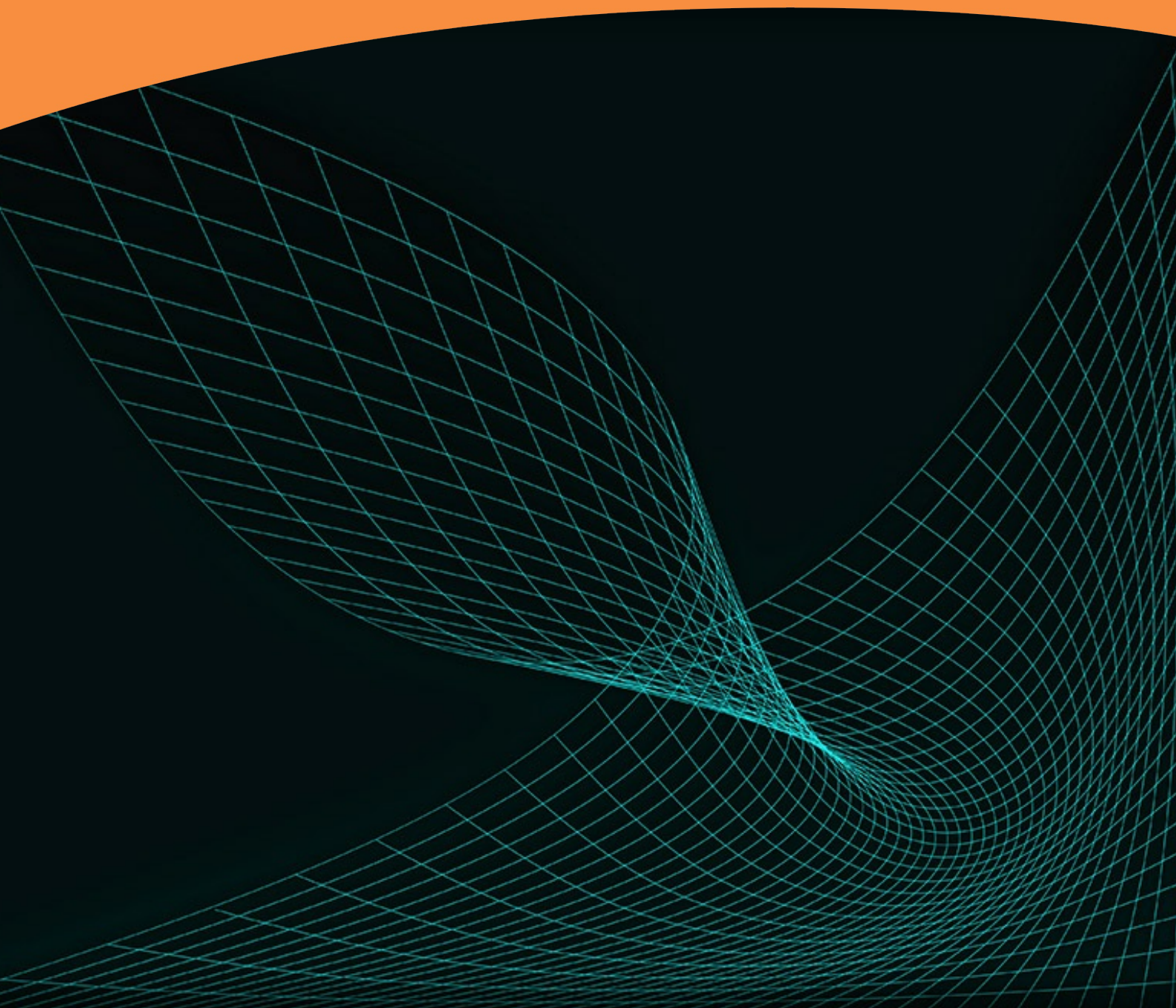


Spectral Theory

Functional Analysis Examples c-4

Leif Mejlbro



Leif Mejlbro

Spectral Theory

Spectral Theory

© Leif Mejlbro & Ventus Publishing ApS

ISBN 978-87-7681-530-1

Disclaimer: The texts of the advertisements are the sole responsibility of Ventus Publishing, no endorsement of them by the author is either stated or implied.

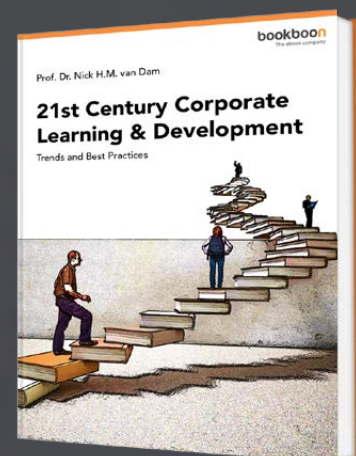
Contents

1.	Spectrum and resolvent	5
2.	The adjoint of a bounded operator	28
3.	Self adjoint operator	42
4.	Isometric operators	57
5.	Unitary and normal operators	61
6.	Positive operators and projections	67
7.	Compact operators	71
	Index	80

Free eBook on Learning & Development

By the Chief Learning Officer of McKinsey

Download Now



1 Spectrum and resolvent

Example 1.1 Define, for $h \in \mathbb{R}$, the operator τ_h on $L^2(\mathbb{R})$ by

$$\tau_h f(x) = f(x - h).$$

Show that τ_h is bounded.

Obviously, τ_h is linear, and it follows from

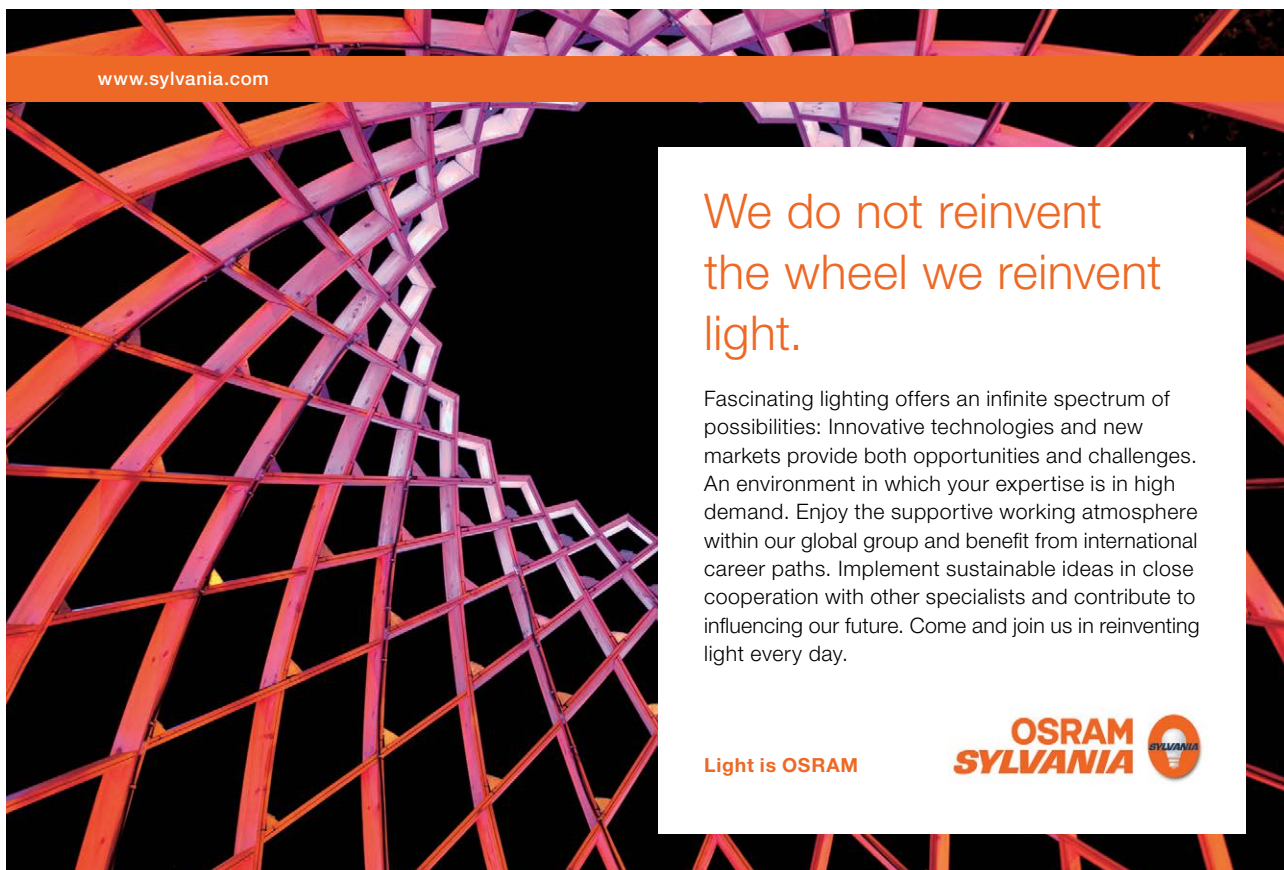
$$\|\tau_h f\|_2^2 = \int_{-\infty}^{+\infty} |f(x - h)|^2 dx = \int_{-\infty}^{+\infty} |f(x)|^2 dx = \|f\|_2^2,$$

that $\|Tf\|_2 = \|f\|_2$ for all $f \in L^2(\mathbb{R})$, hence $\|T\| = 1$.

Remark 1.1 Here we add that τ_h is also regular. In fact, if $\tau_h f = 0$, then $f(x - h) = 0$ for all $x \in \mathbb{R}$, thus $f \equiv 0$. This shows that τ_h is injective, hence the inverse operator exists. Then we get by the change of variable $y = x - h$, i.e. $x = y + h$, that $\tau_h f(x + h) = f(x)$, and we infer that

$$(\tau_h)^{-1} f(x) = f(x + h) = \tau_{-h} f(x),$$

so also $\|(\tau_h)^{-1}\| = 1$, and we have proved that τ_h is regular for every $h \in \mathbb{R}$. \diamond




www.sylvania.com

We do not reinvent
the wheel we reinvent
light.

Fascinating lighting offers an infinite spectrum of possibilities: Innovative technologies and new markets provide both opportunities and challenges. An environment in which your expertise is in high demand. Enjoy the supportive working atmosphere within our global group and benefit from international career paths. Implement sustainable ideas in close cooperation with other specialists and contribute to influencing our future. Come and join us in reinventing light every day.

Light is OSRAM

OSRAM
SYLVANIA



Example 1.2 Consider in $L^2(\mathbb{R})$ the operator Q defined by

$$Qf(x) = xf(x),$$

with

$$D(Q) = \{f \in L^2(\mathbb{R}) \mid Qf \in L^2(\mathbb{R})\}.$$

Determine $\varrho(Q)$ and $\sigma_p(Q)$.

A qualified guess is that $\varrho(Q) = \mathbb{C} \setminus \mathbb{R}$. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. We shall prove that $Q_\lambda = Q - \lambda I$ is regular. Write $\lambda = \xi + i\eta$, where $\xi, \eta \in \mathbb{R}$ and $\eta \neq 0$. It follows from the equation

$$Q_\lambda f(x) = Qf(x) - \lambda f(x) = (x - \lambda)f(x) = g(x),$$

that

$$Q_\lambda^{-1}g(x) = f(x) = \frac{g(x)}{x - \lambda} = \frac{g(x)}{(x - \xi) + i\eta}.$$

It follows for $\eta \neq 0$ that

$$|Q_\lambda^{-1}g(x)|^2 = \frac{|g(x)|^2}{|(x - \xi) + i\eta|^2} \leq \frac{1}{|\eta|^2} |g(x)|^2,$$

and we infer that Q_λ^{-1} is defined on all of $L^2(\mathbb{R})$, and

$$\|Q_\lambda^{-1}\|_2 \leq \frac{1}{|\eta|} \|g\|_2.$$

Hence,

$$\|Q_\lambda^{-1}\| \leq \frac{1}{|\eta|} = \frac{1}{|\operatorname{Im} \lambda|},$$

and we have proved that $\mathbb{C} \setminus \mathbb{R} \subseteq \varrho(Q)$.

Then let $\lambda \in \mathbb{R}$. As before, Q_λ^{-1} is defined by

$$Q_\lambda^{-1}g(x) = \frac{g(x)}{\lambda - x},$$

only the domain is now given by

$$D(Q_\lambda^{-1}) = \left\{ g \in L^2(\mathbb{R}) \mid \frac{g(x)}{\lambda - x} \in L^2(\mathbb{R}) \right\}.$$

Due to the singularity at $x = \lambda$, the inverse Q_λ^{-1} is not defined in all of $L^2(\mathbb{R})$. However, it is easily seen that the subspace

$$U = \{f \in L^2(\mathbb{R}) \mid \exists \varepsilon > 0 \forall x \in [\lambda - \varepsilon, \lambda + \varepsilon] : f(x) = 0\}$$

of $L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, so we conclude from $U \subseteq D(Q_\lambda^{-1})$ that Q_λ^{-1} is densely defined and unbounded, hence $\lambda \in \sigma_c(Q)$ for every $\lambda \in \mathbb{R}$. Utilizing that the splitting of the spectral sets is disjoint, we conclude that

$$\varrho(Q) = \mathbb{C} \setminus \mathbb{R}, \quad \sigma_p(Q) = \emptyset, \quad \sigma_c(Q) = \mathbb{R}, \quad \sigma_r(Q) = \emptyset.$$

Example 1.3 Let (e_n) denote an orthonormal basis in a Hilbert space H , and consider the operator

$$T \left(\sum_{k=1}^{\infty} a_k e_k \right) = \sum_{k=1}^{\infty} a_k e_{k+1}.$$

Determine $\|T\|$ and $\sigma(T)$.

It is well-known that T is called the *shift operator*. We first analyze $T_\lambda = T - \lambda I$, thus

$$T_\lambda x = T_\lambda \left(\sum_{k=1}^{+\infty} a_k e_k \right) = \sum_{k=1}^{+\infty} a_k e_{k+1} - \sum_{k=1}^{+\infty} \lambda a_k e_k = -\lambda a_1 e_1 + \sum_{k=2}^{+\infty} \{a_{k-1} - \lambda a_k\} e_k.$$

Hence, if $T_\lambda x = 0$, then

$$\lambda a_1 = 0 \quad \text{and} \quad \lambda a_k = a_{k-1}, \quad k \geq 2.$$

We have two possibilities:

- 1) If $\lambda = 0$, then $a_1 = \lambda a_2 = 0$, and $a_{k-1} = \lambda a_k = 0$, thus $x = 0$, and $T_0 = T$ is injective, so $\lambda = 0$ is not an eigenvalue.
- 2) If $\lambda \neq 0$, then $a_1 = 0$ and $a_k = \frac{1}{\lambda} a_{k-1}$, hence we get by recursion that all $a_k = 0$, which means that $x = 0$. This proves that every T_λ is injective.

Summing up we have proved that T_λ^{-1} exists for every $\lambda \in \mathbb{C}$, sã $\sigma_p(T) = \emptyset$.

It follows from

$$\|Tx\|^2 = \left\| T \left(\sum_{k=1}^{+\infty} a_k e_k \right) \right\|^2 = \left\| \sum_{k=1}^{+\infty} a_k e_{k+1} \right\|^2 = \sum_{k=1}^{+\infty} |a_k|^2 = \|x\|^2$$

for all x that $\|T\| = 1$, hence

$$\varrho(T) \supseteq \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}.$$

Let $\lambda \neq 0$, $|\lambda| < 1$ and

$$y = \sum_{k=1}^{+\infty} b_k e_k \in H.$$

We shall try to solve the equation $T_\lambda x = y$. It follows immediately from the above that

$$-\lambda x_1 = b_1 \quad \text{and} \quad x_{k-1} - \lambda x_k = b_k, \quad k \geq 2,$$

thus

$$x_1 = -\frac{b_1}{\lambda} \quad \text{and} \quad x_k = \frac{1}{\lambda} x_{k-1} - \frac{1}{\lambda} b_k, \quad k \geq 2,$$

from which e.g. $x_2 = -\frac{b_1}{\lambda^2} - \frac{b_2}{\lambda}$. Choosing in particular $y = e_1$ we get $x_1 = -\frac{1}{\lambda}$, $x_2 = -\frac{1}{\lambda^2}$, and in general,

$$x_n = -\frac{1}{\lambda^n}, \quad n \in \mathbb{N}.$$

From $0 < |\lambda| < 1$ follows that $|x_n| \rightarrow +\infty$ for $n \rightarrow +\infty$, so the only possible solution is

$$x = \sum_{n=1}^{+\infty} x_n e_n = - \sum_{n=1}^{+\infty} \frac{1}{\lambda^n} e_n \notin H,$$

which, however, does *not* belong to H . This shows that

$$e_1 \notin T_\lambda(D(T_\lambda)) = T_\lambda(H).$$

Hence we conclude that T_λ^{-1} exists, but it is unbounded, when $0 < |\lambda| < 1$, so

$$\{\lambda \in \mathbb{C} \mid 0 < |\lambda| < 1\} \subseteq \sigma(T).$$

The set $\sigma(T)$ is closed, so it follows from $\sigma(T) \cap \varrho(T) = \emptyset$ that

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\} \quad \text{and} \quad \varrho(T) = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}.$$

Example 1.4 Consider in ℓ^2 the operator

$$(x_1, x_2, x_3, \dots) \mapsto \left(x_1, \frac{1}{2}(x_1 + x_2), \frac{1}{4}(x_1 + x_2 + x_3), \dots, \frac{1}{2^{n-1}}(x_1 + x_2 + \dots + x_n), \dots \right).$$

Show that the operator is bounded and not surjective.

Let (e_n) denote an orthonormal basis in a Hilbert space H , and consider the operator

$$T \left(\sum_{k=1}^{\infty} a_k e_k \right) = \sum_{k=2}^{\infty} \sqrt{k} a_k e_{k-1}.$$

Determine the spectrum $\sigma(T)$, and find for each eigenvalue the corresponding eigenvectors.

Assume that

$$Tx = \left(x_1, \frac{1}{2}(x_1 + x_2), \frac{1}{4}(x_1 + x_2 + x_3), \dots \right) = (0, 0, 0, \dots).$$

Then $x_1 = 0$, $\frac{1}{2}x_2 = 0$, thus $x_2 = 0$, and we get by induction that $x_n = 0$ for all $n \in \mathbb{N}$. It follows that $Tx = 0$ implies that $x = 0$, hence T is injective.

Then we get

$$\|Tx\|_2^2 = \sum_{n=1}^{+\infty} \frac{1}{4^{n-1}} |x_1 + x_2 + \dots + x_n|^2 \leq \sum_{n=1}^{+\infty} \frac{1}{4^{n-1}} \sum_{j=1}^n n^2 |x_j|^2 \leq \sum_{n=1}^{+\infty} \frac{n^2}{4^{n-1}} \|x\|_2^2,$$

from which we conclude that

$$\|T\| \leq \sqrt{\sum_{n=1}^{+\infty} \frac{n^2}{4^{n-1}}} < +\infty,$$

and T is bounded.

If

$$y_0 = 0 \quad \text{and} \quad y_n = \frac{1}{2^{n-1}} (x_1 + x_2 + \cdots + x_n),$$

then

$$x_1 + x_2 + \cdots + x_n = 2^{n-1}y_n, \quad \text{thus} \quad x_n = 2^{n-1}y_n - 2^{n-2}y_{n-1}, \quad n \in \mathbb{N}.$$

Choose in particular, $y = \frac{1}{n}$, $n \in \mathbb{N}$. Then $(y_n) \in \ell^2$ with $\|y\| = \frac{\pi}{\sqrt{6}}$, while

$$x_n = \frac{2^{n-1}}{n} - \frac{2^{n-2}}{n-1} = 2^{n-2} \cdot \frac{n-2}{n(n-1)} \rightarrow +\infty,$$

according to the rule of magnitudes. In particular, the necessary condition of convergence of $\sum |x_n|^2$ is not fulfilled. We conclude that T is not surjective, $T\ell^2 \neq \ell^2$, hence T is singular.

Let us first find the point spectrum, i.e. let $\lambda \in \sigma_p(T)$ be an eigenvalue. Then there exists a vector $x \neq 0$, such that $Tx = \lambda x$, which can also be written

$$T \left(\sum_{k=1}^{+\infty} x_k e_k \right) = \sum_{k=2}^{+\infty} \sqrt{k} \cdot x_k e_{k-1} = \sum_{k=1}^{+\infty} \sqrt{k-1} \cdot x_{k+1} e_k = \sum_{k=1}^{+\infty} \lambda x_k e_k.$$



Discover the truth at www.deloitte.ca/careers

Deloitte.

© Deloitte & Touche LLP and affiliated entities.



Then

$$x_{k+1} = \frac{\lambda}{\sqrt{k+1}} x_k = \cdots = \frac{\lambda^k}{\sqrt{(k+1)!}} \cdot x_1.$$

Choosing $x_1 = 1$ we see that if x is an eigenvector with $x_1 = 1$, then x necessarily has the form

$$x = \sum_{k=1}^{+\infty} \frac{\lambda^{k-1}}{\sqrt{k!}} e_k.$$

It only remains to check if the constructed x belongs to H . We get

$$\|x\|^2 = \sum_{k=1}^{+\infty} |x_k|^2 = \sum_{k=1}^{+\infty} \frac{|\lambda^2|^{k-1}}{k!} = \frac{1}{|\lambda|^2} \left\{ e^{|\lambda|^2} - 1 \right\},$$

because the series is convergent for all $\lambda \in \mathbb{C}$, and the sum function above has a removable singularity for $\lambda = 0$. (Notice that e_1 is an eigenvector corresponding to $\lambda = 0$). We infer that

$$\sigma(T) = \sigma_p(T) = \mathbb{C},$$

and the given linear operator has every complex $\lambda \in \mathbb{C}$ as an eigenvalue.

Example 1.5 Let (e_n) denote an orthonormal basis in a Hilbert space H . We define the sequence $(f_k)_{k \in \mathbb{Z}}$ by

$$\begin{aligned} f_0 &= e_1, \\ f_k &= e_{2k+1} \quad \text{for } k > 0, \\ f_k &= e_{-2k} \quad \text{for } k < 0. \end{aligned}$$

In this way $(f_k)_{k \in \mathbb{Z}}$ is an orthonormal basis. We define the double sided shift operator S by

$$S \left(\sum_{k=-\infty}^{\infty} a_k f_k \right) = \sum_{k=-\infty}^{\infty} a_k f_{k+1}.$$

Show that S is a bounded operator and show that S has no eigenvalues.

First notice that

$$\sum_{k=-\infty}^{+\infty} a_k f_k = \sum_{k=0}^{+\infty} a_k e_{2k+1} + \sum_{k=1}^{+\infty} a_{-k} e_{2k},$$

and

$$T \left(\sum_{k=-\infty}^{+\infty} a_k f_k \right) = \sum_{k=-\infty}^{+\infty} a_k f_{k+1} = \sum_{k=-\infty}^{+\infty} a_{k-1} f_k = \sum_{k=0}^{+\infty} a_{k-1} e_{2k+1} + \sum_{k=1}^{+\infty} a_{-k-1} e_{2k}.$$

From $(f_k)_{k \in \mathbb{Z}}$ being an orthonormal basis follows that

$$\left\| T \left(\sum_{k=-\infty}^{+\infty} a_k f_k \right) \right\|^2 = \left\| \sum_{k=-\infty}^{+\infty} a_k f_{k+1} \right\|^2 = \sum_{k=-\infty}^{+\infty} |a_k|^2 = \left\| \sum_{k=-\infty}^{+\infty} a_k f_k \right\|^2,$$

from which $\|T\| = 1$ and $T \in B(H)$.

Assume that the equation $Tx = \lambda x$ is fulfilled. It follows from the above that

$$\lambda a_k = a_{k-1} \quad \text{for } k \in \mathbb{N}_0, \quad \text{and} \quad \lambda a_{-k} = a_{-k-1} \quad \text{for } k \in \mathbb{N}.$$

If $\lambda = 0$, then $Tx = 0$, and we get from $\|Tx\| = \|x\| = 0$ that $x = 0$, hence $\lambda = 0 \notin \sigma_p(T)$.

If $\lambda \neq 0$, then we get by recursion,

$$a_k = \frac{1}{\lambda^{k+1}} a_{-1} \quad \text{for } k \in \mathbb{N}_0, \quad \text{and} \quad a_{-k-1} = \lambda^k a_{-1} \quad \text{for } k \in \mathbb{N}.$$

Thus, if $a_{-1} \neq 0$, then all possible $a_k \neq 0$, and we get

$$\begin{aligned} \sum_{k=-\infty}^{+\infty} |a_k|^2 &= \sum_{k=0}^{+\infty} \frac{1}{|\lambda^{k+1}|^2} |a_{-1}|^2 + \sum_{k=1}^{+\infty} |\lambda^k|^2 \cdot |a_{-1}|^2 \\ &= |a_{-1}|^2 \sum_{k=-\infty}^{+\infty} |\lambda^k|^2, \end{aligned}$$

which of course is divergent for every $\lambda \in \mathbb{C}$. We conclude that T does not have eigenvalues, hence $\sigma_p(T) = \emptyset$.

Example 1.6 Define, for $h \in \mathbb{R}_+$, the operator τ_h on $L^2(\mathbb{R})$ by

$$\tau_h f(x) = f(x - h).$$

Show that τ_h has no eigenvalues and that

$$\sigma(\tau_h) \subset \{z \in \mathbb{C} \mid |z| = 1\}.$$

(It is in fact true that $\sigma(\tau_h) = \{z \in \mathbb{C} \mid |z| = 1\}$.)

Remark 1.2 Note that if $h = 0$, then $\tau_0 = I$, and $\lambda = 1$ is trivially an eigenvalue with all of $L^2(\mathbb{R})$ as its eigenspace. For that reason we assume that $h > 0$. \diamond

It follows from

$$\|\tau_h f\|_2^2 = \int_{-\infty}^{+\infty} |f(x - h)|^2 dx = \int_{-\infty}^{+\infty} |f(x)|^2 dx = \|f\|_2^2,$$

that $\|\tau_h\| = 1$, hence

$$\sigma(\tau_h) \subseteq \{z \in \mathbb{C} \mid |z| \leq 1\}.$$

Assume that

$$\tau_h f(x) = f(x - h) = \lambda f(x), \quad \text{where } |\lambda| \leq \|\tau_h\| = 1.$$

If $|\lambda| = 1$, then $|f(x-h)| = |f(x)|$, $h > 0$. Thus the function $|f(x)|$ is periodic of period $h > 0$, hence

$$\|f\|_2^2 = \int_{-\infty}^{+\infty} |f(x)|^2 dx = \sum_{n=-\infty}^{+\infty} \int_0^h |f(x)|^2 dx < +\infty.$$

This is of course only possible, if $\int_0^h |f(x)|^2 dx = 0$, i.e. if $f(x) = 0$ for almost every $x \in [0, h]$, and hence for almost every $x \in \mathbb{R}$. Then x is represented by the zero function, and we infer that no $\lambda \in \mathbb{C}$ satisfying $|\lambda| = 1$ can be an eigenvalue.

It has previously been proven in EXAMPLE 1.1 that $(\tau_h)^{-1} = \tau_{-h}$. Of course, this can also be proved directly,

$$\tau_{-h}\tau_h f(x) = \tau_{-h}f(x-h) = f(x-h+h) = f(x) = If(x),$$

and

$$\tau_h\tau_{-h}f(x) = \tau_hf(x+h) = f(x+h-h) = f(x) = If(x).$$

It is also obvious that $\|(\tau_h)^{-1}\| = \|\tau_{-h}\| = 1$, and $(\tau_h)^{-1} \in B(H)$. Thus if $|\lambda| < 1$, then

$$(\tau_h - \lambda I)^{-1} = (\tau_h)^{-1} (I - \lambda (\tau_h)^{-1})^{-1} \in B(H),$$

SIMPLY CLEVER

ŠKODA



We will turn your CV into
an opportunity of a lifetime



Do you like cars? Would you like to be a part of a successful brand?
We will appreciate and reward both your enthusiasm and talent.
Send us your CV. You will be surprised where it can take you.

Send us your CV on
www.employerforlife.com



because $\|\lambda(\tau_h)^{-1}\| = |\lambda| < 1$. Therefore, $\{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \subset \varrho(\tau_h)$, and thus

$$\varrho(\tau_h) \supseteq \{\lambda \in \mathbb{C} \mid |\lambda| \neq 1\},$$

which implies that

$$\sigma(\tau_h) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}.$$

Finally,

$$\sigma_p(\tau_h) \subseteq \sigma(\tau_h) \quad \text{og} \quad \sigma_p(\tau_h) \cap \{\lambda \in \mathbb{C} \mid |\lambda| = 1\} = \emptyset,$$

from which follows that $\sigma_p(\tau_h) = \emptyset$.

Example 1.7 Given below some closed linear operators from ℓ^2 into ℓ^2 . Check in each case if the operator is singular.

1) $T_1 x = (x_2, x_3, \dots)$.

2) $T_2 z = \left(\frac{1}{2}x_1, \frac{1}{2^2}x_2, \frac{1}{2^3}x_3, \dots\right)$.

3) $T_3 x = (0, x_1, x_2, \dots)$.

4) $T_4 x = (0, x_2, x_3, \dots)$.

A linear operator is singular, if at least one of the following three conditions is satisfied:

- 1) There exists an $f \in D(T) \setminus \{0\}$, such that $Tf = 0$.
- 2) The inverse T^{-1} exists, and $\overline{D(T^{-1})} = \overline{TD(T)} = Y$, while T^{-1} itself is unbounded.
- 3) The inverse T^{-1} exists, but it is not densely defined in Y , thus $\overline{TD(T)} \neq Y$.

We shall below check these three conditions.

- 1) It follows by choosing $x = (1, 0, 0, \dots) \neq 0$ that $T_1 x = 0$, hence T_1 is singular of type (1). This means that $0 \in \sigma_p(T_1)$, i.e. 0 is an eigenvalue of T_1 .
- 2) Clearly, $T_2 x = 0$ implies that $x = 0$, so T_2 is injective and the inverse exists. Then we solve the equation $T_2 x = y$, thus

$$T_2 x = \left(\frac{1}{2}x_1, \frac{1}{2^2}x_2, \frac{1}{2^3}x_3, \dots\right) = (y_1, y_2, y_2, \dots) = y.$$

When we identify the coordinates we get $\frac{1}{2^n}x_n = y_n$, hence $x_n = 2^n y_n$, and the inverse operator T_2^{-1} is given by

$$T_2^{-1}y = (2y_1, 2^2y_2, 2^3y_3, \dots)$$

for

$$y \in D(T_2^{-1}) = \left\{ y \in \ell^2 \mid \sum_{n=1}^{+\infty} 2^{2n}|y_n|^2 < +\infty \right\} \subset \ell^2.$$

Let U be the subspace consisting of all sequences which are 0 eventually. Then clearly,

$$U \subset D(T_2^{-1}) \subset \ell^2.$$

The subspace U is dense in ℓ^2 , so this is also the case for the larger subspace $D(T_2^{-1})$. Furthermore, it follows from the definition of the inverse T_2^{-1} that it is unbounded, i.e. T_2 is singular of type (2). This means that $0 \in \sigma_c(T_2)$ lies in the continuous spectrum for T_2 .

3) It is obvious that T_3 is injective and that

$$T_3^{-1}y = (y_2, y_2, y_4, \dots)$$

for

$$y \in D(T_3^{-1}) \{y \in \ell^2 \mid y_1 = 0\}.$$

Clearly, T_3^{-1} is bounded, though not densely defined, so T_3 is of type (3), corresponding to that $0 \in \sigma_r(T_3)$ lies in the residual spectrum for T_3 .

4) We infer from $T_4x = 0$ for $x = (1, 0, 0, \dots) \neq 0$ that 0 is an eigenvalue, $0 \in \sigma_p(T_4)$, hence T_4 is singular of type (1).

Example 1.8 Let V denote the Banach space $(C([0, 1]), \|\cdot\|_\infty)$, and let the operator T be given by

$$Tf(x) = \int_0^x f(t) dt, \quad f \in V.$$

Check if T is regular.

The inverse operator of T is the differential operator \mathcal{D} , given by

$$D(\mathcal{D}) = \{f \in C^1([0, 1]) \mid f(0) = 0\},$$

$$\mathcal{D}f = \frac{df}{dx} = f' \quad \text{for } f \in C^1([0, 1]), \quad f(0) = 0.$$

It is easily seen (e.g. by using Weierstraß's Approximation Theorem) that $D(\mathcal{D})$ is dense in V . On the other hand, \mathcal{D} is unbounded. In fact, choose

$$f_n(x) = \sin(\pi nx), \quad x \in [0, 1], \quad f_n \in D(\mathcal{D}).$$

then

$$\mathcal{D}f_n(x) = \pi n \cdot \cos(\pi nx), \quad x \in [0, 1],$$

hence $\|f_n\|_\infty = 1$ and $\|\mathcal{D}f_n\|_\infty = \pi n$.

Remark 1.3 A simpler example is of course $g_n(x) = x^n$, $x \in [0, 1]$. However, the f_n occur very frequently as an example in other cases, so we have chosen to present it here. \diamond

We have proved that T is singular of type (2), i.e. $0 \in \sigma_c(T)$ lies in the continuous spectrum for T .

Example 1.9 Let $H = L^2(\mathbb{R})$, and let g be a bounded continuous real function defined on \mathbb{R} . Prove that the operator T given by

$$Tf(x) = g(x)f(x), \quad f \in L^2(\mathbb{R}),$$

belongs to $B(H)$.

Find a necessary and sufficient condition on g that T is regular.

When g is bounded, $\|g\|_\infty < +\infty$, then

$$\|Tf\|_2^2 = \int_{-\infty}^{+\infty} g(x)^2 |f(x)|^2 dx \leq \|g\|_\infty^2 \int_{-\infty}^{+\infty} |f(x)|^2 dx = \|g\|_\infty^2 \cdot \|f\|_2^2,$$

hence $\|Tf\|_2 \leq \|g\|_\infty \cdot \|f\|_2$ for all $f \in H$, and we infer that $T \in B(H)$ with $\|T\| \leq \|g\|_\infty$.

Then we shall find when T is regular, i.e. when T fulfils the following three conditions:

- 1) The equation $Tf = 0$ has only the trivial solution $f = 0$, so T^{-1} exists.
- 2) The inverse operator T^{-1} is densely defined, i.e.

$$D(T^{-1}) = T(L^2(\mathbb{R}))$$

is dense in $L^2(\mathbb{R})$.

- 3) The inverse operator T^{-1} is bounded.

We now check each of these conditions:

- 1) It follows from $Tf(x) = g(x) \cdot f(x)$ that $Tf = 0$, if and only if $g(x) \cdot f(x) = 0$ for almost every $x \in \mathbb{R}$. Therefore, if we want always to conclude that $f = 0$ (in $L^2(\mathbb{R})$), then we must assume that $g(x) \neq 0$ for almost every $x \in \mathbb{R}$.
- 2) Then we want that T^{-1} is bounded. It follows from $Tf(x) = g(x)f(x) = h(x)$ that

$$f(x) = T^{-1}h(x) = \frac{1}{g(x)} h(x),$$

and then the same consideration as above shows that we must require that

$$\left\| \frac{1}{g} \right\|_\infty < +\infty.$$

- 3) Based on the conditions above, assume that

$$0 < b \leq |g(x)| < a < +\infty, \quad \text{for all } x \in \mathbb{R}.$$

Then clearly all three conditions are fulfilled, so these conditions are sufficient that both T and $T^{-1} \in B(H)$.

Example 1.10 Let (e_k) denote an orthonormal basis in a Hilbert space H , and let the operator T be defined by

$$T \left(\sum_{k=1}^{+\infty} a_k e_k \right) = \sum_{k=2}^{+\infty} a_k e_{k-1}.$$

Prove that λ is an eigenvalue for T , if and only if $|\lambda| < 1$.
Find $\sigma(T)$ and $\rho(T)$.

Assume that $\lambda \in \sigma_p(T)$, thus there exists

$$x = \sum_{k=1}^{+\infty} x_k e_k, \quad \text{where} \quad 0 < \sum_{k=1}^{+\infty} |x_k|^2 < +\infty,$$

such that $Tx = \lambda x$, i.e.

$$\sum_{k=2}^{+\infty} x_k e_{k-1} = \sum_{k=1}^{+\infty} x_{k+1} e_k = \lambda \sum_{k=1}^{+\infty} x_k e_k.$$

When we identify the coordinates we get

$$x_{k+1} = \lambda x_k, \quad k \in \mathbb{N}.$$



e-learning for kids

- The number 1 MOOC for Primary Education
- Free Digital Learning for Children 5-12
- 15 Million Children Reached

About e-Learning for Kids Established in 2004, e-Learning for Kids is a global nonprofit foundation dedicated to fun and free learning on the Internet for children ages 5 - 12 with courses in math, science, language arts, computers, health and environmental skills. Since 2005, more than 15 million children in over 190 countries have benefitted from eLessons provided by EFK! An all-volunteer staff consists of education and e-learning experts and business professionals from around the world committed to making difference. eLearning for Kids is actively seeking funding, volunteers, sponsors and courseware developers; get involved! For more information, please visit www.e-learningforkids.org.

Choosing $x_1 = 1$, we get by either induction or by recursion – both methods can be applied – that $x_k = \lambda^{k-1}$, and an eigenvector corresponding to the eigenvalue λ must *necessarily* be of the form

$$x = x_1 \sum_{k=1}^{+\infty} \lambda^{k-1} e_k.$$

This *candidate* belongs to the Hilbert space, if and only if

$$\sum_{k=1}^{+\infty} |\lambda^{k-1}|^2 = \sum_{k=0}^{+\infty} |\lambda|^{2k} < +\infty,$$

i.e. if and only if $|\lambda| < 1$. We infer that

$$\sigma_p(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}.$$

If on the other hand $\lambda \in \mathbb{C}$ satisfies $|\lambda| < 1$, then we get by insertion that $x = \sum_{k=1}^{+\infty} \lambda^{k-1} e_k$ is an eigenvector, so $\lambda \in \sigma_p(T)$, and we have proved that

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}.$$

Then assume that $\lambda \in \mathbb{C}$ satisfies $|\lambda| > 1$. We shall prove that $\lambda \in \varrho(T)$.

Remark 1.4 If here one tries directly to find the inverse operator T_λ^{-1} , thus try to solve the equation $T_\lambda x = y$ with respect to $x \in H$ for given $y \in H$, then we end up with an unpleasant infinite system of equations of the form

$$(1) \quad x_{k+1} - \lambda x_k = y_k, \quad k \in \mathbb{N},$$

where the solution also must satisfy

$$\sum_{k=1}^{+\infty} |x_k|^2 < +\infty.$$

Even this is possible, it is very difficult to solve this system of equations. Hence we search an alternative method of solution. \diamond

We note that

$$\|Tx\| = \left\| \sum_{k=1}^{+\infty} x_{k+1} e_k \right\| \leq \|x\|,$$

where we get equality, when $x_1 = 0$. This shows that $\|T\| = 1$.

It follows from

$$T_\lambda = T - \lambda I = -\lambda I \left(I - \frac{1}{\lambda} T \right), \quad |\lambda| > 1,$$

and

$$\left\| \frac{1}{\lambda} T \right\| = \frac{1}{|\lambda|} < 1,$$

by using the Neumann series that

$$T_\lambda^{-1} = -\frac{1}{\lambda} \left(I - \frac{1}{\lambda} T \right)^{-1} \in B(H).$$

Remark 1.5 The explicit solution is given by the Neumann series

$$x = T_\lambda^{-1} y = -\frac{1}{\lambda} \sum_{j=0}^{+\infty} \frac{1}{\lambda^j} T^j y,$$

which can also be found directly, if we work on (1). However, the precise solution is not so interesting in this connection. \diamond

We infer that

$$\{\lambda \in \mathbb{C} \mid |\lambda| > 1\} \subseteq \varrho(T).$$

Now, $\sigma(T)$ is *closed* and disjoint from $\varrho(T)$, and

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \subseteq \sigma(T),$$

hence

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\} \quad \text{og} \quad \varrho(T) = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}.$$

Example 1.11 Consider the Banach space $(C([0, 1]), \|\cdot\|_\infty)$. Let $v \in C([0, 1])$ be real, and let the operator T be defined by

$$Tf(x) = v(x)f(x).$$

Find $\sigma(T)$ and $\varrho(T)$.

We conclude from

$$\|Tf\|_\infty = \|v(x)f(x)\|_\infty \leq \|v\|_\infty \|f\|_\infty,$$

where we get equality by choosing $f = v$, that $\|T\| = \|v\|_\infty$. Then it follows that

$$\sigma(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|v\|_\infty\}.$$

Now, v is continuous, and $[0, 1]$ is compact, hence $v([0, 1])$ is also compact. Let $\lambda \notin v([0, 1])$. Then there exists a $b_\lambda > 0$, such that

$$|v(x) - \lambda| \geq b_\lambda \quad \text{for all } x \in [0, 1].$$

Then

$$T_\lambda f(x) = \{v(x) - \lambda\}f(x) = g(x) \in C([0, 1])$$

for

$$f(x) = T_\lambda^{-1}g(x) = \frac{g(x)}{v(x) - \lambda} \in C([0, 1]).$$

It follows that $\|T_\lambda\| \leq \frac{1}{b_\lambda}$, hence $T_\lambda \in B(C([0, 1]))$, and

$$\varrho(T) \supseteq \mathbb{C} \setminus v([0, 1]) \quad \text{and} \quad \sigma(T) \subseteq v([0, 1]).$$

If conversely $\lambda \in v([0, 1])$, then there exists an $x_0 \in [0, 1]$, such that $v(x_0) = \lambda$. Then the equation $T_\lambda f = g$ cannot be solved for any g , for which $f(x_0) \neq 0$, because then the candidate f then will not be continuous at x_0 . Hence we finally get

$$\sigma(T) = v([0, 1]) \quad \text{and} \quad \varrho(T) = \mathbb{C} \setminus v([0, 1]).$$

Example 1.12 Consider in the Banach space ℓ^∞ the operator T given by

$$T(x_1, x_2, \dots) = (x_2, x_3, \dots).$$

Find $\varrho(T)$, $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$.

We get from $\|Tx\|_\infty \leq \|x\|_\infty$ with equality for

$$|x_1| \leq \sup_{i \geq 2} |x_i|,$$

that $\|T\| = 1$, hence $\sigma(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$.

Therefore, if $\lambda \in \sigma_p(T)$, then $|\lambda| \leq 1$, and there exists an $x \neq 0$, such that $Tx = \lambda x$, i.e.

$$x_{k+1} = \lambda x_k = \dots = \lambda^k x_1.$$

We can therefore put $x_1 = 1$ for an eigenvector, and thus any eigenvector has the form of a constant times

$$(1, \lambda, \lambda^2, \dots, \lambda^{n-1}, \dots).$$

It follows by insertion that this candidate indeed is an eigenvector, if it belongs to ℓ^∞ , i.e. if $|\lambda| \leq 1$. We conclude that

$$\sigma_p(T) = \sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\},$$

and

$$\varrho(T) = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\},$$

and $\sigma_c(T) = \sigma_r(T) = \emptyset$.

Example 1.13 Let $T : \ell^2 \rightarrow \ell^2$ denote the operator

$$T(x_1, x_2, \dots, x_n, \dots) = (x_2, x_4, \dots, x_{2n}, \dots).$$

Find $\|T\|$.

Find all eigenvalues for T .

Show that the eigenspace corresponding to any eigenvalue is infinite dimensional.

Determine the operators T^* , TT^* and T^*T .

Determine $\sigma(T)$ and $\varrho(T)$.

1) We infer from


$$\|Tx\|^2 = \sum_{n=1}^{+\infty} |x_{2n}|^2 \leq \sum_{n=1}^{+\infty} |x_n|^2 = \|x\|^2$$

for every $x \in \ell^2$ that $\|T\| \leq 1$.

For $x = (0, x_2, 0, x_4, 0, x_6, 0, \dots)$ we get in particular that

$$\|Tx\|^2 = \|T(0, x_2, 0, x_4, 0, x_6, \dots)\|^2 = \sum_{n=1}^{+\infty} |x_{2n}|^2 = \sum_{n=1}^{+\infty} |x_n|^2 = \|x\|^2,$$


and we conclude that $\|T\| = 1$.



Cynthia | AXA Graduate

AXA Global Graduate Program

Find out more and apply

redefining / standards 

- 2) Assume that $\lambda \in \sigma_p(T)$. Then there exists an $x \in \ell^2 \setminus \{0\}$, such that $Tx = \lambda x$. We get for the n -th coordinate of this equation that

$$(2) \quad x_{2n} = \lambda x_n, \quad n \in \mathbb{N}.$$

If $\lambda = 0$, then we get the conditions $x_{2n} = 0$, $n \in \mathbb{N}$. It follows that if

$$\sum_{n=0}^{+\infty} |x_{2n+1}|^2 < +\infty,$$

then $(x_1, 0, x_3, 0, x_5, 0, \dots)$ is an eigenvector corresponding to the eigenvalue $\lambda = 0$, hence $0 \in \sigma_p(T)$, and the eigenspace corresponding to $\lambda = 0$ is spanned by $\{e_{2n-1} \mid n \in \mathbb{N}\}$, hence it is infinite dimensional, cf. the third question.

Assume that $\lambda \in \sigma_p(T) \setminus \{0\}$. Then it follows from (2) with $n = 2^{m-1}q$ that

$$x_{2^m q} = \lambda x_{x^{n-1}q} = \lambda^2 x_{2^{m-2}q} = \dots = \lambda^m x_q, \quad m \in \mathbb{N}.$$

We get in particular for $q = 1$,

$$x_{2^m} = \lambda^m x_1.$$

If we put $x_1 = 1$ and $x^r = 0$, when r is not of the form 2^n , we get an eigenvector, if and only if

$$\sum_{n=0}^{+\infty} |\lambda^n|^2 < +\infty.$$

This condition is fulfilled if and only if $|\lambda| < 1$. Hence we conclude that the point spectrum is given by

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}.$$

- 3) Assume that $\lambda \in \sigma_p(T)$, so $|\lambda| < 1$. Then we get by a simple computation that every odd index $2q + 1$, $q \in \mathbb{N}_0$, determines an eigenvector x by

$$x_{(2q+1) \cdot 2^n} = \lambda^n, \quad n \in \mathbb{N}_0, \quad \text{og} \quad x_r = 0 \text{ otherwise.}$$

All these eigenvectors are linearly independent, so we conclude that the eigenspace corresponding to an eigenvalue $\lambda \in \sigma_p(T)$ is infinite dimensional.

- 4) Now, $T \in B(\ell^2)$, and $\|T\| = 1$, so $T^* \in B(\ell^2)$ and $\|T^*\| = 1$.

We have for every $x \in \ell^2$ and every $y \in \ell^2$ that

$$\begin{aligned} (Tx, y) &= ((x_2, x_4, x_6, \dots), (y_1, y_2, y_3, \dots)) = \sum_{n=1}^{+\infty} x_{2n} \overline{y_n} \\ &= ((0, x_2, 0, x_4, 0, \dots), (0, y_1, 0, y_2, 0, \dots)) = (x, T^*y), \end{aligned}$$

so we infer that

$$T^*y = T^*(y_1, y_2, y_3, \dots) = (0, y_1, 0, y_2, 0, y_3, \dots), \quad y \in \ell^2.$$

Furthermore,

$$TT^*x = T(T^*(x_1, x_2, x_3, \dots)) = T(0, x_1, 0, x_2, 0, x_3, \dots) = (x_1, x_2, x_3, \dots) = x,$$

i.e. $TT^* = I$, and

$$T^*Tx = T^*(T(x_1, x_2, x_3, \dots)) = T^*(x_2, x_4, x_6, \dots) = (0, x_2, 0, x_4, 0, x_6, \dots),$$

proving that $T^*T = P$ is the projection onto the subspace of ℓ^2 which is spanned by $\{e_{2n} \mid n \in \mathbb{N}\}$.

5) It follows from $\|T\| = 1$ that

$$\sigma(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\} = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}.$$

Furthermore,

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \subseteq \sigma(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\},$$

and the spectrum is closed, hence

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\} \quad \text{og} \quad \varrho(T) = \mathbb{C} \setminus \sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}.$$

Remark 1.6 It is also easy to prove that

$$\sigma_p(T^*) = \emptyset.$$

In fact, we get from $T^*y = \lambda y$ that

$$(0, y_1, 0, y_2, 0, y_3, \dots) = \lambda(y_1, y_2, y_3, y_4, y_5, y_6, \dots).$$

If $\lambda = 0$, then the right hand side is 0, and this implies that $y_n = 0$, thus $y = 0$, and $0 \notin \sigma_p(T^*)$. If $\lambda \neq 0$, then

$$0 = \lambda y_{2n+1}, \quad n \in \mathbb{N}_0, \quad \text{and} \quad y_n = \lambda y_{2n}, \quad n \in \mathbb{N}.$$

The former equation gives $y_{2n+1} = 0$, which is then inserted into the latter (follows by an iteration, when n is even) to give $y_{2n} = 0$, hence $y = 0$, and we have proved that $\sigma_p(T^*) = \emptyset$.

Now, $\sigma_p(T^*) = \emptyset$, hence also $\sigma_r(T) = \emptyset$. Since $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$ is a disjoint splitting of the spectrum, we conclude that

$$\begin{aligned} \varrho(T) &= \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}, \\ \sigma(T) &= \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}, \\ \sigma_p(T) &= \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}, \\ \sigma_c(T) &= \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}, \\ \sigma_r(T) &= \emptyset. \end{aligned} \quad \diamond$$

Example 1.14 Let X denote the Banach space of $C([-1, 1])$ -functions equipped with the usual sup-norm $\|\cdot\|_\infty$, and let $T \in B(X)$ be given by

$$Tf = f(0) + f.$$

- 1) Find the norm of T .
- 2) Determine the resolvent set $\rho(T)$ for T and find

$$T_\lambda^{-1} = (T - \lambda I)^{-1}$$

for all $\lambda \in \rho(T)$.

- 3) Show that the spectrum for T is a pure point spectrum and find all eigenvalues and corresponding eigenfunctions.
- 4) Show that all $f \in X$ can be written as a sum of eigenfunctions belonging to different eigenspaces, and show that this decomposition is unique.

- 1) Clearly,

$$\|Tf\|_\infty \leq |f(0)| + \|f\|_\infty \leq \|f\|_\infty + \|f\|_\infty = 2\|f\|_\infty,$$

where we obtain equality if e.g. f is a real function with maximum at 0, i.e. $\|T\| = 2$.

- 2) Then we shall check when it is possible for all $g \in X$ to solve the equation

$$(T - \lambda I)g = g, \quad f \in X.$$

We get

$$(3) \quad g(x) = Tf(x) - \lambda f(x) = f(0) + f(x) - \lambda f(x).$$

In particular for $x = 0$,

$$g(0) = f(0) + f(0) - \lambda f(0) = (2 - \lambda) f(0).$$

Now, the solution f must be continuous, so this equation *cannot* be solved for arbitrary $g \in X$, when $\lambda = 2$, hence $2 \in \sigma(T)$.

If $\lambda \neq 2$, then

$$f(0) = \frac{1}{2 - \lambda} g(0),$$

which gives by insertion into (3),

$$g(x) - \frac{1}{2 - \lambda} g(0) = (1 - \lambda) f(x).$$

Hence, if $\lambda = 1$, then this equation *cannot* be solved for an arbitrary $g \in X$, so $1 \in \sigma(T)$. If we assume that $\lambda \neq 1$, then we get the candidate of the solution

$$f(x) = T_\lambda^{-1} g(x) = \frac{1}{1 - \lambda} g(x) - \frac{1}{(1 - \lambda)(2 - \lambda)} g(0),$$

which is clearly continuous, when g is continuous. Finally,

$$\|T_\lambda^{-1}g\|_\infty \leq \left\{ \frac{1}{|1-\lambda|} + \frac{1}{|1-\lambda| \cdot |2-\lambda|} \right\} \|g\|_\infty = C(\lambda) \|g\|_\infty.$$

This implies that $\varrho(T) \supseteq \mathbb{C} \setminus \{1, 2\}$, and because we have proved above that $\{1, 2\} \subseteq \sigma(T)$, it follows that

$$\varrho(T) = \mathbb{C} \setminus \{1, 2\} \quad \text{and} \quad \sigma(T) = \{1, 2\}.$$

3) Here we shall prove that $\lambda = 1$ and $\lambda = 2$ are eigenvalues, i.e. we shall prove that the equation

$$Tf = f(0) + f(x) = \lambda f(x)$$

has non-trivial solutions for $\lambda = 1$ and $\lambda = 2$.

If $\lambda = 1$, then a check gives

$$f(0) + f(x) = f(x),$$

and the condition becomes $f(0) = 0$. Any function $f \in C([-1, 1])$, for which $f(0) = 0$, is therefore an eigenfunction corresponding to the eigenvalue $\lambda = 1$.

I joined MITAS because
I wanted **real responsibility**

The Graduate Programme
for Engineers and Geoscientists
www.discovermitas.com



Month 16

I was a construction
supervisor in
the North Sea
advising and
helping foremen
solve problems

Real work
International opportunities
Three work placements





If $\lambda = 2$, then

$$f(0) + f(x) = 2f(x),$$

and we get the condition $f(x) = f(0)$ for all $x \in [-1, 1]$. This shows that every constant function $f(x) = c$ is an eigenfunction corresponding to the eigenvalue $\lambda = 2$, and we have proved that

$$\sigma(T) = \{1, 2\} = \sigma_p(T).$$

4) Let $f \in C([-1, 1])$. Then we have the following splitting of f ,

$$f(x) = \{f(x) - f(0)\} + f(0) = g(x) + h(x),$$

where $g(x) = f(x) - f(0)$ satisfies $g(0) = 0$, so g belongs to the eigenspace corresponding to $\lambda = 1$, and where $h(x) = f(0)$ is constant, hence $h(x)$ belongs to the eigenspace of the eigenvalue. This proves the existence.

If conversely

$$f(x) = g(x) + h(x)$$

is such a splitting, then

$$Tf(x) = f(x) + f(0) = Tg(x) + Th(x) = g(x) + 2h(x),$$

and we get the two equations

$$\begin{cases} g(x) + 2h(x) = f(x) + f(0), \\ g(x) + h(x) = f(x), \end{cases}$$

from which we get $h(x) = f(0)$ by subtraction, and then

$$g(x) = f(x) - h(x) = f(x) - f(0),$$

and we have proved the uniqueness.

Example 1.15 Let H denote a Hilbert space and let $T \in B(H)$. Assume that we have for some $m \in \mathbb{N}$ that $T^m = 0$. Show that

$$(I - \lambda T)^{-1} = \sum_{n=0}^{m-1} \lambda^n T^n \in B(H),$$

and deduce that $\mathbb{C} \setminus \{0\} \subset \varrho(T)$.

Show next that $\sigma(T) = \sigma_p(T) = \{0\}$.

We have $T^m = 0$, and

$$\begin{aligned} (I - \lambda T) \sum_{n=0}^{m-1} \lambda^n T^n &= \sum_{n=0}^{m-1} \lambda^n T^n - \sum_{n=0}^{m-1} \lambda^{n+1} T^{n+1} = I + \sum_{n=1}^{m-1} \lambda^n T^n - \sum_{n=1}^m \lambda^n T^n \\ &= I - \lambda^m T^m = I, \end{aligned}$$

and analogously because T is defined everywhere,

$$\sum_{n=0}^{m-1} \lambda^n T^n (I - \lambda T) = I.$$

We therefore conclude that

$$\sum_{n=0}^{m-1} \lambda^n T^n = I + \sum_{n_1}^{m-1} \lambda^n T^n = (I - \lambda T)^{-1} \quad \text{for every } \lambda \in \mathbb{C}.$$

If $\mu \neq 0$, then

$$(T - \mu I)^{-1} = -\frac{1}{\mu} \left(I - \frac{1}{\mu} T \right)^{-1} = -\frac{1}{\mu} \sum_{n=0}^{m-1} \frac{1}{\mu^n} T^n \in B(H),$$

proving that $\varrho(T) \supseteq \mathbb{C} \setminus \{0\}$.

Clearly, $T^m = 0$ implies that $T^m f = T(T^{m-1}f) = 0$ for every $f \in H$. Hence if $T^{m-1}f \neq 0$ for some $f \in H$, then $T^{m-1}f$ is an eigenvector for T , corresponding to $\lambda = 0$.

First find the smallest $m \in \mathbb{N}$, such that $T^m = 0$ and $T^{m-1} \neq 0$. It follows from this that

$$\sigma(T) = \sigma_p(T) = \{0\},$$

and hence

$$\varrho(T) = \mathbb{C} \setminus \{0\}.$$

Example 1.16 Let E be a Banach space and let $P \in B(E)$ satisfy $P^2 = P$.

- 1) Show that $P - \lambda I$ is injective for $\lambda \in \mathbb{C} \setminus \{0, 1\}$.
- 2) Show that $P - \lambda I$ is surjective for $\lambda \in \mathbb{C} \setminus \{0, 1\}$, and find $(P - \lambda I)^{-1}$.
- 3) Show that $\sigma(P) = \sigma_p(P) = \{0, 1\}$.

Remark 1.7 The latter claim of the example is not true, if $P = 0$ or I . In fact, it is well-known that

$$\sigma(0) = \sigma_p(0) = \{0\} \quad \text{and} \quad \sigma(I) = \sigma_p(I) = \{1\},$$

and it is obvious that both $0^2 = 0$ and $I^2 = I$. Of a similar reason we must assume in (2) that $\lambda \notin \{0, 1\}$, while (1) also holds for 0 and I . \diamond

- 1) Let $\lambda \in \mathbb{C} \setminus \{0, 1\}$, and assume that

$$(P - \lambda I)x = Px - \lambda x = 0,$$

i.e. $Px = \lambda x$. Then also

$$Px = P^2x = \lambda Px.$$

Because $\lambda \neq 1$, we must have $Px = 0$, and since also $\lambda \neq 0$, we get

$$x = \frac{1}{\lambda} Px = 0,$$

and we have proved that $P - \lambda I$ is injective.

- 2) Let again $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Because $P^2 = P$, the formal Neumann series for $(P - \lambda I)^{-1}$ can in principle be reduced to $\mu P - \frac{1}{\lambda} I$, where we shall find μ and then prove that this is indeed the inverse operator. A check gives

$$\begin{aligned} \left(\mu P - \frac{1}{\lambda} I\right)(P - \lambda I) &= (P - \lambda I) \left(\mu P - \frac{1}{\lambda} I\right) = I + \mu P^2 - \lambda \mu P - \frac{1}{\lambda} P \\ &= I + \left\{ \mu - \lambda \mu - \frac{1}{\lambda} \right\} P = I + \left\{ \mu(1 - \lambda) - \frac{1}{\lambda} \right\} P. \end{aligned}$$

Choosing $\mu = \frac{1}{\lambda(1 - \lambda)}$ we get that the inverse operator is given by

$$(P - \lambda I)^{-1} = \frac{1}{\lambda(1 - \lambda)} P - \frac{1}{\lambda} I \in B(E)$$

and that in particular, $P - \lambda I$ is surjective.

- 3) It follows from (2) that $\varrho(P) \supseteq \mathbb{C} \setminus \{0, 1\}$, hence $\sigma(P) \subseteq \{0, 1\}$. We have also assumed that $P \neq 0$ and $P \neq I$, hence

$$\{x \in M \mid Px = 0\} \neq \{0\}, M,$$

and

$$\{x \in M \mid Px = x\} \neq \{0\}, M,$$

are the eigenspaces corresponding to $\lambda = 0$ and $\lambda = 1$, respectively, hence

$$\sigma(P) = \sigma_p(P) = \{0, 1\}.$$



ie business school

93%
OF MIM STUDENTS ARE
WORKING IN THEIR SECTOR 3 MONTHS
FOLLOWING GRADUATION

MASTER IN MANAGEMENT

- STUDY IN THE CENTER OF MADRID AND TAKE ADVANTAGE OF THE UNIQUE OPPORTUNITIES THAT THE CAPITAL OF SPAIN OFFERS
- PROPEL YOUR EDUCATION BY EARNING A DOUBLE DEGREE THAT BEST SUITS YOUR PROFESSIONAL GOALS
- STUDY A SEMESTER ABROAD AND BECOME A GLOBAL CITIZEN WITH THE BEYOND BORDERS EXPERIENCE

Length: 10 MONTHS
Av. Experience: 1 YEAR
Language: ENGLISH / SPANISH
Format: FULL-TIME
Intakes: SEPT / FEB

5 SPECIALIZATIONS
PERSONALIZE YOUR PROGRAM

#10 WORLDWIDE
MASTER IN MANAGEMENT
FINANCIAL TIMES

55 NATIONALITIES
IN CLASS

www.ie.edu/master-management | mim.admissions@ie.edu | [f](#) [t](#) [in](#) Follow us on IE MIM Experience

2 The adjoint of a bounded operator

Example 2.1 Let $T \in B(H)$ where H is a Hilbert (or just Banach) space. Show that $\|R_\lambda(T)\| \rightarrow 0$ for $|\lambda| \rightarrow \infty$.

Since $T \in B(H)$, we see that $R_\lambda(T) = (T - \lambda I)^{-1}$ exists for every $\lambda \in \mathbb{C}$, for which $|\lambda| > \|T\|$. Then by the Neumann series,

$$R_\lambda(T) = (T - \lambda I)^{-1} = -\frac{1}{\lambda} \left(I - \frac{1}{\lambda} T \right)^{-1} = -\frac{1}{\lambda} \sum_{n=0}^{+\infty} \frac{1}{\lambda^n} T^n.$$

We get the estimate

$$\|R_\lambda(T)\| \leq \frac{1}{|\lambda|} \sum_{n=0}^{+\infty} \left\{ \frac{\|T\|}{|\lambda|} \right\}^n = \frac{1}{|\lambda|} \cdot \frac{1}{1 - \frac{\|T\|}{|\lambda|}} \rightarrow 0 \quad \text{for } |\lambda| \rightarrow +\infty,$$

and the claim is proved.

Example 2.2 Let T be a self adjoint operator in a Hilbert space H . Show that if $D(T) = H$, then T is bounded.

When T is self adjoint, then T is closed, and since $D(T) = H$ is closed, it follows from the *Closed Graph Theorem* that T is bounded.

Example 2.3 Let T be a bounded operator on a Hilbert space H and assume that N and M are closed subspaces of H . Show that

$$T(M) \subset N \quad \text{if and only if} \quad T^*(N^\perp) \subset M^\perp.$$

Show moreover that

$$\ker(T) = T^*(H)^\perp \quad \text{and} \quad \ker(T)^\perp = \overline{T^*(H)}.$$

We assume that $T(M) \subseteq N$, and we shall prove that $T^*(N^\perp) \subseteq M^\perp$.

Let $x \in M$ and $y \in N^\perp$. By the assumption, $Tx \in N$, thus

$$0 = (Tx, y) = (x, T^*y).$$

Now, $x \in M$ was arbitrary, so it follows that $T^*y \in M^\perp$. This holds for every $y \in N^\perp$, hence

$$T^*(N^\perp) \subseteq M^\perp.$$

Then by iteration, $T^{**}(M^{\perp\perp}) \subseteq N^{\perp\perp}$. However, $T^{**} = T$ and $M^{\perp\perp} = M$, and $N^{\perp\perp} = N$, so we conclude that

$$T(M) \subseteq N \quad \text{if and only if} \quad T^*(N^\perp) \subseteq M^\perp.$$

If $x \in \ker(T)$, then $Tx = 0$, and $\ker(T)$ is a closed subspace. Then put $M = \ker(T)$ and $N = \{0\}$, and it follows from the above that

$$T^*(N^\perp) = T^*(H) \subseteq \ker(T)^\perp, \quad \text{thus} \quad \{T^*(H)\}^\perp \supseteq \ker(T).$$

If conversely $x \in \{T^*(H)\}^\perp$, then for every $y \in H$,

$$0 = (x, T^*y) = (Tx, y),$$

so $Tx = 0$, and we have $x \in \ker(T)$. We have proved that

$$\ker(T) = \{T^*(H)\}^\perp.$$

Finally, it follows from this equation that

$$\ker(T)^\perp = \{T^*(H)\}^{\perp\perp} = \overline{T^*(H)},$$

where the bar means the closure of the set.

Example 2.4 Let T be a bounded operator on a Hilbert space H with $\|T\| = 1$, and assume that we can find $x_0 \in H$ such that $Tx_0 = x_0$. Show that also $T^*x_0 = x_0$.

First we get

$$\begin{aligned} 0 &\leq \|T^*x_0 - x_0\|^2 = (T^*x_0 - x_0, T^*x_0 - x_0) \\ &= (T^*x_0, T^*x_0) - (x_0, T^*x_0) - (T^*x_0, x_0) + (x_0, x_0) \\ &= \|T^*x_0\|^2 - (Tx_0, x_0) - (x_0, Tx_0) + \|x_0\|^2 \\ &= \|T^*x_0\|^2 - (x_0, x_0) - (x_0, x_0) + \|x_0\|^2 \\ &= \|T^*x_0\|^2 - \|x_0\|^2, \end{aligned}$$

from which $\|T^*x_0\|^2 \geq \|x_0\|^2$, or

$$\|x_0\| \leq \|T^*x_0\| \leq \|T^*\| \cdot \|x_0\| = \|T\| \cdot \|x_0\| = \|x_0\|.$$

Thus we must have equality everywhere, and therefore in particular, $\|x_0\| = \|T^*x_0\|$, hence by insertion,

$$\|T^*x_0 - x_0\|^2 = \|T^*x_0\|^2 - \|x_0\|^2 = \|x_0\|^2 - \|x_0\|^2 = 0.$$

This shows that $T^*x_0 - x_0$, or after a rearrangement, $T^*x_0 = x_0$.

Example 2.5 Let (e_n) denote an orthonormal basis in a Hilbert space H , and consider the operator

$$T \left(\sum_{k=1}^{\infty} a_k e_k \right) = \sum_{k=1}^{\infty} a_k e_{k+1}.$$

Find the adjoint T^* and show that T^* is an extension of T^{-1} .

Put

$$x = \sum_{k=1}^{+\infty} x_k e_k \in H \quad \text{and} \quad y = \sum_{k=1}^{+\infty} y_k e_k \in D(T^*) = H.$$

then

$$\begin{aligned} (Tx, y) &= \left(\sum_{k=1}^{+\infty} x_k e_{k+1}, \sum_{k=1}^{+\infty} y_k e_k \right) = \left(\sum_{k=2}^{+\infty} x_{k-1} e_k, \sum_{k=1}^{+\infty} y_k e_k \right) = \sum_{k=2}^{+\infty} x_{k-1} \overline{y_k} = \sum_{k=1}^{+\infty} x_k \overline{y_{k+1}} \\ &= \left(\sum_{k=1}^{+\infty} x_k e_k, \sum_{k=1}^{+\infty} y_{k+1} e_k \right) = (x, T^*y), \end{aligned}$$

from which

$$T^*y = T^* \left(\sum_{k=1}^{+\infty} y_k e_k \right) = \sum_{k=1}^{+\infty} y_{k+1} e_k.$$

It follows from $D(T^{-1}) = \{e_1\}^\perp$ and

$$T^{-1}y = T^{-1} \left(\sum_{k=2}^{+\infty} y_k e_k \right) = \sum_{k=1}^{+\infty} y_{k+1} e_k \quad \text{for } y \in D(T^{-1}),$$

that $T^{-1}y = T^*y$ for all $y \in D(T^{-1}) \subset H$, hence $T^{-1} \subset T^*$.

Finally, we notice that $T^*e_1 = 0$, thus $T^{-1} \neq T^*$.

Example 2.6 Let (e_n) denote an orthonormal basis in a Hilbert space H , and consider the operator

$$T \left(\sum_{k=1}^{\infty} a_k e_k \right) = \sum_{k=2}^{\infty} \sqrt{k-1} a_k e_{k-1}.$$

Show that T is a densely defined unbounded operator, and find T^* .

It follows from $\|e_n\| = 1$ and

$$\|Te_n\| = \sqrt{n-1} \rightarrow +\infty \quad \text{for } n \rightarrow +\infty,$$

that T is unbounded.

Put

$$x = \sum_{k=1}^{+\infty} x_k e_k \quad \text{and} \quad y = \sum_{n=1}^{+\infty} y_n e_n.$$

Then

$$\begin{aligned} (Tx, y) &= \left(\sum_{k=1}^{+\infty} \sqrt{k} x_{k+1} e_k, \sum_{n=1}^{+\infty} y_n e_n \right) = \sum_{n=1}^{+\infty} \sqrt{n} \cdot x_{n+1} \overline{y_n} \\ &= (x, T^*y) = \left(\sum_{n=1}^{+\infty} x_{n+1} e_{n+1}, \sum_{k=1}^{+\infty} \sqrt{k} \cdot y_k e_{k+1} \right) = \left(x, \sum_{k=2}^{+\infty} \sqrt{k-1} \cdot y_{k-1} e_k \right), \end{aligned}$$

and we infer that

$$T^*y = T^* \left(\sum_{k=1}^{+\infty} y_k e_k \right) = \sum_{k=2}^{+\infty} \sqrt{k-1} \cdot y_{k-1} e_k.$$

Then we shall explain that the formal computations above are legal. Thus, we shall prove that

$$D(T) = \left\{ x \in H \mid \sum_{k=2}^{+\infty} k |a_k|^2 < +\infty \right\}$$

is dens in H . Let $x \in H$ be arbitrary. To any $\varepsilon > 0$ there exists an N , such that

$$\sum_{k=N+1}^{+\infty} |a_k|^2 < \varepsilon^2.$$

Choose $x_N = (a_1, a_2, \dots, a_N, 0, 0, \dots) \in D(T)$. Then $\|x - x_N\| < \varepsilon$. This proves that $D(T)$ is dense in H , thus T^* exists and the formal computations above are correct, when $x \in D(T)$ and $y \in D(T^*)$.

We infer from

$$\|T^*y\|^2 = \sum_{k=2}^{+\infty} (k-1) |y_{k-1}|^2 = \sum_{k=1}^{+\infty} k |y_k|^2 \quad (= \|Ty\|^2),$$

that $D(T^*) = D(T)$.

Example 2.7 Consider the operator $T : \ell^2 \rightarrow \ell^2$ given by

$$T(x_1, x_2, \dots, x_n, \dots) = \left(\frac{1}{2}x_2, \frac{2}{3}x_3, \dots, \frac{n}{n+1}x_n, \dots \right).$$

- 1) Determine $\|T\|$.
- 2) Find all eigenvalues $\sigma_p(T)$ and corresponding eigenvectors.
- 3) Determine the adjoint T^* and $\sigma_p(T^*)$ and the resolvent $\varrho(T)$.

1) It is obvious that $\|Tx\| \leq \|x\|$. Then it follows from

$$\|T(e_n)\| = \frac{n}{n+1} \rightarrow 1 \quad \text{for } n \rightarrow +\infty,$$

that $\|T\| = 1$.

2) Assume that $\lambda \in \sigma_p(T)$ is an eigenvalue, and let $x \in \ell^2$ be a corresponding eigenvector. Then we get for the coordinates,

$$\lambda x_n = \frac{n}{n+1} x_{n+1}, \quad n \in \mathbb{N},$$

"I studied English for 16 years but...
...I finally learned to speak it in just six lessons"

Jane, Chinese architect

ENGLISH OUT THERE

Click to hear me talking before and after my unique course download

hence by a rearrangement and recursion,

$$x_{n+1} = \lambda \cdot \frac{n+1}{n} x_n = \dots = \lambda^n \cdot \frac{n+1}{n} \frac{n}{n-1} \dots \frac{2}{1} \cdot x_1 = \lambda^n (n+1) x_1,$$

hence

$$x_n = n \cdot \lambda^{n-1} x_1, \quad n \in \mathbb{N}.$$

It follows that

$$\sum_{n=1}^{+\infty} |x_n|^2 = \sum_{n=1}^{+\infty} n^2 |\lambda|^{2(n-1)} |x_1|^2 = |x_1|^2 \sum_{n=1}^{+\infty} n^2 |\lambda|^{2(n-1)},$$

where the series is convergent, if and only if $|\lambda| < 1$, thus

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\},$$

and a corresponding eigenvector is

$$x_\lambda = (1, 2\lambda, 3\lambda^2, \dots, n\lambda^{n-1}, \dots).$$

3) We see that T^* exists in $B(\ell^2)$, so

$$\begin{aligned} (x, T^*y) &= (Tx, y) = \sum_{n=1}^{+\infty} \frac{n}{n+1} x_{n+1} \overline{y_n} = \sum_{n=2}^{+\infty} x_n \cdot \frac{n-1}{n} \overline{y_{n-1}} \\ &= \left((x_1, x_2, \dots, x_n, \dots), \left(0, \frac{1}{2} y_1, \frac{2}{3} y_2, \dots, \frac{n-1}{n} y_{n-1}, \dots \right) \right), \end{aligned}$$

and we get

$$T^*y = \left(0, \frac{1}{2} y_1, \frac{2}{3} y_2, \dots, \frac{n-1}{n} y_{n-1}, \dots \right), \quad y \in \ell^2.$$

Assume that $\lambda \in \sigma_p(T^*)$ is an eigenvalue for T^* . Then

$$\lambda y_1 = 0, \quad \lambda y_n = \frac{n-1}{n} y_{n-1}, \quad n \in \mathbb{N} \setminus \{1\}.$$

We have two possibilities: Either $\lambda = 0$, or $y_1 = 0$.

(a) $\lambda = 0$. It follows from the latter equation that $y_{n-1} = 0$ for $n \in \mathbb{N} \setminus \{1\}$, meaning that $y = 0$, and we conclude that $0 \notin \sigma_p(T^*)$.

(b) $\lambda \neq 0$. In this case, $y_1 = 0$, and then it follows by induction on

$$y_n = \frac{n-1}{n\lambda} y_{n-1}, \quad n \in \mathbb{N} \setminus \{1\},$$

that $y_n = 0$, and hence $y = 0$. We conclude that $\lambda \notin \sigma_p(T^*)$.

Summing up,

$$\sigma_p(T^*) = \emptyset.$$

Hence $\sigma_r(T) = \emptyset$. Furthermore,

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \subseteq \sigma(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\| = 1\},$$

and because $\sigma(T)$ is closed, we must have

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}.$$

Utilizing that

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T) = \sigma_p(T) \cup \sigma_c(T)$$

is a disjoint splitting, we finally find the continuous spectrum

$$\sigma_c(T) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\},$$

and the resolvent set

$$\varrho(T) = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}.$$

Example 2.8 Let $T : \ell^2 \rightarrow \ell^2$ be the linear operator given by

$$T(x_1, x_2, \dots, x_n, \dots) = (x_1 + x_2, x_2 + x_3, \dots, x_n + x_{n+1}, \dots).$$

- 1) Find the point spectrum $\sigma_p(T)$ and determine all eigenvectors associated to $\lambda \in \sigma_p(T)$.
- 2) Determine $\|T\|$.
- 3) Determine the adjoint T^* and find also the point spectrum $\sigma_p(T^*)$.
- 4) Let $S = T - I$. Determine $\|S\|$.
- 5) Find $\sigma_c(T)$ and $\sigma_r(T)$ with the help of S above.

- 1) We shall find the non-trivial solutions of the equation

$$Tx = \lambda x.$$

The coordinate equation of this equation becomes

$$x_n + x_{n+1} = \lambda x_n, \quad n \in \mathbb{N},$$

thus

$$(4) \quad x_{n+1} = (\lambda - 1)x_n, \quad n \in \mathbb{N}.$$

If $\lambda = 1$, then $x_{n+1} = 0$, so we can only choose $x_1 \neq 0$. On the other hand, e_1 is clearly an eigenvector and $1 \in \sigma_p(T)$.

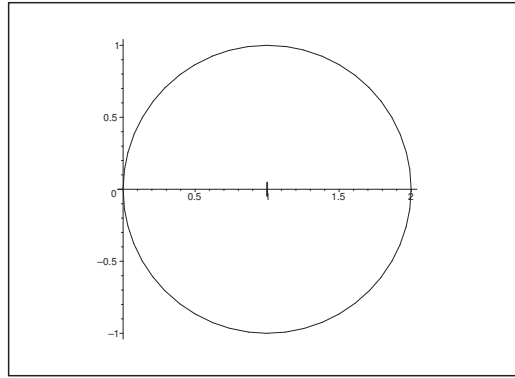


Figure 1: The point spectrum $\sigma_p(T)$ is the open set inside the circle.

If $\lambda \neq 1$, then we can divide (4) by $(\lambda - 1)^{n+1} \neq 0$. Then it follows by recursion that

$$\frac{x_{n+1}}{(\lambda - 1)^{n+1}} = \frac{x_n}{(\lambda - 1)^n} = \cdots = \frac{x_1}{\lambda - 1},$$

so $x_n = (\lambda)^{n-1}x_1$. Choosing $x_1 = 1$ we see that one *candidate* of an eigenvector is given by its coordinates $x_n = (\lambda - 1)^{n-1}$. Because

$$\sum_{n=1}^{+\infty} |x_n|^2 = \sum_{n=1}^{+\infty} |\lambda - 1|^{2(n-1)} = \sum_{n=0}^{+\infty} |\lambda - 1|^{2n}$$

is convergent, if and only if $|\lambda - 1| < 1$, it follows that

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda - 1| < 1\}$$

with the eigenvectors

$$(1, \lambda - 1, (\lambda - 1)^2, \dots, (\lambda - 1)^{n-1}, \dots), \quad \text{for } |\lambda - 1| < 1.$$

We notice for $\lambda = 1$ that we get precisely $(1, 0, 0, \dots)$.

2) From

$$2 \in \sigma(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\},$$

and a consideration of the figure, it follows that $\|T\| \geq 2$.

On the other hand, an application of Minkowski's inequality gives

$$\|Tx\| = \|x + (0, x_1, x_2, \dots)\| \leq \|x\| + \|x\| = 2\|x\|,$$

proving that $\|T\| \leq 2$.

Summing up, $\|T\| = 2$.

3) It follows from

$$\begin{aligned}(Tx, y) &= \sum_{n=1}^{+\infty} (x_n + x_{n+1}) \overline{y_n} = \sum_{n=1}^{+\infty} x_n \overline{y_n} + \sum_{n=2}^{+\infty} x_n \overline{y_{n-1}} \\ &= x_1 \overline{y_1} + \sum_{n=2}^{+\infty} x_n \overline{(y_{n-1} + y_n)} = (x, T^*y) = \sum_{n=1}^{+\infty} x_n \overline{(T^*y)_n},\end{aligned}$$

that

$$T^*y = (y_1, y_1 + y_2, y_2 + y_3, \dots, y_{n-1} + y_n, \dots),$$

or written in coordinates,

$$(T^*y)_1 = y_1, \quad (T^*y)_n = y_{n-1} + y_n \quad \text{for } n \geq 2.$$

The equation $T^*y = \lambda y$ is written in coordinates as

$$y_1 = \lambda y_1 \quad \text{and} \quad y_{n-1} + y_n = \lambda y_n \quad \text{for } n \geq 2,$$

thus

$$(\lambda - 1)y_1 = 0 \quad \text{and} \quad (\lambda - 1)y_n = y_{n-1} \quad \text{for } n \geq 2.$$

We get from the first equation that either $\lambda = 1$ or $y_1 = 0$. If $\lambda = 1$, then it follows from the last equations that $y_{n-1} = 0$ for all $n \geq 2$, hence $y = 0$, and $\lambda = 1$ is not an eigenvalue for T^* .

Excellent Economics and Business programmes at:



**university of
 groningen**





**“The perfect start
of a successful,
international career.”**

www.rug.nl/feb/education

CLICK HERE
to discover why both socially
and academically the University
of Groningen is one of the best
places for a student to be



If $\lambda \neq 1$ and $y_1 = 0$, then we see by recursion on

$$y_n = \frac{1}{\lambda - 1} y_{n-1}$$

that the only solution is $y = 0$.

Summing up, $\sigma_p(T^*) = \emptyset$.

Then of course, $\sigma_r(T) = \emptyset$.

4) Because

$$(Sx)_n = (Tx)_n - x_n = x_{n+1},$$

and $\|Sx\| \leq \|x\|$ with equality for $x_1 = 0$, it follows immediately that $\|S\| = 1$.

5) We get from $T = S + I$ that $T - \lambda I = S - (\lambda - 1)I$, so

$$\begin{array}{llll} \lambda \in \sigma_p(T) & \text{if and only if} & \lambda - 1 \in \sigma_p(S), & \text{thus. } \sigma_p(T) = 1 + \sigma_p(S), \\ \lambda \in \sigma_c(T) & \text{if and only if} & \lambda - 1 \in \sigma_c(S), & \text{thus } \sigma_c(T) = 1 + \sigma_c(S), \\ \lambda \in \sigma_r(T) & \text{if and only if} & \lambda - 1 \in \sigma_r(S), & \text{thus } \sigma_r(T) = 1 + \sigma_r(S). \end{array}$$

It is not surprising that the various parts of the spectrum for T is obtained by translating the corresponding parts of the spectrum for S . We now conclude from

$$\sigma_p(S) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\},$$

and

$$\sigma_r(S) = \emptyset, \quad (\text{because } \sigma_r(T) = \emptyset),$$

and from $\sigma(S)$ being closed, and

$$\sigma_p(S) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \subseteq \sigma(S) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|S\|\} = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\},$$

that

$$\sigma(S) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\},$$

and hence that

$$\sigma_c(S) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}.$$

Finally, by utilizing the translation, we get

$$\begin{array}{ll} \sigma(T) &= \{\lambda \in \mathbb{C} \mid |\lambda - 1| \leq 1\}, \\ \sigma_p(T) &= \{\lambda \in \mathbb{C} \mid |\lambda - 1| < 1\}, \\ \sigma_c(T) &= \{\lambda \in \mathbb{C} \mid |\lambda - 1| = 1\} \\ \sigma_r(T) &= \emptyset. \end{array}$$

Example 2.9 We consider in ℓ^2 the operator

$$T(x_1, x_2, \dots, x_n, \dots) = \left(2x_2, \frac{3}{2}x_3, \dots, \frac{n+1}{n}x_{n+1}, \dots \right).$$

- 1) Find $\|T\|$.
- 2) Find $\sigma_p(T)$ and find the eigenspace associated to all $\lambda \in \sigma_p(T)$.
- 3) Determine the adjoint T^* .
- 4) Determine $\sigma_r(T)$.
- 5) Let $\lambda \notin \sigma_p(T) \cup \sigma_r(T)$. For $k \in \mathbb{N}$ we define an operator I_k on ℓ^2 by

$$I_k((x_1, x_2, \dots, x_k, x_{k+1}, \dots)) = (0, 0, \dots, 0, x_k, x_{k+1}, \dots),$$

and we define $T_k = I_k T$. Show that there is a $k \in \mathbb{N}$ such that

$$\|T_k\| < \lambda.$$

Use this to solve the equation

$$(T_k - \lambda I_k)x = y$$

for a given $y \in \ell^2$. Finally, show that the equation

$$(T - \lambda I)x = y$$

has a solution $x = (T - \lambda T)^{-1}y$ for all $y \in \ell^2$.

- 6) Find $\sigma(T)$ and $\varrho(T)$ (e.g. by use of the Closed Graph Theorem).

- 1) From $1 + \frac{1}{n} \leq 2$ for all $n \in \mathbb{N}$, follows for every $x \in \ell^2$ that

$$\|Tx\|^2 = \sum_{n=1}^{+\infty} \left(1 + \frac{1}{n}\right)^2 |x_{n+1}|^2 \leq 2^2 \sum_{n=1}^{\infty} |x_{n+1}|^2 \leq \{2\|x\|\}^2,$$

proving that $\|T\| \leq 2$.

On the other hand,

$$T(0, 1, 0, 0, \dots) = (2, 0, 0, 0, \dots),$$

and we infer that $\|T\| = 2$.

- 2) Assume that $Tx = \lambda x$, thus

$$\frac{n+1}{n}x_{n+1} = \lambda x_n, \quad n \in \mathbb{N}.$$

For $\lambda = 0$ we get $x = (1, 0, 0, \dots)$ as an eigenvector, and 0 is an eigenvalue, $0 \in \sigma_p(T)$.

If $\lambda \neq 0$, then a multiplication by $n \lambda^{-(n+1)}$ follows by a recursion gives that

$$(n+1) \lambda^{-(n+1)} x_{n+1} = n \lambda^{-n} x_n = \cdots = 1 \cdot \lambda^{-1} x_1,$$

and we get the coordinates of the candidate

$$x_n = \frac{1}{n} \lambda^{n-1} x_1, \quad n \in \mathbb{N}.$$

the corresponding sequence lies in ℓ^2 for $x_1 \neq 0$, if and only if

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} |\lambda|^{2(n-1)} < +\infty.$$

It is well-known that $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < +\infty$, so this condition is equivalent to $|\lambda| \leq 1$, and we conclude that

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\},$$

and an eigenvector corresponding to $\lambda \in \sigma_p(T)$ is given by

$$x_1 \left(1, \frac{\lambda}{2}, \frac{\lambda^2}{3}, \dots, \frac{\lambda^{n-1}}{n}, \dots \right).$$

3) If $x, y \in \ell^2$, then

$$(Tx, y) = \sum_{n=1}^{+\infty} (Tx)_n \overline{y_n} = \sum_{n=1}^{+\infty} \frac{n+1}{n} x_{n+1} \overline{y_n} = \sum_{n=2}^{+\infty} x_n \cdot \overline{\frac{n}{n-1} y_{n-1}} = (x, T^*y),$$

hence

$$T^*(y_1, y_2, \dots, y_n, \dots) = \left(0, 2y_1, \frac{3}{2} y_2, \dots, \frac{n}{n-1} y_{n-1}, \dots \right),$$

or written in coordinates,

$$\begin{cases} (T^*y)_1 = 0, & \text{for } n = 1, \\ (T^*y)_n = \frac{n}{n-1} y_{n-1}, & \text{for } n \in \mathbb{N} \setminus \{1\}. \end{cases}$$

4) We prove that $\sigma_p(T^*) = \emptyset$, because this will imply that $\sigma_r(T) = \emptyset$.

Assume that $\lambda \in \sigma_p(T^*)$. It follows from the equation $T^*y = \lambda y$ that

$$\begin{cases} 0 = \lambda y_1, & \text{for } n = 1, \\ \frac{n}{n-1} y_{n-1} = \lambda y_n, & \text{for } n \in \mathbb{N} \setminus \{1\}. \end{cases}$$

If $\lambda = 0$, then clearly $y = 0$, so $0 \notin \sigma_p(T^*)$.

If $\lambda \neq 0$, then $y_1 = 0$. Multiply the last coordinate equation by $\frac{1}{n} \lambda^{n-1}$. Then it follows by recursion that

$$\frac{\lambda^n}{n} y_n = \frac{\lambda^{n-1}}{n-1} y_{n-1} = \cdots = \frac{\lambda}{1} y_1 = 0,$$

from which $y_n = 0$ for all $n \in \mathbb{N}$, and there is no eigenvectors. Hence, $\sigma_p(T^*) = \emptyset$, and therefore $\sigma_r(T) = \emptyset$.

5) If

$$\lambda \notin \sigma_p(T) \cup \sigma_r(T) = \sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\},$$

then $|\lambda| > 1$. It follows from

$$\|T_k x\|^2 = \sum_{n=k}^{+\infty} \left(\frac{n+1}{n}\right)^2 |x_{n+1}|^2 \leq \left(\frac{k+1}{k}\right)^2 \|x\|^2,$$

that

$$\|T_k\| \leq \frac{k+1}{k} = 1 + \frac{1}{k},$$

where we can obtain equality, so

$$\|T_k\| = 1 + \frac{1}{k}.$$

American online LIGS University

is currently enrolling in the
Interactive Online **BBA, MBA, MSc,**
DBA and PhD programs:

- ▶ enroll **by September 30th, 2014** and
- ▶ **save up to 16%** on the tuition!
- ▶ pay in 10 installments / 2 years
- ▶ Interactive **Online education**
- ▶ visit www.ligsuniversity.com to
find out more!

Note: LIGS University is not accredited by any
nationally recognized accrediting agency listed
by the US Secretary of Education.
More info [here](#).



Because $|\lambda| > 1$, we can choose k so big that

$$\|T_k\| = 1 + \frac{1}{k} < |\lambda|.$$

Now, $\lambda \notin \sigma_p(T) \cup \sigma_r(T)$, so $(T - \lambda I)^{-1}$ exists and is densely defined.

It follows from $\|T_k\| < |\lambda|$, that

$$(T_k - \lambda I_k)^{-1} \in B(I_k \ell^2, \ell^2),$$

where $I_k \ell^2$ is a Hilbert space which is isomorphic to ℓ^2 .

The equation

$$Tx - \lambda x = y, \quad y \in \ell^2,$$

has the coordinate form

$$\frac{n+1}{n} x_{n+1} - \lambda x_n = y_n, \quad n \in \mathbb{N}.$$

Thus it follows from $\lambda \neq 0$ that

$$\begin{cases} x_n = \frac{1}{\lambda} \left(\frac{n+1}{n} x_{n+1} - y_n \right), & n \in \{1, \dots, k-1\}, \\ T_k x - \lambda I_k x = I_k y. \end{cases}$$

It follows from the above that the latter equation can be solved,

$$I_k x = (T_k - \lambda I_k)^{-1} I_k y \quad \text{for all } y \in \ell^2.$$

Hence for a given $y \in \ell^2$,

$$I_k x = (0, \dots, 0, x_k, x_{k+1}, \dots) = (T_k - \lambda I_k)^{-1} I_k y$$

is uniquely determined. The recursion formula

$$x_n = \frac{1}{\lambda} \left\{ \frac{n+1}{n} x_{n+1} - y_n \right\}, \quad \text{for } n \in \{1, \dots, k-1\},$$

determines the remaining elements of x , so $(T - \lambda I)^{-1}$ is defined everywhere.

- 6) If $|\lambda| > 1$, then it follows from the above that $(T - \lambda I)^{-1}$ is defined everywhere. Now, $T - \lambda I$ is closed, so $(T - \lambda I)^{-1}$ is also closed. Then it follows from the Closed Graph Theorem that $\lambda \in \varrho(T)$ for every $\lambda \in \mathbb{C}$, for which $|\lambda| > 1$. Hence

$$\sigma(T) = \sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}, \quad \sigma_r(T) = \sigma_c(T) = \emptyset,$$

and

$$\varrho(T) = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}.$$

Remark 2.1 This example shows that it is possible that every λ for which $\lambda \in \mathbb{C} \setminus \sigma(T)$ belongs to the resolvent set, $\lambda \in \varrho(T)$. So far we have only seen examples, in which there is always at least one $\lambda \in \sigma(T)$, such that $|\lambda| = \|T\|$. This is not the case in the present example. \diamond

3 Self adjoint operators

Example 3.1 Let $T \in B(H)$. Show that we can write T as

$$T = A + iB,$$

where A and B are uniquely determined, bounded self adjoint operators.

First assume that T can be written in the form $T = A + iB$, where A and B are self adjoint. Then

$$\begin{aligned}(Tx, y) &= (Ax + iBx, y) = (Ax, y) + i(Bx, y) \\ &= (x, Ay) + i(x, By) = (x, Ay - iBy) = (x, (A - iB)y) = (x, T^*y),\end{aligned}$$

and it follows that if

$$T = A + iB \quad \text{then} \quad T^* = A - iB.$$

We get by simple addition or subtraction,

$$A = \frac{1}{2} (T + T^*) \quad \text{and} \quad B = \frac{1}{2i} (T - T^*).$$

Conversely, if

$$A = \frac{1}{2} (T + T^*) \quad \text{and} \quad B = \frac{1}{2i} (T - T^*),$$

then clearly, $T = A + iB$. Furthermore, A and B are obviously linear and

$$\|A\| \leq \frac{1}{2} \{\|T\| + \|T^*\|\} = \|T\|, \quad \|B\| \leq \frac{1}{2} \{\|T\| + \|T^*\|\} = \|T\|,$$

so A and B are bounded. Finally,

$$(Ax, y) = \left(\frac{1}{2} \{T + T^*\}x, y \right) = \left(x, \frac{1}{2} \{T^* + T^{**}\}y \right) = \left(x, \frac{1}{2} \{T + T^*\}y \right) = (x, Ay),$$

and

$$(Bx, y) = \left(\frac{1}{2i} \{T - T^*\}x, y \right) = \left(x, -\frac{1}{2i} \{T^* - T^{**}\}y \right) = \left(x, \frac{1}{2i} \{T - T^*\}y \right) = (x, By),$$

shows that both A and B are self adjoint.

Example 3.2 Show that $T \in B(H)$ is self adjoint if and only if one of the following conditions is satisfied:

$$(Tx, x) = (x, Tx) \quad \text{for all } x \in H,$$

and

$$(Tx, x) \in \mathbb{R} \quad \text{for all } x \in H.$$

We assume implicitly that H is a complex Hilbert space.

We have $T \in B(H)$, thus T is self adjoint if and only if $T^* = T$, thus if and only if

$$(5) \quad (Tx, y) = (x, Ty) \quad \text{for all } x, y \in H.$$

Choosing $y = x$ in (5) we get in particular the first condition above, thus

$$(6) \quad (Tx, x) = (x, Tx) \quad \text{for all } x \in H.$$

This condition is equivalent with

$$(Tx, x) = (x, Tx) = \overline{(Tx, x)} \quad (\in \mathbb{R}),$$

and it follows that the two conditions are equivalent. It only remains to prove that (6) implies that T is self adjoint.

Assume (6). We shall prove (5). We get by replacing x in (6) by $x + y$ that

$$\begin{array}{rclcl} (T(x+y), x+y) & = & (Tx, x) & + (Tx, y) + (Ty, x) + & (Ty, y), \\ (x+y, T(x+y)) & = & (x, Tx) & + (x, Ty) + (y, Tx) + & (y, Ty), \\ \star & & \star & & \star \end{array}$$

It follows from the assumption (6) that the three columns marked with a \star inside each column are mutually equal. Hence by a subtraction and a rearrangement,

$$(7) \quad (Tx, y) + (Ty, x) = (x, Ty) + (y, Tx).$$

If we write $x + iy$ in (6) instead of x , then we get analogously,

$$\begin{array}{rclcl} (T(x+iy), x+iy) & = & (Tx, x) & -i(Tx, y) + i(Ty, x) + & (Ty, y), \\ (x+iy, T(x+iy)) & = & (x, Tx) & -i(x, Ty) + i(y, Tx) + & (y, Ty), \\ \star & & \star & & \star \end{array}$$

We conclude as before by utilizing that the columns marked with a \star by the assumption (6) are identical that

$$(8) \quad (Tx, y) - (Ty, x) = (x, Ty) - (y, Tx).$$

We get by adding (7) and (8), followed by a division by 2,

$$(Tx, y) = (x, Ty).$$

This is true for all $x, y \in H$, so we have proved (5), thus T is self adjoint.

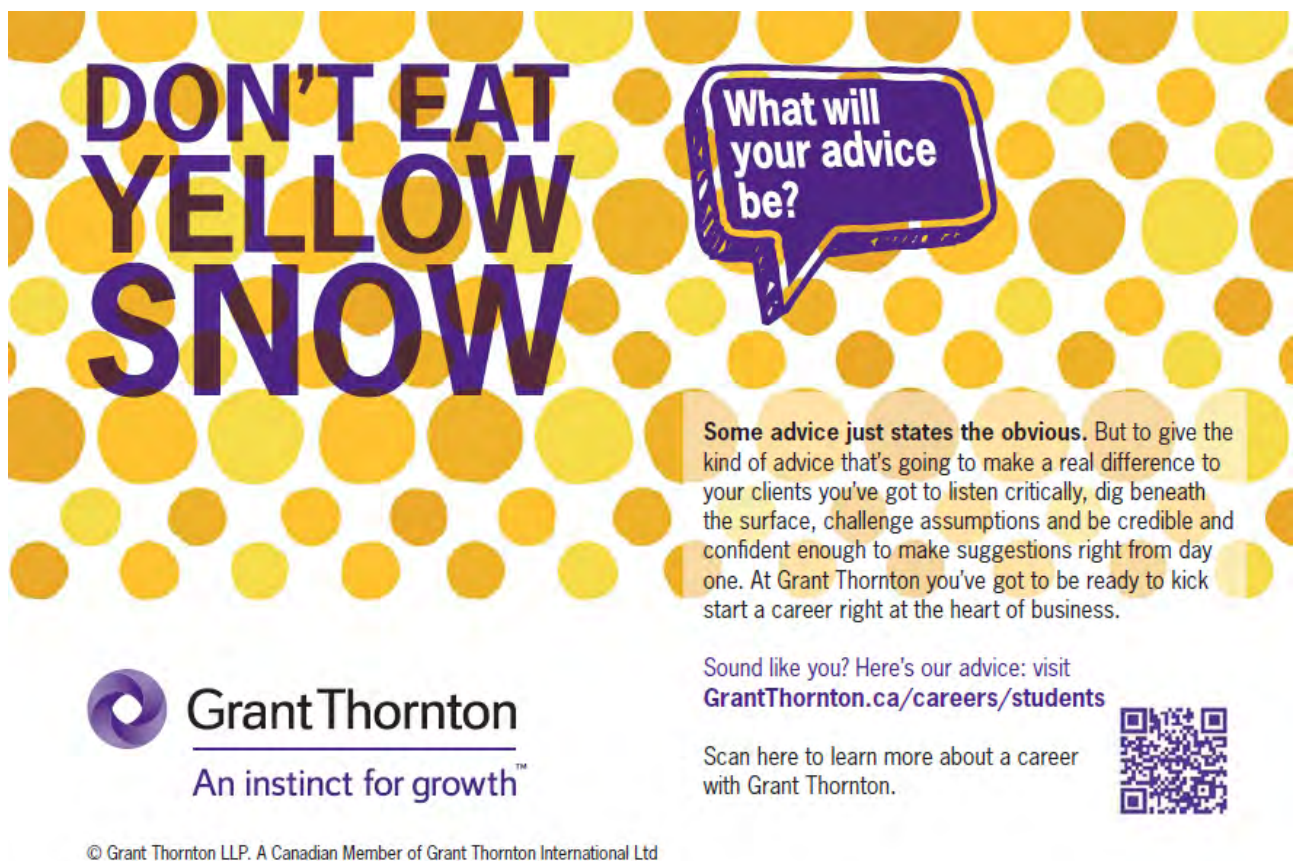
Example 3.3 Let S and T be bounded, self adjoint operators on a Hilbert space. Show that $ST + TS$ and $i(ST - TS)$ are self adjoint.

The proof is simple, because $S, T \in B(H)$ and

$$(ST + TS)^* = (ST)^* + (TS)^* = T^*S^* + S^*T^* = ST + TS,$$

and

$$\{i(ST - TS)\}^* = -i\{(ST)^* - (TS)^*\} = -i\{T^*S^* - S^*T^*\} = i(ST - TS).$$




DON'T EAT YELLOW SNOW

What will your advice be?

Some advice just states the obvious. But to give the kind of advice that's going to make a real difference to your clients you've got to listen critically, dig beneath the surface, challenge assumptions and be credible and confident enough to make suggestions right from day one. At Grant Thornton you've got to be ready to kick start a career right at the heart of business.

Sound like you? Here's our advice: visit [GrantThornton.ca/careers/students](https://www.grantthornton.ca/careers/students)

Scan here to learn more about a career with Grant Thornton.

 **Grant Thornton**
An instinct for growth™

© Grant Thornton LLP. A Canadian Member of Grant Thornton International Ltd

Example 3.4 Let T be a bounded self adjoint operator. Define the numbers

$$m = \inf\{(Tx, x) \mid \|x\| = 1\},$$

and

$$M = \sup\{(Tx, x) \mid \|x\| = 1\}.$$

Show that $\sigma(T) \subset [m, M]$, and show that both m and M belong to $\sigma(T)$.

Show that $\|T\| = \max\{|m|, |M|\}$.

We deduce from the definitions of m and M that

$$m \|x\|^2 \leq (Tx, x) \leq M \|x\|^2 \quad \text{for all } x \in H.$$

Now, $T \in B(H)$ is self adjoint, so $\sigma(T) \subseteq \mathbb{R}$. Choose $\lambda \in \mathbb{R} \setminus [m, M]$. We shall prove that $\lambda \in \varrho(T)$. First assume that $\lambda < m$. Then

$$\begin{aligned} \|(T - \lambda I)x\|^2 &= (Tx - \lambda x, Tx - \lambda x) \\ &= (Tx - mx + (m - \lambda)x, Tx - mx + (m - \lambda)x) \\ &= \|Tx - mx\|^2 + (m - \lambda)^2 \|x\|^2 + 2\{m - \lambda\} (Tx - mx, x). \end{aligned}$$

It follows from $m - \lambda > 0$ and $(Tx - mx, x) = (Tx, x) - m(x, x) \leq 0$ that we have the estimate,

$$\|(T - \lambda I)x\|^2 \geq 0 + (m - \lambda)^2 \|x\|^2 + 0 = (m - \lambda)^2 \|x\|^2,$$

which implies that $T - \lambda I$ is injective, and $(T - \lambda I)^{-1}$ exists and is bounded. Then

$$\lambda \in \varrho(T) \cup \sigma_r(T).$$

Because T is self adjoint, the residual spectrum is $\sigma_r(T) = \emptyset$, hence $\lambda \in \varrho(T)$.

If instead $\lambda > M$, then we get analogously

$$\begin{aligned} \|(T - \lambda I)x\|^2 &= (Tx - \lambda x, Tx - \lambda x) \\ &= (Mx - Tx + (\lambda - M)x, Mx - Tx + (\lambda - M)x) \\ &= \|Mx - Tx\|^2 + (\lambda - M)^2 \|x\|^2 + 2\{\lambda - M\} (Mx - Tx, x) \\ &\geq (\lambda - M)^2 \|x\|^2, \end{aligned}$$

because $\lambda - M > 0$ and $(Mx - Tx, x) = M\|x\|^2 - (Tx, x) \geq 0$. As before we infer that $(T - \lambda I)^{-1}$ exists and is bounded. We have proved that $\mathbb{C} \setminus [m, M] \subseteq \varrho(T)$, and it follows that $\sigma(T) \subseteq [m, M]$.

Using a well-known formula we get

$$\|T\| = \sup\{|(Tx, x)| \mid \|x\| = 1\} = \max\{|m|, |M|\}.$$

Assume e.g. that $\|T\| = |M| = M \geq 0$, and let $\lambda = M$. Then

$$M \in \sigma_p(T) \cup \sigma_c(T) \cup \varrho(T).$$

We shall prove that $M \notin \varrho(T)$. This is done INDIRECTLY.

Assume that $M \in \varrho(T)$, thus $(T - M I)^{-1} \in B(H)$. Then there exists a $c > 0$, such that

$$\|(T - M I)^{-1}x\| \leq \frac{1}{c} \|x\| \quad \text{for all } x \in H.$$

If we put $y = (T - M I)^{-1}x$, then $x = (T - M I)y$, hence

$$\|(T - M I)y\| \geq c\|y\| \quad \text{for all } y \in H.$$

This implies that $\|T - M I\| \geq c > 0$.

From $M = \sup\{\langle Tx, x \rangle \mid \|x\| = 1\}$ follows the existence of a sequence x_n , $\|x_n\| = 1$, of unit vectors, such that

$$\langle Tx_n, x_n \rangle \rightarrow M = \|T\| \quad \text{for } n \rightarrow +\infty,$$

and we conclude from

$$\langle Tx_n, x_n \rangle \leq \|Tx_n\| \cdot \|x_n\| = \|Tx_n\| \leq \|T\| = M,$$

that also $\|Tx_n\| \rightarrow M$. Then for every such sequence,

$$\begin{aligned} 0 &\leq \|(T - M I)x_n\|^2 = \langle Tx_n - Mx_n, Tx_n - Mx_n \rangle \\ &= \|Tx_n\|^2 + M^2\|x_n\|^2 - 2M\langle Tx_n, x_n \rangle \\ &\rightarrow M^2 + M^2 - 2M^2 = 0, \end{aligned}$$

which shows that the estimate $\|(T - M I)x_n\| \geq c\|x_n\| = c > 0$ is not true, and we have derived a contradiction. Therefore, $M \notin \varrho(T)$, i.e. $M \in \sigma(T)$.

An analogous argument shows that if $\|T\| = |m| = -m$, then $m \in \sigma(T)$.

Finally, assume that $|m| = -m < M$. It follows from the above that $M \in \sigma(T)$. We shall prove that also $m \in \sigma(T)$. First notice that $T - M I$ of course is self adjoint. Then it follows from

$$\langle (T - M I)x, x \rangle = \langle Tx, x \rangle - M\|x\|^2,$$

and

$$m\|x\|^2 \leq \langle Tx, x \rangle \leq M\|x\|^2,$$

that

$$(m - M)\|x\|^2 \leq \langle (T - M I)x, x \rangle \leq (M - M)\|x\|^2 = 0,$$

and

$$\inf\{\langle (T - M I)x, x \rangle \mid \|x\| = 1\} = m - M < 0.$$

Then from the above, $m - M \in \sigma(T - M I)$, which means that

$$(T - M I) - (m - M)I = T - m I$$

is not regular, so $m \in \sigma(T)$.

Example 3.5 Consider in $L^2(\mathbb{R})$ the operator Q defined by

$$Qf(x) = x f(x),$$

with

$$D(Q) = \{f \in L^2(\mathbb{R}) \mid Qf \in L^2(\mathbb{R})\}.$$

Show that Q is self adjoint.

Let $f, g \in D(Q)$, thus $f, g \in L^2(\mathbb{R})$ and $x \cdot f(x), x \cdot g(x) \in L^2(\mathbb{R})$. Because Q is densely defined, we get

$$(Qf, g) = \int_{-\infty}^{+\infty} x f(x) \overline{g(x)} dx = \int_{-\infty}^{+\infty} f(x) \cdot \overline{x g(x)} dx = (f, Qg),$$

proving that Q is symmetric, $Q \subseteq Q^*$. It remains to prove that $D(Q) = D(Q^*)$. To do this it suffices to prove that Q is a closed operator.

Assume that $(f_n) \subset D(Q)$ and $f_n \rightarrow f \in L^2(\mathbb{R})$, and $x f_n \rightarrow g \in L^2(\mathbb{R})$. We shall prove that $g(x) = x \cdot f(x)$ almost everywhere. We find

$$\|g - x f\|_2^2 = \int_{-1}^1 |g(x) - x f(x)|^2 dx + \left\{ \int_{-\infty}^{-1} + \int_1^{+\infty} |g(x) - x f(x)|^2 dx \right\}.$$

Here, $\int_{-1}^1 |g(x) - x f(x)|^2 dx = 0$, because $f \in L^2([-1, 1])$ implies that also $x \cdot f \in L^2([-1, 1])$, noting that the interval is bounded. This means that $g(x) = x \cdot f(x)$ for almost every $x \in [-1, 1]$. If $|x| \geq 1$, then we get $f_n \rightarrow f$ and $f_n \rightarrow \frac{g(x)}{x}$, both in the sense of L^2 , because

$$\int_{|x| \geq 1} \left| \frac{g(x)}{x} \right|^2 dx \leq \int_{|x| \geq 1} |g(x)|^2 dx < +\infty.$$

The limit value is unique, hence $f(x) = \frac{g(x)}{x}$ almost everywhere for $|x| \geq 1$. Hence we conclude that $g(x) = x f(x)$ for almost every $x \in \mathbb{R}$.

This proves that Q is closed, which again implies by the above that $Q = Q^*$, and we have proved that Q is self adjoint.

Example 3.6 Show that the set of self adjoint operators is closed in $B(H)$.

We shall only prove that if $(T_n) \subset B(H)$ is a sequence of self adjoint operators, converging towards $T \in B(H)$, then T is also self adjoint. The condition $T_n \rightarrow T$ for $n \rightarrow +\infty$ means that

$$Tx = \lim_{n \rightarrow +\infty} T_n x \quad \text{for all } x \in H.$$

Therefore, if $x, y \in H$, then

$$(Tx, y) = \lim_{n \rightarrow +\infty} (T_n, y) = \lim_{n \rightarrow +\infty} (x, T_n y) = (x, Ty),$$

proving that $T \subseteq T^*$. Because $D(T) = H$, we have $T = T^*$, hence T is self adjoint.

Example 3.7 Let (e_n) denote an orthonormal basis in a Hilbert space H , and let (r_k) be all the rational numbers in $]0, 1[$, arranged as a sequence. Consider the operator

$$T \left(\sum_{k=1}^{\infty} a_k e_k \right) = \sum_{k=1}^{\infty} r_k a_k e_k.$$

Show that T is self adjoint and that $\|T\| = 1$. Find $\rho(T)$ and determine the point spectrum and the continuous spectrum for T .

First note that

$$\|Tx\|^2 = \sum_{k=1}^{+\infty} r_k^2 |x_k|^2 \leq \sum_{k=1}^{+\infty} |x_k|^2 = \|x\|^2,$$

thus $T \in B(H)$ and $\|T\| \leq 1$. Furthermore,

$$(Tx, y) = \sum_{k=1}^{+\infty} r_k x_k \overline{y_k} = \sum_{k=1}^{+\infty} x_k \overline{r_k y_k} = (x, Ty),$$

proving that T is self adjoint. This implies that the residual spectrum is empty, $\sigma_r(T) = \emptyset$.

From $Te_k = r_k e_k$ follows that every $r_k \in \sigma_p(T)$, and we conclude further from $0 < r_k \leq \|T\|$ that

$$\|T\| \geq \sup_{k \in \mathbb{N}} r_k = 1,$$

hence $\|T\| = 1$.

.....Alcatel-Lucent 

www.alcatel-lucent.com/careers

What if
you could
build your
future and
create the
future?

One generation's transformation is the next's status quo. In the near future, people may soon think it's strange that devices ever had to be "plugged in." To obtain that status, there needs to be "The Shift".



Conversely, if $Tx = \lambda x$, then

$$Tx - \lambda x = \sum_{k=1}^{+\infty} (r_k - \lambda) x_k e_k = 0,$$

so either $\lambda = r_k$ or $x_k = 0$. This shows that

$$\sigma_p(T) = \mathbb{Q} \cap]0, 1[= \{r_k \mid k \in \mathbb{N}\}.$$

Assume that $\lambda < 0$. Then

$$\|Tx - \lambda x\|^2 = \left\| \sum_{k=1}^{+\infty} (r_k + |\lambda|) x_k e_k \right\|^2 = \sum_{k=1}^{+\infty} (r_k + |\lambda|)^2 |x_k|^2 \geq |\lambda|^2 \sum_{k=1}^{+\infty} |x_k|^2 = |\lambda|^2 \cdot \|x\|^2,$$

from which we infer that $\|Tx - \lambda x\| \geq |\lambda| \cdot \|x\|$, hence $\lambda \in \varrho(T)$. It follows that

$$\varrho(T) \supseteq \mathbb{C} \setminus [0, 1].$$

On the other hand, $\sigma(T)$ is closed, so it follows from

$$\sigma(T) \supseteq \sigma_p(T) = \mathbb{Q} \cap]0, 1[,$$

that $\sigma(T) \supseteq [0, 1]$. From $\varrho(T)$ and $\sigma(T)$ being disjoint we conclude that

$$\varrho(T) = \mathbb{C} \setminus [0, 1] \quad \text{and} \quad \sigma(T) = [0, 1].$$

Now, $\sigma_r(T) = \emptyset$ for self adjoint operators, and $\sigma_p(T) = \mathbb{Q} \cap]0, 1[$, hence the continuous spectrum is

$$\sigma_c(T) = \sigma(T) \setminus \sigma_p(T) = ([0, 1] \setminus \mathbb{Q}) \cup \{0, 1\}.$$

Example 3.8 Let (e_k) be an orthonormal basis in a Hilbert space H , and assume that $T \in B(H)$ has the matrix representation $\mathbf{T} = (t_{jk})$ with respect to the orthonormal basis (e_k) (see VENTUS, HILBERT SPACES, EXAMPLE 2.7). Derive a necessary and sufficient condition on the t_{jk} that T is self adjoint.

In VENTUS, HILBERT SPACES, EXAMPLE 2.7 we derived that $t_{jk} = (Te_j, e_k)$, and

$$T \left(\sum_{j=1}^{+\infty} x_j e_j \right) = \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} x_j t_{jk} e_k = \sum_{k=1}^{+\infty} \left\{ \sum_{j=1}^{+\infty} x_j t_{jk} \right\} e_k.$$

If

$$x = \sum_{j=1}^{+\infty} x_j e_j \quad \text{and} \quad y = \sum_{k=1}^{+\infty} y_k e_k,$$

then

$$\begin{aligned} (Tx, y) &= \left(\sum_{k=1}^{+\infty} \left\{ \sum_{j=1}^{+\infty} x_j t_{jk} \right\} e_k, \sum_{k=1}^{+\infty} y_k e_k \right) = \sum_{k=1}^{+\infty} \sum_{j=1}^{+\infty} x_j t_{jk} \overline{y_k} = \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} \overline{y_k} t_{jk} x_j \\ &= \left(\sum_{j=1}^{+\infty} x_j e_j, \sum_{j=1}^{+\infty} \left\{ \sum_{k=1}^{+\infty} \overline{t_{jk}} y_k \right\} e_j \right) = (x, T^* y). \end{aligned}$$

Hence

$$T^*y = \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} y_k t_{jk}^* e_j = \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} y_k \overline{t_{kj}} e_j,$$

so $\mathbf{T}^* = (t_{jk}^*) = (\overline{t_{kj}})$. This means that \mathbf{T}^* is obtained from \mathbf{T} by taking the transpose and apply complex conjugating.

It follows from the above that T is self adjoint if and only if $\mathbf{T}^* = \mathbf{T}$, i.e. if and only if

$$\overline{t_{kj}} = t_{jk} \quad \text{for all } j, k = 1, 2, 3, \dots$$

Note that

$$\overline{t_{kj}} = \overline{(Te_k, e_j)} = (e_j, Te_k),$$

so the example shows that in this case T is self adjoint, if

$$(Te_j, e_k) = (e_j, Te_k) \quad \text{for all } j, k \in \mathbb{N},$$

and there is nothing new in that statement.

Example 3.9 Let $H = L^2(\mathbb{R})$, and let V denote a bounded real continuous function. We define the operator T by

$$Tf(x) = V(x) \cdot f(x), \quad f \in L^2(\mathbb{R}).$$

Prove that T is a bounded self adjoint operator.

In Quantum Mechanics the operator T is called a potential operator.

It follows from $\|V\|_\infty < +\infty$ that

$$\begin{aligned} \|Tf\|_2^2 &= \int_{-\infty}^{+\infty} V(x)f(x) \cdot \overline{V(x)f(x)} dx = \int_{-\infty}^{+\infty} V(x)^2 |f(x)|^2 dx \\ &\leq \|V\|_\infty^2 \int_{-\infty}^{+\infty} |f(x)|^2 dx = \|V\|_\infty^2 \cdot \|f\|_2^2, \end{aligned}$$

hence

$$\|Tf\|_2 \leq \|V\|_\infty \cdot \|f\|_2 \quad \text{for every } f \in L^2(\mathbb{R}).$$

We conclude that $T \in B(V)$ and $\|T\| \leq \|V\|_\infty$.

Utilizing that $V(x)$ is real we see that

$$(Tf, g) = \int_{-\infty}^{+\infty} V(x)f(x) \cdot \overline{g(x)} dx = \int_{-\infty}^{+\infty} f(x) \cdot \overline{V(x)g(x)} dx = (f, Tg),$$

which shows that T is self adjoint.

Example 3.10 Let H denote a Hilbert space. Introduce in the set of all self adjoint operators from $B(H)$ a relation \leq by

$$S \leq T, \quad \text{if } T - S \geq 0,$$

cf. EXAMPLE 6.1. Prove that \leq is a partial relation.

It follows from $S - S = 0 \geq 0$ that $S \leq S$.

Assume that $S \leq T$ and $T \leq U$, thus $T - S \geq 0$ and $U - T \geq 0$.

We shall prove that $S \leq U$, i.e. that $U - S \geq 0$.

We have

$$\begin{aligned} ((U - S)x, x) &= ((U - T) + (T - S)x, x) \\ &= ((U - T)x, x) + ((T - S)x, x) \geq 0. \end{aligned}$$

This holds for every $x \in H$, hence the claim is proved.

Example 3.11 Let H be a Hilbert space and let $T \in B(H)$ be positive and self adjoint. Show that

$$\|(Tx, y)\|^2 \leq (Tx, x)(Ty, y),$$

for all $x, y \in H$.

We shall here be aware of two possible obstacles. First, (Tx, y) could be a complex number, and secondly (Tx, x) could be 0, so we must never divide by (Tx, x) .

Let $x, y \in H$ be given, and choose $\alpha \in \mathbb{R}$ such that

$$(Tx, y) = |(Tx, y)| e^{i\alpha}.$$

Using the assumption it follows for any $\lambda \in \mathbb{C}$ that

$$\begin{aligned} 0 &\leq (T(\lambda x + y), \lambda x + y) \\ &= |\lambda|^2 (Tx, x) + \lambda (Tx, y) + \bar{\lambda} (Ty, x) + (Ty, y) \\ &= |\lambda|^2 (Tx, x) + \lambda (Tx, y) + \bar{\lambda} (y, Tx) + (Ty, y) \\ &= |\lambda|^2 (Tx, x) + 2 \operatorname{Re}\{\lambda (Tx, y)\} + (Ty, y), \end{aligned}$$

where we have used that T is self adjoint, hence

$$(Ty, x) = (x, Ty) = \overline{(Tx, y)}.$$

Choosing in particular $\lambda = \mu e^{-i\alpha}$, $\mu \in \mathbb{R}$, then

$$\mu^2 (Tx, x) + 2\mu |(Tx, y)| + (Ty, y) \geq 0 \quad \text{for all } \mu \in \mathbb{R}.$$

All coefficients are real, so the condition of the discriminant $B^2 - AC \leq 0$ holds, thus

$$|(Tx, y)|^2 \leq (Tx, x)(Ty, y) \quad \text{for all } x, y \in H,$$

and the claim is proved.

Example 3.12 1) Let V denote a normed space. Show that

$$\|x - y\| \geq |\|x\| - \|y\|| \quad \text{for all } x, y \in V.$$

- 2) Let T be a bounded, linear and self adjoint operator on a Hilbert space. Assume that T is surjective and show that T is then injective.
- 3) Assume that T is a closed linear operator on a normed space X . Show that $\ker(T)$ is closed in X .
- 4) Let H denote a Hilbert space and assume that (x_n) and (y_n) are two sequences in the closed unit ball of H such that $(x_n, y_n) \rightarrow 1$. Show that $\|x_n - y_n\| \rightarrow 0$.
- 5) Let (x_n) and (y_n) denote two orthonormal sequences in a Hilbert space H , and assume that

$$\sum_{n=1}^{\infty} \|x_n - y_n\|^2 < 1.$$

Show that if (x_n) is an orthonormal basis, then so is (y_n) .

- 1) It follows from the triangle inequality that

$$\|x\| = \|(x - y) + y\| \leq \|x - y\| + \|y\|,$$

and analogously (or just by interchanging letters)

$$\|y\| \leq \|x - y\| + \|x\|.$$



Maastricht University

Leading in Learning!

**Join the best at
the Maastricht University
School of Business and
Economics!**

Top master's programmes

- 33rd place Financial Times worldwide ranking: MSc International Business
- 1st place: MSc International Business
- 1st place: MSc Financial Economics
- 2nd place: MSc Management of Learning
- 2nd place: MSc Economics
- 2nd place: MSc Econometrics and Operations Research
- 2nd place: MSc Global Supply Chain Management and Change

Sources: Keuzegids Master ranking 2013; Elsevier 'Beste Studies' ranking 2012; Financial Times Global Masters in Management ranking 2012

**Maastricht
University is
the best specialist
university in the
Netherlands
(Elsevier)**

**Visit us and find out why we are the best!
Master's Open Day: 22 February 2014**

www.mastersopenday.nl



By a rearrangement,

$$\left. \begin{array}{l} \|x\| - \|y\| \\ \|y\| - \|x\| \end{array} \right\} \leq \|x - y\|,$$

hence

$$\|x - y\| \geq |\|x\| - \|y\||.$$

2) We shall prove that if $Tx = 0$, then $x = 0$. We get for every $y \in H$ that

$$0 = (0, y) = (Tx, y) = (x, Ty).$$

From T being surjective follows that the image of T is all of H , so x is perpendicular to H , thus $x = 0$, and T is injective.

3) Let T be closed, thus the graph $\mathcal{G}(T)$ is closed as a subset of $X \times X$. Let $(x_n) \subset \ker(T)$ denote a convergent sequence in X , i.e. $x_n \rightarrow x$. Then $((x_n, 0)) \subset \mathcal{G}(T)$, and

$$(x_n, 0) \rightarrow (x, 0) \in \overline{\mathcal{G}(T)} = \mathcal{G}(T),$$

which shows that $x \in \ker(T)$.

4) Here,

$$\begin{aligned} \|x_n - y_n\|^2 &= (x_n - y_n, x_n - y_n) = (x_n, x_n) - (y_n, x_n) - (x_n, y_n) + (y_n, y_n) \\ &= \|x_n\|^2 + \|y_n\|^2 - 2 \operatorname{Re} \{(x_n, y_n)\}, \end{aligned}$$

and since all x_n and y_n belong to the unit ball, we have

$$0 \leq \|x_n - y_n\|^2 \leq 1 + 1 - 2 \operatorname{Re} \{(x_n, y_n)\} \rightarrow 2 - 2 = 0 \quad \text{for } n \rightarrow \infty,$$

proving that

$$\|x_n - y_n\| \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

5) Let $x \in H$ be perpendicular to all y_n . From (x_n) being an orthonormal basis and $(x, y_n) = 0$ we get

$$x = \sum_{n=1}^{\infty} (x, x_n) x_n = \sum_{n=1}^{\infty} \{(x, y_n) + (x, x_n - y_n)\} x_n = \sum_{n=1}^{\infty} (x, x_n - y_n) x_n.$$

This implies the estimate, when we apply that (x_n) is orthonormal and the Cauchy-Schwarz inequality,

$$\|x\| = \sum_{n=1}^{\infty} |(x, x_n - y_n)| \leq \sum_{n=1}^{\infty} \|x\|^2 \cdot \|x_n - y_n\|^2 = \|x\|^2 \sum_{n=1}^{\infty} \|x_n - y_n\|^2.$$

It follows from the assumption that $\sum_{n=1}^{\infty} \|x_n - y_n\|^2 < 1$, so the only possibility for this inequality is when $x = 0$, hence $x = 0$ is the only vector in H , which is perpendicular on all y_n . This shows that (y_n) is an orthonormal basis.

Example 3.13 Let $(x_n) \subset \ell^2$ and define the sequence $y = (y_n)$ by

$$y_n = x_{n+1} + n x_n + x_{n-1},$$

where we put $x_0 = 0$ whenever it is necessary.

1. Show that $y \in \ell^2$ if and only if $(n x_n) \in \ell^2$.

Let

$$D = \{x \in \ell^2 \mid (n x_n) \in \ell^2\},$$

and define a linear operator $T : D \rightarrow \ell^2$ by $Tx = y$, where y is given above.

2. Show that D is dense in ℓ^2 .

3. Show that T is self adjoint.

1) It follows from

$$(y_n) = (x_{n+1}) + (n x_n) + (x_{n-1}),$$

and that ℓ^2 is a vector space that if (x_n) and $n x_n \in \ell^2$, then $(y_n) \in \ell^2$.

If conversely (x_n) and $(y_n) \in \ell^2$, then it follows from

$$(n x_n) = (y_n) - (x_{n+1}) - (x_{n-1}),$$

that $(n x_n) \in \ell^2$.

ALTERNATIVELY, we have the following possible, though not very brilliant variant,

$$\begin{aligned} \sum_{n=1}^{+\infty} y_n^2 &= \sum_{n=1}^{+\infty} (x_{n+1} + n x_n + x_{n-1})^2 \\ &= \sum_{n=1}^{+\infty} x_{n+1}^2 + \sum_{n=1}^{+\infty} (n x_n)^2 + \sum_{n=1}^{+\infty} x_{n-1}^2 + 2 \sum_{n=1}^{+\infty} x_{n+1} n x_n + 2 \sum_{n=1}^{+\infty} n x_n x_{n-1} + 2 \sum_{n=1}^{+\infty} x_{n+1} x_{n-1} \\ &\leq \|x\|_2^2 + \sum_{n=1}^{+\infty} (n x_n)^2 + 2\|x\|_2 \left\{ \sum_{n=1}^{+\infty} (n x_n)^2 \right\}^{\frac{1}{2}} + 2 \left\{ \sum_{n=1}^{+\infty} (n x_n)^2 \right\}^2 \|x\|_2 + 2\|x\|_2 \|x\|_2 \\ &= 4\|x\|_2^2 + 4\|x\| \left\{ \sum_{n=1}^{+\infty} (n x_n)^2 \right\}^{\frac{1}{2}} + \sum_{n=1}^{+\infty} (n x_n)^2 \left(\left\{ \sum_{n=1}^{+\infty} (n x_n)^2 \right\}^{\frac{1}{2}} + 2\|x\|_2 \right)^2. \end{aligned}$$

Hence, if $\sum_{n=1}^{+\infty} (n x_n)^2 < +\infty$, then $\sum_{n=1}^{+\infty} y_n^2 < +\infty$, so $y \in \ell^2$.

Conversely, if $y \in \ell^2$, then by a rearrangement,

$$n x_n = y_n - x_{n+1} - x_{n-1},$$

hence

$$\begin{aligned}
\sum_{n=1}^{+\infty} (n x_n)^2 &= \sum_{n=1}^{+\infty} (y_n - x_{n+1} - x_{n-1})^2 \\
&= \sum_{n=1}^{+\infty} y_n^2 + \sum_{n=1}^{+\infty} x_{n+1}^2 + \sum_{n=1}^{+\infty} x_{n-1}^2 - 2 \sum_{n=1}^{+\infty} y_n x_{n+1} - 2 \sum_{n=1}^{+\infty} y_n x_{n-1} + 2 \sum_{n=1}^{+\infty} x_{n+1} x_{n-1} \\
&\leq \|y\|_2^2 + \|x\|_2^2 + \|x\|_2^2 + 2\|y\|_2 \|x\|_2 + 2\|y\|_2 \|x\|_2 + 2\|x\|_2 \|x\|_2 \\
&= \|y\|_2^2 + 4\|y\|_2 \|x\|_2 + 4\|x\|_2^2 = \{\|y\|_2 + 2\|x\|_2\}^2 < +\infty.
\end{aligned}$$

We conclude that $(n x_n) \in \ell^2$.

- 2) Let $D = \{x \in \ell^2 \mid (n x_n) \in \ell^2\}$, and let $z \in \ell^2$ be arbitrary, i.e. $\sum_{n=1}^{+\infty} z_n^2 < +\infty$. To any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$, such that

$$\sum_{n=N+1}^{+\infty} z_n^2 < \varepsilon^2.$$

Define $x = (x_n)$ by

$$x_n = \begin{cases} z_n & \text{for } n = 1, 2, \dots, N, \\ 0 & \text{for } n > N. \end{cases}$$

Then

$$\sum_{n=1}^{+\infty} (n x_n)^2 = \sum_{n=1}^N n^2 x_n^2 < +\infty,$$

because the sum is finite, so $x \in D$, and

$$\|z - x\|_2 = \left\{ \sum_{n=1}^{+\infty} (z_n - x_n)^2 \right\}^{\frac{1}{2}} = \left\{ \sum_{n=N+1}^{+\infty} z_n^2 \right\}^{\frac{1}{2}} < (\varepsilon^2)^{\frac{1}{2}} = \varepsilon,$$

which shows that x approximates z , and we get that D is dense in ℓ^2 . Clearly, D is a subspace, because $(x_n), (y_n), (n x_n), (n y_n) \in \ell^2$ for every $\lambda \in \mathbb{R}$ imply that $(x_n + \lambda y_n)$ and $(n(x_n + \lambda y_n)) = (n x_n + \lambda n y_n) \in \ell^2$. Finally, it is obvious that T is linear.

- 3) Because T is densely defined, the adjoint T^* exists. Let $x \in D$, and let $y \in \mathcal{D}(T^*)$. Then

$$(Tx, y) = (x, T^*y),$$

thus


$$\begin{aligned}
 (Tx, y) &= \sum_{n=1}^{+\infty} (x_{n+1} + n x_n + x_{n-1}) y_n \\
 &= \sum_{n=1}^{+\infty} x_{n+1} y_n + \sum_{n=1}^{+\infty} n x_n y_n + \sum_{n=1}^{+\infty} x_{n-1} y_n \\
 &= \sum_{n=2}^{+\infty} x_n y_{n-1} + \sum_{n=1}^{+\infty} x_n n y_n + \sum_{n=0}^{+\infty} x_n y_{n+1} \\
 &= \sum_{n=1}^{+\infty} x_n y_{n-1} + \sum_{n=1}^{+\infty} x_n n y_n + \sum_{n=1}^{+\infty} x_n y_{n+1} \\
 &= \sum_{n=1}^{+\infty} x_n (y_{n+1} + n y_n + y_{n-1}) = (x, T^* y).
 \end{aligned}$$

The splitting of the sums in the second equality is legal, because each of the three series on the right hand side is absolutely convergent by the Cauchy-Schwarz inequality. Hence we conclude that

$$T^* y = (y_{n+1} + n y_n + y_{n-1}),$$

thus $D \subseteq \mathcal{D}(T^*)$, and $T \subseteq T^*$, so T is at least symmetric.

It follows from the result of (1) that $(y_{n+1} + n y_n + y_{n-1}) \in \ell^2$, when $(y_n) \in \ell^2$, if and only if $(n y_n) \in \ell^2$. Hence $\mathcal{D}(T^*) = D$, and $T = T^*$, and we have proved that T is self adjoint.



Empowering People. Improving Business.

BI Norwegian Business School is one of Europe's largest business schools welcoming more than 20,000 students. Our programmes provide a stimulating and multi-cultural learning environment with an international outlook ultimately providing students with professional skills to meet the increasing needs of businesses.

BI offers four different two-year, full-time Master of Science (MSc) programmes that are taught entirely in English and have been designed to provide professional skills to meet the increasing need of businesses. The MSc programmes provide a stimulating and multi-cultural learning environment to give you the best platform to launch into your career.

- MSc in Business
- MSc in Financial Economics
- MSc in Strategic Marketing Management
- MSc in Leadership and Organisational Psychology

BI NORWEGIAN BUSINESS SCHOOL

EFMD
EQUIS
ACCREDITED

www.bi.edu/master



4 Isometric operators

Example 4.1 Let $T \in B(H)$. An operator is called isometric if $\|Tx\| = \|x\|$ for all $x \in H$. Show that the following conditions are equivalent for $T \in B(H)$.

- 1) T is isometric.
- 2) $T^*T = I$.
- 3) $(Tx, Ty) = (x, y)$ for all $x, y \in H$.

(3) \Rightarrow (2). This is almost trivial:

$$(x, y) = (Tx, Ty) = (T^*Tx, y) \quad \text{for all } x, y \in H,$$

thus $T^*Tx = x$ for all $x \in H$, and hence $T^*T = I$.

(2) \Rightarrow (1). If $T^*T = I$, then

$$\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) = (x, x) = \|x\|^2,$$

proving that T is isometric.

(1) \Rightarrow (3). If T is isometric, we get as above,

$$(T^*Tx, x) = (Ix, x), \quad \text{thus} \quad ((T^*T - I)x, x) = 0,$$

for all $x \in H$. Then it follows from EXAMPLE 1.8 in VENTUS, FUNCTIONAL ANALYSIS, HILBERT SPACES that $T^*T - I = 0$, if H is a complex Hilbert space, hence $T^*T = I$.

Example 4.2 Let $T \in B(H)$ be an isometric operator. Show that $T(H)$ is a closed subspace. Show that $T(H) = H$ if H is finite dimensional. Give an example of an isometric operator with $T(H) \neq H$.

- 1) When $T \in B(H)$ is isometric, i.e. $\|Tx\| = \|x\|$ for all $x \in H$, then in particular T is injective, thus $T^{-1} : T(H) \rightarrow H$ exists.
Put $y = Tx$. Then it follows from the above that $\|T^{-1}y\| = \|x\|$, and T^{-1} is continuous (though not necessarily defined in all of H).
Now, H is closed, so $T(H) = (T^{-1})^{-1}(H)$ is also closed.
- 2) Let H be finite dimensional, $\dim H = n$, and denote by $\{e_1, \dots, e_n\}$ a basis of H .
When T is isometric, then T is injective. In fact, $0 = \|Tx\| = \|x\|$ implies trivially that $x = 0$.
We claim that the images $\{Te_1, \dots, Te_n\}$ of the basis vectors are linearly independent. Assume that

$$0 = \lambda_1 Te_1 + \dots + \lambda_n Te_n \quad (= T(\lambda_1 e_1 + \dots + \lambda_n e_n)).$$

The operator T is injective, so also $\lambda_1 e_1 + \dots + \lambda_n e_n = 0$. Here $\{e_1, \dots, e_n\}$ is a basis, so $\lambda_1 = \dots = \lambda_n = 0$. It follows that Te_1, \dots, Te_n are linearly independent, so $n \leq \dim T(H) \leq n$, thus $\dim T(H) = n$. This is only possible, if $T(H) = H$, because $T : H \rightarrow H$.

- 3) Let $(e_k)_{k \in \mathbb{N}}$ denote an orthonormal basis in an infinite dimensional Hilbert space. Define $T \in B(H)$ by

$$Tx = T \left(\sum_{k=1}^{\infty} x_k e_k \right) = \sum_{k=1}^{\infty} x_k e_{k+1}.$$

Then clearly T is isometric, $\|Tx\| = \|x\|$ for all $x \in H$, and

$$T(H) = \{e_1\}^{\perp} \neq H.$$

Example 4.3 Let $T \in B(H)$ be an isometric operator and let M and N denote closed subspaces of the Hilbert space H . Show that

$$T(M) = N \implies T(M^{\perp}) \subset N^{\perp}.$$

Show that T is isometric if and only if for any orthonormal basis (e_k) , (Te_k) is an orthonormal sequence.

Assume that $T \in B(H)$ is isometric, and let M and $N \subseteq H$ be closed subspaces, and assume that $T(M) = N$. We shall prove that for every $x \in M^{\perp}$ and for every $y \in N$ we have that $(Tx, y) = 0$.

From $y \in N = T(M)$ follows that there exists a $z \in M$, such that $y = Tz$, and then we get from EXAMPLE 4.1, (3) that

$$(Tx, y) = (Tx, Tz) = (x, z) = 0,$$

because $x \in M^{\perp}$ and $z \in M$. It follows that $T(M^{\perp}) \subseteq N^{\perp}$.

Let (e_k) denote an orthonormal basis, and assume that T is isometric. We get again from EXAMPLE 4.1, (3) that

$$(Te_j, Te_k) = (e_j, e_k) = \delta_{jk},$$

(Kronecker symbol), which shows that (Te_k) is an orthonormal sequence. Of course (Te_k) needs *not* be a basis. An example is given in EXAMPLE 4.2.

If conversely there exists an orthonormal basis (e_k) , such that (Te_k) is an orthonormal sequence, then

$$Tx = \sum_{k=1}^{+\infty} x_k Te_k, \quad \text{thus} \quad \|Tx\|^2 = \sum_{k=1}^{+\infty} |x_k|^2 = \|x\|^2,$$

and T is isometric.

Remark 4.1 The answer of the latter question above shows that if there is just one orthonormal basis (e_k) , such that (Te_k) is an orthonormal sequence, then every orthonormal basis has this property. \diamond

Example 4.4 Let $T \in B(H)$ be an isometric operator. Show that TT^* is a projection and determine its range.

Assume that $T \in B(H)$ is isometric. We shall prove that TT^* is a projection, i.e. TT^* must satisfy the two conditions,

$$(TT^*x, y) = (x, TT^*y) \quad \text{for all } x, y \in H,$$

and

$$(TT^*)^2 = TT^*.$$

We get

$$(TT^*x, y) = (T^*x, T^*y) = (x, TT^*y),$$

and the first condition is fulfilled. Then apply the result $T^*T = I$ from EXAMPLE 4.1, (2),

$$(TT^*)^2 = TT^*TT^* = T(T^*T)T^* = TIT^* = TT^*,$$

and it follows that $P = TT^*$ is a projection.

The range of the projection $P = TT^*$ is given by $Px = TT^*x = x$, i.e. TT^*H . Now,

$$T^*(H) = \overline{T^*(H)} = \ker(T)^\perp,$$

thus $TT^*(H) = T(\ker(T)^\perp)$. It follows from

$$H = \ker(T) \oplus \ker(T)^\perp,$$

that

$$TT^*(H) = T(\ker(T)^\perp) = T(\ker(T) \oplus \ker(T)^\perp) = T(H),$$

and the range is TH .

Example 4.5 Consider the Hilbert space $L^2([0, \infty))$. Let $h > 0$ and define the operator T by

$$\begin{aligned} Tf(x) &= 0 & \text{for } 0 \leq x < h, \\ Tf(x) &= f(x - h) & \text{for } h \leq x. \end{aligned}$$

Show that T is isometric and determine T^* . Find TT^* and T^*T .

First notice that

$$\|Tf\|_2^2 = \int_0^{+\infty} |Tf(x)|^2 dx = \int_h^{+\infty} |f(x - h)|^2 dx = \int_0^{+\infty} |f(x)|^2 dx = \|f\|_2^2,$$

which shows that T is isometric. Then it follows from EXAMPLE 4.1, (2) that $T^*T = I$.

Let $f, g \in H$. Then

$$\begin{aligned}(Tf, g) &= \int_0^{+\infty} Tf(x)\overline{g(x)} dx = \int_h^{+\infty} f(x-h)\overline{g(x)} dx \\ &= \int_0^{+\infty} f(x)\overline{g(x+h)} dx = (f, T^*g),\end{aligned}$$

and we conclude that

$$T^*g(x) = g(x+h) \quad \text{for } x \in [0, +\infty[.$$

Then finally we get

$$TT^*g(x) = Tg(x+h) = \begin{cases} g(x+h-h) = g(x) & \text{for } x \in [h, +\infty[, \\ 0 & \text{for } x \in [0, h[, \end{cases}$$

thus $TT^*g = 1_{[h, +\infty[} \cdot g$.

Need help with your dissertation?

Get in-depth feedback & advice from experts in your topic area. Find out what you can do to improve the quality of your dissertation!

Get Help Now



Go to www.helpmyassignment.co.uk for more info



Helpmyassignment



5 Unitary and normal operators

Example 5.1 An operator $T \in B(H)$ is called unitary if it is isometric and surjective. Show that the following conditions are equivalent for an operator $T \in B(H)$,

- (a) T is unitary.
- (b) T is bijective and $T^{-1} = T^*$.
- (c) $T^* = TT^* = I$.
- (d) T and T^* are isometric.
- (e) T is isometric and T^* is injective.
- (f) T^* is unitary.

(a) \Rightarrow (b). Assume that T is unitary, thus $T(H) = H$, and $\|Tx\| = \|x\|$ for $x \in H$. Clearly, $Tx = 0$ implies that $x = 0$, so T is injective, and T^{-1} exists and is continuous with $\|T\|^{-1} = 1$. (Sketch of proof: Put $y = Tx$, etc.) From $D(T^{-1}) = T(H) = H$, we even get that $T^{-1} \in B(H)$, and we conclude that T is bijective.

Then it follows from EXAMPLE 4.1, (2) that $T^*T = I$, and from the definition of T^{-1} we get $T^{-1}T = I$. Hence,

$$0 = (T^* - T^{-1})T, \quad \text{thus} \quad (T^* - T^{-1})T(H) = \{0\}.$$

From $T(H) = H$ follows that $T^* - T^{-1}$ is identically 0 on all of H , thus $T^* = T^{-1}$.

(b) \Rightarrow (c). Assume that T is bijective and that $T^{-1} = T^*$. Then

$$T^*T = T^{-1}T = I \quad \text{and} \quad TT^* = TT^{-1} = I.$$

(c) \Rightarrow (d). Let $T^*T = TT^* = I$. It follows from EXAMPLE 4.1, (2) that T is isometric. Then we conclude from

$$(TT^*)^* = (T^*)^*T^* = I^* = I,$$

that T^* is also isometric by EXAMPLE 4.1, (2).

(d) \Rightarrow (e). If T and T^* are isometric, then T^* is in particular injective.

(e) \Rightarrow (a). Assume that T is isometric and that T^* is injective. We shall prove (a), so it only remains to prove that $T(H) = H$.

Because $T(H)$ is closed, it suffices to prove that if

$$(Ty, x) = 0 \quad \text{for all } y \in H,$$

then $x = 0$. We have

$$0 = (Ty, x) = (y, T^*x) \quad \text{for all } y \in H.$$

When we in particular choose $y = T^*x$, then

$$(T^*x, T^*x) = \|T^*x\|^2 = 0, \quad \text{thus} \quad T^*x = 0.$$

Now, T^* is injective, so $x = 0$.

Summing up we have proved that (a)–(e) are equivalent. We shall only prove that we can add (f) to this family of equivalent conditions.

(a) \wedge (d) \Rightarrow (f). If T is unitary, then T^* and $T^{**} = T$ are isometric, so T^* is unitary by (d).

(f) \wedge (d) \Rightarrow (a). If T^* is unitary, then T^* and $T^{**} = T$ are isometric, and T is unitary by (d).

Example 5.2 Let (e_k) denote an orthonormal basis in a Hilbert space H and let $T \in B(H)$ be given by

$$T \left(\sum_{k=1}^{\infty} a_k e_k \right) = \sum_{k=1}^{\infty} \lambda_k a_k e_k.$$

Show that T is unitary if and only if $|\lambda_k| = 1$ for all k .

We conclude from

$$\|Tx\|^2 = \left\| \sum_{k=1}^{\infty} \lambda_k x_k e_k \right\|^2 = \sum_{k=1}^{\infty} |\lambda_k|^2 |x_k|^2,$$

that if $|\lambda_k| = 1$ for all k , then $\|Tx\| = \|x\|$, hence T is isometric.

If there exists a k , such that $|\lambda_k| \neq 1$, then $\|Te_k\| = |\lambda_k| \neq 1 = \|e_k\|$, and T is not isometric.

We have proved that T is isometric, if and only if $|\lambda_k| = 1$ for all $k \in \mathbb{N}$. We shall only prove that if $|\lambda_k| = 1$ for all $k \in \mathbb{N}$, then $T(H) = H$, because this implies by EXAMPLE 5.1 that T is unitary.

Let $y \in H$, i.e.

$$y = \sum_{k=1}^{\infty} y_k e_k \quad \text{and} \quad \sum_{k=1}^{\infty} |y_k|^2 < \infty.$$

If there exists an $x \in H$, such that $Tx = y$, then

$$\sum_{k=1}^{\infty} \lambda_k x_k e_k = \sum_{k=1}^{\infty} y_k e_k \quad \text{and} \quad \sum_{k=1}^{\infty} |x_k|^2 < \infty.$$

It is seen by the identification that since $\lambda_k \cdot \overline{\lambda_k} = |\lambda_k|^2 = 1$, we have only the possibility that $\lambda_k x_k = y_k$, thus

$$x_k = \frac{y_k}{\lambda_k} = \overline{\lambda_k} y_k.$$

We shall only prove that the candidate

$$x = \sum_{k=1}^{\infty} \overline{\lambda_k} y_k e_k$$

belongs to H . This is trivial, because

$$\sum_{k=1}^{\infty} |x_k|^2 = \sum_{k=1}^{\infty} |\overline{\lambda_k}|^2 |y_k|^2 = \sum_{k=1}^{\infty} |y_k|^2 = \|y\|^2 < \infty,$$

so $x \in H$, and $Tx = y$. This proves that $T(H) = H$, and it then follows from EXAMPLE 5.1 that T is unitary.

Example 5.3 Let $T \in B(H)$ be unitary. Show that

$$\sigma(T) \subset \{z \in \mathbb{C} \mid |z| = 1\}.$$

Let $|\lambda| \neq 1$. Because T is unitary, we get in particular that $\|T\| = \|x\|$, hence

$$\|Tx - \lambda x\| \geq |\|Tx\| - \|\lambda x\|| = |1 - |\lambda|| \cdot \|x\|.$$

It follows that $(T - \lambda I)^{-1}$ exists for every $\lambda \in \mathbb{C}$, for which $|\lambda| \neq 1$. We shall finish the proof by showing that $(T - \lambda I)^{-1}$ is densely defined in H , because then

$$\varrho(T) \supseteq \mathbb{C} \setminus \{z \in \mathbb{C} \mid |z| = 1\} \quad \text{and} \quad \sigma(T) \subseteq \{z \in \mathbb{C} \mid |z| = 1\}.$$

Assume that $(T - \lambda I)^{-1}$ is not densely defined for some $\lambda \in \mathbb{C}$. Then there exists an $y \neq 0$, such that

$$y \perp (T - \lambda I)D(T - \lambda I) = (T - \lambda I)(H),$$

thus

$$0 = ((T - \lambda I)x, y) = (x, (T^* - \bar{\lambda}I)y) = (x, 0) \quad \text{for all } x \in H.$$

We conclude that $T^*y - \bar{\lambda}y = 0$, hence $\bar{\lambda}$ is even an eigenvalue for $T^* = T^{-1}$.

By EXAMPLE 5.1, T^* is also unitary, thus $|\bar{\lambda}| = 1$, and hence also $|\lambda| = 1$. Then it follows by contraposition that if $|\lambda| \neq 1$, then $(T - \lambda I)^{-1}$ is densely defined. Then

$$\varrho(T) \supseteq \mathbb{C} \setminus \{z \in \mathbb{C} \mid |z| = 1\} \quad \text{and} \quad \sigma(T) \subseteq \{z \in \mathbb{C} \mid |z| = 1\}.$$

Example 5.4 An operator $T \in B(H)$ is normal if

$$TT^* = T^*T.$$

Show that T is normal if and only if $\|T^*x\| = \|Tx\|$ for all $x \in H$.

If $T \in B(H)$ is normal, i.e. $T^*T = TT^*$, then

$$\|T^*x\|^2 = (T^*x, T^*x) = (TT^*x, x) = (T^*Tx, x) = (Tx, Tx) = \|Tx\|^2,$$

and we conclude that $\|T^*x\| = \|Tx\|$ for all $x \in H$.

Assume conversely that $\|T^*x\| = \|Tx\|$ for all $x \in H$. Then

$$0 = \|T^*x\|^2 - \|Tx\|^2 = (T^*x, T^*x) - (Tx, Tx) = (TT^*x, x) - (T^*Tx, x) = ((TT^* - T^*T)x, x).$$

The space H is complex. so it follows that $TT^* - T^*T = 0$, hence $T^*T = TT^*$ as required.

Example 5.5 Let $T \in B(H)$ be normal. Show that

$$\|(T - \lambda I)x\| = \|(T^* - \bar{\lambda} I)x\|$$

for all $x \in H$. Show that $\sigma_r(T)$ is empty.

If T is normal, then $T^*T = TT^*$, and we get

$$\begin{aligned} \|(T - \lambda I)x\|^2 &= ((T - \lambda I)x, (T - \lambda I)x) \\ &= (Tx, Tx) - \lambda(x, Tx) - \bar{\lambda}(Tx, x) + |\lambda|^2(x, x) \\ &= (T^*Tx, x) - \lambda(T^*x, x) - \bar{\lambda}(x, T^*x) + |\lambda|^2(x, x) \\ &= (TT^*x, x) - (T^*x, \bar{\lambda}x) - (\bar{\lambda}x, T^*x) + (\bar{\lambda}x, \bar{\lambda}x) \\ &= (T^*x, T^*x) - (T^*x, \bar{\lambda}x) - (\bar{\lambda}x, T^*x) + (\bar{\lambda}x, \bar{\lambda}x) \\ &= ((T^* - \bar{\lambda}I)x, (T^* - \bar{\lambda}I)x) = \|(T^* - \bar{\lambda}I)x\|^2, \end{aligned}$$

and the first claim is proved.

It follows that λ is an eigenvalue for T (of eigenvector x), if and only if $\bar{\lambda}$ is an eigenvalue for T^* (the same eigenvector x), thus

$$\sigma_p(T^*) = \overline{\sigma_p(T)}.$$

On the other hand, $\sigma_r(T) \subseteq \overline{\sigma_p(T^*)} = \sigma_p(T)$, and because $\sigma_r(T)$ and $\sigma_p(T)$ are disjoint, we must have $\sigma_r(T) = \emptyset$.



Brain power

By 2020, wind could provide one-tenth of our planet's electricity needs. Already today, SKF's innovative know-how is crucial to running a large proportion of the world's wind turbines.

Up to 25 % of the generating costs relate to maintenance. These can be reduced dramatically thanks to our systems for on-line condition monitoring and automatic lubrication. We help make it more economical to create cleaner, cheaper energy out of thin air.

By sharing our experience, expertise, and creativity, industries can boost performance beyond expectations. Therefore we need the best employees who can meet this challenge!

The Power of Knowledge Engineering

Plug into The Power of Knowledge Engineering.
Visit us at www.skf.com/knowledge

SKF



Example 5.6 Let $H = L^2([0, 1])$ and consider the operator

$$Tf(x) = \sqrt{3}x f(x^3).$$

- 1) Show that $T \in B(H)$ and find $\|T\|$.
- 2) Show that T^{-1} exists and that $T^{-1} \in B(H)$.
Determine $T^{-1}g(y)$ for $g \in H$, and find $\|T^{-1}\|$.
- 3) Show that $\sigma(T) \subset \{\lambda \in \mathbb{C} \mid |\lambda| = \|T\|\}$.

- 1) The operator T is obviously linear.

Then by the change of variable $y = x^3$,

$$\|Tf\|_2^2 = \int_0^1 |Tf(x)|^2 dx = \int_0^1 3x^2 |f(x^3)|^2 dx = \int_0^1 |f(y)|^2 dy = \|f\|_2^2,$$

hence T is isometric ($\|Tf\|_2 = \|f\|_2$), thus $T \in B(H)$ and $\|T\| = 1$.

- 2) We shall prove that the equation

$$Tf(x) = g(x), \quad g \in L^2([0, 1]),$$

always has a uniquely determined solution, thus $T^{-1} : H \rightarrow H$. It follows by the definition that we shall solve

$$Tf(x) = \sqrt{3}x f(x^3) = g(x).$$

Utilizing the monotone change of variable $x = \sqrt[3]{y}$, we get

$$f(y) = \frac{1}{\sqrt{3}} \cdot \frac{1}{x} g(x) = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt[3]{y}} \cdot g(\sqrt[3]{y}) = T^{-1}g(y),$$

hence

$$T^{-1}g(x) = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt[3]{x}} g(\sqrt[3]{x}), \quad g \in H \in L^2([0, 1]).$$

We get from the computation in (1) that $Tf = g$ and $f = T^{-1}g$ that

$$\|Tf\|_2 = \|g\|_2 = \|f\|_2 = \|T^{-1}g\|_2, \quad T^{-1} \in B(H),$$

and T^{-1} is also isometric, $\|T^{-1}g\|_2 = \|g\|_2$, and $\|T^{-1}\| = 1$.

We say that T is *unitary*, cf. EXAMPLE 5.1.

- 3) This has already been proved in EXAMPLE 5.3. However, let us do it again. If $|\lambda| > 1$, then

$$T - \lambda I = -\lambda \left(I - \frac{1}{\lambda} T \right), \quad \text{where } \left\| \frac{1}{\lambda} T \right\| = \frac{1}{|\lambda|} < 1,$$

thus $(T - \lambda I)^{-1} \in B(H)$, and $(T - \lambda I)^{-1}$ is given by the Neumann series

$$(T - \lambda I)^{-1} = -\frac{1}{\lambda} \sum_{n=0}^{+\infty} \frac{1}{\lambda^n} T^n.$$

Then let $|\lambda| < 1$. From $T^{-1} \in B(H)$ follows that $T - \lambda I = T(I - \lambda T^{-1})$. From $\|\lambda T^{-1}\| = |\lambda| < 1$ follows by a Neumann series that

$$(T - \lambda I)^{-1} = (I - \lambda T^{-1})^{-1} T^{-1} = \left(\sum_{n=0}^{+\infty} \lambda^n (T^{-1})^n \right) T^{-1} = \sum_{n=0}^{+\infty} \lambda^n (T^{-1})^{n+1},$$

hence $(T - \lambda I)^{-1} \in B(H)$, and we conclude that

$$\rho(T) \supseteq \{\lambda \in \mathbb{C} \mid |\lambda| \neq 1\} \quad \text{and} \quad \sigma(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}.$$

TURN TO THE EXPERTS FOR SUBSCRIPTION CONSULTANCY

Subscribe is one of the leading companies in Europe when it comes to innovation and business development within subscription businesses.

We innovate new subscription business models or improve existing ones. We do business reviews of existing subscription businesses and we develop acquisition and retention strategies.

Learn more at [linkedin.com/company/subscribe](https://www.linkedin.com/company/subscribe) or contact Managing Director Morten Suhr Hansen at mha@subscribe.dk

SUBSCRIBE - to the future



6 Positive operators and projections

Example 6.1 An operator $T \in B(H)$ is positive if

$$(Tx, x) \geq 0 \quad \text{for all } x \in H,$$

and we write $T \geq 0$.

Prove the following:

- 1) $T \geq 0$ implies that T is self adjoint.
- 2) If $S, T \geq 0, \alpha \geq 0$, then $S + \alpha T \geq 0$.
- 3) If $T \geq 0$ and $S \in B(H)$, then $S^*TS \geq 0$.
- 4) If $T \in B(H)$ then $T^*T \geq 0$,
- 5) If T is an orthogonal projection then $T \geq 0$.

- 1) Assume that $T \in B(H)$ is positive, i.e. $(Tx, x) \geq 0$ for every $x \in H$. Then

$$(T^*x, x) = (x, Tx) = \overline{(Tx, x)} = (Tx, x) \geq 0,$$

and T^* is also positive, and

$$((T^* - T)x, x) = 0 \quad \text{for every } x \in H.$$

Then assume that the vector space is complex. Then it follows that $T^* - T = 0$, i.e. $T^* = T$, and we have proved that T is self adjoint.

- 2) This is trivial: For every $x \in H$,

$$((S + \alpha T)x, x) = (Sx, x) + \alpha(Tx, x) \geq 0 + \alpha \cdot 0 = 0.$$

- 3) It follows from $Sx \in H$ for every $x \in H$ that

$$(S^*TSx, x) = (T(Sx), Sx) \geq 0.$$

- 4) This is again trivial. In fact, for every $x \in H$,

$$(T^*Tx, x) = (Tx, Tx) = \|Tx\|^2 \geq 0.$$

- 5) Let T denote an orthogonal projection. Then

$$T^* = T \quad \text{and} \quad T^2 = T.$$

It follows from (4) that

$$T^*T = TT = T^2 = T$$

is positive, hence $T \geq 0$.

Example 6.2 Let P_M and P_N denote the orthogonal projections of the closed subspaces M and N of a Hilbert space H . Show that $M \subset N$ implies that $P_M \leq P_N$.

If $M \subseteq N$, then

$$H = N \oplus N^\perp = M \oplus (M^\perp \cap N) \oplus N^\perp,$$

which means that every element $x \in H$ has a unique decomposition

$$x = x_M + x_N + x^\perp, \quad \text{where } x_M \in M, \quad x_N \in M^\perp \cap N, \quad x^\perp \in N^\perp.$$

Then

$$P_M x = P_M (x_M + x_N + x^\perp) = x_M \quad \text{and} \quad P_N x = P_N (x_M + x_N + x^\perp) = x_M + x_N.$$

It follows that

$$\begin{aligned} ((P_N - P_M)x, x) &= (x_M + x_N - x_M, x_M + x_N + x^\perp) = (x_N, x_M + x_N + x^\perp) \\ &= (x_N, x_M) + (x_N, x_N) + (x_N, x^\perp) \\ &= 0 + \|x_N\|^2 + 0 = \|x_N\|^2 \geq 0, \end{aligned}$$

hence $P_N - P_M \geq 0$, and whence $P_M \leq P_N$.



What do you want to do?

No matter what you want out of your future career, an employer with a broad range of operations in a load of countries will always be the ticket. Working within the Volvo Group means more than 100,000 friends and colleagues in more than 185 countries all over the world. We offer graduates great career opportunities – check out the Career section at our web site www.volvogroup.com. We look forward to getting to know you!

VOLVO
AB Volvo (publ)
www.volvogroup.com

VOLVO TRUCKS | RENAULT TRUCKS | MACK TRUCKS | VOLVO BUSES | VOLVO CONSTRUCTION EQUIPMENT | VOLVO PENTA | VOLVO AERO | VOLVO IT
VOLVO FINANCIAL SERVICES | VOLVO 3P | VOLVO POWERTRAIN | VOLVO PARTS | VOLVO TECHNOLOGY | VOLVO LOGISTICS | BUSINESS AREA ASIA



Example 6.3 An operator $T \in B(H)$ is called a contraction

$$\|Tx\| \leq \|x\| \quad \text{for all } x \in H.$$

Show that the following conditions are equivalent for an operator $T \in B(H)$:

- 1) T is a contraction,
- 2) $\|T\| \leq 1$,
- 3) $T^*T \leq I$,
- 4) $TT^* \leq I$,
- 5) T^* is a contraction,
- 6) T^*T is a contraction.

(1) \Rightarrow (2). Let $T \in B(H)$ denote a contraction, thus $\|Tx\| \leq \|x\|$ for all $x \in H$. Then

$$\|T\| = \sup\{\|Tx\| \mid \|x\| \leq 1\} \leq \sup\{\|x\| \mid \|x\| \leq 1\} = 1,$$

and we have proved (2).

(2) \Rightarrow (3). Assume that $\|T\| \leq 1$. Then

$$\begin{aligned} (9) \quad ((I - T^*T)x, x) &= (x, x) - (T^*Tx, x) = \|x\|^2 - (Tx, Tx) \\ &= \|x\|^2 - \|Tx\|^2 \geq \|x\|^2 - 1 \cdot \|x\|^2 = 0, \end{aligned}$$

and we have proved that $I - T^*T \geq 0$, hence $T^*T \leq I$, and we have proved that (3).

(3) \Rightarrow (1). Assume that $T^*T \leq I$. By repeating (9) we see that $\|x\|^2 - \|Tx\|^2 \geq 0$, thus $\|Tx\| \leq \|x\|$, and we have proved (1).

It follows from the above that the former three conditions (1)–(3) are equivalent.

(1) \Leftrightarrow (5). If T is a contraction, then by (2), $\|T^*\| = \|T\| \leq 1$, and we infer that T^* is a contraction.

If conversely T^* is a contraction, then $T^{**} = T$ is contraction.

We have proved that the conditions (1)–(3) and (5) are equivalent.

(1) \Leftrightarrow (4). If (1) is fulfilled, then also (3) and (5), and it follows that (5) is equivalent with

$$(T^*)^*T^* = TT^* \leq I,$$

and (1)–(5) are all equivalent.

(1) \Rightarrow (6). If T is a contraction, then we have proved that $\|T^*\| = \|T\| \leq 1$, and it follows that

$$\|TT^*\| \leq \|T^*\| \cdot \|T\| \leq 1^1 = 1,$$

thus T^*T is a contraction by (2), and we have proved (6).

(6) \Rightarrow (1). If T^*T is a contraction, then

$$\|T^*Tx\| \leq \|x\| \quad \text{for all } x \in H,$$

hence by the Cauchy-Schwarz inequality

$$\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) \leq \|T^*Tx\| \cdot \|x\| \leq \|x\|^2.$$

We infer that $\|Tx\| \leq \|x\|$ for every $x \in H$, and T is by the definition a contraction.

We have proved that the six conditions (1)–(6) are equivalent.



gaiteye[®]
Challenge the way we run

**EXPERIENCE THE POWER OF
FULL ENGAGEMENT...**

.....

**RUN FASTER.
RUN LONGER..
RUN EASIER...**

READ MORE & PRE-ORDER TODAY
WWW.GAITEYE.COM

7 Compact operators

Example 7.1 Let S and T be linear and bounded operators and assume that S is compact. Show that ST and TS are compact.

According to the definition, $S \in B(H)$ is compact, if $\overline{S(X)}$ is compact for every bounded set $X \subset H$.

Consider $S, T \in B(H)$, and let S be compact. If X is bounded, then $T(X)$ is also bounded. In fact, if

$$M = \sup\{\|x\| \mid \|x\| \in X\},$$

then

$$\|Tx\| \leq \|T\| \cdot \|x\| \leq \|T\| \cdot M \quad \text{for all } x \in X.$$

It follows that $\overline{ST(X)} = \overline{S(T(X))}$ is compact, hence the composite operator ST is compact.

Since T is continuous, it follows that $\overline{TS(X)} \subseteq \overline{T(\overline{S(X)})}$. Now, $\overline{S(X)}$ is compact for every bounded set X , and T is continuous, hence $\overline{T(\overline{S(X)})}$ is also compact. Now every closed subset of a compact set is compact, hence $\overline{TS(X)}$ is compact, and the composite operator TS is compact.

Example 7.2 Let S and T be compact operators in $B(H)$, and let $\alpha \in \mathbb{C}$. Show that $S + \alpha T$ is compact.

Denote by X a bounded set. Then $\overline{S(X)}$ and $\overline{T(X)}$ are both compact sets, because S and T are compact operators. Choose any sequence $(x_n) \subseteq (S + \alpha T)(X)$. Then we can find other sequences $(y_n) \subseteq \overline{S(X)}$ and $(z_n) \subseteq \overline{T(X)}$, such that

$$x_n = Sy_n + \alpha Tz_n.$$

The set $\overline{S(X)}$ is compact, hence there exists a subsequence (y_{n_j}) , such that $Sy_{n_j} \rightarrow y$, and we obtain the subsequence (x_{n_j}) by

$$x_{n_j} = Sy_{n_j} + \alpha Tz_{n_j}.$$

If $\alpha = 0$, there is nothing to prove. If $\alpha \neq 0$, it follows by a rearrangement that

$$Tz_{n_j} = \frac{1}{\alpha} x_{n_j} - \frac{1}{\alpha} Sy_{n_j} \in T(X).$$

The set $\overline{T(X)}$ is compact, so there is a subsequence (n_{j_k}) , such that $Tz_{n_{j_k}} \rightarrow z$. This implies that the subsequence $(x_{n_{j_k}})$ is convergent,

$$x_{n_{j_k}} = Sy_{n_{j_k}} + \alpha Tz_{n_{j_k}} \rightarrow y + \alpha z.$$

We have proved that any sequence (x_n) from $(S + \alpha T)(X)$ has a convergent subsequence, hence $\overline{(S + \alpha T)(X)}$ is compact. Furthermore, X is any bounded set in H , so we infer that $S + \alpha T$ is compact.

Remark 7.1 This result shows that the set of compact operators in $B(H)$ is a subspace of $B(H)$. Then it follows from the result of EXAMPLE 7.1 that the subspace of compact operators is even a so-called two-sided ideal in $B(H)$ with the composition of operators as multiplication. \diamond

Example 7.3 Let (e_k) denote an orthonormal basis in a Hilbert space H , and define the operator T by

$$T \left(\sum_{k=1}^{\infty} a_k e_k \right) = \sum_{k=2}^{\infty} \frac{1}{k} a_k e_{k-1}.$$

Show that T is compact and find T^* . Find $\sigma_p(T)$ and $\sigma_p(T^*)$.

Define T_n , $n \geq 2$, by

$$T_n \left(\sum_{k=1}^{+\infty} a_k e_k \right) = \sum_{k=2}^n \frac{1}{k} a_k e_{k-1}.$$

Then T_n is of finite rank, thus also compact. It follows from

$$(T - T_n) \left(\sum_{k=1}^{\infty} a_k e_k \right) = \sum_{k=n+1}^{+\infty} \frac{1}{k} a_k e_{k-1},$$

that

$$\left\| (T - T_n) \left(\sum_{k=1}^{+\infty} a_k e_k \right) \right\|^2 = \sum_{k=n+1}^{+\infty} \frac{1}{k^2} |a_k|^2 \leq \frac{1}{(n+1)^2} \sum_{k=n+1}^{+\infty} |a_k|^2 \leq \frac{1}{(n+1)^2} \left\| \sum_{k=1}^{+\infty} a_k e_k \right\|^2.$$

thus $\|(T - T_n)x\| \leq \frac{1}{n+1} \|x\|$ for all $x \in H$, and we have proved that $\|T - T_n\| \leq \frac{1}{n+1}$, hence $\|T - T_n\| \rightarrow 0$ for $n \rightarrow +\infty$. It follows that T is compact.

Then we check when $T_\lambda = T - \lambda I$ is injective. It follows by recursion from

$$T_\lambda \left(\sum_{k=1}^{+\infty} a_k e_k \right) = \sum_{k=1}^{+\infty} \left\{ \frac{1}{k+1} a_{k+1} - \lambda a_k \right\} e_k = 0,$$

that

$$a_{k+1} = (k+1)\lambda a_k = \cdots = (k+1)!\lambda^k a_1, \quad k \in \mathbb{N}.$$

If $\lambda \neq 0$, then

$$\sum_{n=1}^{+\infty} |a_k|^2 = \sum_{k=1}^{+\infty} |a_1|^2 (k!|\lambda|^{k-1})^2.$$

Now, $(k!|\lambda|^{k-1})^2 \rightarrow +\infty$ for $k \rightarrow +\infty$, thus this series is only convergent, if $a_1 = 0$, and hence all $a_k = 0$. Therefore, when $\lambda \neq 0$, then $T_\lambda x = 0$ implies that $x = 0$, thus T_λ is injective for $\lambda \neq 0$. In

particular we get for the point spectrum $\sigma_p(T) \subseteq \{0\}$. On the other hand $Te_1 = 0 = 0 \cdot e_1$, thus 0 is an eigenvalue, and $\sigma_p(T) = \{0\}$.

Then we search the adjoint operator T^* . Let

$$x = \sum_{k=1}^{+\infty} x_k e_k \quad \text{og} \quad y = \sum_{k=1}^{+\infty} y_k e_k.$$

Then

$$(Tx, y) = \left(\sum_{k=2}^{+\infty} \frac{1}{k} x_k e_{k-1}, \sum_{n=1}^{+\infty} y_n e_n \right) = \sum_{k=2}^{+\infty} \frac{1}{k} x_k \cdot \overline{y_{k-1}} = \left(\sum_{k=2(1)}^{+\infty} x_k e_k, \sum_{n=2}^{+\infty} \frac{1}{n} y_{n-1} e_n \right) = (x, T^*y),$$

from which

$$T^* \left(\sum_{n=1}^{+\infty} y_n e_n \right) = \sum_{n=2}^{+\infty} \frac{1}{n} y_{n-1} e_n = \sum_{n=1}^{+\infty} \frac{1}{n+1} y_n e_{n+1}.$$

Assume that $\mu \in \sigma_p(T^*)$ is an eigenvalue for T^* . Then there is a $y = \sum_{n=1}^{+\infty} y_n e_n \neq 0$, for which

$$(10) \quad (T^* - \mu I) \left(\sum_{n=1}^{+\infty} y_n e_n \right) = -\mu y_1 e_1 + \sum_{n=2}^{+\infty} \left\{ \frac{1}{n} y_{n-1} - \mu y_n \right\} e_n = 0.$$

Here we derive the conditions

$$\mu y_1 = 0 \quad \text{and} \quad \frac{1}{n} y_{n-1} = \mu y_n, \quad n \geq 2.$$

If $\mu = 0$, then it follows immediately from (10) that $y = 0$, thus $0 \notin \sigma_p(T^*)$.

If $\mu \neq 0$, then

$$y_1 = 0 \quad \text{and} \quad y_n = \frac{1}{n\mu} y_{n-1}, \quad n \geq 2,$$

and it follows by either induction or by recursion that $y = 0$, contradiction the assumption. We therefore conclude that $\sigma_p(T^*) = \emptyset$. This implies that the residual spectrum for T is empty, $\sigma_r(T) = \emptyset$.

Remark 7.2 It is also possible here to find $\sigma(T)$ and $\sigma(T^*)$, though this is not an easy task. For completeness the derivations are given in the following.

It follows immediately from the expressions of T and T^* that

$$\|T\| = \|T^*\| = \frac{1}{2},$$

hence

$$\sigma(T) \subseteq \left\{ z \in \mathbb{C} \mid |z| \leq \frac{1}{2} \right\} \quad \text{and} \quad \sigma(T^*) \subseteq \left\{ z \in \mathbb{C} \mid |z| \leq \frac{1}{2} \right\}.$$

It follows from the expression of T^* ,

$$T^* \left(\sum_{n=1}^{+\infty} y_n e_n \right) = \sum_{n=1}^{+\infty} \frac{1}{n+1} y_n e_{n+1},$$

that T^* is injective, so $(T^*)^{-1}$ exists. Then from $e_1 \perp T^* D(T^*)$ follows that $(T^*)^{-1}$ is not densely defined. This means that $0 \in \sigma_r(T^*)$.

It follows from $T^* \in B(H)$ and $T \in B(H)$, that $T^{**} = \overline{T} = T$. We have already proved that

$$\sigma_p(T) = \sigma_p(T^{**}) = \{0\}.$$

so it follows by contraposition that $\sigma_r(T^*) = \{0\}$. We have proved

$$\sigma_p(T) = \{0\}, \quad \sigma_r(T) = \emptyset, \quad \sigma_p(T^*) = \emptyset, \quad \sigma_r(T^*) = \{0\}.$$

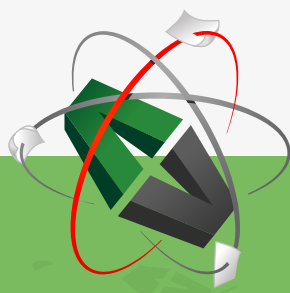
Then we *claim* that

$$(11) \quad \sigma_c(T) = \sigma_c(T^*) = \emptyset.$$

First notice that if (11) holds, then it easily follows that

$$\sigma(T) = \sigma(T^*) = \{0\} \quad \text{and} \quad \varrho(T) = \varrho(T^*) = \mathbb{C} \setminus \{0\}.$$

This e-book
is made with
SetaPDF



PDF components for PHP developers

www.setasign.com



In order to prove (11) we shall need the following theorem:

Theorem 7.1 *Assume that $T \in B(H)$ is compact, and choose $\lambda \neq 0$. If $T_\lambda = T - \lambda I$ is injective, then the range $(T - \lambda I)(H)$ is closed.*

First assume that Theorem 7.1 holds. Let $\lambda \in \sigma_c(T)$. Then $\sigma_p(T) = \{0\}$, and because $\sigma_p(T)$ and $\sigma_c(T)$ are disjoint, we must have $\lambda \neq 0$. Then it follows from the definition of $\sigma_c(T)$ that $T - \lambda I$ is injective and that $(T - \lambda I)(H)$ is dense in H . Theorem 7.1 shows that $(T - \lambda I)(H)$ is closed, hence $(T - \lambda I)(H) = H$, and whence $(T - \lambda I)^{-1}$ is bounded by the theorem of bounded inverse. This means that $\lambda \in \rho(T)$, contradicting the assumption that $\lambda \in \sigma_c(T)$. We conclude that $\sigma_c(T) = \emptyset$.

The proof of $\sigma_c(T^*) = \emptyset$ is apart from a very small modification exactly the same as that above. This modification is that we this time shall use that because $\sigma_r(T^*) = \{0\}$, we must have $\lambda \neq 0$ for any possible $\lambda \in \sigma_c(T)$. \diamond

PROOF OF THEOREM 7.1. Let $y = \lim_{n \rightarrow +\infty} y_n$, where $y_n = (T - \lambda I)x_n$.

- 1) Assume that (x_n) has a bounded subsequence. Because T is compact, there must exist another subsequence (x_{n_i}) such that the image sequence (Tx_{n_i}) is convergent. From follows

$$x_{n_i} = \frac{1}{\lambda} (Tx_{n_i} - y_{n_i}),$$

that $x_{n_i} \rightarrow x$ and $y = (T - \lambda I)x$, hence $y \in (T - \lambda I)(H)$, and we have proved that $(T - \lambda I)(H)$ is closed in this case.

- 2) Then assume that (x_n) does *not* have any bounded subsequence. Then $\|x_n\| \rightarrow +\infty$. We define

$$z_n = \frac{x_n}{\|x_n\|}, \quad \|z_n\| = 1,$$

thus $(T - \lambda I)z_n \rightarrow 0$. There is a subsequence (z_{n_i}) , such that (Tz_{n_i}) is convergent. However, $\left(z_{n_i} - \frac{1}{\lambda} Tz_{n_i}\right)$ is convergent, so $z_{n_i} \rightarrow z$, where $\|z\| = 1$ and $(T - \lambda I)z = 0$, contradicting that $T - \lambda I$ is injective. Hence the sequence (x_n) must have a bounded subsequence, and we are back in case (1) above, and the claim is proved. \square

Example 7.4 *Let T be a bounded operator on a Hilbert space H . Show that:*

- 1) *If T is compact, then T^* is also compact.*
- 2) *If T^*T is compact, then T is compact.*
- 3) *If T is self adjoint and T^n is compact for some n , then T is compact.*

- 1) Assume that T is compact. Let X be a bounded set, and let $(y_n) \subseteq T^*(X)$ be any sequence, thus there exists a sequence $(x_n) \subseteq X$, such that $y_n = T^*x_n$.

We shall prove that there exists a subsequence (x_{n_j}) , such that $(T^*x_{n_j})$ is convergent. This is done INDIRECTLY. Assume that T^* is not compact. Then there exists a bounded sequence (φ_n) ,

which converges weakly towards φ , such that $(T^* \varphi_n)$ does not converge strongly towards $T^* \varphi$, thus there exist a subsequence (f_n) and an $\eta > 0$, such that

$$\|T^* f_n - T^* \varphi\| > \eta \quad \text{for all } n \in \mathbb{N},$$

hence

$$\eta \leq \|T^* f_n - T^* \varphi\| \leq \|T^*\| \cdot \|f_n - \varphi\| \quad (< M),$$

and whence

$$\|f_n - \varphi\| \geq \frac{\eta}{\|T^*\|}.$$

Now, $(T^* f_n - T^* \varphi)$ is bounded and it converges weakly towards 0, hence $TT^* f_n$ converges strongly towards $TT^* \varphi$, i.e.

$$\eta^2 \leq \|T^* (f_n - \varphi)\|^2 = (TT^* (f_n - \varphi), f_n - \varphi) \leq \|TT^* (f_n - \varphi)\| \cdot \|f_n - \varphi\| \rightarrow 0$$

for $n \rightarrow +\infty$. This gives a contradiction, $\eta > 0$ being fixed, and our assumption that T^* is not compact, must be wrong. We therefore conclude that T^* is compact as claimed above.

2) It follows trivially from EXAMPLE 7.1 that if T is compact, then T^*T is also compact.

Assume that T^*T is compact, and also *assume* (thus an INDIRECT proof) that T is *not* compact. Then there exists a bounded sequence (φ_n) , which converges weakly towards φ , such that (cf. (1))

$$\|T\varphi_n - T\varphi\| \geq \eta \quad \text{for all } n \in \mathbb{N}.$$

Because $(\varphi_n - \varphi)$ is bounded and weakly convergent, it follows that $(T^*T\varphi_n - T^*T\varphi)$ is strongly convergent, and we get

$$\begin{aligned} \eta^2 &\leq \|T(\varphi_n - \varphi)\|^2 = (T(\varphi_n - \varphi), T(\varphi_n - \varphi)) \\ &= (T^*T(\varphi_n - \varphi), \varphi_n - \varphi) \leq \|T^*T(\varphi_n - \varphi)\| \cdot \|\varphi_n - \varphi\| \\ &\leq \|T^*T(\varphi_n - \varphi)\| \cdot M \rightarrow 0 \quad \text{for } n \rightarrow +\infty, \end{aligned}$$

which is a contradiction, because $\eta > 0$ is a given constant. We therefore conclude that T is compact.

3) Finally, assume that T is self adjoint, $T^* = T$, and that T^n is compact for some given $n \in \mathbb{N}$.

If $n = 2m$ is even, then it follows from T being self adjoint that

$$T^n = T^{2m} = (T^m)^* (T^m)$$

is compact. Then we infer from (2) that T^m is compact, where $m = \frac{n}{2} < n$.

If instead $n = 2m - 1$ is odd, then

$$T^{n+1}T^nT = T^{2m} = (T^m)^* (T^m)$$

is compact, cf. EXAMPLE 7.1, and we infer as above that T^m is compact, where $m = \frac{n+1}{2} < n$, when $n > 1$.

By recursion we get after a finite number of steps that T^3 is compact, and hence that $T^2 = T \star T$ is also compact, which by (2) implies that T is compact.

Example 7.5 Let $T : \ell^2 \rightarrow \ell^2$ be the linear operator given by

$$T(x_1, x_2, \dots, x_{2n-1}, x_{2n}, \dots) = \left(x_2, x_1, \frac{1}{2}x_4, \frac{1}{2}x_3, \dots, \frac{1}{n}x_{2n}, \frac{1}{n}x_{2n-1}, \dots \right).$$

- 1) Find $\|T\|$.
- 2) Find T^* .
- 3) Prove that T is compact.
- 4) Find the spectrum and resolvent set for T , and determine a set of basis vectors for the eigenspace associated to $\lambda \in \sigma_p(T)$.

1) In general,

$$\|Tx\|^2 = \sum_{n=1}^{+\infty} \frac{1}{n^2} \left\{ |x_{2n}|^2 + |x_{2n-1}|^2 \right\} \leq \sum_{n=1}^{+\infty} |x_n|^2 = \|x\|^2,$$

thus $\|T\| \leq 1$.

On the other hand,

$$\|Te_1\| = \|e_2\| = 1 = \|e_1\| \quad \text{and} \quad \|Te_2\| = \|e_1\| = 1 = \|e_2\|,$$

so $\|T\| = 1$, and $T \in B(\ell^2)$.

2) Because $T \in B(\ell^2)$, we also have $T^* \in B(\ell^2)$, and $\|T^*\| = \|T\|$. Then

$$\begin{aligned} (Tx, y) &= \sum_{n=1}^{+\infty} \left\{ \frac{1}{n} x_{2n} \overline{y_{2n-1}} + \frac{1}{n} x_{2n-1} \overline{y_{2n}} \right\} \\ &= \sum_{n=1}^{+\infty} \left\{ x_{2n-1} \overline{\frac{1}{n} y_{2n}} + x_{2n} \overline{\frac{1}{n} y_{2n-1}} \right\} = (x, T^*y) = (x, Ty), \end{aligned}$$

hence $T = T^*$, and T is self adjoint.

3) We get that T is compact from $T_n \rightarrow T$, where

$$T_n(x_1, x_2, \dots) = \left(x_2, x_1, \frac{1}{2}x_4, \frac{1}{2}x_3, \dots, \frac{1}{n}x_{2n}, \frac{1}{n}x_{2n-1}, 0, 0, \dots \right)$$

is of finite rank, thus compact, and where

$$\|(T - T_n)x\|^2 = \sum_{k=n+1}^{+\infty} \frac{1}{k^2} \left\{ |x_{2k}|^2 + |x_{2k-1}|^2 \right\} \leq \frac{1}{(n+1)!} \|x\|^2,$$

i.e.

$$\|T - T_n\| \leq \frac{1}{n+1} \rightarrow 0 \quad \text{for } n \rightarrow +\infty.$$

4) Because T is self adjoint and compact, we can apply the main theorem, thus

$$\sigma_p(T) = \{\lambda_n \mid n \in \mathbb{N}\}.$$

Now $Tx = 0$ implies that $x = 0$, hence $0 \notin \sigma_p(T)$, which means that $\sigma_c(T) = \{0\}$ and $\sigma_r(T) = \emptyset$, because T is self adjoint.

The eigenvalue problem $Tx = \lambda x$, $\lambda \neq 0$, is now written in coordinates

$$\begin{cases} \frac{1}{n} x_{2n} = \lambda x_{2n-1}, \\ \frac{1}{n} x_{2n-1} = \lambda x_{2n}, \end{cases} \quad \text{i.e.} \quad \begin{cases} -\lambda x_{2n-1} + \frac{1}{n} x_{2n} = 0, \\ \frac{1}{n} x_{2n-1} - \lambda x_{2n} = 0, \end{cases} \quad n \in \mathbb{N},$$

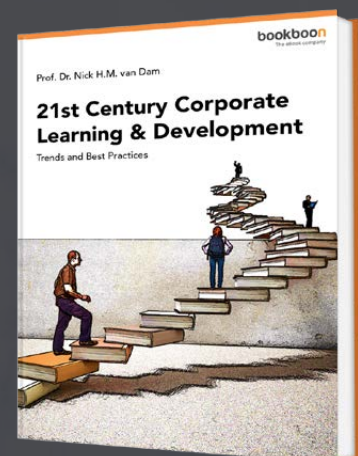
which has non-trivial solutions, if and only if there exists an $n \in \mathbb{N}$, such that

$$\begin{vmatrix} -\lambda & \frac{1}{n} \\ \frac{1}{n} & -\lambda \end{vmatrix} = 0, \quad \text{i.e.} \quad \lambda^2 = \frac{1}{n^2}.$$

Free eBook on Learning & Development

By the Chief Learning Officer of McKinsey

Download Now



We get the eigenvalues $\lambda = \pm \frac{1}{n}$, $n \in \mathbb{N}$, corresponding to e.g. the eigenvectors

$$\begin{cases} e_{2n-1} + e_{2n}, & \lambda_n = \frac{1}{n}, \\ e_{2n-1} - e_{2n}, & \lambda_{-n} = -\frac{1}{n}, \end{cases} \quad n \in \mathbb{N}.$$

We finally get

$$\sigma_p(T) = \left\{ \frac{1}{n} \mid n \in \mathbb{Z} \setminus \{0\} \right\}, \quad \sigma_c(T) = \{0\}, \quad \sigma_r(T) = \emptyset,$$

and

$$\varrho(T) = \mathbb{C} \setminus \left(\{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{Z} \setminus \{0\} \right\} \right).$$

Index

adjoint operator, 28

Cauchy-Schwarz inequality, 53, 56, 70

Closed Graph Theorem, 28, 38

compact operator, 71

complement, 29

contraction, 69

double sided shift operator, 10

isometric operator, 57

kernel, 29, 52

Minkowski's inequality, 35

Neumann series, 18, 27, 28, 65

normal operator, 61, 63

positive operator, 67

potential operator, 50

projection, 22, 59, 67

regular operator, 5

resolvent, 5

self adjoint operator, 42, 75

shift operator, 7

spectrum, 5

translation operator, 5

unitary operator, 61, 65

Weierstraß's Approximation Theorem, 14