

Real Functions in Several Variables: Volume VII

Space Integrals

Leif Mejlbro



Leif Mejlbro

Real Functions in Several Variables

Volume VII Space Integrals



Real Functions in Several Variables: Volume VII Space Integrals

2nd edition

© 2015 Leif Mejlbro & bookboon.com

ISBN 978-87-403-0914-0

Contents

Volume I, Point Sets in \mathbb{R}^n	1
Preface	15
Introduction to volume I, Point sets in \mathbb{R}^n. The maximal domain of a function	19
1 Basic concepts	21
1.1 Introduction	21
1.2 The real linear space \mathbb{R}^n	22
1.3 The vector product	26
1.4 The most commonly used coordinate systems	29
1.5 Point sets in space	37
1.5.1 Interior, exterior and boundary of a set	37
1.5.2 Starshaped and convex sets	40
1.5.3 Catalogue of frequently used point sets in the plane and the space	41
1.6 Quadratic equations in two or three variables. Conic sections	47
1.6.1 Quadratic equations in two variables. Conic sections	47
1.6.2 Quadratic equations in three variables. Conic sectional surfaces	54
1.6.3 Summary of the canonical cases in three variables	66
2 Some useful procedures	67
2.1 Introduction	67
2.2 Integration of trigonometric polynomials	67
2.3 Complex decomposition of a fraction of two polynomials	69
2.4 Integration of a fraction of two polynomials	72
3 Examples of point sets	75
3.1 Point sets	75
3.2 Conics and conical sections	104
4 Formulae	115
4.1 Squares etc.	115
4.2 Powers etc.	115
4.3 Differentiation	116
4.4 Special derivatives	116
4.5 Integration	118
4.6 Special antiderivatives	119
4.7 Trigonometric formulae	121
4.8 Hyperbolic formulae	123
4.9 Complex transformation formulae	124
4.10 Taylor expansions	124
4.11 Magnitudes of functions	125
Index	127

Volume II, Continuous Functions in Several Variables	133
Preface	147
Introduction to volume II, Continuous Functions in Several Variables	151
5 Continuous functions in several variables	153
5.1 Maps in general	153
5.2 Functions in several variables	154
5.3 Vector functions	157
5.4 Visualization of functions	158
5.5 Implicit given function	161
5.6 Limits and continuity	162
5.7 Continuous functions	168
5.8 Continuous curves	170
5.8.1 Parametric description	170
5.8.2 Change of parameter of a curve	174
5.9 Connectedness	175
5.10 Continuous surfaces in \mathbb{R}^3	177
5.10.1 Parametric description and continuity	177
5.10.2 Cylindric surfaces	180
5.10.3 Surfaces of revolution	181
5.10.4 Boundary curves, closed surface and orientation of surfaces	182
5.11 Main theorems for continuous functions	185
6 A useful procedure	189
6.1 The domain of a function	189
7 Examples of continuous functions in several variables	191
7.1 Maximal domain of a function	191
7.2 Level curves and level surfaces	198
7.3 Continuous functions	212
7.4 Description of curves	227
7.5 Connected sets	241
7.6 Description of surfaces	245
8 Formulae	257
8.1 Squares etc.	257
8.2 Powers etc.	257
8.3 Differentiation	258
8.4 Special derivatives	258
8.5 Integration	260
8.6 Special antiderivatives	261
8.7 Trigonometric formulæ	263
8.8 Hyperbolic formulæ	265
8.9 Complex transformation formulæ	266
8.10 Taylor expansions	266
8.11 Magnitudes of functions	267
Index	269

Volume III, Differentiable Functions in Several Variables	275
Preface	289
Introduction to volume III, Differentiable Functions in Several Variables	293
9 Differentiable functions in several variables	295
9.1 Differentiability	295
9.1.1 The gradient and the differential	295
9.1.2 Partial derivatives	298
9.1.3 Differentiable vector functions	303
9.1.4 The approximating polynomial of degree 1	304
9.2 The chain rule	305
9.2.1 The elementary chain rule	305
9.2.2 The first special case	308
9.2.3 The second special case	309
9.2.4 The third special case	310
9.2.5 The general chain rule	314
9.3 Directional derivative	317
9.4 C^n -functions	318
9.5 Taylor's formula	321
9.5.1 Taylor's formula in one dimension	321
9.5.2 Taylor expansion of order 1	322
9.5.3 Taylor expansion of order 2 in the plane	323
9.5.4 The approximating polynomial	326
10 Some useful procedures	333
10.1 Introduction	333
10.2 The chain rule	333
10.3 Calculation of the directional derivative	334
10.4 Approximating polynomials	336
11 Examples of differentiable functions	339
11.1 Gradient	339
11.2 The chain rule	352
11.3 Directional derivative	375
11.4 Partial derivatives of higher order	382
11.5 Taylor's formula for functions of several variables	404
12 Formulæ	445
12.1 Squares etc.	445
12.2 Powers etc.	445
12.3 Differentiation	446
12.4 Special derivatives	446
12.5 Integration	448
12.6 Special antiderivatives	449
12.7 Trigonometric formulæ	451
12.8 Hyperbolic formulæ	453
12.9 Complex transformation formulæ	454
12.10 Taylor expansions	454
12.11 Magnitudes of functions	455
Index	457

Volume IV, Differentiable Functions in Several Variables	463
Preface	477
Introduction to volume IV, Curves and Surfaces	481
13 Differentiable curves and surfaces, and line integrals in several variables	483
13.1 Introduction	483
13.2 Differentiable curves	483
13.3 Level curves	492
13.4 Differentiable surfaces	495
13.5 Special C^1 -surfaces	499
13.6 Level surfaces	503
14 Examples of tangents (curves) and tangent planes (surfaces)	505
14.1 Examples of tangents to curves	505
14.2 Examples of tangent planes to a surface	520
15 Formulæ	541
15.1 Squares etc.	541
15.2 Powers etc.	541
15.3 Differentiation	542
15.4 Special derivatives	542
15.5 Integration	544
15.6 Special antiderivatives	545
15.7 Trigonometric formulæ	547
15.8 Hyperbolic formulæ	549
15.9 Complex transformation formulæ	550
15.10 Taylor expansions	550
15.11 Magnitudes of functions	551
Index	553
Volume V, Differentiable Functions in Several Variables	559
Preface	573
Introduction to volume V, The range of a function, Extrema of a Function in Several Variables	577
16 The range of a function	579
16.1 Introduction	579
16.2 Global extrema of a continuous function	581
16.2.1 A necessary condition	581
16.2.2 The case of a closed and bounded domain of f	583
16.2.3 The case of a bounded but not closed domain of f	599
16.2.4 The case of an unbounded domain of f	608
16.3 Local extrema of a continuous function	611
16.3.1 Local extrema in general	611
16.3.2 Application of Taylor's formula	616
16.4 Extremum for continuous functions in three or more variables	625
17 Examples of global and local extrema	631
17.1 MAPLE	631
17.2 Examples of extremum for two variables	632
17.3 Examples of extremum for three variables	668

17.4 Examples of maxima and minima	677
17.5 Examples of ranges of functions	769
18 Formulæ	811
18.1 Squares etc.	811
18.2 Powers etc.	811
18.3 Differentiation	812
18.4 Special derivatives	812
18.5 Integration	814
18.6 Special antiderivatives	815
18.7 Trigonometric formulæ	817
18.8 Hyperbolic formulæ	819
18.9 Complex transformation formulæ	820
18.10 Taylor expansions	820
18.11 Magnitudes of functions	821
Index	823
Volume VI, Antiderivatives and Plane Integrals	829
Preface	841
Introduction to volume VI, Integration of a function in several variables	845
19 Antiderivatives of functions in several variables	847
19.1 The theory of antiderivatives of functions in several variables	847
19.2 Templates for gradient fields and antiderivatives of functions in three variables	858
19.3 Examples of gradient fields and antiderivatives	863
20 Integration in the plane	881
20.1 An overview of integration in the plane and in the space	881
20.2 Introduction	882
20.3 The plane integral in rectangular coordinates	887
20.3.1 Reduction in rectangular coordinates	887
20.3.2 The colour code, and a procedure of calculating a plane integral	890
20.4 Examples of the plane integral in rectangular coordinates	894
20.5 The plane integral in polar coordinates	936
20.6 Procedure of reduction of the plane integral; polar version	944
20.7 Examples of the plane integral in polar coordinates	948
20.8 Examples of area in polar coordinates	972
21 Formulæ	977
21.1 Squares etc.	977
21.2 Powers etc.	977
21.3 Differentiation	978
21.4 Special derivatives	978
21.5 Integration	980
21.6 Special antiderivatives	981
21.7 Trigonometric formulæ	983
21.8 Hyperbolic formulæ	985
21.9 Complex transformation formulæ	986
21.10 Taylor expansions	986
21.11 Magnitudes of functions	987
Index	989

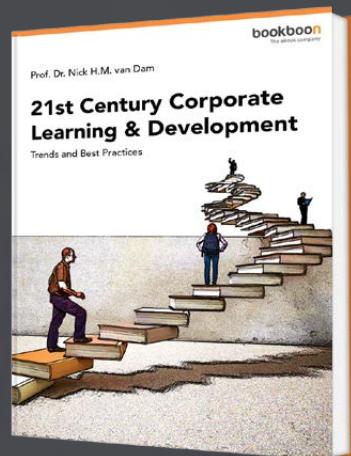
Volume VII, Space Integrals	995
Preface	1009
Introduction to volume VII, The space integral	1013
22 The space integral in rectangular coordinates	1015
22.1 Introduction	1015
22.2 Overview of setting up of a line, a plane, a surface or a space integral	1015
22.3 Reduction theorems in rectangular coordinates	1021
22.4 Procedure for reduction of space integral in rectangular coordinates	1024
22.5 Examples of space integrals in rectangular coordinates	1026
23 The space integral in semi-polar coordinates	1055
23.1 Reduction theorem in semi-polar coordinates	1055
23.2 Procedures for reduction of space integral in semi-polar coordinates	1056
23.3 Examples of space integrals in semi-polar coordinates	1058
24 The space integral in spherical coordinates	1081
24.1 Reduction theorem in spherical coordinates	1081
24.2 Procedures for reduction of space integral in spherical coordinates	1082
24.3 Examples of space integrals in spherical coordinates	1084
24.4 Examples of volumes	1107
24.5 Examples of moments of inertia and centres of gravity	1116
25 Formulæ	1125
25.1 Squares etc.	1125
25.2 Powers etc.	1125
25.3 Differentiation	1126
25.4 Special derivatives	1126
25.5 Integration	1128
25.6 Special antiderivatives	1129
25.7 Trigonometric formulæ	1131
25.8 Hyperbolic formulæ	1133
25.9 Complex transformation formulæ	1134
25.10 Taylor expansions	1134
25.11 Magnitudes of functions	1135
Index	1137
Volume VIII, Line Integrals and Surface Integrals	1143
Preface	1157
Introduction to volume VIII, The line integral and the surface integral	1161
26 The line integral	1163
26.1 Introduction	1163
26.2 Reduction theorem of the line integral	1163
26.2.1 Natural parametric description	1166
26.3 Procedures for reduction of a line integral	1167
26.4 Examples of the line integral in rectangular coordinates	1168
26.5 Examples of the line integral in polar coordinates	1190
26.6 Examples of arc lengths and parametric descriptions by the arc length	1201

27 The surface integral	1227
27.1 The reduction theorem for a surface integral	1227
27.1.1 The integral over the graph of a function in two variables	1229
27.1.2 The integral over a cylindric surface	1230
27.1.3 The integral over a surface of revolution	1232
27.2 Procedures for reduction of a surface integral	1233
27.3 Examples of surface integrals	1235
27.4 Examples of surface area	1296
28 Formulæ	1315
28.1 Squares etc.	1315
28.2 Powers etc.	1315
28.3 Differentiation	1316
28.4 Special derivatives	1316
28.5 Integration	1318
28.6 Special antiderivatives	1319
28.7 Trigonometric formulæ	1321
28.8 Hyperbolic formulæ	1323
28.9 Complex transformation formulæ	1324
28.10 Taylor expansions	1324
28.11 Magnitudes of functions	1325
Index	1327

Free eBook on Learning & Development

By the Chief Learning Officer of McKinsey

Download Now



Volume IX, Transformation formulæ and improper integrals	1333
Preface	1347
Introduction to volume IX, Transformation formulæ and improper integrals	1351
29 Transformation of plane and space integrals	1353
29.1 Transformation of a plane integral	1353
29.2 Transformation of a space integral	1355
29.3 Procedures for the transformation of plane or space integrals	1358
29.4 Examples of transformation of plane and space integrals	1359
30 Improper integrals	1411
30.1 Introduction	1411
30.2 Theorems for improper integrals	1413
30.3 Procedure for improper integrals; bounded domain	1415
30.4 Procedure for improper integrals; unbounded domain	1417
30.5 Examples of improper integrals	1418
31 Formulae	1447
31.1 Squares etc.	1447
31.2 Powers etc.	1447
31.3 Differentiation	1448
31.4 Special derivatives	1448
31.5 Integration	1450
31.6 Special antiderivatives	1451
31.7 Trigonometric formulæ	1453
31.8 Hyperbolic formulæ	1455
31.9 Complex transformation formulæ	1456
31.10 Taylor expansions	1456
31.11 Magnitudes of functions	1457
Index	1459
Volume X, Vector Fields I; Gauß's Theorem	1465
Preface	1479
Introduction to volume X, Vector fields; Gauß's Theorem	1483
32 Tangential line integrals	1485
32.1 Introduction	1485
32.2 The tangential line integral. Gradient fields.	1485
32.3 Tangential line integrals in Physics	1498
32.4 Overview of the theorems and methods concerning tangential line integrals and gradient fields	1499
32.5 Examples of tangential line integrals	1502
33 Flux and divergence of a vector field. Gauß's theorem	1535
33.1 Flux	1535
33.2 Divergence and Gauß's theorem	1540
33.3 Applications in Physics	1544
33.3.1 Magnetic flux	1544
33.3.2 Coulomb vector field	1545
33.3.3 Continuity equation	1548
33.4 Procedures for flux and divergence of a vector field; Gauß's theorem	1549
33.4.1 Procedure for calculation of a flux	1549
33.4.2 Application of Gauß's theorem	1549
33.5 Examples of flux and divergence of a vector field; Gauß's theorem	1551
33.5.1 Examples of calculation of the flux	1551
33.5.2 Examples of application of Gauß's theorem	1580

34 Formulæ	1619
34.1 Squares etc.	1619
34.2 Powers etc.	1619
34.3 Differentiation	1620
34.4 Special derivatives	1620
34.5 Integration	1622
34.6 Special antiderivatives	1623
34.7 Trigonometric formulæ	1625
34.8 Hyperbolic formulæ	1627
34.9 Complex transformation formulæ	1628
34.10 Taylor expansions	1628
34.11 Magnitudes of functions	1629
Index	1631
Volume XI, Vector Fields II; Stokes's Theorem	1637
Preface	1651
Introduction to volume XI, Vector fields II; Stokes's Theorem; nabla calculus	1655
35 Rotation of a vector field; Stokes's theorem	1657
35.1 Rotation of a vector field in \mathbb{R}^3	1657
35.2 Stokes's theorem	1661
35.3 Maxwell's equations	1669
35.3.1 The electrostatic field	1669
35.3.2 The magnetostatic field	1671
35.3.3 Summary of Maxwell's equations	1679
35.4 Procedure for the calculation of the rotation of a vector field and applications of Stokes's theorem	1682

www.sylvania.com

We do not reinvent
the wheel we reinvent
light.

Fascinating lighting offers an infinite spectrum of possibilities: Innovative technologies and new markets provide both opportunities and challenges. An environment in which your expertise is in high demand. Enjoy the supportive working atmosphere within our global group and benefit from international career paths. Implement sustainable ideas in close cooperation with other specialists and contribute to influencing our future. Come and join us in reinventing light every day.

Light is OSRAM

OSRAM
SYLVANIA

Click on the ad to read more

35.5 Examples of the calculation of the rotation of a vector field and applications of Stokes's theorem	1684
35.5.1 Examples of divergence and rotation of a vector field	1684
35.5.2 General examples	1691
35.5.3 Examples of applications of Stokes's theorem	1700
36 Nabla calculus	1739
36.1 The vectorial differential operator ∇	1739
36.2 Differentiation of products	1741
36.3 Differentiation of second order	1743
36.4 Nabla applied on \mathbf{x}	1745
36.5 The integral theorems	1746
36.6 Partial integration	1749
36.7 Overview of Nabla calculus	1750
36.8 Overview of partial integration in higher dimensions	1752
36.9 Examples in nabla calculus	1754
37 Formulæ	1769
37.1 Squares etc.	1769
37.2 Powers etc.	1769
37.3 Differentiation	1770
37.4 Special derivatives	1770
37.5 Integration	1772
37.6 Special antiderivatives	1773
37.7 Trigonometric formulæ	1775
37.8 Hyperbolic formulæ	1777
37.9 Complex transformation formulæ	1778
37.10 Taylor expansions	1778
37.11 Magnitudes of functions	1779
Index	1781
Volume XII, Vector Fields III; Potentials, Harmonic Functions and Green's Identities	1787
Preface	1801
Introduction to volume XII, Vector fields III; Potentials, Harmonic Functions and Green's Identities	1805
38 Potentials	1807
38.1 Definitions of scalar and vectorial potentials	1807
38.2 A vector field given by its rotation and divergence	1813
38.3 Some applications in Physics	1816
38.4 Examples from Electromagnetism	1819
38.5 Scalar and vector potentials	1838
39 Harmonic functions and Green's identities	1889
39.1 Harmonic functions	1889
39.2 Green's first identity	1890
39.3 Green's second identity	1891
39.4 Green's third identity	1896
39.5 Green's identities in the plane	1898
39.6 Gradient, divergence and rotation in semi-polar and spherical coordinates	1899
39.7 Examples of applications of Green's identities	1901
39.8 Overview of Green's theorems in the plane	1909
39.9 Miscellaneous examples	1910

40 Formulæ	1923
40.1 Squares etc.	1923
40.2 Powers etc.	1923
40.3 Differentiation	1924
40.4 Special derivatives	1924
40.5 Integration	1926
40.6 Special antiderivatives	1927
40.7 Trigonometric formulæ	1929
40.8 Hyperbolic formulæ	1931
40.9 Complex transformation formulæ	1932
40.10 Taylor expansions	1932
40.11 Magnitudes of functions	1933
Index	1935



Deloitte.

Discover the truth at www.deloitte.ca/careers

© Deloitte & Touche LLP and affiliated entities.



Click on the ad to read more

Preface

The topic of this series of books on “*Real Functions in Several Variables*” is very important in the description in e.g. *Mechanics* of the real 3-dimensional world that we live in. Therefore, we start from the very beginning, modelling this world by using the coordinates of \mathbb{R}^3 to describe e.g. a motion in space. There is, however, absolutely no reason to restrict ourselves to \mathbb{R}^3 alone. Some motions may be rectilinear, so only \mathbb{R} is needed to describe their movements on a line segment. This opens up for also dealing with \mathbb{R}^2 , when we consider plane motions. In more elaborate problems we need higher dimensional spaces. This may be the case in *Probability Theory* and *Statistics*. Therefore, we shall in general use \mathbb{R}^n as our abstract model, and then restrict ourselves in examples mainly to \mathbb{R}^2 and \mathbb{R}^3 .

For rectilinear motions the familiar *rectangular coordinate system* is the most convenient one to apply. However, as known from e.g. *Mechanics*, circular motions are also very important in the applications in engineering. It becomes natural alternatively to apply in \mathbb{R}^2 the so-called *polar coordinates* in the plane. They are convenient to describe a circle, where the rectangular coordinates usually give some nasty square roots, which are difficult to handle in practice.

Rectangular coordinates and polar coordinates are designed to model each their problems. They supplement each other, so difficult computations in one of these coordinate systems may be easy, and even trivial, in the other one. It is therefore important always in advance carefully to analyze the geometry of e.g. a domain, so we ask the question: Is this domain best described in rectangular or in polar coordinates?

Sometimes one may split a problem into two subproblems, where we apply rectangular coordinates in one of them and polar coordinates in the other one.

It should be mentioned that in *real life* (though not in these books) one cannot always split a problem into two subproblems as above. Then one is really in trouble, and more advanced mathematical methods should be applied instead. This is, however, outside the scope of the present series of books.

The idea of polar coordinates can be extended in two ways to \mathbb{R}^3 . Either to *semi-polar* or *cylindric coordinates*, which are designed to describe a cylinder, or to *spherical coordinates*, which are excellent for describing spheres, where rectangular coordinates usually are doomed to fail. We use them already in daily life, when we specify a place on Earth by its longitude and latitude! It would be very awkward in this case to use rectangular coordinates instead, even if it is possible.

Concerning the contents, we begin this investigation by modelling point sets in an n -dimensional Euclidean space E^n by \mathbb{R}^n . There is a subtle difference between E^n and \mathbb{R}^n , although we often identify these two spaces. In E^n we use *geometrical methods* without a coordinate system, so the objects are independent of such a choice. In the coordinate space \mathbb{R}^n we can use ordinary calculus, which in principle is not possible in E^n . In order to stress this point, we call E^n the “abstract space” (in the sense of calculus; not in the sense of geometry) as a warning to the reader. Also, whenever necessary, we use the colour black in the “abstract space”, in order to stress that this expression is theoretical, while variables given in a chosen coordinate system and their related concepts are given the colours blue, red and green.

We also include the most basic of what mathematicians call *Topology*, which will be necessary in the following. We describe what we need by a function.

Then we proceed with limits and continuity of functions and define continuous curves and surfaces, with parameters from subsets of \mathbb{R} and \mathbb{R}^2 , resp..

Continue with (partial) differentiable functions, curves and surfaces, the chain rule and Taylor's formula for functions in several variables.

We deal with maxima and minima and extrema of functions in several variables over a domain in \mathbb{R}^n . This is a very important subject, so there are given many worked examples to illustrate the theory.

Then we turn to the problems of integration, where we specify four different types with increasing complexity, *plane integral, space integral, curve (or line) integral and surface integral*.

Finally, we consider *vector analysis*, where we deal with vector fields, Gauß's theorem and Stokes's theorem. All these subjects are very important in theoretical Physics.

The structure of this series of books is that each subject is *usually* (but not always) described by three successive chapters. In the first chapter a brief theoretical theory is given. The next chapter gives some practical guidelines of how to solve problems connected with the subject under consideration. Finally, some worked out examples are given, in many cases in several variants, because the standard solution method is seldom the only way, and it may even be clumsy compared with other possibilities.

I have as far as possible structured the examples according to the following scheme:

A Awareness, i.e. a short description of what is the problem.

D Decision, i.e. a reflection over what should be done with the problem.

I Implementation, i.e. where all the calculations are made.

C Control, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

From high school one is used to immediately to proceed to **I. Implementation**. However, examples and problems at university level, let alone situations in real life, are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be executed.

This is unfortunately not the case with **C Control**, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of \wedge I shall either write "and", or a comma, and instead of \vee I shall write "or". The arrows \Rightarrow and \Leftrightarrow are in particular misunderstood by the students, so they should be totally avoided. They are not telegram short hands, and from a logical point of view they usually do not make sense at all! Instead, write in a plain language what you mean or want to do. This is difficult in the beginning, but after some practice it becomes routine, and it will give more precise information.

When we deal with multiple integrals, one of the possible pedagogical ways of solving problems has been to colour variables, integrals and upper and lower bounds in blue, red and green, so the reader by the colour code can see in each integral what is the variable, and what are the parameters, which

do not enter the integration under consideration. We shall of course build up a hierarchy of these colours, so the order of integration will always be defined. As already mentioned above we reserve the colour black for the theoretical expressions, where we cannot use ordinary calculus, because the symbols are only shorthand for a concept.

The author has been very grateful to his old friend and colleague, the late Per Wennerberg Karlsson, for many discussions of how to present these difficult topics on real functions in several variables, and for his permission to use his textbook as a template of this present series. Nevertheless, the author has felt it necessary to make quite a few changes compared with the old textbook, because we did not always agree, and some of the topics could also be explained in another way, and then of course the results of our discussions have here been put in writing for the first time.

The author also adds some calculations in MAPLE, which interact nicely with the theoretic text. Note, however, that when one applies MAPLE, one is forced first to make a geometrical analysis of the domain of integration, i.e. apply some of the techniques developed in the present books.

The theory and methods of these volumes on "Real Functions in Several Variables" are applied constantly in higher Mathematics, Mechanics and Engineering Sciences. It is of paramount importance for the calculations in *Probability Theory*, where one constantly integrate over some point set in space.

It is my hope that this text, these guidelines and these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
March 21, 2015

SIMPLY CLEVER

ŠKODA



We will turn your CV into
an opportunity of a lifetime



Do you like cars? Would you like to be a part of a successful brand?
We will appreciate and reward both your enthusiasm and talent.
Send us your CV. You will be surprised where it can take you.

Send us your CV on
www.employerforlife.com



Click on the ad to read more

Introduction to volume VII, The space integral

This is the seventh volume in the series of books on *Real Functions in Several Variables*.

We continue the investigation of how to integrate a real function in several variables in three dimensions, i.e. how we can reduce abstract *space integrals*. We start with the reduction theorems in rectangular coordinates. This is just an extension of the theory of the plane integral in rectangular coordinates. There are some small complications in the Geometry, when we have to visualize sets A in \mathbb{R}^3 , but in principle, it is the same theory.

In the next chapter we use the theory of the reduction of the plane integral in polar coordinates to the reduction of a space integral in semi-polar coordinates. The reduction takes place in a parameter space, where we introduce a weight function and then integrate as in the case of the rectangular coordinates with respect to the semi-polar coordinates in the parameter domain.

The same idea is used in Chapter 24, where we reduce in spherical coordinates. We add a weight function as a factor to the integrand and then integrate as in the rectangular case with respect to the spherical coordinates in the parameter space, which must not be confused with the body itself.

Since we here are dealing with three dimensions, there are lots of variants, which cannot all be coined as theorems. However, once the coordinate system has been chosen, and we have identified the corresponding weight function, then the problem is always reduced to a geometric analysis of the body under consideration.

We add some examples of how to find a volume, the centre of gravity of a body, or the moment of inertia.

22 The space integral in rectangular coordinates

22.1 Introduction

The extension of the plane integral to the *space integral* follows the pattern known from the plane integral. First we must define a *volume element* $d\Omega$ in \mathbb{R}^3 . As a guideline we see in rectangular coordinates (x, y, z) that it is most reasonable that we interpret

$$d\Omega = dx dy dz,$$

because $dx dy dz$ represents the volume of an axiparallel infinitesimal box of edge lengths dx , dy and dz . Given a continuous function $f_A : \mathbb{R}^3 \rightarrow \mathbb{R}$, where $A \subset \mathbb{R}^3$ is a closed and bounded set in \mathbb{R}^3 . Then the space integral of f over A is denoted by the (abstract) symbol

$$\int_A f(x, y, z) d\Omega,$$

where the green colour indicates that we are dealing with an abstract integral in three dimensions.

When $f(x, y, z) \equiv 1$, then the space integral is the *volume* of A ,

$$\text{vol}(A) = \int_A 1 d\Omega = \int_A d\Omega,$$

because we, roughly speaking, fill up A with mutually disjoint infinitesimal boxes and then add all their volumes, which intuitively give us the volume of A . Actually, this construction is not quite correct, but close to. It represents the idea of the volume and can immediately be extended to the idea of the more general space integral by multiplying each infinitesimal volume with the value of f at some point in this box, and then add the results. By letting $d\Omega \rightarrow 0$ in some controlled sense (which is not obvious here) we obtain the (idea of the) space integral.

We shall not be concerned with correcting the intuitive abstract considerations above. Instead we shall – without proofs – in the next section quote some *reduction theorem* of an abstract space integral, so it in practice becomes possible to calculate its value.

22.2 Overview of setting up of a line, a plane, a surface or a space integral

We start with a general section which explains how one analyzes the setting up of an integral in order to obtain a reduction formula, which can be applied in practice. The material is covering all types of integrals considered in this series of books. For clarity we shall not use colours in this section.

When we set up an integral, we have two possible approaches:

- 1) A geometrical analysis.
- 2) Measure theory (i.e. concerning integration).

In elementary books on Calculus the *geometrical analysis* is dominating, although there may occur examples, where the *measure theory* plays a bigger role than the geometry. The latter is often the case when we apply the transformation theorems. It may also occur when we shall choose between semi-polar and spherical coordinates. We shall in the following not consider these exceptions, so usually we start with a *geometrical analysis* of the domain of integration.

This analysis is depending on

1a. Dimension.

1b. Choice of coordinates.

The classical coordinates are:

Dimension 2:

Rectangular: (x, y) , or (u, v) , (ϱ, φ) or similarly in the parametric domain.

Polar: $x = \varrho \cos \varphi$, $y = \varrho \sin \varphi$.

General: $x = X(u, v)$, $y = Y(u, v)$, injective almost everywhere.

Dimension 3: Rectangular: (x, y, z) , or (u, v, w) , (ϱ, φ, z) , (r, θ, φ) or similarly in the *parametric domain*.

Semi-polar: $z = \varrho \cos \varphi$, $y = \varrho \sin \varphi$, $z = z$,

Spherical: $x = r \sin \theta \cos \varphi$, $y = r \sin \theta \sin \varphi$, $z = r \cos \theta$.

General: $x = X(u, v, w)$, $y = Y(u, v, w)$, $z = Z(u, v, w)$, injective almost everywhere.

These are our building stones in the ongoing geometrical analysis.

The advertisement features a woman teacher smiling and interacting with two young students (a boy and a girl) who are looking at a laptop screen. The background is yellow with orange swirling graphic elements. In the top left corner is the e-Learning for Kids logo. In the bottom right corner, there is a green oval containing text and a hand cursor icon pointing towards it.

About e-Learning for Kids Established in 2004, e-Learning for Kids is a global nonprofit foundation dedicated to fun and free learning on the Internet for children ages 5 - 12 with courses in math, science, language arts, computers, health and environmental skills. Since 2005, more than 15 million children in over 190 countries have benefitted from eLessons provided by EFK! An all-volunteer staff consists of education and e-learning experts and business professionals from around the world committed to making difference. eLearning for Kids is actively seeking funding, volunteers, sponsors and courseware developers; get involved! For more information, please visit www.e-learningforkids.org.

The number 1 MOOC for Primary Education
• Free Digital Learning for Children 5-12
• 15 Million Children Reached

Remark 22.1 If the example does not contain any hint, then choose among the *rectangular*, *polar*, *semi-polar* or *spherical* coordinates. On the other hand, if more general coordinates are needed in an exercise, then these will always be given, usually in their *inverse* form:

$$\text{dimension 2 : } u = U(x, y), \quad v = V(x, y),$$

$$\text{dimension 3 : } u = U(x, y, z), \quad v = V(x, y, z), \quad w = W(x, y, z).$$

If they are given in their inverse form, we start by solving them with respect to x , y (and z), cf. Chapter 17. ◇

Solution strategy:

Choose the coordinates, such that

- a) the geometry of the *parametric domain* becomes simple,
- b) the *integrand* becomes simple.

This is an order of priority, so the *geometry* is most important in a), while the *measure theory* is dominating in b). Both cases can be found in elementary textbooks, though most of the examples are of the type given in a).

In the following we set up an overview guided by the dimension of the classical coordinates. In each case the structure will be given by:

- i) **Formula**, where the right hand side is the reduced expression.
- ii) **Geometry**, where the domain is compared with the parametric domain.
- iii) **Measure theory**, which briefly describes the weight function.
- iv) **Possible comments.**

It is seen that we are aiming at the reduction of a given abstract integral to a *rectangular* integration over a convenient *parametric domain*.

Dimension 1.

Characteristics: There is only one variable of integration t .

We have two cases, a) **An ordinary integral** and b) **A line integral**.

a) The ordinary integral.

Formula: $\int_a^b f(t) dt.$

Geometry: The domain of integration = the parametric interval.

Measure theory: Weight = 1.

Comment: Basic form known from high school and previous courses in Calculus.

b) The line integral.

Formula: $\int_{\mathcal{K}} f(x, y, z) ds = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| dt$, where $\mathcal{K} : (x, y, z) = \mathbf{r}(t)$.

Geometry: Curve \neq parametric interval.

Measure theory: Weight $= \|\mathbf{r}'(t)\|$.

Comments: By the transform given above b) is transferred back to the basic form a). Note that there is a *hidden square root* in the weight function, so pocket calculators cannot always be successfully applied. Note that the curve is embedded in the *space* \mathbb{R}^3 .

Dimension 2.

Characteristics: There are *two* variables of integration, e.g. (u, v) .

Here we have four cases:

a) Rectangular plane integral.

b) Polar plane integral.

c) General transform.

d) Surface integral.

These cases are treated one by one in the following.

a) Rectangular plane integral.

Formula: $\int_A f(x, y) dx dy$.

Geometry: domain of integration $=$ parametric domain.

Measure theory: Weight $= 1$.

Comment: Basic form.

b) Polar plane integral.

Formula: $\int_A f(x, y) dx dy = \int_B f(\varrho \cos \varphi, \varrho \sin \varphi) \varrho d\varrho d\varphi$.

Geometry: domain of integration \neq parametric domain.

Measure theory: Weight $= \varrho$.

Comment: Note that b) is reduced to a).

c) General.

Formula: $\int_A f(x, y) dx dy = \int_B f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv.$

Geometry: domain of integration \neq parametric domain.

Measure theory: Weight $= \left| \frac{\partial(x, y)}{\partial(u, v)} \right|.$

d) Surface integral.

Formula: $\int_A f(x, y, z) dS = \int_B f(\mathbf{r}(u, v)) \|\mathbf{N}(u, v)\| du dv,$

where the surface \mathcal{F} is given by $(x, y, z) = \mathbf{r}(u, v)$.

Geometry: Surface \neq parametric domain.

Measure theory: Weight $= \|\mathbf{N}(u, v)\|.$

Comment: Note that for *rotational surfaces* we usually apply the *semi-polar* coordinates with the weight function ϱ , where ϱ is a function of one parameter t .
Note also that the surface is embedded in \mathbb{R}^3 .

Dimension 3.

Characteristics: There are *three* variables of integration, e.g. (u, v, w) .

Here we have five cases, of which only four usually are treated in elementary courses in Calculus.

a) Rectangular space integral.

Formula: $\int_A f(x, y, z) dx dy dz.$

Geometry: domain of integration $=$ parametric domain.

Measure theory: Weight $= 1.$

b) Semi-polar space integral.

Formula: $\int_A f(x, y, z) d\Omega = \int_B f(\varrho \cos \varphi, \varrho \sin \varphi, z) \varrho d\varrho d\varphi dz.$

Geometry: domain of integration \neq parametric domain.

Measure theory: Weight $= \varrho.$

Comment: The method is in particular applied on
rotational bodies.

c) Spherical space integral.

Formula: $\int_A f(x, y, z) d\Omega = \int_B f(r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta) r^2 \sin \theta dr d\theta d\varphi.$

Geometry: domain of integration \neq parametric domain.

Measure theory: Weight $= r^2 \sin \theta.$

Comment: The method is usually applied on *spherical shells*.

d) General.

Formula:
$$\begin{aligned} \int_A f(x, y, z) d\Omega \\ = \int_B f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw. \end{aligned}$$

Geometry: domain of integration \neq parametric domain.

Measure theory: Weight $= \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right|.$

Cynthia | AXA Graduate

AXA Global Graduate Program

Find out more and apply

redefining / standards **AXA**

A green oval button with a hand cursor icon and the text "Click on the ad to read more".

e) Curved space integral.

One may come across integration over a curved 3-dimensional space in the Theory of Relativity (embedded in \mathbb{R}^4) in Physics. This is of course analogous to the surface integral embedded in \mathbb{R}^3 . Usually it does not occur in elementary textbooks of Calculus.

22.3 Reduction theorems in rectangular coordinates

The idea of the reduction theorems is to reduce an abstract (green) space integral to an abstract (blue) plane integral and an ordinary one dimensional integral (either in black or in red). Then the abstract (blue) plane integral is reduced further by the methods already given in Chapter 20, so whenever convenient, we may even consider a triple integral.

There are two variants of the reduction theorem for space integrals in rectangular coordinates.

- 1) Either the order of integration is first (e.g.) *vertically* with respect to z (the inner integral), and then the outer integral is an abstract (blue) plane integral.
- 2) Or we start with (the inner integration) an abstract (blue) plane integration for each fixed z , and then integrate (black) with respect to z .

vs

Theorem 22.1 First reduction theorem for the space integral in rectangular coordinates.

Let $A \subset \mathbb{R}^3$, and let $f : A \rightarrow \mathbb{R}$ be a continuous function. We assume that there is a bounded and closed set $B \subset \mathbb{R}^2$ and two functions

$$Z_1, Z_2 \in C^0(B) \quad \text{and} \quad Z_1(x, y) < Z_2(x, y) \quad \text{for } (x, y) \in B^\circ.$$

Then the abstract space integral is reduced in the following way,

$$\int_A f(x, y, z) d\Omega = \int_B \left\{ \int_{Z_1(x, y)}^{Z_2(x, y)} f(x, y, z) dz \right\} dS.$$

Theorem 22.2 Second reduction theorem for the space integral in rectangular coordinates.

Let $A \subset \mathbb{R}^3$, and let $f : A \rightarrow \mathbb{R}$ be a continuous function. Assume that A lies in the horizontal slab defined by $a \leq z \leq b$, and that for every fixed $z \in [a, b]$ the set

$$B(z) := \{(x, y) \in \mathbb{R}^2 \mid (x, y, z) \in A\}, \quad z \in [a, b],$$

is bounded and closed. Then the abstract space integral is reduced in the following way,

$$\int_A f(x, y, z) d\Omega = \int_a^b \left\{ \int_{B(z)} f(x, y, z) dS \right\} dz.$$

We mention the following reduction theorem.

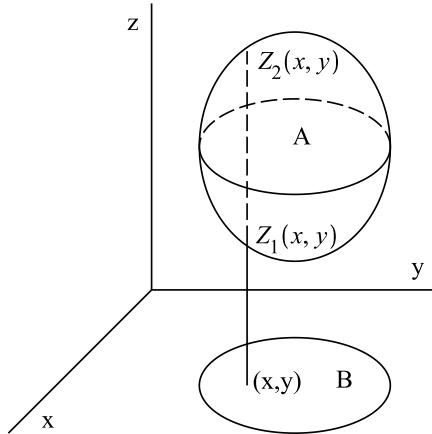


Figure 22.1: Illustration of Theorem 22.1. For fixed $(x, y) \in B^\circ$ the vertical line $\{(x, y, z) \mid z \in\}$ cuts A in an interval $[Z_1(x, y), Z_2(x, y)]$, over which $f(x, y, z)$ is integrated with respect to z . Collect the result $F(x, y)$ as the value of the new function F , and then integrate $F(x, y)$ over B as in Chapter 20.

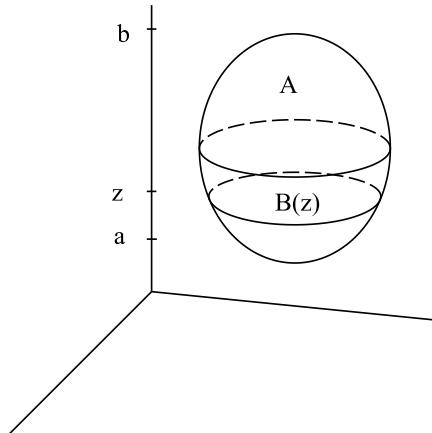


Figure 22.2: Illustration of Theorem 22.1. For fixed $z \in [a, b]$ we cut A in a plane domain $B(z)$. First perform a plane integration over $B(z)$. This defines an ordinary function $F(z)$, which is then integrated over the interval $[a, b]$.

Theorem 22.3 Reduction of a space integral as a triple integral *Let the closed and bounded domain A have the following special structure,*

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid a \leq x \leq b, Y_1(x) \leq y \leq Y_2(x), Z_1(x, y) \leq z \leq Z_2(x, y)\},$$

where Y_1 , Y_2 , Z_1 and Z_2 are continuous functions in their respective domains. Then the abstract space integral is reduced in the following way as a triple integral,

$$\int_A f(x, y, z) d\Omega = \int_a^b \left\{ \int_{Y_1(x)}^{Y_2(x)} \left\{ \int_{Z_1(x, y)}^{Z_2(x, y)} f(x, y, z) dz \right\} dy \right\} dx.$$

The bounds Z_1 and Z_2 of z depend on both x and y , while the bounds Y_1, Y_2 of y only depend on x . In Theorem 22.3 we have used the colour code

green — blue — black — red

to illustrate the order of integration. We go backwards. First we integrate with respect to the red variable, then with respect to the black one, and finally, with respect to the blue variable.

Note that the order of x, y, z may be changed everywhere in the theorems above, causing only an interchange in letter.

I joined MITAS because
I wanted **real responsibility**

The Graduate Programme
for Engineers and Geoscientists
www.discovermitas.com



Month 16

I was a construction supervisor in the North Sea advising and helping foremen solve problems

Real work
International opportunities
Three work placements



22.4 Procedure for reduction of space integral in rectangular coordinates

We shall here more explicitly describe the procedure, when we reduce an abstract space integral in rectangular coordinates. The methods are similar to those given in Section 20.3. The only new is that the dimension 3 (a number) can be divided as a sum of integers in three different ways:

- 1) The method of vertical posts: $3 = 2 + 1$,
- 2) The method of cutting into slabs: $3 = 1 + 2$,
- 3) The triple integral: $3 = 1 + 1 + 1$.

These three cases are treated separately in the following.

The method of posts In this case it follows from a figure that one of the variables, e.g. z , lies between the graphs of two C^0 functions $Z_1(x, y)$ and $Z_2(x, y)$ in the other two variables (x, y) . Furthermore, these variables lie in a specified domain B in the (x, y) plane, $(x, y) \in B$. The graphs of the two functions are surfaces, which cut A from the cylinder over B .

Procedure.

- 1) Write the set A in the form

$$A = \{(x, y, z) \mid (x, y) \in B, Z_1(x, y) \leq z \leq Z_2(x, y)\} \subseteq \mathbb{R}^3.$$

We identify the set $B \subseteq \mathbb{R}^2$ in the (x, y) plane and the functions $Z_1(x, y)$ and $Z_2(x, y)$. Then we set up the reduction formula

$$\int_A f(x, y, z) d\Omega = \int_B \left\{ \int_{Z_1(x, y)}^{Z_2(x, y)} f(x, y, z) dz \right\} dS.$$

The colour code is the usual one. The green integral is the abstract space integral. The blue integral is the abstract plane integral of lower dimension, while the red and innermost integral is a usual integral, which can be calculated by elementary methods.

- 2) For fixed $(x, y) \in B$ we first integrate with respect to z , i.e. along a vertical post,

$$\varphi(x, y) := \int_{Z_1(x, y)}^{Z_2(x, y)} f(x, y, z) dz,$$

where the right hand side is a usual integral.

- 3) The abstract space integral is then by insertion reduced to a simpler abstract plane integral,

$$\int_A f(x, y, z) d\Omega = \int_B \varphi(x, y) dS.$$

Finally, the right hand side is calculated by one of the methods given in Chapter 20; either rectangular or polar.

The method of cutting A into slabs. This method is in particular applied, when the integrand only depends on one of the three variables, e.g. z , though it can also be applied in other cases. The idea is that A at height z is cut into a slab $B(z)$, so we first integrate with respect to $(x, y) \in B(z)$, and afterwards with respect to z . Roughly speaking, one first find the total “mass” in the slab $B(z)$, and then we collect all these “masses” by integrating with respect to the “parameter” z .

Procedure.

- 1) Write the set A in the form

$$A = \{(\mathbf{x}, \mathbf{y}, z) \mid a \leq z \leq b, (\mathbf{x}, \mathbf{y}) \in B(z)\},$$

so we identify (e.g. by analyzing a figure) the cut (intersection of A with the plane at height z), or slab $B(z)$ for every relevant z .

Then set up the reduction formula

$$\int_A f(x, y, z) d\Omega = \int_a^b \left\{ \int_{B(z)} f(x, y, z) dS \right\} dz.$$

Here the green integral is the abstract space integral. The inner blue integral is the abstract plane integral of lower dimension, and the outmost black integral is an ordinary integral.

- 2) For fixed $z \in [a, b]$ we calculate the abstract plane integral

$$\psi(z) := \int_{B(z)} f(x, y, z) dS$$

by one of the methods from Chapter 20, in either rectangular or polar coordinates.

- 3) By insertion of the result the abstract space integral is reduced to an ordinary integral in one variable,

$$\int_A f(x, y, z) d\Omega = \int_a^b \psi(z) dz.$$

REMARK. If the integrand only depends on z , the 2) above is reduced to

$$\psi(z) := f(z) \cdot \text{area } B(z),$$

so 3) can be written

$$\int_A f(z) d\Omega = \int_a^b f(z) \cdot \text{area } B(z) dz,$$

where area $B(z)$ quite often can be found by an alternative simple geometrical argument.

Triple integral. This is a special case of the method of posts above, because we assume that the domain B is also bounded by graphs of functions, this time in one variable.

Procedure.

- 1) Write the set A in the form,

$$A = \{(\mathbf{x}, \mathbf{y}, z) \mid a \leq x \leq b, Y_1(\mathbf{x}) \leq y \leq Y_2(\mathbf{x}), Z_1(\mathbf{x}, \mathbf{y}) \leq z \leq Z_2(\mathbf{x}, \mathbf{y})\}.$$

This is the most difficult point of this process, so one should always afterwards check if the bounds $Y_1(\mathbf{x})$, $Y_2(\mathbf{x})$, $Z_1(\mathbf{x}, \mathbf{y})$, $Z_2(\mathbf{x}, \mathbf{y})$, are right. Usually, errors occur at this step of the process.

2) Set up the reduction formula

$$\int_A f(x, y, z) d\Omega = \int_a^b \left\{ \int_{Y_1(x)}^{Y_2(x)} \left\{ \int_{Z_1(x,y)}^{Z_2(x,y)} f(x, y, z) dz \right\} dy \right\} dx.$$

3) For fixed (x, y) we calculate the innermost integral,

$$f(x, y) = \int_{Z_1(x,y)}^{Z_2(x,y)} f(x, y, z) dz.$$

After this calculation, the black z should have disappeared (check!) If not, we have made an error.

4) After insertion we calculate for fixed x the middle integral,

$$h(x) := \int_{Y_1(x)}^{Y_2(x)} g(x, y) dy.$$

Check, that the red variable y has disappeared.

5) Finally, insert the result and calculate the outer integral,

$$\int_A f(x, y, z) d\Omega = \int_a^b h(x) dx.$$

22.5 Examples of space integrals in rectangular coordinates

A. Calculate the space space integral,

$$I = \int_A (3 + y - z) x d\Omega,$$

where

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in B, 0 \leq z \leq 2y\},$$

and B is the upper triangle on the figure.

D Apply Theorem 22.1.

It follows that

$$(22.1) \quad I = \int_A (3 + y - z) x d\Omega = \int_B \left\{ \int_0^{2y} (3 + y - z) x dz \right\} dS.$$

In the inner integral, x and y are considered as constants, so

$$\begin{aligned} \int_0^{2y} (3 + y - z) x dz &= x \int_0^{2y} (3 + y - z) dz = x \left[(3 + y)z - \frac{1}{2}z^2 \right]_{z=0}^{2y} \\ &= x \{ (3 + y) \cdot 2y - 2y^2 \} = x \cdot 2y \{ 3 + y - y \} = 6xy. \end{aligned}$$

By insertion into (22.1), we reduce to an abstract plane integral over B , i.e. of lower dimension,

$$I = \int_B 6xy dS = 6 \int_B xy dS.$$

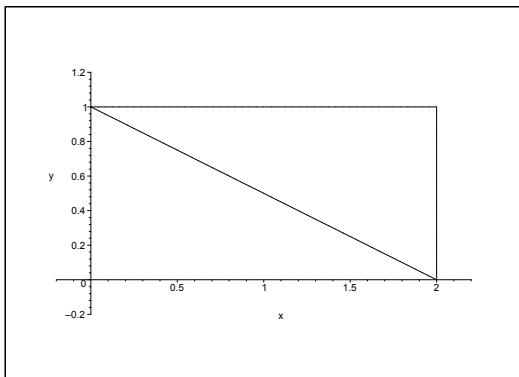


Figure 22.3: The domain B , i.e. the perpendicular projection of the body A onto the (x, y) plane.

We have previously in Section 20.4 found that

$$\int_B xy \, dS = \frac{5}{6},$$

so

$$I = \int_A (3 + y - z)x \, d\Omega = 6 \int_B xy \, dS = 6 \cdot \frac{5}{6} = 5.$$



IE
business
school

93%

OF MIM STUDENTS ARE
WORKING IN THEIR SECTOR 3 MONTHS
FOLLOWING GRADUATION

MASTER IN MANAGEMENT

- STUDY IN THE CENTER OF MADRID AND TAKE ADVANTAGE OF THE UNIQUE OPPORTUNITIES THAT THE CAPITAL OF SPAIN OFFERS
- PROPEL YOUR EDUCATION BY EARNING A DOUBLE DEGREE THAT BEST SUITS YOUR PROFESSIONAL GOALS
- STUDY A SEMESTER ABROAD AND BECOME A GLOBAL CITIZEN WITH THE BEYOND BORDERS EXPERIENCE

Length: 10 MONTHS
Av. Experience: 1 YEAR
Language: ENGLISH / SPANISH
Format: FULL-TIME
Intakes: SEPT / FEB

5 SPECIALIZATIONS
PERSONALIZE YOUR PROGRAM

#10 WORLDWIDE
MASTER IN MANAGEMENT
FINANCIAL TIMES

55 NATIONALITIES
IN CLASS

www.ie.edu/master-management | mim.admissions@ie.edu | [Follow us on IE MIM Experience](#)



Click on the ad to read more

A. Calculate the space integral

$$\int_A (x + 2y + z) \exp(z^4) d\Omega,$$

where

$$A = \{(x, y, z) \mid z \in [2, 0], (x, y) \in B(z)\}$$

and where the cut at the height z is given by

$$B(z) = [0, z] \times \left[0, \frac{z}{2}\right], \quad z \in [0, 2].$$

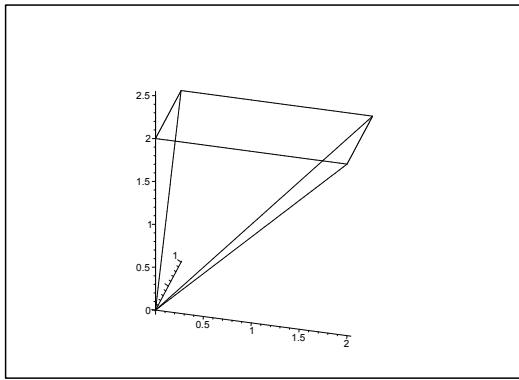


Figure 22.4: Legemet A .

D. Apply Theorem 22.2.

I. We get by insertion into Theorem 22.2,

$$(22.2) \quad I = \int_A (x + 2y + z) \exp(z^4) d\Omega \\ = \int_0^2 \exp(z^4) \left\{ \int_{B(z)} (x + 2y + z) dS \right\} dz.$$

Then we reduce for every fixed z the innermost abstract plane integral,

$$\begin{aligned} \int_{B(z)} (x + 2y + z) dS &= \int_{B(z)} x dS + \int_{B(z)} 2y dS + z \int_{B(z)} dS \\ &= \int_0^z x dx \cdot \int_0^{\frac{z}{2}} dy + \int_0^z dx \cdot \int_0^{\frac{z}{2}} 2y dy + z \cdot \text{area } B(z) \\ &= \frac{z^2}{2} \cdot \frac{z}{2} + z \cdot \frac{z^2}{4} + z \cdot \left\{ z \cdot \frac{1}{2} z \right\} = z^3. \end{aligned}$$

We insert this result into (22.2) and apply the substitution

$$t = z^4, \quad dt = 4z^3 dz, \quad \text{i.e.} \quad z^3 dz = \frac{1}{4} dt,$$

to get the result

$$I = \int_0^2 \exp(z^4) \cdot z^3 dz = \int_0^{2^4} e^t \cdot \frac{1}{4} dt = \frac{1}{4} (e^{16} - 1).$$

**"I studied English for 16 years but...
...I finally learned to speak it in just six lessons"**

Jane, Chinese architect

ENGLISH OUT THERE

Click to hear me talking before and after my unique course download



Click on the ad to read more

Example 22.1 Calculate in each of the following cases the given space integral over a point set

$$A = \{(x, y, z) \mid (x, y) \in B, Z_1(x, y) \leq z \leq Z_2(x, y)\}.$$

- 1) The space integral $\int_A xy^2 z d\Omega$, where the plane point set B is given by $x \geq 0, y \geq 0$ and $x + y \leq 1$, and where $Z_1(x, y) = 0$ and $Z_2(x, y) = 2 - x - y$.
- 2) The space integral $\int_A xy^2 z^3 d\Omega$, where the plane point set B is given by $0 \leq x \leq y \leq 1$, and where $Z_1(x, y) = 0$ and $Z_2(x, y) = xy$.
- 3) The space integral $\int_A z d\Omega$, where the plane point set B is given by $0 \leq x \leq 6$ and $2-x \leq y \leq 3 - \frac{x}{2}$, and where $Z_1(x, y) = 0$ and $Z_2(x, y) = \sqrt{16 - y^2}$.
- 4) The space integral $\int_A y d\Omega$, where the plane point set B is given by $-2 \leq y \leq 1$ and $y^2 \leq x \leq 2 - y$, and where $Z_1(x, y) = 0$ and $Z_2(x, y) = 4 - 2x - 2y$.
- 5) The space integral $\int_A \frac{1}{x^2 y^2 z^2} d\Omega$, where the plane point set B is given by $1 \leq x \leq \sqrt{3}$ and $\frac{1}{1+x^2} \leq y \leq 1$, and where $Z_1(x, y) = \frac{1}{1+x^2}$ and $Z_2(x, y) = 1 + x^2$.
- 6) The space integral $\int_A yz d\Omega$, where the plane point set B is given by $0 \leq x \leq 1$ and $0 \leq y \leq x$, and where $Z_1(x, y) = 0$ and $Z_2(x, y) = 2 - 2x$.
[**Cf. Example 22.2.6.**]
- 7) The space integral $\int_A xz d\Omega$, where the plane point set B is given by $0 \leq x \leq 1$ and $0 \leq y \leq 1$, and where $Z_1(x, y) = 0$ and $Z_2(x, y) = 1 - y$.
[**Cf. Example 22.2.7.**]
- 8) The space integral $\int_A z d\Omega$, where the plane point set B is given by $\sqrt{x^2 + y^2} \leq 2$, and where $Z_1(x, y) = 0$ and $Z_2(x, y) = 2 - \sqrt{x^2 + y^2}$.
[**Cf. Example 22.2.8**]

A Space integral in rectangular coordinates.

D Apply the first theorem of reduction.

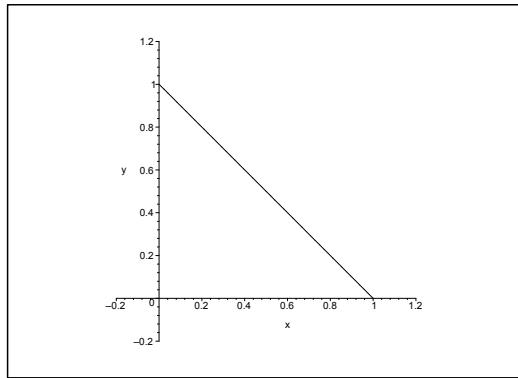


Figure 22.5: The domain B of **Example 22.1.1.**

I 1) By the first theorem of reduction,

$$\begin{aligned}
 \int_A xy^2 z \, d\Omega &= \int_B xy^2 \left\{ \int_0^{2-x-y} z \, dz \right\} \, dS = \frac{1}{2} \int_B x^2 y (2-x-y)^2 \, dS \\
 &= \frac{1}{2} \int_B x^2 y \{ (2-x)^2 - 2(2-x)y + y^2 \} \, dS \\
 &= \frac{1}{2} \int_0^1 x^2 \left\{ \int_0^{1-x} [(2-x)^2 y - 2(2-x)y^2 + y^3] \, dy \right\} \, dx \\
 &= \frac{1}{2} \int_0^1 x^2 \left[\frac{1}{2}(2-x)^2 y^2 - \frac{2}{3}(2-x)y^3 + \frac{1}{4}y^4 \right]_{y=0}^{1-x} \, dx \\
 &= \frac{1}{24} \int_0^1 x^2 \{ 6(2-x)^2(1-x)^2 - 8(2-x)(1-x)^3 + 3(1-x)^4 \} \, dx \\
 &= \frac{1}{24} \int_0^1 x^2 (1-x)^2 \{ 6(4-4x+x^2) - 8(2-3x+x^2) + 3(1-2x+x^2) \} \, dx \\
 &= \frac{1}{24} \int_0^1 (x^4 - 2x^3 + x^2)(x^2 - 6x + 11) \, dx \\
 &= \frac{1}{24} \int_0^1 \{ x^6 - 8x^5 + 24x^4 - 28x^3 + 11x^2 \} \, dx \\
 &= \frac{1}{24} \left[\frac{1}{7}x^7 - \frac{4}{3}x^6 + \frac{24}{5}x^5 - 7x^4 + \frac{11}{3}x^3 \right]_0^1 \\
 &= \frac{1}{24} \left(\frac{1}{7} - \frac{4}{3} + \frac{24}{5} - 7 + \frac{11}{3} \right) = \frac{1}{24} \left(\frac{1}{7} + 2 + \frac{1}{3} + 5 - \frac{1}{5} - 7 \right) \\
 &= \frac{1}{24} \left(\frac{1}{3} - \frac{1}{5} + \frac{1}{7} \right) = \frac{1}{24} \left(\frac{2}{15} + \frac{1}{7} \right) = \frac{29}{24 \cdot 105} = \frac{29}{2520}.
 \end{aligned}$$

MAPLE. This is of course very easy for MAPLE. We use the commands,
with(Student[MultivariateCalculus]):

$$\frac{1}{2} \cdot \text{MultiInt} (x^2 \cdot y \cdot ((2-x)^2 - 2(2-x) \cdot y + y^2), y = 0..1-x, x = 0..1)$$

$\frac{29}{2520}$

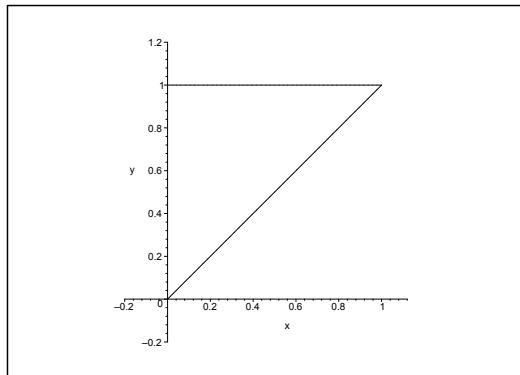


Figure 22.6: The domain B of **Eksempel 22.1.2.**

Excellent Economics and Business programmes at:

 university of
groningen





**“The perfect start
of a successful,
international career.”**

www.rug.nl/feb/education

CLICK HERE
to discover why both socially
and academically the University
of Groningen is one of the best
places for a student to be



Click on the ad to read more

2) By the first theorem of reduction,

$$\begin{aligned}\int_A xy^2 z^3 d\Omega &= \int_B xy^2 \left\{ \int_0^{xy} z^3 dz \right\} dS = \frac{1}{4} \int_B xy [z^4]_{z=0}^{xy} dS = \frac{1}{4} \int_B x^5 y^6 dS \\ &= \frac{1}{4} \int_0^1 y^6 \left\{ \int_0^y x^5 dx \right\} dy = \frac{1}{24} \int_0^1 y^{12} dy = \frac{1}{24 \cdot 13} = \frac{1}{312}.\end{aligned}$$

MAPLE. This is of course very easy for MAPLE. We use the commands,

with(Student[MultivariateCalculus]):

$$\frac{1}{4} \cdot \text{MultiInt}(x^5 \cdot y^6, x = 0..y, y = 0..1)$$

$$\frac{1}{312}$$

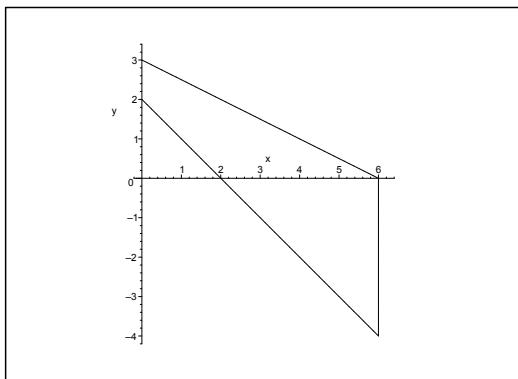


Figure 22.7: The domain B of Example 22.1.3.

3) By the theorem of reduction,

$$\begin{aligned}\int_A z d\Omega &= \int_B \left\{ \int_0^{\sqrt{16-y^2}} z dz \right\} dS = \frac{1}{2} \int_B (16 - y^2) dS \\ &= \frac{1}{2} \int_0^6 \left\{ \int_{2-x}^{3-\frac{x}{2}} (16 - y^2) dy \right\} dx = \frac{1}{2} \int_0^6 \left[16y - \frac{1}{3} y^3 \right]_{y=2-x}^{3-\frac{x}{2}} dx \\ &= \frac{1}{2} \int_0^6 \left\{ 16 \left(3 - \frac{x}{2} \right) - \frac{1}{3} \left(3 - \frac{x}{2} \right)^3 - 16(2-x) + \frac{1}{3}(2-x)^3 \right\} dx \\ &= \frac{1}{2} \int_0^6 \left\{ 16 + 8x + \frac{1}{24} (x-6)^2 - \frac{1}{3} (x-2)^3 \right\} dx \\ &= \frac{1}{2} \left[16x + 4x^2 + \frac{1}{96} (x-6)^4 - \frac{1}{12} (x-2)^4 \right]_0^6 \\ &= \frac{1}{2} \left\{ 96 + 144 + 0 - \frac{4^4}{12} - \frac{6^4}{96} + \frac{2^4}{12} \right\} = \frac{1}{2} \left\{ 240 - \frac{64}{3} - \frac{216}{16} + \frac{4}{3} \right\} \\ &= \frac{1}{2} \left\{ 220 - \frac{27}{2} \right\} = \frac{413}{4}.\end{aligned}$$

MAPLE. This is very easy for MAPLE. We use the commands,
with(Student[MultivariateCalculus]):

$$\frac{1}{2} \cdot \text{MultiInt}(16 - y^2, y = 2 - x..3 - 0.5x, x = 0..6)$$

$$\frac{413}{4}$$

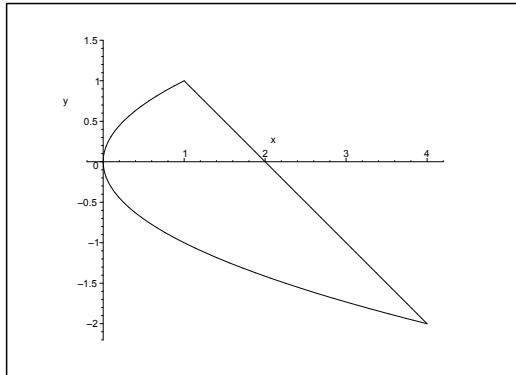


Figure 22.8: The domain B of Example 22.1.4.

4) By the theorem of reduction,

$$\begin{aligned} \int_A y \, d\Omega &= \int_B y \left\{ \int_0^{4-2x-2y} dz \right\} dS = \int_B y(4-2x-2y) \, dS \\ &= \int_{-2}^1 y \left\{ \int_{y^2}^{2-y} (4-2x-2y) \, dx \right\} dy = \int_{-2}^1 y [4x - x^2 - 2xy]_{x=y^2}^{2-y} dy \\ &= \int_{-2}^1 y \{4(2-y) - (2-y)^2 - 2y(2-y) - 4y^2 + y^4 + 2y^3\} dy \\ &= \int_{-2}^1 y \{8 - 4y - 4 + 4y - y^2 - 4y + 2y^2 - 4y^2 + 2y^3 + y^4\} dy \\ &= \int_{-2}^1 (y^5 + 2y^4 - 3y^3 - 4y^2 + 4y) dy = \left[\frac{1}{6}y^6 + \frac{2}{5}y^5 - \frac{3}{4}y^4 - \frac{4}{3}y^3 + 2y^2 \right]_{-2}^1 \\ &= \frac{1}{6} + \frac{2}{5} - \frac{3}{4} - \frac{4}{3} + 2 - \frac{2^6}{6} + \frac{2}{5} \cdot 2^5 + \frac{3}{4} \cdot 2^4 - \frac{4}{3} \cdot 2^3 - 8 = -\frac{81}{20}. \end{aligned}$$

MAPLE. This is very easy for MAPLE. We use the commands,

with(Student[MultivariateCalculus]):

$$\text{MultiInt}(y \cdot (4 - 2x - 2y), x = y^2..2 - y, y = -2..1)$$

$$-\frac{81}{20}$$

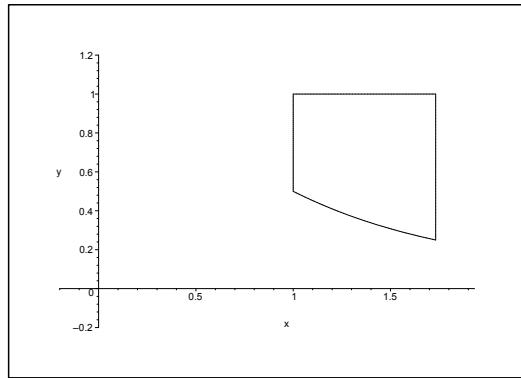


Figure 22.9: The domain B of **Example 22.1.5**.

- 5) First note that the z -integral does not depend on y . When we exploit this observation we get by the theorem of reduction,

$$\begin{aligned}
 \int_A \frac{1}{x^2 y^2 z^2} d\Omega &= \int_1^{\sqrt{3}} \frac{1}{x^2} \left\{ \int_{\frac{-1}{1+x^2}}^1 \frac{1}{y^2} \left(\int_{\frac{-1}{1+x^2}}^{1+x^2} \frac{1}{z^2} dz \right) dy \right\} dx \\
 &= \int_1^{\sqrt{3}} \frac{1}{x^2} \left[-\frac{1}{y} \right]_{\frac{-1}{1+x^2}}^1 \cdot \left[-\frac{1}{z} \right]_{\frac{-1}{1+x^2}}^{1+x^2} dx \\
 &= \int_1^{\sqrt{3}} \frac{1}{x^2} (1 + x^2 - 1) \cdot \left(1 + x^2 - \frac{1}{1+x^2} \right) dx \\
 &= \int_1^{\sqrt{3}} \left(1 + x^2 - \frac{1}{1+x^2} \right) dx = \left[\frac{x^3}{3} + x - \text{Arctan } x \right]_1^{\sqrt{3}} \\
 &= \frac{3\sqrt{3}}{3} + \sqrt{3} - \text{Arctan } \sqrt{3} - \frac{1}{3} - 1 + \text{Arctan } 1 \\
 &= 2\sqrt{3} - \frac{4}{3} - \frac{\pi}{3} + \frac{\pi}{4} = 2\sqrt{3} - \frac{4}{3} - \frac{\pi}{12}.
 \end{aligned}$$

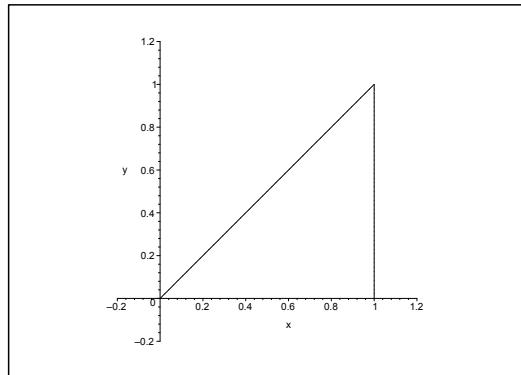


Figure 22.10: The domain B of **Example 22.1.6**.

MAPLE. This shows that MAPLE can also handle triple integrals. We use the commands,
with(Student[MultivariateCalculus]):

$$\frac{1}{2} \cdot \text{MultiInt} \left(\frac{1}{x^2 \cdot y^2 \cdot z^2}, z = \frac{1}{1+x^2}..1+x^2, y = \frac{1}{1+x^2}..1, x = 0..\sqrt{3} \right)$$

$$-\frac{1}{12}\pi - \frac{4}{3} + 2\sqrt{3}$$

6) By the theorem of reduction,

$$\begin{aligned} \int_A yz \, d\Omega &= \int_0^1 \left\{ \int_0^x y \left(\int_0^{2-2x} z \, dz \right) dy \right\} dx = \int_0^1 \left[\frac{y^2}{2} \right]_0^x \cdot \left[\frac{z^2}{2} \right]_0^{2-2x} dx \\ &= \frac{1}{4} \int_0^1 x^2 \cdot (2-2x)^2 dx = \int_0^1 x^2(1-x^2) dx = \int_0^1 x^2(x^2-2x+1) dx \\ &= \int_0^1 (x^4 - 2x^3 + x^2) dx = \frac{1}{5} - \frac{2}{4} + \frac{1}{3} = \frac{6-15+10}{30} = \frac{1}{30}. \end{aligned}$$

American online LIGS University

is currently enrolling in the
Interactive Online BBA, MBA, MSc,
DBA and PhD programs:

- ▶ enroll by **September 30th, 2014** and
- ▶ save up to **16%** on the tuition!
- ▶ pay in 10 installments / 2 years
- ▶ Interactive Online education
- ▶ visit www.ligsuniversity.com to find out more!

Note: LIGS University is not accredited by any nationally recognized accrediting agency listed by the US Secretary of Education.
[More info here.](#)



Click on the ad to read more

MAPLE. This is easy for MAPLE. We use the commands,
with(Student[MultivariateCalculus]):

MultiInt ($y \cdot z, z = 0..2 - 2x, y = 0..x, x = 0..1$)

$$\frac{1}{30}$$

REMARK. The domain is also described by

$$0 \leq z \leq 2, \quad 0 \leq y \leq x \leq 1 - \frac{z}{2},$$

cf. **Example 22.2.6.** The two examples therefore give the same result. ◇

- 7) Here, $B = [0, 1] \times [0, 1]$, It therefore follows by the theorem of reduction that

$$\int_A xz \, d\Omega = \int_0^1 x \, dx \cdot \int_0^1 \left\{ \int_0^{1-y} z \, dz \right\} dy = \frac{1}{2} \int_0^1 \frac{1}{2} (1-y)^2 dy = \frac{1}{4} \int_0^1 t^2 dt = \frac{1}{12}.$$

- 8) Here, B is the closed disc of centrum $(0, 0)$ and radius 2. By using the theorem of reduction in semi-polar coordinates,

$$\begin{aligned} \int_A z \, d\Omega &= 2\pi \int_0^2 \left\{ \int_0^{2-\varrho} z \, dz \right\} \varrho \, d\varrho = \pi \int_0^2 (2-\varrho)^2 \varrho \, d\varrho = \pi \int_0^2 (\varrho^3 - 4\varrho^2 + 4\varrho) \, d\varrho \\ &= \pi \left[\frac{\varrho^4}{4} - \frac{4}{3} \varrho^3 + 2\varrho^2 \right]_0^2 = \pi \left\{ 4 - \frac{32}{3} + 8 \right\} = \frac{4\pi}{3}. \end{aligned}$$

DON'T EAT YELLOW SNOW

What will your advice be?

Some advice just states the obvious. But to give the kind of advice that's going to make a real difference to your clients you've got to listen critically, dig beneath the surface, challenge assumptions and be credible and confident enough to make suggestions right from day one. At Grant Thornton you've got to be ready to kick start a career right at the heart of business.



Grant Thornton
An instinct for growth™

© Grant Thornton LLP. A Canadian Member of Grant Thornton International Ltd

Sound like you? Here's our advice: visit
GrantThornton.ca/careers/students

Scan here to learn more about a career
with Grant Thornton.



Click on the ad to read more

Example 22.2 Calculate in each of the following cases the given space integral over a point set

$$A = \{(x, y, z) \mid \alpha \leq z \leq \beta, (x, y) \in B(z)\}.$$

- 1) The space integral $\int_A z^2 d\Omega$, where $B(z)$ is given by $|x| \leq z$ and $|y| \leq 2z$ for $z \in [0, 1]$.
- 2) The space integral $\int_A xz d\Omega$, where $B(z)$ is given by $0 \leq x, 0 \leq y$ and $x + y \leq z^2$ for $z \in [0, 1]$.
- 3) The space integral $\int_A xy^2 z d\Omega$, where $B(z)$ is given by $0 \leq x, 0 \leq y$ and $x + y \leq 2 - z$ for $z \in [1, 2]$.
- 4) The space integral $\int_A \frac{1}{xy^2} d\Omega$, where $B(z)$ is given by $1 \leq x \leq z$ and $z \leq y \leq z$ for $z \in [1, 3]$.
- 5) The space integral $\int_A \left(\frac{\sin z}{z}\right)^2 d\Omega$, where $B(z)$ is given by $|x| + |y| \leq |z|$ for $z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.
- 6) The space integral $\int_A yz d\Omega$, where $B(z)$ is given by $0 \leq y \leq x \leq 1 - \frac{z}{2}$ for $z \in [0, 2]$.
[Cf. Example 22.1.6.]
- 7) The space integral $\int_A xz d\Omega$, where $B(z)$ is given by $0 \leq x \leq 1$ and $0 \leq y \leq 1 - z$ for $z \in [0, 1]$.
[Cf. Example 22.1.7.]
- 8) The space integral $\int_A z d\Omega$, where $B(z)$ is given by $x^2 + y^2 \leq (2 - z)^2$ for $z \in [0, 2]$.
[Cf. Example 22.1.8.]

A Space integrals in rectangular coordinates, where the domain is sliced, $B(z)$, with the height z as parameter.

D Whenever it is necessary, sketch $B(z)$. Then apply the second theorem of reduction.

I 1) Here,

$$B(z) = \{(x, y) \mid -z \leq x \leq z, -2z \leq y \leq 2z\} = [-z, z] \times [-2z, 2z],$$

which is a rectangle for every $z \in]0, 1]$ of the area

$$\text{area}\{B(z)\} = 8z^2.$$

We get by reduction,

$$\int_A z^2 d\Omega = \int_0^1 z^2 \left\{ \int_{B(z)} dS \right\} dz = \int_0^1 z^2 \cdot \text{area}\{B(z)\} dz = \int_0^1 8z^4 dz = \frac{8}{5}.$$

2) Here

$$B(z) = \{(x, t) \mid 0 \leq x, 0 \leq y, x + y \leq z^2\}$$

is a triangle for every $z \in]0, 1]$, namely the lower triangle of the square $[0, z^2] \times [0, z^2]$, when this is cut by a diagonal from the upper left corner to the lower right corner.

We get by the theorem of reduction,

$$\begin{aligned} \int_A xz d\Omega &= \int_0^1 z \left\{ \int_{B(z)} x dS \right\} dz = \int_0^1 z \left\{ \int_0^{z^2} x \left[\int_0^{z^2-x} dy \right] dx \right\} dz \\ &= \int_0^1 z \left\{ \int_0^{z^2} (xz^2 - x^2) dx \right\} dz = \int_0^1 z \left[\frac{1}{2} x^2 z^2 - \frac{1}{3} x^3 \right]_0^{z^2} dz = \int_0^1 \frac{1}{6} z^7 dz = \frac{1}{48}. \end{aligned}$$

MAPLE. This is easy for MAPLE. We use the commands,
with(Student[MultivariateCalculus]):

$$\text{MultiInt}(x \cdot z, y = 0..z^2 - x, x = 0..z^2, z = 0..1)$$

$$\frac{1}{48}$$

3) Here

$$B(z) = \{(x, y) \mid 0 \leq x, 0 \leq y, x + y \leq 2 - z\}$$

is a triangle for every $z \in [1, 2]$, namely the lower triangle of the square $[0, 2 - z] \times [0, 2 - z]$, when this is cut by a diagonal from the upper left corner to the lower right corner.

Then by the theorem of reduction,

$$\begin{aligned} \int_A xy^2 z \, d\Omega &= \int_1^2 z \left\{ \int_{B(z)} xy^2 \, dS \right\} dz = \int_1^2 z \left\{ \int_0^{2-z} y^2 \left[\int_0^{2-z-y} x \, dx \right] dy \right\} dz \\ &= \frac{1}{2} \int_1^2 z \left\{ \int_0^{2-z} y^2 (z - 2 + y)^2 dy \right\} dz \\ &= \frac{1}{2} \int_1^2 \{(z - 2) + 2\} \left\{ \int_0^{2-z} [(z - 2)^2 y^2 + 2(z - 2)y^3 + y^4] dy \right\} dz \\ &= \frac{1}{2} \int_1^2 \{(z - 2) + 2\} \left[\frac{1}{3}(z - 2)^2 y^3 + \frac{1}{2}(z - 2)y^4 + \frac{1}{5}y^5 \right]_{y=0}^{2-z} dz \\ &= \frac{1}{2} \int_1^2 \{(z - 2) + 2\} \cdot \left(-\frac{1}{3} + \frac{1}{2} - \frac{1}{5} \right) (z - 2)^5 dz \\ &= -\frac{1}{60} \int_1^2 \{(z - 2)^6 + 2(z - 2)^5\} dz \\ &= -\frac{1}{60} \left[\frac{1}{7}(z - 2)^7 + \frac{1}{3}(z - 2)^6 \right]_1^2 = -\frac{1}{60} \left(\frac{1}{7} - \frac{1}{3} \right) = \frac{4}{3 \cdot 7 \cdot 60} = \frac{1}{315}. \end{aligned}$$

MAPLE. This is easy for MAPLE. We use the commands,

with(Student[MultivariateCalculus]):

$$\text{MultiInt}(y \cdot y^2 \cdot z, x = 0..2 - z - y, y = 0..2 - z, z = 1..2)$$

$$\frac{1}{315}$$

4) Here

$$B(z) = \{(x, y) \mid 1 \leq x \leq z, z \leq y \leq 2z\}, \quad z \in [1, 3],$$

which is sketched on the figure.

We get by the theorem of reduction,

$$\begin{aligned} \int_A \frac{1}{xy^2} \, d\Omega &= \int_1^3 \left\{ \int_{B(z)} \frac{1}{xy^2} \, dS \right\} dz = \int_1^3 \left\{ \int_1^z \frac{1}{x} \left[\int_z^{2z} \frac{1}{y^2} dy \right] dx \right\} dz \\ &= \int_1^3 \left\{ \int_1^z \frac{1}{x} \left[-\frac{1}{y} \right]_z^{2z} dx \right\} dz = \int_1^3 \frac{1}{2z} [\ln x]_{x=1}^z dz = \frac{1}{4} [(\ln z)^2]_1^3 = \frac{1}{4} (\ln 3)^2. \end{aligned}$$

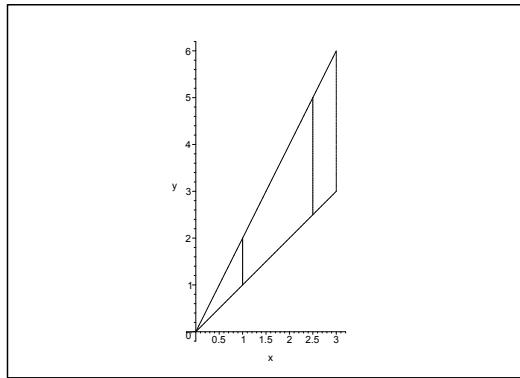


Figure 22.11: The domain $B(z)$ of **Example 22.2.4**.

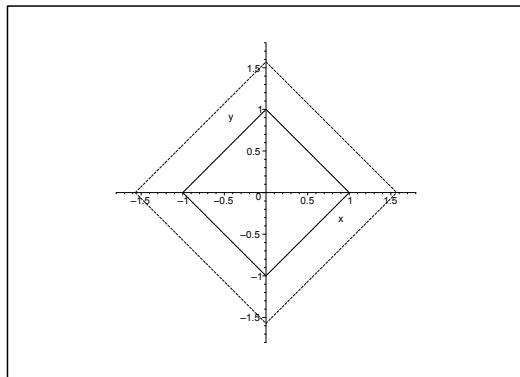


Figure 22.12: The domain $B(z)$ of **Example 22.2.5**.

MAPLE. This is easy for MAPLE. We use the commands,

`with(Student[MultivariateCalculus]):`

$$\text{MultiInt} \left(\frac{1}{x \cdot y^2}, y = z..2z, x = 1..z, z = 1..3 \right)$$

$$\frac{1}{4}(\ln(3))^2$$

- 5) By a continuous extension the integrand is put equal to 1 for $z = 0$. Note that $B(z)$ is a square of edge length $\sqrt{2}|z|$, hence of the area

$$\text{area}\{B(z)\} = 2z^2.$$

Then by the theorem of reduction,

$$\begin{aligned} \int_A \left(\frac{\sin z}{z} \right)^2 d\Omega &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\sin z}{z} \right)^2 \left\{ \oint_{B(z)} dS \right\} dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\sin z}{z} \right) \text{area}\{B(z)\} dz \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\sin z}{z} \right)^2 \cdot 2z^2 dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2 \sin^2 z dz \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \cos 2z) dz = \pi - \left[\frac{1}{2} \sin 2z \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi. \end{aligned}$$

..... Alcatel-Lucent 

www.alcatel-lucent.com/careers

What if you could build your future and create the future?

One generation's transformation is the next's status quo.
In the near future, people may soon think it's strange that devices ever had to be "plugged in." To obtain that status, there needs to be "The Shift".



Click on the ad to read more

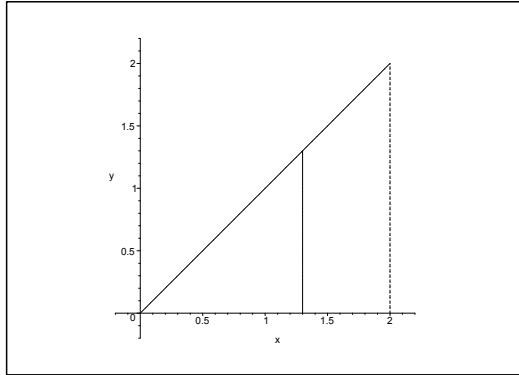


Figure 22.13: The domain $B(z)$ of **Example 22.2.6**.

6) Here

$$B(z) = \left\{ (x, y) \mid 0 \leq y \leq x \leq 1 - \frac{z}{2} \right\}$$

is a triangle for every $z \in [0, 2[$.

We get by the second theorem of reduction,

$$\begin{aligned} \int_A yz \, d\Omega &= \int_0^2 z \left\{ \int_{B(z)} y \, dS \right\} dz = \int_0^2 z \left\{ \int_0^{1-\frac{z}{2}} \left(\int_0^x y \, dy \right) dx \right\} dz \\ &= \int_0^2 z \left\{ \int_0^{1-\frac{z}{2}} \frac{1}{2} x^2 \, dx \right\} dz = \frac{1}{6} \int_0^2 z \left(-\frac{z}{2} \right)^3 dz \\ &= \frac{1}{6} \int_0^2 z \left(1 - \frac{3}{2}z + \frac{3}{4}z^2 - \frac{1}{8}z^3 \right) dz = \frac{1}{6} \int_0^2 \left(z - \frac{3}{2}z^2 + \frac{3}{4}z^3 - \frac{1}{8}z^4 \right) dz \\ &= \frac{1}{6} \left[\frac{1}{2}z^2 - \frac{1}{2}z^3 + \frac{3}{16}z^4 - \frac{1}{40}z^5 \right]_0^2 = \frac{1}{6} \left(\frac{4}{2} - \frac{8}{4} + \frac{3}{16} \cdot 16 - \frac{1}{40} \cdot 32 \right) \\ &= \frac{1}{6} \left(2 - 4 + 3 - \frac{4}{5} \right) = \frac{1}{6} \left(1 - \frac{4}{5} \right) = \frac{1}{30}. \end{aligned}$$

MAPLE. This is easy for MAPLE. We use the commands,

with(Student[MultivariateCalculus]):

$$\text{MultiInt} \left(y \cdot z, y = 0..x, x = 0..1 - \frac{z}{2}, z = 0..2 \right)$$

$$\frac{1}{30}$$

REMARK. The domain is also described by

$$0 \leq x \leq 1, \quad 0 \leq y \leq x, \quad 0 \leq z \leq 2 - 2x,$$

cf. **Example 22.1.6**, and we have computed the integral in two different ways (and luckily obtained the same result). \diamond

- 7) They have the same integrand and the same domain as in **Example 22.1.7**, so we must get the same result. The only difference is that we here cut the domain into slices, while we in **Example 22.1** used the “method of vertical posts”.

We get by the theorem of reduction,

$$\begin{aligned}\int_A xz \, d\Omega &= \int_0^1 x \, dx \cdot \int_0^1 z \left\{ \int_0^{1-z} dy \right\} dz = \frac{1}{2} \int_0^1 z(1-z) \, dz \\ &= \frac{1}{2} \int_0^1 \{z - z^2\} \, dz = \frac{1}{2} \left\{ \frac{1}{2} - \frac{1}{3} \right\} = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}.\end{aligned}$$

- 8) We have the same integrand and the same set as in **Example 22.1.8**, so we must get the same result. The only difference is that we here cut the domain into slices, while we in **Example 22.1** used the “method of vertical posts”. Also note that we use polar coordinates in each slice, so the example should actually be moved to **Example 24.2**.

We get by the theorem of reduction in semi-polar coordinates and the change of variables $u = 2 - z$ that

$$\begin{aligned}\int_A z \, d\Omega &= \int_0^2 z \cdot \pi(2 - z^2) \, dz = \pi \int_0^2 (2 - u)u^2 \, du = \pi \int_0^2 (2u^2 - u^3) \, du = \pi \left[\frac{2}{3}u^3 - \frac{1}{4}u^4 \right] \\ &= \pi \left\{ \frac{16}{3} - \frac{16}{4} \right\} = \frac{16}{12} \pi = \frac{4\pi}{3}.\end{aligned}$$



**Join the best at
the Maastricht University
School of Business and
Economics!**

**Visit us and find out why we are the best!
Master's Open Day: 22 February 2014**

Top master's programmes

- 33rd place Financial Times worldwide ranking: MSc International Business
- 1st place: MSc International Business
- 1st place: MSc Financial Economics
- 2nd place: MSc Management of Learning
- 2nd place: MSc Economics
- 2nd place: MSc Econometrics and Operations Research
- 2nd place: MSc Global Supply Chain Management and Change

Sources: Keuzegids Master ranking 2013; Elsevier 'Beste Studies' ranking 2012; Financial Times Global Masters in Management ranking 2012

Maastricht
University is
the best specialist
university in the
Netherlands
(Elsevier)

www.mastersopenday.nl



Click on the ad to read more

Example 22.3 Let A be the tetrahedron of the vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Compute in each of the following cases the space integral

$$\int_A f(x, y, z) d\Omega,$$

where

- 1) $f(x, y, z) = x + y + z$,
- 2) $f(x, y, z) = \cos(x + y + z)$,
- 3) $f(x, y, z) = \exp(x + y + z)$,
- 4) $f(x, y, z) = (1 + x + y + z)^{-3}$,
- 5) $f(x, y, z) = x^2 + y^2 + z^2$,
- 6) $f(x, y, z) = xy - yz$.

A Space integrals over a tetrahedron.

D Consider the tetrahedron as a cone with $(0, 0, 0)$ as its top point in the first four questions, where the natural variable is $x + y + z$. Therefore, first analyze this special case. Calculate the space integral with respect to this variable. Alternatively, calculate the triple integral. There is also the possibility of some arguments of symmetry.

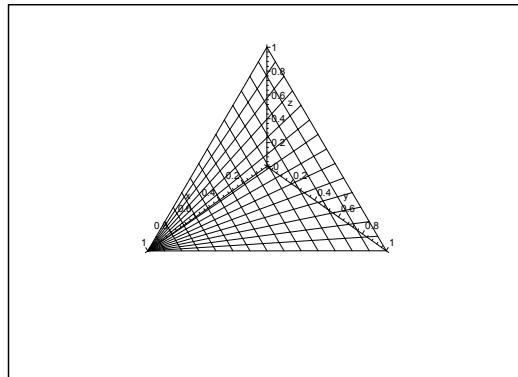


Figure 22.14: The tetrahedron of the vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

I PREPARATIONS. The distance from $(0, 0, 0)$ to the plane $x + y + z = 1$ is

$$\sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{1}{\sqrt{3}}.$$

Hence we can consider the tetrahedron as a cone of height $h = \frac{1}{\sqrt{3}}$ and with the surface where $x + y + z = 1$ as its base. the area of this base is

$$\frac{1}{2} |(1, 0, 0) - (0, 1, 0)| \cdot \left| (0, 0, 1) - \left(\frac{1}{2}, \frac{1}{2}, 0\right) \right| = \frac{1}{2} \sqrt{2} \cdot \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \frac{1}{2} \sqrt{2} \cdot \sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{2}.$$

Intersect the tetrahedron by the plane $x + y + z = t$, $t \in [0, 1]$, parallel to the base. Then the distance from the new triangle $B(t)$ to the top point $(0, 0, 0)$ is $\frac{t}{\sqrt{3}}$, thus the area of this triangle $B(t)$ is due to the similarity given by

$$\text{area}(B(t)) = \left(\frac{1/\sqrt{3}}{1/\sqrt{3}} \right)^2 t^2 \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} t^2, \quad t \in [0, 1].$$

If the integrand $f(x, y, z) = g(x + y + z)$ is a function in $t = x + y + z$, we even get the simpler formula

$$(22.3) \int_A f(x, y, z) d\Omega = \frac{1}{\sqrt{3}} \int_0^1 g(t) \text{area}(B(t)) dt = \frac{1}{2} \int_0^1 t^2 g(t) dt.$$

Clearly, (22.3) can be applied in the first four questions.

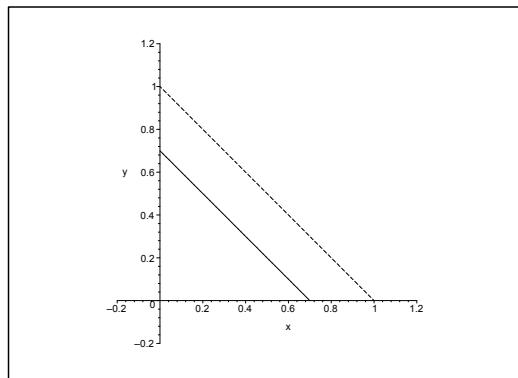


Figure 22.15: The projection of $B'(z)$ onto the XY -plane.

1) From $f(x, y, z) = x + y + z = t = g(t)$ and (22.3) follows that

$$\int_A (x + y + z) d\Omega = \frac{1}{2} \int_0^1 t^2 g(t) dt = \frac{1}{2} \int_0^1 t^3 dt = \frac{1}{8}.$$

ALTERNATIVELY the plane $z = \text{constant}$, $z \in [0, 1[$, intersects the tetrahedron in a set, the projection of which onto the XY -plane is

$$B'(z) = \{(x, y) \mid x \geq 0, y \geq 0, x + y \leq 1 - z\}.$$

Hence by more traditional calculations,

$$\int_A (x + y + z) d\Omega = \int_0^1 z \cdot \text{area}(B'(z)) dz + \int_0^1 \left\{ \int_{B'(z)} (x + y) dS \right\} dz.$$

It follows from the symmetry that

$$\int_{B'(z)} x dS = \int_{B'(z)} y dS,$$

hence

$$\begin{aligned}
 \int_A (x + y + z) d\Omega &= \int_0^1 z \cdot \frac{1}{2}(1-z)^2 dz + 2 \int_0^1 \left\{ \int_{B'(z)} x dS \right\} dz \\
 &= \frac{1}{2} \int_0^1 (1-t)t^2 dt + 2 \int_0^1 \left\{ \int_0^{1-z} x \left\{ \int_0^{1-x-z} dy \right\} dx \right\} dz \\
 &= \frac{1}{2} \left[\frac{1}{3} t^3 - \frac{1}{4} t^4 \right]_0^1 + 2 \int_0^1 \left\{ \int_0^{1-z} x[(1-z)-x] dx \right\} dz \\
 &= \frac{1}{24} + 2 \int_0^1 \left[\frac{1}{2} x^2 (1-z) - \frac{1}{3} x^3 \right]_0^{1-z} dz \\
 &= \frac{1}{24} + \frac{2}{6} \int_0^1 (1-z)^3 dz = \frac{1}{24} + \frac{1}{12} [(1-z)^4]_0^1 = \frac{1}{24} + \frac{1}{12} = \frac{1}{8}.
 \end{aligned}$$

2) It follows from $f(x, y, z) = \cos(x + y + z) = \cos t = g(t)$ and (22.3) that

$$\begin{aligned}
 \int_A \cos(x + y + z) d\Omega &= \frac{1}{2} \int_0^1 t^2 \cos t dt = \frac{1}{2} [t^2 \sin t + 2t \cos t - 2 \sin t]_0^1 \\
 &= \frac{1}{2} \sin 1 + \cos 1 - \sin 1 = \cos 1 - \frac{1}{2} \sin 1.
 \end{aligned}$$

ALTERNATIVELY, we get by more traditional calculations, where we use the same set $B'(z)$ as in 1),

$$\begin{aligned}
 \int_A \cos(x + y + z) d\Omega &= \int_0^1 \left\{ \int_{B'(z)} \cos(x + y + z) dS \right\} dz \\
 &= \int_0^1 \left\{ \int_0^{1-z} \left\{ \int_0^{1-z-x} \cos(x + y + z) dy \right\} dx \right\} dz \\
 &= \int_0^1 \left\{ \int_0^{1-z} [\sin(x + y + z)]_{y=0}^{1-z-x} dx \right\} dz = \int_0^1 \left\{ \int_0^{1-z} \{\sin 1 - \sin(x + z)\} dx \right\} dz \\
 &= \sin 1 \cdot \int_0^1 (1-z) dz + \int_0^1 [\cos(x + z)]_{x=0}^{1-z} dz = \frac{1}{2} \sin 1 + \int_0^1 \{\cos 1 - \cos z\} dz \\
 &= \frac{1}{2} \sin 1 + \cos 1 - \sin 1 = \cos 1 - \frac{1}{2} \sin 1.
 \end{aligned}$$

3) It follows from $f(x, y, z) = \exp(x + y + z) = e^t = g(t)$ and (22.3) that

$$\begin{aligned}
 \int_A \exp(x + y + z) d\Omega &= \frac{1}{2} \int_0^1 t^2 e^t dt = \frac{1}{2} [t^2 e^t - 2e^t + 2e^t]_0^1 \\
 &= \frac{1}{2} (e - 2e + 2e - 2) = \frac{1}{2} (e - 2).
 \end{aligned}$$

ALTERNATIVELY, by traditional computations,

$$\begin{aligned}
 \int_A \exp(x+y+z) d\Omega &= \int_0^1 \left\{ \int_0^{1-z} \left\{ \int_0^{1-z-x} \exp(x+y+z) dy \right\} dx \right\} dz \\
 &= \int_0^1 \left\{ \int_0^{1-z} [\exp(x+y+z)]_{y=0}^{1-z-x} dx \right\} dz = \int_0^1 \left\{ \int_0^{1-z} (e - e^{x+z}) dx \right\} dz \\
 &= e \int_0^1 \left\{ \int_0^{1-z} dx \right\} dz - \int_0^1 \left\{ \int_0^{1-z} e^{x+z} dx \right\} dz \\
 &= e \int_0^1 (1-z) dz - \int_0^1 [e^{x+z}]_{x=0}^{1-z} dz = \frac{1}{2} e - \int_0^1 (e - e^z) dz \\
 &= \frac{1}{2} e - e + [e^z]_0^1 = \frac{e}{2} - 1 = \frac{1}{2}(e-2).
 \end{aligned}$$

4) From $f(x, y, z) = (1+x+y+z)^{-3} = (1+t)^{-3} = g(t)$ and (22.3) follows that

$$\begin{aligned}
 \int_A (1+x+y+z)^{-3} d\Omega &= \frac{1}{2} \int_0^1 \frac{t^2}{(1+t)^3} dt = \frac{1}{2} \int_0^1 \left\{ \frac{1}{(t+1)^3} - \frac{2}{(t+1)^2} + \frac{1}{t+1} \right\} dt \\
 &= \frac{1}{2} \left[-\frac{1}{2} \cdot \frac{1}{(t+1)^2} + \frac{2}{t+1} + \ln(t+1) \right]_0^1 = \frac{1}{2} \left\{ -\frac{1}{2} \cdot \frac{1}{4} + \frac{2}{2} + \ln 2 + \frac{1}{2} - 2 \right\} \\
 &= \frac{1}{2} \left\{ \ln 2 - \frac{5}{8} \right\} = \frac{1}{2} \ln 2 - \frac{5}{16}.
 \end{aligned}$$

**Empowering People.
Improving Business.**

BI Norwegian Business School is one of Europe's largest business schools welcoming more than 20,000 students. Our programmes provide a stimulating and multi-cultural learning environment with an international outlook ultimately providing students with professional skills to meet the increasing needs of businesses.

BI offers four different two-year, full-time Master of Science (MSc) programmes that are taught entirely in English and have been designed to provide professional skills to meet the increasing need of businesses. The MSc programmes provide a stimulating and multi-cultural learning environment to give you the best platform to launch into your career.

- MSc in Business
- MSc in Financial Economics
- MSc in Strategic Marketing Management
- MSc in Leadership and Organisational Psychology

www.bi.edu/master

BI NORWEGIAN BUSINESS SCHOOL

EFMD
EQUIS
ACCREDITED

ALTERNATIVELY, by traditional calculations,

$$\begin{aligned}
 \int_A (1+x+y+z)^{-3} d\Omega &= \int_0^1 \left\{ \int_0^{1-z} \left\{ \int_0^{1-z-x} (1+x+y+z)^{-3} dy \right\} dx \right\} dz \\
 &= \int_0^1 \left\{ \int_0^{1-z} \left[-\frac{1}{2} (1+x+y+z)^{-2} \right]_{y=0}^{1-z-x} dx \right\} dz \\
 &= \frac{1}{2} \int_0^1 \left\{ \int_0^{1-z} \left\{ (1+x+z)^{-2} - \frac{1}{4} \right\} dx \right\} dz \\
 &= \frac{1}{2} \int_0^1 [-(1+x+z)^{-1}]_{x=0}^{1-z} dz - \frac{1}{8} \int_0^1 \left\{ \int_0^{1-z} dx \right\} dz \\
 &= \frac{1}{2} \int_0^1 \left\{ \frac{1}{1+z} - \frac{1}{2} \right\} dz - \frac{1}{16} = \frac{1}{2} \ln 2 - \frac{1}{4} - \frac{1}{16} = \frac{1}{2} \ln 2 - \frac{5}{16}.
 \end{aligned}$$

5) In this case we can no longer apply (22.3). We note by symmetry that

$$\int_{B'(z)} x^2 dS = \int_{B'(z)} y^2 dS,$$

hence by traditional calculations,

$$\begin{aligned}
 \int_A (x^2 + y^2 + z^2) d\Omega &= \int_0^1 z^2 \text{area}(B'(z)) dz + 2 \int_0^1 \left\{ \int_{B'(z)} x^2 dS \right\} dz \\
 &= \frac{1}{2} \int_0^1 z^2 (1-z)^2 dz + 2 \int_0^1 \left\{ \int_0^{1-z} x^2 \left\{ \int_0^{1-z-x} dy \right\} dx \right\} dz \\
 &= \frac{1}{2} \int_0^1 (z^2 - 2z^3 + z^4) dz + 2 \int_0^1 \left\{ \int_0^{1-z} x^2 (1-z-x) dx \right\} dz \\
 &= \frac{1}{2} \left[\frac{1}{3} z^3 - \frac{2}{4} z^4 + \frac{1}{5} z^5 \right]_0^1 + 2 \int_0^1 \left[\frac{1}{3} x^3 (1-z) - \frac{1}{4} x^4 \right]_{x=0}^{1-z} dz \\
 &= \frac{1}{2} \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) + 2 \cdot \frac{1}{12} \int_0^1 (1-z)^4 dz \\
 &= \frac{1}{60} (10 - 15 + 6) + \frac{1}{6} \left[-\frac{1}{5} (1-z)^5 \right]_0^1 = \frac{1}{60} + \frac{1}{30} = \frac{1}{20}.
 \end{aligned}$$

6) Put

$$B''(y) = \{(x, z) \mid 0 \leq x, 0 \leq z, x+z \leq 1-y\}.$$

It follows from symmetric reasons that

$$\int_{B''(y)} xy dS = \int_{B''(y)} yz dS.$$

Hence

$$\int_A (xy - yz) d\Omega = \int_0^1 \left\{ \int_{B''(y)} (xy - yz) dS \right\} dy = 0.$$

ALTERNATIVELY we get by traditional calculations,

$$\begin{aligned}
 \int_A (xy - yz) d\Omega &= \int_0^1 \left\{ \int_{B'(z)} (x-z)y dS \right\} dz \\
 &= \int_0^1 \left\{ \int_0^{1-z} (x-z) \left\{ \int_0^{1-(x+z)} y dy \right\} dx \right\} dz \\
 &= \frac{1}{2} \int_0^1 \left\{ \int_0^{1-z} (x-z)[1-(x+z)]^2 dx \right\} dz \\
 &= \frac{1}{2} \int_0^1 \left\{ \int_0^{1-z} (x+z)[(x+z)-1]^2 dx \right\} dz - \int_0^1 z \left\{ \int_0^{1-z} [(x+z)-1]^2 dx \right\} dz.
 \end{aligned}$$

In order to avoid too complicated expressions we compute the two double integrals one by one:

$$\begin{aligned}
 \frac{1}{2} \int_0^1 \left\{ \int_0^{1-z} (x+z)[(x+z)-1]^2 dx \right\} dz &= \frac{1}{2} \int_0^1 \left\{ \int_0^{1-z} \{(x+z)^3 - 2(x+z)^2 + (x+z)\} dx \right\} dz \\
 &= \frac{1}{2} \int_0^1 \left[\frac{1}{4}(x+z)^4 - \frac{2}{3}(x+z)^3 + \frac{1}{2}(x+z)^2 \right]_{x=0}^{1-z} dz \\
 &= \frac{1}{2} \int_0^1 \left\{ \frac{1}{4} - \frac{2}{3} + \frac{1}{2} - \frac{1}{4} z^4 + \frac{2}{3} z^3 - \frac{1}{2} z^2 \right\} dz \\
 &= \frac{1}{24} + \frac{1}{2} \left[-\frac{1}{20} + \frac{1}{6} - \frac{1}{6} \right] = \frac{1}{24} - \frac{1}{40} = \frac{5-3}{120} = \frac{1}{60},
 \end{aligned}$$

and

$$\begin{aligned}
 \int_0^1 z \left\{ \int_0^{1-z} [(x+z)-1]^2 dx \right\} dz &= \int_0^1 z \left[\frac{1}{3}(x+z-1)^3 \right]_{x=0}^{1-z} dz \\
 &= \frac{1}{3} \int_0^1 z(1-z)^3 dz = \frac{1}{3} \int_0^1 (1-t)t^3 dt = \frac{1}{3} \left[\frac{1}{4}t^4 - \frac{1}{5}t^5 \right]_0^1 = \frac{1}{60}.
 \end{aligned}$$

Finally, we get by insertion,

$$\int_A (xy - yz) d\Omega = \frac{1}{60} - \frac{1}{60} = 0.$$

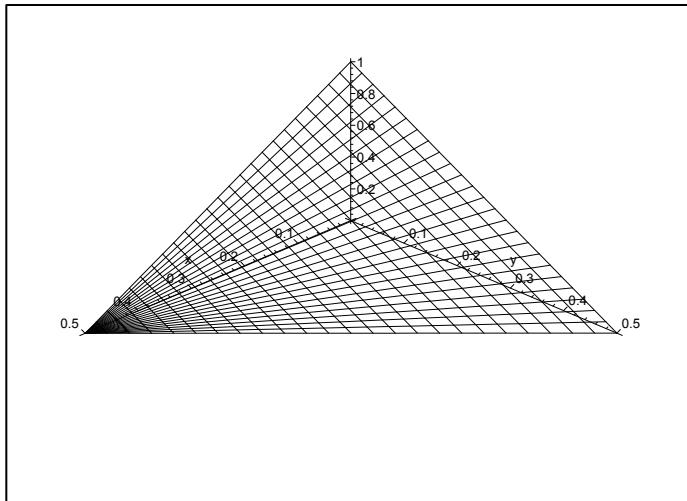


Figure 22.16: The tetrahedron A with its projection B onto the (x, y) -plane.

Example 22.4 Let B be the triangle which is bounded by the X -axis and the Y -axis and the line of the equation $x + y = \frac{1}{2}$. Furthermore, let A be the tetrahedron bounded by the three coordinate planes and the plane of the equation $2x + 2y + z = 1$. Compute the integrals

$$\int_B (1 - 2x - 2y) dS \quad \text{and} \quad \int_A (x + y + z) d\Omega.$$

A Plane integral and space integral.

D Sketch B and A . Then compute the integrals.

I It follows immediately that B is that surfaces of A , which lies in the (x, y) -plane.

First calculate the plane integral (it is actually the volume of the tetrahedron A),

$$\begin{aligned} \int_B (1 - 2x - 2y) dS &= \int_0^{\frac{1}{2}} \left\{ \int_0^{\frac{1}{2}-x} (1 - 2x - 2y) dy \right\} dx \\ &= \int_0^{\frac{1}{2}} \left\{ (1 - 2x) \left(\frac{1}{2} - x \right) - \left(\frac{1}{2} - x \right)^2 \right\} dx = \int_0^{\frac{1}{2}} \left(\frac{1}{2} - x \right)^2 dx \\ &= \left[\frac{1}{3} \left(x - \frac{1}{2} \right)^3 \right]_0^{\frac{1}{2}} = 0 - \frac{1}{3} \left(-\frac{1}{2} \right)^3 = \frac{1}{24}. \end{aligned}$$

Then calculate the space integral,

$$\begin{aligned}\int_A (x + y + z) d\Omega &= \int_B \left\{ \int_0^{1-2x-2y} (x + y + z) dz \right\} dS \\&= \int_B \left\{ (x + y)(1 - 2x - 2y) + \frac{1}{2} (1 - 2x - 2y)^2 \right\} dS \\&= \frac{1}{2} \int_B (1 - 2x - 2y) \{2x + 2y + (1 - 2x - 2y)\} dS \\&= \frac{1}{2} \int_B (1 - 2x - 2y) dS = \frac{1}{2} \cdot \frac{1}{24} = \frac{1}{48},\end{aligned}$$

where we have inserted the value of the plane integral.

Need help with your dissertation?

Get in-depth feedback & advice from experts in your topic area. Find out what you can do to improve the quality of your dissertation!

Get Help Now



Go to www.helpmyassignment.co.uk for more info

Helpmyassignment

Example 22.5 Given the tetrahedron

$$T = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x, 0 \leq y, 0 \leq z, z + 2x + 4y \leq 8\}.$$

Calculate the space integral

$$\int_T x \, d\Omega.$$

A Space integral.

D First find the base of T in the (x, y) -plane.

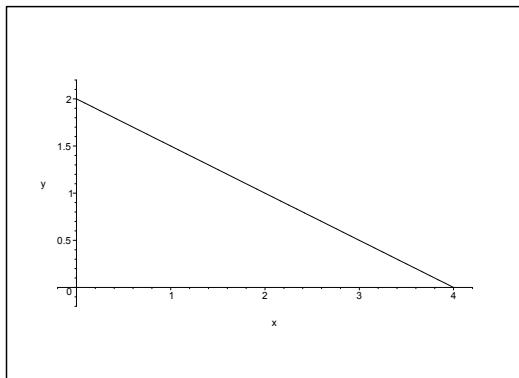


Figure 22.17: The base B of T in the plane $z = 0$.

I The base B is given by

$$0 \leq x, \quad 0 \leq y, \quad 2x + 4y \leq 8,$$

i.e.

$$B = \{(x, y) \mid 0 \leq x, 0 \leq y, x + 2y \leq 4\}.$$

Then we get the space integral

$$\begin{aligned} \int_T x \, d\Omega &= \int_0^4 \left\{ \int_0^{\frac{1}{2}(4-x)} x \cdot (8 - 2x - 4y) \, dy \right\} dx \\ &= -\frac{1}{8} \int_0^4 x [(8 - 2x - 4y)^2]_{y=0}^{\frac{1}{2}(4-x)} dx = \frac{1}{8} \int_0^4 x(8 - 2x)^2 dx \\ &= \frac{4}{8} \int_0^4 \{(x - 4) + 4\}(x - 4)^2 dx = \frac{1}{2} \left[\frac{1}{4} (x - 4)^4 + \frac{4}{3} (x - 4)^3 \right]_0^4 \\ &= \frac{1}{2} \left\{ -\frac{1}{4} \cdot 4^4 + \frac{4}{3} \cdot 4^3 \right\} = \frac{4^3}{2} \left(\frac{4}{3} - 1 \right) = \frac{32}{3}. \end{aligned}$$

ALTERNATIVELY, start by integrating with respect to x . Then

$$\begin{aligned}\int_T x \, d\Omega &= \int_0^2 \left\{ \int_0^{4-2y} (8x - 2x^2 - 4xy) \, dx \right\} dy = \int_0^2 \left[4x^2 - \frac{2}{3}x^3 - 2x^2y \right]_{x=0}^{4-2y} dy \\ &= \int_0^2 \left\{ (4-2y) \cdot (4-2y)^2 - \frac{2}{3}(4-2y)^3 \right\} dy = \frac{1}{3} \int_0^2 (4-2y)^3 dy = \frac{8}{3} \int_0^2 (2-y)^3 dy \\ &= \frac{8}{3} \int_0^2 t^3 dt = \frac{8}{3} \left[\frac{t^4}{4} \right]_0^2 = \frac{32}{3}.\end{aligned}$$

MAPLE. In MAPLE this is easy.

with(Student[MultivariateCalculus])

$$\text{MultiInt}(8x - 2x^2 - 4x \cdot y, x = 0..4 - 2y, y = 0..2)$$

$$\frac{32}{3}$$

Example 22.6 Let a be a positive constant, and let

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in B, \sqrt{ax} \leq z \leq \sqrt{ax + y^2}\},$$

where

$$B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq a, -x \leq y \leq 2x\}.$$

Calculate the space integral

$$\int_A xyz \, d\Omega.$$

A Space integral.

D Reduce the integral by first integrating with respect to z .

I When we reduce as a triple integral, we get

$$\begin{aligned}\int_A xyz \, d\Omega &= \int_0^a \left\{ \int_{-x}^{2x} \left(\int_{\sqrt{ax}}^{\sqrt{ax+y^2}} xyz \, dz \right) dy \right\} dx = \int_0^a x \left\{ \int_{-x}^{2x} y \left[\frac{1}{2}z^2 \right]_{\sqrt{ax}}^{\sqrt{ax+y^2}} dy \right\} dx \\ &= \frac{1}{2} \int_0^a x \left\{ \int_{-x}^{2x} y^3 dy \right\} dx = \frac{1}{2} \int_0^a x \left[\frac{1}{4}y^4 \right]_{-x}^{2x} dx = \frac{1}{8} \int_0^a x \{2^4 - 1\} x^4 dx \\ &= \frac{15}{8} \int_0^a x^5 dx = \frac{15}{8} \cdot \frac{a^6}{6} = \frac{5}{16} a^6.\end{aligned}$$

MAPLE. In MAPLE this is easy.

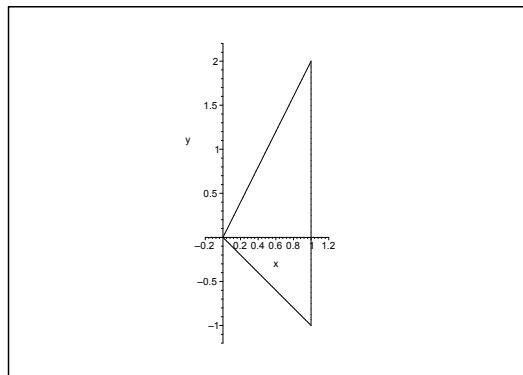


Figure 22.18: The domain B for $a = 1$.

with(Student[MultivariateCalculus])

$$\text{MultiInt}\left(x \cdot y \cdot z, z = \sqrt{a \cdot x}.. \sqrt{a \cdot x + y^2}, y = -x..2x\right)$$

$$\frac{5}{16} a^6$$

Brain power

By 2020, wind could provide one-tenth of our planet's electricity needs. Already today, SKF's innovative know-how is crucial to running a large proportion of the world's wind turbines.

Up to 25 % of the generating costs relate to maintenance. These can be reduced dramatically thanks to our systems for on-line condition monitoring and automatic lubrication. We help make it more economical to create cleaner, cheaper energy out of thin air.

By sharing our experience, expertise, and creativity, industries can boost performance beyond expectations.

Therefore we need the best employees who can meet this challenge!

The Power of Knowledge Engineering

Plug into The Power of Knowledge Engineering.
Visit us at www.skf.com/knowledge



Click on the ad to read more

23 The space integral in semi-polar coordinates

23.1 Reduction theorem in semi-polar coordinates

In Chapter 22 we obtained some reduction theorems in rectangular coordinates. We shall here derive a reduction theorem in semi-polar coordinates. Let us consider the polar coordinates in the 2-dimensional plan. Then we found in Chapter 20 that the area of a small, almost rectangular domain in polar coordinates is $\Delta S = \varrho \Delta \varrho \Delta \varphi$, cf. Figure 23.1.

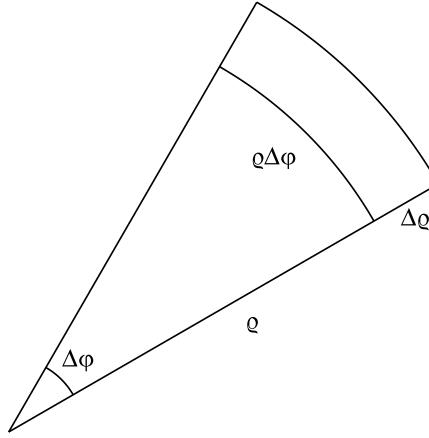


Figure 23.1: Analysis of the area element in polar coordinates.

The z -axis is perpendicular to the (x, y) -plane, so a small body of height Δz and of base above the small plane set of area ΔS would therefore have the volume

$$\Delta \Omega = \Delta S \cdot \Delta z = \varrho \Delta \varrho \Delta \varphi \Delta z.$$

One would therefore guess that the volume element is in semi-polar coordinates given by

$$d\Omega = \varrho d\varrho dz d\varphi,$$

and it can actually be proved that this is indeed the case.

The space integral in semi-polar coordinates is in particular useful, when we integrate over a *rotational body* with the z -axis as its axis of revolution. To see this we first in general consider the following situation. For every fixed φ we define a half-plane, called the *meridian half-plane*. It is characterized by its two coordinates, $(\varrho, z) \in \overline{\mathbb{R}}_+ \times \mathbb{R}$, which together with the chosen φ form the semi-polar coordinates.

For every fixed φ this meridian half-plan cut the body $A \subset \mathbb{R}^3$ in a plane point set, which we denote $B(\varphi)$. If $f : A \rightarrow \mathbb{R}$ is a continuous function on the closed and bounded set $A \subset \mathbb{R}^3$, then the idea is first to integrate f over the set $B(\varphi)$ [where we must not forget some weight function to be derived later on], which defines a function $F(\varphi)$, which only depends on φ . So in order to calculate the space integral of f over A we first find $F(\varphi)$, and then integrate $F(\varphi)$ with respect to φ .

Then note that if A is a *rotational body* with respect to the z -axis, then all $B(\varphi) = B$ are equal, so we can in this case expect some simplifications.

First recall that the rectangular coordinates (x, y, z) in semi-polar coordinates are described by

$$(x, y, z) = (\varrho \cos \varphi, \varrho \sin \varphi, z).$$

After these remarks we formulate without proof the following

Theorem 23.1 The reduction theorem of the space integral in semi-polar coordinates. *Assume that $A \subset \mathbb{R}^3$ is a closed and bounded set, and that $f : A \rightarrow \mathbb{R}$ is a continuous function. If A is described in semi-polar coordinates by its corresponding parameter set*

$$\tilde{A} = \{(\varrho, \varphi, z) \mid \alpha \leq \varphi \leq \beta, (\varrho, z) \in B(\varphi), 0 < \beta - \alpha \leq 2\pi\},$$

where $B(\varphi)$ for every $\varphi \in [\alpha, \beta]$ is a closed and bounded plane set in the meridian half-plane, then the space integral of f over A is reduced in the following way in semi-polar coordinates,

$$(23.1) \quad \int_A f(x, y, z) d\Omega = \int_{\alpha}^{\beta} \left\{ \int_{B(\varphi)} f(\varrho \cos \varphi, \varrho \sin \varphi, z) \varrho d\varrho dz \right\} d\varphi.$$

So we just insert

$$x = \varrho \cos \varphi, \quad y = \varrho \sin \varphi, \quad \text{and add the weight function } \varrho \text{ as a factor,}$$

to get the integrand right.

In case of a *rotational body*, where $B(\varphi) = B$ for all φ , formula (23.1) is also written

$$\int_A f(x, y, z) d\Omega = \int_B \left\{ \int_{\alpha}^{\beta} f(\varrho \cos \varphi, \varrho \sin \varphi, z) d\varphi \right\} \varrho d\varrho dz.$$

One should of course be aware of that the body A itself is *not* equal to its corresponding parameter domain \tilde{A} , so these two sets must never be confused. Note in particular that we after the introduction of the weight function ϱ in the parameter space \tilde{A} , we integrate here as if we had rectangular coordinates. This is *not* the case in the body A itself.

There are of course other variants of the reduction theorem above, but they are difficult to formulate in general, and in practice one will never doubt what should be done, so there is no need to formulate such results. Only Theorem 23.1 above is nice, because it refers to the meridian half-plane, which can be visualized.

23.2 Procedures for reduction of space integral in semi-polar coordinates

This method resembles the “*disc*” method. The difference is that the domain this time is cut like a *pie* with respect to the z -axis.

The formal reduction formula is

$$\int_A f(x, y, z) d\Omega = \int_{\alpha}^{\beta} \left\{ \int_{B(\varphi)} f(\varrho \cos \varphi, \varrho \sin \varphi, z) \varrho d\varrho dz \right\} d\varphi,$$

where we have used *semi-polar* coordinates

$$x = \varrho \cos \varphi, \quad y = \varrho \sin \varphi, \quad z = z,$$

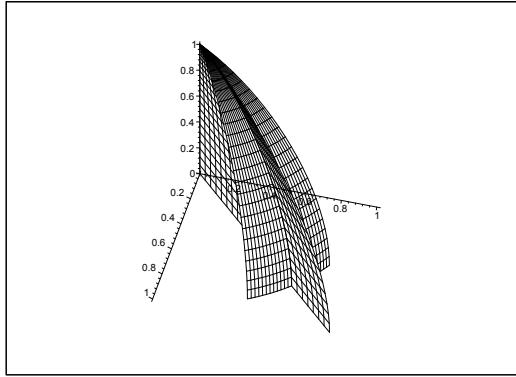


Figure 23.2: A fixed angle φ determines a half-plane through A with the z -axis as axis. This half-plane is then transferred into the meridian half-plane.

and the corresponding *volume element*

$$d\Omega = \varrho d\varrho dz d\varphi.$$

The pie cut is $B(\varphi)$, which for every fixed $\varphi \in [\alpha, \beta]$ represents a domain in the meridian half-plane, i.e. in the (ϱ, z) -plane.

Procedure:

- 1) Sketch a figure (at least in the meridian half-plane).
- 2) Describe the domain A in *semi-polar* coordinates by its parameter domain

$$\tilde{A} = \{(\varrho, \varphi, z) \mid \alpha \leq \varphi \leq \beta, (\varrho, z) \in B(\varphi)\}.$$

Identify the *meridian cut* $B(\varphi)$ for every fixed $\varphi \in [\alpha, \beta]$.

- 3) Keep $\varphi \in [\alpha, \beta]$ fixed and calculate the abstract (inner) plane integral

$$\Phi(\varphi) := \int_{B(\varphi)} f(\varrho \cos \varphi, \varrho \sin \varphi, z) \varrho d\varrho dz,$$

by applying one of the methods from Chapter 20. Here one must *not* forget the weight function ϱ which should be put as an extra factor in the integrand.

- 4) Insert and calculate finally the *ordinary integral* on the right hand side,

$$\int_A f(x, y, z) d\Omega = \int_\alpha^\beta \Phi(\varphi) d\varphi.$$

Remark 23.1 If Ω is a *rotational domain* then $B(\varphi) = B$ is independent of φ . In this case we get a better procedure by interchanging the order of integration,

$$\int_A f(x, y, z) d\Omega = \int_B \left\{ \int_\alpha^\beta f(\varrho \cos \varphi, \varrho \sin \varphi, z) d\varphi \right\} \varrho d\varrho dz,$$

hence one starts in this case by calculating the inner *ordinary integral*. \diamond

23.3 Examples of space integrals in semi-polar coordinates

A. Calculate the space integral

$$I = \int_A x^2 y z \, d\Omega,$$

where

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq a^2, y \geq 0, 0 \leq z \leq h\}.$$

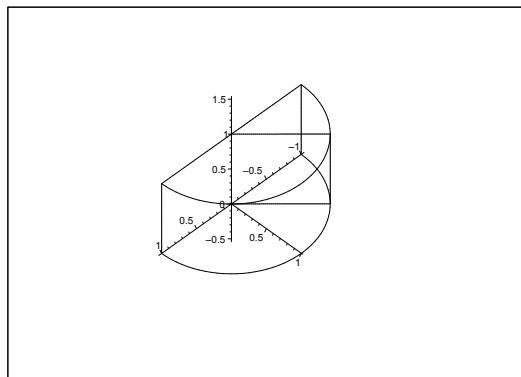


Figure 23.3: The domain (body) A for $a = h = 1$ with a cut $B(\varphi)$.

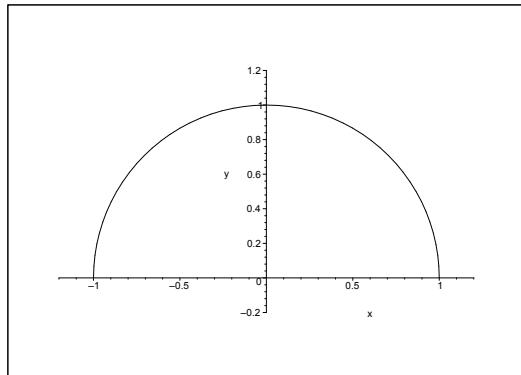


Figure 23.4: Projection of A onto the (x, y) -plane ($a = 1$).

D. It is possible here to calculate in rectangular coordinates, but the calculations are far from easy to perform, so we shall skip this variant here. Instead we shall try the *semi-polar* version (where we must *not* forget the weight function ϱ).

When we use semi-polar coordinates A is represented by the parameter domain

$$\tilde{A} = \{(\varrho, \varphi, z) \mid 0 \leq \varrho \leq a, 0 \leq \varphi \leq \pi, 0 \leq z \leq h\}.$$

We shall go through a couple of variants of the reduction.

- I 1.** For fixed φ the set A is cut into $B(\varphi) = [0, a] \times [0, h]$. Therefore, we get the following reduction, where φ is the “outer” variable,

$$\begin{aligned} I &= \int_0^\pi \left\{ \int_{B(\varphi)} z \cdot \varrho^2 \cos^2 \varphi \cdot \varrho \sin \varphi \cdot \varrho d\varrho \right\} d\varphi \\ &= \int_0^\pi \cos^2 \varphi \sin \varphi d\varphi \cdot \int_0^h z dz \cdot \int_0^a \varrho^4 d\varrho \\ &= \left[-\frac{1}{3} \cos^3 \varphi \right]_0^\pi \cdot \left[\frac{1}{2} z^2 \right]_0^h \cdot \left[\frac{1}{5} \varrho^5 \right]_0^a = \frac{2}{3} \cdot \frac{1}{2} h^2 \cdot \frac{1}{5} a^5 = \frac{1}{15} h^2 a^5. \end{aligned}$$

The calculations in MAPLE are with (Student[MultivariateCalculus])

MultiInt ($\cos(t)^2 \cdot \sin(t) \cdot z \cdot r^4, t = 0.. \pi, z = 0..h, r = 0..a$)

$$\frac{1}{15} h^2 a^5$$

TURN TO THE EXPERTS FOR SUBSCRIPTION CONSULTANCY

Subscrybe is one of the leading companies in Europe when it comes to innovation and business development within subscription businesses.

We innovate new subscription business models or improve existing ones. We do business reviews of existing subscription businesses and we develop acquisition and retention strategies.

Learn more at [linkedin.com/company/subscrybe](https://www.linkedin.com/company/subscrybe) or contact Managing Director Morten Suhr Hansen at mta@subscrybe.dk

SUBSCR✓BE - to the future



Click on the ad to read more

I 2. If instead we integrate innermost with respect to z , then we get with B as the half disc,

$$\begin{aligned} I &= \int_B x^2 y \left\{ \int_0^h z \, dz \right\} dS \\ &= \left[\frac{1}{2} z^2 \right]_0^h \cdot \int_0^\pi \left\{ \int_0^a \varrho^2 \cos^2 \varphi \cdot \varrho \sin \varphi \cdot \varrho \, d\varrho \right\} d\varphi \\ &= \frac{1}{2} h^2 \int_0^\pi \cos^2 \varphi \sin \varphi \, d\varphi \cdot \int_0^a \varrho^4 \, d\varrho \\ &= \frac{1}{2} h^2 \left[-\frac{1}{3} \cos^3 \varphi \right]_0^\pi \cdot \frac{a^5}{5} = \frac{1}{15} h^2 a^5. \end{aligned}$$

C. Weak control (Check of dimension). We get from

$$x \sim a, \quad y \sim a, \quad z \sim h, \quad \int \cdots d\Omega = \int \cdots dx dy dz \sim a \cdot a \cdot h = a^2 h$$

that

$$I = \int_A x^2 y^2 z \, d\Omega \sim a^2 \cdot a^2 \cdot h \cdot (a^2 h) = h^2 a^5,$$

so the result *must* have the form constant $\cdot h^2 a^5$. If this is not the case then we have made an error. Note, however, that even if we get $c \cdot h^2 a^2$, we may not have found the right constant c , so the method gives only a weak control.

A. Calculate the space integral

$$I = \int_A xy^2 z \, d\Omega,$$

where

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq a^2, x \geq 0, \sqrt{x^2 + y^2} \leq z \leq a\}.$$

When we consider the dimensions, we see that $x, y, z \sim a$ and $\int \cdots d\Omega \sim a^3$, so

$$I = \int_A xy^2 z \, d\Omega \sim a^2 \cdot a \cdot a^3 = a^7.$$

Hence the result *must* have the form constant $\cdot a^7$.

D. The shape of A (which is a part of a rotational body) invites to the application of *semi-polar* coordinates (*do not forget the weight function ϱ !*), where A is represented by the parameter domain

$$\tilde{A} = \left\{ (\varrho, \varphi, z) \mid 0 \leq \varrho \leq a, -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}, \varrho \leq z \leq a \right\}.$$

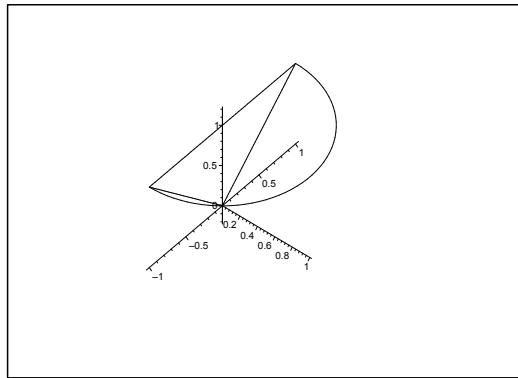


Figure 23.5: The body A for $a = 1$.

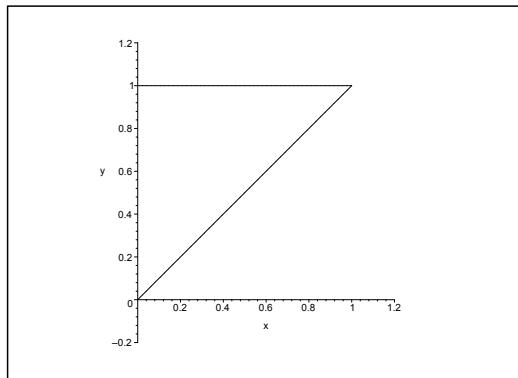


Figure 23.6: The cut in the meridian half-plane for $a = 1$.

- I.** The cut $B(\varphi)$, which is rotated around the z -axis, must be independent of φ . We get in the meridian half-plane

$$B(\varphi) = \{(\varrho, z) \mid 0 \leq z \leq a, 0 \leq \varrho \leq z\}.$$

Since the φ -integral can be separated from the rest, it follows from Theorem 23.1 that

$$\begin{aligned} I &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_{B(\varphi)} \varrho \cos \varphi \cdot \varrho^2 \sin^2 \varphi \cdot z \cdot \varrho d\varrho dz \right\} d\varphi \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 \varphi \cdot \cos \varphi d\varphi \cdot \int_{B(\varphi)} \varrho^4 z d\varrho dz \\ &= \left[\frac{1}{3} \sin^3 \varphi \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdot \int_0^a z \left\{ \int_0^z \varrho^4 d\varrho \right\} dz = \frac{2}{3} \cdot \int_0^a z \cdot \frac{1}{5} z^5 dz \\ &= \frac{2}{15} \int_0^a z^6 dz = \frac{2}{15} \cdot \frac{1}{7} a^7 = \frac{2}{105} a^7. \end{aligned}$$

The calculations in MAPLE are with `with(Student[MultivariateCalculus])`

$$\text{MultiInt} \left(\sin(t)^2 \cdot \cos(t) \cdot z \cdot r^4, t = -\frac{\pi}{2}.. \frac{\pi}{2}, r = 0..z, z = 0..a \right)$$

$$\frac{2}{105} a^7$$

C. A weak control shows that the result has the form constant· a^7 as foreseen in A.

What do you want to do?

No matter what you want out of your future career, an employer with a broad range of operations in a load of countries will always be the ticket. Working within the Volvo Group means more than 100,000 friends and colleagues in more than 185 countries all over the world. We offer graduates great career opportunities – check out the Career section at our web site www.volvolgroup.com. We look forward to getting to know you!

VOLVO
AB Volvo (publ)
www.volvolgroup.com

VOLVO TRUCKS | RENAULT TRUCKS | MACK TRUCKS | VOLVO BUSES | VOLVO CONSTRUCTION EQUIPMENT | VOLVO PENTA | VOLVO AERO | VOLVO IT
VOLVO FINANCIAL SERVICES | VOLVO 3P | VOLVO POWERTRAIN | VOLVO PARTS | VOLVO TECHNOLOGY | VOLVO LOGISTICS | BUSINESS AREA ASIA



Click on the ad to read more

Example 23.1 Compute in each of the following cases the given space integral over a point set A , which in semi-polar coordinates is bounded by

$$\alpha \leq \varphi \leq \beta \quad \text{and} \quad (\varrho, z) \in B(\varphi).$$

One shall first from the given description of the domain of integration find α , β and $B(\varphi)$.

1) The space integral $\int_A \sqrt{x^2 + y^2} d\Omega$, where the domain of integration A is given by

$$\sqrt{x^2 + y^2} \leq z \leq 1.$$

2) The space integral $\int_A \ln(1 + x^2 + y^2) d\Omega$, where the domain of integration A is given by

$$\frac{1}{2}(x^2 + y^2) \leq z \leq 2.$$

3) The space integral $\int_A (x + y^2)z d\Omega$, where the domain of integration A is given by

$$x^2 + y^2 \leq 1 \quad \text{and} \quad x^2 + y^2 \leq z \leq \sqrt{2 - x^2 - y^2}.$$

4) The space integral $\int_A (x^2 + y^2) d\Omega$, where the domain of integration A is given by

$$\frac{x^2 + y^2}{a} \leq z \leq h.$$

5) The space integral $\int_A xy d\Omega$, where the domain of integration A is given by the conditions

$$x \geq 0, \quad y \geq 0 \quad \text{and} \quad \frac{x^2 + y^2}{a} \leq z \leq h.$$

6) The space integral $\int_A xz d\Omega$, where the domain of integration A is given by

$$x^2 + y^2 \leq 2x \quad \text{and} \quad 0 \leq z \leq \sqrt{x^2 + y^2}.$$

7) The space integral $\int_A (z^2 + y^2) d\omega$, where the domain of integration A is given by

$$0 \leq z \leq h - \frac{h}{a} \sqrt{x^2 + y^2}.$$

8) The space integral $\int_A (x^2 + y^2) d\Omega$, where the domain of integration A is given by

$$x^2 + y^2 \leq 3 \quad \text{and} \quad 0 \leq z \leq \sqrt{1 + x^2 + y^2}.$$

9) The space integral $\int_A xy d\Omega$, where the domain of integration A is given by

$$x^2 + y^2 \leq 3 \quad \text{and} \quad 0 \leq z \leq \sqrt{1 + x^2 + y^2}.$$

10) The space integral $\int_A (x^2 z + z^3) d\Omega$, where the domain of integration A is given by

$$0 \leq z \leq \sqrt{a^2 - x^2 - y^2}.$$

11) The space integral $\int_A |y|z \, d\Omega$, where the domain of integration A is given by

$$x^2 + y^2 \leq ax \quad \text{and} \quad 0 \leq z \leq \frac{x^2}{a}.$$

12) The space integral $\int_A xz \, d\Omega$, where the domain of integration is one half cone of revolution of vertex $(0, 0, h)$ and its base in the plane $z = 0$ given by

$$x^2 + y^2 \leq a^2 \quad \text{for } x \geq 0.$$

13) The space integral $\int_A z \, d\Omega$, where the domain of integration A is given by $x^2 + y^2 \leq (2 - z)^2$ for $z \in [0, 2]$.

[This is also Example 22.2.8, so we may compare the results. Cf. also Example 22.1.8.]

A Space integrals in semi-polar coordinates.

D Find the interval $[\alpha, \beta]$ for φ . Describe $B(\varphi)$ in semi-polar coordinates and sketch if necessary $B(\varphi)$ in the meridian half plane. Finally, compute the space integral by using the theorem of reduction in semi-polar coordinates.

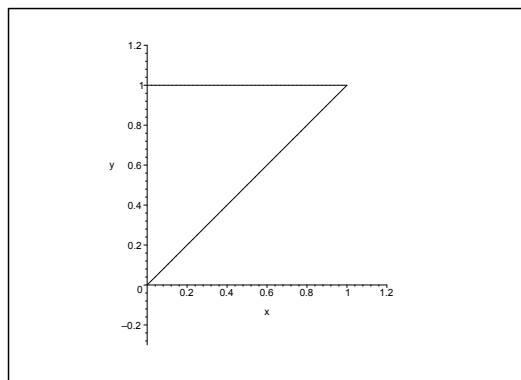


Figure 23.7: The meridian cut $B(\varphi)$ in Example 23.1.1.

I 1) Here $\varphi \in [0, 2\pi]$ and

$$B(\varphi) = \{(\varrho, z) \mid 0 \leq \varrho \leq 1, \varrho \leq z \leq 1\} = \{(\varrho, z) \mid 0 \leq z \leq 1, 0 \leq \varrho \leq z\}.$$

Then by the reduction theorem,

$$\begin{aligned} \int_A \sqrt{x^2 + y^2} \, d\Omega &= \int_0^{2\pi} \left\{ \int_{B(\varphi)} \varrho \cdot \varrho \, d\varrho \, dz \right\} d\varphi \\ &= 2\pi \int_0^1 \left\{ \int_0^z \varrho^2 \, d\varrho \right\} dz = \frac{2\pi}{3} \int_0^1 z^3 \, dz = \frac{\pi}{6}. \end{aligned}$$

The calculations in MAPLE are

with(Student[MultivariateCalculus])

$$2\pi \cdot \text{MultiInt}(r^2, r = 0..z, z = 0..1)$$

$$\frac{1}{6}\pi$$

2) Here $\varphi \in [0, 2\pi]$, and

$$B(\varphi) = \left\{ (\varrho, z) \mid 0 \leq \varrho \leq 2, \frac{1}{2} \varrho^2 \leq z \leq 2 \right\} = \{(\varrho, z) \mid 0 \leq z \leq 2, 0 \leq \sqrt{2z}\}$$

The advertisement features a background photograph of a runner from behind, set against a warm, golden sunset or sunrise. The Gaiteye logo, consisting of a yellow square with a white checkmark-like shape and the word "gaiteye" in lowercase, is positioned in the upper left. Below the logo, the tagline "Challenge the way we run" is written in a smaller font. In the lower left, the text "EXPERIENCE THE POWER OF FULL ENGAGEMENT..." is displayed above a dotted line. To the right of the runner, there is a graphic of a running shoe with several lines radiating from it, suggesting motion and data collection. At the bottom right, a yellow button contains the text "READ MORE & PRE-ORDER TODAY" and the website "WWW.GAITEYE.COM". A white hand cursor icon is pointing towards this button. The overall theme is performance and technology in running.

which does not depend on φ . Then by the reduction theorem,

$$\begin{aligned}
 \int_A \ln(1 + x^2 + y^2) d\Omega &= \int_0^{2\pi} \left\{ \int_{B(\varphi)} \ln(1 + \varrho^2) \cdot \varrho d\varrho dz \right\} d\varphi \\
 &= 2\pi \int_0^2 \left\{ \int_0^{\sqrt{2z}} \ln(1 + \varrho^2) \varrho d\varrho \right\} dz = 2\pi \int_0^2 \left[\frac{1}{2} \{(1 + \varrho^2) \ln(1 + \varrho^2) - \varrho^2\} \right]_{\varrho=0}^{\sqrt{2z}} dz \\
 &= \pi \int_0^2 \{(1 + 2z) \ln(1 + 2z) - 2z\} dz = \frac{\pi}{2} \int_0^4 (1 + t) \ln(1 + t) dt - \pi [z^2]_{z=0} \\
 &= \frac{\pi}{2} \left[\frac{1}{2} (1 + t)^2 \ln(1 + t) - \frac{1}{4} (1 + t)^2 \right]_0^4 - 4\pi = \frac{\pi}{2} \left\{ \frac{25}{2} \ln 5 - \frac{25}{4} + \frac{1}{4} \right\} - 4\pi \\
 &= \pi \left\{ \frac{25}{4} \ln 5 - 7 \right\}.
 \end{aligned}$$

The calculations in MAPLE are

with(Student[MultivariateCalculus])

$$2\pi * \text{MultiInt} \left(\ln(1 + r^2), r = 0.. \sqrt{2z}, z = 0..2 \right)$$

$$2\pi \left(-\frac{7}{2} + \frac{25}{8} \ln(5) \right)$$

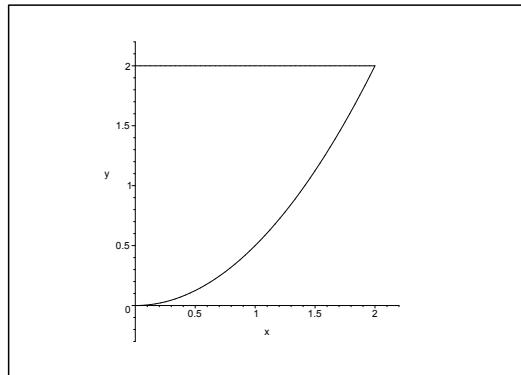


Figure 23.8: The meridian cut $B = B(\varphi)$ of Example 23.1.2.

- 3) Here $\varphi \in [0, 2\pi]$, and $B(\varphi)$ does not depend on φ ,

$$B = B(\varphi) = \{(\varrho, z) \mid 0 \leq \varrho \leq 1, \varrho^2 \leq z \leq \sqrt{2 - \varrho^2}\}.$$

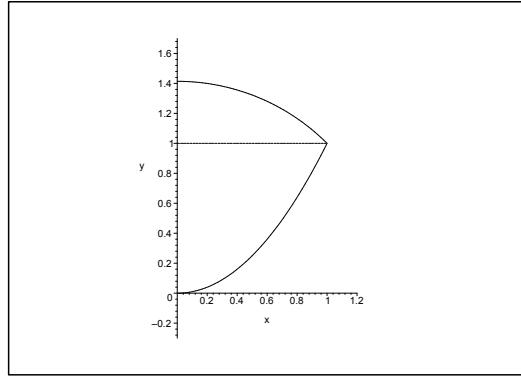


Figure 23.9: The meridian cut $B(\varphi)$ of Example 23.1.3.

It follows by the symmetry that

$$\begin{aligned}
 \int_A (x + y^2)z \, d\Omega &= \int_A xz \, d\Omega + \int_A y^2 z \, d\Omega = 0 + \int_A y^2 z \, d\Omega \\
 &= \int_0^{2\pi} \left\{ \int_{B(\varphi)} \varrho^2 \sin^2 \varphi \cdot z \cdot \varrho \, d\varrho dz \right\} dz = \left\{ \int_0^{2\pi} \sin^2 \varphi \, d\varphi \right\} \cdot \left\{ \int_B z \varrho^2 \, d\varrho dz \right\} \\
 &= \pi \int_0^1 \varrho^3 \left\{ \int_{\varrho^2}^{\sqrt{2-\varrho^2}} z \, dz \right\} d\varrho = \frac{\pi}{2} \int_0^1 \varrho^3 [z^3]_{z=\varrho^2}^{\sqrt{2-\varrho^2}} \, d\varrho = \frac{\pi}{2} \int_0^1 \varrho^3 (2 - \varrho^2 - \varrho^4) \, d\varrho \\
 &= \frac{\pi}{2} \int_0^1 \{2\varrho^3 - \varrho^5 - \varrho^7\} \, d\varrho = \frac{\pi}{2} \left(\frac{1}{2} - \frac{1}{6} - \frac{1}{8} \right) = \frac{5\pi}{48}.
 \end{aligned}$$

The calculations in MAPLE are

with(Student[MultivariateCalculus])

$$\pi \cdot \text{MultiInt} \left(r^3 \cdot z, z = r^2 .. \sqrt{2 - r^2}, r = 0 .. 1 \right)$$

$$\frac{5}{48} \pi$$

- 4) Here $\varphi \in [0, 2\pi]$, and $B(\varphi)$ does not depend on φ ,

$$B = B(\varphi) = \left\{ (\varrho, z) \mid \frac{\varrho^2}{a} \leq z \leq h \right\} = \{(\varrho, z) \mid 0 \leq z \leq h, 0 \leq \varrho \leq \sqrt{az}\}.$$

Then by the reduction theorem,

$$\begin{aligned}
 \int_A (x^2 + y^2) \, d\Omega &= \int_0^{2\pi} \left\{ \int_B \varrho^2 \cdot \varrho \, d\varrho dz \right\} d\varphi = 2\pi \int_0^h \left\{ \int_0^{\sqrt{az}} \varrho^3 \, d\varrho \right\} dz \\
 &= \frac{2\pi}{4} \int_0^h a^2 z^2 \, dz = \frac{\pi a^2 h^3}{6}.
 \end{aligned}$$

The calculations in MAPLE are

with(Student[MultivariateCalculus])

$$2\pi \cdot \text{MultiInt}(r^3, r = 0..\sqrt{a \cdot z}, z = 0..h)$$

$$\frac{1}{6} a^2 h^3$$

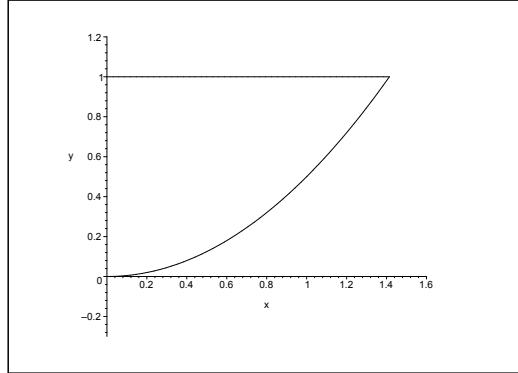


Figure 23.10: The meridian cut $B(\varphi)$ for $a = 2$ and $h = 1$ in **Example 23.1.4** and **Example 23.1.5**.

5) Here $\varphi \in [0, \frac{\pi}{2}]$. Note that $B = B(\varphi)$ is the same set as in 4),

$$B = B(\varphi) = \{(\varrho, z) \mid 0 \leq z \leq h, 0 \leq \varrho \leq \sqrt{az}\}.$$

Then by the reduction theorem,

$$\begin{aligned} \int_A xy \, da \Omega &= \int_0^{\frac{\pi}{2}} \left\{ \int_B \varrho^2 \cos \varphi \cdot \sin \varphi \cdot \varrho \, d\varrho \, dz \right\} d\varphi \\ &= \int_0^{\frac{\pi}{2}} \cos \varphi \cdot \sin \varphi \, d\varphi \cdot \int_0^h \left\{ \int_0^{\sqrt{az}} \varrho^3 \, d\varrho \right\} dz = \left[\frac{\sin^2 \varphi}{2} \right]_0^{\frac{\pi}{2}} \cdot \frac{1}{4} \int_0^h a^2 z^2 \, dz \\ &= \frac{a^2 h^3}{24}. \end{aligned}$$

The calculations in MAPLE are

with(Student[MultivariateCalculus])

$$\text{MultiInt}(\cos(t) \cdot \sin(t) \cdot r^3, 1 = 0..\frac{\pi}{2}, r = 0..\sqrt{a \cdot z}, z = 0..h)$$

$$\frac{1}{24} a^2 h^3$$

6) It follows from $x^2 + y^2 \leq 2x$ that

$$\varrho \leq 2 \cos \varphi, \quad \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

corresponding to the disc $(x - 1)^2 + y^2 \leq 1$ in the XY -plane. Furthermore,

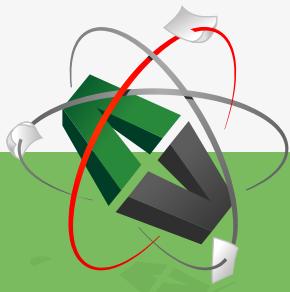
$$B(\varphi) = \{(\varrho, z) \mid 0 \leq \varrho \leq 2 \cos \varphi, 0 \leq z \leq \varrho\},$$

which depends on φ . The domain of integration A is obtained by removing the open cone $z > \sqrt{x^2 + y^2}$ from the half infinite $(x - 1)^2 + y^2 \leq 1$.

We get by using the reduction theorem,

$$\begin{aligned} \int_A xz \, d\Omega &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_{B(\varphi)} \varrho \cos \varphi \cdot z \cdot \varrho \, d\varrho \, dz \right\} d\varphi \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \varphi \left\{ \int_0^{2 \cos \varphi} \varrho^2 \left[\int_0^\varrho z \, dz \right] d\varrho \right\} d\varphi = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_0^{2 \cos \varphi} \varrho^4 d\varrho \right\} d\varphi \\ &= \frac{1}{2 \cdot 5} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \varphi [\varrho^5]_{\varrho=0}^{2 \cos \varphi} d\varphi = \frac{1}{5} \int_0^{\frac{\pi}{2}} 32 \cdot \cos^6 \varphi d\varphi \\ &= \frac{4}{5} \int_0^{\frac{\pi}{2}} (\cos 2\varphi + 1)^3 d\varphi = \frac{2}{5} \int_0^{\pi} (\cos t + 1)^3 dt \\ &= \frac{2}{5} \int_0^{\pi} \{\cos^3 t + 3 \cos^2 t + 3 \cos t + 1\} dt \\ &= \frac{2}{5} \int_0^{\pi} \{(1 - \sin^2 t) \cos t + \frac{3}{2}(\cos 2t + 1) + 3 \cos t + 1\} dt \\ &= \frac{2}{5} \left[-\frac{1}{3} \sin^3 t + 4 \sin t + \frac{3}{4} \sin 2t + \frac{5}{2} t \right]_0^{\pi} = \pi. \end{aligned}$$

This e-book
is made with
SetaPDF



PDF components for **PHP** developers

www.setasign.com

The calculations in MAPLE are

with(Student[MultivariateCalculus])

$$\text{MultiInt} \left(\cos(t) \cdot r^2 \cdot z, z = 0..r, r = 0..2 \cos(t), t = -\frac{\pi}{2}..\frac{\pi}{2} \right)$$

π

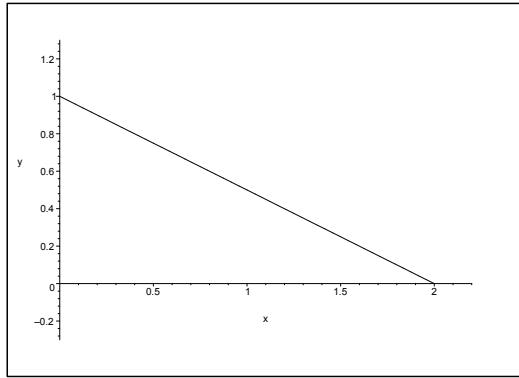


Figure 23.11: The meridian cut $B = B(\varphi)$ for $a = 2$ and $h = 1$ in **Example 23.1.7.**

- 7) Here $\varphi \in [0, 2\pi]$, and $\varrho \in [0, a]$, and $B = B(\varphi)$ does not depend on φ ,

$$B = B(\varphi) = \left\{ (\varrho, z) \mid 0 \leq \varrho \leq a, 0 \leq z \leq h - \frac{h}{a} \varrho \right\}.$$

We get by the reduction theorem,

$$\begin{aligned} \int_A (z^2 + y^2) d\Omega &= \int_0^{2\pi} \left\{ \int_B z^2 \cdot \varrho d\varrho dz \right\} d\varphi + \int_0^{2\pi} \left\{ \int_B \varrho^2 \sin^2 \varphi \cdot \varrho d\varrho dz \right\} d\varphi \\ &= 2\pi \int_0^a \varrho \left\{ \int_0^{h(1-\frac{\varrho}{a})} z^2 dz \right\} d\varrho + \int_0^{2\pi} \sin^2 \varphi d\varphi \cdot \int_0^a \varrho^3 \left\{ \int_0^{h(1-\frac{\varrho}{a})} dz \right\} d\varrho \\ &= \frac{2\pi}{3} \int_0^a \varrho \cdot h^3 \left(1 - \frac{\varrho}{a}\right)^3 d\varrho + \pi \int_0^a \varrho^3 \cdot h \left(1 - \frac{\varrho}{a}\right) d\varrho \\ &= \frac{2\pi h^3}{3} \cdot a^2 \int_0^a \left\{1 - \left(1 - \frac{\varrho}{a}\right)\right\} \left(1 - \frac{\varrho}{a}\right)^3 \frac{1}{a} d\varrho + \pi h a^4 \int_0^a \left(\frac{\varrho}{a}\right)^3 \cdot \left(1 - \frac{\varrho}{a}\right) \frac{1}{a} d\varrho \\ &= \frac{2\pi h^3}{3} \cdot a^2 \int_0^1 (1-t)t^3 dt + \pi h a^4 \int_0^1 t^3(1-t) dt = \pi h a^2 \left(\frac{2}{3} h^2 + a^2\right) \int_0^1 (t^3 - t^4) dt \\ &= \frac{\pi h a^2}{20} \left(\frac{2}{3} h^2 + a^2\right). \end{aligned}$$

The calculations in MAPLE are with(Student[MultivariateCalculus])

$$2\pi \cdot \text{MultiInt} \left(r \cdot z^2, r = 0..h \cdot \left(1 - \frac{r}{a}\right), r = 0..a \right) + \text{MultiInt} \left(\sin(t)^2 \cdot r^3, z = 0..h \cdot \left(1 - \frac{r}{a}\right), r = 0..a \right)$$

$$\frac{1}{30} \pi a^2 h^3 + \frac{1}{20} \pi h a^4$$

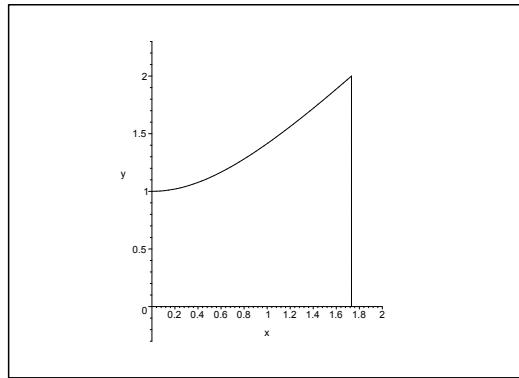


Figure 23.12: The meridian cut $B = B(\varphi)$ in **Example 23.1.8** and **Example 23.1.9**.

- 8) Here $\varphi \in [0, 2\pi]$ and $0 \leq \varrho \leq \sqrt{3}$, and

$$B = B(\varphi) = \{(\varrho, z) \mid 0 \leq \varrho \leq \sqrt{3}, 0 \leq z \leq \sqrt{1 + \varrho^2}\},$$

which is independent of φ .

By the reduction theorem,

$$\begin{aligned} \int_A (x^2 + y^2) d\Omega &= \int_0^{2\pi} \left\{ \int_B \varrho^2 \cdot \varrho d\varrho dz \right\} d\varphi = 2\pi \int_0^{\sqrt{3}} \varrho^3 \left\{ \int_0^{\sqrt{1+\varrho^2}} dz \right\} d\varrho \\ &= 2\pi \int_0^{\sqrt{3}} \varrho^2 \sqrt{1 + \varrho^2} d\varrho = \pi \int_0^3 t \sqrt{1+t} dt = \pi \int_0^3 \left\{ (1+t)^{\frac{3}{2}} - (1+t)^{\frac{1}{2}} \right\} dt \\ &= \pi \left[\frac{2}{5} (1+t)^{\frac{5}{2}} - \frac{2}{3} (1+t)^{\frac{3}{2}} \right]_0^3 = 2\pi \left(\frac{1}{5} \left\{ 4^{\frac{5}{2}} - 1 \right\} - \frac{1}{3} \left\{ 4^{\frac{3}{2}} - 1 \right\} \right) \\ &= 2\pi \left(\frac{31}{5} - \frac{7}{3} \right) = \frac{116\pi}{15}. \end{aligned}$$

The calculations in MAPLE are

with(Student[MultivariateCalculus])

$$2\pi \cdot \text{MultiInt} \left(r^3, z = 0.. \sqrt{1+r^2}, r = 0..\sqrt{3} \right)$$

$$\frac{116}{15} \pi$$

- 9) The domain of integration is the same as in **Example 23.1.8**, so $\varphi \in [0, 2\pi]$, and

$$B = B(\varphi) = \{(\varrho, z) \mid 0 \leq \varrho \leq \sqrt{3}, 0 \leq z \leq \sqrt{1 + \varrho^2}\}.$$

Now A is symmetric with respect to e.g. the plane $y = 0$, so

$$\int_A xy d\Omega = 0.$$

ALTERNATIVELY we have the following calculation

$$\int_A xy \, d\Omega = \int_0^{2\pi} \left\{ \int_B \varrho^2 \sin \varphi \cdot \cos \varphi \cdot \varrho \, d\varrho \, dz \right\} d\varphi = \int_0^{2\pi} \sin \varphi \cdot \cos \varphi \, d\varphi \cdot \int_B \varrho^3 \, d\varrho \, dz = 0,$$

where we have used that B does not depend on φ and also that

$$\int_0^{2\pi} \sin \varphi \cdot \cos \varphi \, d\varphi = \left[\frac{\sin^2 \varphi}{2} \right]_0^{2\pi} = 0.$$

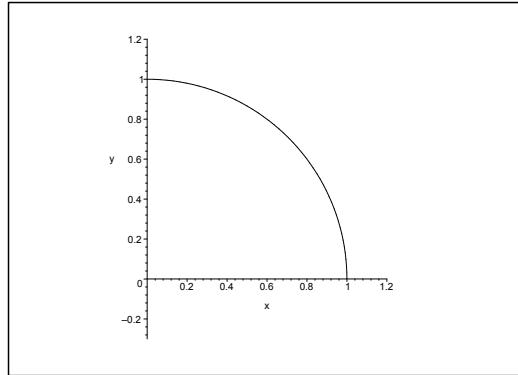


Figure 23.13: The meridian cut $B = B(\varphi)$ in Example 23.1.10.

- 10) Here A is the half ball in the half space $z \geq 0$ of centrum $(0, 0, 0)$ and radius a , thus $\varphi \in [0, 2\pi]$, and $B(\varphi)$ does not depend on φ ,

$$B = B(\varphi) = \{(\varrho, z) \mid 0 \leq \varrho \leq a, 0 \leq z \leq \sqrt{a^2 - \varrho^2}\}.$$

By the reduction theorem,

$$\begin{aligned} \int_A (x^2 z + z^3) \, d\Omega &= \int_0^{2\pi} \left\{ \int_B (\varrho^2 \cos^2 \varphi \cdot z + z^3) \varrho \, d\varrho \, dz \right\} d\varphi \\ &= \int_0^{2\pi} \cos^2 \varphi \, d\varphi \cdot \int_0^a \varrho^3 \left\{ \int_0^{\sqrt{a^2 - \varrho^2}} z \, dz \right\} d\varrho + 2\pi \int_0^a \varrho \left\{ \int_0^{\sqrt{a^2 - \varrho^2}} z^3 \, dz \right\} d\varrho \\ &= \pi \cdot 12 \int_0^a (a^2 \varrho^3 - \varrho^5) \, d\varrho + \frac{2\pi}{4} \int_0^a \varrho (a^2 - \varrho^2)^2 \, d\varrho \\ &= \frac{\pi}{2} \left[\frac{a^2}{4} \varrho^4 - \frac{1}{6} \varrho^6 \right]_0^a + \frac{\pi}{2} \cdot \frac{1}{2} \int_0^{a^2} (a^2 - t)^2 \, dt \\ &= \frac{\pi}{2} \left(\frac{a^6}{4} - \frac{a^6}{6} \right) + \frac{\pi}{12} [-(a^2 - t)^3]_0^{a^2} = \frac{\pi a^6}{24} + \frac{\pi a^6}{12} = \frac{\pi a^6}{8}. \end{aligned}$$

The calculations in MAPLE are with (Student[MultivariateCalculus])

$$\begin{aligned} \text{MultiInt} \left((r^2 \cdot \cos(t)^2 \cdot z + z^3) \cdot r, t = 0..2\pi, z = 0..\sqrt{a^2 - r^2}, r = 0..a \right) \\ \frac{1}{8} \pi a^6 \end{aligned}$$

11) Here $x^2 + y^2 \leq ax$, hence $\varrho \leq a \cos \varphi$, and $0 \leq z \leq \frac{1}{a} \varrho^2 \cos^2 \varphi$, and $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so

$$B(\varphi) = \left\{ (\varrho, z) \mid 0 \leq \varrho \leq a \cos \varphi, 0 \leq z \leq \frac{1}{a} \varrho^2 \cos^2 \varphi \right\}.$$

Clearly, $B(\varphi)$ depends on φ , so we can only conclude that any meridian curve for fixed φ is a parabola in the PZ -plane, and there is no need to sketch it.

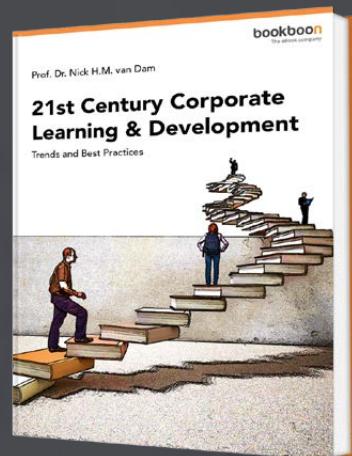
The set A is symmetric with respect to the plane $y = 0$, so by the reduction theorem,

$$\begin{aligned} \int_A |y|z \, d\Omega &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_{B(\varphi)} \varrho |\sin \varphi| \cdot z \cdot \varrho \, d\varrho \, dz \right\} d\varphi = 2 \int_0^{\frac{\pi}{2}} \left\{ \int_{B(\varphi)} \varrho \sin \varphi \cdot z \cdot \varrho \, d\varrho \, dz \right\} d\varphi \\ &= 2 \int_0^{\frac{\pi}{2}} \left\{ \int_0^{a \cos \varphi} \varrho^2 \sin \varphi \left\{ \int_0^{\frac{1}{a} \varrho^2 \cos^2 \varphi} z \, dz \right\} d\varrho \right\} d\varphi \\ &= 2 \int_0^{\frac{\pi}{2}} \sin \varphi \left\{ \int_0^{a \cos \varphi} \varrho^2 \left[\frac{z^2}{2} \right]_{z=0}^{\frac{1}{a} \varrho^2 \cos^2 \varphi} d\varrho \right\} d\varphi \\ &= \int_0^{\frac{\pi}{2}} \sin \varphi \left\{ \int_0^{a \cos \varphi} \frac{1}{a^2} \varrho^6 \cos^4 \varphi d\varrho \right\} d\varphi \\ &= \frac{1}{a^2} \int_0^{\frac{\pi}{2}} \sin \varphi \cdot \cos^4 \varphi \left\{ \int_0^{a \cos \varphi} \varrho^6 d\varrho \right\} d\varphi \\ &= \frac{1}{7a^2} \int_0^{\frac{\pi}{2}} \sin \varphi \cdot \cos^4 \varphi \cdot a^7 \cos^7 \varphi d\varphi = \frac{a^5}{7} \left[-\frac{\cos^{12} \varphi}{12} \right]_0^{\frac{\pi}{2}} = \frac{a^5}{84}. \end{aligned}$$

Free eBook on Learning & Development

By the Chief Learning Officer of McKinsey

Download Now



Click on the ad to read more

The calculations in MAPLE are

with(Student[MultivariateCalculus])

$$2 \cdot \text{MultiInt} \left(r^2 \cdot \sin(t) \cdot z, z = 0.. \frac{1}{a} \cdot r^2 \cdot \cos(t)^2, r = 0..a \cdot \cos(t), t = 0..\frac{\pi}{2} \right)$$

$$\frac{1}{84} a^5$$

12) Here $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$, and we have for any fixes φ that

$$B(\varphi) = \left\{ (\varrho, z) \mid 0 \leq x \leq h, 0 \leq \varrho \leq a \left(1 - \frac{z}{h} \right) \right\}.$$

The meridian cut does not depend on $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$. Note that it is identical with the meridian cut in **Example 23.1.7**.

We get by the reduction theorem in semi-polar coordinates,

$$\begin{aligned} \int_A xz \, d\Omega &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_{B(\varphi)} \varrho \cos \varphi \cdot z \cdot \varrho \, d\varrho \, dz \right\} \, d\varphi \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \varphi \left\{ \int_0^h z \left(\int_0^{a(1-\frac{z}{h})} \varrho^2 \, d\varrho \right) \, dz \right\} \, d\varphi = [\sin \varphi]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^h z \left[\frac{1}{3} \varrho^3 \right]_0^{a(1-\frac{z}{h})} \, dz \\ &= 2 \cdot \frac{a^3}{3} \int_0^h z \left(1 - \frac{z}{h} \right)^3 \, dz = \frac{2}{3} a^3 \int_0^h z \left(1 - \frac{3}{h} z + \frac{3}{h^2} z^2 - \frac{1}{h^3} z^3 \right) \, dz \\ &= \frac{2}{3} a^3 \int_0^h \left(z - \frac{3}{h} z^2 + \frac{3}{h^2} z^3 - \frac{1}{h^3} z^4 \right) \, dz = \frac{2}{3} a^3 \left[\frac{1}{2} z^2 - \frac{1}{h} z^3 + \frac{3}{4h^2} z^4 - \frac{1}{5h^3} z^5 \right]_0^h \\ &= \frac{2}{3} a^3 h^2 \left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5} \right) = \frac{2}{3} \left(\frac{1}{4} - \frac{1}{5} \right) a^3 h^2 = \frac{1}{30} a^3 h^2. \end{aligned}$$

The calculations in MAPLE are with(Student[MultivariateCalculus])

$$\text{MultiInt} \left(\cos(t) \cdot z \cdot r^2, r = 0..a \cdot \left(1 - \frac{z}{h} \right), z = 0..h, t = -\frac{\pi}{2}..\frac{\pi}{2} \right)$$

$$\frac{1}{30} a^3 h^2$$

13) In this case we integrate over the same set as in **Example 22.1.8**. Then by the reduction theorem in semi-polar coordinates followed by the change of variables $u = 2 - z$,

$$\begin{aligned} \int_A z \, d\Omega &= \int_0^2 z \cdot \pi (2 - z)^2 \, dz = \pi \int_0^2 (2 - u) u^2 \, du = \pi \int_0^2 (2u^2 - u^3) \, du \\ &= \pi \left[\frac{2}{3} u^3 - \frac{1}{4} u^4 \right]_0^2 = \pi \left[\frac{16}{3} - \frac{16}{4} \right] = \frac{16}{12} \pi = \frac{4\pi}{3}. \end{aligned}$$

Example 23.2 Consider two balls and their intersection

$$\Omega_1 = \overline{K}((0, 0, 0); a), \quad \Omega_2 = \overline{K}\left((0, 0, a); \frac{a}{2}\right), \quad \Omega = \Omega_1 \cap \Omega_2.$$

- 1) Sketch the three point sets by means of a meridian half plane, and describe the position of the intersection circle $\partial\Omega_1 \cap \partial\Omega_2$.
- 2) Find the volume of Ω .
- 3) Compute the space integral

$$\int_{\Omega} (2 - xy) d\Omega.$$

A Space integrals.

D Follow the given guidelines.

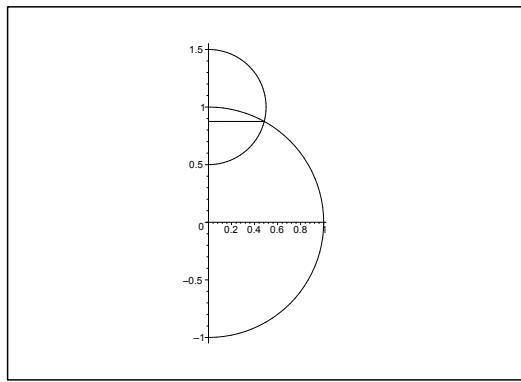


Figure 23.14: The situation in the meridian half plane for $a = 1$.

I 1) The two circles cut each other at height $z \in \left[\frac{a}{2}, a\right]$. Then by Pythagoras's theorem,

$$r^2 = a^2 - z^2 = \left(\frac{a}{2}\right)^2 - (a - z)^2 = -\frac{3}{4}a^2 + 2az - z^2.$$

A reduction gives $2az = \frac{7}{4}a^2$, thus $z = \frac{7}{8}a$, which indicates the whereabouts of the plane, in which the intersection circle $\partial\Omega_1 \cap \partial\Omega_2$ lies.

2) Then split $\Omega = \omega_1 \cup \omega_2$ into its two natural subregions, where ω_1 lies above the plane $z = \frac{7}{8}a$, and ω_2 lies below the same plane. We use in each of the subregions ω_1 and ω_2 the “method of slices”, where each slice is parallel to the (x, y) -plane. By translating the subregion ω_2 in a

convenient way we finally get

$$\begin{aligned}
 \text{vol}(\Omega) &= \text{vol}(\omega_1) + \text{vol}(\omega_2) = \int_{\frac{7}{8}a}^a \pi (a^2 - z^2) \, dz + \int_{-\frac{1}{2}a}^{-\frac{1}{8}a} \pi \left(\frac{a^2}{4} - z^2 \right) \, dz \\
 &= \pi \left[a^2 z - \frac{1}{3} z^3 \right]_{\frac{7}{8}a}^a + \pi \left[\frac{a^2}{4} z - \frac{1}{3} z^3 \right]_{-\frac{1}{2}a}^{-\frac{1}{8}a} \\
 &= \pi a^3 \left\{ \left(1 - \frac{1}{3} - \frac{7}{8} + \frac{1}{3} \cdot \left(\frac{7}{8} \right)^3 \right) + \left(\frac{1}{4} \cdot \frac{1}{2} - \frac{1}{3} \cdot \frac{1}{8} - \frac{1}{4} \cdot \frac{1}{8} + \frac{1}{3} \cdot \left(\frac{1}{8} \right)^3 \right) \right\} \\
 &= \frac{\pi a^3}{8} \left\{ 1 - \frac{8}{3} + \frac{7}{3} \cdot \left(\frac{7}{8} \right)^2 + 1 - \frac{1}{3} - \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{8^2} \right\} \\
 &= \frac{\pi a^3}{24} \left\{ -3 - \frac{3}{4} + \frac{1}{64} (343+1) \right\} = \frac{\pi a^3}{24} \left\{ -\frac{15}{4} + \frac{43}{8} \right\} = \frac{13}{192} \pi a^3.
 \end{aligned}$$

3) Of symmetric reasons, $\int_{\Omega} xy \, d\Omega = 0$, hence

$$\int_{\Omega} (2 - xy) \, d\Omega = 2 \cdot \text{vol}(\Omega) = \frac{13}{96} \pi a^3.$$



www.sylvania.com

We do not reinvent the wheel we reinvent light.

Fascinating lighting offers an infinite spectrum of possibilities: Innovative technologies and new markets provide both opportunities and challenges. An environment in which your expertise is in high demand. Enjoy the supportive working atmosphere within our global group and benefit from international career paths. Implement sustainable ideas in close cooperation with other specialists and contribute to influencing our future. Come and join us in reinventing light every day.

Light is OSRAM

OSRAM
SYLVANIA



Click on the ad to read more

Example 23.3 Given a curve \mathcal{K} in the (z, x) -plane of the equation

$$x = \cos z, \quad z \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

The curve \mathcal{K} is rotated once around the z -axis in the (x, y, z) -space, creating the surface of revolution \mathcal{F} . Let A denote the bounded domain in the (x, y, z) -space with \mathcal{F} as its boundary surface.

1) Find the volume of A .

2) Compute the space integral

$$\int_A \sqrt{x^2 + y^2} d\Omega.$$

A Body of revolution an space integral.

D Sketch a figure and then just compute.

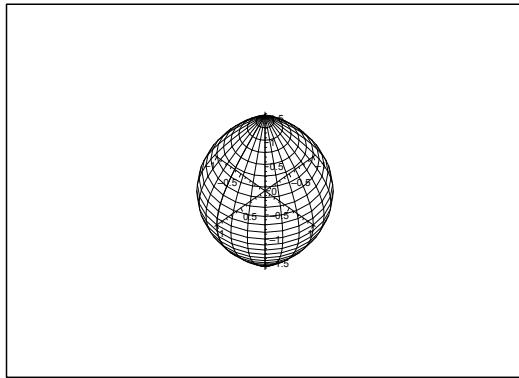


Figure 23.15: The domain A with the boundary surface \mathcal{F} .

I 1) The domain A is the spindle shaped body on the figure.

We get by slicing the body,

$$\text{vol}(A) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi \cos^2 z dz = 2\pi \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2z}{2} dz = 2\pi \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{2}.$$

2) If we put

$$B_z = \{(x, y) \mid \sqrt{x^2 + y^2} \leq \cos z\}, \quad z \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

then

$$\begin{aligned} \int_A \sqrt{x^2 + y^2} d\Omega &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_{B_z} \sqrt{x^2 + y^2} dx dy \right\} dz = 2 \int_0^{\frac{\pi}{2}} \left\{ 2\pi \int_0^{\cos z} \varrho \cdot \varrho d\varrho \right\} dz \\ &= 4\pi \int_0^{\frac{\pi}{2}} \left[\frac{\varrho^3}{3} \right]_0^{\cos z} dz = \frac{4\pi}{3} \int_0^{\frac{\pi}{2}} \cos^3 z dz \\ &= \frac{4\pi}{3} \int_0^{\frac{\pi}{2}} (1 - \sin^2 z) \cos z dz = \frac{4\pi}{3} \left[\sin z - \frac{1}{3} \sin^3 z \right]_0^{\frac{\pi}{2}} = \frac{8\pi}{9}. \end{aligned}$$

Example 23.4 Let c be a positive constant. Consider the half ball A given by the inequalities

$$x^2 + y^2 + z^2 \leq c^2, \quad z \geq 0.$$

1) Compute the space integral

$$J = \int_A z \, d\Omega.$$

2) Show that both the space integrals $\int_A x \, d\Omega$ and $\int_A y \, d\Omega$ are zero.

A Space integrals.

D Apply the slicing method and convenient symmetric arguments.

Alternatively, reduce in

- 1) spherical coordinates,
- 2) semi-polar coordinates,
- 3) rectangular coordinates.

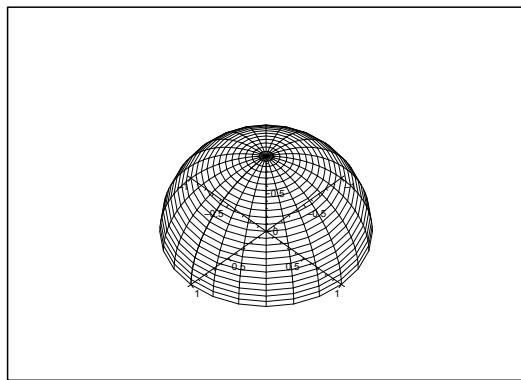


Figure 23.16: The half ball A for $c = 1$.

I 1) First variant. The slicing method.

At the height z the body A is cut into a disc $B(z)$ of radius $\sqrt{c^2 - z^2}$, hence of area $(c^2 - z^2)\pi$. Then we get by the slicing method,

$$J = \int_A z \, d\Omega = \int_0^c z \, \text{area}(B(z)) \, dz = \pi \int_0^c \{c^2 z - z^3\} \, dz = \pi \left[c^2 \cdot \frac{z^2}{2} - \frac{z^4}{4} \right]_0^c = \frac{\pi}{4} c^4.$$

Second variant. Spherical coordinates.

The set A is in spherical coordinates described by

$$\begin{cases} x = r \sin \theta \cos \varphi, \\ y = r \sin \theta \sin \varphi, \\ z = r \cos \theta, \end{cases} \quad \begin{cases} \varphi \in [0, 2\pi], \\ \theta \in \left[0, \frac{\pi}{2}\right], \\ r \in [0, c], \end{cases}$$

and $d\Omega = r^2 \sin \theta dr d\theta d\varphi$. Thus we get by reduction

$$\begin{aligned} J &= \int_A z d\Omega = \int_0^{2\pi} \left\{ \int_0^{\frac{\pi}{2}} \left(\int_0^c r \cos \theta \cdot r^2 \sin \theta dr \right) d\theta \right\} d\varphi \\ &= 2\pi \cdot \left[\frac{1}{2} \sin^2 \theta \right]_0^{\frac{\pi}{2}} \cdot \left[\frac{r^4}{4} \right]_0^c = 2\pi \cdot \frac{1}{2} \cdot \frac{1}{4} c^4 = \frac{\pi}{4} c^4. \end{aligned}$$

Third variant. *Semi-polar coordinates.*

In semi-polar coordinates A is described by

$$\begin{cases} x = \rho \cos \varphi, & \varphi \in [0, 2\pi], \\ y = \rho \sin \varphi, & z \in [0, c], \\ z = z, & \rho \in [0, \sqrt{c^2 - z^2}], \end{cases}$$

and $d\Omega = \rho d\rho d\varphi dz$. We therefore get by reduction

$$\begin{aligned} J &= \int_A z d\Omega = \int_0^{2\pi} \left\{ \int_0^c \left(\int_0^{\sqrt{c^2 - z^2}} z \cdot \rho d\rho \right) dz \right\} d\varphi \\ &= 2\pi \int_0^c z \left[\frac{1}{2} \rho^2 \right]_0^{\sqrt{c^2 - z^2}} dz = \pi \int_0^c (c^2 z - z^3) dz = \frac{\pi}{4} c^4. \end{aligned}$$

Fourth variant. *Rectangular coordinates.*

Here A is described by

$$0 \leq z \leq c, \quad |x| \leq \sqrt{c^2 - z^2}, \quad |y| \leq \sqrt{c^2 - z^2 - x^2},$$

hence

$$\begin{aligned} J &= \int_A z d\Omega = \int_0^c z \left\{ \int_{-\sqrt{c^2 - z^2}}^{\sqrt{c^2 - z^2}} \left(\int_{-\sqrt{c^2 - z^2 - x^2}}^{\sqrt{c^2 - z^2 - x^2}} dy \right) dx \right\} dz \\ &= 2 \int_0^c z \left\{ \int_{-\sqrt{c^2 - z^2}}^{\sqrt{c^2 - z^2}} \sqrt{c^2 - z^2 - x^2} dx \right\} dz. \end{aligned}$$

We then get by the substitution $x = \sqrt{c^2 - z^2} \cdot t$,

$$\begin{aligned} J &= 4 \int_0^c z \left\{ \int_0^{\sqrt{c^2 - z^2}} \sqrt{(c^2 - z^2) - x^2} dx \right\} dz \\ &= 4 \int_0^c z \left\{ \int_0^1 (\sqrt{c^2 - z^2})^2 \cdot \sqrt{1 - t^2} dt \right\} dz \\ &= 4 \int_0^c z (c^2 - z^2) dz \cdot \int_0^1 \sqrt{1 - t^2} dt = c^4 \cdot \frac{\pi}{4}, \end{aligned}$$

where there are lots of similar variants.

2) First variant. A symmetric argument.

The set A is symmetric with respect to the planes $y = 0$ and $x = 0$, and the integrand x , resp. y , is an odd function. Hence,

$$\int_A x d\Omega = 0 \quad \text{and} \quad \int_A y d\Omega = 0.$$

Second variant. *Spherical coordinates.*

By insertion,

$$\begin{aligned}\int_A x \, d\Omega &= \int_0^{2\pi} \left\{ \int_0^{\frac{\pi}{2}} \left(\int_0^c r \sin \theta \cos \varphi \cdot r^2 \sin \theta \, dr \right) d\theta \right\} d\varphi \\ &= \int_0^{2\pi} \cos \varphi \, d\varphi \cdot \int_0^{\frac{\pi}{2}} \sin^2 \theta \, d\theta \cdot \int_0^c r^3 \, dr = [\sin \varphi]_0^{2\pi} \cdot \frac{\pi}{4} \cdot \frac{c^4}{4} = 0,\end{aligned}$$

and similarly.

Third and fourth variant. These are similar to the previous semi-polar and rectangular cases.



Deloitte.

Discover the truth at www.deloitte.ca/careers

© Deloitte & Touche LLP and affiliated entities.



Click on the ad to read more

24 The space integral in spherical coordinates

24.1 Reduction theorem in spherical coordinates

We shall finally introduce the *spherical coordinates* in the reduction of space integrals. We shall here only sketch the idea and then quote the theorem without proof.

When we considered the semi-polar coordinates, we used rectangular coordinates (ϱ, z) in the *meridian half-plane*, and the weight function is $\varrho d\varrho dz$ in \mathbb{R}^3 , when we use semi-polar coordinates.

If we now instead use *polar coordinates* (r, θ) in the meridian half-plane, where θ for convenience is measured from the *z-axis*, and the usual orientation is changed to the opposite one, so we always have $\theta \in [0, \pi]$), then – cf. Chapter 20 – we shall replace the area element $d\varrho dz$ with $r dr d\theta$. Since $\varrho = r \sin \theta$ and $z = r \cos \theta$, because θ is measured from the *z-axis*, we therefore see that we have the formal calculation of the volume element in spherical coordinates,

$$\varrho d\varrho dz d\varphi = \{\varrho\} \{d\varrho dz\} d\varphi = \{r \sin \theta\} \{r dr d\theta\} d\varphi = r^2 \sin \theta dr d\theta d\varphi,$$

which also turns up to be the right volume for infinitesimal small bodies. In particular, we see that the *weight function* here is $r^2 \sin \theta$.

In semi-polar coordinates we fixed φ in order to get the meridian half-plane. This meridian half-plane then cuts A in some set $B(\varphi)$, over which we integrate with respect to the rectangular coordinates (ϱ, z) . When we instead use spherical coordinates, this domain of integration is transformed into another domain of integration $\tilde{B}(\varphi)$ in the variables (r, θ) . This may seem very abstract and strange, but in the applications one will never doubt, what to do. We shall use the notation $\tilde{B}(\varphi)$ in the formulation of the reduction theorem below.

Theorem 24.1 Reduction theorem for a space integral in spherical coordinates. *Let $A \subset \mathbb{R}^3$ be a closed and bounded set, and assume that A in spherical coordinates is described by its parameter domain in the form*

$$\tilde{A} = \{(r, \theta, \varphi) \mid \alpha \leq \varphi \leq \beta, (r, \theta) \in \tilde{B}(\varphi)\},$$

where the constants α and β satisfy $0 < \beta - \alpha \leq 2\pi$, and where we for each fixed φ have given the domain of integration $\tilde{B}(\varphi)$, as described above.

If $f : A \rightarrow \mathbb{R}$ is continuous, then the space integral of f over A is reduced in the following way,

$$\int_A f(x, y, z) d\Omega = \int_{\alpha}^{\beta} \left\{ \int_{\tilde{B}\varphi} f(r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta) r^2 \sin \theta dr d\theta \right\} d\varphi,$$

This is not the only version of the reduction theorem. We mention among others the following. With some more information on the description of A in spherical coordinates we also have a version with a triple integral.

Theorem 24.2 Triple integral for a space integral in spherical coordinates. Let $A \subset \mathbb{R}^3$ be a closed and bounded set, and assume that A in spherical coordinates is described by its parameter domain in the form

$$\tilde{A} = \{(r, \theta, \varphi) \mid \alpha \leq \varphi \leq \beta, \Theta_1(\varphi) \leq \theta \leq \Theta_2(\varphi), R_1(\theta, \varphi) \leq r \leq R_2(\theta, \varphi)\},$$

where the constants α and β satisfy $0 < \beta - \alpha \leq 2\pi$. If $f : A \rightarrow \mathbb{R}$ is continuous, then the space integral of f over A is reduced in the following way,

$$\int_A f(x, y, z) d\Omega = \int_{\alpha}^{\beta} \left\{ \int_{\Theta_1(\varphi)}^{\Theta_2(\varphi)} \left\{ \int_{R_1(\theta, \varphi)}^{R_2(\theta, \varphi)} f(r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta) r^2 dr \right\} \sin \theta d\theta \right\} d\varphi.$$

24.2 Procedures for reduction of space integral in spherical coordinates

We use here the *spherical* coordinates

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta,$$

with the corresponding *volume element*

$$d\Omega = r^2 \sin \theta dr d\theta d\varphi,$$

i.e. the *weight function* is $r^2 \sin \theta$.

The reduction formula is not easy to comprehend,

$$\int_A f(x, y, z) d\Omega = \int_{\alpha}^{\beta} \left\{ \int_{B^*(\varphi)} f(r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta) r^2 \sin \theta dr d\theta \right\} d\varphi.$$

Procedure:

- 1) If A is a reasonable subset of a ball, then write in *spherical* coordinates

$$A = \{(r, \theta, \varphi) \mid \alpha \leq \varphi \leq \beta, (r, \theta) \in B^*(\varphi)\},$$

where $B^*(\varphi)$ is the *meridian cut* for fixed $\varphi \in [\alpha, \beta]$, expressed in the *spherical* coordinates. Identify and sketch $B^*(\varphi)$.

Remark 24.1 This is the most difficult part of this version. \diamond

- 2) Keep $\varphi \in [\alpha, \beta]$ fixed and apply the methods from Chapter 20 in the calculation of the *abstract (inner) plane integral*,

$$H(\varphi) := \int_{B^*(\varphi)} f(r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta) r^2 \sin \theta dr d\theta.$$

Do not forget the weight function $r^2 \sin \theta$ here as a factor of the integrand.

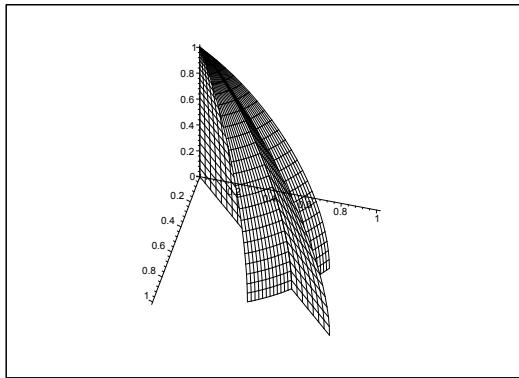


Figure 24.1: The meridian cut $B^*(\varphi)$ for fixed φ .

- 3) Insert the result and calculate the ordinary integral on the right hand side,

$$\int_A f(x, y, z) d\Omega = \int_{\alpha}^{\beta} H(\varphi) d\varphi.$$

Remark 24.2 If Ω is a *rotational domain*, then $B^*(\varphi) = B^*$ is independent of φ . In this case one gets simpler calculations by interchanging the order of integration

$$\int_A f(x, y, z) d\Omega = \int_{B^*} \left\{ \int_{\alpha}^{\beta} f(r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta) d\varphi \right\} r^2 \sin \theta dr d\theta.$$

Notice that the inner integral (which is calculated first)

$$\int_{\alpha}^{\beta} f(r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta) d\varphi$$

becomes much simpler, because the weight $r^2 \sin \theta$ only is added as a factor *after* the calculation of this integral. ◇

24.3 Examples of space integrals in spherical coordinates

Example 24.1

A. Let A be an upper half sphere of radius $2a$, from which we have removed a cylinder of radius a and then halved the resulting domain by the plane $x + y = 0$. We shall only consider that part for which $x + y \geq 0$. Calculate the space integral

$$\int_A xz \, d\Omega.$$

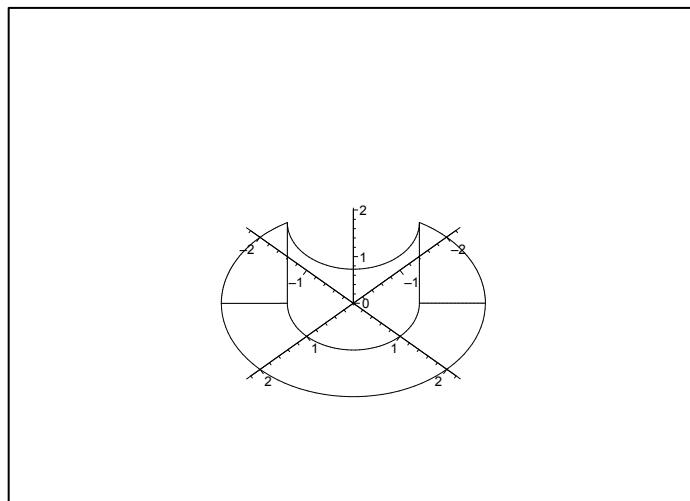


Figure 24.2: The domain A for $a = 1$ in the (x, y, z) -space.

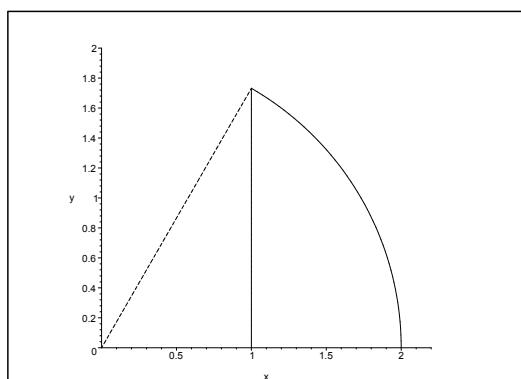


Figure 24.3: The cut in the meridian half-plane for $a = 1$, i.e. in the (ϱ, z) -half-plane.

When we consider the dimensions (i.e. a rough overview) we get

$$x \sim a, \quad y \sim a, \quad z \sim a, \quad \int_A \cdots d\Omega \sim a^3,$$

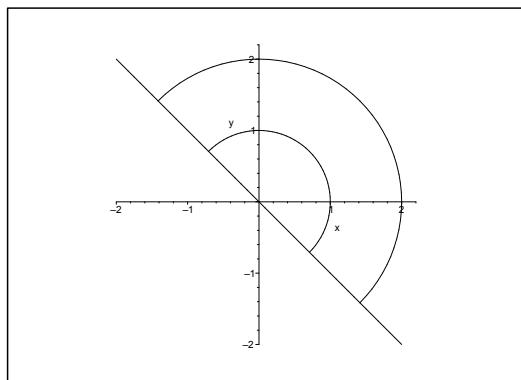


Figure 24.4: The projection of A onto the (x, y) -plane for $a = 1$.

from which $\int_A xz \, d\Omega \sim a^5$, and thus

$$\int_A xz \, d\Omega = \text{constant} \cdot a^5.$$

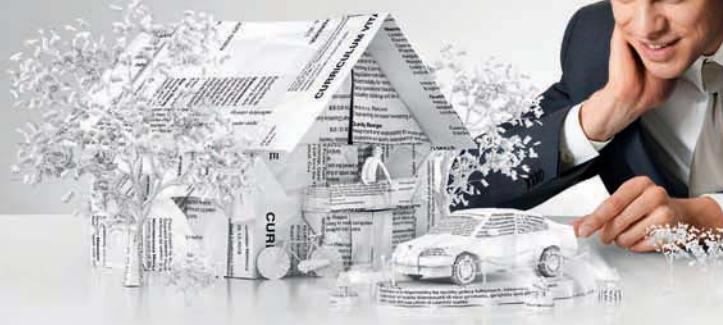
- D. The geometrical structure of revolution and the sphere indicates that one either should apply **I 1. semi-polar coordinates** or **I 2. spherical coordinates**. We shall in the following go through both possibilities for comparison.

SIMPLY CLEVER

ŠKODA



We will turn your CV into
an opportunity of a lifetime



Do you like cars? Would you like to be a part of a successful brand?
We will appreciate and reward both your enthusiasm and talent.
Send us your CV. You will be surprised where it can take you.

Send us your CV on
www.employerforlife.com



Click on the ad to read more

I 1. In *semi-polar* coordinates the domain A is represented by

$$\tilde{A} = \left\{ (\varrho, \varphi, z) \mid a \leq \varrho \leq 2a, -\frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4}, 0 \leq z \leq \sqrt{4a^2 - \varrho^2} \right\}.$$

Hence by the reduction theorem (where the weight function is ϱ),

$$\begin{aligned} I &= \int_A xz \, d\Omega = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left\{ \int_a^{2a} \left\{ \int_0^{\sqrt{4a^2 - \varrho^2}} \varrho \cos \varphi \cdot z \, dz \right\} \varrho \, d\varrho \right\} d\varphi \\ &= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos \varphi \, d\varphi \cdot \int_a^{2a} \varrho^2 \left\{ \int_0^{\sqrt{4a^2 - \varrho^2}} z \, dz \right\} \, d\varrho \\ &= [\sin \varphi]_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \cdot \int_a^{2a} \varrho^2 \left[\frac{1}{2} z^2 \right]_0^{\sqrt{4a^2 - \varrho^2}} \, d\varrho = \sqrt{2} \cdot \frac{1}{2} \int_a^{2a} \varrho^2 (4a^2 - \varrho^2) \, d\varrho \\ &= \frac{\sqrt{2}}{2} \int_a^{2a} (4a^2 \varrho^2 - \varrho^4) \, d\varrho = \frac{\sqrt{2}}{2} \left[\frac{4}{3} a^2 \varrho^3 - \frac{1}{5} \varrho^5 \right]_a^{2a} \\ &= \frac{\sqrt{2}}{2} \left\{ \left(\frac{4}{3} a^2 \cdot 8a^3 - \frac{32}{5} a^5 \right) - \left(\frac{4}{3} a^2 \cdot a^3 - \frac{1}{5} a^5 \right) \right\} \\ &= \frac{\sqrt{2}}{2} a^5 \left\{ \frac{32}{3} - \frac{32}{5} - \frac{4}{3} + \frac{1}{5} \right\} \\ &= \frac{\sqrt{2}}{2} a^5 \cdot \left\{ \frac{28}{3} - \frac{31}{5} \right\} = \frac{47\sqrt{2} a^5}{30}. \end{aligned}$$

I 2. If we instead choose *spherical* coordinates then

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta,$$

where θ is *measured from the z -axis* (and *not* from the (x, y) -plane, which one might expect), and the weight function is $r^2 \sin \theta$, and the domain A is represented by the parametric space

$$\hat{A} = \left\{ (r, \varphi, \theta) \mid -\frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4}, \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}, \frac{a}{\sin \theta} \leq r \leq 2a \right\},$$

where the vertical bounding line for B_0 is described by $r \sin \theta = a$, so the lower bound for r is $\frac{a}{\sin \theta} \leq r$.

Then we get by the reduction theorem

$$\begin{aligned}
 \int_A xz \, d\Omega &= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left\{ \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left\{ \int_{\frac{a}{\sin \theta}}^{2a} r \sin \theta \cos \varphi \cdot r \cos \theta \, r^2 \sin \theta \, dr \right\} d\theta \right\} d\varphi \\
 &= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos \varphi \, d\varphi \cdot \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 \theta \cos \theta \left\{ \int_{\frac{a}{\sin \theta}}^{2a} r^4 \, dr \right\} d\theta \\
 &= [\sin \varphi]_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \cdot \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 \theta \cos \theta \left[\frac{1}{5} r^5 \right]_{\frac{a}{\sin \theta}}^{2a} d\theta \\
 &= \sqrt{2} \cdot 5 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 \theta \cos \theta \cdot \left\{ 32a^5 - \frac{a^5}{\sin^5 \theta} \right\} d\theta \\
 &= \frac{\sqrt{2}}{5} a^5 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left\{ 32 \sin^2 \theta - \frac{1}{\sin^3 \theta} \right\} \cos \theta \, d\theta \\
 &= \frac{\sqrt{2}}{5} a^5 \left[\frac{32}{3} \sin^3 \theta + \frac{1}{2} \frac{1}{\sin^2 \theta} \right] \frac{\pi}{6}^{\frac{\pi}{2}} \\
 &= \frac{\sqrt{2}}{5} a^5 \left\{ \left(\frac{32}{3} + \frac{1}{2} \right) - \left(\frac{32}{3} \cdot \frac{1}{8} + \frac{1}{2} \cdot 4 \right) \right\} \\
 &= \frac{\sqrt{2}}{5} a^5 \left\{ \frac{32}{3} + \frac{1}{2} - \frac{4}{3} - 2 \right\} = \frac{47\sqrt{2}}{60} a^5.
 \end{aligned}$$

C. We see in both variants that the result is $\sim a^5$, so we get a weak control, cf. the examination of the dimensions in A. In MAPLE we use the following commands,

with(Student[MultivariateCalculus])

$$\text{MultiInt}\left(r^2 \cdot \sin(v) \cdot \cos(t) \cdot r \cdot r^2 \cdot \sin(v), r = \frac{a}{\sin(v)}..2a, v = \frac{\pi}{6}..\frac{\pi}{2}, t = -\frac{\pi}{4}..\frac{3\pi}{4}\right)$$

$$\frac{47}{30} a^5 \sqrt{2}$$

Example 24.2 Calculate in each of the following cases the given space integral over a point set A , which in spherical coordinates is bounded by

$$\alpha \leq \varphi \leq \beta \quad \text{and} \quad (r, \theta) \in B^*(\varphi);$$

1) The space integral $\int_A \sqrt{x^2 + y^2} d\Omega$, where the domain of integration A is given by

$$x^2 + y^2 + z^2 \leq 2.$$

2) The space integral $\int_A (x^2 + y^2 + z^2)^2 d\Omega$, where the domain of integration A is given by

$$x^2 + y^2 + z^2 \leq 1.$$

3) The space integral $\int_A xyz d\Omega$, where the domain of integration A is given by

$$x^2 + y^2 + z^2 \leq 1, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0.$$

4) The space integral $\int_A (x^2 + y^2 + z^2)^{-\frac{3}{2}} d\Omega$, where the domain of integration A is given by

$$a^2 \leq x^2 + y^2 + z^2 \leq b^2, \quad \text{where } b > a.$$

5) The space integral $\int_A (x^2 z + z^3) d\Omega$, where the domain of integration A is given by

$$x^2 + y^2 + z^2 \leq a^2 \quad \text{and} \quad z \geq 0.$$

6) The space integral $\int_A \frac{y}{z^2} d\Omega$, where the domain of integration A is given by

$$x^2 + y^2 + z^2 \leq (2a)^2, \quad a \leq z, \quad 0 \leq y \leq x.$$

A Space integrals in spherical coordinates.

D Identify the point set. Sketch if necessary the meridian cut. Finally, compute the space integral by reduction in spherical coordinates.

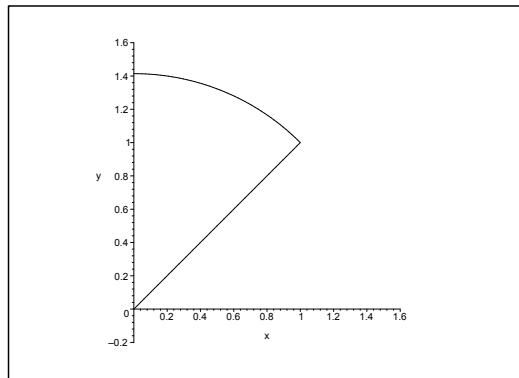


Figure 24.5: The meridian cut B^* in **Example 24.2.1**.

I 1) It is obvious that A is a conic slice of the ball of centrum $(0, 0, 0)$ and radius $\sqrt{2}$. Thus $0 \leq \varphi \leq 2\pi$, and the meridian cut

$$B^* = B^*(\varphi) = \left\{ (r, \theta) \mid 0 \leq r \leq \sqrt{2}, 0 \leq \theta \leq \frac{\pi}{4} \right\}$$

does not depend on φ . Then by the reduction theorem in spherical coordinates,

$$\begin{aligned} \int_A \sqrt{x^2 + y^2} d\Omega &= \int_0^{2\pi} \left\{ \int_{B^*(\varphi)} r \sin \theta \cdot r^2 \sin \theta dr d\theta \right\} d\varphi = 2\pi \int_0^{\sqrt{2}} r^3 dr \cdot \int_0^{\frac{\pi}{4}} \sin^2 \theta d\theta \\ &= 2\pi \left[\frac{r^4}{4} \right]_0^{\sqrt{2}} \int_0^{\frac{\pi}{4}} \frac{1 - \cos 2\theta}{2} d\theta = 2\pi \cdot \frac{4}{4} \cdot \frac{1}{2} \left[\theta - \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{4}} \\ &= \pi \left(\frac{\pi}{4} - \frac{1}{2} \right) = \frac{\pi^2}{4} - \frac{\pi}{2}. \end{aligned}$$

In MAPLE we use the following commands,

with(Student[MultivariateCalculus])

$$\text{MultiInt}\left(r \cdot \sin(v) \cdot r^2 \cdot \sin(v), t = 0..2\pi, r = 0..\sqrt{2}, v = 0..\frac{\pi}{4}\right)$$

$$-\frac{1}{2}\pi + \frac{1}{4}\pi^2$$

e-learning for kids

The number 1 MOOC for Primary Education
Free Digital Learning for Children 5-12
15 Million Children Reached

About e-Learning for Kids Established in 2004, e-Learning for Kids is a global nonprofit foundation dedicated to fun and free learning on the Internet for children ages 5 - 12 with courses in math, science, language arts, computers, health and environmental skills. Since 2005, more than 15 million children in over 190 countries have benefitted from eLessons provided by EFK! An all-volunteer staff consists of education and e-learning experts and business professionals from around the world committed to making difference. eLearning for Kids is actively seeking funding, volunteers, sponsors and courseware developers; get involved! For more information, please visit www.e-learningforkids.org.



Click on the ad to read more

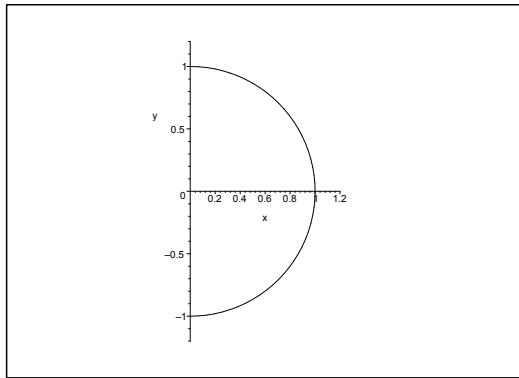


Figure 24.6: The meridian cut B^* in **Example 24.2.2**.

- 2) The set A is the unit ball, so $\varphi \in [0, 2\pi]$, and $B^* = B^*(\varphi)$ is the unit half circle in the right half plane which does not depend on φ ,

$$B^* = B^*(\varphi) = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq \pi\}.$$

Then by the reduction theorem in spherical coordinates,

$$\int_A (x^2 + y^2 + z^2)^2 d\Omega = 2\pi \int_{B^*} r^4 \cdot r^2 \sin \theta dr d\theta = 2\pi \int_0^1 r^6 \cdot \int_0^\pi \sin \theta d\theta = \frac{4\pi}{7}.$$

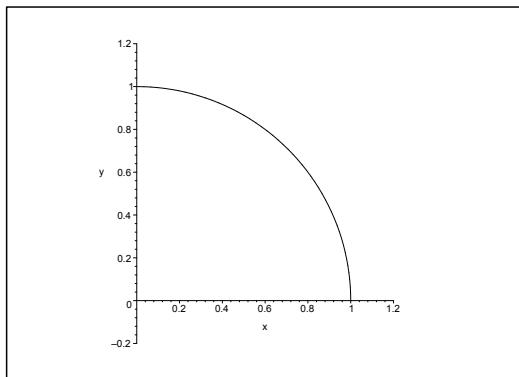


Figure 24.7: The meridian cut $B^*(\varphi)$, $\varphi \in \left[0, \frac{\pi}{2}\right]$ in **Example 24.2.3**.

- 3) The domain of integration is that part of the unit ball which lies in the first octant, thus $0 \leq \varphi \leq \frac{\pi}{2}$ and

$$B^*(\varphi) = \left\{ (r, \theta) \mid 0 \leq \theta \leq \frac{\pi}{2} \right\} \quad \text{for } 0 \leq \varphi \leq \frac{\pi}{2}.$$

By the reduction theorem in spherical coordinates,

$$\begin{aligned}\int_A xyz \, d\Omega &= \int_0^{\frac{\pi}{2}} \left\{ \int_{B^*} r^3 \sin^2 \theta \cos \theta \cdot \sin \varphi \cos \varphi \cdot r^2 \sin \theta \, dr \, d\theta \right\} \, d\varphi \\ &= \int_0^{\frac{\pi}{2}} \sin \varphi \cdot \cos \varphi \, d\varphi \cdot \int_0^1 r^5 \, dr \cdot \int_0^{\frac{\pi}{2}} \sin^3 \theta \cos \theta \, d\theta \\ &= \left[\frac{1}{2} \sin^2 \varphi \right]_0^{\frac{\pi}{2}} \cdot \left[\frac{r^6}{6} \right]_0^1 \cdot \left[\frac{1}{4} \sin^4 \theta \right]_0^{\frac{\pi}{2}} = \frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{4} = \frac{1}{48}.\end{aligned}$$

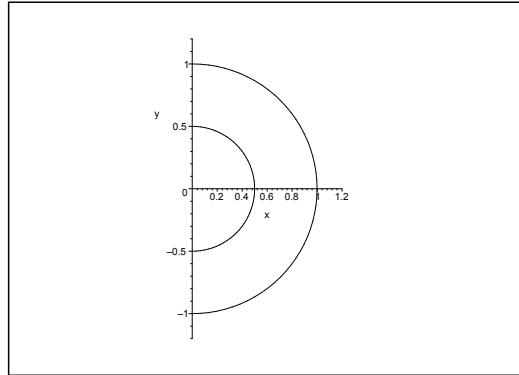


Figure 24.8: The meridian cut B^* in **Example 24.2.4** for $a = \frac{1}{2}$ and $b = 1$.

In MAPLE we use the following commands,

`with(Student[MultivariateCalculus])`

$$\text{MultiInt} \left(r^3 \cdot \sin(v)^2 \cdot \cos(v) \cdot \sin(t) \cdot \cos(t) \cdot r^2 \cdot \sin(v), t = 0.. \frac{\pi}{2}, r = 0..1, v = 0.. \frac{\pi}{2} \right)$$

$$\frac{1}{48}$$

- 4) Here A is a shell, so $\varphi \in [0, 2\pi]$, and

$$B^* = B^*(\varphi) = \{(r, \theta) \mid a \leq r \leq b, 0 \leq \theta \leq \pi\}$$

does not depend on φ .

By the reduction theorem in spherical coordinates,

$$\begin{aligned}\int_A (x^2 + y^2 + z^2)^{-\frac{3}{2}} \, d\Omega &= \int_0^{2\pi} \left\{ \int_{B^*} r^{-3} r^2 \sin \theta \, dr \, d\theta \right\} \, d\varphi \\ &= 2\pi \int_1^b \frac{1}{r} \, dr \cdot \int_0^\pi \sin \theta \, d\theta = 2\pi [\ln r]_a^b \cdot [-\cos \theta]_0^\pi = 4\pi \ln \left(\frac{b}{a} \right).\end{aligned}$$

- 5) Here A is that part of the ball of centrum $(0, 0, 0)$ and radius a , which lies in the upper half space, thus $0 \leq \varphi \leq 2\pi$, and

$$B^* = B^*(\varphi) = \left\{ (r, \theta) \mid 0 \leq r \leq a, 0 \leq \theta \leq \frac{\pi}{2} \right\}.$$

By the reduction theorem in spherical coordinates,

$$\begin{aligned}
 \int_A (x^2 z + z^3) d\Omega &= \int_0^{2\pi} \left\{ \int_{B^*} (r^2 \sin^2 \theta \cos^2 \varphi r \cos \theta + r^3 \cos^3 \theta) r^2 \sin \theta dr d\theta \right\} d\varphi \\
 &= \int_0^{2\pi} \left\{ \int_0^a r^5 \left\{ \int_0^{\frac{\pi}{2}} (\cos^2 \varphi \sin^2 \theta \cos \theta + \cos^3 \theta) \sin \theta d\theta \right\} dr \right\} d\varphi \\
 &= \left[\frac{r^6}{6} \right]_0^a \int_0^{2\pi} \left\{ \int_0^{\frac{\pi}{2}} \{ \cos^2 \varphi (\cos \theta - \cos^3 \theta) + \cos^3 \theta \} \sin \theta d\theta \right\} d\varphi \\
 &= \frac{a^6}{6} \int_0^{2\pi} \left\{ \int_0^{\frac{\pi}{2}} \{ \cos^2 \varphi \cos \theta + \sin^2 \varphi \cos^3 \theta \} \sin \theta d\theta \right\} d\varphi \\
 &= \frac{a^6}{5} \int_0^{2\pi} \left[-\cos^2 \varphi \cdot \frac{1}{2} \cos^2 \theta - \sin^2 \varphi \cdot \frac{1}{4} \cos^4 \theta \right]_{\theta=0}^{\frac{\pi}{2}} d\varphi \\
 &= \frac{a^6}{24} \int_0^{2\pi} \{ 2 \cos^2 \varphi + \sin^2 \varphi \} d\varphi = \frac{a^6}{24} \int_0^{2\pi} \left\{ \frac{3}{2} + \frac{1}{2} \cos 2\varphi \right\} d\varphi \\
 &= \frac{a^6}{24} \cdot \frac{3}{2} \cdot 2\pi = \frac{a^6 \pi}{8}.
 \end{aligned}$$

In MAPLE we use the following commands,

with(Student[MultivariateCalculus])

$$\text{MultiInt}\left((r^2 \cdot \sin(v)^2 \cdot \cos(t)^2 \cdot r \cdot \cos(v) + r^3 \cdot \cos(v)^3) \cdot r^2 \cdot \sin(v), t = 0..2\pi, r = 0..a, v = 0..\frac{\pi}{2}\right)$$

$$\frac{1}{8} \pi a^6$$

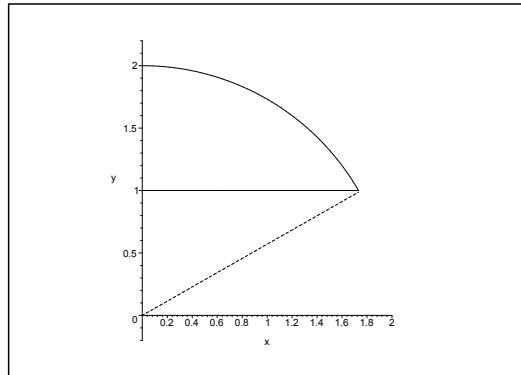


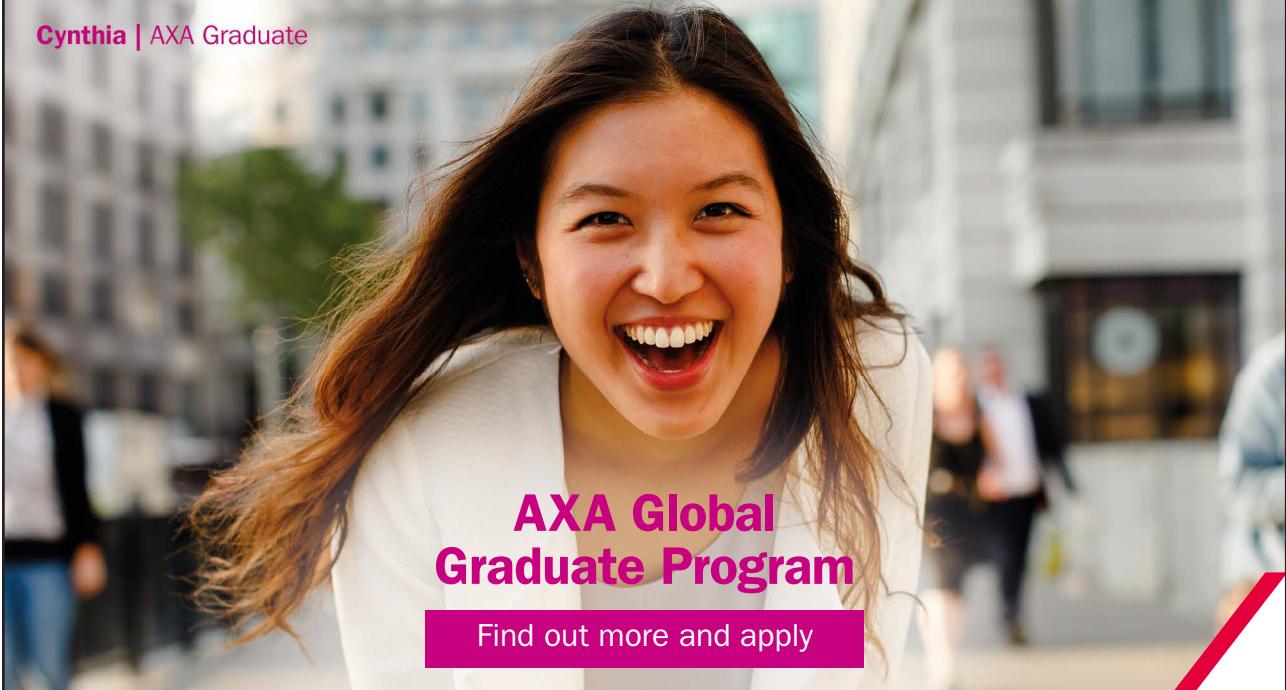
Figure 24.9: The meridian cut $B^*(\varphi)$ for $\varphi \in \left[0, \frac{\pi}{4}\right]$ and $a = 1$ in Example 24.2.6.

- 6) The domain of integration is in spherical coordinates described by $0 \leq \varphi \leq \frac{\pi}{4}$ (from the request $0 \leq y \leq x$) and

$$\begin{aligned}
 B^*(\varphi) &= \left\{ (r, \theta) \mid 0 \leq \theta \leq \frac{\pi}{3}, a \leq r \cos \theta, r \leq 2a \right\} \\
 &= \left\{ (r, \theta) \mid 0 \leq \theta \leq \frac{\pi}{3}, \frac{a}{\cos \theta} \leq r \leq 2a \right\},
 \end{aligned}$$

for $\varphi \in [0, \frac{\pi}{4}]$. We see that $B^* = B^*(\varphi)$ does not change in this φ -interval, hence by a reduction in spherical coordinates,

$$\begin{aligned}
 \int_A \frac{y}{z^2} d\Omega &= \int_0^{\frac{\pi}{4}} \left\{ \int_{B^*} \frac{r \sin \theta \sin \varphi}{r^2 \cos^2 \theta} \cdot r^2 \sin \theta dr d\theta \right\} d\varphi \\
 &= \int_0^{\frac{\pi}{4}} \left\{ \int_0^{\frac{\pi}{3}} \left(\int_{\frac{a}{\cos \theta}}^{2a} r \cdot \frac{\sin^2 \theta}{\cos^2 \theta} \cdot \sin \varphi dr \right) d\theta \right\} d\varphi \\
 &= [-\cos \varphi]_0^{\frac{\pi}{4}} \cdot \int_0^{\frac{\pi}{3}} \frac{\sin^2 \theta}{\cos^2 \theta} \left[\frac{1}{2} r^2 \right]_{\frac{a}{\cos \theta}}^{2a} d\theta \\
 &= \left(1 - \frac{1}{\sqrt{2}} \right) \int_0^{\frac{\pi}{3}} \frac{\sin^2 \theta}{\cos^2 \theta} \left(2a^2 - \frac{a^2}{2} \frac{1}{\cos^2 \theta} \right) d\theta \\
 &= \left(1 - \frac{1}{\sqrt{2}} \right) a^2 \left\{ 2 \int_0^{\frac{\pi}{3}} \frac{1 - \cos^2 \theta}{\cos^2 \theta} d\theta - \frac{1}{2} \int_0^{\frac{\pi}{3}} \tan^2 \theta \cdot \frac{1}{\cos^2 \theta} d\theta \right\} \\
 &= \left(1 - \frac{1}{\sqrt{2}} \right) a^2 \left\{ 2[\tan \theta - \theta]_0^{\frac{\pi}{3}} - \frac{1}{2} \left[\frac{1}{3} \tan^3 \theta \right]_0^{\frac{\pi}{3}} \right\} \\
 &= \left(1 - \frac{1}{\sqrt{2}} \right) \left\{ 2 \left(\sqrt{3} - \frac{\pi}{3} \right) - \frac{1}{6} \cdot 3\sqrt{3} \right\} = \left(1 - \frac{\sqrt{2}}{2} \right) \left(\frac{3}{2} \sqrt{3} - \frac{2}{3} \pi \right) a^2.
 \end{aligned}$$



Cynthia | AXA Graduate

AXA Global Graduate Program

Find out more and apply

redefining / standards 



Click on the ad to read more

In MAPLE we use the following commands,

`with(Student[MultivariateCalculus])`

$$\text{MultiInt} \left(\frac{r \cdot \sin(v) \cdot \sin(t)}{r^2 \cdot \cos(v)^2} \cdot r^2 \cdot \sin(v), r = \frac{a}{\cos(v)}..2a, v = 0..\frac{\pi}{3}, t = 0..\frac{\pi}{4} \right)$$

$$-\frac{2}{3} a^2 \pi + \frac{3}{2} a^2 \sqrt{3} + \frac{1}{3} \sqrt{2} a^2 \pi - \frac{3}{4} \sqrt{2} a^2 \sqrt{3}$$

Example 24.3 Let a denote a positive constant. Let K_0 denote the closed half ball of centrum $(0, 0, 0)$, of radius $2a$, and where $z \geq 0$. Finally, let K_1 denote the open ball of centrum $(0, 0, a)$ and radius a . We define a closed body of revolution A by removing K_1 from K_0 . Thus $A = K_0 \setminus K_1$. Let B denote a meridian cut in A .

- 1) Sketch B , and explain why A in spherical coordinates (r, θ, φ) is given by

$$\varphi \in [0, 2\pi], \quad \theta \in \left[0, \frac{\pi}{2}\right], \quad r \in [2a \cos \theta, 2a].$$

- 2) Calculate the space integral $\int_A z^2 d\Omega$.

A Space integral in spherical coordinates.

D Analyze geometrically the meridian half plane (add a line perpendicular to the radius vector). Then use the spherical reduction of the space integral.

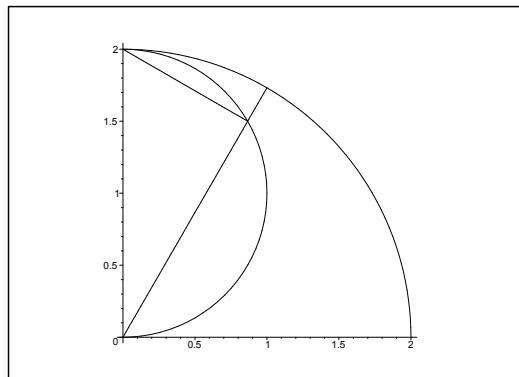


Figure 24.10: The meridian cut for $a = 1$ with a radius vector and a perpendicular line.

I 1) The figure shows that we have a rectangular triangle with the hypotenuse of length $2a$ along the Z -axis and the angle θ between radius vector and the Z -axis. Then a geometrical consideration shows that the distance from origo to the intersection point with the circle of radius a and centrum $(0, a)$ is give by $2a \cos \theta$. This gives us the lower limit for r , thus

$$r \in [2a \cos \theta, 2a].$$

The domains of the other coordinates are obvious.

2) The integrand is written in spherical coordinates in the following way,

$$f(x, y, z) = z^2 = r^2 \cos^2 \theta.$$

Then by the reduction theorem for space integrals in spherical coordinates,

$$\begin{aligned} \int_A z^2 d\Omega &= 2\pi \int_0^{\frac{\pi}{2}} \left\{ \int_{2a \cos \theta}^{2a} r^2 \cos^2 \theta \cdot r^2 \sin \theta dr \right\} d\theta \\ &= 2\pi \int_0^{\frac{\pi}{2}} \cos^2 \theta \sin \theta \cdot \left[\frac{r^5}{5} \right]_{2a \cos \theta}^{2a} d\theta \\ &= \frac{64\pi a^5}{5} \int_0^{\frac{\pi}{2}} \{ \cos^2 - \cos^7 \theta \} \sin \theta d\theta = \frac{64\pi a^5}{5} \left[\frac{1}{8} \cos^8 \theta - \frac{1}{3} \cos^3 \theta \right]_0^{\frac{\pi}{2}} \\ &= \frac{64\pi a^5}{5} \cdot \frac{8-3}{8 \cdot 3} = \frac{8\pi a^5}{3}. \end{aligned}$$

In MAPLE we use the following commands,

with(Student[MultivariateCalculus])

$$\text{MultiInt} \left(r^2 \cdot \cos(v)^2 \cdot r^2 \cdot \sin(v), r = 2a \cdot \cos(v)..2a, v = 0..\frac{\pi}{2} \right)$$

$$\frac{8}{3} \pi a^3$$

Example 24.4 Let

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0, x^2 + y^2 + z^2 \leq 4, \frac{1}{3}(x^2 + y^2) \leq z^2 \leq 3(x^2 + y^2)\}.$$

1) Sketch the curve of the intersection with the (x, z) -plane.

2) Compute the space integral

$$\int_A z d\Omega.$$

(The best method here is to use spherical coordinates).

A Space integral in spherical coordinates.

D Follow the guidelines of the text.

I 1) If $y = 0$, then we get the limitations $z \geq 0$, $x^2 + z^2 \leq 2^2$ and $\frac{1}{3}x^2 \leq z^2 \leq 3x^2$, thus

$$\frac{|x|}{\sqrt{3}} \leq z \leq \sqrt{3}|x|.$$

The intersection curve is given in spherical coordinates (r, θ) by

$$\left\{ (r\theta) \mid r \in [0, 2], \theta \in \left[\frac{\pi}{6}, \frac{\pi}{3} \right] \cup \left[-\frac{\pi}{3}, -\frac{\pi}{6} \right] \right\},$$

where θ is measured from the Z -axis and positive towards the X -axis.

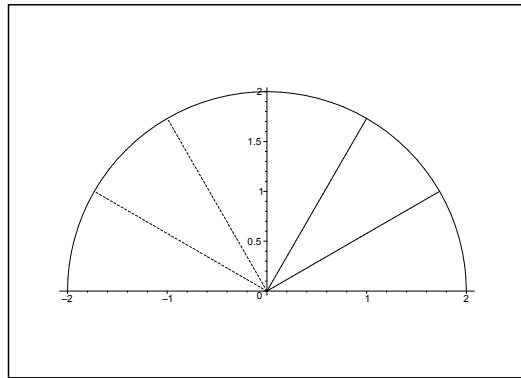


Figure 24.11: The intersection curve with the (x, z) -plane. It follows from the symmetry that we are only interested in the sector of the first quadrant.

2) The space integral is then calculated by reduction in spherical coordinates,

$$\begin{aligned} \int_A z \, d\Omega &= \int_0^{2\pi} \left\{ \int_0^2 \left\{ \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r \cos \theta \cdot r^2 \sin \theta \, d\theta \right\} dr \right\} d\varphi \\ &= 2\pi \int_0^2 r^3 dr \cdot \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin \theta \cdot \cos \theta \, d\theta = 2\pi \left[\frac{r^4}{4} \right]_0^2 \left[\frac{\sin^2 \theta}{2} \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= 2\pi \cdot \frac{16}{4} \cdot \frac{1}{2} \left(\frac{3}{4} - \frac{1}{4} \right) = 2\pi \cdot 4 \cdot \frac{1}{2} \cdot \frac{1}{2} = 2\pi. \end{aligned}$$

Example 24.5 Let a and c be positive constants, and let A denote the half shell given by the inequalities

$$a^2 \leq x^2 + y^2 + z^2 \leq 4a^2, \quad z \geq 0,$$

Calculate the space integral

$$\int_A \frac{z}{c^2 + x^2 + y^2 + z^2} \, d\Omega.$$

A Space integral.

D We give here four variants:

- 1) Reduction in spherical coordinates.
- 2) Reduction in semi-polar coordinates.
- 3) Reduction by the slicing method.
- 4) Reduction in rectangular coordinate.

These methods are here numbered according to their increasing difficulty. The fourth variant is possible, but it is not worth here to produce all the steps involved, because the method cannot be recommended in this particular case.

I First variant. *Spherical coordinates.*

The set A is described in spherical coordinates by

$$\left\{ (r, \varphi, \theta) \mid r \in [a, 2a], \varphi \in [0, 2\pi], \theta \in \left[0, \frac{\pi}{2}\right] \right\},$$

hence by the reduction of the space integral,

$$\begin{aligned} \int_A \frac{z}{c^2 + x^2 + y^2 + z^2} d\Omega &= \int_0^{2\pi} \left\{ \int_0^{\frac{\pi}{2}} \left(\int_a^{2a} \frac{r \cos \theta}{c^2 + r^2} \cdot r^2 \sin \theta dr \right) d\theta \right\} d\varphi \\ &= 2\pi \int_0^{\frac{\pi}{2}} \cos \theta \sin \theta d\theta \cdot \int_a^{2a} \frac{r^2}{c^2 + r^2} \cdot r dr \quad [t = r^2] \\ &= 2\pi \left[\frac{\sin^2 \theta}{2} \right]_0^{\frac{\pi}{2}} \cdot \int_{a^2}^{4a^2} \frac{t + c^2 - c^2}{c^2 + t} \cdot \frac{1}{2} dt \\ &= \frac{\pi}{2} \int_{a^2}^{4a^2} \left\{ 1 - \frac{c^2}{c^2 + t} \right\} dt = \frac{\pi}{2} [t - c^2 \ln(c^2 + t)]_{t=a^2}^{4a^2} = \frac{\pi}{2} \left\{ 3a^2 - c^2 \ln \left(\frac{4a^2 + c^2}{a^2 + c^2} \right) \right\}. \end{aligned}$$

I joined MITAS because
I wanted **real responsibility**

The Graduate Programme
for Engineers and Geoscientists
www.discovermitas.com

Month 16

I was a construction supervisor in the North Sea advising and helping foremen solve problems

Real work
International opportunities
Three work placements

MAERSK

Click on the ad to read more

2. variant. *Semi-polar coordinates.*

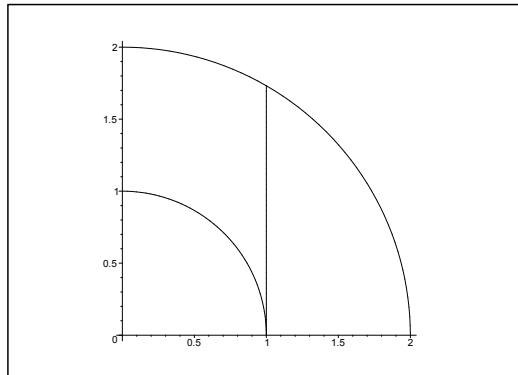


Figure 24.12: The meridian cut for $a = 1$ with the line $x = a = 1$.

We must here split the investigation into two according to whether $\varrho \in [0, a[$ or $\varrho \in [a, 2a]$, cf. the figure.

That part A_1 of A , which is given by $\varrho \in [0, a[$, is described in semi-polar coordinates by

$$\{(\varrho, \varphi, z) \mid \varrho \in [0, a[, \varphi \in [0, 2\pi], \sqrt{a^2 - \varrho^2} \leq z \leq \sqrt{4a^2 - \varrho^2}\}.$$

That part A_2 of A , which is given by $\varrho \in [a, 2a]$, is described in semi-polar coordinates by

$$\{(\varrho, \varphi, z) \mid \varrho \in [a, 2a], \varphi \in [0, 2\pi], 0 \leq z \leq \sqrt{4a^2 - \varrho^2}\}.$$

Then by reduction in semi-polar coordinates,

$$\begin{aligned} \int_A \frac{z}{c^2 + x^2 + y^2 + z^2} d\Omega &= \int_{A_1} \frac{z}{c^2 + x^2 + y^2 + z^2} d\Omega + \int_{A_2} \frac{z}{c^2 + x^2 + y^2 + z^2} d\Omega \\ &= \int_0^{2\pi} \left\{ \int_0^a \left(\int_{\sqrt{a^2 - \varrho^2}}^{\sqrt{4a^2 - \varrho^2}} \frac{z}{c^2 + \varrho^2 + z^2} \varrho dz \right) d\varrho \right\} d\varphi \\ &\quad + \int_0^{2\pi} \left\{ \int_1^{2a} \left(\int_0^{\sqrt{4a^2 - \varrho^2}} \frac{z}{c^2 + \varrho^2 + z^2} \varrho dz \right) d\varrho \right\} d\varphi \\ &= 2\pi \int_0^a \left[\frac{1}{2} \ln(c^2 + \varrho^2 + z^2) \right]_{z=\sqrt{a^2 - \varrho^2}}^{\sqrt{4a^2 - \varrho^2}} \varrho d\varrho + 2\pi \int_a^{2a} \left[\frac{1}{2} \ln(c^2 + \varrho^2 + z^2) \right]_{z=0}^{\sqrt{4a^2 - \varrho^2}} \varrho d\varrho \\ &= \pi \int_0^a \{ \ln(c^2 + 4a^2) - \ln(c^2 + a^2) \} \varrho d\varrho + \pi \int_a^{2a} \{ \ln(c^2 + 4a^2) - \ln(c^2 + \varrho^2) \} \varrho d\varrho, \end{aligned}$$

thus

$$\begin{aligned}
 & \int_A \frac{z}{c^2 + x^2 + y^2 + z^2} d\Omega \\
 &= \pi \{ \ln(c^2 + 4a^2) - \ln(c^2 + a^2) \} \cdot \frac{a^2}{2} + \pi \ln(c^2 + 4a^2) \cdot \frac{1}{2} \{ 4a^2 - a^2 \}, \\
 &\quad - \frac{\pi}{2} \int_{a^2}^{4a^2} \ln(c^2 + t) dt \quad (\text{putting } t = \varrho^2) \\
 &= \frac{\pi}{2} \cdot 4a^2 \ln(c^2 + 4a^2) - \frac{\pi}{2} a^2 \ln(c^2 + a^2) - \frac{\pi}{2} [(c^2 + t) \ln(c^2 + t) - t]_{t=a^2}^{4a^2} \\
 &= \frac{\pi}{2} \cdot 4a^2 \ln(c^2 + 4a^2) - \frac{\pi}{2} a^2 \ln(c^2 + a^2) - \frac{\pi}{2} (c^2 + 4a^2) \ln(c^2 + 4a^2) \\
 &\quad + \frac{\pi}{2} \cdot 4a^2 + \frac{\pi}{2} (c^2 + a^2) \ln(c^2 + a^2) - \frac{\pi}{2} \cdot a^2 \\
 &= \frac{\pi}{2} \cdot \left\{ 3a^2 - c^2 \ln\left(\frac{c^2 + 4a^2}{c^2 + a^2}\right) \right\}.
 \end{aligned}$$

Third variant. *The slicing method.*

The plane at height $z = [0, a]$ intersects A in an annulus $B(z)$, which is described in polar coordinates by

$$\{(\varrho, \varphi) \mid \varphi \in [0, 2\pi], \sqrt{a^2 - z^2} \leq \varrho \leq \sqrt{4a^2 - z^2}\}.$$

The plane at height $z \in [a, 2a]$ intersects A in a disc $B(z)$, which is described in polar coordinates by

$$\{(\varrho, \varphi) \mid \varphi \in [0, 2\pi], 0 \leq \varrho \leq \sqrt{4a^2 - z^2}\}.$$

If we first integrate over $B(z)$ and then with respect to z , we get the following reduction,

$$\begin{aligned}
 & \int_A \frac{z}{c^2 + x^2 + y^2 + z^2} d\Omega \\
 &= \int_0^a \left\{ \int_{B(z)} \frac{z}{c^2 + x^2 + y^2 + z^2} dS \right\} dz + \int_a^{2a} \left\{ \int_{B(z)} \frac{z}{c^2 + x^2 + y^2 + z^2} dS \right\} dz \\
 &= \int_0^a \left\{ \int_0^{2\pi} \left(\int_{\sqrt{a^2 - z^2}}^{\sqrt{4a^2 - z^2}} \frac{z}{c^2 + z^2 + \varrho^2} \varrho d\varrho \right) d\varphi \right\} dz \\
 &\quad + \int_a^{2a} \left\{ \int_0^{2\pi} \left(\int_0^{\sqrt{4a^2 - z^2}} \frac{z}{c^2 + z^2 + \varrho^2} \varrho d\varrho \right) d\varphi \right\} dz,
 \end{aligned}$$

hence

$$\begin{aligned}
 & \int_A \frac{z}{c^2 + x^2 + y^2 + z^2} d\Omega \\
 &= 2\pi \int_0^a \left[\frac{1}{2} \ln(c^2 + z^2 + \varrho^2) \right]_{\varrho=\sqrt{a^2-z^2}}^{\sqrt{4a^2-z^2}} z dz + 2\pi \int_a^{2a} \left[\frac{1}{2} \ln(c^2 + z^2 + \varrho^2) \right]_{\varrho=0}^{\sqrt{4a^2-z^2}} z dz \\
 &= \pi \int_0^a \{ \ln(c^2 + 4a^2) - \ln(c^2 + a^2) \} z dz + \pi \int_a^{2a} \{ \ln(c^2 + 4a^2) - \ln(c^2 + z^2) \} dz \\
 &= \pi \cdot \frac{a^2}{2} \ln\left(\frac{c^2 + 4a^2}{c^2 + a^2}\right) + \pi \ln(c^2 + 4a^2) \cdot \left[\frac{z^2}{2}\right]_a^{2a} - \frac{\pi}{2} \int_{a^2}^{4a^2} \ln(c^2 + t) dt \quad ("t = z^2") \\
 &= \frac{\pi}{2} \cdot a^2 \ln\left(\frac{c^2 + 4a^2}{c^2 + a^2}\right) + \frac{\pi}{2} \cdot 3a^2 \ln(c^2 + 4a^2) - \frac{\pi}{2} [(c^2 + t) \ln(c^2 + t) - t]_{t=a^2}^{4a^2} \\
 &= 2\pi a^2 \ln(c^2 + 4a^2) - \frac{\pi}{2} a^2 \ln(c^2 + a^2) \\
 &\quad - \frac{\pi}{2} (c^2 + 4a^2) \ln(c^2 + 4a^2) + \frac{\pi}{2} (c^2 + a^2) \ln(c^2 + a^2) + \frac{\pi}{2} \cdot 3a^2 \\
 &= \frac{\pi}{2} \left\{ 3a^2 - c^2 \ln\left(\frac{c^2 + 4a^2}{c^2 + a^2}\right) \right\}.
 \end{aligned}$$



93%
OF MIM STUDENTS ARE
WORKING IN THEIR SECTOR 3 MONTHS
FOLLOWING GRADUATION

MASTER IN MANAGEMENT

- STUDY IN THE CENTER OF MADRID AND TAKE ADVANTAGE OF THE UNIQUE OPPORTUNITIES THAT THE CAPITAL OF SPAIN OFFERS
- PROPEL YOUR EDUCATION BY EARNING A DOUBLE DEGREE THAT BEST SUITS YOUR PROFESSIONAL GOALS
- STUDY A SEMESTER ABROAD AND BECOME A GLOBAL CITIZEN WITH THE BEYOND BORDERS EXPERIENCE

5 SPECIALIZATIONS PERSONALIZE YOUR PROGRAM	#10 WORLDWIDE MASTER IN MANAGEMENT FINANCIAL TIMES	Length: 10 MONTHS Av. Experience: 1 YEAR Language: ENGLISH / SPANISH Format: FULL-TIME Intakes: SEPT / FEB
--	---	--

www.ie.edu/master-management | mim.admissions@ie.edu | [f](#) [t](#) [g](#) Follow us on IE MIM Experience



Click on the ad to read more

Fourth variant. *Rectangular coordinates.*

This is a very difficult variant, which I have only been through once. The computations here are only sketchy just to scare people away, because it cannot be recommended.

Let

$$A_0 = \{(x, y, z) \mid a^2 \leq x^2 + y^2 + z^2 \leq 4a^2, x \geq 0, y \geq 0, z \geq 0\}$$

be that part of A , which lies in the first octant. Then by an argument of symmetry on the integrand we conclude that

$$\int_A \frac{z}{c^2 + x^2 + y^2 + z^2} d\Omega = 4 \int_{A_0} \frac{z}{c^2 + x^2 + y^2 + z^2} d\Omega.$$

When $x \in [0, a]$ is fixed, the corresponding plane intersects the set A_0 in a domain $B(x)$, which is given in rectangular coordinates by

$$\begin{aligned} & \{(y, z) \mid y \in [0, \sqrt{a^2 - x^2}], \sqrt{a^2 - x^2 - y^2} \leq z \leq \sqrt{4a^2 - x^2 - y^2}\} \\ & \cup \{(y, z) \mid y \in [\sqrt{a^2 - x^2}, \sqrt{4a^2 - x^2}], 0 \leq z \leq \sqrt{4a^2 - x^2 - y^2}\}. \end{aligned}$$

REMARK. We see that the description in polar coordinates would be easier here, but I shall here demonstrate how bad things can be if one only uses rectangular coordinates. \diamond

Similarly, A_0 is cut for $x \in [a, 2a]$ into a quarter disc

$$\{(y, z) \mid y \in [0, \sqrt{4a^2 - x^2}], 0 \leq z \leq \sqrt{4a^2 - x^2 - y^2}\}.$$

Then by reduction in rectangular coordinates

$$\begin{aligned} \int_A \frac{z}{c^2 + x^2 + y^2 + z^2} d\Omega &= 4 \int_{A_0} \frac{z}{c^2 + x^2 + y^2 + z^2} d\Omega \\ &= 4 \int_0^a \left\{ \int_0^{\sqrt{a^2 - x^2}} \left(\int_{\sqrt{a^2 - x^2 - y^2}}^{\sqrt{4a^2 - x^2 - y^2}} \frac{z}{c^2 + x^2 + y^2 + z^2} dz \right) dy \right\} dx \\ &\quad + 4 \int_0^a \left\{ \int_{\sqrt{a^2 - x^2}}^{\sqrt{4a^2 - x^2}} \left(\int_0^{\sqrt{4a^2 - x^2 - y^2}} \frac{z}{c^2 + x^2 + y^2 + z^2} dz \right) dy \right\} dx \\ &\quad + 4 \int_a^{2a} \left\{ \int_0^{\sqrt{4a^2 - x^2}} \left(\int_0^{\sqrt{4a^2 - x^2 - y^2}} \frac{z}{c^2 + x^2 + y^2 + z^2} dz \right) dy \right\} dx \\ &= 2 \int_0^a \left\{ \int_0^{\sqrt{a^2 - x^2}} \{ \ln(c^2 + 4a^2) - \ln(c^2 + a^2) \} dy \right\} dx \\ &\quad + 2 \int_0^a \left\{ \int_{\sqrt{a^2 - x^2}}^{\sqrt{4a^2 - x^2}} \{ \ln(c^2 + 4a^2) - \ln(c^2 + x^2 + y^2) \} dy \right\} dx \\ &\quad + 2 \int_a^{2a} \left\{ \int_0^{\sqrt{4a^2 - x^2}} \{ \ln(c^2 + 4a^2) - \ln(c^2 + x^2 + y^2) \} dy \right\} dx. \end{aligned}$$

The former of these integrals is easy to compute, because it is a constant integrated over a quarter circle,

$$2 \int_0^a \left\{ \int_0^{\sqrt{a^2 - x^2}} \{ \ln(c^2 + 4a^2) - \ln(c^2 + a^2) \} dy \right\} dx = \frac{\pi}{2} a^2 \ln \left(\frac{c^2 + 4a^2}{c^2 + a^2} \right).$$

The following two integrals are very difficult, if one only sticks to rectangular coordinates. But even in polar coordinates each of these two integrals are very difficult to compute, though nothing in comparison with the rectangular variant.

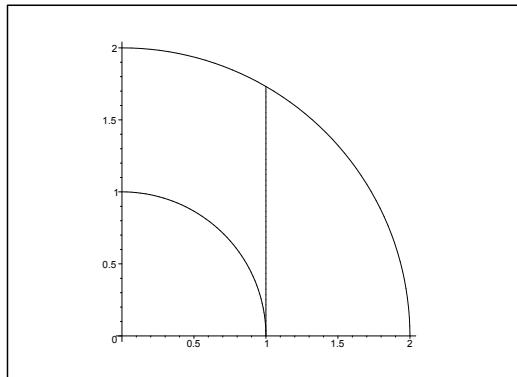


Figure 24.13: The domains B_1 and B_2 in the meridian half plane.

We shall of course start with a geometric analysis, because the integrand is the same in both cases. We can therefore join the two integrations over one single one over the domain $B_1 \cup B_2$, which again is more suitable for a polar description:

$$\left\{ (\varrho, \varphi) \mid \varrho \in [a, 2a], \varphi \in \left[0, \frac{\pi}{2}\right] \right\}.$$

By using this trick we get by insertion,

$$\begin{aligned} & \int_A \frac{z}{c^2 + x^2 + y^2 + z^2} d\Omega \\ &= \frac{\pi}{2} a^2 \ln \left(\frac{c^2 + 4a^2}{c^2 + a^2} \right) + 2 \int_0^{\frac{\pi}{2}} \left(\int_a^{2a} \{ \ln(c^2 + 4a^2) - \ln(c^2 + \varrho^2) \} \varrho d\varrho \right) d\varphi, \end{aligned}$$

and the following computations are reduced to variants of those from the second and the third variant.

REMARK. To my knowledge the full computation in rectangular coordinates without any trick has only been carried through once. We also tried to use MAPLE in an earlier version, at that did not work at all. The reason is that one has to apply a dirty rectangular trick at some place, which cannot be foreseen by the computer. ◇

Example 24.6 Let a be a positive constant, and let

$$A = \left\{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq a^2, z \leq \sqrt{\frac{x^2 + y^2}{3}} \right\}.$$

1) Sketch a meridian half plane, and explain why A is given in spherical coordinates (r, θ, φ) by

$$r \in [0, a], \quad \theta \in \left[\frac{\pi}{3}, \pi \right], \quad \varphi \in [0, 2\pi].$$

2) Compute the space integral

$$\int_A (x^2 + z^2) d\Omega.$$

A Space integral in spherical coordinate.

D The space integral is here calculated in four variants.

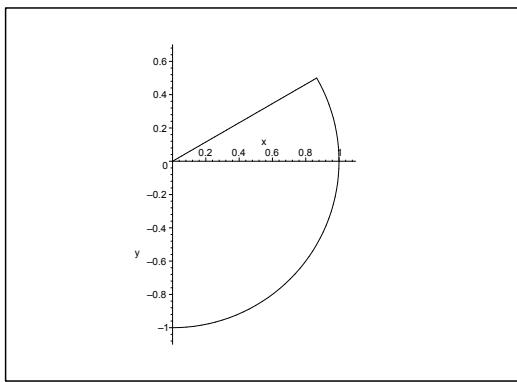


Figure 24.14: The meridian cut A^* for $a = 1$.

I 1) In the meridian half plane the cut A^* has the line $z = \frac{1}{\sqrt{3}} \varrho$ as an upper bound, corresponding to $\theta \in \left[\frac{\pi}{3}, \pi \right]$. The other variables are not restricted further, so A is given in spherical coordinates by

$$r \in [0, a], \quad \theta \in \left[\frac{\pi}{3}, \pi \right], \quad \varphi \in [0, 2\pi].$$

2) The space integral is here computed in four variants.

First variant. Direct insertion:

$$\begin{aligned}
 \int_A (x^2 + z^2) d\Omega &= \int_0^{2\pi} \left\{ \int_{\frac{\pi}{3}}^{\pi} \left(\int_0^a (r^2 \sin^2 \theta \cos^2 \varphi + r^2 \cos^2 \theta) r^2 \sin \theta dr \right) d\theta \right\} d\varphi \\
 &= \frac{a^5}{5} \int_0^{2\pi} \left\{ \int_{\frac{\pi}{3}}^{\pi} \{(1-\cos^2 \theta) \cos^2 \varphi + \cos^2 \theta\} \sin \theta d\theta \right\} d\varphi \\
 &= \frac{a^5}{5} \int_0^{2\pi} \left\{ \left[-\cos \theta + \frac{1}{3} \cos^3 \theta \right]_{\frac{\pi}{3}}^{\pi} \cos^2 \varphi + \left[-\frac{1}{3} \cos^3 \theta \right]_{\frac{\pi}{3}}^{\pi} \right\} d\varphi \\
 &= \frac{a^5}{5} \int_0^{2\pi} \left\{ \left(1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{24} \right) \cos^2 \varphi + \left(\frac{1}{3} + \frac{1}{24} \right) \right\} d\varphi \\
 &= \frac{a^5}{5} \left\{ \left(\frac{3}{2} - \frac{9}{24} \right) \pi + \frac{9}{24} \cdot 2\pi \right\} = \frac{a^5}{5} \pi \left\{ \frac{3}{2} - \frac{3}{8} + \frac{3}{4} \right\} \\
 &= \frac{3\pi}{5} a^5 \left\{ \frac{1}{2} + \frac{1}{8} \right\} = \frac{3\pi}{5} \cdot a^5 \cdot \frac{5}{8} = \frac{3\pi}{8} a^5.
 \end{aligned}$$

**"I studied English for 16 years but...
...I finally learned to speak it in just six lessons"**

Jane, Chinese architect

ENGLISH OUT THERE

Click to hear me talking before and after my unique course download



Click on the ad to read more

Second variant. A small reduction:

It follows from $x^2 + z^2 = r^2 - y^2$ that

$$\begin{aligned}
 \int_A (x^2 + z^2) d\Omega &= \int_A (r^2 - y^2) d\Omega \\
 &= \int_0^{2\pi} \left\{ \int_{\frac{\pi}{3}}^{\pi} \left(\int_0^a r^2 (1 - \sin^2 \theta \sin^2 \varphi) r^2 \sin \theta dr \right) d\theta \right\} d\varphi \\
 &= \frac{a^5}{5} \int_{\frac{\pi}{3}}^{\pi} \{2\pi \sin \theta - \pi \sin^3 \theta\} d\theta \\
 &= \pi \cdot \frac{a^5}{5} \left\{ [-2 \cos \theta]_{\frac{\pi}{3}}^{\pi} - \int_{\frac{\pi}{3}}^{\pi} (1 - \cos^2 \theta) \sin \theta d\theta \right\} \\
 &= \pi \cdot \frac{a^5}{5} \left\{ 2 + 1 + \left[\cos \theta - \frac{1}{3} \cos^3 \theta \right]_{\frac{\pi}{3}}^{\pi} \right\} = \pi \cdot \frac{a^5}{5} \cdot \left\{ 3 + \left(-1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{24} \right) \right\} \\
 &= \pi \cdot \frac{a^5}{5} \left(\frac{3}{2} + \frac{3}{8} \right) = \pi \cdot \frac{a^5}{5} \cdot \frac{15}{8} = \frac{3\pi}{8} a^5.
 \end{aligned}$$

Third variant. A symmetric argument:

For symmetric reasons,

$$\begin{aligned}
 \int_A (x^2 + z^2) d\Omega &= \int_A (y^2 + z^2) d\Omega = \frac{1}{2} \int_A \{(x^2 + y^2 + z^2) + z^2\} d\Omega \\
 &= \frac{1}{2} \int_0^{2\pi} \left\{ \int_{\frac{\pi}{3}}^{\pi} \left(\int_0^a (r^2 + r^2 \cos^2 \theta) r^2 \sin \theta dr \right) d\theta \right\} d\varphi \\
 &= \frac{1}{2} \cdot 2\pi \int_{\frac{\pi}{3}}^{\pi} (1 + \cos^2 \theta) \sin \theta d\theta \cdot \int_0^a r^4 dr = \pi \left[-\cos \theta - \frac{1}{3} \cos^3 \theta \right]_{\frac{\pi}{3}}^{\pi} \cdot \frac{a^5}{5} \\
 &= \pi \cdot \frac{a^5}{5} \left\{ 1 + \frac{1}{3} + \frac{1}{2} + \frac{1}{3 \cdot 8} \right\} = \pi \cdot \frac{a^5}{5} \left(\frac{3}{2} + \frac{3}{8} \right) = \pi \cdot \frac{a^5}{5} \cdot \frac{15}{8} = \frac{3\pi}{8} a^5.
 \end{aligned}$$

Fourth variant. The slicing method.

At the height $z \in]-a, 0]$ the body A is cut into a disc D_z given by

$$0 \leq \rho \leq \sqrt{a^2 - z^2}.$$

If instead $z \in]0, \frac{a}{2}[$, then A is cut into an annulus D_z given by

$$\sqrt{3}z \leq \rho \leq \sqrt{a^2 - z^2}.$$

For symmetric reasons we have for any $z \in]-a, \frac{a}{2}[$ that

$$\int_{D_z} (x^2 + z^2) dS = \int_{D_z} (y^2 + z^2) dS = \int_{D_z} \left\{ \frac{1}{2}(x^2 + y^2) + z^2 \right\} dS.$$

Then we get

$$\begin{aligned}
 & \int_A (x^2 + z^2) d\Omega \\
 &= \int_{-a}^0 \left\{ \int_{D_z} \left\{ \frac{1}{2}(x^2 + y^2) + z^2 \right\} dS \right\} dz + \int_0^{\frac{a}{2}} \left\{ \int_{D_z} \left\{ \frac{1}{2}(x^2 + y^2) + z^2 \right\} dS \right\} dz \\
 &= \int_{-a}^0 \left\{ \int_0^{2\pi} \left(\int_0^{\sqrt{a^2 - z^2}} \frac{1}{2} \varrho^2 \cdot \varrho d\varrho \right) d\varphi + z^2 \text{area}(D_z) \right\} dz \\
 &\quad + \int_0^{\frac{a}{2}} \left\{ \int_0^{2\pi} \left(\int_{\sqrt{3}z}^{\sqrt{a^2 - z^2}} \frac{1}{2} \varrho^2 \cdot \varrho d\varrho \right) d\varphi + z^2 \text{area}(D_z) \right\} dz,
 \end{aligned}$$

i.e.

$$\begin{aligned}
 & \int_A (x^2 + z^2) d\Omega \\
 &= \int_{-a}^0 \left\{ 2\pi \left[\frac{1}{8} \varrho^4 \right]_0^{\sqrt{a^2 - z^2}} + z^2 \pi (a^2 - z^2) \right\} dz \\
 &\quad + \int_0^{\frac{a}{2}} \left\{ 2\pi \left[\frac{1}{8} \varrho^4 \right]_{\sqrt{3}z}^{\sqrt{a^2 - z^2}} + z^2 \pi \{(a^2 - z^2) - 3z^2\} \right\} dz \\
 &= \frac{\pi}{4} \int_{-a}^0 \{(a^2 - z^2)^2 + 4z^2(a^2 - z^2)\} dz \\
 &\quad + \frac{\pi}{4} \int_0^{\frac{a}{2}} \{(a^2 - z^2)^2 - 9z^4 + 4z^2(a^2 - 4z^2)\} dz \\
 &= \frac{\pi}{2} \int_{-a}^0 \{a^4 - 2a^2z^2 + z^4 + 4a^2z^2 - 4z^4\} dz \\
 &\quad + \frac{\pi}{4} \int_0^{\frac{a}{2}} \{a^4 - 2a^2z^2 + z^4 - 9z^4 + 4a^2z^2 - 16z^4\} dz \\
 &= \frac{\pi}{4} \int_{-a}^0 \{a^4 + 2a^2z^2 - 3z^4\} dz + \frac{\pi}{4} \int_0^{\frac{a}{2}} \{a^4 + 2a^2z^2 - 24z^4\} dz \\
 &= \frac{\pi}{4} \left\{ \left[a^4 z + \frac{2}{3} a^2 z^3 - \frac{3}{5} z^5 \right]_{-a}^0 + \left[a^4 z + \frac{2}{3} a^2 z^3 - \frac{24}{5} z^5 \right]_0^{\frac{a}{2}} \right\} \\
 &= \frac{\pi}{4} \left\{ a^5 + \frac{2}{3} a^5 - \frac{3}{5} a^5 + \frac{1}{2} a^5 - \frac{24}{5 \cdot 32} a^5 \right\} \\
 &= \frac{\pi}{4} a^5 \left\{ 1 + \frac{2}{3} - \frac{3}{5} + \frac{1}{2} + \frac{1}{12} - \frac{3}{20} \right\} = \frac{\pi}{4} a^5 \left\{ \frac{3}{2} + \frac{9}{12} - \frac{3}{4} \right\} = \frac{3\pi}{8} a^5.
 \end{aligned}$$

24.4 Examples of volumes

Example 24.7 Set up a formula for the volume of the ellipsoid by applying that an ellipse of half axes a and b has the area πab .

- A Volume of an ellipsoid found by a space integral.
- D Use the slicing method and describe the ellipse for every z , and continue by computing the corresponding space integral.
- I Let the ellipsoid be given by the inequality

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.$$

For any fixed $z \in [-c, c]$ let $B(z)$ denote the ellipse (in (x, y) -coordinates) given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 - \frac{z^2}{c^2}.$$

When $z \in]-c, c[$, this describes an ellipse with the half axes

$$a\sqrt{1 - \frac{z^2}{c^2}} \quad \text{and} \quad b\sqrt{1 - \frac{z^2}{c^2}}.$$

Then by the slicing method,

$$\begin{aligned} \text{vol}(B) &= \int_{-c}^c \left\{ \iint_{B(z)} dx dy \right\} dz = \int_{-c}^c \text{area}(B(z)) dz \\ &= \int_{-c}^c \pi a \sqrt{1 - \frac{z^2}{c^2}} \cdot b \sqrt{1 - \frac{z^2}{c^2}} dz = \pi ab \int_{-c}^c \left(1 - \frac{z^2}{c^2}\right) dz \\ &= 2\pi abc \int_0^1 (1 - t^2) dt = 2\pi abc \left[t - \frac{1}{3}t^3\right]_0^1 = \frac{4\pi}{3} abc. \end{aligned}$$

- C As a weak control we know that for the solid ball of radius $r = a = b = c$ we get the well-known volume $\frac{4\pi}{3} r^3$.

Example 24.8 Let B be a closed domain in the (x, y) -plane, and let P_0 be a point of z -coordinate h . Draw linear segments through P_0 and the points of B . The union of these are making up a (solid) cone. One calls B the base of the cone and h is the height of the cone.

- 1) A plane of constant $z \in [0, h]$ intersects the cone in a plane domain $B(z)$. Show that the area of $B(z)$ is equal to the area of B multiplied by the factor $\left(1 - \frac{z}{h}\right)^2$. (Consider e.g. elements of area which correspond to each other by the segments mentioned above).
- 2) Prove that the volume of the cone is $\frac{1}{3}hA$, where A is the area of the base B .
- 3) Prove that the z -coordinate of the centre of gravity is given by $\frac{1}{4}h$.

A The volume of a cone found by a space integral.

D Follow the guidelines given above.

I 1) By considering a rectangular element of area in B we see by using similar triangles that every length in the corresponding element of area in $B(z)$ is diminished by the factor

$$\frac{h-z}{h} = 1 - \frac{z}{h}.$$

Excellent Economics and Business programmes at:

 university of
groningen





**“The perfect start
of a successful,
international career.”**

www.rug.nl/feb/education

CLICK HERE
to discover why both socially
and academically the University
of Groningen is one of the best
places for a student to be



Click on the ad to read more

The element of area is determined by two lengths (“length” and “breadth”), so the area is reduced by the factor $\left(1 - \frac{z}{h}\right)^2$, i.e.

$$\text{area}(B(z)) = \left(1 - \frac{z}{h}\right)^2 \text{area } B = \left(1 - \frac{z}{h}\right)^2 A.$$

2) Using the result from 1) we get by the slicing method,

$$\begin{aligned} \text{vol}(K) &= \int_K d\Omega = \int_0^h \left\{ \int_{B(z)} dx dy \right\} dz = \int_0^h \text{area}(B(z)) dz \\ &= \int_0^h \left(1 - \frac{z}{h}\right)^2 A dz = Ah \left[-\frac{1}{3} \left(1 - \frac{z}{h}\right)^3 \right]_0^h = \frac{1}{3} h A. \end{aligned}$$

3) Let the cone be homogeneously coated (density $\mu > 0$). Then the mass is

$$M = \mu \text{vol}(K) = \frac{1}{3} \mu h A.$$

The z -coordinate ζ of the centre of gravity is given by

$$M \cdot \zeta = \mu \int_K z d\Omega,$$

thus

$$\begin{aligned} \zeta &= \frac{\mu}{M} \int_K z d\Omega = \frac{\mu}{\frac{1}{3} \mu h A} \int_0^h z \cdot \text{area}(B(z)) dz = \frac{3}{h A} \int_0^h z \left(1 - \frac{z}{h}\right)^2 A dz \\ &= 3 \int_0^h \frac{z}{h} \left(1 - \frac{z}{h}\right)^2 dz = 3h \int_0^1 (1-t)t^2 dt = 3h \int_0^1 (t^2 - t^3) dt \\ &= 3h \left[\frac{1}{3} t^3 - \frac{1}{4} t^4 \right]_0^1 = 3h \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{h}{4}. \end{aligned}$$

Example 24.9 Find the volume of the point set

$$\Omega = \{(x, y, z) \mid x^2 + y^2 \leq a^2, |x| \leq a + y, 0 \leq z \leq x^2 + y^2\}.$$

Then compute the space integral

$$\int_{\Omega} (xy + 1) d\Omega.$$

A Volume and space integral.

D Sketch the projection B of Ω onto the (x, y) -plane. Find the volume and the space integral.

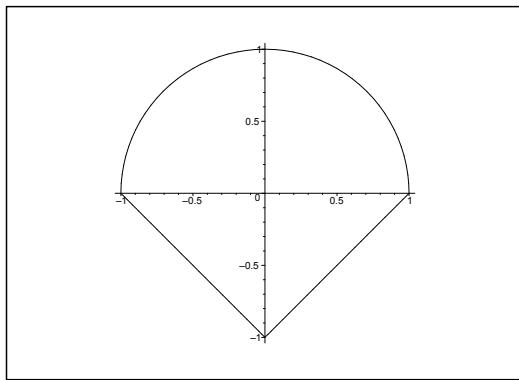


Figure 24.15: The projection B of Ω for $a = 1$.

I The volume is

$$\begin{aligned} \text{vol}(\Omega) &= \int_B (x^2 + y^2) dx dy = \int_0^\pi \left\{ \int_0^a \varrho^2 \cdot \varrho d\varrho \right\} d\varphi + \int_{-a}^0 \left\{ \int_{-a-y}^{a+y} (x^2 + y^2) dx \right\} dy \\ &= \pi \cdot \frac{a^4}{4} + \int_{-a}^0 \left[\frac{1}{3} x^3 + y^2 x \right]_{x=-a-y}^{a+y} dy = \frac{\pi a^4}{4} + \int_{-a}^0 \left\{ \frac{2}{3} (a+y)^3 + 2y^2(a+y) \right\} dy \\ &= \frac{\pi a^4}{4} + \left[\frac{1}{6} (a+y)^4 + \frac{2}{3} ay^3 + \frac{1}{2} y^4 \right]_{-a}^0 = \frac{\pi a^4}{4} + \frac{1}{6} a^4 + \frac{2}{3} a^4 - \frac{1}{2} a^4 \\ &= a^4 \left(\frac{\pi}{4} + \frac{1}{6} + \frac{2}{3} - \frac{1}{2} \right) = a^4 \left(\frac{\pi}{4} + \frac{1}{3} \right). \end{aligned}$$

Due to the symmetry with respect to the Y -axis, we get for the space integral that

$$\int_{\Omega} (xy + 1) d\Omega = \int_B xy(x^2 + y^2) dx dy + \text{vol}(\Omega) = 0 + \text{vol}(\Omega) = a^4 \left(\frac{\pi}{4} + \frac{1}{3} \right).$$

Example 24.10 Let \mathcal{C} denote the cylindric surface the generators of which are parallel to the Z -axis and the intersection curve of which with the (x, y) -plane has the equation $y^2 = x$. Find the volume of the point set Ω , which is bounded by

- 1) the cylindric surface \mathcal{C} ,
- 2) the (x, y) plane, and
- 3) the plane of the equation $2x + 2y + z = 4$.

A Volume.

D Sketch Ω , or at least the projection D of Ω onto the (x, y) -plane.

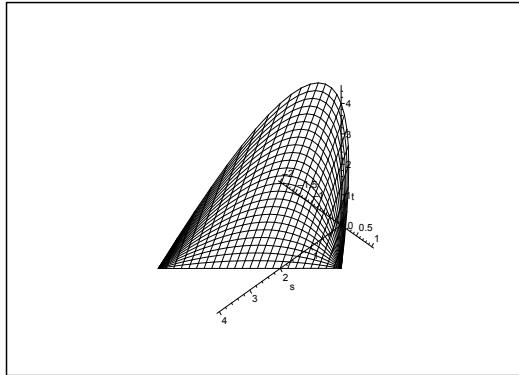


Figure 24.16: The body Ω .

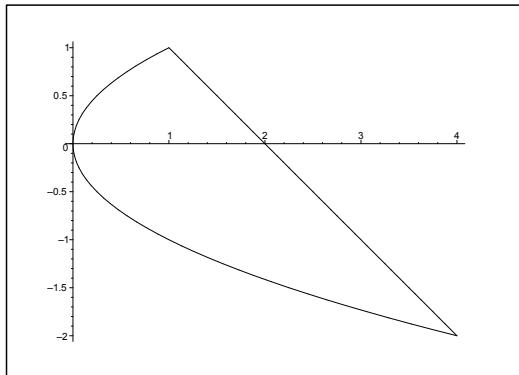


Figure 24.17: The projection D of Ω onto the (x, y) -plane.

I It follows from

$$D = \{(x, y) \mid -2 \leq y \leq 1, y^2 \leq x \leq 2 - y\},$$

and

$$0 \leq z \leq 4 - 2x - 2y = 2(2 - x - y),$$

that

$$\begin{aligned}
 \text{vol}(\Omega) &= \int_D 2(2-x-y) dx dy = \int_{-2}^1 \left\{ \int_{y^2}^{2-y} 2(2-x-y) dx \right\} dy \\
 &= \int_{-2}^1 [-(2-x-y)^2]_{x=y^2}^{2-y} dy = \int_{-2}^1 (2-y-y^2)^2 dy \\
 &= \int_{-2}^1 (y+2)^2(y-1)^2 dy = \int_{-\frac{3}{2}}^{\frac{3}{2}} \left(t + \frac{3}{2}\right)^2 \left(t - \frac{3}{2}\right)^2 dt = 2 \int_0^{\frac{3}{2}} \left(t^2 - \frac{9}{4}\right)^2 dt \\
 &= 2 \int_0^{\frac{3}{2}} \left(t^4 - \frac{9}{2}t^2 + \frac{81}{16}\right) dt = 2 \left\{ \frac{1}{5} \left(\frac{3}{2}\right)^5 - \frac{3}{2} \left(\frac{3}{2}\right)^3 + \frac{81}{16} \cdot \frac{3}{2} \right\} \\
 &= \frac{2}{32} \left\{ \frac{1}{5} \cdot 3^5 - 2 \cdot 3^4 + 3^5 \right\} = \frac{3^4}{16} \cdot \left\{ \frac{3}{5} + 1 \right\} = \frac{81}{16} \cdot \frac{8}{5} = \frac{81}{10}.
 \end{aligned}$$

American online LIGS University

is currently enrolling in the
Interactive Online BBA, MBA, MSc,
DBA and PhD programs:

- ▶ enroll by September 30th, 2014 and
- ▶ save up to 16% on the tuition!
- ▶ pay in 10 installments / 2 years
- ▶ Interactive Online education
- ▶ visit www.ligsuniversity.com to find out more!

Note: LIGS University is not accredited by any nationally recognized accrediting agency listed by the US Secretary of Education.
[More info here.](#)



Click on the ad to read more

Example 24.11 Let

$$f(x, y) = \ln(2 - 2x^2 - 3y^2) + 2 - 4x^2 - 6y^2,$$

and let B be that part of the (x, y) -plane, in which $f(x, y) \geq 0$. Let L denote the point set in the space which is given by

$$(x, y) \in B, \quad 0 \leq z \leq f(x, y).$$

Find the volume of L by the slicing method.

A Volume.

D Consider $f(x, y) = \ln(2 - 2x^2 - 3y^2) + 2 - 4x^2 - 6y^2$ as a function in one single variable.

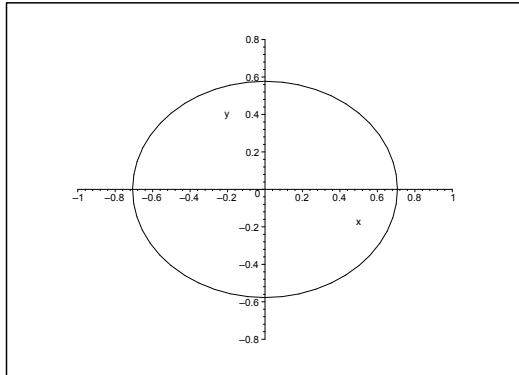


Figure 24.18: The domain B , in which $f(x, y) \geq 0$.

I Since $f(x, y) = 0$, for $2x^2 + 3y^2 = 1$, the domain B is bounded by the ellipse

$$\left(\frac{x}{\frac{1}{\sqrt{2}}}\right)^2 + \left(\frac{y}{\frac{1}{\sqrt{3}}}\right)^2 = 1.$$

This ellipse has the half axes $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{3}}$ and the area $\frac{\pi}{\sqrt{6}}$.

The function $f(x, y)$ is in reality only a function in $t = 1 - 2x^2 - 3y^2$, $t \in [0, 1]$, since we have by this substitution

$$z = f(x, y) = F(t) = \ln(1 + t) + 2t, \quad t \in [0, 1].$$

When $t \in [0, 1]$ is fixed, then $2x^2 + 3y^2 \leq 1 - t$ describes an elliptic disc A_t of area

$$\text{area}(A_t) = \frac{\pi}{\sqrt{6}}(1 - t),$$

thus we get the volume by the slicing method,

$$\begin{aligned} \text{vol}(L) &= \int_0^1 \text{area}(A_t) \cdot \frac{dz}{dt} dt = \int_0^1 \frac{\pi}{\sqrt{6}}(1 - t) \cdot \left\{ \frac{1}{1+t} + 2 \right\} dt \\ &= \frac{\pi}{\sqrt{6}} \int_0^1 \left\{ \frac{2 - (1+t)}{1+t} + 2 - 2t \right\} dt = \frac{\pi}{\sqrt{6}} [\ln(1+t) - t + 2t - t^2]_0^1 = \frac{\pi}{\sqrt{6}} \ln 2. \end{aligned}$$

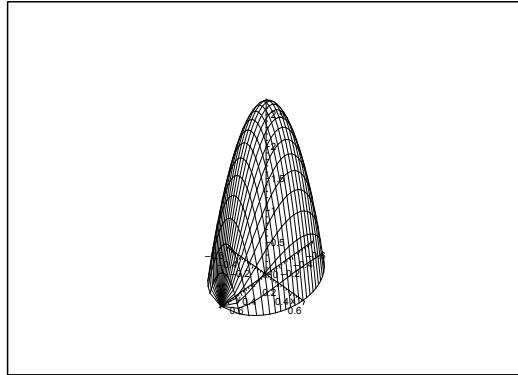


Figure 24.19: The body L .

Example 24.12 Find the volume of the point set

$$Q = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 2, 0 \leq z \leq 4 - (x^2 + y^4)^2\}.$$

A Volume.

D Sketch the set and just compute.

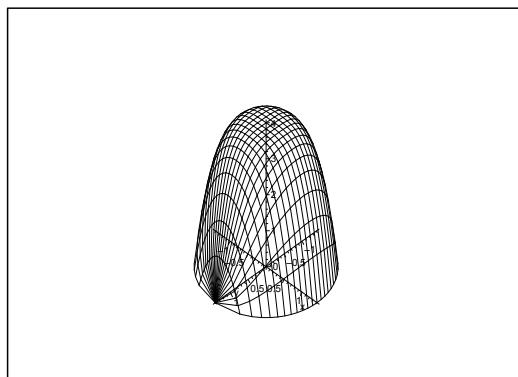


Figure 24.20: The point set Q .

I The set Q is cut at height $z \in [0, 4]$ in a disc of radius $\sqrt[4]{4-z}$. Then by the slicing method,

$$\text{vol}(Q) = \int_0^4 \pi \sqrt{4-z} dz = \pi \left[-\frac{2}{3} (\sqrt{4-z})^3 \right]_0^4 = \frac{2}{3} \pi \cdot (\sqrt{4})^3 = \frac{16}{3} \pi.$$

ALTERNATIVELY we first integrate with respect to z ,

$$\begin{aligned} \text{vol}(Q) &= \int_{\overline{K}(\mathbf{0}; \sqrt{2})} \{4 - (x^2 + y^2)^2\} dx dy = 4 \text{area}(\overline{K}(\mathbf{0}; \sqrt{2})) - 2\pi \int_0^{\sqrt{2}} \varrho^4 \cdot \varrho d\varrho \\ &= 4 \cdot 2\pi - 2\pi \left\{ \frac{(\sqrt{2})^6}{6} \right\} = 8\pi - \frac{8\pi}{3} = \frac{16\pi}{3}. \end{aligned}$$

Example 24.13 Let $B(a)$ denote the bounded point set in the plane which is bounded by the parabola $y = x^2$ and the line $y = a$. Let B denote the unbounded point set which is defined by the inequalities $y \geq x^2$. Let the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by

$$f(x, y) = |x| \exp(x^2 - 2y),$$

and put

$$A(a) = \{(x, y, z) \mid (x, y) \in B(a), 0 \leq z \leq f(x, y)\}.$$

- 1) Find the volume of $A(a)$.
- 2) Prove that the improper plane integral

$$\int_B f(x, y) dS$$

is convergent, and find its value.

A Volume and improper plane integral.

D Sketch $B(a)$ and B ; find vol $A(a)$, and compute the improper plane integral.

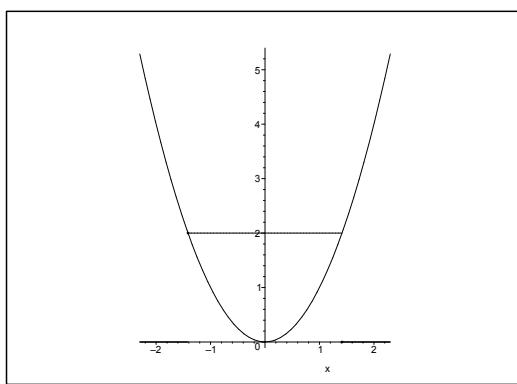


Figure 24.21: The parabola with the truncation at $y = a = 2$.

I 1) We get by direct computation,

$$\begin{aligned} \text{vol}(A(a)) &= \int_{B(a)} f(x, y) dS = 2 \int_0^a \left\{ \int_0^{\sqrt{y}} x \cdot e^{x^2} e^{-2y} dx \right\} dy \\ &= 2 \int_0^a e^{-2y} \left[\frac{1}{2} e^{x^2} \right]_{x=0}^{\sqrt{y}} dy = \int_0^a e^{-2y} (e^y - 1) dy \\ &= \int_0^a (e^{-y} - e^{-2y}) dy = \left[-e^{-y} + \frac{1}{2} e^{-2y} \right]_0^a = \frac{1}{2} - e^{-a} + \frac{1}{2} e^{-2a}. \end{aligned}$$

2) Since $f(x, y) \geq 0$, we get

$$\int_B f(x, y) dS = \lim_{a \rightarrow +\infty} \int_{B(a)} f(x, y) dS = \lim_{a \rightarrow +\infty} \left\{ \frac{1}{2} - e^{-a} + \frac{1}{2} e^{-2a} \right\} = \frac{1}{2}.$$

24.5 Examples of moments of inertia and centres of gravity

Example 24.14 Given the solid ellipsoid

$$\Omega = \left\{ (x, y, z) \mid \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1 \right\}.$$

1) Compute the space integral

$$\int_{\Omega} \sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2} d\Omega.$$

2) Let Ω be homogeneously coated by a mass, where M denotes the total mass. Find the moment of inertia I_x of Ω with respect to the X -axis expressed by a, b, c and M .

A Space integral; moment of inertia.

D Follow the guidelines.

I 1) By putting

$$(x, y, z) = (au, bv, cw), \quad u^2 + v^2 + w^2 \leq 1,$$

and then applying spherical coordinates in the (u, v, w) -space we get

$$\begin{aligned} \int_{\Omega} \sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2} d\Omega &= abc \int_{K(0;1)} \sqrt{u^2 + v^2 + w^2} du dv dw \\ &= abc \int_0^{2\pi} \left\{ \int_0^\pi \left\{ \int_0^1 \varrho \cdot \varrho^2 \sin \theta d\varrho \right\} d\theta \right\} d\varphi = abc \cdot 2\pi \cdot 2 \cdot \frac{1}{4} = abc\pi. \end{aligned}$$

DON'T EAT YELLOW SNOW

What will your advice be?

Some advice just states the obvious. But to give the kind of advice that's going to make a real difference to your clients you've got to listen critically, dig beneath the surface, challenge assumptions and be credible and confident enough to make suggestions right from day one. At Grant Thornton you've got to be ready to kick start a career right at the heart of business.



Grant Thornton
An instinct for growth™

Sound like you? Here's our advice: visit
GrantThornton.ca/careers/students

Scan here to learn more about a career with Grant Thornton.



Click on the ad to read more

- 2) It is well-known that the volume is $\text{vol}(\Omega) = \frac{4\pi}{3} abc$, hence the mass can be written $M = \frac{4\pi}{3} abc \cdot \mu$, from which we get the density $\mu = \frac{3M}{4\pi abc}$.

Due to the symmetry, the moment of inertia with respect to the X -axis is given by

$$\begin{aligned} I_x &= \mu \int_{\Omega} (y^2 + z^2) d\Omega = \mu \int_{\Omega} y^2 d\Omega + \mu \int_{\Omega} z^2 d\Omega \\ &= \mu b^2(abc) \int_{K(0;1)} v^2 dv dw + \mu c^2(abc) \int_{K(0;1)} w^2 dw dv \\ &= \mu abc(b^2 + c^2) \int_{K(0;1)} u^2 du dv dw = \mu(b^2 + c^2)abc \int_{-1}^1 u^2 \pi(1-u^2) du \\ &= 2\mu\pi abc(b^2 + c^2) \int_0^1 (u^2 - u^4) du = 2 \cdot \frac{3M}{4\pi abc} abc(b^2 + c^2) \left(\frac{1}{3} - \frac{1}{5} \right) \\ &= \frac{3}{2} M (b^2 + c^2) \cdot \frac{2}{15} = \frac{1}{5} M(b^2 + c^2). \end{aligned}$$

Example 24.15 Find the centre of gravity for the part of the intersection of the ball of centrum $(0, 0, 0)$ and of radius $a > 0$ in the first octant, i.e. given by the inequalities

$$x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad x^2 + y^2 + z^2 \leq a^2.$$

A Centre of gravity.

D First reduce to the case $a = 1$. Find $\text{vol}(\Omega)$. Compute

$$\xi = \frac{1}{\text{vol}(\Omega)} \int_{\Omega} x d\Omega.$$

It follows by the symmetry that $\xi = \eta = \zeta$.

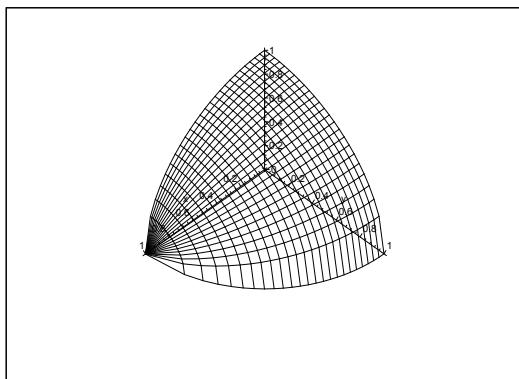


Figure 24.22: The set Ω for $a = 1$.

I We may of geometrical reasons assume that $a = 1$, thus

$$\Omega = \{(x, y, z) \mid x \geq 0, y \geq 0, z \geq 0, x^2 + y^2 + z^2 \leq 1\}.$$

If (ξ, η, ζ) denotes the centre of gravity for Ω , then $(a\xi, a\eta, a\zeta)$ is the centre of gravity for the initial set of radius a .

It follows clearly by the symmetry that $\xi = \eta = \zeta$.

Finally,

$$\text{vol}(\Omega) = \frac{1}{8} \cdot \frac{4\pi}{3} \cdot 1^3 = \frac{\pi}{6}.$$

It follows that

$$\begin{aligned}\xi &= \frac{1}{\text{vol}(\Omega)} = \frac{6}{\pi} \int_0^1 x \left\{ \int_{y^2+z^2 \leq 1-x^2} dy dz \right\} dx \\ &= \frac{6}{\pi} \int_0^1 x \cdot \frac{1}{4} \pi (1-x^2) dx = \frac{3}{2} \int_0^1 (x-x^3) dx = \frac{3}{2} \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{3}{8}.\end{aligned}$$

Therefore, if $a = 1$, then

$$(\xi, \eta, \zeta) = \left(\frac{3}{8}, \frac{3}{8}, \frac{3}{8} \right).$$

We get for a general $a > 0$,

$$(\xi, \eta, \zeta) = \left(\frac{3}{8}a, \frac{3}{8}a, \frac{3}{8}a \right).$$

..... Alcatel-Lucent 

www.alcatel-lucent.com/careers

What if you could build your future and create the future?

One generation's transformation is the next's status quo.
In the near future, people may soon think it's strange that devices ever had to be "plugged in." To obtain that status, there needs to be "The Shift".



Click on the ad to read more

Example 24.16 Let R denote a positive constant. Consider the point set

$$T = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z, x^2 + y^2 + z^2 \leq R^2, x^2 + y^2 \leq 3z^2\}.$$

1) Explain why T is given in spherical coordinates by

$$r \in [0, R], \quad \theta \in \left[0, \frac{\pi}{3}\right], \quad \varphi \in [0, 2\pi].$$

2) Compute the space integrals $\int_T 1 \, dx \, dy \, dz$ and $\int_T z \, dx \, dy \, dz$.

3) Find the coordinates of the centre of gravity of T .

4) Find the area of the boundary surface of T .

A Spherical coordinates, space integrals, centre of gravity and surface area.

D First make a sketch in the meridian half plane.

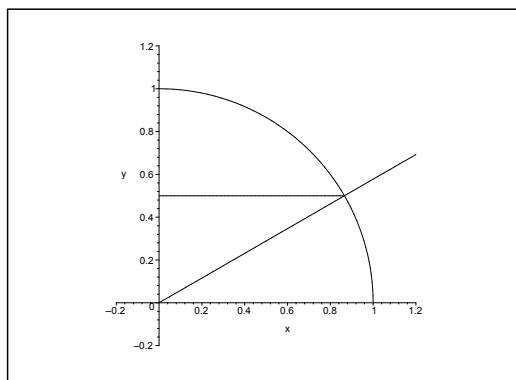


Figure 24.23: The sketch in the meridian half plane for $R = 1$.

I 1) The sketch of the meridian half plane shows that

$$z \geq 0, \quad r^2 \leq R^2, \quad \rho^2 \leq 3z^2, \quad r^2 = \rho^2 + z^2,$$

in spherical coordinates is expressed by

$$r \in [0, R], \quad \theta \in \left[0, \frac{\pi}{3}\right], \quad \varphi \in [0, 2\pi].$$

2) The volume is

$$\begin{aligned} \text{vol}(T) &= \int_T 1 \, dx \, dy \, dz = \int_0^{\frac{R}{2}} \pi \cdot 3z^2 \, dz + \int_{\frac{R}{2}}^R \pi (R^2 - z^2) \, dz \\ &= \pi \left(\frac{R}{2}\right)^3 + \pi \left[R^2 z - \frac{1}{3} z^3\right]_{\frac{R}{2}}^R = \frac{\pi}{8} R^3 + \pi \left\{R^3 - \frac{R^3}{3} - \frac{R^3}{2} + \frac{1}{24} R^3\right\} \\ &= \pi R^3 \left\{\frac{1}{8} + 1 - \frac{1}{3} - \frac{1}{2} + \frac{1}{24}\right\} = \frac{\pi R^3}{24} \{3 + 24 - 8 - 12 + 1\} = \frac{\pi R^3}{3}. \end{aligned}$$

Similarly,

$$\begin{aligned}
 \int_T z \, dx \, dy \, dz &= \int_0^{\frac{R}{2}} z \cdot \pi \cdot 3z^2 \, dz + \int_{\frac{R}{2}}^R z \cdot \pi (R^2 - z^2) \, dz \\
 &= \frac{3\pi}{4} \cdot \left(\frac{R}{2}\right)^4 + \pi \left[\frac{R^2}{2} z^2 - \frac{1}{4} z^4 \right]_{\frac{R}{2}}^R \\
 &= \frac{3\pi}{64} R^4 + \pi \left\{ \frac{R^4}{2} - \frac{R^4}{4} - \frac{R^4}{8} + \frac{R^4}{64} \right\} \\
 &= \frac{\pi R^4}{64} \{3 + 32 - 16 - 8 + 1\} = \frac{12\pi R^4}{64} = \frac{3\pi}{16} R^4.
 \end{aligned}$$

- 3) Of symmetric reasons the centre of gravity must lie on the Z -axis, so $\xi = \eta = 0$, and

$$\zeta = \frac{1}{\text{vol}(T)} \int_T z \, dx \, dy \, dz = \frac{3}{\pi R^3} \cdot \frac{3\pi}{16} R^4 = \frac{9}{16} R,$$

where we have used the results of 2).

- 4) The boundary curve \mathcal{M} in the meridian half plane is now split up into

$$\begin{aligned}
 \mathcal{M}_1 : \quad \varrho &= \sqrt{3} \cdot z, \quad ds = \sqrt{1+3} \, dz = 2 \, dz, \quad z \in \left[0, \frac{R}{2}\right], \\
 \mathcal{M}_2 : \quad \varrho &= \sqrt{R^2 - z^2}, \quad ds = \frac{R}{\sqrt{R^2 - z^2}} \, dz, \quad z \in \left[\frac{R}{2}, R\right],
 \end{aligned}$$

so the surface area becomes

$$\begin{aligned}
 2\pi \int_{\mathcal{M}} P \, ds &= 2\pi \int_0^{\frac{R}{2}} \sqrt{3} \cdot z \cdot 2 \, dz + 2\pi \int_{\frac{R}{2}}^R \sqrt{R^2 - z^2} \cdot \frac{R}{\sqrt{R^2 - z^2}} \, dz \\
 &= 2\pi\sqrt{3} [z^2]_0^{\frac{R}{2}} + 2\pi R \cdot \frac{R}{2} = 2\sqrt{3} \pi \cdot \frac{R^2}{4} + \pi R^2 = \left(1 + \frac{\sqrt{3}}{2}\right) \pi R^2.
 \end{aligned}$$

Example 24.17 Let Ω denote that part of the closed ball $\overline{K}(\mathbf{0}; a)$, which lies above the (x, y) -plane and inside a cylindric surface with its generators parallel to the Z -axes through the curve in the (x, y) -plane given by the equation

$$\varrho = a\sqrt{\cos(2\varphi)}, \quad -\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4}.$$

- 1) Find the volume of $\overline{\Omega}$.
- 2) Find the z -coordinate of the centre of gravity for $\overline{\Omega}$.

A Volume; centre of gravity.

D Sketch Ω and compute $\text{vol}(\Omega)$. Find the centre of gravity.

I 1) Since $z = +\sqrt{a^2 - \varrho^2}$ on the shell, we get

$$\begin{aligned} \text{vol}(\Omega) &= \int_B \sqrt{a^2 - \varrho^2} dS = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left\{ \int_0^{a\sqrt{\cos 2\varphi}} \sqrt{a^2 - \varrho^2} \cdot \varrho d\varrho \right\} d\varphi \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[-\frac{1}{2} \cdot \frac{1}{3} (a^2 - \varrho^2)^{\frac{3}{2}} \right]_{\varrho=0}^{a\sqrt{\cos 2\varphi}} d\varphi = \frac{1}{3} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left\{ a^3 - a^3(1 - \cos 2\varphi)^{\frac{3}{2}} \right\} d\varphi \\ &= 2 \cdot \frac{a^3}{3} \int_0^{\frac{\pi}{4}} \left\{ 1 - (2 \sin^2 \varphi)^{\frac{3}{2}} \right\} d\varphi = \frac{2}{3} a^3 \int_0^{\frac{\pi}{4}} \left\{ 1 - 2\sqrt{2} \sin^3 \varphi \right\} d\varphi \\ &= \frac{2}{3} a^3 \cdot \frac{\pi}{4} - \frac{2}{3} a^3 \cdot 2\sqrt{2} \int_0^{\frac{\pi}{4}} (1 - \cos^2 \varphi) \sin \varphi d\varphi \\ &= \frac{\pi}{6} a^3 + \frac{4\sqrt{2}}{3} a^3 \left[\cos \varphi - \frac{1}{3} \cos^3 \varphi \right]_0^{\frac{\pi}{4}} = \frac{\pi}{6} a^3 + \frac{4\sqrt{2}}{3} a^3 \left(\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}} - \frac{2}{3} \right) \\ &= \frac{\pi}{6} a^3 + \frac{4}{3} \cdot \frac{5}{6} a^3 - \frac{8\sqrt{2}}{9} a^3 = a^3 \left(\frac{\pi}{6} + \frac{10}{9} - \frac{8\sqrt{2}}{9} \right). \end{aligned}$$

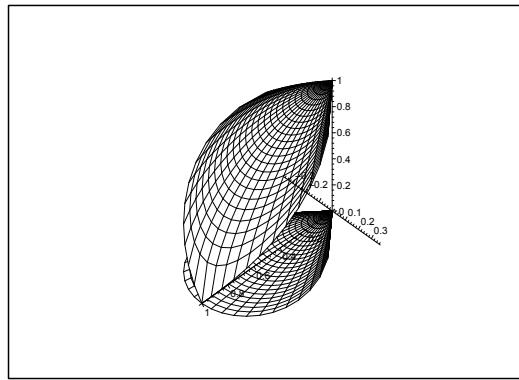


Figure 24.24: The domain Ω for $a = 1$.

2) We have due to the symmetry,

$$\int_{\Omega} y d\Omega = \int_B y \sqrt{a^2 - \varrho^2} dS = 0.$$

Furthermore,

$$\begin{aligned}
 \int_{\Omega} z \, d\Omega &= \int_B \left\{ \int_0^{\sqrt{a^2 - \varrho^2}} z \, dz \right\} dS = \frac{1}{2} \int_B (a^2 - \varrho^2) \, dS \\
 &= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left\{ \int_0^{a\sqrt{\cos 2\varphi}} (a^2 - \varrho^2) \varrho \, d\varrho \right\} d\varphi = \int_0^{\frac{\pi}{4}} \left[\frac{1}{2} a^2 \varrho^2 - \frac{1}{4} \varrho^4 \right]_0^{a\sqrt{\cos 2\varphi}} d\varphi \\
 &= \frac{a^4}{4} \int_0^{\frac{\pi}{4}} \{2 \cos 2\varphi - \cos^2 2\varphi\} d\varphi \\
 &= \frac{a^4}{4} [\sin 2\varphi]_0^{\frac{\pi}{4}} - \frac{a^4}{4} \cdot \frac{1}{2} \int_0^{\frac{\pi}{4}} \{1 + \cos 4\varphi\} d\varphi = \frac{a^4}{4} - \frac{a^4 \pi}{32} = \frac{a^4}{32} (8 - \pi).
 \end{aligned}$$



Maastricht University *Leading in Learning!*

**Join the best at
the Maastricht University
School of Business and
Economics!**



Top master's programmes

- 33rd place Financial Times worldwide ranking: MSc International Business
- 1st place: MSc International Business
- 1st place: MSc Financial Economics
- 2nd place: MSc Management of Learning
- 2nd place: MSc Economics
- 2nd place: MSc Econometrics and Operations Research
- 2nd place: MSc Global Supply Chain Management and Change

Sources: Keuzegids Master ranking 2013; Elsevier 'Beste Studies' ranking 2012;
Financial Times Global Masters in Management ranking 2012

Maastricht
University is
the best specialist
university in the
Netherlands
(Elsevier)

**Visit us and find out why we are the best!
Master's Open Day: 22 February 2014**

www.mastersopenday.nl



Click on the ad to read more

Finally, we get by interchanging the order of integration that

$$\begin{aligned}
 \int_{\Omega} x \, d\Omega &= \int_B x \sqrt{a^2 - \varrho^2} \, dS = 2 \int_0^{\frac{\pi}{4}} \cos \varphi \left\{ \int_0^{a \sqrt{\cos 2\varphi}} \sqrt{a^2 - \varrho^2} \cdot \varrho^2 \, d\varrho \right\} d\varphi \\
 &= 2a^4 \int_0^{\frac{\pi}{4}} \cos \varphi \left\{ \int_0^{\sqrt{\cos 2\varphi}} t^2 \sqrt{1-t^2} \, dt \right\} d\varphi \\
 &= 2a^4 \int_0^1 t^2 \sqrt{1-t^2} \left\{ \int_0^{\frac{1}{2} \operatorname{Arccos}(t^2)} \cos \varphi \, d\varphi \right\} dt \\
 &= 2a^4 \int_0^1 t^2 \sqrt{1-t^2} \sin \left(\frac{1}{2} \operatorname{Arccos}(t^2) \right) dt \\
 &= 2a^4 \int_0^1 t^2 \sqrt{1-t^2} \cdot \sqrt{1-\cos \left(2 \cdot \frac{1}{2} \operatorname{Arccos}(t^2) \right)} dt \\
 &= 2a^4 \int_0^1 t^2 \sqrt{1-t^2} \cdot \sqrt{1-t^2} dt = 2a^4 \int_0^1 t^2 (1-t^2) dt = 2a^4 \left\{ \frac{1}{3} - \frac{1}{5} \right\} = \frac{4}{15} a^4.
 \end{aligned}$$

The centre of gravity is

$$\xi = \frac{1}{\operatorname{vol}(\Omega)} \int_{\Omega} \mathbf{x} \, d\Omega = \frac{a}{\frac{\pi}{6} + \frac{10}{9} - \frac{8\sqrt{2}}{9}} \left(\frac{4}{15}, 0, \frac{8-\pi}{32} \right).$$

25 Formulæ

Some of the following formulæ can be assumed to be known from high school. It is highly recommended that one *learns most of these formulæ in this appendix by heart*.

25.1 Squares etc.

The following simple formulæ occur very frequently in the most different situations.

$$\begin{aligned} (a+b)^2 &= a^2 + b^2 + 2ab, & a^2 + b^2 + 2ab &= (a+b)^2, \\ (a-b)^2 &= a^2 + b^2 - 2ab, & a^2 + b^2 - 2ab &= (a-b)^2, \\ (a+b)(a-b) &= a^2 - b^2, & a^2 - b^2 &= (a+b)(a-b), \\ (a+b)^2 &= (a-b)^2 + 4ab, & (a-b)^2 &= (a+b)^2 - 4ab. \end{aligned}$$

25.2 Powers etc.

Logarithm:

$$\begin{aligned} \ln |xy| &= \ln |x| + \ln |y|, & x, y \neq 0, \\ \ln \left| \frac{x}{y} \right| &= \ln |x| - \ln |y|, & x, y \neq 0, \\ \ln |x^r| &= r \ln |x|, & x \neq 0. \end{aligned}$$

Power function, fixed exponent:

$$(xy)^r = x^r \cdot y^r, x, y > 0 \quad (\text{extensions for some } r),$$

$$\left(\frac{x}{y} \right)^r = \frac{x^r}{y^r}, x, y > 0 \quad (\text{extensions for some } r).$$

Exponential, fixed base:

$$\begin{aligned} a^x \cdot a^y &= a^{x+y}, & a > 0 & \quad (\text{extensions for some } x, y), \\ (a^x)^y &= a^{xy}, & a > 0 & \quad (\text{extensions for some } x, y), \end{aligned}$$

$$a^{-x} = \frac{1}{a^x}, a > 0, \quad (\text{extensions for some } x),$$

$$\sqrt[n]{a} = a^{1/n}, a \geq 0, \quad n \in \mathbb{N}.$$

Square root:

$$\sqrt{x^2} = |x|, \quad x \in \mathbb{R}.$$

Remark 25.1 It happens quite frequently that students make errors when they try to apply these rules. They must be mastered! In particular, as one of my friends once put it: “If you can master the square root, you can master everything in mathematics!” Notice that this innocent looking square root is one of the most difficult operations in Calculus. Do not forget the *absolute value!* ◇

25.3 Differentiation

Here are given the well-known rules of differentiation together with some rearrangements which sometimes may be easier to use:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x),$$

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) = f(x)g(x) \left\{ \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} \right\},$$

where the latter rearrangement presupposes that $f(x) \neq 0$ and $g(x) \neq 0$.

If $g(x) \neq 0$, we get the usual formula known from high school

$$\left\{ \frac{f(x)}{g(x)} \right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

It is often more convenient to compute this expression in the following way:

$$\left\{ \frac{f(x)}{g(x)} \right\} = \frac{d}{dx} \left\{ f(x) \cdot \frac{1}{g(x)} \right\} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f(x)}{g(x)} \left\{ \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)} \right\},$$

where the former expression often is *much easier* to use in practice than the usual formula from high school, and where the latter expression again presupposes that $f(x) \neq 0$ and $g(x) \neq 0$. Under these assumptions we see that the formulæ above can be written

$$\frac{\{f(x)g(x)\}'}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

$$\frac{\{f(x)/g(x)\}'}{f(x)/g(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Since

$$\frac{d}{dx} \ln |f(x)| = \frac{f'(x)}{f(x)}, \quad f(x) \neq 0,$$

we also name these the *logarithmic derivatives*.

Finally, we mention the rule of **differentiation of a composite function**

$$\{f(\varphi(x))\}' = f'(\varphi(x)) \cdot \varphi'(x).$$

We first differentiate the function itself; then the insides. This rule is a 1-dimensional version of the so-called *Chain rule*.

25.4 Special derivatives.

Power like:

$$\frac{d}{dx} (x^\alpha) = \alpha \cdot x^{\alpha-1}, \quad \text{for } x > 0, \text{ (extensions for some } \alpha\text{).}$$

$$\frac{d}{dx} \ln |x| = \frac{1}{x}, \quad \text{for } x \neq 0.$$

Exponential like:

$$\begin{aligned}\frac{d}{dx} \exp x &= \exp x, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} (a^x) &= \ln a \cdot a^x, && \text{for } x \in \mathbb{R} \text{ and } a > 0.\end{aligned}$$

Trigonometric:

$$\begin{aligned}\frac{d}{dx} \sin x &= \cos x, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \cos x &= -\sin x, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \tan x &= 1 + \tan^2 x = \frac{1}{\cos^2 x}, && \text{for } x \neq \frac{\pi}{2} + p\pi, p \in \mathbb{Z}, \\ \frac{d}{dx} \cot x &= -(1 + \cot^2 x) = -\frac{1}{\sin^2 x}, && \text{for } x \neq p\pi, p \in \mathbb{Z}.\end{aligned}$$

Hyperbolic:

$$\begin{aligned}\frac{d}{dx} \sinh x &= \cosh x, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \cosh x &= \sinh x, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \tanh x &= 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \coth x &= 1 - \coth^2 x = -\frac{1}{\sinh^2 x}, && \text{for } x \neq 0.\end{aligned}$$

Inverse trigonometric:

$$\begin{aligned}\frac{d}{dx} \text{Arcsin } x &= \frac{1}{\sqrt{1-x^2}}, && \text{for } x \in]-1, 1[, \\ \frac{d}{dx} \text{Arccos } x &= -\frac{1}{\sqrt{1-x^2}}, && \text{for } x \in]-1, 1[, \\ \frac{d}{dx} \text{Arctan } x &= \frac{1}{1+x^2}, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \text{Arccot } x &= \frac{1}{1+x^2}, && \text{for } x \in \mathbb{R}.\end{aligned}$$

Inverse hyperbolic:

$$\begin{aligned}\frac{d}{dx} \text{Arsinh } x &= \frac{1}{\sqrt{x^2+1}}, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \text{Arcosh } x &= \frac{1}{\sqrt{x^2-1}}, && \text{for } x \in]1, +\infty[, \\ \frac{d}{dx} \text{Artanh } x &= \frac{1}{1-x^2}, && \text{for } |x| < 1, \\ \frac{d}{dx} \text{Arcoth } x &= \frac{1}{1-x^2}, && \text{for } |x| > 1.\end{aligned}$$

Remark 25.2 The derivative of the trigonometric and the hyperbolic functions are to some extent exponential like. The derivatives of the inverse trigonometric and inverse hyperbolic functions are power like, because we include the logarithm in this class. ◇

25.5 Integration

The most obvious rules are dealing with linearity

$$\int \{f(x) + \lambda g(x)\} dx = \int f(x) dx + \lambda \int g(x) dx, \quad \text{where } \lambda \in \mathbb{R} \text{ is a constant,}$$

and with the fact that differentiation and integration are “inverses to each other”, i.e. modulo some arbitrary constant $c \in \mathbb{R}$, which often tacitly is missing,

$$\int f'(x) dx = f(x).$$

If we in the latter formula replace $f(x)$ by the product $f(x)g(x)$, we get by reading from the right to the left and then differentiating the product,

$$f(x)g(x) = \int \{f(x)g(x)\}' dx = \int f'(x)g(x) dx + \int f(x)g'(x) dx.$$

Hence, by a rearrangement

The rule of partial integration:

$$\int f'(x)g(x) dx = f(x)g(x) - \int f(x)g'(x) dx.$$

The differentiation is moved from one factor of the integrand to the other one by changing the sign and adding the term $f(x)g(x)$.

Remark 25.3 This technique was earlier used a lot, but is almost forgotten these days. It must be revived, because MAPLE and pocket calculators apparently do not know it. It is possible to construct examples where these devices cannot give the exact solution, unless you first perform a partial integration yourself. ◇

Remark 25.4 This method can also be used when we estimate integrals which cannot be directly calculated, because the antiderivative is not contained in e.g. the catalogue of MAPLE. The idea is by a succession of partial integrations to make the new integrand smaller. ◇

Integration by substitution:

If the integrand has the special structure $f(\varphi(x)) \cdot \varphi'(x)$, then one can change the variable to $y = \varphi(x)$:

$$\int f(\varphi(x)) \cdot \varphi'(x) dx = " \int f(\varphi(x)) d\varphi(x) " = \int_{y=\varphi(x)} f(y) dy.$$

Integration by a monotonous substitution:

If $\varphi(y)$ is a *monotonous* function, which maps the y -interval *one-to-one* onto the x -interval, then

$$\int f(x) dx = \int_{y=\varphi^{-1}(x)} f(\varphi(y))\varphi'(y) dy.$$

Remark 25.5 This rule is usually used when we have some “ugly” term in the integrand $f(x)$. The idea is to put this ugly term equal to $y = \varphi^{-1}(x)$. When e.g. x occurs in $f(x)$ in the form \sqrt{x} , we put $y = \varphi^{-1}(x) = \sqrt{x}$, hence $x = \varphi(y) = y^2$ and $\varphi'(y) = 2y$. \diamond

25.6 Special antiderivatives

Power like:

$$\int \frac{1}{x} dx = \ln|x|, \quad \text{for } x \neq 0. \quad (\text{Do not forget the numerical value!})$$

$$\int x^\alpha dx = \frac{1}{\alpha+1} x^{\alpha+1}, \quad \text{for } \alpha \neq -1,$$

$$\int \frac{1}{1+x^2} dx = \arctan x, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{1-x^2} dx = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|, \quad \text{for } x \neq \pm 1,$$

$$\int \frac{1}{1-x^2} dx = \operatorname{Artanh} x, \quad \text{for } |x| < 1,$$

$$\int \frac{1}{1-x^2} dx = \operatorname{Arcoth} x, \quad \text{for } |x| > 1,$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x, \quad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = -\arccos x, \quad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{x^2+1}} dx = \operatorname{Arsinh} x, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2+1}} dx = \ln(x + \sqrt{x^2+1}), \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{x}{\sqrt{x^2-1}} dx = \sqrt{x^2-1}, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2-1}} dx = \operatorname{Arcosh} x, \quad \text{for } x > 1,$$

$$\int \frac{1}{\sqrt{x^2-1}} dx = \ln|x + \sqrt{x^2-1}|, \quad \text{for } x > 1 \text{ eller } x < -1.$$

There is an error in the programs of the pocket calculators TI-92 and TI-89. The *numerical signs* are missing. It is obvious that $\sqrt{x^2-1} < |x|$ so if $x < -1$, then $x + \sqrt{x^2-1} < 0$. Since you cannot take the logarithm of a negative number, these pocket calculators will give an error message.

Exponential like:

$$\int \exp x \, dx = \exp x, \quad \text{for } x \in \mathbb{R},$$

$$\int a^x \, dx = \frac{1}{\ln a} \cdot a^x, \quad \text{for } x \in \mathbb{R}, \text{ and } a > 0, a \neq 1.$$

Trigonometric:

$$\int \sin x \, dx = -\cos x, \quad \text{for } x \in \mathbb{R},$$

$$\int \cos x \, dx = \sin x, \quad \text{for } x \in \mathbb{R},$$

$$\int \tan x \, dx = -\ln |\cos x|, \quad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \cot x \, dx = \ln |\sin x|, \quad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos x} \, dx = \frac{1}{2} \ln \left(\frac{1 + \sin x}{1 - \sin x} \right), \quad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin x} \, dx = \frac{1}{2} \ln \left(\frac{1 - \cos x}{1 + \cos x} \right), \quad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos^2 x} \, dx = \tan x, \quad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin^2 x} \, dx = -\cot x, \quad \text{for } x \neq p\pi, \quad p \in \mathbb{Z}.$$

Hyperbolic:

$$\int \sinh x \, dx = \cosh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \cosh x \, dx = \sinh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \tanh x \, dx = \ln \cosh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \coth x \, dx = \ln |\sinh x|, \quad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh x} \, dx = \operatorname{Arctan}(\sinh x), \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\cosh x} \, dx = 2 \operatorname{Arctan}(e^x), \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh x} \, dx = \frac{1}{2} \ln \left(\frac{\cosh x - 1}{\cosh x + 1} \right), \quad \text{for } x \neq 0,$$

$$\int \frac{1}{\sinh x} dx = \ln \left| \frac{e^x - 1}{e^x + 1} \right|, \quad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh^2 x} dx = \tanh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh^2 x} dx = -\coth x, \quad \text{for } x \neq 0.$$

25.7 Trigonometric formulæ

The trigonometric formulæ are closely connected with circular movements. Thus $(\cos u, \sin u)$ are the coordinates of a point P on the unit circle corresponding to the angle u , cf. figure A.1. This geometrical interpretation is used from time to time.

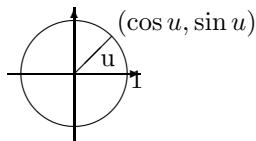


Figure 25.1: The unit circle and the trigonometric functions.

The fundamental trigonometric relation:

$$\cos^2 u + \sin^2 u = 1, \quad \text{for } u \in \mathbb{R}.$$

Using the previous geometric interpretation this means according to *Pythagoras's theorem*, that the point P with the coordinates $(\cos u, \sin u)$ always has distance 1 from the origo $(0, 0)$, i.e. it is lying on the boundary of the circle of centre $(0, 0)$ and radius $\sqrt{1} = 1$.

Connection to the complex exponential function:

The *complex exponential* is for imaginary arguments defined by

$$\exp(iu) := \cos u + i \sin u.$$

It can be checked that the usual functional equation for \exp is still valid for complex arguments. In other word: The definition above is extremely conveniently chosen.

By using the definition for $\exp(iu)$ and $\exp(-iu)$ it is easily seen that

$$\cos u = \frac{1}{2}(\exp(iu) + \exp(-iu)),$$

$$\sin u = \frac{1}{2i}(\exp(iu) - \exp(-iu)),$$

Moivre's formula: We get by expressing $\exp(inu)$ in two different ways:

$$\exp(inu) = \cos nu + i \sin nu = (\cos u + i \sin u)^n.$$

Example 25.1 If we e.g. put $n = 3$ into Moivre's formula, we obtain the following typical application,

$$\begin{aligned} \cos(3u) + i \sin(3u) &= (\cos u + i \sin u)^3 \\ &= \cos^3 u + 3i \cos^2 u \cdot \sin u + 3i^2 \cos u \cdot \sin^2 u + i^3 \sin^3 u \\ &= \{\cos^3 u - 3 \cos u \cdot \sin^2 u\} + i\{3 \cos^2 u \cdot \sin u - \sin^3 u\} \\ &= \{4 \cos^3 u - 3 \cos u\} + i\{3 \sin u - 4 \sin^3 u\} \end{aligned}$$

When this is split into the real- and imaginary parts we obtain

$$\cos 3u = 4 \cos^3 u - 3 \cos u, \quad \sin 3u = 3 \sin u - 4 \sin^3 u. \quad \diamond$$

Addition formulæ:

$$\begin{aligned} \sin(u+v) &= \sin u \cos v + \cos u \sin v, \\ \sin(u-v) &= \sin u \cos v - \cos u \sin v, \\ \cos(u+v) &= \cos u \cos v - \sin u \sin v, \\ \cos(u-v) &= \cos u \cos v + \sin u \sin v. \end{aligned}$$

Products of trigonometric functions to a sum:

$$\begin{aligned} \sin u \cos v &= \frac{1}{2} \sin(u+v) + \frac{1}{2} \sin(u-v), \\ \cos u \sin v &= \frac{1}{2} \sin(u+v) - \frac{1}{2} \sin(u-v), \\ \sin u \sin v &= \frac{1}{2} \cos(u-v) - \frac{1}{2} \cos(u+v), \\ \cos u \cos v &= \frac{1}{2} \cos(u-v) + \frac{1}{2} \cos(u+v). \end{aligned}$$

Sums of trigonometric functions to a product:

$$\begin{aligned} \sin u + \sin v &= 2 \sin\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right), \\ \sin u - \sin v &= 2 \cos\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right), \\ \cos u + \cos v &= 2 \cos\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right), \\ \cos u - \cos v &= -2 \sin\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right). \end{aligned}$$

Formulæ of halving and doubling the angle:

$$\sin 2u = 2 \sin u \cos u,$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2 \cos^2 u - 1 = 1 - 2 \sin^2 u,$$

$$\sin \frac{u}{2} = \pm \sqrt{\frac{1 - \cos u}{2}} \quad \text{followed by a discussion of the sign,}$$

$$\cos \frac{u}{2} = \pm \sqrt{\frac{1 + \cos u}{2}} \quad \text{followed by a discussion of the sign,}$$

25.8 Hyperbolic formulæ

These are very much like the trigonometric formulæ, and if one knows a little of Complex Function Theory it is realized that they are actually identical. The structure of this section is therefore the same as for the trigonometric formulæ. The reader should compare the two sections concerning similarities and differences.

The fundamental relation:

$$\cosh^2 x - \sinh^2 x = 1.$$

Definitions:

$$\cosh x = \frac{1}{2} (\exp(x) + \exp(-x)), \quad \sinh x = \frac{1}{2} (\exp(x) - \exp(-x)).$$

“Moivre’s formula”:

$$\exp(x) = \cosh x + \sinh x.$$

This is trivial and only rarely used. It has been included to show the analogy.

Addition formulæ:

$$\begin{aligned}\sinh(x+y) &= \sinh(x)\cosh(y) + \cosh(x)\sinh(y), \\ \sinh(x-y) &= \sinh(x)\cosh(y) - \cosh(x)\sinh(y), \\ \cosh(x+y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y), \\ \cosh(x-y) &= \cosh(x)\cosh(y) - \sinh(x)\sinh(y).\end{aligned}$$

Formulæ of halving and doubling the argument:

$$\begin{aligned}\sinh(2x) &= 2\sinh(x)\cosh(x), \\ \cosh(2x) &= \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1, \\ \sinh\left(\frac{x}{2}\right) &= \pm\sqrt{\frac{\cosh(x)-1}{2}} \quad \text{followed by a discussion of the sign,} \\ \cosh\left(\frac{x}{2}\right) &= \sqrt{\frac{\cosh(x)+1}{2}}.\end{aligned}$$

Inverse hyperbolic functions:

$$\begin{aligned}\text{Arsinh}(x) &= \ln\left(x + \sqrt{x^2 + 1}\right), \quad x \in \mathbb{R}, \\ \text{Arcosh}(x) &= \ln\left(x + \sqrt{x^2 - 1}\right), \quad x \geq 1, \\ \text{Artanh}(x) &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad |x| < 1, \\ \text{Arcoth}(x) &= \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \quad |x| > 1.\end{aligned}$$

25.9 Complex transformation formulæ

$$\begin{aligned}\cos(ix) &= \cosh(x), & \cosh(ix) &= \cos(x), \\ \sin(ix) &= i \sinh(x), & \sinh(ix) &= i \sin x.\end{aligned}$$

25.10 Taylor expansions

The generalized binomial coefficients are defined by

$$\binom{\alpha}{n} := \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{1 \cdot 2 \cdots n},$$

with n factors in the numerator and the denominator, supplied with

$$\binom{\alpha}{0} := 1.$$

The Taylor expansions for *standard functions* are divided into *power like* (the radius of convergency is finite, i.e. = 1 for the standard series) and *exponential like* (the radius of convergency is infinite).

Power like:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1,$$

$$(1+x)^n = \sum_{j=0}^n \binom{n}{j} x^j, \quad n \in \mathbb{N}, x \in \mathbb{R},$$

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad \alpha \in \mathbb{R} \setminus \mathbb{N}, |x| < 1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad |x| < 1,$$

$$\text{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad |x| < 1.$$

Exponential like:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \quad x \in \mathbb{R}$$

$$\exp(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n, \quad x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \quad x \in \mathbb{R},$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \quad x \in \mathbb{R},$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \quad x \in \mathbb{R},$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \quad x \in \mathbb{R}.$$

25.11 Magnitudes of functions

We often have to compare functions for $x \rightarrow 0+$, or for $x \rightarrow \infty$. The simplest type of functions are therefore arranged in an hierarchy:

- 1) logarithms,
- 2) power functions,
- 3) exponential functions,
- 4) faculty functions.

When $x \rightarrow \infty$, a function from a higher class will always dominate a function from a lower class. More precisely:

A) A *power function* dominates a *logarithm* for $x \rightarrow \infty$:

$$\frac{(\ln x)^\beta}{x^\alpha} \rightarrow 0 \quad \text{for } x \rightarrow \infty, \quad \alpha, \beta > 0.$$

B) An *exponential* dominates a *power function* for $x \rightarrow \infty$:

$$\frac{x^\alpha}{a^x} \rightarrow 0 \quad \text{for } x \rightarrow \infty, \quad \alpha, a > 1.$$

C) The *faculty function* dominates an *exponential* for $n \rightarrow \infty$:

$$\frac{a^n}{n!} \rightarrow 0, \quad n \rightarrow \infty, \quad n \in \mathbb{N}, \quad a > 0.$$

D) When $x \rightarrow 0+$ we also have that a *power function* dominates the *logarithm*:

$$x^\alpha \ln x \rightarrow 0-, \quad \text{for } x \rightarrow 0+, \quad \alpha > 0.$$

Index

- absolute value 162
- acceleration 490
- addition 22
- affinity factor 173
- Ampère-Laplace law 1671
- Ampère-Maxwell's law 1678
- Ampère's law 1491, 1498, 1677, 1678, 1833
- Ampère's law for the magnetic field 1674
- angle 19
- angular momentum 886
- angular set 84
- annulus 176, 243
- anticommutative product 26
- antiderivative 301, 847
- approximating polynomial 304, 322, 326, 336, 404, 488, 632, 662
- approximation in energy 734
- Archimedes's spiral 976, 1196
- Archimedes's theorem 1818
- area 887, 1227, 1229, 1543
- area element 1227
- area of a graph 1230
- asteroid 1215
- asymptote 51
- axial moment 1910
- axis of revolution 181
- axis of rotation 34, 886
- axis of symmetry 49, 50, 53
- barycentre 885, 1910
- basis 22
- bend 486
- bijective map 153
- body of revolution 43, 1582, 1601
- boundary 37–39
- boundary curve 182
- boundary curve of a surface 182
- boundary point 920
- boundary set 21
- bounded map 153
- bounded set 41
- branch 184
- branch of a curve 492
- Brownian motion 884
- cardioid 972, 973, 1199, 1705
- Cauchy-Schwarz's inequality 23, 24, 26
- centre of gravity 1108
- centre of mass 885
- centrum 66
- chain rule 305, 333, 352, 491, 503, 581, 1215, 1489, 1493, 1808
- change of parameter 174
- circle 49
- circular motion 19
- circulation 1487
- circulation theorem 1489, 1491
- circumference 86
- closed ball 38
- closed differential form 1492
- closed disc 86
- closed domain 176
- closed set 21
- closed surface 182, 184
- closure 39
- clothoid 1219
- colour code 890
- compact set 186, 580, 1813
- compact support 1813
- complex decomposition 69
- composite function 305
- conductivity of heat 1818
- cone 19, 35, 59, 251
- conic section 19, 47, 54, 239, 536
- conic sectional conic surface 59, 66
- connected set 175, 241
- conservation of electric charge 1548, 1817
- conservation of energy 1548, 1817
- conservation of mass 1548, 1816
- conservative force 1498, 1507
- conservative vector field 1489
- continuity equation 1548, 1569, 1767, 1817
- continuity 162, 186
- continuous curve 170, 483
- continuous extension 213
- continuous function 168
- continuous surfaces 177
- contraction 167
- convective term 492
- convex set 21, 22, 41, 89, 91, 175, 244
- coordinate function 157, 169
- coordinate space 19, 21

- Cornu's spiral 1219
Coulomb field 1538, 1545, 1559, 1566, 1577
Coulomb vector field 1585, 1670
cross product 19, 163, 169, 1750
cube 42, 82
current density 1678, 1681
current 1487, 1499
curvature 1219
curve 227
curve length 1165
curved space integral 1021
cusp 486, 487, 489
cycloid 233, 1215
cylinder 34, 42, 43, 252
cylinder of revolution 500
cylindric coordinates 15, 21, 34, 147, 181, 182, 289, 477, 573, 841, 1009, 1157, 1347, 1479, 1651, 1801
cylindric surface 180, 245, 247, 248, 499, 1230
degree of trigonometric polynomial 67
density 885
density of charge 1548
density of current 1548
derivative 296
derivative of inverse function 494
Descartes'a leaf 974
dielectric constant 1669, 1670
difference quotient 295
differentiability 295
differentiable function 295
differentiable vector function 303
differential 295, 296, 325, 382, 1740, 1741
differential curves 171
differential equation 369, 370, 398
differential form 848
differential of order p 325
differential of vector function 303
diffusion equation 1818
dimension 1016
direction 334
direction vector 172
directional derivative 317, 334, 375
directrix 53
Dirichlet/Neumann problem 1901
displacement field 1670
distribution of current 886
divergence 1535, 1540, 1542, 1739, 1741, 1742
divergence free vector field 1543
dodecahedron 83
domain 153, 176
domain of a function 189
dot product 19, 350, 1750
double cone 252
double point 171
double vector product 27
eccentricity 51
eccentricity of ellipse 49
eigenvalue 1906
elasticity 885, 1398
electric field 1486, 1498, 1679
electrical dipole moment 885
electromagnetic field 1679
electromagnetic potentials 1819
electromotive force 1498
electrostatic field 1669
element of area 887
elementary chain rule 305
elementary fraction 69
ellipse 48–50, 92, 113, 173, 199, 227
ellipsoid 56, 66, 110, 197, 254, 430, 436, 501, 538, 1107
ellipsoid of revolution 111
ellipsoidal disc 79, 199
ellipsoidal surface 180
elliptic cylindric surface 60, 63, 66, 106
elliptic paraboloid 60, 62, 66, 112, 247
elliptic paraboloid of revolution 624
energy 1498
energy density 1548, 1818
energy theorem 1921
entropy 301
Euclidean norm 162
Euclidean space 19, 21, 22
Euler's spiral 1219
exact differential form 848
exceptional point 594, 677, 920
expansion point 327
explicit given function 161
extension map 153
exterior 37–39
exterior point 38
extremum 580, 632
Faraday-Henry law of electromagnetic induction 1676
Fick's first law of diffusion 297

- Fick's law 1818
field line 160
final point 170
fluid mechanics 491
flux 1535, 1540, 1549
focus 49, 51, 53
force 1485
Fourier's law 297, 1817
function in several variables 154
functional matrix 303
fundamental theorem of vector analysis 1815

Gaussian integral 938
Gauß's law 1670
Gauß's law for magnetism 1671
Gauß's theorem 1499, 1535, 1540, 1549, 1580, 1718,
 1724, 1737, 1746, 1747, 1749, 1751, 1817,
 1818, 1889, 1890, 1913
Gauß's theorem in \mathbb{R}^2 1543
Gauß's theorem in \mathbb{R}^3 1543
general chain rule 314
general coordinates 1016
general space integral 1020
general Taylor's formula 325
generalized spherical coordinates 21
generating curve 499
generator 66, 180
geometrical analysis 1015
global minimum 613
gradient 295, 296, 298, 339, 847, 1739, 1741
gradient field 631, 847, 1485, 1487, 1489, 1491,
 1916
gradient integral theorem 1489, 1499
graph 158, 179, 499, 1229
Green's first identity 1890
Green's second identity 1891, 1895
Green's theorem in the plane 1661, 1669, 1909
Green's third identity 1896
Green's third identity in the plane 1898

half-plane 41, 42
half-strip 41, 42
half disc 85
harmonic function 426, 427, 1889
heat conductivity 297
heat equation 1818
heat flow 297
height 42
helix 1169, 1235

Helmholtz's theorem 1815
homogeneous function 1908
homogeneous polynomial 339, 372
Hopf's maximum principle 1905
hyperbola 48, 50, 51, 88, 195, 217, 241, 255, 1290
hyperbolic cylindric surface 60, 63, 66, 105, 110
hyperbolic paraboloid 60, 62, 66, 246, 534, 614,
 1445
hyperboloid 232, 1291
hyperboloid of revolution 104
hyperboloid of revolution with two sheets 111
hyperboloid with one sheet 56, 66, 104, 110, 247,
 255
hyperboloid with two sheets 59, 66, 104, 110, 111,
 255, 527
hysteresis 1669

identity map 303
implicit given function 21, 161
implicit function theorem 492, 503
improper integral 1411
improper surface integral 1421
increment 611
induced electric field 1675
induction field 1671
infinitesimal vector 1740
infinity, signed 162
infinity, unspecified 162
initial point 170
injective map 153
inner product 23, 29, 33, 163, 168, 1750
inspection 861
integral 847
integral over cylindric surface 1230
integral over surface of revolution 1232
interior 37–40
interior point 38
intrinsic boundary 1227
isolated point 39
Jacobian 1353, 1355

Kronecker symbol 23

Laplace equation 1889
Laplace force 1819
Laplace operator 1743
latitude 35
length 23
level curve 159, 166, 198, 492, 585, 600, 603

- level surface 198, 503
limit 162, 219
line integral 1018, 1163
line segment 41
Linear Algebra 627
linear space 22
local extremum 611
logarithm 189
longitude 35
Lorentz condition 1824
- Maclaurin's trisectrix 973, 975
magnetic circulation 1674
magnetic dipole moment 886, 1821
magnetic field 1491, 1498, 1679
magnetic flux 1544, 1671, 1819
magnetic force 1674
magnetic induction 1671
magnetic permeability of vacuum 1673
magnostatic field 1671
main theorems 185
major semi-axis 49
map 153
MAPLE 55, 68, 74, 156, 171, 173, 341, 345, 350, 352–354, 356, 357, 360, 361, 363, 364, 366, 368, 374, 384–387, 391–393, 395–397, 401, 631, 899, 905–912, 914, 915, 917, 919, 922–924, 926, 934, 935, 949, 951, 954, 957–966, 968, 971–973, 975, 1032–1034, 1036, 1037, 1039, 1040, 1042, 1053, 1059, 1061, 1064, 1066–1068, 1070–1072, 1074, 1087, 1089, 1091, 1092, 1094, 1095, 1102, 1199, 1200
matrix product 303
maximal domain 154, 157
maximum 382, 579, 612, 1916
maximum value 922
maximum-minimum principle for harmonic functions 1895
Maxwell relation 302
Maxwell's equations 1544, 1669, 1670, 1679, 1819
mean value theorem 321, 884, 1276, 1490
mean value theorem for harmonic functions 1892
measure theory 1015
Mechanics 15, 147, 289, 477, 573, 841, 1009, 1157, 1347, 1479, 1651, 1801, 1921
meridian curve 181, 251, 499, 1232
meridian half-plane 34, 35, 43, 181, 1055, 1057, 1081
- method of indefinite integration 859
method of inspection 861
method of radial integration 862
minimum 186, 178, 579, 612, 1916
minimum value 922
minor semi-axis 49
mmf 1674
Möbius strip 185, 497
Moivre's formula 122, 264, 452, 548, 818, 984, 1132, 1322, 1454, 1626, 1776, 1930
monopole 1671
multiple point 171
- nabla 296, 1739
nabla calculus 1750
nabla notation 1680
natural equation 1215
natural parametric description 1166, 1170
negative definite matrix 627
negative half-tangent 485
neighbourhood 39
neutral element 22
Newton field 1538
Newton-Raphson iteration formula 583
Newton's second law 1921
non-oriented surface 185
norm 19, 23
normal 1227
normal derivative 1890
normal plane 487
normal vector 496, 1229
- octant 83
Ohm's law 297
open ball 38
open domain 176
open set 21, 39
order of expansion 322
order relation 579
ordinary integral 1017
orientation of a surface 182
orientation 170, 172, 184, 185, 497
oriented half line 172
oriented line 172
oriented line segment 172
orthonormal system 23
- parabola 52, 53, 89–92, 195, 201, 229, 240, 241
parabolic cylinder 613

- parabolic cylindric surface 64, 66
paraboloid of revolution 207, 613, 1435
parallelepipedum 27, 42
parameter curve 178, 496, 1227
parameter domain 1227
parameter of a parabola 53
parametric description 170, 171, 178
parfrac 71
partial derivative 298
partial derivative of second order 318
partial derivatives of higher order 382
partial differential equation 398, 402
partial fraction 71
Peano 483
permeability 1671
piecewise C^k -curve 484
piecewise C^n -surface 495
plane 179
plane integral 21, 887
point of contact 487
point of expansion 304, 322
point set 37
Poisson's equation 1814, 1889, 1891, 1901
polar coordinates 15, 19, 21, 30, 85, 88, 147, 163,
172, 213, 219, 221, 289, 347, 388, 390,
477, 573, 611, 646, 720, 740, 841, 936,
1009, 1016, 1157, 1165, 1347, 1479, 1651,
1801
polar plane integral 1018
polynomial 297
positive definite matrix 627
positive half-tangent 485
positive orientation 173
potential energy 1498
pressure 1818
primitive 1491
primitive of gradient field 1493
prism 42
Probability Theory 15, 147, 289, 477, 573, 841,
1009, 1157, 1347, 1479, 1651, 1801
product set 41
projection 23, 157
proper maximum 612, 618, 627
proper minimum 612, 613, 618, 627
pseudo-sphere 1434
Pythagoras's theorem 23, 25, 30, 121, 451, 547,
817, 983, 1131, 1321, 1453, 1625, 1775,
1929
quadrant 41, 42, 84
quadratic equation 47
range 153
rectangle 41, 87
rectangular coordinate system 29
rectangular coordinates 15, 21, 22, 147, 289, 477,
573, 841, 1009, 1016, 1079, 1157, 1165,
1347, 1479, 1651, 1801
rectangular plane integral 1018
rectangular space integral 1019
rectilinear motion 19
reduction of a surface integral 1229
reduction of an integral over cylindric surface 1231
reduction of surface integral over graph 1230
reduction theorem of line integral 1164
reduction theorem of plane integral 937
reduction theorem of space integral 1021, 1056
restriction map 153
Riccati equation 369
Riesz transformation 1275
Rolle's theorem 321
rotation 1739, 1741, 1742
rotational body 1055
rotational domain 1057
rotational free vector field 1662
rules of computation 296
saddle point 612
scalar field 1485
scalar multiplication 22, 1750
scalar potential 1807
scalar product 169
scalar quotient 169
second differential 325
semi-axis 49, 50
semi-definite matrix 627
semi-polar coordinates 15, 19, 21, 33, 147, 181,
182, 289, 477, 573, 841, 1009, 1016, 1055,
1086, 1157, 1231, 1347, 1479, 1651, 1801
semi-polar space integral 1019
separation of the variables 853
signed curve length 1166
signed infinity 162
simply connected domain 849, 1492
simply connected set 176, 243
singular point 487, 489
space filling curve 171
space integral 21, 1015

- specific capacity of heat 1818
sphere 35, 179
spherical coordinates 15, 19, 21, 34, 147, 179, 181, 289, 372, 477, 573, 782, 841, 1009, 1016, 1078, 1080, 1081, 1157, 1232, 1347, 1479, 1581, 1651, 1801
spherical space integral 1020
square 41
star-shaped domain 1493, 1807
star shaped set 21, 41, 89, 90, 175
static electric field 1498
stationary magnetic field 1821
stationary motion 492
stationary point 583, 920
Statistics 15, 147, 289, 477, 573, 841, 1009, 1157, 1347, 1479, 1651, 1801
step line 172
Stokes's theorem 1499, 1661, 1676, 1679, 1746, 1747, 1750, 1751, 1811, 1819, 1820, 1913
straight line (segment) 172
strip 41, 42
substantial derivative 491
surface 159, 245
surface area 1296
surface integral 1018, 1227
surface of revolution 110, 111, 181, 251, 499
surjective map 153
- tangent 486
tangent plane 495, 496
tangent vector 178
tangent vector field 1485
tangential line integral 861, 1485, 1598, 1600, 1603
Taylor expansion 336
Taylor expansion of order 2, 323
Taylor's formula 321, 325, 404, 616, 626, 732
Taylor's formula in one dimension 322
temperature 297
temperature field 1817
tetrahedron 93, 99, 197, 1052
Thermodynamics 301, 504
top point 49, 50, 53, 66
topology 15, 19, 37, 147, 289, 477, 573, 841, 1009, 1157, 1347, 1479, 1651, 1801
torus 43, 182–184
transformation formulæ 1353
transformation of space integral 1355, 1357
transformation theorem 1354
trapeze 99
- triangle inequality 23, 24
triple integral 1022, 1053
- uniform continuity 186
unit circle 32
unit disc 192
unit normal vector 497
unit tangent vector 486
unit vector 23
unspecified infinity 162
- vector 22
vector field 158, 296, 1485
vector function 21, 157, 189
vector product 19, 26, 30, 163, 169, 1227, 1750
vector space 21, 22
vectorial area 1748
vectorial element of area 1535
vectorial potential 1809, 1810
velocity 490
volume 1015, 1543
volumen element 1015
- weight function 1081, 1229, 1906
work 1498
- zero point 22
zero vector 22
- (r, s, t)-method 616, 619, 633, 634, 638, 645–647, 652, 655
 C^k -curve 483
 C^n -functions 318
1-1 map 153