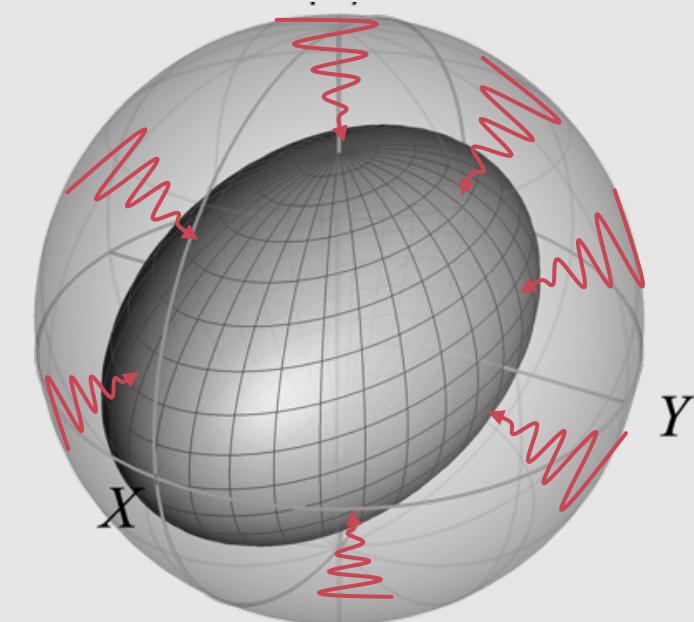


Introduction to Quantum Noise

by Zlatko K. Minev



Qiskit | Global Summer
School 2023



zlatko-minev.com



@zlatko_minev

Introduction to noise in quantum computers

2 Quantum gates meets coherent noise

the state $|0\rangle$ as a vector in the Bloch sphere that points from the origin of Bloch sphere all the way to the surface (radius equal to 1) in the Z direction.

Fig. 2-3 also shows a black vector along the X axis. The black vector shows the axis around which the operator X , and its corresponding unitary $R_X(\theta)$, will rotate a state vector on the Bloch sphere.

If we now look at the way that the X gate operates on the initial state $|\psi_0\rangle = |0\rangle$, we can see that the ideal X gate operation will flip $|0\rangle$ to $|1\rangle$ and $|0\rangle$ to $|1\rangle$. The physical implementation of an X gate ($R_X(\theta)$) can behave as the ideal X gate (up to an ignorable global phase). As θ increases, the $R_X(\theta)$ operator evolves the ground state $|0\rangle$ around the X axis by the right-hand rule as illustrated in Fig. 2-4 with the color changing path from $|0\rangle$ to $|1\rangle$.

Supporting this illustration with some math, $R_X(\theta)$'s action on $|\psi_0\rangle$ from $\theta = 0$ to $\theta = \pi$, evolves the state as

$$R_X(\theta = \pi)|\psi_0\rangle = [\cos(\pi/2)I - i\sin(\pi/2)X]|0\rangle \quad (2.6a)$$

$$|\psi_\theta(\theta = \pi)\rangle = \cos(\pi/2)|0\rangle - i\sin(\pi/2)X|0\rangle \quad (2.6b)$$

$$= \cos(\pi/2)|0\rangle - i\sin(\pi/2)|1\rangle \quad (2.6c)$$

$$= -i|1\rangle \quad (2.6d)$$

$$= |1\rangle. \quad (2.6e)$$

In Eq. (2.6d), we see that the rotation has introduced a global phase factor of $-i$ to the state $|1\rangle$, which we will ignore from now on.

Let's look at the expectation value of the X , Y , and Z quadratures for the state as it evolves as for any angle θ

$$\langle X \rangle = \langle \psi_\theta(\theta) | X | \psi_\theta(\theta) \rangle = 0, \quad (2.7a)$$

$$\langle Y \rangle = \langle \psi_\theta(\theta) | Y | \psi_\theta(\theta) \rangle = -\sin(\theta), \quad (2.7b)$$

$$\langle Z \rangle = \langle \psi_\theta(\theta) | Z | \psi_\theta(\theta) \rangle = \cos(\theta). \quad (2.7c)$$

The three expectation values match the view of the Bloch sphere. The X expectation value is zero the entire evolution, and there are oscillations $-\sin(\theta)$ and $\cos(\theta)$ along Y and Z axes respectively as θ increases. Now we can see the reason as to why the factor of $1/2$ in the definition of the $R_X(\theta)$. The $1/2$ allows the angle by which a vector on the Bloch sphere will rotate to be the same as the angle θ in $R_X(\theta)$.

Of course, you can continue this angle θ under the $R_X(\theta)$ operation. You'll notice that for increasing the angle of θ , it will trace a full circle all the way around the Bloch sphere, back into the ground state when $\theta = 2\pi$. In other words, $R_X(2\pi) = I$.

To summarize, the initial state $|0\rangle$ starts off in the ground state, which is pointing in the plus Z direction (Fig. 2-4). The first rotation, an X gate, takes the state all the way around to the south pole $|1\rangle$. Applying a second gate X evolves the state from $|1\rangle$ all the way around to $|0\rangle$. Applying more and more X gates, the state will continue to rotate around. This visualization helps us keep in mind what's happening, and it's a particularly useful one for understanding coherent errors.

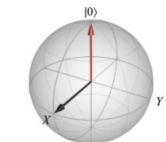


Fig. 2-3. Bloch sphere with the ground state $|0\rangle$ visualized as a red vector pointing from the origin to the surface. The red vector overlaps the Z -axis. The operator $R_X(\theta)$ is represented as the black vector pointing along the X axis.

Right-hand rule

Open your palm and point the thumb of your right hand along the X axis. The direction your fingers curl as you close your palm is the direction of θ . Your fingers will track out the trajectory illustrated by the colored path in Fig. 2-4. (TODO: add pic of right hand rule.)

Practice

On your own, draw the path $|0\rangle$ takes on the surface of the Bloch sphere for an angle θ not equal to π .

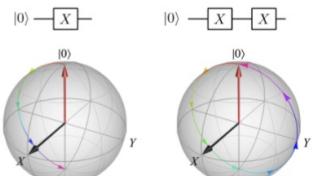
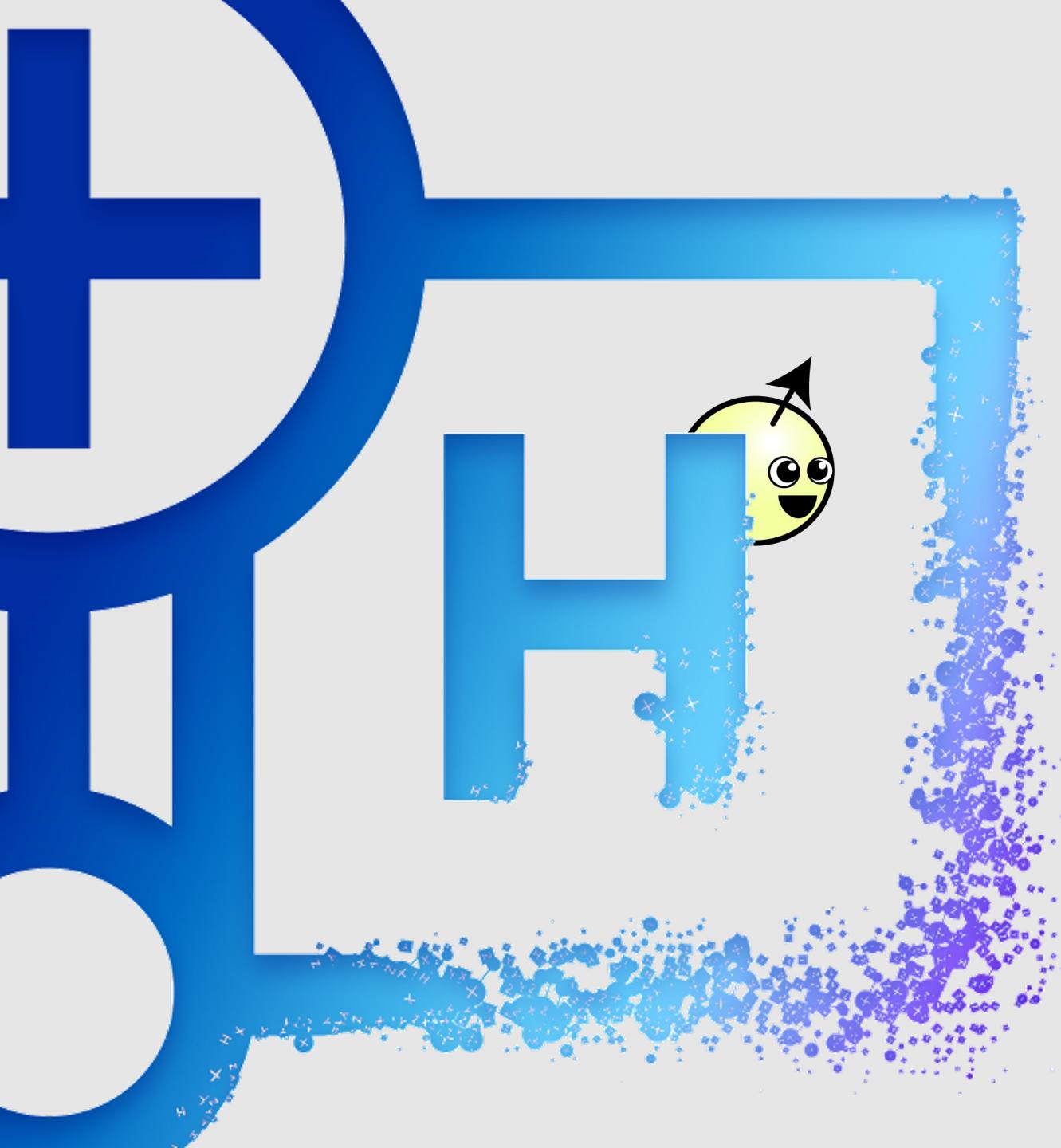


Fig. 2-4. Bloch spheres showing the initial state $|0\rangle$ as a red vector. The color changing path marks out the state's evolution during the corresponding circuit diagram above the spheres. On the left, one X gate is applied leaving the state in $|1\rangle$. On the right, two X gates are applied returning the state to where it began.

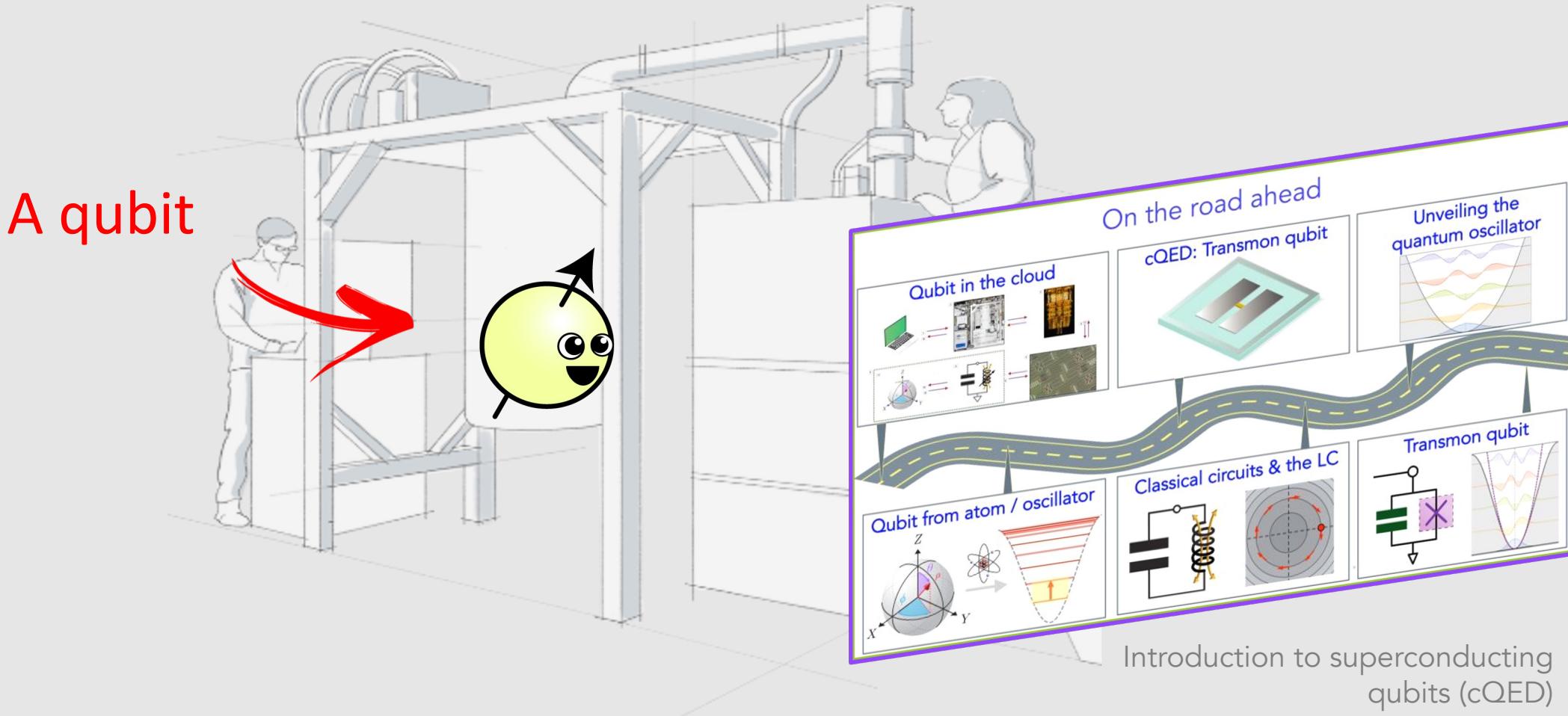




*What do you need to
know before putting
theory to practice on a
real, noisy quantum
processor?*



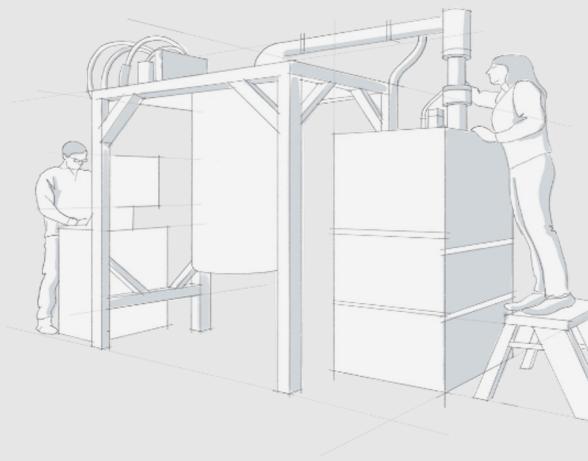
Chapter 1: Hello World with a real experiment!



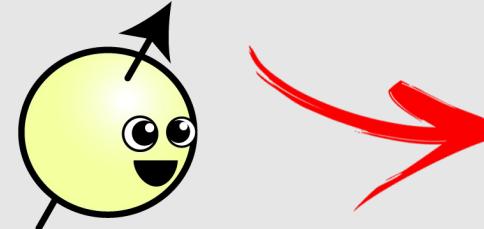
Introduction to superconducting
qubits (cQED)
Lecs. 16-21 Minev
QGSS 2020 at qiskit.org/learn



Hello World! building blocks



A qubit



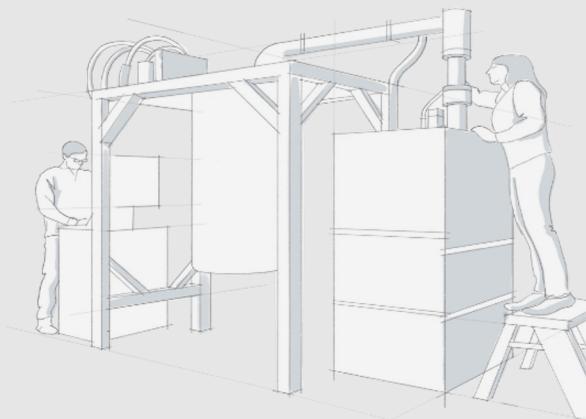
$|1\rangle$

$|0\rangle$

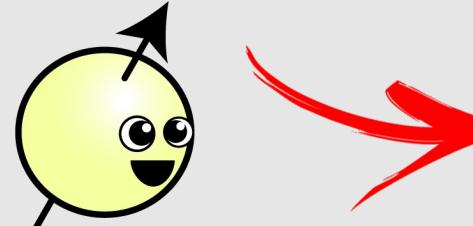
Computational
basis states



Hello World! building blocks



A qubit

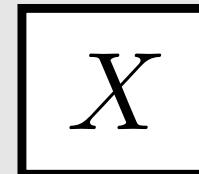


$|1\rangle$

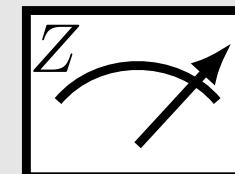
$|0\rangle$

Computational
basis states

Operations: qubit gate



Measurements: qubit observable



refresher:

$$X |0\rangle = |1\rangle$$

$$X |1\rangle = |0\rangle$$

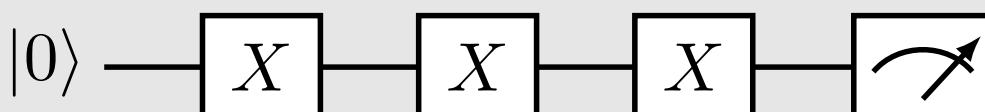
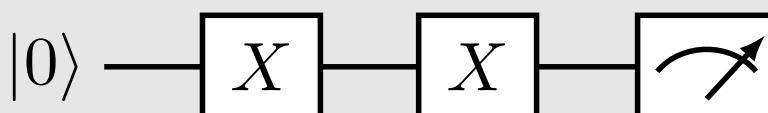
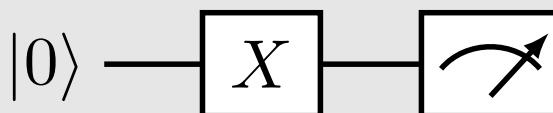
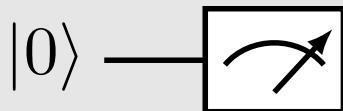
$$Z |0\rangle = +1 |0\rangle$$

$$Z |1\rangle = -1 |1\rangle$$



Hello World! qubit flipper quantum circuits

depth



⋮

refresher:

$$X |0\rangle = |1\rangle$$

$$X |1\rangle = |0\rangle$$

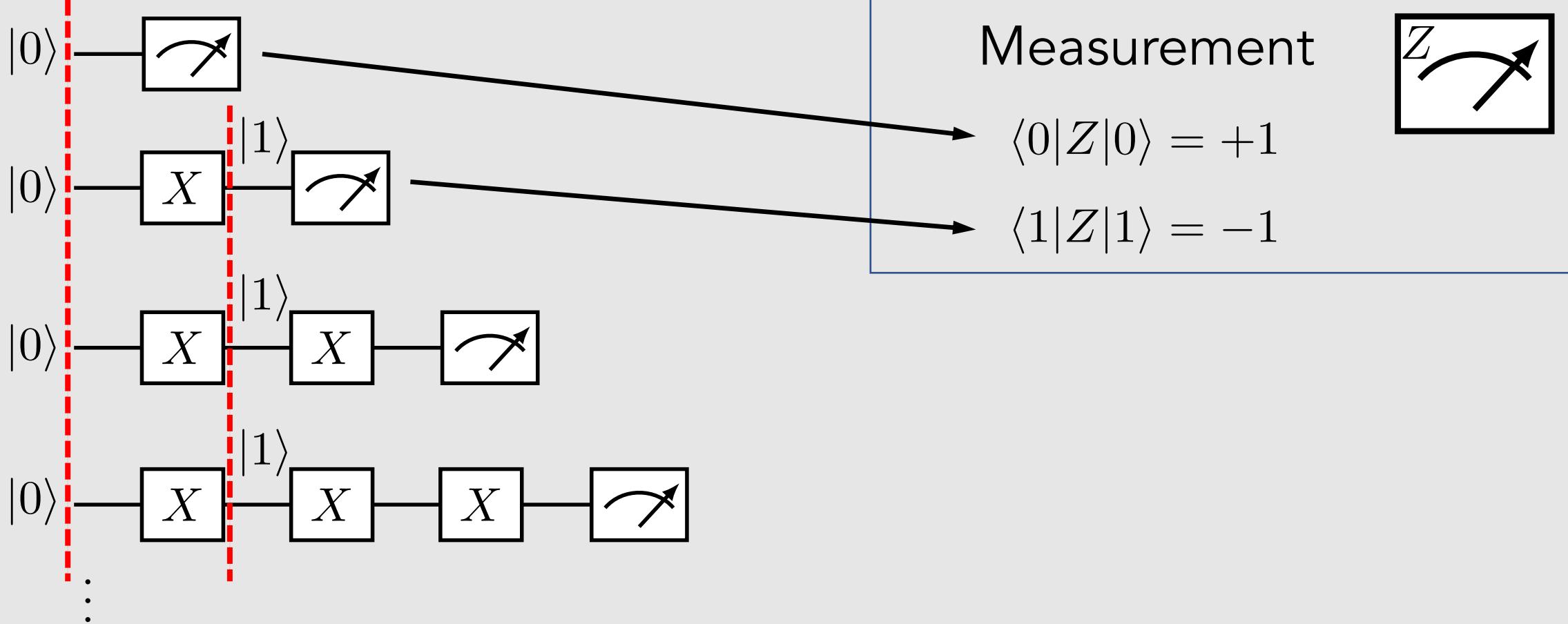
$$Z |0\rangle = +1 |0\rangle$$

$$Z |1\rangle = -1 |1\rangle$$



Hello World! “debugger” step through

depth



refresher:

$$X |0\rangle = |1\rangle$$

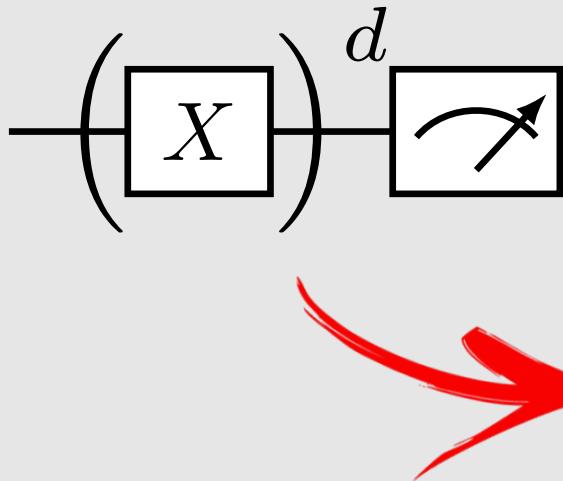
$$X |1\rangle = |0\rangle$$

$$Z |0\rangle = +1 |0\rangle$$

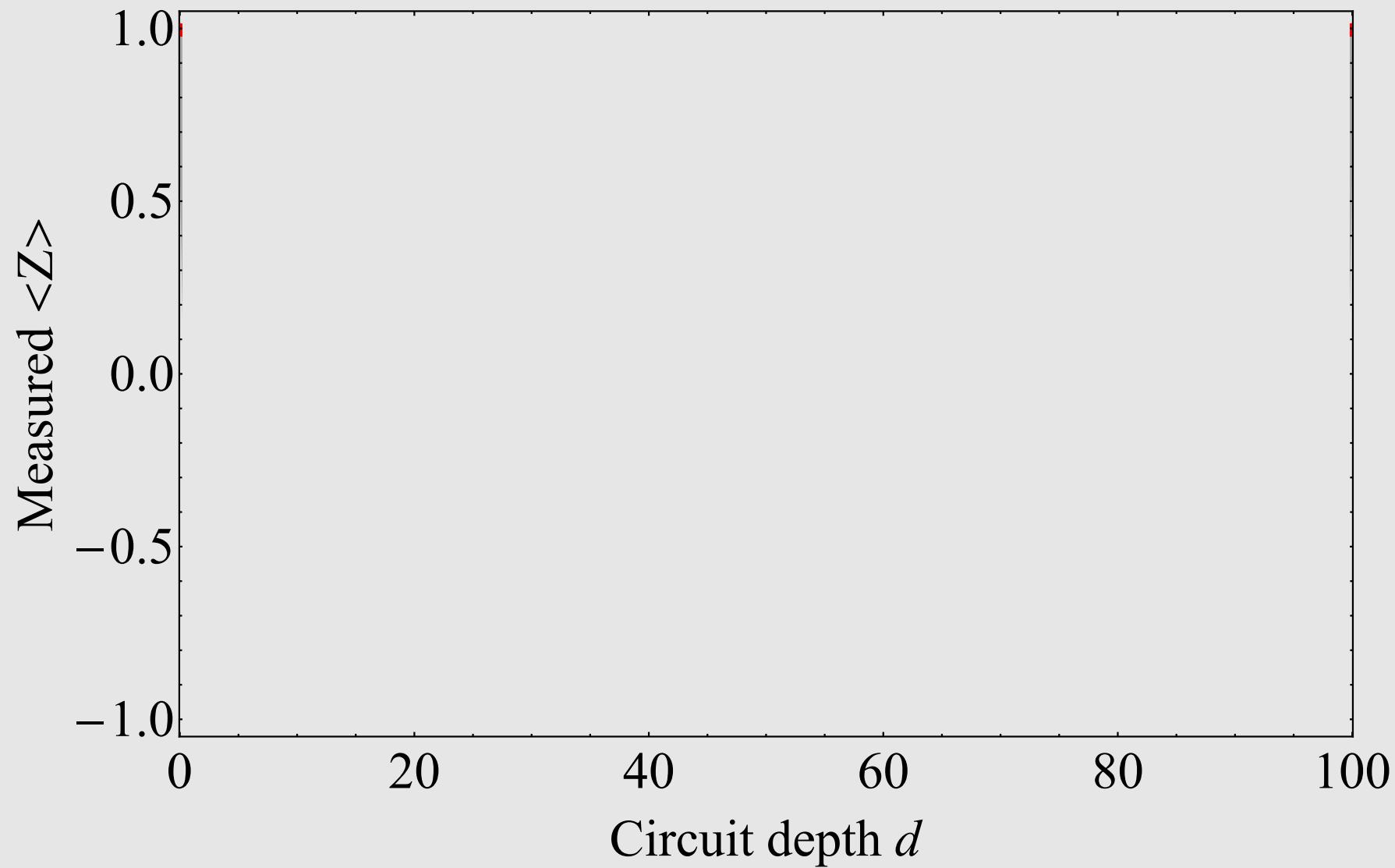
$$Z |1\rangle = -1 |1\rangle$$



Hello World! Ideal expectation results

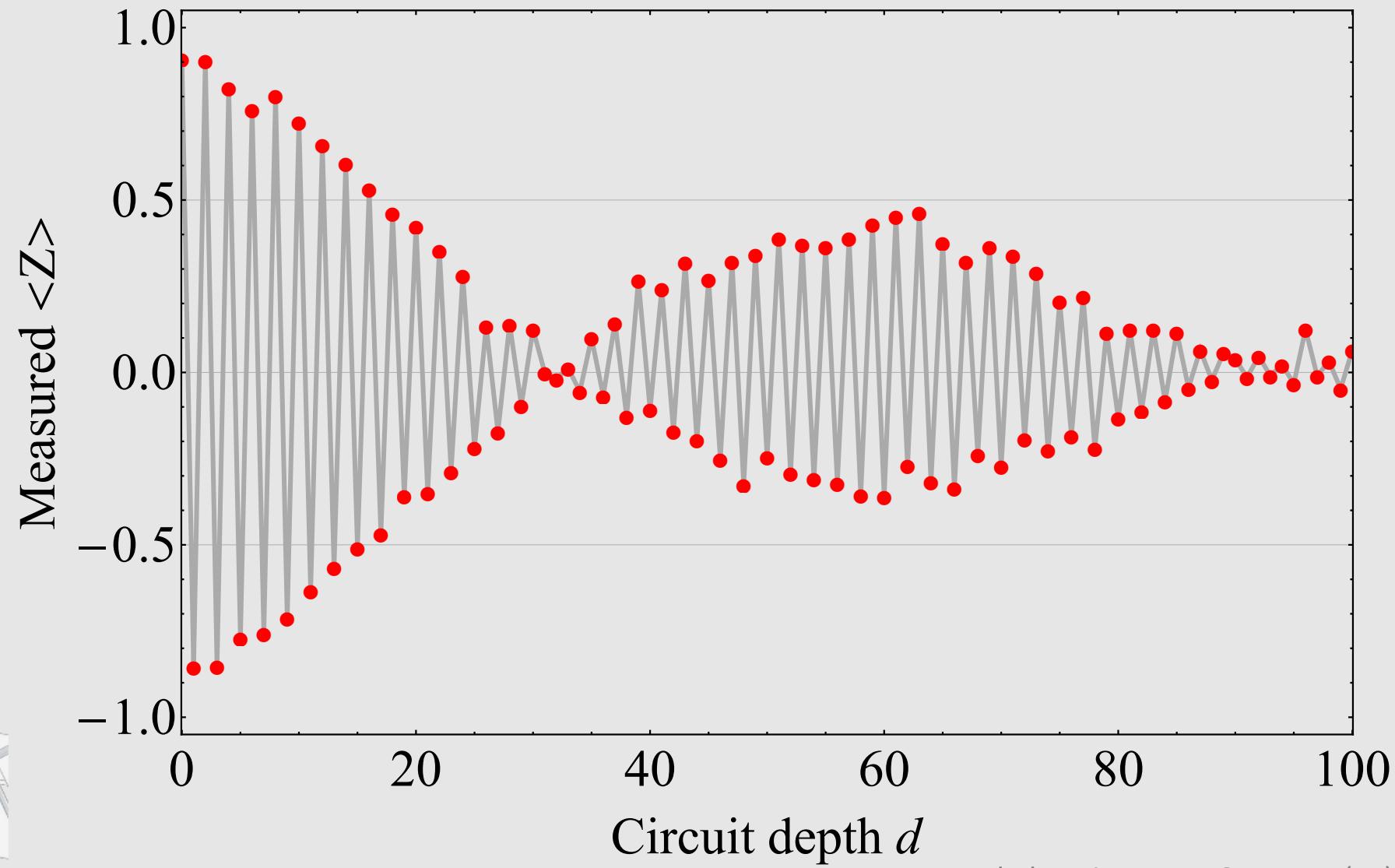
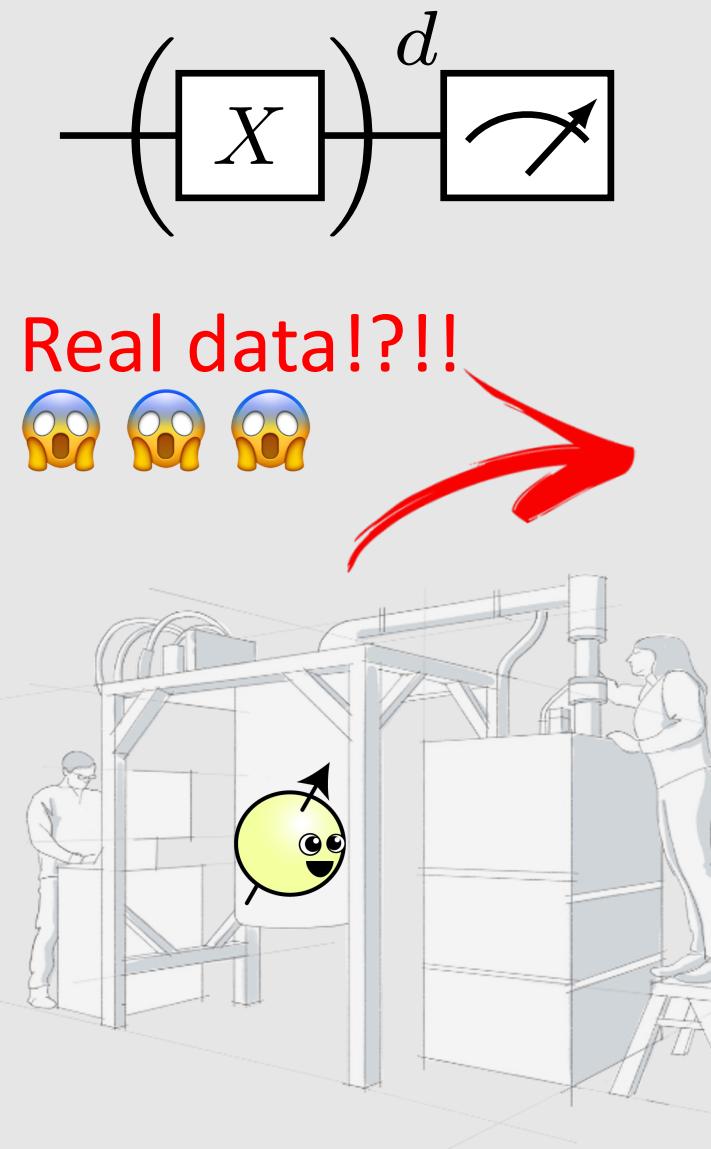


$$\langle Z \rangle = (-1)^d$$





Hello World! Real expectation results





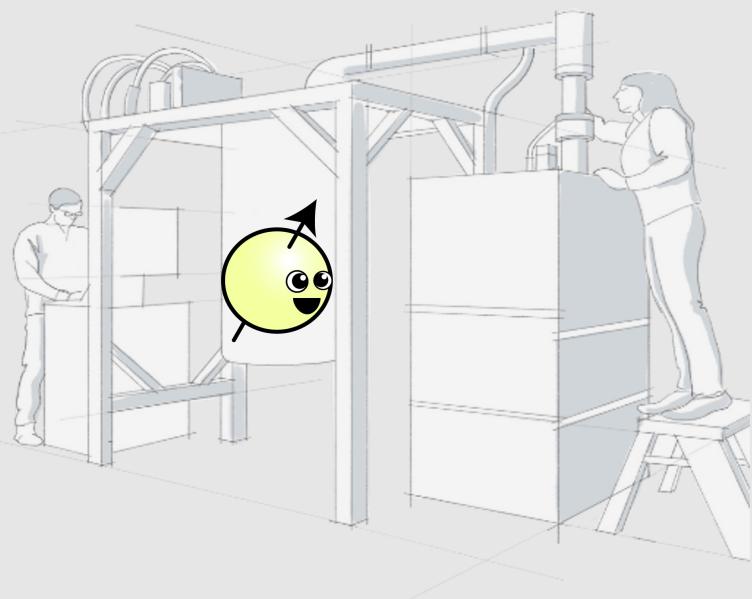
Real & noisy quantum processors: Why study noise?



“Well, your quantum computer is broken in every way possible simultaneously.”

PAUL
NOTH



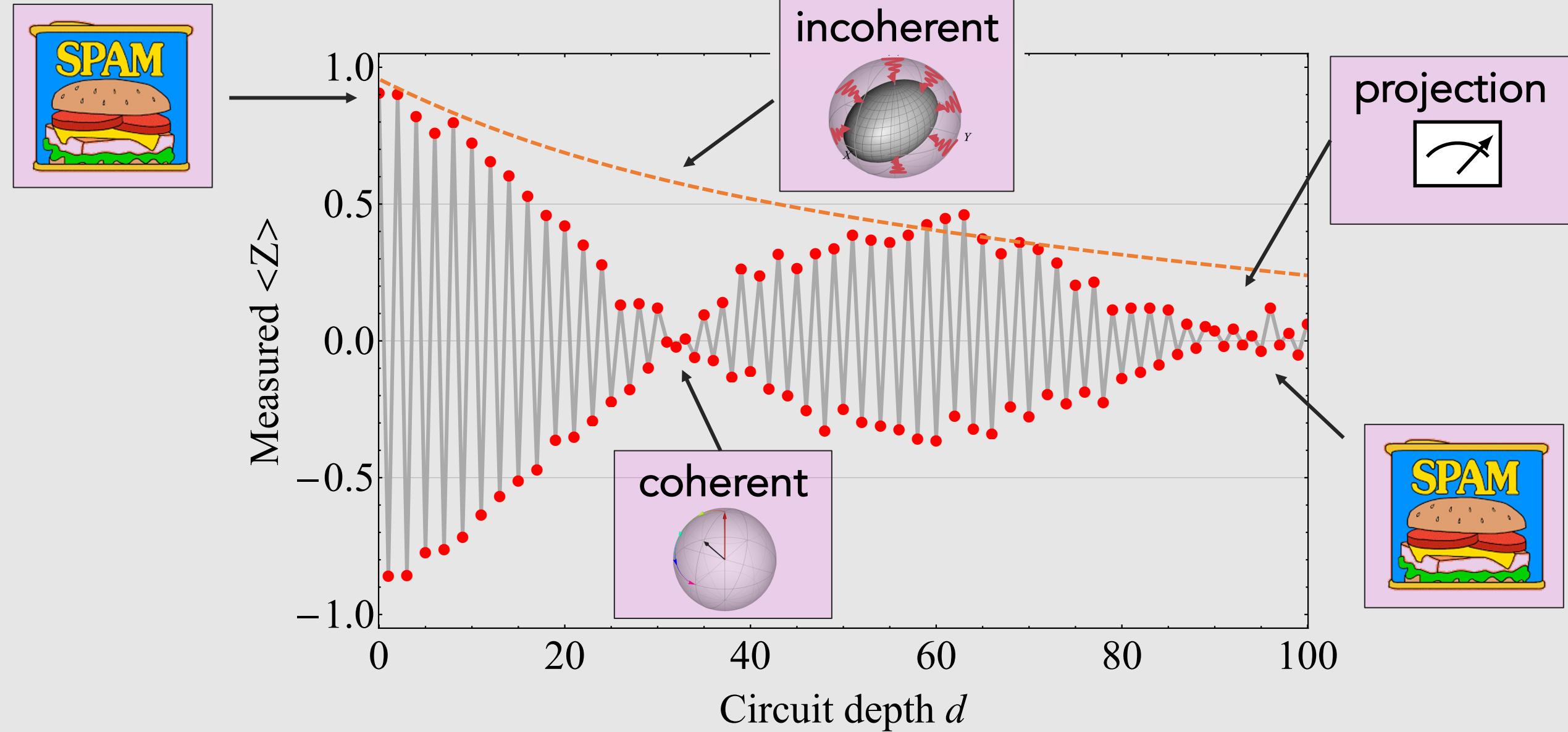


*“Quantum phenomena
do not occur in a Hilbert space,
they occur in a laboratory.”*

Asher Peres



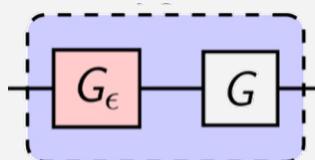
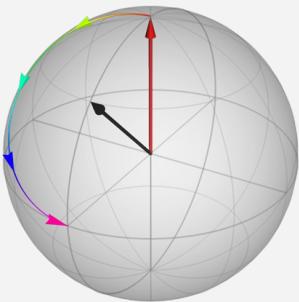
Elements of 😱 noise



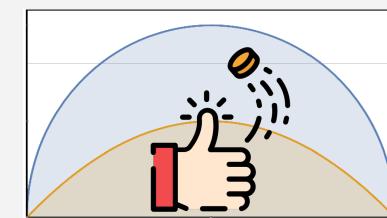


The road ahead

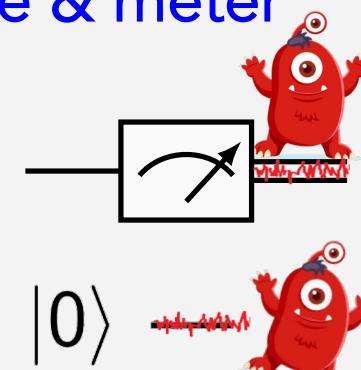
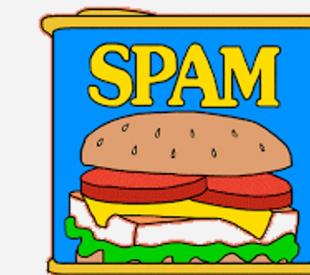
Coherent noise



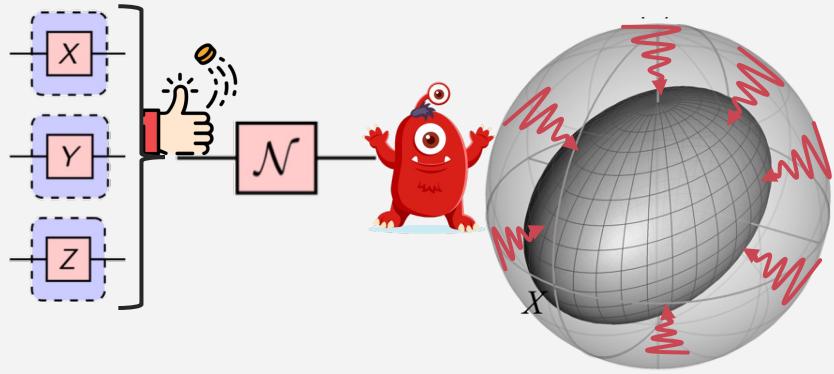
Measurements in quantum Projection noise



SPAM: State & meter



Incoherent noise



Bonus content

Coherent ZZ noise

...



coin-toss icon: Good Ware, flaticon; spam: make it move;
road based on: freepik; Monster image by jcomp on Freepik



Guideposts



Evil monster

Eve the Evil monster is here to terrorize the quantum world



Random Sorcery

Randomness and unpredictability abounds



Green box

Key result of section



Purple box

Key challenge highlighted



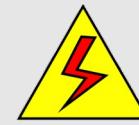
Try it yourself

Pause the video and try to work it out, before I share the answer



Dangerous bend sign

Denotes extra information, with appendix-like depth or digression. You can skip this material on a first reading



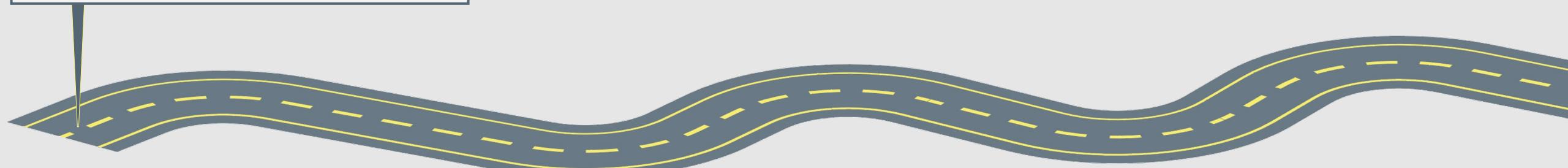
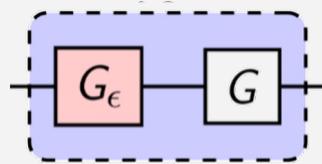
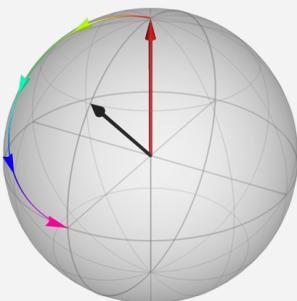
Caution

Caution or common pitfall



Chapter 2: Coherent noise

Coherent

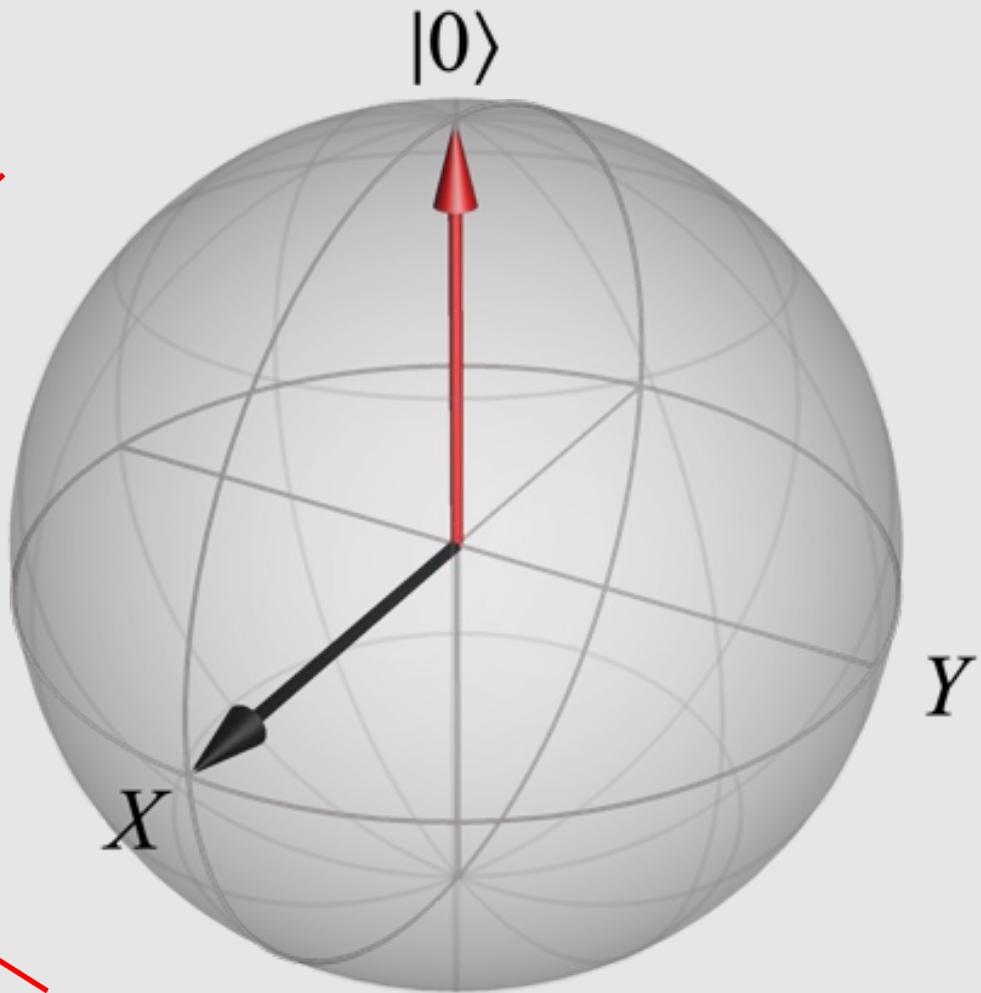
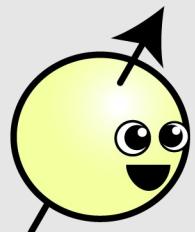
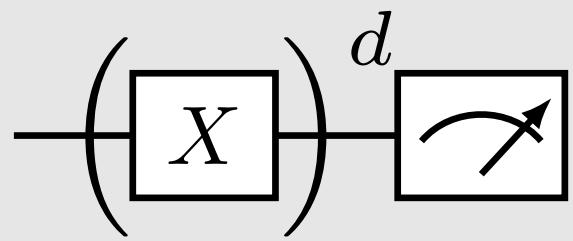


road based on: freepik

Zlatko Minev, IBM Quantum (20)



Return to the Hello World example



Try it yourself

$$\langle \hat{Z} \rangle = 1$$

$$\langle \hat{X} \rangle = 0$$

$$\langle \hat{Y} \rangle = 0$$

Show that
for $|0\rangle$:

$$\langle \hat{Z} \rangle = -1$$

$$\langle \hat{X} \rangle = 0$$

$$\langle \hat{Y} \rangle = 0$$



Bloch sphere cardinal points



Solution For $|\psi\rangle = |0\rangle$, and recall from earlier lectures the spin Pauli matrices and states:

$$\hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \text{ and } |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$\langle 0| = |0\rangle^\dagger = (1 \ 0), \quad \langle 1| = |1\rangle^\dagger = (0 \ 1),$$

then

$$\begin{aligned}\langle \hat{Z} \rangle &= \langle \psi | \hat{Z} | \psi \rangle \\&= \langle 0 | \hat{Z} | 0 \rangle \\&= \langle 0 | (\hat{Z} | 0 \rangle) \\&\quad (\text{recall } \hat{Z} | 0 \rangle = (+1) | 0 \rangle) \\&= \langle 0 | (+1 | 0 \rangle) \\&= (+1) \langle 0 | \cdot | 0 \rangle \\&\quad (\text{recall } \langle 0 | \cdot | 0 \rangle = \langle 0 | 0 \rangle = 1) \\&= 1\end{aligned}$$

$$\begin{aligned}\langle \hat{X} \rangle &= \langle \psi | \hat{X} | \psi \rangle \\&= \langle 0 | \hat{X} | 0 \rangle \\&\quad (\text{recall } \hat{X} | 0 \rangle = | 1 \rangle) \\&= \langle 0 | 1 \rangle \\&= 0\end{aligned}$$

$$\begin{aligned}\langle \hat{Y} \rangle &= \langle \psi | \hat{Y} | \psi \rangle \\&= \langle 0 | \hat{Y} | 0 \rangle \\&\quad (\text{recall } \hat{Y} | 0 \rangle = i | 1 \rangle) \\&= i \langle 0 | 1 \rangle \\&= 0\end{aligned}$$

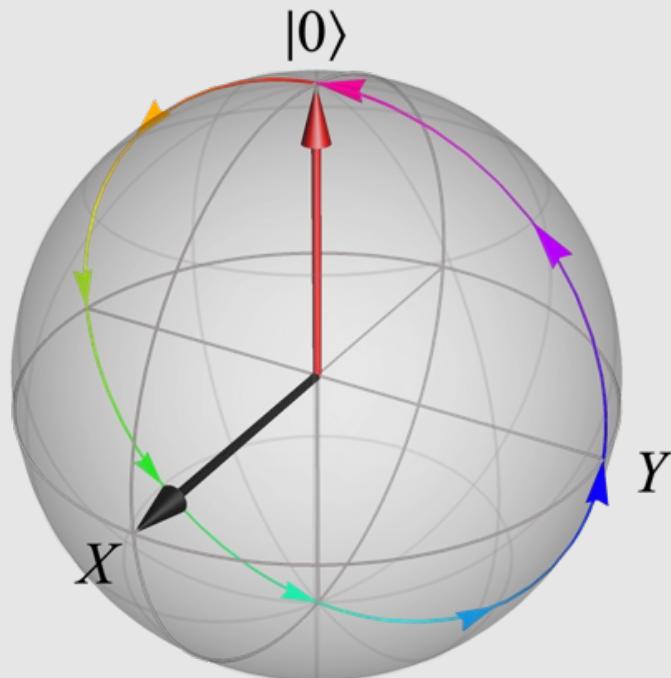
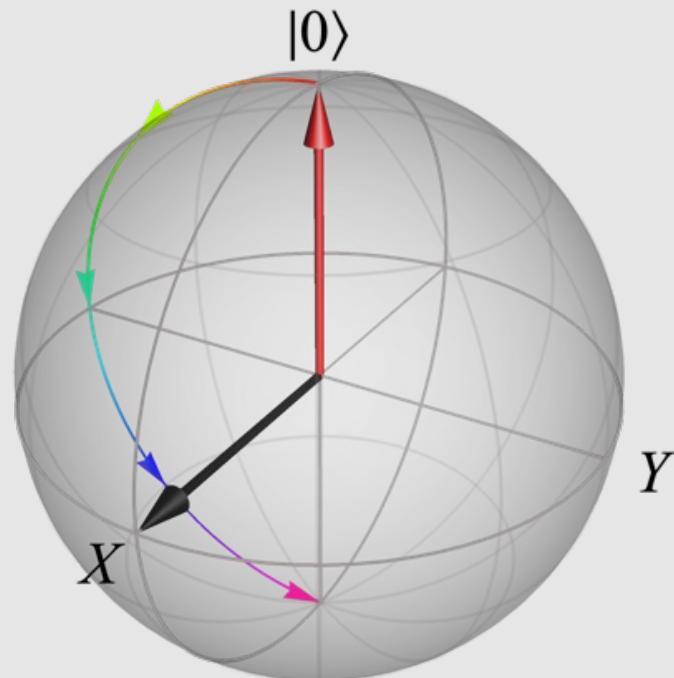
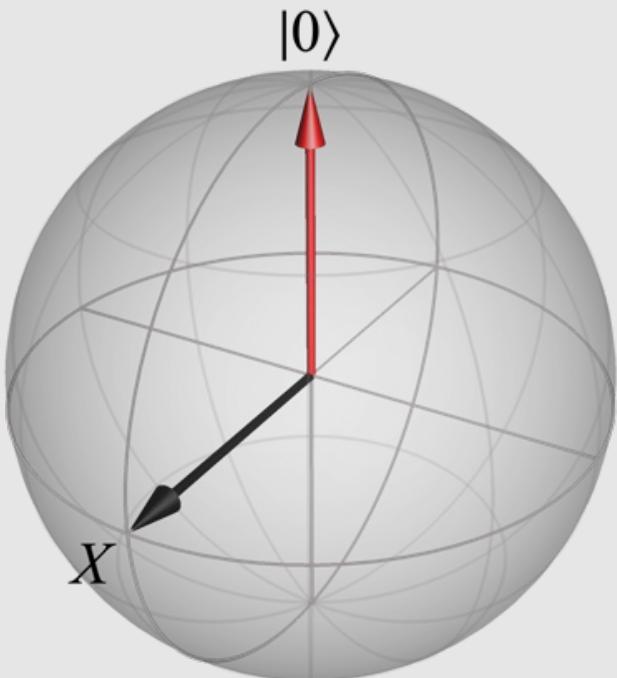
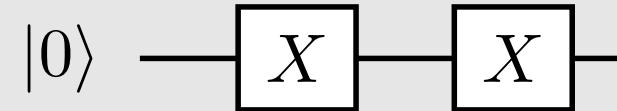
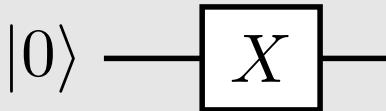
Recall that bra-ket is a dot product between the two vectors.



(Ideal) Evolution on the Bloch sphere

$\left(\begin{matrix} X \\ \vdots \\ X \end{matrix} \right)^d \rightarrow \text{Circuit}$

$|0\rangle$

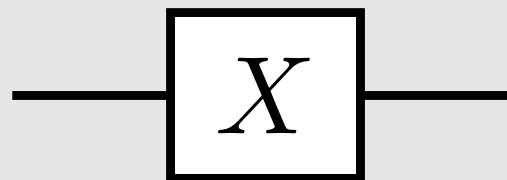




Origin of our X gate: time evolution

$$X = R_X(\pi)$$

(up to global phase)



Refresher:

$$\hat{H} = \frac{\hbar\omega}{2} X$$

$$= \frac{\hbar\omega}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$U(t) = \exp\left(-it\hat{H}/\hbar\right)$$

$$\theta := \omega t$$

$$R_X(\theta) = \exp\left(-\frac{i\theta}{2} X\right)$$

$$= \cos(\theta/2)I - i \sin(\theta/2)X$$

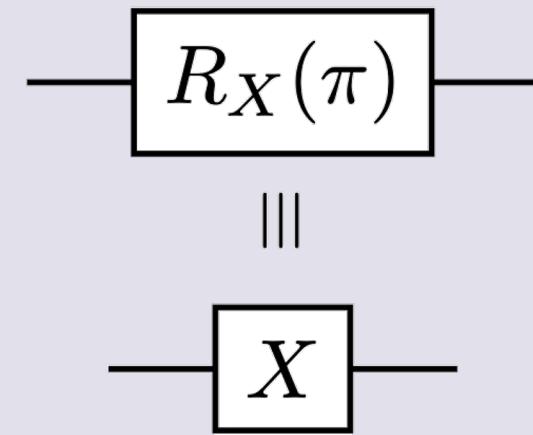
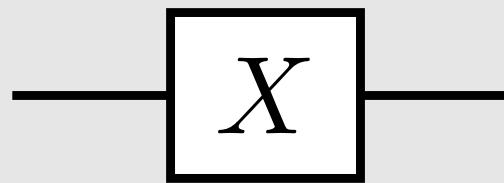
* We will often drop hats on Paulis I, X, Y, Z



Minor footnote

$$X = R_X(\pi)$$

(up to global phase)

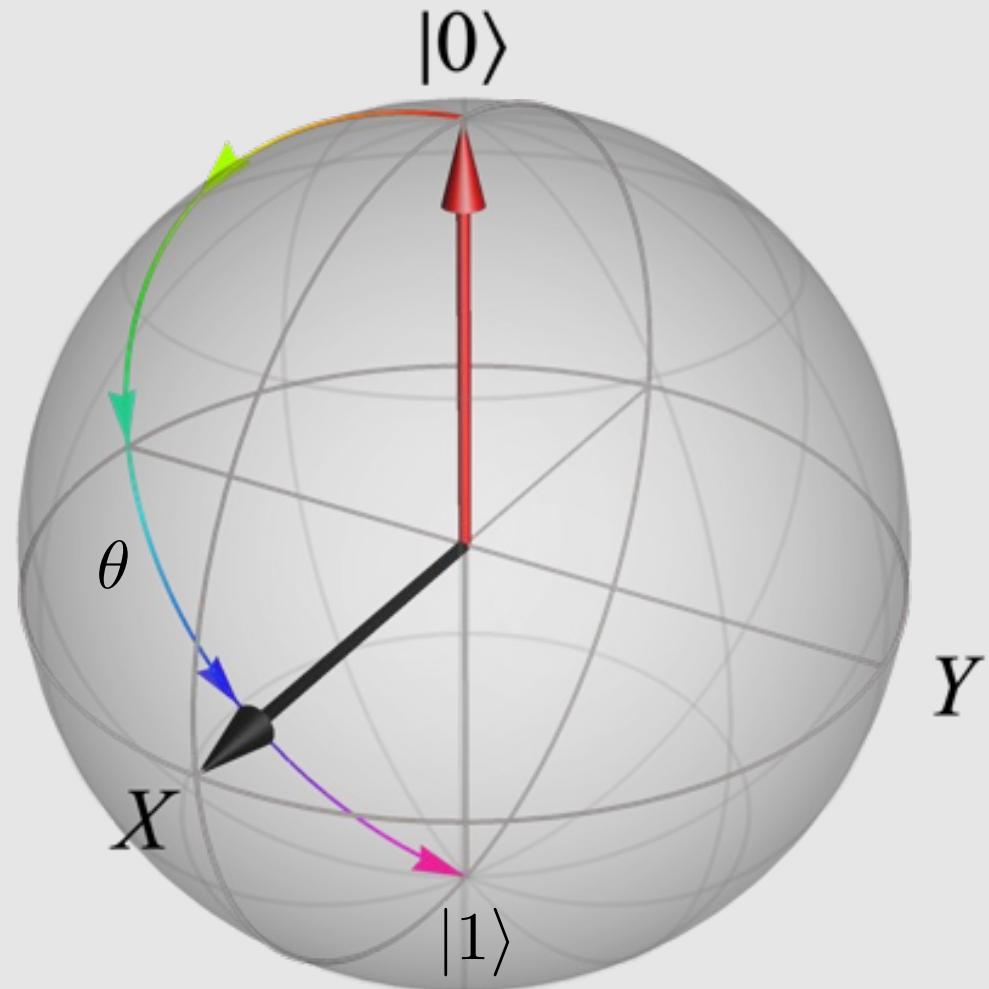


Minor footnote. There is a global phase factor between the way X and $R_X(\pi)$ are defined; i.e., $R_X(\pi) = -iX$.



Visualize: Evolution on the Bloch sphere

$$R_X(\theta) = \exp\left(-\frac{i\theta}{2}X\right)$$



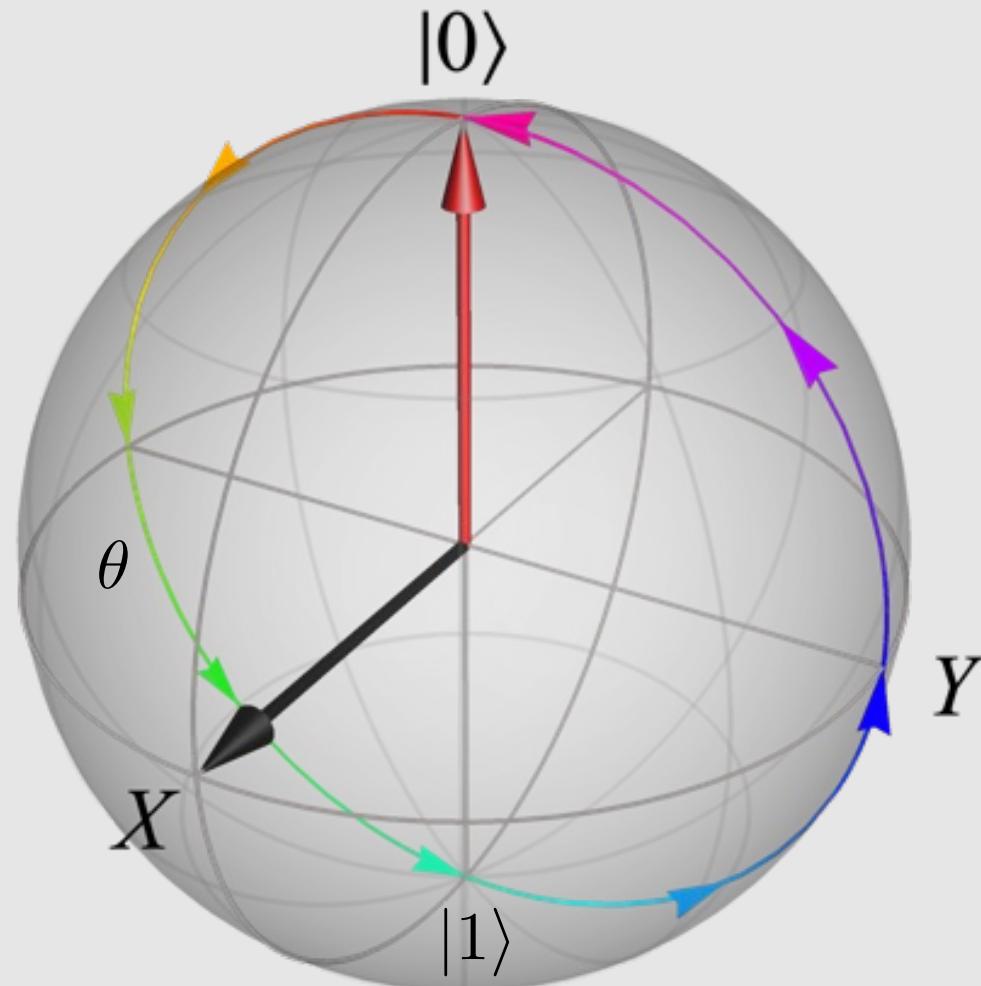


Visualize: Evolution on the Bloch sphere

$$R_X(\theta) = \exp\left(-\frac{i\theta}{2}X\right)$$

$$X|0\rangle = |1\rangle$$

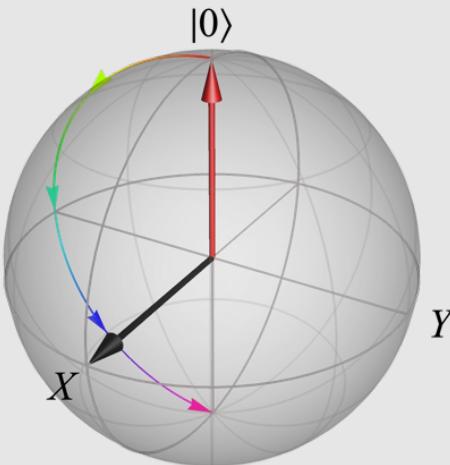
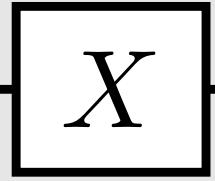
$$X|1\rangle = |0\rangle$$



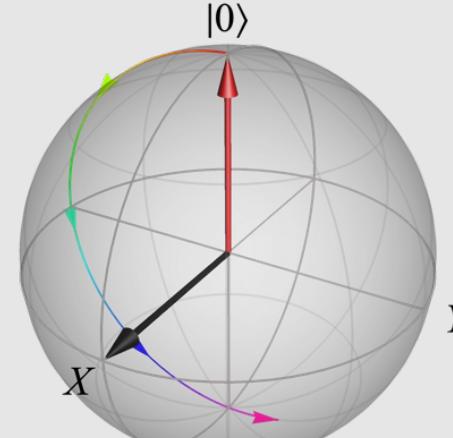
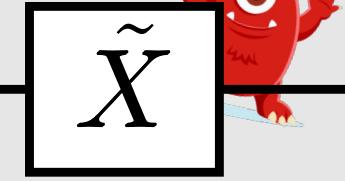


Miscalibrated gate

Ideal gate

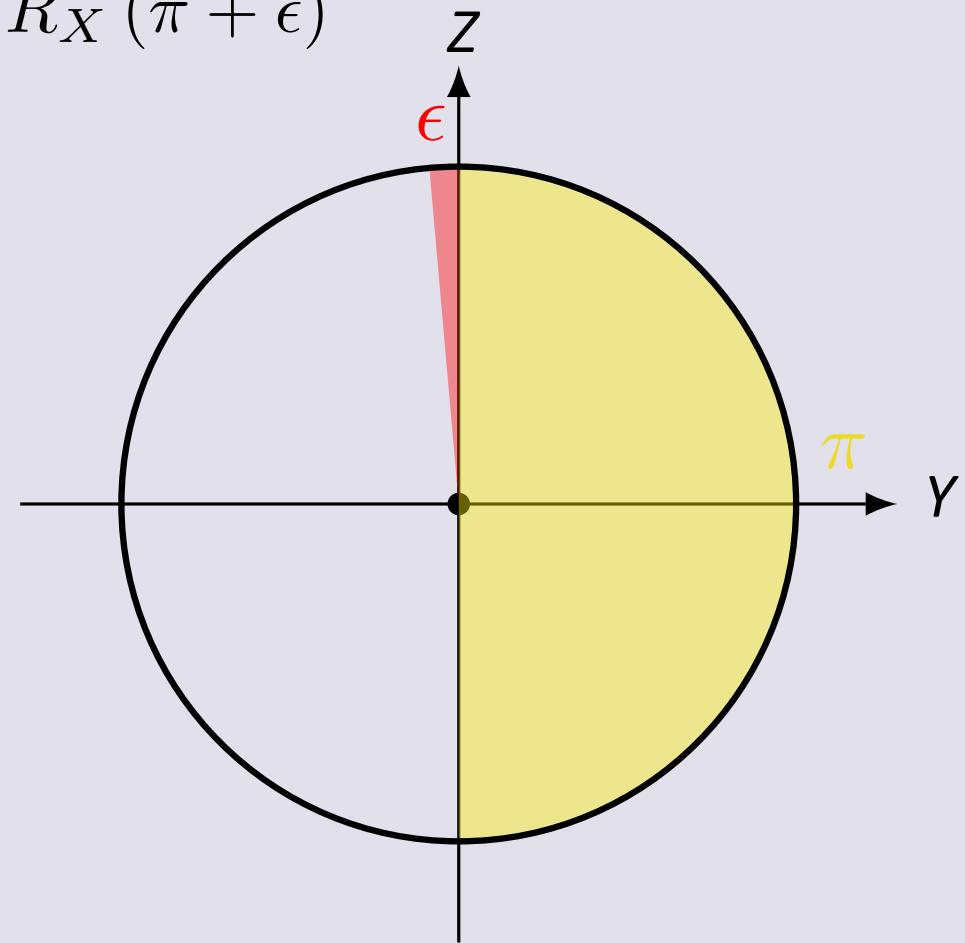


Noisy gate



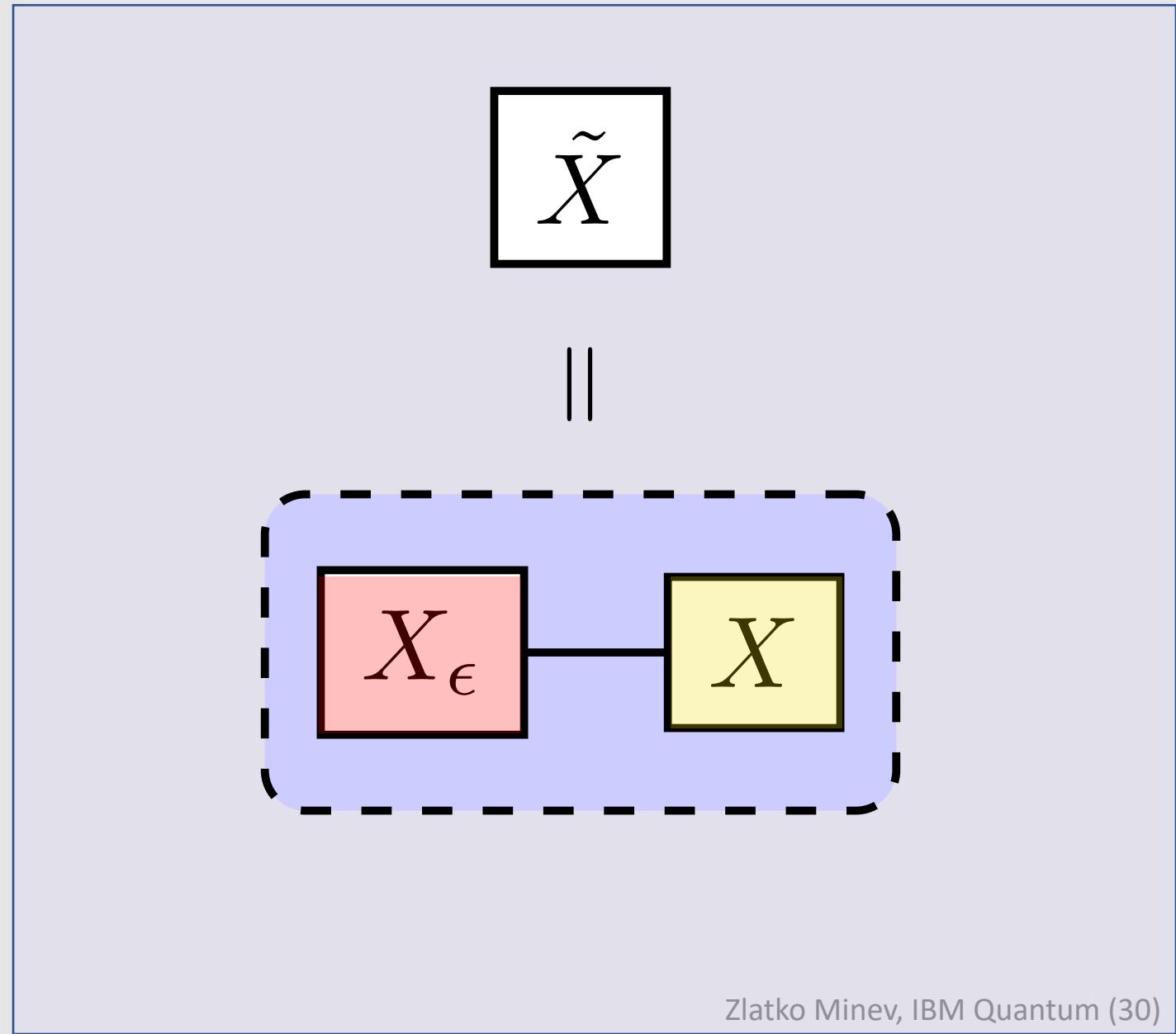
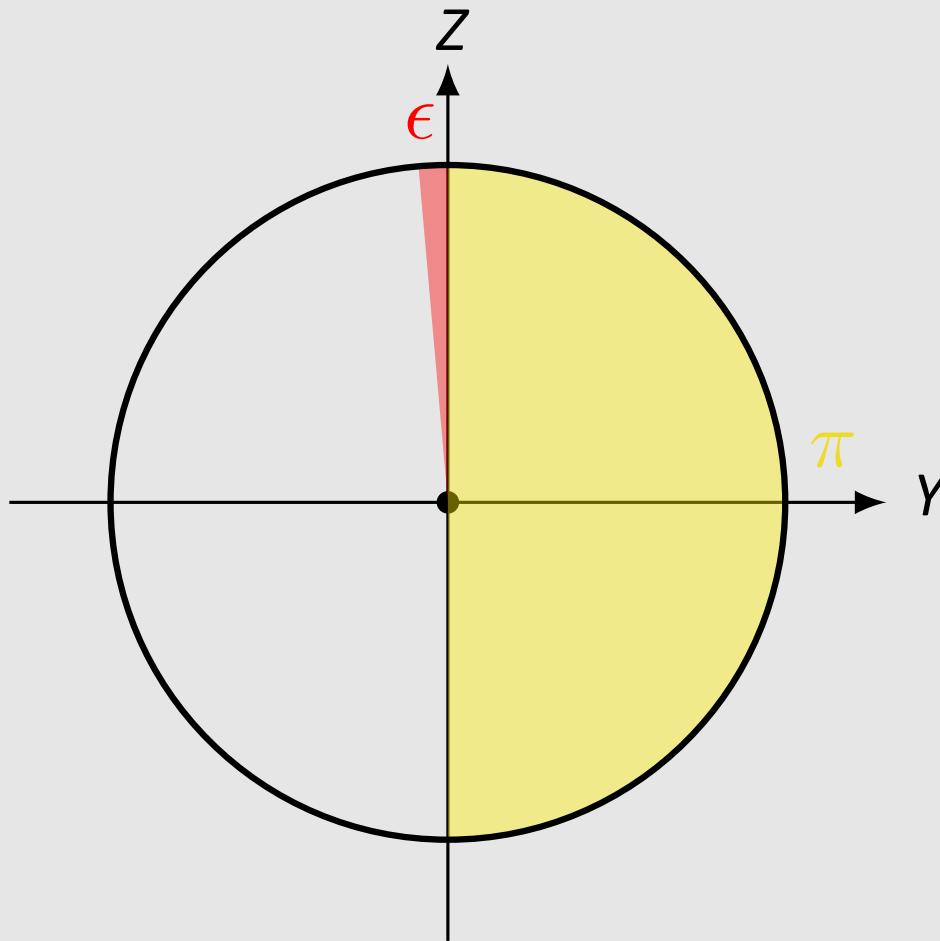
$$X = R_X(\pi)$$

$$\tilde{X} := R_X(\pi + \epsilon)$$





Noisy gate decomposition



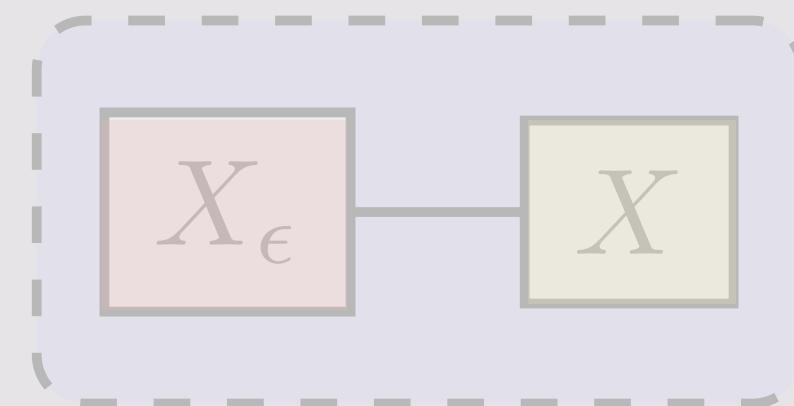
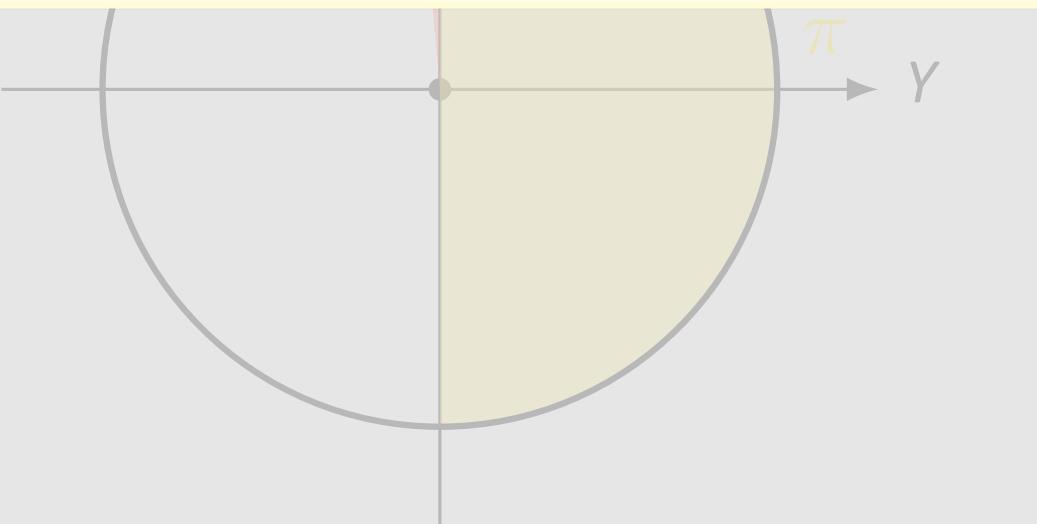
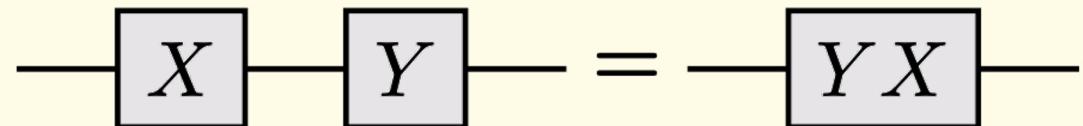


Noisy gate decomposition



Common pitfall

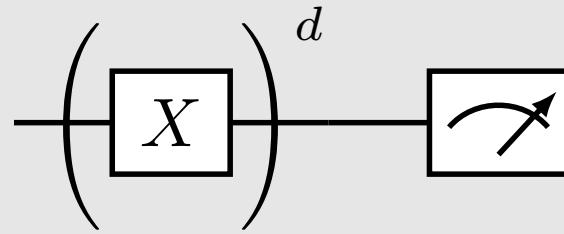
The order in which gates appear in a schematic is the reverse of how they appear in the algebra.





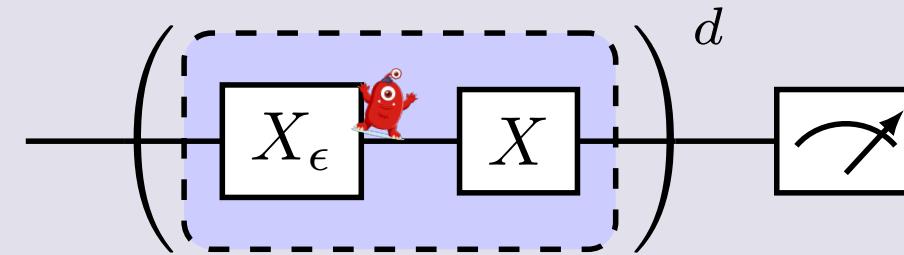
Using a noisy gate in a quantum circuit: final state

Ideal



$$U_{\text{total}} = X^d = R_X(d\pi)$$

Noisy

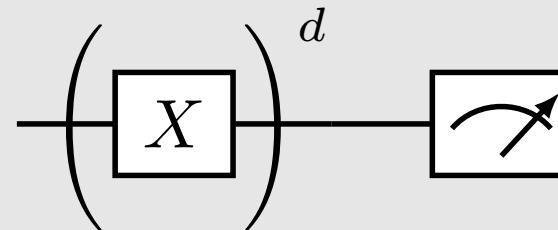


$$\tilde{U}_{\text{total}} = R_X(d\epsilon) U_{\text{total}}$$

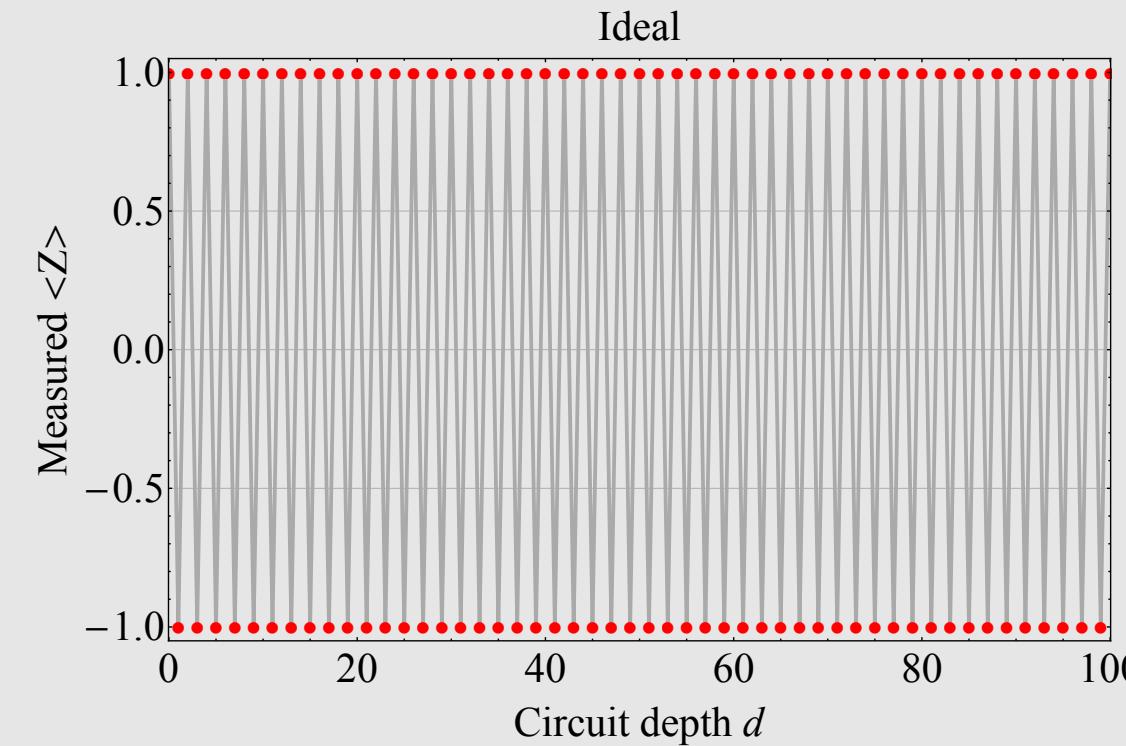


Ideal vs. noisy observable

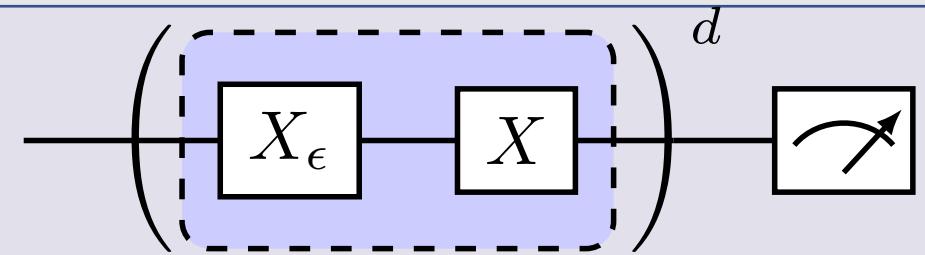
Ideal



$$\langle \psi_f | Z | \psi_f \rangle = \cos(d\pi) = (-1)^d$$

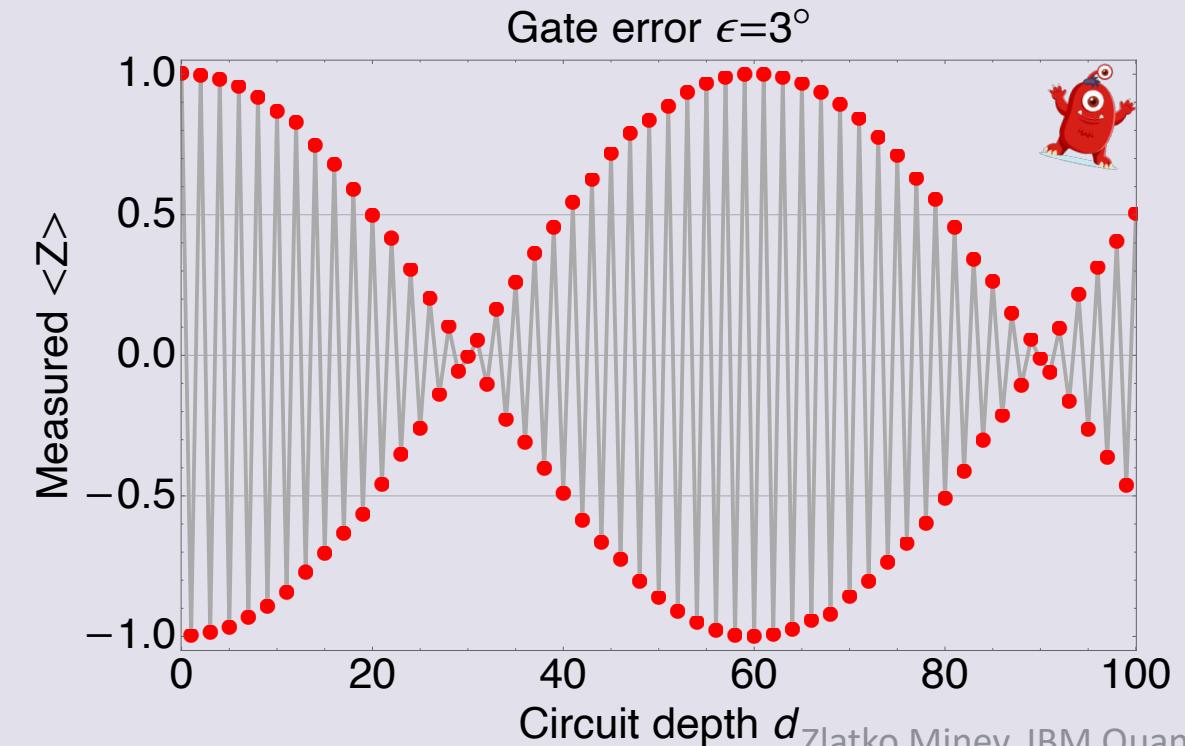


Noisy



$$\langle \tilde{\psi}_f | Z | \tilde{\psi}_f \rangle = \cos(d\pi + d\epsilon)$$

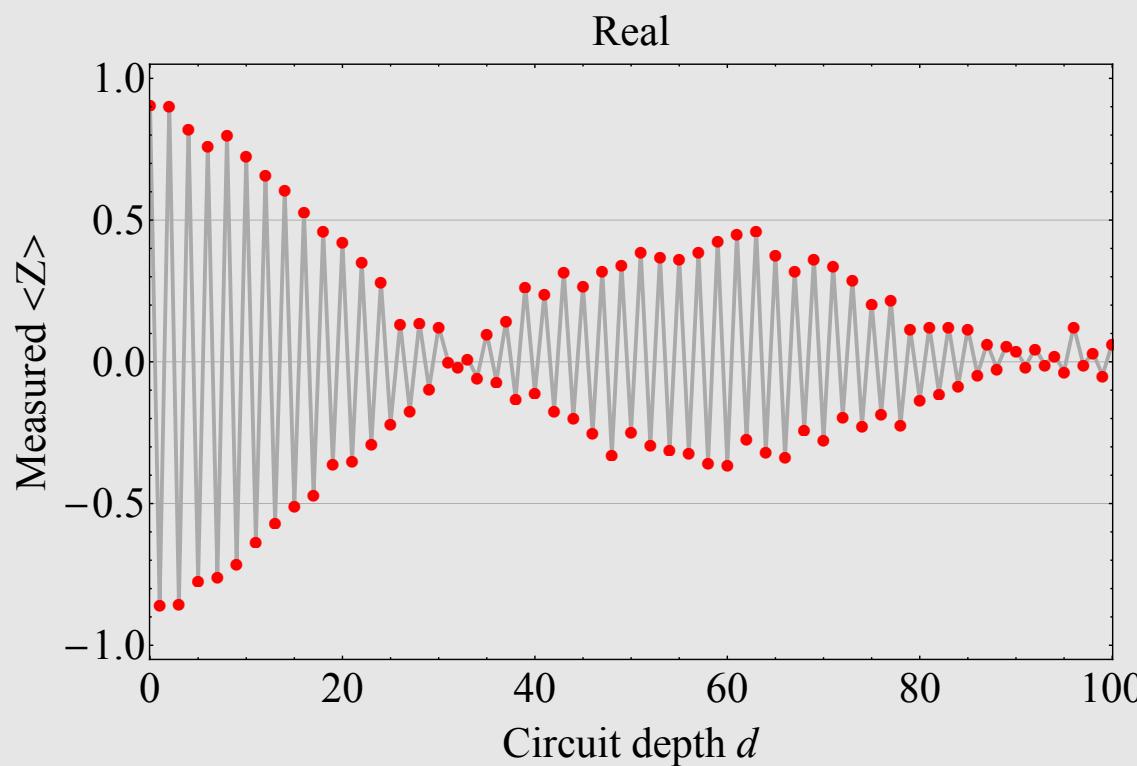
Gate error $\epsilon = 3^\circ$



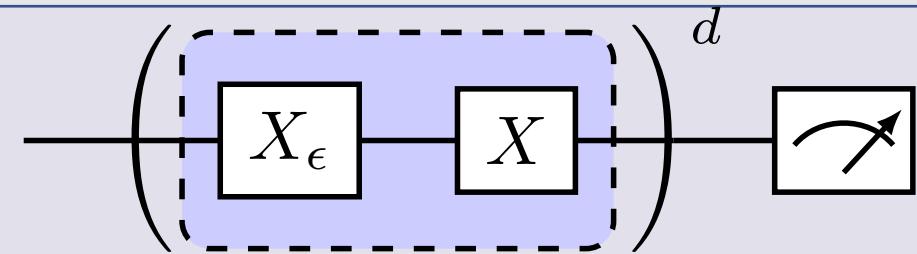


Compare to full experiment

Full experiment

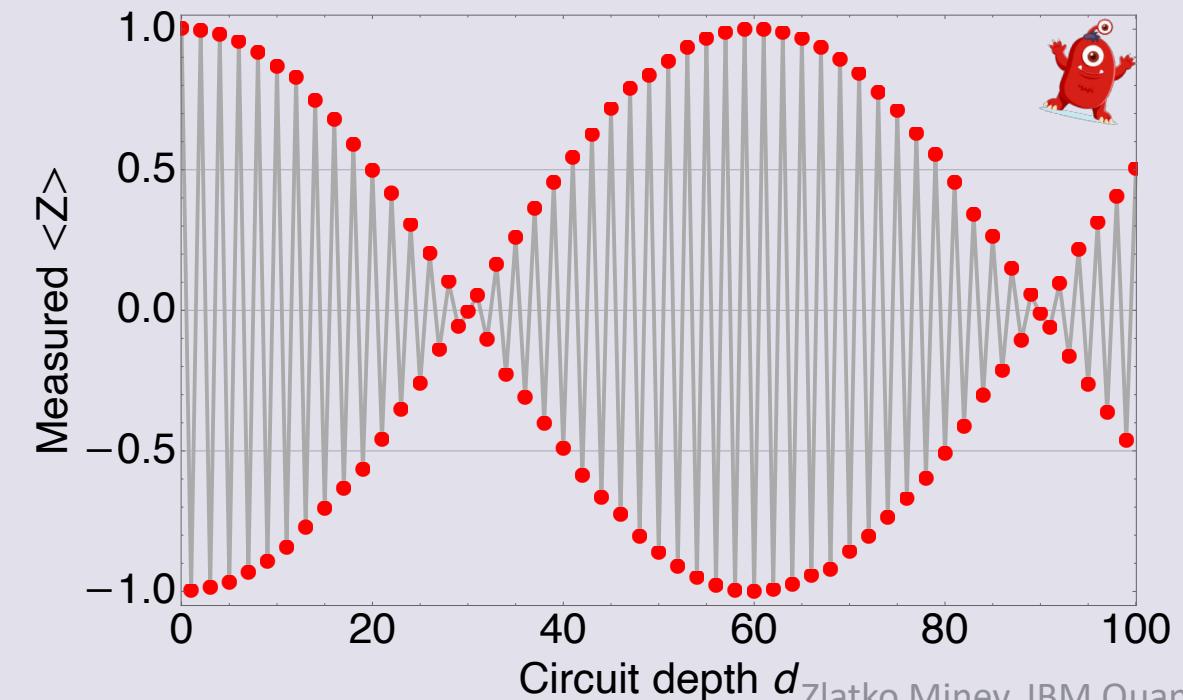


Noisy



$$\langle \tilde{\psi}_f | Z | \tilde{\psi}_f \rangle = \cos(d\pi + d\epsilon)$$

Gate error $\epsilon=3^\circ$



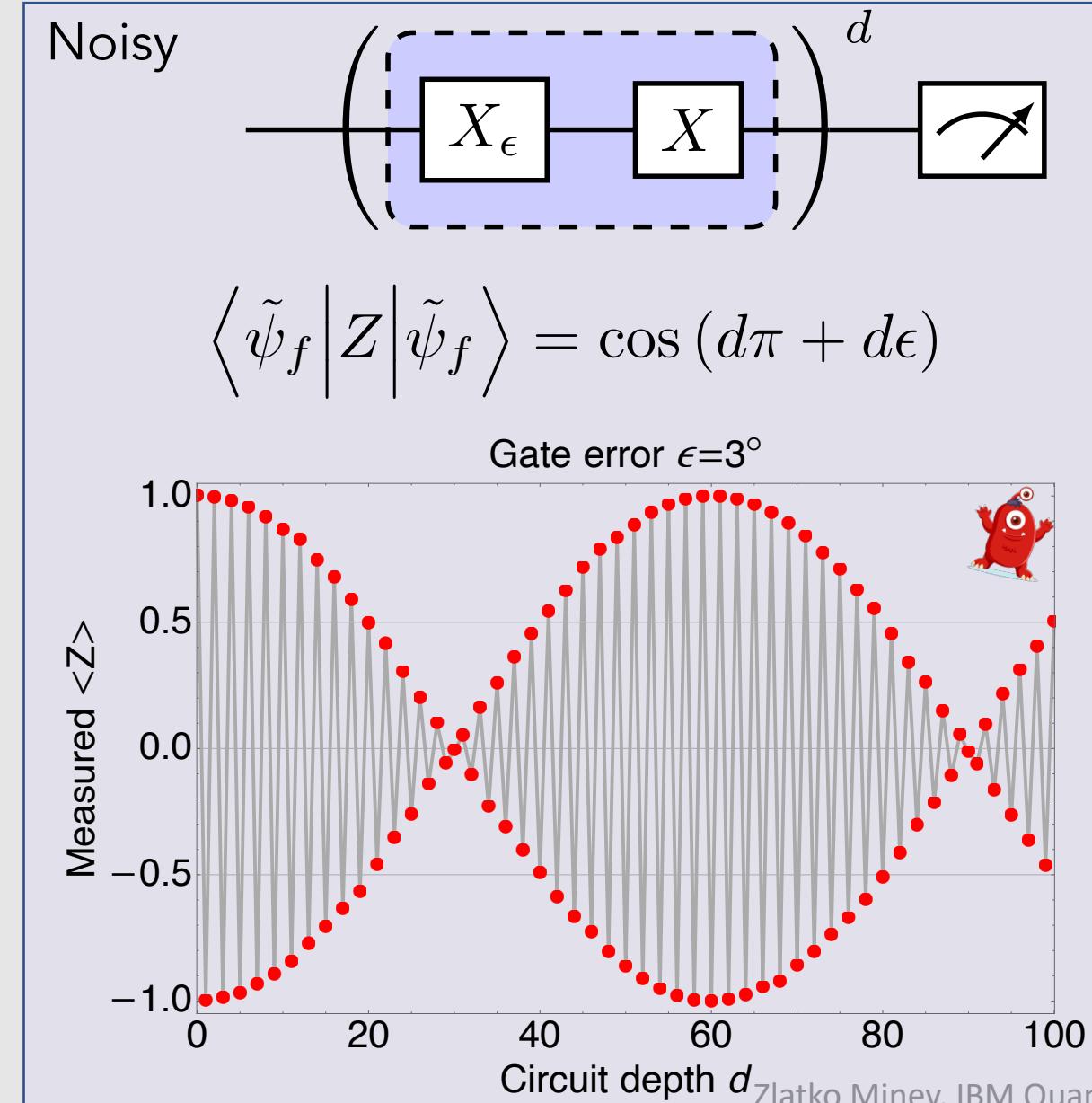


Coherent error is bad, quadratically so

Coherent errors have a quadratic impact on algorithmic accuracy (worst-case error)

$$\langle \tilde{\psi}_f | Z | \tilde{\psi}_f \rangle - \langle \psi_f | Z | \psi_f \rangle$$

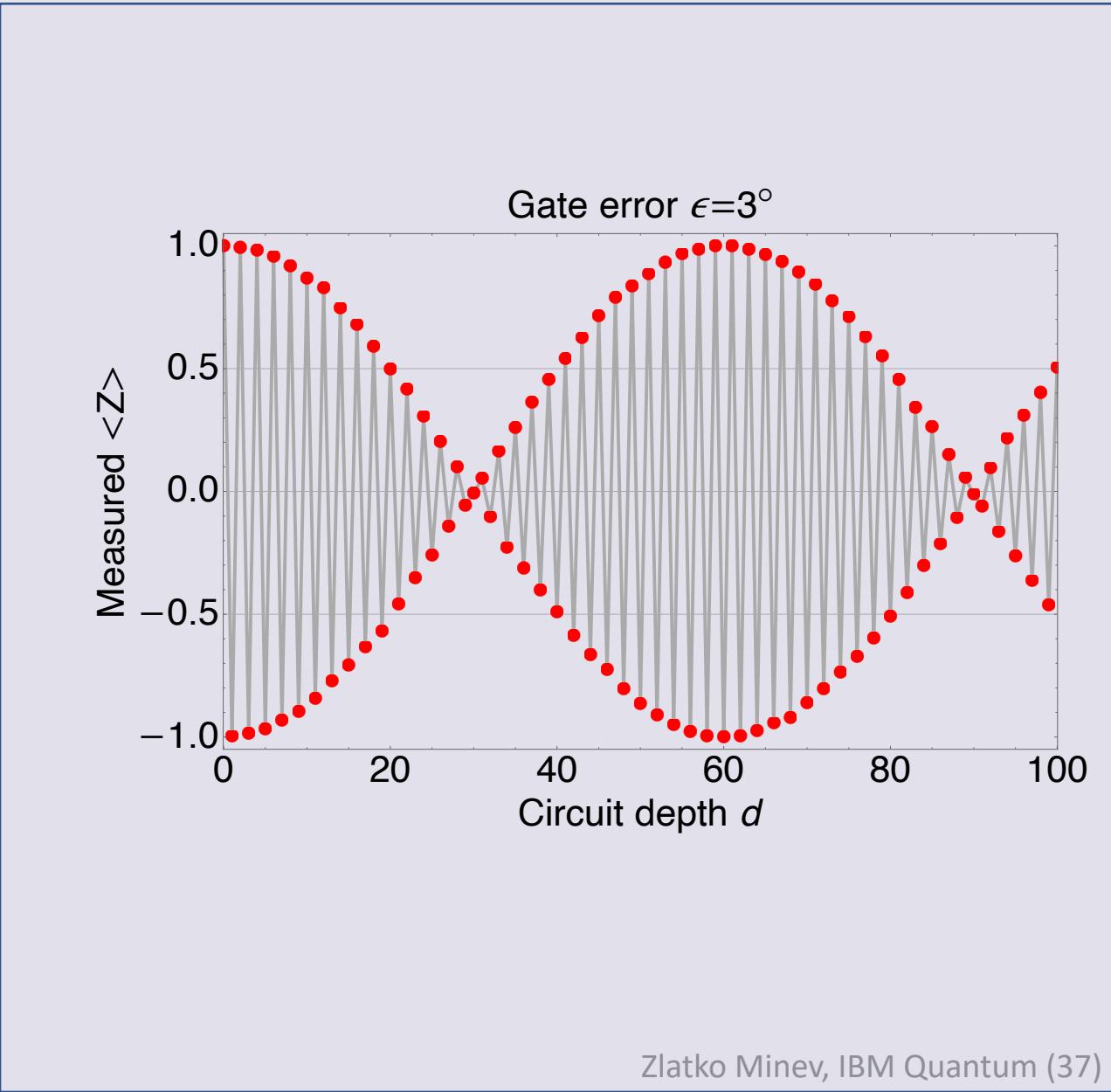
$$\cos(x) = 1 - \frac{1}{2}x^2 + \mathcal{O}(x^3)$$





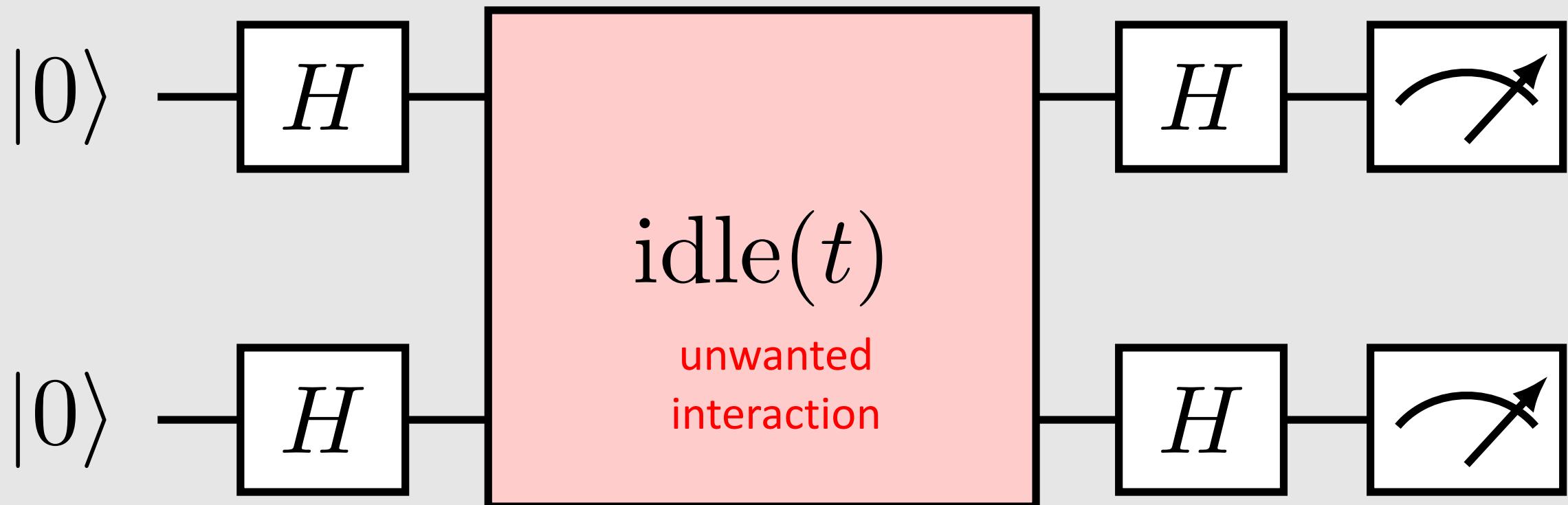
Coherent errors: brief summary

- are ubiquitous
- can be described by unitary operations
- do not loose quantum information
- data: can create oscillations in the data
- data: do not yield exponential decays
- have a *quadratic* impact on algorithmic accuracy (worst-case error)





Bonus content: two-qubit coherent ZZ error





Questions



Answer these multiple-choice questions
in the chat; for example, type “1a 2b.”

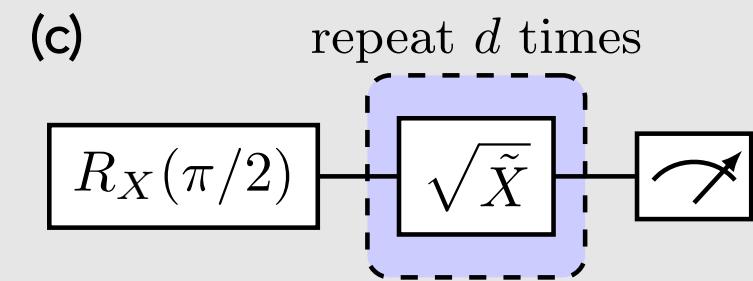
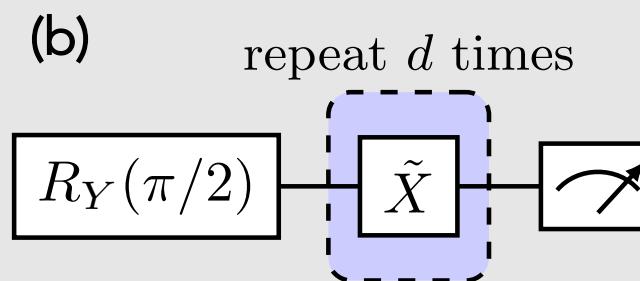
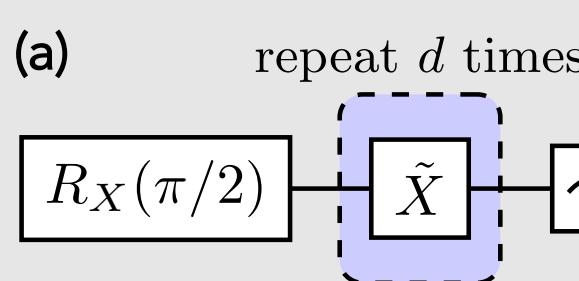
1. Coherent noise can be caused by
 - a) loss of energy of the qubit
 - b) miscalibration, such as over-rotation
 - c) wanted coupling to neighboring qubit
2. Coherent noise can be really bad because
 - a) it results in loss of information
 - b) you cannot undo it
 - c) the worst-case error often grows quadratically



Advanced questions to dive deeper

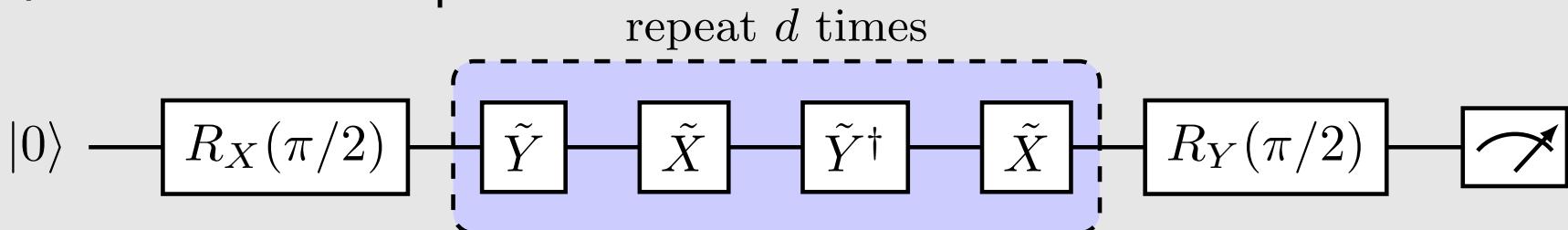
1. Amplitude-error amplification sequences.

- (A) Calculate the expectation value $\langle Z(d) \rangle$ as a function of depth d of the sequence for the following sequences. Assume the noise model $\tilde{X} := R_X(\pi + \epsilon) = X_\epsilon X$
- (B) How could you use this result to fine-tune your gates?
- (C) Can you come up with an alternative or more clever error-amplification sequence?



2. Phase-error amplification sequences. Use the noise model: Repeat (A), (B), and (C) for Exercise (1) for the following sequences and assuming a phase error between the X and Y gates, rather than an amplitude error.

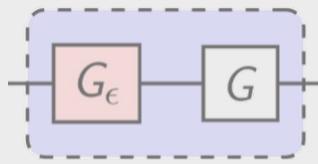
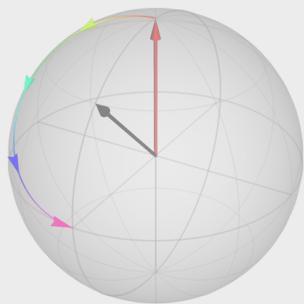
$$\tilde{X} = X ,$$
$$\tilde{Y} = \exp \left[-i \frac{\pi}{2} (\cos \epsilon Y + \sin \epsilon X) \right] .$$



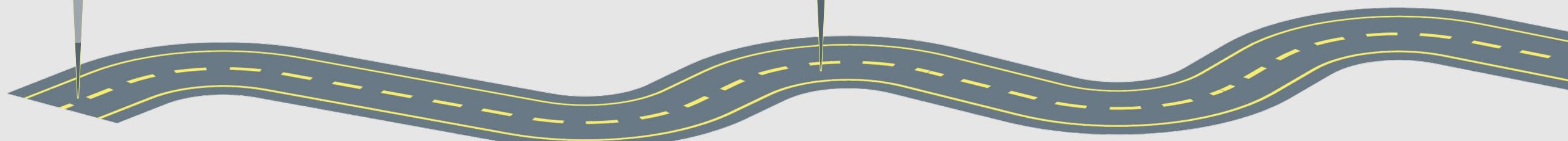
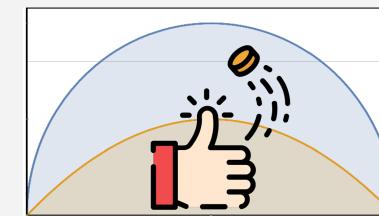


Chapter 3

Coherent noise



Measurements in quantum Projection noise

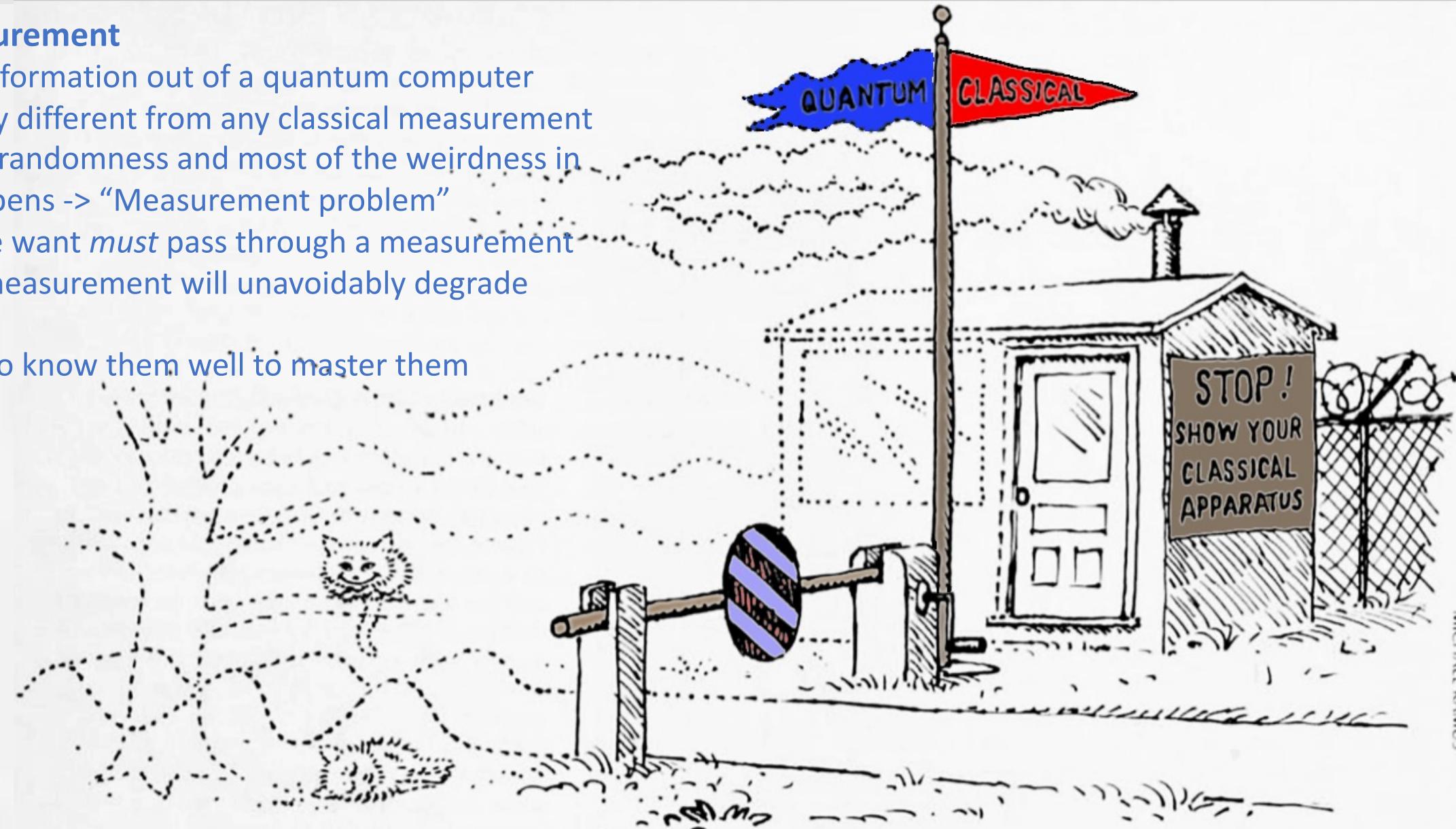




Quantum-to-classical interface

Quantum measurement

- How we get information out of a quantum computer
- Fundamentally different from any classical measurement
- Where all the randomness and most of the weirdness in quantum happens -> “Measurement problem”
- Any results we want *must* pass through a measurement
- Any noise in measurement will unavoidably degrade our results
- We must get to know them well to master them

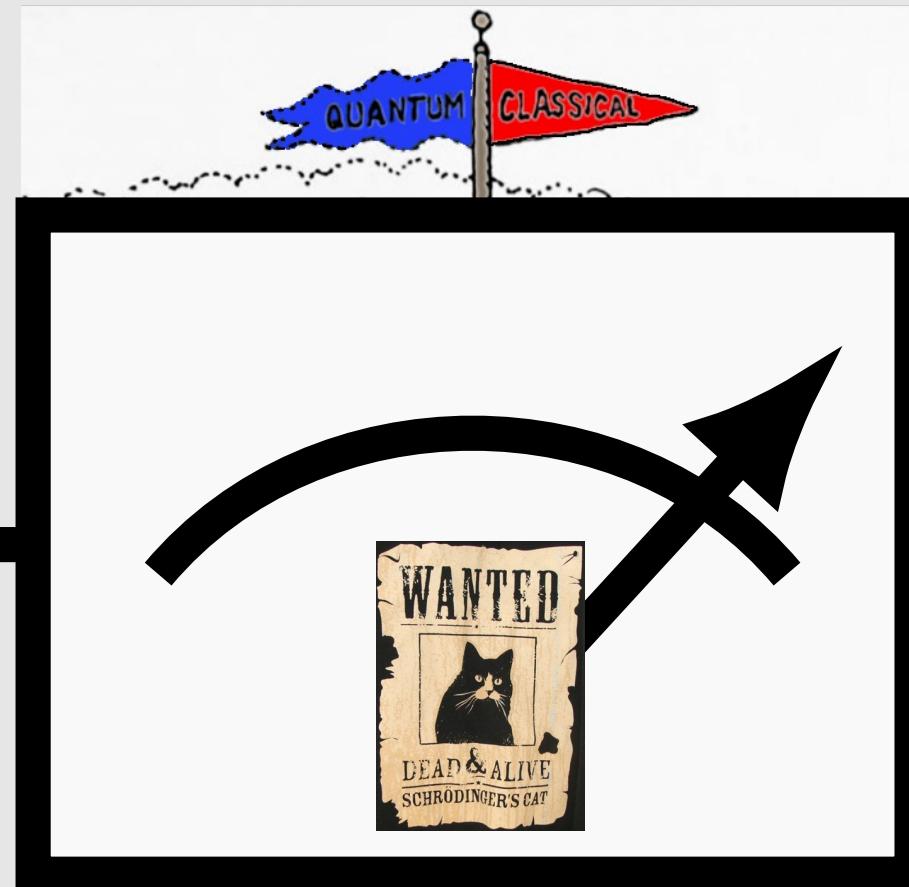




Quantum-to-classical interface

Quantum

$$\frac{1}{\sqrt{2}} \left(| \text{alive cat} \rangle + | \text{dead cat} \rangle \right)$$



Classical





Meter in quantum computing

Quantum

Quantum wire / register

$$|\psi\rangle \in \mathcal{H}$$



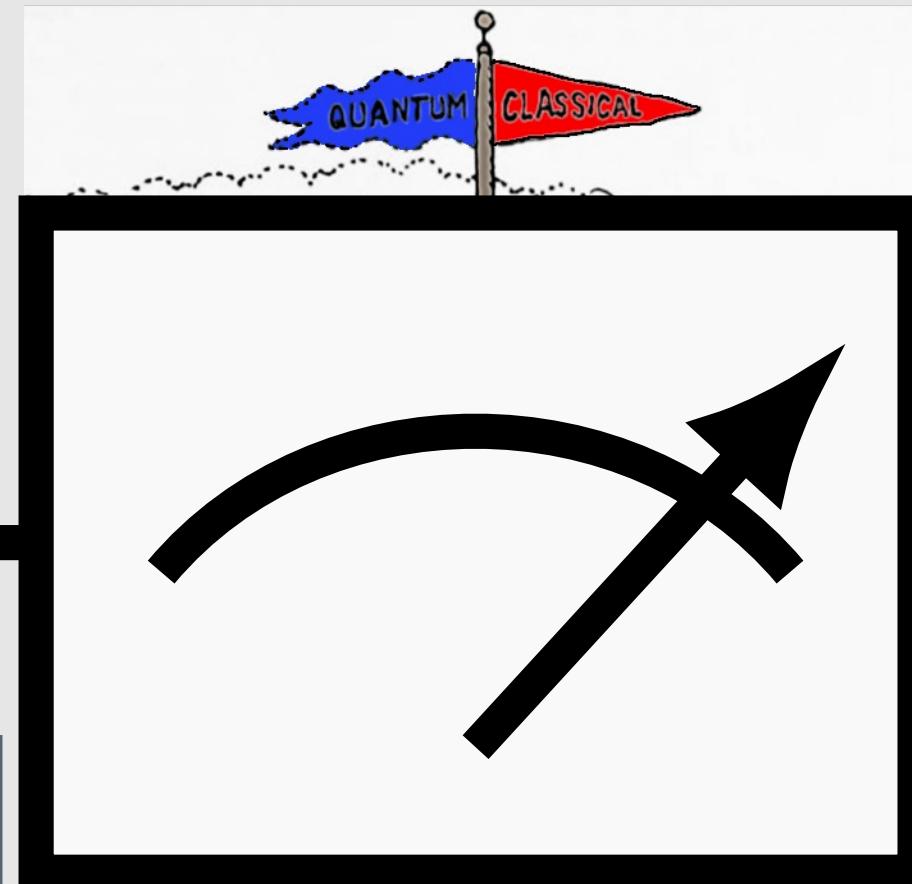
John von Neumann



Max Born

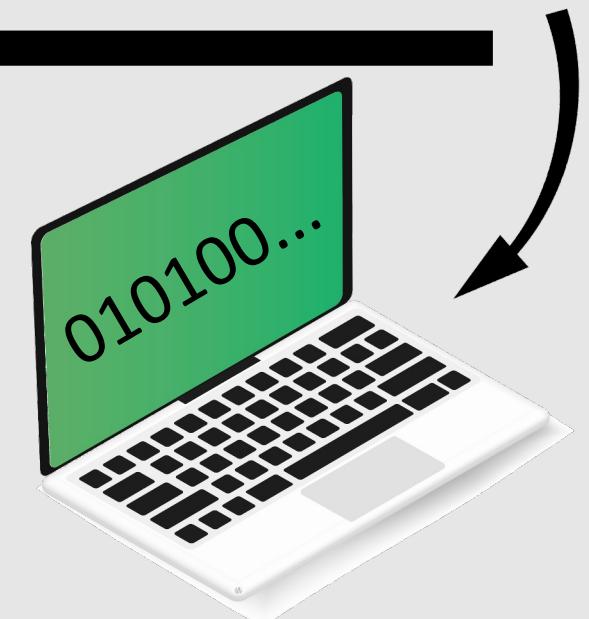
[Image source: LANL](#)

[Image source: Public domain](#)



Classical

Classical wire / register



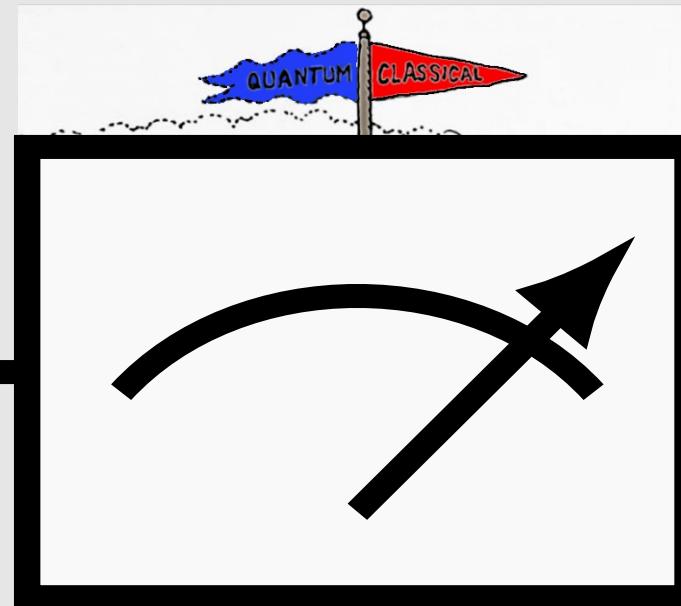


Meter in quantum computing

Quantum

$|\psi\rangle \in \mathcal{H}$

Quantum wire /
register



Classical

Classical wire /
register

X

Measure in
computational basis

$|0\rangle, |1\rangle$

outcome

outcome probability

$$\left\{ \begin{array}{l} X=0 \quad : \quad p(X=0) = |\langle 0|\psi \rangle|^2 \\ X=1 \quad : \quad p(X=1) = |\langle 1|\psi \rangle|^2 \end{array} \right.$$



Try it yourself! Resolution of the identity

$$|\psi\rangle \xrightarrow{\text{gate}} X$$

Try it yourself! (Pause the video) Show that

$$p(X = 0) + p(X = 1) = 1$$

outcome outcome probability

$$\begin{cases} X=0 : p(X=0) = |\langle 0|\psi\rangle|^2 \\ X=1 : p(X=1) = |\langle 1|\psi\rangle|^2 \end{cases}$$



Solution: Resolution of the identity

$$|\psi\rangle \xrightarrow{\text{ } \square \text{ }} X$$

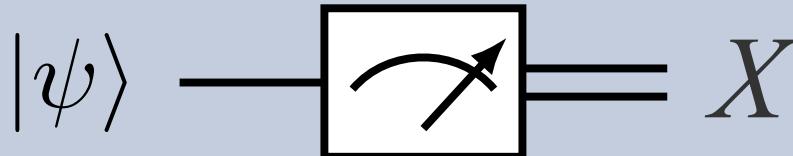
outcome	outcome probability
$X=0$	$p(X=0) = \langle 0 \psi \rangle ^2$
$X=1$	$p(X=1) = \langle 1 \psi \rangle ^2$

Try it yourself! (Pause the video) Show that

$$\begin{aligned} p(X=0) + p(X=1) &= 1 \\ &= |\langle 0|\psi \rangle|^2 + |\langle 1|\psi \rangle|^2 \\ &= \langle \psi|0\rangle \langle 0|\psi \rangle + \langle \psi|1\rangle \langle 1|\psi \rangle \\ &= \langle \psi|(|0\rangle \langle 0| + |1\rangle \langle 1|)|\psi \rangle \\ &= \langle \psi| \hat{I} |\psi \rangle \\ &= ||\psi\rangle|^2 \\ &= 1 \end{aligned}$$



Outcome probability distribution

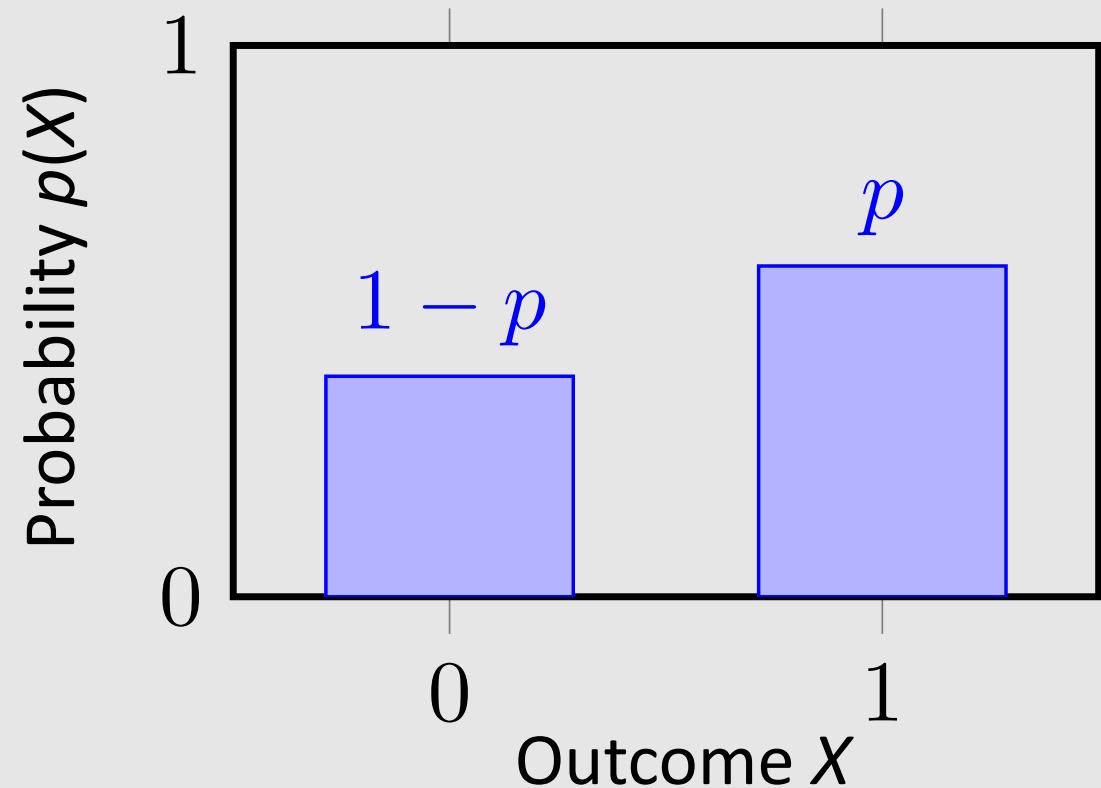


outcome outcome probability

$$\begin{cases} X=0 : p(X=0) = |\langle 0|\psi \rangle|^2 \\ X=1 : p(X=1) = |\langle 1|\psi \rangle|^2 \end{cases}$$

The entire classical distribution is dictated by a single parameter p determined from the quantum state according to

$$\begin{aligned} p &:= \Pr [X = 1] \\ &= \langle |1\rangle \langle 1| \rangle \\ &= |\langle 1|\psi \rangle|^2 \end{aligned}$$





Outcome distribution in terms of projectors and overlaps

$$|\psi\rangle \xrightarrow{\text{ } \square \text{ }} X$$

outcome	outcome probability
$X = 0$	$p(X = 0) = \langle 0 \psi \rangle ^2$
$X = 1$	$p(X = 1) = \langle 1 \psi \rangle ^2$

The resolution of the total probability

$$p(X = 0) + p(X = 1) = 1$$

$$= \langle \psi | (|0\rangle\langle 0| + |1\rangle\langle 1|) |\psi \rangle$$

In terms of projectors

$$\hat{\mu}(0) = |0\rangle\langle 0| \quad \hat{\mu}(1) = |1\rangle\langle 1|$$

General formulation (overlap between states)

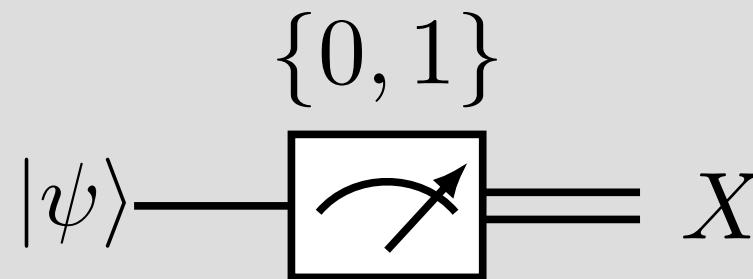
$$p(X = x) = \langle \psi | \hat{\mu}(x) | \psi \rangle$$

as an inner product in operator space

= $\langle \hat{\mu}(x), |\psi\rangle\langle\psi| \rangle$ as trace, to be introduced later



Classical expectation values



Set of possible outcomes

$$\Sigma = \{0, 1\} \quad X \in \Sigma$$

Measurement operators

$$\begin{aligned} x & \quad \hat{\mu}(x) \\ 0 : \quad \hat{\mu}(0) &= |0\rangle\langle 0| \\ 1 : \quad \hat{\mu}(1) &= |1\rangle\langle 1| \end{aligned}$$

$$\Pr[X = x] = \langle \hat{\mu}(x) \rangle$$

Classical expectation value of classical random variable

$$E[X] = \sum_{x \in \{0,1\}} x \Pr[X = x]$$



$$= 0 \Pr[X = 0] + 1 \Pr[X = 1]$$

$$= \Pr[X = 1]$$

$$= p.$$

In terms of quantum operators

$$E[X] = \sum_{x \in \{0,1\}} x \Pr[X = x]$$



$$= \sum_{x \in \{0,1\}} x \langle \hat{\mu}(x) \rangle$$

$$= \sum_{x \in \{0,1\}} \langle x \hat{\mu}(x) \rangle$$

$$= \left\langle \sum_{x \in \{0,1\}} x \hat{\mu}(x) \right\rangle$$

$$= \langle \hat{M} \rangle$$

The classical expectation value gives us the expectation value of the quantum operator

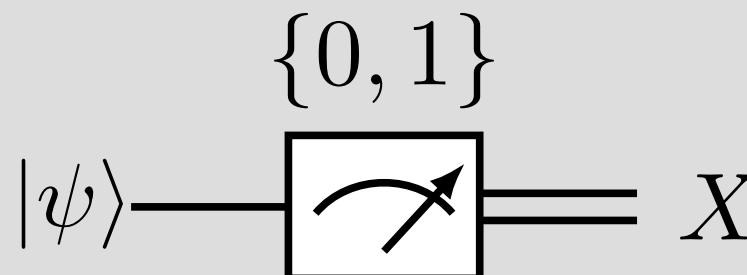
$$\hat{M} := \sum_{x \in \{0,1\}} x \hat{\mu}(x)$$

$$= 0 |0\rangle\langle 0| + 1 |1\rangle\langle 1|$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$



Classical variance



Set of possible outcomes

$$\Sigma = \{0, 1\} \quad X \in \Sigma$$

Measurement operators

$$\begin{aligned} x & \quad \hat{\mu}(x) \\ 0 : \quad \hat{\mu}(0) &= |0\rangle\langle 0| \\ 1 : \quad \hat{\mu}(1) &= |1\rangle\langle 1| \end{aligned}$$

$$\Pr[X = x] = \langle \hat{\mu}(x) \rangle$$

Calculate the variance of the classical random variable

$$\begin{aligned}\mathbb{V}[X] &= E[(X - E[X^2])^2] \\ &= E[X^2] - E[X]^2\end{aligned}$$

Calculate the expectation of the square

“Classical path”

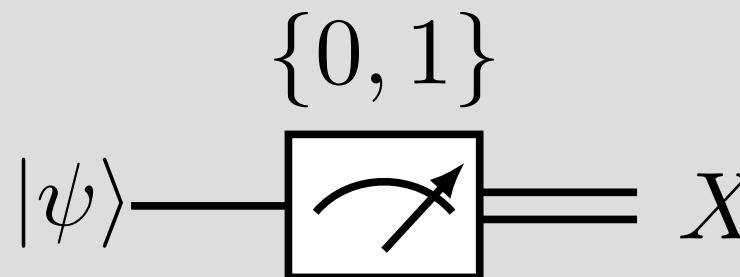
$$\begin{aligned}E[X^2] &= \sum_{x \in \{0,1\}} x^2 \Pr[X = x] \\ &= 0 \Pr[X = 0] + 1 \Pr[X = 1] \\ &= p\end{aligned}$$

“Quantum path”

$$\begin{aligned}E[X^2] &= \sum_{x \in \{0,1\}} x^2 \Pr[X = x] \\ &= \sum_{x \in \{0,1\}} x^2 \langle \hat{\mu}(x) \rangle \\ &= \left\langle \sum_{x \in \{0,1\}} x^2 \hat{\mu}(x) \right\rangle \\ &= \langle \hat{M}^2 \rangle\end{aligned}$$



Classical variance



Set of possible outcomes

$$\Sigma = \{0, 1\} \quad X \in \Sigma$$

Measurement operators

$$x \quad \hat{\mu}(x)$$

$$0 : \hat{\mu}(0) = |0\rangle\langle 0|$$

$$1 : \hat{\mu}(1) = |1\rangle\langle 1|$$

$$\Pr[X = x] = \langle \hat{\mu}(x) \rangle$$

Calculate the variance of the classical random variable

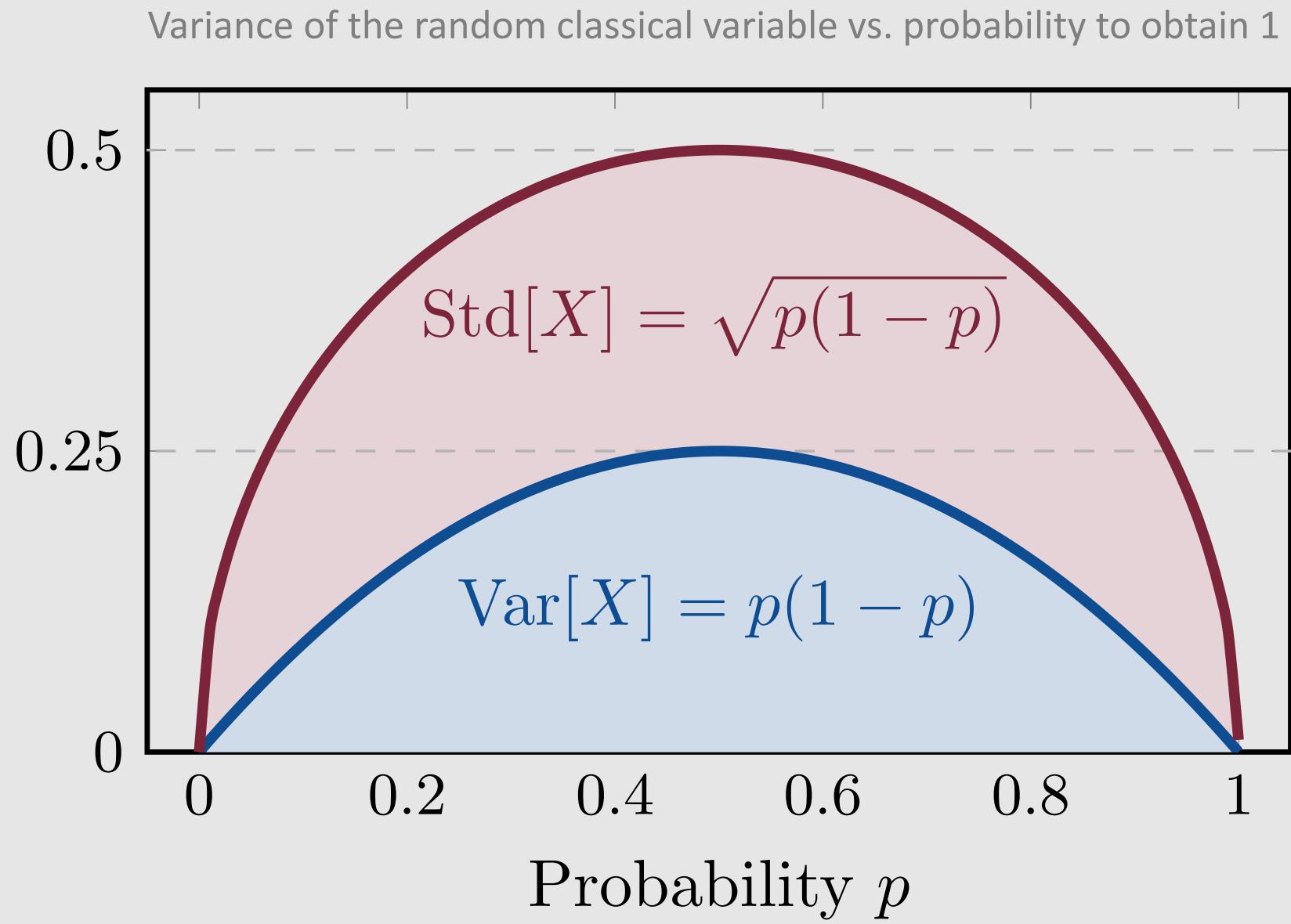
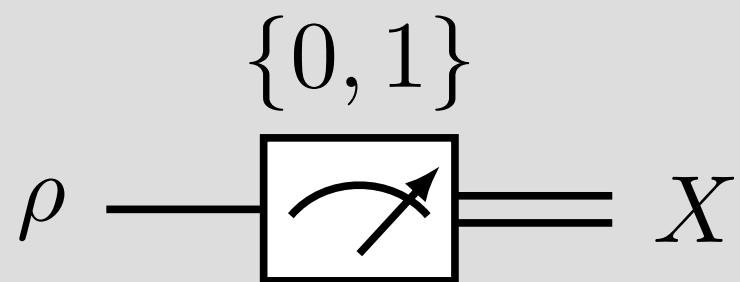
$$\begin{aligned}\mathbb{V}[X] &= E[(X - E[X^2])^2] \\ &= E[X^2] - E[X]^2\end{aligned}$$

Putting it together

$$\mathbb{V}[X] = \langle \hat{M}^2 \rangle - \langle \hat{M} \rangle^2$$

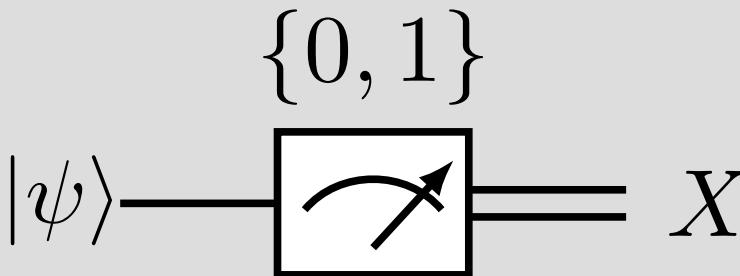
$$\begin{aligned}\mathbb{V}[X] &= p - p^2 \\ &= p(1 - p)\end{aligned}$$

Quantum projection noise





Summary: Qubit measured in the computational basis



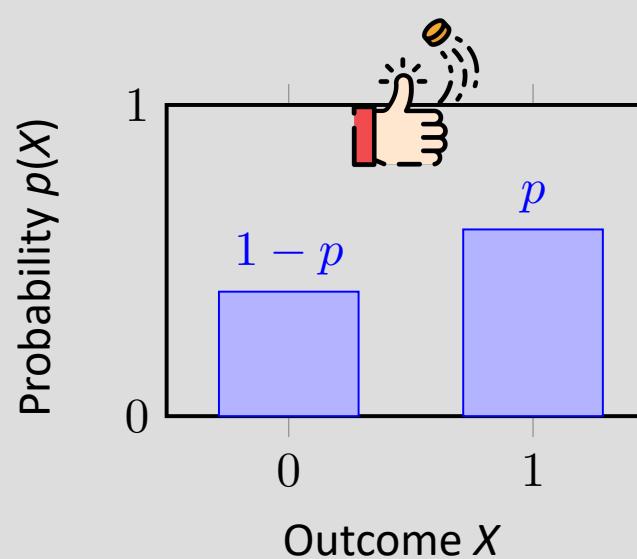
Set of possible outcomes

$$\Sigma = \{0, 1\}$$

$$X \in \Sigma$$

Measurement operators

$$\begin{array}{ll} x & \hat{\mu}(x) \\ 0 : & \hat{\mu}(0) = |0\rangle\langle 0| \\ 1 : & \hat{\mu}(1) = |1\rangle\langle 1| \end{array}$$



Probability to measure outcome

$$\begin{cases} X = 0 : & p(X = 0) = \text{Tr}(\hat{\mu}(0)^\dagger \rho) = \text{Tr}(|0\rangle\langle 0| \rho) = \frac{1}{2}(1 + \langle Z \rangle) \\ X = 1 : & p(X = 1) = \text{Tr}(\hat{\mu}(1)^\dagger \rho) = \text{Tr}(|1\rangle\langle 1| \rho) = \frac{1}{2}(1 - \langle Z \rangle) \end{cases}$$

$$\rho = |\psi\rangle\langle\psi| \quad \text{to be introduced later in detail}$$

Bernoulli distribution. Single shot outcome follows a Bernoulli distribution:

$$X \sim \text{Bernoulli}(p)$$

$$p := \text{Tr}(|1\rangle\langle 1| \rho) \in [0, 1]$$

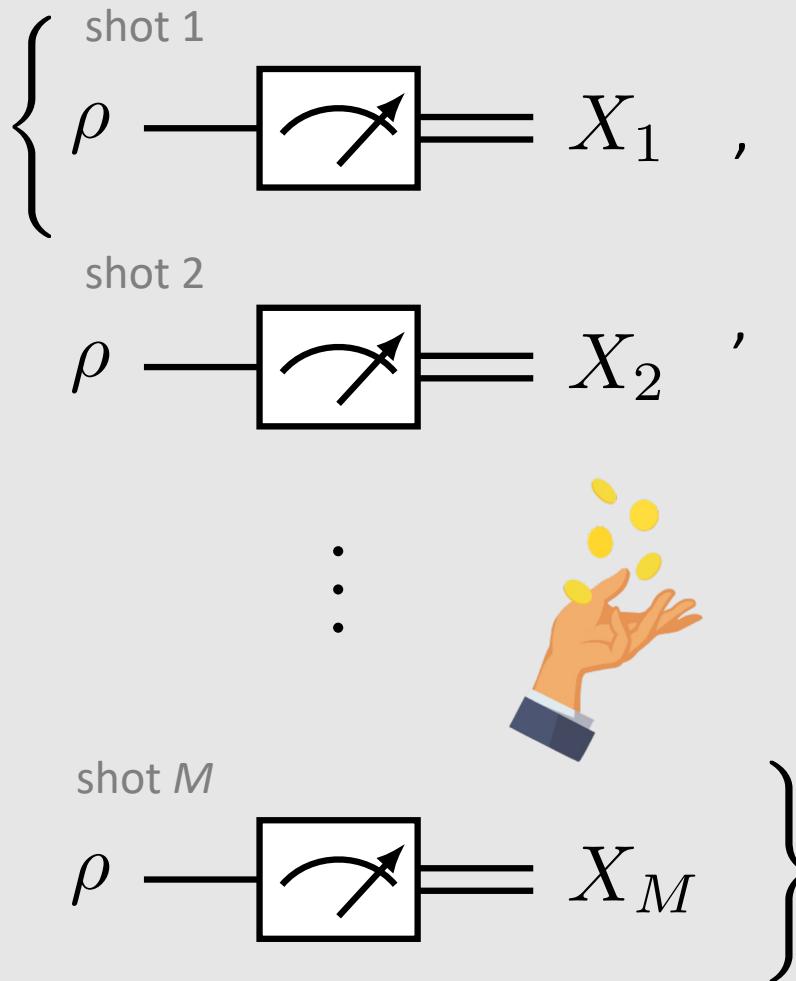
$$\text{E}[X] = p$$

$$\text{Var}[X] = p(1 - p)$$





Ideal single qubit measurement with M shots



M shots with IID distribution

M outcomes: independent and identically distributed (iid) random variables

$$X_1, X_2, \dots, X_M \in \Sigma$$

$$X_1, X_2, \dots, X_M \sim \Pr [X = x] = \langle \hat{\mu} (x) \rangle$$

Empirical mean random variable

$$S = \frac{1}{M} \sum_{m=1}^M X_m$$

Find the expectation value and variance of the empirical mean



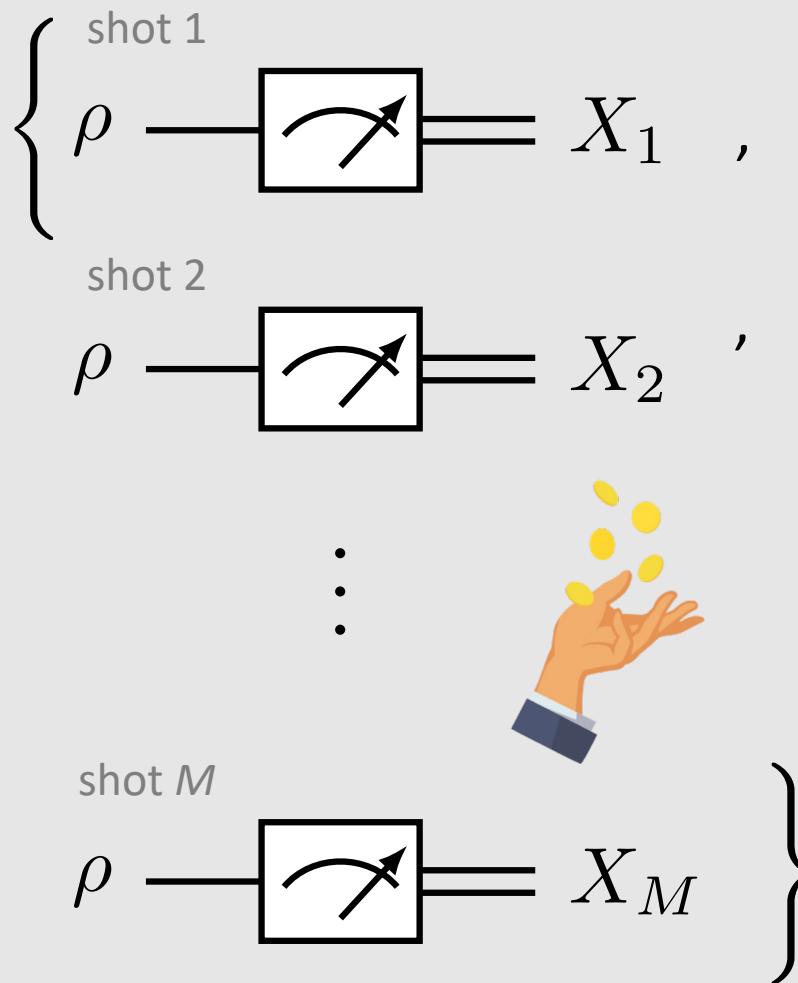
$$\mathbb{E} [X_m] = \mathbb{E} [X] = p \quad \forall m \in \{1, \dots, M\}$$

$$\mathbb{V} [X_m] = \mathbb{V} [X] = p \quad \forall m \in \{1, \dots, M\}$$

$$\mathbb{E} [aX_m + bX_n] = a\mathbb{E} [X_m] + b\mathbb{E} [X_n]$$
$$\forall m, n, \quad a, b \in \mathbb{C}$$



Empirical mean: an unbiased estimator



$$S = \frac{1}{M} \sum_{m=1}^M X_m$$

Find the expectation value and variance of the empirical mean



$$\mathbb{E}[S] = \mathbb{E}\left[\frac{1}{M} \sum_{m=1}^M X_m\right]$$

$$= \frac{1}{M} \sum_{m=1}^M \mathbb{E}[X]$$

linear functional

$$= \mathbb{E}[X]$$

expectation value of a single shot

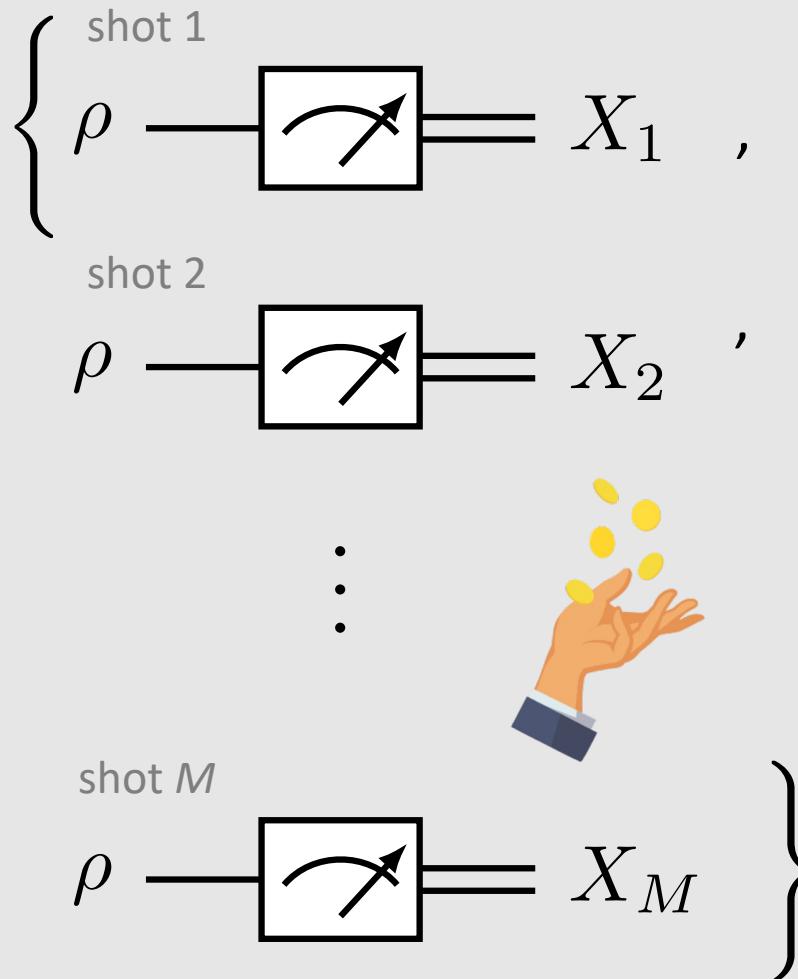
$$\begin{aligned} &= \langle \hat{M} \rangle \\ &= p \end{aligned}$$

relation to quantum operator we derived earlier (unbiased estimator)

relation to probability to measure 1



How noisy is our estimate of the empirical mean?



$$S = \frac{1}{M} \sum_{m=1}^M X_m$$

$$\begin{aligned} \mathbb{V}[S] &= \mathbb{V}\left[\frac{1}{M} \sum_{m=1}^M X_m\right] \\ &= \frac{1}{M^2} \sum_{m=1}^M \mathbb{V}[X] \end{aligned}$$

$$\begin{aligned} &= \frac{1}{M} \mathbb{V}[X] \\ &= \frac{p(1-p)}{M} \end{aligned}$$

$$\begin{aligned} \sigma_S &= \sqrt{\mathbb{V}[S]} \\ &= \sqrt{\frac{\mathbb{V}[X]}{M}} \\ &= \sqrt{\frac{p(1-p)}{M}} \end{aligned}$$

Find the expectation value and variance of the empirical mean



Use key identity for variance

$$\mathbb{V}[aX_m + bX_n] = a^2\mathbb{V}[X_m] + b^2\mathbb{V}[X_n]$$

(you can derive this from the definition)

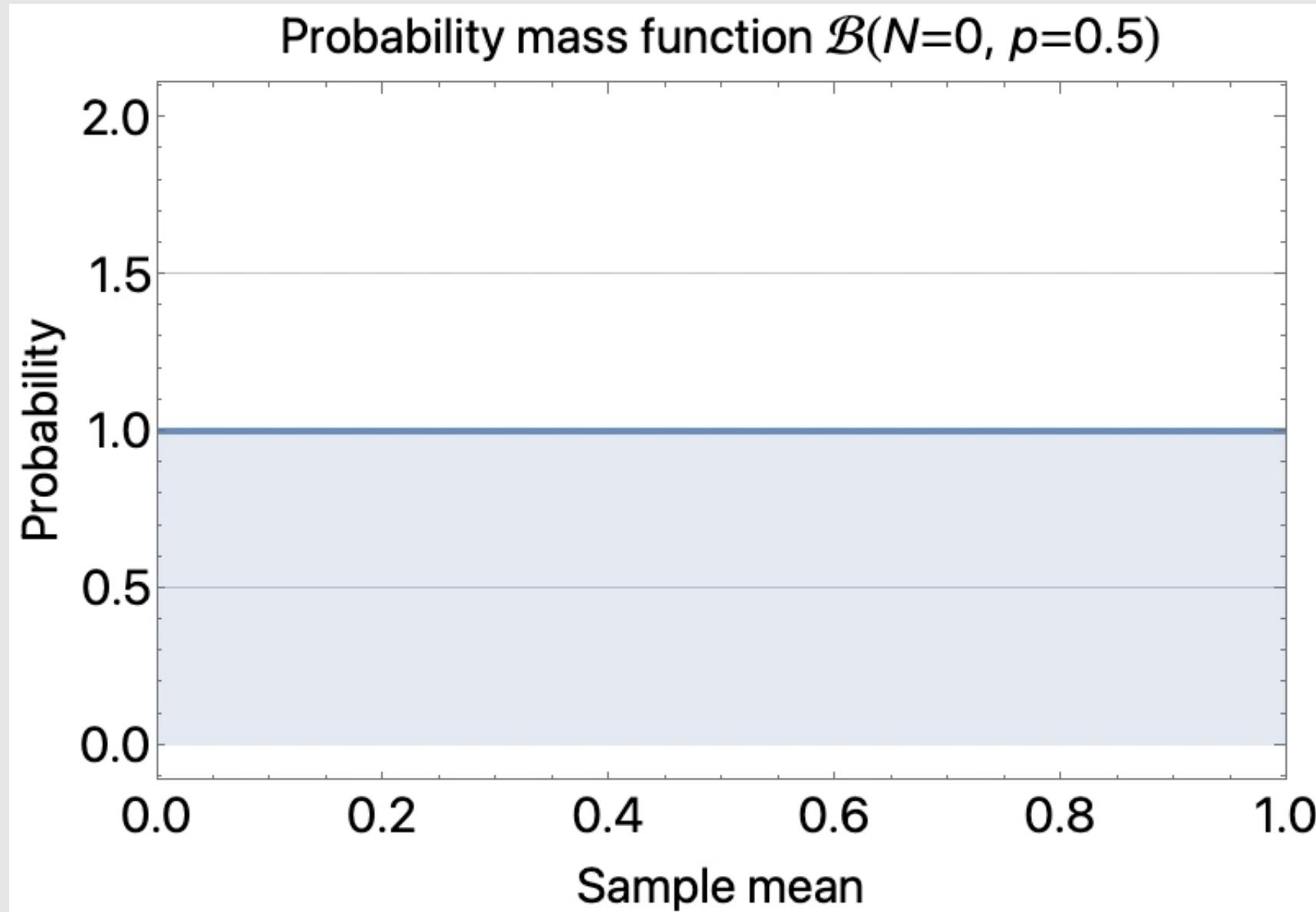
The variance is reduced by the number of sampler we take!

Thus we can suppress the quantum projection noise with enough shots.

The standard deviation drop as one over square root of the number of shots

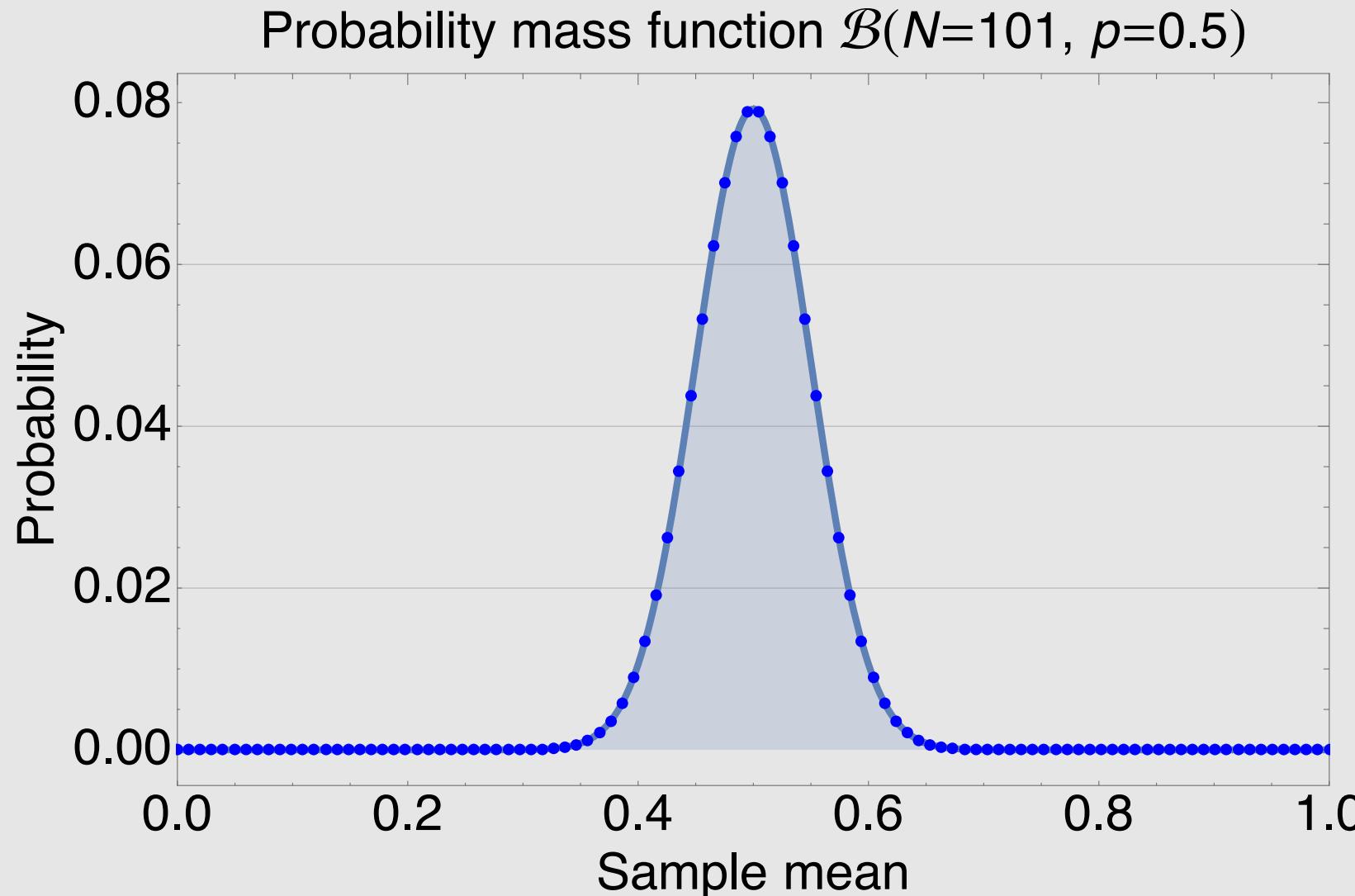


Animation of convergence of shots expectation value and mean





Sampled output distribution





Concentration inequalities and tail bounds

*Making a list,
checking it twice,
going to see
which inequality
is nice!*

*Markov? Hoeffding?
Jensen? Chebyshev?
Chernoff?*

<https://www.zlatko-minev.com/blog/inequalities>

1. Probability (Technical note 11.9 v0.6)

1A. Concentration inequalities and tail bounds

Unless otherwise specified, all variables are real \mathbb{R} . Inequalities come as one-sided $\Pr(\dots \leq \dots)$ and two-sided $\Pr(|\dots| \leq \dots)$. Notation: X is a random variable, $\mu := \mathbb{E}[X]$, $\sigma^2 := \text{Var}[X]$, $S_n := X_1 + \dots + X_n$.

Inequality	Conditions	Common form	Notes / Alternate form	
Markov ¹	Non-negative $X \geq 0$	$\Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}$	$\forall a > 0$	$\Pr[X \geq k\mathbb{E}[X]] \leq \frac{1}{k} \quad k > 1$ [3, Sec. 5.1][6, Thm 1.13]
extension	+ non-negative, strictly increasing func Φ $\Phi(X) \geq 0$ increasing	$\Pr[X \geq a] = \Pr[\Phi(X) \geq \Phi(a)] \leq \frac{\mathbb{E}(\Phi(X))}{\Phi(a)}$	$\forall a > 0$	[Wiki]
Reverse Markov	upper-bounded by U $\max X = U$ (can be positive)	$\Pr[X \leq a] \leq \frac{U - \mathbb{E}[X]}{U - a}$	$\forall a > 0$	[1, Sec. 3.1]
Chebyshev ²	Finite mean and variance $\mathbb{E}[X]$, $\text{Var}[X]$ finite	$\Pr[X - \mathbb{E}[X] \geq a] \leq \frac{\sigma^2}{a^2}$	$\Pr[X - \mathbb{E}[X] \geq a \cdot \sigma] \leq \frac{1}{a^2}$ $\forall a > 0, \sigma^2 = \text{Var}[X]$	[1, Sec. 3.2][3, Sec. 5.1][2, Thm 18.11]
Cantelli	Improved Chebyshev (same; but one-sided)	$\Pr[X - \mathbb{E}[X] \geq a] \leq \frac{\sigma^2}{\sigma^2 + a^2}$	$\forall a > 0, \sigma^2 = \text{Var}[X]$	[Wiki]
Chernoff ³	Generic	$\Pr[X \geq a] = \Pr[e^{tX} \geq e^{ta}]$	$\forall t > 0, a \in \mathbb{R}$	[1, Sec. 3.3]
Jensen	$f : \mathbb{R} \rightarrow \mathbb{R}; f$ convex	$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$		[3, Prob. 5.3][6, Thm 1.14]
Hoeffding's lemma	$\mathbb{E}[X] = \mu$ $a \leq X \leq b$	$\mathbb{E}[e^{\lambda X}] \leq e^{\lambda \mu} e^{\frac{\lambda^2(b-a)^2}{8}}$	$\lambda \in \mathbb{R}$	[1, Sec. 3.4]
Sum of random variables				
Chernoff-Hoeffding (one-sided)	n independent random vars X_1, \dots, X_n indep $S_n = X_1 + \dots + X_n$ $X_i \in [a_i, b_i] \quad \forall i$	$\Pr[S_n - \mathbb{E}[S_n] \geq t] \leq \exp\left(\frac{-2t^2 n^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$		[1, Sec. 3.5]
(two-sided) ⁴	(same as above)	$\Pr[S_n - \mathbb{E}[S_n] > t] \leq 2 \exp\left(\frac{-2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$	$\forall t \in (0, \frac{1}{2})$	[5, Thm.1.1]
(two-sided iid)	same plus iid, range, mean μ for each $X_1, \dots, X_n \in [0, 1]$ $\mathbb{E}[X_i] = \mu$ iid	$\Pr\left[\left \frac{S_n}{n} - \mu\right \geq \epsilon\right] \leq 2 \exp(-2n\epsilon^2)$	$\forall \epsilon > 0$	[6, Thm 1.16]
Thm 1.3	n independent random vars X_1, \dots, X_n indep $S_n = X_1 + \dots + X_n$	$\Pr[S_n - \mathbb{E}[S_n] > \epsilon] \leq 2 \exp\left(\frac{-\epsilon^2}{4 \sum_{i=1}^n \text{Var}[X_i]}\right)$	$\epsilon \in (0, 2 \text{Var}[S_n] / (\max_i X_i - \mathbb{E}[X_i]))$	[5, Thm. 1.3]
Azuma				
Weak law of large numbers	n independent iid random vars X_1, \dots, X_n indep $\mathbb{E}[X_i] = \mu$ iid	$\lim_{n \rightarrow \infty} \Pr[\frac{1}{n} S_n - \mu \geq \epsilon] = 0$	$\forall \epsilon > 0$	[3, Sec. 5.2][6, Thm 1.15]
Strong law of large numbers	(same)	$\Pr[\lim_{n \rightarrow \infty} \frac{1}{n} S_n = \mu] = 1$		[3, Sec. 5.5]
Advanced				
Bennett	n independent zero-mean X_1, \dots, X_n indep $\mathbb{E}[X_i] = 0$ iid	$\Pr[S_n > \epsilon] \leq \exp\left(-n\sigma^2 h\left(\frac{\epsilon}{n\sigma^2}\right)\right)$	$\sigma^2 := \frac{1}{n} \sum_{i=1}^n \text{Var}[X_i], \forall \epsilon > 0,$ $h(a) := (1+a) \log(1+a) - a$ for $a \geq 0$	[1, 4.1]
Bernstein	(same)	$\Pr[S_n > \epsilon] \leq \exp\left(\frac{-ne^2}{2(\sigma^2 + \epsilon/3)}\right)$	(same)	[1, 4.2]
Efron-Stein	scalar func of vars $f: \chi^n \rightarrow \mathbb{R}$ w/ values in set χ	$\text{Var}[Z] \leq \sum_{i=1}^n \mathbb{E}[(Z - \mathbb{E}_i[Z])^2]$	$Z := g(X_1, \dots, X_n)$ $\mathbb{E}_i[Z] := \mathbb{E}[Z X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n]$	[1, 4.3]
McDiarmid's	scalar func of vars $f: \chi^n \rightarrow \mathbb{R}$ w/ values in set χ	$\Pr[f(X_1, \dots, X_n) - \mathbb{E}[f(X_1, \dots, X_n)] \geq \epsilon] \leq \exp\left(\frac{-2\epsilon^2}{\sum_{i=1}^n c_i^2}\right)$	condition: c -bounded difference property $\forall \epsilon > 0$ $ f(X_1, \dots, X_i, \dots, X_n) - f(X_1, \dots, X'_i, \dots, X_n) \leq c_i$	[1, 4.4]

¹Markov's inequality bounds the first moment of random variable. Use it when a constant probability bound is sufficient [1, Sec. 3.3].

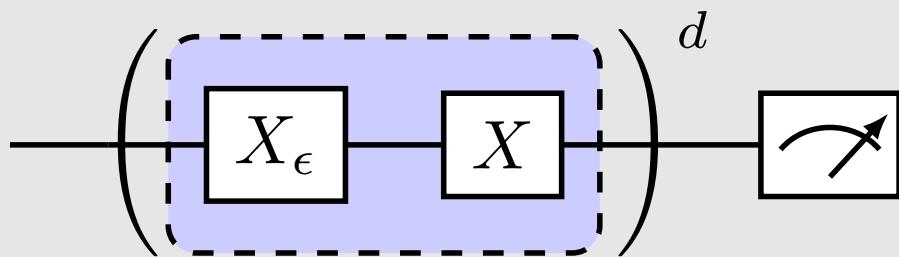
²Chebyshev is derived from Markov. It bounds the second moment. It is the appropriate one when the variance σ is known. If σ is unknown, we can use the bounds of $X \in [a, b]$.

³Chernoff bound is used to bound the tails of the distribution for a sum of independent random variables. By far the most useful tool in randomized algorithms [1, Sec. 3.3].

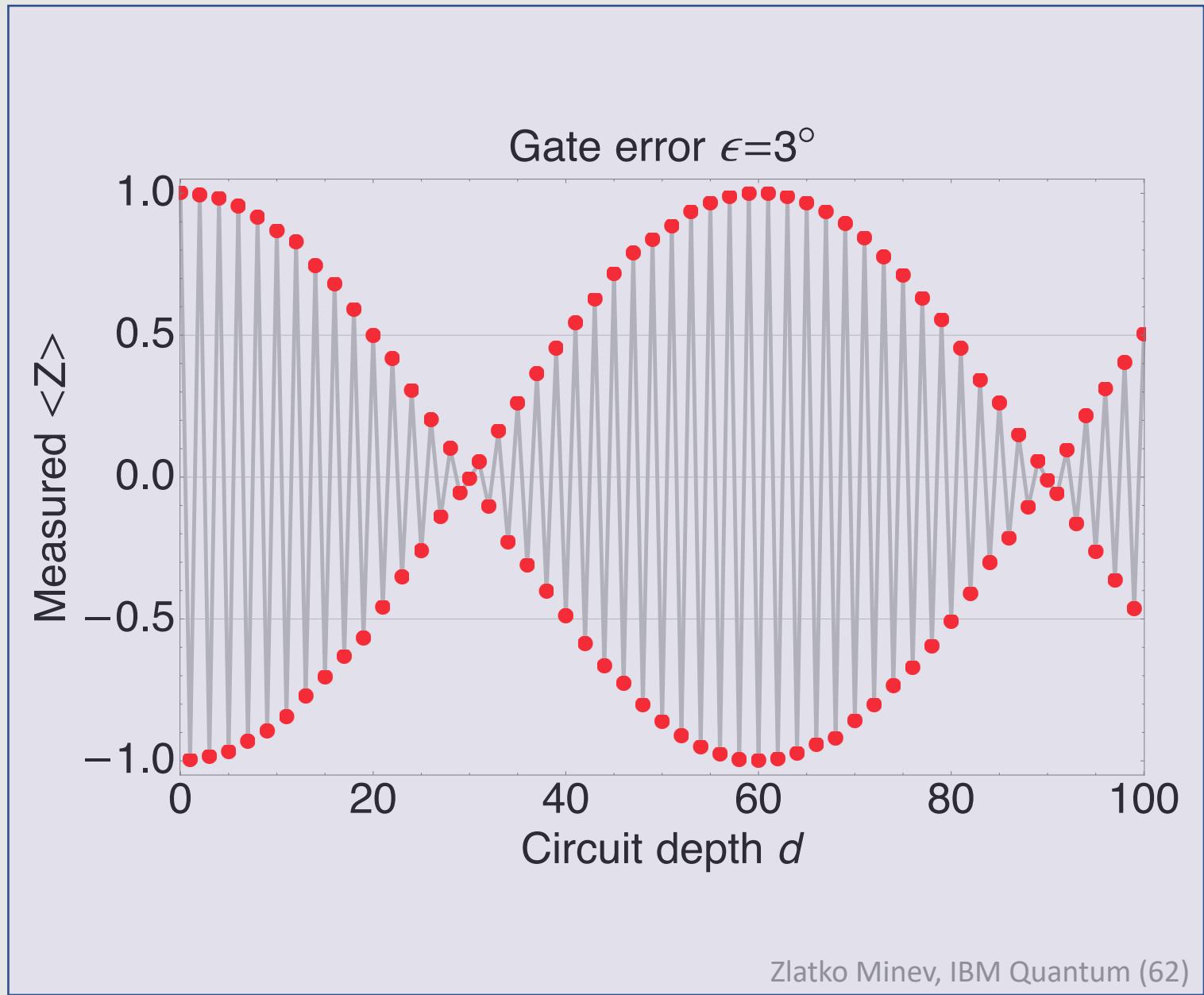
⁴This probability can be interpreted as the level of significance ϵ (probability of making an error) for a confidence interval around the mean of size 2ϵ . Therefore, we require at least $\log(2\alpha)/2t^2$ samples to acquire $1 - \alpha$ confidence interval $\mathbb{E}[X] \pm t$.



Recall gate error result

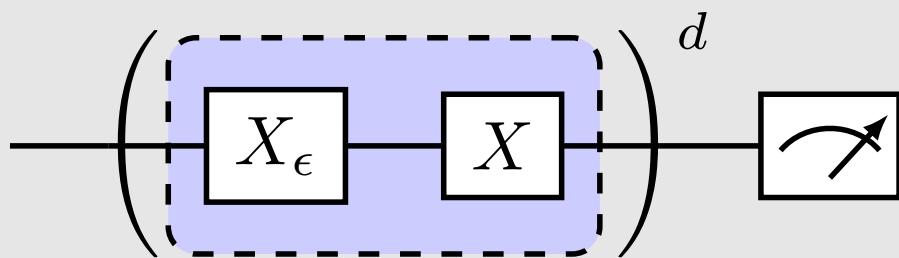


$$\langle \tilde{\psi}_f | Z | \tilde{\psi}_f \rangle = \cos(d\pi + d\epsilon)$$

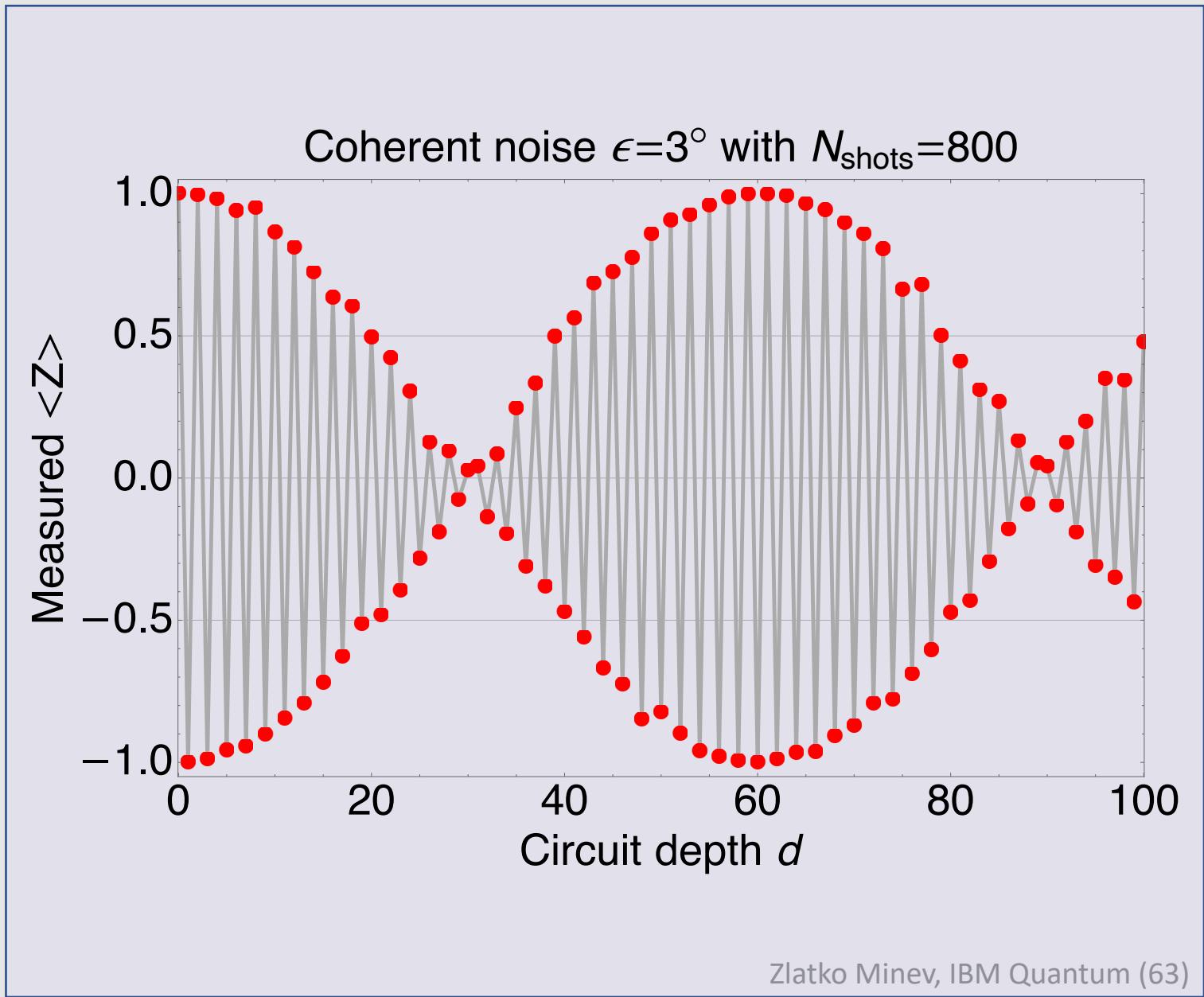




Projection & sampling noise



$$\langle \tilde{\psi}_f | Z | \tilde{\psi}_f \rangle = \cos(d\pi + d\epsilon)$$





Questions



Answer these multiple-choice questions
in the chat; for example, type “1a 2b.”

1. Projection noise is due to

- a) measurement apparatus that could be made more efficient
- b) classical limitations
- c) core nature of quantum physics

2. To reduce projection noise

- a) increase the number of sample
- b) you cannot undo it
- c) apply readout error mitigation



Dive deeper? Try the following



1. Calculate the following for a qubit

1. The expectation value of the sample variance for N shots of the observable $|1\rangle\langle 1|$.

where the sample mean is defined as

$$S = \frac{1}{N} \sum_{n=1}^N M_n$$

and the sample variance is defined as

$$V = \frac{1}{N} \sum_{n=1}^N (M_n - S)^2$$

2. Is the estimate biased?

3. What is the variance of V?

4. Can you find an expression for an unbiased estimate of the variance of S?

2. What about two qubits?

1. Can you find the projection operators for the observable ZZ?

2. Find the probability distribution for the observables ZI, IZ, and ZZ for a general state.

3. If you take 10 shots and find all 10 outcomes to be 1, what is the probability the qubit is in the $|0\rangle$ state? (hint: it's not zero!)

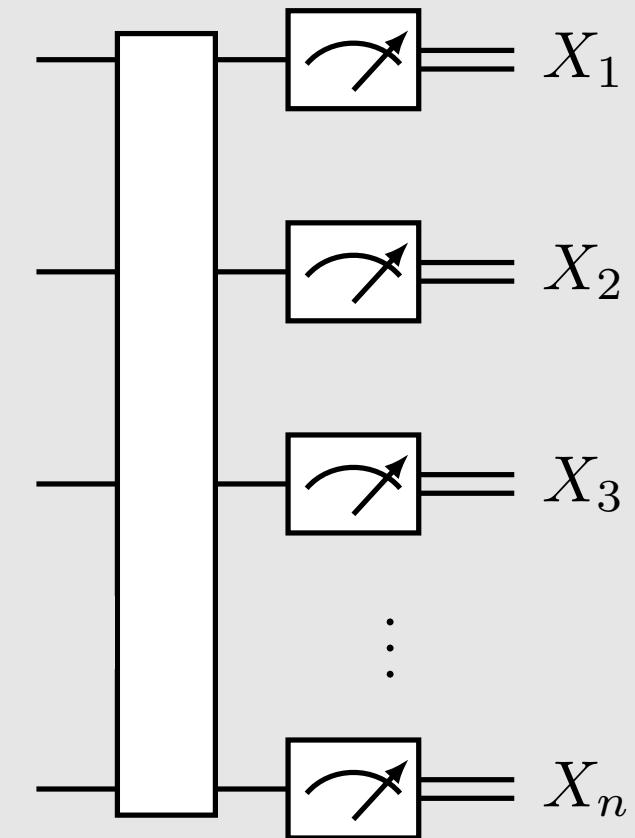


n qubits: preliminaries



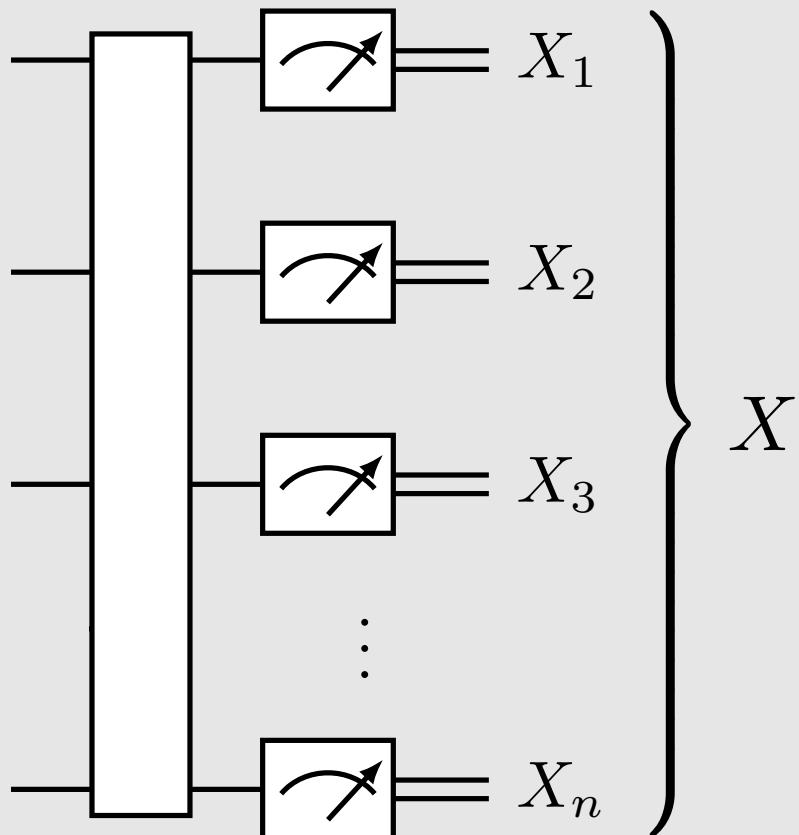
A horizontal double-headed arrow icon, indicating a two-way relationship or comparison.

meter





n qubits: binary strings



binary
string

$$X = X_1 X_2 \cdots X_n$$

bits of string

$$X = X_n X_{n-1} \cdots X_1$$

(one convention)

(Qiskit convention)

Example:

$$00 \cdots 0$$

$$00 \cdots 1$$

$$11 \cdots 0$$

$$11 \cdots 1$$

What do the variables belong to?

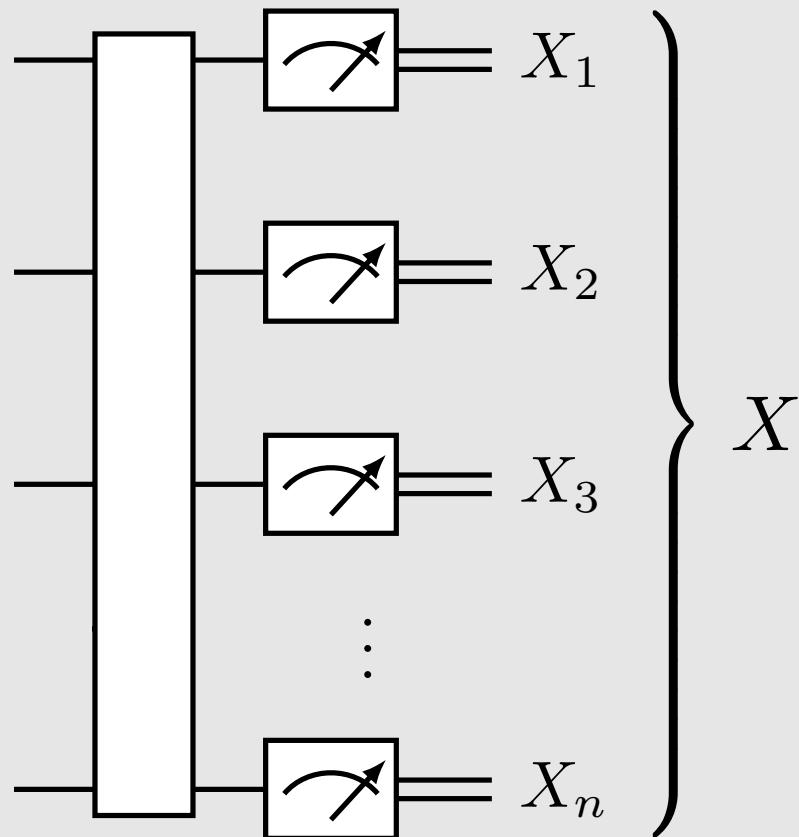
$$X_i \in \{0, 1\} \quad \text{for } i = 1, \dots, n$$

$$X \in \Sigma = \{0, 1\}^n$$





Binary strings: integer representation



integer
representation

little-endian
binary expansion

binary
string

bits

$$X = X_1 X_2 \cdots X_n$$

$$X_{\text{int}} = 0, 1, \dots, 2^n - 1$$

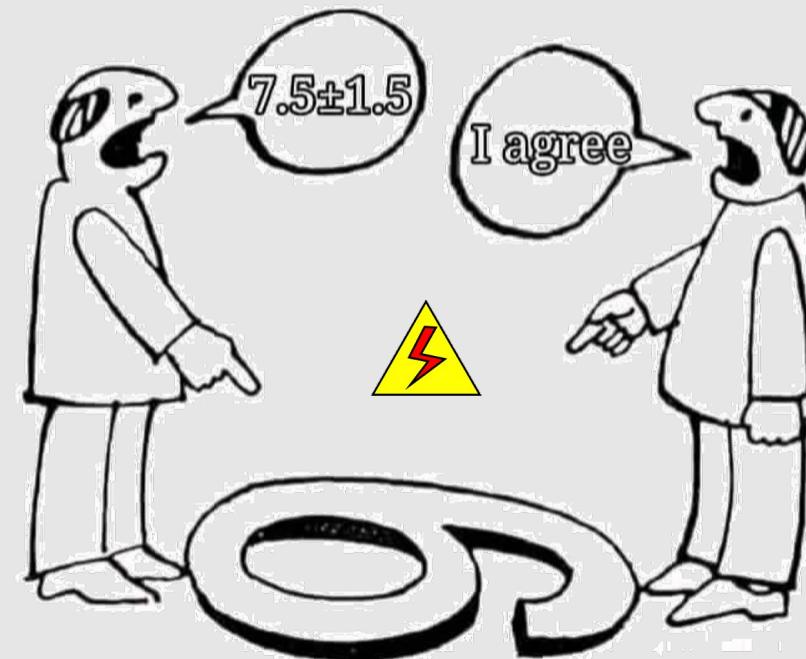
$$X_{\text{int}} = \sum_{j=1}^n X_j 2^{j-1}$$

note, this is one
convention

Increasing numeric significance with increasing memory addresses (index number j) is known as little-endian.
See also: bijective numeration.

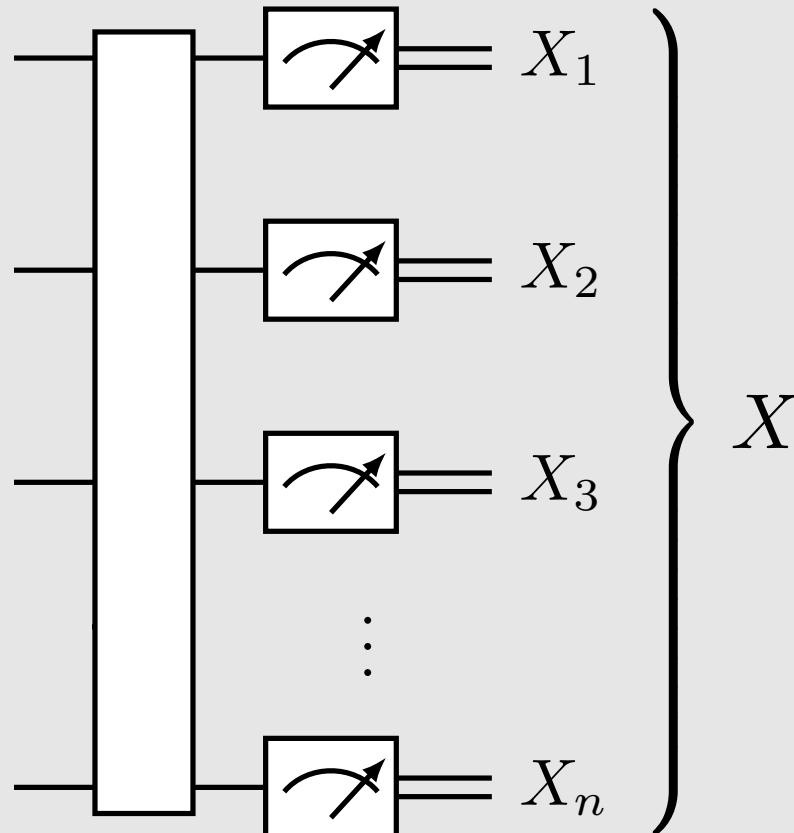


Beware tool's conventions





Binary and Pauli strings



For 1 qubit

$$\langle I \rangle = p(0) + p(1)$$

$$\langle Z \rangle = p(0) - p(1)$$

For 2 qubits

$$\langle II \rangle = p(00) + p(01) + p(10) + p(11)$$

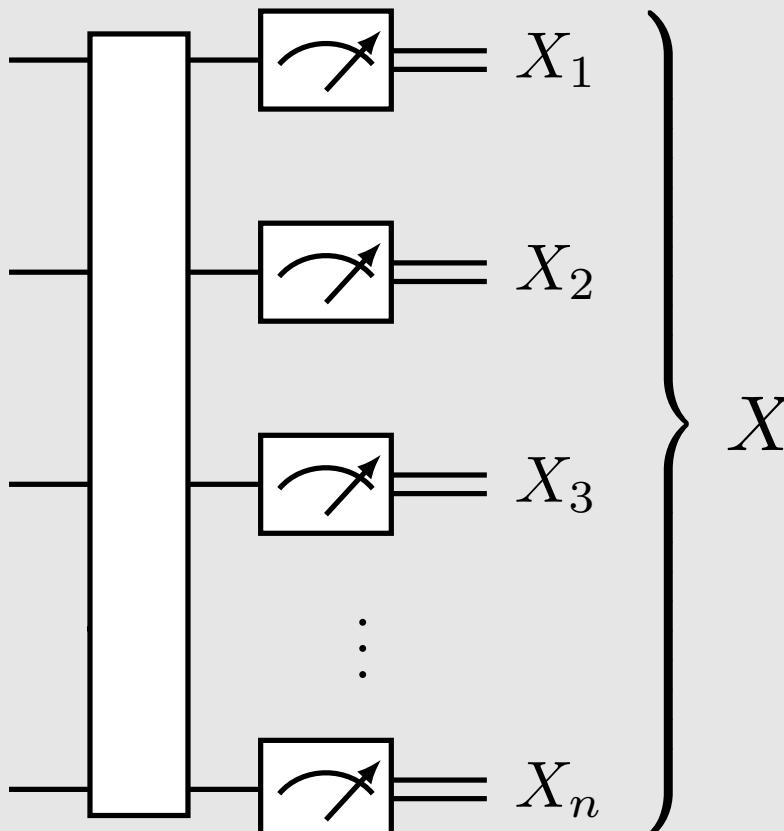
$$\langle IZ \rangle = p(00) - p(01) + p(10) - p(11)$$

$$\langle ZI \rangle = p(00) + p(01) - p(10) - p(11)$$

$$\langle ZZ \rangle = p(00) - p(01) - p(10) + p(11)$$



How to go between bitstrings and Pauli expectation values?



For 2 qubits

In vector form

$$\mathbf{z} = \begin{pmatrix} \langle II \rangle \\ \langle IZ \rangle \\ \langle ZI \rangle \\ \langle ZZ \rangle \end{pmatrix}$$

$$\begin{aligned}\langle II \rangle &= p(00) + p(01) + p(10) + p(11) \\ \langle IZ \rangle &= p(00) - p(01) + p(10) - p(11) \\ \langle ZI \rangle &= p(00) + p(01) - p(10) - p(11) \\ \langle ZZ \rangle &= p(00) - p(01) - p(10) + p(11)\end{aligned}$$

$$\mathbf{W} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad \mathbf{p} = \begin{pmatrix} p(00) \\ p(01) \\ p(10) \\ p(11) \end{pmatrix}$$

Pauli expectation values from binary string probabilities

$$\mathbf{z} = \mathbf{Wp}$$

$$\mathbf{p} = \frac{1}{2^n} \mathbf{Wz}$$



Walsh-Hadamard transform

inner product over Z_2^n

$$W_{ax} = (-1)^{\langle x, a \rangle}$$

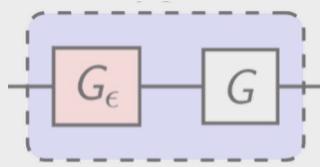
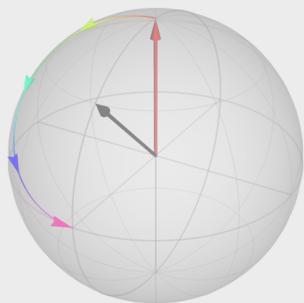
$$x \cdot a = \langle x, a \rangle = \sum_{i=1}^n x_i a_i$$

For deeper dive, see zlatko-minev.com/blog/t21-bit-strings



Chapter 4

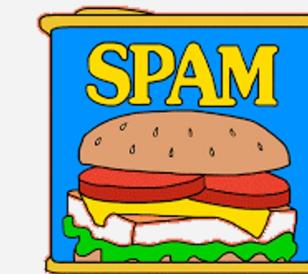
Coherent noise



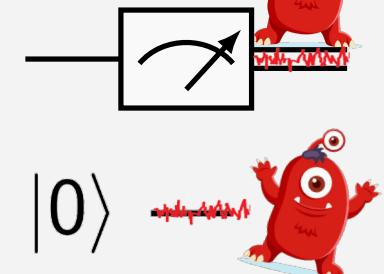
Measurements in quantum Projection noise



SPAM: State & meter

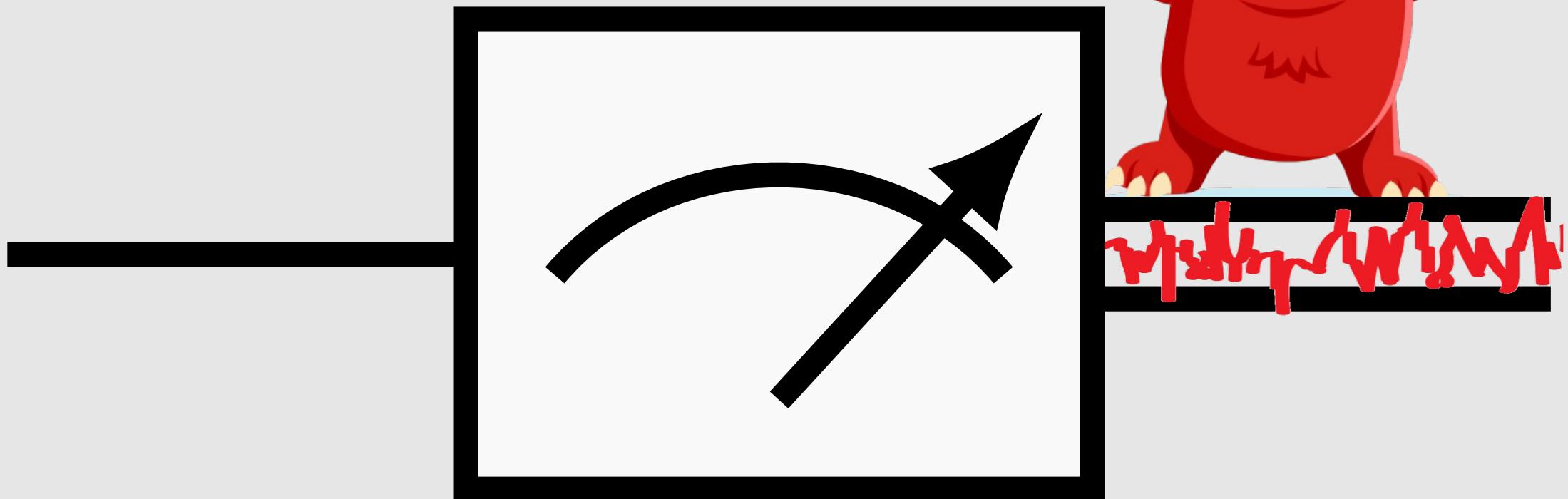


$|0\rangle$

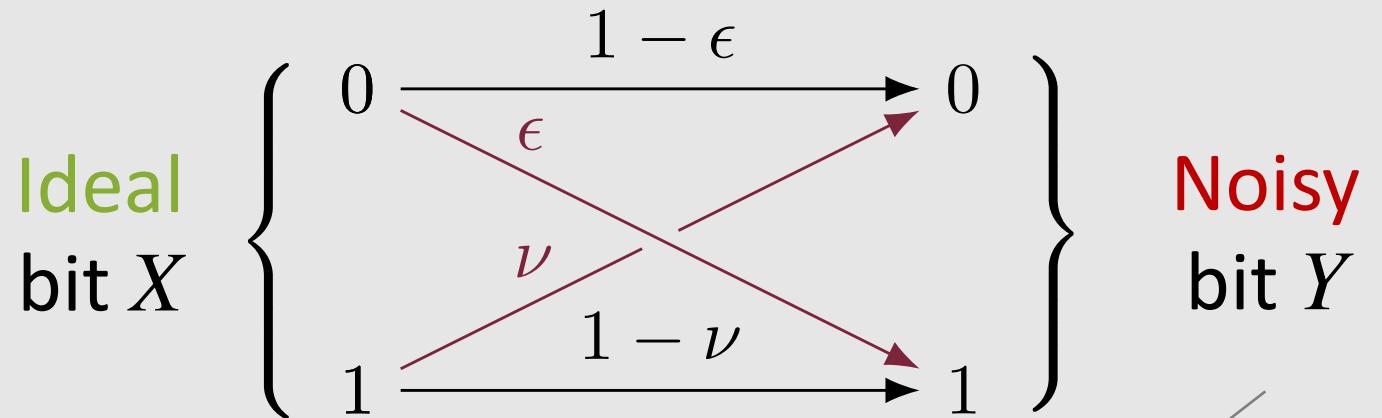
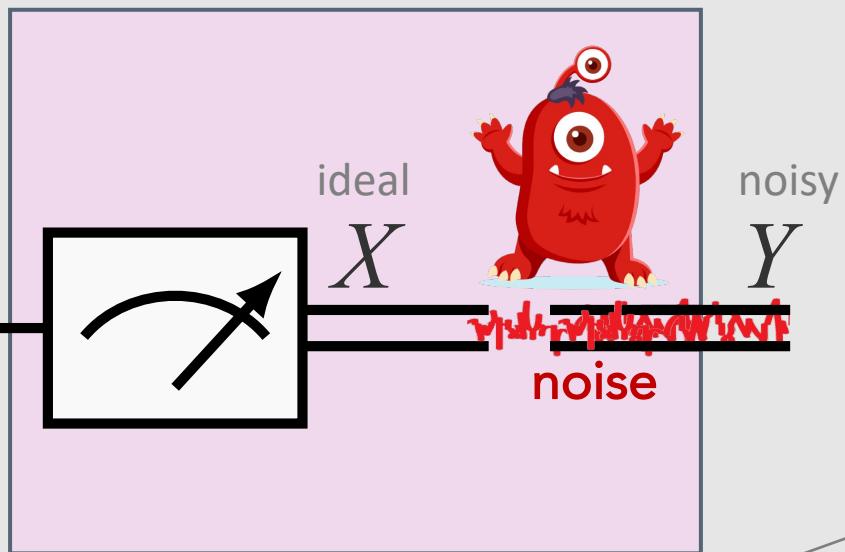




Measurement noise



Toy model of measurement noise

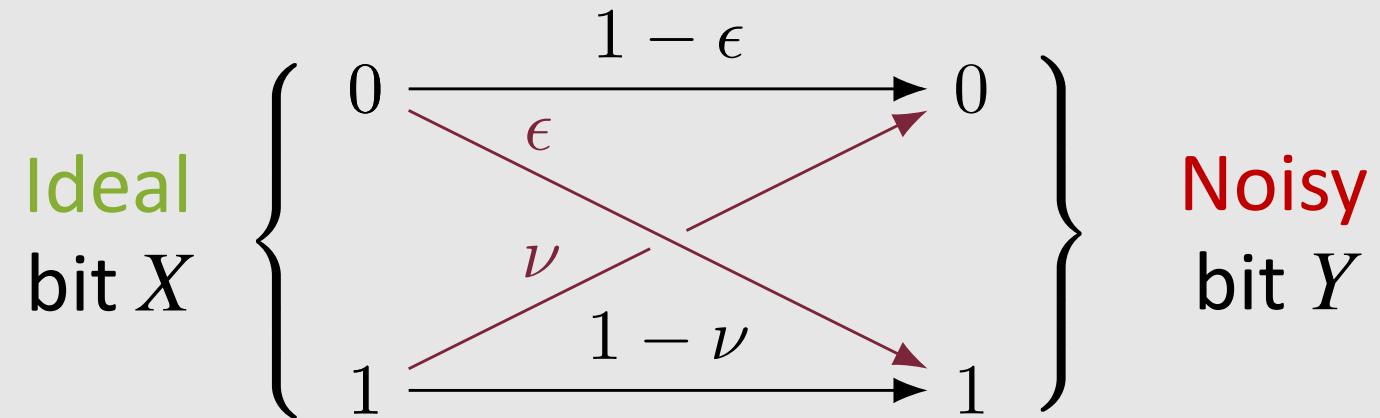
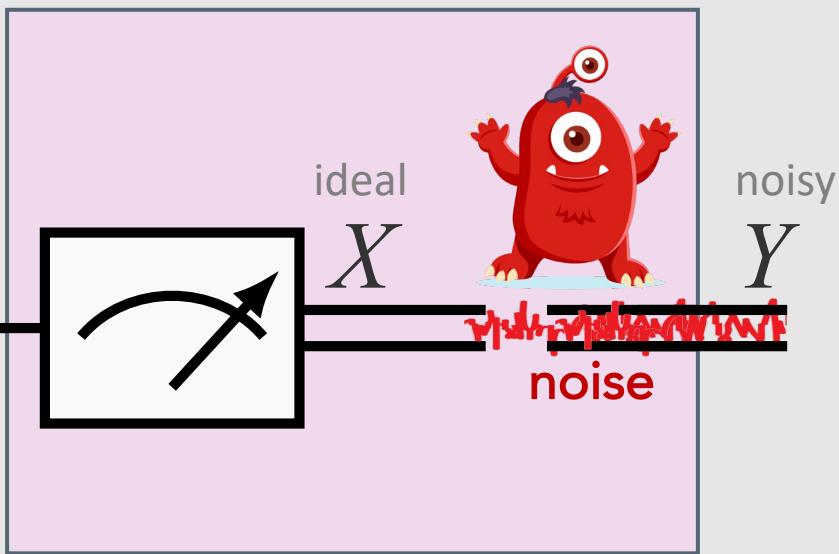


$$\mathbf{p}_{\text{ideal}} = \begin{pmatrix} \mathbb{P}(X=0) \\ \mathbb{P}(X=1) \end{pmatrix}$$

$$\mathbf{A} = \begin{matrix} & \begin{matrix} X=0 & X=1 \end{matrix} \\ \begin{matrix} Y=0 \\ Y=1 \end{matrix} & \begin{pmatrix} 1-\epsilon & \nu \\ \epsilon & 1-\nu \end{pmatrix} \end{matrix}$$

$$\mathbf{p}_{\text{noisy}} = \begin{pmatrix} \mathbb{P}(Y=0) \\ \mathbb{P}(Y=1) \end{pmatrix}$$

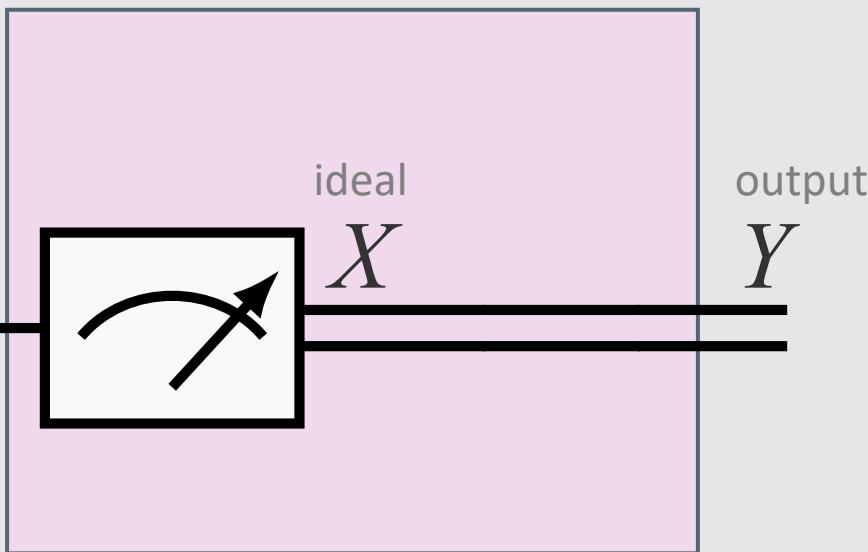
The A matrix of the noisy meter



$$\mathbf{p}_{\text{noisy}} = \mathbf{A} \mathbf{p}_{\text{ideal}}$$

$$\mathbf{A} = \begin{matrix} & X = 0 & X = 1 \\ Y = 0 & 1 - \epsilon & \nu \\ Y = 1 & \epsilon & 1 - \nu \end{matrix}$$

Ideal case

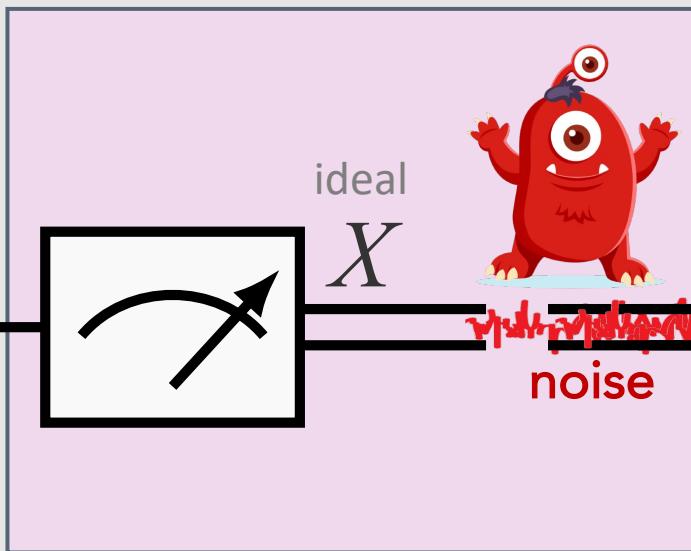


$$\mathbf{A}_{\text{ideal}} = \begin{matrix} & X = 0 & X = 1 \\ Y = 0 & \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \\ Y = 1 & & \end{matrix}$$

$$\mathbf{p}_{\text{noisy}} = \mathbf{A} \mathbf{p}_{\text{ideal}}$$

$$\mathbf{A} = \begin{matrix} & X = 0 & X = 1 \\ Y = 0 & \left(\begin{array}{cc} 1 - \epsilon & \nu \\ \epsilon & 1 - \nu \end{array} \right) \\ Y = 1 & & \end{matrix}$$

The A matrix of the noisy meter



What are the properties of A you can observe?

Properties

- The element $A_{y,x} \in [0, 1]$ with indices $x, y \in \{0, 1\}$ is a probability vector element. (i.e., A is a stochastic matrix)
- $\sum_{y \in \Sigma} A_{y,x} = 1 \quad \forall x \in \{0, 1\}$ is the unit normalization of the probability vector (columns of the A matrix).



What is the probability of Y?

$$\mathbf{p}_{\text{noisy}} = \mathbf{A} \mathbf{p}_{\text{ideal}}$$

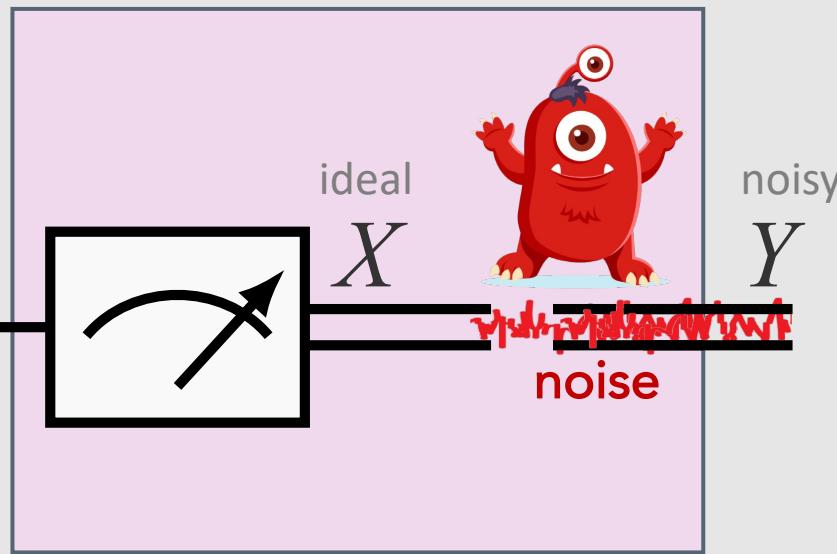
$$\mathbf{A} = \begin{matrix} & X = 0 & X = 1 \\ Y = 0 & \left(\begin{array}{cc} 1 - \epsilon & \nu \\ \epsilon & 1 - \nu \end{array} \right) \end{matrix}$$

$$p_X := \mathbb{P}[X = 1] ,$$

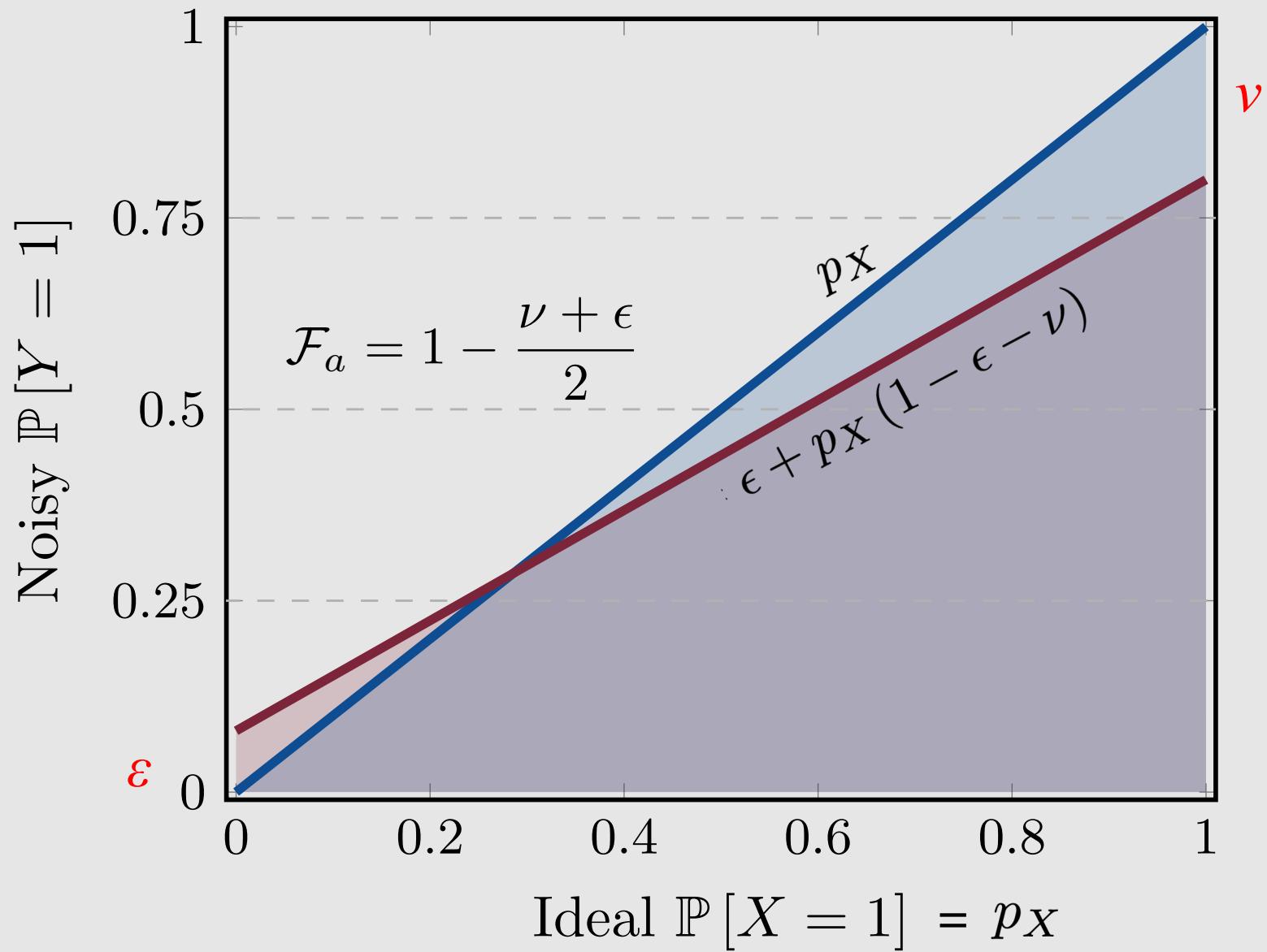
$$p_Y := \mathbb{P}[Y = 1] ,$$

$$\begin{aligned} \mathbb{P}[Y = 1] &= \mathbb{P}[Y = 1|X = 0] \mathbb{P}[X = 0] + \mathbb{P}[Y = 1|X = 1] \mathbb{P}[X = 1] \\ &= \epsilon(1 - p_X) + (1 - \nu)p_X \\ &= \epsilon + p_X(1 - \epsilon - \nu) \end{aligned}$$

The effect of readout error on the output probability

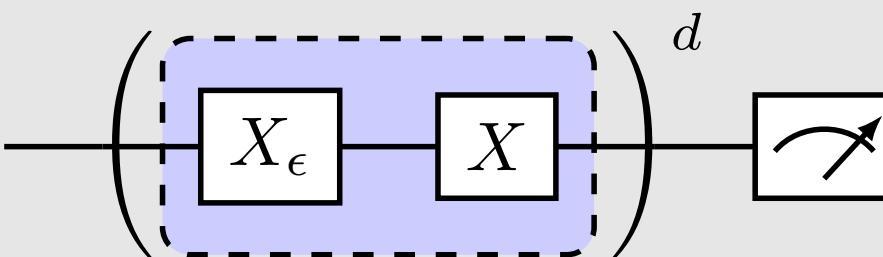


$$\mathbf{p}_{\text{noisy}} = \mathbf{A} \mathbf{p}_{\text{ideal}}$$
$$\mathbf{A} = \begin{matrix} & X = 0 & X = 1 \\ Y = 0 & \left(\begin{matrix} 1 - \epsilon & \nu \\ \epsilon & 1 - \nu \end{matrix} \right) \\ Y = 1 & & \end{matrix}$$



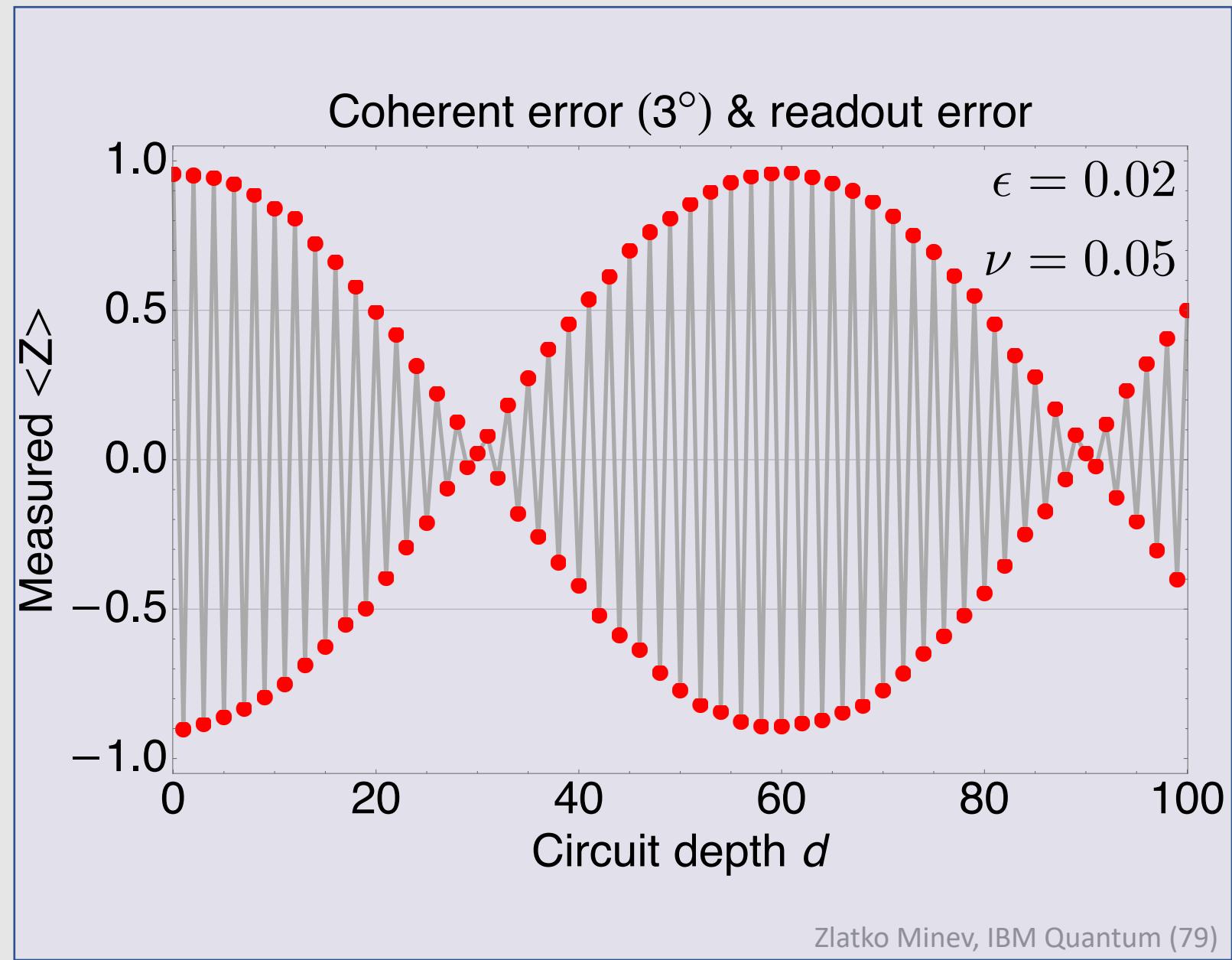


Projection & sampling noise



$$\mathbf{p}_{\text{noisy}} = \mathbf{A} \mathbf{p}_{\text{ideal}}$$

$$\mathbf{A} = \begin{matrix} Y=0 & X=0 & X=1 \\ Y=1 & \begin{pmatrix} 1-\epsilon & \nu \\ \epsilon & 1-\nu \end{pmatrix} \end{matrix}$$





Questions



Answer these multiple-choice questions
in the chat; for example, type “1a 2b.”

1. Readout error is due to

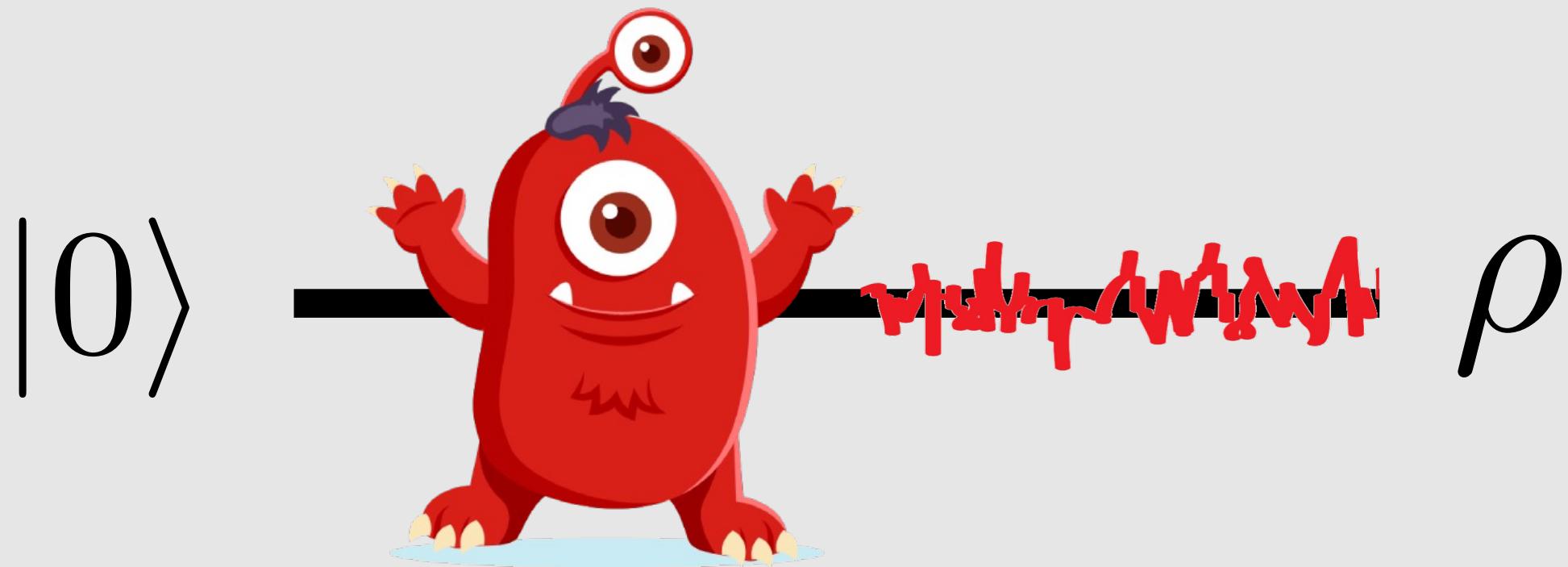
- a) measurement apparatus that could be made more efficient
- b) classical limitations
- c) core nature of quantum physics

2. To reduce readout error bias

- a) increase the number of sample
- b) you cannot undo it
- c) apply readout error mitigation

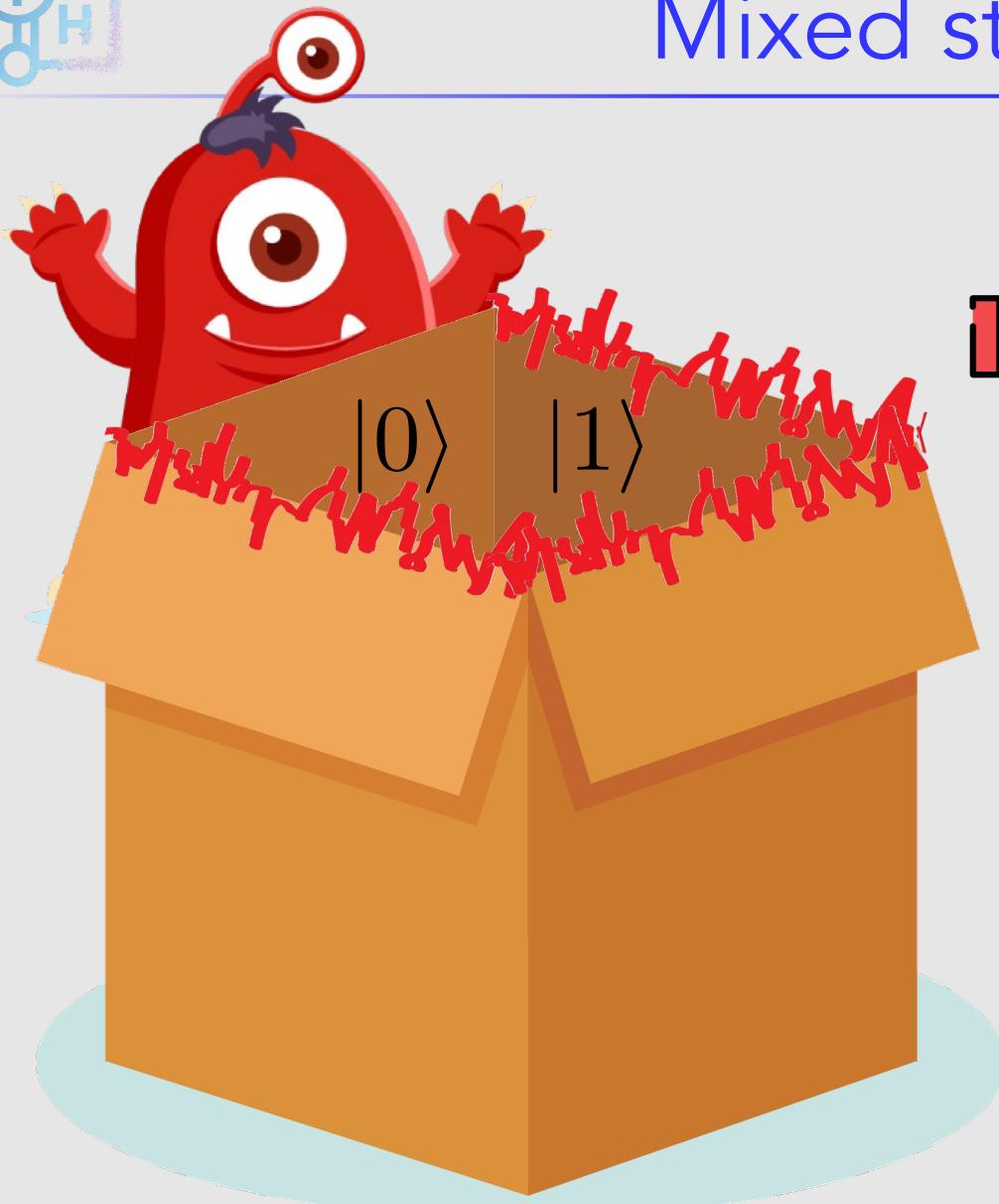


State preparation noise





Mixed state (density matrix)



Probability $1-p$ $|0\rangle$
Probability p $|1\rangle$

$$\begin{aligned}\rho &= \mathbb{P}[|0\rangle] |0\rangle \langle 0| + \mathbb{P}[|1\rangle] |1\rangle \langle 1| \\ &= (1 - p) |0\rangle \langle 0| + p |1\rangle \langle 1|\end{aligned}$$



Mixed state (density matrix)

Recall

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{zero state ket}$$
$$\langle 0| = |0\rangle^\dagger = (1 \ 0), \quad \text{zero state bra}$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{one state ket}$$
$$\langle 1| = |1\rangle^\dagger = (0 \ 1), \quad \text{one state bra}$$

Dyad

$$|a\rangle \langle b| = |a\rangle \otimes \langle b|$$

$$|0\rangle \langle 0| = (1 \ 0) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|1\rangle \langle 1| = (0 \ 1) \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Probability $1-p$ $|0\rangle$

Probability p $|1\rangle$ Equivalent to randomly applying X gate

$$\rho = \mathbb{P}[|0\rangle] |0\rangle \langle 0| + \mathbb{P}[|1\rangle] |1\rangle \langle 1|$$
$$= (1 - p) |0\rangle \langle 0| + p |1\rangle \langle 1|$$

$$= \begin{vmatrix} \langle 0| & \langle 1| \\ |0\rangle & |1\rangle \end{vmatrix} \begin{pmatrix} 1 - p & 0 \\ 0 & p \end{pmatrix}$$



Mixed state expectation values

Expectation values

$$\langle \hat{A} \rangle_{\text{pure}} = \langle \psi | \hat{A} | \psi \rangle$$

$$\begin{aligned}\langle \hat{A} \rangle_{\text{mixed}} &= \sum_{\psi} \mathbb{P}[|\psi\rangle] \langle \psi | \hat{A} | \psi \rangle \\ &= (1-p) \langle 0 | \hat{A} | 0 \rangle + p \langle 1 | \hat{A} | 1 \rangle \\ &= \text{Tr} [\rho \hat{A}]\end{aligned}$$

Trace of a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\text{Tr} [A] = \sum_{i=1}^3 a_{ii} = a_{11} + a_{22} + a_{33}$$

$$\begin{array}{ll} \text{Probability } 1-p & |0\rangle \\ & \\ \text{Probability } p & |1\rangle \end{array}$$

$$\begin{aligned}\rho &= \mathbb{P}[|0\rangle] |0\rangle \langle 0| + \mathbb{P}[|1\rangle] |1\rangle \langle 1| \\ &= (1-p) |0\rangle \langle 0| + p |1\rangle \langle 1|\end{aligned}$$

$$= \begin{vmatrix} \langle 0 | & \langle 1 | \\ | 0 \rangle & \left(\begin{matrix} 1-p & 0 \\ 0 & p \end{matrix} \right) \\ | 1 \rangle & \end{vmatrix}$$



Mixed state Bloch vector

Expectation value of Z

$$\begin{aligned}\langle \hat{Z} \rangle &= \text{Tr} [\hat{Z} \rho] \\ &= (1-p) \langle 0 | \hat{Z} | 0 \rangle + p \langle 1 | \hat{Z} | 1 \rangle \\ &= (1-p)(1) + (p)(-1) \\ &= 1 - 2p\end{aligned}$$



Trace of a matrix

$$\begin{aligned}A &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\ \text{Tr}[A] &= \sum_{i=1}^3 a_{ii} = a_{11} + a_{22} + a_{33}\end{aligned}$$

Probability $1-p$ $|0\rangle$

Probability p $|1\rangle$

$$\begin{aligned}\rho &= \mathbb{P}[|0\rangle] |0\rangle \langle 0| + \mathbb{P}[|1\rangle] |1\rangle \langle 1| \\ &= (1-p) |0\rangle \langle 0| + p |1\rangle \langle 1|\end{aligned}$$

$$= \begin{vmatrix} |0\rangle & |1\rangle \\ |0\rangle & (1-p) & 0 \\ |1\rangle & 0 & p \end{vmatrix}$$



Mixed state Bloch vector

Expectation value of Z

$$\begin{aligned}\langle \hat{Z} \rangle &= \text{Tr} [\hat{Z} \rho] \\ &= (1-p) \langle 0 | \hat{Z} | 0 \rangle + p \langle 1 | \hat{Z} | 1 \rangle \\ &= (1-p)(1) + (p)(-1) \\ &= 1 - 2p\end{aligned}$$

Show that

$$\begin{aligned}\langle \hat{X} \rangle &= 0 && \text{and that generally for a qubit} \\ \langle \hat{Y} \rangle &= 0 && \text{Tr} [\rho] = 1 \\ \langle \hat{I} \rangle &= 1 && 1/2^n \leq \text{Tr} [\rho^2] \leq 1\end{aligned}$$



Probability $1-p$ $|0\rangle$

Probability p $|1\rangle$

$$\begin{aligned}\rho &= \mathbb{P}[|0\rangle] |0\rangle \langle 0| + \mathbb{P}[|1\rangle] |1\rangle \langle 1| \\ &= (1-p) |0\rangle \langle 0| + p |1\rangle \langle 1|\end{aligned}$$

$$= \begin{vmatrix} \langle 0 | & \langle 1 | \\ | 0 \rangle & | 1 \rangle \end{vmatrix} \begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix}$$



Mixed state Bloch vector

Expectation value of Z

$$\begin{aligned}\langle \hat{Z} \rangle &= \text{Tr} [\hat{Z} \rho] \\ &= (1-p) \langle 0 | \hat{Z} | 0 \rangle + p \langle 1 | \hat{Z} | 1 \rangle \\ &= (1-p)(1) + (p)(-1) \\ &= 1 - 2p\end{aligned}$$

Show that

$$\begin{array}{ll}\langle \hat{X} \rangle = 0 & \text{and that generally for a qubit} \\ \langle \hat{Y} \rangle = 0 & \text{Tr} [\rho] = 1 \\ \langle \hat{I} \rangle = 1 & 1/2^n \leq \text{Tr} [\rho^2] \leq 1\end{array}$$

Show that

$$\rho = \begin{vmatrix} |0\rangle & \langle 0| \\ |1\rangle & \langle 1| \end{vmatrix} = \frac{1}{2} (\hat{I} + 0\hat{X} + 0\hat{Y} + (1-2p)\hat{Z})$$

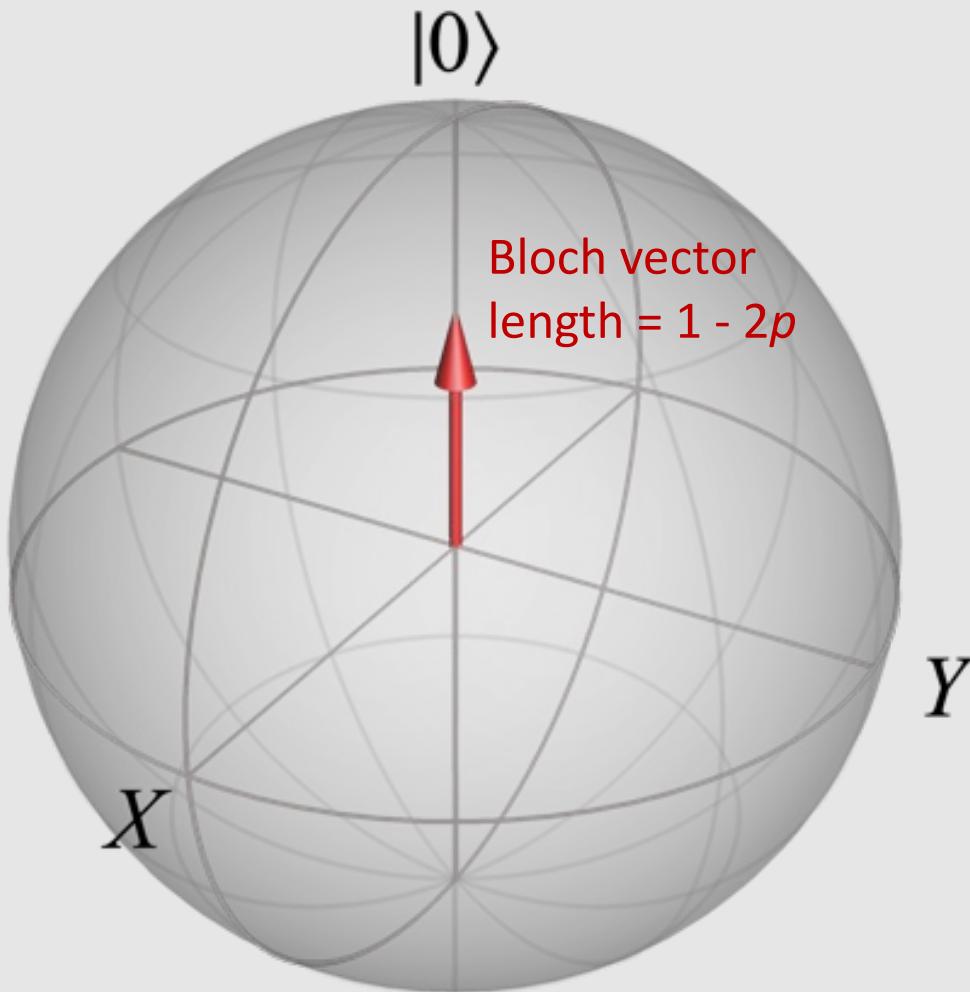


And more generally for any qubit mixed state

$$\rho = \frac{1}{2} (\hat{I} + \langle \hat{X} \rangle \hat{X} + \langle \hat{Y} \rangle \hat{Y} + \langle \hat{Z} \rangle \hat{Z})$$



Visualizing the mixed-state Bloch vector



For our state prep noisy monster, the state is

$$\rho = \begin{vmatrix} |0\rangle & \langle 0| \\ |1\rangle & \langle 1| \end{vmatrix} = \frac{1}{2} \left(\hat{I} + 0\hat{X} + 0\hat{Y} + (1 - 2p)\hat{Z} \right)$$

Keep in mind that a qubit mixed state in general is

$$\rho = \frac{1}{2} \left(\hat{I} + \langle \hat{X} \rangle \hat{X} + \langle \hat{Y} \rangle \hat{Y} + \langle \hat{Z} \rangle \hat{Z} \right)$$

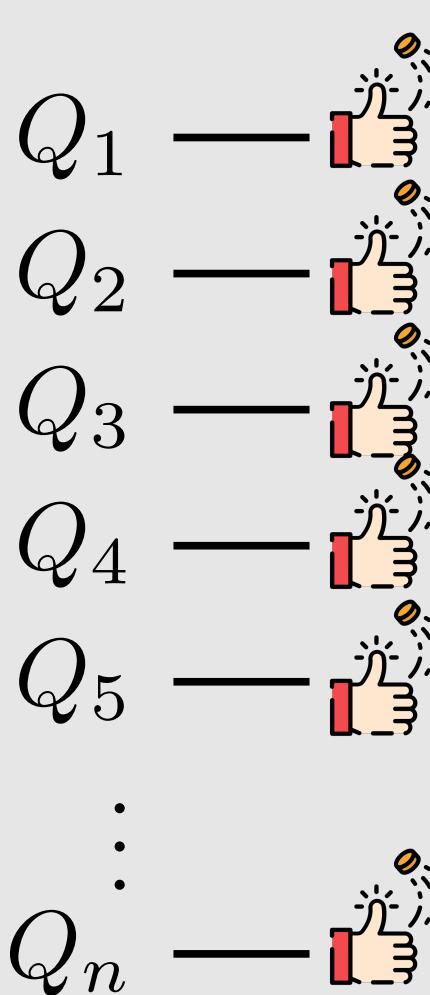
Purity

$$\begin{aligned} \text{Tr} [\rho^2] &= \frac{1}{2} \left(1 + \langle \hat{X} \rangle^2 + \langle \hat{Y} \rangle^2 + \langle \hat{Z} \rangle^2 \right) \\ &= \frac{1}{2} \left(1 + (1 - 2p)^2 \right) \end{aligned}$$





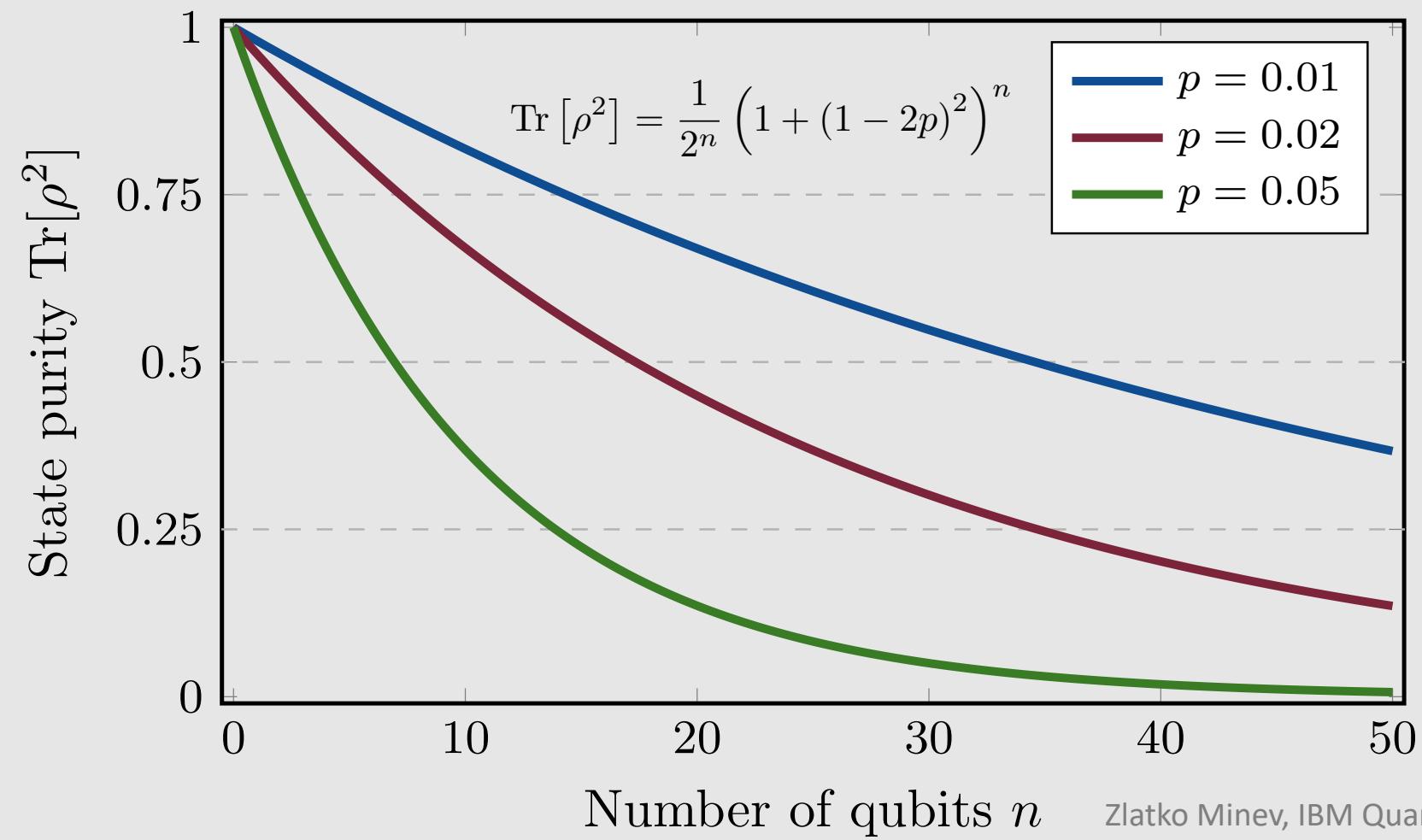
Noisy state preparation for multiple qubits



Harder, but see if you can try the derivation

$$\rho = [(1 - p) |0\rangle\langle 0| + p |1\rangle\langle 1|]_{Q_1} \otimes \cdots \otimes [(1 - p) |0\rangle\langle 0| + p |1\rangle\langle 1|]_{Q_N}$$

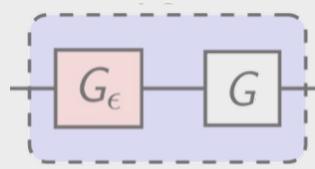
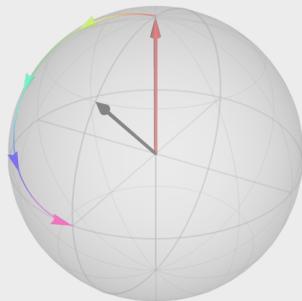
where p is individual qubit thermal probability to be in the 1 state.





Chapter 5

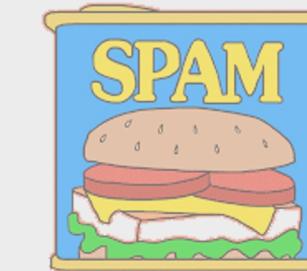
Coherent noise



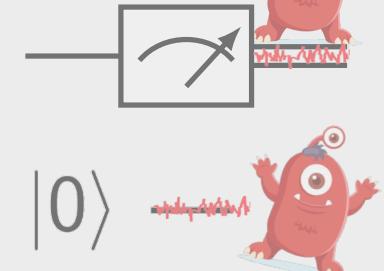
Measurements in quantum Projection noise



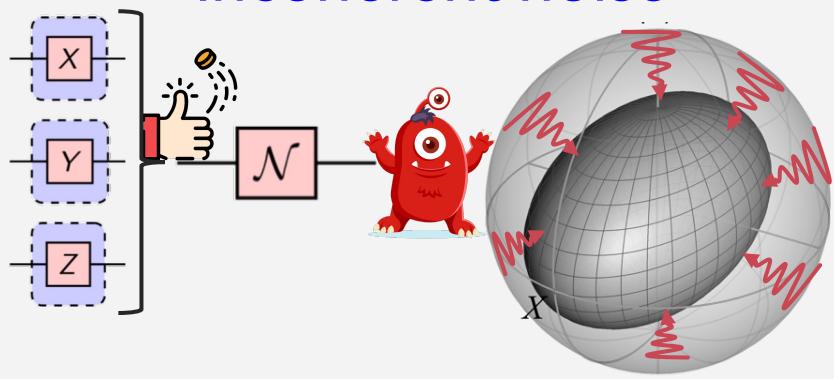
SPAM: State & meter



$|0\rangle$



Incoherent noise

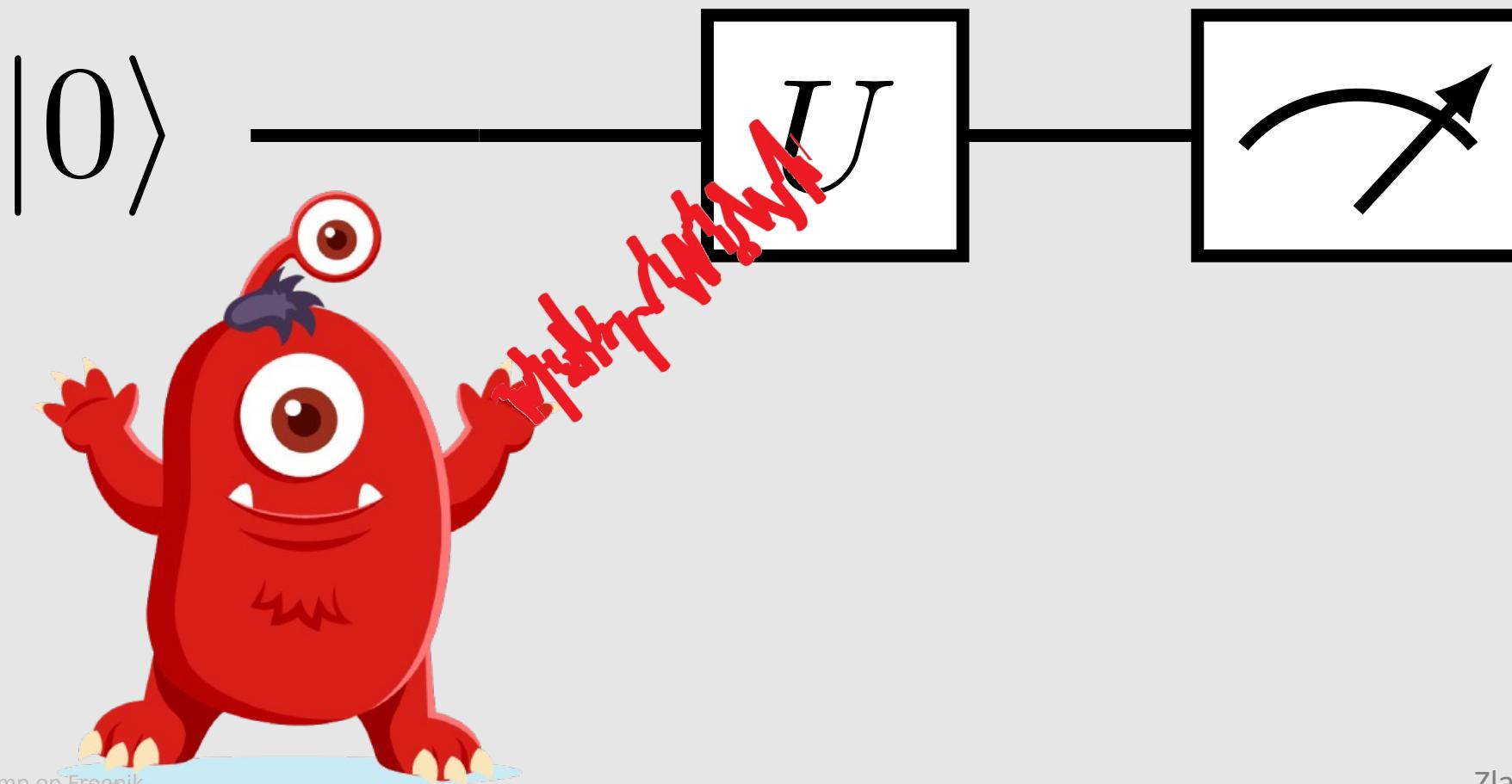


In this chapter, we will show more results, and leave more of the calculations for you to try

coin-toss icon: Good Ware, flaticon; spam: make it move;
road based on: freepik; Monster image by jcomp on Freepik

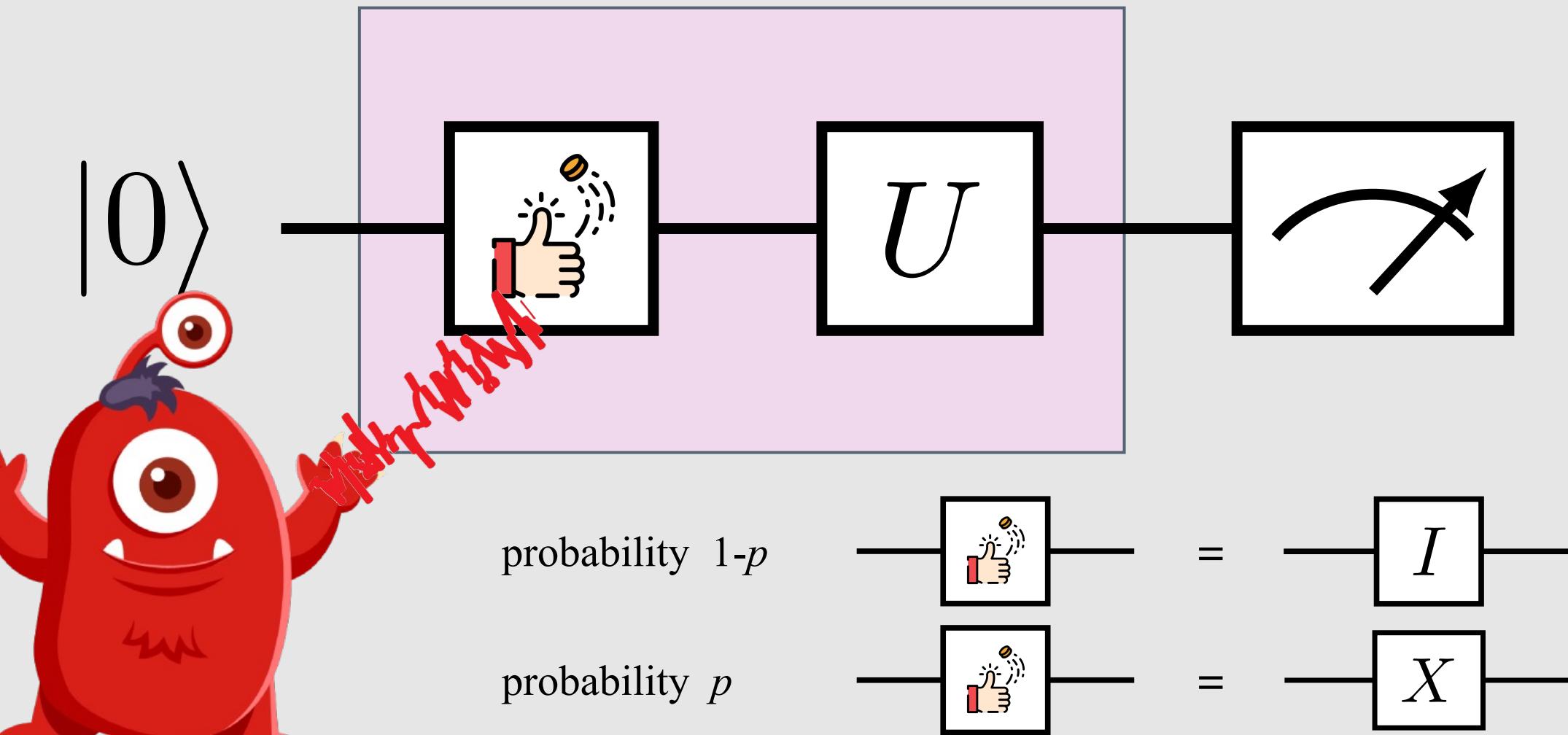


Quantum noise Toy model example



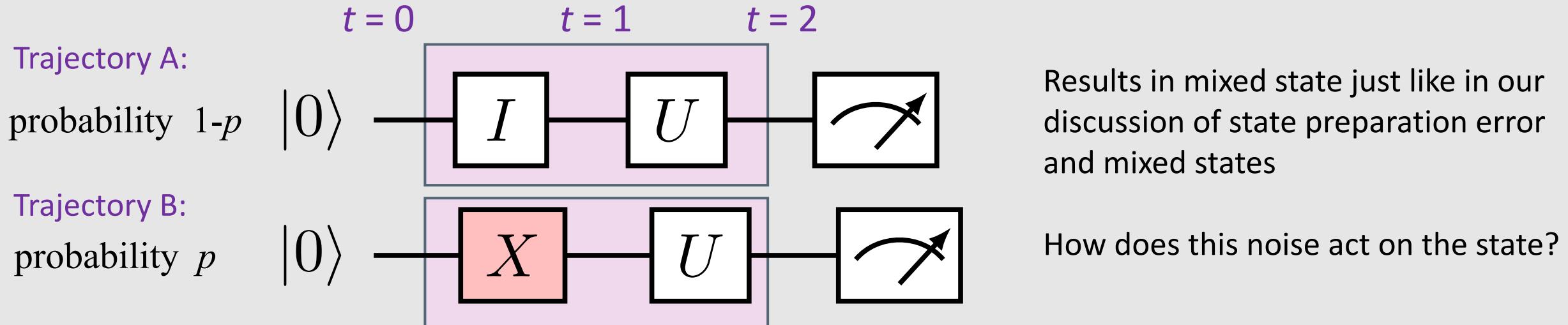


Quantum noise Toy model example





Noise raveling of quantum trajectories



For trajectory A:

$$\rho_0^A = |0\rangle \langle 0|$$

$$\rho_1^A = I |0\rangle \langle 0| I^\dagger$$

$$\rho_2^A = UI |0\rangle \langle 0| I^\dagger U^\dagger$$

Average evolution

$$\rho_t = (1 - p) \rho_t^A + p \rho_t^B$$

For trajectory B:

$$\rho_0^B = |0\rangle \langle 0|$$

$$\rho_1^B = X |0\rangle \langle 0| X^\dagger$$

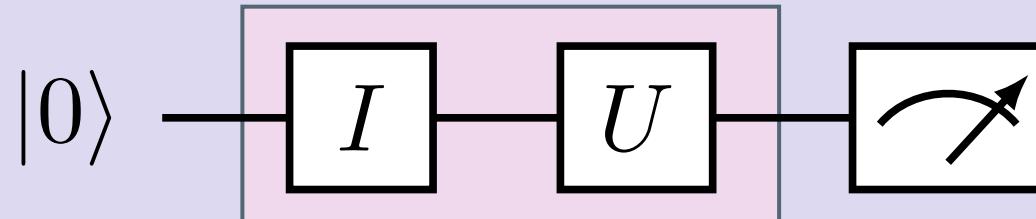
$$\rho_2^B = UX |0\rangle \langle 0| X^\dagger U^\dagger$$



Average evolution: Quantum noise channel

Trajectory A:

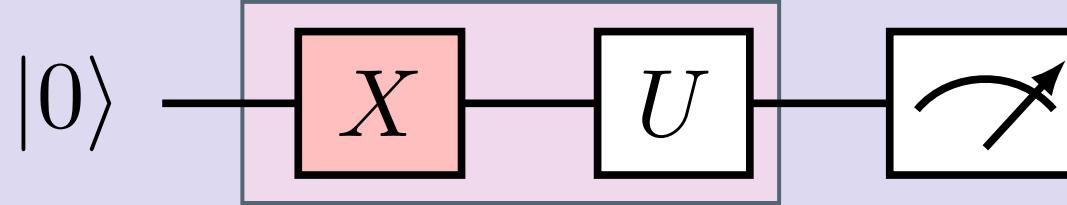
probability $1-p$



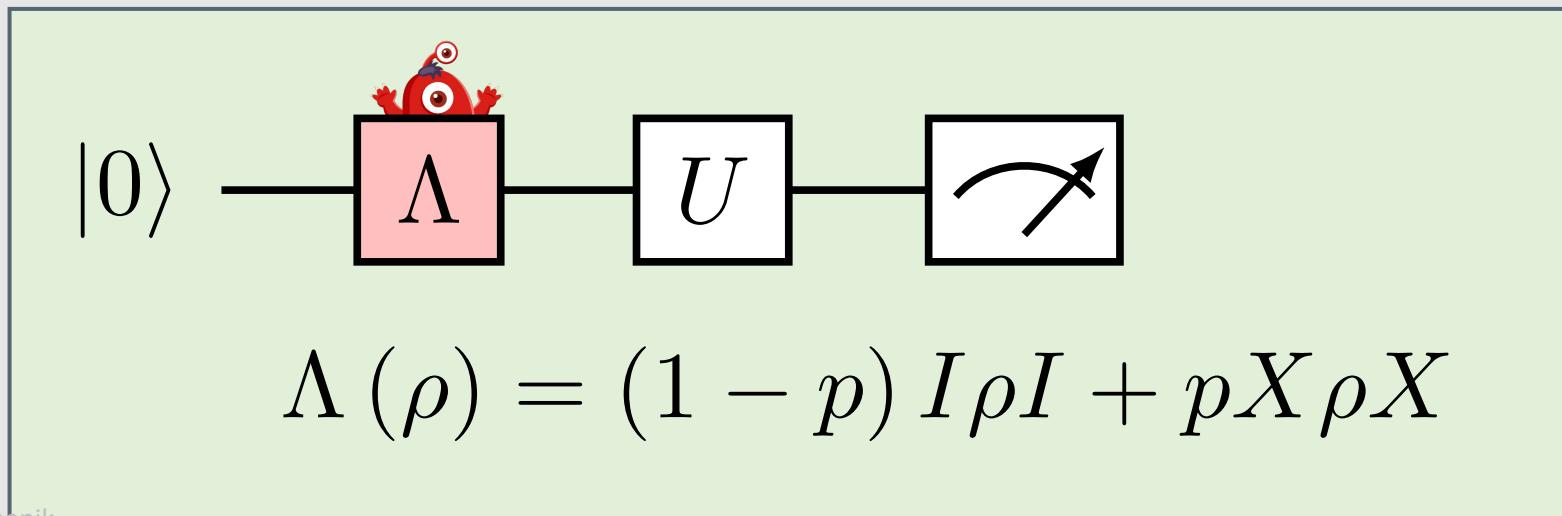
+

Trajectory B:

probability p

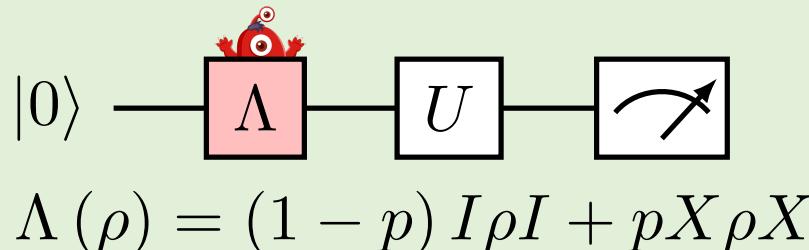


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Bit flip channel effects on the Bloch vector



Show the effect of the noise channel on the Bloch vector of the state $\rho_0 = \frac{1}{2}(\hat{I} + x\hat{X} + y\hat{Y} + z\hat{Z})$

Calculate $\rho_1 = \Lambda(\rho_0)$ and its $\langle \hat{I} \rangle, \langle \hat{X} \rangle, \langle \hat{Y} \rangle, \langle \hat{Z} \rangle$ for ρ_0, ρ_1



For the initial state ρ_0

$$\langle \hat{I} \rangle =$$

$$\langle \hat{X} \rangle =$$

$$\langle \hat{Y} \rangle =$$

$$\langle \hat{Z} \rangle =$$

For the final state ρ_1

$$\langle \hat{I} \rangle =$$

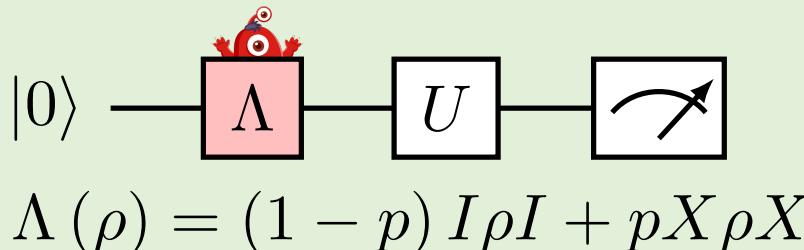
$$\langle \hat{X} \rangle =$$

$$\langle \hat{Y} \rangle =$$

$$\langle \hat{Z} \rangle =$$



Bit flip channel effects on the Bloch vector



Tool bag tricks:

For $\hat{A}, \hat{B} \in \{\hat{I}, \hat{X}, \hat{Y}, \hat{Z}\}$ Paulis,

$$\hat{A}^n = \begin{cases} \hat{A}, & n \text{ odd}, \\ \hat{I}, & n \text{ even}, \end{cases}$$

$$\text{Tr} [\hat{A}] = 0$$

$$\text{Tr} [\hat{A}\hat{B}] = 2\delta_{AB}$$

The trace is a linear functional

For $a, b \in \mathbb{C}$ and $\hat{C}, \hat{D} \in L(\mathcal{H})$,

$$\text{Tr} [a\hat{C} + b\hat{D}] = a \text{Tr} [\hat{C}] + b \text{Tr} [\hat{D}]$$

Show the effect of the noise channel on the Bloch vector of the state $\rho_0 = \frac{1}{2} (\hat{I} + x\hat{X} + y\hat{Y} + z\hat{Z})$

Calculate $\rho_1 = \Lambda(\rho_0)$ and its $\langle \hat{I} \rangle, \langle \hat{X} \rangle, \langle \hat{Y} \rangle, \langle \hat{Z} \rangle$ for ρ_0, ρ_1



$$\begin{aligned} \langle \hat{I} \rangle &= \text{Tr} [\hat{I}\rho] \\ &= \text{Tr} \left[\hat{I} \cdot \frac{1}{2} (\hat{I} + x\hat{X} + y\hat{Y} + z\hat{Z}) \right] \\ &= \text{Tr} \left[\frac{1}{2} (\hat{I} + x\hat{X} + y\hat{Y} + z\hat{Z}) \right] \\ &= \frac{1}{2} \text{Tr} [\hat{I} + x\hat{X} + y\hat{Y} + z\hat{Z}] \\ &= \frac{1}{2} \left(\text{Tr} [\hat{I}] + x \text{Tr} [\hat{X}] + y \text{Tr} [\hat{Y}] + z \text{Tr} [\hat{Z}] \right) \\ &= \frac{1}{2} (2 + 0 + 0 + 0) \\ &= 1 \end{aligned}$$

For the initial state ρ_0

Identity times anything is itself

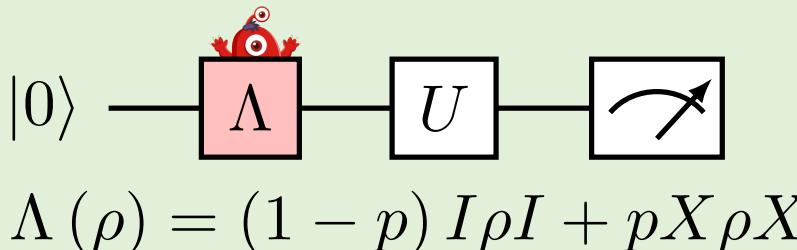
trace is linear functional, move constant outside

trace is linear functional, expand sum and distribute

all Pauli operators are traceless, except for the identity
we have shown the trace of the mixed state is 1 as expected



Bit flip channel effects on the Bloch vector



Tool bag tricks:

For $\hat{A}, \hat{B} \in \{\hat{I}, \hat{X}, \hat{Y}, \hat{Z}\}$ Paulis,

$$\hat{A}^n = \begin{cases} \hat{A}, & n \text{ odd}, \\ \hat{I}, & n \text{ even}, \end{cases}$$

$$\text{Tr} [\hat{A}] = 0$$

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The trace is a linear functional

For $a, b \in \mathbb{C}$ and $\hat{C}, \hat{D} \in L(\mathcal{H})$,

$$\text{Tr} [a\hat{C} + b\hat{D}] = a \text{Tr} [\hat{C}] + b \text{Tr} [\hat{D}]$$

Show the effect of the noise channel on the Bloch vector of the state $\rho_0 = \frac{1}{2} (\hat{I} + x\hat{X} + y\hat{Y} + z\hat{Z})$

Calculate $\rho_1 = \Lambda(\rho_0)$ and its $\langle \hat{I} \rangle, \langle \hat{X} \rangle, \langle \hat{Y} \rangle, \langle \hat{Z} \rangle$ for ρ_0, ρ_1



$$\begin{aligned} \langle \hat{X} \rangle &= \text{Tr} [\hat{X}\rho_0] \\ &= \text{Tr} [\hat{X} \cdot \frac{1}{2} (\hat{I} + x\hat{X} + y\hat{Y} + z\hat{Z})] \\ &= \frac{1}{2} \text{Tr} [\hat{X}\hat{I}] + \frac{1}{2}x \text{Tr} [\hat{X}\hat{X}] + \frac{1}{2}y \text{Tr} [\hat{X}\hat{Y}] + \frac{1}{2}z \text{Tr} [\hat{X}\hat{Z}] \\ &= \frac{1}{2} \cancel{\text{Tr} [\hat{X}]} + \frac{1}{2}x \text{Tr} [\hat{I}] + \frac{1}{2}y \text{const} \cancel{\text{Tr} [\hat{Z}]} + \frac{1}{2}z \text{const} \cancel{\text{Tr} [\hat{Y}]} \\ &= \frac{1}{2}x \text{Tr} [\hat{I}] \\ &= x \end{aligned}$$

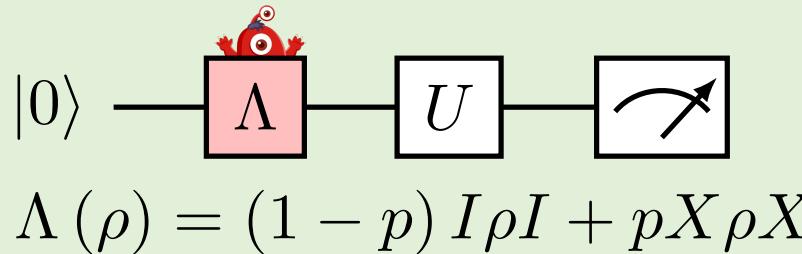
For the initial state ρ_0

trace is linear functional,
expand sum and distribute

* Some constants we don't need to
calculate since the terms are zero



Bit flip channel effects on the Bloch vector



Show the effect of the noise channel on the Bloch vector of the state

$$\rho_0 = \frac{1}{2} (\hat{I} + x\hat{X} + y\hat{Y} + z\hat{Z})$$

Calculate $\rho_1 = \Lambda(\rho_0)$ and its $\langle \hat{I} \rangle, \langle \hat{X} \rangle, \langle \hat{Y} \rangle, \langle \hat{Z} \rangle$ for ρ_0, ρ_1



For the initial state ρ_0

$$\langle \hat{I} \rangle = 1$$

$$\langle \hat{X} \rangle = x$$

$$\langle \hat{Y} \rangle = y$$

$$\langle \hat{Z} \rangle = z$$

For the final state ρ_1

$$\langle \hat{I} \rangle = 1$$

$$\langle \hat{X} \rangle = x$$

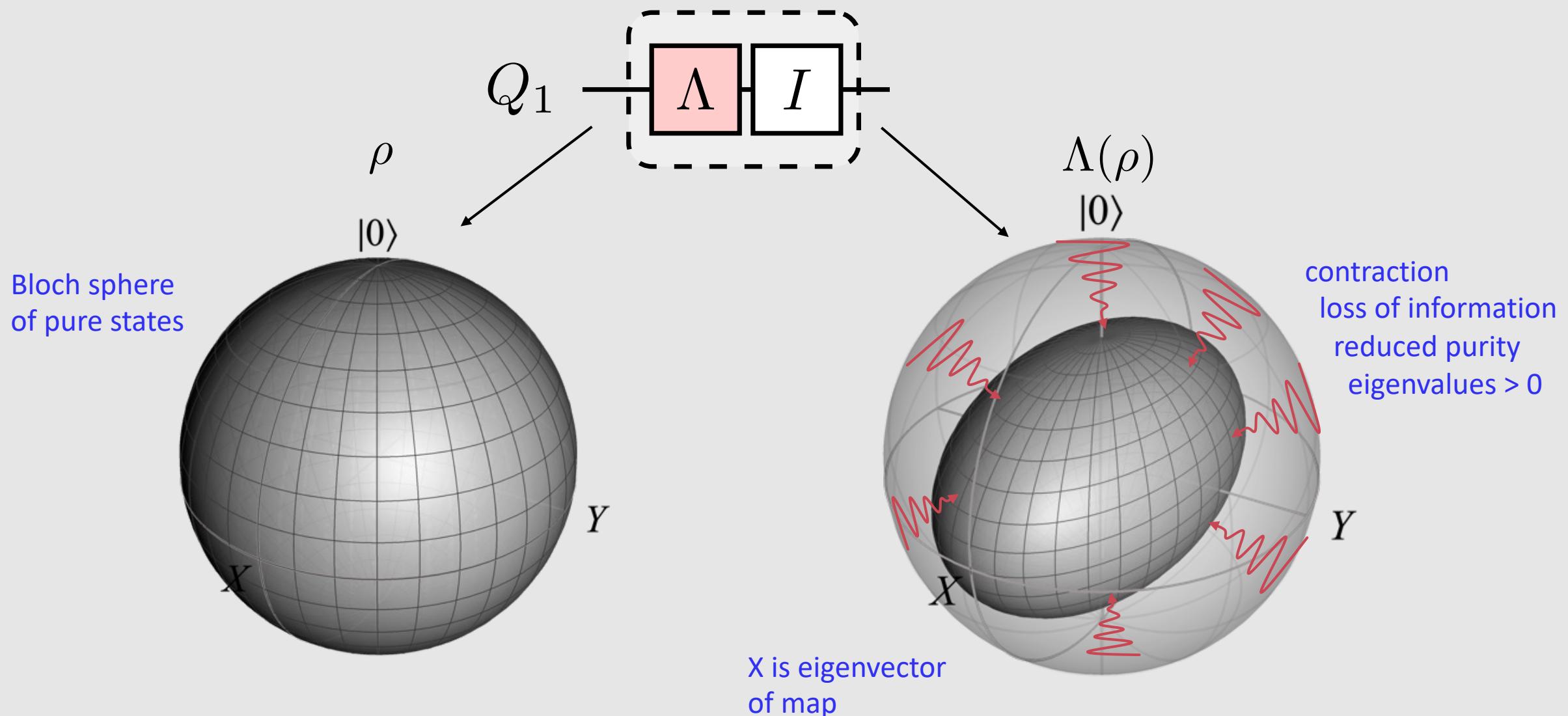
$$\langle \hat{Y} \rangle = (1 - 2p)y$$

$$\langle \hat{Z} \rangle = (1 - 2p)z$$

The noisy damping factor for each Pauli is often called the *Pauli fidelity* for the relevant Pauli component

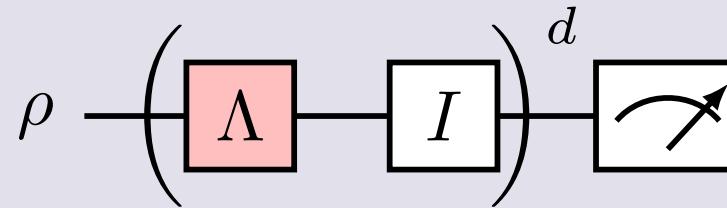


Visual explanation





Effect of repeated noise



For the bit flip channel, find the effect on the Bloch vector of repeating the noise d times with an identity gate as the unitary

$$\Lambda(\rho) = (1 - p) I \rho I + p X \rho X$$



Initial state

$$\rho_0 = \frac{1}{2} (\hat{I} + x \hat{X} + y \hat{Y} + z \hat{Z})$$

$$\rho_1 = \Lambda(\rho_0)$$

⋮

$$\rho_d = \Lambda(\rho_{d-1})$$

Use the result 2 slides back for the effect of the channel on the Bloch components with the recursive relation on the left:

$$\langle \hat{I} \rangle = 1$$

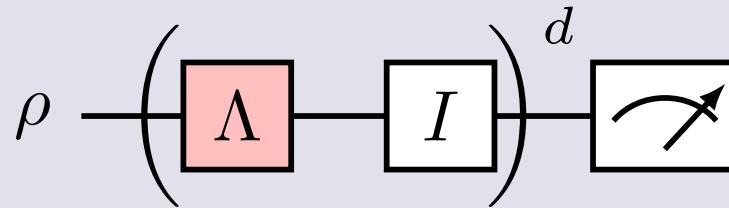
$$\langle \hat{X} \rangle = x$$

$$\langle \hat{Y} \rangle = (1 - 2p) y$$

$$\langle \hat{Z} \rangle = (1 - 2p) z$$



Effect of repeated noise



For the bit flip channel, find the effect on the Bloch vector of repeating the noise d times with an identity gate as the unitary

$$\Lambda(\rho) = (1 - p)I\rho I + pX\rho X$$



It follows that the noise causes the Bloch vector components to shift exponentially at a rate given by the Pauli fidelities of the noise channel, where the fidelity is

$$f = 1 - 2p$$

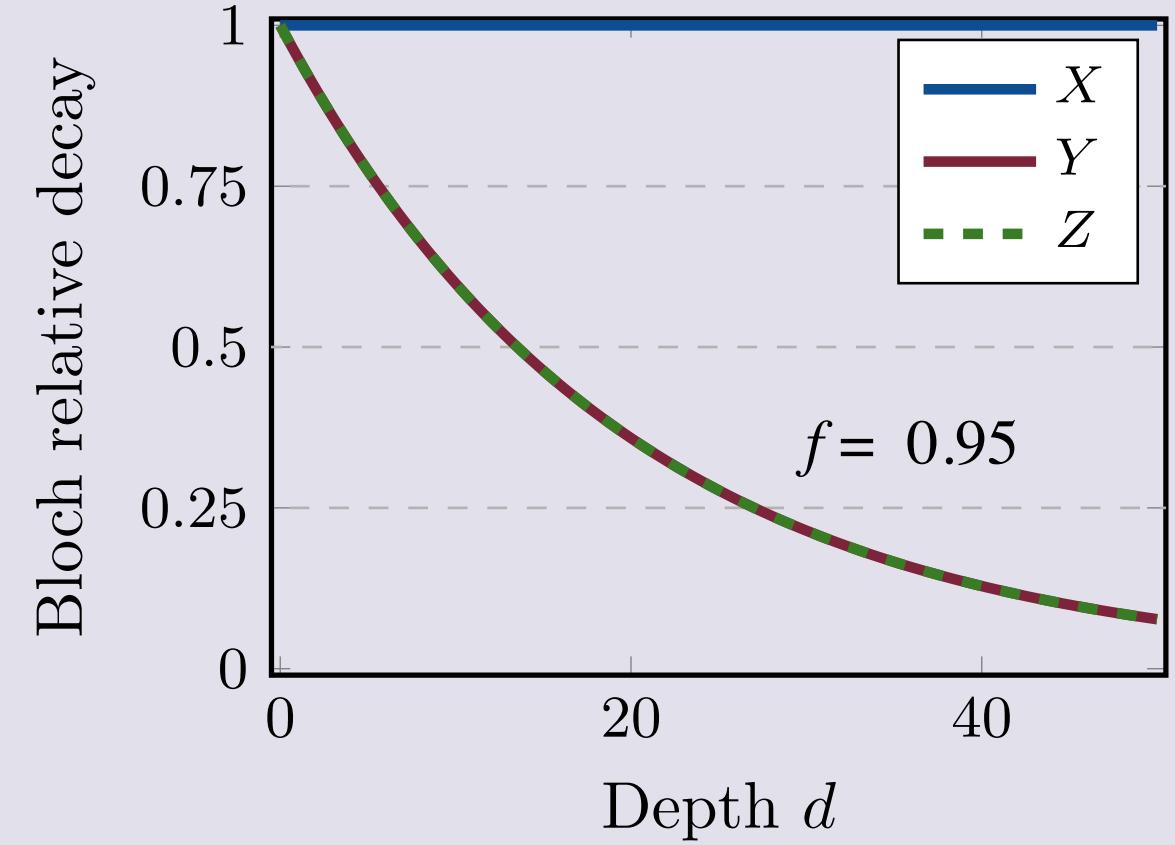
and the Bloch components are

$$\langle \hat{I} \rangle(d) = 1$$

$$\langle \hat{X} \rangle(d) = x$$

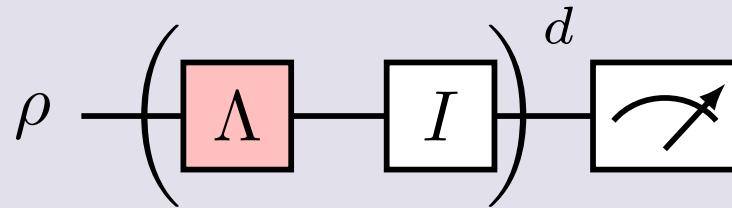
$$\langle \hat{Y} \rangle(d) = f^d y$$

$$\langle \hat{Z} \rangle(d) = f^d z$$





Effect of repeated noise



For the bit flip channel, find the effect on the Bloch vector of repeating the noise d times with an identity gate as the unitary

$$\Lambda(\rho) = (1 - p) I \rho I + p X \rho X$$

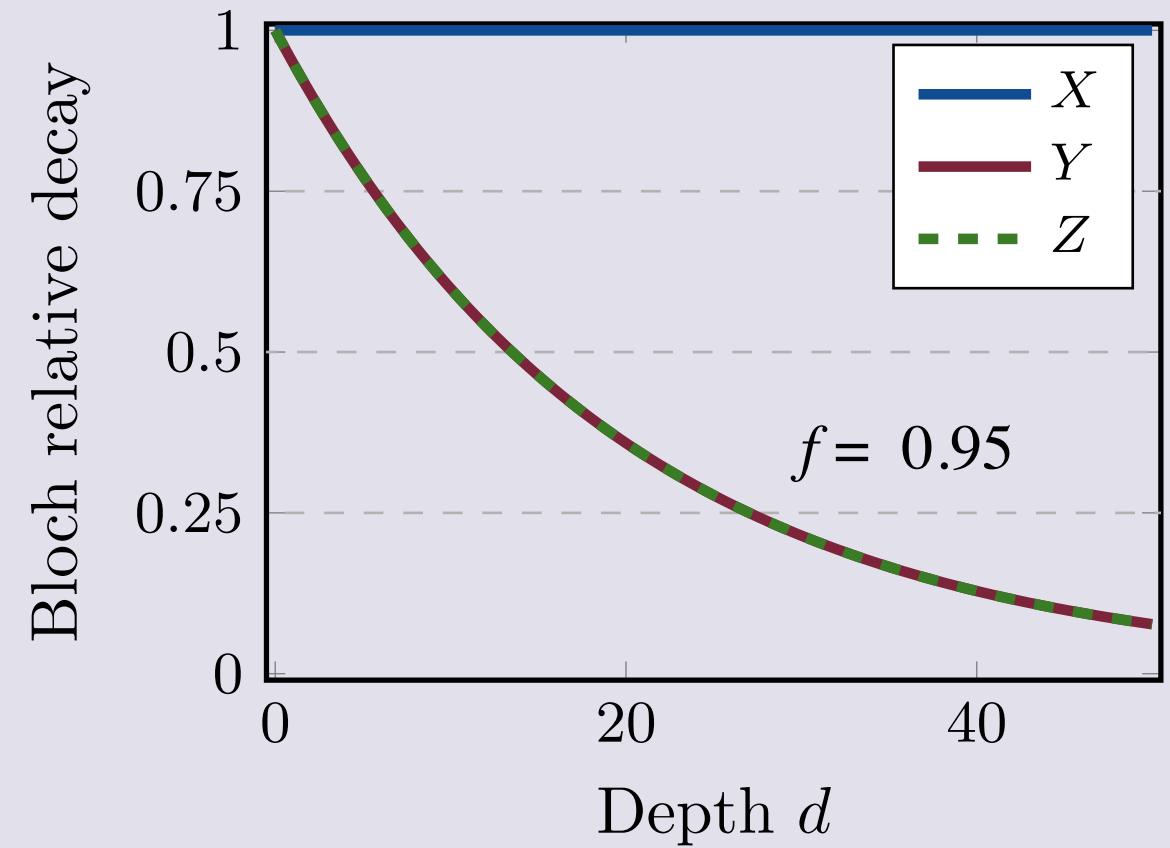


It follows that the noise causes the Bloch vector components to shift exponentially at a rate given by the Pauli fidelities of the noise channel, where the fidelity is

$$f = 1 - 2p$$

Note, to first order the errors add up linearly rather than quadratically as for coherent noise, in other words, slower:

$$\langle \hat{Z} \rangle(d) = (1 - 2p)^d z \approx (1 - 2pd) z + \mathcal{O}(d^2)$$





Hello World example: all the pieces explained

Coherent error

$\varepsilon = 3$ degrees over rotation

Projection noise

800 shots

Readout error

$\varepsilon = p(1|0) = 0.02$

$\nu = p(0|1) = 0.05$

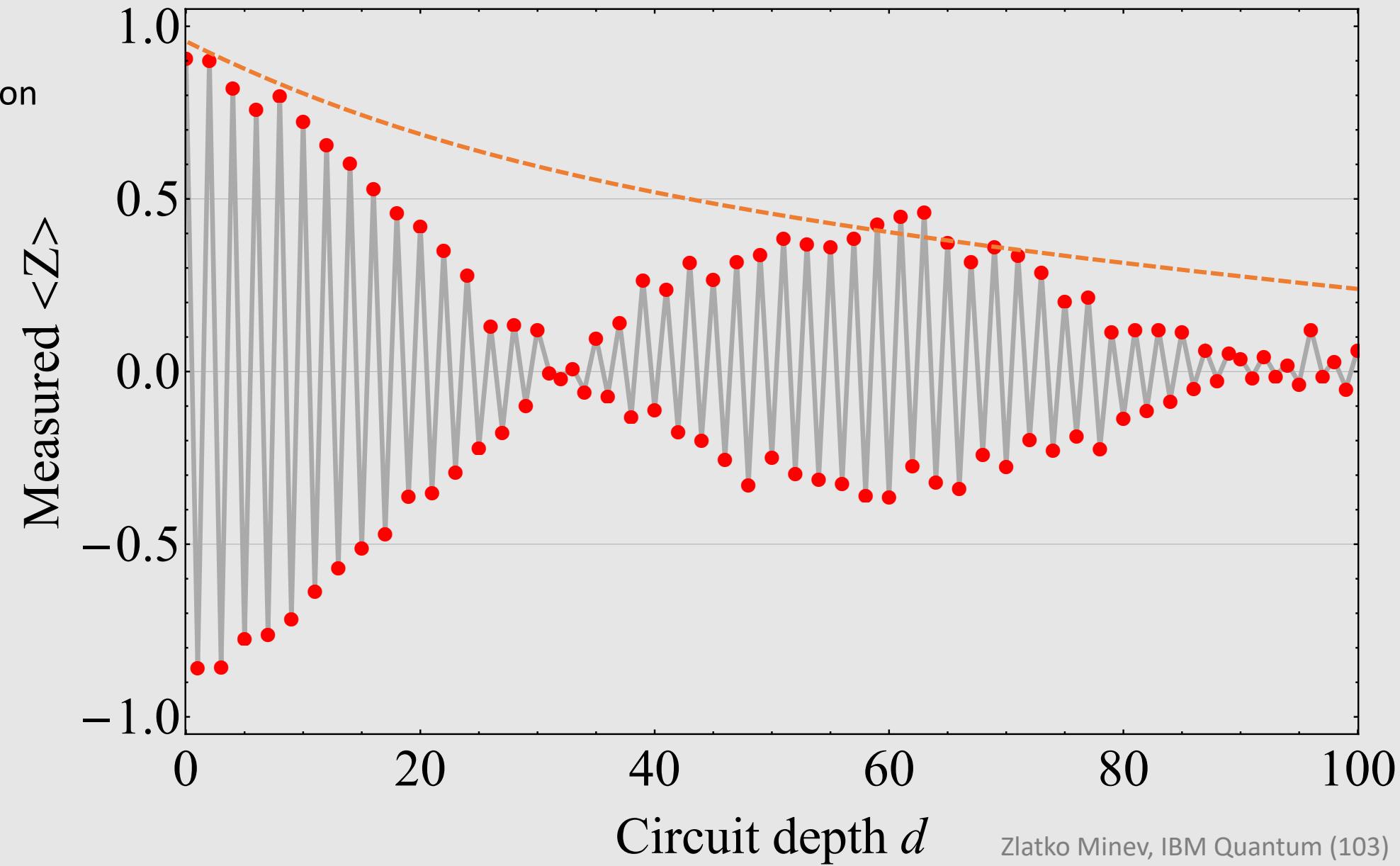
State prep error

$p = 0.025$

Incoherent bit flip noise

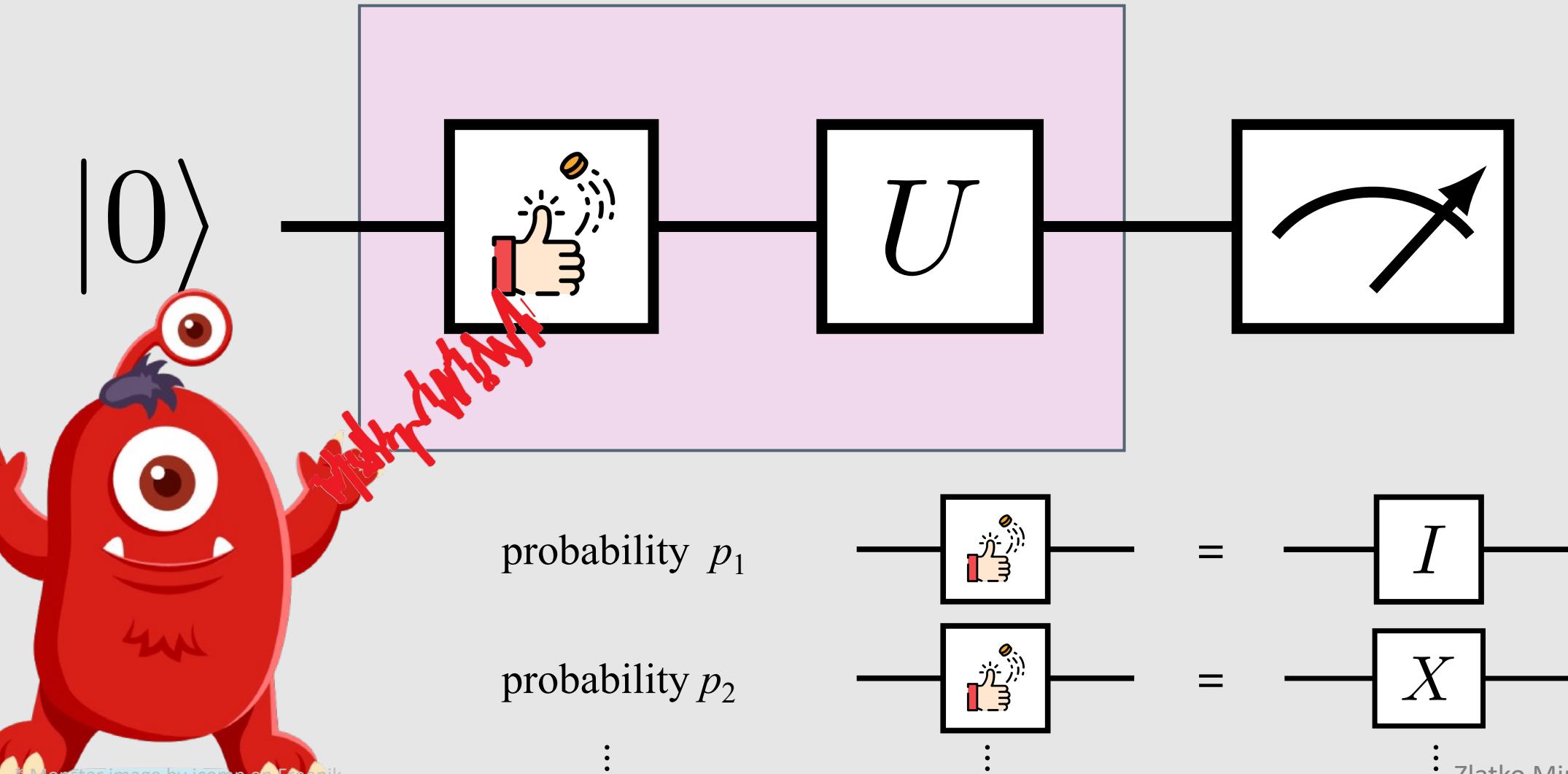
$p = 0.008$

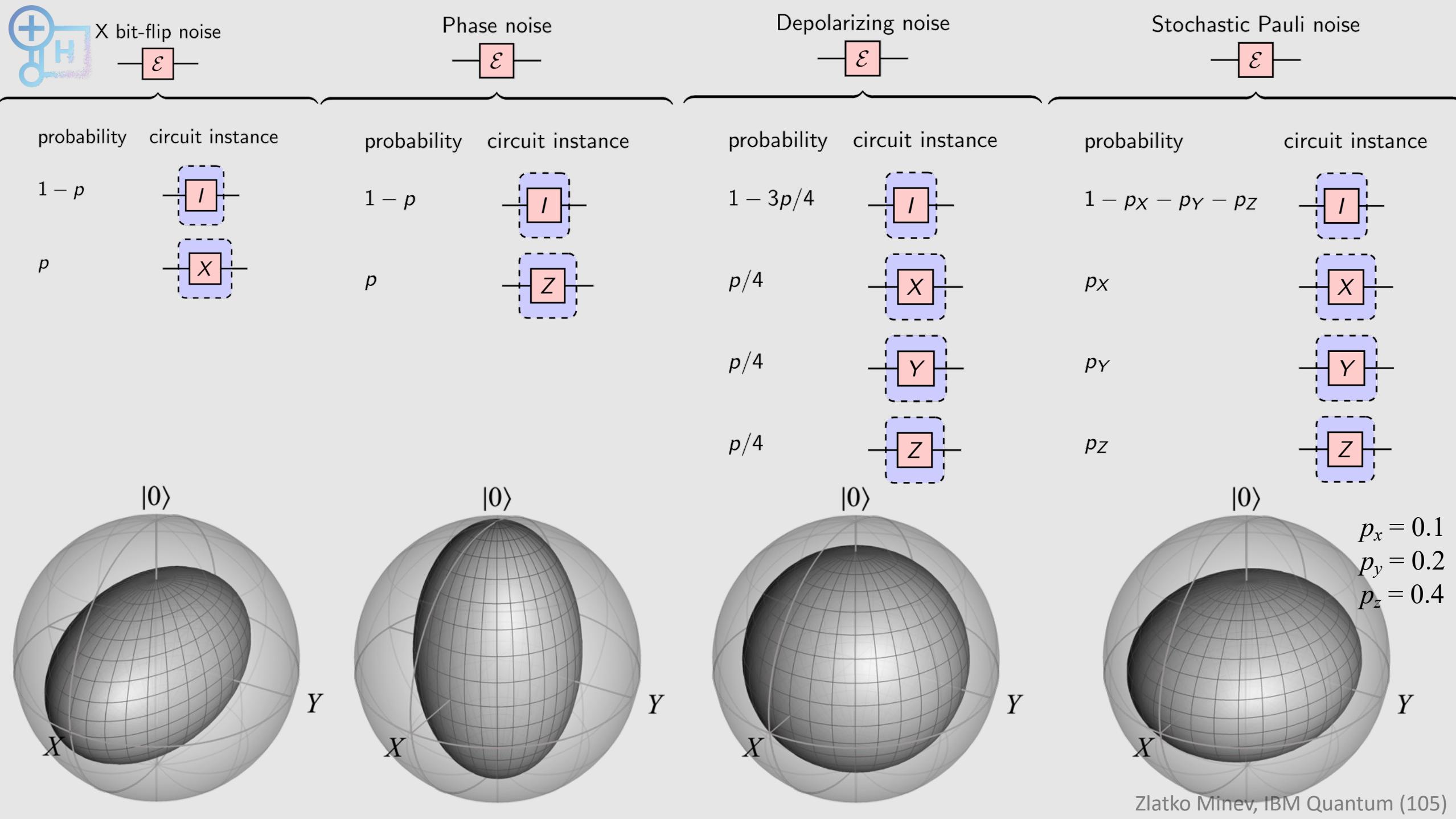
$\Lambda(\rho) = (1 - p) I \rho I + p X \rho X$





Other quantum noise models?







(hard) Exercise on channels



Extra credit, harder problem to challenge yourself

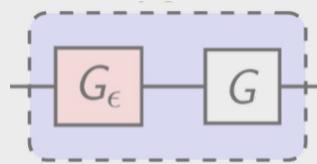
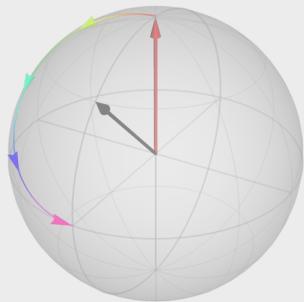
1. Calculate the expectation values of $\langle I \rangle$, $\langle X \rangle$, $\langle Y \rangle$, $\langle Z \rangle$ for the state after the following noise channels acted on a general qubit state
(hint: see bit flip channel example and last slide)
 - a) Phase noise
 - b) Depolarizing noise
 - c) Stochastic Pauli noise
 - d) For (a) – (c) summarize the Pauli fidelities.
2. Repeat the same calculation as (1) for the following channel known as *amplitude damping* and plot the effect on the Bloch sphere

$$\Lambda(\rho) = K_0 \rho K_0^\dagger + K_1 \rho K_1^\dagger, K_0 := \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, K_1 := \begin{pmatrix} 1 & \sqrt{p} \\ 0 & 0 \end{pmatrix}.$$



We now understand noise, how do we overcome it?

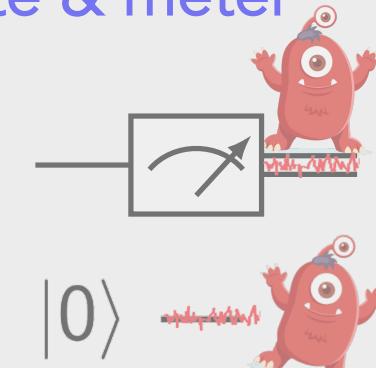
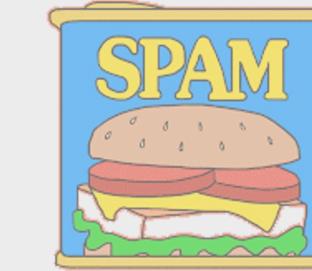
Coherent noise



Measurements in quantum Projection noise

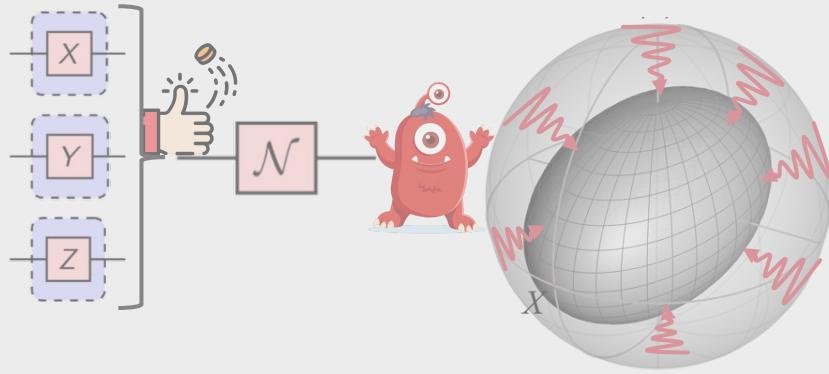


SPAM: State & meter



$|0\rangle$

Incoherent noise

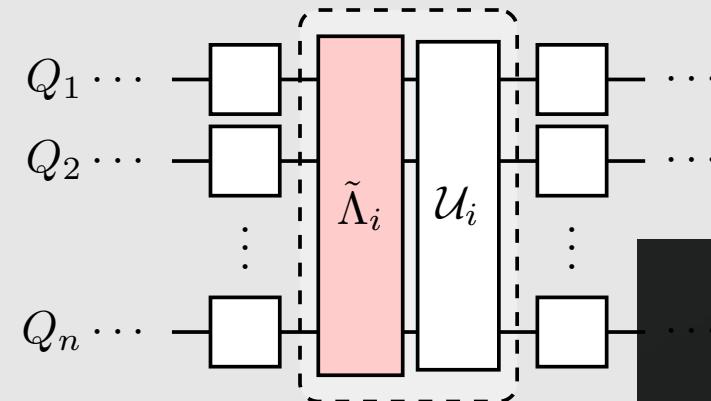


coin-toss icon: Good Ware, flaticon; spam: make it move;
road based on: freepik; Monster image by jcomp on Freepik

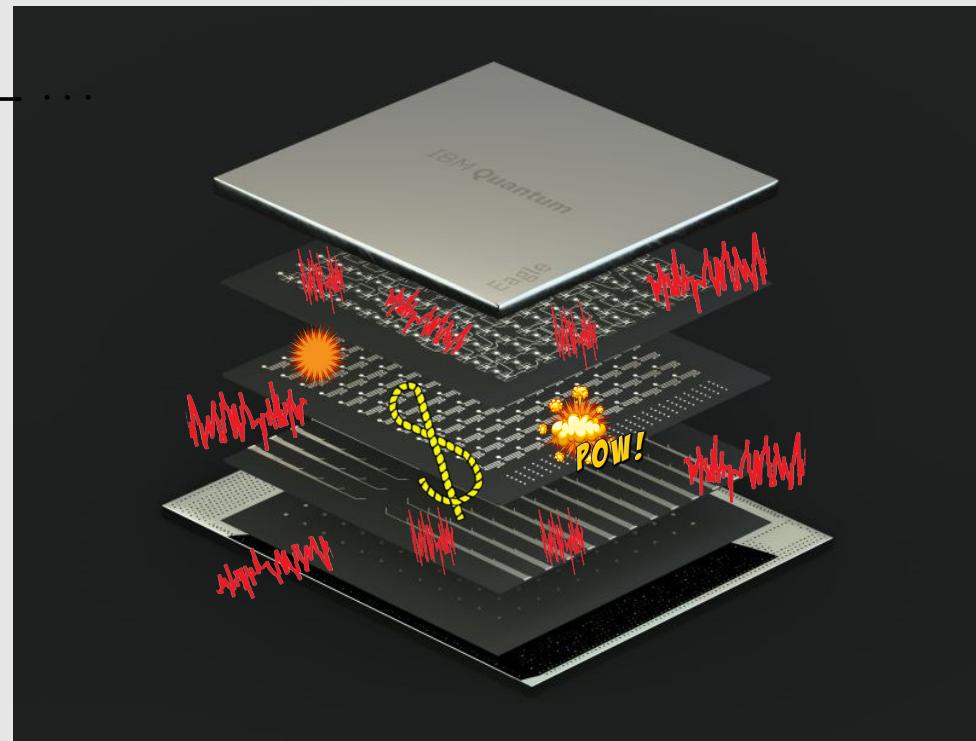




Is it possible to understand & overcome quantum noise with accuracy, efficiency, and scalability?



Energy relaxation T_1
Dephasing T_2
Coherent errors ZZ
Classical crosstalk
Quantum crosstalk
State preparation error
Measurement correlated errors
...



Control errors
Photon shot noise
1/f charge noise
1/f flux noise
Nonequilibrium quasiparticles
Leakage
Cosmic rays
...



How to deal with errors due to noise?

Monitor

Error occurs
Error detect



Quantum error correction

Shor, PRA (1995), ...

Monitor

Error anticipated
Tell signal detected



Catch and reverse

Minev, Nature (2019), ...

No monitor

Error occurs
Error undetected

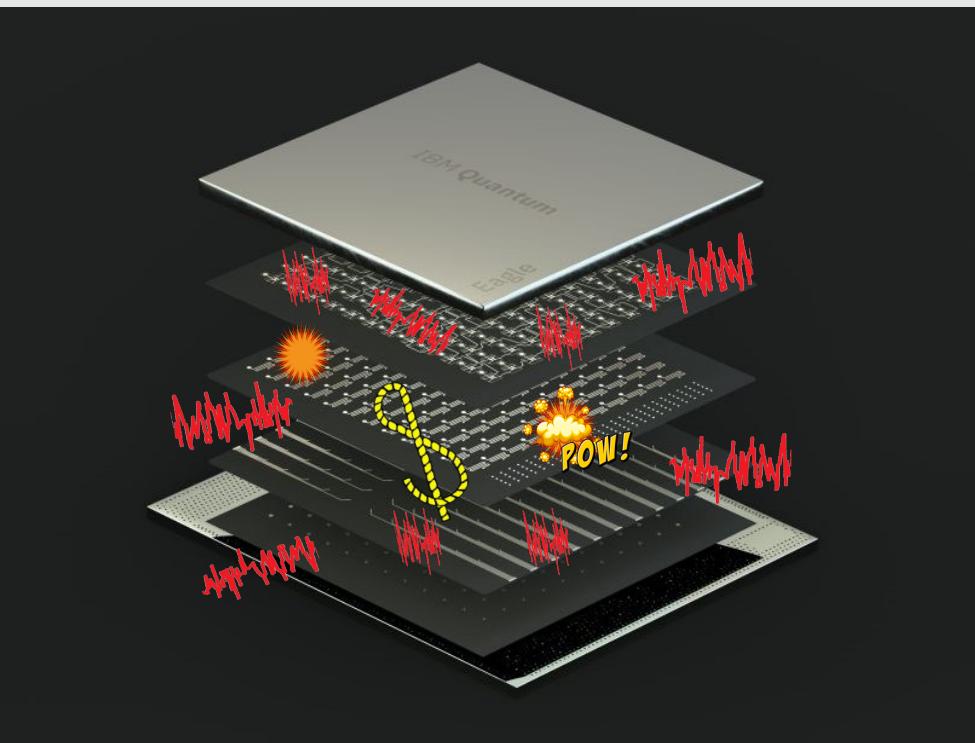


Error mitigation

... subject of today



Error mitigation and error correction



Error mitigation: working with what you have

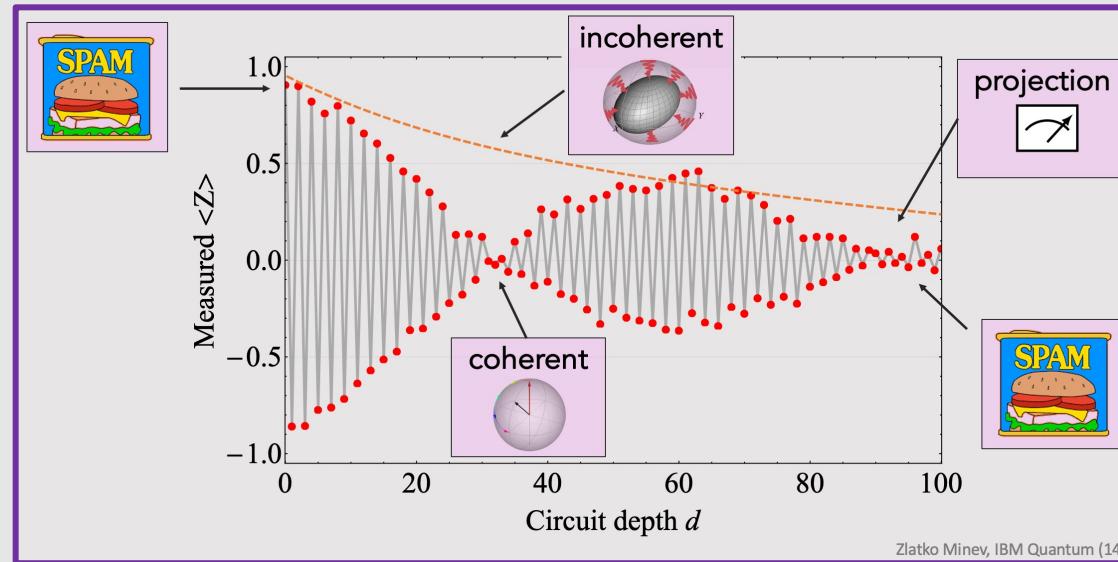
- **benefit** suppress errors on classical results (expectation values)
- **q-cost** no extra qubits or hardware resources needed
- **c-cost** trades classical resources (post-processing) for lower error
- **limitation** bad asymptotic scaling: high number of samples & circuits

Error correction: protecting quantum information

- **benefit** suppress & correct errors to arbitrarily small level
- **q-cost** very large qubit and hardware overhead
- **c-cost** decoding and encoding can be classically costly
- **challenge** requires fault-tolerant operations and readout



Error mitigation landscape



more speed



more information, accuracy



Zlatko Minev, IBM Quantum (111)



Latest: Qiskit Quantum Seminar YouTube Series

140+ years in 120+ seminars: Quantum research community connects on YouTube



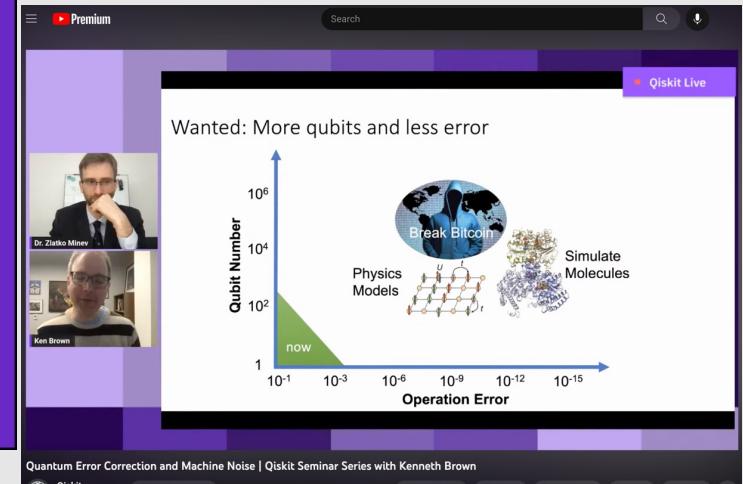
qiskit.org/events/seminar-series

Live every Friday

120+ seminar

244,461+ hours of total viewer
watch time

~ 1 million viewers (people who
watched the video)

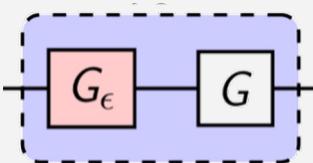
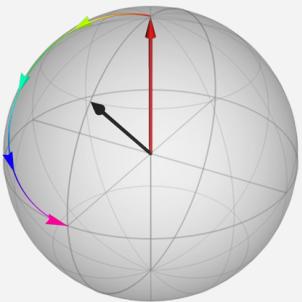


Zlatko Minev, IBM Quantum (112)

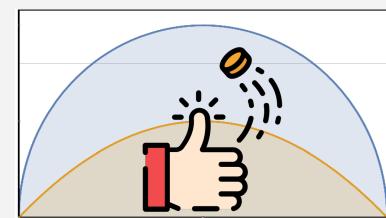


We come the end of our journey today

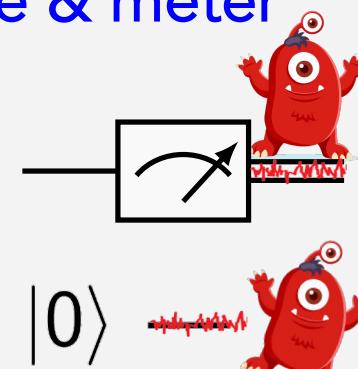
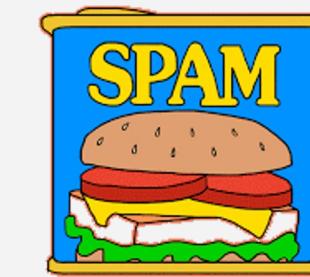
Coherent noise



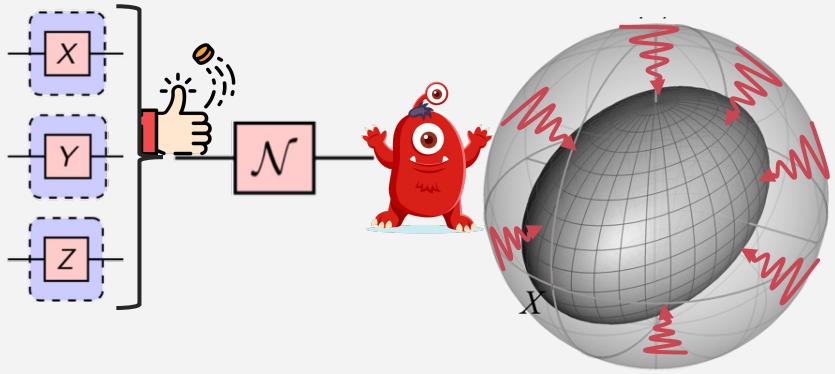
Measurements in quantum Projection noise



SPAM: State & meter



Incoherent noise



Bonus content

Coherent ZZ noise

...



coin-toss icon: Good Ware, flaticon; spam: make it move;
road based on: freepik; Monster image by jcomp on Freepik



The important thing is not to stop questioning.
Curiosity has its own reason for existence.

One cannot help but be in awe when they
contemplate the mysteries of eternity, of life, of the
marvelous structure of reality.

It is enough if one tries merely to comprehend a
little of this mystery each day.

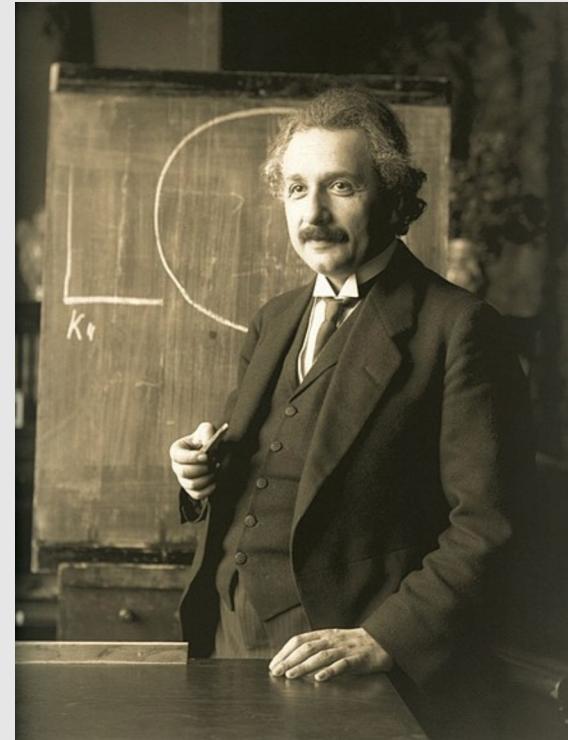


Photo: F. Schmutz

Albert Einstein



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IBM Quantum



Introduction to Quantum Noise

by Zlatko K. Minev

Thank you!

Next:
Lab work with Qiskit
Run experiments on real devices



Check out references, problems given in
the lecture, dangerous bends

Stay in touch and have fun!

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