Lecture : Beta and Gamma Functions

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January 31, 2015

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Introduction I

$$\Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx$$

$$(x_1, x_2, \ldots, x_n) \times (y_1, y_2, \ldots, y_n)$$

Gamma Functions I

Definition

If $n > 0$, the gamma function Γ is defined by the improper integral

$$\Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx$$

This integral converge for all $n > 0$.
It is sometimes called the *second Eulerian integral*.

Gamma Functions II

The most useful factorial property of gamma function are as follows.

Theorem

If $n > 0$, *then* $\Gamma(n+1) = n\Gamma(n)$

***Proof*:**
By definition of gamma function

$$\Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx \tag{1}$$

If $0 < a < b$, then we can integrate by parts

$$\int_a^b x^n\, e^{-x}\, dx = [x(-e^{-x})]_a^b - \int_a^b nx^{n-1}(-1)e^{-x}\, dx$$
$$= -b^n e^{-b} + a^n e^{-a} + n\int_a^b x^{n-1}e^{-x}\, dx$$

Take the limit of this equation as $a \to 0$ and $b \to \infty$ to get

$$= \Gamma(n+1) = n\int_0^\infty x^{n-1}\, e^{-x}\, dx = n\Gamma(n)$$

Gamma Functions III

By repeating this result, we get

$$\begin{aligned}\Gamma(n+1) n\Gamma(n) \\ &= n(n-1)\Gamma(n-1) \\ &= n(n-1)(n-2)\Gamma(n-3) \\ &= \ldots = n(n-1)(n-2)\ldots(2)(1)\Gamma(1) \\ &= n!\Gamma(1)\end{aligned}$$

But

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = 1 \tag{2}$$

So

$$\Gamma(n+1) = n!$$

Gamma Functions IV

From this result we can find the gamma function of negative integer.

Theorem

If n is zero of a negative integer then $\Gamma(n) = \infty$.

***Proof*:**

Transformation of Gamma Function I

We will prove some important results of gamma function using the improper integral formulation.

Theorem

If $n > 0$, then $\dfrac{\Gamma(n)}{z^n} = \int_0^\infty e^{-zx}\, x^{n-1}\, dx$

***Proof*:**

We know that

$$\Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx \tag{3}$$

On putting $x = az$ in the above equation (3), we get

$$\begin{aligned} \Gamma(n) &= \int_0^\infty e^{-az}(az)^{n-1} a\, dz \\ &= a^n \int_0^\infty e^{-az} z^{n-1} a\, dz \\ &= a^n \int_0^\infty e^{-ax} x^{n-1}\, dx \end{aligned} \tag{4}$$

Replacing a by z in above equation, we get

$$\frac{\Gamma(n)}{z^n} = \int_0^\infty e^{-zx}\, x^{n-1}\, dx$$

Theorem

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

***Proof*:** By definition of gamma function

$$\Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx \tag{5}$$

Put $x^n = y$ and $nx^{n-1}dx = dy$, we get

$$\begin{aligned}
\Gamma(n) &= \int_0^\infty y^{n-1/n} e^{-y^{1/n}} \frac{1}{nx^{n-1}}\, dy \\
&= \int_0^\infty y^{n-1/n} e^{-y^{1/n}} \frac{1}{ny^{n-1/n}}\, dy \\
\Gamma(n) &= \frac{1}{n} \int_0^\infty e^{-y^{1/n}}\, dy
\end{aligned}$$

Now put $n = \frac{1}{2}$ in above equation, we get

$$\Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \int_0^\infty e^{-y^{1/2}}\, dy = 2\left(\frac{1}{2}\sqrt{\pi}\right) = \sqrt{\pi}$$

Hence

Example

Evaluate $\int_0^\infty \frac{x^c}{c^x}\,dx \quad$ where $c > 1$.

Solution:

Putting $c^x = e^t$ and $dx = \frac{dt}{\log c}$, we obtain

$$
\begin{aligned}
I &= \int_0^\infty \left(\frac{t}{\log c}\right)^c e^{-t}\,\frac{dt}{\log c} \\
&= \frac{1}{(\log)^{c+1}} \int_0^\infty e^{-t} t^c\,dt \\
&= \frac{1}{(\log c)^{c+1}} \Gamma(c+1)
\end{aligned}
$$

Example

Evaluate $\Gamma\left(-\frac{7}{2}\right)$.

Solution: As we now that in n is negative but not integer, then

$$\Gamma(n) = \frac{1}{n}\Gamma(n+1) \tag{6}$$

Therefore

$$\Gamma\left(-\frac{7}{2}\right) = \frac{1}{\left(-\frac{7}{2}\right)}\Gamma\left(-\frac{7}{2}+1\right) = \left(-\frac{2}{7}\right)\Gamma\left(-\frac{5}{2}\right)$$

Again repeating the above formula. we get

$$\begin{aligned}\Gamma\left(-\frac{7}{2}\right) &= \left(-\frac{2}{7}\right)\left(-\frac{2}{5}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{1}\right)\Gamma\left(\frac{1}{2}\right)\\ &= \frac{16}{105}\Gamma\left(\frac{1}{2}\right)\\ \Gamma\left(-\frac{7}{2}\right) &= \frac{16}{105}\sqrt{\pi}\end{aligned}$$

Exercise I

- Evaluate $\int_0^\infty x^{2n-1} e^{-ax^2}\, dx$.
- Prove that $\int_0^\infty e^{-y^{1/m}}\, dy = m\Gamma(m)$.
- Evaluate $\Gamma\left(-\frac{3}{2}\right)$.

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Feel free to mail me for any type of suggestion for improvement of this lecture notes: drsanjaymishra1@gmail.com


Thank you!