin t\}.
$$

Thus

$$
\begin{aligned}
f(t) &= \frac{1}{2}\int_0^t (t-\tau)^2\sin\tau\,d\tau=\frac{1}{2}\left[-(t-\tau)^2\cos\tau\right]_0^t-\int_0^t (t-\tau)\cos\tau\,d\tau \\
&= \frac{1}{2}\,t^2-\left[(t-\tau)\sin\tau\right]_0^t-\int_0^t \sin\tau\,d\tau=\frac{1}{2}\,t^2+\left[\cos\tau\right]_0^t=\frac{1}{2}\,t^2+\cos t-1.
\end{aligned}
$$

If we instead apply the *residuum formula*, then

$$\begin{aligned}
f(t) &= \operatorname{res}\left(\frac{e^{zt}}{z^3\,(z^2+1)}\,;\,0\right)+\operatorname{res}\left(\frac{e^{zt}}{z^3\,(z^2+1)}\,;\,i\right)+\operatorname{res}\left(\frac{e^{zt}}{z^3\,(z^2+1)}\,;\,-i\right)\\
&= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\left\{\frac{e^{zt}}{z^2+1}\right\}+\frac{e^{it}}{i^3\cdot 2i}+\frac{e^{-it}}{(-i)^3(-2i)}\\
&= \frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left\{t\cdot\frac{e^{zt}}{z^2+1}-\frac{2z\,e^{zt}}{\left(z^2+1\right)^2}\right\}+\frac{1}{2}\left\{e^{it}+e^{-it}\right\}\\
&= \frac{1}{2}\lim_{z\to 0}\left\{t^2\,\frac{e^{zt}}{z^2+1}-\frac{4z\,t\,e^{zt}}{\left(z^2+1\right)^2}-\frac{2\,e^{zt}}{\left(z^2+1\right)^2}+\frac{2z\,e^{zt}\cdot 2\cdot 2z}{\left(z^2+1\right)^3}\right\}+\cos t\\
&= \frac{1}{2}t^2-\frac{1}{2}\cdot 2+\cos t=\frac{1}{2}t^2-1+\cos t.
\end{aligned}$$

Example 2.27 *Find the inverse Laplace transform of*

$(a)\ \dfrac{z}{(z+1)^2\,(z^2+3z-10)},\qquad (b)\ \dfrac{2z^2-4}{(z-2)(z-3)(z+1)},\qquad (c)\ \dfrac{5z^2-15z-11}{(z+1)(z-2)^3}.$

We may use the residuum formula in all three cases.

(a) We get by a *decomposition*,

$$\begin{aligned}
\frac{z}{(z+1)^2\,(z^2+3z-10)} &= \frac{z}{(z+1)^2(z+5)(z-2)}\\
&= \frac{-1}{4(-3)}\cdot\frac{1}{(z+1)^2}+\frac{A}{z+1}+\frac{-5}{4^2\cdot(-7)}\cdot\frac{1}{z+5}+\frac{2}{3^2\cdot 7}\cdot\frac{1}{z-2}\\
&= \frac{1}{2}\cdot\frac{1}{(z+1)^2}-\frac{11}{144}\cdot\frac{1}{z+1}+\frac{5}{112}\cdot\frac{1}{z+5}+\frac{2}{63}\cdot\frac{1}{z-2}.
\end{aligned}$$

Hence,

$$f(t)=\mathcal{L}^{\circ -1}\left\{\frac{z}{(z+1)^2\,(z^2+3z-10)}\right\}(t)=\frac{1}{12}\,t\,e^{-t}-\frac{11}{144}\,e^{-t}+\frac{5}{112}\,e^{-5t}+\frac{2}{63}\,e^{2t}.$$

Alternatively, we get by the *residuum formula*, where the poles are -1, -5 and 2,

$$\begin{aligned}
f(t) &= \operatorname{res}\left(\frac{z\,e^{zt}}{(z+1)^2(z+5)(z-2)}\,;\,-1\right)+\operatorname{res}\left(\frac{z\,e^{zt}}{(z+1)^2(z+5)(z-2)}\,;\,-5\right)\\
&\qquad+\operatorname{res}\left(\frac{z\,e^{zt}}{(z+1)^2(z+5)(z-2)}\,;\,2\right)\\
&= \lim_{z\to -1}\frac{d}{dz}\left\{\frac{z\,e^{zt}}{z^2+3z-10}\right\}+\frac{-5\,e^{-5t}}{(-4)^2(-5-2)}+\frac{2\,e^{2t}}{3^2(2+5)}\\
&= \lim_{z\to -1}\left\{\frac{e^{zt}+tze^{zt}}{z^2+3z-10}-\frac{ze^{zt}(2z+3)}{\left(z^2+3z-10\right)^2}\right\}+\frac{5}{7\cdot 16}\,e^{-5t}+\frac{2}{7\cdot 9}\,e^{2t}\\
&= \frac{e^{-t}-t\,e^{-t}}{1-3-10}+\frac{e^{-t}(-2+3)}{(-12)^2}+\frac{5}{112}\,e^{-5t}+\frac{2}{63}\,e^{2t}\\
&= \frac{1}{12}\,t\,e^{-t}-\frac{11}{144}\,e^{-t}+\frac{5}{112}\,e^{-5t}+\frac{2}{63}\,e^{2t}.
\end{aligned}$$

(b) We *decompose*

$$
\begin{aligned}
\frac{2z^2-4}{(z-2)(z-3)(z+1)} &= \frac{2\cdot 4-4}{(-1)\cdot 3}\cdot\frac{1}{z-2}+\frac{2\cdot 9-4}{1\cdot 4}\cdot\frac{1}{z-3}+\frac{2-4}{(-3)(-4)}\cdot\frac{1}{z+1} \\
&= -\frac{4}{3}\cdot\frac{1}{z-2}+\frac{7}{2}\cdot\frac{1}{z-3}-\frac{1}{6}\cdot\frac{1}{z+1},
\end{aligned}
$$

from which

$$
f(t)=\mathcal{L}^{\circ -1}\left\{\frac{2z^2-4}{(z-2)(z-3)(z+1)}\right\}(t)=-\frac{1}{6}\,e^{-t}-\frac{4}{3}\,e^{2t}+\frac{7}{2}\,e^{3t}.
$$

ALTERNATIVELY we apply the *residuum formula*. Then

$$
\begin{aligned}
f(t) &= \operatorname{res}\left(\frac{(2z^2-4)\,e^{zt}}{(z-2)(z-3)(z+1)};-1\right)+\operatorname{res}\left(\frac{(2z^2-4)\,e^{zt}}{(z-2)(z-3)(z+1)};2\right) \\
&\qquad +\operatorname{res}\left(\frac{(2z^2-4)\,e^{zt}}{(z-2)(z-3)(z+1)};3\right) \\
&= \frac{(2-4)e^{-t}}{(-3)(-4)}+\frac{(8-4)e^{2t}}{2-3)3}+\frac{18-4)e^{3t}}{(3-2)(3+1)}=-\frac{1}{6}\,e^{-t}-\frac{4}{3}\,e^{2t}+\frac{7}{2}\,e^{3t}.
\end{aligned}
$$

(c) A *decomposition* gives

$$\frac{5z^2-15z-11}{(z+1)(z-2)^3}=\frac{5+15-11}{(-3)^3}\cdot\frac{1}{z+1}+\frac{P_1(z)}{(z-2)^3}=-\frac{1}{3}\cdot\frac{1}{z+1}+\frac{P_1(z)}{(z-2)^3},$$

where

$$\begin{aligned}P_1(z) &= \frac{1}{z+1}\left\{5z^2-15z-11+\frac{1}{3}(z-2)^3\right\}=\frac{1}{3}\cdot\frac{1}{z+1}\left\{z^2+8z-41\right\}(z+1)\\ &= \frac{1}{3}\left(z^2+8z-41\right).\end{aligned}$$

Then by an insertion,

$$\begin{aligned}\frac{5z^2-15z-11}{(z+1)(z-2)^3} &= -\frac{1}{3}\frac{1}{z+1}+\frac{1}{3}\frac{z^2+8z-41}{(z-2)^3}=-\frac{1}{3}\frac{1}{z+1}+\frac{1}{3}\frac{\left(z^2-4z+4\right)+(12z-24)-21}{(z-2)^3}\\ &= -\frac{1}{3}\frac{1}{z+1}+\frac{1}{3}\frac{1}{z-2}+\frac{4}{(z-2)^2}-\frac{7}{(z-2)^2},\end{aligned}$$

so

$$f(t)=\mathcal{L}^{\circ-1}\left\{\frac{5z^2-15z-11}{(z+1)(z-2)^3}\right\}(t)=-\frac{1}{3}e^{-t}-\frac{7}{2}t^2e^{2t}+4t\,e^{2t}+\frac{1}{3}e^{2t}.$$

ALTERNATIVELY we apply the *residuum formula*,

$$\begin{aligned}f(t) &= \text{res}\left(\frac{5z^2-15z-11}{(z+1)(z-2)^3}e^{zt};-1\right)+\text{res}\left(\frac{5z^2-15z-11}{(z+1)(z-2)^3}e^{zt};2\right)\\ &= \frac{5+15-11}{(-1-2)^3}e^{-t}+\frac{1}{2!}\lim_{z\to 2}\frac{d^2}{dz^2}\left\{\frac{5z^2-15z-11}{z+1}e^{zt}\right\}\\ &= -\frac{1}{3}e^{-t}+\frac{1}{2}\lim_{z\to 2}\frac{d^2}{dz^2}\left\{5z\,e^{zt}-20\,e^{zt}+\frac{9\,e^{zt}}{z+1}\right\},\end{aligned}$$

where we use that it is easier to differentiate after the division by a polynomial. Then

$$\begin{aligned}f(t) &= -\frac{1}{3}e^{-t}+\frac{1}{2}\lim_{z\to 2}\frac{d}{dz}\left\{5e^{zt}+5tze^{zt}-20te^{zt}+\frac{9te^{zt}}{z+1}-\frac{9e^{zt}}{(z+1)^2}\right\}\\ &= -\frac{1}{3}e^{-t}++\frac{1}{2}\lim_{z\to 2}\left\{10te^{zt}+5t^2ze^{zt}-20t^2e^{zt}+\frac{9t^2e^{zt}}{z+1}-\frac{18te^{zt}}{(z+1)^2}+\frac{18e^{zt}}{(z+1)^3}\right\}\\ &= -\frac{1}{3}e^{-t}+5te^{2t}+5t^2e^{2t}-10t^2e^{2t}+\frac{3}{2}t^2e^{2t}-te^{2t}+\frac{1}{3}e^{2t}\\ &= -\frac{1}{3}e^{-t}-\frac{7}{2}t^2e^{2t}+4t\,e^{2t}+\frac{1}{3}e^{2t}.\end{aligned}$$

Example 2.28 *Find the inverse Laplace transforms of*

$$(a)\ \frac{6z-4}{z^2-4z+20}, \qquad (b)\ \frac{3z+7}{z^2-2z-3}, \qquad (c)\ \frac{4z+12}{z^2+8z+16}.$$

In all three cases it is easier to make an *inspection* than to use the residuum formula, although the assumptions of this use are all fulfilled.

(a) We get by a small reformulation,

$$\begin{aligned}\mathcal{L}\{f\}(z) &= \frac{6z-4}{z^2-4z+20} = \frac{6z-4}{(z-2)^2+4^2} = \frac{6(z-2)+2\cdot 4}{(z-2)^2+4^2}\\ &= 6\,\mathcal{L}\{\cos 4t\}(z-2)+2\,\mathcal{L}\{\sin 4t\}(z-2) = \mathcal{L}\left\{6\,e^{2t}\cos 4t+2\,e^{2t}\sin 4t\right\}(z),\end{aligned}$$

so we conclude that

$$f(t) = 6\,e^{2t}\cos 4t + 2\,e^{2t}\sin 4t.$$

(b) Here it follows by a *decomposition*,

$$\begin{aligned}\mathcal{L}\{f\}(z) &= \frac{3z+7}{z^2-2z-3} = \frac{3z+7}{(z+1)(z-3)} = \frac{-3+7}{-1-3}\cdot\frac{1}{z+1}+\frac{3\cdot 3+7}{3+1}\cdot\frac{1}{z-3} = -\frac{1}{z+1}+\frac{4}{z-3}\\ &= \mathcal{L}\left\{4\,e^{3t}-e^{-t}\right\}(z),\end{aligned}$$

hence

$$f(t) = 4\,e^{3t} - e^{-t}.$$

(c) Here we get directly,

$$\begin{aligned}\mathcal{L}\{f\}(z) &= \frac{4z+12}{z^2+8z+16} = \frac{4z+12}{(z+4)^4} = \frac{4z+16-4}{(z+4)^2} = \frac{4}{z+4}-\frac{4}{(z+4)^2}\\ &= 4\,\mathcal{L}\left\{e^{-4t}\right\}-4\,\mathcal{L}\left\{t\,e^{-4t}\right\}(z),\end{aligned}$$

hence

$$f(t) = 4\,e^{-4t} - 4t\,e^{-4t}.$$

Example 2.29 *Find the Laplace transforms of*

$$(a)\quad \frac{3z+1}{(z-1)\left(z^2+1\right)}, \qquad (b)\quad \frac{z^2+2z+3}{\left(z^2+2z+2\right)\left(z^2+2z+5\right)}, \qquad (c)\quad \frac{z^3+5z^2+4z+20}{z^2\left(z^2+9\right)}.$$

It is possible in all three cases to apply the residuum formula, but it will be easier to use other methods.

(a) We get by a *decomposition*,

$$\frac{3z+1}{(z-1)\left(z^2+1\right)} = \frac{3\cdot 1+1}{1+1}\cdot\frac{1}{z-1}+\frac{Az+B}{z^2+1} = \frac{2}{z-1}+\frac{Az+B}{z^2+1},$$

where

$$\frac{Az+B}{z^2+1} = \frac{3z+1}{(z-1)\left(z^2+1\right)}-\frac{2}{z-1} = \frac{3z+1-2z^2-2}{(z-1)\left(z^2+1\right)} = \frac{-2z^2+3z-1}{(z-1)\left(z^2+1\right)} = \frac{-2z+1}{z^2+1}.$$

Hence

$$\mathcal{L}\{f\} = \frac{3z+1}{(z-1)\left(z^2+1\right)} = \frac{2}{z-1}-2\,\frac{z}{z^2+1}+\frac{1}{z^2+1},$$

and we conclude that

$$f(t) = 2\,e^t - 2\,\cos t + \sin t.$$

(b) Using here the *change of variable* $w=(z+1)^2$ followed by a *decomposition*, we get

$$\begin{aligned}
\mathcal{L}\{f\} &= \frac{z^2+2z+3}{\left(z^2+2z+2\right)\left(z^2+2z+5\right)} = \frac{w+2}{(w+1)(w+4)} \\
&= \frac{-1+2}{-1+4}\cdot\frac{1}{w+1}+\frac{-4+2}{-4+1}\cdot\frac{1}{w+4} = \frac{1}{3}\,\frac{1}{w+1}+\frac{2}{3}\,\frac{1}{w+4} \\
&= \frac{1}{3}\,\frac{1}{(z+1)^2+1}+\frac{1}{3}\,\frac{2}{(z+1)^2+2^2},
\end{aligned}$$

and we conclude that

$$f(t) = \frac{1}{3}\,e^{-t}\sin t+\frac{1}{3}\,e^{-t}\sin 2t.$$

(c) The *decomposition* needed here is a little tricky. It is based on the fact that z only occurs in the form z^2 in the denominator. We get

$$\begin{aligned}
\mathcal{L}\{f\}(z) &= \frac{z^3+5z^2+4z+20}{z^2\left(z^2+9\right)} = z\cdot\frac{z^2+4}{z^2\left(z^2+9\right)}+\frac{5z^2+20}{z^2\left(z^2+9\right)} \\
&= z\left\{\frac{4}{9}\cdot\frac{1}{z^2}+\frac{-9+4}{-9}\cdot\frac{1}{z^2+9}\right\}+\frac{20}{9}\cdot\frac{1}{z^2}+\frac{-45+20}{-9}\cdot\frac{1}{z^2+9} \\
&= \frac{20}{9}\cdot\frac{1}{z^2}+\frac{4}{9}\cdot\frac{1}{z}+\frac{5}{9}\cdot\frac{z}{z^2+3^2}+\frac{25}{27}\cdot\frac{3}{z^2+3^2} \\
&= \mathcal{L}\left\{\frac{20}{9}\,t+\frac{4}{9}+\frac{5}{9}\,\cos 3t+\frac{25}{27}\,\sin 3t\right\}(z).
\end{aligned}$$

Finally, this implies that the inverse Laplace transform is given by

$$f(t) = \frac{20}{9}\,t+\frac{4}{9}+\frac{5}{9}\,\cos 3t+\frac{25}{27}\,\sin 3t.$$

Example 2.30 *Find the inverse Laplace transforms of*

$$(a)\quad \frac{e^{-z}}{z^2+1}, \qquad (b)\quad \frac{z}{(z+1)^2}+\frac{e^{-z}}{z}, \qquad (c)\quad \frac{e^{-2z}}{(z-1)^4}.$$

We cannot apply the residuum formula in any of these cases, because the assumptions are not satisfied. Instead one should first try the Rule of Shifting.

(a) Since

$$\mathcal{L}\{[\sin t\}(z) = \frac{1}{z^2+1},$$

if follows from the Rule of Shifting that if

$$f(t) = \begin{cases} \sin(t-1) & \text{for } t \geq 1, \\ 0 & \text{for } t < 1, \end{cases}$$

then

$$\mathcal{L}\{f\}(z) = e^{-z}\,\mathcal{L}\{\sin t\}(z) = \frac{e^{-z}}{z^2+1},$$

so we conclude that

$$\mathcal{L}^{\circ -1}\left\{\frac{e^{-z}}{z^2+1}\right\}(t) = \sin(t-1)\cdot \chi_{[1,+\infty[}(t).$$

(b) It follows from

$$\mathcal{L}\left\{\chi_{[1,+\infty[}\right\}(z) = \frac{e^{-z}}{z},$$

that

$$\begin{aligned}\mathcal{L}^{\circ -1}\left\{\frac{z}{(z+1)^2}+\frac{e^{-z}}{z}\right\}(t) &= \mathcal{L}^{\circ -1}\left\{\frac{z+1-1}{(z+1)^2}\right\}(t)+\chi_{[1,+\infty[}(t)\\ &= \mathcal{L}^{\circ -1}\left\{\frac{1}{z+1}-\frac{1}{(z+1)^2}\right\}(t)+\chi_{[1,+\infty[}(t)\\ &= e^{-t}-t\,e^{-t}+\chi_{[1,+\infty[}(t).\end{aligned}$$

(c) Since

$$\mathcal{L}\left\{\frac{1}{(z-1)^4}\right\}(t) = \frac{1}{3!}\,t^3e^t,$$

we may apply the Rule of Shifting,

$$\mathcal{L}^{\circ -1}\left\{\frac{e^{-2z}}{(z-1)^4}\right\}(t) = \frac{1}{6}\,(t-2)^3e^{t-2}\,\chi_{[2,+\infty[}(t) = \frac{1}{6e^2}\,(t-2)^3e^t\,\chi_{[2,+\infty[}(t).$$

Example 2.31 *Find the inverse Laplace transforms of*

$$(a)\;\; \frac{e^{-2z}}{z^2}, \qquad (b)\;\; \frac{8\,e^{-3z}}{z^2+4}, \qquad (c)\;\; \frac{z\,e^{-2z}}{z^2+3z+2}.$$

(a) Since

$$\mathcal{L}^{\circ -1}\left\{\frac{1}{z^2}\right\}(t) = t,$$

it follows from the *Rule of Shifting* that

$$\mathcal{L}^{\circ -1}\left\{\frac{e^{-2z}}{z^2}\right\}(t) = (t-2)\,\chi_{[2,+\infty[}(t).$$

(b) Since

$$\mathcal{L}^{\circ -1}\left\{\frac{8}{z^2+4}\right\}(t) = 4\,\sin 2t,$$

it follows from the *Rule of Shifting* that

$$\mathcal{L}^{\circ -1}\left\{\frac{8\,e^{-3z}}{z^2+4}\right\}(t) = 4\,\sin(2t-6)\cdot\chi_{[3,+\infty[}(t).$$

(c) From

$$\frac{1}{z^2+3z+2} = \frac{z}{(z+1)(z+2)} = -\frac{1}{z+1} + \frac{2}{z+2},$$

follows that

$$\mathcal{L}^{\circ -1}\left\{\frac{z}{z^2+3z+2}\right\}(t) = \mathcal{L}^{\circ -1}\left\{-\frac{1}{z+1} + \frac{2}{z+2}\right\}(t) = 2\,e^{-2t} - e^{-t}.$$

Then apply the *Rule of Shifting*,

$$\mathcal{L}^{\circ -1}\left\{\frac{z\,e^{-2z}}{z^2+3z+2}\right\}(t) = \left\{2\,e^{-2(t-2)} - e^{-(t-2)}\right\} \cdot \chi_{[2,+\infty[}(t).$$

Example 2.32 *Find the inverse Laplace transforms of*

$$(a)\quad \frac{e^{-5z}}{(z-2)^4}, \qquad (b)\quad \frac{z\,\exp\left(-\frac{4\pi}{5}\,z\right)}{z^2+25}, \qquad (c)\quad \frac{(z+1)e^{-\pi z}}{z^2+z+1}.$$

(a) Since

$$\frac{1}{(z-2)^4} = \frac{1}{6}\,\mathcal{L}\left\{t^3 e^{2t}\right\}(z),$$

if follows from the *Rule of Shifting* that

$$\mathcal{L}^{\circ -1}\left\{\frac{e^{-5z}}{(z-2)^4}\right\}(t) = \frac{1}{6}\,(t-5)^3 e^{2t-10} \cdot \chi_{[5,+\infty[}(t).$$

(b) Since

$$\mathcal{L}\{\cos 5t\}(z) = \frac{z}{z^2+25},$$

it follows from the *Rule of Shifting* that

$$\mathcal{L}^{\circ -1}\left\{\frac{z\,\exp\left(-\frac{4\pi}{5}\,z\right)}{z^2+25}\right\}(t) = \cos\left(5\left(t+\frac{4\pi}{5}\right)\right) \cdot \chi_{[\frac{4\pi}{5},+\infty[}(t) = \cos(5t) \cdot \chi_{[\frac{4\pi}{5},+\infty[}(t).$$

(c) Since

$$\frac{z+1}{z^2+z^1} = \frac{\left(z+\frac{1}{2}\right)+\frac{1}{2}}{\left(z+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \mathcal{L}\left\{e^{-\frac{1}{2}\,t}\left(\cos\left(\frac{\sqrt{3}}{2}\,t\right)+\frac{1}{\sqrt{3}}\,\sin\left(\frac{\sqrt{3}}{2}\,t\right)\right)\right\}(z),$$

it follows from the *Rule of Shifting* that

$$f(t) = \mathcal{L}^{\circ -1}\left\{\frac{(z+1)e^{-\pi z}}{z^2+z+1}\right\}(t) = e^{-\frac{1}{2}\,(t-\pi)}\left\{\cos\left(\frac{\sqrt{3}}{2}\,(t-\pi)\right)+\frac{1}{\sqrt{3}}\,\sin\left(\frac{\sqrt{3}}{2}\,(t-\pi)\right)\right\}\cdot\chi_{[\pi,+\infty[}(t).$$

Example 2.33 *Find the inverse Laplace transforms of*

$$(a)\quad \frac{z}{\left(z^2+1\right)^2}, \qquad (b)\quad \frac{e^{-z}\left(1-e^{-z}\right)}{z\left(z^2+1\right)}.$$

(a) We get by the *Rule of Convolution*,

$$\begin{aligned}\mathcal{L}\{f\}(z) &= \frac{z}{\left(z^2+1\right)^2} = \frac{z}{z^2+1}\cdot\frac{1}{z^2+1} = \mathcal{L}\{\cos t\}(z)\cdot\mathcal{L}\{\sin t\}(z)\\ &= \mathcal{L}\{(\sin\star\cos)(t)\}(z).\end{aligned}$$

Thus we conclude that

$$\begin{aligned}f(t) &= \int_0^t \sin\tau\cdot\cos(t-\tau)\,d\tau = \frac{1}{2}\int_0^t\{\sin t-\sin(2\tau-t)\}\,d\tau = \frac{1}{2}\,t\,\sin t+\frac{1}{4}\left[\cos(2\tau-t)\right]_0^t\\ &= \frac{1}{2}\,t\,\sin t+\frac{1}{4}\,\cos t-\frac{1}{4}\,\cos t = \frac{1}{2}\,t\,\sin t.\end{aligned}$$

Since

$$\mathcal{L}\{\sin t\}(z) = \frac{1}{z^2+1},$$

we ALTERNATIVELY get by the *Rule of Multiplication by* t, where the formula should be read from the right to the left,

$$\frac{z}{\left(z^2+1\right)^2} = -\frac{1}{2}\,\frac{d}{dz}\left\{\frac{1}{z^2+1}\right\} = \frac{1}{2}\cdot(-1)^1\,\frac{d}{dz}\,\mathcal{L}\{\sin t\}(z) = \mathcal{L}\left\{\frac{1}{2}\,t\,\sin t\right\}(z),$$

hence

$$f(t) = \mathcal{L}^{\circ-1}\left\{\frac{z}{\left(z^2+1\right)^2}\right\}(t) = \frac{1}{2}\,t\,\sin t.$$

(b) We get by a *decomposition* that

$$\frac{1}{z\left(z^2+1\right)} = \frac{1}{z}-\frac{z}{z^2+1} = \mathcal{L}\{1-\cos t\}(z),$$

so it follows by the *Rule of Shifting* that

$$\begin{aligned}&\mathcal{L}^{\circ-1}\left\{\frac{e^{-z}\left(1-e^{-z}\right)}{z\left(z^2+1\right)}\right\}(t) = \mathcal{L}^{\circ-1}\left\{\frac{e^{-z}}{z\left(z^2+1\right)}-\frac{e^{-2z}}{z\left(z^2+1\right)}\right\}(t)\\ &\quad = \{1-\cos(t-1)\}\,\chi_{[1,+\infty[}(t)-\{1-\cos(t-2)\}\,\chi_{[2,+\infty[}(t).\end{aligned}$$

Example 2.34 *Find a series expansion of the inverse Laplace transform*

$$f(t) = \sum_{n=0}^{+\infty} a_n t^n$$

of

$$F(z) = \frac{1}{z} \exp\left(\frac{1}{z}\right), \qquad z \in \mathbb{C} \setminus \{0\}.$$

The function $f(t)$ is a transcendental function, which cannot be expressed by elementary functions.

We get

$$\begin{aligned}\mathcal{L}\{f\}(z) &= F(z) = \frac{1}{z} \exp\left(\frac{1}{z}\right) = \sum_{n=0}^{+\infty} \frac{1}{n!} \cdot \frac{1}{z^{n+1}} = \sum_{n=0}^{+\infty} \frac{1}{(n!)^2} \cdot \frac{n!}{z^{n+1}} = \sum_{n=0}^{+\infty} \frac{1}{(n!)^2} \mathcal{L}\left\{t^n\right\}(z) \\ &= \mathcal{L}\left\{\sum_{n=0}^{+\infty} \frac{1}{(n!)^2} t^n\right\}(z).\end{aligned}$$

Formally we have

$$f(t) = \sum_{n=0}^{+\infty} \frac{1}{(n!)^2} t^n.$$

It is, however, obvious that this series is convergent for every $t \in \mathbb{R}$.

Remark 2.3 It is possible to express the sum by a Bessel function. $\Diamond$

An ALTERNATIVE procedure is the following: First we note that since

$$\exp\left(\frac{1}{z}\right) \to \exp(0) = 1 \qquad \text{for } z \to \infty,$$

we have the estimate

$$|F(z)| \leq \frac{c}{|z|} \qquad \text{for } |z| \geq R.$$

This means that the inverse Laplace transform $f(t)$ of $F(z)$ can be expressed by a residuum formula,

$$f(t) = \text{res}\left(\frac{1}{z}\exp\left(\frac{1}{z}\right)\cdot e^{zt}\,;\,0\right).$$

Here 0 is an essential singularity, so we have to use a series expansion. Clearly, a_{-1} is the constant term of

$$\exp\left(\frac{1}{z}\right)\cdot\exp(zt) = \sum_{n=0}^{+\infty}\frac{1}{n!}\cdot\frac{1}{z^n}\cdot\sum_{m=0}^{+\infty}\frac{t^m}{m!}\,z^m,$$

so $a_{-1}(t)$ is obtained by putting $m = n$, and then

$$f(t) = a_{-1}(t) = \sum_{n=0}^{+\infty}\frac{1}{(n!)^2}\,t^n, \qquad t \in \mathbb{R},$$

because one as before notes that the series has the radius of convergence $+\infty$.

Example 2.35 *Find the inverse Laplace transforms of*

$$(a)\ \sin\frac{1}{z}, \quad (b)\ \exp\left(\frac{1}{z}\right) - 1, \quad (c)\ \sinh\left(\frac{1}{z}\right), \quad (d)\ \cosh\left(\frac{1}{z}\right) - 1.$$

(a) It follows by a series expansion that

$$\begin{aligned}\mathcal{L}\{f\}(z) &= \sin\frac{1}{z} = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!}\cdot\frac{1}{z^{2n+1}} = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!(2n)!}\cdot\frac{(2n)!}{z^{2n+1}} \\ &= \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!(2n)!}\,\mathcal{L}\left\{t^{2n}\right\}(z) = \mathcal{L}\left\{\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!(2n)!}\,t^{2n}\right\}(z),\end{aligned}$$

and since the latter series is convergent for every $t \in \mathbb{R}$, we get

$$f(t) = \mathcal{L}^{\circ-1}\left\{\sin\frac{1}{z}\right\}(t) = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!(2n)!}\,t^{2n}, \qquad t \in \mathbb{R}.$$

ALTERNATIVELY, the series expansion

$$\sin\frac{1}{z}=\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!}\,\frac{1}{z^{2n+1}}=\frac{1}{z}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!}\,\frac{1}{z^{2n}},\qquad\text{for } z\neq 0,$$

gives an estimate of the type

$$\left|\sin\frac{1}{z}\right|\leq\frac{C}{|z|}\qquad\text{for } |z|\geq R.$$

Here, $z=0$ is the only singularity (unfortunately an essential singularity), so we get by the inverse Laplace transform by a residuum formula,

$$\mathcal{L}^{\circ-1}\left\{\sin\frac{1}{z}\right\}(t)=\operatorname{res}\left(e^{zt}\sin\frac{1}{z}\,;\,0\right)=a_{-1}(t),$$

where $a_{-1}(t)$ is the coefficient of $\dfrac{1}{z}$ in the Laurent series expansion of $e^{zt}\sin\dfrac{1}{z}$. Since

$$e^{zt}\sin\frac{1}{z}=\sum_{m=0}^{+\infty}\frac{t^m}{m!}\,z^m\cdot\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!}\cdot\frac{1}{z^{2n+1}},$$

it follows from *Weierstraß's double series theorem* that $a_{-1}(t)$ corresponds to $m=2n$, thus

$$\mathcal{L}^{\circ-1}\left\{\sin\frac{1}{z}\right\}(t)=a_{-1}(t)=\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!(2n+1)!}\,t^{2n}.$$

(b) By a series expansion,

$$\begin{aligned}\mathcal{L}\{f\}(z)&=\exp\left(\frac{1}{z}\right)-1=\sum_{n=1}^{+\infty}\frac{1}{n!}\cdot\frac{1}{z^n}=\sum_{n=0}^{+\infty}\frac{1}{(n+1)!n!}\cdot\frac{n!}{z^{n+1}}\\&=\sum_{n=0}^{+\infty}\frac{1}{n!(n+1)!}\,\mathcal{L}\left\{t^n\right\}(z)=\mathcal{L}\left\{\sum_{n=0}^{+\infty}\frac{1}{(n)!(n+1)!}\,t^n\right\}(z),\end{aligned}$$

and since the latter series is convergent for every $t\in\mathbb{R}$, we get

$$f(t)=\mathcal{L}^{\circ-1}\left\{\exp\left(\frac{1}{z}\right)-1\right\}(t)=\sum_{n=0}^{+\infty}\frac{1}{n!(n+1)!}\,t^n,\qquad t\in\mathbb{R}.$$

ALTERNATIVELY we conclude from the series expansion

$$\exp\left(\frac{1}{z}\right)-1=\sum_{n=1}^{+\infty}\frac{1}{n!}\cdot\frac{1}{z^n}\qquad\text{for } z\neq 0,$$

that the limit value for $z\left\{\exp\left(\dfrac{1}{z}\right)-1\right\}$ is 1 when $z\to\infty$. Hence we have estimates of the type

$$\left|\exp\left(\frac{1}{z}\right)-1\right|\leq\frac{C}{|z|}\qquad\text{for } |z|\geq R.$$

Here $z=0$ is the only singularity (an essential singularity), so we get by the residuum formula that

$$\mathcal{L}^{\circ -1}\left\{\exp\left(\frac{1}{z}\right)-1\right\}(t)=\operatorname{res}\left(e^{zt}\left\{\exp\left(\frac{1}{z}\right)-1\right\};0\right)=a_{-1}(t),$$

where $a_{-1}(t)$ is the coefficient of $\dfrac{1}{z}$ in the power series expansion

$$e^{zt}\left\{\exp\left(\frac{1}{z}\right)-1\right\}=\sum_{m=0}^{+\infty}\frac{t^m}{m!}\,z^m\cdot\sum_{n=1}^{+\infty}\frac{1}{n!}\,\frac{1}{z^n}=\frac{1}{z}\sum_{m=0}^{+\infty}\frac{t^m}{m!}\,z^m\cdot\sum_{n=0}^{+\infty}\frac{1}{(n+1)!}\cdot\frac{1}{z^n}.$$

It follows from this that we get precisely $a_{-1}(t)$, when $m=n$, so

$$\mathcal{L}^{\circ -1}\left\{\exp\left(\frac{1}{z}\right)-1\right\}(t)=\sum_{n=0}^{+\infty}\frac{1}{n!(n+1)!}\,t^n,$$

which is convergent for every $t\in\mathbb{R}$ (the radius of convergence is clearly $+\infty$).

(c) We get by a series expansion,

$$\begin{aligned}\mathcal{L}\{f\}(z) &= \sinh\left(\frac{1}{z}\right)=\sum_{n=0}^{+\infty}\frac{1}{(2n+1)!}\cdot\frac{1}{z^{2n+1}}=\sum_{n=0}^{+\infty}\frac{1}{(2n)!(2n+1)!}\,\mathcal{L}\left\{t^{2n}\right\}(z)\\ &= \mathcal{L}\left\{\sum_{n=0}^{+\infty}\frac{1}{(2n)!(2n+1)!}\,t^{2n}\right\}(z).\end{aligned}$$

The latter series is convergent for every $t\in\mathbb{R}$, so

$$f(t)=\mathcal{L}^{\circ -1}\left\{\sinh\left(\frac{1}{z}\right)\right\}(t)=\sum_{n=0}^{+\infty}\frac{1}{(2n)!(2n+1)!}\,t^{2n},\qquad t\in\mathbb{R}.$$

ALTERNATIVELY one proves as above that we have the estimate

$$\left|\sinh\left(\frac{1}{z}\right)\right|\le\frac{C}{|z|}\qquad\text{for } |z|\ge R.$$

It follows from the series expansions

$$e^{zt}\sinh\left(\frac{1}{z}\right)=\sum_{m=0}^{+\infty}\frac{t^m}{m!}\,z^m\cdot\sum_{n=0}^{+\infty}\frac{1}{(2n+1)!}\cdot\frac{1}{z^{2n+1}},$$

that the coefficient $a_{-1}(t)$ corresponds to $m=2n$, thus

$$\mathcal{L}^{\circ -1}\left\{\sinh\left(\frac{1}{z}\right)\right\}(t)=a_{-1}(t)=\sum_{n=0}^{+\infty}\frac{1}{(2n)!(2n+1)!}\,t^{2n}.$$

(d) We get by a series expansion,

$$\begin{aligned}\mathcal{L}\{f\} &= \cosh\left(\frac{1}{z}\right) - 1 = \sum_{n=1}^{+\infty} \frac{1}{(2n)!} \cdot \frac{1}{z^{2n}} = \sum_{n=1}^{+\infty} \frac{1}{(2n)!(2n-1)!} \cdot \frac{(2n-1)!}{z^{2n}} \\ &= \sum_{n=1}^{+\infty} \frac{1}{(2n)!(2n-1)!}\, \mathcal{L}\left\{t^{2n-1}\right\} = \mathcal{L}\left\{\sum_{n=1}^{+\infty} \frac{1}{(2n)!(2n-1)!}\, t^{2n-1}\right\}(z).\end{aligned}$$

The latter series is convergent for every $t \in \mathbb{R}$, so

$$f(t) = \sum_{n=1}^{+\infty} \frac{1}{(2n)!(2n-1)!}\, t^{2n-1}, \qquad t \in \mathbb{R}.$$

ALTERNATIVELY we even have the estimate

$$\left|\cosh\left(\frac{1}{z}\right) - 1\right| \le \frac{C}{|z|^2} \qquad \text{for } |z| \ge R.$$

It follows from the series expansions

$$e^{zt}\left\{\cosh\left(\frac{1}{z}\right)-1\right\}=\sum_{m=0}^{+\infty}\frac{t^m}{m!}\,z^m\cdot\sum_{n=1}^{+\infty}\frac{1}{(2n)!}\cdot\frac{1}{z^{2n}}=\frac{1}{z}\sum_{m=0}^{+\infty}\frac{t^m}{m!}\,z^m\cdot\sum_{n=1}^{+\infty}\frac{1}{(2n)!}\cdot\frac{1}{z^{2n-1}},$$

that the coefficient $a_{-1}(t)$ corresponds to $m=2n-1$, i.e.

$$\mathcal{L}^{\circ -1}\left\{\cosh\left(\frac{1}{z}\right)-1\right\}(t)=a_{-1}(t)=\sum_{n=1}^{+\infty}\frac{1}{(2n-1)!(2n)!}\,t^{2n-1}.$$

Example 2.36 *Find by using the Convolution Theorem the inverse Laplace transform of* $\dfrac{1}{(z+3)(z-1)}$.

Since

$$\mathcal{L}\{f\}(z)=\frac{1}{(z+3)(z-1)}=\mathcal{L}\left\{e^{-3t}\right\}(z)\cdot\mathcal{L}\left\{e^{t}\right\}(z),$$

we get by the Convolution Theorem that

$$f(t)=\int_0^t e^{-3\tau}r^{t-\tau}\,d\tau=e^t\int_0^t e^{-4\tau}\,d\tau=e^t\cdot\left(-\frac{1}{4}\right)\left(e^{-4t}-1\right)=\frac{1}{4}\,e^t-\frac{1}{4}\,e^{-3t}.$$

Remark 2.4 For comparison we get by a decomposition that

$$\frac{1}{(z+3)(z-1)}=\frac{1}{4}\,\frac{1}{z-1}-\frac{1}{4}\,\frac{1}{z+3},$$

from which we immediately get

$$f(t)=\mathcal{L}^{\circ -1}\left\{\frac{1}{(z+3)(z-1)}\right\}(t)=\frac{1}{4}\,e^t-\frac{1}{4}\,e^{-3t}.\qquad\Diamond$$

Example 2.37 *Use the Convolution Theorem to prove that*

$$\int_0^t \sin u\cdot\cos(t-u)\,du=\frac{1}{2}\,t\,\sin t.$$

We compute the Laplace transform of the right hand side $\frac{1}{2}\,t\,\sin t$ with some rewriting. By the Rule of Multiplication by t we get

$$\begin{aligned}\mathcal{L}\left\{\frac{1}{2}\,t\,\sin t\right\}&=\frac{1}{2}\,(-1)^1\,\frac{d}{dz}\,\mathcal{L}\{\sin t\}(z)=-\frac{1}{2}\,\frac{d}{dz}\left\{\frac{1}{z^2+1}\right\}\\&=-\frac{1}{2}\cdot\frac{-2z}{\left(z^2+1\right)^2}=\frac{z}{\left(z^2+1\right)^2}=\frac{1}{z^2+1}\cdot\frac{z}{z^2+1}\\&=\mathcal{L}\{\sin t\}(z)\cdot\mathcal{L}\{\cos t\}(z)=\mathcal{L}\{\sin\star\cos(t)\}(z)\\&=\mathcal{L}\left\{\int_0^t\sin u\cdot\cos(t-u)\,du\right\}(z),\end{aligned}$$

hence by the inverse Laplace transform,

$$\int_0^t \sin u \cdot \cos(t-u)\,du = \frac{1}{2}\,t\,\sin t.$$

Example 2.38 *Solve the convolution equation*

$$\int_0^t f(u)f(t-u)\,du = 2\,f(t)+t-2, \qquad t \in \mathbb{R}_+,$$

where $f \in \mathbb{F}$ is assumed to be continuous.

Put $F(z) = \mathcal{L}\{f\}(z)$. Then the equation is by the *Rule of Convolution* transformed by the Laplace transform into

$$F(z)^2 = 2\,F(z) + \frac{1}{z^2} - 2 \cdot \frac{1}{z},$$

hence

$$F(z) = 1 \pm \left(1 - \frac{1}{z}\right) = \begin{cases} 2 - \dfrac{1}{z} & (= \mathcal{L}\{2\delta - 1\}(z)), \\[2mm] \dfrac{1}{z} = \mathcal{L}\{1\}(z). & \end{cases}$$

Here "δ" denotes Dirac's delta-"function", which is not a true function, and which has not been introduced into the Calculus courses. Now, we have required that $f \in \mathbb{F}$ is continuous, so we must reject the solution $2\delta - 1$, and we get

$$f(t) \equiv 1 \in \mathbb{E} \subset \mathbb{F}.$$

CHECK. By a small computation the left hand side becomes

$$\int_0^t f(u)f(t-u)\,du = \int_0^t t \cdot 1\,du = t,$$

and the right hand side becomes

$$2\,f(t) + t - 2 = 2 + t - 2 = t.$$

We get the same result, and we have checked our computations.

Example 2.39 *Solve the convolution equation*

$$f(t) = t^2 + \int_0^t f(u)\sin(t-u)\,du, \qquad t \in \mathbb{R}_+,$$

where $f \in \mathbb{F}$.

Put $F(z) = \mathcal{L}\{f\}(z)$. Then by the *Rule of Convolution* the equation is transformed by the Laplace transform into

$$F(z) = \mathcal{L}\left\{t^2\right\}(z) + F(z)\cdot\mathcal{L}\{\sin t\}(z) = \frac{2!}{z^3} + \frac{1}{z^2+1}\,F(z),$$

thus by a reduction,

$$\frac{2}{z^3} = \left\{1 - \frac{1}{z^2+1}\right\} F(z) = \frac{z^2}{z^2+1}\,F(z),$$

and we conclude that

$$F(z) = \frac{2}{z^3}\cdot\frac{z^2+1}{z^2})\frac{2}{z^3} + \frac{2}{z^5} = \frac{2!}{z^{2+1}} + \frac{2}{4!}\cdot\frac{4!}{z^{4+1}} = \mathcal{L}\left\{t^2\right\}(z) + \frac{1}{12}\,\mathcal{L}\left\{t^4\right\}(z) = \mathcal{L}\left\{t^2 + \frac{t^4}{12}\right\}(z).$$

It follows that the solution is

$$f(t) = t^2 + \frac{1}{12}\,t^4 \in \mathbb{E} \subset \mathbb{F}.$$

Check. If we put the solution

$$f(t) = t^2 + \frac{1}{12}\,t^4$$

into the right hand side, then

$$\begin{aligned}
t^2 + \int_0^t f(u)\sin(t-u)\,du &= t^2 + \int_0^t u^2\sin(t-u)\,du + \frac{1}{12}\int_0^t u^4\sin(t-u)\,du \\
&= t^2 + \int_0^t u^2\sin(t-u)\,du + \left[\frac{1}{12}\,u^4\cos(t-u)\right]_0^t - \frac{4}{12}\int_0^t u^3\cos(t-u)\,du \\
&= t^2 + \frac{t^4}{12} + \frac{1}{3}\int_0^t \left\{3u^2\sin(t-u) - u^3\cos(t-u)\right\}\,du \\
&= t^2 + \frac{t^4}{12} + \frac{1}{3}\left[u^3\sin(t-u)\right]_0^t = t^2 + \frac{t^4}{12} = f(t),
\end{aligned}$$

where we have used that

$$\frac{\partial}{\partial u}\left\{u^3\sin(t-u)\right\} = 3u^2\sin(t-u) - u^3\cos(t-u). \qquad \Diamond$$

Example 2.40 *Solve the convolution equation*

$$f(t) = t + 2\int_0^t f(u)\,\cos(t-u)\,du, \qquad t \in \mathbb{R}_+,$$

where $f \in \mathbb{F}$.

Put $F(z) = \mathcal{L}\{f\}(z)$. Using the *Rule of Convolution* the equation is transferred into

$$F(z) = \frac{1}{z^2} + 2\,F(z)\cdot\frac{z}{z^2+1},$$

hence by a reduction,

$$\frac{1}{z^2} = \left\{1 - \frac{2z}{z^2+1}\right\} F(z) = \frac{z^2-2z+1}{z^2+1}\,F(z) = \frac{(z-1)^2}{z^2+1}\,F(z),$$

and the only possibility of the inverse Laplace transform is

$$F(z) = \frac{z^2+1}{z^2(z-1)^2}.$$

It is obvious that the inverse Laplace transform exists in this case and that it is given by a residuum formula,

$$\begin{aligned}
f(t) &= \operatorname{res}\left(\frac{z^2+1}{z^2(z-1)^2}\,e^{zt}\,;\,0\right)+\operatorname{res}\left(\frac{z^2+1}{z^2(z-1)^2}\,e^{zt}\,;\,1\right)\\
&= \lim_{z\to 0}\frac{d}{dz}\left\{\frac{z^2+1}{(z-1)^2}\,e^{zt}\right\}+\lim_{z\to 1}\frac{d}{dz}\left\{\frac{z^2+1}{z^2}\,e^{zt}\right\}\\
&= \lim_{z\to 0}\left\{\frac{2z}{(z-1)^2}\,e^{zt}-\frac{2\left(z^2+1\right)}{(z-1)^3}\,e^{zt}+t\,\frac{z^2+1}{(z-1)^2}\,e^{zt}\right\}\\
&\qquad+\lim_{z\to 1}\left\{\frac{2z}{z^2}\,e^{zt}-\frac{2\left(z^2+1\right)}{z^3}\,e^{zt}+t\cdot\frac{z^2+1}{z^2}\,e^{zt}\right\}\\
&= -\frac{2\cdot 1}{(-1)^3}+t+2\,e^t-4\,e^t+t\cdot 2\cdot e^t\\
&= 2+t-2\,e^t+2\,t\,e^t.
\end{aligned}$$

Check. It is not a nice task directly to put the function into the convolution equation. Instead we consider the Laplace transforms. In fact, if they agree and we are still in $\mathbb{F}$, then the result follows from the uniqueness theorem. It follows immediately that

$$\mathcal{L}\{f\}(z)=\frac{2}{z}+\frac{1}{z^2}-\frac{2}{z-1}+\frac{2}{(z-1)^2}.$$

We get for the right hand side,

$$\begin{aligned}
&\mathcal{L}\left\{t+2\int_0^t f(u)\,\cos(t-u)\,du\right\}(z)=\frac{1}{z^2}+2\left\{\frac{2}{z}+\frac{1}{z^2}-\frac{2}{z-1}+\frac{2}{(z-1)^2}\right\}\cdot\frac{z}{z^2+1}\\
&=\mathcal{L}\{f\}(z)+\frac{1}{z^2}+\left\{\frac{2}{z}+\frac{1}{z^2}-\frac{2}{z-1}+\frac{2}{(z-1)^2}\right\}\left\{\frac{2z}{z^2+1}-1\right\}\\
&=\mathcal{L}\{f\}(z)+\frac{1}{z^2}-\frac{(z-1)^2}{z^2+1}\left\{\frac{2}{z}+\frac{1}{z^2}-\frac{2}{z-1}+\frac{2}{(z-1)^2}\right\}\\
&=\mathcal{L}\{f\}(z)+\frac{1}{z^2}-\frac{1}{z^2+1}\left\{\frac{(2z+1)(z-1)^2}{z^2}-2(z-1)+2\right\}\\
&=\mathcal{L}\{f\}(z)+\frac{1}{z^2\left(z^2+1\right)}\,\left\{z^2+1-(2z+1)(z-1)^2+2(z-1)z^2-2z^2\right\}\\
&=\mathcal{L}\{f\}(z)+\frac{1}{z^2\left(z^2+1\right)}\,\left\{z^2+1-(2z+1)\left(z^2-2z+1\right)+2z^3-4z^2\right\}\\
&=\mathcal{L}\{f\}(z)+\frac{1}{z^2\left(z^2+1\right)}\,\left\{z^2+1-2z^3+4z^2-z^2-2z+2z-1+2z^3-4z^2\right\}\\
&=\mathcal{L}\{f\}(z),
\end{aligned}$$

and we see that we have an agreement of the two sides of the equation. $\Diamond$

Example 2.41 *Solve the integro-differential equation*

$$\int_0^t f(u)\,\cos(t-u)\,du = f'(t), \qquad t \in \mathbb{R}_+,$$

where $f(0)=1$, *and* f *is differentiable in* $\mathbb{R}_+$ *and* $f' \in \mathbb{F}$.

We put $F(z)=\mathcal{L}\{f\}(z)$. The equation is transferred by the Laplace transform into

$$F(z)\cdot\frac{z}{z^2+1} = \mathcal{L}\left\{f'\right\}(z) = z\cdot F(z)-f(0) = z\cdot F(z)-1.$$

Hence by a rearrangement,

$$1 = z\cdot F(z) - \frac{z}{z^2+1}\,F(z) = z\left(1-\frac{1}{z^2+1}\right)F(z) = \frac{z^3}{z^2+1}\,F(z),$$

and

$$F(z) = \frac{z^2+1}{z^3} = \frac{1}{z}+\frac{1}{z^3} = \mathcal{L}\{1\}(z)+\frac{1}{2}\,\mathcal{L}\left\{t^2\right\}(z).$$

Finally, we get by the inverse Laplace transform,

$$f(t) = 1+\frac{1}{2}\,t^2.$$

CHECK. Clearly, $f(0)=1$ and

$$f'(t)=t.$$

Then by computing the convolution integral,

$$\begin{aligned}\int_0^t f(u)\,\cos(t-u)\,du &= \int_0^t\left(1+\frac{1}{2}\,u^2\right)\cos(u-t)\,du = \left[\left(1+\frac{1}{2}\,u^2\right)\right]_0^t - \int_0^t u\cdot\sin(u-t)\,du\\ &= \sin t + [u\cdot\cos(u-t)]_0^t - \int_0^t \cos(u-t)\,du = \sin t + t - [\sin(u-t)]_0^t = \sin t + t - \sin t\\ &= t = f'(t),\end{aligned}$$

and we have found the right solution.

ALTERNATIVELY we may prove that the Laplace transforms of the left hand side and the right hand side agree for the function

$$f(t) = 1+\frac{1}{2}\,t^2.$$

Thus we have

$$\begin{aligned}\mathcal{L}\left\{\int_0^t f(u)\,\cos(t-u)\,du\right\} &= \mathcal{L}\left\{1+\frac{1}{2}\,t^2\right\}(z)\cdot\frac{z}{z^2+1} = \left\{\frac{1}{z}+\frac{1}{z^3}\right\}\cdot\frac{z}{z^2+1}\\ &= \frac{z^2+1}{z^3}\cdot\frac{z}{z^2+1} = \frac{1}{z^2},\end{aligned}$$

and

$$\mathcal{L}\left\{f'\right\}(z) = \mathcal{L}\{t\} = \frac{1}{z^2},$$

and we have tested our solution. $\Diamond$

Example 2.42 *Find $f \in \mathbb{F}$, such that*

$$\int_0^t u\, f(u)\, \cos(t-u)\, du = t\, e^{-t} - \sin t, \qquad t \in \mathbb{R}_+.$$

We put $F(z) = \mathcal{L}\{f\}(z)$. Then the equation is transferred by the Laplace transform into

$$\mathcal{L}\{t \cdot f(t)\}(z) \cdot \frac{z}{z^2+1} = \frac{1}{(z+1)^2} - \frac{1}{z^2+1},$$

so we get by the Rule of Multiplication by t and reduction

$$\begin{aligned} \frac{dF}{dz} &= -\mathcal{L}\{t\, f(t)\}(z) = \frac{z^2+1}{z}\left\{\frac{1}{z^2+1} - \frac{1}{(z+1)^2}\right\} = \frac{1}{z}\left\{1 - \frac{z^2+1}{(z+1)^2}\right\} \\ &= \frac{1}{z(z+1)^2}\left\{(z+1)^2 - \left(z^2+1\right)\right\} = \frac{2}{(z+1)^2}. \end{aligned}$$

There exists an arbitrary constant C, such that

$$F(z) = -\frac{2}{z+1} + C.$$

However, since $f \in \mathbb{F}$, we must have $F(z) \to 0$ for $z \to \infty$, which implies that $C = 0$, and we have

$$F(z) = -\frac{2}{z+1} = -\mathcal{L}\left\{2\, e^{-t}\right\}(z).$$

Finally, the solution is given by

$$f(t) = -2\, e^{-t}.$$

CHECK. If we put $f(t) = -2\, e^{-t}$, then

$$\begin{aligned} \int_0^t u\, f(u)\, \cos(t-u)\, du &= -2\int_0^t u\, e^{-u} \cos(t-u)\, du = -2\,\mathrm{Re}\left\{\int_0^t u\, e^{-u} e^{i(t-u)}\, du\right\} \\ &= -2\,\mathrm{Re}\left\{e^{it}\int_0^t u\, e^{-(1+i)u}\, du\right\} = -2\,\mathrm{Re}\left\{e^{it}\left[\frac{u\, e^{-(1+i)u}}{-(1+i)}\right]_0^t + \frac{e^{it}}{1+i}\int_0^t e^{-(1+i)u}\, du\right\} \\ &= -2\,\mathrm{Re}\left\{e^{it} \cdot \frac{-1+i}{2}\, t\, e^{-(1+i)t}\right\} + 2\,\mathrm{Re}\left\{\frac{e^{it}}{(1+i)^2}\left(e^{-(1+i)t} - 1\right)\right\} \\ &= -2\,\mathrm{Re}\left\{\frac{e^{-t}}{2} \cdot t \cdot (-1+i)\right\} + 2\,\mathrm{Re}\left\{\frac{1}{2i}\, e^{-t} - \frac{1}{2i}\, e^{it}\right\} = t\, e^{-y} - \sin t, \end{aligned}$$

and we have tested our solution. $\Diamond$

Example 2.43 *Find a function $f \in \mathbb{F}$, such that*

$$\int_0^t f(u)\, f(t-u)\, du = 8(\sin t - t\, \cos t), \qquad t \in \mathbb{R}_+.$$

We put $F(z) = \mathcal{L}\{f\}(z)$. Then we get by the Laplace transform that

$$\begin{aligned} F(z)^2 &= 8\left\{\frac{1}{z^2+1} - \mathcal{L}\{t \cdot \cos t\}(z)\right) = 8\left(\frac{1}{z^2+1} + \frac{d}{dz}\,\mathcal{L}\{\cos t\}(z)\right) \\ &= 8\left(\frac{1}{z^2+1} + \frac{d}{dz}\left\{\frac{z}{z^2+1}\right\}\right) = \frac{8}{\left(z^2+1\right)^2}\,\left\{z^2+1+z^2+1-z\cdot 2z\right\} \\ &= \frac{16}{\left(z^2+1\right)^2} = \left(\frac{4}{z^2+1}\right)^2, \end{aligned}$$

and we see that we have the two solutions

$$F(z) = \pm\frac{4}{z^2+1} = \pm\mathcal{L}\{4\, \sin t\}(z),$$

corresponding to

$$f(t) = \pm 4\, \sin t.$$

CHECK. The sign is of course of no importance. It follows by insertion that

$$\begin{aligned} \int_0^t f(u)\, f(t-u)\, du &= 16\int_0^t \sin u \cdot \sin(t-u)\, du = 8\int_0^t \{\cos(2u-t) - \cos t\}\, du \\ &= 4\left[\sin(2u-t)\right]_0^t - 8\, t\, \cos t = 8\, \sin t - 8\, t\, \cos t, \end{aligned}$$

and we have tested our solution. $\Diamond$

3 The Mellin transform

Example 3.1 *Assume that* $0 < a < 2$ *is a constant. Compute*

$$\int_0^{+\infty} \frac{x^a}{x^2+1}\,\frac{dx}{x}.$$

If $a = 1$, then it is well-known that

$$\int_0^{+\infty} \frac{x^1}{x^2+1}\,\frac{dx}{x} = \int_0^{+\infty} \frac{dx}{x^2+1} = \frac{\pi}{2}.$$

We assume in the following that $a \neq 1$. The function $f(z) = \dfrac{1}{z^2+1}$, $z \in \mathbb{C} \setminus \{-i, i\}$ has no pole on $\mathbb{R}_+$, and there exist constants $0 < r_0 < R_0$, such that we have the estimates

$$\left|z^0 f(z)\right| \leq 2 \quad \text{for } |z| \leq r_0, \qquad \text{og} \qquad \left|z^2 f(z)\right| \leq 2 \quad \text{for } |z| \geq R_0.$$

Then the integral is convergent by the theorem of the Mellin transform, and its value is given by a residuum formula (where we use that $a \in]0,1[$ and $a \neq 1$),

$$\begin{aligned}
\int_0^{+\infty} \frac{x^a}{x^2+1}\,\frac{dx}{x} &= -\frac{\pi e^{-i\pi a}}{\sin \pi a}\left\{\operatorname{res}\left(\frac{z^{a-1}}{z^2+1}\,;\,i\right) + \operatorname{res}\left(\frac{z^{a-1}}{z^2+1}\,;\,-i\right)\right\} \\
&= -\frac{\pi e^{-i\pi a}}{\sin \pi a}\left\{\frac{1}{2i}\,\exp\left((a-1)i\,\frac{\pi}{2}\right) - \frac{1}{2i}\,\exp\left((a-1)i\,\frac{3\pi}{4}\right)\right\} \\
&= -\frac{\pi e^{-i\pi a}}{\sin \pi a}\cdot\frac{1}{2}\left\{\frac{1}{i}\,\exp\left(ia\,\frac{\pi}{2}\right)\cdot(-i) - \frac{1}{i}\,\exp\left(ia\,\frac{3\pi}{2}\right)\cdot i\right\} \\
&= \frac{\pi}{\sin \pi a}\cdot\frac{1}{2}\left\{\exp\left(-ia\,\frac{\pi}{2}\right) 0 \exp\left(ia\,\frac{\pi}{2}\right)\right\} = \frac{\pi\cos\left(a\,\dfrac{\pi}{2}\right)}{\sin \pi a} = \frac{\pi}{2\sin\left(a\,\dfrac{\pi}{2}\right)}.
\end{aligned}$$

We have derived this expression for $a \neq 1$ and $0 < a < 2$. However, a simple check shows that it is also true for $a = 1$. Summing up we have

$$\int_0^{+\infty} \frac{x^{a-1}}{x^2+1}\,dx = \frac{\pi}{2\sin\left(a\,\dfrac{\pi}{2}\right)}, \qquad \text{for } a \in]0,2[.$$

Example 3.2 *Prove that*

$$\int_0^{+\infty} \frac{x^a}{1+x} \cdot \frac{dx}{x} = \frac{\pi}{\sin \pi a} \qquad \textit{for } a \in]0,1[.$$

We shall check the assumptions of the theorem of the Mellin transform. First we note that the function

$$f(z) = \frac{1}{1+z}$$

is analytic $\mathbb{C} \setminus \{-1\}$. Then we shall estimate at the endpoints of the interval

$$]\alpha, \beta[=]0,1[.$$

If $\alpha = 0$, then we have the estimate

$$|z^\alpha f(z)| = |f(z)| = \left| \frac{1}{1+z} \right| \le 2 \qquad \text{for } |z| \le \frac{1}{2}.$$

If $\beta = 1$, then we have the estimate

$$\left|z^\beta f(z)\right| = |z\, f(z)| = \left| \frac{z}{1+z} \right| = \frac{1}{\left|1 + \dfrac{1}{z}\right|} \le 2 \qquad \text{for } |z| \ge 2.$$

Then we conclude from a theorem that the integral

$$\int_0^{+\infty} \frac{x^a}{1+x} \, \frac{dx}{x}$$

is convergent for $a \in]0,1[$, and its value is given by a residuum formula,

$$\begin{aligned}\int_0^{+\infty} \frac{x^a}{1+x} \, \frac{dx}{x} &= -\frac{\pi\, e^{-\pi i\, a}}{\sin \pi a} \operatorname{res}\left(\frac{z^{a-1}}{1+z} \,;\, -1 \right) = -\frac{\pi\, e^{-\pi i\, a}}{\sin \pi a} \cdot (-1)^{a-1} := -\frac{\pi\, e^{-\pi i\, a}}{\sin \pi a} \cdot e^{(a-1) i \pi} \\ &= -\frac{\pi}{\sin \pi a} \cdot e^{-i\pi a + i\pi a - i\pi} = \frac{\pi}{\sin \pi a} \cdot (-1) \cdot (-1) = \frac{\pi}{\sin \pi a}.\end{aligned}$$

Example 3.3 *Prove that*

$$\int_0^{+\infty} \frac{x^a}{1+x^3} \, \frac{dx}{x} = \frac{\pi}{3 \, \sin\left(\dfrac{\pi a}{3}\right)} \qquad \textit{for } a \in]0,3[.$$

We shall again use a convenient theorem with $]\alpha, \beta[=]0,3[$ and with the function

$$f(z) = \frac{1}{1+z^3},$$

which is analytic for

$$z \in \mathbb{C} \setminus \left\{ -1, \, \frac{1}{2} + i\, \frac{\sqrt{3}}{2}, \, \frac{1}{2} - i\, \frac{\sqrt{3}}{2} \right\}.$$

If $\alpha = 0$, then we get the estimate

$$|f(z)| = \left|\frac{1}{1+z^3}\right| \le 2 \qquad \text{for } |z| \le \frac{1}{2}.$$

If $\beta = 3$, then we get the estimate

$$\left|z^3 f(z)\right| = \left|\frac{z^3}{1+z^3}\right| = \frac{1}{\left|1+\dfrac{1}{z^3}\right|} \le 2 \qquad \text{for } |z| \ge 2.$$

Then it follows from a theorem that the integral

$$\int_0^{+\infty} \frac{x^a}{1+x^3}\,\frac{dx}{x}$$

is convergent for $a \in]0,3[$, and its value is given by a residuum formula. Since we have for every pole z_j that $z_j^3 = -1$, we get

$$\begin{aligned}
\int_0^{+\infty} \frac{x^a}{1+x^3}\,\frac{dx}{x} &= -\frac{\pi\,e^{-\pi i\,a}}{\sin \pi a}\sum_{z_j} \operatorname{res}\left(\frac{z^{a-1}}{z^3+1}\,;\,z_j\right) = -\frac{\pi\,e^{-\pi i\,a}}{\sin \pi a}\sum_{z_j}\frac{z_j^{a-1}}{3z_j^2} \\
&= \frac{\pi\,e^{-i\pi a}}{3\,\sin \pi a}\sum_{z_j} a_j^a := \frac{\pi\,e^{-\pi i a}}{3\,\sin \pi a}\left\{e^{i\,a\,\frac{\pi}{3}} + e^{i\pi a} + e^{i\,a\,\frac{5\pi}{3}}\right\} \\
&= \frac{\pi}{3}\cdot\frac{e^{-\pi\,i\,a}}{\frac{1}{2i}\left(e^{i\pi a}-e^{-i\pi a}\right)}\cdot\left\{e^{i\,a\,\frac{\pi}{3}} + e^{i\pi a} + e^{i\,a\,\frac{5\pi}{3}}\right\} \\
&= \frac{\pi}{3}\cdot\frac{1}{\sin\frac{\pi a}{3}}\cdot\frac{e^{i\,a\,\frac{\pi}{3}}-e^{-i\,a\,\frac{\pi}{3}}}{e^{i\pi a}-e^{-i\pi\,a}}\cdot\left\{e^{i\,a\,\frac{\pi}{3}} + e^{i\pi a} + e^{i\,a\,\frac{5\pi}{3}}\right\}e^{-i\pi a} \\
&= \frac{\pi}{3}\cdot\frac{1}{\sin\frac{\pi a}{3}}\cdot\frac{e^{i\,a\,\frac{\pi}{3}}-e^{-i\pi a}}{e^{-i\pi a}}\cdot e^{i\,a\,\frac{\pi}{3}}\cdot\frac{\left(e^{i\,a\,\frac{2\pi}{3}}-1\right)\left(1+e^{i\,a\,\frac{2\pi}{3}}+e^{i\,a\,\frac{4\pi}{3}}\right)}{e^{2i\pi a}-1} \\
&= \frac{\pi}{3\sin\frac{\pi a}{3}}\cdot\frac{e^{i\,a\cdot 1\pi}-1}{e^{2i\pi a}-1} = \frac{\pi}{3\,\sin\frac{\pi a}{3}}.
\end{aligned}$$

4 The $\mathfrak{z}$-transform

Example 4.1 *Find the $\mathfrak{z}$-transform of the sequence*

$$\left(\frac{1}{n!}\right)_{n\ge 0}.$$

By the definition,

$$\mathfrak{z}\left\{\frac{1}{n!}\right\}(z)=\sum_{n=0}^{+\infty}\frac{1}{n!}\,z^{-n}=\exp\left(\frac{1}{z}\right),\qquad |z|>0,$$

because we have

$$\limsup_{n\to+\infty}\sqrt[n]{\frac{1}{n!}}=0<+\infty.$$

Example 4.2 *Find the $\mathfrak{z}$-transforms of $f(t) = \sin t$, $t \geq 0$, when*

$$(a)\ T = \pi, \qquad (b)\ T = \frac{\pi}{2}.$$

Since $\sin t$ is continuous and bounded, we have $R < +\infty$, so

(a)

$$\mathfrak{z}_\pi\{\sin(t)\}(z) = \sum_{n=0}^{+\infty} \sin n\pi \cdot z^{-n} = 0.$$

(b)

$$\begin{aligned}
\mathfrak{z}_{\frac{\pi}{2}}\{\sin(t)\}(z) &= \sum_{n=0}^{+\infty} \sin \frac{n\pi}{2} \cdot z^{-n} = \sum_{n=0}^{+\infty} \sin \frac{(4n+1)\pi}{2} \cdot z^{-4n-1} + \sum_{n=0}^{+\infty} \sin \frac{(4n+3)\pi}{2} \cdot z^{-4n-3} \\
&= \sum_{n=0}^{+\infty} z^{-4n-1} - \sum_{n=0}^{+\infty} z^{-4n-3} = \left(\frac{1}{z} - \frac{1}{z^3}\right) \sum_{n=0}^{+\infty} \left\{\frac{1}{z^4}\right\}^n = \frac{z^2-1}{z^3} \cdot \frac{1}{1 - \frac{1}{z^4}} \\
&= \frac{z^3 - z}{z^4 - 1} = \frac{z\left(z^2-1\right)}{\left(z^2-1\right)\left(z^2+1\right)} = \frac{z}{z^2+1}, \qquad \text{for } |z| > 1.
\end{aligned}$$

Remark 4.1 It is obvious in **(a)** that we lose too much information. However, also in **(b)** the choice of the sample interval is questionable. $\Diamond$

Example 4.3 *Find the $\mathfrak{z}$-transform with sample interval $T = 1$ of the sequence*

$$\left(\sum_{k=1}^{n+1} \frac{1}{k}\right)_{n \geq 0}.$$

Choose

$$f(t) = \frac{1}{t+1} \qquad \text{for } t \geq 0,$$

as our auxiliary function. Then

$$R = \limsup_{n \to +\infty} \sqrt[n]{|f(n)|} = \lim_{n \to +\infty} \sqrt[n]{\frac{1}{n+1}} = 1,$$

hence

$$\mathfrak{z}_1\{f\} = \sum_{n=0}^{+\infty} \frac{1}{n+1}\, z^{-n} = z \sum_{n=1}^{+\infty} \frac{1}{n} \left(\frac{1}{z}\right)^n = z\,\mathrm{Log}\left(\frac{1}{1 - \frac{1}{z}}\right) = z\,\mathrm{Log}\left(\frac{z}{z-1}\right), \qquad |z| > 1.$$

Then by some theorem we have for $|z| > 1$,

$$\mathfrak{z}\left\{\sum_{k=1}^{n+1}\frac{1}{k}\right\}(z) = \mathfrak{z}\left\{\sum_{k=0}^{n} f(k)\right\}(z) = \frac{z}{z-1}\,\mathfrak{z}_1\{f\}(z) = \frac{z^2}{z-1}\,\text{Log}\left(\frac{z}{z-1}\right).$$

ALTERNATIVELY we consider directly the $\mathfrak{z}$-transform of

$$\left(\sum_{k=1}^{n+1}\frac{1}{k}\right)_{n\geq 0}.$$

It follows from

$$a_n = \sum_{k=1}^{n+1}\frac{1}{k},$$

that

$$1 \leq a_n \leq n+1,$$

so

$$1 \leq \limsup_{n\to+\infty} \sqrt[n]{|a_n|} \leq \lim_{n\to+\infty} \sqrt[n]{n+1} = 1,$$

and $R = 1$.

Assume that $|z| > 1$. The $\mathfrak{z}$-transform is *analytic* for $|z| > 1$, so we may interchange the summations in the following, when we use the definition:

$$\begin{aligned}
\mathfrak{z}\left\{\sum_{k=1}^{n+1}\frac{1}{k}\right\}(z) &= \sum_{n=0}^{+\infty}\left(\sum_{k=1}^{n+1}\frac{1}{k}\right)z^{-n} = \sum_{n=0}^{+\infty}\sum_{k=0}^{n}\frac{1}{k+1}\,z^{-n} = \sum_{k=0}^{+\infty}\sum_{n=k}^{+\infty}\frac{1}{k+1}\,z^{-n} = \sum_{k=0}^{+\infty}\frac{1}{k+1}\sum_{n=k}^{+\infty}\left(\frac{1}{z}\right)^n \\
&= \sum_{k=0}^{+\infty}\frac{1}{k+1}\left(\frac{1}{z}\right)^k\cdot\frac{1}{1-\dfrac{1}{z}} = \frac{z}{z-1}\cdot z\sum_{k=1}^{+\infty}\frac{1}{k}\,z^{-k} = \frac{z^2}{z-1}\,\text{Log}\left(\frac{1}{1-\dfrac{1}{z}}\right) \\
&= \frac{z^2}{z-1}\,\text{Log}\left(\frac{z}{z-1}\right).
\end{aligned}$$

Example 4.4 *Find the inverse $\mathfrak{z}$-transform with the sample interval T of* $\exp\left(\frac{1}{z}\right)$.

It follows by identification from

$$\mathfrak{z}_T\left(\{a_n\}\right)(z) = \sum_{n=0}^{+\infty} a_n z^{-n} = \exp\left(\frac{1}{z}\right) = \sum_{n=0}^{+\infty}\frac{1}{n!}\,z^{-n}, \qquad z \neq 0,$$

that

$$f(nT) = a_n = \frac{1}{n!}, \qquad n \in \mathbb{N}_0.$$

Example 4.5 *Find the inverse* $\mathfrak{z}$*-transform with sample interval* T *of* $\dfrac{z+2}{z^4-1}$.

First find the Laurent series of

$$F(z) = \frac{z+2}{z^4-1} \qquad \text{for } |z|>1.$$

We get by the usual technique, where we use that $\left|\dfrac{1}{z^4}\right| < 1$ for $|z|>1$,

$$\begin{aligned} F(z) &= \sum_{n=0}^{+\infty} f(nT)\, z^{-n} = \frac{z+2}{z^4-1} = \frac{z+2}{z^4}\cdot\frac{1}{1-\dfrac{1}{z^4}} = \frac{z+2}{z^4}\sum_{n=0}^{+\infty} z^{-4n} = (z+2)\sum_{n=1}^{+\infty} z^{-4n} \\ &= \sum_{n=1}^{+\infty} z^{-4n+1} + 2\sum_{n=1}^{+\infty} z^{-4n}, \end{aligned}$$

hence

$$\begin{array}{lll} f((4n-1)T) = 1 & \text{for } n\in\mathbb{N}, & \text{dvs. for } n = 3,\, 7,\, 11,\, 15,\, \dots, \\ f(nT) = 2 & \text{for } n\in\mathbb{N}, & \text{dvs. for } n)4,\, 8,\, 12,\, 16,\, \dots, \\ f(kT) = 0 & \text{ellers,} & \text{dvs. for } n = 0,\, 1,\, 2,\, 5,\, 6,\, 9,\, 10,\, \dots. \end{array}$$

Alternatively,

$$F(z) = \frac{z+2}{z^4-1}$$

has the four simple roots $z_k = i^k$, so $z_k^4 = 1$. Then for $n\in\mathbb{N}$ by a residuum formula and Rule II,

$$\begin{aligned} f(nT) &= \sum_{k=1}^{4} \operatorname{res}\left(\frac{z+2}{z^4-1}\cdot z^{n-1}\,;\, z_k\right) = \sum_{k=1}^{4} \lim_{z\to z_k} \frac{z+2}{4z^3}\cdot z^{n-1} = \frac{1}{4}\sum_{k=1}^{4} (z_k+2)\, z_k^{n-4} \\ &= \frac{1}{4}\sum_{k=1}^{4} (z_k+2)\, z_k^{n} = \frac{1}{4}\sum_{k=1}^{4}\left\{ i^{(n+1)k} + 2i^{nk}\right\}. \end{aligned}$$

Now $n>0$, so it is possible to reduce the result to the previous one.

We note that if $n=0$, then we must add a correction term,

$$f(0) = \frac{1}{4}\sum_{k=1}^{4}(z_k+2) + \operatorname{res}\left(\frac{z+2}{z^4-1}\cdot\frac{1}{z}\,;\,0\right) = 2 + \frac{2}{-1} = 0.$$

Example 4.6 *Find the $\mathfrak{z}$-transform $F(z)$ of the sequence*

$$\left(\sum_{k=0}^{n} \frac{1}{k!}\right)_{n\in\mathbb{N}_0},$$

expressed by elementary functions.
Then compute $res(F(z);\infty)$.

We shall first indicate the domain of convergence. We have here

$$1 \le a_n := \sum_{k=0}^{n} \frac{1}{k!} < e,$$

so

$$R = \limsup_{n\to+\infty} \sqrt[n]{a_n} \quad \begin{cases} \ge 1, \\ \le \lim_{n\to+\infty} \sqrt[n]{e} = 1. \end{cases}$$

We conclude that $R=1$, so

$$(1)\quad F(z)=\mathfrak{z}\left\{\sum_{k=0}^{n}\frac{1}{k!}\right\}(z):=\sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}\frac{1}{k!}\right)z^{-n}\qquad\text{for } |z|>1.$$

If $|z|>1$, then

$$F(z)=\mathfrak{z}\left\{\sum_{k=0}^{n}\frac{1}{k!}\right\}(z)=\frac{z}{z-1}\,\mathfrak{z}\left\{\frac{1}{n!}\right\}(z)=\frac{z}{z-1}\sum_{n=0}^{+\infty}\frac{1}{n!}\left(\frac{1}{z}\right)^n=\frac{z}{z-1}\exp\left(\frac{1}{z}\right),\qquad |z|>1.$$

ALTERNATIVELY, though not so smart, we compute the right hand side of (1) for $|z|>1$. If we in the computation interchange the order of summation and implicitly use that $|z|>1$, i.e. $\left|\frac{1}{z}\right|<1$, so the series is convergent, then

$$\begin{aligned}F(z)&:=\sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}\frac{1}{k!}\right)z^{-n}=\sum_{n=0}^{+\infty}\sum_{k=0}^{n}\frac{1}{k!}\left(\frac{1}{z}\right)^n=\sum_{k=0}^{+\infty}\left(\frac{1}{z}\right)^n=\sum_{k=0}^{+\infty}\frac{1}{k!}\left(\frac{1}{z}\right)^k\cdot\sum_{n=0}^{+\infty}\left(\frac{1}{z}\right)^n\\&=\frac{1}{1-\frac{1}{z}}\exp\left(\frac{1}{z}\right)=\frac{z}{z-1}\exp\left(\frac{1}{z}\right).\end{aligned}$$

Now

$$F(z)=\sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}\frac{1}{k!}\right)z^{-n}$$

is defined outside a disc $|z|>1$, so

$$\operatorname{res}(F(z);\infty)=-a_{-1}=-\sum_{k=0}^{1}\frac{1}{2!}=-2.$$

ALTERNATIVELY, though more clumsy, we find the residuum at ∞ of $F(z)$ given by the expression

$$F(z)=\frac{z}{z-1}\exp\left(\frac{1}{z}\right),$$

so we shall find the coefficient a_{-1} of $\frac{1}{z}$ in the Laurent series expansion of $F(z)$. Since $|z|>1$, the most natural computation is

$$F(z)=\frac{z}{z-1}\exp\left(\frac{1}{z}\right)=\frac{1}{1-\frac{1}{z}}\exp\left(\frac{1}{z}\right)=\sum_{n=0}^{+\infty}\frac{1}{z^n}\cdot\sum_{k=0}^{+\infty}\frac{1}{k!}\cdot\frac{1}{z^k},$$

and the coefficient of $\frac{1}{z}$ corresponds to $(n,k)=(0,1)$ and $(1,0)$, thus

$$\operatorname{res}\left(\frac{z}{z-1}\exp\left(\frac{1}{z}\right);\infty\right)=-a_{-1}=-(1+1)=-2.$$

ALTERNATIVELY we first apply RULE V, followed by RULE I (because the order of the pole at 0 of the transformed expression is $q = 2$):

$$\begin{aligned}
\operatorname{res}(F(z);\infty) &= \operatorname{res}\left(\frac{z}{z-1}\,\exp\left(\frac{1}{z}\right);\infty\right) = -\operatorname{res}\left(\frac{1}{z^2}\cdot\frac{\frac{1}{z}}{\frac{1}{z}-1}\cdot\exp(z);0\right) = -\operatorname{res}\left(\frac{e^z}{z^2(1-z)};0\right)\\
&= -\frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left\{\frac{e^z}{1-z}\right\} = -\lim_{z\to 0}\left\{\frac{e^z}{1-z}+\frac{e^z}{(1-z)^2}\right\} = -2.
\end{aligned}$$

Example 4.7 *Given*

$$f(t) = \frac{1}{(t+1)(t+2)} \qquad \textit{for } t \geq 0.$$

(a) *Find the $\mathfrak{z}$-transform $F(z)$ of the sequence*

$$\left(\sum_{k=0}^{n} f(k)\right)_{n\in\mathbb{N}_0}$$

for $|z| > R$, where one shall find the smallest possible R.

(b) *Find*

$$\lim_{z\to 1+}(x-1)\,F(x),$$

where x runs through the positive real numbers > 1.

(a) We get by the definition for $T = 1$,

$$\mathfrak{z}_1\{f\}(z) := \sum_{n=0}^{+\infty} f(n) z^{-n} \qquad \text{for } |z| > R,$$

where

$$R = \limsup_{n\to+\infty}\sqrt[n]{|f(n)|} = \limsup_{n\to+\infty}\frac{1}{\sqrt[n]{(n+1)(n+2)}} = 1,$$

so if $|z| > 1$, then

$$\begin{aligned}
(2)\ \mathfrak{z}_1\{f\} &= \sum_{n=0}^{+\infty}\left\{\frac{1}{n+1}-\frac{1}{n+2}\right\}\left(\frac{1}{z}\right)^n = z\sum_{n=0}^{+\infty}\frac{1}{n+1}\left(\frac{1}{z}\right)^{n+1} - z^2\sum_{n=0}^{+\infty}\frac{1}{n+2}\left(\frac{1}{z}\right)^{n+2}\\
&= z\sum_{n=1}^{+\infty}\frac{1}{n}\left(\frac{1}{z}\right)^n - z^2\sum_{n=1}^{+\infty}\frac{1}{n}\left(\frac{1}{z}\right)^n + z = \left(z^2-z\right)\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\left(-\frac{1}{z}\right)^n + z\\
&= z(z-1)\operatorname{Log}\left(1-\frac{1}{z}\right) + z = z(z-1)\operatorname{Log}\left(\frac{z-1}{z}\right) + z.
\end{aligned}$$

Then for $|z| > \max\{1,1\} = 1$,

$$\mathfrak{z}\left\{\sum_{k=0}^{n} f(k)\right\}(z) = \frac{z}{z-1}\,\mathfrak{z}_1\{f\}(z) = z^2 \mathrm{Log}\left(\frac{z-1}{z}\right) + \frac{z^2}{z-1}.$$

Note that the splitting by (2) can be made for $|z| > 1$.

ALTERNATIVELY we have

$$a_n = \sum_{k=0}^{n} f(k) = \sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} = \sum_{k=0}^{n}\left\{\frac{1}{k+1} - \frac{1}{k+2}\right\} = 1 - \frac{1}{n+1},$$

so

$$\frac{1}{2} \le a_n \le 1$$

and

$$1 = \lim_{n\to+\infty} \sqrt[n]{\frac{1}{2}} \le R = \limsup_{n\to+\infty} \sqrt[n]{a_n} \le 1,$$

giving $R = 1$. Then for $|z| > 1$,

$$\begin{aligned}
\mathfrak{z}\left\{\sum_{k=0}^{n} f(k)\right\}(z) &:= \sum_{n=0}^{+\infty}\left(1 - \frac{1}{n+2}\right)\frac{1}{z^n} = \sum_{n=0}^{+\infty}\frac{1}{z^n} - \sum_{n=0}^{+\infty}\frac{1}{n+2}\cdot\frac{1}{z^n} \\
&= \sum_{n=0}^{+\infty}\frac{1}{z^n} + z - \frac{1}{(-1)+2}\cdot\frac{1}{z^{-1}} - \sum_{n=0}^{+\infty}\frac{1}{z^n} = \sum_{n=-1}^{+\infty}\frac{1}{z^n} - z^2\sum_{n=-1}^{+\infty}\frac{1}{n+2}\cdot\frac{1}{z^{n+2}} \\
&= \frac{z}{1-\frac{1}{z}} - z^2\sum_{n=1}^{+\infty}\frac{1}{n}\cdot\left(\frac{1}{z}\right)^n = \frac{z^2}{z-1} + z^2\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\left(-\frac{1}{z}\right)^n \\
&= \frac{z^2}{z-1} + z^2\,\mathrm{Log}\left(1 - \frac{1}{z}\right) = z^2\,\mathrm{Log}\left(\frac{z-1}{z}\right) + \frac{z^2}{z-1}.
\end{aligned}$$

(b) It follows directly from the result of **(a)** that

$$\lim_{z\to1+}(x-1)\,\mathfrak{z}\left\{\sum_{k=0}^{n} f(k)\right\}(x) = \lim_{x\to1+} x^2(x-1)\ln\left(\frac{x-1}{x}\right) + \lim_{x\to1+} x^2 = 0 + 1 = 1.$$

ALTERNATIVELY, by the *Final Value Theorem*,

$$\begin{aligned}
\lim_{x\to1+}(x-1)\,\mathfrak{z}\left\{\sum_{k=0}^{n} f(k)\right\}(x) &= \lim_{n\to+\infty}\sum_{k=0}^{n} f(k) = \sum_{n=0}^{+\infty}\frac{1}{(n+1)(n+2)} = \sum_{n=0}^{+\infty}\left\{\frac{1}{n+1} - \frac{1}{n+2}\right\} \\
&= \lim_{n\to+\infty}\left\{1 - \frac{1}{n+2}\right\} = 1,
\end{aligned}$$

because the series of course is convergent.

Example 4.8 *Given*

$$f(t) = (t+1)^2, \qquad t \geq 0.$$

Find the domain of convergence of the $\mathfrak{z}$-transform $F(z)$ of f with the sample period $T = 1$, and express $F(z)$ by elementary functions without using sum signs.
The $\mathfrak{z}$-transform $F(z)$ is naturally extended to an analytic function $F_1(z)$ in $\mathbb{C}\setminus\{1\}$. Find $res\,(F_1(z);1)$ and $res\,(F_1(z);\infty)$.

It follows from

$$\lim_{n\to+\infty} \sqrt[n]{|f(n)|} = \lim_{n\to+\infty} \sqrt[n]{(n+1)^2} = 1,$$

that the $\mathfrak{z}$-transform of f is convergent for $|z| > 1$, and we have

$$\mathfrak{z}_1\{f\}(z) = \sum_{n=0}^{+\infty} (n+1)^2 \frac{1}{z^n}, \qquad \text{for } |z| > 1.$$

Then note that we have for $|w| < 1$,

$$\begin{aligned}\sum_{n=0}^{+\infty}(n+1)^2w^n &= \sum_{n=0}^{+\infty}(n+1)n\,w^n+\sum_{n=0}^{+\infty}(n+1)w^n = w\sum_{n=1}^{+\infty}(n+1)n\,w^{n-1}+\sum_{n=0}^{+\infty}(n+1)w^n\\ &= w\,\frac{d^2}{dw^2}\left\{\frac{1}{1-w}\right\}+\frac{d}{dw}\left\{\frac{1}{1-w}\right\} = w\,\frac{d}{dw}\left\{\frac{1}{(1-w)^2}\right\}+\frac{1}{(1-w)^2}\\ &= \frac{2w}{(1-w)^3}+\frac{1-w}{(1-w)^3}=\frac{1+w}{(1-w)^3}.\end{aligned}$$

Putting $w = \dfrac{1}{z}$, we get for $|z| > 1$ that

$$\mathfrak{z}_1\{f\}(z) = \sum_{n=0}^{+\infty}(n+1)^2\,\frac{1}{z^n} = \frac{1+\frac{1}{z}}{\left(1-\frac{1}{z}\right)^3} = \frac{z^2(z+1)}{(z-1)^3}.$$

The function

$$F(z) = \frac{z^2(z+1)}{(z-1)^3}$$

is clearly analytic in $\mathbb{C}\setminus\{1\}$. Using that the sum of residues is zero, we get

$$\begin{aligned}\operatorname{res}\left(\frac{z^2(z+1)}{(z-1)^3}\,;\,1\right) &= -\operatorname{res}\left(\frac{z^2(z+1)}{(z-1)^3}\,;\,\infty\right) = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left\{z^3+z^2\right\}\\ &= \frac{1}{2}\lim_{z\to 1}\{3\cdot 2\cdot z+2\cdot 1\} = 4,\end{aligned}$$

and we conclude that

$$\operatorname{res}\left(\frac{z^2(z+1)}{(z-1)^3}\,;\,1\right) = 4 \qquad \text{og} \qquad \operatorname{res}\left(\frac{z^2(z+1)}{(z-1)^3}\,;\,\infty\right) = -4.$$

ALTERNATIVELY we introduce $w = z-1$, i.e. $z = w+1$, and then

$$\begin{aligned}F(z) &= \frac{z^2(z+1)}{(z-1)^3} = \frac{(w+1)^2(w+2)}{w^3} = \frac{\left(w^2+2w+1\right)(w+2)}{w^3}\\ &= 1+\frac{4}{w}+\frac{5}{w^2}+\frac{2}{w^3} = 1+\frac{4}{z-1}+\frac{5}{(z-1)^2}+\frac{2}{(z-1)^3}, \qquad z\neq 1,\end{aligned}$$

so we conclude that

$$\operatorname{res}\left(\frac{z^2(z+1)}{(z-1)^3}\,;\,1\right) = a_{-1} = 4.$$

Finally, we have the following variant of the computation of the residuum at ∞:

$$\begin{aligned}\operatorname{res}\left(\frac{z^2(z+1)}{(z-1)^3}\,;\,\infty\right) &= -\operatorname{res}\left(\frac{1}{z^2}\cdot\frac{\frac{1}{z^2}\left(\frac{1}{z}+1\right)}{\left(\frac{1}{z}-1\right)^3}\,;\,0\right) = -\operatorname{res}\left(\frac{1}{z^2}\cdot\frac{z+1}{(1-z)^3}\,;\,0\right)\\ &= -\frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left\{\frac{z+1}{(1-z)^3}\right\} = -\lim_{z\to 0}\left\{\frac{1}{(1-z)^3}+\frac{3(z+1)}{(1-z)^4}\right\} = -4.\end{aligned}$$

5 The Fourier transform

Example 5.1 *Find the Fourier transform of the function*

$$f(x) = \begin{cases} 1-|x| & \text{for } |x| \le 1, \\ \\ 0 & \text{otherwise.} \end{cases}$$

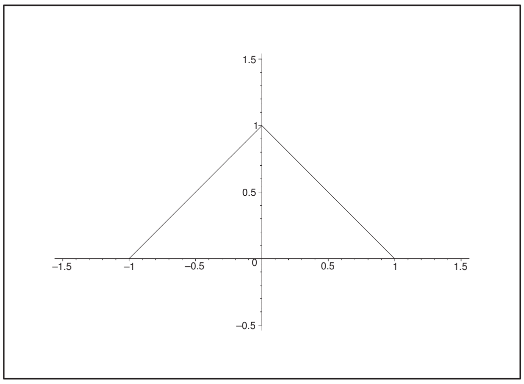

Figure 5: The graph of $f(x)$.

Clearly, $f \in L^1(\mathbb{R})$, so the Fourier transform exists.
If $\xi = 0$, then

$$\hat{f}(0) = \int_{-\infty}^{+\infty} f(x)\,dx = \frac{1}{2}\cdot 2 \cdot 1 = 1.$$

Now, $f(x)$ is an even function, so if $\xi \neq 0$, then

$$\begin{aligned} \hat{f}(\xi) &= \int_{-\infty}^{+\infty} f(x)\,e^{-ix\xi}\,dx = \int_{-1}^{1}(1-|x|)\{\cos(x\xi) - i\,\sin(x\xi)\}\,dx = 2\int_0^1 (1-x)\cos(x\xi)\,dx \\ &= \frac{2}{\xi}\left[(1-x)\sin(\xi x)\right]_0^1 + \frac{2}{\xi}\int_0^1 \sin(x\xi)\,dx = \frac{2}{\xi^2}\,(1-\cos\xi). \end{aligned}$$

Example 5.2 *Find the Fourier transform of* $\dfrac{1}{x^2+1}$.

By the definition,

$$\mathcal{F}\left\{\frac{1}{x^2+1}\right\}(\xi) = \int_{-\infty}^{+\infty} \frac{e^{-ix\xi}}{1+x^2}\,dx, \qquad \xi \in \mathbb{R}.$$

Since $f(x) = \dfrac{1}{x^2+1}$ has no pole on the real axis, and since $\dfrac{1}{z^2+1}$ has a zero of second order at ∞, we can compute the integral by using residues, *assuming that* $-\xi > 0$, i.e. $\xi < 0$. This gives us the following splitting of the computations:

1) If $\xi < 0$, then

$$\begin{aligned}\mathcal{F}\left\{\frac{1}{x^2+1}\right\}(\xi) &= \int_{-\infty}^{+\infty} \frac{e^{-ix\xi}}{x^2+1}\,dx = \int_{-\infty}^{+\infty} \frac{e^{i|\xi|x}}{x^2+1}\,dx = 2\pi i \operatorname{res}\left(\frac{e^{i|\xi|z}}{z^2+1}\,;\,i\right) \\ &= 2\pi i \lim_{z\to i} \frac{e^{i|\xi|z}}{2z} = 2\pi i\cdot\frac{e^{-|\xi|}}{2i} = \pi\,e^{-|\xi|} = \pi\,e^{\xi}.\end{aligned}$$

2) If $\xi > 0$, then it follows from **(1)** that

$$\begin{aligned}\mathcal{F}\left\{\frac{1}{x^2+1}\right\}(\xi) &= \int_{-\infty}^{+\infty} \frac{e^{-ix\xi}}{1+x^2}\,dx = \overline{\int_{-\infty}^{+\infty} \frac{e^{ix\xi}}{x^2+1}\,dx} = \overline{\int_{-\infty}^{+\infty} \frac{e^{i|\xi|x}}{x^2+1}\,dx} \\ &= \overline{\pi\,e^{-|\xi|}} = \pi\,e^{-\xi}.\end{aligned}$$

3) If $\xi = 0$, then we of course get

$$\mathcal{F}\left\{\frac{1}{x^2+1}\right\}(\xi))\int_{-\infty}^{+\infty}\frac{1}{1+x^2}\,dx = \pi \qquad \left(=\pi\,e^0\right).$$

Summing up,

$$\mathcal{F}\left\{\frac{1}{x^2+1}\right\}(\xi) = \pi\,e^{-|\xi|}, \qquad \xi\in\mathbb{R}.$$

Example 5.3 *Find the Fourier transform of* $\dfrac{\sin x}{x}$.

First note that $\dfrac{\sin x}{x}$ is a linear function. Note also that

$$\frac{\sin x}{x} \notin L^1,$$

so we have not proved that the Fourier transform exists. However, *if* it does exist, then

$$\begin{aligned}\mathcal{F}\left\{\frac{\sin x}{x}\right\}(\xi) &= \int_{-\infty}^{+\infty}\frac{\sin x}{x}\cdot e^{-ix\xi}\,dx = 2\int_0^{+\infty}\frac{\sin x}{x}\cdot\cos(x\xi)\,dx \\ &= \int_0^{+\infty}\frac{1}{x}\left\{\sin((1+\xi)x)+\sin((1-\xi)x)\right\}dx.\end{aligned}$$

If $|\xi| < 1$, then $1+\xi > 0$ and $1-\xi > 0$, and we get that

$$\begin{aligned}\mathcal{F}\left\{\frac{\sin x}{x}\right\}(\xi) &= \int_0^{+\infty}\frac{\sin((1+\xi)x)}{(1+\xi)x}\cdot(1+\xi)\,dx + \int_0^{+\infty}\frac{\sin((1-\xi)x)}{(1-\xi)x}\cdot(1-\xi)\,dx \\ &= \frac{\pi}{2}+\frac{\pi}{2} = \pi.\end{aligned}$$

If instead $|\xi| > 1$, then we may assume that $\xi > 1$, because the computations are analogous for $\xi < -1$. Then

$$\begin{aligned}
\mathcal{F}\left\{\frac{\sin x}{x}\right\}(\xi) &= \int_0^{+\infty} \frac{\sin((1+\xi)x)}{(1+\xi)x}\cdot(1+\xi)\,dx + \int_0^{+\infty} \frac{\sin((1-\xi)x)}{(1-\xi)x}\cdot(1-\xi)\,dx \\
&= \int_0^{+\infty} \frac{\sin t}{t}\,dt + \int_0^{-\infty} \frac{\sin t}{t}\,dt = \int_0^{+\infty} \frac{\sin t}{t}\,dt - \int_0^{+\infty} \frac{\sin t}{t}\,dt = 0.
\end{aligned}$$

If $\xi = \pm 1$, e.g. $\xi = 1$, then

$$\mathcal{F}\left\{\frac{\sin x}{x}\right\}(1) = \int_0^{+\infty} \frac{\sin 2x + 0}{x}\,dx = \frac{\pi}{2} \qquad \left(= \mathcal{F}\left\{\frac{\sin x}{x}\right\}(-1)\right).$$

Since $\xi = \pm 1$ is a null set, we can neglect the value at these points, so we get summing up,

$$\mathcal{F}\left\{\frac{\sin x}{x}\right\}(\xi) = \begin{cases} \pi & \text{for } \xi \in]-1,1[, \\ 0 & \text{otherwise.} \end{cases}$$

Example 5.4 *Find the Fourier transform of* $e^{-a|x|}$, $a>0$.

Since $a>0$, it follows trivially that $e^{-a|x|}\in L^1$. Since the function is even, we get

$$\begin{aligned}
\mathcal{F}\left\{e^{-a|x|}\right\}(\xi) &= \int_{-\infty}^{+\infty} e^{-a|x|}e^{-ix\xi}\,dx = 2\int_0^{+\infty} e^{-ax}\cos(x\xi)\,dx \\
&= 2\,\mathrm{Re}\left\{\int_0^{+\infty} e^{-ax+i\xi x}\,dx\right\} = 2\,\mathrm{Re}\left\{\left[\frac{1}{-a+i\xi}\,e^{-ax+i\xi x}\right]_0^{+\infty}\right\} \\
&= 2\,\mathrm{Re}\left\{-\frac{1}{-a+i\xi}\right\} = 2\,\mathrm{Re}\left\{\frac{1}{a-i\xi}\right\} = 2\,\mathrm{Re}\left\{\frac{a+i\xi}{a^2+\xi^2}\right\} = \frac{2a}{a^2+\xi^2}.
\end{aligned}$$

Example 5.5 *Find the Fourier transform of* $\dfrac{1}{x^4+5x^2+6}$.

Clearly,

$$f(x)=\frac{1}{x^4+5x^2+6}\in L^1(\mathbb{R}).$$

Now, $f(x)$ is an even function, so

$$\begin{aligned}
\mathcal{F}\{f\}(\xi) &= \int_{-\infty}^{+\infty}\frac{e^{-ix\xi}}{x^4+5x^2+6}\,dx = \int_{-\infty}^{+\infty}\frac{e^{ix\xi}}{x^4+5x^2+6}\,dx \\
&= \int_{-\infty}^{+\infty}\frac{e^{i|\xi|x}}{x^4+5x^2+6}\,dx = \mathcal{F}\{f\}(-\xi).
\end{aligned}$$

We get by a decomposition that

$$\frac{1}{x^4+5x^2+6}=\frac{1}{x^2+2}-\frac{1}{x^2+3},$$

hence by residuum computations for $\xi\neq 0$,

$$\begin{aligned}
\mathcal{F}\left\{\frac{1}{x^4+5x^2+6}\right\}(\xi) &= \int_{-\infty}^{+\infty}\frac{e^{i|\xi|x}}{x^4+5x^2+6}\,dx = \int_{-\infty}^{+\infty}\frac{e^{i|\xi|x}}{x^2+2}\,dx - \int_{-\infty}^{+\infty}\frac{e^{i|xi|x}}{x^2+3}\,dx \\
&= 2\pi i\left\{\mathrm{res}\left(\frac{e^{i|\xi|z}}{z^2+2}\,;\,i\sqrt{2}\right) - \mathrm{res}\left(\frac{e^{i|\xi|z}}{z^2+3}\,;\,i\sqrt{3}\right)\right\} \\
&= 2\pi i\left\{\frac{e^{i|\xi|\cdot i\sqrt{2}}}{2i\sqrt{2}} - \frac{e^{i|\xi|\cdot i\sqrt{3}}}{2i\sqrt{3}}\right\} = \pi\left\{\frac{e^{-|\xi|\sqrt{2}}}{\sqrt{2}} - \frac{e^{-|\xi|\sqrt{3}}}{\sqrt{3}}\right\}.
\end{aligned}$$

Since we have a zero of second order at ∞, the integral is also convergent for $\xi=0$, and it follows by the continuity that the expressions above holds for all $\xi\in\mathbb{R}$.

Example 5.6 *Compute*

$$(a)\quad \int_{-\infty}^{+\infty} \frac{\sin x}{x}\, e^{-|x|}\, dx, \qquad (b)\quad \int_{0}^{+\infty} \frac{\sin x}{x}\, e^{-x}\, dx.$$

The integrand is an even function, so

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x}\, e^{-|x|}\, dx = 2\int_{0}^{+\infty} \frac{\sin x}{x}\, e^{-x}\, dx,$$

which is the double of the value of the integral in **(b)**.

We get by the *Laplace transform* that

$$\int_{0}^{+\infty} \frac{\sin x}{x}\, e^{-x}\, dx = \mathcal{L}\left\{\frac{\sin x}{x}\right\}(1) = \int_{1}^{+\infty} \mathcal{L}\{\sin x\}(t)\, dt = \int_{1}^{+\infty} \frac{1}{1+t^2}\, dt = \frac{\pi}{4}.$$

It follows that

$$(a)\quad \int_{-\infty}^{+\infty} \frac{\sin x}{x}\, e^{-|x|}\, dx = \frac{\pi}{2} \qquad \text{og} \qquad (b)\quad \int_{0}^{+\infty} \frac{\sin x}{x}\, e^{-x}\, dx = \frac{\pi}{4}.$$

Example 5.7 *Find the Fourier transform of*

$$f(x) = e^{-x}\int_0^x \frac{\sin t}{t}\, e^t\, dt.$$

Since $f(x)$ is differentiable with the derivative

$$f'(x) = -f(x) + e^x \cdot \frac{\sin x}{x} \cdot e^x = -f(x) + \frac{\sin x}{x},$$

we have

$$f'(x) + f(x) = \frac{\sin x}{x}.$$

Then by the Fourier transform,

$$i\,\xi\,\hat{f}(\xi) + \hat{f}(\xi) = \mathcal{F}\left\{\frac{\sin x}{x}\right\}(\xi) = \pi\,\chi_{[-1,1]}(\xi),$$

(either by using a table, or by referring to Example 5.3), so

$$\hat{f}(\xi) = \frac{\pi}{1+i\xi}\,\chi[-1,1](\xi) = \pi\cdot\frac{1-i\xi}{1+\xi^2}\,\chi_{[-1,1]}(\xi).$$

Example 5.8 *Find the Fourier transform of*

$$f(x) = \frac{1}{x^2+1} \cdot \frac{\sin x}{x}.$$

It follows by the Fourier transform from

$$x^2 f(x) + f(x) = \frac{\sin x}{x},$$

that

$$\mathcal{F}\left\{x^2 f\right\}(\xi) + \mathcal{F}\{f\}(\xi) = -\frac{d^2 \hat{f}}{d\xi^2} + \hat{f}(\xi) = \mathcal{F}\left\{\frac{\sin x}{x}\right\}(\xi) = \pi \cdot \chi_{[-1,1]}(\xi).$$

We shall for $\xi \in]-1,1[$ solve the differential equation

$$\frac{d^2 \hat{f}}{d\xi^2} - \hat{f}(\xi) = -\pi, \qquad \xi \in]-1,1[.$$

The complete solution is

$$\hat{f}(\xi) = \pi + a_1 e^{\xi} + a_2 e^{-\xi}, \qquad \xi \in]-1,1[.$$

When $\xi \in \mathbb{R} \setminus [-1,1]$, we shall instead solve the differential equation

$$\frac{d^2 \hat{f}}{d\xi^2} - \hat{f}(\xi) = 0, \qquad \xi \in \mathbb{R} \setminus [-1,1].$$

Since $f \in L^1(\mathbb{R})$, we have $\hat{f}(\xi) \to 0$ for $\xi \to \pm\infty$, so the set of solutions is

$$\hat{f}(\xi) = \begin{cases} b_1 e^{\xi} & \text{for } \xi < -1, \\ b_2 e^{-\xi} & \text{for } \xi > 1. \end{cases}$$

Since f and $x\,f \in L^1(\mathbb{R})$, both $\hat{f}(\xi)$ and $\dfrac{d\hat{f}}{d\xi}$ are *uniformly continuous*, hence the solution

$$\hat{f}(\xi) = \begin{cases} b_1 e^{\xi}, & \text{for } \xi < -1, \\ \pi + a_1 e^{\xi} + a_2 e^{-\xi}, & \text{for } \xi \in]-1,1[, \\ b_2 e^{-\xi}, & \text{for } \xi > 1, \end{cases}$$

is continuously differentiable, even at the points $\xi = \pm 1$. It follows from

$$\frac{d\hat{f}}{d\xi} = \begin{cases} b_1 e^{\xi}, & \text{for } \xi < -1, \\ a_1 e^{\xi} - a_2 e^{-\xi}, & \text{for } \xi \in]-1,1[, \\ -b_2 e^{-\xi}, & \text{for } \xi > 1, \end{cases}$$

when $\xi = -1$ that

$$\begin{cases} b_1 e^{-1} = a_1 e^{-1} + a_2 e + \pi, \\ b_1 e^{-1} - a_2 e, \end{cases}$$

hence by subtraction,

$$a_2 = -\frac{\pi}{2e}.$$

If $\xi = 1$, then we get the conditions

$$\begin{cases} b_2 e^{-1} &= a_1 e + a_2 e^{-1} + \pi, \\ -b_2 e^{-1} &= a_1 e - a_2 e^{-1}, \end{cases}$$

hence by addition,

$$a_1 = -\frac{\pi}{2e} = a_2.$$

Then

$$b_1 = a_1 - a_2 e^2 = -\frac{\pi}{2e}\left(1 - e^2\right) = \frac{\pi}{2e}\left(e^2 - 1\right) = \pi \sinh 1,$$

and

$$b_2 = a_2 - a_1 e^2 = \frac{\pi}{2e}\left(e^2 - 1\right) = \pi\ \sinh 1.$$

We conclude that

$$\hat{f}(\xi) = \begin{cases} \pi \cdot \sinh 1 \cdot e^{\xi}, & \text{for } \xi < -1, \\ \pi - \frac{\pi}{2e}\, e^{\xi} - \frac{\pi}{2e}\, e^{-\xi} = \pi - \frac{\pi}{e}\ \cosh \xi, & \text{for } \xi \in]-1,1[, \\ \pi \cdot \cdot \sinh 1 \cdot e^{-\xi}, & \text{for } \xi > 1. \end{cases}$$

6 Linear difference equations

Example 6.1 *Solve the difference equation*

$$x_{n+2} + x_{n+1} - \frac{3}{4}\,x_n = 2^{-n}, \qquad n \in \mathbb{N}_0.$$

The characteristic polynomial is

$$z^2 + z - \frac{3}{4} = \left\{z - \frac{1}{2}\right\}\left\{z + \frac{3}{4}\right\},$$

so we conclude that the corresponding homogeneous difference equation has the complete solution

$$a \cdot \left\{\frac{1}{2}\right\}^n + b \cdot \left\{-\frac{3}{2}\right\}^n, \qquad n \in \mathbb{N}_0,\ \text{and } a,\, b \in \mathbb{C} \text{ arbitrary constants.}$$

Then we shall find the particular solution for which $x_0 = 0$ and $x_1 = \frac{1}{2}$. One may of course choose other initial values. However, the initial conditions above have been chosen such that the computations become as simple as possible.

We get by the $\mathfrak{z}$-transform,

$$A(z) = \sum_{n=0}^{+\infty} a_n\, z^{-n} = \sum_{n=0}^{+\infty} 2^n\, z^- n = \frac{2z}{2z-1} \qquad \text{for } |z| > \frac{1}{2}.$$

Then it follows from the solution formula for the linear inhomogeneous difference equation of second order that the $\mathfrak{z}$-transform of the particular solution is given by

$$\begin{aligned}
X(z) &= \frac{1}{z^2 + z - \frac{3}{4}}\left\{\frac{z}{z-\frac{1}{2}} + x_0 z^2 + (c_1 x_0 + x_1)\, z\right\} = \frac{1}{\left(z-\frac{1}{2}\right)\left(z+\frac{3}{2}\right)}\left\{\frac{z}{z-\frac{1}{2}} + \frac{1}{2}\,z\right\} \\
&= \frac{1}{\left(z-\frac{1}{2}\right)^2\left(z+\frac{3}{2}\right)}\left\{z + \frac{1}{2}\,z^2 - \frac{1}{4}\,z\right\} = \frac{1}{\left(z-\frac{1}{2}\right)^2\left(z+\frac{3}{2}\right)} \cdot \frac{1}{2} \cdot z\left(z + \frac{3}{2}\right) = \frac{\frac{1}{2}\,z}{\left(z-\frac{1}{2}\right)^2} \\
&= \frac{1}{2z}\left(1 - \frac{1}{2z}\right)^{-2} = \frac{1}{2z}\sum_{n=1}^{+\infty} n\left\{\frac{1}{2z}\right\}^{n-1} = \sum_{n=0}^{+\infty} \frac{n}{2^n}\, z^{-n},
\end{aligned}$$

where the series are convergent for $|z| > \frac{1}{2}$. Since $X(z)$ is the $\mathfrak{z}$-transform of the particular solution $\{x_n\}$, we conclude that the complete solution is given by

$$x_n = \frac{n}{2}^n + a\left\{\frac{1}{2}\right\}^2 + b\left\{-\frac{3}{2}\right\}^n, \qquad \text{for } n \in \mathbb{N}_0 \text{ og } a,\, b \in \mathbb{C} \text{ arbitrary constants.}$$

Example 6.2 *Find the complete solution of the difference equation*

$$x_{n+2} = x_{n+1} + x_n, \qquad n \in \mathbb{N}.$$

Then find the solution for which $x_1 = x_2 = 1$. (Fibonacci's sequence).
Prove for Fibonacci's sequence that

$$\left(\frac{x_{n+1}}{x_n} \right)_{n \in \mathbb{N}}$$

is convergent and find its limit.

The characteristic equation

$$\lambda^2 - \lambda - 1 = 0$$

has the roots

$$\lambda = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1 \pm \sqrt{5}}{2},$$

so we conclude that the complete solution is given by

$$x_n = a \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + b \cdot \left(-\frac{\sqrt{5}-1}{2}\right)^n, \qquad n \in \mathbb{N}_0,$$

where a and b are arbitrary constants.

When $x_1 = x_2 = 1$, then it follows from the above that

$$1 = a \cdot \frac{1+\sqrt{5}}{2} - b \cdot \frac{\sqrt{5}-1}{2}, \qquad 1 = a\left(\frac{1+\sqrt{5}}{2}\right)^2 + b\left(\frac{\sqrt{5}-1}{2}\right)^2.$$

When we subtract the former equation from the latter one, we get

$$\begin{aligned} 0 &= a\left\{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1+\sqrt{5}}{2}\right)\right\} + b\left\{\left(\frac{\sqrt{5}-1}{2}\right)^2 + \left(\frac{\sqrt{5}-1}{2}\right)\right\} \\ &= \frac{1+\sqrt{5}}{2} \cdot \frac{\sqrt{5}-1}{2} \cdot a + \frac{\sqrt{5}-1}{2} \cdot \frac{\sqrt{5}+1}{2} \cdot b = 2(a+b), \end{aligned}$$

and we conclude that $b = -a$, hence

$$1 = a\left(\frac{1+\sqrt{5}}{2} + \frac{\sqrt{5}-1}{2}\right) = \sqrt{5} \cdot a,$$

thus

$$a = \frac{1}{\sqrt{5}} \qquad \text{and} \qquad b = -\frac{1}{\sqrt{5}},$$

and the solution is given by

$$x_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(-\frac{\sqrt{5}-1}{2}\right)^n\right\}, \qquad n \in \mathbb{N}_0.$$

Now put $\alpha = \dfrac{1+\sqrt{5}}{2} > 1$. Then $\dfrac{\sqrt{5}-1}{2} = \dfrac{1}{\alpha} < 1$, so

$$x_n = \frac{1}{\sqrt{5}}\left\{\alpha^n - (-1)^n \cdot \frac{1}{\alpha^n}\right\} > 0,$$

from which we get

$$\begin{aligned} \frac{x_{n+1}}{x_n} &= \frac{\frac{1}{\sqrt{5}} \cdot \left\{\alpha^{n+1} - (-1)^{n+1} \cdot \frac{1}{\alpha^{n+1}}\right\}}{\frac{1}{\sqrt{5}}\left\{\alpha^n - (-1)^n \cdot \frac{1}{\alpha^n}\right\}} = \frac{\alpha - (-1)^{n+1} \cdot \frac{1}{\alpha^{2n+1}}}{1 - (-1)^n \cdot \frac{1}{\alpha^{2n}}} \\ &\to \alpha = \frac{1+\sqrt{5}}{2} \qquad \text{for } n \to +\infty. \end{aligned}$$

Remark 6.1 The number $\alpha = \dfrac{1+\sqrt{5}}{2}$ is called the *golden section*. When $n = 10$, we get

$$\frac{x_{11}}{x_{10}} \approx 1,618\,181\,8,$$

which is a reasonable approximation of

$$\alpha = \frac{1+\sqrt{5}}{2} \approx 1,618\,034\,0. \qquad \Diamond$$

Example 6.3 *Find the complete solution of the difference equation*

$$x_{n+2} = 4\,x_{n+1} - 4\,x_n, \qquad n \in \mathbb{N}.$$

Then find the solution, for which $x_1 = 0$ *and* $x_2 = 4$.

Write the equation in the form

$$x_{n+2} - 4\,x_{n+1} + 4\,x_n = 0, \qquad n \in \mathbb{N}.$$

The characteristic equation is

$$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0,$$

where $\lambda = 2$ is a double root. The complete solution is then

$$x_n = a \cdot 2^n + b\,n \cdot 2^n, \qquad n \in \mathbb{N}_0,$$

where a and b are arbitrary constants.

Put $x_1 = 0$ and $x_2 = 4$. Then we have the equations

$$\begin{cases} 0 = x_1 = 2a + 2b, \\ \\ 4 = x_2 = 4a + 8b, \end{cases}$$

so $a = -b$ and $4b = 4$, thus $b = 1$ and $a = -1$. The solution is

$$x_n = -2^n + n \cdot 2^n = (n-1)2^n, \qquad n \in \mathbb{N}.$$

Example 6.4 *Find the complete solution of the difference equation*

$$x_{n+2} = 2\,x_{n+1} - 2\,x_n, \qquad n \in \mathbb{N}.$$

Find the solution (expressed in the real), for which $x_1 = 2$ and $x_2 = 0$.

The characteristic equation

$$\lambda^2 - 2\lambda + 2 = 0$$

has the roots $\lambda = 1 \pm i$, so the complete solution is

$$x_n = a \cdot (1+i)^n + b \cdot (1-i)^n, \qquad n \in \mathbb{N}_0,$$

where a and b are complex arbitrary constants.

Put $x_1 = 2$ and $x_2 = 0$. Then

$$\begin{cases} 2 = x_1 = a \cdot (1+i) + b \cdot (1-i) = (a+b) + i(a-b), \\ \\ 0 = x_2 = a \cdot 2i + b \cdot (-2i) = 2i(a-b), \end{cases}$$

hence $a - b = 0$ and $a + b = 2$, thus $a = b = 1$, and the solution is

$$\begin{aligned} x_n &= (1+i)^n + (1-i)^n = 2\,\mathrm{Re}\,\{(1+i)^n\} = 2\,\mathrm{Re}\left\{\left(\sqrt{2}\right)^n \exp\left(i\,n\frac{\pi}{4}\right)\right\} \\ &= 2 \cdot \left(\sqrt{2}\right)^n \cos\left(n \cdot \frac{\pi}{4}\right), \qquad n \in \mathbb{N}. \end{aligned}$$

Remark 6.2 If we choose $x_1 = 2$ and $x_2 = 4$ instead, then we get (cf. the above)

$$\begin{cases} (a+b) + i(a-b) = 2, \\ \\ 2i(a-b) = 4, \end{cases} \quad \text{thus} \quad \begin{cases} a+b = 0, \\ \\ a-b = -2i, \end{cases}$$

so $a = -i$ and $b = i$. Then we get by insertion,

$$\begin{aligned} x_n &= -i(1+i)^n + i(1-i)^n = 2\,\mathrm{Re}\,\{-i(1+i)^n\} = 2\,\mathrm{Re}\left\{-i\left\{\sqrt{2}\right)^n \exp\left(i\,n\,\frac{\pi}{4}\right)\right\} \\ &= 2\,\mathrm{Re}\left\{\left(\sqrt{2}\right)^n \cdot (-i)\left(\cos\frac{n\pi}{4} + i\,\sin\frac{n\pi}{4}\right)\right\} = 2\left(\sqrt{2}\right)^n \sin\left(n \cdot \frac{\pi}{4}\right). \end{aligned}$$

Note in particular that the constants a and b are complex, even if the solution itself

$$x_n = 2\left(\sqrt{2}\right)^n \sin\left(n \cdot \frac{\pi}{4}\right), \qquad n \in \mathbb{N},$$

is real. $\Diamond$

Example 6.5 *Prove that every difference equation of the type*

$$x_{n+2} + a\, x_{n+1} + x_n = 0, \qquad n \in \mathbb{N},$$

where a is any constant, has a bounded solution (x_n).

The characteristic equation is

$$\lambda^2 + a \cdot \lambda + 1 = 0.$$

The product of the roots is $\alpha \cdot \beta = 1$, so we can assume that $|\alpha| \le 1$. Then a bounded solution is

$$x_n = c \cdot \alpha^n, \qquad n \in \mathbb{N},$$

because

$$|x_n| \le |c| \qquad \text{for every } n \in \mathbb{N}.$$

The argument is unchanged, no matter if $\alpha \neq \beta$ or $\alpha = \beta$.

Example 6.6 *1) Find the complete solution of the difference equation*

$$a_{n+2} - 3\, a_{n+1} + 2\, a_n = 0, \qquad n \in \mathbb{N}_0.$$

2) Solve the linear differential equation

$$\left(2z^2 - 3z + 1\right) f''(z) + (8z - 6) f'(z) + 4f(z) = 0$$

by the power series solution method from the point $z_0 = 0$,

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n, \qquad |z| < R,$$

Find the largest possible R.
Then express $f(z)$ by elementary functions.

1) The characteristic equation

$$\lambda^2 - 3\,\lambda + 2 = 0$$

has the roots 1 and 2. The difference equation is homogeneous, so the complete solution is given by

$$a_n = a + b \cdot 2^n, \qquad n \in \mathbb{N}_0,$$

where a, $b \in \mathbb{C}$ are arbitrary constants.

2) **Inspection.** We rearrange the differential equation in the following way,

$$\begin{aligned}
0 &= \left(2z^2-3z+1\right) f''(z)+(8z-6)f'(z)+4f(z) \\
&= \left(2z^2-3z+1\right) f''(z)+(4z-3)f'(z)+(4z-3)f'(z)+4f(z) \\
&= \left\{\left(2z^2-3z+1\right) \frac{d}{dz}\, f'(z) + \frac{d}{dz}\, \left(2z^2-3z+1\right)\cdot f'(z)\right\} + \left\{(4z-3)\,\frac{df}{dz} + \frac{d}{dz}\,(4z-3)\cdot f(z)\right\} \\
&= \frac{d}{dz}\,\left\{\left(2z^2-3z+1\right) f'(z) + (4z-3)f(z)\right\} \\
&= \frac{d}{dz}\,\left\{\left(2z^2-3z+1\right)\,\frac{d}{dz}\, f(z) + \frac{d}{dz}\,\left(2z^2-3z+1\right)\cdot f(z)\right\} \\
&= \frac{d^2}{dz^2}\,\left\{\left(2z^2-3z+1\right) f(z)\right\}.
\end{aligned}$$

Then it follows by two successive integrations,

$$\left(2z^2-3z+1\right) f(z) = c_1 z + c_2,$$

so the complete solution is

$$f(z) = \frac{c_1 z + c_2}{2z^2-3z+1} = \cdots = -\frac{c_1+2x_2}{2z-1} + \frac{c_1+c_2}{z-1}$$

for $z \in \mathbb{C} \setminus \left\{\frac{1}{2}\,;\,1\right\}$, when the arbitrary constants c_1 and c_2 are chosen, such that $c_1 + 2c_2 \neq 0$ and $c_1 + c_2 \neq 0$, with trivial modifications, if one of them is 0.

Power series method. ALTERNATIVELY we insert a formal power series and its formal derivatives,

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n, \quad f'(z) = \sum_{n=1}^{+\infty} n\, a_n z^{n-1}, \quad f''(z) = \sum_{n=2}^{+\infty} n(n-1)a_n z^{n-2},$$

where we assume that the series are convergent for $|z| < R$. Since

$$2z^2 - 3z + 1 = 0$$

for $z = \frac{1}{2}$ and for $z = 1$, we conclude that $R \geq \frac{1}{2}$, where $R = \frac{1}{2}$ is our guess. This will become more clear in the following investigation.

We get by insertion of the series,

$$
\begin{aligned}
0 &= \left(2z^2-3z+1\right) f''(z)+(8z-6)f'(z)+4f(z) \\
&= \sum_{\substack{n=2\\(n=0)}}^{+\infty} 2n(n-1)a_n z^n - \sum_{\substack{n=2\\(n=1)}}^{+\infty} 3n(n-1)a_n z^{n-1} + \sum_{n=2}^{+\infty} n(n-1)a_n z^{n-2} \\
&\qquad + \sum_{\substack{n=1\\(n=0)}}^{+\infty} 8na_n z^n - \sum_{n=1}^{+\infty} 6na_n z^{n-1} + \sum_{n=0}^{+\infty} 4a_n z^n \\
&= \sum_{n=0}^{+\infty} \left\{2n^2-2n+8n+4\right\} a_n z^n - \sum_{n=1}^{+\infty} 3n(n+1)a_n z^{n-1} + \sum_{n=2}^{+\infty} n(n-1)a_n z^{n-2} \\
&= \sum_{n=0}^{+\infty} 2\left\{n^2+3n+2\right\} a_n z^n - \sum_{n=0}^{+\infty} 3(n+1)(n+2)a_{n+1} z^n + \sum_{n=0}^{+\infty}(n+2)(n+1)a_{n+2} z^n \\
&= \sum_{n=0}^{+\infty}(n+1)(n+2)\left\{2a_n-3a_{n+1}+a_{n+2}\right\} z^n .
\end{aligned}
$$

We conclude from the *identity theorem* that this equation is fulfilled, if and only if the following recursion formula holds (and z belongs to the domain of convergence),

$$(n+1)(n+2)\left\{2a_n - 3a_{n+1} + 1_{n+2}\right\} = 0, \qquad n \in \mathbb{N}_0 .$$

Since $(n+1)(n+2) \neq 0$ for every $n \in \mathbb{N}_0$, this recursion formula is equivalent to the difference equation

$$a_{n+2} - 3\,a_{n+1} + 2\,a_n = 0, \qquad n \in \mathbb{N}_0,$$

which we solved in **(1)**.

According to **(1)** the complete solution is

$$a_n = a + b \cdot 2^n, \qquad n \in \mathbb{N}_0,$$

so we get the *formal* power series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n = \sum_{n=0}^{+\infty} \left\{a + b \cdot 2^n\right\} z^n.$$

If $b \neq 0$, then

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{a + b \cdot 2^n}{a + b \cdot 2^{n+1}}\right| = \frac{1}{2}\left|\frac{b + a \cdot 2^{-n}}{b + a \cdot 2^{-n-1}}\right| \to \frac{1}{2} \qquad \text{for } n \to +\infty,$$

so the radius of convergence is $\dfrac{1}{2}$ in this case.

If $b = 0$ and $a \neq 0$, then

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{a}{a}\right| = 1 \to 1 \qquad \text{for } n \to +\infty,$$

and the radius of convergence is 1 in this case.

If both a and b are zero, we get the zero series of radius of convergence $+\infty$.

When $|z| < \dfrac{1}{2}$, then $|2z| < 1$, so

$$f(z) = \sum_{n=0}^{+\infty} \left\{a + b \cdot 2^n\right\} z^n = a \sum_{n=0}^{+\infty} z^n + b \sum_{n=0}^{+\infty} 2^n z^n = a \sum_{n=0}^{+\infty} z^n + b \sum_{n=0}^{+\infty} (2z)^n = \frac{a}{1-z} + \frac{b}{1-2z}.$$

It is obvious that if $z \in \mathbb{C} \setminus \left\{\dfrac{1}{2}\,,\,1\right\}$, then

$$f(z) = \frac{a}{1-z} + \frac{b}{1-2z}$$

satisfies the differential equation.

Example 6.7 *By the solution of a difference equation of the solution*

$$y_n, \qquad n = 0, 1, 2, \cdots,$$

it has been derived that the $\mathfrak{z}$*-transform of* y_n*, i.e.* $\mathfrak{z}\{y_n\}$*, is*

$$Y(z) = \frac{z}{(z-\alpha)(z-\beta)(z-1)},$$

where z *is a complex variable, and* $\beta < \alpha < 0$.

(a) *Find the singularities of* $Y(z)$ *and their type for* $|z| < \infty$.
Compute the residues in the poles.

(b) *Using the calculus of residues one shall find* y_n, $n = 0, 1, 2, \cdots$, *and prove that*

$$y_n = \frac{1}{\alpha-\beta}\left\{\frac{\alpha^n - 1}{\alpha - 1} - \frac{\beta^n - 1}{\beta}\right\}.$$

(a) The singularities are the simple poles 1, α and β and

$$\operatorname{res}(Y(z);1) = \frac{1}{(1-\alpha)(1-\beta)},$$

$$\operatorname{res}(Y(z);\alpha) = \frac{\alpha}{(\alpha-\beta)(\alpha-1)}.$$

$$\operatorname{res}(Y(z);\beta) = \frac{\beta}{(\beta-\alpha)(\beta-1)}.$$

(b) We shall use the residuum formula

$$y_n = \operatorname{res}\left(Y(z)z^{n-1};1\right) + \operatorname{res}\left(Y(z)z^{n-1};\alpha\right) + \operatorname{res}\left(Y(z)z^{n-1};\beta\right),$$

where

$$\operatorname{res}\left(Y(z)z^{n-1};1\right) = \operatorname{res}\left(\frac{z^n}{(z-\alpha)(z-\beta)(z-1)};1\right) = \frac{1}{(1-\alpha)(1-\beta)},$$

$$\operatorname{res}\left(Y(z)z^{n-1};\alpha\right) = \operatorname{res}\left(\frac{z^n}{(z-\alpha)(z-\beta)(z-1)};\alpha\right) = \frac{\alpha^n}{(\alpha-\beta)(\alpha-1)},$$

$$\operatorname{res}\left(Y(z)z^{n-1};\beta\right) = \operatorname{res}\left(\frac{z^n}{(z-\alpha)(z-\beta)(z-1)};\beta\right) = \frac{\beta^n}{(\beta-\alpha)(\beta-1)}.$$

Since

$$\operatorname{res}\left(Y(z)z^{n-1};1\right) = \frac{1}{(\alpha-1)(\beta-1)} = \frac{1}{\alpha-\beta}\left\{\frac{1}{\beta-1} - \frac{1}{\alpha-1}\right\},$$

it follows by insertion and reduction,

$$y_n = \frac{1}{\alpha-\beta}\left\{\frac{\alpha^n - 1}{\alpha - 1} - \frac{\beta^n - 1}{\beta - 1}\right\}, \qquad n \in \mathbb{N}.$$

If $n = 0$, then

$$y_0 = Y(0) = 0,$$

in accordance with the formula above.

7 Distribution theory

Example 7.1 *Compute* $x^n\delta^{(n)}$, $n \in \mathbb{N}$.

Assuming that $\varphi \in C_0^\infty(\mathbb{R})$, then

$$\begin{aligned}\left\langle x^n\delta^{(n)}\,,\,\varphi\right\rangle &= \left\langle \delta^{(n)}\,,\,x^n\varphi\right\rangle = (-1)^n\left\langle \delta\,,\,\frac{d^2}{dx^n}\left(x^n\varphi\right)\right\rangle = (-1)^n\left\langle\delta\,,\,\sum_{j=0}^{+\infty}\binom{n}{j}\frac{d^j}{dx^j}\,x^n\cdot\frac{d^{n-j}}{dx^{n-j}}\,\varphi\right\rangle\\ &= (-1)^n\left\langle\delta\,,\,\frac{d^n x^n}{dx^n}\cdot\varphi\right\rangle = (-1)^n\cdot n!\langle\delta\,,\,\varphi\rangle,\end{aligned}$$

and we conclude that

$$x^n\delta^{(n)} = (-1)^n n!\,\delta.$$

Example 7.2 *Compute the derivative in the sense of distributions of the function*

$$f(x) = \begin{cases}\cos x & \textit{for } x \in \left[0\,,\,\dfrac{\pi}{2}\right],\\[2mm] 0 & \textit{ellers.}\end{cases}$$

Since

$$f(x) = \cos x\cdot\chi_{[0,\frac{\pi}{2}]}(x),$$

it follows for every $\varphi \in C_0^\infty(\mathbb{R})$ that

$$\begin{aligned}\langle f'\,,\,\varphi\rangle &= -\langle f\,,\,\varphi'\rangle = -\int_0^{\frac{\pi}{2}}\cos x\cdot\varphi'(x)\,dx = \left[-\cos x\cdot\varphi(x)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\sin x\cdot\varphi(x)\,dx\\ &= \varphi(0) - \left\langle\sin x\cdot\chi_{[0,\frac{\pi}{2}]}(x)\,,\,\varphi(x)\right\rangle = \left\{\delta - \sin x\cdot\chi_{[0,\frac{\pi}{2}]}(x)\,,\,\varphi\right\rangle.\end{aligned}$$

This holds for every test function $\varphi \in C_0^\infty(\mathbb{R})$, so it follows that

$$f' = \delta - \sin x\cdot\chi_{[0,\frac{\pi}{2}]}(x).$$

Alternatively (and perhaps a little more dangerous?) we shall use the rules of computations, because we have

$$\frac{d}{dx}\,\chi_{[0,\frac{\pi}{2}]} = \delta_{(0)} - \delta_{\left(\frac{\pi}{2}\right)},$$

which either can be seen directly or by the rearrangement

$$\frac{d}{dx}\,\chi_{[0,\frac{\pi}{2}]} = \frac{d}{dx}\left\{\chi_{[0,+\infty[} - \chi_{]\frac{\pi}{2},+\infty[}\right\} = \delta_{(0)} - \delta_{\left(\frac{\pi}{2}\right)}.$$

Then it follows by the rules of computations that

$$\begin{aligned}f' &= \frac{d}{dx}\left\{\cos x\cdot\chi_{[0,\frac{\pi}{2}]}(x)\right\} = -\sin x\cdot\chi_{[0,\frac{\pi}{2}]} + \cos x\cdot\frac{d}{dx}\,\chi_{[0,\frac{\pi}{2}]}\\ &= -\sin x\cdot\chi_{[0,\frac{\pi}{2}]}(x) + \cos x\cdot\delta_{(0)} - \cos x\cdot\delta_{\left(\frac{\pi}{2}\right)} = \cos 0\cdot\delta_{(0)} - \cos\left(\frac{\pi}{2}\right)\cdot\delta_{\left(\frac{\pi}{2}\right)} - \sin x\cdot\chi_{[0,\frac{\pi}{2}]}(x)\\ &= \delta - \sin x\cdot\chi_{[0,\frac{\pi}{2}]}(x).\end{aligned}$$

Example 7.3 *Find the Fourier transform of* $\cos x$*.*

Formally/ the Fourier transform of $\cos x$ is given by

$$\begin{aligned}\mathcal{F}\{\cos\}(\xi) &= \int_{-\infty}^{+\infty} \cos x \cdot e^{-ix\xi}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-x(\xi-1)}\,dx + \frac{1}{2}\int_{-\infty}^{+\infty} e^{-ix(\xi+1)}dx \\ &= \frac{1}{2}\left\{2\pi\delta_{(1)} + 2\pi\delta_{(-1)}\right\} = \pi\left\{\delta_{(1)} + \delta_{(-1)}\right\}.\end{aligned}$$

We shall not *prove* this result by using the inversion formula, i.e. we shall only show that

$$2\pi \cdot \cos(-\xi) = 2\pi\,\cos\xi = \pi \cdot \mathcal{F}\left\{\delta_{(1)} + \delta_{(-1)}\right\}(\xi).$$

Now

$$\mathcal{F}\left\{\delta_{(1)} + \delta_{(-1)}\right\}(\xi) = \delta_{(1)}\left(e^{-ix\xi}\right) + \delta_{(-1)}\left(e^{-ix\xi}\right) = e^{-i\xi} + e^{i\xi} = 2\cos\xi,$$

from which follows that

$$\mathcal{F}\{\cos\} = \pi\left\{\delta_{(1)} + \delta_{(-1)}\right\}.$$

Example 7.4 *Discuss why Parseval's relation does not hold for δ.*

Since $\mathcal{F}\{\delta\} = 1$, and since neither 1 nor δ lie in $L^2(\mathbb{R})$, Parseval's relation does not make sense at all.

Example 7.5 *Express the Fourier series*

$$\sum_{n=-\infty}^{+\infty} \left(2n^2 + 7n + 1\right) e^{inx}$$

as a train of impulses of δ and its derivatives.

We get

$$\begin{aligned}
\sum_{n=-\infty}^{+\infty} \left(2n^2 + 7n + 1\right) e^{inx} &= 2\pi \left\{ \frac{2}{i^2} \cot \frac{i^2}{2\pi} \sum_{n=-\infty}^{+\infty} n^2 e^{inx} + \frac{7}{i} \cdot \frac{i}{2\pi} \sum_{n=-\infty}^{+\infty} n\, e^{inx} + \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} e^{inx} \right\} \\
&= 2\pi \left\{ -2 \sum_{n=-\infty}^{+\infty} \delta''_{(2n\pi)} - 7i \sum_{n=-\infty}^{+\infty} \delta'_{(2n\pi)} + \sum_{n=-\infty}^{+\infty} \delta_{(2n\pi)} \right\} \\
&= 2\pi \sum_{n=-\infty}^{+\infty} \left\{ -2\, \delta''_{(2n\pi)} - 7i\, \delta'_{(2n\pi)} + \delta_{(2n\pi)} \right\}.
\end{aligned}$$

Example 7.6 *Prove that if*

$$|c_n(f)| \le \frac{c}{|n|^p} \qquad \text{for } n \in \mathbb{Z} \setminus \{0\},$$

where $p \in \mathbb{N} \setminus \{1\}$ is a constant, then

$$\sum_{n=-\infty}^{+\infty} c_n(f)\, e^{inx}$$

is the Fourier series of a function f, which is at least of class $C_{2\pi}^{p-2}$.
Then find the Fourier series of

$$f(x) = \begin{cases} 1 & \text{for } x \in [0,\pi[, \\ -1 & \text{for } x \in [\pi, 2\pi[, \end{cases}$$

and show that the result cannot be improved to $C_{2\pi}^{p-1}$ for $p \in \mathbb{N}$.

When

$$|c_n(f)| \le \frac{c}{|n|^p} \qquad \text{for } n \in \mathbb{Z} \setminus \{0\} \quad \text{and} \quad p \in \mathbb{N} \setminus \{1\},$$

then we have the estimate

$$\left| \sum_{n=-\infty}^{+\infty} c_n(f)\, e^{inx} \right| \le |c_0(f)| + \sum_{\substack{n=-\infty \\ n \ne 0}}^{+\infty} |c_n(f)| \le |c_0(f)| + 2c \sum_{n=1}^{+\infty} \frac{1}{n^p} < +\infty,$$

proving that $\sum_{n=-\infty}^{+\infty} c_n(f)\, e^{inx}$ has a convergent majoring series, so it is *uniformly convergent.* Since all the terms $c_n(f)e^{inx}$ are continuous, the sum function

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n(f)\, e^{inx}$$

is also continuous.

Then assume that $p > 2$. Let $k \in \{1, \dots, p-2\}$. Then the formally k times differentiated series

$$\sum_{n=-\infty}^{+\infty} i^k n^k c_n(f)\, e^{inx}$$

satisfies the estimate

$$\left| \sum_{n=-\infty}^{+\infty} i^k n^k c_n(f)\, e^{inx} \right| \le \sum_{n=-\infty}^{+\infty} |n|^k \cdot |c_n(f)| \le 2c \sum_{n=1}^{+\infty} \frac{n^k}{n^p} \le 2c \sum_{n=1}^{+\infty} \frac{1}{n^2} = 2c \cdot \frac{\pi^2}{6} < +\infty,$$

proving that the formally differentiated series is also *uniformly convergent.* Then it must converge towards the function $f^{(k)}$, which at the same time is proved to be continuous,

$$f^{(k)}(x) = \sum_{n=-\infty}^{+\infty} i^k n^k c_n(f)\, e^{inx} \in C_{2\pi}^{p-2}, \qquad k = 1, \dots, p-2.$$

Finally, if

$$f(x) = \begin{cases} 1 & \text{for } x \in [0, \pi[, \\ -1 & \text{for } x \in [\pi, 2\pi[, \end{cases}$$

then

$$c_0(f) = \int_0^{2\pi} f(x)\, dx = 0.$$

If $n \neq 0$, then

$$\begin{aligned} c_n(f) &= \frac{1}{2\pi} \int_0^{2\pi} f(x)\, e^{-inx}\, dx = \frac{1}{2\pi} \int_0^{\pi} e^{-inx}\, dx - \frac{1}{2\pi} \int_\pi^{2\pi} e^{-inx}\, dx \\ &= \frac{1}{2\pi} \left\{ \frac{1}{-in} \left[e^{-inx} \right]_0^\pi - \frac{1}{-in} \left[e^{-inx} \right]_\pi^{2\pi} \right\} \\ &= \frac{i}{2\pi n} \left\{ e^{-in\pi} - 1 - e^{-2in\pi} + e^{-in\pi} \right\} = \frac{i}{2\pi n} \cdot 2 \left\{ (-1)^n - 1 \right\} \\ &= -\frac{i}{\pi n} \left\{ 1 - (-1)^n \right\} = \begin{cases} -\dfrac{2i}{\pi n} & \text{for } n \text{ odd}, \\ 0 & \text{for } n \text{ even}, \end{cases} \end{aligned}$$

thus

$$f(x) \sim \sum_{n=-\infty}^{+\infty} \left(-\frac{2i}{\pi(2n+1)} \right) e^{i(2n+1)x},$$

and it follows that

$$|c_n(f)| \leq \frac{2}{\pi} \cdot \frac{1}{|n|}, \qquad \text{for } n \in \mathbb{Z} \setminus \{0\},$$

where this estimate cannot be improved for $p = 1$.

Obviously, f does not belong to $C_{2\pi}^0 = C_{2\pi}^{p-1}$.