Lecture Notes of Mathematics-I for Integral Calculus, Improper Integrals, Beta and Gamma functions

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Chapter 1

Integral Calculus

1.1 Jacobian Matrix

Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a vector valued function given by $F(x_1, \ldots, x_n) = (F_1(x_1, \ldots, x_n), \ldots, F_m(x_1, \ldots, x_n))$. Then Jacobian matrix is the matrix of all first order partial derivatives defined as

$$J(F) = \begin{pmatrix} \dfrac{\partial F_1}{\partial x_1} & \cdots & \dfrac{\partial F_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial F_m}{\partial x_1} & \cdots & \dfrac{\partial F_m}{\partial x_n} \end{pmatrix}$$

Jacobian matrix is important because if the function F is differentiable at point $p = (x_1, \ldots, x_n)$, the Jacobian matrix defines a linear map $\mathbb{R}^n \longrightarrow \mathbb{R}^m$, which is the best linear approximation of the function F near point p. The Jacobian generalizes the gradient of a scalar valued function of several variables, which is generalization of derivative of a scalar valued function of single variable. Jacobian can be thought of as describing the amount of stretching, rotating or transforming and that transformation imposes locally.

Definition 1.1.1. *If $F_1, F_2, \ldots, F_n$ are functions of $x_1, x_2, \ldots, x_n$, then the de-*

terminant

$$J = \begin{vmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{vmatrix}$$

is called Jacobian of $F_1, F_2, \ldots, F_n$ with respect to $x_1, x_2, \ldots, x_n$, In short it is written as $\frac{\partial(F_1, \ldots, F_n)}{\partial(x_1, \ldots, x_n)}$.

Remark: Determinant of square Jacobian matrix (called as Jacobian) gives important information about the behaviour of F near that point. If the Jacobian determinant at p is non-zero, then the continuously differentiable function F is invertible near a point $p \in \mathbb{R}^n$. This is the inverse function theorem. Further if $Det(J) > 0$, then F preserves orientation near p. If $Det(J) < 0$, then F reverses orientation. Absolute value of the Jacobian determinant gives the factor by which the function F expands or shrinks volumes near p.

Properties of Jacobian:

1. If f and g are functions of u and v and u, v are functions of x and y, then $\frac{\partial(f,g)}{\partial(x,y)} = \frac{\partial(f,g)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(x,y)}$

2. If J is the Jacobian of the system u, v with respect to x, y and $J^{'}$ is the Jacobian of x, y with respect to u and v, then $JJ^{'} = 1$.

Example 1.1.2. *If $x = rcos\ \theta$ and $y = r\ sin\ \theta$, then (i) $\frac{\partial(x,y)}{\partial(r,\theta)} = r$ and (ii) $\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$.*

1.1.1 Change of variables

Suppose $z = f(x,y)$ and $x = \phi(u,v)$, $y = \psi(u,v)$. Then by chain rule

$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$ and $\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$.

Solving the system of equations by Crammer rule

$$\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial f}{\partial v}\frac{\partial y}{\partial u}}=\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial v}\frac{\partial x}{\partial u}-\frac{\partial f}{\partial u}\frac{\partial x}{\partial v}}=\frac{1}{\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}}$$

$$\frac{\frac{\partial f}{\partial x}}{\frac{\partial (f,y)}{\partial (u,v)}}=\frac{\frac{\partial f}{\partial y}}{\frac{\partial (f,x)}{\partial (u,v)}}=\frac{1}{\frac{\partial (x,y)}{\partial (u,v)}}$$

$$Determinant\ J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\frac{\partial (x,y)}{\partial (u,v)}$$

is called the Jacobian of variables of transformation. Therefore $\frac{\partial f}{\partial x}=\frac{1}{J}\frac{\partial (f,y)}{\partial (u,v)}$ and $\frac{\partial f}{\partial y}=-\frac{1}{J}\frac{\partial (f,x)}{\partial (u,v)}$.

In case of three variables, let $S=f(x,y,z)$ and $x=F(u,v,w)$, $y=G(u,v,w)$ & $z=H(u,v,w)$. Then

$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$.

$\frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$.

$\frac{\partial f}{\partial w}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial w}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial w}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial w}$.

$\frac{\partial f}{\partial x}=\frac{1}{J}[\frac{\partial (f,y,z)}{\partial (u,v,w)}]$, $\frac{\partial f}{\partial y}=-\frac{1}{J}[\frac{\partial (f,x,z)}{\partial (u,v,w)}]$ and $\frac{\partial f}{\partial z}=\frac{1}{J}[\frac{\partial (f,x,y)}{\partial (u,v,w)}]$, where

$$J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}$$

Example 1.1.3. *Let $z=f(x,y)$, $x=r\cos\theta$, $y=r\sin\theta$. Then*

$(\frac{\partial f}{\partial x})^2+(\frac{\partial f}{\partial y})^2=(\frac{\partial f}{\partial r})^2+\frac{1}{r^2}(\frac{\partial f}{\partial \theta})^2$.

Here $J=\frac{\partial (x,y)}{\partial (r,\theta)}=r$,

$\frac{\partial f}{\partial x}=\frac{1}{J}\frac{\partial (f,y)}{\partial (r,\theta)}$................(1) *and*

$\frac{\partial f}{\partial y} = -\frac{1}{J}\frac{\partial(f,x)}{\partial(r,\theta)}$................(2).
Also $\frac{\partial(f,y)}{\partial(r,\theta)} = rcos\ \theta\frac{\partial f}{\partial r} - sin\ \theta\frac{\partial f}{\partial\ \theta}$........(3) *and*
$\frac{\partial(f,x)}{\partial(r,\theta)} = -rsin\ \theta\frac{\partial f}{\partial r} - cos\ \theta\frac{\partial f}{\partial\ \theta}$...........(4)
Squaring and adding (1) & (2) and using (3) & (4) we get the required.

Cartesian coordinates to cylindrical coordinates

Cylindrical coordinates (r, θ, z) are given by $x = r\ cos\ \theta$, $y = r\ sin\ \theta$, $z = z$ and Jacobian of transformation is

$$J = J(\frac{(x,y,z)}{(r,\theta,z)}) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} cos\ \theta & -r\ sin\ \theta & z \\ sin\ \theta & r\ cos\ \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

1.2 Volume of solid of revolution

Let AB be the portion of the curve $y = f(x)$, $f(x) > 0$, $x = a$, $x = b$. Consider the area bounded by the arc AB of the curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$. Volume of the solid generated by revolving this area about the x-axis is $V = \int_a^b \pi y^2 dx$. Similarly volume of the solid generated by revolving this area about the y-axis and lines $y = c$, $y = d$ is $V = \int_a^b \pi x^2 dy$.

Example 1.2.1. *Find the volume of the solid generated by revolving the finite region bounded by the curve* $y = x^2 + 1$, $y = 5$ *about the line* $x = 3$.
Solution: $V = \int_1^5 \pi({x_1}^2 - {x_2}^2)dy = \int_1^5 \pi[(3 + \sqrt{y-1})^2 - (3 - \sqrt{y-1})^2]dy = \int_1^5 12\pi(\sqrt{y-1})dy = 64\pi$.

Example 1.2.2. *Consider the element $\delta x \delta y$ at $P(x, y)$ of plane area A. As this elementary area revolves about x-axis, we get a ring of volume* $\pi[(y + \delta y)^2 -$

$y^2]\delta x = 2\pi y\delta x\delta y$. Total volume= $\int\int_A 2\pi y\ dxdy = \int\int_A 2\pi r\ sin\theta\ rdr\ d\theta = \int\int_A 2\pi r^2\ sin\theta\ dr\ d\theta$.

Example 1.2.3. *Find the volume of the solid generated by revolving the finite region bounded by the curve $y = 3 - x^2$, $y = -1$ about the line $y = -1$.*

Solution: $V = 2[$ *volume of solid generated about the* $y = -1]$

$$=\int_0^2 \pi(1+y)^2 dx = \int_0^2 2\pi(1+y)^2 dx = \int_0^2 2\pi(1+3-x^2)^2 dx = \frac{512\pi}{15}.$$

Example 1.2.4. *Find the volume of the solid generated by revolving the arc of cycloid $x = a(t - sint)$, $y = a(1 - cost)$, about $x-$axis.*

Solution: $V = \int_0^{2\pi a} \pi y^2 dx = \int_0^{2\pi} \pi a^2(1-cos\ t)^2 a(1-cos\ t)dt = 16\pi a^3 \int_0^{\pi} sin^6\ tdt = 5\pi^2 a^3$.

1.3 Double Integral

Notion of double integral is an extension of the notion of definite integral on the real line to the case of two dimensional space $\mathbb{R}^2$. Let $f(x, y)$ be a continuous function in a simply connected, closed bounded region R in two variables x and y. Divide the region R into subregions (rectangles) by drawing lines $x = x_k$, $y = y_k$, $k = 1, 2, \ldots, m$ parallel to coordinate axes. Let (x_i, y_i) be an arbitrary point inside the ith rectangle, whose area is $\triangle A_i$. Let $S_n = \sum_{i=1}^n f(x_i, y_i)\triangle A_i$. When $n \to \infty$, the number of subregions increase indefinitely such that the largest of areas $\triangle A_i$ approaches zero. Then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^n f(x_i, y_i)\triangle A_i$, if exists, is called the double integral of the function $f(x, y)$ over the region R and is denoted by $\int\int_R f(x, y)dxdy$.

Properties of double integrals

If $f(x, y)$ and $g(x, y)$ are integrable functions, then

1. $\int\int_R f(x, y) \pm g(x, y)]dxdy = \int\int_R f(x, y)dxdy \pm \int\int_R g(x, y)]dxdy$.

2. $\int\int_R k\ f(x, y)dxdy = k\ \int\int_R f(x, y)dxdy$.

3. $|\int\int_R k\ f(x, y)dxdy| \leq \int\int_R |f(x, y)|dxdy$.

4. **Mean Value Theorem**: $\int\int_R f(x,y)dxdy = f(u,v)A$, where A is the area of the region R and (u,v) is any arbitrary point in the region R. If $m \leq f(x,y) \leq M$ for all (x,y) in R, then $m\ A \leq \int\int_R f(x,y)dxdy \leq M\ A$.

5. If $0 < f(x,y) \leq g(x,y)$ for all $(x,y) \in R$, then $\int\int_R f(x,y)dxdy \leq \int\int_R g(x,y)dxdy$.

6. If $f(x,y) \geq 0$, then $\int\int_R f(x,y)dxdy \geq 0$.

Example 1.3.1. *Evaluate* $\int_0^1 \int_0^{\sqrt{1+x^2}} \dfrac{dydx}{1+x^2+y^2}$.

Solution: $\int_0^1 \int_0^{\sqrt{1+x^2}} \dfrac{dydx}{1+x^2+y^2} = \int_0^1 dx \int_0^{\sqrt{1+x^2}} \dfrac{dy}{1+x^2+y^2}$

$= \int_0^1 dx[\dfrac{1}{\sqrt{1+x^2}} tan^{-1} \dfrac{y}{1+x^2}]_0^{\sqrt{1+x^2}} = \int_0^1 dx[\dfrac{1}{\sqrt{1+x^2}} \dfrac{\pi}{4}]$

$= \dfrac{\pi}{4} \int_0^1 \dfrac{1}{\sqrt{1+x^2}} dx = \dfrac{\pi}{4}[log(x+\sqrt{x^2+1}]]_0^1 = \dfrac{\pi}{4}[log(\sqrt{2}+1)]$.

Example 1.3.2. *Evaluate* $\int\int_A xy\ dx\ dy$, *where* A *is the region bounded by the ordinate* $x = 2a$, *curve* $x^2 = 4ay$.

Solution: *Limit of integration for* x *are from* 0 *to* $2a$ *and for* y *from* 0 *to* $\dfrac{x^2}{4a}$.

$\int\int_A xy\ dx\ dy = \int_0^{2a} \int_0^{x^2/4a} xy\ dx\ dy = \int_0^{2a}\ x\ dx \int_0^{x^2/4a}\ y\ dy = a^4/3.$

Example 1.3.3. *Evaluate* $\int\int_R e^{x^2} dxdy$, *where* $R : 2y \leq x \leq 2$ *and* $0 \leq y \leq 1$. *First integrate with respect to* y *and then with respect* x.

$\int_0^2 [\int_0^{\frac{x}{2}} e^{x^2} dy]dx = \int_0^2 [ye^{x^2}]_0^{\frac{x}{2}} dx = \dfrac{1}{2} \int_0^2 xe^{x^2} dx = \dfrac{1}{4}(e^4 - 1).$

Applications of double integrals

1. If $f(x,y) = 1$, then $\int\int_R dxdy$= Area of the region R.

2. If $z = f(x,y)$ is a surface, then $\int\int_R f(x,y)dxdy$ = Volume of the region beneath the surface $z = f(x,y)$ and above the $x-y$ plane.

3. If $f(x,y) = \rho(x,y)$ is the density function (mass per unit area), then

   (i) $\int\int_R \rho(x,y)dxdy = M$ (Total mass of the region R).

   (ii) Centre of gravity $(\bar{x}, \bar{y})$ is given by

   $\bar{x} = \dfrac{1}{M} \int\int_R x\rho(x,y)dxdy$, $\bar{y} = \dfrac{1}{M} \int\int_R y\rho(x,y)dxdy$.

   (iii) Moment of inertia of the mass in R about x-axis, $I_x = \int\int_R y^2\rho(x,y)dxdy$

   Moment of inertia of the mass in R about y-axis, $I_y = \int\int_R x^2\rho(x,y)dxdy$

1.3.1 Change of variables in double and triple integral

Let R be the domain of integration $\int\int_R f(x,y)\ dxdy$ in $x-y$ plane Let $x=\phi(u,v)$, $y=\psi(u,v)$ and R^* be the domain in the $u-v$ plane. Then
$\int\int_R f(x,y)\ dxdy= \int\int_{R^*} F(u,v)\ |J|\ dudv$, where

$$Determinant\ J=\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial(x,y)}{\partial(u,v)}.$$

If $x=\phi(u,v,w)$, $y=\psi(u,v,w)$,$z=\eta(u,v,w)$, then
$\int\int_R f(x,y,z)\ dxdydz = \int\int\int_{R^*} F(u,v,w)\ |J|\ dudvdw$, where

$$J=\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

Example 1.3.4. *Let R be rhombus with successive vertices at $(1,0),(2,1),(1,2)$ and $(0,1)$. Then find the value of the integral $\int\int_R (x-y)^2 cos^2(x+y)dxdy$.*

***Solution**: Equations of sides AB, BC, CD, and DA are given by $x-y=1$, $x+y=3$, $x-y=-1$ and $x+y=1$ respectively. Put $x-y=u$, $y+x=v$. Then $-1\le u\le 1$, $1\le v\le 3$ and $x=(v-u)/2$, $y=(u+v)/2$. Jacobian of transformation is given by*

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = -1/2$$

Then $I=\int\int_R (x-y)^2 cos^2(x+y)\ dxdy = \int_1^3\int_{-1}^1 u^2 cos^2 v |J|\ dudv$
$= \dfrac{1}{2}\int_1^3\int_{-1}^1 u^2 cos^2 v\ dudv = \dfrac{1}{3}\int_1^3 cos^2 v dv = \dfrac{1}{6}\int_1^3 (1+cos\ 2v)dv$
$= \dfrac{1}{3} + \dfrac{sin\ 6 - sin\ 2}{12}.$

1.4 Triple Integral

Triple integral is an extension of the notion of double integral to three dimensional space $\mathbb{R}^3$. Let $f(x,y,z)$ be a continuous function in a simply connected, closed bounded volume V. Divide the region V into small volume elements by drawing planes $x = x_k,\ y = y_k,\ z = z_k,\ k = 1, 2, \ldots, n$ parallel to three coordinate planes. Let (x_i, y_i, z_i) be an arbitrary point inside the ith volume element $\delta V_i = \delta x_i \delta y_i \delta z_i$. Let $S_n = \sum_{i=1}^{n} f(x_i, y_i, z_i)\delta V_i$. When $n \to \infty$, the number of subregions increase indefinitely such that the largest of volumes δV_i approaches zero. Then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\delta V_i$, if exists, is called the triple integral of the function $f(x,y,z)$ over the region V and is denoted by $\int\int\int_V f(x,y,z)dxdydz$.

Let $f(x,y,z)$ be a continuous function over a regular solid V defined by $a < x < b$, $h_1(x) < y < h_2(x)$, and $g_1(x,y) < z < g_2(x,y)$. Then the triple integral is equal to the triple iterated integral given by

$$\int\int\int_V f(x,y,z)dxdydz = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)dxdydz.$$

Applications of triple integrals

1. If $f(x,y,z) = 1$, then $\int\int\int_R dxdydz$= Volume of the region R.

2. If $f(x,y,z) = \rho(x,y,z)$ is the density of mass, then

   (i) $\int\int\int_R \rho(x,y)dxdydz = M$ (Total mass of the solid of region R).

   (ii) Centre of gravity $(\bar{x}, \bar{y}, \bar{z})$ is given by
   $\bar{x} = \dfrac{1}{M}\int\int\int_R x\ \rho(x,y,z)dxdydz$, $\bar{y} = \dfrac{1}{M}\int\int_R y\ \rho(x,y,z)dxdydz$,
   $\bar{z} = \dfrac{1}{M}\int\int_R z\ \rho(x,y,z)dxdydz$.

   (iii) Moment of inertia of the mass in R about x-axis,

   $I_x = \int\int\int_R (y^2 + z^2)\rho(x,y,z)dxdydz$

   Moment of inertia of the solid of mass in R about y-axis,

   $I_y = \int\int\int\int_R (x^2 + z^2)\rho(x,y,z)dxdydz$ and

   moment of inertia of the solid of mass in R about z-axis,

   $I_z = \int\int\int\int_R (x^2 + y^2)\rho(x,y,z)dxdydz$.

Example 1.4.1. *Find the volume of the solid bounded by the plane $x = 0$, $y = 0$, $x + y + z = a$ and $z = 0$.*

Volume $V = \int_0^a \int_0^{a-x} \int_0^{a-x-y} dx\ dy\ dz = \int_0^a \int_0^{a-x} (a - x - y)dy\ dx = \frac{1}{2} \int_0^a (a - x)^2\ dx = a^3/6.$

Example 1.4.2. *Find the volume generated by the revolution of cardioid $r = a(1 - cos\ \theta)$ about its axis.*

Volume $V = \int_0^\pi \int_0^{a(1-cos\ \theta)} 2\pi r^2\ drsin\ \theta d\theta$
$= \frac{2\pi a^3}{3} \int_0^\pi (1 - cos\ \theta)^3 sin\ \theta d\theta = \frac{\pi a^3}{6}.$

Example 1.4.3. *Calculate the volume of the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

Using the transformation of spherical coordinates $x = arsin\theta\ cos\phi$, $y = brsin\theta\ sin\phi$, $z = crcos\theta$. Jacobian of transformation

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = abc\ r^2 sin\ \theta$$

$dxdydz = Jdr\ d\theta\ d\phi = abc\ r^2 sin\ \theta dr\ d\theta\ d\phi$

$V = 8\times$ *volume of sphere in positive octant*
$= 8 \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} abc\ r^2 sin\ \theta dr d\theta\ d\phi$
$= 8 \int_0^1 abc\ r^2 dr \int_0^{\pi/2} sin\ \theta d\theta \int_0^{\pi/2}\ d\phi = \frac{4}{3}\pi abc.$

Example 1.4.4. *Calculate the volume of the solid surrounded by the surface $(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} = 1$.*

Using the transformation of coordinates $x = aX^3$, $y = bY^3$, $z = cZ^3$. Jacobian of transformation $J = \frac{\partial(x, y, z)}{\partial(X, Y, Z)} = 27abcX^2Y^2Z^2$. *Required volume* $= \int\int\int dx\ dy\ dz = 27abc \int\int\int X^2Y^2Z^2 dX\ dY\ dZ$

Now use the transformation $X = rsin\ \theta cos\ \phi$, $Y = rsin\ \ \theta sin\ \phi$, $Z = rcos\ \theta$. Then Jacobian of transformation $J' = \frac{\partial(X, Y, Z)}{\partial(r, \theta, \phi)} = r^2 sin\ \theta\ drd\theta d\phi$. *Thus*

$dXdYdZ = r^2 sin\ \theta drd\theta d\phi$ *and* $dxdydz = 27abcX^2Y^2Z^2r^2 sin\ \theta\ drd\theta\ d\phi$.

$V = 8\times$ *volume in positive octant*$=$

$8\int_0^1\int_0^{\pi/2}\int_0^{\pi/2} 27abc\ (rsin\ \theta\ cos\ \phi)^2(rsin\ \ \theta\ sin\ \phi)^2(rcos\ \theta)^2r^2sin\ \theta drd\theta\ d\phi.$

$= 216abc\int_0^1 r^8dr\int_0^{\pi/2} sin^5\ \theta cos^2\ \theta d\theta\int_0^{\pi/2} sin^2\ \phi\ cos^2\ \phi\ d\phi = \dfrac{4\pi abc}{35}.$

Example 1.4.5. *Find total mass of the solid bounded by the surfaces* $x^2+y^2=16$, $z=2$ *and* $z=4$ *if its density function is* $\rho(x,y,z)=x^2+y^2$.

Convert the integral $\int_{x=-4}^{4}\int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{z=2}^{z=4}\rho(x,y,z)\ dx\ dy\ dz$ *in cylindrical coordinates* $x=rcos\ \theta,\ ,y=rsin\ \theta,\ z=z$, *then Jacobian* $J=r$ *and*

$\int_{r=0}^{4}\int_{\theta=0}^{2\pi}\int_{z=2}^{4} r^2\ r\ dr\ d\theta\ dz = \int_{r=0}^{4} r^3dr\int_{\theta=0}^{2\pi} d\theta\int_{z=2}^{4}\ dz = \{\dfrac{r^4}{4}\}_0^4\times 2\pi\times 2 = 256\pi.$

Example 1.4.6. *Show that in the first octant, paraboloid* $z=36-4x^2-9y^2$ *has the volume* 27π *cubic units.*

Solution: *Projection of the given paraboloid in the x y plane is the first quadrant of the ellipse* $4x^2+9y^2=36$. *Then region R is given by* $0\le z\le 36-(4x^2+9y^2)$, $0\le y\le \dfrac{1}{3}\sqrt{36-4x^2}$, $0\le x\le 3$. *Therefore volume*

$V=\int_0^3[\int_0^{\frac{2}{3}\sqrt{(9-x^2)}}(36-4x^2-9y^2)dy]dx$

$=\int_0^3[4(9-x^2)y-3y^3]_0^{\frac{2}{3}\sqrt{(9-x^2)}}dx = \dfrac{16}{9}\int_0^3(9-x^2)^{3/2}dx.$

Now put $x=3sin\ \theta$. *Then* $V=144\int_0^{\pi/2} cos^4\ \theta d\theta = 144(\dfrac{3}{4}.\dfrac{1}{2}.\dfrac{\pi}{2}) = 27\pi$ *cubic unit.*

Example 1.4.7. *Find the volume of the solid which is contained in between the cone* $z^2=2(x^2+y^2)$ *and the hyperboloid* $z^2=x^2+y^2+a^2$.

Solution: *Put* $x=rcos\ \theta$, $y=rsin\ \theta$. *Then* $dxdy=rdrd\theta$ *and intersection of* $z^2=2(x^2+y^2)$ *and* $z^2=x^2+y^2+a^2$ *is* $2(x^2+y^2)=x^2+y^2+a^2$ *i.e.* $x^2+y^2=a^2$. *Therefore*

$V=\int_0^{2\pi}\int_0^a\int_{\sqrt{2(x^2+y^2)}}^{\sqrt{x^2+y^2+a^2}} dz\ dy\ dx = \int_0^{2\pi}\int_0^a \sqrt{(x^2+y^2+a^2)}-\sqrt{2(x^2+y^2)}\ dy\ dx =$

$\int_0^{2\pi}\int_0^a(\sqrt{(r^2+a^2)}-\sqrt{2}r)r\ dr\ d\theta = \dfrac{4\pi a^3(\sqrt{2}-1)}{3}.$

Example 1.4.8. *Find mass of a solid lying between spheres of radius* 3 *and* 4 *in the region* $y\ge 0$, $z\ge 0$. *Density at any point* (x,y,z) *is given by* $\rho(x,y,z)=\sqrt{x^2+y^2+z^2}$.

Solution: *Convert the integral into spherical coordinates* $x=r\ sin\ \theta\ cos\ \phi$,

$y = r\ sin\ \theta\ sin\ \phi$, $z = r\ cos\ \theta$, Region occupied by the solid is described by $3 \leq r \leq 4$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq \frac{\pi}{2}$. Also

$\rho(r, \theta, \phi) = \rho(r\ sin\ \theta\ cos\ \phi, r\ sin\ \theta\ sin\ \phi, r\ cos\ \theta)$

$= \sqrt{(r\ sin\ \theta\ cos\ \phi)^2 + (r\ sin\ \theta\ sin\ \phi)^2 + (r\ cos\ \theta)^2} = r.$

Jacobian of transformation $J = r^2\ sin\ \phi$.

Hence mass$= \int_0^\pi \int_0^{\pi/2} \int_3^4 r\ r^2\ sin\ \phi\ dr\ d\phi\ d\theta = \int_0^\pi\ d\theta\ \int_0^{\pi/2} sin\ \phi\ d\phi \int_3^4 r^3\ dr\ = \frac{175\pi}{4}$

Exercise 1.4.1.

1. Evaluate $\int\int\int sin(x+y+z)\ dx\ dy\ dz$ over portion cut off by the plane $x + y + z = \pi$.

2. By a suitable change of variables calculate the integral $\int\int_R log\ \frac{x-y}{x+y}\ dx\ dy$, where R is the triangular region bounded the vertices $(1, 0)$, $(4, -3)$, $(4, 1)$.
Ans: $\frac{1}{4}(49\ log\ 7 - \frac{75}{2} log\ 5 - 27\ log\ 3 + 6)$.

3. Evaluate $\int\int_R (x+y)^2\ dx\ dy$, where R is the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, using transformation of variables.
Ans: $\frac{\pi ab}{4}(a^2 + b^2)$.

4. Show that the area of the surface of paraboloid $\frac{x^2}{a} + \frac{y^2}{b} = 2z$ inside the cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = k$ is $\frac{2}{3}\pi\{(1+k)^{3/2} - 1\}ab$.

5. Find the area of the part of the cylinder $x^2 + y^2 = a^2$ which is cut by the cylinder $x^2 + z^2 = a^2$.

6. Let $f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$, $(x, y) \neq (0, 0)$, $f(0, 0) = 0$. Then show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

Chapter 2

Improper Integrals, Beta and Gamma Functions

2.1 Improper Integral

For the existence of Riemann integral (definite integral) $\int_a^b f(x)dx$, we require that the limit of integration a and b are finite and function $f(x)$ is bounded. In case

(i) limit of integration a or b or both become infinite (improper integral of first kind),

(ii) integrand $f(x)$ has singular points (discontinuity) i.e. $f(x)$ becomes infinite at some points in the interval $a \leq x \leq b$ (improper integral of second kind),

then the integral $\int_a^b f(x)dx$ is called improper integral. Note that improper integral are evaluated by limiting process.

Example 2.1.1. $\int_{-1}^{\infty} \frac{-1}{x^2}\, dx$, $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$, $\int_0^1 \frac{dx}{x(1-x)}$ *are improper integrals.*

Definition 2.1.2. *If a is the only point of infinite discontinuity of $f(x)$, then*

$$\int_a^b f(x)dx = \lim_{\epsilon \to 0+} \int_{a+\epsilon}^b f(x)dx \quad (if\ finite\ limit\ exists)$$

If b is the only point of infinite discontinuity, then

$$\int_a^b f(x)dx = \lim_{\epsilon\to 0+}\int_a^{b-\epsilon} f(x)dx, \quad 0<\epsilon<b-a$$

If end points a and b are the only points of infinite discontinuity,

$$\int_a^b f(x)dx = \lim_{\epsilon\to 0+\ \&\ \mu\to 0+}\int_{a-\epsilon}^{b-\mu} f(x)dx,$$

Improper integral is convergent if limit exists and is finite otherwise it is divergent.

If an interior point c, $a<c<b$, is the only point of infinite discontinuity, then we write

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$

Improper integral is convergent if both the limits exist and are finite otherwise it is divergent.

Example 2.1.3. *Examine the convergence of* $\int_a^b \dfrac{dx}{(x-a)^n}$

Solution: It is a proper integral if $n \leq 0$ and improper for other values of n. For $n \neq 1$,

$\int_a^b \dfrac{dx}{(x-a)^n} = \lim_{p\to 0+}\int_{a+p}^b \dfrac{dx}{(x-a)^n}, \qquad 0<p<b-a$

$$= \lim_{p\to 0+}\frac{1}{(1-n)}[\frac{1}{(b-a)^{n-1}} - \frac{1}{p^{n-1}}] = \begin{cases} \dfrac{1}{(1-n)(b-a)^{n-1}}, & if\ n<1 \\ \infty & if\ n>1, \end{cases}$$

For $n=1$

$\int_a^b \dfrac{dx}{(x-a)^n} = \lim_{p\to 0+}\int_{a+p}^b \dfrac{dx}{(x-a)}$

$= \lim_{p\to 0+} log(b-a) - log\ p = \infty$

Thus integral converges only if $n<1$.

Comparison Test I

If $f(x)$ and $g(x)$ are two functions such that $f(x) \leq g(x)$, for all $x \in [a,b]$, then

1. $\int_a^b f(x)\ dx$ converges if $\int_a^b g(x)\ dx$ converges.

2. $\int_a^b g(x)\ dx$ diverges if $\int_a^b f(x)\ dx$ diverges.

Comparison Test II

If $f(x)$ and $g(x)$ are two functions such that $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = l$ (nonzero & finite), then $\int_a^\infty f(x)\ dx$ and $\int_a^\infty g(x)\ dx$ converge or diverge together.

Example 2.1.4. *Examine the convergence of (i)* $\int_0^1 \dfrac{dx}{\sqrt{(1-x^3)}}$ *(ii)* $\int_0^{\pi/2} \dfrac{sin\ x\ dx}{x^n}$ *(iii)* $\int_1^\infty \dfrac{dx}{x^3(e^{-x}+1)}$

Solution: (i) Let $f(x) = \dfrac{1}{\sqrt{(1-x^3)}} = \dfrac{1}{\sqrt{(1-x)}\sqrt{(1+x+x^2)}}$.

Note that $\dfrac{1}{\sqrt{(1+x+x^2)}}$ is a bounded function and let M be its upper bound. Then $f(x) \leq \dfrac{M}{(1-x)^{1/2}}$ and $\int_0^1 \dfrac{1}{(1-x)^{1/2}}\ dx$ is convergent. Therefore by comparison test $\int_0^1 \dfrac{dx}{\sqrt{(1-x^3)}}$ is convergent.

(ii) For $n \leq 1$, it is a proper integral. For $n > 1$, it is an integral and 0 is the point of infinite discontinuity. Now $\dfrac{sin\ x}{x^n} = \dfrac{sin\ x}{x}\dfrac{1}{x^{n-1}}$. Function $\dfrac{sin\ x}{x}$ is bounded and $\dfrac{sin\ x}{x} \leq 1$. Thus $\dfrac{sin\ x}{x^n} \leq \dfrac{1}{x^{n-1}}$ and $\int_0^{\pi/2} \dfrac{dx}{x^{n-1}}$ converges only if $n-1 < 1$ or $n < 2$ and diverges for $n \geq 2$.

(iii) Let $f(x) = \dfrac{1}{x^3(e^{-x}+1)}$ and $g(x) = \dfrac{1}{x^3}$. Also

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{x^3(e^{-x}+1)}\frac{x^3}{1} = \lim_{x\to\infty} \frac{1}{e^{-x}+1} = 1.$$

Also $\int_1^\infty \dfrac{1}{x^3}\ dx$ converges. Therefore $\int_1^\infty \dfrac{dx}{x^3(e^{-x}+1)}$ converges.

Example 2.1.5. *Examine the convergence of (i)* $\int_{-\pi/2}^{\pi/2} tan\ x\ dx$ *(ii)* $\int_0^{\pi/2} \dfrac{cos^n\ x}{x^m}\ dx$, *if* $m < 1$.

Solution:

$$(i)\ \int_{-\pi/2}^{\pi/2} tan\ x\ dx = \lim_{\epsilon\to 0}\int_{-\pi/2+\epsilon}^{c} tan\ x\ dx + \lim_{\eta\to 0}\int_{c}^{\pi/2-\eta} tan\ x\ dx$$

$$= \lim_{\epsilon\to 0} ln[cos\ (-\pi/2+\epsilon)] - ln[cos\ c] - \lim_{\eta\to 0} ln[cos\ (\pi/2-\eta)] - ln[cos\ c].$$

Limits do not exist. Hence improper integral diverges.
(ii) Note that $\dfrac{cos^n\ x}{x^m} < \dfrac{1}{x^m}$ *for* $0 < x < \pi/2$. *Also* $x = 0$ *is the point of infinite discontinuity and* $\int_0^{\pi/2} \dfrac{1}{x^m}\ dx$ *is convergent if* $m < 1$ *by comparison test.*

Absolute Convergence of Improper Integrals

If the function $f(x)$ changes sign within the interval of integration, we consider absolute convergence. The improper integral $\int_a^b f(x)\ dx$ is called absolutely convergent if $\int_a^b |f(x)|\ dx$ converges. Since $f(x) \le |f(x)|$, $\forall\ x$, absolutely convergent improper integral is convergent.

Example 2.1.6. *Examine the convergence of improper integral* $\int_{-\infty}^{\infty} \dfrac{sin\ x}{1+x^2}\ dx$.
Solution: *Note that* $|I| = |\int_{-\infty}^{\infty} \dfrac{sin\ x}{1+x^2}|\ dx \le \int_{-\infty}^{\infty} |\dfrac{sin\ x}{1+x^2}|\ dx$
$= \lim_{a\to-\infty} \int_a^c |\dfrac{sin\ x}{1+x^2}|\ dx + \lim_{b\to\infty} \int_c^b |\dfrac{sin\ x}{1+x^2}|\ dx = I_1 + I_2.$
$I_1 = \lim_{a\to-\infty} \int_a^c |\dfrac{sin\ x}{1+x^2}|\ dx \le \lim_{a\to-\infty} \int_a^c |\dfrac{1}{1+x^2}|\ dx = tan^{-1}c + \dfrac{\pi}{2}.$
$I_2 = \lim_{b\to\infty} \int_c^b |\dfrac{sin\ x}{1+x^2}|\ dx \le \lim_{b\to\infty} \int_c^b |\dfrac{1}{1+x^2}|\ dx = \dfrac{\pi}{2} - tan^{-1}c.$
Thus $|I| \le |I_1| + |I_2| \le \pi$ *and hence convergent.*

2.2 Beta Function

The improper integral defined by

$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ for $m > 0$, $n > 0$ is called as beta function.

Note that

$\int_0^1 x^{m-1}(1-x)^{n-1}dx$ converges if $m > 0$, $n > 0$.

Properties:

1. $\beta(m,n) = \beta(n,m)$

   Put $x = 1-y$. Then

   $\beta(m,n) = -\int_1^0 (1-y)^{m-1}y^{n-1}dy = \int_0^1 y^{n-1}(1-y)^{m-1}dy = \beta(n,m).$

2. $\beta(m,n) = \int_0^{\pi/2} sin^{2m-1}\ \theta cos^{2n-1}\theta\ d\theta$, $n > 0$.

   Put $x = sin^2\theta$, $dx = 2sin\ \theta\ cos\ \theta\ d\theta$ in $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$.

3. $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

2.2.1 Gamma Functions

The improper integral of the form $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$, $n > 0$ is called gamma function.

Properties:

1. $\Gamma(1) = \int_0^\infty e^{-x}dx = 1.$

2. Reduction formula $\Gamma(n+1) = n\Gamma(n)$

   Proof: $\Gamma(n+1) = \int_0^\infty e^{-x}x^n dx = [-x^n e^{-x}]_0^\infty + n\int_0^\infty e^{-x}x^{n-1}dx$

   $= n\Gamma(n) = n!.$

3. $\Gamma(1/2) = \sqrt{\pi}.$

   Proof: $\Gamma(1/2) = \int_0^\infty e^{-x}x^{1/2}dx = 2\int_0^\infty e^{-y^2}dy$ (by putting $x = y^2$).

   $(\Gamma(1/2))^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy$

   $= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}rdrd\theta = \frac{4\pi}{2}\int_0^\infty e^{-r^2}rdr = \pi$

4. Relation between β and Γ functions

   $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

   Proof: $\Gamma(m) = \int_0^\infty e^{-t}t^{m-1}dt = 2\int_0^\infty e^{-x^2}x^{2m-1}dx$

   $\Gamma(n) = 2\int_0^\infty e^{-y^2}y^{2n-1}dy$

   $\Gamma(m)\Gamma(n) = 4\int_0^\infty e^{-x^2}x^{2m-1}dx\int_0^\infty e^{-y^2}y^{2n-1}dy$

   $= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}x^{2m-1}y^{2n-1}dxdy$

   $= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}r^{2(m+n)-1}cos^{2m-1}\theta\ sin^{2n-1}\ \theta\ drd\theta$

   $= 2\int_0^\infty e^{-r^2}r^{2(m+n)-1}dr \times 2\int_0^{\pi/2} cos^{2m-1}\theta\ sin^{2n-1}\ \theta d\theta$

   $= \Gamma(m+n)\beta(m,n).$

5. $\int_0^{\pi/2} cos^m\theta\ sin^n\ \theta d\theta$

   $= \frac{1}{2}\beta(\frac{m+1}{2}, \frac{n+1}{2}) = \frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})}{2\Gamma(\frac{m+n+2}{2})}$

Example 2.2.1. *If* $\int_0^\infty \frac{x^{m-1}}{1-x}dx = \frac{\pi}{sin\ m\pi}$, $0 < m < 1$, *then* $\Gamma(m)\Gamma(1-m) = \frac{\pi}{sin\ m\pi}$.

Put $\frac{x}{1+x} = y$. *Then* $x = \frac{y}{1-y}$ *and* $dx = \frac{1}{(1-y)^2}dy$. *Thus*
$\int_0^\infty \frac{x^{m-1}}{1-x}dx = \int_0^1 y^{m-1}(1-y)^{m-1}dy = \beta(m, 1-m)$
$= \frac{\Gamma(m)\Gamma(1-m)}{\Gamma(m+1-m)} = \Gamma(m)\Gamma(1-m)$. *Hence* $\int_0^\infty \frac{x^{m-1}}{1-x}dx = \frac{\pi}{sin\ m\pi}$.

Example 2.2.2. (*Legendre Duplication formula*)

$\sqrt{\pi}\ \Gamma(2m) = 2^{2m-1}\Gamma(m)\Gamma(m+\frac{1}{2})$.
Since $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \beta(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta\ cos^{2n-1}\theta\ d\theta$..............(1)
Putting $n = m$, *we have*
$\frac{\Gamma(m)^2}{\Gamma(2m)} = \beta(m,m) = 2\int_0^{\pi/2} sin^{2m-1}\theta\ cos^{2m-1}\theta\ d\theta$.
$= \frac{1}{2^{2m-2}}\int_0^{\pi/2} sin^{2m-1}2\theta\ d\theta$. *Put* $2\theta = \phi$. *Then*
$= \frac{1}{2^{2m-1}}\int_0^{\pi/2} sin^{2m-1}\phi\ d\phi$.......................(2).
In (1) take $n = 1/2$, *we get*
$\frac{\Gamma(m)\Gamma(1/2)}{\Gamma(m+1/2)} = 2\int_0^{\pi/2} sin^{2m-1}\theta d\theta$..............(3).
From (2) *and* (3), *we get,*
$\frac{\Gamma(m)^2}{\Gamma(2m)} = \frac{1}{2^{2m-1}}\frac{\Gamma(m)\Gamma(1/2)}{\Gamma(m+1/2)}$. *Since* $\Gamma(1/2) = \sqrt{\pi}$, *we have*
$\sqrt{\pi}\ \Gamma(2m) = 2^{2m-1}\Gamma(m)\Gamma(m+\frac{1}{2})$.

Example 2.2.3. $\beta(m+\frac{1}{2}, m+\frac{1}{2}) = \frac{\pi}{m2^{4m-1}\beta(m,m)}$.

Solution: *By definition* $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m > 0$, $n > 0$. *By substituting* $x = sin^2\ \theta$, *we have* $\beta(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta\ cos^{2n-1}\theta\ d\theta$.

First show that $\beta(m,m) = \frac{1}{2^{2m-1}}\beta(m, 1/2)$ ....*(1)*
Then $\Gamma(m+1/2)\Gamma(m) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$ ....*(2)*
Since $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \beta(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta\ cos^{2n-1}\theta\ d\theta$, *we have*
$\beta(m+1/2, m+1/2) = \frac{\Gamma(m+1/2)\Gamma(m+1/2)}{\Gamma(2m+1)}$. *Using (2) we have*
$\beta(m+\frac{1}{2}, m+\frac{1}{2}) = \frac{\pi}{m2^{4m-1}\beta(m,m)}$.

Example 2.2.4. *(i) Let* $\phi(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)dx$. *Then by Leibniz formula*
$\frac{d\phi}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f(x,\alpha)}{\partial\alpha}dx + f(b,\alpha)\frac{db}{\partial\alpha} - f(a,\alpha)\frac{da}{\partial\alpha}$ *show that* $\frac{d\phi}{da} = \frac{\pi}{2(a+1)}$.

(ii) $\phi(a) = \int_0^\infty \frac{tan^{-1}(ax)}{x(1+x^2)}dx.$

Solution: *Let* $\phi(a) = \int_0^\infty \frac{tan^{-1}(ax)}{x(1+x^2)}dx$. *Then by Leibniz formula*

$\frac{d\phi}{da} = \int_0^\infty \frac{\partial}{\partial\ a}[\frac{tan^{-1}(ax)}{x(1+x^2)}]dx = \int_0^\infty \frac{dx}{(1+x^2)(1+a^2x^2)}$

$= \frac{1}{a^2-1}\int_0^\infty[\frac{a^2}{a^2x^2+1} - \frac{1}{1+x^2}]dx = \frac{1}{a^2-1}[\{a\ tan^{-1}(ax)\}_0^\infty - \{tan^{-1}\ x\}_0^\infty]$

$= \frac{\pi}{2}[\frac{a-1}{a^2-1}],\ a > 0,\ a \neq 1$

$= \frac{\pi}{2(a+1)}$. *Integrating with respect to a, we have*

$\phi(a) = \frac{\pi}{2}ln(a+1) + c$. *Since* $\phi(0) = 0$, $c = 0$.

Therefore $\phi(a) = \int_0^\infty \frac{tan^{-1}(ax)}{x(1+x^2)}dx = \frac{\pi}{2}ln(a+1).$

2.3 Dirichlet Integral

Theorem 2.3.1. $\int\int_D x^{l-1}y^{m-1}dxdy = \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m+1)}h^{l+m}$, where D is domain $x \geq 0$, $y \geq 0$, $x + y \leq h$.

Proof. Put $x = Xh$, $y = Yh$, $dxdy = h^2dXdY$. Then

$$\begin{aligned}\int\int_D x^{l-1}y^{m-1}dxdy &= \int\int_D (Xh)^{l-1}(Yh)^{m-1}dXdY \\ &= h^{l+m}\int_0^1\int_0^{1-X} X^{l-1}Y^{m-1}dXdY \\ &= h^{l+m}\int_0^1 X^{l-1}dX\int_0^{1-X} Y^{m-1}dY \\ &= h^{l+m}\int_0^1 X^{l-1}dX[\frac{Y^m}{m}]_0^{1-X} \\ &= \frac{h^{l+m}}{m}\int_0^1 X^{l-1}(1-X)^m dX \\ &= \frac{h^{l+m}}{m}\beta(l, m+1) \\ &= \frac{h^{l+m}}{m}\frac{\Gamma(l)\Gamma(m+1)}{\Gamma(l+m+1)} \\ &= h^{l+m}\frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m+1)}\end{aligned}$$

□

Theorem 2.3.2. $\int\int\int_V x^{l-1}y^{m-1}z^{n-1}dxdydz = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}$, where V is region $x \geq 0$, $y \geq 0$, $z \geq 0$, $x + y + z \leq 1$.

Proof. Put $x + z \leq 1 - x = h$. then $z \leq h - y$.

$$\begin{aligned}\int\int\int_V x^{l-1}y^{m-1}z^{n-1}dxdydz &= \int_0^1 x^{l-1}dx\int_0^{1-x} y^{m-1}dy\int_0^{1-x-y} z^{n-1}dz \\ &= \int_0^1 x^{l-1}dx[\int_0^h\int_0^{h-y} y^{m-1}dyz^{n-1}dz]\end{aligned}$$

$$\begin{aligned}
&= \int_0^1 x^{l-1}dx[h^{m+n}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)}] \\
&= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)}\int_0^1 x^{l-1}(1-x)^{m+n}dx \\
&= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)}\beta(l,m+n+1) \\
&= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)}\frac{\Gamma(l)\Gamma(m+n+1)}{\Gamma(l+m+n+1)} \\
&= \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}
\end{aligned}$$

□

Corollary 2.3.3. $\int\int\int_V x^{l-1}y^{m-1}z^{n-1}dxdydz = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}h^{l+m+n}$, where V is region $x \geq 0$, $y \geq 0$, $z \geq 0$ $x+y+z \leq h$.

Exercise 2.3.4.

1. Discuss the convergence of (i) $\int_1^2 \frac{\sqrt{x}}{ln\ x}\ dx$ (ii) $\int_0^{\pi/2} \frac{sin\ x}{x\sqrt{x}}dx$.
2. Examine the convergence of improper integrals (i) $\int_2^\infty \frac{dx}{ln\ x}$ (ii) $\int_1^\infty e^{-x^2}\ dx$ (iii) $\int_1^\infty \frac{dx}{x^2(e^{-x}+1)}$ (iv) $\int_1^\infty \frac{dx}{x^p}$.
3. Show that for $m,n > 0$,
   $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$. Hint: Put $x = \frac{t}{1-t}$
4. Show that $\int_0^\infty \frac{x^c}{c^x}\ dx = \frac{\Gamma(c+1)}{(log\ c)^{c+1}}$.
5. Show that $\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}}\Gamma(n+\frac{1}{2})\Gamma(n)$ and hence $\Gamma(1/4)\Gamma(3/4) = \pi\sqrt{2}$.
6. Show that $\int_0^{\pi/2} \sqrt{tan\ \theta}\ d\theta = \frac{1}{2}\Gamma(\frac{1}{4})\Gamma(\frac{3}{4}) = \pi\sqrt{2}$. (Hint: $\int_0^{\pi/2} \sqrt{tan\ \theta}\ dx = \int_0^{\pi/2} sin^{1/2}\ \theta\ cos^{-1/2}\ \theta\ d\theta = \frac{1}{2}\frac{\Gamma(1/4)\Gamma(3/4)}{\Gamma(1)} = \frac{\pi\sqrt{2}}{2} = \frac{\pi}{\sqrt{2}}$.
7. Evaluate the improper integral $\int_0^\infty \sqrt{x}\ e^{-x^2}dx$.

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