Calcul of integral

$$\int_0^{\frac{\pi}{2}} x \ln(\tan(x))\ dx$$

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0.1 Subject

Calculate the value of the integral proposed by **mathshofo** in mathlinks community exactly in this url : Clique here!.
may be my english language are bad but, i thinks that the mathematics formula are very intersting. If there is any intervention sign it.
Thank you :))!.

0.2 Values of some integrals

$$\boxed{\int_0^{\frac{\pi}{2}} \ln(\cos(x))\ dx = \int_0^{\frac{\pi}{2}} \ln(\sin(x))\ dx}$$

Proof :
Very easy :

$$\int_0^{\frac{\pi}{2}} \ln(\cos(x))\ dx = \int_0^{\frac{\pi}{2}} \ln\left(\cos\left(\frac{\pi}{2} - x\right)\right)\ dx = \int_0^{\frac{\pi}{2}} \ln(\sin(x))\ dx$$

$\square$

$$\boxed{\int_0^{\frac{\pi}{2}} \ln(\sin(x))\ dx = -\frac{\pi}{2}\ln(2)}$$

Proof :
Put us the integral :

$$I := \int_0^{\pi} \ln(sin(x))\ dx$$

By integration by substitution $t = x - \frac{\pi}{2}$ we got :

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln(\cos(t))\ dt = 2\int_0^{\frac{\pi}{2}} \ln(\cos(t))\ dt = 2\int_0^{\frac{\pi}{2}} \ln(\sin(t))\ dt$$

Than :

$$\int_0^{\frac{\pi}{2}} \ln(\sin(t))\ dt = \frac{1}{2}I$$

In other part, by integration by substitution $x = 2t$, we find :

$$I = 2\int_0^{\frac{\pi}{2}} \ln(\sin(2t))\ dt = 2\int_0^{\frac{\pi}{2}} \ln(2)\ dt + 2\int_0^{\frac{\pi}{2}} \ln(\cos(t))\ dt + 2\int_0^{\frac{\pi}{2}} \ln(\sin(t))\ dt$$

than :

$$I = \pi\ln(2) + 4\int_0^{\frac{\pi}{2}} \ln(\sin(t))\ dt = \pi\ln(2) + 2I$$

finaly :

$$I = -\pi\ln(2)$$

also :

$$\int_0^{\frac{\pi}{2}} \ln(\sin(t))\ dt = -\frac{\pi}{2}\ln(2)$$

□

$$\boxed{\int_0^{\frac{\pi}{2}} x\ln(\tan(x))\ dx = 2\int_0^{\frac{\pi}{2}} \ln(\sin(x))\ dx + \frac{\pi^2}{4}\ln(2)}$$

Proof :

That's obvious that :

$$\int_0^{\frac{\pi}{2}} x\ln(\tan(x))\ dx = \int_0^{\frac{\pi}{2}} x\ln(\sin(x))\ dx - \int_0^{\frac{\pi}{2}} x\ln(\cos(x))\ dx$$

And if we considered the integral :

$$\int_0^{\frac{\pi}{2}} x\ln(\cos(x))\ dx = \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)\ln\left(\cos\left(\frac{\pi}{2} - x\right)\right)\ dx = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \ln(\sin(x))\ dx - \int_0^{\frac{\pi}{2}} x\ln(\sin(x))\ dx$$

Than :

$$\int_0^{\frac{\pi}{2}} x\ln(\cos(x))\ dx = -\frac{\pi^2}{4}\ln(2) - \int_0^{\frac{\pi}{2}} x\ln(\sin(x))\ dx$$

We concluse :

$$\int_0^{\frac{\pi}{2}} x\ln(\tan(x))\ dx = 2\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\ dx + \frac{\pi^2}{4}\ln(2)$$

□

Comment :

This last formula prouve us that if we can calculate the integral :

$$\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\ dx$$

we could easly find the value of the our subject integral :

$$\int_0^{\frac{\pi}{2}} x\ln(\tan(x))\ dx$$

Calculing than the integral :

$$\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\ dx$$

The integral

$$\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\ dx$$

converge and :

$$\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\ dx = -\frac{1}{2}\int_0^{\frac{\pi}{2}} x^2\cot(x)\ dx$$

Proof :

That's clear that the function :

$$f(x) = \ln(\sin(x))$$

are continuous in all $]0,\frac{\pi}{2}]$ and $\lim_{x\to 0^+} f(x) = 0$ than we extending f by continuity into $[0,\frac{\pi}{2}]$ prouve that the integral $\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\ dx$ converge, than by integration by part we found :

$$\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\ dx = \left[\frac{1}{2}x^2\ln(\sin(x))\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}} x^2\cot(x)\ dx$$

that is :

$$\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\ dx = -\frac{1}{2}\int_0^{\frac{\pi}{2}} x^2\cot(x)\ dx$$

□

We find a sequence of integrals tow to tow dependant, using this laste integral !
we know that for all $x\in]0,\frac{\pi}{2}]$:

$$\cot(x) = i\frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}} = i\left(1+\frac{2}{e^{2ix}-1}\right) = i - 2i\sum_{n\geq 0} e^{2inx}$$

than for all $x\in[0,\pi/2]$:

$$x^2\cot(x) = x^2 i - 2i\sum_{n\geq 0} x^2 e^{2inx} = -x^2 i - 2i\sum_{n\geq 1} x^2 e^{2inx}$$

We integrate and using the Dominant Lebesgue theorem we get :

$$\int_0^{\frac{\pi}{2}} x^2\cot(x)\ dx = -\frac{\pi^3}{24}i - 2i\sum_{n\geq 1}\int_0^{\frac{\pi}{2}} x^2 e^{2inx}\ dx$$

Easly we find that :

$$\int_0^{\frac{\pi}{2}} x^2 e^{2inx} dx = -\frac{1}{8in^3}\int_0^{i\pi n} t^2 e^t dt = -\frac{1}{8in^3}\left[e^t\left(t^2-2t+2\right)\right]_0^{i\pi n}$$

After some simplifications we get :

$$\int_0^{\frac{\pi}{2}} x^2 e^{2inx} dx = \frac{(-1)^n}{8in}\pi^2 + \frac{(-1)^n}{4n^2}\pi - \frac{(-1)^n}{4in^3} + \frac{1}{4in^3}$$

We substate the diffirent result that we found we get :

$$\int_0^{\frac{\pi}{2}} x^2\cot(x)\ dx = -\frac{\pi^3}{24}i + \frac{\pi^2}{4}\sum_{n\geq 1}\frac{(-1)^{n-1}}{n^2} - \frac{1}{2}\sum_{n\geq 1}\frac{(-1)^{n-1}}{n^3} - \frac{1}{2}\sum_{n\geq 1}\frac{1}{n^3}$$

we such that :

$$\eta(s)=\sum_{n\geq 1}\frac{(-1)^{n-1}}{n^s}=(1-2^{1-s})\zeta(s)$$

than :

$$\int_0^{\frac{\pi}{2}} x^2\cot(x)dx=-\frac{\pi^3}{24}i+\frac{\pi^2}{4}\ln(2)+i\frac{\pi}{4}\zeta(2)-\frac{3}{8}\zeta(3)-\frac{1}{2}\zeta(3)$$

after a small Simplification, we find finaly :

$$\int_0^{\frac{\pi}{2}} x^2\cot(x)dx=\frac{\pi^2}{4}\ln(2)-\frac{7}{8}\zeta(3)$$

Other integral :

$$\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\ dx=-\frac{\pi^2}{8}\ln(2)+\frac{7}{16}\zeta(3)$$

The integral subject is :

$$\int_0^{\frac{\pi}{2}} x\ln(\tan(x))\ dx=2\int_0^{\frac{\pi}{2}}\ln(\sin(x))\ dx+\frac{\pi^2}{4}\ln(2)=-2\left(\frac{\pi^2}{8}\ln(2)+\frac{7}{16}\zeta(3)\right)+\frac{\pi^2}{4}\ln(2)$$

Finaly :

$$\int_0^{\frac{\pi}{2}} x\ln(\tan(x))\ dx=\frac{7}{8}\zeta(3)$$

The end

If there is some remarque sign it and thank you ♡
May be my language are bad because in our university we study mathematics using the French language, than i'm sorry for the differents mistakes. Ⓢ
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