

# Quantitative Analysis

Algebra with a Business Perspective

Donna M. Wacha



DONNA M. WACHA

---

**QUANTITATIVE  
ANALYSIS  
ALGEBRA WITH A  
BUSINESS PERSPECTIVE**

Quantitative Analysis: Algebra with a Business Perspective

2<sup>nd</sup> edition

© 2017 Donna M. Wacha & [bookboon.com](http://bookboon.com)

ISBN 978-87-403-1830-2

Peer review by Lynn Dietrich, director of Math Tutoring Lab, Monmouth University

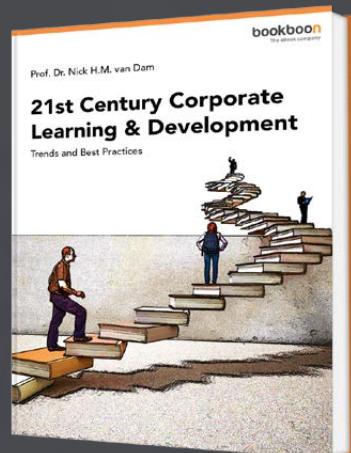
# CONTENTS

<b>Preface</b>	<b>11</b>
<b>1 Equations, Inequalities &amp; Linear Programming</b>	<b>12</b>
Tutorial 1.1: Solve one-step linear equations in one variable	12
Tutorial 1.2: Solve two-step linear equations in one variable	14
Tutorial 1.3: Solve multi-step linear equations involving like-terms	17
Tutorial 1.4: Solve multi-step linear equations involving parentheses	18
Tutorial 1.5: Solve multi-step linear equations in one variable where the variable appears on each side of the equation	20
Tutorial 1.6: Solve multi-step linear inequalities in one variable	22
Tutorial 1.7: Graph the solution of multi-step linear equations or inequalities in one variable on a number line	26
Tutorial 1.8: Graph a line that passes through two given points	30
Tutorial 1.9: Graph a linear equation using its x- and y-intercepts	33
Tutorial 1.10: Graph an equation (in slope-intercept form) using its slope and y-intercept	36

## Free eBook on Learning & Development

By the Chief Learning Officer of McKinsey

**Download Now**



Download free eBooks at [bookboon.com](http://bookboon.com)



**Click on the ad to read more**

Tutorial 1.11: Graph a linear equation (in standard form) using its slope and y-intercept	39
Tutorial 1.12: Determine the slope of a line passing through 2 given points	42
Tutorial 1.13: Use the slope-intercept formula to write the equation of a line with a given slope and point	48
Tutorial 1.14: Use the point-slope formula to write the slope-intercept form of an equation of a line with a given slope and point	50
Tutorial 1.15: Write the standard form of an equation of a line with a given slope and point	52
Tutorial 1.16: Write an equation of a line that passes through two given points	56
Tutorial 1.17: Determine whether the lines for a pair of equations are parallel, perpendicular or coincide with each other	59
Tutorial 1.18: Write the equation of a line through a given point that is parallel to a given line	63
Tutorial 1.19: Write the equation of a line through a given point that is perpendicular to a given line	65
Tutorial 1.20: Solve a system of linear equations by graphing	68
Tutorial 1.21: Solve a system of linear equations by substitution	69
Tutorial 1.22: Solve a system of linear equations by elimination	72
Tutorial 1.23: Solve a system of three equations in three variables	77
Tutorial 1.24: Solve applications involving systems of equations	86
Tutorial 1.25: Find the marginal cost, the marginal cost revenue and the marginal cost profit of given linear total cost functions, linear total revenue functions and linear profit functions	91
Tutorial 1.26: Evaluate linear total cost functions, linear total revenue functions and linear profit functions	94
Tutorial 1.27: Write the equations for linear total cost functions, linear total revenue functions and linear profit functions by using information given about the functions	96
Tutorial 1.28: Find the break-even point for the given cost and revenue functions	97
Tutorial 1.29: Evaluate and graph supply and demand functions	100
Tutorial 1.30: Write the equation (in slope-intercept form) of a supply or a demand function	103
Tutorial 1.31: Determine the market equilibrium for a given scenario	108
Tutorial 1.32: Graph a linear inequality in two variables	112
Tutorial 1.33: Solve a system of linear inequalities in two variables	116

Tutorial 1.34: Name the vertices for the feasible solution region for the given system of linear inequalities	118
Tutorial 1.35: Use the vertices for the feasible solution region for a given system of linear inequalities to determine the optimal (minimum or maximum) value of a given linear function that is subject to constraints as defined by the given system of inequalities	121
Tutorial 1.36: Use graphing to determine the optimal maximum value of a linear function subject to constraints	126
Tutorial 1.37: Use graphing to determine the optimal minimum value of a linear function subject to constraints	128
Tutorial 1.38: Solve applications involving linear programming	130
<b>2 Matrices &amp; Array Operations</b>	<b>137</b>
Tutorial 2.1: Determine the number of rows or columns in a matrix	137
Tutorial 2.2: Determine the order (dimensions) of a matrix	138
Tutorial 2.3: Identify square matrices	139
Tutorial 2.4: Identify the element (entry) at a given location ( $M_{RC}$ )	139
Tutorial 2.5: Determine a zero matrix with a given order (dimensions)	140
Tutorial 2.6: Find the transpose of a matrix	142
Tutorial 2.7: Determine the negative of a matrix	143
Tutorial 2.8: Use matrices to present and interpret given data	144
Tutorial 2.9: Add matrices	146
Tutorial 2.10: Subtract matrices	148
Tutorial 2.11: Use matrix addition/subtraction to solve real-world business applications	149
Tutorial 2.12: Multiply a matrix by a scalar (real number)	151
Tutorial 2.13: Find the product of given matrices	152
Tutorial 2.14: Complete matrix multiplication involving zero matrices	158
Tutorial 2.15: Complete multiplication involving identity matrices	160
Tutorial 2.16: Use matrix multiplication to solve business applications	162
Tutorial 2.17: Write the augmented matrix for a system of equations	165
Tutorial 2.18: Use a reduced matrix to determine the solutions for a given system	166
Tutorial 2.19: Use matrices to solve systems with unique solutions	167
Tutorial 2.20: Use matrices to solve systems with non-unique solutions	177
Tutorial 2.21: Find the inverse of a $2 \times 2$ square matrix	181
Tutorial 2.22: Use $2 \times 2$ inverse matrices to solve systems of equations	183
Tutorial 2.23: Use row operations to find the inverse matrix of a square matrix	186

Tutorial 2.24: Use $N \times N$ inverse matrices to solve systems of equations	189
Tutorial 2.25: Use matrix inverses to solve real-world applications	191
Tutorial 2.26: Use a computer spreadsheet to determine the number of rows and columns contained in a matrix and thereby determine the order of the matrix	194
Tutorial 2.27: Use a computer spreadsheet to determine the transpose and/or the negative of a matrix	196
Tutorial 2.28: Use a computer spreadsheet to add or subtract matrices	198
Tutorial 2.29: Use a computer spreadsheet to complete a scalar multiplication	200
Tutorial 2.30: Use a computer spreadsheet to complete a matrix multiplication	201
Tutorial 2.31: Use a computer spreadsheet to solve systems of equations involving row operations	202
Tutorial 2.32: Use a computer spreadsheet to determine the inverse and/or determinant of a matrix	206
Tutorial 2.33: Use a computer spreadsheet to solve a system of equations involving inverse matrices	208
<b>3 Quadratic &amp; other Special Functions</b>	<b>210</b>
Tutorial 3.1: Factor a polynomial expression by finding its greatest common monomial factor	210
Tutorial 3.2: Factor a polynomial expression by grouping	212
Tutorial 3.3: Factor a polynomial expression ( $x^2 + bx + c$ ) as a product of binomials	214
Tutorial 3.4: Factor a polynomial expression ( $ax^2 + bx + c$ ) as a product of binomials	217
Tutorial 3.5: Factor “special” polynomial expressions through patterning	221
Tutorial 3.6: Use factoring to complete business applications	227
Tutorial 3.7: Solve quadratic equations by factoring	229
Tutorial 3.8: Solve quadratic equations by completing the square	232
Tutorial 3.9: Derive the quadratic formula by completing the square	236
Tutorial 3.10: Solve quadratic equations using the quadratic formula	237
Tutorial 3.11: Determine the number and the nature of the solutions for a quadratic equation	239
Tutorial 3.12: Solve business applications involving quadratic functions	243
Tutorial 3.13: For any quadratic function:	247
Tutorial 3.14: Use quadratic functions to find market equilibrium	254
Tutorial 3.15: Use quadratic functions to find break-even point(s)	259

Tutorial 3.16: Maximize quadratic revenue and/or profit functions.	
Minimize quadratic cost functions	264
Tutorial 3.17: Use quadratic functions to solve real-world applications	267
Tutorial 3.18: Identify the graphs of basic functions	269
<b>4 Mathematics of Finance</b>	<b>283</b>
Tutorial 4.1: Find the amount of interest for a simple interest loan or investment	283
Tutorial 4.2: Find the future value for a simple interest investment	284
Tutorial 4.4: Find the simple interest rate earned on an investment	285
Tutorial 4.5: Find the time needed for an account with a simple interest rate to reach a desired future value	287
Tutorial 4.6: Find the future value for an account that is compounded periodically	288
Tutorial 4.7: Find the amount of interest earned for an account that is compounded periodically	289
Tutorial 4.8: Find the amount that was invested into an account that is compounded periodically	291
Tutorial 4.9: Find the interest rate for a periodically-compounded account	293
Tutorial 4.10: Solve an exponential equation using logarithms	296
Tutorial 4.11: Find the time needed for an account that is compounded periodically to reach a certain future value	297
Tutorial 4.12: Find the future value for an account when interest is compounded continuously	300
Tutorial 4.13: Find the amount that was invested into an account that is compounded continuously	301
Tutorial 4.14: Find the interest rate for an account that is compounded continuously	303
Tutorial 4.15: Find the time needed for an account that is compounded continuously to reach a specified amount	305
Tutorial 4.16: Find the annual percentage yield (APY) of money in an account where the interest is compounded periodically	307
Tutorial 4.17: Find the annual percentage yield (APY) of money in an account where the interest is compounded continuously	308
Tutorial 4.18: Compute the future value of an ordinary annuity	310
Tutorial 4.19: Compute the payment amounts required in order for an ordinary annuity to have a specified future value	312

Tutorial 4.20: Find out how long it will take for an ordinary annuity to reach a savings goal	315
Tutorial 4.21: Compute the future values of an annuity due	319
Tutorial 4.22: Compute the payment amounts required in order for an annuity due to have specified future values	322
Tutorial 4.23: Find out how long it will take for an annuity due to reach a savings goal	324
Tutorial 4.24: Compute the present value of an ordinary annuity	327
Tutorial 4.25: Compute the payments for a specified present value for an ordinary annuity	329
Tutorial 4.26: Compute the present value of an annuity due	331
Tutorial 4.27: Compute the payments for a specified present value for an annuity due	333
Tutorial 4.28: Compute the present value of a deferred annuity	335
Tutorial 4.29: Compute the payments for a specified present value for a deferred annuity	337
Tutorial 4.30: Find the regular payments to amortize a debt	341
Tutorial 4.31: Find the amount that can be borrowed for a specified payment	342
Tutorial 4.32: Find the unpaid balance of a loan	344
<b>About the Author</b>	<b>347</b>

# PREFACE

- ***acknowledgements...***

*First of all, I would like to thank Lynn Dietrich (director of the Math Learning Center at Monmouth University, West Long Branch, New Jersey, USA) for her help in producing this tutorial text. Without her help, many typographical errors might not have been caught before submitting the final draft.*

*I would also like to extend my thanks to Sophie Tergeist (bookboon.com Ltd. – Davenport House, 16 Pepper Street, London E14 9RP, United Kingdom) for her constant patience in answering my numerous questions during the production of this text.*

- ***explanation of how the textbook is organized...***

*This tutorial textbook has been organized into 4 chapters (units) with several individual tutorial lessons within each chapter. As presented in the table of contents, each of the tutorials has been listed separately with its objective and its starting page. For coding purposes: Tutorial #N.M means that the tutorial is the Mth lesson in chapter N.*

*I attempted to present each of the tutorials as if a person (teacher, tutor) was sitting next to the reader talking each of the concepts through. My goal was to make sure that each of the lessons are fully explained but still fully understandable. Hopefully this goal was met.*

- ***description of whom I believe might be interested in using these tutorials...***

*The purpose of this tutorial textbook is to present mathematical skills (algebraic concepts) and their various applications that may be important to students of management (business) science. The applications included should allow readers to view math in a practical setting relevant to their intended careers.*

# 1 EQUATIONS, INEQUALITIES & LINEAR PROGRAMMING

## TUTORIAL 1.1: SOLVE ONE-STEP LINEAR EQUATIONS IN ONE VARIABLE

The equations below are considered one-step linear equations because they are each expressed using one variable ( $x$ ) and take exactly one step to solve:

$$x + 5 = 12$$

$$x - 7 = 18$$

$$8x = 72$$

$$x \div 6 = 8$$

Whenever we need to solve a one-step linear equation, we can talk ourselves through the process by asking these questions:

1. What value do we wish to address?
2. Within the given equation, what operation (addition, subtraction, multiplication, division) are we doing with the value that needs to be addressed?
3. What is the opposite of the operation answered in question #2? (Apply the operation named here to both sides of the equation and we will derive the solution to the given equation.)

**example 1.1a**
**Solve for the variable:  $x + 5 = 12$** 

$$x + 5 = 12 \rightarrow x + (5 - 5) = (12 - 5) \rightarrow x = 7$$

We wish to address the 5 that is being added to the variable  $x$ . Since the 5 is being added, we need to subtract it from both sides of the equation. And, since  $12 - 5 = 7$ , the answer to the equation is:  $x = 7$ .

**example 1.1b**
**Solve for the variable:  $x - 7 = 18$** 

$$x - 7 = 18 \rightarrow x + (-7 + 7) = (18 + 7) \rightarrow x = 25$$

We wish to address the 7 that is being subtracted from the variable  $x$ . Since the 7 is being subtracted, we need to add it to both sides of the equation. And, since  $18 + 7 = 25$ , the answer to the equation is:  $x = 25$ .

**example 1.1c****Solve for the variable:  $8x = 72$** 

$$8x = 72 \rightarrow \frac{8x}{8} = \frac{72}{8} \rightarrow x = 9$$

We wish to address the 8 that is being multiplied with the variable  $x$ . Since the 8 is being multiplied, we need to divide it on both sides of the equation. And, since  $72 \div 8 = 9$ , the answer to the equation is:  $x = 9$ .

**example 1.1d****Solve for the variable:  $x \div 6 = 8$** 

$$x \div 6 = 8 \rightarrow x (\div 6 \times 6) = (8 \times 6) \rightarrow x = 48$$

We wish to address the 6 where the variable  $x$  is being divided by 6. Since the 6 is being divided, we need to multiply it on both sides of the equation. And, since  $8 \times 6 = 48$ , the answer to the equation is:  $x = 48$ .



[www.sylvania.com](http://www.sylvania.com)

We do not reinvent the wheel we reinvent light.

Fascinating lighting offers an infinite spectrum of possibilities: Innovative technologies and new markets provide both opportunities and challenges. An environment in which your expertise is in high demand. Enjoy the supportive working atmosphere within our global group and benefit from international career paths. Implement sustainable ideas in close cooperation with other specialists and contribute to influencing our future. Come and join us in reinventing light every day.

Light is OSRAM

**OSRAM** **SYLVANIA**



## TUTORIAL 1.2: SOLVE TWO-STEP LINEAR EQUATIONS IN ONE VARIABLE

The equations below are considered two-step linear equations because they are each expressed using one variable ( $x$ ) and take exactly two steps to solve:

$$2x + 5 = 17$$

$$3x - 7 = 26$$

$$\frac{x}{7} + 4 = -3$$

$$\frac{x}{5} - 2 = -10$$

When solving a two-step equation, we will be performing the order of operation (PEMDAS) in reverse order...which means that we should address any values that are being added or subtracted first and then address any values that are being multiplied or divided. Therefore, whenever we need to solve a two-step linear equation, we can talk ourselves through the process by asking these questions:

1. What value is being added to or subtracted from the variable term? (This is the value that needs to be addressed first.)
2. How should we undo the value that is being added to or subtracted from the variable term? (We will need to transform the given equation into a one-step equation by adding or subtracting the value that needs to be addressed as we did in a previous tutorial. Remember to apply the opposing operation to both sides of the equation and we will derive a transformed equation that will lead to the solution to the given equation.)
3. Within the transformed equation, how should we undo the value that is being multiplied with the variable? Or, how should we undo the value that is the divisor of the variable? (Remember to apply the opposing operation named here to both sides of the equation and we will derive the solution to the given equation.)

**example 1.2a      Solve for the variable:  $2x + 5 = 17$**

$$\begin{aligned} 2x + 5 = 17 &\rightarrow 2x + (5 - 5) = (17 - 5) \rightarrow 2x = 12 \rightarrow \frac{2x}{2} = \frac{12}{2} \rightarrow \\ x &= 6 \end{aligned}$$

We should first address the 5 that is being added to the variable term of  $2x$ . Since the 5 is being added, we need to subtract it from both sides of the equation. And, since  $17 - 5 = 12$ , the transformed equation should read:  $2x = 12$ . Secondly, since the 2 is being multiplied with the variable  $x$  in the transformed equation, we will need to divide each side to the transformed equation by 2. After doing this, the solution to this equation is:  $x = 6$ .

**example 1.2b****Solve for the variable:**  $3x - 7 = 26$ 

$$3x - 7 = 26 \rightarrow 3x + (-7 + 7) = (26 + 7) \rightarrow$$

$$3x = 33 \rightarrow \frac{3x}{3} = \frac{33}{3} \rightarrow x = 11$$

We should first address the 7 that is being subtracted from the variable term of  $3x$ . Since the 7 is being subtracted, we need to add it to both sides of the equation. And, since  $26 + 7 = 33$ , the transformed equation should read:  $3x = 33$ . Secondly, since the 3 is being multiplied with the variable  $x$  in the transformed equation, we will need to divide each side to the transformed equation by 3. After doing this, the solution to this equation is:  $x = 11$ .

**example 1.2c****Solve for the variable:**  $\frac{x}{7} + 4 = -3$ 

$$\frac{x}{7} + 4 = -3 \rightarrow \frac{x}{7} + (4 - 4) = (-3 - 4) \rightarrow \frac{x}{7} = -7 \rightarrow$$

$$\left(\frac{x}{7}\right)\left(\frac{7}{1}\right) = (-7)(7) \rightarrow x = -49$$

We should first address the 4 that is being added to the variable term of  $\frac{x}{7}$ . Since the 4 is being added, we need to subtract it from both sides of the equation. And, since  $-3 - 4 = -7$ , the transformed equation should read:  $\frac{x}{7} = -7$ . Secondly, since the  $x$  is being divided by the 7, we will need to multiply each side of the transformed equation by 7. After doing this, the solution to this equation is:  $x = -49$ .

**example 1.2d**

**Solve for the variable:**  $\frac{x}{5} - 2 = -10$

$$\frac{x}{5} - 2 = -10 \rightarrow \frac{x}{5} + (-2 + 2) = (-10 + 2) \rightarrow \frac{x}{5} = -8 \rightarrow$$

$$\left(\frac{x}{5}\right)\left(\frac{5}{1}\right) = (-8)(5) \rightarrow x = -40$$

We should first address the 2 that is being subtracted from the variable term of  $\frac{x}{5}$ . Since the 2 is being subtracted, we need to add it to both sides of the equation. And, since  $-10 + 2 = -8$ , the transformed equation should read:  $\frac{x}{5} = -8$ . Secondly, since the  $x$  is being divided by the 5, we will need to multiply each side of the transformed equation by 5. After doing this, the solution to this equation is:  $x = -40$ .



**Deloitte.**

Discover the truth at [www.deloitte.ca/careers](http://www.deloitte.ca/careers)

© Deloitte & Touche LLP and affiliated entities.

Download free eBooks at [bookboon.com](http://bookboon.com)



**Click on the ad to read more**

## TUTORIAL 1.3: SOLVE MULTI-STEP LINEAR EQUATIONS INVOLVING LIKE-TERMS

Terms that have exactly the same variable combination (or lack thereof) are considered ***like-terms***...which can then be combined into a transformed term. For example, consider the like-terms of  $2x$  and  $3x$ . When added:  $2x + 3x = 5x$ . Or, when subtracted:  $2x - 3x = -1x$ . We can use this combining like-terms strategy to simplify linear equations as those given below:

$$3x + 5 + 5x - 9 = 12 \quad 4x - 3 - 7x + 3 = 12 \quad 2x - 6 + 3x + 8 = -33$$

**example 1.3a**      **Solve for the variable:**  $3x + 5 + 5x - 9 = 12$

$$\begin{aligned} 3x + 5 + 5x - 9 &= 12 \rightarrow (3x + 5x) + (5 - 9) = 12 \rightarrow 8x - 4 = 12 \\ \rightarrow 8x + (-4 + 4) &= (12 + 4) \rightarrow 8x = 16 \rightarrow \frac{8x}{8} = \frac{16}{8} \rightarrow x = 2 \end{aligned}$$

In the given equation,  $3x$  and  $5x$  are considered like-terms as are  $5$  and  $-9$ . Combining like-terms,  $3x + 5x = 8x$  and  $5 - 9 = -4$ . We have now transformed the given equation into a two-step equation:  $8x - 4 = 12$ . We should next address the  $4$  that is being subtracted from  $8x$  by adding  $4$  to both sides of the equation...which results into the one-step equation:  $8x = 16$ . To determine the final solution of the given equation, divide both sides of the equation by  $8$ . Therefore the solution to the given equation is:  $x = 2$ .

**example 1.3b**      **Solve for the variable:**  $4x - 3 - 7x + 3 = 12$

$$\begin{aligned} 4x - 3 - 7x + 3 &= 12 \rightarrow (4x - 7x) + (-3 + 3) = 12 \rightarrow -3x = 12 \rightarrow \\ \frac{-3x}{-3} &= \frac{12}{-3} \rightarrow x = -4 \end{aligned}$$

In the given equation,  $4x$  and  $-7x$  are considered like-terms as are  $3$  and  $-3$ . Combining like-terms,  $4x - 7x = -3x$  and  $-3 + 3 = 0$ . We have now transformed the given equation into a one-step equation:  $-3x = 12$ . To determine the final solution of the given equation, divide both sides of the equation by  $-3$ . Therefore the solution to the given equation is:  $x = -4$ .

**example 1.3c****Solve for the variable:**  $2x - 6 + 3x + 8 = -33$ 

$$\begin{aligned} 2x - 6 + 3x + 8 &= -33 \rightarrow (2x + 3x) + (-6 + 8) = -33 \rightarrow 5x + 2 = -33 \rightarrow \\ 5x + (2 - 2) &= (-33 - 2) \rightarrow 5x = -35 \rightarrow \frac{5x}{5} = \frac{-35}{5} \rightarrow x = -7 \end{aligned}$$

In the given equation,  $2x$  and  $3x$  are considered like-terms as are  $-6$  and  $8$ . Combining like-terms,  $2x + 3x = 5x$  and  $-6 + 8 = 2$ . We have now transformed the given equation into a two-step equation:  $5x + 2 = -33$ . We should next address the  $2$  that is being added to  $5x$  by subtracting  $2$  from both sides of the equation...which results into the one-step equation:  $5x = -35$ . To determine the final solution of the given equation, divide both sides of the equation by  $5$ . Therefore the solution to the given equation is:  $x = -7$ .

## **TUTORIAL 1.4: SOLVE MULTI-STEP LINEAR EQUATIONS INVOLVING PARENTHESES**

Occasionally, linear equations will contain parentheses or other grouping symbols. Following the order of operations (PEMDAS), we need to address the grouping symbols first. Therefore, we will use the Distributive Property to clear these parentheses (or other grouping symbols).

**Distributive Property over Addition:**

$$a(b + c) = ab + ac$$

**Distributive Property over Subtraction:**

$$a(b - c) = ab - ac$$

**example 1.4a****Solve for the variable:**  $5(3x - 8) = -70$ 

$$\begin{aligned} 5(3x - 8) &= -70 \rightarrow 5(3x) - 5(8) = -70 \rightarrow 15x - 40 = -70 \rightarrow 15x \\ &+ (-40 + 40) = (-70 + 40) \rightarrow 15x = -30 \rightarrow \frac{15x}{15} = \frac{-30}{15} \rightarrow x = -2 \end{aligned}$$

Since the  $5$  is being multiplied with the entire contents of the parentheses, we need to apply the Distributive Property...multiplying the  $5$  with the  $3x$  and then the  $5$  with the  $-8$ . We have transformed the given equation into a two-step equation:  $15x - 40 = -70$ . We must than address the  $40$  that is being subtracted from  $15x$  by adding  $40$  to both sides of the equation...which results into a one-step equation:  $15x = -30$ . To determine the final solution of the given equation, divide both sides of the equation by  $15$ . Therefore, the solution of the given equation is:  $x = -2$ .

**example 1.4b****Solve for the variable:**  $-7(4x - 8) = 140$ 

$$\begin{aligned}
 -7(4x - 8) &= 140 \rightarrow -7(4x) -7(-8) = 140 \rightarrow -28x + 56 = 140 \\
 \rightarrow -28x + (56 - 56) &= (140 - 56) \rightarrow -28x = 84 \rightarrow \frac{-28x}{-28} = \frac{84}{-28} \\
 \rightarrow x &= -3
 \end{aligned}$$

Since the  $-7$  is being multiplied with the entire contents of the parentheses, we need to apply the Distributive Property...multiplying the  $-7$  with the  $4x$  and then the  $-7$  with the  $-8$ . We have transformed the given equation into a two-step equation:  $-28x + 56 = 140$ . We must than address the  $56$  that is being added to  $-28x$  by subtracting it from both sides of the equation...which results into a one-step equation:  $-28x = 84$ . To determine the final solution, divide both sides of the equation by  $-28$ ...resulting in the solution:  $x = -3$ .

SIMPLY CLEVER

ŠKODA



We will turn your CV into  
an opportunity of a lifetime



Do you like cars? Would you like to be a part of a successful brand?  
We will appreciate and reward both your enthusiasm and talent.  
Send us your CV. You will be surprised where it can take you.

Send us your CV on  
[www.employerforlife.com](http://www.employerforlife.com)



**example 1.4c****Solve for the variable:**  $10x - 3(8 - 5x) = 26$ 

$$\begin{aligned}
 10x - 3(8 - 5x) &= 26 \rightarrow 10x - 3(8) - 3(-5x) = 26 \rightarrow 10x - 24 + \\
 15x &= 26 \rightarrow (10x + 15x) - 24 = 26 \rightarrow 25x + (-24 + 24) = (26 + \\
 24) \rightarrow 25x &= 50 \rightarrow \frac{25x}{25} = \frac{50}{25} \rightarrow x = 2
 \end{aligned}$$

Since the  $-3$  is being multiplied with the entire contents of the parentheses, we need to apply the Distributive Property...multiplying the  $-3$  with the  $8$  and then the  $-3$  with the  $-5x$ . We have transformed the given equation into a multi-step equation:  $10x - 24 + 15x = 26$ . After combining the like-terms of  $10x$  and  $15x$ , we now have the transformed two-step equation:  $25x - 24 = 26$ . We must then address the  $24$  that is being subtracted from  $25x$  by adding  $24$  to both sides of the equation...which results into the one-step equation:  $25x = 50$ . To determine the final solution, divide both sides of the equation by  $25$ ...resulting in the solution:  $x = 2$ .

### **TUTORIAL 1.5: SOLVE MULTI-STEP LINEAR EQUATIONS IN ONE VARIABLE WHERE THE VARIABLE APPEARS ON EACH SIDE OF THE EQUATION**

Occasionally, linear equations will have variables on both sides of the equation. We will need to transform the given equation so that the variable appears on only one side of the new equation.

**example 1.5a****Solve for the variable:**  $7x = 2x - 25$ 

$$\begin{aligned}
 7x = 2x - 25 \rightarrow (7x - 2x) &= (2x - 2x) - 25 \rightarrow 5x = -25 \rightarrow \frac{5x}{5} = \frac{-25}{5} \\
 \rightarrow x &= -5
 \end{aligned}$$

Since the variable  $x$  appears on both sides of the given equation, we need to bring one of the variable terms to the other side. Since it is usually preferable (although not absolutely necessary) to move the smaller coefficient of the variable terms to the larger, we should subtract  $2x$  from both sides of the equation...resulting in the one-step equation:  $5x = -25$ . Dividing both sides of the transformed equation by  $5$ , we obtain the final solution:  $x = -5$ .

**example 1.5b****Solve for the variable:**  $16x - 4 = 38x + 84$ 

$$\begin{aligned}
 16x - 4 &= 38x + 84 \rightarrow (16x - 16x) - 4 = (38x - 16x) + 84 \rightarrow \\
 -4 &= 22x + 84 \rightarrow (-4 - 84) = 22x + (84 - 84) \rightarrow \\
 -88 &= 22x \rightarrow \frac{-88}{22} = \frac{22x}{22} \rightarrow -4 = x
 \end{aligned}$$

Since the variable  $x$  appears on both sides of the given equation, we need to bring one of the variable terms to the other side. And, since there are numeral terms on both sides, we need to bring one of them to the other side. It is usually preferable (although not absolutely necessary) to move the smaller coefficient of the variable terms to the larger...so, we should subtract  $16x$  from both sides of the equation to handle the variable terms and subtract 84 from both sides of the equation to handle the numeral terms. These actions result in the one-step equation:  $-88 = 22x$ . Dividing both sides of the transformed equation by 22, we obtain the final solution:  $x = -4$ .

**example 1.5c****Solve for the variable:**  $8(3x - 1) = 7(2x - 4)$ 

$$\begin{aligned}
 8(3x - 1) &= 7(2x - 4) \rightarrow 8(3x) - 8(1) = 7(2x) - 7(4) \rightarrow \\
 24x - 8 &= 14x - 28 \rightarrow (24x - 14x) - 8 = -28 \rightarrow 10x - 8 = -28 \rightarrow \\
 10x + (-8 + 8) &= (-28 + 8) \rightarrow 10x = -20 \rightarrow \frac{10x}{10} = \frac{-20}{10} \rightarrow x = -2
 \end{aligned}$$

Since the given equation contains parentheses, we need to apply the Distributive Property to both sets of parentheses...resulting in the equation:  $24x - 8 = 14x - 28$ . Since the variable  $x$  appears on both sides of the given equation, we need to bring one of the variable terms to the other side. And, since there are numeral terms on both sides, we need to bring one of them to the other side. It is usually preferable (although not absolutely necessary) to move the smaller coefficient of the variable terms to the larger...so, we should subtract  $14x$  from both sides of the equation to handle the variable terms and add 8 to both sides of the equation to handle the numeral terms. These actions result in the one-step equation:  $10x = -20$ . Dividing both sides of the transformed equation by 10, we obtain the final solution:  $x = -2$ .

## TUTORIAL 1.6: SOLVE MULTI-STEP LINEAR INEQUALITIES IN ONE VARIABLE

We can apply the strategies of solving multi-step equations to solve multi-step inequalities... especially since the only difference between these number sentences is the relationship symbols used. Equations will always use the “is equal to” (=) sign while inequalities will use one of these comparison signs:

- < ... which is read: “is less than”
- > ... which is read: “is greater than”
- $\leq$  ... which is read: “is less than or equal to”
- $\geq$  ... which is read: “is greater than or equal to”
- $\neq$  ... which is read: “is not equal to”

The advertisement features a woman teacher smiling and interacting with two young students (a boy and a girl) who are looking at a laptop screen. The background is yellow with orange and white swirling graphic elements. The e-Learning for Kids logo is in the top left corner. In the bottom right corner, there is a green oval containing text: "The number 1 MOOC for Primary Education", "Free Digital Learning for Children 5-12", and "15 Million Children Reached".

**About e-Learning for Kids** Established in 2004, e-Learning for Kids is a global nonprofit foundation dedicated to fun and free learning on the Internet for children ages 5 - 12 with courses in math, science, language arts, computers, health and environmental skills. Since 2005, more than 15 million children in over 190 countries have benefitted from eLessons provided by EFK! An all-volunteer staff consists of education and e-learning experts and business professionals from around the world committed to making difference. eLearning for Kids is actively seeking funding, volunteers, sponsors and courseware developers; get involved! For more information, please visit [www.e-learningforkids.org](http://www.e-learningforkids.org).



**example 1.6a****Solve this inequality:**  $8x + 3 < 27$ 

$$8x + 3 < 27 \rightarrow 8x + (3 - 3) < (27 - 3) \rightarrow 8x < 24 \rightarrow \frac{8x}{8} < \frac{24}{8} \rightarrow x < 3$$

This inequality can be solved in just two-steps by:

1. Subtracting 3 from both sides...resulting in:  $8x < 24$
2. Dividing both sides by 8...resulting in:  $x < 3$

*NOTE: Any value less than 3 would be considered a solution.*

**example 1.6b****Solve this inequality:**  $5x - 3 > 32$ 

$$5x - 3 > 32 \rightarrow 5x + (-3 + 3) > (32 + 3) \rightarrow 5x > 35 \rightarrow \frac{5x}{5} > \frac{35}{5} \rightarrow x > 7$$

This inequality can be solved in just two-steps by:

1. Adding 3 to both sides...resulting in:  $5x > 35$
2. Dividing both sides by 5...resulting in:  $x > 7$

*NOTE: Any value greater than 7 would be considered a solution.*

**example 1.6c****Solve this inequality:**  $3(3x + 5) \leq 24$ 

$$3(3x + 5) \leq 24 \rightarrow 3(3x) + 3(5) \leq 24 \rightarrow 9x + 15 \leq 24 \rightarrow 9x + (15 - 15) \leq (24 - 15) \rightarrow 9x \leq 9 \rightarrow \frac{9x}{9} \leq \frac{9}{9} \rightarrow x \leq 1$$

Since this inequality contains a set of parentheses, we need to apply the Distributive Property...resulting in:  $9x + 15 \leq 24$ . Now we have a two-step inequality that can be solved by:

1. Subtracting 15 from both sides...resulting in:  $9x \leq 9$
2. Dividing both sides by 9...resulting in:  $x \leq 1$

*NOTE: Any value that is less than or equal to 1 would be a solution.*

**example 1.6d****Solve this inequality:  $4(9x - 4) \neq 56$** 

$$4(9x - 4) \neq 56 \rightarrow 4(9x) - 4(4) \neq 56 \rightarrow 36x - 16 \neq 56 \rightarrow$$

$$36x + (-16 + 16) \neq (56 + 16) \rightarrow 36x \neq 72 \rightarrow \frac{36x}{36} \neq \frac{72}{36} \rightarrow x \neq 2$$

Since this inequality contains a set of parentheses, we need to apply the Distributive Property...resulting in:  $36x - 16 \neq 56$ . Now we have a two-step inequality that can be solved by:

1. Adding 16 to both sides...resulting in:  $36x \neq 72$
2. Dividing both sides by 36...resulting in:  $x \neq 2$

*NOTE: Any value other than 2 would be a solution.*

Whenever we multiply or divide by a negative number in the solution of an inequality, we must “flip” the given relationship...so that  $<$  (is less than) will become  $>$  (is greater than) or vice-versa. The same thing would happen when the relationship symbol was an  $=$  (is equal to) or a  $\neq$  (is not equal to)...but, when we flip these symbols we don’t see any physical change.

**example 1.6e****Solve this inequality:  $-5x + 8 \geq 68$** 

$$-5x + 8 \geq 68 \rightarrow -5x + (8 - 8) \geq (68 - 8) \rightarrow$$

$$-5x \geq 60 \rightarrow \frac{-5x}{-5} \leq \frac{60}{-5} \rightarrow x \leq -12$$

This inequality can be solved in just two-steps by:

1. Subtracting 8 from both sides...resulting in:  $-5x \geq 60$ .
2. Dividing both sides by  $-5$ ...resulting in:  $x \leq -12$ . Since we divided both sides by a negative, the relationship symbol had to be “flipped” from  $\geq$  to  $\leq$ .

*NOTE: Any number value that is less than or equal to  $-12$  or smaller would be considered a correct solution.*

**example 1.6f**

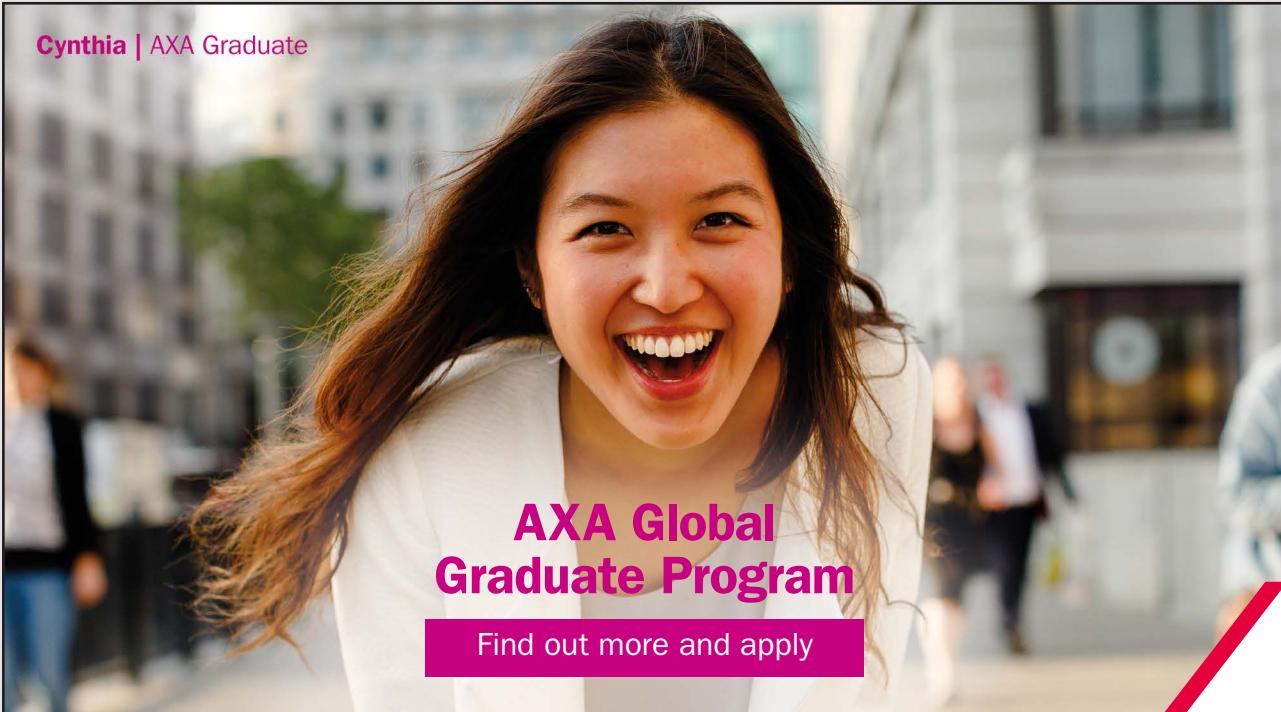
**Solve this inequality:**  $8 - \frac{x}{5} \leq 6$

$$8 - \frac{x}{5} \leq 6 \rightarrow (8 - 8) - \frac{x}{5} \leq (6 - 8) \rightarrow \frac{-x}{5} \leq -2 \rightarrow$$

$$\left(\frac{-x}{5}\right)\left(\frac{-5}{1}\right) \leq (-2)(-5) \rightarrow x \geq 10$$

This inequality can be solved in just two-steps by:

1. Subtracting 8 from both sides...resulting in:  $\frac{-x}{5} \leq -2$
2. Multiplying both sides by  $-5$ ...resulting in:  $x \geq 10$ . Since we multiplied both sides by a negative, the relationship symbol had to be “flipped” from  $\leq$  to  $\geq$ . (NOTE: Any value that is greater than or equal to 10 would be considered a solution.)



Cynthia | AXA Graduate

**AXA Global Graduate Program**

Find out more and apply

**redefining / standards** 

## TUTORIAL 1.7: GRAPH THE SOLUTION OF MULTI-STEP LINEAR EQUATIONS OR INEQUALITIES IN ONE VARIABLE ON A NUMBER LINE

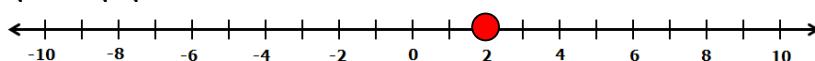
Occasionally, we will need to graph the solutions to a linear equation or inequality on a number line. How we do this depends on the relationship symbol used in the given number sentence. In general, solve each given number sentence so that the variable is on the left of the solution sentence and then use the relationship symbol to determine how the solution graph will look. If the reference value is included within the solution set, we need to use a solid dot and if the reference value is not included in the solution set, we need to use an open dot.

**example 1.7a**      **Graph the solution on a number line:  $45 - 13x = 19$**

$$\begin{aligned} 45 - 13x = 19 &\rightarrow (45 - 45) - 13x = (19 - 45) \rightarrow -13x = -26 \rightarrow \\ \frac{-13x}{-13} &= \frac{-26}{-13} \rightarrow x = 2 \end{aligned}$$

This equation can be solved by:

1. Subtracting 45 from both sides...resulting in:  $-13x = -26$ .
2. Dividing both sides by  $-13$ ...resulting in:  $x = 2$ .
3. Since just  $+2$  is the only number value that will satisfy the given equation, we just need to place a solid dot on  $+2$  as



**example 1.7b****Graph the solution on a number line:  $16x - 55 < -7$** 

$$16x - 55 < -7 \rightarrow 16x + (-55 + 55) < (-7 + 55) \rightarrow$$

$$16x < 48 \rightarrow \frac{16x}{16} < \frac{48}{16} \rightarrow x < 3$$

This inequality can be solved by:

1. Adding 55 to both sides...resulting in:  $16x < 48$ .
2. Dividing both sides by 16...resulting in:  $x < 3$ .
3. Since the solution set for the given inequality is  $x < 3$ , we need to place an open dot on  $+3$  and an arrow going to the left (the same direction that the relationship symbol points) from the open dot as shown below:

**example 1.7c****Graph the solution on a number line:  $7x + 15 > -6$** 

$$7x + 15 > -6 \rightarrow 7x + (15 - 15) > (-6 - 15) \rightarrow$$

$$7x > -21 \rightarrow \frac{7x}{7} > \frac{-21}{7} \rightarrow x > -3$$

This inequality can be solved by:

1. Subtracting 15 from both sides...resulting in:  $7x > -21$ .
2. Dividing both sides by 7...resulting in:  $x > -3$ .
3. Since the solution set for the given inequality is  $x > -3$ , we need to place an open dot on  $-3$  and an arrow going to the right (the same direction that the relationship symbol points) from the open dot as shown below:



**example 1.7d****Graph the solution on a number line:  $11 \leq 2x + 25$** 

$$11 \leq 2x + 25 \rightarrow (11 - 25) \leq 2x + (25 - 25) \rightarrow$$

$$-14 \leq 2x \rightarrow \frac{-14}{2} \leq \frac{2x}{2} \rightarrow -7 \leq x$$

This inequality can be solved by:

1. Subtracting 25 from both sides...resulting in:  $-14 \leq 2x$ .
2. Dividing both sides by 2...resulting in:  $-7 \leq x$ .
3. We need to have the variable on the left-hand side of the solution. So, we need to graph:  $x \geq -7$ ...meaning that we need to place an solid dot on  $-7$  and an arrow going to the right (the same direction that the relationship symbol points) from the solid dot as shown below:



I joined MITAS because  
I wanted **real responsibility**

The Graduate Programme  
for Engineers and Geoscientists  
[www.discovermitas.com](http://www.discovermitas.com)

**Month 16**

I was a construction supervisor in the North Sea advising and helping foremen solve problems

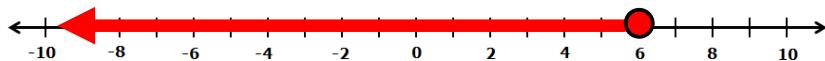
Real work  
International opportunities  
Three work placements

**example 1.7e****Graph the solution on a number line:  $10 - 4x \geq -14$** 

$$\begin{aligned}
 10 - 4x \geq -14 &\rightarrow (10 - 10) - 4x \geq (-14 - 10) \rightarrow \\
 -4x \geq -24 &\rightarrow \frac{-4x}{-4} \geq \frac{-24}{-4} \rightarrow x \leq 6
 \end{aligned}$$

This inequality can be solved by:

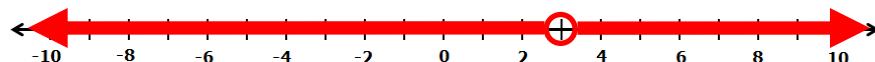
1. Subtracting 10 from both sides...resulting in:  $-4x \geq -24$ .
2. Dividing both sides by  $-4$ ...resulting in:  $x \leq 6$ .
3. Since the solution set for the given inequality is  $x \leq 6$ , we need to place a solid dot on  $+6$  and an arrow going to the left (the same direction that the relationship symbol points) from the solid dot as shown below:

**example 1.7f****Graph the solution on a number line:  $8x - 3 \neq 21$** 

$$8x - 3 \neq 21 \rightarrow 8x + (-3 + 3) \neq (21 + 3) \rightarrow 8x \neq 24 \rightarrow \frac{8x}{8} \geq \frac{24}{8} \rightarrow x \neq 3$$

This inequality can be solved by:

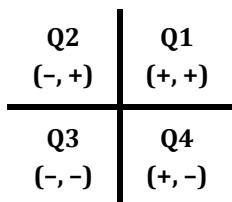
1. Adding 3 to both sides...resulting in:  $8x \neq 24$ .
2. Dividing both sides by 8...resulting in:  $x \neq 3$ .
3. Since the solution set for the given inequality is  $x \neq 3$ , we need to place an open dot on  $+3$  and 2 arrows (one to the left and one to the right) from the open dot as shown below:



## TUTORIAL 1.8: GRAPH A LINE THAT PASSES THROUGH TWO GIVEN POINTS

To graph a line that passes through two given points, we must first plot the given points as described by the ordered pairs. Every ordered pair  $(x, y)$  refers to a specific movement on the coordinate grid (sometimes called a *Cartesian Plane*). The first number in the ordered pair (sometimes called the *abscissa*) indicates a horizontal movement from the *origin*  $(0, 0)$  which is the point at which the two perpendicular number lines that define the coordinate grid intersects each other. The second number in the ordered pair (sometimes called the *ordinate*) indicates a vertical movement from the desired horizontal position.

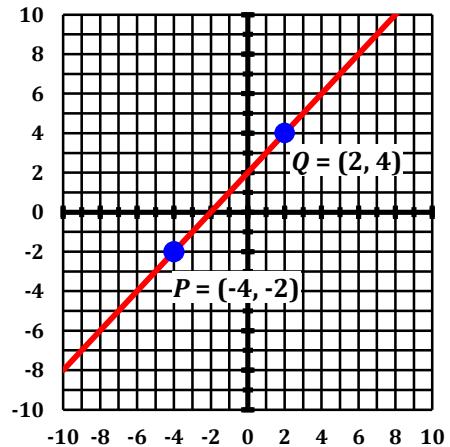
The two perpendicular number lines divide the coordinate plane into four *quadrants* (sections) that are designated as shown in the diagram below:



- Any point in the first quadrant (Q1) has a positive  $x$  and a positive  $y$ .
- Any point in the second quadrant (Q2) has a negative  $x$  and a positive  $y$ .
- Any point in the third quadrant (Q3) has a negative  $x$  and a negative  $y$ .
- Any point in the fourth quadrant (Q4) has a positive  $x$  and a negative  $y$ .
- Any point on the  $x$ -axis (the horizontal number line) has a  $y$  of zero (0).
- Any point on the  $y$ -axis (the vertical number line) has an  $x$  of zero (0).

**example 1.8a****Graph the line that passes through  $P = (-4, -2)$  &  $Q = (2, 4)$** 

1. Plot point  $P = (-4, -2)$  by moving 4 units to the left of the origin  $(0, 0)$  and then 2 units down. Make a mark at this position. (See the blue point located in the third quadrant.)
2. Plot point  $Q = (+2, +4)$  by moving 2 units to the right of the origin  $(0, 0)$  and then 4 units up. Make a mark at this position. (See the blue point located in the first quadrant.)
3. Connect the two plotted points with a nice long straight line that extends to the outer portion of the coordinate grid. (The red line is the desired result that passes through the given points.)



**ie business school**

**93%**  
OF MIM STUDENTS ARE  
WORKING IN THEIR SECTOR 3 MONTHS  
FOLLOWING GRADUATION

**MASTER IN MANAGEMENT**

- STUDY IN THE CENTER OF MADRID AND TAKE ADVANTAGE OF THE UNIQUE OPPORTUNITIES THAT THE CAPITAL OF SPAIN OFFERS
- PROPEL YOUR EDUCATION BY EARNING A DOUBLE DEGREE THAT BEST SUITS YOUR PROFESSIONAL GOALS
- STUDY A SEMESTER ABROAD AND BECOME A GLOBAL CITIZEN WITH THE BEYOND BORDERS EXPERIENCE

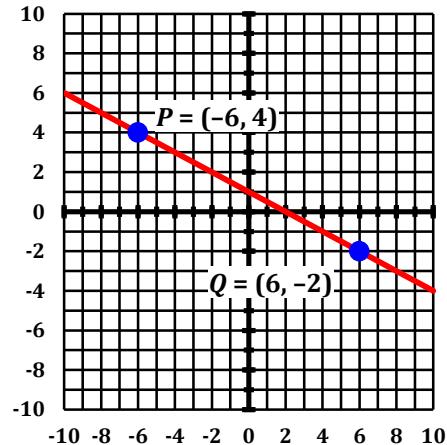
Length: 10 MONTHS  
Av. Experience: 1 YEAR  
Language: ENGLISH / SPANISH  
Format: FULL-TIME  
Intakes: SEPT / FEB

<b>5 SPECIALIZATIONS</b> PERSONALIZE YOUR PROGRAM	<b>#10 WORLDWIDE</b> MASTER IN MANAGEMENT <small>FINANCIAL TIMES</small>	<b>55 NATIONALITIES</b> IN CLASS
--	--	-------------------------------------

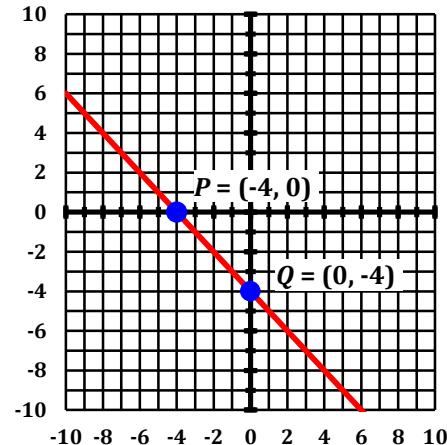
www.ie.edu/master-management | mim.admissions@ie.edu | [Follow us on IE MIM Experience](#)

**example 1.8b      Graph the line that passes through  $P = (-6, 4)$  &  $Q = (6, -2)$** 

1. Plot point  $P = (-6, +4)$  by moving 6 units to the left of the origin  $(0, 0)$  and then 4 units up. Make a mark at this position. (See the blue point located in the second quadrant.)
2. Plot point  $Q = (+6, -2)$  by moving 6 units to the right of the origin  $(0, 0)$  and then 2 units down. Make a mark at this position. (See the blue point located in the fourth quadrant.)
3. Connect the two plotted points with a nice long straight line that extends to the outer portion of the coordinate grid. (The red line is the desired result that passes through the given points.)

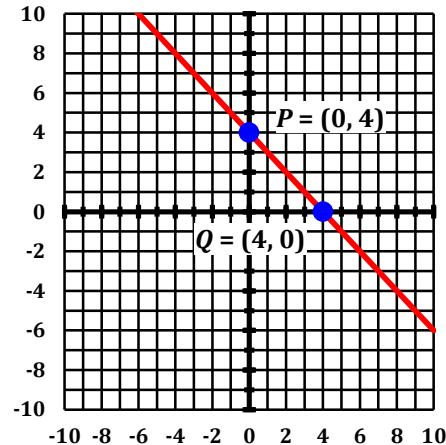
**example 1.8c      Graph the line that passes through  $P = (-4, 0)$  &  $Q = (0, -4)$** 

1. Plot point  $P = (-4, 0)$  by moving 4 units to the left of the origin  $(0, 0)$  but then not making any vertical move. Make a mark at this position. (See the blue point located on the left-hand portion of the horizontal axis.)
2. Plot point  $Q = (0, -4)$  by not moving horizontally from the origin  $(0, 0)$  but then making a vertical move of 4 units down. Make a mark at this position. (See the blue point located on the lower portion of the vertical axis.)
3. Connect the two plotted points with a nice long straight line that extends to the outer portion of the coordinate grid. (The red line is the desired result that passes through the given points.)



**example 1.8d      Graph the line that passes through  $P = (0, 4)$  &  $Q = (4, 0)$** 

1. Plot point  $P = (0, +4)$  by not moving horizontally from the origin  $(0, 0)$  but then making a vertical move of 4 units up. Make a mark at this position. (See the blue point located on the upper portion of the vertical axis.)
2. Plot point  $Q = (+4, 0)$  by moving 4 units to the right of the origin  $(0, 0)$  but then not making any vertical move. Make a mark at this position. (See the blue point located on the right-hand portion of the horizontal axis.)
3. Connect the two plotted points with a nice long straight line that extends to the outer portion of the coordinate grid. (The red line is the desired result that passes through the given points.)

**TUTORIAL 1.9: GRAPH A LINEAR EQUATION USING ITS X- AND Y-INTERCEPTS**

The ***x-intercept*** of a linear equation is the point where its graphed line will cross the horizontal ( $x$ ) axis. All  $x$ -intercepts are similar in that the *ordinates* are zero (0). Thus, the coordinate of an  $x$ -intercept takes the form of:  $(x, 0)$ .

The ***y-intercept*** of a linear equation is the point where its graphed line will cross the vertical ( $y$ ) axis. All  $y$ -intercepts are similar in that the *abscissas* are zero (0). Thus, the coordinate of a  $y$ -intercept takes the form of:  $(0, y)$ .

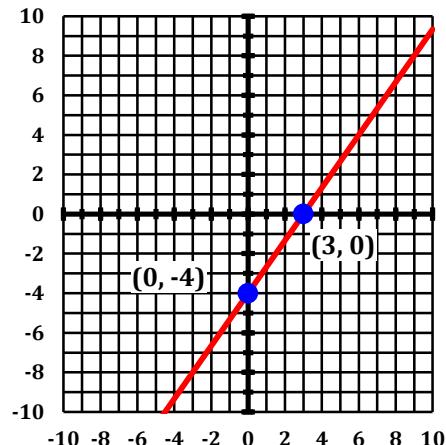
To use the  $x$ - and  $y$ -intercepts to graph a linear equation, determine the ordered pairs (coordinates) for the points of intersection on each of the axes of the coordinate grid. Plot these calculated points. Use these points to draw the desired linear graph.

**example 1.9a****Graph this equation on a coordinate grid:  $4x - 3y = 12$** 

- Determine the  $x$ -intercept by replacing the  $y$  in the given equation with zero (0) and solving the equation for  $x$ :

$$\begin{aligned}4x - 3y &= 12 \rightarrow 4x - 3(0) = 12 \\&\rightarrow 4x = 12 \rightarrow x = 3 \\x\text{-intercept} &= (3, 0)\end{aligned}$$

- Plot the calculated  $x$ -intercept on the coordinate grid by moving 3 units to the right of the origin  $(0, 0)$  but then not making any vertical move. (See the blue point located on the right-hand portion of the horizontal axis.)



**"I studied English for 16 years but...  
...I finally learned to speak it in just six lessons"**

Jane, Chinese architect

ENGLISH OUT THERE

Click to hear me talking  
before and after my unique course download



Download free eBooks at [bookboon.com](http://bookboon.com)

**Click on the ad to read more**

3. Determine the  $y$ -intercept by replacing the  $x$  in the given equation with zero (0) and solving the equation for  $y$ :  

$$4x - 3y = 12 \rightarrow 4(0) - 3y = 12 \rightarrow -3y = 12 \rightarrow y = -4 \rightarrow y\text{-intercept} = (0, -4)$$
4. Plot the calculated  $y$ -intercept on the coordinate grid by not moving horizontally from the origin (0, 0) but then making a vertical move of 4 units down. (See the blue point located on the lower portion of the vertical axis.)
5. Connect the two plotted points with a nice long straight line that extends to the outer portion of the coordinate grid. (The red line is the desired result that passes through the given points.)

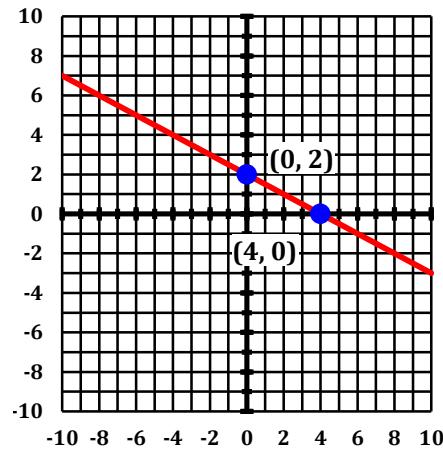
**example 1.9b      Graph this equation on a coordinate grid:  $2x + 4y = 8$**

1. Determine the  $x$ -intercept by replacing the  $y$  in the given equation with zero (0) and solving the equation for  $x$ :

$$\begin{aligned} 2x + 4y = 8 &\rightarrow 2x + 4(0) = 8 \rightarrow \\ 2x = 8 &\rightarrow x = 4 \\ x\text{-intercept} &= (4, 0) \end{aligned}$$

2. Plot the calculated  $x$ -intercept on the coordinate grid by moving 4 units to the right of the origin (0, 0) but then not making any vertical move. (See the blue point located on the right-hand portion of the horizontal axis.)
3. Determine the  $y$ -intercept by replacing the  $x$  in the given equation with zero (0) and solving the equation for  $y$ :  

$$2x + 4y = 8 \rightarrow 2(0) + 4y = 8 \rightarrow 4y = 8 \rightarrow y = 2 \rightarrow y\text{-intercept} = (0, 2)$$
4. Plot the calculated  $y$ -intercept on the coordinate grid by not moving horizontally from the origin (0, 0) but then making a vertical move of 2 units up. (See the blue point located on the upper portion of the vertical axis.)
5. Connect the two plotted points with a nice long straight line that extends to the outer portion of the coordinate grid. (The red line is the desired result that passes through the given points.)



## TUTORIAL 1.10: GRAPH AN EQUATION (IN SLOPE-INTERCEPT FORM) USING ITS SLOPE AND Y-INTERCEPT

Linear equations can be expressed using two different formats:

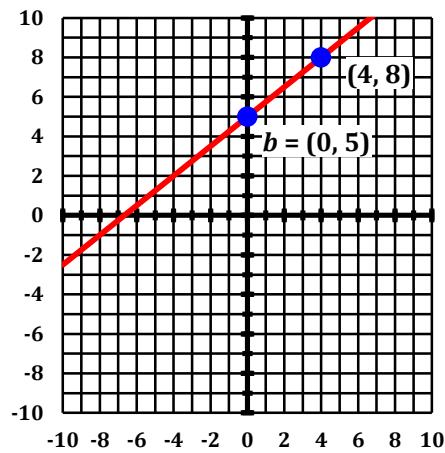
- Slope-Intercept form ( $y = mx + b$ ) where the  $m$  represents the slant (slope; rise over run) and  $b$  represents the  $y$ -intercept of the graphed line.
- Standard form ( $Ax + By = C$ ) where  $A$  represents any non-negative integer while  $B$  and  $C$  represent any integers.

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{vertical}}{\text{horizontal}} = \frac{v}{h}$$

When a linear equation is formatted in its slope-intercept format, the  $b$  value that is being added or subtracted from the  $x$  term can be rewritten into coordinates of:  $(0, b)$ . As done previously, we can use this point to graph the linear equation. The coefficient (value in front) of the  $x$  will aid us in determining a second point to be used in our graphing task. The numerator (top value) of a fractional slope indicates the rise (vertical movement) that the desired line will take while the denominator (bottom value) indicates the run (horizontal movement). Therefore, a second plot point can be calculated by:  $(0 + h, b + v)$

**example 1.10a**      Use the slope and the y-intercept to graph:  $y = \frac{3}{4}x + 5$

1. Plot the  $y$ -intercept  $(0, 5)$  on the coordinate grid by not moving horizontally from the origin  $(0, 0)$  but then making a vertical move of 5 units up. (See the blue point located on the upper portion of the vertical axis.)
2. Use the fractional slope to determine the needed vertical (rise) and horizontal (run) moves off of the  $y$ -intercept. For this example, we will need to move 3 units up and 4 units to the right from the  $y$ -intercept of  $(0, 5)$ . Make a mark at this position:  $(0 + 4, 5 + 3)$  or  $(4, 8)$ . (See the blue point located in the first quadrant.)



3. Connect the  $y$ -intercept point with the point that was determined after the “rise over run” movements (in step #2) with a nice long straight line that extends to the outer portion of the coordinate grid. (The red line is the graph of the given equation that passes through the  $y$ -intercept and the second point determined by the “rise over run” movements.)

Excellent Economics and Business programmes at:

 / **university of  
groningen**





**“The perfect start  
of a successful,  
international career.”**

[www.rug.nl/feb/education](http://www.rug.nl/feb/education)

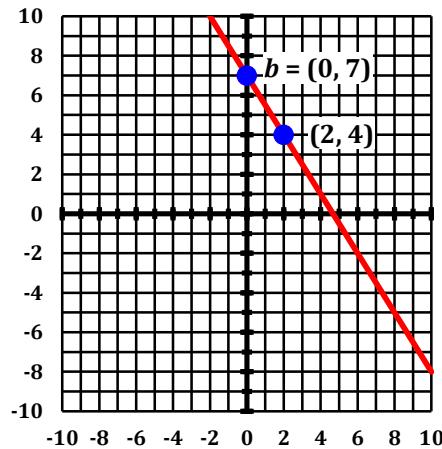
**CLICK HERE**  
to discover why both socially  
and academically the University  
of Groningen is one of the best  
places for a student to be



**example 1.10b**

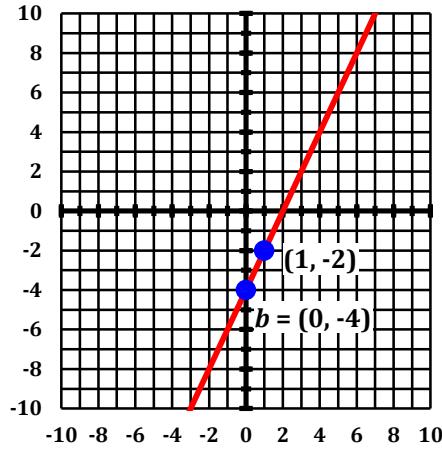
**Use the slope and the y-intercept to graph:**  $y = 7 - \frac{3}{2}x$

1. Plot the  $y$ -intercept  $(0, 7)$  on the coordinate grid by not moving horizontally from the origin  $(0, 0)$  but then making a vertical move of 7 units up. (See the blue point located on the upper portion of the vertical axis.)
2. Use the fractional slope to determine the needed vertical (rise) and horizontal (run) moves off of the  $y$ -intercept. For this example, we will need to move 3 units down and 2 units to the right from the  $y$ -intercept of  $(0, 7)$ . Make a mark at this position:  $(0 + 2, 7 - 3)$  or  $(2, 4)$ . (See the blue point located in the first quadrant.)
3. Connect the  $y$ -intercept point with the point that was determined after the “rise over run” movements (in step #2) with a nice long straight line that extends to the outer portion of the coordinate grid. (The red line is the graph of the given equation that passes through the  $y$ -intercept and the second point determined by the “rise over run” movements.)

**example 1.10c**

**Use the slope and the y-intercept to graph:**  $y = 2x - 4$

1. Plot the  $y$ -intercept  $(0, -4)$  on the coordinate grid by not moving horizontally from the origin  $(0, 0)$  but then making a vertical move of 4 units down. (See the blue point located on the lower portion of the vertical axis.)
2. Use the fractional slope to determine the needed vertical (rise) and horizontal (run) moves off of the  $y$ -intercept. For this example, we will need to move 2 units up and 1 unit to the right from the  $y$ -intercept of  $(0, -4)$ . Make a mark at this position:  $(0 + 1, -4 + 2)$  or  $(1, -2)$ . (See the blue point located in the fourth quadrant.)



3. Connect the  $y$ -intercept point with the point that was determined after the “rise over run” movements (in step #2) with a nice long straight line that extends to the outer portion of the coordinate grid. (The red line is the graph of the given equation that passes through the  $y$ -intercept and the second point determined by the “rise over run” movements.)

## TUTORIAL 1.11: GRAPH A LINEAR EQUATION (IN STANDARD FORM) USING ITS SLOPE AND Y-INTERCEPT

The easiest way to graph a linear equation when it is presented in its standard form ( $Ax + By = C$ ) is to use its  $x$ - and  $y$ -intercepts as we did in a previous tutorial. However, there may be times when the  $x$ -intercept is not easily plotted. Therefore, another way to graph a standard form equation is to convert it into its slope-intercept form ( $y = mx + b$ ).

### example 1.11a Use the slope and the $y$ -intercept to graph: $4x + 3y = 12$

Using the axis intercepts for this equation, we can see that the  $x$ -intercept is  $(3, 0)$  while the  $y$ -intercept is  $(0, 4)$ . So, this equation can be easily graphed using these points. However, let's use the slope-intercept conversion so that we can compare methods. To rewrite the given standard form equation into its slope-intercept format:

1. Subtract  $4x$  from both sides of the equation:  $4x + 3y = 12 \rightarrow 3y = 12 - 4x$
2. Divide both sides of the equation by 3:

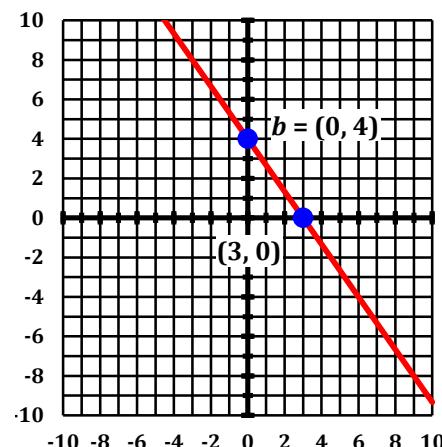
$$3y = 12 - 4x \rightarrow y = \left(\frac{12}{3}\right) - \left(\frac{4}{3}\right)(x) = 4 - \left(\frac{4}{3}\right)(x)$$

3. Reformat the equation into proper slope-intercept format:

$$y = 4 - \left(\frac{4}{3}\right)x \rightarrow y = \left(-\frac{4}{3}\right)x + 4$$

We can see from our result of step #3 that the  $y$ -intercept of this given equation is  $(0, 4)$  and the fractional slope is  $\frac{-4}{3}$ . Now, to graph the given equation using its slope-intercept form.

4. Plot the  $y$ -intercept  $(0, 4)$  on the coordinate grid by not moving horizontally from the origin  $(0, 0)$  but then making a vertical move of 4 units up. (See the blue point located on the upper portion of the vertical axis.)



5. Use the fractional slope to determine the needed vertical (rise) and horizontal (run) moves off of the  $y$ -intercept. For this example, we will need to move 4 units down and 3 units to the right from the  $y$ -intercept of  $(0, 4)$ . Make a mark at this position:  $(0 + 3, 4 - 4)$  or  $(3, 0)$ .
6. Connect the  $y$ -intercept point with the point that was determined after the “rise over run” movements (in step #5) with a nice long straight line that extends to the outer portion of the coordinate grid. (The red line is the graph of the given equation that passes through the  $y$ -intercept and the second point determined by the “rise over run” movements.)

Although completing the standard form to slope-intercept form conversion steps is not that difficult for just one problem, it becomes time-consuming if we are attempting to graph several standard form equations. Therefore, let's see if we can derive an algebraic short-cut.

American online  
**LIGS University**  
 is currently enrolling in the  
 Interactive Online **BBA, MBA, MSc,**  
**DBA and PhD** programs:

- ▶ **enroll by September 30th, 2014** and
- ▶ **save up to 16%** on the tuition!
- ▶ pay in 10 installments / 2 years
- ▶ Interactive Online education
- ▶ visit [www.ligsuniversity.com](http://www.ligsuniversity.com) to find out more!

Note: LIGS University is not accredited by any nationally recognized accrediting agency listed by the US Secretary of Education.  
[More info here.](#)



$$\begin{array}{rcl}
 Ax + By = C \\
 -Ax \\
 \hline
 By = C - Ax \\
 \frac{By}{B} = \frac{C - Ax}{B} = \frac{C}{B} - \frac{Ax}{B} \\
 y = \frac{-Ax}{B} + \frac{C}{B}
 \end{array}$$

**given: standard form**

1. Subtract  $Ax$  from both sides of the equation.
2. Divide both sides of the equation by  $B$ .
3. Reformat the equation into proper slope-intercept format...result: **slope-intercept format**

**For every linear equation expressed in its standard form ( $Ax + By = C$ ):**

**The slope of the given equation is:  $\frac{-A}{B}$  while the  $y$ -intercept of the given equation is:  $\frac{C}{B}$**

### example 1.11b Use the slope and the $y$ -intercept to graph: $2x - 4y = 8$

1. From the given equation we can see that:

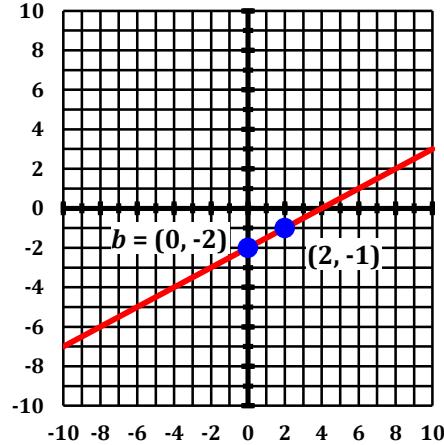
$$C = 8, B = -4 \rightarrow y\text{-intercept} = \frac{C}{B} = \frac{8}{-4} = -2$$

2. Plot the  $y$ -intercept  $(0, -2)$  on the coordinate grid by not moving horizontally from the origin  $(0, 0)$  but then making a vertical move of 2 units down. (See the blue point located on the lower portion of the vertical axis.)

3. From the given equation we can see that:  $A = 2, B = -4$ . Thus, the slope of this line will be:  $\frac{-A}{B} = \frac{-2}{-4} = \frac{1}{2}$

4. Use the fractional slope to determine the needed vertical (rise) and horizontal (run) moves off of the  $y$ -intercept. For this example, we will need to move 1 unit up and 2 units to the right from the  $y$ -intercept of  $(0, -2)$ . Make a mark at this position:  $(0 + 2, -2 + 1)$  or  $(2, -1)$ .

5. Connect the  $y$ -intercept point with the point that was determined after the "rise over run" movements (in step #4) with a nice long straight line that extends to the outer portion of the coordinate grid. (The red line is the graph of the given equation that passes through the  $y$ -intercept and the second point determined by the "rise over run" movements.)



**example 1.11c      Use the slope and the y-intercept to graph:  $3y - 4x = 12$** 

1. From the given equation we can see that:

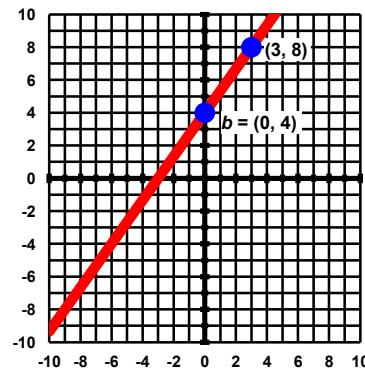
$$C = 12, B = 3 \rightarrow y\text{-intercept} = \frac{C}{B} = \frac{12}{3} = 4$$

2. Plot the  $y$ -intercept  $(0, 4)$  on the coordinate grid by not moving horizontally from the origin  $(0, 0)$  but then making a vertical move of 4 units up. (See the blue point located on the upper portion of the vertical axis.)

3. From the given equation we can see that:

$A = -4, B = 3$ . Thus, the slope of this line

$$\text{will be: } \frac{-A}{B} = \frac{-(-4)}{3} = \frac{4}{3}$$



4. Use the fractional slope to determine the needed vertical (rise) and horizontal (run) moves off of the  $y$ -intercept. For this example, we will need to move 4 units up and 3 units to the right from the  $y$ -intercept of  $(0, 4)$ . Make a mark at this position:  $(0 + 3, 4 + 4)$  or  $(3, 8)$ .
5. Connect the  $y$ -intercept point with the point that was determined after the “rise over run” movements (in step #4) with a nice long straight line that extends to the outer portion of the coordinate grid. (The red line is the graph of the given equation that passes through the  $y$ -intercept and the second point determined by the “rise over run” movements.)

**TUTORIAL 1.12: DETERMINE THE SLOPE OF A LINE PASSING THROUGH 2 GIVEN POINTS**

We've seen how to use the slope to graph a linear equation...but...let's take a deeper look into the concept of the slope or slant of a line. A line graphed on a coordinate grid can have one of four possible slants (slope) as seen in the diagrams below:

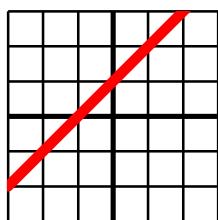


diagram #1  
positive slope

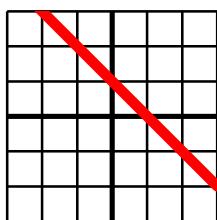


diagram #2  
negative slope

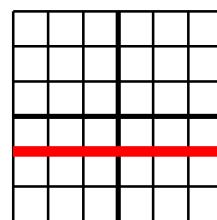


diagram #3  
zero slope

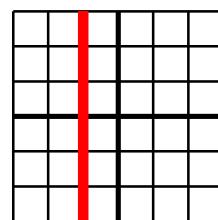


diagram #4  
undefined slope

- It can be higher on its right side than its left and will then have a ***positive*** slope (diagram #1).
- It can be lower on its right side than its left and will then have a ***negative*** slope (diagram #2).
- It can run horizontally (where the right and left sides are on the same level and the graphed line is parallel to the  $x$ -axis) and will then have a slope of ***zero*** (diagram #3).
- It can run vertically (where the top and the bottom portions are on the same level and the graphed line is parallel to the  $y$ -axis) and will then have a slope that is ***undefined*** (diagram #4).

The slant of any graphed line is determined by the  $(x, y)$  coordinates that are either given or are derived by a companion linear equation. So, let's take a look at the coordinate sets that might produce each of the diagrams above.

**DON'T EAT YELLOW SNOW**

What will your advice be?

Some advice just states the obvious. But to give the kind of advice that's going to make a real difference to your clients you've got to listen critically, dig beneath the surface, challenge assumptions and be credible and confident enough to make suggestions right from day one. At Grant Thornton you've got to be ready to kick start a career right at the heart of business.

Sound like you? Here's our advice: visit [GrantThornton.ca/careers/students](http://GrantThornton.ca/careers/students)

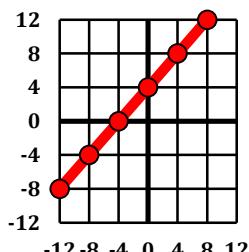
Scan here to learn more about a career with Grant Thornton.

 **Grant Thornton**  
An instinct for growth™

© Grant Thornton LLP. A Canadian Member of Grant Thornton International Ltd



For diagram #1:



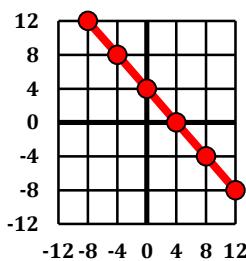
$x$	$y$
8	12
4	8
0	4
-4	0
-8	-4
-12	-8

$4 - 8 = -4$   
 $0 - 4 = -4$   
 $-4 - 0 = -4$   
 $-8 - (-4) = -4$   
 $-12 - (-8) = -4$

Notice that for diagram #1 each successive *ordinate* ( $y$ ) has a common difference of  $-4$  and that each successive *abscissa* ( $x$ ) has a common difference of  $-4$ . To go from point to point we would need to “rise”  $-4$  units (which means go down 4 units) and then “run”  $-4$  units (which means go to the left). Thus, the slope (slant) for this line:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{-4}{-4} = +1$$

For diagram #2:



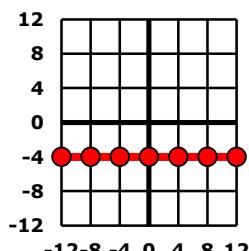
$x$	$y$
-8	12
-4	8
0	4
4	0
8	-4
12	-8

$-4 - (-8) = 4$   
 $0 - (-4) = 4$   
 $4 - 0 = 4$   
 $8 - 4 = 4$   
 $12 - 8 = 4$

Notice that for diagram #2 each successive *ordinate* ( $y$ ) has a common difference of  $-4$  and that each successive *abscissa* ( $x$ ) has a common difference of  $+4$ . To go from point to point we would need to “rise”  $-4$  units (which means go down 4 units) and then “run” 4 units to the right. Thus:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{-4}{4} = -1$$

For diagram #3:



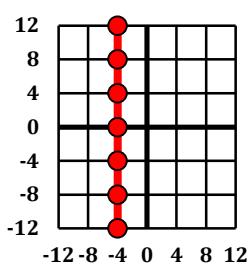
<i>x</i>	<i>y</i>
-12	-4
-8	-4
-4	-4
0	-4
4	-4
8	-4

$-8 - (-12) = 4$   
 $-4 - (-8) = 4$   
 $0 - (-4) = 4$   
 $4 - 0 = 4$   
 $8 - 4 = 4$

Notice that each successive *ordinate* (*y*) has a common difference of zero (0) and that each successive *abscissa* (*x*) has a common difference of +4. To go from point to point we would need to “rise” zero (0) units (which means not to rise at all) and then “run” 4 units to the right. Thus, the line is completely horizontal and its slope is:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{0}{+4} = \text{zero}$$

For diagram #4:



<i>x</i>	<i>y</i>
-4	12
-4	8
-4	4
-4	0
-4	-4
-4	-8

$-4 - (-4) = 0$   
 $-4 - (-4) = 0$   
 $-4 - (-4) = 0$   
 $-4 - (-4) = 0$   
 $-4 - (-4) = 0$

Notice that for diagram #4 each successive *ordinate* (*y*) has a common difference of +4 and that each successive *abscissa* (*x*) has a common difference of zero (0). To go from point to point we would need to “rise” +4 units (which means go up 4 units) and then “run” zero (0) units (which means not to run at all). Thus, the line is completely vertical and its slope is:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{+4}{0} = \text{undefined}$$

As we can see from the four sets of coordinates that produced the graphed lines in diagrams #1–4:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{difference of ordinates}}{\text{difference of abscissas}}$$

Further, it makes no difference which two ordered pairs are used or in what order these ordered pairs are considered...the ratio of “rise over run” will remain the same. For example, suppose we take any two coordinates listed for each line:

<i>diagram #1: (8, 12) &amp; (-12, -8)</i>	<i>diagram #2: (-4, 8) &amp; (4, 0)</i>	<i>diagram #3: (8, -4) &amp; (-4, -4)</i>	<i>diagram #4: (-4, 0) &amp; (-4, -8)</i>
$\frac{\text{rise}}{\text{run}} = \frac{12 - (-8)}{8 - (-12)} = \frac{20}{20} = +1$ -or- $\frac{\text{rise}}{\text{run}} = \frac{-8 - 12}{-12 - 8} = \frac{-20}{-20} = +1$	$\frac{\text{rise}}{\text{run}} = \frac{8 - 0}{-4 - 4} = \frac{8}{-8} = -1$ -or- $\frac{\text{rise}}{\text{run}} = \frac{0 - 8}{4 - (-4)} = \frac{-8}{8} = -1$	$\frac{\text{rise}}{\text{run}} = \frac{-4 - (-4)}{8 - (-4)} = \frac{0}{12} = 0$ -or- $\frac{\text{rise}}{\text{run}} = \frac{-4 - (-4)}{-4 - 8} = \frac{0}{-12} = 0$	$\frac{\text{rise}}{\text{run}} = \frac{0 - (-8)}{-4 - (-4)} = \frac{8}{0}$ -or- $\frac{\text{rise}}{\text{run}} = \frac{-8 - 0}{-4 - (-4)} = \frac{-8}{0}$
<b>slope = +1</b>	<b>slope = -1</b>	<b>slope = 0</b>	<b>slope = undefined</b>

..... Alcatel-Lucent   
[www.alcatel-lucent.com/careers](http://www.alcatel-lucent.com/careers)



**What if  
you could  
build your  
future and  
create the  
future?**

One generation's transformation is the next's status quo.  
In the near future, people may soon think it's strange that  
devices ever had to be "plugged in." To obtain that status, there  
needs to be "The Shift".

So, we can generalize the process of determining the slope of any line given two points by:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{difference of ordinates}}{\text{difference of abscissas}} = \frac{y_1 - y_2}{x_1 - x_2} \text{ or } \frac{y_2 - y_1}{x_2 - x_1}$$

**example 1.12a** Find the slope through:  $P = (-3, -2)$ ,  $Q = (2, 3)$

$$\frac{\text{rise}}{\text{run}} = \frac{-2 - 3}{-3 - 2} = \frac{-5}{-5} = +1 \dots \text{OR} \dots \frac{\text{rise}}{\text{run}} = \frac{3 - (-2)}{2 - (-3)} = \frac{5}{5} = +1 \rightarrow \text{slope} = +1$$

**example 1.12b** Find the slope through:  $P = (-5, 4)$ ,  $Q = (5, -2)$

$$\frac{\text{rise}}{\text{run}} = \frac{4 - (-2)}{-5 - 5} = \frac{6}{-10} \dots \text{OR} \dots \frac{\text{rise}}{\text{run}} = \frac{-2 - 4}{5 - (-5)} = \frac{-6}{10} \rightarrow \text{slope} = \frac{-6}{10} = \frac{-3}{5}$$

**example 1.12c** Find the slope through:  $P = (-3, 4)$ ,  $Q = (-3, -4)$

$$\frac{\text{rise}}{\text{run}} = \frac{4 - (-4)}{-3 - (-3)} = \frac{8}{0} \dots \text{OR} \dots \frac{\text{rise}}{\text{run}} = \frac{-4 - 4}{-3 - (-3)} = \frac{-8}{0} \rightarrow \text{slope} = \text{undefined}$$

**example 1.12d** Find the slope through:  $P = (3, -4)$ ,  $Q = (-3, -4)$

$$\frac{\text{rise}}{\text{run}} = \frac{-4 - (-4)}{3 - (-3)} = \frac{0}{6} = 0 \dots \text{OR} \dots \frac{\text{rise}}{\text{run}} = \frac{-4 - (-4)}{-3 - 3} = \frac{0}{-6} = 0 \rightarrow \text{slope} = \text{zero (0)}$$

## TUTORIAL 1.13: USE THE SLOPE-INTERCEPT FORMULA TO WRITE THE EQUATION OF A LINE WITH A GIVEN SLOPE AND POINT

If we know the slope and the  $y$ -intercept of a linear equation, we can easily write the slope-intercept ( $y = mx + b$ ) form of the desired equation...

**example 1.13a** Write the slope-intercept equation for the line that has the given slope and passes through the given point: slope =  $-2$ ,  $P = (0, 3)$   
 $slope = -2 \rightarrow m = -2 \dots P = (0, 3) \rightarrow P = (0, b) \rightarrow b = 3$

Since we know the slope and the  $y$ -intercept, all we need to do is to substitute these values into the slope-intercept ( $y = mx + b$ ) form of the equation:  $y = mx + b \rightarrow y = -2x + 3$  **equation:  $y = -2x + 3$**

**example 1.13b** Write the slope-intercept equation for the line that has the given slope and passes through the given point: slope =  $\frac{3}{4}$ ,  $P = (0, -5)$   
 $slope = \frac{3}{4} \rightarrow m = \frac{3}{4} \dots P = (0, -5) \rightarrow P = (0, b) \rightarrow b = -5$

Since we know the slope and the  $y$ -intercept, all we need to do is to substitute these values into the slope-intercept ( $y = mx + b$ ) form of the equation:  $y = mx + b \rightarrow y = \frac{3}{4}x + -5 \rightarrow$  **equation:  $y = \frac{3}{4}x - 5$**

If we know the slope and a point that is not a  $y$ -intercept, we can still use the slope-intercept ( $y = mx + b$ ) format to write the desired equation...

**example 1.13c** Write the slope -intercept equation for the line that has the given slope and passes through the given point: slope =  $\frac{2}{3}$ ,  $P = (3, 2)$

Since we know the slope, we know that  $m = \frac{2}{3}$ ...and, since we know a point through which the graphed line will pass, we know  $(x, y)$  and that  $x = 3$  when  $y = 2$ . All we need to do is substitute these values into the slope-intercept ( $y = mx + b$ ) form of the equation to solve for the value of  $b$ :  $y = mx + b \rightarrow 2 = (\frac{2}{3})(3) + b \rightarrow b = 2 - (\frac{2}{3})(3) = 2 - 2 = 0$

With the value of  $b = 0$  and the value of  $m = \frac{2}{3}$ , we can write the slope-intercept form of the equation:  $y = \frac{2}{3}x + 0 \rightarrow$  **equation:  $y = \frac{2}{3}x$**

**example 1.13d**

**Write a slope-intercept equation for the line that has the given slope and passes through the given point: slope = 0, P = (1, 2)**

Since we know the slope, we know that  $m = 0$ ...and, since we know a point through which the graphed line will pass, we know  $(x, y)$  and that  $x = 1$  when  $y = 2$ . All we need to do is substitute these values into the slope-intercept ( $y = mx + b$ ) form of the equation to solve for the value of  $b$ :

$$y = mx + b \rightarrow 2 = (0)(1) + b \rightarrow b = 2 - (0)(1) = 2 - 0 = 2$$

With the value of  $b = 2$  and the value of  $m = 0$ , we can write the slope-intercept form of the equation:  $y = 0x + 2 \rightarrow$  **equation:  $y = 2$**  (NOTE: Since the slope was zero, the graphed line will be horizontal. Any horizontal line graphed on a coordinate grid will have the general equation of:  $y = b$ .)



**Maastricht University** *Leading in Learning!*

**Join the best at  
the Maastricht University  
School of Business and  
Economics!**



**Top master's programmes**

- 33<sup>rd</sup> place Financial Times worldwide ranking: MSc International Business
- 1<sup>st</sup> place: MSc International Business
- 1<sup>st</sup> place: MSc Financial Economics
- 2<sup>nd</sup> place: MSc Management of Learning
- 2<sup>nd</sup> place: MSc Economics
- 2<sup>nd</sup> place: MSc Econometrics and Operations Research
- 2<sup>nd</sup> place: MSc Global Supply Chain Management and Change

Sources: Keuzegids Master ranking 2013; Elsevier 'Beste Studies' ranking 2012; Financial Times Global Masters in Management ranking 2012

**Maastricht  
University is  
the best specialist  
university in the  
Netherlands  
(Elsevier)**

**Visit us and find out why we are the best!  
Master's Open Day: 22 February 2014**

[www.mastersopenday.nl](http://www.mastersopenday.nl)



**Click on the ad to read more**

**example 1.13e**

**Write a slope-intercept equation for the line that has the given slope and passes through the given point: slope = undefined,  $P = (-3, -1)$**

When the slope of the line is undefined, we cannot really use the slope-intercept formula to determine the desired equation. Since we know that the slope is undefined, we know that the difference of the abscissas is zero (0)...which means that the  $y$ -values of all graphed points are identical. And, since the slope is undefined, the graphed line will be vertical. Any vertical line graphed on a coordinate grid will have the general equation of:  $x = N$  (which is the given abscissa value). So, for this particular example, the equation is:  $\textcolor{red}{x = -3}$

### **TUTORIAL 1.14: USE THE POINT-SLOPE FORMULA TO WRITE THE SLOPE-INTERCEPT FORM OF AN EQUATION OF A LINE WITH A GIVEN SLOPE AND POINT**

Another way to obtain the slope-intercept ( $y = mx + b$ ) form of an equation is to work with the ***point-slope formula*** which is derived from the slope formula:

$$\text{slope} = m = \frac{y_2 - y_1}{x_2 - x_1} \rightarrow \text{point-slope format: } y_2 - y_1 = m(x_2 - x_1)$$

**example 1.14a**

**Write a slope-intercept equation for the line that has the given slope and passes through the given point: slope =  $-2$ ,  $P = (0, 3)$**

Since we know the slope, we know that  $m = -2$ ...and, since we know a point through which the graphed line will pass, we know  $(x, y)$  and that  $x_1 = 0$  when  $y_1 = 3$ . All we need to do is substitute these values into the point-slope formula and make sure that the final equation is in its slope-intercept format:  $y - \textcolor{blue}{y1} = \textcolor{blue}{m}(x - \textcolor{blue}{x1}) \rightarrow y - 3 = -2(x - 0) \rightarrow y = -2(x - 0) + 3 = -2x + 0 + 3 = -2x + 3 \rightarrow \text{equation: } \textcolor{red}{y = -2x + 3}$

**example 1.14b**

**Write a slope-intercept equation for the line that has the given slope and passes through the given point: slope =  $\frac{3}{4}$ ,  $P = (0, -5)$**

Since we know the slope, we know that  $m = \frac{3}{4}$ ...and, since we know a point through which the graphed line will pass, we know  $(x, y)$  and that  $x_1 = 0$  when  $y_1 = -5$ . All we need to do is substitute these values into the point-slope formula and make sure that the final equation is in its slope-intercept format:  $y - y_1 = m(x - x_1) \rightarrow y - (-5) = \frac{3}{4}(x - 0) \rightarrow$

$$y + 5 = \frac{3}{4}(x - 0) \rightarrow y = \frac{3}{4}x + 0 - 5 \rightarrow \text{equation: } y = \frac{3}{4}x - 5$$

**example 1.14c**

**Write a slope-intercept equation for the line that has the given slope and passes through the given point: slope =  $\frac{2}{3}$ ,  $P = (3, 2)$**

Since we know the slope, we know that  $m = \frac{2}{3}$ ...and, since we know a point through which the graphed line will pass, we know  $(x, y)$  and that  $x_1 = 3$  when  $y_1 = 2$ . All we need to do is substitute these values into the point-slope formula and make sure that the final equation is in its slope-intercept format:  $y - y_1 = m(x - x_1) \rightarrow y - 2 = \frac{2}{3}(x - 3) \rightarrow$

$$y = \frac{2}{3}(x - 3) + 2 \rightarrow y = \frac{2}{3}x - 2 + 2 \rightarrow \text{equation: } y = \frac{2}{3}x$$

**example 1.14d**

**Write a slope-intercept equation for the line that has the given slope and passes through the given point: slope = 0,  $P = (1, 2)$**

Since we know the slope, we know that  $m = 0$ ...and, since we know a point through which the graphed line will pass, we know  $(x, y)$  and that  $x_1 = 1$  when  $y_1 = 2$ . All we need to do is substitute these values into the point-slope formula and make sure that the final equation is in its slope-intercept format:  $y - y_1 = m(x - x_1) \rightarrow y - 2 = 0(x - 1) \rightarrow$

$$y = 0(x - 1) + 2 \rightarrow y = 0x - 0 + 2 \rightarrow \text{equation: } y = 2$$

**example 1.14e**

**Write a slope-intercept equation for the line that has the given slope and passes through the given point: slope = undefined,  $P = (-3, -1)$**

When the slope of the line is undefined, we cannot really use the point-slope formula to determine the desired equation. Since we know that the slope is undefined, we know that the difference of the abscissas is zero (0)...which means that the  $y$ -values of all graphed points are identical. And, since the slope is undefined, the graphed line will be vertical. Any vertical line graphed on a coordinate grid will have the general equation of:  $x = N$  (which is the given abscissa value). So, for this particular example, the equation is:  $x = -3$

## TUTORIAL 1.15: WRITE THE STANDARD FORM OF AN EQUATION OF A LINE WITH A GIVEN SLOPE AND POINT

Although the slope-intercept ( $y = mx + b$ ) form is used most often when using graphing technology, there may be times when we want to express the desired equation in its standard ( $Ax + By = C$ ) form. We can accomplish this desired format in one of three ways:

- Reformat the slope-intercept ( $y = mx + b$ ) form into standard ( $Ax + By = C$ ) form.
- Reformat the point-slope ( $y_2 - y_1 = m(x_2 - x_1)$ ) form into standard ( $Ax + By = C$ ) form.
- Use the slope and some arithmetic to determine the standard ( $Ax + By = C$ ) form.

**Empowering People. Improving Business.**

BI Norwegian Business School is one of Europe's largest business schools welcoming more than 20,000 students. Our programmes provide a stimulating and multi-cultural learning environment with an international outlook ultimately providing students with professional skills to meet the increasing needs of businesses.

BI offers four different two-year, full-time Master of Science (MSc) programmes that are taught entirely in English and have been designed to provide professional skills to meet the increasing need of businesses. The MSc programmes provide a stimulating and multi-cultural learning environment to give you the best platform to launch into your career.

- MSc in Business
- MSc in Financial Economics
- MSc in Strategic Marketing Management
- MSc in Leadership and Organisational Psychology

[www.bi.edu/master](http://www.bi.edu/master)

**BI NORWEGIAN BUSINESS SCHOOL**

EFMD  
**EQUIS**  
ACCREDITED

- example 1.15a** Write a standard form equation for the line that has the given slope and passes through the given point: slope = -2, P = (0, 3)
- method #1: reformatting the slope-intercept form**

Since all of the values in the slope-intercept ( $y = mx + b$ ) form of the equation are integers, just rearrange the terms so that the  $x$  and  $y$  terms are on the left-hand side of the equation while the non-variable term is on the right-hand side:

$$\text{slope} = -2 \text{ and } P = (0, 3) \rightarrow \text{slope-intercept: } y = -2x + 3$$

Adding  $2x$  to both sides of the equation, we get:  **$2x + y = 3$**

**method #2: reformatting the point-slope form**

Use the point-slope format to structure an equation:

$$y - y_1 = m(x - x_1) \rightarrow y - 3 = -2(x - 0) \rightarrow$$

$$y = -2(x - 0) + 3 = -2x + 0 + 3 = -2x + 3 \rightarrow y = -2x + 3$$

Reformat the equation by adding  $2x$  to both sides of the equation:

$$\mathbf{2x + y = 3}$$

**method #3: using the slope**

Since the slope of any linear equation is  $-A/B$  and the slope for this example is -2, we know that  $A = 2$  when  $B = 1$ . Since the given point is (0, 3), we know that  $x = 0$  when  $y = 3$ . Using all of this information, we can determine the value of  $C$  in the standard ( $Ax + By = C$ ) form of the desired equation:

$$(A)(x) + (B)(y) = C \rightarrow C = (2)(0) + (1)(3) = 0 + 3 = 3$$

If  $A = 2$ ,  $B = 1$  and  $C = 3$ , then the standard form equation is:  **$2x + y = 3$**

**example 1.15b** Write a standard form equation for the line that has the given slope and passes through the given point: slope =  $\frac{3}{4}$ ,  $P = (0, -5)$

***method #1: reformatting the slope-intercept form***

slope =  $\frac{3}{4}$  and  $P = (0, -5) \rightarrow$  slope-intercept:  $y = \frac{3}{4}x - 5$

Since the slope value in the slope-intercept ( $y = mx + b$ ) form of the equation is not an integer, we will need to multiply the entire equation through by the denominator of the slope...for this example, that would be 4. After multiplying both sides of the equation by 4, we get the transformed equation:  $4y = 3x - 20$ . We can now rearrange the terms so that the  $x$  and  $y$  terms are on the left-hand side of the equation while the non-variable term is on the right-hand side. Subtracting  $3x$  from both sides of the transformed equation, we get:  $-3x + 4y = -20$ . However, recall that in standard form, the value of  $A$  should be positive...therefore, we need to multiply the entire equation by  $-1$ ... which results in the standard form equation:  **$3x - 4y = 20$**

***method #2: reformatting the point-slope form***

Use the point-slope format to structure an equation:

$$\begin{aligned} y - y_1 &= m(x - x_1) \rightarrow y - (-5) = \left(\frac{3}{4}\right)(x - 0) \rightarrow \\ y + 5 &= \left(\frac{3}{4}\right)(x - 0) \rightarrow y = \left(\frac{3}{4}\right)(x - 0) - 5 \rightarrow y = \left(\frac{3}{4}\right)(x) - 5 \end{aligned}$$

Multiply both sides of the equation by the denominator (4):  $4y = 3x - 20$

Subtract  $4y$  from both sides of the equation:  $0 = 3x - 4y - 20$

Add 20 to both sides of the equation: **equation:  $3x - 4y = 20$**

***method #3: using the slope***

Since the slope of any linear equation is  $-A/B$  and the slope for this example is  $\frac{3}{4}$ , we know that  $A = 3$  when  $B = -4$ . (NOTE: Since the  $A$  value needs to be positive, make it positive now rather than later.) Since the given point is  $(0, -5)$ , we know that  $x = 0$  when  $y = -5$ . Using all of this information, we can determine the value of  $C$  in the standard ( $Ax + By = C$ ) form of the desired equation:

$$(A)(x) + (B)(y) = C \rightarrow C = (3)(0) + (-4)(-5) = 0 + 20 = 20$$

If  $A = 3$ ,  $B = -4$  and  $C = 20$ , then the standard form equation is:

$$\boxed{3x - 4y = 20}$$

**example 1.15c** Write a standard form equation for the line that has the given slope and passes through the given point: slope =  $\frac{2}{3}$ ,  $P = (3, 2)$

*method #1: reformatting the slope-intercept form*

$$\text{slope} = \frac{2}{3} \text{ and } P = (3, 2) \rightarrow y = (\frac{2}{3})x + b \rightarrow$$

$$2 = (\frac{2}{3})(3) + b \rightarrow b = 2 - 2 = 0$$

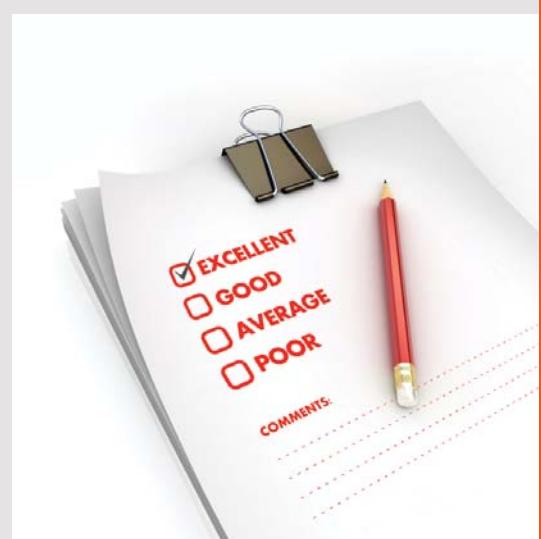
$$\text{slope-intercept form: } y = \frac{2}{3}x$$

Since the slope value in the slope-intercept ( $y = mx + b$ ) form of the equation is not an integer, we will need to multiply the entire equation through by the denominator of the slope...for this example, that would be 3. After multiplying both sides of the equation by 3, we get the transformed equation:  $3y = 2x$ . We can now rearrange the terms so that the  $x$  and  $y$  terms are on the left-hand side of the equation while the non-variable term is on the right-hand side. Since we want  $2x$  to be positive, subtract  $3x$  from both sides of the transformed equation, to get:  $0 = 2x - 3y$  (which can also be written as: **2x – 3y = 0**).

## Need help with your dissertation?

Get in-depth feedback & advice from experts in your topic area. Find out what you can do to improve the quality of your dissertation!

**Get Help Now**



Go to [www.helpmyassignment.co.uk](http://www.helpmyassignment.co.uk) for more info



**Helpmyassignment**

***method #2: reformatting the point-slope form***

Use the point-slope format to structure an equation:

$$\begin{aligned}y - y_1 &= m(x - x_1) \rightarrow y - 2 = (\frac{2}{3})(x - 3) \rightarrow \\y &= (\frac{2}{3})(x - 3) + 2 \rightarrow y = (\frac{2}{3})(x) - 2 + 2 \rightarrow y = (\frac{2}{3})(x) + 0 \rightarrow y = \frac{2}{3}x\end{aligned}$$

Multiply both sides of the equation by the denominator (3):  $3y = 2x$   
 Subtract  $3y$  from both sides of the equation:  $0 = 2x - 3y$  (which can also be written as:  **$2x - 3y = 0$** .)

***method #3: using the slope***

Since the slope of any linear equation is  $-A/B$  and the slope for this example is  $\frac{2}{3}$ , we know that  $A = 2$  when  $B = -3$ . Since the given point is  $(3, 2)$ , we know that  $x = 3$  when  $y = 2$ . Using all of this information, we can determine the value of  $C$  in the standard  $(Ax + By = C)$  form of the desired equation:

$$(A)(x) + (B)(y) = C \rightarrow C = (2)(3) + (-3)(2) = 6 - 6 = 0$$

If  $A = 2$ ,  $B = -3$  and  $C = 0$ , then the standard form equation is:

$$\mathbf{2x - 3y = 0}$$

### **TUTORIAL 1.16: WRITE AN EQUATION OF A LINE THAT PASSES THROUGH TWO GIVEN POINTS**

When we are attempting to write an equation for the line that passes through two given points, we must first calculate the slope (as we did in a previous tutorial) between the given points.

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{difference of ordinates}}{\text{difference of abscissas}} = \frac{y_1 - y_2}{x_1 - x_2} \text{ or } \frac{y_2 - y_1}{x_2 - x_1}$$

Then, we will use the calculated slope and one of the given points to determine the desired equation. Since we have two given points, we can choose either of them to accomplish the task...it makes no difference as to which point we choose. Depending on which format we would like to write the desired equation, we will follow the steps as outlined in previous tutorials.

**example 1.16a      Write an equation that passes through:  $P = (5, 5)$ ,  $Q = (15, -5)$** 

Just follow these basic steps:

- Determine the slope between the given points:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5 - 5}{15 - 5} = \frac{-10}{10} = -1$$

- Choose which of the given points to use.
- Select the format in which to format the desired equation and complete the necessary steps to accomplish the task.

***slope-intercept format:  $y = mx + b$*** 

using point  $P$  ...  $P = (5, 5)$

$$m = -1, x = 5, y = 5$$

$$y = mx + b \rightarrow b = y - mx$$

$$b = 5 - (-1)(5) = 5 + 5 = 10$$

$$y = -1x + 10 \rightarrow \mathbf{y = 10 - x}$$

using point  $Q$  ...  $Q = (15, -5)$

$$m = -1, x = 15, y = -5$$

$$y = mx + b \rightarrow b = y - mx$$

$$b = -5 - (-1)(15) = -5 + 15 = 10$$

$$y = -1x + 10 \rightarrow \mathbf{y = 10 - x}$$

***point-slope format:  $y - y_1 = m(x - x_2)$*** 

using point  $P$  ...  $P = (5, 5)$

$$m = -1, x_1 = 5, y_1 = 5$$

$$y - y_1 = m(x - x_2)$$

$$y - 5 = (-1)(x - 5)$$

$$\mathbf{y - 5 = -1x + 5}$$

using point  $Q$  ...  $Q = (15, -5)$

$$m = -1, x_1 = 15, y_1 = -5$$

$$y - y_1 = m(x - x_2)$$

$$y - (-5) = (-1)(x - 15)$$

$$\mathbf{y + 5 = -1x + 15}$$

which can be reformatted into:

$$y = -1x + 5 + 5 = -1x + 10$$

which can also be reformatted

$$\text{into: } x + y = 10$$

which can be reformatted into:

$$y = -1x + 15 - 5 = -1x + 10$$

which can also be reformatted

$$\text{into: } x + y = 10$$

***standard format:  $Ax + By = C$*** 

using point  $P$  ...  $P = (5, 5)$

$$m = -1 \rightarrow A = 1, B = -1$$

$$x = 5, y = 5$$

$$Ax + By = C$$

$$(1)(5) + (1)(5) = 5 + 5 = 10 = C$$

$$\mathbf{x + y = 10}$$

using point  $Q$  ...  $Q = (15, -5)$

$$m = -1 \rightarrow A = 1, B = -1$$

$$x = 15, y = -5$$

$$Ax + By = C$$

$$(1)(15) + (1)(-5) = 15 - 5 = 10 = C$$

$$\mathbf{x + y = 10}$$

**example 1.16b      Write an equation that passes through:  $P = (10, -1)$ ,  $Q = (-2, 2)$** 

Just follow these basic steps:

1. Determine the slope between the given points:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 2}{10 - (-2)} = \frac{-3}{12} = \frac{-1}{4}$$

2. Choose which of the given points to use.
3. Select the format in which to format the desired equation and complete the necessary steps to accomplish the task.

**Brain power**

By 2020, wind could provide one-tenth of our planet's electricity needs. Already today, SKF's innovative know-how is crucial to running a large proportion of the world's wind turbines.

Up to 25 % of the generating costs relate to maintenance. These can be reduced dramatically thanks to our systems for on-line condition monitoring and automatic lubrication. We help make it more economical to create cleaner, cheaper energy out of thin air.

By sharing our experience, expertise, and creativity, industries can boost performance beyond expectations. Therefore we need the best employees who can meet this challenge!

The Power of Knowledge Engineering

Plug into The Power of Knowledge Engineering.  
Visit us at [www.skf.com/knowledge](http://www.skf.com/knowledge)

**SKF**

**slope-intercept format:  $y = mx + b$** using point  $P \dots P = (10, -1)$ 

$$m = -\frac{1}{4}, x = 10, y = -1$$

$$y = mx + b \rightarrow b = y - mx$$

$$b = -1 - (-\frac{1}{4})(10) = -1 + 2.5 = 1.5$$

$$\textcolor{red}{y = -\frac{1}{4}x + 1.5}$$

using point  $Q \dots Q = (-2, 2)$ 

$$m = -\frac{1}{4}, x = -2, y = 2$$

$$y = mx + b \rightarrow b = y - mx$$

$$b = 2 - (-\frac{1}{4})(-2) = 2 - 0.5 = 1.5$$

$$\textcolor{red}{y = -\frac{1}{4}x + 1.5}$$

**point-slope format:  $y - y_1 = m(x - x_1)$** using point  $P \dots P = (10, -1)$ 

$$m = -\frac{1}{4}, x_1 = 10, y_1 = -1$$

$$y - y_1 = m(x - x_1)$$

$$y - (-1) = (-\frac{1}{4})(x - 10)$$

$$\textcolor{red}{y + 1 = -\frac{1}{4}(x - 10)}$$

which can be reformatted into:

$$y = -\frac{1}{4}x + 2.5 - 1$$

$$y = -\frac{1}{4}x + 1.5$$

using point  $Q \dots Q = (-2, 2)$ 

$$m = -\frac{1}{4}, x_1 = -2, y_1 = 2$$

$$y - y_1 = m(x - x_1)$$

$$y - 2 = (-\frac{1}{4})(x + 2)$$

$$\textcolor{red}{y = -\frac{1}{4}(x + 2) + 2}$$

which can be reformatted into:

$$y = -\frac{1}{4}x - 0.5 + 2$$

$$y = -\frac{1}{4}x + 1.5$$

which can be reformatted as:

$$4y = -1x + 6 \rightarrow x + 4y = 6$$

which can be reformatted as:

$$4y = -1x + 6 \rightarrow x + 4y = 6$$

**standard format:  $Ax + By = C$** using point  $P \dots P = (10, -1)$ 

$$m = -\frac{1}{4} \rightarrow A = 1, B = 4$$

$$x = 10, y = -1$$

$$Ax + By = C$$

$$(1)(10) + (4)(-1) = 10 - 4 = 6 = C$$

$$1x + 4y = 6 \rightarrow \textcolor{red}{x + 4y = 6}$$

using point  $Q \dots Q = (-2, 2)$ 

$$m = -\frac{1}{4} \rightarrow A = 1, B = 4$$

$$x = -2, y = 2$$

$$Ax + By = C$$

$$(1)(-2) + (4)(2) = -2 + 8 = 6 = C$$

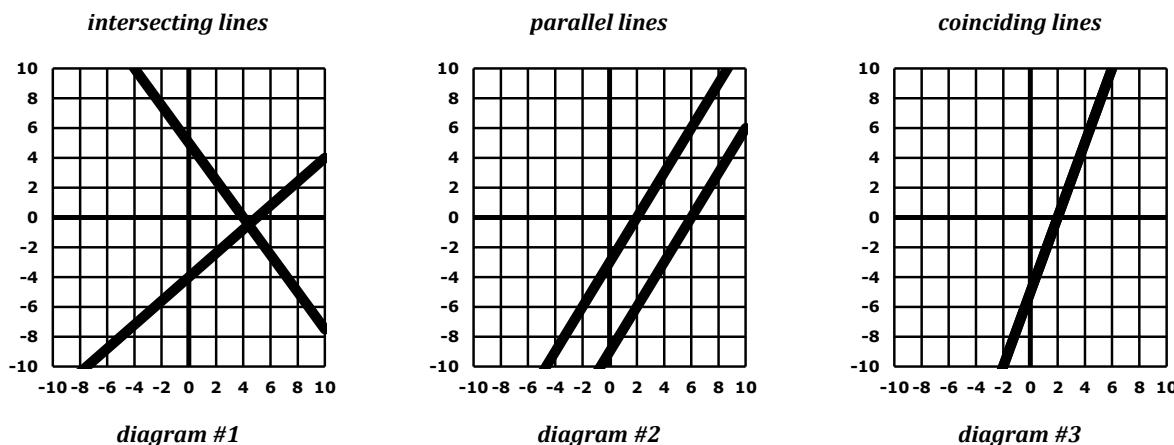
$$1x + 4y = 6 \rightarrow \textcolor{red}{x + 4y = 6}$$

## **TUTORIAL 1.17: DETERMINE WHETHER THE LINES FOR A PAIR OF EQUATIONS ARE PARALLEL, PERPENDICULAR OR COINCIDE WITH EACH OTHER**

When two linear equations are graphed on the same coordinate grid, these lines can interact with each other in one of three ways:

- The two graphed lines can intersect each other...meeting at exactly one point. If the two lines form right angles, they are considered perpendicular with each other... but, the lines will still only have one point of intersection. (See diagram #1 below for an example.)

- The two graphed lines can be parallel with each other...meaning that they do not touch each (intersect with) other at all. (See diagram #2 below for an example.)
- The two graphed lines can coincide...meaning that the lines will lie on top of each other. Although there will be two equations, the finalized graph will look as if there was only one line drawn. (See diagram #3 below for an example.)



Here are the equations that produced the graphs above:

$5x + 4y = 20$	$4x - 5y = 20$	$6x - 4y = 12$	$3x - 2y = 18$	$5x - 2y = 10$	$10x - 4y = 20$
slope = $-5/4$	slope = $4/5$	slope = $6/4$	slope = $3/2$	slope = $5/2$	slope = $10/4$
$(0, y) = (0, 5)$	$(0, y) = (0, -4)$	$(0, y) = (0, 3)$	$(0, y) = (0, -9)$	$(0, y) = (0, -5)$	$(0, y) = (0, -5)$
$(x, 0) = (4, 0)$	$(x, 0) = (5, 0)$	$(x, 0) = (2, 0)$	$(x, 0) = (6, 0)$	$(x, 0) = (2, 0)$	$(x, 0) = (2, 0)$

We can analyze what elements of the linear equations will produce each of the possible interactions. Below each of the graphed equations, we have listed the slope,  $y$ -intercept and  $x$ -intercept for the graphed lines. Notice that:

- The two graphed lines will intersect when the two slopes are not equal (diagram #1).
- The two graphed lines will be perpendicular to each other when the two slopes are ***negative reciprocals*** of each other...meaning that the product of the two slopes will be  $-1$ . (diagram #1)
- The two graphed lines will be parallel to each other when the two slopes are equal to each other but the  $y$ -intercepts are not. (diagram #2)...NOTE: Remember to simplify fractional slopes to their lowest terms to compare them.
- The two graphed lines will coincide when both the two slopes and the  $y$ -intercepts are equal to each other. (diagram #3)...NOTE: Remember to simplify fractional slopes to their lowest terms to compare them.

We can use these observations to determine whether two lines will be parallel, perpendicular, coincide or just intersect directly from their equations. Just follow these steps:

1. Find the slope for each given equation.
2. Find the  $y$ -intercept for each given equation.
3. Use the information obtained to determine how the lines will interact with each other.

**example 1.17a**    **Determine whether the lines of the given equations will be parallel, perpendicular, coincide, or just intersect:**

$$5x - 2y = 20$$

$$10x - 4y = 20$$

1. Find the slope for each given equation:  
 $5x - 2y = 20 \rightarrow m = -A/B = -5/-2 = 5/2 = 2.5$   
 $10x - 4y = 20 \rightarrow m = -A/B = -10/-4 = 5/2 = 2.5$
2. Find the  $y$ -intercept for each given equation:  
 $5x - 2y = 20 \rightarrow b = C/B = 20/-2 = -10 \rightarrow (0, -10)$   
 $10x - 4y = 20 \rightarrow b = C/B = 20/-4 = -5 \rightarrow (0, -5)$
3. Use the information obtained to determine how the lines will interact with each other: *The slopes are equal but the  $y$ -intercepts are not. Thus, the lines will be **parallel** to each other.*

## TURN TO THE EXPERTS FOR SUBSCRIPTION CONSULTANCY

Subscrybe is one of the leading companies in Europe when it comes to innovation and business development within subscription businesses.

We innovate new subscription business models or improve existing ones. We do business reviews of existing subscription businesses and we develop acquisition and retention strategies.

Learn more at [linkedin.com/company/subscrybe/](https://www.linkedin.com/company/subscrybe/) or contact Managing Director Morten Suhr Hansen at [mha@subscrybe.dk](mailto:mha@subscrybe.dk)

**SUBSCRYBE** - to the future



Click on the ad to read more

**example 1.17b**

**Determine whether the lines of the given equations will be parallel, perpendicular, coincide, or just intersect:**

$$\begin{aligned} 5x + 4y &= 9 \\ y &= (\frac{4}{5})x + 9 \end{aligned}$$

- Find the slope for each given equation:

$$\begin{aligned} 5x + 4y = 9 &\rightarrow m = -A/B = -5/4 = -1.25 \\ y = (\frac{4}{5})x + 9 &\rightarrow m = 4/5 = 0.8 \end{aligned}$$

- Find the  $y$ -intercept for each given equation:

$$\begin{aligned} 5x + 4y = 9 &\rightarrow b = C/B = 9/4 = 2.25 \rightarrow (0, 2.25) \\ y = (\frac{4}{5})x + 9 &\rightarrow b = 9 \rightarrow (0, 9) \end{aligned}$$

- Use the information obtained to determine how the lines will interact with each other: *The slopes are not equal but  $(-5/4)$   $(4/5) = -1$ . Thus, the lines will be **perpendicular** to each other.*

**example 1.17c**

**Determine whether the lines of the given equations will be parallel, perpendicular, coincide, or just intersect:**

$$\begin{aligned} 6x + 4y &= 24 \\ 3x + 2y &= 12 \end{aligned}$$

- Find the slope for each given equation:

$$\begin{aligned} 6x + 4y = 24 &\rightarrow m = -A/B = -6/4 = -3/2 = -1.5 \\ 3x + 2y = 12 &\rightarrow m = -A/B = -3/2 = -1.5 \end{aligned}$$

- Find the  $y$ -intercept for each given equation:

$$\begin{aligned} 6x + 4y = 24 &\rightarrow b = C/B = 24/4 = 6 \rightarrow (0, 6) \\ 3x + 2y = 12 &\rightarrow b = C/B = 12/2 = 6 \rightarrow (0, 6) \end{aligned}$$

- Use the information obtained to determine how the lines will interact with each other: *Both the slopes and the  $y$ -intercepts are equal. Thus, the lines will **coincide**.*

**example 1.17d**

**Determine whether the lines of the given equations will be parallel, perpendicular, coincide, or just intersect:**

$$3x + 2y = 10$$

$$2x - y = 6$$

- Find the slope for each given equation:

$$3x + 2y = 10 \rightarrow m = -A/B = -3/2 = -1.5$$

$$2x - y = 6 \rightarrow m = -A/B = -2/-1 = 2/1 = 2$$

- Find the  $y$ -intercept for each given equation:

$$3x + 2y = 10 \rightarrow b = C/B = 10/2 = 5 \rightarrow (0, 5)$$

$$2x - y = 6 \rightarrow b = C/B = 6/-1 = -6 \rightarrow (0, -6)$$

- Use the information obtained to determine how the lines will interact with each other: *The slopes are not equal and they are not negative reciprocals of each other. Thus, the lines will intersect with each other at one specific point.*

### TUTORIAL 1.18: WRITE THE EQUATION OF A LINE THROUGH A GIVEN POINT THAT IS PARALLEL TO A GIVEN LINE

For two graphed lines to be parallel to each other, they must have equal slopes but different  $y$ -intercepts. Thus, to find the slope-intercept equation for the line that is parallel to a given line and passes through a given point, just do these steps:

- Determine the slope of the given line.
- Determine the values of  $x$  and  $y$  to be used to complete the task.
- Verify that the given point is not a solution for the given line...otherwise, the new line will coincide with the given line and will not be parallel to the given line.
- Decide in what format to express the desired equation and use the given point to determine the desired equation.

**example 1.18a**

**Write the equation for the line that is parallel to line for  $3x + 5y = 15$  and passes through point  $P = (5, 3)$ .**

- Determine the slope of the given line:  $3x + 5y = 15 \rightarrow m =$

$$\frac{-A}{B} = \frac{-3}{5}$$

- Determine the values of  $x$  and  $y$  to be used:  $P = (5, 3) \rightarrow x = 5, y = 3$

- verification:  $3(5) + 5(3) = 15 + 15 = 30 \rightarrow 15 \neq 30 \rightarrow$  continue

4. Write the equation in its desired format:

**slope-intercept format:  $y = mx + b$**

$$b = y - mx \rightarrow b = 3 - (-\frac{3}{5})(5)$$

$$b = 3 + 3 = 6$$

equation:  $\textcolor{red}{y = (-\frac{3}{5})x + 6}$

**standard format:  $Ax + By = C$**

$$\text{slope} = \frac{-A}{B} = \frac{-3}{5} \rightarrow A = 3 \text{ when } B = 5$$

$$P = (5, 3) \rightarrow x = 5 \text{ when } y = 3$$

$$C = Ax + By = (3)(5) + (5)(3)$$

$$C = 15 + 15 = 30$$

equation:  $\textcolor{red}{3x + 5y = 30}$



**What do you want to do?**

No matter what you want out of your future career, an employer with a broad range of operations in a load of countries will always be the ticket. Working within the Volvo Group means more than 100,000 friends and colleagues in more than 185 countries all over the world. We offer graduates great career opportunities – check out the Career section at our web site [www.volvologroup.com](http://www.volvologroup.com). We look forward to getting to know you!

**VOLVO**

AB Volvo (publ)  
[www.volvologroup.com](http://www.volvologroup.com)

VOLVO TRUCKS | RENAULT TRUCKS | MACK TRUCKS | VOLVO BUSES | VOLVO CONSTRUCTION EQUIPMENT | VOLVO PENTA | VOLVO AERO | VOLVO IT  
VOLVO FINANCIAL SERVICES | VOLVO 3P | VOLVO POWERTRAIN | VOLVO PARTS | VOLVO TECHNOLOGY | VOLVO LOGISTICS | BUSINESS AREA ASIA



**example 1.18b** Write the equation for the line that is parallel to line for  $6x - 5y = 7$  and passes through point  $P = (5, -6)$ .

- Determine the slope of the given line:  $6x - 5y = 7 \rightarrow$

$$m = \frac{-A}{B} = \frac{-6}{-5} = \frac{6}{5} = 1.2$$

- Determine the values of  $x$  and  $y$  to be used:  $P = (5, -6) \rightarrow$

$$x = 5, y = -6$$

- verification:  $6(5) - 5(-6) = 30 + 30 = 60 \rightarrow 7 \neq 60 \rightarrow$  continue

- Write the equation in its desired format:

slope-intercept format:  $y = mx + b$

$$b = y - mx \rightarrow b = -6 - (1.2)(5)$$

$$b = -6 - 6 = -12$$

$$\text{equation: } \color{red}{y = (1.2)x - 12}$$

standard format:  $Ax + By = C$

$$\text{slope} = \frac{-A}{B} = \frac{6}{5} \rightarrow A = 6 \text{ when } B = -5$$

$$P = (5, -6) \rightarrow x = 5 \text{ when } y = -6$$

$$C = Ax + By = (6)(5) + (-5)(-6)$$

$$C = 30 + 30 = 60$$

$$\text{equation: } \color{red}{6x - 5y = 60}$$

## TUTORIAL 1.19: WRITE THE EQUATION OF A LINE THROUGH A GIVEN POINT THAT IS PERPENDICULAR TO A GIVEN LINE

For two graphed lines to be perpendicular to each other they must have slopes that are negative reciprocals of each other...that is, the product of the two slopes must be  $-1$ . Thus, to find the slope-intercept equation for the line that is perpendicular to a given line and passes through a given point, just do these steps:

- Determine the slope of the given line.
- Determine the negative reciprocal of the slope of the given line.
- Determine the values of  $x$  and  $y$  to be used to complete the task.
- Verify that the given point is not a solution for the given line...otherwise, the new line will coincide with the given line and will not be perpendicular to the given line.
- Decide in what format to express the desired equation and then determine the desired equation.

**example 1.19a** Write the equation for the line that is perpendicular to line for  $3x + 5y = 15$  and passes through point  $P = (3, 10)$ .

1. Determine the slope of the given line:  $3x + 5y = 15 \rightarrow m =$

$$\frac{-A}{B} = \frac{-3}{5}$$

2. slope to be used: negative reciprocal of  $\frac{-3}{5} = \frac{5}{3}$

3. Determine the values of  $x$  and  $y$  to be used:  $P = (3, 10) \rightarrow$

$$x = 3, y = 10$$

4. verification:  $3(3) + 5(10) = 9 + 50 = 59 \rightarrow 15 \neq 59 \rightarrow$  continue

5. Write the equation in its desired format:

slope-intercept format:  $y = mx + b$

$$b = y - mx \rightarrow b = 10 - (5/3)(3)$$

$$b = 10 - 5 = 5$$

$$\text{equation: } \mathbf{y = (5/3)x + 5}$$

standard format:  $Ax + By = C$

$$\text{slope} = \frac{-A}{B} = \frac{5}{3} \rightarrow$$

$$A = 5 \text{ when } B = -3$$

$$P = (3, 10) \rightarrow x = 3 \text{ when } y = 10$$

$$C = Ax + By = (5)(3) + (-3)(10)$$

$$C = 15 - 30 = -15$$

$$\text{equation: } \mathbf{5x - 3y = -15}$$

**example 1.19b** Write the equation for the line that is perpendicular to line for  $6x - 5y = 7$  and passes through point  $P = (-6, 15)$ .

1. Determine the slope of the given line:  $6x - 5y = 7 \rightarrow m =$

$$\frac{-A}{B} = \frac{-6}{-5} = \frac{6}{5}$$

2. slope to be used: negative reciprocal of  $\frac{6}{5} = \frac{-5}{6}$

3. Determine the values of  $x$  and  $y$  to be used:  $P = (-6, 15) \rightarrow$

$$x = -6, y = 15$$

4. verification:  $6(-6) - 5(15) = -36 - 75 = -111 \rightarrow 7 \neq -111$

$\rightarrow$  continue

5. Write the equation in its desired format:

**slope-intercept format:  $y = mx + b$**

$$b = y - mx \rightarrow b = 15 - (-\frac{5}{6})(-6)$$

$$b = 15 - 5 = 10$$

equation:  **$y = (-\frac{5}{6})x + 10$**

**standard format:  $Ax + By = C$**

$$\text{slope} = \frac{-A}{B} = \frac{-5}{6} \rightarrow$$

$$A = 5 \text{ when } B = 6$$

$$P = (-6, 15) \rightarrow x = -6 \text{ when } y = 15$$

$$C = Ax + By = (5)(-6) + (6)(15)$$

$$C = -30 + 90 = 60$$

equation:  **$5x + 6y = 60$**

The advertisement features a runner in a field at sunset, with the GaitEye logo and slogan "Challenge the way we run" in the upper left. The text "EXPERIENCE THE POWER OF FULL ENGAGEMENT..." is overlaid on the image. Below the runner, three lines of text read "RUN FASTER.", "RUN LONGER..", and "RUN EASIER...". A yellow call-to-action button in the bottom right corner contains the text "READ MORE & PRE-ORDER TODAY" and "WWW.GAITEYE.COM", with a hand cursor icon pointing towards it.

## TUTORIAL 1.20: SOLVE A SYSTEM OF LINEAR EQUATIONS BY GRAPHING

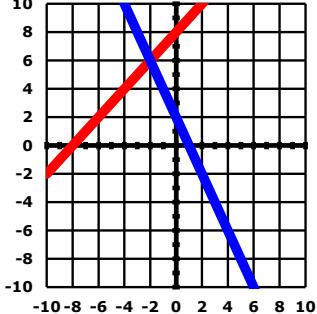
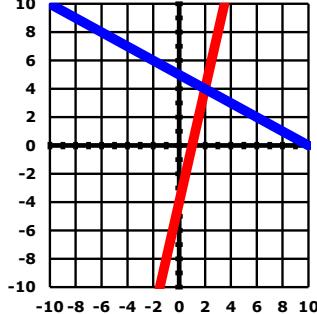
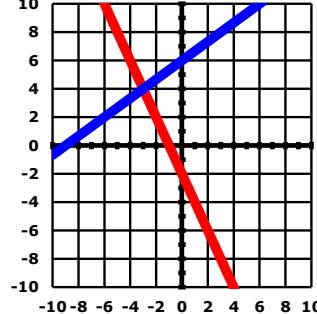
When we solve a system of linear equations, we are looking for *the* one point (ordered pair) that will be a solution for all of the equations in the system. When we solve a system by graphing, we are looking for the point of intersection contained in the graphs of the equations in the system. Therefore, we can accomplish the desired task by:

1. Graph each of the equations contained within the given system on the same coordinate grid. (Remember that we can use whatever graphing method we prefer: the  $x$ - and  $y$ -intercepts, the slope and  $y$ -intercept, the slope and any solution point, or a full set of solution points.)
2. Determine the coordinates for the point of intersection (if it exists). This will be the solution of the given system.

NOTE #1: If the lines do not intersect at all, then there is no solution to the system.

NOTE #2: If the lines coincide, then there are infinitely many solutions...all of the points that lie on the graphed line(s).

**Solve the given system of linear equations by graphing on a coordinate grid:**

example 1.20a:	example 1.20b:	example 1.20c:			
$\begin{aligned}x - y &= -8 \\2x + y &= 2\end{aligned}$  <p><b>solution = (-2, 6)</b></p>	$\begin{aligned}4x - y &= 4 \\2x + 4y &= 20\end{aligned}$  <p><b>solution = (2, 4)</b></p>	$\begin{aligned}6x + 3y &= -6 \\2x - 3y &= -18\end{aligned}$  <p><b>solution = (-3, 4)</b></p>			
<b>red line:</b> $\begin{aligned}x - y &= -8 \\m &= \frac{-1}{-1} = +1 \\(0, y) &= (0, 8) \\(x, 0) &= (-8, 0)\end{aligned}$	<b>blue line:</b> $\begin{aligned}2x + y &= 2 \\m &= \frac{-2}{1} = -2 \\(0, y) &= (0, 2) \\(x, 0) &= (1, 0)\end{aligned}$	<b>red line:</b> $\begin{aligned}4x - y &= 4 \\m &= \frac{-4}{-1} = +4 \\(0, y) &= (0, -4) \\(x, 0) &= (1, 0)\end{aligned}$	<b>blue line:</b> $\begin{aligned}2x + 4y &= 20 \\m &= \frac{-2}{4} = \frac{-1}{2} \\(0, y) &= (0, 5) \\(x, 0) &= (10, 0)\end{aligned}$	<b>red line:</b> $\begin{aligned}6x + 3y &= -6 \\m &= \frac{-6}{3} = -2 \\(0, y) &= (0, -2) \\(x, 0) &= (-1, 0)\end{aligned}$	<b>blue line:</b> $\begin{aligned}2x - 3y &= -18 \\m &= \frac{-2}{-3} = \frac{2}{3} \\(0, y) &= (0, 6) \\(x, 0) &= (-9, 0)\end{aligned}$

## TUTORIAL 1.21: SOLVE A SYSTEM OF LINEAR EQUATIONS BY SUBSTITUTION

Although solving a system of equations by graphing is not that difficult, not too many people walk around with graph paper. So, an algebraic solution method might be preferable. One such algebraic method is using substitution...where one of the variables (either  $x$  or  $y$ ) is replaced with a value and/or an expression containing the remaining variable. The goal here is to work with an equation that contains only one variable rather than two. Basically, just follow these steps to solve a system of equations by substitution:

1. Solve one of the given equations (no difference which one) for one of the variables (again, no difference which one).
2. Replace the result from step #1 into the second equation...solving the second equation for the remaining variable.
3. Take the answer from step #2 and substitute back into the first equation to find the value of the second variable.
4. Form the solution for the system by putting the values into an ordered pair  $(x, y)$ .
5. Check the proposed solution ordered pair by substituting the values into each of the given equations.

**example 1.21a      Solve this system through algebraic substitution:**  $3x = 12$

$$4x - 3y = 10$$

1.  $3x = 12 \rightarrow x = 12 \div 3 = 4$   
(For this example, the solution is a value rather than an expression.)
2.  $4x - 3y = 10 \rightarrow 4(4) - 3y = 10 \rightarrow 16 - 3y = 10 \rightarrow -3y = 10 - 16 = -6 \rightarrow y = -6 \div -3 = 2$
3. There is no need to complete step #3 for this particular example since we already know the value of  $x$  (as determined in step #1) as well as the value of  $y$  (as determined in step #2).
4. system solution:  $x = 4, y = 2 \rightarrow (4, 2)$
5.  $3x = 12 \rightarrow x = 4 \rightarrow 3(4) = 12 \dots$  passed 1<sup>st</sup> test  
 $4x - 3y = 10 \rightarrow x = 4, y = 2 \rightarrow 4(4) - 3(2) = 16 - 6 = 10 \dots$   
passed 2<sup>nd</sup> test

**example 1.21b**    **Solve this system through algebraic substitution:**

$$\begin{aligned}x + 3y &= 3 \\3x - 2y &= 31\end{aligned}$$

$$1. \quad x + 3y = 3 \rightarrow x = 3 - 3y$$

(For this example, the solution is an expression in terms of  $y$ .)

$$2. \quad 3x - 2y = 31 \rightarrow 3(3 - 3y) - 2y = 31 \rightarrow 9 - 9y - 2y = 31 \rightarrow 9 - 11y = 31 \rightarrow -11y = 31 - 9 = 22 \rightarrow y = 22 \div -11 = -2$$

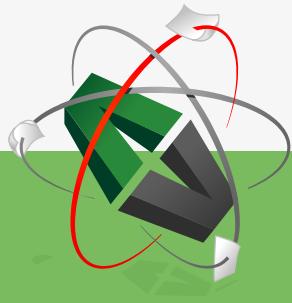
$$3. \quad x + 3y = 3 \rightarrow y = -2 \rightarrow x + 3(-2) = 3 \rightarrow x - 6 = 3 \rightarrow x = 3 + 6 = 9$$

$$4. \text{ system solution: } x = 9, y = -2 \rightarrow (9, -2)$$

$$5. \quad x + 3y = 3 \rightarrow x = 9, y = -2 \rightarrow (9) + 3(-2) = 9 - 6 = 3 \dots$$

$$\begin{aligned}\text{passed 1}^{\text{st}} \text{ test } 3x - 2y &= 31 \rightarrow x = 9, y = -2 \rightarrow 3(9) - 2(-2) \\&= 27 + 4 = 31 \dots \text{passed 2}^{\text{nd}} \text{ test}\end{aligned}$$

This e-book  
*is made with*  
**SetaPDF**



PDF components for PHP developers

**www.setasign.com**



Click on the ad to read more

**example 1.21c**    **Solve this system through algebraic substitution:**     $4x - y = -23$   
 $2x + 3y = -1$

1.  $4x - y = -23 \rightarrow -y = -23 - 4x \rightarrow y = 23 + 4x$   
 (For this example, the solution is an expression in terms of  $x$ .)
2.  $2x + 3y = -1 \rightarrow 2x + 3(23 + 4x) = -1 \rightarrow 2x + 69 + 12x = -1 \rightarrow 14x + 69 = -1 \rightarrow 14x = -1 - 69 = -70 \rightarrow x = -70 \div 14 = -5$
3.  $4x - y = -23 \rightarrow x = -5 \rightarrow 4(-5) - y = -23 \rightarrow -20 - y = -23 \rightarrow -y = -23 + 20 \rightarrow -y = -3 \rightarrow y = 3$
4. system solution:  $x = -5, y = 3 \rightarrow (-5, 3)$
5.  $4x - y = -23 \rightarrow 4(-5) - (3) = -20 - 3 = -23 \dots \text{passed 1}^{\text{st}} \text{ test}$   
 $2x + 3y = -1 \rightarrow 2(-5) + 3(3) = -10 + 9 = -1 \dots \text{passed 2}^{\text{nd}} \text{ test}$

In some “special” cases, all of the variables will be eliminated leaving a numeric equation. When all of the variables disappear, we need to determine whether the numeric equation is a true statement or a false statement. If true, then the given system has **infinitely many** solutions because the lines will coincide on a coordinate grid. If false, then the given system will have **no solution** because the lines will be parallel to each other and not intersect each other at all.

**example 1.21d**    **Solve this system through algebraic substitution:**     $6x + 4y = 24$   
 $y = 6 - 1.5x$

1.  $y = 6 - 1.5x$  (already solved for  $y$ )
2.  $6x + 4y = 24 \rightarrow 6x + 4(6 - 1.5x) = 24 \rightarrow 6x + 24 - 6x = 24 \rightarrow 24 = 24$
3. This is a “special” case. The numeric statement is true...thus, this system has infinitely many solutions.
4. system solution: **infinitely many**  $\rightarrow \{(x, y) \mid y = 6 - 1.5x\}$   
 (When there are infinitely many solutions, we need to define the ordered pairs that will be part of the solution set by using the given notation.)

**example 1.21e**    **Solve this system through algebraic substitution:**     $10x - 4y = 20$   
 $y = 2.5x - 5$

1.  $y = 2.5x - 5$  (already solved for  $y$ )
2.  $10x - 4y = 20 \rightarrow 10x - 4(2.5x - 5) = 20 \rightarrow 10x - 10x = 20$   
 $= 20 \rightarrow -20 = 20$
3. This is a “special” case. The numeric statement is false...thus,  
 this system has no solutions.
4. system solution: **no solution**

## TUTORIAL 1.22: SOLVE A SYSTEM OF LINEAR EQUATIONS BY ELIMINATION

Substitution is only one of the algebraic methods to solve a system of linear equations. The second method is an **elimination** process using addition and/or subtraction (which is sometimes called **linear combination**). Again the goal is to work with an equation that contains only one variable rather than two. Basically, just follow these steps to solve a system of equations by elimination:

1. Determine which of the given variables is the easiest to eliminate. (It makes no difference which of the two variable is eliminated first.)
2. Determine the best strategy to eliminate the chosen variable. (This strategy may involve just adding straight down, just subtracting straight down, or multiplying then adding straight down.)
3. After applying the strategy from step #2, solve the resulting equation for its variable.
4. Using the result from step #3, substitute back into one of the original equations (no difference which one) and solve for the remaining variable.
5. Form the solution for the system by putting the values into an ordered pair  $(x, y)$ .
6. Check the proposed solution ordered pair by substituting the values into each of the given equations.

**example 1.22a** Solve this system through algebraic elimination:  $2x + y = 8$

$$5x - y = 13$$

1. The easiest variable to eliminate is the  $y$ . Take advantage of the opposite signs in front of the  $y$ -terms and add straight down:  $7x = 21$

$$\begin{array}{r} 2x + y = 8 \\ 5x - y = 13 \\ \hline 7x = 21 \end{array}$$

2. Solve for  $x$  by dividing both sides by 7:  $\frac{7x}{7} = \frac{21}{7} \rightarrow x = 3$

3. Substituting back into either of the original equations, solve for  $y$ :

$$2x + y = 8 \rightarrow 2(3) + y = 8 \rightarrow y = 8 - 2(3) = 8 - 6 \rightarrow y = 2$$

$$5x - y = 13 \rightarrow 5(3) - y = 13 \rightarrow -y = 13 - 15 = -2 \rightarrow y = 2$$

4. **solution:**  $(x, y) = (3, 2)$

5. Checking to make sure that this solution works for the system:

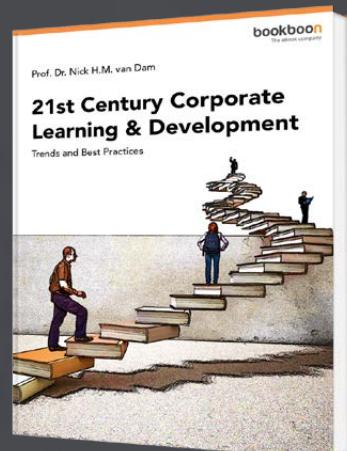
$$2x + y = 8 \rightarrow 2(3) + (2) = 8 \rightarrow 6 + 2 = 8 \dots \text{passed first test}$$

$$5x - y = 13 \rightarrow 5(3) - (2) = 13 \rightarrow 15 - 2 = 13 \dots \text{passed second test}$$

## Free eBook on Learning & Development

By the Chief Learning Officer of McKinsey

Download Now



Download free eBooks at [bookboon.com](http://bookboon.com)



Click on the ad to read more

**example 1.22b      Solve this system through algebraic elimination:**  $x + 3y = 5$

$$x - 5y = -3$$

1. The easiest variable to eliminate is the  $x$ .

Since the signs in front of the  $x$ -terms are the same (positive), we need to make on one of them into the opposite sign by subtracting straight down:  $8y = 8$

2. Solve for  $y$  by dividing both sides by 8:  $\frac{8y}{8} = \frac{8}{8} \rightarrow y = 1$
3. Substituting back into either of the original equations, solve for  $y$ :

$$x + 3y = 5 \rightarrow x + 3(1) = 5 \rightarrow x = 5 - 3(1) = 5 - 3 \rightarrow x = 2$$

$$x - 5y = -3 \rightarrow x - 5(1) = -3 \rightarrow x = -3 + 5 \rightarrow x = 2$$

4. **solution:**  $(x, y) = (2, 1)$
5. Checking to make sure that this solution works for the system:

$$x + 3y = 5 \rightarrow (2) + 3(1) = 5 \rightarrow 2 + 3 = 5 \dots \text{passed first test}$$

$$x - 5y = -3 \rightarrow (2) - 5(1) = -3 \rightarrow 2 - 5 = -3 \dots \text{passed second test}$$

Occasionally, just adding or subtracting down is not enough to eliminate the desired variable. Therefore, we will need to multiply one or both of the equations so that we convert the given system into a system where we can just add or subtract straight down.

**example 1.22c      Solve this system through algebraic elimination:  $5x + 2y = 26$** 

$$x - 3y = -5$$

1. The easiest variable to eliminate is the  $x$ .

Since we need both  $x$  coefficients to be the same number, multiply the second equation by  $-5$ :

$$-5(x - 3y = -5) \rightarrow -5x + 15y = +25$$

2. Adding straight down:  $17y = 51$

$$\begin{array}{r} 5x + 2y = 26 \\ x - 3y = -5 \\ \hline 5x + 2y = 26 \\ -5x + 15y = 25 \\ \hline 17y = 51 \end{array}$$

3. Solve for  $y$  by dividing both sides by 17:  $\frac{17y}{17} = \frac{51}{17} \rightarrow y = 3$   
 4. Substituting back into either of the original equations, solve for  $x$ :

$$5x + 2y = 26 \rightarrow 5x + 2(3) = 26 \rightarrow 5x = 26 - 6 = 20 \rightarrow x = 4$$

$$x - 3y = -5 \rightarrow x - 3(3) = -5 \rightarrow x = -5 + 9 \rightarrow x = 4$$

5. **solution:**  $(x, y) = (4, 3)$

6. Checking to make sure that this solution works for the system:  
 $5x + 2y = 26 \rightarrow 5(4) + 2(3) = 26 \rightarrow 20 + 6 = 26$ ...passed first test  
 $x - 3y = -5 \rightarrow (4) - 3(3) = -5 \rightarrow 4 - 9 = -5$ ...passed second test

**example 1.22d      Solve this system through algebraic elimination:  $2x - y = 9$** 

$$3x + 2y = 17$$

1. The easiest variable to eliminate is the  $y$ .

Since we need both  $y$  coefficients to be the same number, multiply the first equation by  $+2$ :

$$+2(2x - y = 9) \rightarrow 4x - 2y = 18$$

$$\begin{array}{r} 2x - y = 9 \\ 3x + 2y = 17 \\ \hline 4x - 2y = 18 \\ 3x + 2y = 17 \\ \hline 7x = 35 \end{array}$$

2. Adding straight down:  $7x = 35$

3. Solve for  $x$  by dividing both sides by 7:  $\frac{7x}{7} = \frac{35}{7} \rightarrow x = 5$

4. Substituting back into either of the original equations, solve for  $x$ :

$$2x - y = 9 \rightarrow 2(5) - y = 9 \rightarrow 10 - y = 9 \rightarrow -y = 9 - 10 = -1 \rightarrow y = 1$$

$$3x + 2y = 17 \rightarrow 3(5) + 2y = 17 \rightarrow 15 + 2y = 17 \rightarrow 2y = 17 - 15 = 2 \rightarrow y = 1$$

5. **solution:**  $(x, y) = (5, 1)$

6. Checking to make sure that this solution works for the system:

$$2x - y = 9 \rightarrow 2(5) - (1) = 10 - 1 = 9$$
...passed first test

$$3x + 2y = 17 \rightarrow 3(5) + 2(1) = 15 + 2 = 17$$
...passed second test

In some “special” cases, all of the variables will be eliminated leaving a numeric equation. When all of the variables disappear, we need to determine whether the numeric equation is a true statement or a false statement. If true, then the given system has ***infinitely many*** solutions because the lines will coincide on a coordinate grid. If false, then the given system will have ***no solution*** because the lines will be parallel to each other and not intersect each other at all.

www.sylvania.com

We do not reinvent  
the wheel we reinvent  
light.

Fascinating lighting offers an infinite spectrum of possibilities: Innovative technologies and new markets provide both opportunities and challenges. An environment in which your expertise is in high demand. Enjoy the supportive working atmosphere within our global group and benefit from international career paths. Implement sustainable ideas in close cooperation with other specialists and contribute to influencing our future. Come and join us in reinventing light every day.

Light is OSRAM

**OSRAM**  
**SYLVANIA**



**example 1.22e      Solve this system through algebraic elimination:  $6x + 4y = 24$** 

$$3x + 2y = 12$$

1. The easiest variable to eliminate is the  $y$ . Since we need both  $y$  coefficients to be the same number, multiply the second equation by  $-2$ :

$$\begin{aligned} -2(3x + 2y = 12) &\rightarrow -6x - 4y \\ &= -24 \end{aligned}$$

2. Adding straight down:  $0 = 0$   
 3. Since all of the variables disappeared, this system is a “special” case and we must determine whether the resulting numeric equation is true or false. Since  $0 = 0$  is true, this system has infinitely many solutions.  
 4. **solution: infinitely many  $\rightarrow \{(x, y) \mid 6x + 4y = 24\}$  –or–  $\{(x, y) \mid 3x + 2y = 12\}$**

$$\begin{array}{rcl} 6x &+& 4y = 24 \\ 3x &+& 2y = 12 \\ \hline 6x &+& 4y = 24 \\ -6x &-& 4y = -24 \\ \hline 0 &=& 0 \end{array}$$

**example 1.22f      Solve this system through algebraic elimination:  $5x - 2y = 20$** 

$$10x - 4y = 20$$

1. The easiest variable to eliminate is the  $y$ . Since we need both  $y$  coefficients to be the same number, multiply the first equation by  $-2$ :

$$-2(5x - 2y = 20) \rightarrow -10x + 4y = -40$$

2. Adding straight down:  $0 = -20$   
 3. Since all of the variables disappeared, this system is a “special” case and we must determine whether the resulting numeric equation is true or false. Since  $0 = -20$  is false, this system has no solutions.  
 4. **solution: no solutions**

$$\begin{array}{rcl} 5x &-& 2y = 20 \\ 10x &-& 4y = 20 \\ \hline -10x &+& 4y = -40 \\ 10x &-& 4y = 20 \\ \hline 0 &=& -20 \end{array}$$

**TUTORIAL 1.23: SOLVE A SYSTEM OF THREE EQUATIONS IN THREE VARIABLES**

We can adapt the elimination (or linear combination) method to solve systems that contain more than two equations. One such adaptation is called the **left to right** method where we eliminate the variable on the far left first and then work our way to the right...

**example 1.23a**    **Solve this system through algebraic elimination:**

$$\begin{aligned}x - 3y - z &= -5 \\x - 2y + z &= 3 \\2x - 6y + z &= -1\end{aligned}$$

Follow these steps:

1. Group the given equations so that they are in pairs.
2. Following the left to right method, we will need to eliminate the  $z$  variable first. Taking advantage of the  $+/-$  sign before the  $z$  terms, add straight down:

$$\begin{array}{rcl}x - 3y - z & = & -5 \\x - 2y + z & = & 3 \\ \hline 2x - 5y & = & -2\end{array} \quad \begin{array}{rcl}x - 3y - z & = & -5 \\2x - 6y + z & = & -1 \\ \hline 3x - 9y & = & -6\end{array}$$

3. Following the left to right method, we will need to eliminate the  $y$  variable next. Pair the two-variable equations from step #2 and then solve for  $x$ :

$$\begin{array}{rcl}2x - 5y & = & -2 \\3x - 9y & = & -6 \\ \hline 18x - 45y & = & -18 \\-15x + 45y & = & 30 \\ \hline 3x & = & 12\end{array}$$

$$x = 4$$

- Multiply the first equation by 9:  
 $9(2x - 5y = -2) \rightarrow 18x - 45y = -18$
- Multiply the second equation by  $-5$ :  
 $-5(3x - 9y = -6) \rightarrow -15x + 45y = 30$
- Add straight down  $\rightarrow 3x = 12$
- Divide by sides of the equation by 3:  $\frac{3x}{3} = \frac{12}{3} \rightarrow x = 4$

4. Using the  $x$  value obtained in step #3, solve for  $y$  using either of the equations obtained in step #2:

$$2x - 5y = -2 \rightarrow 2(4) - 5y = -2 \rightarrow -5y = -2 - 8 = -10 \rightarrow$$

$$\frac{-5y}{-5} = \frac{-10}{-5} \rightarrow y = 2$$

$$3x - 9y = -6 \rightarrow 3(4) - 9y = -6 \rightarrow -9y = -6 - 12 = -18 \rightarrow$$

$$\frac{-9y}{-9} = \frac{-18}{-9} \rightarrow y = 2$$

5. Using the  $x$  value obtained in step #3 and the  $y$  value obtained in step #4, solve for  $z$  using any of the original equations:

$$x - 3y - z = -5 \rightarrow (4) - 3(2) - z = -5 \rightarrow -z = -5 - 4 + 6 = -3 \\ \rightarrow z = 3$$

$$x - 2y + z = 3 \rightarrow (4) - 2(2) + z = 3 \rightarrow z = 3 - 4 + 4 = 3 \rightarrow z = 3$$

$$2x - 6y + z = -1 \rightarrow 2(4) - 6(2) + z = -1 \rightarrow z = -1 - 8 + 12 = 3 \rightarrow z = 3$$

6. **solution:**  $(x, y, z) = (4, 2, 3)$

7. verification: all three tests are passed

$$x - 3y - z = -5 \rightarrow (4) - 3(2) - (3) = -5 \rightarrow 4 - 6 - 3 = -5$$

$$x - 2y + z = 3 \rightarrow (4) - 2(2) + (3) = 3 \rightarrow 4 - 4 + 3 = 3$$

$$2x - 6y + z = -1 \rightarrow 2(4) - 6(2) + (3) = -1 \rightarrow 8 - 12 + 3 = -1$$



**Deloitte.**

Discover the truth at [www.deloitte.ca/careers](http://www.deloitte.ca/careers)

© Deloitte & Touche LLP and affiliated entities.

Download free eBooks at [bookboon.com](http://bookboon.com)



**Click on the ad to read more**

**example 1.23b**      **Solve this system through algebraic elimination:**

$$\begin{aligned}x + 4y - 2z &= 3 \\x + 5y + 2z &= 25 \\x + 4y - z &= 8\end{aligned}$$

Follow these steps:

1. Group the given equations so that they are in pairs.
2. Following the left to right method, we will need to eliminate the  $z$  variable first. Taking advantage of the  $+/-$  sign before the  $z$  terms, add straight down for the first pair. We will need to multiply the second equation for the second pair by 2 and then add down:

$$\begin{array}{rcl}x &+& 4y &-& 2z &=& 3 \\x &+& 5y &+& 2z &=& 25 \\ \hline 2x &+& 9y && &=& 28\end{array} \quad \begin{array}{rcl}x &+& 5y &+& 2z &=& 25 \\x &+& 4y &-& z &=& 8 \\ \hline x &+& 5y &+& 2z &=& 25 \\2x &+& 8y &-& 2z &=& 16 \\ \hline 3x &+& 13y && &=& 41\end{array}$$

3. Following the left to right method, we will need to eliminate the  $y$  variable next. Pair the two-variable equations from step #2 and then solve for  $x$ :
  - Multiply the first equation by  $-13$ :  $-13(2x + 9y = 28) \rightarrow -26x - 117y = -364$
  - Multiply the second equation by  $+9$ :  $9(3x + 13y = 41) \rightarrow 27x + 117y = 369$
  - Add straight down  $\rightarrow x = 5$
4. Using the  $x$  value obtained in step #3, solve for  $y$  using either of the equations obtained in step #2:
 
$$2x + 9y = 28 \rightarrow 2(5) + 9y = 28 \rightarrow 9y = 28 - 10 = 18 \rightarrow$$

$$\frac{9y}{9} = \frac{18}{9} \rightarrow y = 2$$

$$3x + 13y = 41 \rightarrow 3(5) + 13y = 41 \rightarrow 13y = 41 - 15 = 26 \rightarrow$$

$$\frac{13y}{13} = \frac{26}{13} \rightarrow y = 2$$

5. Using the  $x$  value obtained in step #3 and the  $y$  value obtained in step #4, solve for  $z$  using any of the original equations:
- $$x + 4y - 2z = 3 \rightarrow (5) + 4(2) - 2z = 3 \rightarrow -2z = 3 - 5 - 8 = -10 \rightarrow$$

$$\frac{-2z}{-2} = \frac{-10}{-2} \rightarrow z = 5$$

$$x + 5y + 2z = 25 \rightarrow (5) + 5(2) + 2z = 25 \rightarrow 2z = 25 - 5 - 10 = 10 \rightarrow$$

$$\frac{2z}{2} = \frac{10}{2} \rightarrow z = 5$$

$$x + 4y - z = 8 \rightarrow (5) + 4(2) - z = 8 \rightarrow -z = 8 - 5 - 8 = -5 \rightarrow$$

$$\frac{-z}{-1} = \frac{-5}{-1} \rightarrow z = 5$$

6. **solution:  $(x, y, z) = (5, 2, 5)$**

7. verification: all three tests are passed

$$x + 4y - 2z = 3 \rightarrow (5) + 4(2) - 2(5) = 3 \rightarrow 5 + 8 - 10 = 3$$

$$x + 5y + 2z = 25 \rightarrow (5) + 5(2) + 2(5) = 25 \rightarrow 5 + 10 + 10 = 25$$

$$x + 4y - z = 8 \rightarrow (5) + 4(2) - (5) = 8 \rightarrow 5 + 8 - 5 = 8$$

Another adaptation of the elimination (or linear combination) method is called the ***right to left*** method where we eliminate the variable on the far right first and then work our way to the left...following the same process that we did with the *left to right* method.

**example 1.23a**      **Solve this system through algebraic elimination:**

$$\begin{aligned}x - 3y - z &= -5 \\x - 2y + z &= 3 \\2x - 6y + z &= -1\end{aligned}$$

Follow these steps:

1. Group the given equations so that they are in pairs.
2. Following the right to left method, we will need to eliminate the  $x$  variable first. Multiply the top equation for the first pair by  $-1$  and then add straight down. Multiply the top equation for the second pair by  $-2$  and then add straight down:

$$\begin{array}{rcl}x & - & 3y & - & z & = & -5 \\x & - & 2y & + & z & = & 3 \\ \hline -x & + & 3y & + & z & = & 5 \\ x & - & 2y & + & z & = & 3 \\ \hline y & + & 2z & = & 8\end{array} \quad \begin{array}{rcl}x & - & 3y & - & z & = & -5 \\2x & - & 6y & + & z & = & -1 \\ \hline -2x & + & 6y & + & 2z & = & 10 \\ 2x & - & 6y & + & z & = & -1 \\ \hline 3z & = & 9\end{array}$$

SIMPLY CLEVER

ŠKODA



We will turn your CV into  
an opportunity of a lifetime



Do you like cars? Would you like to be a part of a successful brand?  
We will appreciate and reward both your enthusiasm and talent.  
Send us your CV. You will be surprised where it can take you.

Send us your CV on  
[www.employerforlife.com](http://www.employerforlife.com)



3. Following the right to left method, we will need to eliminate the  $y$  variable next. Since the  $y$  was eliminated in step #2 for the second pair of equations, all we need to do is use the second pairing to solve for  $z$ . Dividing both sides of the equation, we get:  $\mathbf{z = 3}$ .
4. Using the  $z$  value obtained in step #3, we would normally use either of the equations obtained in step #2...however, since the second equation already has  $y$  eliminated, just use the first equation obtained in step #2 to solve for  $y$ :

$$z = 3 \rightarrow y + 2z = 8 \rightarrow y + 2(3) = 8 \rightarrow y = 8 - 6 = 2 \rightarrow \mathbf{y = 2}$$

5. Using the  $z$  value obtained in step #3 and the  $y$  value obtained in step #4, solve for  $x$  using any of the original equations:

$$x - 3y - z = -5 \rightarrow x - 3(2) - (3) = -5 \rightarrow x = -5 + 6 + 3 = 4 \rightarrow \mathbf{x = 4}$$

$$x - 2y + z = 3 \rightarrow x - 2(2) + (3) = 3 \rightarrow x = 3 + 4 - 3 = 4 \rightarrow \mathbf{x = 4}$$

$$2x - 6y + z = -1 \rightarrow 2x - 6(2) + (3) = -1 \rightarrow 2x = -1 + 12 - 3 = 8 \rightarrow$$

$$\frac{2x}{2} = \frac{8}{2} \rightarrow \mathbf{x = 4}$$

6. **solution:  $(x, y, z) = (4, 2, 3)$**

7. verification: all three tests are passed

$$x - 3y - z = -5 \rightarrow (4) - 3(2) - (3) = -5 \rightarrow 4 - 6 - 3 = -5$$

$$x - 2y + z = 3 \rightarrow (4) - 2(2) + (3) = 3 \rightarrow 4 - 4 + 3 = 3$$

$$2x - 6y + z = -1 \rightarrow 2(4) - 6(2) + (3) = -1 \rightarrow 8 - 12 + 3 = -1$$

**example 1.23b**      Solve this system through algebraic elimination:

$$\begin{aligned}x + 4y - 2z &= 3 \\x + 5y + 2z &= 25 \\x + 4y - z &= 8\end{aligned}$$

Follow these steps:

1. Group the given equations so that they are in pairs.
2. Following the right to left method, we will need to eliminate the  $x$  variable first. Multiply the top equation for the first pair by  $-1$ , the bottom equation for the second pair by  $-1$  and then add straight down for both pairs of equations:

$$\begin{array}{rcl}x + 4y - 2z &=& 3 \\x + 5y + 2z &=& 25 \\ \hline -x - 4y + 2z &=& -3 \\ x + 5y + 2z &=& 25 \\ \hline y + 4z &=& 22\end{array} \quad \begin{array}{rcl}x + 5y + 2z &=& 25 \\x + 4y - z &=& 8 \\ \hline x + 5y + 2z &=& 25 \\ -x - 4y + z &=& -8 \\ \hline y + 3z &=& 17\end{array}$$

3. Following the right to left method, we will need to eliminate the  $y$  variable next. Pair the two-variable equations from step #2 and then solve for  $x$ :
 
$$\begin{array}{rcl}y + 4z &=& 22 \\y + 3z &=& 17 \\ \hline y + 4z &=& 22 \\ -1y - 3z &=& -17 \\ \hline z &=& 5\end{array}$$
  - Multiply the second equation by  $-1$ :  $-1(y + 3z = 17) \rightarrow -1y - 3z = -17$
  - Add straight down  $\rightarrow z = 5$
4. Using the  $z$  value obtained in step #3, solve for  $y$  using either of the equations obtained in step #2:
 
$$\begin{aligned}y + 4z &= 22 \rightarrow y + 4(5) = 22 \rightarrow y = 22 - 20 = 2 \rightarrow y = 2 \\y + 3z &= 17 \rightarrow y + 3(5) = 17 \rightarrow y = 17 - 15 = 2 \rightarrow y = 2\end{aligned}$$

5. Using the  $z$  and  $y$  values obtained in steps #3-4, solve for  $x$  using any of the original equations:

$$x + 4y - 2z = 3 \rightarrow x + 4(2) - 2(5) = 3 \rightarrow x = 3 - 8 + 10 = 5 \rightarrow \mathbf{x = 5}$$

$$x + 5y + 2z = 25 \rightarrow x + 5(2) + 2(5) = 25 \rightarrow x = 25 - 10 - 10 = 5 \rightarrow \mathbf{x = 5}$$

$$x + 4y - z = 8 \rightarrow x + 4(2) - (5) = 8 \rightarrow x = 8 - 8 + 5 = 5 \rightarrow \mathbf{x = 5}$$

6. **solution:**  $(x, y, z) = (5, 2, 5)$

7. verification: all three tests are passed

$$x + 4y - 2z = 3 \rightarrow (5) + 4(2) - 2(5) = 3 \rightarrow 5 + 8 - 10 = 3$$

$$x + 5y + 2z = 25 \rightarrow (5) + 5(2) + 2(5) = 25 \rightarrow 5 + 10 + 10 = 25$$

$$x + 4y - z = 8 \rightarrow (5) + 4(2) - (5) = 8 \rightarrow 5 + 8 - 5 = 8$$

Notice that whether we use the left to right method or the right to left method, we are really doing the same steps. The only difference between these two methods is which variable is the first to be eliminated in the solution process.

**e-learning for kids**

- The number 1 MOOC for Primary Education
- Free Digital Learning for Children 5-12
- 15 Million Children Reached

**About e-Learning for Kids** Established in 2004, e-Learning for Kids is a global nonprofit foundation dedicated to fun and free learning on the Internet for children ages 5 - 12 with courses in math, science, language arts, computers, health and environmental skills. Since 2005, more than 15 million children in over 190 countries have benefitted from eLessons provided by EFK! An all-volunteer staff consists of education and e-learning experts and business professionals from around the world committed to making difference. eLearning for Kids is actively seeking funding, volunteers, sponsors and courseware developers; get involved! For more information, please visit [www.e-learningforkids.org](http://www.e-learningforkids.org).



## TUTORIAL 1.24: SOLVE APPLICATIONS INVOLVING SYSTEMS OF EQUATIONS

When we are solving real-world applications, we will need to translate the written word into an appropriate (equivalent) set of equations that can then be solved as a system (as was explained in previous tutorials).

**example 1.24a** A woman has \$450,000 invested in two rental properties. One ( $x$ ) yields an annual return of 10% of her investment and the other ( $y$ ) returns 15% per year on her investment. Her total amount return from the two investments is \$55,000. How much is invested in each property?

1. Translate the given information sentences into their equivalent equations:

- *A woman has \$450,000 invested in two rental properties.*

$$\text{translation} \rightarrow x + y = 450,000$$

- *One ( $x$ ) yields an annual return of 10% of her investment and the other ( $y$ ) returns 15% per year on her investment. Her total amount return from the two investments is \$55,000.*

$$\text{translation} \rightarrow 0.10x + 0.15y = 55,000$$

2. Solve the system of equations obtained from the translations using the elimination method:

$$\begin{array}{rcl}
 x & + & y = 450,000 \\
 0.10x & + & 0.15y = 55,000 \quad \text{Multiply this equation by } -10 \\
 \hline
 x & + & y = 450,000 \\
 -x & - & 1.5y = -550,000 \quad \text{Add straight down} \\
 \hline
 & -0.5y & = -100,000 \quad \text{Divide both sides of the} \\
 & y & = 200,000 \quad \text{equation by } -0.5
 \end{array}$$

3. Use the value of  $y$  obtained above and either of the original equations to solve for  $x$ :

$$x + y = 450,000 \rightarrow x + 200,000 = 450,000 \rightarrow x = 450,000 - 200,000 \\ \rightarrow x = 250,000$$

$$0.10x + 0.15y = 55,000 \rightarrow 0.10x + 0.15(200,000) = 55,000 \rightarrow \\ 0.10x = 55,000 - 30,000 \rightarrow 0.10x = 25,000 \rightarrow \\ x = (25,000)(10) = 250,000$$

4. Use the derived solutions to answer the word problem:

**solution: The woman invested \$250,000 at the 10% rate and \$200,000 at the 15% rate.**

**example 1.24b** A bank lent \$250,000 to a company for the development of two products. If the loan for product “A” was for \$75,000 more than that for product “B”, how much was lent for each product?

1. Translate the given information sentences into their equivalent equations:

- *A bank lent \$250,000 to a company for the development of two products.*

$$\text{translation} \rightarrow A + B = 250,000$$

- *The loan for product “A” was for \$75,000 more than that for product “B”.*

$$\text{translation} \rightarrow A = B + 75,000$$

2. Solve the system of equations obtained from the translations using the substitution method:

$$A + B = 250,000$$

$$A = B + 75,000$$

$$(B + 75,000) + B = 250,000$$

*Substitute the expression for A into the other equation.*

$$2B + 75,000 = 250,000$$

*Combine like-terms.*

$$2B = 250,000 - 75,000 = 175,000$$

*Subtract 75,000 from both sides of the equation.*

$$B = \frac{175,000}{2} = 87,500$$

*Divide both sides of the equation by 2.*

3. Use the value of  $B$  obtained above and either of the original equations to solve for  $A$ :

$$A = B + 75,000 \rightarrow A = 87,500 + 75,000 \rightarrow A = 162,500$$

$$A + B = 250,000 \rightarrow A + 87,500 = 250,000 \rightarrow A = 250,000 - 87,500 \rightarrow A = 162,500$$

4. Use the derived solutions to answer the word problem:

**solution: The bank lent \$162,500 for product “A” and \$87,500 for product “B”.**

Cynthia | AXA Graduate

**AXA Global Graduate Program**

Find out more and apply

redefining / standards **AXA**

**example 1.24c**

The Monmouth Emergency Management Agency is charged with providing services to three types of clients ( $A$ ,  $B$ ,  $C$ ) after a major crisis. A total of 1000 clients are to be served, with \$445,000 available for counseling and \$285,000 available for food/shelter. Type  $A$  clients require an average of \$300 for counseling and \$400 for food/shelter. Type  $B$  clients require an average of \$600 for counseling and \$300 for food/shelter. Type  $C$  clients require an average of \$400 for counseling and \$200 for food/shelter. Under these constraints, how many of each type of client can be served?

1. Translate the given information sentences into their equivalent equations:

- *A total of 1000 clients are to be served.*

$$\text{translation} \rightarrow A + B + C = 1000$$

- *counseling:  $A$  @ \$300,  $B$  @ \$600,  $C$  @ 400,*

$$\text{total available} = \$400,000$$

$$\text{translation} \rightarrow 300A + 600B + 400C = 445,000$$

- *food/shelter:  $A$  @ \$400,  $B$  @ \$300,  $C$  @ \$200,*

$$\text{total available} = \$312,500$$

$$\text{translation} \rightarrow 400A + 300B + 200C = 285,000$$

2. Solve the system of equations obtained from the translations using the substitution method:

- Pair the equations:

$$300A + 600B + 400C = 445,000 \text{ with } A + B + C = 1000$$

$$400A + 300B + 200C = 285,000 \text{ with } A + B + C = 1000$$

- Using the right to left method, eliminate  $C$  first by multiplying the second equations so that the  $C$  terms cancel each other out:

$$\begin{array}{rcccl}
 300A & + & 600B & + & 400C = 445,000 \\
 A & + & B & + & C = 1000 \\
 \hline
 300A & + & 600B & + & 400C = 445,000 \\
 -400A & + & -400B & + & -400C = -400,000 \\
 \hline
 -100A & + & 200B & & = 45,000
 \end{array}$$

$$\begin{array}{rcccl}
 400A & + & 300B & + & 200C = 285,000 \\
 A & + & B & + & C = 1000 \\
 \hline
 400A & + & 300B & + & 200C = 285,000 \\
 -200A & + & -200B & + & -200C = -200,000 \\
 \hline
 200A & + & 100B & & = 85,000
 \end{array}$$

- Pair the 2-variable equations that were the result of the previous step. Since the numbers all have at least two zeroes, we can make the numbers smaller by dividing each of these equations:

$$\begin{array}{rcccl}
 -100A & + & 200B & = & 45,000 \\
 200A & + & 100B & = & 85,000 \\
 \hline
 A & - & 2B & = & -450 \\
 2A & + & B & = & 850 \\
 \hline
 A & - & 2B & = & -450 \\
 4A & + & 2B & = & 1700 \\
 \hline
 5A & & & = & 1250 \\
 A & & & = & 250
 \end{array}$$

- ✓ Divide the top equation by  $-100 \rightarrow A - 2B = -450$
- ✓ Divide the bottom equation by  $+100 \rightarrow 2A + B = 850$
- ✓ Multiply the new second equation by 2  $\rightarrow 4A + 2B = 1700$
- ✓ Add straight down  $\rightarrow 5A = 1250$
- ✓ Divide by 5:  $\frac{5A}{5} = \frac{1250}{5} \rightarrow A = 250$

- Use the  $A$  value to determine the  $B$  value:

$$\begin{aligned}-100A + 200B &= 45,000 \rightarrow -100(250) + 200B = 45,000 \rightarrow \\ 200B &= 45,000 + 100(250) \rightarrow 200B = 70,000 \rightarrow \\ \frac{200B}{200} &= \frac{70,000}{200} \rightarrow B = 350\end{aligned}$$

$$\begin{aligned}200A + 100B &= 85,000 \rightarrow 200(250) + 100B = 85,000 \rightarrow \\ 100B &= 85,000 - 200(250) \rightarrow 100B = 35,000 \rightarrow \\ \frac{100B}{100} &= \frac{35,000}{100} \rightarrow B = 350\end{aligned}$$

- Use one of the original equations and the values of  $A$  and  $B$  to determine the value of  $C$ :

$$A + B + C = 1000 \rightarrow C = 1000 - A - B \rightarrow C = 1000 - 250 - 350 \rightarrow C = 400$$

$$\begin{aligned}300A + 600B + 400C &= 445,000 \rightarrow \\ 400C &= 445,000 - 300A - 600B \rightarrow \\ 400C &= 445,000 - 300(250) - 600(350) \rightarrow \\ 400C &= 160,000 \rightarrow \frac{400C}{400} = \frac{160,000}{400} \rightarrow C = 400\end{aligned}$$

$$\begin{aligned}400A + 300B + 200C &= 285,000 \rightarrow \\ 200C &= 285,000 - 400A - 300B \rightarrow \\ 200C &= 285,000 - 400(250) - 300(350) \rightarrow \\ 200C &= 80,000 \rightarrow \frac{200C}{200} = \frac{80,000}{200} \rightarrow C = 400\end{aligned}$$

3. Use the derived solutions to answer the word problem:

**solution: The Monmouth Emergency Management Agency has the capability of serving 250 type "A" clients, 350 type "B" clients and 400 type "C" clients.**

## TUTORIAL 1.25: FIND THE MARGINAL COST, THE MARGINAL COST REVENUE AND THE MARGINAL COST PROFIT OF GIVEN LINEAR TOTAL COST FUNCTIONS, LINEAR TOTAL REVENUE FUNCTIONS AND LINEAR PROFIT FUNCTIONS

When a company produces an item for sale in a retail market, the company must pay attention to the cost of production (money going out), the amount of revenue (money coming in) and the profit made.

**Total Cost Function**

$$C(x) = \begin{pmatrix} \text{variable} \\ \text{costs} \end{pmatrix} x + \begin{pmatrix} \text{fixed} \\ \text{costs} \end{pmatrix}$$

**Total Revenue Function**

$$R(x) = \begin{pmatrix} \text{variable} \\ \text{revenue} \end{pmatrix} x$$

**Total Profit Function**

$$P(x) = \begin{pmatrix} \text{variable} \\ \text{profits} \end{pmatrix} x + \begin{pmatrix} \text{fixed} \\ \text{profits} \end{pmatrix}$$

In a linear total cost function, the  $x$  represents the number of units being produced. The *variable costs* can be thought of as the *rate of change* or *slope* of the function while the *fixed costs* can be thought of as the vertical ( $y$ ) intercept. Occasionally, the *variable costs* are also referred to as *marginal costs*.

In a linear total revenue function, the  $x$  represents the number of units being produced. The *variable revenue* can be thought of as the *rate of change* or *slope* of the function. Notice that a linear total revenue function will have a vertical ( $y$ ) intercept of zero. The *variable revenue* may also be referred to as *marginal revenue*.

Profit is the difference between the amount received from sales (revenue) and the total amount spent to produce the product (cost). Thus, when the revenue and the costs are known, profit can be calculated by:

$$\text{Profit} = (\text{Total Revenue}) - (\text{Total Cost}) \dots \text{or} \dots P(x) = R(x) - C(x)$$

I joined MITAS because  
I wanted **real responsibility**

The Graduate Programme  
for Engineers and Geoscientists  
[www.discovermitas.com](http://www.discovermitas.com)



**Month 16**

I was a construction supervisor in the North Sea advising and helping foremen solve problems

Real work  
International opportunities  
Three work placements





In a linear profit function, the  $x$  represents the number of units being produced. The *variable profits* can be thought of as the *rate of change* or *slope* of the function while the *fixed profits* can be thought of as the vertical ( $y$ ) intercept. The *variable profits* may also be referred to as *marginal profits*.

**example 1.25a****Use this information to answer the questions:**

Suppose a stereo receiver manufacturer has the given total cost function and the given total revenue function:

$$C(x) = 200x + 3500 \text{ and } R(x) = 450x$$

a) What is the variable cost for this product?

**answer: 200** (*The variable cost is the slope of the given cost function.*)

b) What is the marginal cost for this product?

**answer: 200** (*The marginal cost is the slope of the given cost function.*)

c) What is the fixed cost for this product?

**answer: 3500** (*The fixed cost is the  $y$ -intercept of the given cost function.*)

d) What is the variable revenue for this product?

**answer: 450** (*The variable revenue is the slope of the given revenue function.*)

e) What is the marginal revenue for this product?

**answer: 450** (*The marginal revenue is the slope of the given revenue function.*)

f) What is the equation for the profit function for this product?

$$P(x) = R(x) - C(x) = (450x) - (200x + 3500) = 250x - 3500$$

**answer:  $P(x) = 250x - 3500$**

g) What is the variable profit for this product?

**answer: 250** (*The variable profit is the slope of the profit function.*)

h) What is the marginal profit for this product?

**answer: 250** (*The marginal profit is the slope of the profit function.*)

i) What is the fixed profit for this product?

**answer: -3500** (*The fixed profit is the  $y$ -intercept of the profit function.*)

**example 1.25b****Use this information to answer the questions:**

Suppose a computer manufacturer has the given total cost function and the given total revenue function:

$$C(x) = 250x + 4000 \text{ and } R(x) = 750x$$

- a) What is the variable cost for this product?

**answer: 250** (*The variable cost is the slope of the given cost function.*)

- b) What is the marginal cost for this product?

**answer: 250** (*The marginal cost is the slope of the given cost function.*)

- c) What is the fixed cost for this product?

**answer: 4000** (*The fixed cost is the y-intercept of the given cost function.*)

- d) What is the variable revenue for this product?

**answer: 750** (*The variable revenue is the slope of the given revenue function.*)

- e) What is the marginal revenue for this product?

**answer: 750** (*The marginal revenue is the slope of the given revenue function.*)

- f) What is the equation for the profit function for this product?

$$P(x) = R(x) - C(x) = (750x) - (250x + 4000) = 500x - 4000$$

**answer:  $P(x) = 500x - 4000$**

- g) What is the variable profit for this product?

**answer: 500** (*The variable profit is the slope of the profit function.*)

- h) What is the marginal profit for this product?

**answer: 500** (*The marginal profit is the slope of the profit function.*)

- i) What is the fixed profit for this product?

**answer: -4000** (*The fixed profit is the y-intercept of the profit function.*)

## TUTORIAL 1.26: EVALUATE LINEAR TOTAL COST FUNCTIONS, LINEAR TOTAL REVENUE FUNCTIONS AND LINEAR PROFIT FUNCTIONS

When we want to evaluate a linear function, we must replace the variable with the given value that is to be used in the evaluation. Following the order of operations (PEMDAS), we just calculate the desired result.

**example 1.26a**

Suppose a stereo receiver manufacturer has the given total cost function and the given total revenue function:

$$C(x) = 200x + 3500 \text{ and } R(x) = 450x$$

- a) If 1000 units are produced, find the total costs for this product.

$$x = 1000 \rightarrow C(1000) = 200(1000) + 3500 = 203,500$$

**answer: \$203,500**

- b) If 1000 units are produced, find the total revenue for this product.

$$x = 1000 \rightarrow R(1000) = 450(1000) = 450,000$$

**answer: \$450,000**

- c) If 1000 units are produced, find the total profits for this product.

$$\text{profit} = \text{revenue} - \text{cost} = \$450,000 - \$203,500 = \$246,500$$

$$P(x) = R(x) - C(x) \rightarrow P(1000) = R(1000) - C(1000) \rightarrow$$

$$P(1000) = 450(1000) - [200(1000) + 3500] \rightarrow$$

$$P(1000) = 450,000 - 200,000 - 3500 = 246,500$$

**answer: \$246,500**

The advertisement features a background image of modern skyscrapers. In the upper left, the IE business school logo is displayed. The central text reads: "93% OF MIM STUDENTS ARE WORKING IN THEIR SECTOR 3 MONTHS FOLLOWING GRADUATION". Below this, the program name "MASTER IN MANAGEMENT" is prominently displayed. To the left of the main text, there is a bulleted list of benefits: "STUDY IN THE CENTER OF MADRID AND TAKE ADVANTAGE OF THE UNIQUE OPPORTUNITIES THAT THE CAPITAL OF SPAIN OFFERS", "PROPEL YOUR EDUCATION BY EARNING A DOUBLE DEGREE THAT BEST SUITS YOUR PROFESSIONAL GOALS", and "STUDY A SEMESTER ABROAD AND BECOME A GLOBAL CITIZEN WITH THE BEYOND BORDERS EXPERIENCE". To the right, detailed information is provided: "Length: 10 MONTHS", "Av. Experience: 1 YEAR", "Language: ENGLISH / SPANISH", "Format: FULL-TIME", and "Intakes: SEPT / FEB". At the bottom, three boxes highlight "5 SPECIALIZATIONS PERSONALIZE YOUR PROGRAM", "#10 WORLDWIDE MASTER IN MANAGEMENT FINANCIAL TIMES", and "55 NATIONALITIES IN CLASS". The footer contains the website "www.ie.edu/master-management", email "mim.admissions@ie.edu", social media links for Facebook, Twitter, and LinkedIn, and the text "Follow us on IE MIM Experience".

**example 1.26b** Suppose a computer manufacturer has the given total cost function and the given total revenue function:

$$C(x) = 250x + 4000 \text{ and } R(x) = 750x$$

- a) If 1500 units are produced, find the total costs for this product.

$$x = 1500 \rightarrow C(1500) = 250(1500) + 4000 = 379,000$$

**answer: \$379,000**

- b) If 1500 units are produced, find the total revenue for this product.

$$x = 1500 \rightarrow R(1500) = 750(1500) = 1,125,000$$

**answer: \$1,125,000**

- c) If 1500 units are produced, find the total profits for this product.

$$\text{profit} = \text{revenue} - \text{cost} = \$1,125,000 - \$379,000 = \$746,000$$

$$P(x) = R(x) - C(x) \rightarrow P(1500) = R(1500) - C(1500) \rightarrow$$

$$P(1500) = 750(1500) - [250(1500) + 4000] \rightarrow$$

$$P(1500) = 1,125,000 - 375,000 - 4000 = 746,000$$

**answer: \$746,000**

## TUTORIAL 1.27: WRITE THE EQUATIONS FOR LINEAR TOTAL COST FUNCTIONS, LINEAR TOTAL REVENUE FUNCTIONS AND LINEAR PROFIT FUNCTIONS BY USING INFORMATION GIVEN ABOUT THE FUNCTIONS

When we want to write the equation for any linear function all we need to do is to use the given information to determine the slope and the  $y$ -intercept of the function (as was described in previous tutorials)...

**example 1.27a**

Suppose that the total cost function for an iPod is linear, that the marginal cost is \$30 and that the total cost for fifty iPods is \$5000.

Write the slope-intercept form for this function.

- marginal cost is \$30  $\rightarrow m = \text{slope} = 30$
- total cost for 50 iPods is \$5000  $\rightarrow (50, 5000) \rightarrow x = 50$   
when  $y = 5000$
- slope-intercept:  $y = mx + b \rightarrow b = y - mx \rightarrow$   
 $b = 5000 - (30)(50) = 5000 - 1500 = 3500$
- write the desired equation  $\rightarrow$  **answer:  $C(x) = 30x + 3500$**

**example 1.27b**

Suppose that the total revenue function for an iPod is linear and that the marginal revenue is \$150. Write the slope-intercept form for this function.

- marginal cost is \$150 →  $m = \text{slope} = 150$
- The  $y$ -intercept for all revenue functions is  $(0, 0)$ ...thus,  $b = 0$
- write the desired equation → **answer:  $R(x) = 150x$**

**example 1.27b**

A company is charting its profits and discovered that the relationship between the number of items sold and the profit is linear. If 200 units sold results in \$3000 profit and 250 units sold results in \$5000 profit, write the slope-intercept form for this function.

- Find the slope of  $(200, 3000)$  and  $(250, 5000)$ :

$$\text{slope } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5000 - 3000}{250 - 200} = \frac{2000}{50} = 40$$

- Use one of the given ordered pairs to determine the value of  $b$ :  
slope-intercept:  $y = mx + b \rightarrow b = y - mx \rightarrow$   
 $(200, 3000) \rightarrow x = 200, y = 3000 \rightarrow b = 3000 - (40)(200) = -5000$   
 $(250, 5000) \rightarrow x = 250, y = 5000 \rightarrow b = 5000 - (40)(250) = -5000$
- write the desired equation → **answer:  $P(x) = 40x - 5000$**

### TUTORIAL 1.28: FIND THE BREAK-EVEN POINT FOR THE GIVEN COST AND REVENUE FUNCTIONS

The phrase ***break-even point*** refers to the point at which the total cost and the total revenue are equal to each other:  $C(x) = R(x)$ . Or, in other words, when there is no profit:  $P(x) = 0$ . When we use the total cost and the total revenue functions to determine the break-even point, we are actually solving a system of linear equations...

**example 1.28a**

Find the number of units ( $x$ ) that must be sold to break even if a manufacturer has these given total cost and total revenue functions:

$$C(x) = 200x + 3500 \text{ and } R(x) = 450x$$

**solution #1:**

$$\text{break-even: } C(x) = R(x) \rightarrow 200x + 3500 = 450x$$

$$\text{Subtracting } 200x \text{ from both sides} \rightarrow 3500 = 450x - 200x = 250x \rightarrow \\ 3500 = 250x$$

$$\text{Dividing both sides by } 250 \rightarrow x = 3500 \div 250 = 14$$

Evaluate the  $C(x)$  function when  $x = 14 \rightarrow$

$$C(14) = 200(14) + 3500 = 6300$$

Evaluate the  $R(x)$  function when  $x = 14 \rightarrow R(14) = 450(14) = 6300$

**answer: The break-even point occurs when  $x = 14$  and  $y = 6300$**

**“I studied English for 16 years but...  
...I finally learned to speak it in just six lessons”**

Jane, Chinese architect

ENGLISH OUT THERE

Click to hear me talking before and after my unique course download



**solution #2:**

$$C(x) = 200x + 3500; R(x) = 450x \rightarrow P(x) = R(x) - C(x) \rightarrow$$

$$P(x) = 450x - (200x + 3500) = 450x - 200x - 3500 = 250x - 3500$$

break-even:  $P(x) = 0 \rightarrow 250x - 3500 = 0 \rightarrow 250x = 3500 \rightarrow x = 14$

Evaluate the  $C(x)$  function when  $x = 14 \rightarrow$

$$C(14) = 200(14) + 3500 = 6300$$

Evaluate the  $R(x)$  function when  $x = 14 \rightarrow R(14) = 450(14) = 6300$

**answer: The break-even point occurs when  $x = 14$  and  $y = 6300$**

**example 1.28b**

A company distributes college logo sweatshirts and sells them for \$74 each. Assuming that the total cost function is linear, the total cost for 100 sweatshirts is \$5000 while the total cost for 250 sweatshirts is \$8600. How many sweatshirts must be sold to break even?

- Determine the total revenue function for this scenario:

*A company distributes sweatshirts and sells them for \$74 each*  $\rightarrow$   
 $R(x) = 74x$

- Determine the total cost function for this scenario:

- Find the slope between the given points:

$$(100, 5000) \text{ and } (250, 8600) \rightarrow$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8600 - 5000}{250 - 100} = \frac{3600}{150} = 24$$

- Use one of the given points to determine  $b \rightarrow$   
 $(100, 5000) \rightarrow b = y - mx = 5000 - 24(100) = 2600$   
 $(250, 8600) \rightarrow b = y - mx = 8600 - 24(250) = 2600$
- Write the slope-intercept form of  $C(x) \rightarrow C(x) = 24x + 2600$

**solution #1:**

break-even:  $C(x) = R(x) \rightarrow 24x + 2600 = 74x$

Subtracting  $24x$  from both sides  $\rightarrow 2600 = 74x - 24x = 50x \rightarrow$

$$2600 = 50x$$

Dividing both sides by 50  $\rightarrow x = 2600 \div 50 = 52$

Evaluate the  $C(x)$  function when  $x = 52 \rightarrow C(52) = 24(52) + 2600 = 3848$

Evaluate the  $R(x)$  function when  $x = 52 \rightarrow R(52) = 74(52) = 3848$

**answer: The break-even point occurs when  $x = 52$  and  $y = 3848$ .**

**solution #2:**

$$C(x) = C(x) = 24x + 2600; R(x) = 74x \rightarrow P(x) = R(x) - C(x) \rightarrow \\ P(x) = 74x - (24x + 2600) = 74x - 24x - 2600 = 50x - 2600$$

break-even:  $P(x) = 0 \rightarrow 50x - 2600 = 0 \rightarrow 50x = 2600 \rightarrow x = 52$

Evaluate the  $C(x)$  function when  $x = 52 \rightarrow C(52) = 24(52) + 2600 = 3848$

Evaluate the  $R(x)$  function when  $x = 52 \rightarrow R(52) = 74(52) = 3848$

**answer: The break-even point occurs when  $x = 52$  and  $y = 3848$ .**

## TUTORIAL 1.29: EVALUATE AND GRAPH SUPPLY AND DEMAND FUNCTIONS

The Laws of Supply and Demand state that:

- The quantity ( $x$ ) supplied will increase as the price of a product increases...or... will decrease as the price of a product decreases.
- The quantity ( $x$ ) demanded will increase as the price of a product decreases...or... will decrease as the price of a product increases.

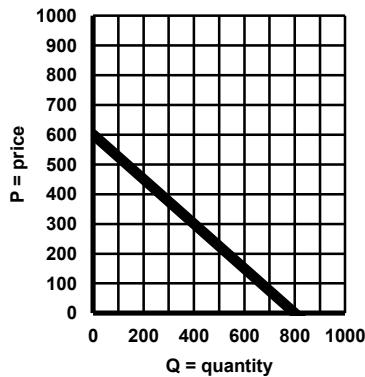
If the supply function for a particular product is linear, then the slope of its graph will be positive since the quantity and the price both go up together or down together. If the demand function for a particular product is linear, then the slope of its graph will be negative since the quantity goes up while the price goes down...or vice-versa: the quantity goes down while the price goes up.

If  $S(x)$  = the supply function and  $D(x)$  = the demand function, comparing  $S(x)$  with  $D(x)$  will help us determine whether there will be a **surplus** (having more than needed to satisfy the demand) or a **shortage** (not having enough to satisfy the demand) of the product in question:

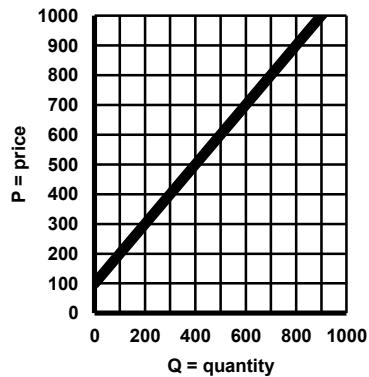
- A surplus will occur when  $S(x) > D(x)$ .
- A shortage will occur when  $S(x) < D(x)$ .

**example 1.29a**

The graphs of the supply function and the demand function for the same product are shown below. Use these graphs to answer the following questions:



graph #1



graph #2

Excellent Economics and Business programmes at:

 / **university of  
groningen**





**“The perfect start  
of a successful,  
international career.”**

[www.rug.nl/feb/education](http://www.rug.nl/feb/education)

**CLICK HERE**  
to discover why both socially  
and academically the University  
of Groningen is one of the best  
places for a student to be



a) Which of the given graphs is the display for the supply function?

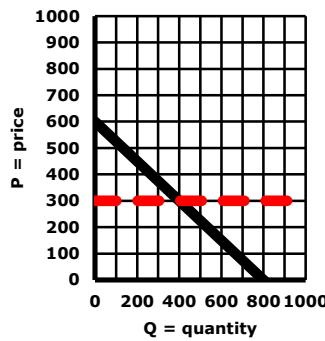
**answer: graph #2** (*Supply functions have positive slopes.*)

b) Which of the given graphs is the display for the demand function?

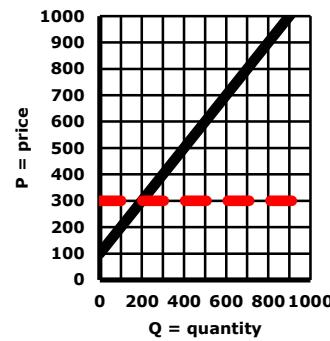
**answer: graph #1** (*Demand functions have negative slopes.*)

c) How many units are supplied when the price is \$300?

**answer: 200**...Using graph #2, we can see that the point of intersection of the supply function (black line) and the  $P = 300$  (red) line is: (200, 300)...which means that 200 units are supplied when the price is \$300.



*graph #1 = demand function*



*graph #2 = supply function*

d) How many units are demanded when the price is \$300?

**answer: 400**...Using graph #1, we can see that the point of intersection of the supply function (black line) and the  $P = 300$  (red) line is: (400, 300)...which means that 400 units are demanded when the price is \$300.

e) Will there be a market surplus or a market shortage when the price is \$300?

**answer: market shortage**... $200 < 400 \rightarrow S(x) < D(x)$

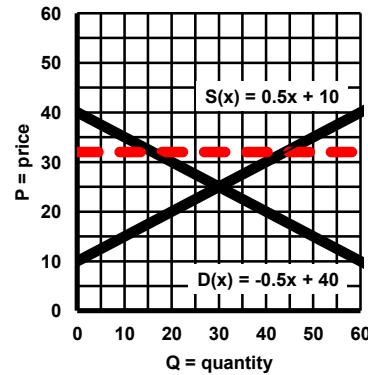
**example 1.29b** Suppose that a certain product has the following supply and demand functions:

$$S(x) = 0.5x + 10$$

$$D(x) = -0.5x + 40$$

Will there be a market surplus or shortage for this product when its price is \$32?

**solution #1:** Both of the given functions have been graphed on the coordinate grid at the right. Since we are interested in the price of \$32, we have drawn a horizontal (red) line where  $P = 32$ . Notice that the horizontal (red) line intersects with the  $D(x)$  function before it intersects with the  $S(x)$ . Thus,  $S(x) > D(x)$  which means that there will be a **market surplus**.



**solution #2:**

$$S(x) = 0.5x + 10 = 32 \rightarrow 0.5x = 32 - 10 = 22 \rightarrow x = 22 \div 0.5 = 44$$

$$D(x) = -0.5x + 40 = 32 \rightarrow -0.5x = 32 - 40 = -8 \rightarrow x = -8 \div -0.5 = +16$$

$S(x) > D(x) \rightarrow$  There will be a **market surplus** for this product

### TUTORIAL 1.30: WRITE THE EQUATION (IN SLOPE-INTERCEPT FORM) OF A SUPPLY OR A DEMAND FUNCTION

The supply function for a given scenario defines the amounts and unit prices that the manufacturer (supplier, wholesaler) is willing to supply a particular product. The demand function for a given scenario defines the amounts and unit prices that the retailer (customer, buyer) is willing to purchase a particular product.

**example 1.30a**

Suppose that the owner of a local shoe store will buy only 10 pairs of a certain shoe if the unit price is \$90 but 20 pairs if the unit price is \$65. The manufacturer of the shoes is willing to provide 75 pairs if the unit price is \$130 but only 45 pairs if the unit price is \$70. If both of the supply and demand functions are linear, write the slope-intercept equation for the supply function as well as the demand function.

***Supply Function:***

To write the equation for the supply function, determine that the sentence about the manufacturer should be used. Then just follow these steps:

- Use the given information in this sentence to create two ordered pairs  $(Q, P)$ :  
 $P_1 = (75, 130)$   
 $P_2 = (45, 70)$
- Use these points to determine the slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{70 - 130}{45 - 75} = \frac{-60}{-30} = 2$$

- Then use either of these points to determine the  $y$ -intercept of the supply function:

$$\begin{aligned} m = 2, P = (75, 130) \rightarrow y = mx + b \rightarrow b = y - mx \rightarrow \\ b = 130 - (2)(75) = -20 \\ m = 2, P = (45, 70) \rightarrow y = mx + b \rightarrow b = y - mx \rightarrow \\ b = 70 - (2)(45) = -20 \end{aligned}$$

- Finally, use the calculated values of  $m$  and  $b$  to write the supply function in its slope-intercept format:

$$m = 2, b = -20 \rightarrow S(x) = 2x - 20$$

***Demand Function:***

To write the equation for the demand function, determine that the sentence about the manufacturer should be used. Then just follow these steps:

- Use the given information in this sentence to create two ordered pairs  $(Q, P)$ :  
 $P_1 = (10, 90)$   
 $P_2 = (20, 65)$

- Use these points to determine the slope:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{65 - 90}{20 - 10} = \frac{-25}{10} = -2.5$$

- Then use either of these points to determine the  $y$ -intercept of the demand function:

$$m = -2.5, P = (10, 90) \rightarrow y = mx + b \rightarrow b = y - mx \rightarrow \\ b = 90 - (-2.5)(10) = 115$$

$$m = -2.5, P = (20, 65) \rightarrow y = mx + b \rightarrow b = y - mx \rightarrow \\ b = 65 - (-2.5)(20) = 115$$

- Finally, use the calculated values of  $m$  and  $b$  to write the demand function in its slope-intercept format:

$$m = -2.5, b = 115 \rightarrow D(x) = 115 - 2.5x$$

## American online LIGS University

is currently enrolling in the  
Interactive Online **BBA, MBA, MSc,**  
**DBA and PhD** programs:

- ▶ **enroll by September 30th, 2014** and
- ▶ **save up to 16%** on the tuition!
- ▶ pay in 10 installments / 2 years
- ▶ Interactive Online education
- ▶ visit [www.ligsuniversity.com](http://www.ligsuniversity.com) to find out more!

Note: LIGS University is not accredited by any nationally recognized accrediting agency listed by the US Secretary of Education.  
[More info here.](#)



**example 1.30b**

Suppose a group of retailers will buy only 72 televisions from a wholesaler if the unit price is \$310 but 120 televisions if the unit price is \$250. The wholesaler is willing to supply only 64 televisions if the unit price is \$282 and 144 televisions if the unit price is \$372. If both of the supply and demand functions are linear, write the slope-intercept equation for the supply function as well as the demand function.

***Supply Function:***

To write the equation for the supply function, determine that the sentence about the manufacturer should be used. Then just follow these steps:

- Use the given information in this sentence to create two ordered pairs  $(Q, P)$ :  
 $P_1 = (64, 282)$   
 $P_2 = (144, 372)$

- Use these points to determine the slope:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{372 - 282}{144 - 64} = \frac{90}{80} = \frac{9}{8} = 1.125$$

- Then use either of these points to determine the  $y$ -intercept of the supply function:

$$m = -1.25, P = (72, 310) \rightarrow y = mx + b \rightarrow b = y - mx \rightarrow \\ b = 282 - (1.125)(64) = 210$$

$$m = 1.125, P = (144, 372) \rightarrow y = mx + b \rightarrow b = y - mx \rightarrow \\ b = 372 - (1.125)(144) = 210$$

- Finally, use the calculated values of  $m$  and  $b$  to write the supply function in its slope-intercept format:

$$m = 1.125, b = 210 \rightarrow S(x) = 1.125x + 210$$

**Demand Function:**

To write the equation for the demand function, determine that the sentence about the manufacturer should be used. Then just follow these steps:

- Use the given information in this sentence to create two ordered pairs  $(Q, P)$ :  
 $P_1 = (72, 310)$   
 $P_2 = (120, 250)$

- Use these points to determine the slope:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{250 - 310}{120 - 72} = \frac{-60}{48} = \frac{-5}{4} = -1.25$$

- Then use either of these points to determine the  $y$ -intercept of the demand function:

$$m = -1.25, P = (72, 310) \rightarrow y = mx + b \rightarrow b = y - mx \rightarrow \\ b = 310 - (-1.25)(72) = 400$$

$$m = -1.25, P = (120, 250) \rightarrow y = mx + b \rightarrow b = y - mx \rightarrow \\ b = 250 - (-1.25)(120) = 400$$

- Finally, use the calculated values of  $m$  and  $b$  to write the demand function in its slope-intercept format:

$$m = -1.25, b = 400 \rightarrow D(x) = 400 - 1.25x$$

Recall that the slope of all linear demand functions will be negative while the slope of all linear supply functions will be positive. If we are not sure which given information should be used for the supply function or which given information should be used for the demand function, just determine the slope for each of the sentences...the negative slope will be the demand function.

**NOTE:** Although the same slope values were used for the supply functions in each of these examples that may not always be the case. And, although the same slope values were used for the demand functions in each of these examples that may not always be the case.

## TUTORIAL 1.31: DETERMINE THE MARKET EQUILIBRIUM FOR A GIVEN SCENARIO

**Market equilibrium** is the point of compromise between the suppliers and the demanders of a product. In other words, the market equilibrium is the point of intersection when the supply function is equal to the demand function:  $S(x) = D(x)$ . The point at which the market equilibrium occurs is sometimes called the **equilibrium point ( $Q, P$ )**.

- example 1.31a** Suppose that the owner of a local shoe store will buy only 20 pairs of a certain shoe if the unit price is \$80 but 30 pairs if the unit price is \$55. The manufacturer of the shoes is willing to provide 60 pairs if the unit price is \$70 but only 50 pairs if the unit price is \$50. If both of the supply and demand functions are linear, find the market equilibrium for this scenario.

**DON'T EAT YELLOW SNOW**

What will your advice be?

Some advice just states the obvious. But to give the kind of advice that's going to make a real difference to your clients you've got to listen critically, dig beneath the surface, challenge assumptions and be credible and confident enough to make suggestions right from day one. At Grant Thornton you've got to be ready to kick start a career right at the heart of business.

Sound like you? Here's our advice: visit [GrantThornton.ca/careers/students](http://GrantThornton.ca/careers/students)

Scan here to learn more about a career with Grant Thornton.

 **Grant Thornton**  
An instinct for growth™

© Grant Thornton LLP. A Canadian Member of Grant Thornton International Ltd





Just follow these steps to answer this word problem...

- Determine which sentence refers to the supply function:  
***manufacturer = supplier***
- Determine which sentence refers to the demand function:  
***owner = retailer***
- Use the given information to create ordered pairs ( $Q, P$ )... two for the supply function and two for the demand function:  
 $\text{supply function: } P_1 = (60, 70); P_2 = (50, 50)$   
 $\text{demand function: } P_1 = (20, 80); P_2 = (30, 55)$
- Use these points to determine the slope ( $m$ ) of each function:  
 $\text{supply function: } \frac{y_2 - y_1}{x_2 - x_1} = \frac{50 - 70}{50 - 60} = \frac{-20}{-10} = 2$   
 $\text{demand function: } \frac{y_2 - y_1}{x_2 - x_1} = \frac{55 - 80}{30 - 20} = \frac{-25}{10} = -2.5$
- Use either of the  $S(x)$  points to determine its  $y$ -intercept:  
 $m = 2, P_1 = (60, 70) \rightarrow y = mx + b \rightarrow b = y - mx \rightarrow$   
 $b = 70 - 2(60) = -50$
- Use either of the  $D(x)$  points to determine its  $y$ - intercept:  
 $m = -2.5, P_1 = (20, 80) \rightarrow y = mx + b \rightarrow b = y - mx \rightarrow$   
 $b = 80 - (-2.5)(20) = 130$
- Use the calculated values of  $m$  and  $b$  to write the supply and the demand functions in slope-intercept format:  
 $\text{supply function: } m = 2, b = -50 \rightarrow S(x) = 2x - 50$   
 $\text{demand function: } m = -2.5, b = 130 \rightarrow D(x) = 130 - 2.5x$

- To find the market equilibrium, set the two functions equal to each other and solve for the variable  $x$  (which represents the quantity agreed upon):

$$2x - 50 = 130 - 2.5x \rightarrow 2x + 2.5x = 130 + 50 \rightarrow 4.5x = 180 \rightarrow \frac{4.5x}{4.5} = \frac{180}{4.5} \rightarrow x = 40$$

- Finally, substitute the value of  $x$  into either the supply function or the demand function to determine the agreed upon price: *supply function:*  $S(x) = 2x - 50 \rightarrow S(40) = 2(40) - 50 \rightarrow$

$$S(x) = 80 - 50 = 30$$

*demand function:*  $D(x) = 130 - 2.5x \rightarrow D(40) = 130 - (2.5)(40) \rightarrow D(x) = 130 - 100 = 30$

- Use the derived values to answer the word problem:

**The equilibrium point  $(Q, P) = (40, 30)$ ...which means that the agreed upon quantity was 40 pairs of shoes at the unit price of \$30.**

**example 1.31b**

Suppose a group of retailers will buy only 74 televisions from a wholesaler if the unit price is \$370 but 130 televisions if the unit price is \$300. The wholesaler is willing to supply only 82 televisions if the unit price is \$436 and 154 televisions if the unit price is \$517. If both of the supply and demand functions are linear, find the market equilibrium for this scenario.

- Just follow these steps to answer this word problem...
- Determine which sentence refers to the supply function:  
***wholesaler = supplier***
- Determine which sentence refers to the demand function:  
***retailers = retailer***
- Use the given information to create ordered pairs  $(Q, P)$ ... two for the supply function and two for the demand function:  
*supply function:*  $P_1 = (82, 436); P_2 = (154, 517)$   
*demand function:*  $P_1 = (74, 370); P_2 = (130, 300)$
- Use these points to determine the slope ( $m$ ) of each function:

$$\text{supply function: } \frac{y_2 - y_1}{x_2 - x_1} = \frac{517 - 436}{154 - 82} = \frac{81}{72} = \frac{9}{8} = 1.125$$

$$\text{demand function: } \frac{y_2 - y_1}{x_2 - x_1} = \frac{300 - 370}{130 - 74} = \frac{-70}{56} = \frac{-5}{4} = -1.25$$

- Use either of the  $S(x)$  points to determine its  $y$ -intercept:

$$m = 1.125, P_1 = (82, 436) \rightarrow y = mx + b \rightarrow b = y - mx \rightarrow \\ b = 436 - (1.125)(82) = 343.75$$

$$m = 1.125, P_2 = (154, 517) \rightarrow y = mx + b \rightarrow b = y - mx \rightarrow \\ b = 517 - (1.125)(154) = 343.75$$

- Use either of the  $D(x)$  points to determine its  $y$ - intercept:

$$m = -1.25, P_1 = (74, 370) \rightarrow y = mx + b \rightarrow b = y - mx \rightarrow \\ b = 370 - (-1.25)(74) = 462.5$$

$$m = -1.25, P_2 = (130, 300) \rightarrow y = mx + b \rightarrow b = y - mx \rightarrow \\ b = 300 - (-1.25)(130) = 462.5$$

- Use the calculated values of  $m$  and  $b$  to write the supply and the demand functions in slope-intercept format:

*supply:  $m = 1.125, b = 343.75 \rightarrow S(x) = 1.125x + 343.75$*

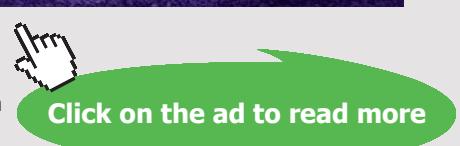
*demand:  $m = -1.25, b = 462.50 \rightarrow D(x) = 462.50 - 1.25x$*

www.alcatel-lucent.com/careers

Alcatel-Lucent

What if you could build your future and create the future?

One generation's transformation is the next's status quo. In the near future, people may soon think it's strange that devices ever had to be "plugged in." To obtain that status, there needs to be "The Shift".



- To find the market equilibrium, set the two functions equal to each other and solve for the variable  $x$  (which represents the quantity agreed upon):

$$1.125x + 343.75 = 462.50 - 1.25x \rightarrow$$

$$1.125x + 1.25x = 462.50 - 343.75 \rightarrow 2.375x = 118.75 \rightarrow$$

$$\frac{2.375x}{2.375} = \frac{118.75}{2.375} \rightarrow x = 50$$

- Finally, substitute the value of  $x$  into either the supply function or the demand function to determine the agreed upon price:  
*supply:*  $S(x) = 1.125x + 343.75 \rightarrow S(50) = 1.125(50) + 343.75 \rightarrow S(x) = 56.25 + 343.75 = 400$   
*demand:*  $D(x) = D(x) = 462.50 - 1.25x \rightarrow D(50) = 462.50 - 1.25(50) \rightarrow D(x) = 462.50 - 62.50 = 400$
- Use the derived values to answer the word problem:
- The equilibrium point (Q, P) = (50, 400)...which means that the agreed upon quantity was 50 televisions at the unit price of \$400.**

Recall that the slope of all linear demand functions will be negative while the slope of all linear supply functions will be positive. If we are not sure which given information should be used for the supply function or which given information should be used for the demand function, just determine the slope for each of the sentences...the negative slope will be the demand function.

**NOTE:** Although the same slope values were used for the supply functions in each of these examples that may not always be the case.

**NOTE:** Although the same slope values were used for the demand functions in each of these examples that may not always be the case.

### TUTORIAL 1.32: GRAPH A LINEAR INEQUALITY IN TWO VARIABLES

When we graph a linear equation, the correct display will only be a straight line with a certain slope and  $y$ -intercept. All of the points on the graphed line will be included in the solution set for the equation.

However, when we graph a linear inequality, there will be a lot more points that should be included in the solution set for the inequality. In order to properly graph a linear inequality, we need to determine a **feasible solution region** by plotting a boundary line and then shading the portion of the coordinate plane that is appropriate for the given linear inequality. Sometimes the boundary line will be included in the feasible solution region and sometimes it won't. The relationship symbol used in the given linear inequality will determine whether or not the boundary line is to be included:

- If the “is less than” symbol ( $<$ ) or the “is greater than” symbol ( $>$ ) is used in the given linear inequality, the boundary line should not be included in the feasible solution region and should be shown as a dotted line on the coordinate graph.
- If the “is less than or equal to” symbol ( $\leq$ ) or the “is greater than or equal to” symbol ( $\geq$ ) is used in the given linear inequality, the boundary line should be included in the feasible solution region and should be shown as a solid line on the coordinate graph

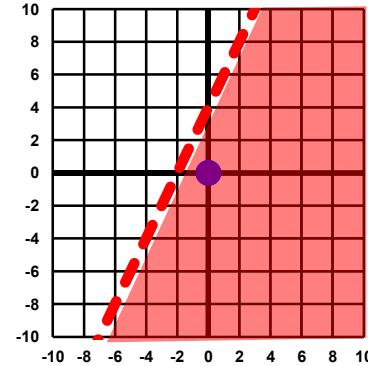
**example 1.32a      Graph this linear inequality on a coordinate grid:  $y < 2x + 4$**

1. Graph the boundary line by plotting:  $y = 2x + 4$

$$m = 2, \text{ y-intercept} = (0, 4) \\ x\text{-intercept} = (-2, 0)$$

Since  $<$  is used, the boundary line should be dotted.

2. Use a **test point** (any point that is not on the boundary line) to determine which side of the coordinate plane should be shaded. The easiest test point to use is  $(0, 0)$ . If  $x = 0$  and  $y = 0$  then substitute these values into the given linear inequality to determine whether  $(0, 0)$  will be part of its solution set:  $y < 2x + 4 \rightarrow 0 < 2(0) + 4 \rightarrow 0 < 4$ ... Since this numeric statement is true, shade over the test point  $(0, 0)$ .



NOTE: If  $(0, 0)$  is on the boundary line, then select any other point that is not on the boundary line for the test point and substitute the  $x$  and  $y$  values into the given linear equation. Suppose we chose  $(6, 4)$  as the test point for this inequality, then:

$$y < 2x + 4 \rightarrow 4 < 2(6) + 4 \rightarrow 4 < 16$$

Since this numeric statement is true, shade over the test point  $(0, 0)$ .

**example 1.32b****Graph this linear inequality on a coordinate grid:  $y > 6 - 3x$** 

1. Graph the boundary line by plotting:  $y = 6 - 3x$

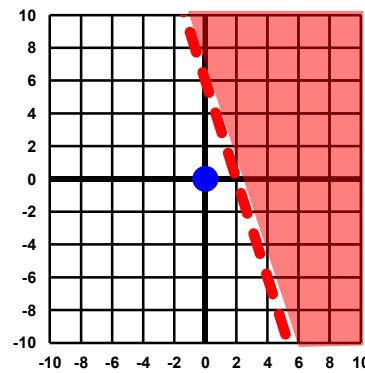
$$m = -3, y\text{-intercept} = (0, 6)$$

$$x\text{-intercept} = (2, 0)$$

Since  $>$  is used, the boundary line should be dotted.

2. Use a **test point** (any point that is not on the boundary line) to determine which side of the coordinate plane should

be shaded. The easiest test point to use is  $(0, 0)$ . If  $x = 0$  and  $y = 0$  then substitute these values into the given linear inequality to determine whether  $(0, 0)$  will be part of its solution set:  $y > 6 - 3x \rightarrow 0 > 6 - 3(0) \rightarrow 0 > 6\dots$  Since this numeric statement is false, shade away from the test point  $(0, 0)$ .



**Maastricht University** *Leading in Learning!*

**Join the best at  
the Maastricht University  
School of Business and  
Economics!**

**Top master's programmes**

- 33<sup>rd</sup> place Financial Times worldwide ranking: MSc International Business
- 1<sup>st</sup> place: MSc International Business
- 1<sup>st</sup> place: MSc Financial Economics
- 2<sup>nd</sup> place: MSc Management of Learning
- 2<sup>nd</sup> place: MSc Economics
- 2<sup>nd</sup> place: MSc Econometrics and Operations Research
- 2<sup>nd</sup> place: MSc Global Supply Chain Management and Change

Sources: Keuzegids Master ranking 2013; Elsevier 'Beste Studies' ranking 2012; Financial Times Global Masters in Management ranking 2012

**Maastricht  
University is  
the best specialist  
university in the  
Netherlands  
(Elsevier)**

**Visit us and find out why we are the best!  
Master's Open Day: 22 February 2014**

[www.mastersopenday.nl](http://www.mastersopenday.nl)

**example 1.32c****Graph this linear inequality on a coordinate grid:  $y \leq 3x + 6$** 

1. Graph the boundary line by plotting:  $y = 3x + 6$

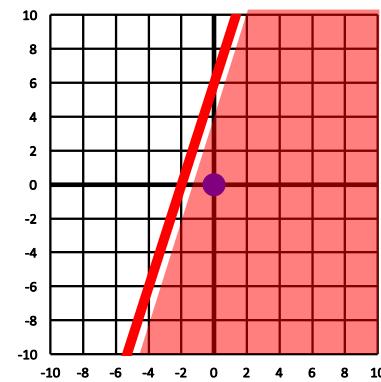
$$m = 3, y\text{-intercept} = (0, 6)$$

$$x\text{-intercept} = (-2, 0)$$

Since  $\leq$  is used, the boundary line should be solid.

2. Use a **test point** (any point that is not on the boundary line) to determine which side of the coordinate plane should

be shaded. The easiest test point to use is  $(0, 0)$ . If  $x = 0$  and  $y = 0$  then substitute these values into the given linear inequality to determine whether  $(0, 0)$  will be part of its solution set:  $y \leq 3x + 6 \rightarrow 0 \leq 3(0) + 6 \rightarrow 0 \leq 6$ ... Since this numeric statement is true, shade over the test point  $(0, 0)$ .

**example 1.32d****Graph this linear inequality on a coordinate grid:  $y \geq 4 - 2x$** 

1. Graph the boundary line by plotting:  $y = 4 - 2x$

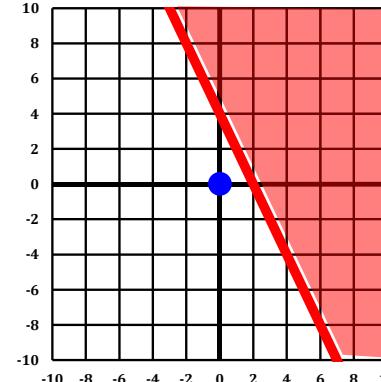
$$m = -2, y\text{-intercept} = (0, 4)$$

$$x\text{-intercept} = (2, 0)$$

Since  $\geq$  is used, the boundary line should be solid.

2. Use a **test point** (any point that is not on the boundary line) to determine which side of the coordinate plane should

be shaded. The easiest test point to use is  $(0, 0)$ . If  $x = 0$  and  $y = 0$  then substitute these values into the given linear inequality to determine whether  $(0, 0)$  will be part of its solution set:  $y \geq 4 - 2x \rightarrow 0 \geq 4 - 2(0) \rightarrow 0 \geq 4$ ... Since this numeric statement is false, shade away from the test point  $(0, 0)$ .



## TUTORIAL 1.33: SOLVE A SYSTEM OF LINEAR INEQUALITIES IN TWO VARIABLES

We can find the solution of a system of linear inequalities by graphing on a coordinate grid just as we can for a system of linear equations. The only difference is that we are not looking for the intersection of lines (as we do with a system of linear equations) but for the intersection of feasible solution regions.

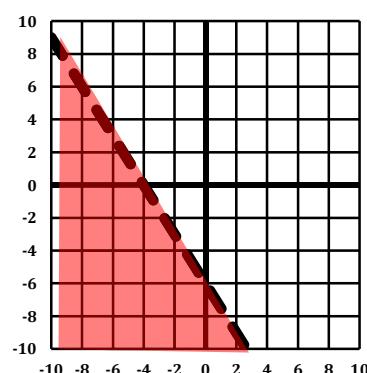
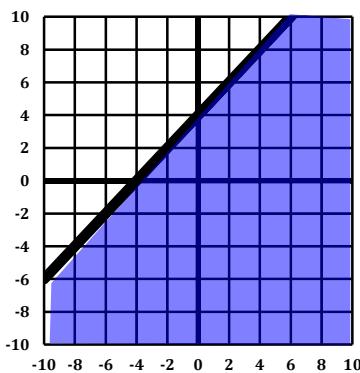
All we need to do to graph any system of linear inequalities is to graph each inequality individually (boundary line plus shading) and see where the shaded regions will overlap. It does not matter how many inequalities are contained in the given system...just repeat the same steps.

**Remember that:**

- The boundary lines may be dotted (when  $<$  or  $>$  are used) or solid (when  $\leq$  or  $\geq$  are used).
- If the boundary line is dotted, then it is not part of the feasible solution region.
- If the boundary line is solid, then it is part of the feasible solution region.
- The shaded area will be determined by using any test point  $(x, y)$ .
- The easiest test point to use is  $(0, 0)$  but any will do – just as long as it is not on the boundary line.
- Shade over the test point  $(x, y)$  if it produces a true statement when substituted into the inequality.
- Shade away from the test point  $(x, y)$  if it produces a false statement.
- When graphing a system of inequalities graph each inequality individually and then see where the shaded parts will overlap.

**example 1.33**      **Graph these linear inequalities on a coordinate grid:**  $x - y \geq -4$   
 $3x + 2y < 2$

1. Graph each of the inequalities on the same coordinate:



$$x - y \geq -4 \rightarrow \text{solid line}$$

$$x - y = -4$$

$$m = -1/-1 = +1$$

$$y\text{-intercept} = (0, y) = (0, 4)$$

$$x\text{-intercept} = (x, 0) = (-4, 0)$$

$$3x + 2y < -12 \rightarrow \text{dotted line}$$

$$3x + 2y = -12$$

$$m = -3/2 = -1.5$$

$$y\text{-intercept} = (0, y) = (0, -6)$$

$$x\text{-intercept} = (x, 0) = (-4, 0)$$

test point = (0, 0)

$$x - y \geq -4$$

$$(0) - (0) > -4$$

$$0 > -4 \rightarrow \text{true}$$

shade over (0, 0)

test point = (0, 0)

$$3x + 2y < -12$$

$$3(0) + 2(0) < -12$$

$$0 < -12 \rightarrow \text{false}$$

shade away from (0, 0)

**Empowering People. Improving Business.**

BI Norwegian Business School is one of Europe's largest business schools welcoming more than 20,000 students. Our programmes provide a stimulating and multi-cultural learning environment with an international outlook ultimately providing students with professional skills to meet the increasing needs of businesses.

BI offers four different two-year, full-time Master of Science (MSc) programmes that are taught entirely in English and have been designed to provide professional skills to meet the increasing need of businesses. The MSc programmes provide a stimulating and multi-cultural learning environment to give you the best platform to launch into your career.

- MSc in Business
- MSc in Financial Economics
- MSc in Strategic Marketing Management
- MSc in Leadership and Organisational Psychology

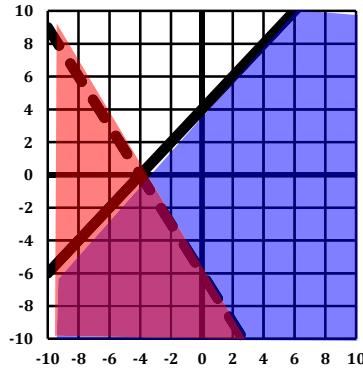
[www.bi.edu/master](http://www.bi.edu/master)

**BI NORWEGIAN BUSINESS SCHOOL**

EFMD **EQUIS** ACCREDITED



2. Determine where the two feasible solution regions (as graphed individually above) will overlap. This overlap is the feasible solution region for the given system of inequalities. The feasible solution region for the given system would be the area shaded in *purple* (as seen at the right).



### **TUTORIAL 1.34: NAME THE VERTICES FOR THE FEASIBLE SOLUTION REGION FOR THE GIVEN SYSTEM OF LINEAR INEQUALITIES**

When we use graphing a system of linear inequalities in a business application, we will often need to use numbers greater than zero as the *domain* (the permissible replacements for  $x$ ) and the *range* (the permissible replacements for  $y$ ). Therefore, we will often restrict the possible values to be used by including these additional inequalities:  $x \geq 0$ ,  $y \geq 0$ ...and so on...one additional inequality for each variable needed to complete the task of graphing a system of linear inequalities. These additional inequalities will guarantee that the entire graph will be displayed in the first quadrant (top right quarter) of the coordinate plane and that all of the numbers used will be greater than zero.

When we graph a system of linear inequalities, the feasible solution region is polygonal in shape. The number of inequalities contained in the given system will determine how many sides the polygonal shape will have. The points (coordinates) at which these sides intersect are called the *vertices* of the feasible solution region.

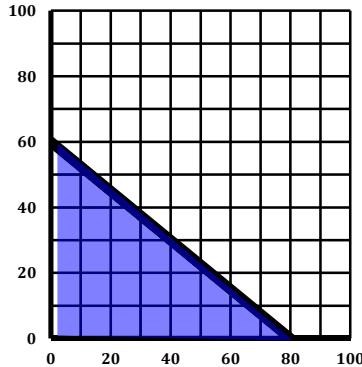
**example 1.34****Name the vertices for the feasible region of:**

$$3x + 4y \leq 240$$

$$3x - 2y \leq 60$$

$$x \geq 0, y \geq 0$$

1. Graph each of the inequalities on the same coordinate:



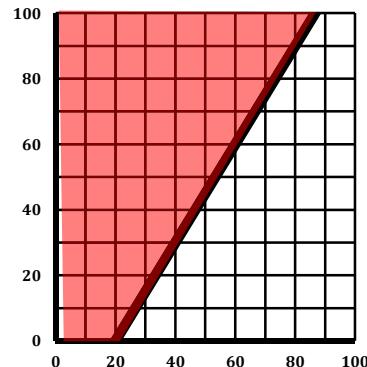
$$3x + 4y \leq 240 \rightarrow \text{solid line}$$

$$3x + 4y = 240$$

$$m = -3/4 = -0.75$$

$$y\text{-intercept} = (0, y) = (0, 60)$$

$$x\text{-intercept} = (x, 0) = (80, 0)$$



$$3x - 2y \leq 60 \rightarrow \text{solid line}$$

$$3x - 2y = 60$$

$$m = -3/2 = -1.5$$

$$y\text{-intercept} = (0, y) = (0, -30)^*$$

$$x\text{-intercept} = (x, 0) = (20, 0)$$

$$\text{test point} = (0, 0)$$

$$3x + 4y < 240$$

$$3(0) + 4(0) < 240$$

$$0 < 240 \rightarrow \text{true}$$

shade toward (0, 0)

$$\text{test point} = (0, 0)$$

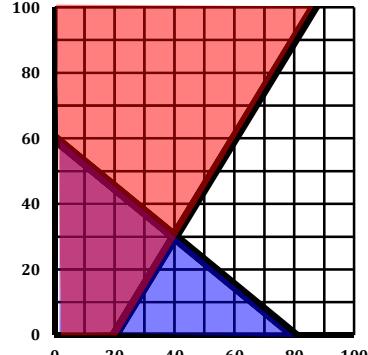
$$3x - 2y < 60$$

$$3(0) - 2(0) < 60$$

$$0 < 60 \rightarrow \text{true}$$

shade toward (0, 0)

2. Determine where the two feasible solution regions (as graphed individually above) will overlap. This overlap is the feasible solution region for the given system of inequalities. The feasible solution region for the given system would be the area shaded in ***purple*** (as seen at the right).

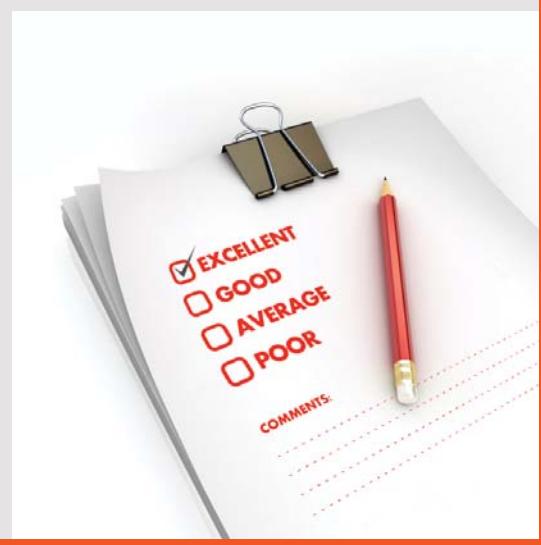


NOTE: The last two inequalities ( $x \geq 0$ ,  $y \geq 0$ ) will guarantee that the final graph will be displayed in the first quadrant (top right quarter) of the coordinate plane. When graphing technology (such as MS Excel) is used, we can format the axes of the graph to reflect that the horizontal and vertical axes begin at (0, 0).

## Need help with your dissertation?

Get in-depth feedback & advice from experts in your topic area. Find out what you can do to improve the quality of your dissertation!

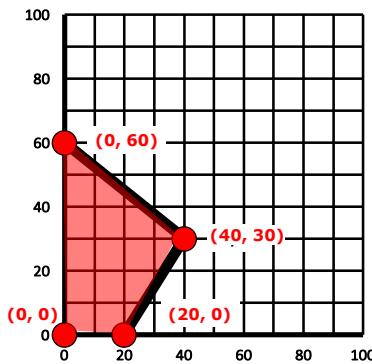
Get Help Now



Go to [www.helpmyassignment.co.uk](http://www.helpmyassignment.co.uk) for more info

**Helpmyassignment**

3. Examine the feasible solution region for the given system of linear inequalities to determine its vertices (the points where the sides of the polygonal region intersect). The feasible solution region for the given system of linear inequalities is displayed at the right without any extra lines or shading. Because of the given **constraints** (inequalities), this feasible solution region is a polygonal region with 4 sides. The red dots are the vertices of the feasible solution region. The coordinates of these vertices can be determined from the graph itself since each one is considered a ***lattice point*** (located at the intersection of two grid lines). Therefore, the vertices of the given system are: **(0, 0)**, **(0, 60)**, **(40, 30)** and **(20, 0)**



However, there may be times when the vertex coordinates will not be located at the intersection of grid lines. In that case, we can always determine these coordinates by solving a series of systems that use pairs of equations that are derived from the given inequalities.

### **TUTORIAL 1.35: USE THE VERTICES FOR THE FEASIBLE SOLUTION REGION FOR A GIVEN SYSTEM OF LINEAR INEQUALITIES TO DETERMINE THE OPTIMAL (MINIMUM OR MAXIMUM) VALUE OF A GIVEN LINEAR FUNCTION THAT IS SUBJECT TO CONSTRAINTS AS DEFINED BY THE GIVEN SYSTEM OF INEQUALITIES**

In business, it is often useful to determine the minimum or the maximum values that can occur for a particular situation: minimum costs, maximum revenue, maximum profits, and the like. If the constraints for the particular task can be represented by linear inequalities, then we can graph the inequalities to determine the vertices for the appropriate feasible solution region. We then can use the vertices to determine the requested minimum or maximum value possible for a given linear objective function.

**example 1.35a****For the given system:**

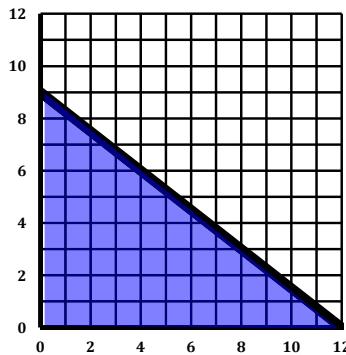
$$3x + 4y \leq 36$$

$$x + y \leq 10$$

$$4x + 3y \leq 36$$

$$x \geq 0, y \geq 0$$

- Determine its feasible solution region.**
- Determine the list of the vertices of the feasible solution region.**
- Determine the minimum value (if it exists) of the system.**
- Determine the maximum value (if it exists) of the system.**



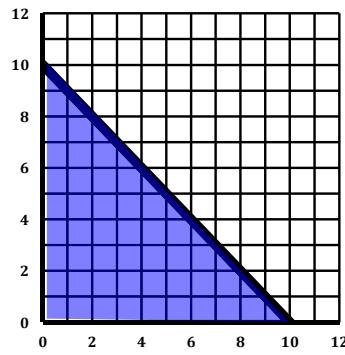
$$3x + 4y \leq 36 \rightarrow \text{solid line}$$

$$m = -3/4 = -0.75$$

$$\begin{aligned} y\text{-intercept} &= (0, y) = (0, 9) \\ x\text{-intercept} &= (x, 0) = (12, 0) \end{aligned}$$

$$\text{test point} = (0, 0)$$

$$\begin{aligned} 3x + 4y &\leq 36 \\ 3(0) + 4(0) &< 36 \\ 0 &< 36 \rightarrow \text{true} \\ \text{shade toward } (0, 0) & \end{aligned}$$



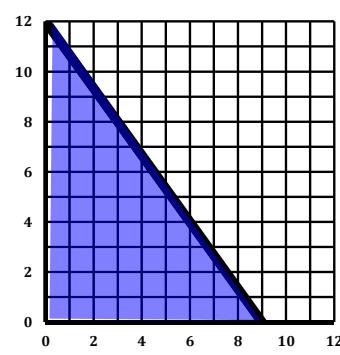
$$x + y \leq 10 \rightarrow \text{solid line}$$

$$m = -1/1 = -1$$

$$\begin{aligned} y\text{-intercept} &= (0, y) = (0, 10) \\ x\text{-intercept} &= (x, 0) = (10, 0) \end{aligned}$$

$$\text{test point} = (0, 0)$$

$$\begin{aligned} x + y &\leq 10 \\ (0) + (0) &< 10 \\ 0 &< 10 \rightarrow \text{true} \\ \text{shade toward } (0, 0) & \end{aligned}$$



$$4x + 3y \leq 36 \rightarrow \text{solid line}$$

$$m = -4/3 = 4/3 = 1.333\dots$$

$$\begin{aligned} y\text{-intercept} &= (0, y) = (0, 12) \\ x\text{-intercept} &= (x, 0) = (9, 0) \end{aligned}$$

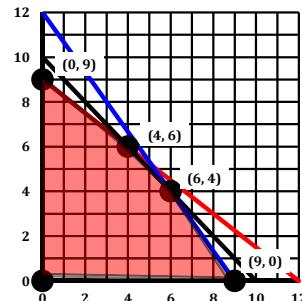
$$\text{test point} = (0, 0)$$

$$\begin{aligned} 4x + 3y &\leq 36 \\ 4(0) + 3(0) &< 36 \\ 0 &< 36 \rightarrow \text{true} \\ \text{shade toward } (0, 0) & \end{aligned}$$

Notice that the last two given inequalities (constraints) will guarantee that all of the desired graphs will be in the first quadrant (top right quarter) of the coordinate plane. Also, since the first three inequalities (constraints) are all using the relationship symbol of “≤” (is less than or equal to), all of the lines should be “solid” (which means that the points on these lines are included in the solution region) and the shading should be towards (0, 0) on the coordinate plane.

Now that we have each of the main inequalities graphed, we need to determine where they overlap. Since the shading of all three main inequalities (constraints) are toward  $(0, 0)$ , the shading that should be displayed in the solution region will also be towards  $(0, 0)$ .

vertices		objective	
x	y	$F = 6x + 4y$	MIN/MAX???
0	0	$F = 6(0) + 4(0) = 0$	<b>minimum</b>
0	9	$F = 6(0) + 4(9) = 36$	
4	6	$F = 6(4) + 4(6) = 48$	
6	4	$F = 6(4) + 4(4) = 40$	
9	0	$F = 6(9) + 4(0) = 54$	<b>maximum</b>



# Brain power

By 2020, wind could provide one-tenth of our planet's electricity needs. Already today, SKF's innovative know-how is crucial to running a large proportion of the world's wind turbines.

Up to 25 % of the generating costs relate to maintenance. These can be reduced dramatically thanks to our systems for on-line condition monitoring and automatic lubrication. We help make it more economical to create cleaner, cheaper energy out of thin air.

By sharing our experience, expertise, and creativity, industries can boost performance beyond expectations.

Therefore we need the best employees who can meet this challenge!

The Power of Knowledge Engineering

Plug into The Power of Knowledge Engineering.  
Visit us at [www.skf.com/knowledge](http://www.skf.com/knowledge)

**SKF**



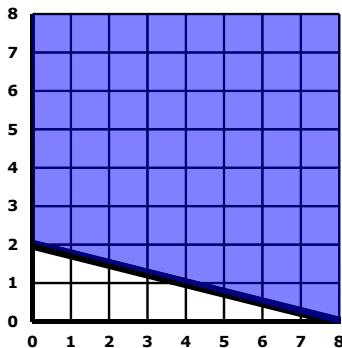
To evaluate the objective function, substitute the values of  $x$  and  $y$  using the coordinates of each vertex. Then, examine the results: 0, 36, 48, 40, 54

- Which one is the smallest? That would be the minimum value possible for the given constraints.
- Which one is the largest? That would be the maximum value possible for the given constraints.

**example 1.35b**    For the given system:

$$\begin{aligned}x + 4y &\geq 8 \\x + 2y &\geq 6 \\2x + y &\geq 6 \\x &\geq 0, y \geq 0\end{aligned}$$

- Determine its feasible solution region.
- Determine the list of the vertices of the feasible solution region.
- Determine the minimum value (if it exists) of the system.
- Determine the maximum value (if it exists) of the system.



$$x + 4y \geq 8 \rightarrow \text{solid line}$$

$$m = -1/4 = -0.25$$

$$y\text{-intercept} = (0, y) = (0, 2)$$

$$x\text{-intercept} = (x, 0) = (8, 0)$$

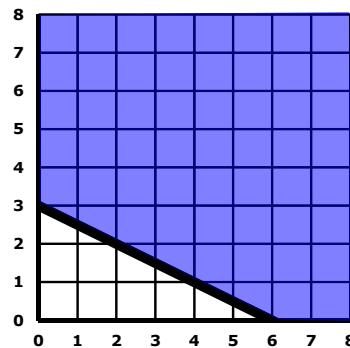
$$\text{test point} = (0, 0)$$

$$x + 4y \geq 8$$

$$(0) + 4(0) \geq 8$$

$$0 \geq 8 \rightarrow \text{false}$$

shade away from (0, 0)



$$x + 2y \geq 6 \rightarrow \text{solid line}$$

$$m = -1/2 = -0.5$$

$$y\text{-intercept} = (0, y) = (0, 3)$$

$$x\text{-intercept} = (x, 0) = (6, 0)$$

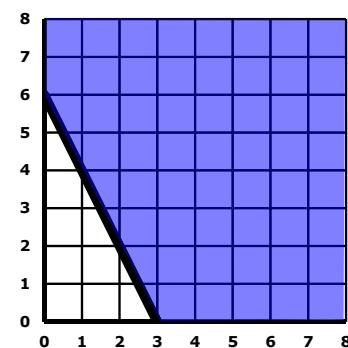
$$\text{test point} = (0, 0)$$

$$x + 2y \geq 6$$

$$(0) + 2(0) \geq 6$$

$$0 \geq 6 \rightarrow \text{false}$$

shade away from (0, 0)



$$2x + y \geq 6 \rightarrow \text{solid line}$$

$$m = -2/1 = -2$$

$$y\text{-intercept} = (0, y) = (0, 6)$$

$$x\text{-intercept} = (x, 0) = (3, 0)$$

$$\text{test point} = (0, 0)$$

$$2x + y \geq 6$$

$$2(0) + (0) \geq 6$$

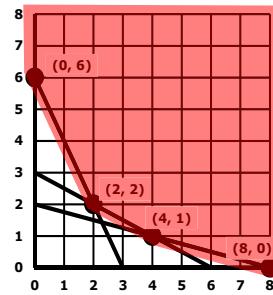
$$0 \geq 6 \rightarrow \text{false}$$

shade away from (0, 0)

Notice that the last two given inequalities (constraints) will guarantee that all of the desired graphs will be in the first quadrant (top right quarter) of the coordinate plane. Also, since the first three inequalities (constraints) are all using the relationship symbol of “ $\leq$ ” (is less than or equal to), all of the lines should be “solid” (which means that the points on these lines are included in the solution region) and the shading should be towards  $(0, 0)$  on the coordinate plane.

Now that we have each of the main inequalities graphed, we need to determine where they overlap. Since the shading of all three main inequalities (constraints) are toward  $(0, 0)$ , the shading that should be displayed in the solution region will also be towards  $(0, 0)$ .

vertices		objective	
x	y	$F = 6x + 4y$	MIN/MAX???
0	6	$F = 4(0) + 5(6) = 30$	
2	2	$F = 4(2) + 5(2) = 18$	minimum
4	1	$F = 4(4) + 5(1) = 21$	
8	0	$F = 4(8) + 5(0) = 32$	



To evaluate the objective function, substitute the values of  $x$  and  $y$  using the coordinates of each vertex. Then, examine the results: 30, 18, 21, 32

- Which one is the smallest? That would be the minimum value possible for the given constraints.
- Since this solution region continues on indefinitely away from  $(0, 0)$ , there will be no maximum value for the given objective function.

When the feasible solution region is contained on all sides (as shown in the final graph for example 1.35a), there will be a minimum as well as a maximum value for the given objective function.

When the feasible solution region is an “unbounded” region and the shading is going infinitely towards the positive regions of the coordinate plane (as shown in the final graph for example 1.35b), there will be no maximum value of the given objective function. On the other hand, when the feasible solution region is an “unbounded” region and the shading is going infinitely towards the negative regions of the coordinate plane, there will be no minimum value of the given objective function.

### **TUTORIAL 1.36: USE GRAPHING TO DETERMINE THE OPTIMAL MAXIMUM VALUE OF A LINEAR FUNCTION SUBJECT TO CONSTRAINTS**

Incorporating all of the tasks involved in graphing a linear inequality we can now complete any *linear programming* problem that will ask us to determine the optimal maximum value of a given linear objective function.

## **TURN TO THE EXPERTS FOR SUBSCRIPTION CONSULTANCY**

Subscrybe is one of the leading companies in Europe when it comes to innovation and business development within subscription businesses.

We innovate new subscription business models or improve existing ones. We do business reviews of existing subscription businesses and we develop acquisition and retention strategies.

Learn more at [linkedin.com/company/subscrybe/](https://www.linkedin.com/company/subscrybe/) or contact Managing Director Morten Suhr Hansen at [mha@subscrybe.dk](mailto:mha@subscrybe.dk)

**SUBSCRYBE** - to the future



**example 1.36**

**Use coordinate grid graphing to determine the optimal maximum value of a linear function that is subject to set constraints:**

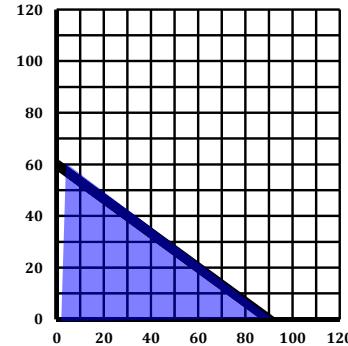
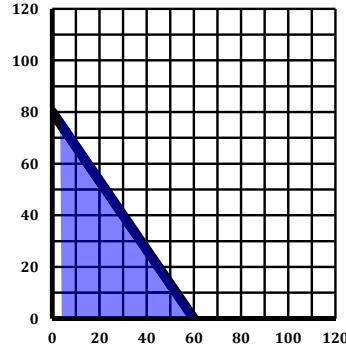
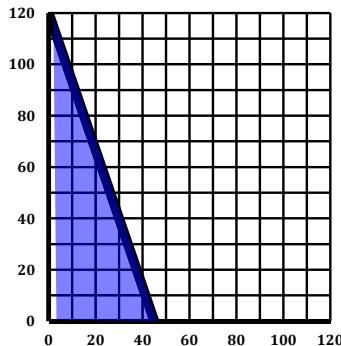
$$8x + 3y \leq 360$$

$$4x + 3y \leq 240$$

$$2x + 3y \leq 180$$

$$x \geq 0, y \geq 0$$

$$F = 8x + 5y$$



$$8x + 3y \leq 360 \rightarrow \text{solid line}$$

$$m = -8/3 = -2.333\dots$$

$$y\text{-intercept} = (0, y) = (0, 120)$$

$$x\text{-intercept} = (x, 0) = (45, 0)$$

$$\text{test point} = (0, 0)$$

$$8x + 3y \leq 360$$

$$8(0) + 3(0) \leq 360$$

$$0 \leq 360 \rightarrow \text{true}$$

shade toward (0, 0)

$$4x + 3y \leq 240 \rightarrow \text{solid line}$$

$$m = -4/3 = -1.333\dots$$

$$y\text{-intercept} = (0, y) = (0, 80)$$

$$x\text{-intercept} = (x, 0) = (60, 0)$$

$$\text{test point} = (0, 0)$$

$$4x + 3y \leq 240$$

$$4(0) + 3(0) \leq 240$$

$$0 \leq 240 \rightarrow \text{true}$$

shade toward (0, 0)

$$2x + 3y \leq 180 \rightarrow \text{solid line}$$

$$m = -2/3 = -0.666\dots$$

$$y\text{-intercept} = (0, y) = (0, 60)$$

$$x\text{-intercept} = (x, 0) = (90, 0)$$

$$\text{test point} = (0, 0)$$

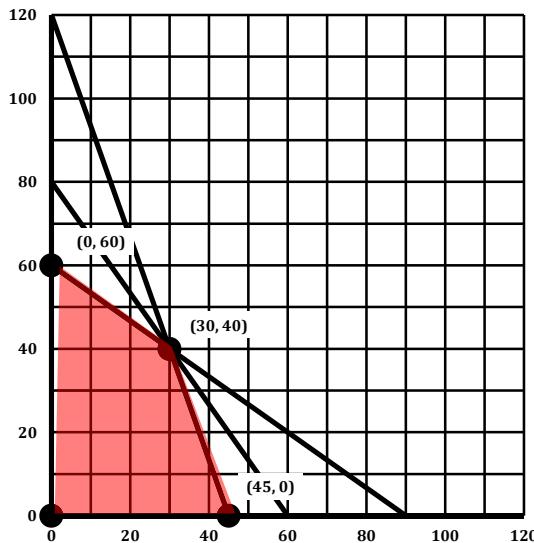
$$2x + 3y \leq 180$$

$$2(0) + 3(0) \leq 180$$

$$0 \leq 180 \rightarrow \text{true}$$

shade toward (0, 0)

The last two inequalities (constraints) guarantee that the graph will be in the first quadrant. We have graphed each of the first three inequalities (constraints). Now we just need to determine where the overlap exists among all of the given inequalities (constraints). Then we need to determine the vertex coordinates of the points where the graphed lines intersect and form the polygonal solution region. Once we have the vertex coordinates, we can evaluate the given objective function. Finally, just find the highest (maximum) value of these evaluations.



vertices		objective
x	y	$F = 8x + 5y$
0	0	$F = 8(0) + 5(0) = 0$
0	60	$F = 8(0) + 5(60) = 300$
30	40	$F = 8(30) + 5(40) = 440$
45	0	$F = 8(45) + 5(0) = 360$

Using the vertex coordinates, we obtain the evaluations of 0, 300, 440 and 360. The highest of these evaluations is 440. Therefore, the optimal maximum value for the given constraints is 440 when  $x = 30$  while  $y = 40$ .

**MAX = 440 when  $x = 30, y = 40$**

### TUTORIAL 1.37: USE GRAPHING TO DETERMINE THE OPTIMAL MINIMUM VALUE OF A LINEAR FUNCTION SUBJECT TO CONSTRAINTS

Incorporating all of the tasks involved in graphing a linear inequality we can now complete any *linear programming* problem that will ask us to determine the optimal minimum value of a given linear objective function.

#### example 1.37

**Use coordinate grid graphing to determine the optimal maximum value of a linear function that is subject to set constraints:**

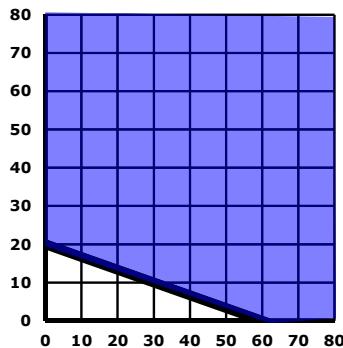
$$x + 3y \geq 60$$

$$2x + y \geq 50$$

$$x + y \geq 40$$

$$x \geq 0, y \geq 0$$

$$F = 3x + y$$



$$x + 3y \geq 60 \rightarrow \text{solid line}$$

$$m = -1/3 = -0.333\dots$$

$$y\text{-intercept} = (0, y) = (0, 20)$$

$$x\text{-intercept} = (x, 0) = (60, 0)$$

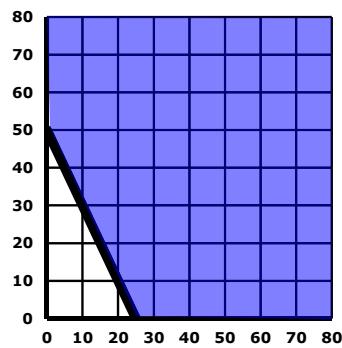
$$\text{test point} = (0, 0)$$

$$x + 3y \geq 60$$

$$(0) + 3(0) \geq 60$$

$$0 \geq 60 \rightarrow \text{false}$$

shade away from (0, 0)



$$2x + y \geq 50 \rightarrow \text{solid line}$$

$$m = -2/1 = -2$$

$$y\text{-intercept} = (0, y) = (0, 50)$$

$$x\text{-intercept} = (x, 0) = (25, 0)$$

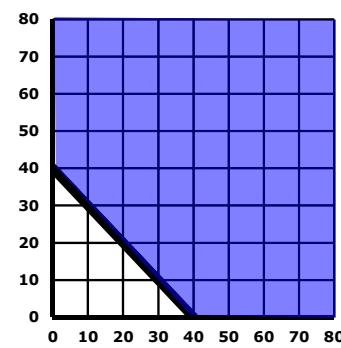
$$\text{test point} = (0, 0)$$

$$2x + y \geq 50$$

$$2(0) + (0) \geq 50$$

$$0 \geq 50 \rightarrow \text{false}$$

shade away from (0, 0)



$$x + y \geq 40 \rightarrow \text{solid line}$$

$$m = -1/1 = -1$$

$$y\text{-intercept} = (0, y) = (0, 40)$$

$$x\text{-intercept} = (x, 0) = (40, 0)$$

$$\text{test point} = (0, 0)$$

$$x + y \geq 40$$

$$(0) + (0) \geq 40$$

$$0 \geq 40 \rightarrow \text{false}$$

shade away from (0, 0)

**What do you want to do?**

No matter what you want out of your future career, an employer with a broad range of operations in a load of countries will always be the ticket. Working within the Volvo Group means more than 100,000 friends and colleagues in more than 185 countries all over the world. We offer graduates great career opportunities – check out the Career section at our web site [www.volvologroup.com](http://www.volvologroup.com). We look forward to getting to know you!

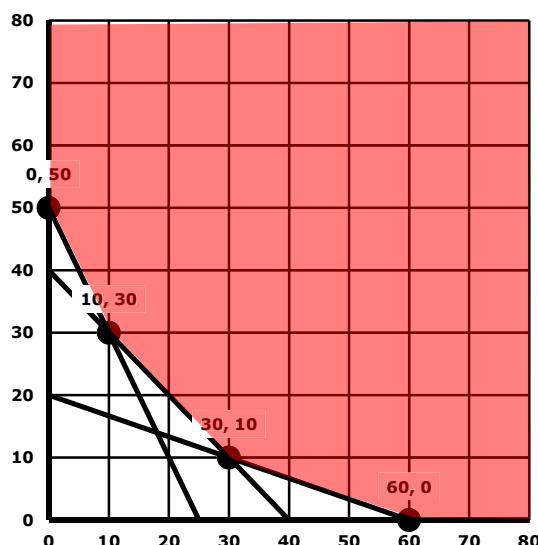
**VOLVO**

AB Volvo (publ)  
[www.volvologroup.com](http://www.volvologroup.com)

VOLVO TRUCKS | RENAULT TRUCKS | MACK TRUCKS | VOLVO BUSES | VOLVO CONSTRUCTION EQUIPMENT | VOLVO PENTA | VOLVO AERO | VOLVO IT  
VOLVO FINANCIAL SERVICES | VOLVO 3P | VOLVO POWERTRAIN | VOLVO PARTS | VOLVO TECHNOLOGY | VOLVO LOGISTICS | BUSINESS AREA ASIA



The last two inequalities (constraints) guarantee that the graph will be in the first quadrant. We have graphed each of the first three inequalities (constraints). Now we just need to determine where the overlap exists among all of the given inequalities (constraints). Then we need to determine the vertex coordinates of the points where the graphed lines intersect and form the polygonal solution region. Once we have the vertex coordinates, we can evaluate the given objective function. Finally, just find the lowest (minimum) value of these evaluations.



vertices		objective
x	y	$F = 3x + y$
0	50	$F = 3(0) + (50) = 50$
10	30	$F = 3(10) + (30) = 60$
30	10	$F = 3(30) + (10) = 100$
60	0	$F = 3(60) + (0) = 180$

Using the vertex coordinates, we obtain the evaluations of 50, 60, 110 and 180. The lowest of these evaluations is 50. Therefore, the optimal minimum value for the given constraints is 50 when  $x = 0$  while  $y = 50$ .

**MAX = 50 when  $x = 0, y = 50$**

## TUTORIAL 1.38: SOLVE APPLICATIONS INVOLVING LINEAR PROGRAMMING

When we are solving real-world applications where linear programming might be useful, we will need to translate the written word into an appropriate (equivalent) set of inequalities that can then be solved as a system (as was explained in previous tutorials).

**example 1.38a**

The Monmouth Bakery Company makes regular cake mixes as well as gourmet pancake mixes that are sold at various groceries stores in the area:

- Each pound of the regular cake mix uses 0.4 pound of flour and 0.1 pound of shortening.
- Each pound of the gourmet pancake mix uses 0.6 pound of flour, 0.1 pound of shortening and 0.4 pound of sugar.
- Suppliers can deliver at most 8400 pounds of flour, at least 800 pounds of shortening, and at most 2000 pounds of sugar.

If the profit per pound is 25¢ for regular cake mix and 35¢ for gourmet pancake mix, how many pounds of each mix should be made to earn maximum profit? What is that maximum profit?

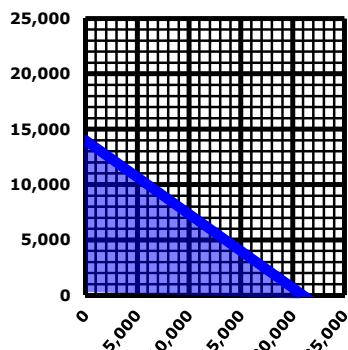
Sometimes, translation is made a bit easier by organizing the given information into a data table (as has been done in the table at the left below). We can then use the organized table to write the constraints for the given word problem. Notice that the supplier can deliver at most one quantity and at least another. In mathematical translations, ***at most*** is translated into the  $\leq$  (is less than or equal to) sign because the amount delivered cannot go over the stated amount. The phrase ***at least*** is translated into the  $\geq$  (is greater than or equal to) sign because the amount delivered cannot be any lower than the stated amount. Also, remember that when completing a real-world business application, we want to use numbers that are greater than or equal to zero...thus:  $x \geq 0$  and  $y \geq 0$ . The information given about the profit will aid us in creating the objective function for this scenario.

	<b>x</b>	<b>y</b>		
	<b>cake</b>	<b>pancake</b>	<b>symbol</b>	<b>totals</b>
<b>flour</b>	<b>0.4</b>	<b>0.6</b>	<b><math>\leq</math></b>	<b>8400</b>
<b>shortening</b>	<b>0.1</b>	<b>0.1</b>	<b><math>\geq</math></b>	<b>800</b>
<b>sugar</b>	<b>0.0</b>	<b>0.4</b>	<b><math>\leq</math></b>	<b>2000</b>
<b>profits</b>	<b>0.25</b>	<b>0.35</b>	<b>=</b>	<b>MAX</b>

<b>constraints (inequalities):</b>	
<b>flour:</b>	<b><math>0.4x + 0.6y \leq 8400</math></b>
<b>shortening:</b>	<b><math>0.1x + 0.1y \geq 800</math></b>
<b>sugar:</b>	<b><math>0.4y \leq 2000</math></b>
<b>x-values:</b>	<b><math>x \geq 0</math></b>
<b>y-values:</b>	<b><math>y \geq 0</math></b>
<b>objective:</b>	<b><math>F = 0.25x + 0.35y</math></b>

Now that we have the system of inequalities and the objective function translated from the given word problem, we can now go about the task of graphing the system on a coordinate grid, shading in the feasible solution region, determining the coordinates of any appropriate vertices and then determining the values of  $x$  and  $y$  that produce the optimal maximum profit for this given situation.

Note that one of the constraints (inequalities) has a different relationship sign than the others...thus, we have a system with **mixed** constraints. When shading, we will need to pay attention to this fact so that we arrive at the correct feasible solution region.



$$0.4x + 0.6y \leq 8400 \rightarrow \text{solid line}$$

$$m = -0.4/0.6 = -2/3 = -0.666\ldots$$

$$y\text{-intercept} = (0, y) = (0, 14000)$$

$$x\text{-intercept} = (x, 0) = (21000, 0)$$

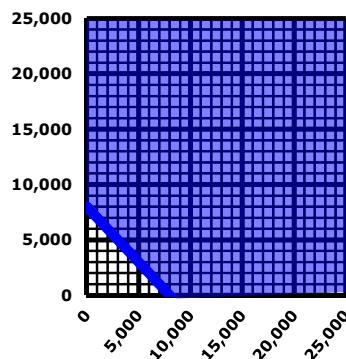
test point = (0, 0)

$$0.4x + 0.6y \leq 8400$$

$$0.4(0) + 0.6(0) \leq 8400$$

$$0 \leq 8400 \rightarrow \text{true}$$

shade toward (0, 0)



$$0.1x + 0.1y \geq 800 \rightarrow \text{solid line}$$

$$m = -0.1/0.1 = -1/1 = -1$$

$$y\text{-intercept} = (0, y) = (0, 8000)$$

$$x\text{-intercept} = (x, 0) = (8000, 0)$$

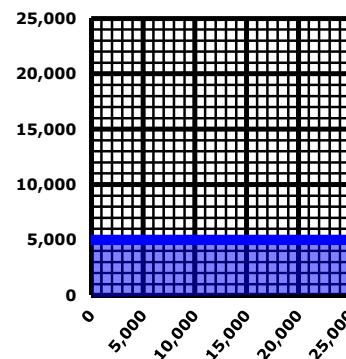
test point = (0, 0)

$$0.1x + 0.1y \geq 800$$

$$0.1(0) + 0.1(0) \geq 800$$

$$0 \geq 800 \rightarrow \text{false}$$

shade away from (0, 0)



$$0.4y \leq 2000 \rightarrow \text{solid line}$$

$$m = -0.0/0.4 = 0 \rightarrow \text{horizontal}$$

$$y\text{-intercept} = (0, y) = (0, 5000)$$

$$x\text{-intercept} = (x, 0) = \text{none}$$

test point = (0, 0)

$$0.4y \leq 2000$$

$$0.4(0) \leq 2000$$

$$0 \leq 2000 \rightarrow \text{true}$$

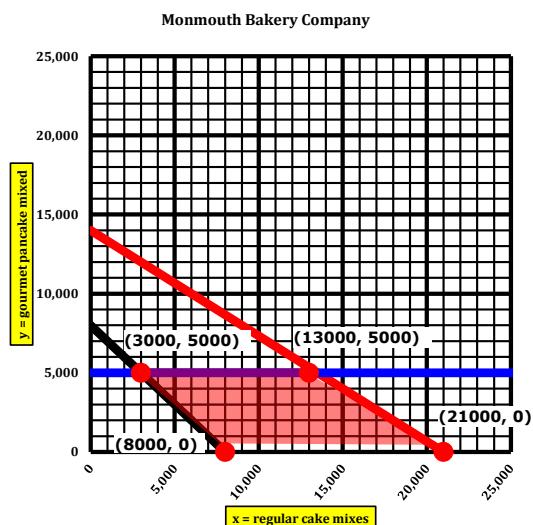
shade toward (0, 0)

**EXPERIENCE THE POWER OF FULL ENGAGEMENT...**

**RUN FASTER.  
RUN LONGER..  
RUN EASIER...**

**READ MORE & PRE-ORDER TODAY  
[WWW.GAITEYE.COM](http://WWW.GAITEYE.COM)**

The last two inequalities (constraints) guarantee that the graph will be in the first quadrant. We have graphed each of the first three inequalities (constraints). Now we just need to determine where the overlap exists among all of the given inequalities (constraints). Then we need to determine the vertex coordinates of the points where the graphed lines intersect and form the polygonal solution region.



Once we have the vertex coordinates, we can evaluate the given objective function:

vertices		objective
x	y	$F = 0.25x + 0.35y$
3000	5000	$F = 0.25(3000) + 0.35(5000) = 2500$
13,000	5000	$F = 0.25(13000) + 0.35(5000) = 5000$
21,000	0	$F = 0.25(21000) + 0.35(0) = 5250$
8000	0	$F = 0.25(8000) + 0.35(0) = 2000$

Finally, just find the highest (maximum) value of these evaluations. Using the vertex coordinates, we obtain the evaluations of: 2500, 5000, 5250 and 2000...the highest of these evaluations being: 5250. Therefore, the optimal maximum value for the given constraints is: **MAX = 5250 when  $x = 21000$  and  $y = 0$**  (*which might mean stopping production of the gourmet pancake mixes*).

**example 1.38b**

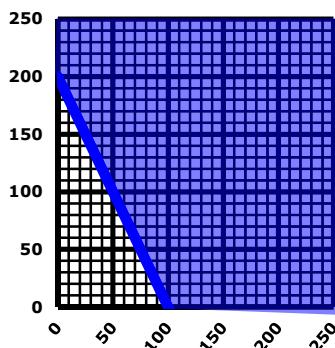
The Monmouth Drapery Company makes 3 types of draperies at 2 different locations:

- At location #1, it can make 20 pairs of deluxe drapes, 15 pairs of better drapes, and 6 pairs of standard drapes per day.
- At location #2, it can make 10 pairs of deluxe drapes, 20 pairs of better drapes, and 12 pairs of standard drapes per day.

The company has orders for 2000 pairs of deluxe drapes, 2250 pairs of better drapes, and 1200 pairs of standard drapes. If the daily costs are \$900 per day at location #1 and \$750 per day at location #2, how many days should be scheduled at each location in order to fill the orders at minimum cost? What is that minimum cost?

	<b>x</b>	<b>y</b>		
	<b>location 1</b>	<b>location 2</b>	<b>symbol</b>	<b>orders</b>
<b>deluxe</b>	<b>20</b>	<b>10</b>	<b><math>\geq</math></b>	<b>2000</b>
<b>better</b>	<b>15</b>	<b>15</b>	<b><math>\geq</math></b>	<b>2250</b>
<b>standard</b>	<b>6</b>	<b>12</b>	<b><math>\geq</math></b>	<b>1200</b>
<b>daily costs</b>	<b>900</b>	<b>750</b>	<b>=</b>	<b>MIN</b>

<b>constraints (inequalities):</b>	
<b>deluxe:</b>	<b><math>20x + 10y \geq 2000</math></b>
<b>better:</b>	<b><math>15x + 15y \geq 2250</math></b>
<b>standard:</b>	<b><math>6x + 12y \geq 1200</math></b>
<b>x-values:</b>	<b><math>x \geq 0</math></b>
<b>y-values:</b>	<b><math>y \geq 0</math></b>
<b>objective:</b>	<b><math>F = 900x + 750y</math></b>



$$20x + 10y \geq 2000 \rightarrow \text{solid line}$$

$$m = -20/10 = -2$$

$$y\text{-intercept} = (0, y) = (0, 200)$$

$$x\text{-intercept} = (x, 0) = (100, 0)$$

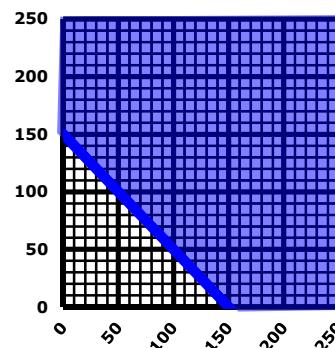
$$\text{test point} = (0, 0)$$

$$20x + 10y \geq 2000$$

$$20(0) + 10(0) \geq 2000$$

$$0 \geq 2000 \rightarrow \text{false}$$

$$\text{shade away from } (0, 0)$$



$$15x + 15y \geq 2250 \rightarrow \text{solid line}$$

$$m = -15/15 = -1$$

$$y\text{-intercept} = (0, y) = (0, 150)$$

$$x\text{-intercept} = (x, 0) = (150, 0)$$

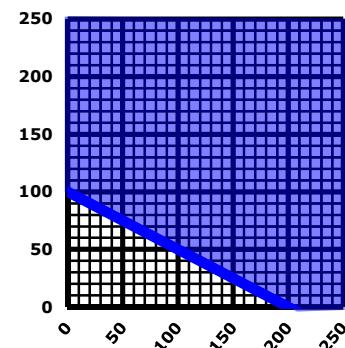
$$\text{test point} = (0, 0)$$

$$15x + 15y \geq 2250$$

$$15(0) + 15(0) \geq 2250$$

$$0 \geq 2250 \rightarrow \text{false}$$

$$\text{shade away from } (0, 0)$$



$$6x + 12y \geq 1200 \rightarrow \text{solid line}$$

$$m = -6/12 = -1/2$$

$$y\text{-intercept} = (0, y) = (0, 100)$$

$$x\text{-intercept} = (x, 0) = (200, 0)$$

$$\text{test point} = (0, 0)$$

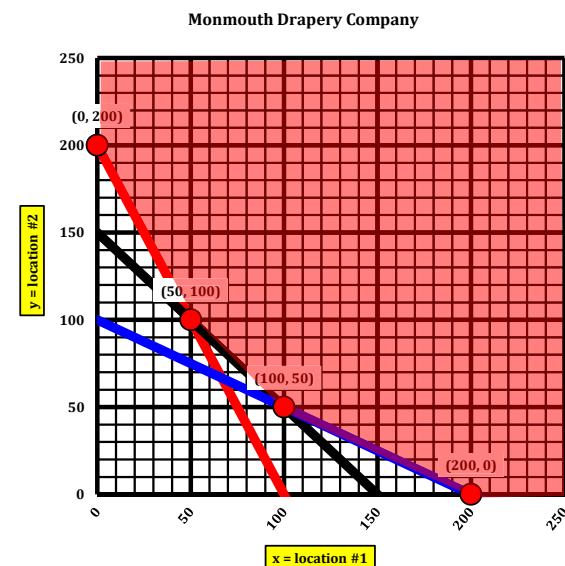
$$6x + 12y \geq 1200$$

$$6(0) + 12(0) \geq 1200$$

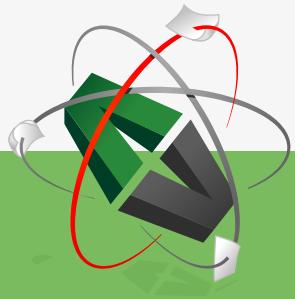
$$0 \geq 1200 \rightarrow \text{false}$$

$$\text{shade away from } (0, 0)$$

The last two inequalities (constraints) guarantee that the graph will be in the first quadrant. We have graphed each of the first three inequalities (constraints). Now we just need to determine where the overlap exists among all of the given inequalities (constraints). Then we need to determine the vertex coordinates of the points where the graphed lines intersect and form the polygonal solution region.



This e-book  
*is made with*  
**SetaPDF**



PDF components for PHP developers

[www.setasign.com](http://www.setasign.com)

Once we have the vertex coordinates, we can evaluate the given objective function:

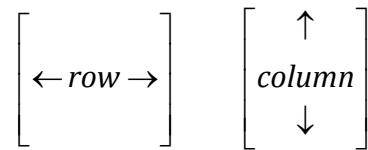
vertices		objective
x	y	$F = 900x + 750y$
0	200	$F = 900(0) + 750(200) = 150,000$
50	100	$F = 900(50) + 750(100) = 120,000$
100	50	$F = 900(100) + 750(50) = 127,500$
200	0	$F = 900(200) + 750(0) = 180,000$

Finally, just find the lowest (minimum) value of these evaluations. Using the vertex coordinates, we obtain the evaluations of: 150,000 120,000 127,500 180,000...the lowest of these evaluations being: 120,000. Therefore, the optimal minimum value for the given constraints is: **MIN = \$120,000 when  $x = 50$  and  $y = 100$**  (*which means that the company should schedule 50 days at location #1 and 100 days at location #2 to make enough draperies to satisfy the given orders*).

## 2 MATRICES & ARRAY OPERATIONS

### TUTORIAL 2.1: DETERMINE THE NUMBER OF ROWS OR COLUMNS IN A MATRIX

Occasionally, numeric data can be organized in a rectangular array called a ***matrix*** where each of the rows (running from left to right) and each of the columns (running up and down) represent very specific information. The number of rows and columns are important pieces of information when we need to compute (add, subtract or multiply) with ***matrices*** (the plural of matrix).



Consider these matrices...

$$M_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \end{bmatrix}$$

NOTE: Being an orderly bunch, mathematicians tend to name matrices with a capital letter and/or subscripted variables as was done above.

**example 2.1a** How many rows are in M1 (matrix #1) from above?

**Matrix #1 has 3 rows.**

**example 2.1b** How many rows are in M2 (matrix #2) from above?

**Matrix #2 has 4 rows.**

**example 2.1c** How many rows are in M3 (matrix #3) from above?

**Matrix #3 has 5 rows.**

**example 2.1d** How many columns are in M1 (matrix #1) from above?

**Matrix #1 has 4 columns.**

**example 2.1e** How many columns are in M2 (matrix #2) from above?

**Matrix #2 has 3 columns.**

**example 2.1f** How many columns are in M3 (matrix #3) from above?

**Matrix #3 has 4 columns.**

## TUTORIAL 2.2: DETERMINE THE ORDER (DIMENSIONS) OF A MATRIX

When we want to add, subtract or multiply matrices, we will need to determine the ***order*** of the given matrices so that the proper calculations can be completed. The ***order*** of any matrix is nothing more than its dimensions expressed by its number of rows by its number of columns ( $R \times C$ ).

Consider these matrices...

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$$\mathbf{M}_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

$$\mathbf{M}_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \end{bmatrix}$$

### example 2.2a

What is the order of  $\mathbf{M}_1$  (matrix #1) from above?

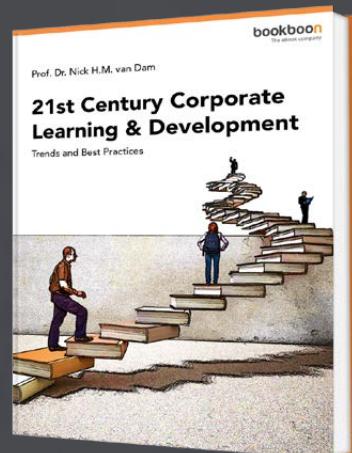
matrix #1 has 3 rows and 4 columns → dimensions =  $R \times C = 3 \times 4$

**order =  $3 \times 4$**

## Free eBook on Learning & Development

By the Chief Learning Officer of McKinsey

**Download Now**



Download free eBooks at [bookboon.com](http://bookboon.com)



**Click on the ad to read more**

**example 2.2b****What is the order of  $M_2$  (matrix #2) from above?**matrix #2 has 4 rows and 3 columns → dimensions =  $R \times C = 4 \times 3$ **order =  $4 \times 3$** **example 2.2c****What is the order of  $M_3$  (matrix #3) from above?**matrix #3 has 5 rows and 4 columns → dimensions =  $R \times C = 5 \times 4$ **order =  $5 \times 4$** 

**NOTE:** When we multiply the number of rows by the number of columns found within a matrix, we obtain the number of **elements** (members, entries) of the data set displayed in the matrix.

**TUTORIAL 2.3: IDENTIFY SQUARE MATRICES**

When the number of rows and the number of columns within a matrix are the same value, the matrix is considered a **square matrix**.

Therefore, each of these matrices is a **square matrix**...

$$M_4 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$M_5 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$M_6 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

*This matrix has 2 rows.*      *This matrix has 3 rows.*      *This matrix has 4 rows.*

*This matrix has 2 columns.*    *This matrix has 3 columns.*    *This matrix has 4 columns.*

**order =  $2 \times 2$**

**order =  $3 \times 3$**

**order =  $4 \times 4$**

**NOTE:** The number of elements (members, entries) in a square matrix is a perfect square.

**TUTORIAL 2.4: IDENTIFY THE ELEMENT (ENTRY) AT A GIVEN LOCATION ( $M_{RC}$ )**

Although what each of the rows/columns represent is important when using matrices, there may be times when we need to address just one element (member, entry) of the data being displayed in a matrix. We can do this by thinking of any location within a matrix as being defined by coordinates that use the row/column position as its basis. Any position (location) within a given matrix can be described by using the name of the matrix and the row/column numbers in which the desired element is located.

Consider these matrices...

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \end{bmatrix}$$

**example 2.4a** Identify the element with this location:  $A_{2,3}$   
*entry that sits in the 2nd row and 3rd column of matrix A  $\rightarrow A_{2,3} = 7$*

**example 2.4b** Identify the element with this location:  $B_{3,2}$   
*entry that sits in the 3rd row and 2nd column of matrix B  $\rightarrow B_{3,2} = 8$*

**example 2.4c** Identify the element with this location:  $C_{3,3}$   
*entry that sits in the 3rd row and 3rd column of matrix C  $\rightarrow C_{3,3} = 11$*

**example 2.4d** Identify the location of:  $10 = A_{R,C}$   
*10 is located in the 3rd row and in the 2nd column of matrix A  $\rightarrow 10 = A_{3,2}$*

**example 2.4e** Identify the location of:  $5 = B_{R,C}$   
*5 is located in the 2nd row and in the 2nd column of matrix B  $\rightarrow 5 = B_{2,2}$*

**example 2.4e** Identify the location of:  $19 = C_{R,C}$   
*19 is located in the 5th row and in the 3rd column of matrix C  $\rightarrow 19 = C_{5,3}$*

## TUTORIAL 2.5: DETERMINE A ZERO MATRIX WITH A GIVEN ORDER (DIMENSIONS)

Just as we use zero as a place-holder when we are writing numbers (such as 101 or 2005), we may need the use of a **zero-matrix**...a matrix where all of the entries are zero (0).

Consider these matrices...

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \end{bmatrix}$$

**example 2.5a** Write the zero-matrix that has the same order as matrix A.

$$\text{order of } [A] = R \times C = 3 \times 4 \rightarrow$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

**example 2.5b** Write the zero-matrix that has the same order as matrix B.

$$\text{order of } [B] = R \times C = 4 \times 3 \rightarrow$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**example 2.5c** Write the zero-matrix that has the same order as matrix C.

$$\text{order of } [C] = R \times C = 5 \times 4 \rightarrow$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

www.sylvania.com

We do not reinvent the wheel we reinvent light.

Fascinating lighting offers an infinite spectrum of possibilities: Innovative technologies and new markets provide both opportunities and challenges. An environment in which your expertise is in high demand. Enjoy the supportive working atmosphere within our global group and benefit from international career paths. Implement sustainable ideas in close cooperation with other specialists and contribute to influencing our future. Come and join us in reinventing light every day.

Light is OSRAM

**OSRAM** SYLVANIA



## TUTORIAL 2.6: FIND THE TRANSPOSE OF A MATRIX

The order of a given matrix is important. However, a given matrix may not have an appropriate order (dimensions) for a particular use. Therefore, there may be times when we want to rearrange or ***transpose*** the order of a matrix. When we perform a transposition for a given matrix, we swap the elements (members, entries) so that the first row becomes the first column, the second row becomes the second column, the third row becomes the third column, and so on.

**The mathematical symbol for a transposed matrix name “M” is:  $M^T$**

Consider these matrices...

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \end{bmatrix}$$

**example 2.6a**      Find  $A^T$

$$\begin{aligned} \text{order of } [A] &= R \times C = 3 \times 4 \\ \text{order of } AT &= R \times C = 4 \times 3 \end{aligned}$$

$$A^T = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}$$

**example 2.6b**      Find  $B^T$

$$\begin{aligned} \text{order of } [B] &= R \times C = 4 \times 3 \\ \text{order of } BT &= R \times C = 3 \times 4 \end{aligned}$$

$$B^T = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

**example 2.6c**      Find  $C^T$

$$\begin{aligned} \text{order of } [C] &= R \times C = 5 \times 4 \\ \text{order of } CT &= R \times C = 4 \times 5 \end{aligned}$$

$$C^T = \begin{bmatrix} 1 & 5 & 9 & 13 & 17 \\ 2 & 6 & 10 & 14 & 18 \\ 3 & 7 & 11 & 15 & 19 \\ 4 & 8 & 12 & 16 & 20 \end{bmatrix}$$

## TUTORIAL 2.7: DETERMINE THE NEGATIVE OF A MATRIX

Just as we have negative numbers (which is really nothing more than a number with a minus sign in front), we also have ***negative matrices***...which is nothing more than putting a minus sign in front of each of the given elements (members, entries).

The mathematical symbol for a negative matrix name “M” is:  $-M$

Consider these matrices...

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \end{bmatrix}$$

**example 2.7a**      Find  $-[A]$

*multiply each element in [A] by -1  $\rightarrow$*   $-A = \begin{bmatrix} -1 & -2 & -3 & -4 \\ -5 & -6 & -7 & -8 \\ -9 & -10 & -11 & -12 \end{bmatrix}$

**example 2.7b**      Find  $-[B]$

*multiply each element in [B] by -1  $\rightarrow$*   $-B = \begin{bmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \\ -10 & -11 & -12 \end{bmatrix}$

**example 2.7c**      Find  $-[C]$

*multiply each element in [C] by -1  $\rightarrow$*   $-C = \begin{bmatrix} -1 & -2 & -3 & -4 \\ -5 & -6 & -7 & -8 \\ -9 & -10 & -11 & -12 \\ -13 & -14 & -15 & -16 \\ -17 & -18 & -19 & -20 \end{bmatrix}$

## TUTORIAL 2.8: USE MATRICES TO PRESENT AND INTERPRET GIVEN DATA

### example 2.8

A poll of 5004 people was completed asking the respondents their registered political party and their opinion about how they felt about the NJ Governor's job performance:

- Of those registered as Democrats (D), 387 approved, 680 did not approve and 139 had no opinion.
- Of those registered as Republicans (R), 1270 approved, 640 did not approve and 1131 had no opinion.
- Of those registered as Independents (I), 204 approved, 192 did not approve and 58 had no opinion.
- Of those who were not registered (NR), 139 approved, 97 did not approve and 67 had no opinion.



**Deloitte.**

© Deloitte & Touche LLP and affiliated entities.

Discover the truth at [www.deloitte.ca/careers](http://www.deloitte.ca/careers)

Download free eBooks at [bookboon.com](http://bookboon.com)



**Click on the ad to read more**

- a) Use the given information to create a  $4 \times 3$  matrix.
- b) Use the given information to create a  $3 \times 4$  matrix.
- c) What is the relationship between the two matrices?
- d) How many Democrats (D) were included in the poll results?
- e) How many Republicans (R) were included in the poll results?
- f) How many Independents (I) were included in the poll results?
- g) How many poll respondents were not registered?
- h) How many poll respondents approved of the Governor's job performance?
- i) How many poll respondents did not approve of the Governor's job performance?
- j) How many poll respondents had no opinion toward the Governor's job performance?

**answers:**

- a) Since the desired matrix is to have 4 rows and 3 columns, we need to determine the

	approve	disapprove	no opinion
D	387	680	139
R	1270	640	1131
I	204	192	58
NR	139	97	67

labels for these rows and columns. We have 4 political affiliations, so these affiliations should be the labels for the rows. We have 3 approval ratings so these ratings should be the labels for the columns.

- b) Since the desired matrix is to have 3 rows and 4 columns, we need to determine the

	D	R	I	NR
approve	387	1270	204	139
disapprove	680	640	192	97
no opinion	139	1131	58	67

labels for these rows and columns. We have 3 approval ratings, so these ratings should be the labels for the rows. We have 4 political affiliations, so these affiliations should be the labels for the columns.

- c) These two matrices are **transposes of each other** since the rows and columns were switched for each other.
- d) Look at either of the matrices created above and add the numbers with the "democrat" label:  $387 + 680 + 139 = 1206$ .
- e) Look at either of the matrices created above and add the numbers with the "republican" label:  $1270 + 640 + 1131 = 3041$ .
- f) Look at either of the matrices created above and add the numbers with the "independent" label:  $204 + 192 + 58 = 454$ .

**answers:**  
**(continued)**

- g) Look at either of the matrices created above and add the numbers with the “not registered” label:  $139 + 97 + 67 = \textcolor{red}{303}$ .
- h) Look at either of the matrices created above and add the numbers with the “approved” label:  $387 + 1270 + 204 + 139 = \textcolor{red}{2000}$ .
- i) Look at either of the matrices created above and add the numbers with the “disapproved” label:  $680 + 640 + 192 + 97 = \textcolor{red}{1609}$ .
- j) Look at either of the matrices created above and add the numbers with the “no opinion” label:  $139 + 1131 + 58 + 67 = \textcolor{red}{1395}$ .

## TUTORIAL 2.9: ADD MATRICES

When we need to add two or more matrices together, we must first make sure that the order of each matrix to be added is the same as the others. We cannot add matrices if these dimensions are different. Once we have established that the given matrices can in fact be added, we just add elements (members, entries) that are located in the same spot of each matrix. The sum of these multiple additions will be written in the common location in the answer matrix:

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & m \end{bmatrix} + \begin{bmatrix} n & p & q & r \\ s & t & u & v \\ w & x & y & z \end{bmatrix} = \begin{bmatrix} a+n & b+p & c+q & d+r \\ e+s & f+t & g+u & h+v \\ i+w & j+x & k+y & m+z \end{bmatrix}$$

**example 2.9a**      Find the sum of:  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{bmatrix}$

Following the general rule for adding matrices:

$$\begin{bmatrix} 1+10 & 2+11 & 3+12 \\ 4+13 & 5+14 & 6+15 \\ 7+16 & 8+17 & 9+18 \end{bmatrix} = \begin{bmatrix} 11 & 13 & 15 \\ 17 & 19 & 21 \\ 23 & 25 & 27 \end{bmatrix}$$

**example 2.9b**      Find the sum of:  $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$

Following the general rule for adding matrices:

$$\begin{bmatrix} 1+1 & 2+2 & 3+3 & 4+4 \\ 5+5 & 6+6 & 7+7 & 8+8 \\ 9+9 & 10+10 & 11+11 & 12+12 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 10 & 12 & 14 & 16 \\ 18 & 20 & 22 & 24 \end{bmatrix}$$

**example 2.9c**

**Find the sum of:** 
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

*1st matrix has an order of  $3 \times 4$  while the 2nd matrix has an order of  $4 \times 3$*

**Since these dimensions are not identical, these matrices cannot be added together.**

**NOTE:**

As long as the dimensions of the matrices that are to be added are the same, it makes no difference which of the matrices comes first. Therefore, the Commutative Property applies to the addition of matrices.

SIMPLY CLEVER
ŠKODA

We will turn your CV into  
an opportunity of a lifetime

Do you like cars? Would you like to be a part of a successful brand?  
We will appreciate and reward both your enthusiasm and talent.  
Send us your CV. You will be surprised where it can take you.

Send us your CV on  
[www.employerforlife.com](http://www.employerforlife.com)

Do you like cars? Would you like to be a part of a successful brand?  
We will appreciate and reward both your enthusiasm and talent.  
Send us your CV. You will be surprised where it can take you.

Send us your CV on  
[www.employerforlife.com](http://www.employerforlife.com)



## TUTORIAL 2.10: SUBTRACT MATRICES

When we need to subtract two or more matrices, we must first make sure that the order of each matrix to be added is the same as the others. We cannot subtract matrices if these dimensions are different. Once we have established that the given matrices can in fact be subtracted, we just subtract elements (members, entries) that are located in the same spot of each matrix. The difference of these multiple subtractions will be written in the common location in the answer matrix:

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & m \end{bmatrix} - \begin{bmatrix} n & p & q & r \\ s & t & u & v \\ w & x & y & z \end{bmatrix} = \begin{bmatrix} a-n & b-p & c-q & d-r \\ e-s & f-t & g-u & h-v \\ i-w & j-x & k-y & m-z \end{bmatrix}$$

**example 2.10a**      **Find the difference of:**  $\begin{bmatrix} -1 & 2 & -3 \\ 4 & -5 & 6 \\ -7 & 8 & -9 \end{bmatrix} - \begin{bmatrix} 10 & -11 & 12 \\ -13 & 14 & -15 \\ 16 & -17 & 18 \end{bmatrix}$

*Following the general rule for subtracting matrices:*

$$\begin{bmatrix} -1-10 & 2-(-11) & -3-12 \\ 4-(-13) & -5-14 & 6-(-15) \\ -7-16 & 8-(-17) & -9-18 \end{bmatrix} = \begin{bmatrix} -11 & 13 & -15 \\ 17 & -19 & 21 \\ -23 & 25 & -27 \end{bmatrix}$$

**example 2.10b**      **Find the difference of:**  $\begin{bmatrix} 0 & -2 & 3 & -4 \\ -5 & 6 & -7 & 8 \\ 9 & -10 & 11 & -12 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 0 & 4 \\ 5 & -6 & 7 & -8 \\ -9 & 10 & -11 & 12 \end{bmatrix}$

*Following the general rule for subtracting matrices:*

$$\begin{bmatrix} 0-(-1) & -2-2 & 3-0 & -4-4 \\ -5-5 & 6-(-6) & -7-7 & 8-(-8) \\ 9-(-9) & -10-10 & 11-(-11) & -12-12 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 3 & -8 \\ -10 & 0 & -14 & 0 \\ 0 & -20 & 0 & -24 \end{bmatrix}$$

**example 2.10c**      **Find the difference of:**  $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$

*1st matrix has an order of  $3 \times 4$  while 2nd matrix has an order of  $4 \times 3$*   
**Since these dimensions are not identical, these matrices cannot be subtracted.**

**NOTE:** Since the second matrix in each matrix subtraction becomes its negative, the order in which the problem is presented makes a difference. Therefore, the Commutative Property does not apply to matrix subtraction.

## TUTORIAL 2.11: USE MATRIX ADDITION/SUBTRACTION TO SOLVE REAL-WORLD BUSINESS APPLICATIONS

### example 2.11

**Complete this real-world application:**

The Monmouth Cabinet Company manufactures bookcases and filing cabinets at two different plants (A, B). Matrix "M" displays the production for the two plants during the month of September. Matrix "N" displays the production for the two plants during the month of October:

	<b>A</b>	<b>B</b>	
<b>M =</b>	100	200	<i>bookcases</i>
	150	300	<i>filing cabinets</i>

	<b>A</b>	<b>B</b>	
<b>N =</b>	200	400	<i>bookcases</i>
	300	600	<i>filing cabinets</i>

Matrix "P" represents the inventories at the plants at the beginning of September and matrix "S" represents the shipments from the plants during September:

	<b>A</b>	<b>B</b>	
<b>P =</b>	150	250	<i>bookcases</i>
	350	450	<i>filing cabinets</i>

	<b>A</b>	<b>B</b>	
<b>S =</b>	250	350	<i>bookcases</i>
	450	550	<i>filing cabinets</i>

- a) Use the proper array (matrix) operation to produce the matrix that represents the total production at the two plants for September and October by performing the proper matrix operation.
- b) Use the proper array (matrix) operations involving matrices "M", "P" and "S" to create the matrix that represents the inventories at the end of September.
- c) If matrix "P" also represents the inventories at the plants at the beginning of October and matrix "S" also represents shipments from the plants during October, use the proper array (matrix) operations to create the matrix that represents the inventories at the end of October.

**e-learning for kids**

- The number 1 MOOC for Primary Education
- Free Digital Learning for Children 5-12
- 15 Million Children Reached

**About e-Learning for Kids** Established in 2004, e-Learning for Kids is a global nonprofit foundation dedicated to fun and free learning on the Internet for children ages 5 - 12 with courses in math, science, language arts, computers, health and environmental skills. Since 2005, more than 15 million children in over 190 countries have benefitted from eLessons provided by EFK! An all-volunteer staff consists of education and e-learning experts and business professionals from around the world committed to making difference. eLearning for Kids is actively seeking funding, volunteers, sponsors and courseware developers; get involved! For more information, please visit [www.e-learningforkids.org](http://www.e-learningforkids.org).



**answers:**

- a) To find the “total” production of the two plants: [M] + [N]

$$\begin{bmatrix} 100 & 200 \\ 150 & 300 \end{bmatrix} + \begin{bmatrix} 200 & 400 \\ 300 & 600 \end{bmatrix} = \begin{bmatrix} 100+200 & 200+400 \\ 150+300 & 300+600 \end{bmatrix} = \begin{bmatrix} 300 & 600 \\ 450 & 900 \end{bmatrix}$$

- b) To find the inventories at the end of September: [M] + [P] – [S]

$$\begin{bmatrix} 100 & 200 \\ 150 & 300 \end{bmatrix} + \begin{bmatrix} 150 & 250 \\ 350 & 450 \end{bmatrix} - \begin{bmatrix} 250 & 350 \\ 450 & 550 \end{bmatrix} = \begin{bmatrix} 100+150-250 & 200+250-350 \\ 150+350-450 & 300+450-550 \end{bmatrix} = \begin{bmatrix} 0 & 100 \\ 50 & 200 \end{bmatrix}$$

- c) To find the inventories at the end of October: [N] + [P] – [S]

$$\begin{bmatrix} 200 & 400 \\ 300 & 600 \end{bmatrix} + \begin{bmatrix} 150 & 250 \\ 350 & 450 \end{bmatrix} - \begin{bmatrix} 250 & 350 \\ 450 & 550 \end{bmatrix} = \begin{bmatrix} 200+150-250 & 400+250-350 \\ 300+350-450 & 600+450-550 \end{bmatrix} = \begin{bmatrix} 100 & 300 \\ 200 & 500 \end{bmatrix}$$

## TUTORIAL 2.12: MULTIPLY A MATRIX BY A SCALAR (REAL NUMBER)

Each element (member, entry) within a matrix can be multiplied by one number called the *scalar*. This scalar multiplication might be useful when we want to increase (decrease) the entries within a matrix by the same rate...such as a 15% mark-up (as will be done in example 2.12b) or a 15% mark-down (as will be done in example 2.12c).

**example 2.12a**

**Complete this scalar multiplication:**  $5 \begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix}$

*Multiply each of the elements by the scalar (5):*

$$\begin{bmatrix} 5(1) & 5(-2) & 5(3) \\ 5(-4) & 5(5) & 5(-6) \\ 5(7) & 5(-8) & 5(9) \end{bmatrix} = \begin{bmatrix} 5 & -10 & 15 \\ -20 & 25 & -30 \\ 35 & -40 & 45 \end{bmatrix}$$

**example 2.12b**

**Complete this scalar multiplication:**  $1.15 \begin{bmatrix} 100 & 200 \\ 300 & 400 \end{bmatrix}$

*Multiply each of the elements by the scalar (1.15):*

$$\begin{bmatrix} 1.15(100) & 1.15(200) \\ 1.15(300) & 1.15(400) \end{bmatrix} = \begin{bmatrix} 115 & 230 \\ 345 & 460 \end{bmatrix}$$

**example 2.12c** Complete this scalar multiplication:  $0.85 \begin{bmatrix} 100 & 200 \\ 300 & 400 \end{bmatrix}$

Multiply each of the elements by the scalar (0.85):

$$\begin{bmatrix} (0.85)(100) & (0.85)(200) \\ (0.85)(300) & (0.85)(400) \end{bmatrix} = \begin{bmatrix} 85 & 170 \\ 255 & 340 \end{bmatrix}$$

**example 2.12d** Complete this matrix problem:  $4 \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} + 3 \begin{bmatrix} 8 & 7 \\ 6 & 5 \end{bmatrix}$

$$\text{complete first scalar multiplication: } 4 \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4(4) & 4(3) \\ 4(2) & 4(1) \end{bmatrix} = \begin{bmatrix} 16 & 12 \\ 8 & 4 \end{bmatrix}$$

$$\text{complete second scalar multiplication: } 3 \begin{bmatrix} 8 & 7 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} 3(8) & 3(7) \\ 3(6) & 3(5) \end{bmatrix} = \begin{bmatrix} 24 & 21 \\ 18 & 15 \end{bmatrix}$$

$$\text{add products: } \begin{bmatrix} 16 & 12 \\ 8 & 4 \end{bmatrix} + \begin{bmatrix} 24 & 21 \\ 18 & 15 \end{bmatrix} = \begin{bmatrix} 16+24 & 12+21 \\ 8+18 & 18+15 \end{bmatrix} = \begin{bmatrix} 40 & 33 \\ 26 & 33 \end{bmatrix}$$

**example 2.12e** Complete this matrix problem:  $3 \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} - 4 \begin{bmatrix} 8 & 7 \\ 6 & 5 \end{bmatrix}$

$$\text{complete first scalar multiplication: } 3 \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3(4) & 3(3) \\ 3(2) & 3(1) \end{bmatrix} = \begin{bmatrix} 12 & 9 \\ 6 & 3 \end{bmatrix}$$

$$\text{complete second scalar multiplication: } -4 \begin{bmatrix} 8 & 7 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} -4(8) & -4(7) \\ -4(6) & -4(5) \end{bmatrix} = \begin{bmatrix} -32 & -28 \\ -24 & -20 \end{bmatrix}$$

$$\text{add products: } \begin{bmatrix} 12 & 9 \\ 6 & 3 \end{bmatrix} + \begin{bmatrix} -32 & -28 \\ -24 & -20 \end{bmatrix} = \begin{bmatrix} 12-32 & 9-28 \\ 6-24 & 3-20 \end{bmatrix} = \begin{bmatrix} -20 & -19 \\ -18 & -17 \end{bmatrix}$$

## TUTORIAL 2.13: FIND THE PRODUCT OF GIVEN MATRICES

Two matrices (of any dimension  $R \times C$ ) may be multiplied if the number of columns (which run up and down) in the first matrix equals the number of rows (which run left to left) in the second matrix.

The two matrices presented at the right may be multiplied together. The order ( $R \times C$ ) of the first matrix is  $3 \times 2$  while the order ( $R \times C$ ) of the second matrix is  $2 \times 4$ . Since the number of columns in the first matrix is equal to the number of rows in the second matrix, the given matrices may be multiplied as presented.

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \times \begin{bmatrix} s & t & u & v \\ w & x & y & z \end{bmatrix}$$

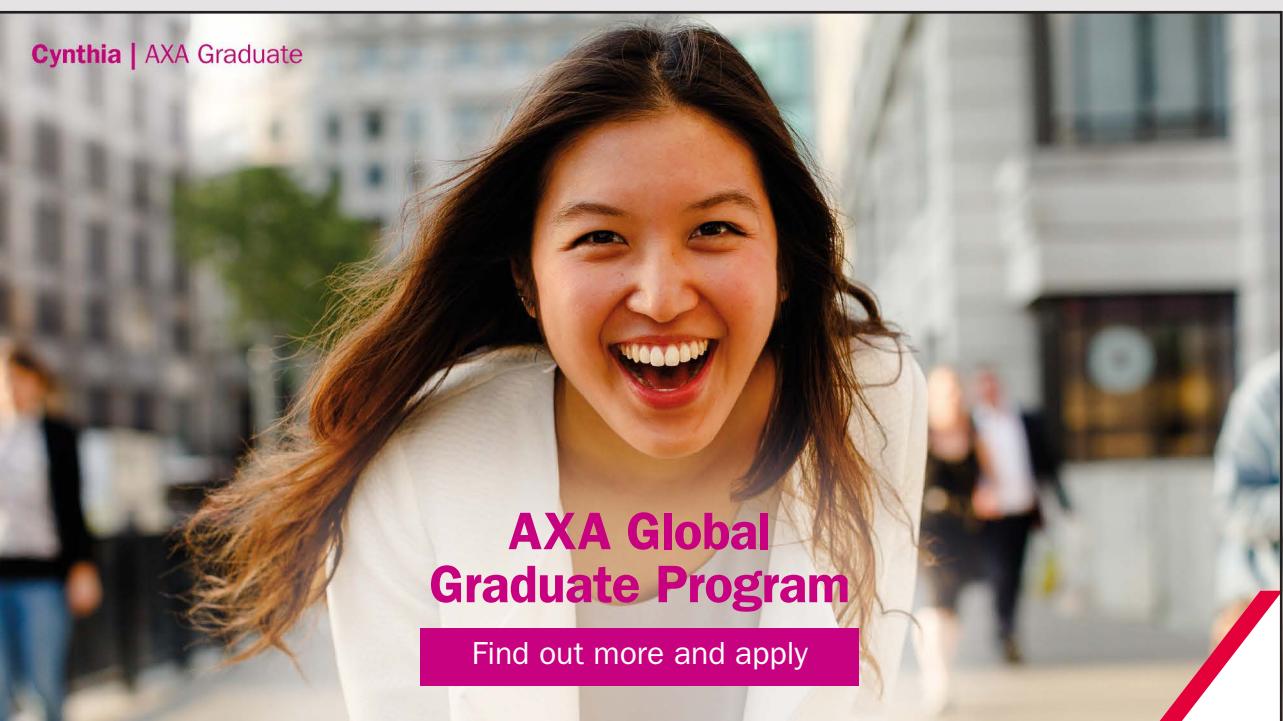
However, if these same matrices are reversed (as presented at the right), The order ( $R \times C$ ) of the first matrix is  $2 \times 4$  while the order ( $R \times C$ ) of the second matrix is  $3 \times 2$ . Since the number of columns in the first matrix no longer equals the number of rows in the second matrix, the given matrices cannot be multiplied as presented.

$$\begin{bmatrix} s & t & u & v \\ w & x & y & z \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

Therefore, it is very important that we pay attention to the dimensions of the given matrices and in the order in which we pose a matrix multiplication problem.

When we multiply two matrices that are appropriately presented, we take the elements (members, entries) from the first row of the first matrix, multiply them with the respective elements (members, entries) from each of the columns from the second matrix and then add the products together. Notice how the product of the matrices from above is achieved:

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \times \begin{bmatrix} s & t & u & v \\ w & x & y & z \end{bmatrix} = \begin{bmatrix} as + bw & at + bx & au + by & av + bz \\ cs + dw & ct + dx & cu + dy & cv + dz \\ es + fw & et + fx & eu + fy & ev + fz \end{bmatrix}$$



**Cynthia | AXA Graduate**

**AXA Global Graduate Program**

[Find out more and apply](#)

**redefining / standards** 

**NOTE:** Since the number of columns in the first matrix is equal to the number of rows in the second matrix ( $C_1 = R_2$ ), these matrices can be multiplied as presented. The product of these matrices will have the dimensions defined by the number of rows in the first matrix by the number of columns in the second matrix:  $R_1 \times C_2$

As shown above, the order in which the matrices appear in the problem makes a difference. Therefore, the Commutative Property does not apply to matrix multiplication unless the two matrices being multiplied are both the same size square matrices.

**example 2.13a** Determine whether these matrices can be multiplied as presented:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \end{bmatrix}$$

1. Determine the matrix orders:  
 $[R_1 \times C_1] \times [R_2 \times C_2] = [4 \times 3] \times [5 \times 4]$
2. Compare  $C_1$  and  $R_2$ ...Are they equal or not equal???
3. Since  $C_1 \neq R_2$ , these matrices **cannot be multiplied** as presented.

**example 2.13b** Determine whether these matrices can be multiplied as presented:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

1. Determine the matrix orders:  
 $[R_1 \times C_1] \times [R_2 \times C_2] = [5 \times 4] \times [4 \times 3]$
2. Compare  $C_1$  and  $R_2$ ...Are they equal or not equal???
3. Since  $C_1 = R_2$ , these matrices **can be multiplied** as presented.  
*NOTE: This product will be a  $5 \times 3$  matrix.*

**example 2.13c** Determine whether these matrices can be multiplied as presented:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

- Determine the matrix orders:

$$[R_1 \times C_1] \times [R_2 \times C_2] = [3 \times 4] \times [4 \times 3]$$

- Compare  $C_1$  and  $R_2$ ...Are they equal or not equal???

- Since  $C_1 = R_2$ , these matrices **can be multiplied** as presented.

*NOTE: This product will be a  $3 \times 3$  matrix.*

**example 2.13d** Determine whether these matrices can be multiplied as presented:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

- Determine the matrix orders:

$$[R_1 \times C_1] \times [R_2 \times C_2] = [4 \times 3] \times [3 \times 4]$$

- Compare  $C_1$  and  $R_2$ ...Are they equal or not equal???

- Since  $C_1 = R_2$ , these matrices **can be multiplied** as presented.

*NOTE: This product will be a  $4 \times 4$  matrix.*

**example 2.13e**

Find this matrix product:  $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$

- Determine the matrix orders:

$$[R_1 \times C_1] \times [R_2 \times C_2] = [3 \times 4] \times [4 \times 3]$$

- Compare  $C_1$  and  $R_2$ ... $C_1 = R_2$ ...OK to multiply.

- Determine the order of the product:  $[3 \times 3]$

- Multiply the entries in the Nth row of the first matrix by the respective entries in the Nth column of the second matrix, entering the results in the appropriate cells.

Generally:

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & m \end{bmatrix} \times \begin{bmatrix} n & p & q \\ r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} an+br+cu+dx & ap+bs+cv+dy & aq+bt+cw+dz \\ en+fr+gu+hx & ep+fs+gv+hy & eq+ft+gw+hz \\ in+jr+ku+mx & ip+js+kv+my & iq+jt+kw+mz \end{bmatrix}$$

Specifically:

$$\left[ \begin{array}{lll} 1(1) + 2(4) + 3(7) + 4(10) = 70 & 1(2) + 2(5) + 3(8) + 4(11) = 80 & 1(3) + 2(6) + 3(9) + 4(12) = 90 \\ 5(1) + 6(4) + 7(7) + 8(10) = 158 & 5(2) + 6(5) + 7(8) + 8(11) = 184 & 5(3) + 6(6) + 7(9) + 8(12) = 210 \\ 9(1) + 10(4) + 11(7) + 12(10) = 246 & 9(2) + 10(5) + 11(8) + 12(11) = 288 & 9(3) + 10(6) + 11(9) + 12(12) = 330 \end{array} \right]$$

**requested product:**  $\begin{bmatrix} 70 & 80 & 90 \\ 158 & 184 & 210 \\ 246 & 288 & 330 \end{bmatrix}$

I joined MITAS because  
I wanted **real responsibility**

The Graduate Programme  
for Engineers and Geoscientists  
[www.discovermitas.com](http://www.discovermitas.com)



**Month 16**

I was a construction supervisor in the North Sea advising and helping foremen solve problems

Real work  
International opportunities  
Three work placements





**example 2.13f****Find this matrix product:**

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

1. Determine the matrix orders:
2.  $[R_1 \times C_1] \times [R_2 \times C_2] = [4 \times 3] \times [3 \times 4]$
3. Compare  $C_1$  and  $R_2 \dots C_1 = R_2 \dots$  OK to multiply.
4. Determine the order of the product:  $[4 \times 4]$
5. Multiply the entries in the Nth row of the first matrix by the respective entries in the Nth column of the second matrix, entering the results in the appropriate cells.

Generally:

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & m \\ x & y & z \end{bmatrix} \times \begin{bmatrix} n & p & q \\ r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} an+br+cu+dx & ap+bs+cv+dy & aq+bt+cw+dz \\ en+fr+gu+hx & ep+fs+gv+hy & eq+ft+gw+hz \\ in+jr+ku+mx & ip+js+kv+my & iq+jt+kw+mz \end{bmatrix}$$

Specifically:

$$\begin{bmatrix} 1(1)+2(5)+3(9)=38 & 1(2)+2(6)+3(10)=44 & 1(3)+2(7)+3(11)=50 & 1(4)+2(8)+3(12)=56 \\ 4(1)+5(5)+6(9)=83 & 4(2)+5(6)+6(10)=98 & 4(3)+5(7)+6(11)=113 & 4(4)+5(8)+6(12)=128 \\ 7(1)+8(5)+9(9)=128 & 7(2)+8(6)+9(10)=152 & 7(3)+8(7)+9(11)=176 & 7(4)+8(8)+9(12)=200 \\ 10(1)+11(5)+12(9)=173 & 10(2)+11(6)+12(10)=206 & 10(3)+11(7)+12(11)=239 & 10(4)+11(8)+12(12)=272 \end{bmatrix}$$

**requested product:**

$$\begin{bmatrix} 38 & 44 & 50 & 56 \\ 83 & 98 & 113 & 128 \\ 128 & 152 & 176 & 200 \\ 173 & 206 & 239 & 272 \end{bmatrix}$$

Notice that the given matrices in example 2.13c have been reversed to create example 2.13d. As seen in example 2.13e and example 2.13f, the reversal of the matrices will produce a very different answer. Since both of these presentations can be multiplied and matrix multiplication is not normally commutative, it is very important that we put the desired matrices into the proper order to achieve the desired product. If these matrices were derived from a real-world application, we would definitely need to pay attention to the order in which the matrix multiplication is presented.

## TUTORIAL 2.14: COMPLETE MATRIX MULTIPLICATION INVOLVING ZERO MATRICES

Although there probably will never be a time that we actually need to multiply by a zero matrix, we should examine what happens when we do...

**example 2.14a**      **Find this matrix product:**  $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- Determine the matrix orders:

$$[R_1 \times C_1] \times [R_2 \times C_2] = [3 \times 4] \times [4 \times 3]$$

- Compare  $C_1$  and  $R_2$ ...  $C_1 = R_2$ ...OK to multiply.

- Determine the order of the product:  $[3 \times 3]$

- Multiply the entries in the Nth row of the first matrix by the respective entries in the Nth column of the second matrix, entering the results in the appropriate cells.

Generally:

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & m \end{bmatrix} \times \begin{bmatrix} n & p & q \\ r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} an+br+cu+dx & ap+bs+cv+dy & aq+bt+cw+dz \\ en+fr+gu+hx & ep+fs+gv+hy & eq+ft+gw+hz \\ in+jr+ku+mx & ip+js+kv+my & iq+jt+kw+mz \end{bmatrix}$$

Specifically:

$$\begin{bmatrix} 1(0)+2(0)+3(0)+4(0)=0 & 1(0)+2(0)+3(0)+4(0)=0 & 1(0)+2(0)+3(0)+4(0)=0 \\ 5(0)+6(0)+7(0)+8(0)=0 & 5(0)+6(0)+7(0)+8(0)=0 & 5(0)+6(0)+7(0)+8(0)=0 \\ 9(0)+10(0)+11(0)+12(0)=0 & 9(0)+10(0)+11(0)+12(0)=0 & 9(0)+10(0)+11(0)+12(0)=0 \end{bmatrix}$$

**requested product:**  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

**example 2.14b****Find this matrix product:**

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

- Determine the matrix orders:

$$[R_1 \times C_1] \times [R_2 \times C_2] = [3 \times 4] \times [4 \times 3]$$

- Compare  $C_1$  and  $R_2$ ... $C_1 = R_2$ ...OK to multiply.

- Determine the order of the product:  $[3 \times 3]$

- Multiply the entries in the Nth row of the first matrix by the respective entries in the Nth column of the second matrix, entering the results in the appropriate cells.

Generally:

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & m \end{bmatrix} \times \begin{bmatrix} n & p & q \\ r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} an+br+cu+dx & ap+bs+cv+dy & aq+bt+cw+dz \\ en+fr+gu+hx & ep+fs+gv+hy & eq+ft+gw+hz \\ in+jr+ku+mx & ip+js+kv+my & iq+jt+kw+mz \end{bmatrix}$$

**93%**  
OF MIM STUDENTS ARE  
WORKING IN THEIR SECTOR 3 MONTHS  
FOLLOWING GRADUATION

## MASTER IN MANAGEMENT

- STUDY IN THE CENTER OF MADRID AND TAKE ADVANTAGE OF THE UNIQUE OPPORTUNITIES THAT THE CAPITAL OF SPAIN OFFERS
- PROPEL YOUR EDUCATION BY EARNING A DOUBLE DEGREE THAT BEST SUITS YOUR PROFESSIONAL GOALS
- STUDY A SEMESTER ABROAD AND BECOME A GLOBAL CITIZEN WITH THE BEYOND BORDERS EXPERIENCE

Length: 10 MONTHS  
Av. Experience: 1 YEAR  
Language: ENGLISH / SPANISH  
Format: FULL-TIME  
Intakes: SEPT / FEB

5 SPECIALIZATIONS PERSONALIZE YOUR PROGRAM	#10 WORLDWIDE MASTER IN MANAGEMENT FINANCIAL TIMES	55 NATIONALITIES IN CLASS
---	--	------------------------------

[www.ie.edu/master-management](http://www.ie.edu/master-management) | [mim.admissions@ie.edu](mailto:mim.admissions@ie.edu) | [f](#) [t](#) [g](#) Follow us on IE MIM Experience

Specifically:

$$\begin{bmatrix} 0(1)+0(4)+0(7)+0(10)=0 & 0(2)+0(5)+0(8)+0(11)=0 & 0(3)+0(6)+0(9)+0(12)=0 \\ 0(1)+0(4)+0(7)+0(10)=0 & 0(2)+0(5)+0(8)+0(11)=0 & 0(3)+0(6)+0(9)+0(12)=0 \\ 0(1)+0(4)+0(7)+0(10)=0 & 0(2)+0(5)+0(8)+0(11)=0 & 0(3)+0(6)+0(9)+0(12)=0 \end{bmatrix}$$

**requested product:**  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Notice that whether the zero matrix comes first or second, the product is still a zero matrix with the same dimensions. Assuming that the presentation of matrix multiplication will allow for the multiplication process, multiplying by a zero matrix is an exception to the fact that matrix multiplication is not commutative.

## TUTORIAL 2.15: COMPLETE MULTIPLICATION INVOLVING IDENTITY MATRICES

When working with matrices, an identity matrix is a square matrix that contains the value of one (1) as the entries on the main diagonal and all of the other entries are zero. Three examples of identity matrices are shown below. Notice where the ones (1) are located.

*2 × 2 identity matrix*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

*3 × 3 identity matrix*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

*4 × 4 identity matrix*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**example 2.15a**

**Find this matrix product:**

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Determine the matrix orders:

$$[R_1 \times C_1] \times [R_2 \times C_2] = [5 \times 4] \times [4 \times 4]$$

- Compare  $C_1$  and  $R_2 \dots C_1 = R_2 \dots$  OK to multiply.
- Determine the order of the product:  $[5 \times 4]$
- Multiply the entries in the Nth row of the first matrix by the respective entries in the Nth column of the second matrix, entering the results in the appropriate cells.

$$\begin{bmatrix} 1(1)+2(0)+3(0)+4(0)=1 & 1(0)+2(1)+3(0)+4(0)=2 & 1(0)+2(0)+3(1)+4(0)=3 & 1(0)+2(0)+3(0)+4(1)=4 \\ 5(1)+6(0)+7(0)+8(0)=5 & 5(0)+6(1)+7(0)+8(0)=6 & 5(0)+6(0)+7(1)+8(0)=7 & 5(0)+6(0)+7(0)+8(1)=8 \\ 9(1)+10(0)+11(0)+12(0)=9 & 9(0)+10(1)+11(0)+12(0)=10 & 9(0)+10(0)+11(1)+12(0)=11 & 9(0)+10(0)+11(0)+12(1)=12 \\ 13(1)+14(0)+15(0)+16(0)=13 & 13(0)+14(1)+15(0)+16(0)=14 & 13(0)+14(0)+15(1)+16(0)=15 & 13(0)+14(0)+15(0)+16(1)=16 \\ 17(1)+18(0)+19(0)+20(0)=17 & 17(0)+18(1)+19(0)+20(0)=18 & 17(0)+18(0)+19(1)+20(0)=19 & 17(0)+18(0)+19(0)+20(1)=20 \end{bmatrix}$$

**requested product:**

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \end{bmatrix}$$

**example 2.15b**      **Find this matrix product:**

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \end{bmatrix}$$

1. Determine the matrix orders:
2.  $[R_1 \times C_1] \times [R_2 \times C_2] = [5 \times 5] \times [5 \times 4]$
3. Compare  $C_1$  and  $R_2 \dots C_1 = R_2 \dots$  OK to multiply.
4. Determine the order of the product:  $[5 \times 4]$
5. Multiply the entries in the Nth row of the first matrix by the respective entries in the Nth column of the second matrix, entering the results in the appropriate cells.

$$\begin{bmatrix} 1(1)+0(5)+0(9)+0(13)+0(17)=1 & 1(2)+0(6)+0(10)+0(14)+0(18)=2 & 1(3)+0(7)+0(11)+0(15)+0(19)=3 & 1(4)+0(8)+0(12)+0(16)+0(20)=4 \\ 0(1)+1(5)+0(9)+0(13)+0(17)=5 & 0(2)+1(6)+0(10)+0(14)+0(18)=6 & 0(3)+1(7)+0(11)+0(15)+0(19)=7 & 0(4)+1(8)+0(12)+0(16)+0(20)=8 \\ 0(1)+0(5)+1(9)+0(13)+0(17)=9 & 0(2)+0(6)+1(10)+0(14)+0(18)=10 & 0(3)+0(7)+1(11)+0(15)+0(19)=11 & 0(4)+0(8)+1(12)+0(16)+0(20)=12 \\ 0(1)+0(5)+0(9)+1(13)+0(17)=13 & 0(2)+0(6)+0(10)+1(14)+0(18)=14 & 0(3)+0(7)+0(11)+1(15)+0(19)=15 & 0(4)+0(8)+0(12)+1(16)+0(20)=16 \\ 0(1)+0(5)+0(9)+0(13)+1(17)=17 & 0(2)+0(6)+0(10)+0(14)+1(18)=18 & 0(3)+0(7)+0(11)+0(15)+1(19)=19 & 0(4)+0(8)+0(12)+0(16)+1(20)=20 \end{bmatrix}$$

**requested product:**

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \end{bmatrix}$$

Notice that when multiplying by an identity matrix, the product will be the matrix that is not the identity. If the identity matrix is second in the multiplication presentation, its dimensions must agree with the number of columns in the first matrix. If the identity matrix is first, then its dimensions must agree with the number of rows in the second matrix.

## TUTORIAL 2.16: USE MATRIX MULTIPLICATION TO SOLVE BUSINESS APPLICATIONS

### example 2.16

**Use matrix multiplication to complete:**

The annual budget of the Monmouth Corporation has the following expenses for the named departments:

	<i>manufacturing</i>	<i>office</i>	<i>sales</i>	<i>shipping</i>	<i>accounting</i>	<i>management</i>
<i>supplies</i>	\$800	\$10,300	\$10,300	\$1200	\$5700	\$3700
<i>communication</i>	\$600	\$6200	\$6200	\$1400	\$300	\$1100
<i>transportation</i>	\$2300	\$8900	\$8900	\$1300	\$1300	\$4900
<i>salaries</i>	\$251,900	\$81,700	\$81,700	\$35,300	\$54,400	\$144,300
<i>utilities</i>	\$30,100	\$1100	\$1100	\$1100	\$1100	\$1100
<i>materials</i>	\$789,000	\$0	\$0	\$0	\$0	\$0

**"I studied English for 16 years but...  
...I finally learned to speak it in just six lessons"**

Jane, Chinese architect

ENGLISH OUT THERE

Click to hear me talking before and after my unique course download



- Create a matrix that is to be multiplied with the given matrix above so that the product of the two matrices will reflect at 15% increase in manufacturing, sales and shipping but a 7.5% decrease in the other departments (office, accounting, management).
- Multiply the given matrix from above and the matrix created in part A:  $[G] \times [A]$
- Create a matrix that is to be multiplied with the given matrix above so that the product of the two matrices will reflect at 25% increase in manufacturing, a 5% increase in office, a 7.5% increase in sales, a 25% increase in shipping, a 7.5% increase in accounting and a 5% decrease in management.
- Multiply the given matrix from above and the matrix created in part C:  $[G] \times [C]$

**answers:**

- a) The matrix that displays the given information is a  $6 \times 6$  matrix. Therefore, the requested matrix should also be a  $6 \times 6$  matrix. Since we want to display a 15% increase in the entire first, third and fourth columns, we should place 1.15 ( $100\% + 15\% = 115\%$ ) in these positions: A1,1 A3,3 A4,4. Since we want to display a 7.5% decrease in the entire second, fifth and sixth columns, we should place 0.925 ( $100\% - 7.5\% = 92.5\%$ ) in these positions: A2,2 A5,5 A6,6. The remaining positions within the requested matrix should be filled with zeroes. Thus, the requested matrix is:

$$\begin{bmatrix} 1.15 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.925 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.925 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.925 \end{bmatrix}$$

- b) To find the product of  $[G] \times [A]$ , multiply the entries in the Nth row of the first matrix by the respective entries in the Nth column of the second matrix:

$$\begin{bmatrix} 800 & 10300 & 10300 & 1200 & 5700 & 3700 \\ 600 & 6200 & 6200 & 1400 & 300 & 1100 \\ 2300 & 8900 & 8900 & 1300 & 1300 & 4900 \\ 251900 & 81700 & 81700 & 35300 & 54400 & 144300 \\ 30100 & 1100 & 1100 & 1100 & 1100 & 1100 \\ 789000 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1.15 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.925 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.925 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.925 \end{bmatrix} = \begin{bmatrix} 920 & 9528 & 11845 & 1380 & 5273 & 3423 \\ 690 & 5735 & 7130 & 1610 & 278 & 1018 \\ 2645 & 8233 & 10235 & 1495 & 1203 & 4533 \\ 289685 & 75573 & 93955 & 40595 & 50320 & 133478 \\ 34615 & 1018 & 1265 & 1265 & 1018 & 1018 \\ 907350 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**answers:**  
**(continued)**

c) The matrix that displays the given information is a  $6 \times 6$  matrix. Therefore, the requested matrix should also be a  $6 \times 6$  matrix. We want to display a 25% increase in the entire first column, a 5% increase in the entire second column, a 7.5% increase in the entire third column, a 25% increase in the entire fourth column, a 7.5% increase in the entire fifth column and a 5% decrease in the entire sixth column. Thus:

- We should place 1.25 ( $100\% + 25\% = 125\%$ ) in position C1, 1.
- We should place 1.05 ( $100\% + 5\% = 105\%$ ) in position C2, 2.
- We should place 1.075 ( $100\% + 7.5\% = 107.5\%$ ) in position C3, 3.
- We should place 1.25 ( $100\% + 25\% = 125\%$ ) in position C4, 4.
- We should place 1.075 ( $100\% + 7.5\% = 107.5\%$ ) in position C5, 5.
- We should place 0.95 ( $100\% - 5\% = 95\%$ ) in position C6, 6.

The remaining positions within the requested matrix should be filled with zeroes. Thus, the requested matrix is::

$$\begin{bmatrix} 1.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.05 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.075 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.075 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.95 \end{bmatrix}$$

d) To find the product of  $[G] \times [C]$ , multiply the entries in the Nth row of the first matrix by the respective entries in the Nth column of the second matrix:

$$\begin{bmatrix} 800 & 10300 & 10300 & 1200 & 5700 & 3700 \end{bmatrix} \times \begin{bmatrix} 1.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.05 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.075 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.075 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.95 \end{bmatrix} = \begin{bmatrix} 1000 & 10815 & 11073 & 1500 & 6128 & 3515 \\ 750 & 6510 & 6665 & 1750 & 323 & 1045 \\ 2875 & 9345 & 9568 & 1625 & 1398 & 4655 \\ 314875 & 85785 & 87828 & 44125 & 58480 & 137085 \\ 37625 & 1155 & 1183 & 1375 & 1183 & 1045 \\ 986250 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

## TUTORIAL 2.17: WRITE THE AUGMENTED MATRIX FOR A SYSTEM OF EQUATIONS

An ***augmented matrix*** removes the variables from a given system and then displays the coefficients (those number that were multiplied with the variables) to the left of a vertical line while the constants that were to the right of the equals signs to the right of the vertical line. Any coefficients that are being multiplied are thought of as negatives numbers. If a variable is missing from an equation, its coefficient should be thought of as zero.

**example 2.17a**

**Write the augmented matrix for:** 
$$\begin{cases} x - 3y + 4z = 10 \\ 2x + 2y + z = 15 \\ 3x - 4y + 2z = 10 \end{cases}$$

*Use just the numbers that appear in the given system of equations:*

$$\begin{cases} 1x + -3y + 4z = 10 \\ 2x + 2y + 1z = 15 \\ 3x + -4y + 2z = 10 \end{cases} \rightarrow \left[ \begin{array}{ccc|c} 1 & -3 & 4 & 10 \\ 2 & 2 & 1 & 15 \\ 3 & -4 & 2 & 10 \end{array} \right]$$

Excellent Economics and Business programmes at:


/

university of  
groningen



**“The perfect start  
of a successful,  
international career.”**

[www.rug.nl/feb/education](http://www.rug.nl/feb/education)

**CLICK HERE**  
to discover why both socially  
and academically the University  
of Groningen is one of the best  
places for a student to be

**example 2.17b**      Write the augmented matrix for: 
$$\begin{cases} 2x - 6y - 12z = 82 \\ 3x - 10y - 20z = 135 \\ 2x - 17z = 129 \end{cases}$$

Use just the numbers that appear in the given system of equations:

$$\begin{cases} 2x + -6y + -12z = 82 \\ 3x + -10y + -20z = 135 \\ 2x + 0y + -17z = 129 \end{cases} \rightarrow \left[ \begin{array}{ccc|c} 2 & -6 & -12 & 82 \\ 3 & -10 & -20 & 135 \\ 2 & 0 & -17 & 129 \end{array} \right]$$

**example 2.17c**      Write the augmented matrix for: 
$$\begin{cases} w + 4x - 6y - 3z = -4 \\ w - x - y + 2z = 16 \\ w + y + z = 12 \\ x + y + z = 6 \end{cases}$$

Use just the numbers that appear in the given system of equations:

$$\begin{cases} 1w + 4x + -6y + -3z = -4 \\ 1w + -1x + -1y + 2z = 16 \\ 1w + 0x + 1y + 1z = 12 \\ 0w + 1x + 1y + 1z = 6 \end{cases} \rightarrow \left[ \begin{array}{cccc|c} 1 & 4 & -6 & -3 & -4 \\ 1 & -1 & -1 & 2 & 16 \\ 1 & 0 & 1 & 1 & 12 \\ 0 & 1 & 1 & 1 & 6 \end{array} \right]$$

## TUTORIAL 2.18: USE A REDUCED MATRIX TO DETERMINE THE SOLUTIONS FOR A GIVEN SYSTEM

We can use the **Gauss-Jordan Row Operations** method to solve systems of equations by:

1. Rewriting the given system into an augmented matrix.
2. Perform one or more of these row operations...
  - a. Interchange any two rows.
  - b. Add a multiple of one row to another.
  - c. Multiply a row by a non-zero constant.
3. Use the reduced matrix to determine the solutions for the given system.

Once the Gauss-Jordan row operations have successfully been applied, an identity matrix will appear to the right of the vertical line while the solutions appear to the left of the vertical line. The first row provides the solution for the first variable, the second row provides the solution for the second variable, the third row provides the solution for the third variable, and so on.

**example 2.18a**      Use the reduced matrix to determine the solutions: 
$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$3 \times 3$  identity matrix  $\rightarrow$  3 solutions to be read  $\rightarrow x = 4, y = 2, z = 3$

**example 2.18b**      Use the reduced matrix to determine the solutions: 
$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -7 \end{array} \right]$$

$3 \times 3$  identity matrix  $\rightarrow$  3 solutions to be read  $\rightarrow x = 5, y = 2, z = -7$

**example 2.18c**      Use the reduced matrix to determine the solutions: 
$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \end{array} \right]$$

$4 \times 4$  identity matrix  $\rightarrow$  4 solutions to be read  $\rightarrow w = 7, x = 1, y = 0, z = 5$

## TUTORIAL 2.19: USE MATRICES TO SOLVE SYSTEMS WITH UNIQUE SOLUTIONS

We can use the **Gauss-Jordan Row Operations** method to solve systems of equations by:

1. Rewriting the given system into an augmented matrix.
2. Perform one or more of these row operations...
  - a. Interchange any two rows.
  - b. Add a multiple of one row to another.
  - c. Multiply a row by a non-zero constant.
3. Use the reduced matrix to determine the solutions for the given system.

Remember that the goal of this process is to transform an augmented matrix into a “reduced matrix” that displays an identity matrix to the left of the vertical line and the proper solutions of the system to the right.

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{array} \right]$$

To that end (in a  $3 \times 3$  coefficient matrix), we need to address these steps:

	C1	C2	C3	C4
R1:	1			
R2:				
R3:				

diagram #1

	C1	C2	C3	C4
R1:	1			
R2:	2			
R3:	3			

diagram #2

	C1	C2	C3	C4
R1:	1			
R2:	0	4		
R3:	0			

diagram #3

	C1	C2	C3	C4
R1:	1	5		
R2:	0	1		
R3:	0	6		

diagram #4

	C1	C2	C3	C4
R1:	1	0		
R2:	0	1		
R3:	0	0	7	

diagram #5

	C1	C2	C3	C4
R1:	1	0	8	
R2:	0	1	9	
R3:	0	0	1	

diagram #6

## American online LIGS University

is currently enrolling in the  
Interactive Online **BBA, MBA, MSc,**  
**DBA and PhD** programs:

- ▶ **enroll by September 30th, 2014** and
- ▶ **save up to 16%** on the tuition!
- ▶ pay in 10 installments / 2 years
- ▶ Interactive Online education
- ▶ visit [www.ligsuniversity.com](http://www.ligsuniversity.com) to find out more!

Note: LIGS University is not accredited by any nationally recognized accrediting agency listed by the US Secretary of Education.  
[More info here.](#)



- 1 Apply a permissible row operation to fill this position (in diagram #1) with a one.
- 2 Apply a permissible row operation to fill this position (in diagram #2) with a zero.
- 3 Apply a permissible row operation to fill this position (in diagram #2) with a zero.
- 4 Apply a permissible row operation to fill this position (in diagram #3) with a one.
- 5 Apply a permissible row operation to fill this position (in diagram #4) with a zero.
- 6 Apply a permissible row operation to fill this position (in diagram #4) with a zero.
- 7 Apply a permissible row operation to fill this position (in diagram #5) with a one.
- 8 Apply a permissible row operation to fill this position (in diagram #6) with a zero.
- 9 Apply a permissible row operation to fill this position (in diagram #6) with a zero.

**NOTE:** If a larger number of variables is found within the given system, just continue the pattern started in these steps.

**example 2.19a**      **Solve using the Gauss-Jordan Row Operations:** 
$$\begin{cases} x - 3y + 4z = 10 \\ 2x + 2y + z = 15 \\ 3x - 4y + 2z = 10 \end{cases}$$

1. Rewrite the given system into an augmented matrix: 
$$\left[ \begin{array}{ccc|c} 1 & -3 & 4 & 10 \\ 2 & 2 & 1 & 15 \\ 3 & -4 & 2 & 10 \end{array} \right]$$

2. Perform permissible row operations in order to achieve the desired “reduced matrix:

	C <sub>1</sub>	C <sub>2</sub>	C <sub>3</sub>	C <sub>4</sub>
R <sub>1</sub>	1	-3	4	10
R <sub>2</sub>	2	2	1	15
R <sub>3</sub>	3	-4	2	10

- Use the “1” in the  $R_1C_1$  position to convert the other values in  $C_1$  into zeroes:

		$C_1$	$C_2$	$C_3$	$C_4$	
<b>R1</b>	$\rightarrow$	$R_1$	1	-3	4	10
<b>R2 - 2(R1)</b>	$\rightarrow$	$R_2$	0	8	-7	-5
<b>R3 - 3(R1)</b>	$\rightarrow$	$R_3$	0	5	-10	-20

old R2:	2	2	1	15
-2(R1):	-2(1)	-2(-3)	-2(4)	-2(10)
new R2:	<b>0</b>	<b>8</b>	<b>-7</b>	<b>-5</b>

old R3:	3	-4	2	10
-3(R1):	-3(1)	-3(-3)	-3(4)	-3(10)
new R3:	<b>0</b>	<b>5</b>	<b>-10</b>	<b>-20</b>

- Swap  $R_3$  and  $R_2$  (as shown below):

		$C_1$	$C_2$	$C_3$	$C_4$	
<b>R1</b>	$\rightarrow$	$R_1$	1	-3	4	10
<b>R3</b>	$\rightarrow$	$R_2$	0	5	-10	-20
<b>R2</b>	$\rightarrow$	$R_3$	0	8	-7	-5

- Convert the “5” in the  $R_2C_2$  position (above) into a “1” by dividing all of  $R_2$  by 5:

		$C_1$	$C_2$	$C_3$	$C_4$	
<b>R1</b>	$\rightarrow$	$R_1$	1	-3	4	10
<b>R2 ÷ 5</b>	$\rightarrow$	$R_2$	0	1	-2	-4
<b>R3</b>	$\rightarrow$	$R_3$	0	8	-7	-5

- Use the “1” in the  $R_2C_2$  position to convert the other values in  $C_2$  into zeroes:

		$C_1$	$C_2$	$C_3$	$C_4$
<b>R1 + 3(R2)</b>	$\rightarrow$	$R_1$	1	0	-2
<b>R2</b>	$\rightarrow$	$R_2$	0	1	-2
<b>R3 - 8(R2)</b>	$\rightarrow$	$R_3$	0	0	9

old R1:	1	-3	4	10
+3(R2):	3(0)	3(1)	3(-2)	3(-4)
new R1:	1	0	-2	-2

old R3:	0	8	-7	-5
-8(R2):	-8(0)	-8(1)	-8(-2)	-8(-4)
new R3:	0	0	9	27

**DON'T EAT YELLOW SNOW**

**What will your advice be?**

Some advice just states the obvious. But to give the kind of advice that's going to make a real difference to your clients you've got to listen critically, dig beneath the surface, challenge assumptions and be credible and confident enough to make suggestions right from day one. At Grant Thornton you've got to be ready to kick start a career right at the heart of business.

Sound like you? Here's our advice: visit [GrantThornton.ca/careers/students](http://GrantThornton.ca/careers/students)

Scan here to learn more about a career with Grant Thornton.

 **Grant Thornton**  
An instinct for growth™

© Grant Thornton LLP. A Canadian Member of Grant Thornton International Ltd



- Convert the “9” in the  $R_3C_3$  position (above) into a “1” by dividing all of  $R_3$  by 9:

		$C_1$	$C_2$	$C_3$	$C_4$
<b>R1</b>	$\rightarrow$	$R_1$	1	0	-2
<b>R2</b>	$\rightarrow$	$R_2$	0	1	-2
<b>R3 ÷ 9</b>	$\rightarrow$	$R_3$	0	0	1
					3

- Use the “1” in the  $R_3C_3$  position to convert the other values in  $C_3$  into zeroes:

		$C_1$	$C_2$	$C_3$	$C_4$
<b>R1 + 2(R3)</b>	$\rightarrow$	$R_1$	1	0	0
<b>R2 + 2(R3)</b>	$\rightarrow$	$R_2$	0	1	0
<b>R3</b>	$\rightarrow$	$R_3$	0	0	1
					3

old R1:	1	0	-2	-2
+2(R3):	2(0)	2(0)	2(1)	2(3)
new R1:	<b>1</b>	<b>0</b>	<b>0</b>	<b>4</b>

old R2:	0	1	-2	-4
+2(R3):	2(0)	2(0)	2(1)	2(3)
new R2:	<b>0</b>	<b>1</b>	<b>0</b>	<b>2</b>

3. Read the solutions from: 
$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right] \rightarrow x = 4, y = 2, z = 3$$

Following these same basic steps, we can solve most systems no matter how many variables are contained in the given system.

**example 2.19b:**

**Solve using the Gauss-Jordan Row Operations:**  $\begin{cases} 2x - 6y - 12z = 82 \\ 3x - 10y - 20z = 135 \\ 2x - 17z = 129 \end{cases}$

**augmented matrix:**

	C1	C2	C3	C4
R1:	2	-6	-12	82
R2:	3	-10	-20	135
R3:	2	0	-17	129

**R1 ÷ 2**

→ R1:	1	-3	-6	41
→ R2:	3	-10	-20	135
→ R3:	2	0	-17	129

get a 1 in  $R_1 C_1$  position**R2**just copy entries from  $R_2$ **R3**just copy entries from  $R_3$ **R1**

→ R1:	1	-3	-6	41
→ R2:	0	-1	-2	12
→ R3:	0	6	-5	47

just copy entries from  $R_1$ **R2 – 3(R1)**get a zero in  $R_2 C_1$  position**R3 – 2(R1)**get a zero in  $R_3 C_1$  position**R1**

→ R1:	1	-3	-6	41
→ R2:	0	1	2	-12
→ R3:	0	6	-5	47

just copy entries from  $R_1$ **(-1)(R2)**get a 1 in  $R_2 C_2$  position**R3**just copy entries from  $R_3$ **R1 + 3(R2)**get a zero in  $R_1 C_2$  position**R2**just copy entries from  $R_2$ **R3 – 6(R2)**get a zero in  $R_3 C_2$  position

<b>R1</b>	$\rightarrow$	R1:	1	0	0	5
<b>R2</b>	$\rightarrow$	R2:	0	1	2	-12
<b>R3 ÷ (-17)</b>	$\rightarrow$	R3:	0	0	1	-7

just copy entries from  $R_1$

just copy entries from  $R_2$

get a 1 in  $R_3C_3$  position

<b>R1</b>	$\rightarrow$	R1:	1	0	0	<b>5</b>
<b>R2 - 2(R3)</b>	$\rightarrow$	R2:	0	1	0	<b>2</b>
<b>R3</b>	$\rightarrow$	R3:	0	0	1	<b>-7</b>

get a zero in  $R_1C_3$  position

get a zero in  $R_2C_3$  position

just copy entries from  $R_3$

Now that the row operations have achieved an identity matrix in place of the coefficients, we can read the solutions for the give system:  $x = 5, y = 2, z = -7$

..... Alcatel-Lucent 
  
[www.alcatel-lucent.com/careers](http://www.alcatel-lucent.com/careers)



What if you could build your future and create the future?

One generation's transformation is the next's status quo. In the near future, people may soon think it's strange that devices ever had to be "plugged in." To obtain that status, there needs to be "The Shift".

**example 2.19c:****Solve using the Gauss-Jordan Row Operations:**

$$\begin{cases} w + 4x - 6y - 3z = -4 \\ w - x - y + 2z = 16 \\ w + y + z = 12 \\ x + y + z = 6 \end{cases}$$

augmented  
matrix:

	C1	C2	C3	C4	C5
R1:	1	4	-6	-3	-4
R2:	1	-1	-1	2	16
R3:	1	0	1	1	12
R4:	0	1	1	1	6

<b>R1</b>	→	R1: 1 4 -6 -3 -4	just copy entries from R <sub>1</sub>
<b>R2 - R1</b>	→	R2: 0 -5 5 5 20	get a zero in R <sub>2</sub> C <sub>1</sub> position
<b>R3 - R1</b>	→	R3: 0 -4 7 4 16	get a zero in R <sub>3</sub> C <sub>1</sub> position
<b>R4</b>	→	R4: 0 1 1 1 6	get a zero in R <sub>4</sub> C <sub>1</sub> position

<b>R1</b>	→	R1: 1 4 -6 -3 -4	just copy entries from R <sub>1</sub>
<b>R4</b>	→	R2: 0 1 1 1 6	get a 1 in R <sub>2</sub> C <sub>2</sub> position
<b>R2</b>	→	R3: 0 -5 5 5 20	just copy entries from R <sub>2</sub>
<b>R3</b>	→	R4: 0 -4 7 4 16	just copy entries from R <sub>3</sub>

<b>R1 - 4(R2)</b>	→	R1: 1 0 -10 -7 -28	get a zero in R <sub>1</sub> C <sub>2</sub> position
<b>R2</b>	→	R2: 0 1 1 1 6	just copy entries from R <sub>2</sub>
<b>R3 + 5(R2)</b>	→	R3: 0 0 10 10 50	get a zero in R <sub>3</sub> C <sub>2</sub> position
<b>R4 + 4(R2)</b>	→	R4: 0 0 11 8 40	get a zero in R <sub>4</sub> C <sub>2</sub> position

<b>R1</b>	$\rightarrow$	R1:	1	0	-10	-7	-28	just copy entries from $R_1$
<b>R2</b>	$\rightarrow$	R2:	0	1	1	1	6	just copy entries from $R_2$
<b>R3 ÷ 10</b>	$\rightarrow$	R3:	0	0	1	1	5	get a 1 in $R_3C_3$ position
<b>R4</b>	$\rightarrow$	R4:	0	0	11	8	40	just copy entries from $R_4$
<b>R1 + 10(R3)</b>	$\rightarrow$	R1:	1	0	0	3	22	get a zero in $R_1C_3$ position
<b>R2 - R3</b>	$\rightarrow$	R2:	0	1	0	0	1	get a zero in $R_2C_3$ position
<b>R3</b>	$\rightarrow$	R3:	0	0	1	1	5	just copy entries from $R_3$
<b>R4 - 11(R3)</b>	$\rightarrow$	R4:	0	0	0	-3	-15	get a zero in $R_4C_3$ position
<b>R1</b>	$\rightarrow$	R1:	1	0	0	3	22	just copy entries from $R_1$
<b>R2</b>	$\rightarrow$	R2:	0	1	0	0	1	just copy entries from $R_2$
<b>R3</b>	$\rightarrow$	R3:	0	0	1	1	5	just copy entries from $R_3$
<b>R4 ÷ (-3)</b>	$\rightarrow$	R4:	0	0	0	1	5	get a 1 in $R_4C_4$ position
<b>R1 - 3(R4)</b>	$\rightarrow$	R1:	1	0	0	0	7	get a zero in $R_1C_4$ position
<b>R2</b>	$\rightarrow$	R2:	0	1	0	0	1	just copy entries from $R_2$
<b>R3 - R4</b>	$\rightarrow$	R3:	0	0	1	0	0	get a zero in $R_3C_4$ position
<b>R4</b>	$\rightarrow$	R4:	0	0	0	1	5	just copy entries from $R_4$

Now that the row operations have achieved an identity matrix in place of the coefficients, we can read the solutions for the given system:  $w = 7, x = 1, y = 0, z = 5$

## TUTORIAL 2.20: USE MATRICES TO SOLVE SYSTEMS WITH NON-UNIQUE SOLUTIONS

We would hope that every system that we encounter has a unique solution. However, that is not always the case. When solving a system of equations, there are actually 3 different outcomes that might happen. We can distinguish these outcomes by looking at their possible “reduced matrix”:

<i>unique solution:</i>	<i>no solution:</i>	<i>general solution:</i>
$\left[ \begin{array}{ccc c} 1 & 0 & 0 & 400 \\ 0 & 1 & 0 & 500 \\ 0 & 0 & 1 & 600 \end{array} \right]$	$\left[ \begin{array}{ccc c} 1 & 0 & 0 & 300 \\ 0 & 1 & 0 & 400 \\ 0 & 0 & 0 & 500 \end{array} \right]$	$\left[ \begin{array}{ccc c} 1 & 0 & 2 & 5 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & 0 & 0 \end{array} \right]$
$1x + 0y + 0z = 400 \rightarrow x = 400$ $0x + 1y + 0z = 500 \rightarrow y = 500$ $0x + 0y + 1z = 600 \rightarrow z = 600$	$1x + 0y + 0z = 300 \rightarrow x = 300$ $0x + 1y + 0z = 400 \rightarrow y = 400$ $0x + 0y + 0z = 500 \rightarrow \text{impossible}$	$1x + 0y + 2z = 5 \rightarrow x = 5 - 2z$ $0x + 1y - 3z = 7 \rightarrow y = 7 + 3z$ $0x + 0y + 0z = 0 \rightarrow z = \text{any real } \#$

When we have a unique solution, there will be an identity matrix to the left of the vertical line and we can read the solution values directly from the reduced matrix.



Maastricht University *Leading in Learning!*

Join the best at  
the Maastricht University  
School of Business and  
Economics!

Visit us and find out why we are the best!  
Master's Open Day: 22 February 2014

### Top master's programmes

- 33<sup>rd</sup> place Financial Times worldwide ranking: MSc International Business
- 1<sup>st</sup> place: MSc International Business
- 1<sup>st</sup> place: MSc Financial Economics
- 2<sup>nd</sup> place: MSc Management of Learning
- 2<sup>nd</sup> place: MSc Economics
- 2<sup>nd</sup> place: MSc Econometrics and Operations Research
- 2<sup>nd</sup> place: MSc Global Supply Chain Management and Change

Sources: Keuzegids Master ranking 2013; Elsevier 'Beste Studies' ranking 2012;  
Financial Times Global Masters in Management ranking 2012

Maastricht  
University is  
the best specialist  
university in the  
Netherlands  
(Elsevier)

[www.mastersopenday.nl](http://www.mastersopenday.nl)



Click on the ad to read more

When we have a “no solution” situation, the bottom row will be an impossible feat. In the display above, the third row reads:  $0x + 0y + 0z = 500$ . There are no numbers that can replace the variables so that their combination will equal 500. This is a definite indication that there is no solution for the given system.

When we have a “general solution” situation, the bottom row will contain all zeroes. In the display above, the third row reads:  $0x + 0y + 0z = 0$ ...which will be true for any number that we use to replace the variables. This is a definite indication that there will be a general solution for the given system. It is normally expected that the last variable (in this case  $z$ ) will be considered any real number...then, the other variables will be expressed in terms of this last variable. So for the third display above,  $x = 5 - 2z$  and  $y = 7 + 3z$  when  $z$  is any real number that is applicable for the system.

So, we attempt to complete the Gauss-Jordan Row Operations as we normally would:

1. Rewrite the given system into an augmented matrix.
2. Perform one or more of the permissible row operations.
3. Use the reduced matrix to determine the solutions for the given system.

It is not until we get close to the reduced matrix that we can then determine what type of solution we will actually have for the given system.

The goal of this process is to transform an augmented matrix into a “reduced matrix” that displays an identity matrix to the left of the vertical line and the proper solutions of the system to the right. We will not be able to determine the actual type of solutions until we have attempted to achieve the desired “reduced matrix”.

**example 2.20a:**

**Solve using the Gauss-Jordan Row Operations:**  $\begin{cases} 2x - y + 3z = 10 \\ x + 2y + 2z = 10 \\ x - 3y + z = 10 \end{cases}$

**augmented  
matrix:**

	C1	C2	C3	C4
R1:	R1:	2	-1	3
R2:	R2:	1	2	2
R3:	R3:	1	-3	1

**R3** $\rightarrow$ 

R1:

1	-3	1	10
1	2	2	10
2	-1	3	10

just copy entries from  $R_3$ **R2** $\rightarrow$ 

R2:

just copy entries from  $R_2$ **R1** $\rightarrow$ 

R3:

just copy entries from  $R_1$ **R1** $\rightarrow$ 

R1:

1	-3	1	10
0	5	1	0
0	5	1	-10

just copy entries from  $R_1$ **R2 – R1** $\rightarrow$ 

R2:

get a zero in  $R_2C_1$  position**R3 – 2(R1)** $\rightarrow$ 

R3:

get a zero in  $R_3C_1$  position**R1** $\rightarrow$ 

R1:

1	-3	1	10
0	1	0.2	0
0	5	1	-10

just copy entries from  $R_1$ **R2 ÷ 5** $\rightarrow$ 

R2:

get a zero in  $R_2C_2$  position**R3** $\rightarrow$ 

R3:

just copy entries from  $R_3$ **R1 + 3(R2)** $\rightarrow$ 

R1:

1	0	1.6	10
0	1	0.2	0
0	0	0	-10

get a zero in  $R_1C_2$  position**R2** $\rightarrow$ 

R2:

just copy entries from  $R_2$ **R3 – 5(R2)** $\rightarrow$ 

R3:

get a zero in  $R_3C_2$  position

The next step for this example would be to get a 1 into the  $R_3C_3$  position...BUT, this is impossible to do with this system. Therefore, the third row of the last matrix reads:  $0x + 0y + 0z = -10$ ...which cannot ever happen. Thus: **This system has no solution.**

**example 2.20b:****Solve using the Gauss-Jordan Row Operations:**

$$\begin{cases} x + y + 2z = 0 \\ x + 2y + 2z = 0 \\ 2x + 3y + 4z = 0 \end{cases}$$

**augmented matrix:**

	C1	C2	C3	C4
R1:	1	1	2	0
R2:	1	2	2	0
R3:	2	3	4	0

**Empowering People. Improving Business.**

BI Norwegian Business School is one of Europe's largest business schools welcoming more than 20,000 students. Our programmes provide a stimulating and multi-cultural learning environment with an international outlook ultimately providing students with professional skills to meet the increasing needs of businesses.

BI offers four different two-year, full-time Master of Science (MSc) programmes that are taught entirely in English and have been designed to provide professional skills to meet the increasing need of businesses. The MSc programmes provide a stimulating and multi-cultural learning environment to give you the best platform to launch into your career.

- MSc in Business
- MSc in Financial Economics
- MSc in Strategic Marketing Management
- MSc in Leadership and Organisational Psychology

[www.bi.edu/master](http://www.bi.edu/master)

**BI NORWEGIAN BUSINESS SCHOOL**

EFMD **EQUIS** ACCREDITED



<b>R1</b>	$\rightarrow$	R1:	<table border="1" style="display: inline-table; vertical-align: middle;"> <tr><td>1</td><td>1</td><td>2</td><td>0</td></tr> <tr><td>0</td><td>1</td><td>0</td><td>0</td></tr> <tr><td>0</td><td>1</td><td>0</td><td>0</td></tr> </table>	1	1	2	0	0	1	0	0	0	1	0	0	just copy entries from $R_1$
1	1	2	0													
0	1	0	0													
0	1	0	0													
<b>R2 - R1</b>	$\rightarrow$	R2:	<table border="1" style="display: inline-table; vertical-align: middle;"> <tr><td>1</td><td>1</td><td>2</td><td>0</td></tr> <tr><td>0</td><td>1</td><td>0</td><td>0</td></tr> <tr><td>0</td><td>1</td><td>0</td><td>0</td></tr> </table>	1	1	2	0	0	1	0	0	0	1	0	0	get a zero in $R_2C_1$ position
1	1	2	0													
0	1	0	0													
0	1	0	0													
<b>R3 - 2(R1)</b>	$\rightarrow$	R3:	<table border="1" style="display: inline-table; vertical-align: middle;"> <tr><td>1</td><td>1</td><td>2</td><td>0</td></tr> <tr><td>0</td><td>1</td><td>0</td><td>0</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>0</td></tr> </table>	1	1	2	0	0	1	0	0	0	0	0	0	get a zero in $R_2C_1$ position
1	1	2	0													
0	1	0	0													
0	0	0	0													

<b>R1 - R2</b>	$\rightarrow$	R1:	<table border="1" style="display: inline-table; vertical-align: middle;"> <tr><td>1</td><td>0</td><td>2</td><td>0</td></tr> <tr><td>0</td><td>1</td><td>0</td><td>0</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>0</td></tr> </table>	1	0	2	0	0	1	0	0	0	0	0	0	get a zero in $R_1C_2$ position
1	0	2	0													
0	1	0	0													
0	0	0	0													
<b>R2</b>	$\rightarrow$	R2:	<table border="1" style="display: inline-table; vertical-align: middle;"> <tr><td>1</td><td>0</td><td>2</td><td>0</td></tr> <tr><td>0</td><td>1</td><td>0</td><td>0</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>0</td></tr> </table>	1	0	2	0	0	1	0	0	0	0	0	0	just copy entries from $R_2$
1	0	2	0													
0	1	0	0													
0	0	0	0													
<b>R3 - R2</b>	$\rightarrow$	R3:	<table border="1" style="display: inline-table; vertical-align: middle;"> <tr><td>1</td><td>0</td><td>2</td><td>0</td></tr> <tr><td>0</td><td>1</td><td>0</td><td>0</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>0</td></tr> </table>	1	0	2	0	0	1	0	0	0	0	0	0	get a zero in $R_3C_2$ position
1	0	2	0													
0	1	0	0													
0	0	0	0													

The next step for this example would be to get a 1 into the  $R_3C_3$  position...BUT, this is impossible to do with this system. Therefore, the third row of the last matrix reads:  $0x + 0y + 0z = 0$ ...which will work for any values chosen for the variables. Thus, this system has a **general solution**.

Reading the top two rows of the last matrix:

$$1x + 0y + 2z = 0 \rightarrow x = -2z$$

$$0x + 1y + 0z = 0 \rightarrow y = 0$$

$$0x + 0y + 0z = 0 \rightarrow z = \text{any real number}$$

## TUTORIAL 2.21: FIND THE INVERSE OF A $2 \times 2$ SQUARE MATRIX

If the product of two given matrices results in the identity matrix, then the two given matrices are “inverses” of each other. And conversely, if two matrices are “inverses” of one another, their product will be an identity matrix. Therefore,  $[A] \times [A^{-1}] = \text{identity matrix}$ . The symbol  $A^{-1}$  is the symbol for the inverse matrix of  $A$ .

$$\begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix} \times \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow A = \begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix}, \text{ then: } A^{-1} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{7} \end{bmatrix}$$

The inverse of any  $2 \times 2$  matrix can be found by using this rule:

$$\text{If: } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then: } A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} (\text{where } ad - bc \neq 0).$$

- The denominator of the fraction above is called the **determinant** because its value will determine whether there actually is an inverse. If the determinant ( $ad - bc$ ) equals zero, there will be no inverse for the given matrix.

**example 2. 21a:** Find the determinant and inverse of:  $\begin{bmatrix} 5 & 9 \\ 3 & 7 \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 3 & 7 \end{bmatrix} \rightarrow \text{determinant} = ad - bc = 5(7) - 9(3) = 35 - 27 = 8$$

**determinant = 8**

$$\text{inverse} = A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 7 & -9 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} \frac{7}{8} & \frac{-9}{8} \\ \frac{-3}{8} & \frac{5}{8} \end{bmatrix}$$

**example 2. 21b:** Find the determinant and inverse of:  $\begin{bmatrix} 4 & 6 \\ 8 & 12 \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 8 & 12 \end{bmatrix} \rightarrow \text{determinant} = ad - bc = 4(12) - 6(8) = 48 - 48 = 0$$

**determinant = 0**

Because the determinant equals zero, there is **no inverse** for this given matrix.

**example 2. 21c:** Find the determinant and inverse of:  $\begin{bmatrix} 1 & 4 \\ 4 & 6 \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 4 & 6 \end{bmatrix} \rightarrow \text{determinant} = ad - bc = 1(3) - 4(4) = 6 - 16 = -10$$

**determinant = -10**

$$\text{inverse} = A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{-10} \begin{bmatrix} 6 & -4 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} \frac{-6}{10} & \frac{4}{10} \\ \frac{4}{10} & \frac{-1}{10} \end{bmatrix}$$

## TUTORIAL 2.22: USE $2 \times 2$ INVERSE MATRICES TO SOLVE SYSTEMS OF EQUATIONS

Any system of equations may be rewritten as a matrix equation where the coefficient matrix is multiplied by the variable matrix to obtain the constant matrix. The matrix equation can then be solved by multiplying the constant matrix by the inverse of the coefficient matrix. Notice that the inverse matrix must appear before the constant matrix in the proper matrix multiplication problem.

$$\begin{cases} 3x + y = 17 \\ 2x - y = 3 \end{cases} \rightarrow \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 17 \\ 3 \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}^{-1} \times \begin{bmatrix} 17 \\ 3 \end{bmatrix}$$

# Need help with your dissertation?

Get in-depth feedback & advice from experts in your topic area. Find out what you can do to improve the quality of your dissertation!

**Get Help Now**



Go to [www.helpmyassignment.co.uk](http://www.helpmyassignment.co.uk) for more info

**Helpmyassignment**

**example 2.22a:** Use matrix inverses to solve:  $\begin{cases} 3x - 4y = -11 \\ 2x + 3y = 4 \end{cases}$

- Find the determinant for the coefficient matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 2 & 3 \end{bmatrix} \rightarrow \text{determinant} = ad - bc = 3(3) - (-4)2 = 9 + 8 = 17$$

- If one exists, find the inverse of the coefficient matrix:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 3 & 4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{17} & \frac{4}{17} \\ -\frac{2}{17} & \frac{3}{17} \end{bmatrix}$$

- Rewrite the system as a matrix equation solved for the variables:

$$\begin{cases} 3x - 4y = -11 \\ 2x + 3y = 4 \end{cases} \rightarrow \begin{bmatrix} 3 & -4 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -11 \\ 4 \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 2 & 3 \end{bmatrix}^{-1} \times \begin{bmatrix} -11 \\ 4 \end{bmatrix}$$

- Multiply the derived inverse by the given constants:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 2 & 3 \end{bmatrix}^{-1} \times \begin{bmatrix} -11 \\ 4 \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{17} & \frac{4}{17} \\ -\frac{2}{17} & \frac{3}{17} \end{bmatrix} \times \begin{bmatrix} -11 \\ 4 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (\frac{3}{17})(-11) + (\frac{4}{17})(4) = -\frac{33}{17} + \frac{16}{17} = -\frac{17}{17} = -1 \\ (-\frac{2}{17})(-11) + (\frac{3}{17})(4) = +\frac{22}{17} + \frac{12}{17} = \frac{34}{17} = 2 \end{bmatrix}$$

- Read the solutions for the system:  $x = -1, y = 2$

**example 2.22b:** Use matrix inverses to solve:  $\begin{cases} 2x - 5y = 4 \\ 3x + 3y = 27 \end{cases}$

- Find the determinant for the coefficient matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 3 & 3 \end{bmatrix} \rightarrow \text{determinant} = ad - bc = 2(3) - (-5)3 = 6 + 15 = 21$$

- If one exists, find the inverse of the coefficient matrix:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 3 & +5 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{21} & \frac{5}{21} \\ \frac{-3}{21} & \frac{2}{21} \end{bmatrix}$$

- Rewrite the system as a matrix equation solved for the variables:

$$\begin{cases} 2x - 5y = 4 \\ 3x + 3y = 27 \end{cases} \rightarrow \begin{bmatrix} 2 & -5 \\ 3 & 3 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 27 \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 3 & 3 \end{bmatrix}^{-1} \times \begin{bmatrix} 4 \\ 27 \end{bmatrix}$$

- Multiply the derived inverse by the given constants:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 3 & 3 \end{bmatrix}^{-1} \times \begin{bmatrix} 4 \\ 27 \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{21} & \frac{5}{21} \\ \frac{-3}{21} & \frac{2}{21} \end{bmatrix} \times \begin{bmatrix} 4 \\ 27 \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (\frac{3}{21})(4) + (\frac{5}{21})(27) = \frac{12}{21} + \frac{135}{21} = \frac{147}{21} = 7 \\ (\frac{-3}{21})(4) + (\frac{2}{21})(27) = \frac{-12}{21} + \frac{54}{21} = \frac{42}{21} = 2 \end{bmatrix}$$

- Read the solutions for the system:  $x = 7, y = 2$

**NOTE:** The desired matrix inverse may contain fractions. Remember that when we add or subtract fractions, we need common denominators. So, it is a good idea not to simplify the fractions within the inverse before we use the inverse in a matrix multiplication.

**NOTE:** If a system does not have an inverse, it does not have a unique solution. However, it may have a general solution or no solution at all. Completing proper row operations will determine which of these alternatives is the better choice.

## TUTORIAL 2.23: USE ROW OPERATIONS TO FIND THE INVERSE MATRIX OF A SQUARE MATRIX

To find the inverse of any  $N \times N$  matrix, we can transform an augmented matrix using “row operations” where the original matrix (on the left of the vertical line) is converted into an identity matrix. The resulting “inverse” will now be displayed to the right of the vertical line within the transformed matrix.

<b><i>original matrix:</i></b> $\left[ \begin{array}{ccc ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array} \right]$	$\rightarrow$	<b><i>transformed matrix:</i></b> $\left[ \begin{array}{ccc ccc} 1 & 0 & 0 & r & s & t \\ 0 & 1 & 0 & u & v & w \\ 0 & 0 & 1 & x & y & z \end{array} \right]$
---	---------------	--



# Brain power

By 2020, wind could provide one-tenth of our planet's electricity needs. Already today, SKF's innovative know-how is crucial to running a large proportion of the world's wind turbines.

Up to 25 % of the generating costs relate to maintenance. These can be reduced dramatically thanks to our systems for on-line condition monitoring and automatic lubrication. We help make it more economical to create cleaner, cheaper energy out of thin air.

By sharing our experience, expertise, and creativity, industries can boost performance beyond expectations.

Therefore we need the best employees who can meet this challenge!

The Power of Knowledge Engineering

Plug into The Power of Knowledge Engineering.  
Visit us at [www.skf.com/knowledge](http://www.skf.com/knowledge)

SKF

**example 2.23a:****Use row operations to find the inverse (if one exists):**

$$\begin{bmatrix} 1 & 3 & -6 \\ 3 & 6 & -15 \\ 4 & 0 & -16 \end{bmatrix}$$

**original setup:**

	C1	C2	C3	C4	C5	C6
→ R1:	1	3	-6	1	0	0
→ R2:	3	6	-15	0	1	0
→ R3:	4	0	-16	0	0	1

**R1**

→ R1:	1	3	-6	1	0	0
→ R2:	0	-3	3	-3	1	0
→ R3:	0	-12	8	-4	0	1

just copy entries from  $R_1$ ,get a zero in  $R_2 C_1$  positionget a zero in  $R_3 C_1$  position**R1**

→ R1:	1	3	-6	1	0	0
→ R2:	0	1	-1	1	-1/3	0
→ R3:	0	-12	8	-4	0	1

just copy entries from  $R_1$ ,get a zero in  $R_2 C_2$  positionjust copy entries from  $R_3$ **R1 – 3(R2)**

→ R1:	1	0	-3	-2	1	0
→ R2:	0	1	-1	1	-1/3	0
→ R3:	0	0	-4	8	-4	1

get a zero in  $R_1 C_2$  positionjust copy entries from  $R_2$ get a zero in  $R_3 C_2$  position**R2**

→ R1:	1	0	-3	-2	1	0
→ R2:	0	1	-1	1	-1/3	0
→ R3:	0	0	1	-2	1	-1/4

just copy entries from  $R_1$ ,just copy entries from  $R_2$ get a one in  $R_3 C_3$  position**(-1/4)R3**

→ R1:	1	0	0	-8	4	-3/4
→ R2:	0	1	0	-1	2/3	-1/4
→ R3:	0	0	1	-2	1	-1/4

get a zero in  $R_1 C_3$  positionget a zero in  $R_2 C_3$  positionjust copy entries from  $R_3$ 

The inverse of the given matrix for this example is:

$$\begin{bmatrix} -8 & 4 & -3/4 \\ -1 & 2/3 & -1/4 \\ -2 & 1 & -1/4 \end{bmatrix}$$

This same process can be used for any  $N \times N$  matrices...that is,  $2 \times 2$  matrices,  $3 \times 3$  matrices,  $4 \times 4$  matrices,  $5 \times 5$  matrices and so on. The only difference will be the number of columns and the number of rows that need to be used during the row operations.

**example 2.23b:** Use row operations to find the inverse (if one exists):  $\begin{bmatrix} 1 & 2 & 2 \\ 1 & -3 & 1 \\ 2 & -1 & 3 \end{bmatrix}$

		C1	C2	C3	C4	C5	C6
	→ R1:	1	2	2	1	0	0
original setup:	→ R2:	1	-3	1	0	1	0
	→ R3:	2	-1	3	0	0	1

R1	→ R1:	1	2	2	1	0	0	just copy entries from $R_1$
R2 – R1	→ R2:	0	-5	-1	-1	1	0	get a zero in $R_2 C_1$ position
R3 – 2(R1)	→ R3:	0	-5	-1	-2	0	1	get a zero in $R_3 C_1$ position

R1	→ R1:	1	2	2	1	0	0	just copy entries from $R_1$
(-1/5)R2	→ R2:	0	1	0.2	0.2	-0.2	0	get a one in $R_2 C_2$ position
R3	→ R3:	0	-5	-1	-2	0	1	just copy entries from $R_3$

R1 – 2(R2)	→ R1:	1	0	1.6	0.6	0.4	0	get a zero in $R_1 C_2$ position
R2	→ R2:	0	1	0.2	0.2	-0.2	0	just copy entries from $R_2$
R3 + 5(R2)	→ R3:	0	0	0	-1	-1	1	get a zero in $R_3 C_2$ position

Since the first three columns of the third row contain zeroes, there is no way to create the desired  $3 \times 3$  identity matrix. Thus, there is **no inverse** for the given matrix for this example.

**NOTE:** When the coefficient matrix for any system does not have an inverse, it does not have a unique solution. Whether the system will have no solution or a general solution will depend on the completion of the proper row operations needed to solve the system.

## TUTORIAL 2.24: USE $N \times N$ INVERSE MATRICES TO SOLVE SYSTEMS OF EQUATIONS

Although we have multiple ways to solve  $2 \times 2$  systems, using a matrix inverse may be one of the only ways to solve larger systems with the use of some technology.

**example 2.24:**      **Solve using matrix inverses:**  $\begin{cases} x - y + 3z = 31 \\ 2x - y + z = 16 \\ 3x - y = 11 \end{cases}$

When solving a system larger than  $2 \times 2$ , we need to use row operations to determine the matrix inverse of the coefficient matrix before we can complete the matrix multiplication necessary to find the solution for the given system:

## TURN TO THE EXPERTS FOR SUBSCRIPTION CONSULTANCY

Subscrybe is one of the leading companies in Europe when it comes to innovation and business development within subscription businesses.

We innovate new subscription business models or improve existing ones. We do business reviews of existing subscription businesses and we develop acquisition and retention strategies.

Learn more at [linkedin.com/company/subscrybe/](https://www.linkedin.com/company/subscrybe/) or contact Managing Director Morten Suhr Hansen at [mha@subscrybe.dk](mailto:mha@subscrybe.dk)

**SUBSCRYBE** - to the future



Click on the ad to read more

**original setup:**

	C1	C2	C3	C4	C5	C6
→ R1:	1	-1	3	1	0	0
→ R2:	2	-1	1	0	1	0
→ R3:	3	-1	0	0	0	1

<b>R1</b>	→ R1:	1	-1	3	1	0	just copy entries from $R_1$
<b>R2 - 2(R1)</b>	→ R2:	0	1	-5	-2	1	get a zero in $R_2 C_1$ position
<b>R3 - 3(R1)</b>	→ R3:	0	2	-9	-3	0	get a zero in $R_3 C_1$ position

<b>R1 + R2</b>	→ R1:	1	0	-2	-1	1	0	get a zero in $R_1 C_2$ position
<b>R2</b>	→ R2:	0	1	-5	-2	1	0	just copy entries from $R_2$
<b>R3 - 2(R2)</b>	→ R3:	0	0	1	1	-2	1	get a zero in $R_3 C_2$ position

<b>R1 + 2(R3)</b>	→ R1:	1	0	0	1	-3	2	get a zero in $R_1 C_3$ position
<b>R2 + 5(R3)</b>	→ R2:	0	1	0	3	-9	5	get a zero in $R_2 C_3$ position
<b>R3</b>	→ R3:	0	0	1	1	-2	1	just copy entries from $R_3$

The inverse of the given matrix for this example is:

$$\begin{bmatrix} 1 & -3 & 2 \\ 3 & -9 & 5 \\ 1 & -2 & 1 \end{bmatrix}$$

Once we have the matrix inverse, we can complete the necessary matrix multiplication:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -9 & 5 \\ 1 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 31 \\ 16 \\ 11 \end{bmatrix} = \begin{bmatrix} 1(31) + (-3)(16) + 2(11) = 5 \\ 3(31) + (-9)(16) + 5(11) = 4 \\ 1(31) + (-2)(16) + 1(11) = 10 \end{bmatrix}$$

$$x = 5, y = 4, z = 10$$

## TUTORIAL 2.25: USE MATRIX INVERSES TO SOLVE REAL-WORLD APPLICATIONS

Just as we solved real-world applications using  $2 \times 2$  systems, we may also need to solve a real-world application that involves larger  $N \times N$  systems...

**example 2.25a:** A woman who has \$270,000 to invest would like to have an annual income of \$33,000 from her investments. One safe investment pays 10% per year while a riskier investment pays 15% per year. How much should she invest in each of these accounts?

Just as we have done in the past, we need to translate the given information into equations that can be solved as a system. First thing to do is to select variables to be used during the solution of the problem:

$x$  = investment at 10% rate;  $y$  = investment at 15% rate

Then, we need to translate sentences into equations:

$$\text{total investment amount} \rightarrow x + y = 270,000$$

$$\text{annual income desired} \rightarrow 0.10x + 0.15y = 33,000$$

We then need to rewrite the system as a matrix equation:

$$\begin{bmatrix} 1 & 1 \\ 0.10 & 0.15 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 270,000 \\ 33,000 \end{bmatrix}$$

Using the  $2 \times 2$  matrix inverse formula, the needed inverse is:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{0.15 - 0.10} \begin{bmatrix} 0.15 & -1 \\ -0.10 & 1 \end{bmatrix} = \frac{1}{0.05} \begin{bmatrix} 0.15 & -1 \\ -0.10 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -20 \\ -2 & 20 \end{bmatrix}$$

Now we just need to multiply the inverse matrix by the constant matrix:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -20 \\ -2 & 20 \end{bmatrix} \times \begin{bmatrix} 270,000 \\ 33,000 \end{bmatrix} = \begin{bmatrix} 3(270,000) - 20(33,000) = 150,000 \\ -2(270,000) + 20(33,000) = 120,000 \end{bmatrix}$$

**solution:** The woman should invest \$150,000 at the 10% rate and \$120,000 at the 15% rate.

**example 2.25b:** A brokerage house offers three stock portfolios:

- Portfolio I consists of 2 blocks of common stock and 1 municipal bond.
- Portfolio II consists of 4 blocks of common stock, 2 municipal bonds, and 3 blocks of preferred stock.
- Portfolio III consists of 7 blocks of common stock, 3 municipal bonds, and 3 blocks of preferred stock.

A customer wants 70 blocks of common stock, 32 municipal bonds, and 33 blocks of preferred stock. How many units of each portfolio should be offered?



What do you want to do?

No matter what you want out of your future career, an employer with a broad range of operations in a load of countries will always be the ticket. Working within the Volvo Group means more than 100,000 friends and colleagues in more than 185 countries all over the world. We offer graduates great career opportunities – check out the Career section at our web site [www.volvogroup.com](http://www.volvogroup.com). We look forward to getting to know you!

**VOLVO**

AB Volvo (publ)  
[www.volvogroup.com](http://www.volvogroup.com)

VOLVO TRUCKS | RENAULT TRUCKS | MACK TRUCKS | VOLVO BUSES | VOLVO CONSTRUCTION EQUIPMENT | VOLVO PENTA | VOLVO AERO | VOLVO IT  
 VOLVO FINANCIAL SERVICES | VOLVO 3P | VOLVO POWERTRAIN | VOLVO PARTS | VOLVO TECHNOLOGY | VOLVO LOGISTICS | BUSINESS AREA ASIA



Select variables to be used during the solution of the problem:

$$x = \text{portfolio I} \quad y = \text{portfolio II} \quad z = \text{portfolio III}$$

Translate sentences into equations:

$$\text{common stocks} \rightarrow 2x + 4y + 7z = 70$$

$$\text{municipal bonds} \rightarrow 1x + 2y + 3z = 32$$

$$\text{preferred stocks} \rightarrow 0x + 3y + 3z = 33$$

Rewrite the system as a matrix equation:

$$\begin{bmatrix} 2 & 4 & 7 \\ 1 & 2 & 3 \\ 0 & 3 & 3 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 70 \\ 32 \\ 33 \end{bmatrix}$$

Use row operations to determine the matrix inverse for the coefficient matrix:

		C1	C2	C3	C4	C5	C6
original setup:	$\rightarrow$	R1:	2	4	7	1	0
	$\rightarrow$	R2:	1	2	3	0	1
	$\rightarrow$	R3:	0	3	3	0	1

R2	$\rightarrow$	R1:	1	2	3	0	1	0	get a one in $R_1C_1$ position
R1	$\rightarrow$	R2:	2	4	7	1	0	0	just copy entries from $R_1$
R3	$\rightarrow$	R3:	0	3	3	0	0	1	just copy entries from $R_3$

R1	$\rightarrow$	R1:	1	2	3	0	1	0	just copy entries from $R_1$
R2 - 2(R1)	$\rightarrow$	R2:	0	0	1	1	-2	0	get a zero in $R_2C_1$ position
R3	$\rightarrow$	R3:	0	3	3	0	0	1	get a zero in $R_3C_1$ position

R1	$\rightarrow$	R1:	1	2	3	0	1	0	just copy entries from $R_1$
R3	$\rightarrow$	R2:	0	3	3	0	0	1	just copy entries from $R_3$
R2	$\rightarrow$	R3:	0	0	1	1	-2	0	just copy entries from $R_2$

<b>R1</b>	$\rightarrow$	R1:	1	2	3	0	1	0	just copy entries from $R_1$
<b>R2 ÷ 3</b>	$\rightarrow$	R2:	0	1	1	0	0	$\frac{1}{3}$	get a one in $R_2 C_1$ position
<b>R3</b>	$\rightarrow$	R3:	0	0	1	1	-2	0	just copy entries from $R_3$

<b>R1 - 2(R2)</b>	$\rightarrow$	R1:	1	0	1	0	1	$-\frac{2}{3}$	get a zero in $R_1 C_2$ position
<b>R2</b>	$\rightarrow$	R2:	0	1	1	0	0	$\frac{1}{3}$	just copy entries from $R_2$
<b>R3</b>	$\rightarrow$	R3:	0	0	1	1	-2	0	just copy entries from $R_3$

<b>R1 - R3</b>	$\rightarrow$	R1:	1	0	0	-1	3	$-\frac{2}{3}$	get a zero in $R_1 C_3$ position
<b>R2 - R3</b>	$\rightarrow$	R2:	0	1	0	-1	2	$\frac{1}{3}$	get a zero in $R_2 C_3$ position
<b>R3</b>	$\rightarrow$	R3:	0	0	1	1	-2	0	just copy entries from $R_3$

The inverse of the given matrix for this example is:

$$\begin{bmatrix} -1 & 3 & -\frac{2}{3} \\ -1 & 2 & \frac{1}{3} \\ 1 & -2 & 0 \end{bmatrix}$$

Multiply the inverse matrix by the constant matrix:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 3 & -\frac{2}{3} \\ -1 & 2 & \frac{1}{3} \\ 1 & -2 & 0 \end{bmatrix} \times \begin{bmatrix} 70 \\ 32 \\ 33 \end{bmatrix} = \begin{bmatrix} -1(70) + 3(32) - \frac{2}{3}(33) = 4 \\ -1(70) + 2(32) + \frac{1}{3}(33) = 5 \\ 1(70) - 2(32) + 0(33) = 6 \end{bmatrix}$$

**solution:** **The number of units to be offered should be 4 units of portfolio I, 5 units of portfolio II, 6 units of portfolio III**

## TUTORIAL 2.26: USE A COMPUTER SPREADSHEET TO DETERMINE THE NUMBER OF ROWS AND COLUMNS CONTAINED IN A MATRIX AND THEREBY DETERMINE THE ORDER OF THE MATRIX

A computer spreadsheet is a series of small blocks (cells) where the columns are labeled with capital letters and the rows are labeled by numbers. To enter a matrix into a *Google Sheets* or a *Microsoft Excel* spreadsheet, simply type each matrix element into its own small block (cell). *NOTE: All screen shots used in this text were created using Microsoft Excel.*

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16
17	18	19	20

→

	A	B	C	D	E
1	1	2	3	4	
2	5	6	7	8	
3	9	10	11	12	
4	13	14	15	16	
5	17	18	19	20	
6					

For example, suppose we want to determine the number of rows and columns contained in the matrix above. First thing we would need to do is to enter the data entries onto a computer spreadsheet (see the partial screen shot on the right above) by typing 1 into cell A1, by typing 2 into cell B1, by typing 3 into cell C1, by typing 4 into cell D1, by typing 5 in cell A2, by typing 6 in cell B2, by typing 7 in cell C2, by typing 8 in cell D2, by typing 9 in cell A3, by typing 10 in cell B3, by typing 11 in cell C3, by typing 12 in cell D3, by typing 13 in cell A4, by typing 14 in cell B4, by typing 15 in cell C4, by typing 16 in cell D4, by typing 17 in cell A5, by typing 18 in cell B5, by typing 19 in cell C5, by typing 20 in cell D5.

To use the computer spreadsheet to determine the number of rows and the number of columns contained in this matrix, highlight all of the entries (as was done on the partial screen shot below) with a computer mouse.

A	B	C	D	E
1	2	3	4	
2	5	6	7	8
3	9	10	11	12
4	13	14	15	16
5	17	18	19	20
6				

While holding down the left button on the mouse, notice that the computer displays the desired information (as is indicated by the red oval). Therefore, this matrix has 5 rows and 4 columns...making its order (dimensions):  $5 \times 4$ .

### TUTORIAL 2.27: USE A COMPUTER SPREADSHEET TO DETERMINE THE TRANSPOSE AND/OR THE NEGATIVE OF A MATRIX

Suppose that we want to use a computer to determine the transpose of:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \end{bmatrix}$$

After entering the data onto the computer spreadsheet, highlight another section of the worksheet that has the dimensions of the transpose of a  $5 \times 4$  matrix...that is (as can be seen in the partial screen shot on the top below), a  $4 \times 5$  section. (Recall that the transpose of a matrix swaps the given rows for columns and the given columns for rows. So, if the given matrix as an order of  $R \times C$  then its transpose will have an order of  $C \times R$ .)

The top screenshot shows a 4x5 matrix in a spreadsheet. The columns are labeled A through E and the rows are labeled 1 through 5. The matrix values are:

1	2	3	4	
2	5	6	7	8
3	9	10	11	12
4	13	14	15	16
5	17	18	19	20
6				

The bottom screenshot shows the same matrix transposed, with the formula =TRANSPOSE(A1:D5) in the formula bar. The columns are labeled A through K and the rows are labeled 1 through 5. The transposed matrix values are:

1	2	3	4						
2	5	6	7	8	1	5	9	13	17
3	9	10	11	12	2	6	10	14	18
4	13	14	15	16	3	7	11	15	19
5	17	18	19	20	4	8	12	16	20
6					=TRANSPOSE(A1:D5)				

While the  $4 \times 5$  section is still highlighted, type this array formula (which will appear in the formula bar): **=TRANSPOSE(A1:D5)**. Since this is an array formula, we will need to do the **CRTL+SHIFT+ENTER** key-stroke combination to enter the desired formula. (Braces will surround the formula indicating that it is an array formula.) As we can see from the partial screen shot on the bottom above, the transpose has swapped the data around so that the entries that were originally in rows are in columns and the entries that were originally in columns are in rows.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \end{bmatrix}$$

Suppose that we want to use a computer to determine the negative of:

After entering the data onto the computer spreadsheet, highlight another section of the worksheet that has the same dimensions as the original  $5 \times 4$  matrix...that is (as can be seen in the partial screen shot on the top below), a  $5 \times 4$  section.

The top screenshot shows a 5x4 matrix in a spreadsheet. The columns are labeled A through D and the rows are labeled 1 through 5. The matrix values are:

1	2	3	4
2	5	6	7
3	9	10	11
4	13	14	15
5	17	18	19
6			

The bottom screenshot shows the same matrix negated, with the formula =-1\*(A1:D5) in the formula bar. The columns are labeled A through J and the rows are labeled 1 through 7. The negated matrix values are:

1	2	3	4	-1	-2	-3	-4
2	5	6	7	-5	-6	-7	-8
3	9	10	11	-9	-10	-11	-12
4	13	14	15	-13	-14	-15	-16
5	17	18	19	-17	-18	-19	-20
6				=-1*(A1:D5)			
7							

While the  $5 \times 4$  section is still highlighted, type this array formula (which will appear in the formula bar): **=-1\*(A1:D5)**. Since this is an array formula, we will need to do the **CRTL+SHIFT+ENTER** key-stroke combination to enter the desired formula. (Braces will surround the formula indicating that it is an array formula.) As we can see from the partial screen shot on the bottom above, each of the original entries has been changed to its negative (opposite).

### TUTORIAL 2.28: USE A COMPUTER SPREADSHEET TO ADD OR SUBTRACT MATRICES

Suppose that we want to use a spreadsheet to add:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{bmatrix}$$

After entering the data from each matrix onto the computer spreadsheet, highlight another section of the worksheet that has the same dimensions as the given matrices, for this example (as can be seen in the partial screen shot on the top below) that is a  $3 \times 3$  section. (Recall that we can only add two matrices together if they both have the same dimensions.)

This e-book  
is made with  
**SetaPDF**



**PDF components for PHP developers**

**[www.setasign.com](http://www.setasign.com)**



**Click on the ad to read more**

	A	B	C	D	E	F	G	H	I	J	K	L
1	1	2	3		10	11	12		11	13	15	
2	4	5	6		13	14	15		17	19	21	
3	7	8	9		16	17	18		23	25	27	
4									= (A1:C3)+(E1:G3)			
5												

While the  $3 \times 3$  section is still highlighted, type this array formula (which will appear in the formula bar):  $=(A1:C3)+(E1:G3)$ . Since this is an array formula, we will need to do the **CRTL+SHIFT+ENTER** key-stroke combination to enter the desired formula. (Braces will surround the formula indicating that it is an array formula.) As we can see from the partial screen shot on the bottom above, the two entries in respective cells have been added together...thus displaying the sum of the given matrices.

Suppose that we want to use a spreadsheet to subtract:

$$\begin{bmatrix} -1 & 2 & -3 \\ 4 & -5 & 6 \\ -7 & 8 & -9 \end{bmatrix} - \begin{bmatrix} 10 & -11 & 12 \\ -13 & 14 & -15 \\ 16 & -17 & 18 \end{bmatrix}$$

After entering the data from each matrix onto the computer spreadsheet, highlight another section of the worksheet that has the same dimensions as the given matrices, for this example (as can be seen in the partial screen shot on the top below) that is a  $3 \times 3$  section. (Recall that we can only subtract two matrices together if they both have the same dimensions.)

	A	B	C	D	E	F	G	H	I	J	K	L
1	-1	2	-3		10	-11	12		-11	13	-15	
2	4	-5	6		-13	14	-15		17	-19	21	
3	-7	8	-9		16	-17	18		-23	25	-27	
4									= (A1:C3)-(E1:G3)			
5												

While the  $3 \times 3$  section is still highlighted, type this array formula (which will appear in the formula bar):  $=(A1:C3) - (E1:G3)$ . Since this is an array formula, we will need to do the **CRTL+SHIFT+ENTER** key-stroke combination to enter the desired formula. (Braces will surround the formula indicating that it is an array formula.) As we can see from the partial screen shot on the bottom above, the two entries in respective cells have been subtracted...thus displaying the difference of the given matrices.

## TUTORIAL 2.29: USE A COMPUTER SPREADSHEET TO COMPLETE A SCALAR MULTIPLICATION

Suppose that we want to use a spreadsheet to complete:  $1.15 \begin{bmatrix} 100 & 200 \\ 300 & 400 \end{bmatrix}$

Enter just the data from the matrix onto the computer spreadsheet. Highlight another section of the worksheet that has the same dimensions as the given matrix, for this example (as can be seen in the partial screen shot on the top below) that is a  $2 \times 2$  section.

	A	B	C	D	E	F
1	100	200				
2	300	400				
3						
4						

	A	B	C	D	E	F
1	100	200		115	230	
2	300	400		345	460	
3				=1.15*(A1:B2)		
4						
5						

While the  $2 \times 2$  section is still highlighted, type this array formula (which will appear in the formula bar):  $=1.15*(A1:B2)$ . Since this is an array formula, we will need to do the **CRTL+SHIFT+ENTER** key-stroke combination to enter the desired formula. (Braces will surround the formula indicating that it is an array formula.) As we can see from the partial screen shot on the bottom above, each of the original entries has been multiplied by 1.15 (which is the scalar in this example).

Suppose that we want to use a spreadsheet to complete:  $4 \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} + 3 \begin{bmatrix} 8 & 7 \\ 6 & 5 \end{bmatrix}$

Enter the data from each of the matrices onto the computer spreadsheet. Highlight another section of the worksheet that has the same dimensions as the given matrices...for this example (as can be seen in the partial screen shot on the top below) that is a  $2 \times 2$  section.

	A	B	C	D	E	F	G	H	I
1	4	3		8	7				
2	2	1		6	5				
3									
4									

	A	B	C	D	E	F	G	H	I	J	K
1	4	3		8	7		40	33			
2	2	1		6	5		26	19			
3							=4*(A1:B2)+3*(D1:E2)				
4											
5											

While the  $2 \times 2$  section is still highlighted, type this array formula (which will appear in the formula bar): **=4\*(A1:B2)+3\*(D1:E2)**. Since this is an array formula, we will need to do the **CRTL+SHIFT+ENTER** key-stroke combination to enter the desired formula. (Braces will surround the formula indicating that it is an array formula.) As we can see from the partial screen shot on the bottom above, the entries from each matrix have been multiplied by their respective scalars and then these products have been added together.

**NOTE:** A scalar multiplication combined with a subtraction would be completed in a similar fashion as described above. The main difference would be the array formula that would be needed to accomplish the task.

### TUTORIAL 2.30: USE A COMPUTER SPREADSHEET TO COMPLETE A MATRIX MULTIPLICATION

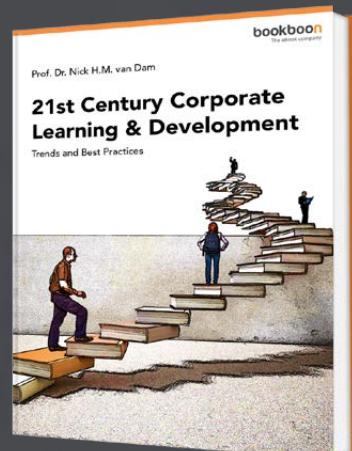
Suppose that we want to use a spreadsheet to complete:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

## Free eBook on Learning & Development

By the Chief Learning Officer of McKinsey

Download Now



Download free eBooks at [bookboon.com](http://bookboon.com)



Click on the ad to read more

Enter the data from each of the given matrices. Highlight another section of the worksheet that has the dimensions of the product for these matrices...for this example (as can be seen in the partial screen shot below) that is a  $5 \times 3$  section. (Recall that when multiplying matrices, the number of columns in the first matrix must be equal to the number of rows in the second matrix and that the product of these matrices will then have the dimensions defined by the number of rows in the first matrix by the number of columns in the second matrix:  $R_1 \times C_2$ . For this example, the order of the first matrix is  $5 \times 4$  while the order of the second matrix is  $4 \times 3$ ...which makes the product of these two arrays a  $5 \times 3$  matrix.)

	5R x 3C														
	A	B	C	D	E	F	G	H	I	J	K	L	M		
1	1	2	3	4		1	2	3							
2	5	6	7	8		4	5	6							
3	9	10	11	12		7	8	9							
4	13	14	15	16		10	11	12							
5	17	18	19	20											
6															

While the  $5 \times 3$  section is still highlighted, type this array formula (which will appear in the formula bar): **=MMULT(A1:D5, F1:H4)**. Since this is an array formula, we will need to do the **CRTL+SHIFT+ENTER** key-stroke combination to enter the desired formula. (Braces will surround the formula indicating that it is an array formula.) As we can see from the partial screen shot below, the cells have been filled with the data produced by multiplying the given matrices.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	
1	1	2	3	4		1	2	3		70	80	90			
2	5	6	7	8		4	5	6		158	184	210			
3	9	10	11	12		7	8	9		246	288	330			
4	13	14	15	16		10	11	12		334	392	450			
5	17	18	19	20						422	496	570			
6															
7															

**NOTE:** Recall that multiplication of matrices is NOT commutative. So, if we reversed the two matrices, we would be attempting to multiply a  $4 \times 3$  matrix by a  $5 \times 4$  matrix. Since the two middle numbers do not match, the multiplication cannot be completed.

### TUTORIAL 2.31: USE A COMPUTER SPREADSHEET TO SOLVE SYSTEMS OF EQUATIONS INVOLVING ROW OPERATIONS

Suppose that we want to use a spreadsheet to solve:  $\begin{cases} x - 3y + 4z = 10 \\ 2x + 2y + z = 15 \\ 3x - 4y + 2z = 10 \end{cases}$

To accomplish this task, apply the Gauss-Jordan row operations. We would need to enter the numbers from the given problem...creating an augmented matrix (as can be seen in the partial screen shot below).

	A	B	C	D	E	F	G
1			C1	C2	C3	C4	
2		R1:	1	-3	4	10	
3		R2:	2	2	1	15	
4		R3:	3	-4	2	10	
5							

Recall that the goal of the Gauss-Jordan row operations method is to convert all but the last column into an  $N \times N$  identity matrix so that the solutions for the given system will appear in the last column. For this example, we need a  $3 \times 3$  identity matrix and the solutions will appear in the fourth column.

	A	B	C	D	E	F	G	H	I
1				C1	C2	C3	C4		
2			R1:	1	-3	4	10	=D2	
3			R2:	2	2	1	15	=D3-2*D2	
4			R3:	3	-4	2	10	=D4-3*D2	
5									
6		R1	→	R1:	1	-3	4	10	=D2
7		R2 - 2(R1)	→	R2:	0	8	-7	-5	=D3-2*D2
8		R3 - 3(R1)	→	R3:	0	5	-10	-20	=D4-3*D2
9									
10		R1	→	R1:	1	-3	4	10	=D6
11		R3	→	R2:	0	5	-10	-20	=D8
12		R2	→	R3:	0	8	-7	-5	=D7
13									
14		R1	→	R1:	1	-3	4	10	=D10
15		R2 + 5	→	R2:	0	1	-2	-4	=D11/5
16		R3	→	R3:	0	8	-7	-5	=D12
17									
18		R1 + 3(R2)	→	R1:	1	0	-2	-2	=D14+3*D15
19		R2	→	R2:	0	1	-2	-4	=D15
20		R3 - 8(R2)	→	R3:	0	0	9	27	=D16-8*D15
21									
22		R1	→	R1:	1	0	-2	-2	=D18
23		R2	→	R2:	0	1	-2	-4	=D19
24		R3 ÷ 9	→	R3:	0	0	1	3	=D20/9
25									
26		R1 + 2(R3)	→	R1:	1	0	0	4	=D22+2*D24
27		R2 + 2(R3)	→	R2:	0	1	0	2	=D23+2*D24
28		R3	→	R3:	0	0	1	3	=D24
29									

Recall that the goal of the first step in the Gauss-Jordan process is to get a 1 in the  $R_1 C_1$  position (cell D2 in the partial screen shot above) of the desired identity matrix. Since we already have that step accomplished, we then need to convert the other entries in C1 (cells D3 and D4) into zeroes:

- Click into cell D6 and type: **=D2**...and then copy this formula from cells D6 to G6.
- Click into cell D7 and type: **=D3-2\*D2**...and then copy this formula from cells D7 to G7.
- Click into cell D8 and type: **=D4-3\*D2**...and then copy this formula from cells D8 to G8.

We now have the first column of the desired identity matrix completed. The next step is to get a 1 into the  $R_2C_2$  position (cell E7 in the partial screen shot on the previous page). Since the non-zero entries of the third row are all multiples of 5, it would be better to have these entries as the second row rather than the third. So, we can swap the last two rows around:

- Click into cell D10 and type: =D6...and then copy this formula from D10 to G10.
- Click into cell D11 and type: =D8...and then copy this formula from D11 to G11.
- Click into cell D12 and type: =D7...and then copy this formula from D12 to G12.

Now that we have the rows swapped around, we still need to get a 1 in the  $R_2C_2$  position of the desired identity matrix. So, we need to divide the entire second row by 5:

- Click into cell D14 and type: =D10...and then copy this formula from D14 to G14.
- Click into cell D15 and type: =D11/5...and then copy this formula from D15 to G15.
- Click into cell D16 and type: =D12...and then copy this formula from D16 to G16.



[www.sylvania.com](http://www.sylvania.com)

We do not reinvent the wheel we reinvent light.

Fascinating lighting offers an infinite spectrum of possibilities: Innovative technologies and new markets provide both opportunities and challenges. An environment in which your expertise is in high demand. Enjoy the supportive working atmosphere within our global group and benefit from international career paths. Implement sustainable ideas in close cooperation with other specialists and contribute to influencing our future. Come and join us in reinventing light every day.

Light is OSRAM

**OSRAM** **SYLVANIA**

We now have to convert the other entries in  $C_2$  (cells E14 and E16) into zeroes:

- Click into cell D18 and type: **=D14+3\*D15**...and then copy this formula from D18 to G18.
- Click into cell D19 and type: **=D15**...and then copy this formula from D19 to G19
- Click into cell D20 and type: **=D16-8\*D15**...and then copy this formula from D20 to G20.

We now have the first and the second columns of the desired identity matrix completed. The next step is to get a 1 into the  $R_3C_3$  position of the desired identity matrix. Since there is a 9 in that position (cell F20 in the partial screen shot on the previous page), we just need to divide the third row by 9:

- Click into cell D22 and type: **=D18**...and then copy this formula from D22 to G22.
- Click into cell D23 and type: **=D19**...and then copy this formula from D23 to G23.
- Click into cell D24 and type: **=D20/9**...and then copy this formula from D24 to G24.

We now have to convert the other entries in  $C_3$  (cells F22 and F23) into zeroes:

- Click into cell D26 and type: **=D22+2\*D24**...and then copy this formula from D26 to G26.
- Click into cell D27 and type: **=D23+2\*D24**...and then copy this formula from D27 to G27.
- Click into cell D28 and type: **=D24**...and then copy this formula from D28 to G28.

We now have the desired identity matrix completely finished. The only thing to do is now read the solutions to the given system:  $x = 4$  (based on the entry in cell G26)

$y = 2$  (based on the entry in cell G27)

$z = 3$  (based on the entry in cell G28).

**NOTE:** This process can be duplicated using any sized system. The only difference will be the number of rows and columns that will be handled based on the size of the system.

## TUTORIAL 2.32: USE A COMPUTER SPREADSHEET TO DETERMINE THE INVERSE AND/OR DETERMINANT OF A MATRIX

Suppose we want to use a spreadsheet to determine the inverse of:

$$\begin{bmatrix} 1 & 3 & -6 \\ 3 & 6 & -15 \\ 4 & 0 & -16 \end{bmatrix}$$

Enter the data from the given matrix. Highlight another section of the worksheet that has the same dimensions as the given matrix...for this example (as can be seen in the partial screen shot on the top below), that would be a  $3 \times 3$  section.

While this  $3 \times 3$  section is still highlighted, type this array formula (which will appear in the formula bar): **=MINVERSE(A1:C3)**. Since this is an array formula, we will need to do the **CRTL+ SHIFT+ ENTER** key-stroke combination to enter the desired formula. (Braces will surround the formula indicating that it is an array formula.) As we can see from the partial screen shot on the bottom above, the cells have been filled with the values of the desired inverse.

Suppose we want to use a spreadsheet to determine the determinant of:

$$\begin{bmatrix} 1 & 3 & -6 \\ 3 & 6 & -15 \\ 4 & 0 & -16 \end{bmatrix}$$

Enter the data from the given matrix. Since the determinant will only be one value, we just need to decide in what cell we would like the information to be displayed. As can be seen in the partial screen shot on the top below, the determinant will be displayed in cell E2. Therefore, just click into cell E2 and type: **=MDETERM(A1:C3)**. Although this is an array formula, only one value will be displayed...so, we can just press **ENTER** rather than the **CRTL+ SHIFT+ ENTER** key-stroke combination used for other array formulas. As can be seen in the partial screen shot on the bottom below, the determinant for the given matrix is 12.

	A	B	C	D	E	F
1	1	3	-6		det	
2	3	6	-15			
3	4	0	-16			
4						

	A	B	C	D	E	F	G	H
1	1	3	-6		det			
2	3	6	-15		12			
3	4	0	-16		=MDETERM(A1:C3)			
4								

NOTE: Recall that if the determinant of any matrix is zero, then the given matrix will not have an inverse.



**Deloitte.**

Discover the truth at [www.deloitte.ca/careers](http://www.deloitte.ca/careers)

© Deloitte & Touche LLP and affiliated entities.



## TUTORIAL 2.33: USE A COMPUTER SPREADSHEET TO SOLVE A SYSTEM OF EQUATIONS INVOLVING INVERSE MATRICES

Although we could use the Gauss-Jordan row operations to solve a system of equations, it is much easier to have a computer spreadsheet to all of the work for us by using inverse matrices.

Suppose that we want to use a computer spreadsheet to solve:  $\begin{cases} x - 3y + 4z = 10 \\ 2x + 2y + z = 15 \\ 3x - 4y + 2z = 10 \end{cases}$

First thing we need to do is to enter the coefficients into one matrix and the constants into another (as can be seen in the partial screen shot on the top below). Then, we need to highlight another section of the worksheet that will contain the inverse of the coefficient matrix. For this example, we would need to highlight a  $3 \times 3$  section (cells G2 to I4 in the partial screen shot on the top below).

	A	B	C	D	E	F	G	H	I	J
1	coefficients:			constants:			inverse:			
2	1	-3	4			10				
3	2	2	1			15				
4	3	-4	2			10				
5										

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	coefficients:			constants:			inverse:			solutions:				
2	1	-3	4			10	-0.18	0.22	0.24	4				
3	2	2	1			15	0.02	0.22	-0.16	2				
4	3	-4	2			10	0.31	0.11	-0.18	3				
5							=MINVERSE(A2:C4)	=MMULT(G2:I4,E2:E4)						
6														

While the section is highlighted, type this array formula: `=MINVERSE(A2:C4)`. Since this is an array formula, we will need to do the **CTRL+ SHIFT+ ENTER** key-stroke combination to enter the desired formula. (Braces will surround the formula indicating that it is an array formula.) As can be seen in the partial screen shot on the bottom above, cells G2 to I4 now contain the inverse matrix for the coefficient matrix.

Now all we need to do is multiply the inverse matrix (cells G2 to I4) with the constant matrix (cells E2 to E4) to obtain the solutions. Since the inverse is a  $3 \times 3$  matrix and the constants form a  $3 \times 1$  matrix, the product of these two matrices will be a  $3 \times 1$  matrix. Highlight a  $3 \times 1$  section on the worksheet (cells K2 to K4 in the partial screen shot to the right above) and type: **=MMULT(G2:I4,E2:E4)**. Remember we need to do the **CRTL+ SHIFT+ ENTER** key-stroke combination to enter the formula. As can be seen in the partial screen shot on the bottom above, cells K2 to K4 now contain the solutions to the given system:

$x = 4$  (based on the entry in cell K2)

$y = 2$  (based on the entry in cell K3)

$z = 3$  (based on the entry in cell K4).

**NOTE:** It may be wise to have the computer spreadsheet find the determinant of the coefficient matrix before proceeding with the steps listed in this tutorial. If the determinant is zero, then there will be no inverse. If there is no inverse, these steps will not help in finding the desired solutions. If necessary, we may need to use row operations to determine whether a given system has a general solution or no solution at all.

SIMPLY CLEVER
ŠKODA

We will turn your CV into  
an opportunity of a lifetime

Do you like cars? Would you like to be a part of a successful brand?  
We will appreciate and reward both your enthusiasm and talent.  
Send us your CV. You will be surprised where it can take you.

Send us your CV on  
[www.employerforlife.com](http://www.employerforlife.com)

Do you like cars? Would you like to be a part of a successful brand?  
We will appreciate and reward both your enthusiasm and talent.  
Send us your CV. You will be surprised where it can take you.

Send us your CV on  
[www.employerforlife.com](http://www.employerforlife.com)

# 3 QUADRATIC & OTHER SPECIAL FUNCTIONS

## TUTORIAL 3.1: FACTOR A POLYNOMIAL EXPRESSION BY FINDING ITS GREATEST COMMON MONOMIAL FACTOR

First, some vocabulary...

- A **monomial** is a one-termed expression. Some sample monomials are:  $2x$ ,  $8a^2b$ ,  $-7$ , and  $12$
- A **binomial** is a two-termed expression joined by an addition or a subtraction sign. Some sample binomials are:  $x + y$ ,  $8a^2b - 4bc^2$ , and  $4x + 5$
- A **trinomial** is a three-termed expression joined together by addition or subtraction signs. Some sample trinomials are:  $x + y - z$ ,  $12a^3b + 4ab^2 - 8a^2b^3$ , and  $5x^2 + 3y - 7$
- A **polynomial** is a multi-termed expression where the terms are joined together by addition or subtraction signs. The set of polynomials consist of all monomials, all binomials, all trinomials as well as any other expressions that might have more than three terms.

Occasionally, we might need to express polynomials in factored form. One way to factor a polynomial expression is to factor out its greatest common monomial factor. We can use prime factorization to determine the greatest common factor (GCF) of any set of numbers. All we need to do is determine what prime factors the given numbers share *at most*:

**example 3.1a**

**Find: GCF(4, 8, 12)**

prime factorization:  $4 = 2^2 \dots 8 = 2^3 \dots 12 = 2^2 \times 3^1$

shared at most:  $2^2 = 4 \rightarrow \text{GCF}(4, 8, 12) = 4$

**example 3.1b**

**Find: GCF(10, 40, 60)**

prime factorization:  $10 = 2^1 \times 5^1 \dots 40 = 2^3 \times 5^1 \dots 60 = 2^2 \times 3^1 \times 5^1$

shared at most:  $2^1 \times 5^1 = 10 \rightarrow \text{GCF}(10, 40, 60) = 10$

**example 3.1c**

**Find: GCF(10, 15, 25)**

prime factorization:  $10 = 2^1 \times 5^1 \dots 15 = 3^1 \times 5^1 \dots 25 = 5^2$

shared at most:  $5^1 = 5 \rightarrow \text{GCF}(10, 15, 25) = 5$

NOTE:

The greatest common factor (GCF) of any set of numbers is the one value that will divide evenly into all of the given numbers.

We can extend the prime factorization idea when variables and exponents are introduced into the mix. In fact, when variables and exponents are included, they are already in “prime factorization” since the variable is a base raised to a power (exponent).

**example 3.1d****Find: GCF( $4x^4$ ,  $8x^3$ ,  $12x^2$ )**

“prime factorization”:  $4x^4 = 2^2 \times x^4$  ...  $8x^3 = 2^3 \times x^3$  ...  $12x^2 = 2^2 \times 3 \times x^2$   
 shared at most:  $2^2 \times x^2 \rightarrow \text{GCF} = 4x^2$

**example 3.1e****Find: GCF( $10x^5y^3$ ,  $40x^4y^4$ ,  $60x^3y^5$ )**

“prime factorization”:  $10x^5y^3 = 2^1 \times 5^1 \times x^5 \times y^3$   
 $40x^4y^4 = 2^3 \times 5^1 \times x^4 \times y^4$   
 $60x^3y^5 = 2^2 \times 3^1 \times 5^1 \times x^3 \times y^5$   
 shared at most:  $2^1 \times 5^1 \times x^3 \times y^3 \rightarrow \text{GCF} = 10x^3y^3$

**example 3.1f****Find: GCF( $-10ab$ ,  $-15a^2b^3$ ,  $-25a^3b^5$ )**

“prime factorization”:  $-10ab = -1 \times 2^1 \times 5^1 \times a^1 \times b^1$   
 $-15a^2b^3 = -1 \times 3^1 \times 5^1 \times a^2 \times b^3$   
 $-25a^3b^5 = -1 \times 5^2 \times a^3 \times b^5$   
 shared at most:  $-1 \times 5^1 \times a^1 \times b^1 \rightarrow \text{GCF} = -5ab$

**example 3.1g****Find: GCF( $12a^2b^4$ ,  $18a^4b^6$ ,  $30a^6b^8$ )**

“prime factorization”:  $12a^2b^4 = 2^2 \times 3^1 \times a^2 \times b^4$   
 $18a^4b^6 = 2^1 \times 3^2 \times a^4 \times b^6$   
 $30a^6b^8 = 2^1 \times 3^1 \times 5^1 \times a^6 \times b^8$   
 shared at most:  $2^1 \times 3^1 \times a^2 \times b^4 \rightarrow \text{GCF} = 6a^2b^4$

So, when we need to factor any given polynomial, the very first thing that we should determine is whether the terms within the given polynomial have a greatest common factor.

**example 3.1h****Factor:  $4x^4 + 8x^3 + 12x^2$** 

- as established in example 3.1d,  $\text{GCF}(4x^4, 8x^3, 12x^2) = 4x^2$
- divide each term by the GCF:  $\frac{4x^4}{4x^2} + \frac{8x^3}{4x^2} + \frac{12x^2}{4x^2} = x^2 + 2x + 3$
- write the GCF first & then the quotient from above for the desired result  $\rightarrow \text{factored form: } 4x^2(x^2 + 2x + 3)$

**example 3.1i****Factor:**  $10x^5y^3 - 40x^4y^4 + 60x^3y^5$ 

- as established in example 3.1e, GCF( $10x^5y^3$ ,  $40x^4y^4$ ,  $60x^3y^5$ )  
 $= 10x^3y^3$
- divide each term by the GCF:

$$\frac{10x^5y^3}{10x^3y^3} + \frac{-40x^4y^4}{10x^3y^3} + \frac{60x^3y^5}{10x^3y^3} = x^2 - 4xy + 6y^2$$

- write the GCF first & then the quotient from above for the desired result → **factored form:**  $10x^3y^3(x^2 - 4xy + 6y^2)$

**example 3.1j****Factor:**  $-10ab - 15a^2b^3 - 25a^3b^5$ 

- as established in example 3.1f, GCF( $-10ab$ ,  $-15a^2b^3$ ,  $-25a^3b^5$ )  
 $= -5ab$  divide each term by the GCF:

$$\frac{-10ab}{-5ab} + \frac{-15a^2b^3}{-5ab} + \frac{-25a^3b^5}{-5ab} = 2 + ab^2 + 5a^2b^4$$

- write the GCF first & then the quotient from above for the desired result → **factored form:**  $-5ab(2 + ab^2 + 5a^2b^4)$

**example 3.1k****Factor:**  $12a^2b^4 - 18a^4b^6 - 30a^6b^8$ 

- as established in example 3.1g, GCF( $12a^2b^4$ ,  $18a^4b^6$ ,  $30a^6b^8$ )  
 $= 6a^2b^4$  divide each term by the GCF:

$$\frac{12a^2b^4}{6a^2b^4} + \frac{-18a^4b^6}{6a^2b^4} + \frac{-30a^6b^8}{6a^2b^4} = 2 - 3a^2b^2 - 5a^4b^4$$

- write the GCF first & then the quotient from above for the desired result → **factored form:**  $6a^2b^4(2 - 3a^2b^2 - 5a^4b^4)$

**NOTE:** If we use the distributive property to multiply the greatest common factor (GCF) by the polynomial that appears in the parentheses, we will go back to the given polynomial that was to be factored.

## TUTORIAL 3.2: FACTOR A POLYNOMIAL EXPRESSION BY GROUPING

Extending the idea of factoring out the greatest common factor (GCF) from a given polynomial, we can establish another method of factoring...that of “grouping”. Sometimes all of the terms in the given polynomial to be factored may not have one common factor... but...some of them may. Therefore, we can “group” the terms into smaller sets and factor out the GCF from them.

**example 3.2a****Factor by grouping:**  $5b^2 - 20b - a^3b + 4a^3$ 

- Decide how to group given polynomial:  $(5b^2 - 20b) + (-a^3b + 4a^3)$
- Factor out the GCF from each group:  $5b(b - 4) - a^3(b - 4)$
- Then, factor out the GCF from the two sub-groups: **( $b - 4$ )**  
**( $5b - a^3$ )**

**example 3.2b****Factor by grouping:**  $a^3 - a^2 - 6a + 6$ 

- Decide how to group given polynomial:  $(a^3 - a^2) + (-6a + 6)$
- Factor out the GCF from each group:  $a^2(a - 1) - 6(a - 1)$
- Then, factor out the GCF from the two sub-groups: **( $a - 1$ )**  
**( $a^2 - 6$ )**

**example 3.2c****Factor by grouping:**  $10x - 10y + 7bx - 7by$ 

- Decide how to group given polynomial:  $(10x - 10y) + (7bx - 7by)$
- Factor out the GCF from each group:  $10(x - y) + 7b(x - y)$
- Then, factor out the GCF from the two sub-groups: **( $x - y$ )**  
**( $10 + 7b$ )**

The advertisement features a central image of a teacher smiling and interacting with two young students (a boy and a girl) who are looking at a laptop screen. The background is yellow with orange swirling patterns. In the top left corner is the e-Learning for Kids logo. In the bottom right corner, there is a green oval containing text about the organization's achievements.

**About e-Learning for Kids** Established in 2004, e-Learning for Kids is a global nonprofit foundation dedicated to fun and free learning on the Internet for children ages 5 - 12 with courses in math, science, language arts, computers, health and environmental skills. Since 2005, more than 15 million children in over 190 countries have benefitted from eLessons provided by EFK! An all-volunteer staff consists of education and e-learning experts and business professionals from around the world committed to making difference. eLearning for Kids is actively seeking funding, volunteers, sponsors and courseware developers; get involved! For more information, please visit [www.e-learningforkids.org](http://www.e-learningforkids.org).

**• The number 1 MOOC for Primary Education  
• Free Digital Learning for Children 5-12  
• 15 Million Children Reached**



**example 3.2d****Factor by grouping:**  $7x^3 - 21x^2 + 2x - 6$ 

- Decide how to group given polynomial:  $(7x^3 - 21x^2) + (2x - 6)$
- Factor out the GCF from each group:  $7x^2(x - 3) + 2(x - 3)$
- Then, factor out the GCF from the two sub-groups:  **$(x - 3)$**   
 **$(7x^2 + 2)$**

NOTE:

If we use extended distributive property (sometimes called FOIL when multiplying two binomials) to multiply the first factor by the second, we will go back to the given polynomial that was to be factored in the first place.

NOTE:

Although the examples above paired the given four terms, we are not restricted to that grouping. It is quite possible that four terms could be grouped into a trio then a single...or, they might be grouped into a single and then a trio. It all depends on how we think to group the given terms. However, notice that after the first greatest common factor has been factored out of the “groups”, there will be a common factor in the parentheses. If this does not happen, then the grouping was done incorrectly.

### TUTORIAL 3.3: FACTOR A POLYNOMIAL EXPRESSION ( $X^2 + BX + C$ ) AS A PRODUCT OF BINOMIALS

When we multiply one binomial by another, we do so by applying extended distributive property. Examining these products will aid us in reversing the process and factoring a trinomial ( $ax^2 + bx + c$ ).

**example 3.3a****Find:**  $(x + 5)(x + 6)$ 

- multiplying:  $x(x + 6) = x^2 + 6x$   
 $5(x + 6) = x^2 + 5x + 30$
- combining like-terms:  $x^2 + [6x + 5x] + 30 = \mathbf{x^2 + 11x + 30}$

**example 3.3b****Find:**  $(x - 5)(x - 6)$ 

- multiplying:  $x(x - 6) = x^2 - 6x$   
 $-5(x - 6) = x^2 - 5x + 30$
- combining like-terms:  $x^2 + [-6x - 5x] + 30 = \mathbf{x^2 - 11x + 30}$

**example 3.3c****Find:**  $(x + 5)(x - 6)$ 

- multiplying:  $x(x - 6) = x^2 - 6x$   
 $5(x - 6) = x^2 + 5x - 30$
- combining like-terms:  $x^2 + [-6x + 5x] + 30 = \mathbf{x^2 - 1x - 30}$

**example 3.3d****Find:**  $(x - 5)(x + 6)$ 

- multiplying:  $x(x + 6) = x^2 + 6x$   
 $-5(x + 6) = x^2 - 5x - 30$
- combining like-terms:  $x^2 + [6x - 5x] + 30 = \textcolor{red}{x^2 + 1x - 30}$

Although we only did four specific problems, we can use them to make some observations that can be applied to other binomial multiplication:

- When the signs within the factors are the same (whether positive or negative, as is the case in example 3.3a or example 3.3b), the last term in the trinomial product will be positive.
- When the signs within the factors are different (as is the case in example 3.3c and example 3.3d), the last term in the trinomial product will be negative.
- No matter what signs are used within the factors, the middle term in the trinomial product will be the sum of the numbers (along with their respective signs) within the binomials being multiplied:

For 3.3a: Since +5 and +6 appear in the binomials, the middle term is:  $5x + 6x = 11x$

For 3.3b: Since -5 and -6 appear in the binomials, the middle term is:  $-5x + -6x = -11x$

For 3.3c: Since +5 and -6 appear in the binomials, the middle term is:  $5x - 6x = -1x$

For 3.3d: Since -5 and +6 appear in the binomials, the middle term is:  $-5x + 6x = +1x$

Now to use the observations and product trinomials from above to reverse the process and to factor each trinomial into its two binomials factors...

**example 3.3e****Factor:**  $x^2 + 11x + 30$ 

To factor this trinomial, we are looking for two factors of +30 that will also add to be +11. The only correct set of numbers that satisfy these two conditions is: +5 and +6. Thus, the factors of the given trinomial are:  $(x + 5)(x + 6)$

**example 3.3f****Factor:**  $x^2 - 11x + 30$ 

To factor this trinomial, we are looking for two factors of +30 that will also add to be -11. The only correct set of numbers that satisfy these two conditions is: -5 and -6. Thus, the factors of the given trinomial are:  $(x - 5)(x - 6)$

**example 3.3g****Factor:  $x^2 - 1x - 30$** 

To factor this trinomial, we are looking for two factors of  $-30$  that will also add to be  $-1$ . The only correct set of numbers that satisfy these two conditions is:  $+5$  and  $-6$ . Thus, the factors of the given trinomial are:  $(x + 5)(x - 6)$

**example 3.3h****Factor:  $x^2 + 1x - 30$** 

To factor this trinomial, we are looking for two factors of  $-30$  that will also add to be  $+1$ . The only correct set of numbers that satisfy these two conditions is:  $-5$  and  $+6$ . Thus, the factors of the given trinomial are:  $(x - 5)(x + 6)$

Now to attempt some examples where we don't already know the factors...

**example 3.3i****Factor:  $x^2 + 9x + 18$** 

To factor this trinomial, we are looking for two factors of  $+18$  that will also add to be  $+9$ . The only correct set of numbers that satisfy these two conditions is:  $+3$  and  $+6$ . Thus, the factors of the given trinomial are:  $(x + 3)(x + 6)$

**Cynthia | AXA Graduate**

**AXA Global Graduate Program**

Find out more and apply

**redefining / standards** **AXA**

**example 3.3j****Factor:  $x^2 - 7x + 12$** 

To factor this trinomial, we are looking for two factors of +12 that will also add to be  $-7$ . The only correct set of numbers that satisfy these two conditions is:  $-3$  and  $-4$ . Thus, the factors of the given trinomial are:  $(x - 3)(x - 4)$

**example 3.3k****Factor:  $x^2 - 2x - 24$** 

To factor this trinomial, we are looking for two factors of  $-24$  that will also add to be  $-2$ . The only correct set of numbers that satisfy these two conditions is:  $-6$  and  $+4$ . Thus, the factors of the given trinomial are:  $(x - 6)(x + 4)$

**example 3.3l****Factor:  $x^2 + 5x - 36$** 

To factor this trinomial, we are looking for two factors of  $-36$  that will also add to be  $+5$ . The only correct set of numbers that satisfy these two conditions is:  $+9$  and  $-4$ . Thus, the factors of the given trinomial are:  $(x + 9)(x - 4)$

### **TUTORIAL 3.4: FACTOR A POLYNOMIAL EXPRESSION ( $AX^2 + BX + C$ ) AS A PRODUCT OF BINOMIALS**

We can apply the grouping method of factoring to help us factor most  $ax^2 + bx + c$  trinomials. Let's look back at some previous examples...

**example 3.4a****Factor:  $x^2 + 9x + 18$** 

Since  $a = 1$  and  $c = 18$  in the given trinomial, we are looking for two numbers that multiply to be  $(+1)(+18) = +18$  and that also add to be  $+9$ . Since the 18 is positive, the two desired numbers will have the same sign...and, since the 9 is also positive, these signs will be positive. Therefore, the desired numbers are:  $+3$ ,  $+6$ . Regrouping the middle term of the given trinomial:  $x^2 + 9x + 18 = x^2 + 3x + 6x + 18$ . Using grouping to factor:

$$(x^2 + 3x) + (6x + 18) = x(x + 3) + 6(x + 3) = (x + 3)(x + 6)$$

**example 3.4b****Factor:  $x^2 - 7x + 12$** 

Since  $a = 1$  and  $c = 12$  in the given trinomial, we are looking for two numbers that multiply to be  $(+1)(+12) = +12$  and that also add to be  $-7$ . Since the 12 is positive, the two desired numbers will have the same sign...and, since the 7 is negative, these signs will be negative. Therefore, the desired numbers are:  $-4, -3$ . Regrouping the middle term of the given trinomial:  $x^2 - 7x + 12 = x^2 - 4x - 3x + 12$ . Using grouping to factor:

$$(x^2 - 4x) + (-3x + 12) = x(x - 4) - 3(x - 4) = (\mathbf{x - 4})(\mathbf{x - 3})$$

**example 3.4c****Factor:  $x^2 - 2x - 24$** 

Since  $a = 1$  and  $c = -24$  in the given trinomial, we are looking for two numbers that multiply to be  $(+1)(-24) = -24$  and that also add to be  $-2$ . Since the 24 is negative, the two desired numbers will have different sign...and, since the 2 is also negative, the larger absolute value of these numbers will be negative while the smaller absolute value will be positive. Therefore, the desired numbers are:  $-6, +4$ . Regrouping the middle term of the given trinomial:  $x^2 - 2x - 24 = x^2 - 6x + 4x - 24$ . Using grouping to factor:

$$(x^2 - 6x) + (4x - 24) = x(x - 6) + 4(x - 6) = (\mathbf{x - 6})(\mathbf{x + 4})$$

**example 3.4d****Factor:  $x^2 + 5x - 36$** 

Since  $a = 1$  and  $c = -36$  in the given trinomial, we are looking for two numbers that multiply to be  $(+1)(-36) = -36$  and that also add to be  $+5$ . Since the 36 is negative, the two desired numbers will have different sign...and, since the 5 is positive, the larger absolute value of these numbers will be positive while the smaller absolute value will be negative. Therefore, the desired numbers are:  $+9, -4$ . Regrouping the middle term of the given trinomial:  $x^2 + 5x - 24 = x^2 + 9x - 4x - 36$ . Using grouping to factor:

$$(x^2 + 9x) + (-4x - 36) = x(x + 9) - 4(x + 9) = (\mathbf{x + 9})(\mathbf{x - 4})$$

Now, you are probably saying that this method took more time than doing the same factoring examples in tutorial #3-3. Although that may be true, it is because  $a = 1$  in these problems. Let's look at some examples where  $a \neq 1$ ...

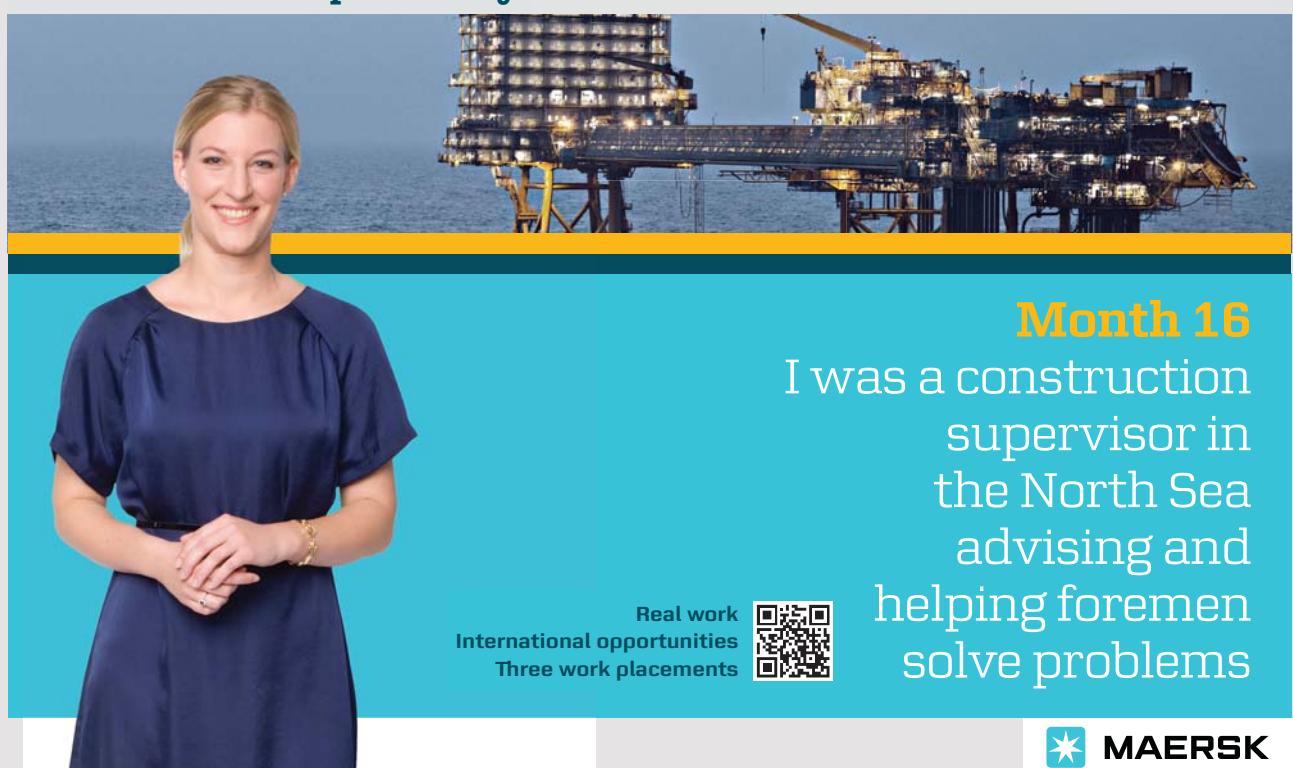
**example 3.4e****Factor:  $12x^2 + 13x + 3$** 

Since  $a = +12$  and  $c = +3$  in the given trinomial, we are looking for two numbers that multiply to be  $(+12)(+3) = +36$  and that also add to be  $+13$ . Since the 36 is positive, the two desired numbers will have the same sign...and, since the 13 is also positive, these signs will be positive. Therefore, the desired numbers are:  $+9, +4$ . Regrouping the middle term of the given trinomial:  $12x^2 + \mathbf{13x} + 3 = 12x^2 + \mathbf{9x} + \mathbf{4x} + 3$ . Using grouping to factor:

$$(12x^2 + 9x) + (4x + 3) = 3x(4x + 3) + 1(4x + 3) = (\mathbf{4x} + \mathbf{3})(\mathbf{3x} + \mathbf{1})$$

I joined MITAS because  
I wanted **real responsibility**

The Graduate Programme  
for Engineers and Geoscientists  
[www.discovermitas.com](http://www.discovermitas.com)



**Month 16**

I was a construction supervisor in the North Sea advising and helping foremen solve problems

Real work  
International opportunities  
Three work placements





Download free eBooks at [bookboon.com](http://bookboon.com)



Click on the ad to read more

**example 3.4f****Factor:  $9x^2 - 18x + 8$** 

Since  $a = +9$  and  $c = +8$  in the given trinomial, we are looking for two numbers that multiply to be  $(+9)(+8) = +72$  and that also add to be  $-18$ . Since the 72 is positive, the two desired numbers will have the same sign...and, since the 18 is negative, these signs will be negative. Therefore, the desired numbers are:  $-12, -6$ . Regrouping the middle term of the given trinomial:  $9x^2 - 18x + 8 = 9x^2 - 12x - 6x + 8$ . Using grouping to factor:

$$(9x^2 - 12x) + (-6x + 8) = 3x(3x - 4) - 2(3x - 4) = (\mathbf{3x - 4})(\mathbf{3x - 2})$$

**example 3.4g****Factor:  $9x^2 - 1x - 10$** 

Since  $a = +9$  and  $c = -10$  in the given trinomial, we are looking for two numbers that multiply to be  $(+9)(-10) = -90$  and that also add to be  $-1$ . Since the 90 is negative, the two desired numbers will have different signs...and, since the 1 is also negative, the larger absolute value of these numbers will be negative while the smaller absolute value will be positive. Therefore, the desired numbers are:  $+9, -10$ . Regrouping the middle term of the given trinomial:  $9x^2 - 1x - 10 = 9x^2 + \mathbf{9x} - \mathbf{10x} - 10$ . Using grouping to factor:

$$(9x^2 + 9x) + (-10x - 10) = 9x(x + 1) - 10(x + 1) = (\mathbf{x + 1})(\mathbf{9x - 10})$$

**example 3.4h****Factor:  $8x^2 + 29x - 12$** 

Since  $a = +8$  and  $c = -12$  in the given trinomial, we are looking for two numbers that multiply to be  $(+8)(-12) = -96$  and that also add to be  $+29$ . Since the 96 is negative, the two desired numbers will have different signs...and, since the 29 is positive, the larger absolute value of these numbers will be positive while the smaller absolute value will be negative. Therefore, the desired numbers are:  $+32, -3$ . Regrouping the middle term of the given trinomial:

$$8x^2 + \mathbf{29x} - 12 = 8x^2 + \mathbf{32x} - \mathbf{3x} - 12$$

Using grouping to factor:

$$(8x^2 + 32x) + (-3x - 12) = 8x(x + 4) - 3(x + 4) = (\mathbf{x + 4})(\mathbf{8x - 3})$$

**NOTE:** To double check the factors, we can always multiply the binomials together. The proper product should be the given trinomial before it was factored.

**NOTE:** When regrouping the middle term of the original trinomial, it makes no difference which of the two “desired” numbers are placed first and which is placed second. The final factors should be the same. Remember that the commutative property works for the multiplication of polynomials.

### TUTORIAL 3.5: FACTOR “SPECIAL” POLYNOMIAL EXPRESSIONS THROUGH PATTERNING

Mathematicians always look for easier or quicker ways to complete tasks. Thus, some factoring tasks can be completed through the use of previously established patterns. If we know these patterns, we can just present the factors without doing too much back-up work.

When we multiply two identical binomial factors, we get a ***perfect square trinomial***:

$$(a + b)^2 = (a + b)(a + b) = a(a) + a(b) + b(a) + b(b) = a^2 + 2ab + b^2$$

$$(a - b)^2 = (a - b)(a - b) = a(a) - a(b) - b(a) - b(-b) = a^2 - 2ab + b^2$$

When we multiply two binomials such as  $(a + b)(a - b)$ , we get a ***difference of squares binomial***:

$$(a + b)(a - b) = a(a) + a(-b) + b(a) + b(-b) = a^2 - b^2$$

**NOTE:** There is no such thing as the sum of squares binomial.

When we multiply three identical binomial factors, we get a ***perfect cube trinomial***:

$$\begin{aligned} (a + b)^3 &= (a + b)(a + b)(a + b) \\ &= (a + b) \times [(a + b)(a + b)] \\ &= (a + b) \times [a(a) + a(b) + b(a) + b(b)] \\ &= (a + b) \times [a^2 + 2ab + b^2] \\ &= a(a^2) + a(2ab) + a(b^2) + b(a^2) + b(2ab) + b(b^2) \\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3 \end{aligned}$$

$$\begin{aligned}
 (a - b)^3 &= (a - b)(a - b)(a - b) \\
 &= (a - b) \times [(a - b)(a - b)] \\
 &= (a - b) \times [a(a) + a(-b) - b(a) - b(-b)] \\
 &= (a - b) \times [a^2 - 2ab + b^2] \\
 &= a(a^2) - a(2ab) + a(b^2) - b(a^2) - b(-2ab) - b(b^2) \\
 &= a^3 - 2a^2b + ab^2 - a^2b + 2ab^2 - b^3 \\
 &= a^3 - 3a^2b + 3ab^2 - b^3
 \end{aligned}$$

Two other special polynomials are displayed below:

**sum of cubes:**

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

**difference of cubes:**

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

The advertisement features a background image of several modern skyscrapers. In the upper left, the IE business school logo is displayed. The central text highlights that 93% of MIM students are working in their sector 3 months following graduation. Below this, the program name "MASTER IN MANAGEMENT" is prominently displayed. To the left, a list of benefits includes studying in Madrid, earning a double degree, and having a semester abroad. To the right, program details are listed: Length: 10 MONTHS, Av. Experience: 1 YEAR, Language: ENGLISH / SPANISH, Format: FULL-TIME, Intakes: SEPT / FEB. At the bottom, three boxes highlight "5 SPECIALIZATIONS PERSONALIZE YOUR PROGRAM", "#10 WORLDWIDE MASTER IN MANAGEMENT FINANCIAL TIMES", and "55 NATIONALITIES IN CLASS". Contact information and social media links are provided at the very bottom.

**IE business school**

**93%**  
OF MIM STUDENTS ARE  
WORKING IN THEIR SECTOR 3 MONTHS  
FOLLOWING GRADUATION

**MASTER IN MANAGEMENT**

- STUDY IN THE CENTER OF MADRID AND TAKE ADVANTAGE OF THE UNIQUE OPPORTUNITIES THAT THE CAPITAL OF SPAIN OFFERS
- PROPEL YOUR EDUCATION BY EARNING A DOUBLE DEGREE THAT BEST SUITS YOUR PROFESSIONAL GOALS
- STUDY A SEMESTER ABROAD AND BECOME A GLOBAL CITIZEN WITH THE BEYOND BORDERS EXPERIENCE

Length: 10 MONTHS  
Av. Experience: 1 YEAR  
Language: ENGLISH / SPANISH  
Format: FULL-TIME  
Intakes: SEPT / FEB

**5 SPECIALIZATIONS**  
PERSONALIZE YOUR PROGRAM

**#10 WORLDWIDE**  
MASTER IN MANAGEMENT  
FINANCIAL TIMES

**55 NATIONALITIES**  
IN CLASS

[www.ie.edu/master-management](http://www.ie.edu/master-management) | [mim.admissions@ie.edu](mailto:mim.admissions@ie.edu) | [f](#) [t](#) [g](#) Follow us on IE MIM Experience

**NOTE:** If the given binomial is a sum of cubes, then the smaller binomial factor will also be a sum and the middle term of the trinomial factor will be subtracted. However, if the given binomial is a difference of cubes, then the smaller binomial factor will also be a difference and the middle term of the trinomial factor will be added.

When we attempt to factor polynomials through patterning, we should follow these guidelines:

1. Factor out the greatest common monomial factor (if one exists).
2. Determine which of the factoring patterns might apply for the given polynomial.
3. Factor the given polynomial using the proper pattern.

**example 3.5a**

**Factor:  $9x^2 - 12x + 4$**

Since  $a = +9$  and  $c = +4$  in the given trinomial, we are looking for two numbers that multiply to be  $(+9)(+4) = +36$  and that also add to be  $-12$ . Since the 36 is positive, the two desired numbers will have the same sign...and, since the 12 is negative, these signs will be negative. Therefore, the desired numbers are:  $-6, -6$ . Since both of these numbers are identical, the given trinomial is a perfect square. Thus, we can use a pattern to complete this task:

$$\text{given polynomial: } 9x^2 - 12x + 4 = (a - b)^2$$

$$\text{perfect square pattern: } a^2 - 2ab + b^2 = (a - b)^2$$

$$\begin{array}{lll} \text{Using the} & a^2 = 9x^2 = (3x)^2 & -2ab = -12x \\ \text{pattern:} & a = 3x & -2(3x)b = -12x \\ & & (-6x)b = -12x \\ & & b = 2 \end{array}$$

$$\text{Therefore: } 9x^2 - 12x + 4 = (3x - 2)^2 = (3x - 2)(3x - 2)$$

**example 3.5b****Factor:  $16x^2 + 40x + 25$** 

Since  $a = +16$  and  $c = +25$  in the given trinomial, we are looking for two numbers that multiply to be  $(+16)(+25) = +400$  and that also add to be  $+40$ . Since the 400 is positive, the two desired numbers will have the same sign...and, since the 40 is also positive, these signs will be positive. Therefore, the desired numbers are:  $+20, +20$ . Since both of these numbers are identical, the given trinomial is a perfect square. Thus, we can use a pattern to complete this task:

$$\begin{array}{ll} \text{given polynomial:} & 16x^2 + 40x + 25 = (a + b)^2 \\ \text{perfect square pattern:} & a^2 + 2ab + b^2 = (a + b)^2 \end{array}$$

$$\begin{array}{lll} \text{Using the} & a^2 = 16x^2 = (4x)^2 & +2ab = +40x \\ \text{pattern:} & a = 4x & +2(4x)b = +40x \\ & & (+8x)b = +40x \\ & & b = 5 \end{array}$$

$$\text{Therefore: } 16x^2 + 40x + 25 = (4x + 5)^2 = (4x + 5)(4x + 5)$$

**example 3.5c****Factor:  $4x^2 - 9$** 

Since  $a = 4$  and  $c = -9$  in the given trinomial, we are looking for two numbers that multiply to be  $(+4)(-9) = -36$  and that also add to be zero (since  $b = 0$ ). Since the 36 is negative, the two desired numbers will have different signs...and, since  $b = 0$ , these two numbers will be opposites of each other. Therefore, the desired numbers are:  $\pm 6$ . Whenever these conditions exist, we have the difference of squares and we can use a pattern to complete this task:

$$\begin{array}{ll} \text{given polynomial:} & 4x^2 - 9 = (a + b)(a - b) \\ \text{difference of squares pattern:} & a^2 - b^2 = (a + b)(a - b) \end{array}$$

$$\begin{array}{lll} \text{Using the pattern:} & a^2 = 4x^2 = (2x)^2 & b^2 = 9 = (3)^2 \\ & a = 2x & b = 3 \end{array}$$

$$\text{Therefore: } 4x^2 - 9 = (2x + 3)(2x - 3)$$

**example 3.5d****Factor:  $x^3 - 12x^2 + 48x - 64$** 

Because this polynomial has four terms, we could attempt to group the terms to see if that strategy would work. However, we would soon discover that grouping is not the way to go for this example. Therefore, let's attempt to see if the given polynomial fits one of our “special” patterns:

$$\begin{array}{ll} \text{given polynomial:} & x^3 - 12x^2 + 48x - 64 = (a - b)^3 \\ \text{perfect cube pattern:} & a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3 \end{array}$$

Using the pattern:

$$\begin{array}{llll} a^3 = & 3a^2b = 12x^2 & +3ab^2 = +48x & b^3 = 64 = (4)^3 \\ x^3 & 3(x^2)b = 12x^2 & +3(x)b^2 = +48x & b = 4 \\ a = x & 3b = 12 & +3b^2 = +48 & \\ b = 4 & b = 4 & b^2 = 16 & \\ & & b = 4 & \end{array}$$

Therefore:  $x^3 - 12x^2 + 48x - 64 = (x - 4)^3 = (x - 4)(x - 4)(x - 4)$

**“I studied English for 16 years but...  
...I finally learned to speak it in just six lessons”**

Jane, Chinese architect

ENGLISH OUT THERE

Click to hear me talking before and after my unique course download

**example 3.5e****Factor:**  $8x^3 + 36x^2 + 54x + 27$ 

Because this polynomial has four terms, we could attempt to group the terms to see if that strategy would work. However, we would soon discover that grouping is not the way to go for this example. Therefore, let's attempt to see if the given polynomial fits one of our “special” patterns:

$$\begin{aligned} \text{given polynomial: } & 8x^3 + 36x^2 + 54x + 27 = (a + b)^3 \\ \text{perfect cube pattern: } & a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3 \end{aligned}$$

Using the pattern:

$$\begin{aligned} a^3 &= 8x^3 = (2x)^3 & 3a^2b &= 36 & 3ab^2 &= 54x & b^3 &= 27 = (3)^3 \\ a &= 2x & 3(2x)^2b &= 36x^2 & 3(2x)b^2 &= 54x & b &= 3 \\ && 3(4x^2)b &= 36x^2 & (6x)b^2 &= 54x \\ && (12x^2)b &= 36x^2 & b^2 &= 9 \\ && b &= 3 & b &= 3 \end{aligned}$$

Therefore:  $8x^3 + 36x^2 + 54x + 27 = (2x + 3)^3 = (2x + 3)(2x + 3)$   
 $(2x + 3)$

**example 3.5f****Factor:**  $64x^3 + 27$ 

Although this polynomial has two terms, it is not an example of the difference of two squares. Therefore, let's attempt to see if the given polynomial fits one of our “special” patterns:

$$\begin{aligned} \text{given polynomial: } & 64x^3 + 27 = (a + b)(a^2 - ab + b^2) \\ \text{sum of cubes pattern: } & a^3 + b^3 = (a + b)(a^2 - ab + b^2) \end{aligned}$$

$$\begin{aligned} \text{Using the pattern: } & a^3 = 64x^3 = (4x)^3 \rightarrow a = 4x \\ & b^3 = 27 = (3)^3 \rightarrow b = 3 \\ & a^2 = (4x)^2 = 16x^2 \\ & b^2 = (3)^2 = 9 \\ & ab = (4x)(3) = 12x \end{aligned}$$

Therefore:  $64x^3 + 27 = (4x + 3)(16x^2 - 12x + 9)$

**example 3.5g****Factor:  $125x^3 - 64$** 

Although this polynomial has two terms, it is not an example of the difference of two squares. Therefore, let's attempt to see if the given polynomial fits one of our "special" patterns:

$$\begin{array}{l} \text{given polynomial: } 125x^3 - 64 = (a - b)(a^2 + ab + b^2) \\ \text{difference of cubes pattern} \quad a^3 - b^3 = (a - b)(a^2 + ab + b^2) \end{array}$$

Using the pattern:  $a^3 = 125x^3 = (5x)^3 \rightarrow a = 5x$   
 $b^3 = 64 = (4)^3 \rightarrow b = 4$   
 $a^2 = (5x)^2 = 25x^2$   
 $b^2 = (4)^2 = 16$   
 $ab = (5x)(4) = 20x$

Therefore:  $125x^3 - 64 = (5x - 4)(25x^2 + 20x + 16)$

### TUTORIAL 3.6: USE FACTORING TO COMPLETE BUSINESS APPLICATIONS

On occasion, various business applications may involve the use of polynomial factoring in order to complete a desired task...

**example 3.6a**

**The revenue ( $R$ ) from the sale of  $x$  units of a product or commodity is found by multiplying the price by the number of items sold:  $R = (\text{price})(\text{units})$ . Suppose that the revenue of a certain product is given by the algebraic expression of:  $R = 500x - x^2$ . If  $x$  is the number of units sold, what is the expression for the price of this item?**

We need to factor out the greatest common factor from the algebraic binomial that appears to the right of the equals sign:  $R = 500x - x^2 = x(500 - x)$ . If  $x$  is the number of units sold, then **500 – x** represents the price of the item.

**example 3.6b**

**Suppose that the revenue ( $R$ ) for a boat tour is:  $R = 1050 - 20x - 0.5x^2$ , where  $x$  is the number of passengers above the minimum of 50. The revenue expression can be factored into two binomials where one of the binomials represents the total number of passengers and the other represents the cost of the boat tour. Express the revenue formula in its factored form.**

We first need to factor out the greatest common factor from the algebraic expression that appears to the right of the equals sign:

$$R = 1050 - 20x - 0.5x^2 = (-0.5)(x^2 + 40x - 2100)$$

We then need to factor the resulting trinomial into its two binomials by obtaining two numbers that multiply to be  $-2100$  and also add to be  $+40$ :  $+70$ ,  $-30$ . Now we just need to complete the factoring through the use of “grouping”:

$$\begin{aligned}x^2 + 40x - 2100 &= x^2 + 70x - 30x - 2100 \\x^2 + 40x - 2100 &= (x^2 + 70x) + (-30x - 2100) \\x^2 + 40x - 2100 &= x(x + 70) - 30(x + 70) \\x^2 + 40x - 2100 &= (x + 70)(x - 30)\end{aligned}$$

Putting the GCF back into the mix:  $(-0.5)(x^2 + 40x - 2100) = (-0.5)(x + 70)(x - 30)$ . So, the final binomial factors are:

$(-0.5x - 35)(x - 30)$  or  $(x + 70)(-0.5x + 15)$

Excellent Economics and Business programmes at:


/

**university of  
groningen**



**“The perfect start  
of a successful,  
international career.”**

[www.rug.nl/feb/education](http://www.rug.nl/feb/education)

**CLICK HERE**  
to discover why both socially  
and academically the University  
of Groningen is one of the best  
places for a student to be

**example 3.6c**

The consumer ( $C$ ) expenditure (in dollars) for a commodity is the product of its market price ( $p$ ) and the number of units demanded. Suppose that for a certain commodity, the consumer expenditure is given by:  $C = 9600p - 150p^2$ . Factor this given expression in order to find an expression for the number of units demanded.

We need to factor out the  $p$  from the algebraic binomial that appears to the right of the equals sign:  $C = 9600p - 150p^2 = p(9600 - 150p)$ . Then, the  $p$  represents the price while the **9600 – 150p** represents the number of units demanded.

### TUTORIAL 3.7: SOLVE QUADRATIC EQUATIONS BY FACTORING

Just as we solved linear equations (standard form:  $Ax + By = C$ ; slope-intercept form:  $y = mx + b$ ) to complete some tasks, we may also need to solve quadratic equations ( $ax^2 + bx + c = 0$ ) to complete other tasks.

One way to solve quadratic equations is through factoring and the application of:

**Zero Product Property:** The product of any two real numbers will equal zero if either or both of these numbers is equal to zero. Expressed differently, if  $(a)(b) = 0$ , then  $a = 0$  (alone),  $b = 0$  (alone), or both  $a$  and  $b$  equal zero.

**example 3.7a****Solve by factoring:  $16x^2 - 25 = 0$** 

1. Set the quadratic equation to zero. (*This step is already completed.*)
2. Factor:  $16x^2 - 25 \rightarrow$  Since this binomial is the difference of squares, we can use the “special pattern” to complete the factoring:  $a^2 - b^2 = (a + b)(a - b) \rightarrow 16x^2 - 25 = (4x)^2 - (5)^2 = (4x + 5)(4x - 5)$
3. Set each factor equal to zero and solve each “mini-equation”:  

$$4x + 5 = 0 \rightarrow 4x = -5 \rightarrow x = \frac{-5}{4} = -1.25$$

$$4x - 5 = 0 \rightarrow 4x = +5 \rightarrow x = \frac{+5}{4} = 1.25$$
4. Determine the solutions for the given quadratic:  $x = \frac{\pm 5}{4} = \pm 1.25$

**example 3.7b****Solve by factoring:  $x^2 - 4x = 5x^2$** 

- Set the quadratic equation to zero:

$$x^2 - 4x = 5x^2 \rightarrow 0 = 5x^2 - x^2 + 4x = 4x^2 + 4x$$

- Factor:  $4x^2 + 4x \rightarrow$  Factor out the greatest common factor:

$$4x(x + 1)$$

- Set each factor equal to zero and solve each “mini-equation”:

$$4x = 0 \rightarrow x = \frac{0}{4} = 0 \quad x + 1 = 0 \rightarrow x = -1$$

- Determine the solutions for the given quadratic:  $\textcolor{red}{x = 0, x = -1}$

**example 3.7c****Solve by factoring:  $x^2 + 12x = 4x - 15$** 

- Set the quadratic equation to zero:

$$x^2 + 12x = 4x - 15 \rightarrow x^2 + 8x + 15 = 0$$

- Factor:  $x^2 + 8x + 15 \rightarrow$  We are looking for two numbers that multiply to be +15 and add to be +8: +3, +5  $\rightarrow$

$$x^2 + 8x + 15 = x^2 + 3x + 5x + 15 = x(x + 3) + 5(x + 3) = (x + 3)(x + 5)$$

- Set each factor equal to zero and solve each “mini-equation”:

$$x + 3 = 0 \rightarrow x = -3 \quad x + 5 = 0 \rightarrow x = -5$$

- Determine the solutions for the given quadratic:  $\textcolor{red}{x = -3, x = -5}$

**example 3.7d****Solve by factoring:  $25x^2 + 20x + 4 = 0$** 

- Set the quadratic equation to zero. (*This step is already completed.*)

- Factor:  $25x^2 + 20x + 4 \rightarrow$  We are looking for two numbers that multiply to be +100 ( $25 \times 4 = 100$ ) and add to be +20: +10, +10  $\rightarrow$

*NOTE: Since these numbers are identical, the trinomials is a perfect square and we can factor it by using one of the “special patterns”:*

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$25x^2 + 20x + 4 = (5x)^2 + 2(5)(2)x + (2)^2 = (5x + 2)^2$$

- Set each factor equal to zero and solve each “mini-equation”:

$$5x + 2 = 0 \rightarrow 5x = -2 \rightarrow x = \frac{-2}{5} = -0.4$$

*NOTE: Since we have a perfect square trinomial, we will have a “double root” as a solution.*

- Determine the solutions for the given quadratic:  $\textcolor{red}{x = -\frac{2}{5} = -0.4}$

**example 3.7e**

$$\text{Solve by factoring: } \frac{4x^2}{15} + \frac{7x}{15} = 1$$

- Set the quadratic equation to zero:

$$\frac{4x^2}{15} + \frac{7x}{15} = 1 \rightarrow \frac{4x^2}{15} + \frac{7x}{15} - 1 = 0$$

- Because of the fractions, multiply through by the lowest common denominator (which is 15 for this example):  $4x^2 + 7x - 15 = 0$
- Factor:  $4x^2 + 7x - 15 \rightarrow$  We are looking for two numbers that multiply to be  $-60$  ( $-15 \times 4 = -60$ ) and add to be  $+7$ :  $+12, -5 \rightarrow$   
 $4x^2 + 7x - 15 = 4x^2 + 12x - 5x - 15 =$

$$4x(x + 3) - 5(x + 3) = (x + 3)(4x - 5)$$

- Set each factor equal to zero and solve each “mini-equation”:

$$x + 3 = 0$$

$$x = -3$$

$$4x - 5 = 0$$

$$4x = 5$$

$$x = \frac{5}{4} = 1.25$$

- Determine the solutions for the given quadratic:

$$x = -3, x = \frac{5}{4} = 1.25$$

## American online LIGS University

is currently enrolling in the  
Interactive Online **BBA, MBA, MSc,**  
**DBA and PhD** programs:

- ▶ **enroll by September 30th, 2014** and
- ▶ **save up to 16%** on the tuition!
- ▶ pay in 10 installments / 2 years
- ▶ Interactive Online education
- ▶ visit [www.ligsuniversity.com](http://www.ligsuniversity.com) to find out more!

Note: LIGS University is not accredited by any nationally recognized accrediting agency listed by the US Secretary of Education.  
[More info here.](#)



**example 3.7f****Solve by factoring:**  $(x - 1)(2x + 1) = 14$ 

- Multiply the two binomials:

$$(x - 1)(2x + 1) = 2x^2 + x - 2x - 1 = 2x^2 - x - 1$$

- Set the quadratic equation to zero:

$$2x^2 - x - 1 = 14 \rightarrow 2x^2 - x - 15 = 0$$

- Factor:  $2x^2 - x - 15 \rightarrow$  We are looking for two numbers that multiply to be  $-30$  ( $-15 \times 2 = -60$ ) and add to be  $-1$ :  
 $+5, -6 \rightarrow$

$$2x^2 - x - 15 = 2x^2 - 6x + 5x - 15 =$$

$$2x(x - 3) + 5(x - 3) = (x - 3)(2x + 5)$$

- Set each factor equal to zero and solve each “mini-equation”:

$$x - 3 = 0$$

$$x = 3$$

$$2x + 5 = 0$$

$$2x = -5$$

$$x = \frac{-5}{2} = -2.5$$

- Determine the solutions for the given quadratic:

$$\textcolor{red}{x = 3, x = \frac{-5}{2} = -2.5}$$

## **TUTORIAL 3.8: SOLVE QUADRATIC EQUATIONS BY COMPLETING THE SQUARE**

Another way to solve quadratic equations is through a method called “completing the square” which follows these steps:

- Add/Subtract the constant from both sides of the equation.
- Divide both sides of the equation by the coefficient of the quadratic ( $x^2$ ) term... simplifying wherever possible.
- Divide the coefficient of the linear ( $x$ ) term by two (2) and square the quotient.  
Add this result to both sides of the equation...simplifying wherever possible.
- Factor the trinomial on the left side of the equation into its  $(a \pm b)^2$  form.
- Take a square root of each side of the equation.
- Solve for  $x$ ...simplifying wherever possible. Normally, there will be two solutions so we will need to solve two “mini-equations” for  $x$ .

**example 3.8a****Solve by completing the square:  $x^2 - 14x = 72$** 

1. This step has already been accomplished in the given example.
2. No need to complete this step since the coefficient of  $ax^2$  is one (1).
3.  $14 \div 2 = 7 \rightarrow 7^2 = 49 \rightarrow x^2 - 14x + 49 = 72 + 49 \rightarrow x^2 - 14x + 49 = 121$
4.  $x^2 - 14x + 49 = (x - 7)^2$
5.  $(x - 7)^2 = 121 \rightarrow \pm\sqrt{121} = \pm 11 \rightarrow x - 7 = \pm 11$
6. Solve each “mini-equation”:

$$x - 7 = -11 \rightarrow x = -11 + 7 = -4$$

$$x - 7 = 11 \rightarrow x = 11 + 7 = 18$$

7. Therefore, the solutions to this quadratic equation are:  $\textcolor{red}{x} = -4, x = 18$

**example 3.8b****Solve by completing the square:  $x^2 + 16x = -60$** 

1. This step has already been accomplished in the given example.
2. No need to complete this step since the coefficient of  $ax^2$  is one (1).
3.  $16 \div 2 = 8 \rightarrow 8^2 = 64 \rightarrow x^2 + 16x + 64 = -60 + 64 \rightarrow x^2 + 16x + 64 = 4$
4.  $x^2 + 16x + 64 = (x + 8)^2$
5.  $(x + 8)^2 = 4 \rightarrow \pm\sqrt{4} = \pm 2 \rightarrow x + 8 = \pm 2$
6. Solve each “mini-equation”:

$$x + 8 = -2 \rightarrow x = -2 - 8 = -10$$

$$x + 8 = 2 \rightarrow x = 2 - 8 = -6$$

7. Therefore, the solutions to this quadratic equation are:  $\textcolor{red}{x} = -10, x = -6$

**example 3.8c****Solve by completing the square:  $x^2 - 9x = -20$** 

1. This step has already been accomplished in the given example.
2. No need to complete this step since the coefficient of  $ax^2$  is one (1).

$$3. \ 9 \div 2 = \frac{9}{2} \rightarrow \left(\frac{9}{2}\right)^2 = \frac{81}{4} \rightarrow x^2 - 9x + \frac{81}{4} = -20 + \frac{81}{4} \rightarrow \\ x^2 - 9x + \frac{81}{4} = \frac{1}{4}$$

$$4. \ x^2 - 9x + \frac{81}{4} = \left(x - \frac{9}{2}\right)^2$$

$$5. \ \left(x - \frac{9}{2}\right)^2 = \frac{1}{4} \rightarrow \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2} \rightarrow x - \frac{9}{2} = \pm\frac{1}{2}$$

6. Solve each “mini-equation”:

$$x - \frac{9}{2} = -\frac{1}{2} \rightarrow x = -\frac{1}{2} + \frac{9}{2} = \frac{-1+9}{2} = \frac{8}{2} = 4$$

$$x - \frac{9}{2} = \frac{1}{2} \rightarrow \frac{1}{2} + \frac{9}{2} = \frac{1+9}{2} = \frac{10}{2} = 5$$

Therefore, the solutions to this quadratic equation are:  **$x = 4, x = 5$**

**DON'T EAT YELLOW SNOW**

**What will your advice be?**

Some advice just states the obvious. But to give the kind of advice that's going to make a real difference to your clients you've got to listen critically, dig beneath the surface, challenge assumptions and be credible and confident enough to make suggestions right from day one. At Grant Thornton you've got to be ready to kick start a career right at the heart of business.

Sound like you? Here's our advice: visit [GrantThornton.ca/careers/students](http://GrantThornton.ca/careers/students)

Scan here to learn more about a career with Grant Thornton.

 **Grant Thornton**  
An instinct for growth™

© Grant Thornton LLP. A Canadian Member of Grant Thornton International Ltd



**example 3.8d****Solve by completing the square:  $2x^2 + 5x - 3 = 0$** 

$$1. \ 2x^2 + 5x - 3 = 0 \rightarrow 2x^2 + 5x - 3 + 3 = 0 + 3 \rightarrow 2x^2 + 5x = 3$$

$$2. \text{ Divide by 2: } \frac{2x^2}{2} + \frac{5x}{2} = \frac{3}{2} \rightarrow \text{simplify: } x^2 + \frac{5x}{2} = \frac{3}{2}$$

$$3. \frac{5}{2} \div 2 = \frac{5}{2} \times \frac{1}{2} = \frac{5}{4} \rightarrow \left(\frac{5}{4}\right)^2 = \frac{25}{16} \rightarrow$$

$$x^2 + \frac{5x}{2} + \frac{25}{16} = \frac{3}{2} + \frac{25}{16} = \frac{24+25}{16} = \frac{49}{16}$$

$$4. \ x^2 + \frac{5x}{2} + \frac{25}{16} = \frac{49}{16} \rightarrow x^2 + \frac{5x}{2} + \frac{25}{16} = \left(x + \frac{5}{4}\right)^2$$

$$5. \left(x + \frac{5}{4}\right)^2 = \frac{49}{16} \rightarrow \pm \sqrt{\frac{49}{16}} = \pm \frac{7}{4} \rightarrow x + \frac{5}{4} = \pm \frac{7}{4}$$

6. Solve each “mini-equation”:

$$x + \frac{5}{4} = -\frac{7}{4} \rightarrow x = -\frac{7}{4} - \frac{5}{4} = \frac{-7-5}{4} = \frac{-12}{4} = -3$$

$$x + \frac{5}{4} = \frac{7}{4} \rightarrow x = \frac{7}{4} - \frac{5}{4} = \frac{7-5}{4} = \frac{2}{4} = \frac{1}{2} = 0.5$$

Therefore, the solutions to this quadratic equation are:  $x = -3, x = 0.5$

**example 3.8e****Solve by completing the square:  $4x^2 - 23x - 35 = 0$** 

$$1. \ 4x^2 - 23x - 35 = 0 \rightarrow 4x^2 - 23x - 35 + 35 = 0 + 35 \rightarrow$$

$$4x^2 - 23x = 35$$

$$2. \text{ Divide by 4: } \frac{4x^2}{4} - \frac{23x}{4} = \frac{35}{4} \rightarrow \text{simplify: } x^2 - \frac{23x}{4} = \frac{35}{4}$$

$$3. \frac{23}{4} \div 2 = \frac{23}{4} \times \frac{1}{2} = \frac{23}{8} \rightarrow \left(\frac{23}{8}\right)^2 = \frac{529}{64} \rightarrow$$

$$x^2 - \frac{23x}{4} + \frac{529}{64} = \frac{35}{4} + \frac{529}{64} = \frac{16(35)+529}{64} = \frac{560+529}{64} = \frac{1089}{64}$$

$$4. \ x^2 - \frac{23x}{4} + \frac{529}{64} = \frac{1089}{64} \rightarrow \left(x - \frac{23}{8}\right)^2 = \frac{1089}{64}$$

$$5. \left(x - \frac{23}{8}\right)^2 = \frac{1089}{64} \rightarrow \pm \sqrt{\frac{1089}{64}} = \frac{33}{8} \rightarrow x - \frac{23}{8} = \pm \frac{33}{8} = \pm 4.125$$

6. Solve each “mini-equation”:

$$x - \frac{23}{8} = -\frac{33}{8} \rightarrow x = -\frac{33}{8} + \frac{23}{8} = \frac{-33+23}{8} = \frac{-10}{8} = \frac{-5}{4} = -1.25$$

$$x - \frac{23}{8} = \frac{33}{8} \rightarrow x = \frac{33}{8} + \frac{23}{8} = \frac{33+23}{8} = \frac{56}{8} = 7$$

7. Therefore, the solutions to this quadratic equation are:

$$x = -1.25, x = 7$$

### TUTORIAL 3.9: DERIVE THE QUADRATIC FORMULA BY COMPLETING THE SQUARE

The “completing the square” method can be used to obtain a formula that will solve most quadratic equations by following these steps:

1. given general form for any quadratic equation:  $ax^2 + bx + c = 0$
2. Subtract the constant from both sides of the equation:  $ax^2 + bx + c = -c$
3. Divide both sides of the equation by the coefficient of the quadratic term:

$$\frac{ax^2}{a} + \frac{bx}{a} = \frac{-c}{a}$$

$$4. \text{ Simplify: } \frac{ax^2}{a} = x^2 \rightarrow x^2 + \frac{bx}{a} = \frac{-c}{a}$$

$$5. \text{ Complete the square by adding } \left(\frac{1}{2} \times \frac{b}{a}\right)^2 \text{ to both sides: } x^2 + \frac{bx}{a} + \left(\frac{1}{2} \times \frac{b}{a}\right)^2 = \frac{-c}{a} + \left(\frac{1}{2} \times \frac{b}{a}\right)^2$$

$$6. \text{ Simplify: } \left(\frac{1}{2} \times \frac{b}{a}\right)^2 = \frac{b^2}{4a^2} \rightarrow x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{-c}{a} + \frac{b^2}{4a^2}$$

$$7. \text{ Simplify: } \frac{-c}{a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2} \rightarrow x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

$$8. \text{ Factor the trinomial on the left side of the equation: } \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$9. \text{ Take a square root of each side of the equation: } x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$10. \text{ Solve for } x: x = \frac{-b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$11. \text{ Simplify: } \sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\sqrt{b^2 - 4ac}}{2a} \rightarrow x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$12. \text{ Quadratic Formula: } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**Quadratic Formula:** If  $ax^2 + bx + c = 0$  (where  $a \neq 0$ ), then  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

## TUTORIAL 3.10: SOLVE QUADRATIC EQUATIONS USING THE QUADRATIC FORMULA

Another way to solve quadratic equations is through the use of the quadratic formula which follows these steps:

1. Set the given quadratic equation to zero.
2. Identify the values for  $a$ ,  $b$  and  $c$ :  $ax^2 + bx + c = 0$
3. Substitute the values for  $a$ ,  $b$  and  $c$  into the quadratic formula:  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
4. Simplify the numeric expression.
5. Branch off the two numeric expressions evaluating each of them separately. These results will be the desired solutions.

www.alcatel-lucent.com/careers

Alcatel-Lucent

What if you could build your future and create the future?

One generation's transformation is the next's status quo. In the near future, people may soon think it's strange that devices ever had to be "plugged in." To obtain that status, there needs to be "The Shift".



**example 3.10a****Solve by using the quadratic formula:  $x^2 - 14x = 72$** 

1.  $x^2 - 14x = 72 \rightarrow x^2 - 14x - 72 = 0$

2.  $a = 1, b = -14, c = -72$

3.  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(1)(-72)}}{2(1)}$

4.  $x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(1)(-72)}}{2(1)} = \frac{14 \pm \sqrt{196 + 288}}{2} =$

$$\frac{14 \pm \sqrt{484}}{2} = \frac{14 \pm 22}{2}$$

5.  $x_1 = \frac{14 - 22}{2} = \frac{-8}{2} = -4$  and  $x_2 = \frac{14 + 22}{2} = \frac{36}{2} = 18$

Therefore, the solutions to this quadratic equation are:  **$x = -4, x = 18$** **example 3.10b****Solve by using the quadratic formula:  $x^2 + 16x = -60$** 

1.  $x^2 + 16x = -60 \rightarrow x^2 + 16x + 60 = 0$

2.  $a = 1, b = 16, c = 60$

3.  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(16) \pm \sqrt{(16)^2 - 4(1)(60)}}{2(1)}$

4.  $x = \frac{-(16) \pm \sqrt{(16)^2 - 4(1)(60)}}{2(1)} = \frac{-16 \pm \sqrt{256 - 240}}{2} =$

$$\frac{-16 \pm \sqrt{16}}{2} = \frac{-16 \pm 4}{2}$$

5.  $x_1 = \frac{-16 - 4}{2} = \frac{-20}{2} = -10$  and  $x_2 = \frac{-16 + 4}{2} = \frac{-12}{2} = -6$

Therefore, the solutions to this quadratic equation are:  **$x = -10, x = -6$**

**example 3.10c**      **Solve by using the quadratic formula:  $2x^2 + 5x = 3$**

$$1. \ 2x^2 + 5x - 3 = 0$$

$$2. \ a = 2, \ b = 5, \ c = -3$$

$$3. \ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(5) \pm \sqrt{(5)^2 - 4(2)(-3)}}{2(2)}$$

$$4. \ x = \frac{-(5) \pm \sqrt{(5)^2 - 4(2)(-3)}}{2(2)} = \frac{-5 \pm \sqrt{25 + 24}}{4} = \frac{-16 \pm \sqrt{16}}{2} = \frac{-16 \pm 4}{2}$$

$$5. \ x_1 = \frac{-5 - 7}{4} = \frac{-12}{4} = -3 \text{ and } x_2 = \frac{-5 + 7}{4} = \frac{2}{4} = \frac{1}{2} = 0.5$$

Therefore, the solutions to this quadratic equation are:  $x = -3, x = 0.5$

**example 3.10d**      **Solve by using the quadratic formula:  $3x^2 + 4 = 8x$**

$$1. \ 3x^2 - 8x + 4 = 0$$

$$2. \ a = 3, \ b = -8, \ c = 4$$

$$3. \ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(4)}}{2(3)}$$

$$4. \ x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(4)}}{2(3)} = \frac{8 \pm \sqrt{64 - 48}}{6} = \frac{8 \pm \sqrt{16}}{6} = \frac{8 \pm 4}{6}$$

$$5. \ x_1 = \frac{8 - 4}{6} = \frac{4}{6} = \frac{2}{3} \text{ and } x_2 = \frac{8 + 4}{6} = \frac{12}{6} = 2$$

Therefore, the solutions to this quadratic equation are:  $x = \frac{2}{3}, x = 2$

### TUTORIAL 3.11: DETERMINE THE NUMBER AND THE NATURE OF THE SOLUTIONS FOR A QUADRATIC EQUATION

Occasionally, it will be useful to know the number and the nature of the solutions for a given quadratic equation. This information can be derived from part of the quadratic formula... specifically, the discriminant:  $b^2 - 4ac$  (the part of the formula that appears beneath the square root sign):

Given  $ax^2 + bx + c = 0$  (where  $a \neq 0$ ),

- If  $b^2 - 4ac > 0$ , the equation has 2 distinct real solutions.
- If  $b^2 - 4ac = 0$ , the equation has exactly 1 real solution.
- If  $b^2 - 4ac < 0$ , the equation has no real solutions.

**example 3.11a****Determine the number and nature of the solutions of:  $3x^2 + 4 = 8x$** 

1. Set the given quadratic equation to zero:  $3x^2 - 8x + 4 = 0$
2. Identify the values of  $a$ ,  $b$  and  $c$ :  $a = 3$ ,  $b = -8$ ,  $c = 4$
3. Substitute the values for  $a$ ,  $b$  and  $c$  into the discriminant and evaluate the numeric expression:

$$b^2 - 4ac = (-8)^2 - 4(3)(4) = 64 - 48 = 16$$

4. Use the value of the discriminant to determine the number and nature of the solutions for the given equation:  $b^2 - 4ac = 16 \rightarrow 16 > 0 \rightarrow$  this equation will have **2 distinct real solutions**. *NOTE:* Look back at example 3.10d, the actually solutions are  $x = \frac{2}{3}$  and  $x = 2$ .



**Maastricht University** *Leading in Learning!*

**Join the best at  
the Maastricht University  
School of Business and  
Economics!**

**Top master's programmes**

- 33<sup>rd</sup> place Financial Times worldwide ranking: MSc International Business
- 1<sup>st</sup> place: MSc International Business
- 1<sup>st</sup> place: MSc Financial Economics
- 2<sup>nd</sup> place: MSc Management of Learning
- 2<sup>nd</sup> place: MSc Economics
- 2<sup>nd</sup> place: MSc Econometrics and Operations Research
- 2<sup>nd</sup> place: MSc Global Supply Chain Management and Change

Sources: Keuzegids Master ranking 2013; Elsevier 'Beste Studies' ranking 2012; Financial Times Global Masters in Management ranking 2012

**Maastricht  
University is  
the best specialist  
university in the  
Netherlands  
(Elsevier)**

**Visit us and find out why we are the best!  
Master's Open Day: 22 February 2014**

[www.mastersopenday.nl](http://www.mastersopenday.nl)



**Click on the ad to read more**

**example 3.11b      Determine the number and nature of the solutions of:  $4x^2 + 25 = 20x$** 

1. Set the given quadratic equation to zero:  $4x^2 - 20x + 25 = 0$
2. Identify the values of  $a$ ,  $b$  and  $c$ :  $a = 4$ ,  $b = -20$ ,  $c = 25$
3. Substitute the values for  $a$ ,  $b$  and  $c$  into the discriminant and evaluate the numeric expression:

$$b^2 - 4ac = (-20)^2 - 4(4)(25) = 400 - 400 = 0$$

4. Use the value of the discriminant to determine the number and nature of the solutions for the given equation:  $b^2 - 4ac = 0 \rightarrow 0 = 0 \rightarrow$  this equation will have **1 distinct real solution**.

*NOTE:* Solving this quadratic equation by factoring, we can see that the quadratic trinomial is a perfect square which can be factored in identical binomials:  $4x^2 - 20x + 25 = (2x - 5)^2 \dots$  which will have one double root solution of  $\frac{5}{2}$  (or 2.5).

**example 3.11c      Determine the number and nature of the solutions of:  $x^2 + 12 = 5x$** 

1. Set the given quadratic equation to zero:  $x^2 - 5x + 12 = 0$
2. Identify the values of  $a$ ,  $b$  and  $c$ :  $a = 1$ ,  $b = -5$ ,  $c = 12$
3. Substitute the values for  $a$ ,  $b$  and  $c$  into the discriminant and evaluate the numeric expression:

$$b^2 - 4ac = (-5)^2 - 4(1)(12) = 25 - 48 = -13$$

4. Use the value of the discriminant to determine the number and nature of the solutions for the given equation:  $b^2 - 4ac = -13 \rightarrow -13 < 0 \rightarrow$  this equation will have **no real solutions**.

There will be times when the discriminant is a perfect square...and, thus the solutions will be rational numbers – meaning that the solutions can be expressed as fractions, decimals, or integers. However, there will also be times when the discriminant is a not perfect square...and, these solutions will be irrational numbers – meaning that the solutions can be expressed using a square root (radical) sign or will have to be rounded to a situation-appropriate place value.

**example 3.11d****Determine the number and nature of the solutions of:  $2x^2 - 15 = x$** 

1. Set the given quadratic equation to zero:  $2x^2 - x - 15 = 0$
2. Identify the values of  $a$ ,  $b$  and  $c$ :  $a = 2$ ,  $b = -1$ ,  $c = -15$
3. Substitute the values for  $a$ ,  $b$  and  $c$  into the discriminant and evaluate the numeric expression:

$$b^2 - 4ac = (-1)^2 - 4(2)(-15) = 1 + 120 = 121$$

4. Use the value of the discriminant to determine the number and nature of the solutions for the given equation:  $b^2 - 4ac = 121 \rightarrow 121 > 0 \rightarrow$  this equation will have **2 distinct real solutions**.
5. Use the quadratic formula to solve:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-15)}}{2(2)} = \frac{1 \pm \sqrt{121}}{4} = \frac{1 \pm 11}{4}$$

$$x_1 = \frac{1 - 11}{4} = \frac{-10}{4} = \frac{-5}{2} = -2.5 \dots x_2 = \frac{1 + 11}{4} = \frac{12}{4} = 3$$

*NOTE:* Since the discriminant for this equation was a perfect square ( $11^2 = 121$ ), these solutions are rational numbers because they can be expressed as fractions, decimals or integers.

**example 3.11e****Determine the number and nature of the solutions of:  $3x^2 - 6 = 15x$** 

1. Set the given quadratic equation to zero:  $3x^2 - 15x - 6 = 0$
2. Identify the values of  $a$ ,  $b$  and  $c$ :  $a = 3$ ,  $b = -15$ ,  $c = -6$
3. Substitute the values for  $a$ ,  $b$  and  $c$  into the discriminant and evaluate the numeric expression:

$$b^2 - 4ac = (-15)^2 - 4(3)(-6) = 225 + 72 = 297$$

4. Use the value of the discriminant to determine the number and nature of the solutions for the given equation:  $b^2 - 4ac = 297 \rightarrow 297 > 0 \rightarrow$  this equation will have **2 distinct real solutions**.
5. Use the quadratic formula to solve:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-15) \pm \sqrt{(-15)^2 - 4(3)(-6)}}{2(3)} = \frac{15 \pm \sqrt{297}}{6}$$

$$x_1 = \frac{15 - \sqrt{297}}{6} \dots x_2 = \frac{15 + \sqrt{297}}{6}$$

*NOTE:* Since the discriminant for this equation was not a perfect square, these solutions are irrational numbers because they cannot be expressed as fractions, decimals or integers.

## TUTORIAL 3.12: SOLVE BUSINESS APPLICATIONS INVOLVING QUADRATIC FUNCTIONS

There may be times when solving a quadratic equation will be necessary to solve real-world business applications. In most of these cases, using the quadratic formula will be the way to go...the only other decision will be whether to use “paper & pencil” or a spreadsheet to complete the computation.

**Empowering People. Improving Business.**

BI Norwegian Business School is one of Europe's largest business schools welcoming more than 20,000 students. Our programmes provide a stimulating and multi-cultural learning environment with an international outlook ultimately providing students with professional skills to meet the increasing needs of businesses.

BI offers four different two-year, full-time Master of Science (MSc) programmes that are taught entirely in English and have been designed to provide professional skills to meet the increasing need of businesses. The MSc programmes provide a stimulating and multi-cultural learning environment to give you the best platform to launch into your career.

- MSc in Business
- MSc in Financial Economics
- MSc in Strategic Marketing Management
- MSc in Leadership and Organisational Psychology

[www.bi.edu/master](http://www.bi.edu/master)

**BI NORWEGIAN BUSINESS SCHOOL**

EFMD **EQUIS** ACCREDITED



**example 3.12a** Suppose the profit from the sale of  $x$  units of a product is:

$$P = 90x - 200 - x^2$$

What level(s) of production will yield a profit of \$1600?

**"paper & pencil" solution:**

1. Replace  $P$  with its given value ( $P = 1600$ ):  $1600 = 90x - 200 - x^2$
2. Set the equation to zero:  $0 = 90x - 200 - 1600 - x^2 \rightarrow 0 = -x^2 + 90x - 1800$
3. Identify the values of  $a$ ,  $b$  and  $c$ :  $a = -1$ ,  $b = 90$ ,  $c = -1800$
4. Use the quadratic formula to determine the solutions:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(90) \pm \sqrt{(90)^2 - 4(-1)(-1800)}}{2(-1)} = \\ &\quad \frac{-90 \pm \sqrt{8100 - 7200}}{-2} = \frac{-90 \pm \sqrt{900}}{-2} = \frac{-90 \pm 30}{-2} \rightarrow \\ x_1 &= \frac{-90 + 30}{-2} = \frac{-60}{-2} = 30 \text{ and } x_2 = \frac{-90 - 30}{-2} = \frac{-120}{-2} = 60 \end{aligned}$$

Based on these solutions, there will be two levels that will yield a profit of \$1600...when  $x = 30$  units or when  $x = 60$  units.

**spreadsheet solution:**

1. Replace  $P$  with its given value ( $P = 1600$ ):  $1600 = 90x - 200 - x^2$
2. Set the equation to zero:  $0 = 90x - 200 - 1600 - x^2 \rightarrow 0 = -x^2 + 90x - 1800$
3. Identify the values of  $a$ ,  $b$  and  $c$ :  $a = -1$ ,  $b = 90$ ,  $c = -1800$
4. Use a spreadsheet to aid in the calculation of the quadratic formula:

A	B	C	D	E	F
1 values:					
2	a	b	c		
	-1	90	-1800		
4 discriminant =	900				
	=C2^2-4*B2*D2				
7 solution #1:	30				
	=(-1*C2+SQRT(B4))/(2*B2)				
10 solution #2:	60				
	=(-1*C2-SQRT(B4))/(2*B2)				

- Enter the values for  $a$ ,  $b$  and  $c$  onto to spreadsheet. (These values have been entered into cells B2, C2 and D2 in the partial screen-shot.)
- In another cell of the spreadsheet, determine the value of the discriminant by typing:  $=C2^2-4*B2*D2$ . (This formula was entered into cell B4 in the partial screen-shot.)
- In yet another cell of the spreadsheet, determine the value of the first solution by typing:  $=(-1*C2+SQRT(B4))/(2*B2)$ . (This formula was entered into cell B7 in the partial screen-shot.)
- Finally, in yet another cell of the spreadsheet, determine the value of the first solution by typing:  $=(-1*C2-SQRT(B4))/(2*B2)$ . (This formula was entered into cell B10 in the partial screen-shot.)

The only difference between arriving at these solutions is that the spreadsheet can be saved to be used for another problem while the “paper & pencil” method must be repeated.

**example 3.12b Suppose the profit from the sale of  $x$  units of a product is:**

$$P = 16x - 100 - 0.1x^2$$

**What level(s) of production will yield a profit of \$500?**

**“paper & pencil” solution:**

1. Replace  $P$  with its given value ( $P = 500$ ):  $500 = 16x - 100 - 0.1x^2$
2. Set the equation to zero:  $0 = 16x - 100 - 500 - 0.1x^2 \rightarrow$
3.  $0 = -0.1x^2 + 16x - 600$
4. Identify the values of  $a$ ,  $b$  and  $c$ :  $a = -0.1$ ,  $b = 16$ ,  $c = -600$
5. Use the quadratic formula to determine the solutions:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(16) \pm \sqrt{(16)^2 - 4(-0.1)(-600)}}{2(-0.1)} = \\ &\quad \frac{-16 \pm \sqrt{256 - 240}}{-0.2} = \frac{-16 \pm \sqrt{16}}{-0.2} = \frac{-16 \pm 4}{-0.2} \rightarrow \end{aligned}$$

$$x_1 = \frac{-16 + 4}{-0.2} = \frac{-12}{-0.2} = \frac{120}{2} = 60 \text{ and } x_2 = \frac{-16 - 4}{-0.2} = \frac{-20}{-0.2} = \frac{200}{2} = 100$$

Based on these solutions, there will be two levels that will yield a profit of \$500...when  $x = 60$  units or when  $x = 100$  units.

***spreadsheet solution:***

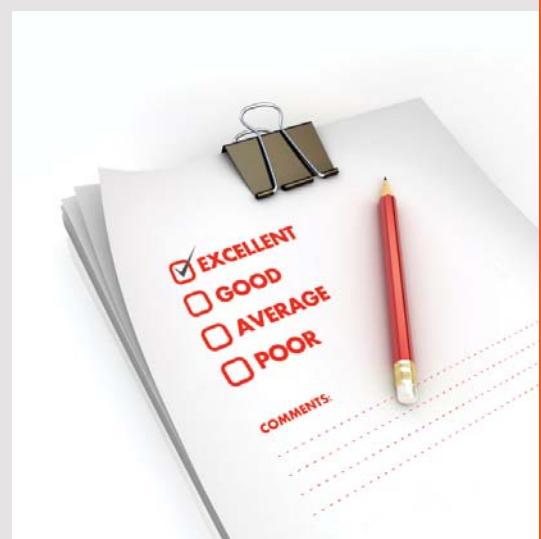
1. Replace  $P$  with its given value ( $P = 500$ ):  $500 = 16x - 100 - 0.1x^2$
2. Set the equation to zero:  $0 = 16x - 100 - 500 - 0.1x^2 \rightarrow 0 = -0.1x^2 + 16x - 600$
3. Identify the values of  $a$ ,  $b$  and  $c$ :  $a = -0.1$ ,  $b = 16$ ,  $c = -600$
4. Use a spreadsheet to aid in the calculation of the quadratic formula:

A	B	C	D	E	F
1    values:					
2	a    -0.1	b    16	c    -600		
3					
4    discriminant =	16				
5	=C2^2-4*B2*D2				
6					
7    solution #1:	60				
8	=(-1*C2+SQRT(B4))/(2*B2)				
9					
10    solution #2:	100				
11	=(-1*C2-SQRT(B4))/(2*B2)				
12					

## Need help with your dissertation?

Get in-depth feedback & advice from experts in your topic area. Find out what you can do to improve the quality of your dissertation!

**Get Help Now**



Go to [www.helpmyassignment.co.uk](http://www.helpmyassignment.co.uk) for more info

**Helpmyassignment**

- Enter the values for  $a$ ,  $b$  and  $c$  onto to spreadsheet. (These values have been entered into cells B2, C2 and D2 in the partial screen-shot.)
- In another cell of the spreadsheet, determine the value of the discriminant by typing:  $=C2^2-4*B2*D2$ . (This formula was entered into cell B4 in the partial screen-shot.)
- In yet another cell of the spreadsheet, determine the value of the first solution by typing:  $=(-1*C2+SQRT(B4))/(2*B2)$ . (This formula was entered into cell B7 in the partial screen-shot.)
- Finally, in yet another cell of the spreadsheet, determine the value of the first solution by typing:  $=(-1*C2-SQRT(B4))/(2*B2)$ . (This formula was entered into cell B10 in the partial screen-shot.)

NOTE: The appearance of the spreadsheet does not have to appear as it is shown for these examples. We can condense the appearance so that the information does not take up so much room

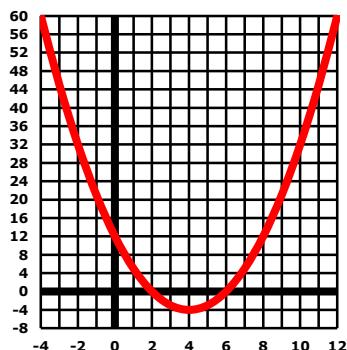
### TUTORIAL 3.13: FOR ANY QUADRATIC FUNCTION:

- Determine whether its graph will open up or down.
- Determine whether its vertex is a maximum or a minimum point.
- Determine its zeroes (horizontal intercepts).
- Define its axis of symmetry.
- Determine the coordinates of its vertex.
- Sketch its graph.

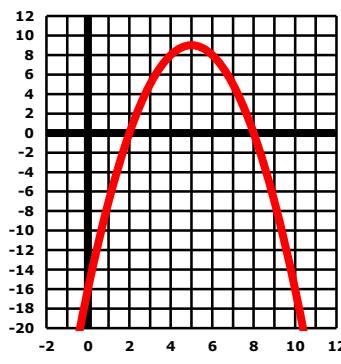
The graph of any quadratic function will be a **parabola** – a U-shaped curved that can open up or down – depending on the value of the coefficient ( $a$ ) of the quadratic ( $ax^2$ ) term:

If  $a > 0$ , the parabola will open upward.

If  $a < 0$ , the parabola will open downward.



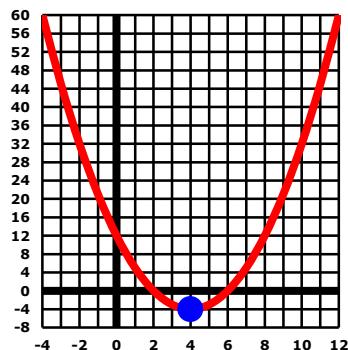
$$y = x^2 - 8x + 12 \rightarrow a = +1$$



$$y = -1x^2 + 10x - 16 \rightarrow a = -1$$

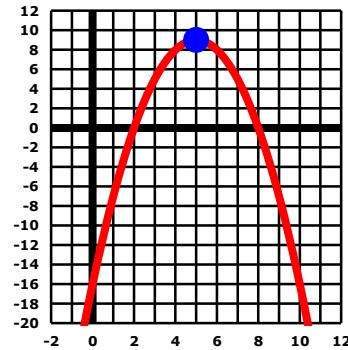
The point at which the curve changes direction is called the **vertex** of the parabola. This vertex will either be the optimal minimum point or the optimal maximum point of the quadratic function – depending on the value of the coefficient ( $a$ ) of the quadratic ( $ax^2$ ) term:

If  $a > 0$ , the vertex will be a minimum point. If  $a < 0$ , the vertex will be a maximum point.



$$y = x^2 - 8x + 12 \rightarrow a = +1$$

$$\text{vertex} = \text{blue point} = (4, -4)$$



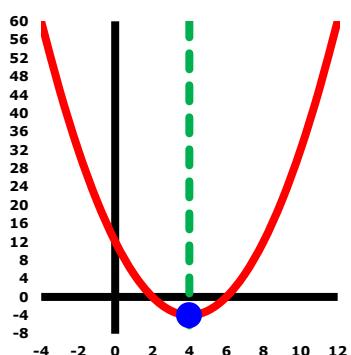
$$y = -1x^2 + 10x - 16 \rightarrow a = -1$$

$$\text{vertex} = \text{blue point} = (5, 9)$$

Another way to determine whether the vertex is the optimal minimum point or the optimal maximum point is to determine whether the parabola will open upward or downward:

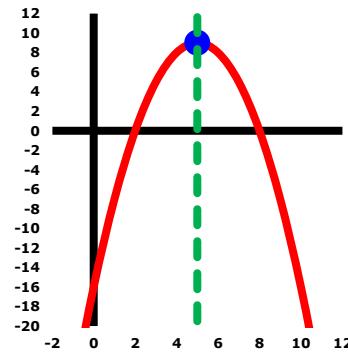
- If the parabola opens upward, then the vertex will be the optimal minimum point.
- If the parabola opens downward, then the vertex will be the optimal maximum point.

The parabola for any given quadratic function will be symmetrical to its **axis of symmetry** which is a vertical line that runs through the vertex. Other than the vertex, all of the other points on the parabola will be equidistant from its **axis of symmetry**.



$$y = x^2 - 8x + 12$$

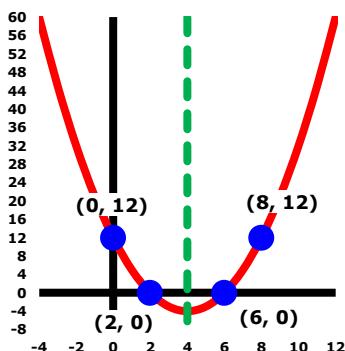
$$\text{axis of symmetry (green line): } x = 4$$



$$y = -1x^2 + 10x - 16$$

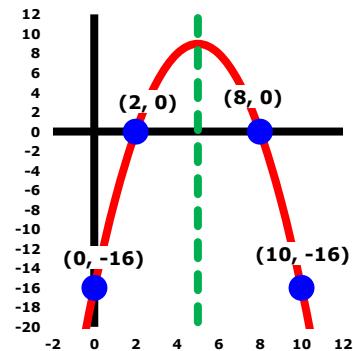
$$\text{axis of symmetry (green line): } x = 5$$

The point at which the parabola crosses the  $y$ -axis is the  $y$ -intercept. The points at which the parabola crosses the  $x$ -axis are the  $x$ -intercepts or the **zeroes** of the parabola.



$$y = x^2 - 8x + 12$$

$y$ -intercept =  $(0, y)$ :  $(0, 12)$   
 zeroes =  $(x, 0)$ :  $(2, 0)$  &  $(6, 0)$



$$y = -x^2 + 10x - 16$$

$y$ -intercept =  $(0, y)$ :  $(0, -16)$   
 zeroes =  $(x, 0)$ :  $(2, 0)$  &  $(8, 0)$

# Brain power



By 2020, wind could provide one-tenth of our planet's electricity needs. Already today, SKF's innovative know-how is crucial to running a large proportion of the world's wind turbines.

Up to 25 % of the generating costs relate to maintenance. These can be reduced dramatically thanks to our systems for on-line condition monitoring and automatic lubrication. We help make it more economical to create cleaner, cheaper energy out of thin air.

By sharing our experience, expertise, and creativity, industries can boost performance beyond expectations. Therefore we need the best employees who can meet this challenge!

The Power of Knowledge Engineering

Plug into The Power of Knowledge Engineering.  
Visit us at [www.skf.com/knowledge](http://www.skf.com/knowledge)

**SKF**

For each of the parabolas displayed above, notice that:

- The zeroes are equidistant (the same distance) from the axis of symmetry. For  $y = x^2 - 8x + 12$  (on the left), the distance between (2, 0) and (6, 0) is 4 units...with the axis of symmetry 2 units from each of these points. For  $y = -1x^2 + 10x - 16$  (on the right), the distance between (2, 0) and (8, 0) is 6 units...with the axis of symmetry 3 units from each of these points.
- The  $y$ -intercept has a “partner point” that is the same distance from the axis of symmetry as it is. For  $y = x^2 - 8x + 12$  (on the left), point (0, 12) is 4 units to the left of the axis of symmetry. Therefore, there is a “partner point” that is 4 units to the right of the axis of symmetry...namely, (8, 12). For  $y = -1x^2 + 10x - 16$  (on the right), point (0, -16) is 5 units to the left of the axis of symmetry. Therefore, there is a “partner point” that is 5 units to the right of the axis of symmetry...namely, (10, -16).

The **zeroes** of the parabola are nothing more than the solutions for the equation that was used to produce the parabola. Therefore, all we need to do is solve the given quadratic equation to determine the zeroes for the given function. Using the quadratic formula, we can determine a general formula to find the axis of symmetry of any given quadratic function. We can also use the quadratic formula to determine a general formula to find the vertex of any given quadratic function. Since the axis of symmetry “divides” the parabola into two equal parts, the zeroes of the parabola are equidistant from each other...with the axis of symmetry running between these points. Therefore, the  $x$ -value for the axis of symmetry is the average between the  $x$ -values of the zeroes.

By completing just two steps, we can generate a general formula to determine the axis of symmetry and vertex for any parabola:

$$\text{Suppose that the two solutions are: } x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

1. Find the average of these “solutions” by adding the two “values” together and then dividing the result by two:

$$\frac{\left( \frac{-b + \sqrt{b^2 - 4ac}}{2a} \right) + \left( \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right)}{2} = \frac{-2b}{2a} \div 2$$

2. Using division of fractions, simplify the expression:  $\frac{-2b}{2a} \div 2 = \frac{-2b}{2a} \times \frac{1}{2} = \frac{-2b}{4a} = \frac{-b}{2a}$

Based on this algebraic manipulation, we can use the  $a$ -value and the  $b$ -value from any quadratic function to determine that:

- The axis of symmetry for any quadratic function will be defined by:  $x = \frac{-b}{2a}$ .
- The vertex for any quadratic function will be defined by:  $\text{vertex } (h, k) = \left( \frac{-b}{2a}, F\left(\frac{-b}{2a}\right) \right)$ .

which instructs us to substitute the  $x$ -value derived for the axis of symmetry back into the given quadratic function to determine the  $y$ -value for the coordinates of the vertex.

- example 3.13a** Consider the graph of this quadratic function:  $y = 6 - 4x - 2x^2$
- Determine whether the graph will open upward or downward.
  - Determine the coordinates of the vertex.
  - Determine whether this vertex is an optimal minimum or maximum point.
  - Define the axis of symmetry for the graph of the given function.
  - Determine the zeroes of the given function.
  - Determine the  $y$ -intercept of the given function.
  - Determine the coordinates of the “partner point” for this  $y$ -intercept.
  - Sketch the parabola for the given function.

a)  $y = -2x^2 - 4x + 6 \rightarrow a = -2 \rightarrow$  parabola will open **downward**

b)  $a = -2, b = -4 \rightarrow h = \frac{-b}{2a} = \frac{-(-4)}{2(-2)} = \frac{4}{-4} = -1 \rightarrow$

$$k = 6 - 4x - 2x^2 = 6 - 4(-1) - 2(-1)^2 = 6 + 4 - 2 = 8 \rightarrow$$

**vertex  $(h, k) = (-1, 8)$**

c)  $a = -2 \rightarrow$  parabola will open down  $\rightarrow$  vertex = **maximum point**

d)  $a = -2, b = -4 \rightarrow h = \frac{-b}{2a} = \frac{-(-4)}{2(-2)} = \frac{4}{-4} = -1 \rightarrow$

**axis of symmetry:  $x = -1$**

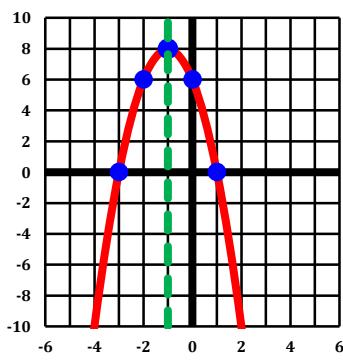
e)  $a = -2, b = -4, c = 6 \rightarrow$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(-2)(6)}}{2(-2)} = \frac{4 \pm \sqrt{16 + 48}}{-4} =$$

$$\frac{4 \pm \sqrt{64}}{-4} = \frac{4 \pm 8}{-4} \rightarrow x_1 = \frac{4+8}{-4} = \frac{12}{-4} = -3 \text{ and } x_2 = \frac{4-8}{-4} = \frac{-4}{-4} = 1$$

**zeroes:  $(-3, 0)$  and  $(1, 0)$**

- f) The  $y$ -intercept is when  $x = 0 \rightarrow y = 6 - 4(0) - 2(0)^2 = 6 \rightarrow \textcolor{red}{(0, 6)}$
- g) Since the  $y$ -intercept  $(0, 6)$  is one (1) unit to the right of the axis of symmetry, its “partner point” will be one (1) unit to the left of the axis of symmetry. Therefore the coordinates of the desired “partner point” is:  $\textcolor{red}{(-2, 6)}$ .
- h) Using all of the derived information from parts A-G:



- Plot the vertex:  $(-1, 8)$
- Plot the zeroes:  $(-3, 0), (1, 0)$
- Plot the  $y$ -intercept:  $(0, 6)$
- Plot the “partner point”:  $(-2, 6)$
- Draw in the axis of symmetry:  $x = -1$
- Sketch a downward parabola curve that passes through all of the mentioned points.

## TURN TO THE EXPERTS FOR SUBSCRIPTION CONSULTANCY

Subscrybe is one of the leading companies in Europe when it comes to innovation and business development within subscription businesses.

We innovate new subscription business models or improve existing ones. We do business reviews of existing subscription businesses and we develop acquisition and retention strategies.

Learn more at [linkedin.com/company/subscrybe/](https://www.linkedin.com/company/subscrybe/) or contact Managing Director Morten Suhr Hansen at [mha@subscrybe.dk](mailto:mha@subscrybe.dk)

**SUBSCRYBE** - to the future

**example 3.13b**

**Consider the graph of this quadratic function:  $y = x^2 - 2x - 8$**

- Determine whether the graph will open upward or downward.
- Determine the coordinates of the vertex.
- Determine whether this vertex is an optimal minimum or maximum point.
- Define the axis of symmetry for the graph of the given function.
- Determine the zeroes of the given function.
- Determine the  $y$ -intercept of the given function.
- Determine the coordinates of the “partner point” for this  $y$ -intercept.
- Sketch the parabola for the given function.

a)  $y = x^2 - 2x - 8 \rightarrow a = +1 \rightarrow$  parabola will open **upward**

b)  $a = 1, b = -2 \rightarrow h = \frac{-b}{2a} = \frac{-(-2)}{2(1)} = \frac{2}{2} = 1 \rightarrow$   
 $k = x^2 + 2x - 8 = (1)^2 - 2(1) - 8 = 1 - 2 - 8 = -9 \rightarrow$   
**vertex ( $b, k$ ) = (1, -9)**

c)  $a = +1 \rightarrow$  parabola will open up  $\rightarrow$  vertex = **minimum** point

d)  $a = 1, b = -2 \rightarrow h = \frac{-b}{2a} = \frac{-(-2)}{2(1)} = \frac{2}{2} = 1 \rightarrow$  **axis of symmetry:  $x = 1$**

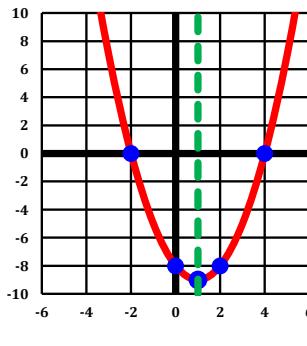
e)  $a = 1, b = -2, c = -8 \rightarrow$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-8)}}{2(1)} = \frac{2 \pm \sqrt{4+32}}{2} =$$

$$\frac{2 \pm \sqrt{36}}{2} = \frac{2 \pm 6}{2} \rightarrow x_1 = \frac{2-6}{2} = \frac{-4}{2} = -2 \text{ and } x_2 = \frac{2+6}{2} = \frac{8}{2} = 4$$

**zeroes: (-2, 0) and (4, 0)**

- The  $y$ -intercept is when  $x = 0 \rightarrow y = (0)^2 - 2(0) - 8 = -8 \rightarrow$  **(0, -8)**
- Since the  $y$ -intercept (0, -8) is one (1) unit to the left of the axis of symmetry, its “partner point” will be one (1) unit to the right of the axis of symmetry. Therefore the coordinates of the desired “partner point” is: **(2, -8)**.
- Using all of the derived information from parts A-G:



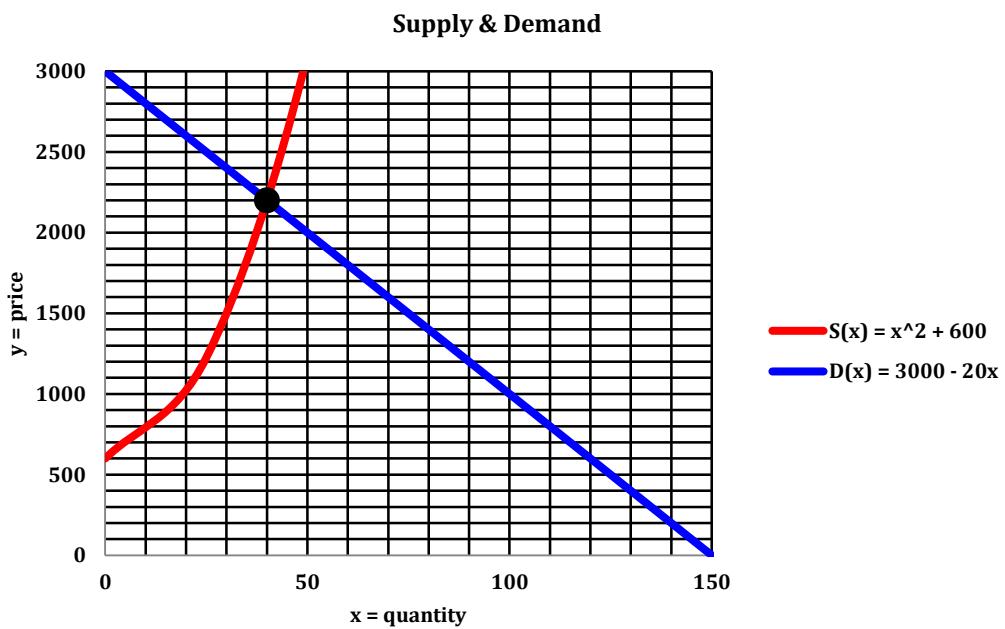
- Plot the vertex:  $(1, -9)$
- Plot the zeroes:  $(4, 0), (-2, 0)$
- Plot the  $y$ -intercept:  $(0, -8)$
- Plot the “partner point”:  $(2, -8)$
- Draw in the axis of symmetry:  $x = 1$
- Sketch an upward parabola curve that passes through all of the mentioned points.

## TUTORIAL 3.14: USE QUADRATIC FUNCTIONS TO FIND MARKET EQUILIBRIUM

Market equilibrium is when the supply function equals the demand function:  $S(x) = D(x)$ . Recall that when we first worked with this concept, we completed a system of equations. We will do the same thing even though we may be working with one or two quadratic functions.

**example 3.14a**      The supply function for a commodity is:  $S(x) = x^2 + 600$   
 The demand function for this same commodity is:  $D(x) = 3000 - 20x$

- a) Use the graph below to find the point of market equilibrium for this commodity.
- b) Use algebraic means to find the point of market equilibrium for this commodity.



a) Market equilibrium occurs when  $S(x) = D(x)$ . As displayed by the graph, this occurs at point (40, 2200)...which means when 40 units of the commodity are sold at \$2200 per unit.

- b)  $S(x) = D(x) \rightarrow x^2 + 600 = 3000 - 20x$
- Set the equation to zero:  $x^2 + 20x - 2400 = 0$
  - Identify the values of  $a$ ,  $b$  and  $c$ :  $a = 1$ ,  $b = 20$ ,  $c = -2400$
  - Use the quadratic formula to solve:

$$x = \frac{-(20) \pm \sqrt{(20)^2 - 4(1)(-2400)}}{2(1)} = \frac{-20 \pm \sqrt{400 + 9600}}{2} =$$

$$\frac{-20 \pm \sqrt{10,000}}{2} = \frac{-20 \pm 100}{2} \rightarrow$$

$$x_1 = \frac{-20 - 100}{2} = \frac{-120}{2} = -60 \dots x_2 = \frac{-20 + 100}{2} = \frac{80}{2} = 40$$

**What do you want to do?**

No matter what you want out of your future career, an employer with a broad range of operations in a load of countries will always be the ticket. Working within the Volvo Group means more than 100,000 friends and colleagues in more than 185 countries all over the world. We offer graduates great career opportunities – check out the Career section at our web site [www.volvologroup.com](http://www.volvologroup.com). We look forward to getting to know you!

**VOLVO**

AB Volvo (publ)  
[www.volvologroup.com](http://www.volvologroup.com)

VOLVO TRUCKS | RENAULT TRUCKS | MACK TRUCKS | VOLVO BUSES | VOLVO CONSTRUCTION EQUIPMENT | VOLVO PENTA | VOLVO AERO | VOLVO IT  
VOLVO FINANCIAL SERVICES | VOLVO 3P | VOLVO POWERTRAIN | VOLVO PARTS | VOLVO TECHNOLOGY | VOLVO LOGISTICS | BUSINESS AREA ASIA

*NOTE: Since this is a business application, negatives cannot be used. Therefore, the -60 is an extraneous solution and must be thrown out. That means that +40 is the only possible value for x.*

- c) Using  $x = 40$ , evaluate  $S(x)$  or  $D(x)$ :

$$S(x) = x^2 + 600 = (40)^2 + 600 = 1600 + 600 = 2200$$

$$D(x) = 3000 - 20x = 3000 - 20(40) = 3000 - 800 = 2200$$

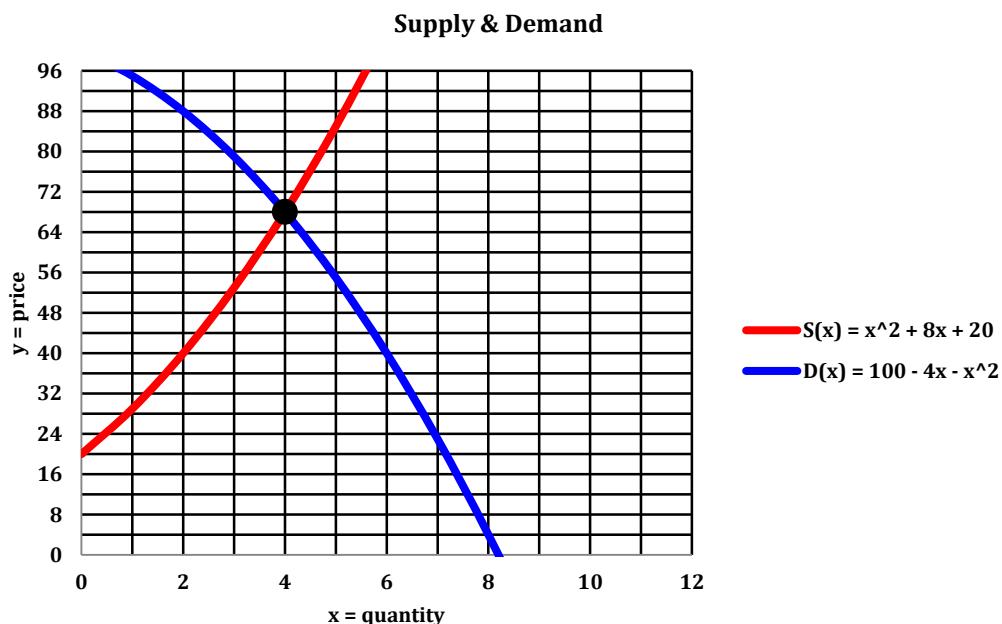
- d) Use the derived values to determine the point of market equilibrium: **(40, 2200)**

**example 3.14b**

The supply function for a commodity is:  $S(x) = x^2 + 8x + 20$

The demand function for this same commodity is:  $D(x) = 100 - 4x - x^2$

- a) Use the graph below to find the point of market equilibrium for this commodity.  
 b) Use algebraic means to find the point of market equilibrium for this commodity.



- a) Market equilibrium occurs when  $S(x) = D(x)$ . As displayed by the graph, this occurs at point **(4, 68)**...which means when 4 units of the commodity are sold at \$68 per unit.
- b)  $S(x) = D(x) \rightarrow x^2 + 8x + 20 = 100 - 4x - x^2$
- Set the equation to zero:  $2x^2 + 12x - 80 = 0$
  - Identify the values of  $a$ ,  $b$  and  $c$ :  $a = 2$ ,  $b = 12$ ,  $c = -80$
  - Use the quadratic formula to solve:

$$x = \frac{-(12) \pm \sqrt{(12)^2 - 4(2)(-80)}}{2(2)} = \frac{-12 \pm \sqrt{144 + 640}}{4} =$$

$$\frac{-12 \pm \sqrt{784}}{4} = \frac{-12 \pm 28}{4} \rightarrow$$

$$x_1 = \frac{-12 - 28}{4} = \frac{-40}{4} = -10 \dots x_2 = \frac{-12 + 28}{4} = \frac{16}{4} = 4$$

*NOTE: Since this is a business application, negatives cannot be used. Therefore, the -10 is an extraneous solution and must be thrown out. That means that +4 is the only possible value for x.*

- Using  $x = 4$ , evaluate  $S(x)$  or  $D(x)$ :
- $$S(x) = x^2 + 8x + 20 = (4)^2 + 8(4) + 20 = 16 + 32 + 20 = 68$$
- $$D(x) = 100 - 4x - x^2 = 100 - 4(4) - (4)^2 = 100 - 16 - 16 = 68$$
- Use the derived values to determine the point of market equilibrium: **(4, 68)**

**example 3.14c**

**Use algebraic means to find the point of market equilibrium:**

The supply function for a commodity is:  $S(x) = x^2 + 8x + 16$

The demand function for this same commodity is:  $D(x) = 436 + 6x - 3x^2$

$$S(x) = D(x) \rightarrow x^2 + 8x + 16 = 436 + 6x - 3x^2$$

- Set the equation to zero:  $4x^2 + 2x - 420 = 0$
- Identify the values of  $a$ ,  $b$  and  $c$ :  $a = 4$ ,  $b = 2$ ,  $c = -420$
- Use the quadratic formula to solve:

$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(4)(-420)}}{2(4)} = \frac{-2 \pm \sqrt{4 + 6720}}{8} =$$

$$\frac{-2 \pm \sqrt{6724}}{8} = \frac{-2 \pm 82}{8} \rightarrow$$

$$x_1 = \frac{-2 - 82}{8} = \frac{-84}{8} = -10.5 \dots x_2 = \frac{-2 + 82}{8} = \frac{80}{8} = 10$$

**Challenge the way we run**

**EXPERIENCE THE POWER OF FULL ENGAGEMENT...**

**RUN FASTER.  
RUN LONGER..  
RUN EASIER...**

**READ MORE & PRE-ORDER TODAY  
[WWW.GAITEYE.COM](http://WWW.GAITEYE.COM)**

*NOTE: Since this is a business application, negatives cannot be used. Therefore, the  $-10.5$  is an extraneous solution and must be thrown out. That means that  $+10$  is the only possible value for  $x$ .*

- Using  $x = 10$ , evaluate  $S(x)$  or  $D(x)$ :

$$S(x) = x^2 + 8x + 16 = (10)^2 + 8(10) + 16 = 100 + 80 + 16 = 196$$

$$D(x) = 436 + 6x - 3x^2 = 436 + 6(10) - 3(10)^2 = 436 + 60 - 300 = 196$$

- Use the derived values to determine the point of market equilibrium: **(10, 196)**...which means that 10 units of the commodity are sold at \$196 per unit.

## TUTORIAL 3.15: USE QUADRATIC FUNCTIONS TO FIND BREAK-EVEN POINT(S)

The break-even points occur when the cost function equals the revenue function:  $C(x) = R(x)$ . Recall that when we first worked with this concept, we completed a system of equations. We will do the same thing even though we may be working with one or two quadratic functions.

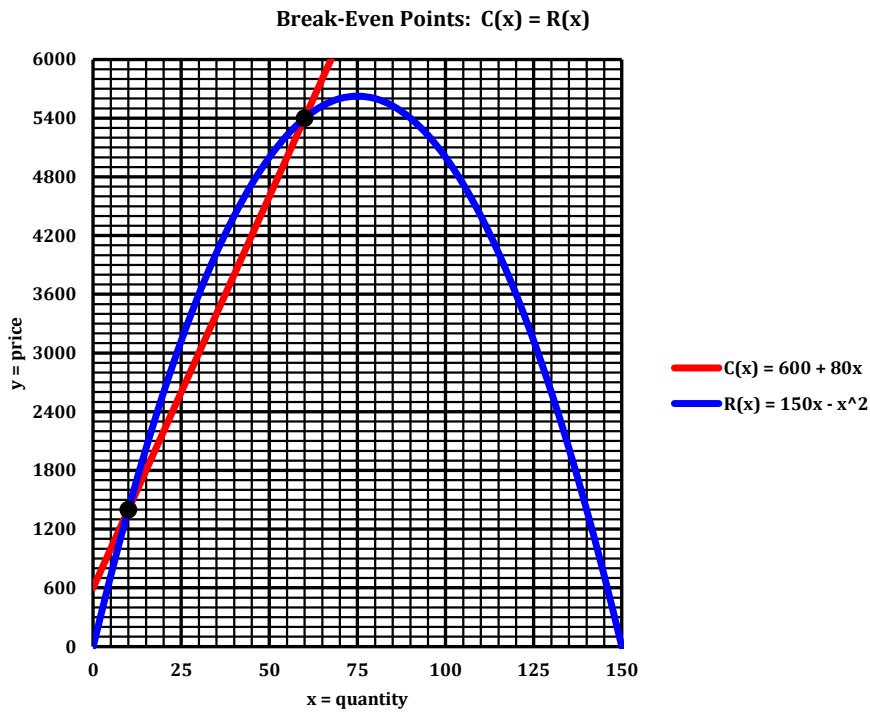
**example 3.15a**      The total cost per week of producing a certain product is:

$$C(x) = 600 + 80x$$

The total revenue per week for this same product is:

$$R(x) = 150x - x^2$$

- a) Use the graph to find the break-even points for this commodity.
- b) Use algebraic means to find the break-even points for this commodity.
- c) Write the profit function for this commodity.
- d) What are the zeroes of this profit function?
- e) Compare the zeroes of the profit function with the break-even points (established in parts A–B) for this commodity.



- a) The break-even points occur when:  $C(x) = R(x)$ . As displayed by the graph, there are two break-even points: **(10, 1400)** and **(60, 5400)**. *NOTE: The area between these two points is where any possible profits will occur.*

- b)  $C(x) = R(x) \rightarrow 600 + 80x = 150x - x^2$
- Set the equation to zero:  $x^2 - 70x + 600 = 0$
  - Identify the values of  $a$ ,  $b$  and  $c$ :  $a = 1$ ,  $b = -70$ ,  $c = 600$
  - Use the quadratic formula to solve:

$$x = \frac{-(-70) \pm \sqrt{(-70)^2 - 4(1)(600)}}{2(1)} = \frac{70 \pm \sqrt{4900 - 2400}}{2} = \frac{70 \pm \sqrt{2500}}{2} = \frac{70 \pm 50}{2} \rightarrow$$

$$x_1 = \frac{70 - 50}{2} = \frac{20}{2} = 10 \dots x_2 = \frac{70 + 50}{2} = \frac{120}{2} = 60$$

- Since both  $x$ -values are positive, this application has two break-even points:

Using  $x = 10$ , evaluate  $C(x)$  or  $R(x)$ :

$$C(x) = 600 + 80x = 600 + 80(10) = 600 + 800 = 1400$$

$$R(x) = 150x - x^2 = 150(10) - (10)^2 = 1500 - 100 = 1400$$

when  $x = 10 \rightarrow$  break-even = **(10, 1400)**

Using  $x = 60$ , evaluate  $C(x)$  or  $R(x)$ :

$$C(x) = 600 + 80x = 600 + 80(60) = 600 + 4800 = 5400$$

$$R(x) = 150x - x^2 = 150(60) - (60)^2 = 9000 - 3600 = 5400$$

when  $x = 60 \rightarrow$  break-even = **(60, 5400)**

c)  $P(x) = R(x) - C(x) \rightarrow P(x) = 150x - x^2 - (600 + 80x) = 150x - x^2 - 600 - 80x = 70x - x^2 - 600 \rightarrow P(x) = 70x - x^2 - 600$

d)  $P(x) = 70x - x^2 - 600 \rightarrow$

- Set the equation to zero:  $-x^2 + 70x - 600 = 0$
- Identify the values of  $a$ ,  $b$  and  $c$ :  $a = -1$ ,  $b = 70$ ,  $c = -600$
- Use the quadratic formula to solve:

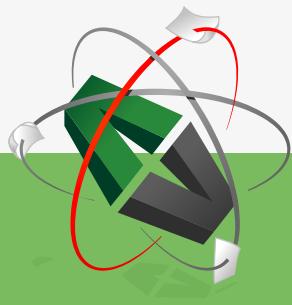
$$x = \frac{-(70) \pm \sqrt{(70)^2 - 4(-1)(-600)}}{2(-1)} = \frac{-70 \pm \sqrt{4900 - 2400}}{-2} = \frac{-70 \pm \sqrt{2500}}{-2} = \frac{-70 \pm 50}{-2}$$

$$x_1 = \frac{-70 + 50}{-2} = \frac{-20}{-2} = 10 \dots x_2 = \frac{-70 - 50}{-2} = \frac{-120}{-2} = 60$$

- Since both  $x$ -values are positive, this profit function for this commodity has two zeroes: **(10, 0) and (60, 0)**

e) **The zeroes of the profit function are the  $x$ -values of the break-even points.**

This e-book  
is made with  
**SetaPDF**



**PDF components for PHP developers**

**[www.setasign.com](http://www.setasign.com)**



**Click on the ad to read more**

**example 3.15b**    The total cost per week of producing a certain product is:

$$C(x) = 1600 + 1500x$$

The total revenue per week for this same product is:

$$R(x) = 1600x - x^2$$

- a) Determine the break-even points for this commodity.
- b) Write the profit function for this commodity.
- c) What are the zeroes of this profit function?
- d) Compare the zeroes of the profit function with the break-even points (established in parts A-B) for this commodity.

- a) break-even:  $C(x) = R(x) \rightarrow 1600 + 1500x = 1600x - x^2$
- Set the equation to zero:  $x^2 - 100x + 1600 = 0$
  - Identify the values of  $a$ ,  $b$  and  $c$ :  $a = 1$ ,  $b = -100$ ,  $c = 1600$
  - Use the quadratic formula to solve:

$$x = \frac{-(-100) \pm \sqrt{(-100)^2 - 4(1)(1600)}}{2(1)} = \frac{100 \pm \sqrt{10,000 - 6400}}{2} = \frac{100 \pm \sqrt{3600}}{2} = \frac{100 \pm 60}{2} \rightarrow$$

$$x_1 = \frac{100 - 60}{2} = \frac{40}{2} = 20 \dots x_2 = \frac{100 + 60}{2} = \frac{160}{2} = 80$$

- Since both  $x$ -values are positive, this application has two break-even points:

Using  $x = 20$ , evaluate  $C(x)$  or  $R(x)$ :

$$C(x) = 1600 + 1500x = 1600 + 1500(20) = 31,600$$

$$R(x) = 1600x - x^2 = 1600(20) - (20)^2 = 31,600$$

Using  $x = 80$ , evaluate  $C(x)$  or  $R(x)$ :

$$C(x) = 1600 + 1500x = 1600 + 1500(80) = 121,600$$

$$R(x) = 1600x - x^2 = 1600(80) - (80)^2 = 121,600$$

- break-even points: **(20, 31600) and (80, 121600)**

- b)  $P(x) = R(x) - C(x) \rightarrow P(x) = 1600x - x^2 - (1600 + 1500x) \rightarrow$
- $$\color{red}P(x) = 100x - x^2 - 1600$$

- c)  $P(x) = 100x - x^2 - 1600$

- Set the equation to zero:  $-x^2 + 100x - 1600 = 0$
- Identify the values of  $a$ ,  $b$  and  $c$ :  $a = -1$ ,  $b = 100$ ,  $c = -1600$

- Use the quadratic formula to solve:

$$x = \frac{-(100) \pm \sqrt{(100)^2 - 4(-1)(-1600)}}{2(-1)} = \frac{-100 \pm \sqrt{10,000 - 6400}}{-2} =$$

$$\frac{-100 \pm \sqrt{3600}}{-2} = \frac{-100 \pm 60}{-2} \rightarrow$$

$$x_1 = \frac{100 - 60}{2} = \frac{40}{2} = 20 \dots x_2 = \frac{100 + 60}{2} = \frac{160}{2} = 80$$

- Since both  $x$ -values are positive, this profit function for this commodity has two zeroes: **(20, 0) and (80, 0)**
- d) **The zeroes of the profit function are the  $x$ -values of the break-even points.**

**example 3.15c****Determine the break-even points for this commodity:**

The total cost per week of producing a certain product is:

$$C(x) = 3600 + 25x + 0.5x^2$$

The total revenue per week for this same product is:

$$R(x) = 175x - 0.5x^2$$

$$\text{break-even: } C(x) = R(x) \rightarrow 3600 + 25x + 0.5x^2 = 175x - 0.5x^2$$

- Set the equation to zero:  $x^2 - 150x + 3600 = 0$
- Identify the values of  $a$ ,  $b$  and  $c$ :  $a = 1$ ,  $b = -150$ ,  $c = 3600$
- Use the quadratic formula to solve:

$$x = \frac{-(-150) \pm \sqrt{(-150)^2 - 4(1)(3600)}}{2(1)} = \frac{150 \pm \sqrt{22,500 - 14,400}}{2} =$$

$$\frac{150 \pm \sqrt{8100}}{2} = \frac{150 \pm 90}{2} \rightarrow$$

$$x_1 = \frac{150 - 90}{2} = \frac{60}{2} = 30 \text{ and } x_2 = \frac{150 + 90}{2} = \frac{240}{2} = 120$$

- both  $x$ -values are positive  $\rightarrow$  two break-even points:

Using  $x = 30$ , evaluate  $C(x)$  or  $R(x)$ :

$$C(x) = 3600 + 25x + 0.5x^2 = 3600 + 25(30) + 0.5(30)^2 = 4800$$

$$R(x) = 175x - 0.5x^2 = 175(30) - 0.5(30)^2 = 4800$$

Using  $x = 120$ , evaluate  $C(x)$  or  $R(x)$ :

$$C(x) = 3600 + 25x + 0.5x^2 = 3600 + 25(120) + 0.5(120)^2 = 13,800$$

$$R(x) = 175x - 0.5x^2 = 175(120) - 0.5(120)^2 = 13,800 \text{ break-even: } \mathbf{(30, 4800) \text{ and } (120, 13800)}$$

## TUTORIAL 3.16: MAXIMIZE QUADRATIC REVENUE AND/OR PROFIT FUNCTIONS. MINIMIZE QUADRATIC COST FUNCTIONS

Businesses will often be interested in maximizing their revenue and/or profit while attempting to minimize any business costs. When we worked with linear functions, we determined the minimum or maximum values through linear programming. However, when working with quadratic functions, we will need to work with the vertex of the appropriate parabolas for the given functions.

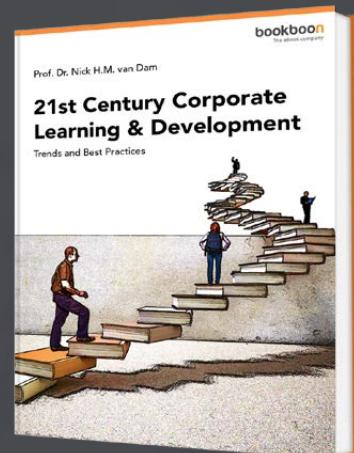
Recall that:

- The vertex of a parabola will be a minimum point when  $a > 0$  or the parabola opens upward.
- The vertex of a parabola will be a maximum point when  $a < 0$  or the parabola opens downward.
- The coordinates of the vertex of a parabola can be determined by:  $\left(\frac{-b}{2a}, F\left(\frac{-b}{2a}\right)\right)$ .

## Free eBook on Learning & Development

By the Chief Learning Officer of McKinsey

**Download Now**



Download free eBooks at [bookboon.com](http://bookboon.com)



**Click on the ad to read more**

**example 3.16a** The daily profit from the sale of  $x$  units of a product is:

$$P(x) = 80x - 0.4x^2 - 500$$

- a) What level of production will maximize profit?
- b) What is the maximum profit for this product?

$a = -0.4 \rightarrow$  parabola will open downward  $\rightarrow$  vertex = optimal maximum point  $\rightarrow$  vertex =  $(h, k) = \left(\frac{-b}{2a}, F\left(\frac{-b}{2a}\right)\right)$ . = (production level, profit)

- a) The production level that will maximize profit will be the  $x$ -coordinate of its vertex:

$$a = -0.4, b = 80 \rightarrow h = \frac{-b}{2a} = \frac{-80}{2(-0.4)} = \frac{-80}{-0.8} = \mathbf{100 \text{ units}}$$

- b) The maximum profit for this product will be the  $y$ -coordinate of the vertex:  $P(x) = 60x - x^2 - 500 \rightarrow$

$$k = 80(100) - 0.4(100)^2 - 500 = 8000 - 4000 - 500 = 3500 \rightarrow \mathbf{\$3500}$$

**example 3.16b** The yield in bushels from a grove of orange trees is given by:

$$F(x) = x(600 - x)$$

- a) How many trees will maximize the yield?
- b) What is the maximum yield for this grove?

$F(x) = x(600 - x) = 600x - x^2 \rightarrow a = -1 \rightarrow$  parabola will open downward

vertex = optimal maximum point

$$\text{vertex} = (h, k) = \left(\frac{-b}{2a}, F\left(\frac{-b}{2a}\right)\right) = (\text{number of trees, yield for grove})$$

- a) The number of trees that will maximize the yield will be the  $x$ -coordinate of its vertex:

$$a = -1, b = 600 \rightarrow h = \frac{-b}{2a} = \frac{-600}{2(-1)} = \frac{-600}{-2} = \mathbf{300 \text{ bushels}}$$

- b) The maximum yield for this grove will be the  $y$ -coordinate of the vertex:  $F(x) = 600x - x^2 \rightarrow$

$$k = 600(300) - (300)^2 = 180,000 - 90,000 = 90,000 \rightarrow \mathbf{\text{maximum yield} = 90,000 \text{ oranges}}$$

**example 3.16c** The daily production costs for  $x$  units of a commodity is:

$$C(x) = 3600 - 100x + 2x^2$$

- a) What level of production will minimize costs?
- b) What are the minimum daily production costs for this commodity?

$a = +2 \rightarrow$  parabola will open upward  $\rightarrow$  vertex = optimal minimum point  $\rightarrow$  vertex =  $(h, k) = \left( \frac{-b}{2a}, F\left(\frac{-b}{2a}\right) \right)$  = (production level, costs)

- a) The level that will minimize production costs will be the  $x$ -coordinate of its vertex:

$$a = 2, b = -100 \rightarrow h = \frac{-b}{2a} = \frac{-(-100)}{2(2)} = \frac{100}{4} = 25 \text{ units}$$

- b) The minimum costs for this commodity will be the  $y$ -coordinate of the vertex:  $C(x) = 3600 - 100x + 2x^2 \rightarrow$

$$k = 3600 - 100(25) + 2(25)^2 = 2350 \rightarrow \$2350$$

**example 3.16d** The daily production costs for  $x$  units of a commodity is:

$$C(x) = 15,000 - 35x + 0.1x^2$$

- a) What level of production will minimize costs?
- b) What are the minimum daily production costs for this commodity?

$a = +0.1 \rightarrow$  parabola will open upward  $\rightarrow$  vertex = optimal minimum point  $\rightarrow$  vertex =  $(h, k) = \left( \frac{-b}{2a}, F\left(\frac{-b}{2a}\right) \right)$  = (production level, costs)

- a) The level that will minimize production costs will be the  $x$ -coordinate of its vertex:

$$a = 0.1, b = -35 \rightarrow h = \frac{-b}{2a} = \frac{-(-35)}{2(0.1)} = \frac{35}{0.2} = \frac{350}{2} = 175 \text{ units}$$

- b) The minimum costs for this commodity will be the  $y$ -coordinate of the vertex:  $C(x) = 15,000 - 35x + 0.1x^2 \rightarrow$

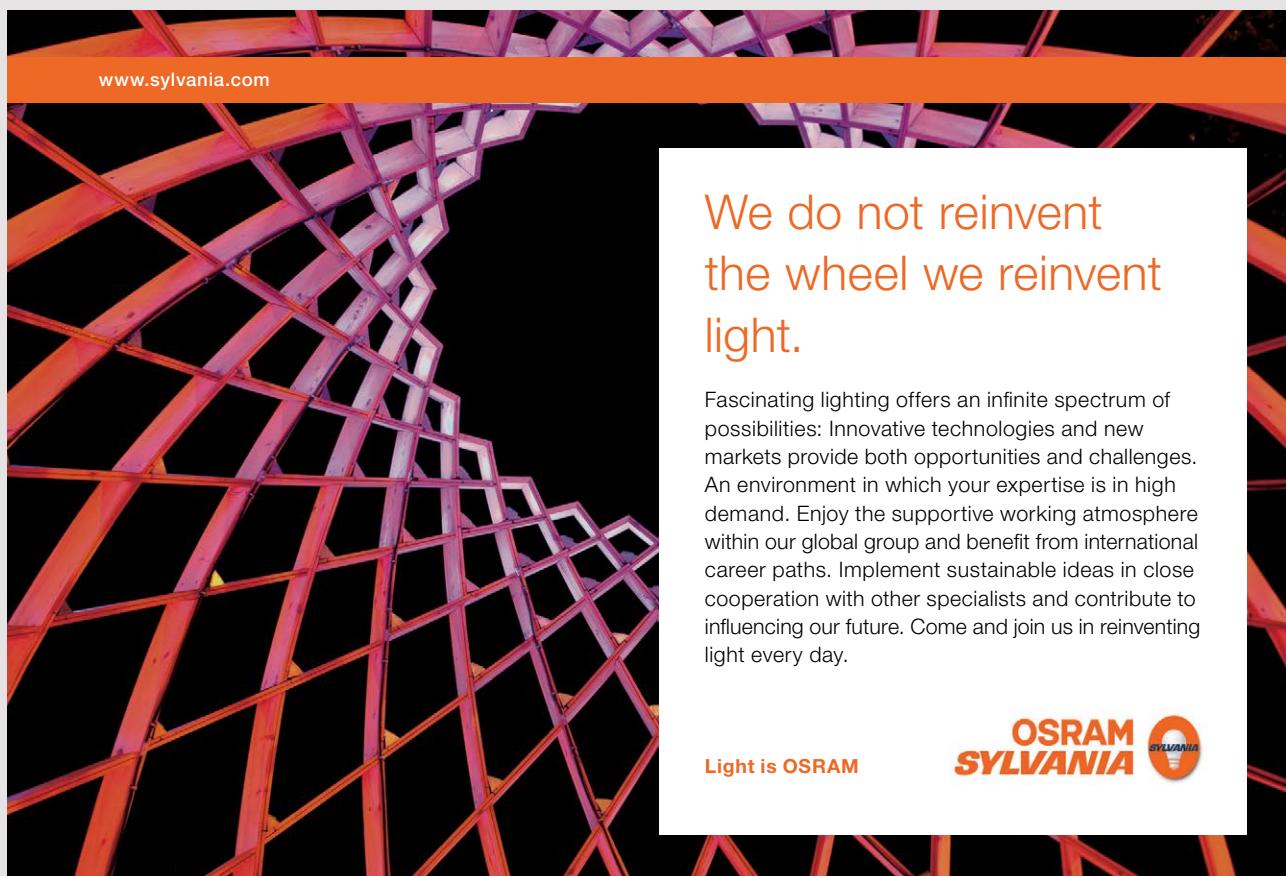
$$k = 15,000 - 35(175) + 0.1(175)^2 = 11,938 \rightarrow \$11,938$$

## TUTORIAL 3.17: USE QUADRATIC FUNCTIONS TO SOLVE REAL-WORLD APPLICATIONS

Although it may not be initially obvious, we can use the concept of quadratics and maximization to solve other real-world applications. As long as the situation can be described by a quadratic function, we can use the vertex coordinates to help answer many maximization problems.

**example 3.17a** Suppose that tickets for a high school concert cost \$4 each and that 800 people are expected to attend the concert. However, for each 25¢ increase in the ticket price attendance will decrease by 20 people.

- What quadratic function is appropriate for this situation?
- What ticket price will yield the maximum income?
- How many people should attend to achieve this maximum income?
- What is the maximum income for this concert?



www.sylvania.com

We do not reinvent the wheel we reinvent light.

Fascinating lighting offers an infinite spectrum of possibilities: Innovative technologies and new markets provide both opportunities and challenges. An environment in which your expertise is in high demand. Enjoy the supportive working atmosphere within our global group and benefit from international career paths. Implement sustainable ideas in close cooperation with other specialists and contribute to influencing our future. Come and join us in reinventing light every day.

Light is OSRAM

**OSRAM**  SYLVANIA

a)  $F(x) = (\text{ticket price})(\text{people}) = (4 + 0.25x)(800 - 20x) = 3200 + 120x - 5x^2 \rightarrow F(x) = 3200 + 120x - 5x^2$

b) vertex =  $(h, k) = \left( \frac{-b}{2a}, F\left(\frac{-b}{2a}\right) \right) \rightarrow a = -5, b = 120 \rightarrow h = \frac{-b}{2a} = \frac{-120}{2(-5)} = \frac{-120}{-10} = 12$

$$x = 12 \rightarrow \text{ticket price} = 4 + 0.25x = 4 + 0.25(12) = 4 + 3 = 7 \rightarrow \$7$$

c)  $x = 12 \rightarrow \text{people} = 800 - 20x = 800 - 20(12) = 800 - 240 = 560$

d)  $F(x) = 3200 + 120x - 5x^2 \rightarrow h = 12 \rightarrow k = 3200 + 120(12) - 5(12)^2 = 3200 + 1440 - 720 = \$3920$

**example 3.17b** Suppose that a chartered bus ride costs \$30 per ticket when there are 20 passengers. However, the bus company will reduce the ticket price by \$1 for each additional passenger greater than 20 passengers.

- a) What quadratic function is appropriate for this situation?
- b) What ticket price will yield the maximum income?
- c) How many people should attend to achieve this maximum income?
- d) What is the maximum income for this concert?

a)  $F(x) = (\text{ticket price})(\text{passengers}) = (30 - x)(20 + x) = 600 + 10x - x^2$

$$\rightarrow F(x) = 600 + 10x - x^2$$

b) vertex =  $(h, k) = \left( \frac{-b}{2a}, F\left(\frac{-b}{2a}\right) \right) \rightarrow a = -1, b = 10 \rightarrow h = \frac{-b}{2a} = \frac{-10}{2(-1)} = \frac{-10}{-2} = 5$

$$x = 5 \rightarrow \text{ticket price} = 30 - x = 30 - (5) = \$25 \text{ per ticket}$$

c)  $x = 5 \rightarrow \text{passengers} = 20 + x = 20 + (5) = 25 \text{ passengers}$

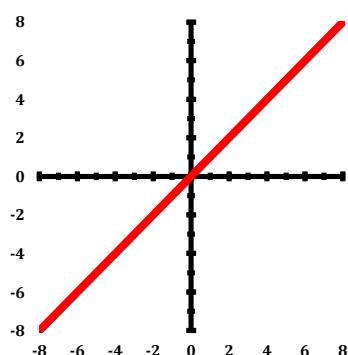
d)  $F(x) = 600 + 10x - x^2 \rightarrow h = 5 \rightarrow$

$$k = 600 + 10(5) - (5)^2 = 600 + 50 - 25 = \$625$$

## TUTORIAL 3.18: IDENTIFY THE GRAPHS OF BASIC FUNCTIONS

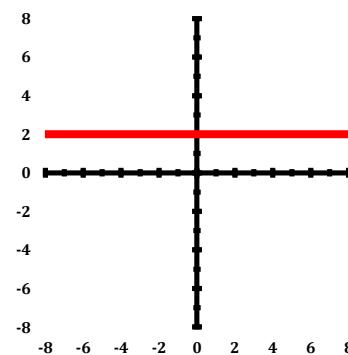
When graphed, a proper function will pass the “vertical line” test. If a vertical line is drawn on top of any graphed equation and the vertical line only intersects the graphed equation in exactly one point, then the graphed equation is a function. Another way to determine a “proper function” is to examine the ordered pairs for the given equation. If the  $x$ -values of these ordered pairs do not repeat and are paired with a unique  $y$ -value, then the given equation will be a function.

The first “basic” functions that we should examine are displayed below...that of an identity function (on the left)...or, a constant function (in the middle). Both of these functions will have equations that will produce straight lines as their graphs. These graphs will also pass the “vertical line” test as described above.



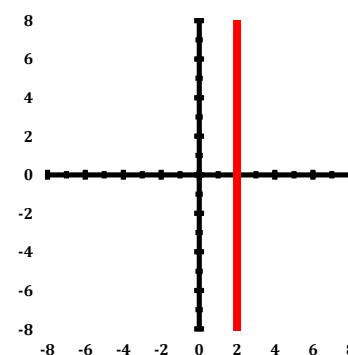
**Identity Function**

- basic function:  $F(x) = x$
- specific function:  $y = x$
- distinctive shape: slanted line
- slope: +1
- $y$ -intercept:  $(0, 0)$
- $x$ -intercept:  $(0, 0)$
- ordered pairs  $(x, y)$ :  $(x, x)$
- degree:  $n = 1$
- domain:  $x = \text{real numbers}$
- range:  $y = \text{real numbers}$



**Constant Function**

- basic function:  $F(x) = C$
- specific function:  $y = 2$
- distinctive shape: horizontal line slope: zero (0)
- $y$ -intercept:  $(0, C)$
- $x$ -intercept: *none*
- ordered pairs  $(x, y)$ :  $(x, C)$
- degree: *none*
- domain:  $x = \text{real numbers}$
- range:  $y = C$



**Not a Function**

- basic equation:  $x = C$
- specific function:  $x = 2$
- distinctive shape: vertical line
- slope: undefined
- $y$ -intercept: *none*
- $x$ -intercept:  $(C, 0)$
- ordered pairs  $(x, y)$ :  $(x, C)$
- degree: *none*
- domain:  $x = C$
- range:  $y = \text{real numbers}$

The **domain** of a function is the set of values that may be used as the “input” (normally, the  $x$ ) for the given function. We can determine the domain of a function by looking at how its graph “covers” the  $x$ -axis. If the graph indicates that positive, negative and zero  $x$ -values are used, then the domain will be the set of real numbers (as is the case for the identity or constant functions). If one of these value-types is not used, then the domain is restricted accordingly.

The **range** of a function is the set of values that are the “output” (normally, the  $y$ ) after a particular domain-value is substituted into the given function. We can determine the domain of a function by looking at how its graph “covers”  $y$ -axis. If the graph indicates that positive, negative and zero  $y$ -values have been achieved by the evaluation of the function, then the range will be the set of real numbers (as it is for the identity function). If one of these value-types is not used, then the range is restricted accordingly (as it is for the constant function).

The graph of a vertical line shown (on the right of the previous page) is being displayed so that we can compare it with the two linear functions. Any graphed vertical line is NOT a function...mainly because it will not pass the “vertical line” test...and, each of the  $x$ -values repeat and is paired with a different  $y$ -value.



**Deloitte.**

© Deloitte & Touche LLP and affiliated entities.

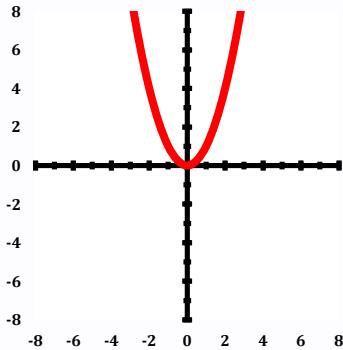
Discover the truth at [www.deloitte.ca/careers](http://www.deloitte.ca/careers)

Download free eBooks at [bookboon.com](http://bookboon.com)



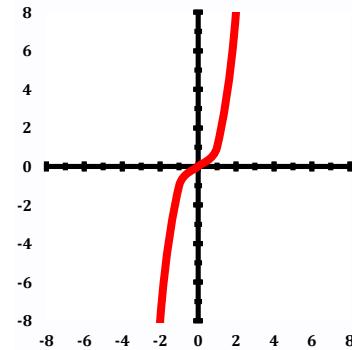
Click on the ad to read more

Another set of functions that we should examine are the ***power functions***. Notice that each of these graphs will pass the “vertical line” test and therefore each of the  $x$ -coordinates are paired with a unique  $y$ - coordinate. Each of these functions will produce a curved graph.



**Quadratic Function**

- basic function:  $F(x) = ax^2$
- specific function:  $y = x^2$
- distinctive shape: parabola curve
- degree:  $n = 2$
- domain:  $x = \text{real numbers}$
- range:  $y \geq 0$



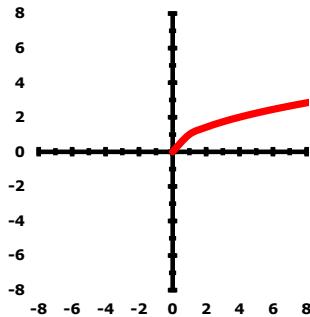
**Cubic Function**

- basic function:  $F(x) = ax^3$
- specific function:  $y = x^3$
- distinctive shape: cubic curve
- degree:  $n = 3$
- domain:  $x = \text{real numbers}$
- range:  $y = \text{real numbers}$

Notice that for the basic quadratic function, we may use any real number (negative, positive or zero) for the  $x$ -value...thus, its domain is the set of real numbers. However, because the parabola appears from the  $x$ -axis up, its range must be restricted to be  $y \geq 0$  (greater than or equal to zero).

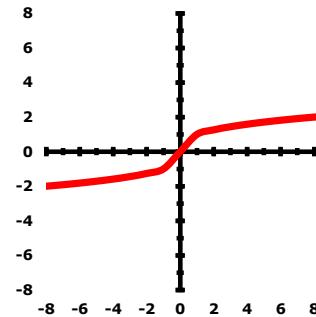
Notice that for the basic cubic function, both the  $x$ -values and  $y$ -values for this function will include negatives, positives and zero...thus, both the domain and the range for these functions will be the set of real numbers.

Notice that for the basic square root function, we need to restrict the  $x$ -values to be used. Since we cannot take the square root of a negative number, the domain for this type of function must be  $x \geq 0$ . And, since we are taking the square root of non-negative numbers, the answers (range) must also be restricted to  $y \geq 0$ .



### ***Square Root Function***

- basic function:  $F(x) = ax^{1/2}$
- specific function:  $y = x^{1/2}$
- distinctive shape: square root curve
- domain:  $x \geq 0$
- range:  $y \geq 0$



### ***Cube Root Function***

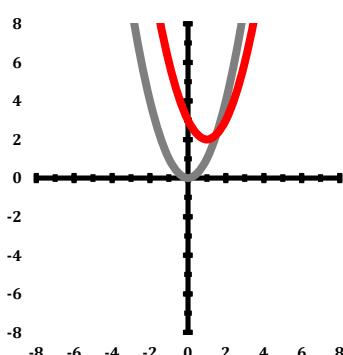
- basic function:  $F(x) = ax^{1/3}$
- specific function:  $y = x^{1/3}$
- distinctive shape: cube root curve
- domain:  $x = \text{real numbers}$
- range:  $y = \text{real numbers}$

Notice that for the basic cube root function, both the  $x$ -values and  $y$ -values for this function will include negatives, positives and zero...thus, both the domain and the range for these functions will be the set of real numbers.

We can take any of the previously mentioned functions and move (*translate*) them around the coordinate grid by changing the appearance of its function equation.

#### ***Parabolic Curve:***

$$F(x) = a(x - b)^2 + k$$

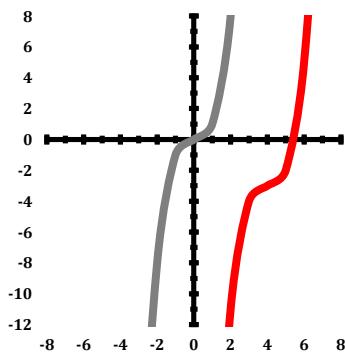


	original (grey):	translated (red):
<b>function:</b>	$F(x) = x^2$	$y = (x - 1)^2 + 2$
<b>shape:</b>	parabola curve	parabola curve
<b>degree:</b>	$n = 2$	$n = 2$
<b>domain:</b>	$x = \text{real numbers}$	$x = \text{real numbers}$
<b>range:</b>	$y \geq 0$	$y \geq 2$
<b>vertex:</b>	(0, 0)	(1, 2)

Notice that the translated (red) parabola has been moved one (1) unit to the right and two (2) units up from the original position as displayed by the grey parabola. Also notice that the vertex of the translated parabola can be determined by adding the values of  $h$  (horizontal movement) and  $k$  (vertical movement) to the respective coordinates of the vertex:  $(0 + h, 0 + k) = (0 + 1, 0 + 2) = (1, 2)$

### Cubic Curve:

$$F(x) = a(x - h)^3 + k$$



	original (grey):	translated (red):
function:	$F(x) = x^3$	$y = (x - 1)^3 + 3$
shape:	cubic curve	cubic curve
degree:	$n = 3$	$n = 3$
domain:	$x = \text{real numbers}$	$x = \text{real numbers}$
range:	$y = \text{real numbers}$	$y = \text{real numbers}$

SIMPLY CLEVER

ŠKODA



We will turn your CV into  
an opportunity of a lifetime



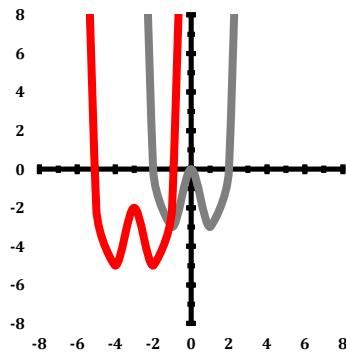
Do you like cars? Would you like to be a part of a successful brand?  
We will appreciate and reward both your enthusiasm and talent.  
Send us your CV. You will be surprised where it can take you.

Send us your CV on  
[www.employerforlife.com](http://www.employerforlife.com)



Notice that the translated (red) cubic curve has been moved four (4) units to the right and three (3) units down from the original position as displayed by the grey cubic curve. Also notice that the values of  $h$  and  $k$  from the translated equation are  $h = 4$ ,  $k = -3$ ...the same number of units translated ( $h$  = horizontally,  $k$  = vertically) from the original position.

**Quartic Curve:**  
 $F(x) = a(x - h)^4 + b(x - h)^2 + k$



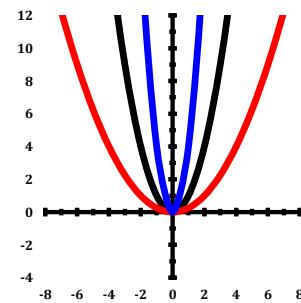
	original (grey):	translated (red):
<b>function:</b>	$F(x) = x^4$	$y = (x + 3)^4 - 4(x + 3)^2 - 2$
<b>shape:</b>	quartic curve	quartic curve
<b>degree:</b>	$n = 4$	$n = 4$
<b>domain:</b>	$x = \text{real numbers}$	$x = \text{real numbers}$
<b>range:</b>	$y = \text{real numbers}$	$y = \text{real numbers}$

Notice that the translated (red) quartic curve has been moved three (3) units to the left and two (2) units down from the original position as displayed by the grey quartic curve. Also notice that the values of  $h$  and  $k$  from the translated equation are:  $h = -3$ ,  $k = -2$ ...the same number of units translated ( $h$  = horizontally,  $k$  = vertically) from the original position.

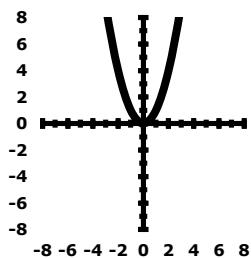
We have seen how the horizontal and vertical translations can be determined by the values of  $h$  and  $k$ . How the graphs can be widened or narrowed from a **parent** (basic original) function can be determined by the value of  $a$  (the number that is multiplied by the variable expression within a function).

Consider the parent function of:  $y = x^2$ ...which is the black parabola ( $a = +1$ ):

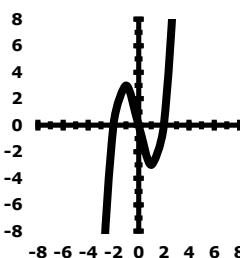
- The red parabola has the equation of:  $y = (\frac{1}{4})x^2$ ...where  $a = \frac{1}{4}$ . Notice that the opening of the red parabola is wider than that of the black one. Whenever the value of  $a$  is a fraction (that is,  $0 < a < 1$ ), the opening of the “new” parabola will be wider than that of the parent function.
- The blue parabola has the equation of:  $y = 4x$  and  $a = 4$ . Whenever the value of  $a$  is larger than one (1), the opening of the “new” parabola will be thinner than that of the parent function.



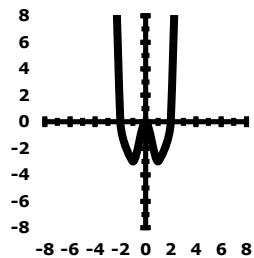
Although each of these examples have  $|a| = 1$ , the important thing to notice here is that whenever  $a < 0$ , the parent graph will be flipped in reference to the axes and the origin (0, 0):

**quadratic functions**

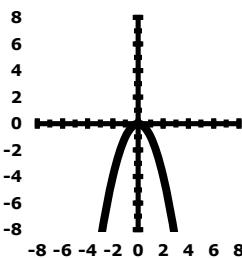
$$y = +1x^2$$

**cubic functions**

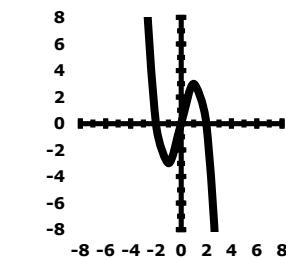
$$y = +1(x^3 - 4x)$$

**quartic functions**

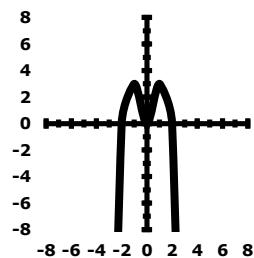
$$y = +1(x^4 - 4x^2)$$



$$y = -1x^2$$



$$y = -1(x^3 - 4x) = -1x^3 + 4x$$



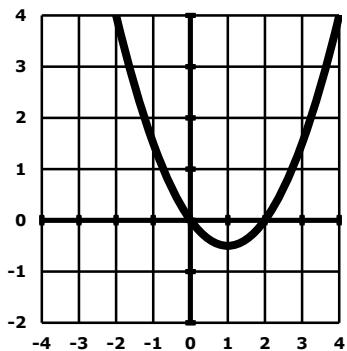
$$y = -1(x^4 - 4x^2) = -1x^4 + 4x^2$$

Let's examine some ***polynomial functions*** that consist of joined monomials and takes the form of:

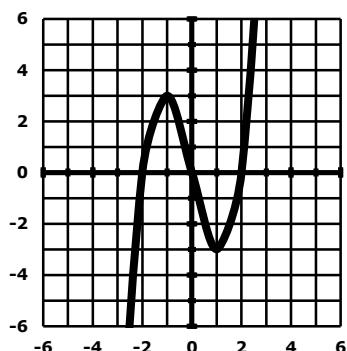
$$y = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x^1 + a_0$$

where  $a_n \neq 0$ , integer  $n \geq 0$  and  $n$  = degree of the function (the polynomial's highest exponent)...

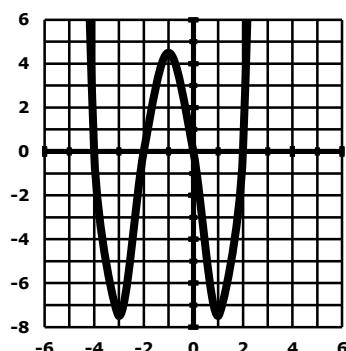
When a polynomial function is expressed in its factored form (as are the three examples below), we can see that the zeroes of the function are where the graph will intersect (cross) the  $x$ -axis.

***polynomial function #1***

$$y = \frac{x(x-2)}{2}$$

***polynomial function #2***

$$y = \frac{x(x+2)(x-2)}{2}$$

***polynomial function #3***

$$y = \frac{x(x+2)(x-2)(x+4)}{2}$$

The advertisement features a large central image of a teacher interacting with two young children at a computer. The background is yellow with orange swirling patterns. In the top left corner is the e-Learning for Kids logo. In the bottom right corner, there is a green oval containing text about the organization's impact.

**About e-Learning for Kids** Established in 2004, e-Learning for Kids is a global nonprofit foundation dedicated to fun and free learning on the Internet for children ages 5 - 12 with courses in math, science, language arts, computers, health and environmental skills. Since 2005, more than 15 million children in over 190 countries have benefitted from eLessons provided by EFK! An all-volunteer staff consists of education and e-learning experts and business professionals from around the world committed to making difference. eLearning for Kids is actively seeking funding, volunteers, sponsors and courseware developers; get involved! For more information, please visit [www.e-learningforkids.org](http://www.e-learningforkids.org).

- The number 1 MOOC for Primary Education
- Free Digital Learning for Children 5-12
- 15 Million Children Reached



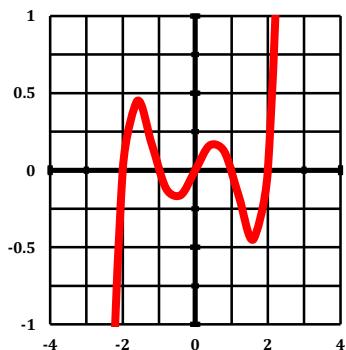
The number of variable factors can lead to the degree of the polynomial function:

- For polynomial function #1, we have two (2) variable factors. If we express the function in its unfactored form, the numerator of the function would be:  $x^2 - 2x$ . Notice that this polynomial has a degree of two (2).
- For polynomial function #2, we have three (3) variable factors. If we express the function in its unfactored form, the numerator of the function would be:  $x^3 - 4x$ . Notice that this polynomial has a degree of three (3).
- For polynomial function #3, we have four (4) variable factors. If we express the function in its unfactored form, the numerator of the function would be:  $x^4 + 4x^3 - 4x^2 - 16x$ . Notice that this polynomial has a degree of four (4).

The change in direction can also lead to the degree of the polynomial function:

- For polynomial function #1, if we started to trace the graph starting on its far left, we would come down the parabola, hit the vertex and then go up the other side. We changed direction twice. Recall that this polynomial function has a degree of two (2).
- For polynomial function #2, if we started to trace the graph starting on its far left, we would go up the curve to about point  $(-1, 3)$ , we would then change directions and come down the curve to about point  $(1, 3)$  and then change directions one more time...for a total of 3 changes. Recall that this polynomial function has a degree of three (3).
- For polynomial function #3, if we started to trace the graph starting on its far left, we would come down the curve, we would then change directions and go up the curve, we would then change directions and go down the curve, and then finally, change direction one more time and go up the curve...for a total of 4 changes. Recall that this polynomial function has a degree of four (4).

Using all of this information, we can estimate a possible polynomial function for a given graph. We can only approximate a polynomial function based on the  $x$ -intercepts. We would need to refine the possible coefficients through examination of the range ( $y$ -values) that was displayed.

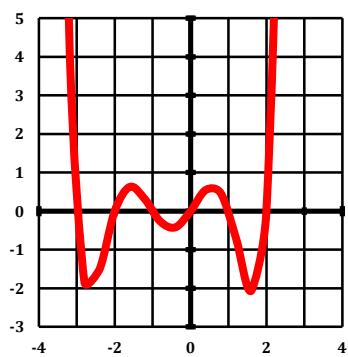
***polynomial function #4***

This graph appears to have five (5) zeroes...five (5) points where the graph intersects the  $x$ -axis:  $(-2, 0)$ ,  $(-1, 0)$ ,  $(0, 0)$ ,  $(1, 0)$ ,  $(2, 0)$ . Therefore, the polynomial function for this graph probably has a degree of at least five (5)...which means that the polynomial has as its highest exponent a five (5) or higher. Using the  $x$ -values from the five (5) points of intersection (in order from smallest to largest), we can obtain a possible polynomial function for this graph:

$$y = (x + 2)(x + 1)(x + 0)(x - 1)(x - 2)$$

In actuality, the polynomial function used to create this graph was:

$$y = \frac{x(x+2)(x-2)(x+1)(x-1)}{8} = \frac{x^5 - 5x^3 + 4x}{8}$$

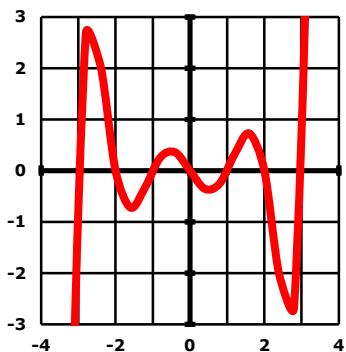
***polynomial function #5***

This graph appears to have six (6) zeroes...six (6) points where the graph intersects the  $x$ -axis:  $(-3, 0)$ ,  $(-2, 0)$ ,  $(-1, 0)$ ,  $(0, 0)$ ,  $(1, 0)$ ,  $(2, 0)$ . Therefore, the polynomial function for this graph probably has a degree of at least six (6)...which means that the polynomial has as its highest exponent a six (6) or higher. Using the  $x$ -values from the six (6) points of intersection (in order from smallest to largest), we can obtain a possible polynomial function for this graph:

$$y = (x + 3)(x + 2)(x + 1)(x + 0)(x - 1)(x - 2)$$

In actuality, the polynomial function used to create this graph was:

$$y = \frac{x(x+3)(x+2)(x-2)(x+1)(x-1)}{8} = \frac{x^6 + 3x^5 - 5x^4 - 15x^3 + 4x^2 + 12x}{8}$$

***polynomial function #6***

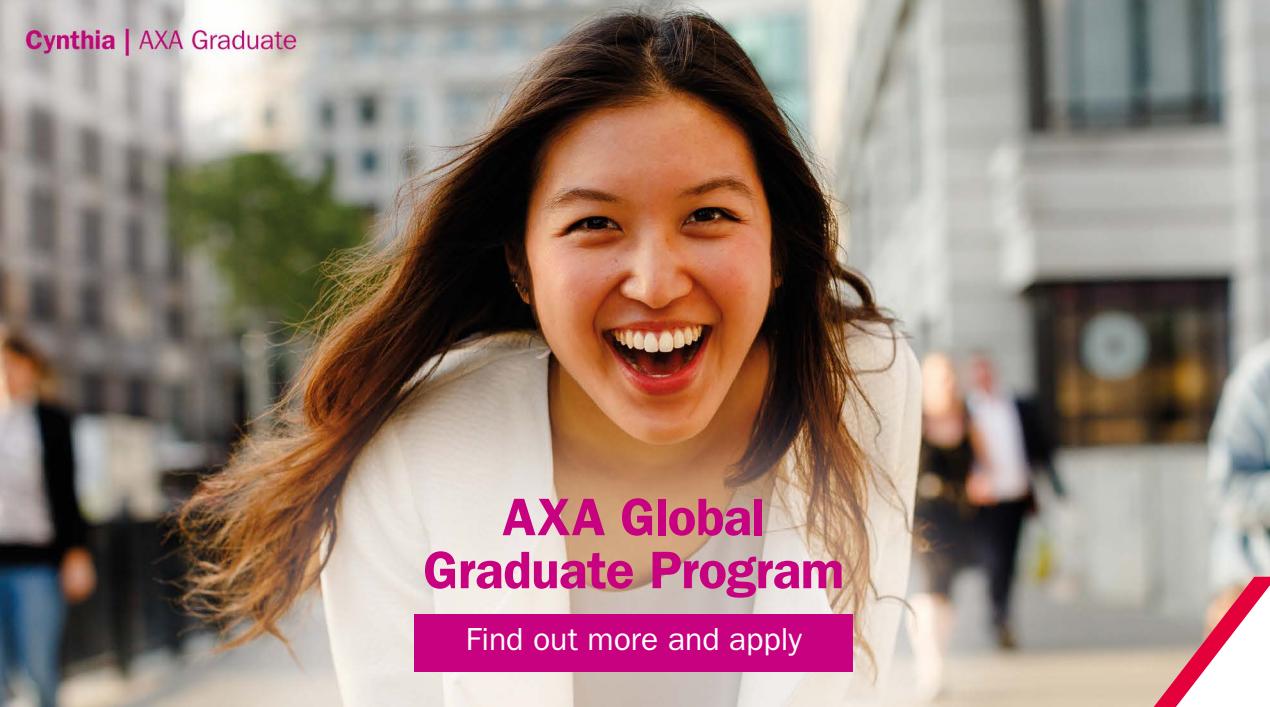
This graph appears to have seven (7) zeroes...seven (7) points where the graph intersects the  $x$ -axis:  $(-3, 0)$ ,  $(-2, 0)$ ,  $(-1, 0)$ ,  $(0, 0)$ ,  $(1, 0)$ ,  $(2, 0)$ ,  $(3, 0)$ . Therefore, the polynomial function for this graph probably has a degree of at least seven (7)...which means that the polynomial has as its highest exponent a seven (7) or higher. Using the  $x$ -values from the seven (7) points of intersection (in order from smallest to largest), we can obtain a possible polynomial function for this graph:

$$y = (x + 3)(x + 2)(x + 1)(x + 0)(x - 1)(x - 2)(x - 3)$$

In actuality, the polynomial function used to create this graph was:

$$y = \frac{x(x+3)(x-3)(x+2)(x-2)(x+1)(x-1)}{32} = \frac{x^7 - 14x^5 - 41x^3 - 36}{32}$$

Cynthia | AXA Graduate



**AXA Global  
Graduate Program**

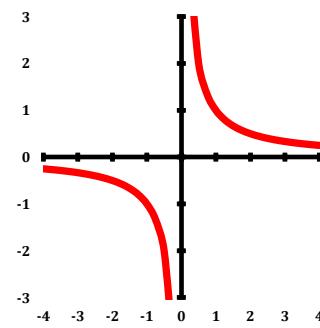
Find out more and apply



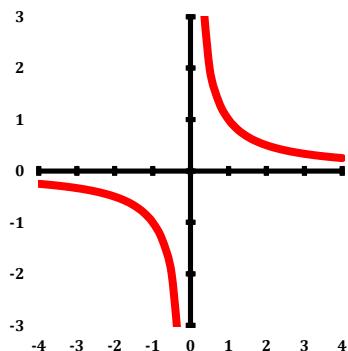
redefining / standards

Another basic function that needs to be addressed is a ***rational function*** that will result in a hyperbola as its graph (as seen at the right):

- basic function:  $F(x) = \frac{a}{x}$  (where  $x \neq 0$ ,  $a \neq 0$ )
- distinctive shape: hyperbola...notice that a hyperbola is made up of two ***branches***. For this parent graph, the ***branches*** use the two axes as ***asymptotes*** (guidelines) but never intersect them
- domain:  $x \neq 0$ ...fraction denominator  $\neq 0$
- range:  $y \neq 0$ ...if  $x \neq 0$ , then  $y \neq 0$
- When  $a > 0$  (as in the display), the hyperbola will exist in the first and third quadrants. However, when  $a < 0$ , the hyperbola will exist in the second and fourth quadrants

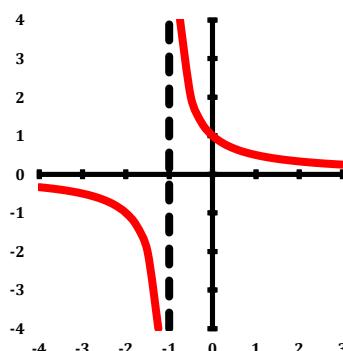


Consider these rational functions:



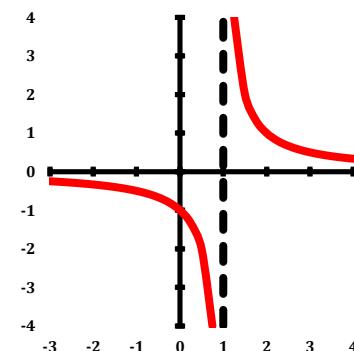
**specific function:**  $y = \frac{1}{x}$

**domain:**  $x \neq 0$ ; **range:**  $y \neq 0$



**specific function:**  $y = \frac{1}{x+1}$

**domain:**  $x \neq -1$ ; **range:**  $y \neq 0$



**specific function:**  $y = \frac{1}{x-1}$

**domain:**  $x \neq 1$ ; **range:**  $y \neq 0$

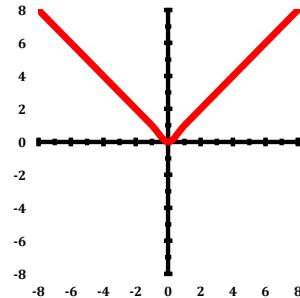
Notice that the main difference among these three rational functions is their domains. Since the denominator of any algebraic fraction cannot ever equal zero, we must determine what value would make that happen and then restrict that value from the domain (values allowed to be used as inputs).

The hyperbola for any rational function will always have two asymptotes – a vertical one as well as a horizontal one – that will be used as guideline in sketching the actual graph. While the horizontal asymptote for each of these hyperbolas is:  $y = 0$ , notice that:

- The vertical asymptote in the first diagram above is:  $x = 0$  (domain:  $x \neq 0$ ).
- The vertical asymptote in the second diagram above is:  $x = -1$  (domain:  $x \neq -1$ )
- The vertical asymptote in the third diagram above is:  $x = +1$  (domain:  $x \neq +1$ ).

Another basic function that needs to be addressed is an ***absolute value function*** that will result in a V-shaped graph (as seen at the right):

- basic function:  $F(x) = a | x |$
- distinctive shape: V-shaped piece-wise function...that has two branches
- domain:  $x = \text{real numbers}$
- range:  $y \geq 0$



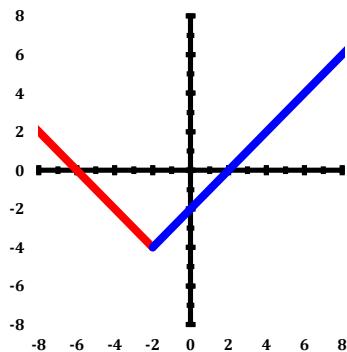
For this specific display, the function is:  $y = | x |$ , with a domain where  $x = \text{real numbers}$ , and a range where  $y \geq 0$ .

An absolute value function is really a ***piece-wise function*** which is defined by two or more constraints (domains, relationships). The absolute value function displayed above can be defined as:

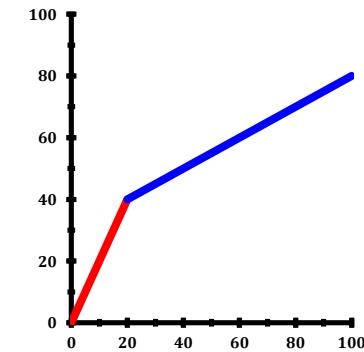
$$F(x) = |x| \dots \text{or, in its piece-wise format, as: } F(x) = \begin{cases} -x & \text{when } x < 0 \\ +x & \text{when } x \geq 0 \end{cases}$$

where the left-hand branch is defined by the top line ( $y = -x$  when  $x < 0$ ) of the piece-wise function while the right-hand branch is defined by the bottom line ( $y = +x$  when  $x \geq 0$ ).

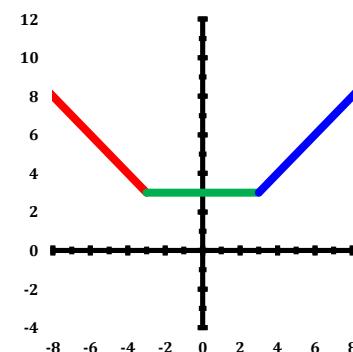
Here are some other piece-wise functions to consider (where each “part” of the piece-wise function is displayed in a different color):



$$F(x) = \begin{cases} -x - 6 & \text{when } x < -2 \\ x - 2 & \text{when } x \geq -2 \end{cases}$$



$$F(x) = \begin{cases} 2x & \text{when } 0 \leq x < 20 \\ 0.5x + 30 & \text{when } x \geq 20 \end{cases}$$



$$F(x) = \begin{cases} -x & \text{when } x < 0 \\ 3 & \text{when } -3 \leq x \leq 3 \\ x & \text{when } x > 3 \end{cases}$$

# 4 MATHEMATICS OF FINANCE

## TUTORIAL 4.1: FIND THE AMOUNT OF INTEREST FOR A SIMPLE INTEREST LOAN OR INVESTMENT

Many financial applications involve algebraic formulas that need to be evaluated to complete the desired task. One such application would be finding the amount of simple interest on a loan or investment which uses the formula that appears below.

**Simple Interest:  $I = Prt$**

where  $I$  = interest

$P$  = principal (dollars)

$r$  = annual interest rate (as a decimal)

$t$  = time (years)

### example 4.1a

**Find the amount of interest that will be earned if \$5000 is invested for 10 years at an annual simple interest rate of 5%.**

This real-world application can be solved by following these steps:

1. Determine the known values for the variables within the formula:

$$P = \text{principal} = \text{dollars invested} = 5000$$

$$r = \text{annual interest rate} = 5\% \rightarrow \text{as a decimal} = 0.05$$

$$t = \text{time in years} = 10 \text{ years} = 10$$

$$I = \text{simple interest amount} = \text{unknown for this problem}$$

2. Using the known values, evaluate the formula...

$$I = Prt = (5000)(0.05)(10) = 2500$$

**\$2500 of simple interest was earned**

### example 4.1b

**Find the amount of interest that will be earned if \$7500 is invested for 3 years at an annual simple interest rate of 1.25%.**

Following the same two steps as mentioned above:

$$P = 7500 \quad r = 1.25\% = 0.0125 \quad t = 3$$

$$I = Prt = (7500)(0.0125)(3) = 281.25$$

**\$281.25 of simple interest was earned**

**example 4.1c**

**Find the amount of interest that will be earned if \$10,000 is invested for 3 months at an annual simple interest rate of 1.5%.**  
Following the same two steps as mentioned above:

$$P = 10,000 \quad r = 1.5\% = 0.015 \quad t = 3 \text{ months} = 3/12 \text{ years} = 0.25$$

$$I = Prt = (10,000)(0.015)(0.25) = 37.5$$

**\$37.50 of simple interest was earned**

## TUTORIAL 4.2: FIND THE FUTURE VALUE FOR A SIMPLE INTEREST INVESTMENT

The future value ( $S$ ) of a loan or investment is the amount that occurs after the simple interest ( $I = Prt$ ) has been added to the amount ( $P$ ) that was invested or borrowed, as is expressed by the formula below.

***Future Value of Investment or Loan:  $S = P + I = P + Prt$***

where  **$S$  = future value (in dollars)**

**$P$  = principal (in dollars)**

**$r$  = annual interest rate (as a decimal)**

**$t$  = time (years)**

**example 4.2a**

**Find the future value amount (in dollars) if \$5000 is invested for 10 years at an annual simple interest rate of 5%.**

- Determine the known values for the variables within the formula:

$P$  = principal = dollars invested = 5000

$r$  = annual interest rate = 5% → as a decimal = 0.05

$t$  = time in years = 10 years = 10

- Using the known values, evaluate the formula for simple interest:

$$I = Prt = (5000)(0.05)(10) = 2500$$

- Add the interest found to the original principal amount:

$$S = P + I = 5000 + 2500 = 7500 \rightarrow \text{future value} = \$7500$$

$$S = P + I = P + Prt = 5000 + 5000(0.05)$$

$$(10) = 5000 + 2500 = 7500$$

**example 4.2b** Find the future value amount (in dollars) if \$7500 is invested for 3 years at an annual simple interest rate of 1.25%.

$$P = 7500 \quad r = 1.25\% = 0.0125 \quad t = 3$$

$$I = Prt = (7500)(0.0125)(3) = 281.25$$

$$S = P + I = 7500 + 281.25 = 7781.25 \rightarrow \text{future value} = \$7781.25$$

$$S = P + I = P + Prt = 7500 + 7500(0.0125)(3) = 7781.25$$

**example 4.2c** Find the future value amount (in dollars) if \$10,000 is invested for 3 months at an annual simple interest rate of 1.5%.

$$P = 10,000 \quad r = 1.5\% = 0.015 \quad t = 3 \text{ months} = 3/12 \text{ years} = 0.25$$

$$I = Prt = (10,000)(0.015)(0.25) = 37.5$$

$$S = P + I = 10,000 + 37.5 = 10,037.5 \rightarrow \text{future value} = \$10,037.50$$

$$S = P + I = P + Prt = 10,000 + 10,000(0.015)(0.25) = 10,037.50$$

### TUTORIAL 4.3: FIND THE AMOUNT THAT WAS INVESTED INTO AN ACCOUNT WHERE SIMPLE INTEREST IS USED

Occasionally, we might know the future value amount along with the simple interest rate and the length of time the simple interest rate was in effect but we do not know the initial amount ( $P$  = principal) that was invested. When this happens, we need to work backwards to determine the desired initial amount. Recall that the future value of a simple interest investment can be determined by the formula on the left below. We can adapt this formula to solve for the principal ( $P$ ) by performing some simple algebraic steps as shown in the box on the right below.

**Future Value of Investment or Loan:**  $S = P + Prt$   
**where**  $S$  = future value (in dollars)  
 $P$  = principal (in dollars)  
 $r$  = annual interest rate (as a decimal)  
 $t$  = time (years)

$$\begin{aligned} S &= P + Prt \\ S &= P(1 + rt) \\ P &= \frac{S}{(1 + rt)} = S \div (1 + rt) \end{aligned}$$

**example 4.3a**

**An investor wants to have \$30,000 in 9 months. If the best available interest rate is 7.05% per year, how much (to the nearest dollar) must be invested now to yield the desired amount?**

- Determine the known values for the variables within the formula:

$S$  = future value = dollars that will be in the account = 30,000

$r$  = annual interest rate = 7.05% → as a decimal = 0.0705

$t$  = time in years = 9 months = 9/12 years = 3/4 years = 0.75 years

- Using the known values, evaluate the adjusted formula version for  $P$ :

$$P = S \div (1 + rt) = 30,000 \div (1 + (0.0705)(0.75)) = 28493.41089$$

*Rounding mathematically to the nearest dollar: \$28,493*

**example 4.3b**

**What is the present value (to the nearest dollar) of an investment at 9% annual simple interest if it is worth \$1060 in 8 months?**

$$S = 1060$$

$$r = 9\% = 0.09$$

$t$  = 8 months = 8/12 years = 2/3 years = 0.666 years

$$P = S \div (1 + rt) = 1060 \div (1 + (0.09)(2/3)) = 1000$$

*Rounding mathematically to the nearest dollar: \$1000*

## **TUTORIAL 4.4: FIND THE SIMPLE INTEREST RATE EARNED ON AN INVESTMENT**

Occasionally, we might know the original principal and the future value amounts along with the length of time the simple interest rate was in effect but we do not know the rate that was used. When this happens, we need to work backwards to determine the desired rate ( $r$ ). Recall that the future value of an investment can be determined by the formula on the left below. We can adapt this formula to solve for the annual interest rate ( $r$ ) by performing some simple algebraic steps as shown in the box on the right below.

**Future Value of Investment or Loan:  $S = P + Prt$**   
**where  $S$  = future value (in dollars)**  
 **$P$  = principal (in dollars)**  
 **$r$  = annual interest rate (as a decimal)**  
 **$t$  = time (years)**

$$\begin{aligned} S &= P + Prt \\ S - P &= Prt \\ r &= \frac{S - P}{(P)(t)} = (S - P) \div P \div t \end{aligned}$$

**example 4.4a**

**Suppose that \$5000 is invested for 10 years at an unknown annual simple interest rate. At the end of the 10 years, the future value of this account is \$7500. What annual simple interest rate was used?**

1. Determine the known values for the variables within the needed formulas:

$$P = \text{principal} = \text{dollars invested} = 5000$$

$$t = \text{time in years} = 10 \text{ years} = 10$$

$$S = \text{future value (in dollars)} = 7500$$

2. Use the adjusted formula (solved for  $r$ ) version to determine  $r$ :

$$r = (S - P) \div P \div t = (7500 - 5000) \div 5000 \div 10 = 0.05 \rightarrow \text{rate} = 5\%$$

**example 4.4b**

**Suppose that \$10,000 is invested for 3 months at an unknown annual simple interest rate. At the end of the 3 months, the future value of this account is \$10,037.50 exactly. What annual simple interest rate was used?**

$$P = 10,000 \quad t = 3 \text{ months} = 3/12 \text{ years} = 0.25 \quad S = 10,037.50$$

$$r = (S - P) \div P \div t = (10,037.50 - 10,000) \div 10,000 \div 0.25 = 0.015$$

$$\text{rate} = 1.5\%$$

**example 4.4c**

**If a \$5000 investment grows to \$5937.50 in 15 months, find the simple interest rate (to the nearest hundredth of a percent) earned on this investment.**

$$P = 5000 \quad t = 15 \text{ months} = 15/12 \text{ years} = 1.25 \quad S = 5937.50$$

$$r = (S - P) \div P \div t = (5937.50 - 5000) \div 5000 \div 1.25 = 0.15$$

$$\text{rate} = 15\%$$

## TUTORIAL 4.5: FIND THE TIME NEEDED FOR AN ACCOUNT WITH A SIMPLE INTEREST RATE TO REACH A DESIRED FUTURE VALUE

Occasionally, we might know the original principal and the future value amounts along with the simple interest rate but we do not know the length of time that was involved. When this happens, we need to work backwards to determine the desired number of years ( $t$ ). Recall that the future value of an investment can be determined by the formula on the left below. We can adapt this formula to solve for the number of years ( $t$ ) by performing some simple algebraic steps as shown in the box on the right below.

**Future Value of Investment or Loan:**  $S = P + Prt$   
**where**  $S$  = future value (in dollars)  
 $P$  = principal (in dollars)  
 $r$  = annual interest rate (as a decimal)  
 $t$  = time (years)

$$\begin{aligned} S &= P + Prt \\ S - P &= Prt \\ t &= \frac{S - P}{(P)r} = (S - P) \div P \div r \end{aligned}$$

**example 4.5a** Suppose that \$5000 is invested at an annual simple interest rate of 5% and now has the future value of \$7500. How long did this account take to achieve the given future value amount?

1. Determine the known values for the variables within the needed formulas:

$$P = \text{principal} = \text{dollars invested} = 5000$$

$$r = \text{annual interest rate} = 5\% = 0.05$$

$$S = \text{future value (in dollars)} = 7500$$

2. Use the adjusted formula (solved for  $t$ ) version to determine  $t$ :

$$t = (S - P) \div P \div r = (7500 - 5000) \div 5000 \div 0.05 = 10$$

**time = 10 years**

**example 4.5b** Suppose that \$10,000 is invested at an annual simple interest rate of 1.5% and now has the future value of \$10,037.50 exactly. How long did this account take to achieve the given future value amount?

$$P = 10,000 \quad r = 1.5\% = 0.015 \quad S = 10,037.50$$

$$t = (S - P) \div P \div r = (10,037.50 - 10,000) \div 10,000 \div 0.015 = 0.25$$

**time = 0.25 years = (0.25)(12 months) = 3 months**

## TUTORIAL 4.6: FIND THE FUTURE VALUE FOR AN ACCOUNT THAT IS COMPOUNDED PERIODICALLY

Many financial applications involve algebraic formulas that need to be evaluated to complete the desired task. One such application would be finding the amount of periodically compounded interest on a loan or investment which uses the formula that appears below. The compounding period might be annually (once a year), semi-annually (twice a year), quarterly (four times a year), monthly (twelve times a year) or any other preset time period within one year.

$$\text{Future Value (Periodic Compounding): } S = P \left( 1 + \frac{r}{m} \right)^{mt}$$

where  $S$  = future value (dollars)

$P$  = principal (dollars)

$r$  = annual interest rate (as a decimal)

$m$  = compounding periods per year

$t$  = time (years)

graphing calculator notation:  $S = P * (1 + r \div m)^{(m * t)}$

### example 4.6a

What is the future value (to the nearest cent) if \$7500 is invested for 5 years at 10% compounded quarterly?

- Determine the known values for the variables within the formula:

$P$  = principal = dollars invested = 7500

$r$  = annual interest rate = 10% → as a decimal = 0.10

$m$  = compounding periods per year = quarterly = 4

$t$  = time in years = 5 years = 5

$S$  = future value amount = unknown for this problem

- Using the known values, evaluate the formula:

$$S = P \left( 1 + \frac{r}{m} \right)^{mt} = 7500 \left( 1 + \frac{0.10}{4} \right)^{(4)(5)} = 7500(1 + 0.025)^{20} = 12289.6233$$

graphing calculator solution:

$$S = 7500 * (1 + 0.10 \div 4)^{(4 * 5)} = 12289.6233$$

future value (to nearest cent) = \$12,289.62

**example 4.6b**      **What is the future value (to the nearest cent) if \$7500 is invested for 10 years at 5% compounded monthly?**

$$P = 7500 \quad r = 5\% = 0.05 \quad m = \text{monthly} = 12 \quad t = 10 \text{ years} = 10$$

$$S = P \left(1 + \frac{r}{m}\right)^{mt} = 7500 \left(1 + \frac{0.05}{12}\right)^{(12)(10)} = 7500 \left(1 + 0.0041666\right)^{120} = 12352.57123$$

**graphing calculator solution:**

$$S = 7500 * (1 + 0.05 \div 12)^{(12 * 10)} = 12352.57123$$

**future value (to nearest cent) = \$12,352.57**

### TUTORIAL 4.7: FIND THE AMOUNT OF INTEREST EARNED FOR AN ACCOUNT THAT IS COMPOUNDED PERIODICALLY

If we subtract the amount invested (principal) from the future value amount, we will be able to determine the exact amount of interest earned for an account that is compounded periodically (yearly, semi-annually, quarterly, monthly or any other preset time period within one year).

**Future Value (Periodic Compounding):**  $S = P \left(1 + \frac{r}{m}\right)^{mt}$

where **S** = future value (dollars)

**P** = principal (dollars)

**r** = annual interest rate (as a decimal)

**m** = compounding periods per year

**t** = time (years)

**interest earned:  $I = S - P$**

where **I** = interest earned

**S** = future value (dollars)

**P** = principal (dollars)

**example 4.7a**

**Determine the amount of interest (to the nearest cent) for an account where \$7500 is invested for 5 years at 10% compounded quarterly.**

1. Determine the known values for the variables within the formula:

$P$  = principal = dollars invested = 7500

$r$  = annual interest rate = 10% → as a decimal = 0.10

$m$  = compounding periods per year = quarterly = 4

$t$  = time in years = 5 years = 5

$S$  = future value amount = unknown for this problem

2. Using the known values, evaluate the periodic compounding formula:

$$S = P \left(1 + \frac{r}{m}\right)^{mt} = 7500 \left(1 + \frac{0.10}{4}\right)^{(4)(5)} = 7500(1 + 0.025)^{20} = 12289.6233$$

3. Determine the amount of interest earned by subtracting:

$S - P$ ...

interest earned =  $S - P = \$12,289.62 - \$7500 = \$4789.62$

**graphing calculator solution:**

$$I = 7500 * (1 + 0.10 / 4)^(4 * 5) - 7500 = 4789.6233$$

I joined MITAS because  
I wanted **real responsibility**

The Graduate Programme  
for Engineers and Geoscientists  
[www.discovermitas.com](http://www.discovermitas.com)

**Month 16**

I was a construction supervisor in the North Sea advising and helping foremen solve problems

Real work  
International opportunities  
Three work placements

**MAERSK**

**example 4.7b**

**Determine the amount of interest (to the nearest cent) for an account where \$7500 is invested for 10 years at 5% compounded monthly?**

$$P = 7500 \quad r = 5\% = 0.05 \quad m = \text{monthly} = 12 \quad t = 10 \text{ years} = 10$$

$$S = P \left(1 + \frac{r}{m}\right)^{mt} = 7500 \left(1 + \frac{0.05}{12}\right)^{(12)(10)} = 12352.57122$$

$$\$12,352.57 - \$7500 = 4852.57 \rightarrow \text{interest earned} = \$4852.57$$

**graphing calculator solution:**

$$S = 7500 * (1 + 0.05 \div 12)^{(12 * 10)} - 7500 = 4852.57123$$

### **TUTORIAL 4.8: FIND THE AMOUNT THAT WAS INVESTED INTO AN ACCOUNT THAT IS COMPOUNDED PERIODICALLY**

Occasionally, we might know the future value amount along with the interest rate, the number of compounding periods per year and the length of time for a particular account that has been periodically compounded but we do not know the initial amount (principal) that was invested. When this happens, we need to work backwards to determine the desired principal ( $P$ ).

Recall that the future value of an investment can be determined by the formula on the left below. We can adapt this formula to solve for the principal ( $P$ ) by performing some simple algebraic steps as shown in the box on the right below.

**Future Value (Periodic Compounding):**  $S = P \left(1 + \frac{r}{m}\right)^{mt}$

where  $S$  = future value (dollars)

$P$  = principal (dollars)

$r$  = annual interest rate (as a decimal)

$m$  = compounding periods per year

$t$  = time (years)

$$S = P \left(1 + \frac{r}{m}\right)^{mt}$$

$$P = S \div \left(1 + \frac{r}{m}\right)^{mt}$$

**graphing calculator notation:**  $P = S \div (1 + r \div m)^{(m * t)}$

**example 4.8a**

**Suppose an unknown amount was invested into an account with an interest rate of 12% compounded quarterly. At the end of 10 years, this account had an accumulated amount of \$50,000. What was the initial amount (principal) that was invested to the nearest dollar?**

- Determine the known values for the variables within the formula...

$P$  = principal = dollars invested = unknown for this problem

$r$  = annual interest rate = 12% → as a decimal = 0.12

$m$  = compounding periods per year = quarterly = 4

$t$  = time in years = 10 years = 10

$S$  = future value amount = \$50,000

- Using the known values, evaluate the formula...

$$P = S \div \left(1 + \frac{r}{m}\right)^{mt} = 50,000 \div \left(1 + \frac{0.12}{4}\right)^{(4)(10)} = 50,000 \div (1.03)^{40} = 15327.84204$$

**graphing calculator solution:**

$$P = 50,000 \div (1 + 0.12 \div 4)^{(4 * 10)} = 15327.84204$$

**principal (to nearest dollar) = \$15,328**

**example 4.8b**

**How much (to the nearest dollar) should be invested into an account that earns an interest rate of 12% compounded monthly 5 years to achieve the future value amount of \$25,000?**

$$S = 25,000 \quad r = 12\% = 0.12 \quad m = \text{monthly} = 12 \quad t = 5 \text{ years} = 5$$

$$P = S \div \left(1 + \frac{r}{m}\right)^{mt} = 25,000 \div \left(1 + \frac{0.12}{12}\right)^{(12)(5)} = 25,000 \div (1.01)^{60} = 13761.2404$$

**graphing calculator solution:**

$$P = 25,000 \div (1 + 0.12 \div 12)^{(12 * 5)} = 13761.2404$$

**principal (to nearest dollar) = \$13,761**

**example 4.8c**

**How much (to the nearest dollar) should be invested into an account that earns an interest rate of 10% compounded semi-annually for 25 years to achieve the future value amount of \$25,000?**

$$r = 10\% \quad \text{semi-annually} \quad 25 \text{ years}$$

$$S = 100,000 \quad r = 0.10 \quad m = 2 \quad t = 25$$

$$P = S \div \left(1 + \frac{r}{m}\right)^{mt} = 100,000 \div \left(1 + \frac{0.10}{2}\right)^{(2)(25)} = 100,000 \div (1.05)^{50} = 8720.372697$$

**graphing calculator solution:**

$$P = 100,000 \div (1 + 0.10 \div 2)^{(2 * 25)} = 8720.372697$$

**principal (to nearest dollar) = \$8720**

## TUTORIAL 4.9: FIND THE INTEREST RATE FOR A PERIODICALLY-COMPOUNDED ACCOUNT

Occasionally, we might know the future value and the principal amounts along with the number of compounding periods per year and the length of time for a particular account that has been periodically compounded but we do not know the annual interest rate at which the principal was invested. When this happens, we need to work backwards to determine the desired interest rate ( $r$ ). Recall that the future value of an investment can be determined by the formula in the first box below. We can adapt this formula to solve for the annual interest rate ( $r$ ) by performing some simple algebraic steps as shown in the second box below.

$$\text{Future Value (Periodic Compounding): } S = P \left( 1 + \frac{r}{m} \right)^{mt}$$

where  $S$  = future value (dollars)

$P$  = principal (dollars)

$r$  = annual interest rate (as a decimal)

$m$  = compounding periods per year

$t$  = time (years)

$$S = P \left( 1 + \frac{r}{m} \right)^{mt} \rightarrow \frac{S}{P} = \left( 1 + \frac{r}{m} \right)^{mt} \rightarrow \left( \frac{S}{P} \right)^{\frac{1}{mt}} = \left( \left( 1 + \frac{r}{m} \right)^{mt} \right)^{\frac{1}{mt}} \rightarrow$$

$$\left( \frac{S}{P} \right)^{\frac{1}{mt}} = 1 + \frac{r}{m} \rightarrow \frac{-r}{m} = 1 - \left( \frac{S}{P} \right)^{\frac{1}{mt}} \rightarrow r = (-m) \left[ 1 - \left( \frac{S}{P} \right)^{\frac{1}{mt}} \right]$$

graphing calculator notation:  $r = (-1 * m) * (1 - (S \div P)^(1 \div (m * t)))$

**example 4.9a**

**Suppose \$30,000 was invested into an account where the interest was compounded quarterly. The future value of this account was \$66,241.19 at the end of 10 years. Find the interest rate that was in effect for this account?**

1. Determine the known values for the variables within the formula:

$$P = \text{principal} = \text{dollars invested} = \$30,000$$

$$r = \text{annual interest rate} = \text{unknown for this problem}$$

$$m = \text{compounding periods per year} = \text{quarterly} = 4$$

$$t = \text{time in years} = 10 \text{ years} = 10$$

$$S = \text{future value amount} = \$66,241.19$$

2. Using the known values, evaluate the formula:

$$\begin{aligned} r &= (-m) \left[ 1 - \left( \frac{S}{P} \right)^{\frac{1}{mt}} \right] = (-4) \left[ 1 - \left( \frac{66,241.19}{30,000} \right)^{\frac{1}{(4)(10)}} \right] = (-4) \left[ 1 - \left( \frac{66,241.19}{30,000} \right)^{\frac{1}{40}} \right] \\ &\quad \left( \frac{66,241.19}{30,000} \right)^{\frac{1}{40}} = 1.02 \end{aligned}$$

$$r = (-4)[1 - 1.02] = (-4)(-0.02) = 0.08$$

**interest rate  $\approx 0.08 \rightarrow 8\%$**

**graphing calculator solution:**

$$\begin{aligned} r &= (-1 * 4) * (1 - (66,241.19 \div 30,000)^{(1 \div (4 * 10))}) = \\ &\quad \text{0.0800000001} \end{aligned}$$

**example 4.9b**

Suppose \$25,000 was invested into an account where the interest was compounded monthly. The future value of this account was \$45,417.42 at the end of 5 years. Find the interest rate that was in effect for this account?

$$\begin{array}{cccc} & \text{monthly} & & 5 \text{ years} \\ P = 25,000 & S = 45,417.42 & m = 12 & t = 5 \\ r = (-m) \left[ 1 - \left( \frac{S}{P} \right)^{\frac{1}{mt}} \right] = (-12) \left[ 1 - \left( \frac{45,417.42}{25,000} \right)^{\frac{1}{(12)(5)}} \right] = (-12) \left[ 1 - \left( \frac{45,417.42}{25,000} \right)^{\frac{1}{60}} \right] \\ & & \left( \frac{45,417.42}{25,000} \right)^{\frac{1}{60}} = 1.010000112 \end{array}$$

$$r = (-12)[1 - 1.01] = (-12)(-0.01) = 0.12$$

**interest rate  $\approx 0.12 \rightarrow 12\%$**

**graphing calculator solution:**

$$r = (-1 * 12) * (1 - (45,417.42 \div 25,000)^{(1 \div (12 * 5))}) = 0.1200000113$$

**example 4.9c**

Suppose \$10,000 was invested into an account where the interest was compounded 6 times a year. The future value of this account was \$14,371.88 at the end of 10 years. Find the interest rate that was in effect for this account?

$$\begin{array}{cccc} & \text{6 times/year} & & 10 \text{ years} \\ P = 10,000 & S = 14,371.88 & m = 6 & t = 10 \\ r = (-m) \left[ 1 - \left( \frac{S}{P} \right)^{\frac{1}{mt}} \right] = (-6) \left[ 1 - \left( \frac{14,371.88}{10,000} \right)^{\frac{1}{(6)(10)}} \right] = (-6) \left[ 1 - \left( \frac{14,371.88}{10,000} \right)^{\frac{1}{60}} \right] \\ & & \left( \frac{14,371.88}{10,000} \right)^{\frac{1}{60}} = 1.006063114 \end{array}$$

$$r = (-6)[1 - 1.006] = (-6)(-0.006) = 0.036$$

**interest rate  $\approx 0.036 \rightarrow 3.6\%$**

**graphing calculator solution:**

$$r = (-1 * 6) * (1 - (14,371.88 \div 10,000)^{(1 \div (6 * 10))}) = 0.0359999711$$

## TUTORIAL 4.10: SOLVE AN EXPONENTIAL EQUATION USING LOGARITHMS

Occasionally, the unknown that we wish to determine appears in the exponent. Therefore, we need to know how to solve an *exponential equation*. To accomplish this task, we may use the *logarithm of a power property*. Two versions of this property are shown below (one for common logarithms and one for natural logarithms):

**logarithm of a power property  
(common logarithms)**  
 $\log b X^P = P(\log b X)$

**logarithm of a power property  
(natural logarithms)**  
 $\ln e^P = P(\ln e)$

### example 4.10a      Solve for the variable: $4(25)^{2x} = 312,500$

1. Isolate  $(25)^{2x}$  by dividing both sides of the given equation by 4:

$$4(25)^{2x} \div 4 = 312,500 \div 4 \rightarrow (25)^{2x} = 78,125$$

- 2) Apply the logarithm of a power property:

*common logarithms version:*    *natural logarithms version:*

$$\begin{aligned} \log(25)^{2x} &= \log(78,125) & \ln(25)^{2x} &= \ln(78,125) \\ (2x)(\log(25)) &= \log(78,125) & (2x)(\ln(25)) &= \ln(78,125) \end{aligned}$$

- 3) Solve for the variable  $x$ :

$$2x = \log(78,125) \div \log(25) \quad 2x = \ln(78,125) \div \ln(25)$$

$$2x = 3.5$$

$$2x = 3.5$$

$$x = 3.5 \div 2 = 1.75 \rightarrow \mathbf{x = 1.75} \quad x = 3.5 \div 2 = 1.75 \rightarrow \mathbf{x = 1.75}$$

Notice that it did not make any difference which version of the logarithm of a power property was used...the final answer came out the same.

**example 4.10b      Solve for the variable (to the nearest hundredth):  $3(9)(2x - 4) = 15$**

- common logarithms version:

$$\begin{aligned} 3(9)^{(2x-4)} &= 15 \rightarrow 3(9)^{(2x-4)} \div 3 = 15 \div 3 \rightarrow 9^{(2x-4)} = 5 \rightarrow \\ \log(9)^{(2x-4)} &= \log(5) \rightarrow (2x-4)(\log(9)) = \log(5) \rightarrow \\ 2x-4 &= \log(5) \div \log(9) \rightarrow 2x = [\log(5) \div \log(9)] + 4 \rightarrow \\ x &= \frac{[\log(5) \div \log(9)] + 4}{2} \approx 2.36624338 \rightarrow \textcolor{red}{x \approx 2.37} \end{aligned}$$

- natural logarithms version:

$$\begin{aligned} 3(9)^{(2x-4)} &= 15 \rightarrow 3(9)^{(2x-4)} \div 3 = 15 \div 3 \rightarrow 9^{(2x-4)} = 5 \rightarrow \\ \ln(9)^{(2x-4)} &= \ln(5) \rightarrow (2x-4)(\ln(9)) = \ln(5) \rightarrow \\ 2x-4 &= \ln(5) \div \ln(9) \rightarrow 2x = [\ln(5) \div \ln(9)] + 4 \rightarrow \\ x &= \frac{[\ln(5) \div \ln(9)] + 4}{2} \approx 2.36624338 \rightarrow \textcolor{red}{x \approx 2.37} \end{aligned}$$

## TUTORIAL 4.11: FIND THE TIME NEEDED FOR AN ACCOUNT THAT IS COMPOUNDED PERIODICALLY TO REACH A CERTAIN FUTURE VALUE

Occasionally, we might know the future value and the principal amounts along with the interest rate and the number of compounding periods per year for a particular account that has been compounded periodically but we do not know the time (number of years) at which the principal has been invested. When this happens, we need to work backwards to determine the desired time ( $t$ ). Recall that the future value of an investment can be determined by the formula in the first box below.

$$\textbf{Future Value (Periodic Compounding): } S = P \left( 1 + \frac{r}{m} \right)^{mt}$$

where  $S$  = future value (dollars)

$P$  = principal (dollars)

$r$  = annual interest rate (as a decimal)

$m$  = compounding periods per year

$t$  = time (years)

We can adapt this formula to solve for the time ( $t$ ) by performing some simple algebraic steps as shown below:

**common logarithms version:**

$$S = P \left( 1 + \frac{r}{m} \right)^{mt} \rightarrow \frac{S}{P} = \left( 1 + \frac{r}{m} \right)^{mt} \rightarrow \log\left(\frac{S}{P}\right) = \log\left(\left(1 + \frac{r}{m}\right)^{mt}\right) \rightarrow$$

$$\log\left(\frac{S}{P}\right) = mt * \log\left(1 + \frac{r}{m}\right) \rightarrow t = \frac{\log\left(\frac{S}{P}\right)}{m * \log\left(1 + \frac{r}{m}\right)} \rightarrow t = \log\left(\frac{S}{P}\right) \div \left(m * \log\left(1 + \frac{r}{m}\right)\right)$$

**natural logarithms version:**

$$S = P \left( 1 + \frac{r}{m} \right)^{mt} \rightarrow \frac{S}{P} = \left( 1 + \frac{r}{m} \right)^{mt} \rightarrow \ln\left(\frac{S}{P}\right) = \ln\left(\left(1 + \frac{r}{m}\right)^{mt}\right) \rightarrow$$

$$\ln\left(\frac{S}{P}\right) = mt * \ln\left(1 + \frac{r}{m}\right) \rightarrow t = \frac{\ln\left(\frac{S}{P}\right)}{m * \ln\left(1 + \frac{r}{m}\right)}$$

**graphing calculator notation:**

**using common logarithms:**  $t = \log(S \div P) \div (m * \log(1 + r \div m))$

**using natural logarithms:**  $t = \ln(S \div P) \div (m * \ln(1 + r \div m))$

Also depending on what calculator is being used, we might have to search for the “log” and the “ln” functions. Each one might be a button on the face of the calculator or might be hidden within a menu list that can be found on the calculator.

Although we may use a different logarithm type to adapt the “future value” formula, the final numeric result will be the same for either version. So, if we wish to use only one type of logarithm function for all problems, the one to choose would be the natural logarithm (ln).

**example 4.11a** How many years (to the nearest tenth) would it take \$10,000 invested in an account with an annual interest rate of 6% (compounded quarterly) to reach \$20,000?

1. Determine the known values for the variables within the formula:

$$P = \text{principal} = \text{dollars invested} = \$10,000$$

$$r = \text{annual interest rate} = 6\% = 0.06$$

$$m = \text{compounding periods per year} = \text{quarterly} = 4$$

$$t = \text{time in years} = \text{unknown for this problem}$$

$$S = \text{future value amount} = \$20,000$$

2. Using the known values, evaluate the adjusted formula for  $t$ :

- common logarithms version:

$$t = \log\left(\frac{S}{P}\right) \div \log\left[\left(1 + \frac{r}{m}\right)^m\right] = \log\left(\frac{20,000}{10,000}\right) \div \log\left[\left(1 + \frac{0.06}{4}\right)^4\right]$$

$$t = \log(2) \div \log(1.015)^4 = \log(2) \div [4 * \log(1.015)] = 11.63888141$$

$$\mathbf{t = 11.6 \text{ years}}$$

- natural logarithms version:

$$t = \ln\left(\frac{S}{P}\right) \div \ln\left[\left(1 + \frac{r}{m}\right)^m\right] = \ln\left(\frac{20,000}{10,000}\right) \div \ln\left[\left(1 + \frac{0.06}{4}\right)^4\right]$$

$$t = \ln(2) \div \ln(1.015)^4 = \ln(2) \div [4 * \ln(1.015)] = 11.63888141$$

$$\mathbf{t = 11.6 \text{ years}}$$

**example 4.11b** How many years (to the nearest tenth) would it take \$10,000 invested in an account with an annual interest rate of 6% (compounded monthly) to reach \$30,000?

$$P = 10,000 \quad S = 30,000 \quad r = 0.06 \quad m = 12$$

- common logarithms version:  $t = \log(S \div P) \div (m * \log(1 + r \div m)) = \log(30,000 \div 10,000) \div (12 * \log(1 + 0.06 \div 12)) = 18.35594227$

- natural logarithms version:  $t = \ln(S \div P) \div (m * \ln(1 + r \div m)) = \ln(30,000 \div 10,000) \div (12 * \ln(1 + 0.06 \div 12)) = 18.35594227$

$$\mathbf{t = \approx 18.4 \text{ years}}$$

## TUTORIAL 4.12: FIND THE FUTURE VALUE FOR AN ACCOUNT WHEN INTEREST IS COMPOUNDED CONTINUOUSLY

Many financial applications involve algebraic formulas that need to be evaluated to complete the desired task. One such application would be finding the amount of continuously compounded interest on a loan or investment which uses the formula that appears below.

**Future Value (Continuous Compounding):  $S = Pe^{rt}$**   
**where  $S$  = future value (dollars)**  
 **$P$  = principal (dollars)**  
 **$e$  = irrational number of approximately 2.718281828**  
 **$r$  = annual interest rate**  
 **$t$  = time (years)**

**graphing calculator notation:  $S = P * e^{(r * t)}$**

Depending on what calculator being used, we might have to type the exponential value ( $rt$ ) into the calculator first and then access the  $e$  function and then multiply by  $P$ .

**example 4.12a** **Find the future value (to the nearest cent) if \$6200 is invested for 5 years at 8% compounded continuously.**

1. Determine the known values for the variables within the formula:

$P$  = principal = dollars invested = 6200

$r$  = annual interest rate = 8% → as a decimal = 0.08

$t$  = time in years = 5 years = 5

$S$  = future value amount = unknown for this problem

2. Using the known values, evaluate the formula:

$$S = Pe^{rt} = 6200 \times e^{(0.08)(5)} = 6200 \times e^{0.4} =$$

$$S = (6200)(1.491824698) = 9249.313125$$

**graphing calculator solution:**

$$S = 6200 * e^{(0.08 * 5)} = 9249.313125$$

**future value (to nearest cent) = \$9249.31**

**example 4.12b** Find the future value (to the nearest cent) if \$1000 is invested for 20 years at 10% compounded continuously.

$$P = 1000 \quad r = 0.10 \quad t = 20$$

$$S = Pe^{rt} = 1000 \times e^{(0.10)(20)} = 1000 \times e^2 =$$

$$S = (1000)(7.389056099) = 7389.056099$$

graphing calculator solution:

$$S = 1000 * e^(0.10 * 20) = 7389.056099$$

future value (to nearest cent) = \$7389.06

**example 4.12c** Find the future value (to the nearest cent) if \$10,000 is invested for 18 years at 5% compounded continuously.

$$P = 1000 \quad r = 0.05 \quad t = 18$$

$$S = Pe^{rt} = 1000 \times e^{(0.05)(18)} = 1000 \times e^{0.9} =$$

$$S = (1000)(2.459603111) = 2459.603111$$

graphing calculator solution:

$$S = 10,000 * e^(0.05 * 18) = 24596.03111$$

future value (to nearest cent) = \$24,596.03

### TUTORIAL 4.13: FIND THE AMOUNT THAT WAS INVESTED INTO AN ACCOUNT THAT IS COMPOUNDED CONTINUOUSLY

Occasionally, we might know the future value amount along with the interest rate and the length of time for a particular account that has been compounded continuously but we do not know the initial amount (principal) that was invested. When this happens, we need to work backwards to determine the desired principal ( $P$ ). Recall that the future value of an investment can be determined by the formula on the left below. We can adapt this formula to solve for the principal ( $P$ ) by performing some simple algebraic steps as shown in the box on the right below.

**Future Value (Continuous Compounding):  $S = Pe^{rt}$**

where  $S$  = future value (dollars)

$P$  = principal (dollars)

$e$  = irrational number of approximately 2.718281828

$r$  = annual interest rate

$t$  = time (years)

$$S = Pe^{rt}$$

$$P = \frac{S}{e^{rt}} = S \div e^{rt}$$

graphing calculator notation:  $P = S \div (e^(r * t))$

**example 4.13a** What amount (to the nearest dollar) must be invested at 6.5% compounded continuously so that it will be worth \$25,000 after 8 years?

1. Determine the known values for the variables within the formula:

$P$  = principal = dollars invested = unknown for this problem

$r$  = annual interest rate = 6.5% = 0.065

$t$  = time in years = 8 years = 8

$S$  = future value amount = \$25,000

2. Using the known values, evaluate the formula:

$$P = \frac{S}{e^{rt}} = \frac{25,000}{e^{(0.065)(8)}} = \frac{25,000}{e^{0.52}} = 14863.0137 \rightarrow$$

**principal = \$14,863**

**graphing calculator solution:  $P = 25,000 \div (e^{(0.065 * 8)}) = 14863.0137$**

**example 4.13b** What amount (to the nearest dollar) must be invested at 2.5% compounded continuously so that it will be worth \$50,000 after 10 years?

$$S = 50,000$$

$$r = 0.025$$

$$t = 10$$

$$P = \frac{S}{e^{rt}} = \frac{50,000}{e^{(0.025)(10)}} = \frac{50,000}{e^{0.25}} = 38940.03915 \rightarrow \text{principal} = \$38,940$$

**graphing calculator solution:  $P = 50,000 \div (e^{(0.025 * 10)}) = 38940.03915$**

## TUTORIAL 4.14: FIND THE INTEREST RATE FOR AN ACCOUNT THAT IS COMPOUNDED CONTINUOUSLY

Occasionally, we might know the future value and the principal amounts along with the length of time for a particular account that has been compounded continuously but we do not know the annual interest rate at which the principal was invested. When this happens, we need to work backwards to determine the desired interest rate ( $r$ ). Recall that the future value of an investment can be determined by the formula in the first box below. We can adapt this formula to solve for the annual interest rate ( $r$ ) by performing some simple algebraic steps as shown in the second box below.

**Future Value (Continuous Compounding):  $S = Pe^{rt}$**

where  $S$  = future value (dollars)

$P$  = principal (dollars)

$e$  = irrational number of approximately 2.718281828

$r$  = annual interest rate

$t$  = time (years)

$$S = Pe^{rt} \rightarrow e^{rt} = \frac{S}{P} \rightarrow \ln\left(e^{rt}\right) = \ln\left(\frac{S}{P}\right) \rightarrow rt = \ln\left(\frac{S}{P}\right) \rightarrow r = \ln\left(\frac{S}{P}\right) \div t$$

**graphing calculator notation:  $r = \ln(S \div P) \div t$**

**NOTE:** Unlike the other exponential equations that we have discussed, it is best to only use the natural logarithm version of the *logarithm of a power property* to alter the original formula. To use the common logarithm version, we would need to use the numeric approximation for “ $e$ ” when completing the process – which would definitely complicate the solution process.

**example 4.14a**

**Find the interest rate (to the nearest percent) for an account where an initial amount of \$6200 grew to about \$9249 over a 5-year period if the interest earned was compounded continuously.**

1. Determine the known values for the variables within the formula:

$$P = \text{principal} = \text{dollars invested} = 6200$$

$$S = \text{future value amount} = \$9249$$

$$t = \text{time in years} = 5 \text{ years} = 5$$

2. Using the known values, evaluate the adjusted formula:

$$3. r = \ln\left(\frac{S}{P}\right) \div t = \ln\left(\frac{9249}{6200}\right) \div 5 = 0.079993229 \rightarrow \text{interest rate } \approx 8\%$$

**graphing calculator notation:  $r = \ln(9249 \div 6200) \div 5 = 0.0799932291$**

**example 4.14b**

**Find the interest rate (to the nearest percent) for an account where an initial amount of \$1000 grew to about \$7389 over a 20-year period if the interest earned was compounded continuously.**

$$P = 1000$$

$$S = 7389$$

$$t = 20$$

$$r = \ln\left(\frac{S}{P}\right) \div t = \ln\left(\frac{7389}{1000}\right) \div 20 = 0.9999962039 \rightarrow \text{interest rate } \approx 10\%$$

**graphing calculator notation:  $r = \ln(7389 \div 1000) \div 20 = 0.0999996204$**

**example 4.14c**

**Find the interest rate for an account where an initial amount of \$10,000 grew to about \$24,596 over an 18-year period if the interest earned was compounded continuously.**

$$P = 10,000$$

$$S = 24,596$$

$$t = 18$$

$$r = \ln\left(\frac{S}{P}\right) \div t = \ln\left(\frac{24,596}{10,000}\right) \div 18 = 0.0499999297 \rightarrow \text{interest rate } \approx 5\%$$

**graphing calculator notation:  $r = \ln(24,596 \div 10,000) \div 18 = 0.0499999297$**

## TUTORIAL 4.15: FIND THE TIME NEEDED FOR AN ACCOUNT THAT IS COMPOUNDED CONTINUOUSLY TO REACH A SPECIFIED AMOUNT

Occasionally, we might know the future value and the principal amounts along with the annual interest for a particular account that has been compounded continuously but we do not know the number of years at which the principal had been invested. When this happens, we need to work backwards to determine the desired time ( $t$ ). Recall that the future value of an investment can be determined by the formula in the first box below. We can adapt this formula to solve for the annual interest rate ( $r$ ) by performing some simple algebraic steps as shown in the second box below.

**Future Value (Continuous Compounding):  $S = Pe^{rt}$**

where  $S$  = future value (dollars)

$P$  = principal (dollars)

$e$  = irrational number of approximately 2.718281828

$r$  = annual interest rate

$t$  = time (years)

$$S = Pe^{rt} \rightarrow e^{rt} = \frac{S}{P} \rightarrow \ln\left(e^{rt}\right) = \ln\left(\frac{S}{P}\right) \rightarrow rt = \ln\left(\frac{S}{P}\right) \rightarrow t = \ln\left(\frac{S}{P}\right) \div r$$

graphing calculator notation:  $t = \ln(S \div P) \div r$

**NOTE:** Unlike the other exponential equations that we have discussed, it is best to only use the natural logarithm version of the *logarithm of a power property* to alter the original formula. To use the common logarithm version, we would need to use the numeric approximation for “ $e$ ” when completing the process – which would definitely complicate the solution process.

**example 4.15a** How long (to the nearest year) would \$6200 have to be invested at 8% (compounded continuously) to amount to about \$9249?

- Determine the known values for the variables within the formula:

$$P = \text{principal} = \text{dollars invested} = 6200$$

$$S = \text{future value amount} = \$9249$$

$$r = \text{interest rate} = 8\% = 0.08$$

- Using the known values, evaluate the adjusted formula:

$$t = \ln\left(\frac{S}{P}\right) \div r = \ln\left(\frac{9249}{6200}\right) \div 0.08 = 4.999576819$$

number of years  $\approx 5$

graphing calculator notation:

$$\textcolor{blue}{t = \ln(9249 \div 6200) \div 0.08 = 4.999576819}$$

**example 4.15b** How long (to the nearest year) would \$1000 have to be invested at an annual rate of 10% (compounded continuously) to amount to about \$7389?

$$P = 1000$$

$$S = 7389$$

$$r = 0.10$$

$$t = \ln\left(\frac{S}{P}\right) \div r = \ln\left(\frac{7389}{1000}\right) \div 0.10 = 19.99992408$$

number of years  $\approx 20$

graphing calculator notation:

$$\textcolor{blue}{t = \ln(7389 \div 1000) \div 0.10 = 19.99992408}$$

**example 4.15c** How long (to the nearest year) would \$10,000 have to be invested at an annual rate of 5% (compounded continuously) to amount to about \$24,596?

$$P = 10,000$$

$$S = 24,596$$

$$r = 0.05$$

$$t = \ln\left(\frac{S}{P}\right) \div r = \ln\left(\frac{24,596}{10,000}\right) \div 0.05 = 17.9999747$$

number of years  $\approx 18$

graphing calculator notation:

$$\textcolor{blue}{t = \ln(24,596 \div 10,000) \div 0.05 = 17.9999747}$$

## TUTORIAL 4.16: FIND THE ANNUAL PERCENTAGE YIELD (APY) OF MONEY IN AN ACCOUNT WHERE THE INTEREST IS COMPOUNDED PERIODICALLY

The annual percentage yield (APY) – which may also be called “effective or true annual rate” – is the rate of interest earned when the effect of compounding is taken into consideration. If the investment earns interest that is compounded periodically, use the formula stated below:

$$\text{Annual Percentage Yield (Periodic): } APY = -1 + \left(1 + \frac{r}{m}\right)^m$$

where APY = annual percentage yield  
 r = annual interest rate (as a decimal)  
 m = compounding periods per year

**graphing calculator notation:  $APY = -1 + (1 + (r \div m))^m$**

**example 4.16a** What is the annual percentage yield (to the nearest hundredth of a percent) for a nominal rate of 8.6% compounded monthly?

1. Determine the known values for the variables within the formula:

$$r = \text{annual interest rate} = 8.6\% = 0.086$$

$$m = \text{compounding periods per year} = 12 \text{ (monthly)}$$

2. Using the known values, evaluate the formula:

$$APY = \left(1 + \frac{r}{m}\right)^m - 1 = \left(1 + \frac{0.086}{12}\right)^{12} - 1 = 0.089472134 \rightarrow APY \approx 8.95\%$$

**graphing calculator solution:**

$$APY = -1 + (1 + (0.086 \div 12))^12 = 0.0894721336$$

**example 4.16b** What is the annual percentage yield (to the nearest hundredth of a percent) for a nominal rate of 8.6% compounded quarterly?

$$r = 8.6\% = 0.086 \quad m = 4 \text{ (quarterly)}$$

$$APY = \left(1 + \frac{r}{m}\right)^m - 1 = \left(1 + \frac{0.086}{4}\right)^4 - 1 = 0.0888134672 \rightarrow APY \approx 8.88\%$$

graphing calculator solution:

$$APY = -1 + (1 + (0.086 \div 4))^4 = 0.0888134672$$

**example 4.16c** What is the annual percentage yield (to the nearest hundredth of a percent) for a nominal rate of 8.6% compounded semi-annually?

$$r = 8.6\% = 0.086 \quad m = 2 \text{ (semi-annually)}$$

$$APY = \left(1 + \frac{r}{m}\right)^m - 1 = \left(1 + \frac{0.086}{2}\right)^2 - 1 = 0.087849 \rightarrow APY \approx 8.78\%$$

graphing calculator solution:

$$APY = -1 + (1 + (0.086 \div 2))^2 = 0.087849$$

### TUTORIAL 4.17: FIND THE ANNUAL PERCENTAGE YIELD (APY) OF MONEY IN AN ACCOUNT WHERE THE INTEREST IS COMPOUNDED CONTINUOUSLY

The annual percentage yield (APY) – which may also be called “effective or true annual rate” – is the rate of interest earned when the effect of compounding is taken into consideration. If the investment earns interest that is compounded continuously, use the formula stated below:

**Annual Percentage Yield (Continuous):  $APY = -1 + e^r$**

where  $APY$  = annual percentage yield

$e$  = irrational number of approximately 2.718281828

$r$  = nominal annual rate (as a decimal)

graphing calculator notation:  $APY = -1 + e^r$

**example 4.17a** **What is the annual percentage yield (to the nearest hundredth of a percent) for a nominal rate of 7.5% compounded continuously?**

1. Determine the known values for the variables within the formula:

$$r = \text{annual interest rate} = 7.5\% = 0.075$$

2. Using the known values, evaluate the formula:

$$APY = -1 + (e)^r = -1 + (e)^{0.075} = 0.077884151 \rightarrow APY \approx 7.79\%$$

**graphing calculator solution:**

$$APY = -1 + e^{(0.075)} = 0.0778841509$$

**example 4.17b** **What is the annual percentage yield (to the nearest hundredth of a percent) for a nominal rate of 8.6% compounded continuously?**

$$r = \text{annual interest rate} = 8.6\% = 0.086$$

$$APY = -1 + (e)^r = -1 + (e)^{0.086} = 0.898063283 \rightarrow APY \approx 8.98\%$$

**graphing calculator solution:**

$$APY = -1 + e^{(0.086)} = 0.898063283$$

**example 4.17c** **What is the annual percentage yield (to the nearest hundredth of a percent) for a nominal rate of 1.25% compounded continuously?**

$$r = \text{annual interest rate} = 1.25\% = 0.0125$$

$$APY = -1 + (e)^r = -1 + (e)^{0.0125} = 0.0125784515 \rightarrow APY \approx 1.26\%$$

**graphing calculator solution:**

$$APY = -1 + e^{(0.0125)} = 0.0125784515$$

## TUTORIAL 4.18: COMPUTE THE FUTURE VALUE OF AN ORDINARY ANNUITY

We often invest (or save) money by depositing small amounts at different times. When we make equal deposits at regular intervals, we are contributing to an annuity. The deposits (or payments) may be made at any preset period: annually, semi-annually, quarterly, monthly and so on. If these deposits (or payments) are made at the **end** of the preset period, the annuity is called an *ordinary annuity*. The sum of all of these deposits (or payments) plus all of the interest earned is call the *future value of the ordinary annuity* and can be calculated by using the formula below:

$$\text{Future Value of an Ordinary Annuity: } S = R \left[ \frac{-1 + (1+i)^n}{i} \right]$$

where: **S** = future value (in dollars) of the ordinary annuity

where: **R** = amount (dollars) deposited at the **end** of each period

where: **i** = interest rate per period (as a decimal) =  $\frac{\text{nominal annual rate as a decimal}}{\text{total compounding periods per year}} = \frac{r}{m}$

where: **n** = total number of compounding periods = (compounding periods per year)  
(years) = (m)(t)

graphing calculator notation:

$$S = R * ((-1 + (1 + i)^n) \div i)$$

$$S = R * ((-1 + (1 + (r \div m))^n) \div (r \div m))$$

**example 4.18a** Suppose that you deposit \$300 at the *end* of each quarter into an account that pays 4% compounded quarterly. How much money (to the nearest dollar) will you have in the account in  $2\frac{1}{4}$  years?

- Determine the known values for the variables within the formula:

$$\begin{array}{lll} \text{quarterly} & & 2.25 \text{ years} \\ R = 300 & r = 4\% = 0.04 & m = 4 \\ & & t = 2.25 \end{array}$$

- Use the values for  $r$ ,  $m$  and  $t$  to determine the appropriate values for  $i$  and  $n$ :

$$i = \frac{r}{m} = \frac{0.04}{4} = 0.01 \quad n = (m)(t) = (4)(2.25) = 9$$

Using the known values, evaluate the formula:

$$S = R \left[ \frac{-1 + (1+i)^n}{i} \right] = 300 \left[ \frac{-1 + (1+0.01)^9}{0.01} \right] = 2810.558181$$

**graphing calculator solution:**

$$\begin{aligned} S &= 300 * ((-1 + (1.01)^9) \div 0.01) = 2810.558181 \\ \text{future value} &\approx \$2811 \end{aligned}$$

**example 4.18b** Find the future value (to the nearest cent) of an ordinary annuity of \$1000 paid monthly for 4 years if the interest rate is 9%, compounded monthly.

$$\begin{array}{lll} \text{monthly} & & 4 \text{ years} \\ R = 1000 & r = 9\% = 0.09 & m = 12 \\ & & t = 4 \end{array}$$

$$i = \frac{r}{m} = \frac{0.09}{12} = 0.0075 \quad n = (m)(t) = (12)(4) = 48$$

$$S = R \left[ \frac{-1 + (1+i)^n}{i} \right] = 1000 \left[ \frac{-1 + (1+0.0075)^{48}}{0.0075} \right] = 57520.71111$$

**graphing calculator solution:**

$$\begin{aligned} S &= 1000 * ((-1 + (1.0075)^{48}) \div 0.0075) = 57520.71111 \\ \text{future value} &\approx \$57,521 \end{aligned}$$

**example 4.18c** Suppose Donald's parents agree to invest \$1000 (at 8%, compounded semi-annually) for him on the December 31st or the June 30th following each semester that he makes the Dean's list during his 4 years in college. If Donald makes the Dean's list in each of the 8 semesters, how much money (to the nearest dollar) will his parents have to give him when he graduates?

$$R = 1000 \quad r = 8\% = 0.08 \quad \begin{matrix} \text{semi-annually} \\ m = 2 \end{matrix} \quad \begin{matrix} 4 \text{ years} \\ t = 4 \end{matrix}$$

$$i = \frac{r}{m} = \frac{0.08}{2} = 0.04 \quad n = (m)(t) = (2)(4) = 8$$

$$S = R \left[ \frac{-1 + (1+i)^n}{i} \right] = 1000 \left[ \frac{-1 + (1+0.04)^8}{0.04} \right] = 9214.22626$$

graphing calculator solution:

$$S = 1000 * ((-1 + (1 + (0.08 \div 2))^8)$$

$$\div (0.08 \div 2)) = 9214.22626$$

future value  $\approx \$9214$

### TUTORIAL 4.19: COMPUTE THE PAYMENT AMOUNTS REQUIRED IN ORDER FOR AN ORDINARY ANNUITY TO HAVE A SPECIFIED FUTURE VALUE

Occasionally, we might know the future value amount needed for an ordinary annuity along with the interest rate, the number of compounding periods per year and the length of time for a particular annuity that has been periodically compounded but we do not know the payment amount needed to accomplish the desired monetary goal. When this happens, we need to work backwards to determine the desired payment ( $R$ ).

Recall that the future value of an ordinary annuity can be determined by the formula in the top box below. We can adapt this formula to solve for the deposit amount ( $R$ ) by performing some simple algebraic steps as shown in the second box below.

$$\text{Future Value of an Ordinary Annuity: } S = R \left[ \frac{-1 + (1+i)^n}{i} \right]$$

where:

$S$  = future value (in dollars) of the ordinary annuity

$R$  = amount (dollars) deposited at the end of each period

$i$  = interest rate per period (as a decimal) =  $\frac{\text{nominal annual rate as a decimal}}{\text{total compounding periods per year}} = \frac{r}{m}$

$n$  = total number of compounding periods = (compounding periods per year)(years) =  $(m)(t)$

$$S = R \left[ \frac{-1 + (1+i)^n}{i} \right] \rightarrow R = S \div \left[ \frac{-1 + (1+i)^n}{i} \right] = \frac{S(i)}{-1 + (1+i)^n}$$

graphing calculator notation:

$$R = S * i \div (-1 + (1 + i)^n)$$

$$R = S * (r \div m) \div (-1 + (1 + (r \div m))^n)$$

**example 4.19a** A family wants to have a \$500,000 college fund for their children at the end of 20 years. What contribution (to the nearest dollar) must be made at the *end* of each quarter if their investment pays 7.8%, compounded quarterly?

- Determine the known values for the variables within the formula:

$$r = 7.8\% \quad \text{quarterly} \quad 20 \text{ years}$$

$$S = 500,000 \quad r = 0.078 \quad m = 4 \quad t = 20$$

- Use the values for  $r$ ,  $m$  and  $t$  to determine the appropriate values for  $i$  and  $n$ :

$$i = \frac{r}{m} = \frac{0.078}{4} = 0.0195 \quad n = (m)(t) = (4)(20) = 80$$

- Using the known values, evaluate the adjusted formula for  $R$ :

$$R = \left[ \frac{S(i)}{-1 + (1+i)^n} \right] = \left[ \frac{(500,000)(0.0195)}{-1 + (1+0.0195)^{80}} \right] = 2643.780471 = 2643.780471$$

**graphing calculator solution:**

$$R = 500,000 * 0.0195 \div (-1 + (1.0195)^{80}) = 2643.780471$$

quarterly deposit  $\approx \$2644$

**example 4.19b** The Jacksons want to save \$50,000 in 3 years for a down payment on a house. If they make deposits at the *end* of each month into an account paying 12% (compounded monthly), what size payments (to the nearest dollar) are required to meet their goal?

$$r = 12\% \quad \text{monthly} \quad 3 \text{ years}$$

$$S = 50,000 \quad r = 0.12 \quad m = 12 \quad t = 3$$

$$i = \frac{r}{m} = \frac{0.12}{12} = 0.01 \quad n = (m)(t) = (12)(3) = 36$$

$$R = \left[ \frac{S(i)}{-1 + (1+i)^n} \right] = \left[ \frac{(50,000)(0.01)}{-1 + (1+0.01)^{36}} \right] = 1160.715491$$

**graphing calculator solution:**

$$R = 50,000 * 0.01 \div (-1 + (1.01)^{36}) = 1160.715491$$

monthly deposit  $\approx \$1161$

## TUTORIAL 4.20: FIND OUT HOW LONG IT WILL TAKE FOR AN ORDINARY ANNUITY TO REACH A SAVINGS GOAL

Occasionally, we might know the future value and the periodic payment amounts along with the interest rate and the number of compounding periods per year for a particular account that has been compounded periodically but we do not know the time (number of years) at which the annuity payments have been invested. When this happens, we need to work backwards to determine the desired time ( $t$ ). Recall that the future value of an ordinary annuity can be determined by the formula in the top box below. We can adapt this formula to solve for the deposit amount ( $R$ ) by performing some simple algebraic steps as shown in the second box below:

$$\text{Future Value of an Ordinary Annuity: } S = R \left[ \frac{-1 + (1+i)^n}{i} \right] = R \left[ \frac{-1 + (1+i)^{mt}}{i} \right]$$

**where:**

**S = future value (in dollars) of the ordinary annuity**

**R = amount (dollars) deposited at the end of each period**

**where:**

**i = interest rate per period (as a decimal) =  $\frac{\text{nominal annual rate as a decimal}}{\text{total compounding periods per year}} = \frac{r}{m}$**

**n = total number of compounding periods = (compounding periods per year)(years) = (m)(t)**

$$S = R \left[ \frac{-1 + (1+i)^{mt}}{i} \right] \rightarrow \frac{S}{R} = \left[ \frac{-1 + (1+i)^{mt}}{i} \right] \rightarrow \frac{S(i)}{R} = -1 + (1+i)^{mt} \rightarrow 1 + \frac{S(i)}{R} = (1+i)^{mt}$$

*common logarithms version:*

$$\log\left(1 + \frac{S(i)}{R}\right) = \log\left((1+i)^{mt}\right) \rightarrow \log\left(1 + \frac{S(i)}{R}\right) = (mt)(\log(1+i)) \rightarrow$$

$$mt = \frac{\log\left(1 + \frac{S(i)}{R}\right)}{\log(1+i)} \rightarrow t = \frac{\log\left(1 + \frac{S(i)}{R}\right)}{(m)(\log(1+i))}$$

*natural logarithms version:*

$$\ln\left(1 + \frac{S(i)}{R}\right) = \ln\left((1+i)^{mt}\right) \rightarrow \ln\left(1 + \frac{S(i)}{R}\right) = (mt)(\ln(1+i)) \rightarrow$$

$$mt = \frac{\ln\left(1 + \frac{S(i)}{R}\right)}{\ln(1+i)} \rightarrow t = \frac{\ln\left(1 + \frac{S(i)}{R}\right)}{(m)(\ln(1+i))}$$

#### graphing calculator notation:

$$t = \log(1 + (S * i / R)) / (m * \log(1 + i)) = \log(1 + (S * (r / m) / R)) / (m * \log(1 + (r / m)))$$

$$t = \ln(1 + (S * i / R)) / (m * \ln(1 + i)) = \ln(1 + (S * (r / m) / R)) / (m * \ln(1 + (r / m)))$$

Recall that depending on what calculator is being used, we might have to search for the “log” and the “ln” functions. Each one might be a button on the face of the calculator or each might be hidden within a menu list on the calculator.

**example 4.20a** If \$5000 is deposited at the end of each half year in an account that earns 6.5% (compounded semi-annually), how many years (to the nearest tenth of a year) will it take before the account contains \$125,000?

1. Determine the known values for the variables within the formula:

$$\begin{array}{lll} r = 6.5\% & \text{semi-annually} \\ R = 5000 & r = 0.065 & m = 2 \\ & & S = 125,000 \end{array}$$

2. Use the values for  $r$  and  $m$  to determine the appropriate value for  $i$ :

$$i = \frac{r}{m} = \frac{0.065}{2} = 0.0325$$

3. Using the known values, evaluate the adjusted formula for  $t$ :  
*common logarithms version:*

$$\begin{aligned} t &= \frac{\log\left(1 + \frac{S(i)}{R}\right)}{(m)(\log(1+i))} = \frac{\log\left(1 + \frac{(125,000)(0.0325)}{5000}\right)}{(2)(\log(1+0.0325))} = \frac{\log(1+0.8125)}{(2)(\log(1.0325))} \\ t &= \frac{\log(1.8125)}{(2)(\log(1.0325))} = 9.297224387 \approx \mathbf{9.3} \end{aligned}$$

*natural logarithms version:*

$$\begin{aligned} t &= \frac{\ln\left(1 + \frac{S(i)}{R}\right)}{(m)(\ln(1+i))} = \frac{\ln\left(1 + \frac{(125,000)(0.0325)}{5000}\right)}{(2)(\ln(1+0.0325))} = \frac{\ln(1+(25)(0.0325))}{(2)(\ln(1.0325))} \\ t &= \frac{\ln(1+0.8125)}{(2)(\ln(1.0325))} = \frac{\ln(1.8125)}{(2)(\ln(1.0325))} = 9.297224387 \approx \mathbf{9.3} \end{aligned}$$

**graphing calculator solution (using log):**

$$t = \log(1 + (125,000 * 0.0325 \div 5000)) \div (2 * \log(1.0325)) = 9.297224387$$

$$t = \log(1 + (125,000 * (0.065 \div 2) \div 5000)) \div (2 * \log(1 + (0.065 \div 2)))$$

**graphing calculator solution (using ln):**

$$t = \ln(1 + (125,000 * 0.0325 \div 5000)) \div (2 * \ln(1.0325)) = 9.297224387$$

$$t = \ln(1 + (125,000 * (0.065 \div 2) \div 5000)) \div (2 * \ln(1 + (0.065 \div 2)))$$

**example 4.20b** If \$2500 is deposited at the *end* of each month in an account that earns an annual rate of 7.5% compounded monthly, how many years (to the nearest tenth of a year) will it take before the account contains \$100,000?

$$\begin{aligned} r &= 7.5\% && \text{monthly} \\ R &= 2500 & r &= 0.075 & m &= 12 & S &= 100,000 \\ i &= \frac{r}{m} = \frac{0.075}{12} = 0.00625 \end{aligned}$$

common logarithms version:

$$\begin{aligned} t &= \frac{\log\left(1 + \frac{S(i)}{R}\right)}{(m)(\log(1+i))} = \frac{\log\left(1 + \frac{(100,000)(0.00625)}{2500}\right)}{(12)(\log(1.00625))} = \frac{\log(1 + (40)(0.00625))}{(12)(\log(1.00625))} \\ t &= \frac{\log(1+0.25)}{(12)(\log(1.00625))} = \frac{\log(1.25)}{(12)(\log(1.00625))} = 2.984535344 \approx \mathbf{3.0} \end{aligned}$$

natural logarithms version:

$$\begin{aligned} t &= \frac{\ln\left(1 + \frac{S(i)}{R}\right)}{(m)(\ln(1+i))} = \frac{\ln\left(1 + \frac{(100,000)(0.00625)}{2500}\right)}{(12)(\ln(1+0.00625))} = \frac{\ln(1 + (40)(0.00625))}{(12)(\ln(1+0.00625))} \\ t &= \frac{\ln(1+0.25)}{(12)(\ln(1.00625))} = \frac{\ln(1.25)}{(12)(\ln(1.00625))} = 2.984535344 \approx \mathbf{3.0} \end{aligned}$$

graphing calculator solution (using log):

$$t = \log(1 + (100,000 * 0.00625 / 2500)) \div (12 * \log(1.00625)) = 2.984535344$$

$$t = \log(1 + (100,000 * (0.075 / 12) / 2500)) \div (12 * \log(1 + (0.075 / 12)))$$

graphing calculator solution (using ln):

$$t = \ln(1 + (100,000 * 0.00625 / 2500)) \div (12 * \ln(1.00625)) = 2.984535344$$

$$t = \ln(1 + (100,000 * (0.075 / 12) / 2500)) \div (12 * \ln(1 + (0.075 / 12)))$$

## TUTORIAL 4.21: COMPUTE THE FUTURE VALUES OF AN ANNUITY DUE

We often invest (or save) money by depositing small amounts at different times. When we make equal deposits at regular intervals, we are contributing to an annuity. The deposits (or payments) may be made at any preset period: annually, semi-annually, quarterly, monthly and so on. If these deposits (or payments) are made at the **beginning** of the preset period, the annuity is called an ***annuity due***. The sum of all of these deposits (or payments) plus all of the interest earned is call the ***future value of the annuity due*** and can be calculated by using the formula below:

$$\text{Future Value of an Annuity Due: } S_{\text{due}} = R \left[ \frac{-1 + (1+i)^n}{i} \right] (1+i)$$

where

$S_{\text{due}}$  = future value (in dollars) of the annuity due

$R$  = amount (dollars) deposited at the **beginning** of each period

$i$  = interest rate per period (as a decimal) =  $\frac{\text{nominal annual rate as a decimal}}{\text{total compounding periods per year}} = \frac{r}{m}$

$n$  = total number of compounding periods = (compounding periods per year)(years) =  $(m)(t)$

graphing calculator notation:

$$S_{\text{due}} = R * (1 + i) * ((-1 + (1 + i)^n) / i)$$

$$S_{\text{due}} = R * (1 + (r / m)) * ((-1 + (1 + (r / m))^n) / (r / m))$$

**example 4.21a** Find the future value (to the nearest dollar) of an annuity due of \$250 each quarter for 3 years at 12%, compounded quarterly.

- Determine the known values for the variables within the formula:

$$\begin{array}{lll} r = 12\% & \text{quarterly} & 3 \text{ years} \\ R = 250 & r = 0.12 & m = 4 \\ & & t = 3 \end{array}$$

- Use the values for  $r$ ,  $m$  and  $t$  to determine the appropriate values for  $i$  and  $n$ :

$$i = \frac{r}{m} = \frac{0.12}{4} = 0.03 \quad n = (m)(t) = (4)(3) = 12$$

- Using the known values, evaluate the adjusted formula for  $t$ :

$$\begin{aligned} S_{due} &= R \left[ \frac{-1 + (1+i)^n}{i} \right] (1+i) = 250 \left[ \frac{-1 + (1+0.03)^{12}}{0.03} \right] (1+0.03) \\ S_{due} &= 250 \left[ \frac{-1 + (1.03)^{12}}{0.03} \right] (1.03) = 3654.447612 \approx \$3654 \end{aligned}$$

**graphing calculator solution:**

$$\begin{aligned} S_{due} &= 250 * (1.03) * ((-1 + (1.03)^{12}) \div 0.03) = 3654.447612 \\ S_{due} &= 250 * (1 + (0.12 \div 4)) * ((-1 + (1 + (0.12 \div 4))^{(4 * 3)}) \div (0.12 \div 4)) \end{aligned}$$

**example 4.21b** Find the future value (to the nearest dollar) of an annuity of \$500 paid at the *beginning* of each period for 10 years if the interest rate is 8%, compounded semi-annually.

$$\begin{array}{lll} r = 8\% & \text{semi-annually} & 10 \text{ years} \\ R = 500 & r = 0.08 & m = 2 \\ & & t = 10 \end{array}$$

$$i = \frac{r}{m} = \frac{0.08}{2} = 0.04 \quad n = (m)(t) = (2)(10) = 20$$

$$S_{due} = R \left[ \frac{-1 + (1+i)^n}{i} \right] (1+i) = 500 \left[ \frac{-1 + (1+0.04)^{20}}{0.04} \right] (1+0.04)$$

$$S_{due} = 500 \left[ \frac{-1 + (1.04)^{20}}{0.04} \right] (1.04) = 15484.60086 \approx \$15,485$$

**graphing calculator solution:**

$$\begin{aligned} S_{due} &= 500 * (1.04) * ((-1 + (1.04)^{20}) \div 0.04) = 15484.60086 \\ S_{due} &= 500 * (1 + (0.08 \div 2)) * ((-1 + (1 + (0.08 \div 2))^{(2 * 10)}) \div (0.08 \div 2)) \end{aligned}$$

**example 4.21c** A house is rented for \$3600 per month, with each month's rent payable in advance. If money is worth 6% (compounded monthly) and the rent is deposited in an account, what is the future value of the rent for one year?

$$\begin{array}{lll} r = 6\% & \text{monthly} & 1 \text{ year} \\ R = 3600 & r = 0.06 & m = 12 \\ & & t = 1 \end{array}$$

$$i = \frac{r}{m} = \frac{0.06}{12} = 0.005 \quad n = (m)(t) = (12)(1) = 12$$

$$S_{due} = R \left[ \frac{-1 + (1+i)^n}{i} \right] (1+i) = 3600 \left[ \frac{-1 + (1+0.005)^{12}}{0.005} \right] (1+0.005)$$

$$S_{due} = 3600 \left[ \frac{-1 + (1.005)^{12}}{0.005} \right] (1.005) = 44630.06466 \approx \$44,630$$

**graphing calculator solution:**

$$S_{due} = 3600 * (1.005) * ((-1 + (1.005)^{12}) \div 0.005) = 44630.06467$$

$$S_{due} = 3600 * (1 + (0.06 \div 12)) * ((-1 + (1 + (0.06 \div 12))^{(12 * 1)}) \div (0.06 \div 12))$$

## TUTORIAL 4.22: COMPUTE THE PAYMENT AMOUNTS REQUIRED IN ORDER FOR AN ANNUITY DUE TO HAVE SPECIFIED FUTURE VALUES

Occasionally, we might know the future value along with the interest rate, the number of compounding periods per year and the length of time for a particular account that has been compounded periodically but we do not know the periodic payment amount ( $R$ ). When this happens, we need to work backwards to determine the desired value. Recall that the future value of an annuity due can be determined by the formula in the top box below. We can adapt this formula to solve for the deposit amount ( $R$ ) by performing some simple algebraic steps as shown in the second box below:

$$\text{Future Value of an Annuity Due: } S_{\text{due}} = R \left[ \frac{-1 + (1+i)^n}{i} \right] (1+i)$$

where

$S_{\text{due}}$  = future value (in dollars) of the annuity due

$R$  = amount (dollars) deposited at the beginning of each period

$i$  = interest rate per period (as a decimal) =  $\frac{\text{nominal annual rate as a decimal}}{\text{total compounding periods per year}} = \frac{r}{m}$

$n$  = total number of compounding periods = (compounding periods per year)(years) =  $(m)(t)$

$$S_{\text{due}} = R \left[ \frac{-1 + (1+i)^n}{i} \right] (1+i) \rightarrow R = S_{\text{due}} \div \left[ \frac{-1 + (1+i)^n}{i} \right] \div (1+i) \rightarrow R = \frac{(S_{\text{due}})(i)}{(-1 + (1+i)^n)(1+i)}$$

graphing calculator notation:

$$R = S_{\text{due}} * i \div (-1 + (1 + i)^n) \div (1 + i)$$

$$R = S_{\text{due}} * (r \div m) \div (-1 + (1 + (r \div m))^n) \div (1 + (r \div m))$$

**example 4.22a** How much must be deposited (to the nearest dollar) at the beginning of each year into an annuity account that pays 8% (compounded annually) so that the account will contain \$100,000 at the end of 10 years?

- Determine the known values for the variables within the formula:

$$\begin{array}{llll} S_{due} = & r = 8\% & \text{annually} & 10 \text{ years} \\ 100,000 & r = 0.08 & m = 1 & t = 10 \end{array}$$

- Use the values for  $r$ ,  $m$  and  $t$  to determine the appropriate values for  $i$  and  $n$ :

$$i = \frac{r}{m} = \frac{0.08}{1} = 0.08 \quad n = (m)(t) = (1)(10) = 10$$

- Using the known values, evaluate the adjusted formula for  $t$ :

$$R = \frac{(S_{due})(i)}{(-1 + (1+i)^n)(1+i)} = \frac{(100,000)(0.08)}{(-1 + (1+0.08)^{10})(1+0.08)} = 6391.619324$$

**initial deposit ≈ \$6392**

**graphing calculator solution:**

$$R = 100,000 * 0.08 \div (-1 + (1.08)^{10}) \div (1.08) = 6391.619324$$

$$R = 100,000 * (0.08 \div 1) \div (-1 + (1 + (0.08 \div 1))^{(1 * 10)}) \div (1 + (0.08 \div 1))$$

**example 4.22b** Determine the size of the payments (to the nearest dollar) that must be deposited at the beginning of each period in an account that pays 7.8% (compounded semi-annually) so that the account will have a future value of \$120,000 at the end of 15 years?

$$\begin{array}{llll} S_{due} = 120,000 & r = 7.8\% & \text{semi-annually} & 15 \text{ years} \\ & r = 0.078 & m = 2 & t = 15 \end{array}$$

$$i = \frac{r}{m} = \frac{0.078}{2} = 0.039 \quad n = (m)(t) = (2)(15) = 30$$

$$R = \frac{(S_{due})(i)}{(-1 + (1+i)^n)(1+i)} = \frac{(120,000)(0.039)}{(-1 + (1+0.039)^{30})(1+0.039)} = 2093.93643$$

**initial deposit ≈ \$2094**

**graphing calculator solution:**

$$R = 120,000 * 0.039 \div (-1 + (1.039)^{30}) \div (1.039) = 2093.93643$$

$$R = 120,000 * (0.078 \div 2) \div (-1 + (1 + (0.078 \div 2))^{(2 * 15)}) \div (1 + (0.078 \div 2))$$

## TUTORIAL 4.23: FIND OUT HOW LONG IT WILL TAKE FOR AN ANNUITY DUE TO REACH A SAVINGS GOAL

Occasionally, we might know the future value and the periodic payment amounts along with the interest rate and the number of compounding periods per year but we do not know the number of years (time) for the given future value to be reached. When this happens, we need to work backwards to determine the desired value. Recall that the future value of an annuity due can be determined by the formula in the top box below. We can adapt this formula to solve for the time ( $t$ ) by performing some simple algebraic steps as shown in the second box below:

$$\text{Future Value of an Annuity Due: } S_{\text{due}} = R \left[ \frac{-1 + (1+i)^n}{i} \right] (1+i) = R \left[ \frac{-1 + (1+i)^{mt}}{i} \right] (1+i)$$

where:

$S_{\text{due}}$  = future value (in dollars) of the annuity due

$R$  = amount (dollars) deposited at the beginning of each period

$i$  = interest rate per period (as a decimal) =  $\frac{\text{nominal annual rate as a decimal}}{\text{total compounding periods per year}} = r$

$n$  = total number of compounding periods = (compounding periods per year)(years) =  $(m)(t)$

$$S_{\text{due}} = R \left[ \frac{-1 + (1+i)^{mt}}{i} \right] (1+i) \rightarrow \frac{S_{\text{due}}}{R(1+i)} = \frac{-1 + (1+i)^{mt}}{i} \rightarrow$$

$$\frac{(S_{\text{due}})(i)}{R(1+i)} = -1 + (1+i)^{mt} \rightarrow 1 + \frac{(S_{\text{due}})(i)}{R(1+i)} = (1+i)^{mt} \rightarrow$$

*common logarithms version:*

$$\log \left( 1 + \frac{(S_{\text{due}})(i)}{R(1+i)} \right) = \log \left( (1+i)^{mt} \right) \rightarrow \log \left( 1 + \frac{(S_{\text{due}})(i)}{R(1+i)} \right) = mt \cdot \log(1+i) \rightarrow t = \frac{\log \left( 1 + \frac{(S_{\text{due}})(i)}{R(1+i)} \right)}{m \cdot \log(1+i)}$$

*natural logarithms version:*

$$\ln \left( 1 + \frac{(S_{\text{due}})(i)}{R(1+i)} \right) = \ln \left( (1+i)^{mt} \right) \rightarrow \ln \left( 1 + \frac{(S_{\text{due}})(i)}{R(1+i)} \right) = mt \cdot \ln(1+i) \rightarrow t = \frac{\ln \left( 1 + \frac{(S_{\text{due}})(i)}{R(1+i)} \right)}{m \cdot \ln(1+i)}$$

**graphing calculator notation:**

$$t = \log(1 + (S_{\text{due}} * i) / (R * (1 + i))) / (m * \log(1 + i))$$

$$t = \ln(1 + (S_{\text{due}} * i) / (R * (1 + i))) / (m * \ln(1 + i))$$

**example 4.23a** Suppose that you deposit \$1000 in an account at the beginning of each 3-month period for a certain number of years. If the account pays interest that is compounded quarterly at the annual rate of 11%, how long (to the nearest tenth of a year) will it take for the account to reach \$125,000?

1. Determine the known values for the variables within the formula:

$$r = 11\% \quad \text{quarterly}$$

$$S_{\text{due}} = 125,000 \quad R = 1000 \quad r = 0.11 \quad m = 4$$

2. Use the values for  $r$  and  $m$  to determine the appropriate value for  $i$ :

$$i = \frac{r}{m} = \frac{0.11}{4} = 0.0275$$

3. Using the known values, evaluate the adjusted formula for  $t$ :

*common logarithms version:*

$$t = \frac{\log\left(1 + \frac{(S_{\text{due}})(i)}{R(1+i)}\right)}{m \cdot \log(1+i)} = \frac{\log\left(1 + \frac{(125,000)(0.0275)}{(1000)(1+0.0275)}\right)}{(4)(\log(1+0.0275))} = \frac{\log\left(1 + \frac{3437.5}{1027.5}\right)}{(4)(\log(1.0275))}$$

$$t = 13.5386354 \approx \mathbf{13.5 \text{ years}}$$

4. *natural logarithms version:*

$$t = \frac{\ln\left(1 + \frac{(S_{\text{due}})(i)}{R(1+i)}\right)}{m \cdot \ln(1+i)} = \frac{\ln\left(1 + \frac{(125,000)(0.0275)}{(1000)(1+0.0275)}\right)}{(4)(\ln(1+0.0275))} = \frac{\ln\left(1 + \frac{3437.5}{1027.5}\right)}{(4)(\ln(1.0275))}$$

$$t = 13.5386354 \approx \mathbf{13.5 \text{ years}}$$

**graphing calculator solution:**

$$t = \log(1 + (125,000 * 0.0275) / (1000 * 1.0275)) / (4 * \log(1.0275))$$

$$t = \ln(1 + (125,000 * 0.0275) / (1000 * 1.0275)) / (4 * \ln(1.0275)) =$$

$$\mathbf{13.5386354}$$

**example 4.23b** A property owner has several rental units and wants to build more. He plans on depositing \$1750 at the beginning of each month in an account that earns an annual rate of 7.2% (compounded monthly). How long (to the nearest tenth of a year) will it take for the account to reach \$150,000?

$$\begin{array}{lll} r = 7.2\% & \text{monthly} \\ S_{due} = 150,000 & R = 1750 & r = 0.072 \\ i = \frac{r}{m} = \frac{0.072}{12} = 0.006 & & m = 12 \end{array}$$

*common logarithms version:*

$$t = \frac{\log\left(1 + \frac{(S_{due})(i)}{R(1+i)}\right)}{m \cdot \log(1+i)} = \frac{\log\left(1 + \frac{(150,000)(0.006)}{(1750)(1+0.006)}\right)}{(12)(\log(1+0.006))} = \frac{\log\left(1 + \frac{900}{1760.5}\right)}{(12)(\log(1.006))}$$

$$t = 5.752135126 \approx \mathbf{5.8 \text{ years}}$$

*natural logarithms version:*

$$t = \frac{\ln\left(1 + \frac{(S_{due})(i)}{R(1+i)}\right)}{m \cdot \ln(1+i)} = \frac{\ln\left(1 + \frac{(150,000)(0.006)}{(1750)(1+0.006)}\right)}{(12)(\ln(1+0.006))} = \frac{\ln\left(1 + \frac{900}{1760.5}\right)}{(12)(\ln(1.006))}$$

$$t = 5.752135126 \approx \mathbf{5.8 \text{ years}}$$

**graphing calculator solution:**

$$t = \log(1 + (150,000 * 0.006) \div (1750$$

$$* 1.006)) \div (12 * \log(1.006))$$

$$t = \ln(1 + (150,000 * 0.006) \div (1750 * 1.006)) \div (12 * \ln(1.006)) = \\ 5.752135126$$

## TUTORIAL 4.24: COMPUTE THE PRESENT VALUE OF AN ORDINARY ANNUITY

Recall that an ***ordinary annuity*** is a set of equal payments that occur at the **end** of evenly spaced periods (intervals). The ***present value of an ordinary annuity*** is the amount invested **today** in an ordinary annuity account at a specified interest rate so that future equal withdrawals (payments) can be made based on the original principal and all of the accumulated interest. The idea here is that these equal withdrawals (payments) will completely exhaust the accumulated amount at the end of the annuity. So, if we know the approximate amount of equal withdrawals (payments) needed, we can use the formula below to determine the present value needed to fund the desired annuity plan.

$$\text{Present Value of an Ordinary Annuity: } A_n = R \left[ \frac{1 - (1+i)^{-n}}{i} \right]$$

where

$A_n$  = present value (dollars)

$R$  = amount (dollars) withdrawn at the **end** of each period

$i$  = interest rate per period (as a decimal) =  $\frac{\text{nominal annual rate as a decimal}}{\text{total compounding periods per year}} = \frac{r}{m}$

$n$  = total number of compounding periods = (compounding periods per year)(years) =  $(m)(t)$

graphing calculator notation:  $A_n = R * (1 - (1 + i)^{-n}) / i$

**example 4.24a**

**How much (to the nearest dollar) is needed in an account that earns 8.4% compounded monthly in order to withdraw \$900 at the end of each month for 20 years?**

- Determine the known values for the variables within the formula:

$$\begin{array}{lll} r = 8.4\% & \text{monthly} & 20 \text{ years} \\ R = 900 & r = 0.084 & m = 12 \\ & & t = 20 \end{array}$$

- Use the values for  $r$ ,  $m$  and  $t$  to determine the appropriate value for  $i$  and  $n$ :

$$i = \frac{r}{m} = \frac{0.084}{12} = 0.007 \quad n = (m)(t) = (12)(20) = 240$$

- Using the known values, evaluate the formula for  $A_n$ :

$$A_n = R \left[ \frac{1 - (1+i)^{-n}}{i} \right] = 900 \left[ \frac{1 - (1.007)^{-240}}{0.007} \right] = 104468.4044 \approx \$104,468$$

**graphing calculator solution:**

$$A_n = 900 * (1 - (1.007)^{-240}) \div 0.007 = 104468.4044$$

**example 4.24b**

**How much (to the nearest dollar) is needed in an account that earns an annual rate of 8.7% compounded monthly in order to withdraw \$1000 at the end of each month for 17 years?**

$$\begin{array}{lll} r = 8.7\% & \text{monthly} & 17 \text{ years} \\ R = 1000 & r = 0.087 & m = 12 \\ & & t = 17 \end{array}$$

$$i = \frac{r}{m} = \frac{0.087}{12} = 0.00725 \quad n = (m)(t) = (12)(17) = 204$$

$$A_n = R \left[ \frac{1 - (1+i)^{-n}}{i} \right] = 1000 \left[ \frac{1 - (1.00725)^{-204}}{0.00725} \right] = 106333.174 \approx \$106,333$$

**graphing calculator solution:**

$$A_n = 1000 * (1 - (1.00725)^{-204}) \div 0.00725 = 106333.174$$

## TUTORIAL 4.25: COMPUTE THE PAYMENTS FOR A SPECIFIED PRESENT VALUE FOR AN ORDINARY ANNUITY

Occasionally, we might know the present value amount along with the interest rate and the length of time the interest rate was in effect but we do not know the payment (withdrawal) amount that will be possible from the accumulated principal and interest of the established ordinary annuity. When this happens, we need to work backwards to determine the desired payment (withdrawal) amount. Recall that the present value of an ordinary annuity can be determined by the formula in the first box below. We can adapt this formula to solve for the payment (withdrawal) amount ( $R$ ) by performing some simple algebraic steps as shown in the second box below:

$$\text{Present Value of an Ordinary Annuity: } A_n = R \left[ \frac{1 - (1+i)^{-n}}{i} \right]$$

**where:**

$A_n$  = present value (dollars)

$R$  = amount (dollars) withdrawn at the **end of each period**

$i$  = interest rate per period (as a decimal) =  $\frac{\text{nominal annual rate as a decimal}}{\text{total compounding periods per year}} = \frac{r}{m}$

$n$  = total number of compounding periods = (compounding periods per year)(years) =  $(m)(t)$

$$A_n = R \left[ \frac{1 - (1+i)^{-n}}{i} \right] \rightarrow R = A_n \div \left[ \frac{1 - (1+i)^{-n}}{i} \right] = \frac{(A_n)(i)}{1 - (1+i)^{-n}}$$

**graphing calculator notation:  $R = A_n * i \div (1 - (1 + i)^{-n})$**

**example 4.25a** With a present value of \$250,000, what size withdrawals (to the nearest dollar) can be made at the *end* of each quarter for the next 10 years if money is worth 6.4%, compounded quarterly?

- Determine the known values for the variables within the formula:

$$\begin{array}{lll} r = 6.4\% & \text{quarterly} & 10 \text{ years} \\ A_n = 250,000 & r = 0.064 & m = 4 \\ & & t = 10 \end{array}$$

- Use the values for  $r$ ,  $m$  and  $t$  to determine the appropriate value for  $i$  and  $n$ :

$$i = \frac{r}{m} = \frac{0.064}{4} = 0.016 \quad n = (m)(t) = (4)(10) = 40$$

- Using the known values, evaluate the adjusted formula for  $R$ :

$$R = \frac{(A_n)(i)}{1 - (1+i)^{-n}} = \frac{(250,000)(0.016)}{1 - (1.016)^{-40}} = 8510.103847 \approx \$8510$$

**graphing calculator solution:**

$$R = 250,000 * 0.016 \div (1 - 1.016^{-40}) = 8510.103847$$

**example 4.25b** If \$100,000 is invested in an annuity that earns an annual rate of 6.8% (compounded quarterly), what payments (to the nearest dollar) will it provide at the *end* of each quarter for the next 5.5 years?

$$\begin{array}{lll} r = 6.8\% & \text{quarterly} & 5.5 \text{ years} \\ A_n = 100,000 & r = 0.068 & m = 4 \\ & & t = 5.5 \end{array}$$

$$i = \frac{r}{m} = \frac{0.068}{4} = 0.017 \quad n = (m)(t) = (4)(5.5) = 22$$

$$R = \frac{(A_n)(i)}{1 - (1+i)^{-n}} = \frac{(100,000)(0.017)}{1 - (1.017)^{-22}} = 5486.400338 \approx \$5486$$

**graphing calculator solution:**

$$R = 100,000 * 0.017 \div (1 - 1.017^{-22}) = 5486.400338$$

## TUTORIAL 4.26: COMPUTE THE PRESENT VALUE OF AN ANNUITY DUE

Recall that an ***annuity due*** is a set of equal payments that occur at the **beginning** of evenly spaced periods (intervals). The ***present value of an annuity due*** is the amount invested **today** in an annuity due account at a specified interest rate so that future equal withdrawals (payments) can be made based on the original principal and all of the accumulated interest. The idea here is that these equal withdrawals (payments) will completely exhaust the accumulated amount at the end of the annuity. So, if we know the approximate amount of equal withdrawals (payments) needed, we can use the formula below to determine the present value needed to fund the desired annuity plan.

$$\text{Present Value of an Annuity Due: } A_{(n, \text{due})} = R \left[ \frac{1 - (1+i)^{-n}}{i} \right] (1+i)$$

where

$A_{(n, \text{due})}$  = present value (dollars)

$R$  = amount (dollars) withdrawn at the **beginning** of each period

$i$  = interest rate per period (as a decimal) =  $\frac{\text{nominal annual rate as a decimal}}{\text{total compounding periods per year}} = \frac{r}{m}$

$n$  = total number of compounding periods = (compounding periods per year)(years) =  $(m)(t)$

graphing calculator notation:  $A_{(n, \text{due})} = R * (1 + i) * (1 - (1 + i)^{-n}) \div i$

**example 4.26a** If the money is worth 5.8% (compounded quarterly), find the present value (to the nearest dollar) of an annuity due that pays \$6000 at the beginning of each quarter for the next 12 years.

- Determine the known values for the variables within the formula:

$$\begin{array}{lll} r = 5.8\% & \text{quarterly} & 12 \text{ years} \\ R = 6000 & r = 0.058 & m = 4 \\ & & t = 12 \end{array}$$

- Use the values for  $r$ ,  $m$  and  $t$  to determine the appropriate value for  $i$  and  $n$ :

$$i = \frac{r}{m} = \frac{0.058}{4} = 0.0145 \quad n = (m)(t) = (4)(12) = 48$$

- Using the known values, evaluate the adjusted formula for  $A_{(n, \text{due})}$ :

$$A_{(n, \text{due})} = R \left[ \frac{1 - (1+i)^{-n}}{i} \right] (1+i) = 6000 \left[ \frac{1 - (1.0145)^{-48}}{0.0145} \right] (1.0145)$$

**present value needed to fund annuity  $\approx \$209,446$**

**graphing calculator solution:**

$$\begin{aligned} A_{(n, \text{due})} &= 6000 * 1.0145 * (1 - 1.0145^{-48}) \div 0.0145 = \\ &209445.8558 \end{aligned}$$

**example 4.26b** Suppose an annuity will pay \$20,000 at the beginning of each year for the next 10 years. How much money (to the nearest dollar) is needed to start this annuity if it earns 6.8%, compounded annually?

$$\begin{array}{lll} r = 6.8\% & \text{annually} & 10 \text{ years} \\ R = 20,000 & r = 0.068 & m = 1 \\ & & t = 10 \end{array}$$

$$i = \frac{r}{m} = \frac{0.068}{1} = 0.068 \quad n = (m)(t) = (1)(10) = 10$$

$$A_{(n, \text{due})} = R \left[ \frac{1 - (1+i)^{-n}}{i} \right] (1+i) = 20,000 \left[ \frac{1 - (1.068)^{-10}}{0.068} \right] (1.068) = 151420.5483$$

**present value needed to fund annuity  $\approx \$151,421$**

**graphing calculator solution:**

$$\begin{aligned} A_{(n, \text{due})} &= 20,000 * 1.068 * (1 - 1.068^{-10}) \div 0.068 = \\ &151420.5483 \end{aligned}$$

**example 4.26c** A trust will provide \$15,000 to a county library at the beginning of each 3-month period for the next 2.5 years. If the money is worth an annual rate of 7.2% (compounded quarterly), find the present value amount (to the nearest dollar) of this trust.

$$\begin{array}{lll} r = 7.2\% & \text{quarterly} & 2.5 \text{ years} \\ R = 15,000 & r = 0.072 & m = 4 \quad t = 2.5 \end{array}$$

$$i = \frac{r}{m} = \frac{0.072}{4} = 0.018 \quad n = (m)(t) = (4)(2.5) = 10$$

$$A_{(n, \text{due})} = R \left[ \frac{1 - (1+i)^{-n}}{i} \right] (1+i) = 15,000 \left[ \frac{1 - (1.018)^{-10}}{0.018} \right] (1.018) = 138610.542$$

**present value needed to fund annuity  $\approx \$138,611$**

**graphing calculator solution:**

$$A_{(n, \text{due})} = 15,000 * 1.018 * (1 - 1.018^{-10}) / 0.018 = 138610.542$$

## TUTORIAL 4.27: COMPUTE THE PAYMENTS FOR A SPECIFIED PRESENT VALUE FOR AN ANNUITY DUE

Occasionally, we might know the present value amount along with the interest rate and the length of time the interest rate was in effect but we do not know the payment (withdrawal) amount that will be possible from the accumulated principal and interest of the established annuity due. When this happens, we need to work backwards to determine the desired payment (withdrawal) amount. Recall that the present value of an annuity due can be determined by the formula in the first box below. We can adapt this formula to solve for the payment (withdrawal) amount ( $R$ ) by performing some simple algebraic steps as shown in the second box below:

$$\text{Present Value of an Annuity Due: } A_{(n, \text{due})} = R \left[ \frac{1 - (1+i)^{-n}}{i} \right] (1+i)$$

**where:**

$A_{(n, \text{due})}$  = present value (dollars)

$R$  = amount (dollars) withdrawn at the **beginning of each period**

$i$  = interest rate per period (as a decimal) =  $\frac{\text{nominal annual rate as a decimal}}{\text{total compounding periods per year}} = \frac{r}{m}$

$n$  = total number of compounding periods = (compounding periods per year)(years) =  $(m)(t)$

$$A_{(n, \text{due})} = R \left[ \frac{1 - (1+i)^{-n}}{i} \right] (1+i) \rightarrow R = A_{(n, \text{due})} \div \left[ \frac{1 - (1+i)^{-n}}{i} \right] \div (1+i) \rightarrow R = \frac{(A_{(n, \text{due})})(i)}{(1+i)(1 - (1+i)^{-n})}$$

**graphing calculator notation:**  $R = A_{(n, \text{due})} * i \div (1 + i) \div (1 - (1 + i)^{-n})$

**example 4.27a** Recent sales of some real estate and record profits make it possible for a manufacturer to set aside \$2,000,000 in a fund to be used for modernization and remodeling. How much (to the nearest dollar) can be withdrawn from this fund at the beginning of each half-year period for the next 3 years if the fund earns 7.7%, compounded semi-annually?

1. Determine the known values for the variables within the formula:

$$\begin{array}{ll} A_{(n, \text{due})} = 2,000,000 & \text{semi-annually} \rightarrow m = 2 \\ r = 7.7\% = 0.077 & 3 \text{ years} \rightarrow t = 3 \end{array}$$

2. Use the values for  $r$ ,  $m$  and  $t$  to determine the appropriate value for  $i$  and  $n$ :

$$i = \frac{r}{m} = \frac{0.077}{2} = 0.0385 \quad n = (m)(t) = (2)(3) = 6$$

3. Using the known values, evaluate the adjusted formula for  $R$ :

$$R = \frac{(A_{(n, \text{due})})(i)}{(1+i)(1 - (1+i)^{-n})} = \frac{(2,000,000)(0.0385)}{(1.0385)(1 - (1.0385)^{-6})} = 365587.6599$$

**possible withdrawal amount  $\approx \$365,588$**

**graphing calculator solution:**

$$R = 2,000,000 * 0.0385 \div 1.0385 \div (1 - 1.0385^{-6}) = 365587.6599$$

**example 4.27b** Suppose a manufacturer has set aside \$925,000 in a fund to be used for modernization and remodeling. How much (to the nearest dollar) can be withdrawn from this fund at the beginning of each quarter for the next 5 years if the fund earns an annual rate of 6.8%, compounded quarterly?

$$A_{(n, \text{ due})} = 925,000$$

$$r = 6.8\% = 0.068$$

quarterly  $\rightarrow m = 4$

5 years  $\rightarrow t = 5$

$$i = \frac{r}{m} = \frac{0.068}{4} = 0.017$$

$$n = (m)(t) = (4)(5) = 20$$

$$R = \frac{(A_{(n, \text{ due})})(i)}{(1+i)\left(1-(1+i)^{-n}\right)} = \frac{(925,000)(0.017)}{(1.017)\left(1-(1.017)^{-20}\right)} = 54027.02337$$

**possible withdrawal amount  $\approx \$54,027$**

**graphing calculator solution:**

$$R = 925,000 * 0.017 \div 1.017 \div (1 - 1.017^{-20}) = 54027.02337$$

## TUTORIAL 4.28: COMPUTE THE PRESENT VALUE OF A DEFERRED ANNUITY

Recall that with an ordinary annuity, payments (withdrawals) are made at the end of the payment period. With an annuity due, payments (withdrawals) are made at the beginning of the payment period. A **deferred annuity** is an annuity where the first payment is not made at the end or the beginning of the first possible period but is postponed for a length of time.

The **present value of a deferred annuity** is the amount invested **today** in an annuity account at a specified interest rate so that future equal withdrawals (payments) can be made based on the original principal and all of the accumulated interest. The idea here is that these equal withdrawals (payments) will completely exhaust the accumulated amount at the end of the annuity. So, if we know the approximate amount of equal withdrawals (payments) needed, we can use the formula below to determine the present value needed to fund the desired annuity plan.

$$\text{Present Value of a Deferred Annuity: } A_{(n,k)} = R \left[ \frac{1 - (1+i)^{-n}}{i} \right] (1+i)^{-k}$$

where:

$A_{(n, \text{ due})}$  = present value (dollars)

$R$  = amount (dollars) withdrawn

$i$  = interest rate per period (as a decimal) =  $\frac{\text{nominal annual rate as a decimal}}{\text{total compounding periods per year}} = \frac{r}{m}$

$n$  = total number of compounding periods = (compounding periods per year)(years) =  $(m)(t)$

$k$  = number of periods to be deferred

graphing calculator notation:  $A_{(n, k)} = R * (1 - (1 + i)^{-n}) * ((1 + i)^{-k}) / i$

**example 4.28a** The terms of a single parent's last will & testament indicate that a niece will receive an annuity of \$50,000 per year from age 18 to age 24 (so the niece can attend college) and that the balance of the estate goes to a daughter. If the single parent dies on the niece's 14th birthday, how much money (to the nearest dollar) must be removed from the estate to purchase the annuity? (Assume an interest rate of 6%, compounded annually.)

- Determine the known values for the variables within the formula:

$$R = 2,000,000 \quad r = 6\% = 0.06 \quad \text{annually} \rightarrow m = 1$$

- Use the values for  $r$  and  $m$  to determine the appropriate value for  $i$ :

$$i = \frac{r}{m} = \frac{0.06}{1} = 0.06$$

- Determine the value of  $n$  (number of payment periods):  
*will receive an annuity per year from age 18 to age 24* →  
ages 17–18, 18–19, 19–20, 20–21, 21–22, 22–23, 23–24 →  $n = 7$
- Determine the value of  $k$  by (number of deferred periods):  
*single parent dies on the niece's 14th birthday* →  
will not receive: ages 14–15, 15–16, 16–17 →  $k = 3$
- Using the known values, evaluate the adjusted formula for  $A_{(n, k)}$ :

$$A_{(n,k)} = R \left[ \frac{1 - (1+i)^{-n}}{i} \right] (1+i)^{-k} = 50,000 \left[ \frac{1 - (1.06)^{-7}}{0.06} \right] (1.06)^{-3} = 234353.7551$$

present value of this deferred annuity ≈ \$234,354

graphing calculator solution:

$$A_{(n, k)} = 50,000 * (1 - 1.06^{-7}) * (1.06^{-3}) / 0.06 = 234353.7551$$

**example 4.28b** Find the present value (to the nearest dollar) of an annuity of \$2500, at the end of each quarter for 8 years after being deferred for 7 years, if the money is worth an annual rate of 8.4% compounded quarterly.

$$\begin{array}{ll} r = 8.4\% & \text{quarterly} \\ R = 2500 & r = 0.084 \\ & m = 4 \\ & t = 8 \end{array}$$

$$i = \frac{r}{m} = \frac{0.084}{4} = 0.021 \quad n = (m)(t) = (4)(8) = 32 \\ k = (4)(7) = 28$$

$$A_{(n,k)} = R \left[ \frac{1 - (1+i)^{-n}}{i} \right] (1+i)^{-k} = 2500 \left[ \frac{1 - (1.021)^{-32}}{0.021} \right] (1.021)^{-28} = 32315.48493$$

**present value of this deferred annuity  $\approx \$32,315$**

**graphing calculator solution:**

$$A_{(n,k)} = 2500 * (1 - 1.021^{-32}) * (1.021^{-28}) / 0.021 = 32315.48493$$

## TUTORIAL 4.29: COMPUTE THE PAYMENTS FOR A SPECIFIED PRESENT VALUE FOR A DEFERRED ANNUITY

Occasionally, we might know the present value amount needed for a deferred annuity along with the interest rate, the number of compounding periods per year, the total number of compounding periods for the life of the annuity and the number of periods that will be deferred...but...we do not know the payment (withdrawal) amount that will be possible from the accumulated principal and interest of the established deferred annuity. When this happens, we need to work backwards to determine the desired payment (withdrawal) amount. Recall that the present value of a deferred annuity can be determined by the formula in the first box below. We can adapt this formula to solve for the payment (withdrawal) amount ( $R$ ) by performing some simple algebraic steps as shown in the second box below:

$$\text{Present Value of a Deferred Annuity: } A_{(n,k)} = R \left[ \frac{1 - (1+i)^{-n}}{i} \right] (1+i)^{-k}$$

**where:**

$A_{(n, \text{ due})}$  = present value (dollars)

$R$  = amount (dollars) withdrawn

$i$  = interest rate per period (as a decimal) =  $\frac{\text{nominal annual rate as a decimal}}{\text{total compounding periods per year}} = \frac{r}{m}$

$n$  = total number of compounding periods = (compounding periods per year)(years) =  $(m)(t)$

$k$  = number of periods to be deferred

$$A_{(n,k)} = R \left[ \frac{1 - (1+i)^{-n}}{i} \right] (1+i)^{-k} \rightarrow R = A_{(n,k)} \div \left[ \frac{1 - (1+i)^{-n}}{i} \right] \div (1+i)^{-k} = \frac{(A_{(n,k)})(i)}{(1+i)^{-k} (1 - (1+i)^{-n})}$$

**graphing calculator notation:**  $R = A_{(n,k)} * i \div ((1 + i)^{-k}) \div (1 - (1 + i)^{-n})$

**example 4.29a** Jack Henderson's grandparents invested \$25,000 when he was born. This money is to be used for Jack's college education and is to be withdrawn in 4 equal annual payments beginning when Jack is age 19. Find the amount (to the nearest dollar) that will be available each year if money is worth 6%, compounded annually.

- Determine the known values for the variables within the formula:

$$A_{(n,k)} = 25,000 \quad r = 6\% = 0.06 \quad \text{annually} \rightarrow m = 1$$

- Use the values for  $r$  and  $m$  to determine the appropriate value for  $i$ :

$$i = \frac{r}{m} = \frac{0.06}{1} = 0.06$$

- Determine the value of  $n$  (number of payment periods):

*to be withdrawn in 4 equal annual payments*  $\rightarrow n = 4$

- Determine the value of  $k$  by (number of deferred periods):  
*payments beginning when Jackson is age 19*  
will not receive: ages 0–1, 1–2, 2–3, 3–4, 4–5, 5–6, 6–7, 7–8, 8–9,  
9–10, 10–11, 11–12, 12–13, 13–14, 14–15, 15–16, 16–17, 17–18  
 $\rightarrow k = 18$

- Using the known values, evaluate the adjusted formula for  $R$ :

$$R \frac{(A_{(n,k)})(i)}{(1+i)^{-k} (1 - (1+i)^{-n})} = \frac{(25,000)(0.06)}{(1.06^{-18})(1 - (1.06)^{-4})} = 20593.4499$$

**possible withdrawal amount  $\approx \$20,593$**

**graphing calculator solution:**

$$R = 25,000 * 0.06 \div (1.06^{-18}) \div (1 - (1.06^{-4})) = 20593.4499$$

**example 4.29b**

**Carolyn received a trust fund inheritance of \$75,000 on her 20th birthday. She plans to use the money to supplement her income with 24 quarterly payments beginning on her 50th birthday. If the money is worth an annual rate of 8% (compounded quarterly), how much will each quarterly payment be?**

$$R = 75,000$$

quarterly  $\rightarrow m = 4$

$$r = 8\% = 0.08$$

$$i = \frac{r}{m} = \frac{0.08}{4} = 0.02$$

*24 quarterly payments  $\rightarrow n = 24$*

*will not receive quarterly payments for ages shown below  $\rightarrow k = 119$*

1.	20.25–20.50	41.	30.25–30.50	81.	40.25–40.50
2.	20.50–20.75	42.	30.50–30.75	82.	40.50–40.75
3.	20.75–21.00	43.	30.75–31.00	83.	40.75–41.00
4.	21.00–21.25	44.	31.00–31.25	84.	41.00–41.25
5.	21.25–21.50	45.	31.25–31.50	85.	41.25–41.50
6.	21.50–21.75	46.	31.50–31.75	86.	41.50–41.75
7.	21.75–22.00	47.	31.75–32.00	87.	41.75–42.00
8.	22.00–22.25	48.	32.00–32.25	88.	42.00–42.25
9.	22.25–22.50	49.	32.25–32.50	89.	42.25–42.50
10.	22.50–22.75	50.	32.50–32.75	90.	42.50–42.75
11.	22.75–23.00	51.	32.75–33.00	91.	42.75–43.00
12.	23.00–23.25	52.	33.00–33.25	92.	43.00–43.25

13.	23.25-23.50	53.	33.25-33.50	93.	43.25-43.50
14.	23.50-23.75	54.	33.50-33.75	94.	43.50-43.75
15.	23.75-24.00	55.	33.75-34.00	95.	43.75-44.00
16.	24.00-24.25	56.	34.00-34.25	96.	44.00-44.25
17.	24.25-24.50	57.	34.25-34.50	97.	44.25-44.50
18.	24.50-24.75	58.	34.50-34.75	98.	44.50-44.75
19.	24.75-25.00	59.	34.75-35.00	99.	44.75-45.00
20.	25.00-25.25	60.	35.00-35.25	100.	45.00-45.25
21.	25.25-25.50	61.	35.25-35.50	101.	45.25-45.50
22.	25.50-25.75	62.	35.50-35.75	102.	45.50-45.75
23.	25.75-26.00	63.	35.75-36.00	103.	45.75-46.00
24.	26.00-26.25	64.	36.00-36.25	104.	46.00-46.25
25.	26.25-26.50	65.	36.25-36.50	105.	46.25-46.50
26.	26.50-26.75	66.	36.50-36.75	106.	46.50-46.75
27.	26.75-27.00	67.	36.75-37.00	107.	46.75-47.00
28.	27.00-27.25	68.	37.00-37.25	108.	47.00-47.25
29.	27.25-27.50	69.	37.25-37.50	109.	47.25-47.50
30.	27.50-27.75	70.	37.50-37.75	110.	47.50-47.75
31.	27.75-28.00	71.	37.75-38.00	111.	47.75-48.00
32.	28.00-28.25	72.	38.00-38.25	112.	48.00-48.25
33.	28.25-28.50	73.	38.25-38.50	113.	48.25-48.50
34.	28.50-28.75	74.	38.50-38.75	114.	48.50-48.75
35.	28.75-29.00	75.	38.75-39.00	115.	48.75-49.00
36.	29.00-29.25	76.	39.00-39.25	116.	49.00-49.25
37.	29.25-29.50	77.	39.25-39.50	117.	49.25-49.50
38.	29.50-29.75	78.	39.50-39.75	118.	49.50-49.75
39.	29.75-30.00	79.	39.75-40.00	119.	49.75-50.00
40.	30.00-30.25	80.	40.00-40.25		

$$R = \frac{(A_{n,k})(i)}{(1+i)^{-k} \left(1 - (1+i)^{-n}\right)} = \frac{(75,000)(0.02)}{(1.02^{-119}) \left(1 - (1.02)^{-24}\right)} = 41850.43983$$

**quarterly payment  $\approx \$41,850$**

**graphing calculator solution:**

$$R = 75,000 * 0.02 \div (1.02^{-119}) \div (1 - (1.02^{-24})) = 41850.43983$$

## TUTORIAL 4.30: FIND THE REGULAR PAYMENTS TO AMORTIZE A DEBT

Most people purchase large-ticket items through installment loans. The process of repaying such a loan is called **amortization**. We can determine the regular payments for an amortized debt through the use of the formula below:

$$\text{Amortization Formula: } R = A_n \left[ \frac{i}{1 - (1+i)^{-n}} \right]$$

where:

$R$  = size of each regular payment (in dollars)

$A_n$  = amount of debt (in dollars)

$i$  = interest rate per period (as a decimal) =  $\frac{\text{nominal annual rate as a decimal}}{\text{total compounding periods per year}} = \frac{r}{m}$

$n$  = total number of equal periodic payments = (compounding periods per year)(years) =  $(m)(t)$

graphing calculator notation:  $R = An * i \div (1 - (1 + i)^{-n})$

**example 4.30a** A debt of \$50,000 is to be amortized by equal payments at the end of each year for 5 years. If the interest charged is 8% (compounded annually), find the periodic payment (to the nearest dollar).

- Determine the known values for the variables within the formula:

$$r = 8\% \quad \text{annually} \quad 5 \text{ years}$$

$$A_n = 50,000 \quad r = 0.08 \quad m = 1 \quad t = 5$$

- Use the values for  $r$ ,  $m$  and  $t$  to determine the appropriate value for  $i$  and  $n$ :

$$i = \frac{r}{m} = \frac{0.08}{1} = 0.08 \quad n = (m)(t) = (1)(5) = 5$$

- Using the known values, evaluate the formula for  $R$ :

$$R = A_n \left[ \frac{i}{1 - (1+i)^{-n}} \right] = 50,000 \left[ \frac{0.08}{1 - (1.08)^{-5}} \right] = 12522.82273$$

regular amortization payment  $\approx \$12,523$

graphing calculator solution:

$$R = 50,000 * 0.08 \div (1 - 1.08^{-5}) = 12522.82273$$

**example 4.30b** A debt of \$20,000 is to be amortized with 8 equal semiannual payments. If the interest rate is 10% (compounded semi-annually), what is the size of each payment (to the nearest dollar)?

$$A_n = 20,000 \quad r = 0.10 \quad r = 10\% \quad \text{semi-annually } m = 2 \quad t = 4$$

$$i = \frac{r}{m} = \frac{0.10}{2} = 0.05 \quad n = (m)(t) = (2)(4) = 8$$

$$R = A_n \left[ \frac{i}{1 - (1+i)^{-n}} \right] = 20,000 \left[ \frac{0.05}{1 - (1.05)^{-8}} \right] = 3094.436273$$

**regular amortization payment  $\approx \$3094$**

**graphing calculator solution:**

$$R = 20,000 * 0.05 \div (1 - 1.05^{-8}) = 3094.436273$$

### TUTORIAL 4.31: FIND THE AMOUNT THAT CAN BE BORROWED FOR A SPECIFIED PAYMENT

When we attempt to live within a preset budget, we might need to determine how much of a repayment amount we can afford before purchasing a large-ticket item that needs to be amortized. Therefore, we might need to use the amortization formula to solve for the amount of the debt...which means that we need to work the formula backwards. The first box below is the amortization formula while the second box is the adjusted formula after applying some simple algebraic steps.

**Amortization Formula:**  $R = A_n \left[ \frac{i}{1 - (1+i)^{-n}} \right]$

**where:**

**R** = size of each regular payment (in dollars)

**A<sub>n</sub>** = amount of debt (in dollars)

**i** = interest rate per period (as a decimal) =  $\frac{\text{nominal annual rate as a decimal}}{\text{total compounding periods per year}} = \frac{r}{m}$

**n** = total number of equal periodic payments = (compounding periods per year)(years) = (m)(t)

$$R = A_n \left[ \frac{i}{1 - (1+i)^{-n}} \right] \rightarrow A_n = R \div \left[ \frac{i}{1 - (1+i)^{-n}} \right] = \frac{R(1 - (1+i)^{-n})}{i}$$

**graphing calculator notation:**  $A_n = R * (1 - (1 + i)^{-n}) \div i$

**example 4.31a** A developer wants to buy a certain parcel of land. The developer feels she can afford payments of \$30,000 each half-year for the next 10 years. How much can she borrow (to the nearest dollar) and hold to this budget at an annual rate of 8.5% compounded semi-annually?

1. Determine the known values for the variables within the formula:

$$\begin{array}{lll} r = 8.5\% & \text{semi-annually} & 10 \text{ years} \\ R = 30,000 & r = 0.085 & m = 2 \\ & & t = 10 \end{array}$$

2. Use the values for  $r$ ,  $m$  and  $t$  to determine the appropriate value for  $i$  and  $n$ :

$$i = \frac{r}{m} = \frac{0.085}{2} = 0.0425 \quad n = (m)(t) = (2)(10) = 20$$

3. Using the known values, evaluate the formula for  $A_n$ :

$$A_n = \frac{R(1 - (1+i)^{-n})}{i} = \frac{30,000(1 - (1.0425)^{-20})}{0.0425} = 398830.9742$$

**amount of affordable debt  $\approx \$398,831$**

**graphing calculator solution:**

$$A_n = 30,000 * (1 - 1.0425^{-20}) \div 0.0425 = 398830.9742$$

**example 4.31b** Monmouth Auto Repair wants to add a new service bay. How much can they borrow (to the nearest dollar) at an annual rate of 5% (compounded quarterly) if the desired quarterly payment is \$5000? Assume a loan period of 5.5 years.

$$\begin{array}{lll} r = 5\% & \text{quarterly} & 5.5 \text{ years} \\ R = 5000 & r = 0.05 & m = 4 \\ & & t = 5.5 \end{array}$$

$$i = \frac{r}{m} = \frac{0.05}{4} = 0.0125 \quad n = (m)(t) = (4)(5.5) = 22$$

$$A_n = \frac{R(1 - (1+i)^{-n})}{i} = \frac{5000(1 - (1.0125)^{-22})}{0.0125} = 95652.81466$$

**amount of affordable debt  $\approx \$95,653$**

**graphing calculator solution:**

$$A_n = 5000 * (1 - 1.0125^{-22}) \div 0.0125 = 95652.81455$$

## TUTORIAL 4.32: FIND THE UNPAID BALANCE OF A LOAN

There may be times when we have purchased a home (vehicle) through an amortization process and are still paying the loan off when we decide that we would like to purchase a different home (vehicle). Before we can make the new purchase, we have to determine the **unpaid balance** (amount still remaining on the loan) so that we can make arrangements for the repayment of the old loan. We can calculate this unpaid balance by using the formula below.

$$\text{Unpaid Balance Formula: } A_{n-k} = R \left[ \frac{1 - (1+i)^{-(n-k)}}{i} \right]$$

**where:**

**A<sub>(n-k)</sub>** = unpaid balance (in dollars)

**R** = size of each periodic payment (in dollars)

**i** = interest rate per period (as a decimal) =  $\frac{\text{nominal annual rate as a decimal}}{\text{total compounding periods per year}} = \frac{r}{m}$

**n** = total number of equal periodic payments = (compounding periods per year)(years) = (m)(t)

**k** = number of equal payments paid

**n - k** = number of remaining payments

**graphing calculator notation:  $A_{(n-k)} = R * (1 - (1 + i)^{-(n - k)}) / i$**

**example 4.32a** A loan is to be amortized for 10 years with quarterly payments of \$1709.74 each. If the interest rate is 6.5% (compounded quarterly), what is the unpaid balance (to the nearest dollar) immediately after the tenth payment?

- Determine the known values for the variables within the formula:

$$R = 1709.74 \quad r = 6.5\% = 0.065 \quad \text{quarterly} \rightarrow m = 4$$

$$t = 10 \quad k = 10$$

- Use the values for  $r$ ,  $m$  and  $t$  to determine the appropriate value for  $i$  and  $n$ :

$$i = \frac{r}{m} = \frac{0.065}{4} = 0.01625 \quad n = (m)(t) = (4)(10) = 40$$

- Using the known values, evaluate the formula for  $A_{n-k}$ :

$$A_{(n-k)} = R \left[ \frac{1 - (1+i)^{-(n-k)}}{i} \right] = 1709.74 \left[ \frac{1 - (1.01625)^{-(40-10)}}{0.01625} \right] = 40342.3717$$

**amount of unpaid balance  $\approx \$40,342$**

**graphing calculator solution:**

$$A_{(n-k)} = 1709.74 * (1 - 1.01625^{-(40 - 10)}) / 0.01625 = 40342.3717$$

**example 4.32b** A 42-month auto loan has monthly payments of \$488.61 each. If the interest rate is 7.5% (compounded monthly), find the unpaid balance (to the nearest dollar) immediately after the 28th payment.

$$R = 488.61 \quad r = 7.5\% = 0.075 \quad \text{monthly} \rightarrow m = 12$$

$$t = 42 \div 12 = 3.5 \quad k = 28$$

$$i = \frac{r}{m} = \frac{0.075}{12} = 0.00625 \quad n = (m)(t) = (12)(3.5) = 42$$

$$A_{(n-k)} = R \left[ \frac{1 - (1+i)^{-(n-k)}}{i} \right] = 488.61 \left[ \frac{1 - (1.00625)^{-(42-28)}}{0.00625} \right] = 6530.300386$$

amount of unpaid balance  $\approx \$6530$

graphing calculator solution:

$$A_{(n-k)} = 488.61 * (1 - 1.00625^{-(42 - 28)}) \div 0.00625 = 6530.300386$$

# ABOUT THE AUTHOR

## \* brief listing of my educational qualifications and experiences...

I received my BS degree from Trenton State College (now known as the College of New Jersey, Trenton, New Jersey, USA) in 1973 with a dual state certification in elementary education and secondary mathematics. I received my MA degree from Georgian Court College (now known as Georgian Court University, Lakewood, New Jersey, USA) in 1983.

From February 1977 to the end of June 2003, I worked as a full-time secondary (grades 9–12) mathematics teacher at Point Pleasant Borough High School (Point Pleasant, New Jersey, USA). Throughout my 26.5 years at this school, I taught several courses offered through the mathematics department: Algebra 1, Algebra 2, Algebra 2 Honors, Basic Skills Math, Computer Programming (BASIC), Computer Programming (QBASIC), Discrete Mathematics, Geometry, Geometry Honors, SAT Preparation. I retired from this school district as of July 01, 2003.

I was an adjunct professor at Monmouth University (West Long Branch, New Jersey, USA) for three semesters from January 1998 to May 1999, teaching through the Mathematics Department. Due to a conflict with my full-time teaching position, I declined any teaching duties at MU until my return to the list of adjunct professors in September 2004. I decided to stop my teaching duties at Monmouth University...my last semester there was Spring 2016. During my time at Monmouth University, I taught several courses offered through the University's mathematics department: Pre-Algebra Mathematics, Quantitative Reasoning and Problem Solving, College Algebra, Mathematical Modeling in the Social Sciences, Quantitative Analysis for Business I, Foundations of Elementary Mathematics I, and Foundations of Elementary Mathematics II.