Geometry
Problem booklet

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1 Week 7: Products of vectors

1.1 The triple scalar product

The *triple scalar product* $(\vec{a}, \vec{b}, \vec{c})$ of the vectors $\vec{a}, \vec{b}, \vec{c}$ is the real number $(\vec{a} \times \vec{b}) \cdot \vec{c}$.

Proposition 1.1. *If $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis and $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$, $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$ şi $\vec{c} = c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k}$ then*

$$(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{1.1}$$

Corollary 1.2. 1. *The free vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent (collinear) if and only if $(\vec{a}, \vec{b}, \vec{c}) = 0$*

2. *The free vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly independent (noncollinear) if and only if $(\vec{a}, \vec{b}, \vec{c}) \neq 0$*

3. *The free vectors $\vec{a}, \vec{b}, \vec{c}$ form a basis of the space $\mathcal{V}$ if and only if $(\vec{a}, \vec{b}, \vec{c}) \neq 0$.*

4. *The correspondence $F : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \to \mathbb{R}$, $F(\vec{a}, \vec{b}, \vec{c}) = (\vec{a}, \vec{b}, \vec{c})$ is a skew-symmetric, i..e*

$$\begin{array}{l} (\alpha \vec{a} + \alpha' \vec{a}\,', \vec{b}, \vec{c}) = \alpha(\vec{a}, \vec{b}, \vec{c}) + \alpha'(\vec{a}\,', \vec{b}, \vec{c}) \\ (\vec{a}, \beta \vec{b} + \beta' \vec{b}\,', \vec{c}) = \beta(\vec{a}, \vec{b}, \vec{c}) + \beta'(\vec{a}, \vec{b}', \vec{c}) \\ (\vec{a}, \vec{b}, \gamma \vec{c} + \gamma' \vec{c}\,') = \gamma(\vec{a}, \vec{b}, \vec{c}) + \gamma'(\vec{a}, \vec{b}, \vec{c}\,') \end{array} \tag{1.2}$$

$\forall\, \alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \mathbb{R}$, $\forall\ \vec{a}, \vec{b}, \vec{c}, \vec{a}\,', \vec{b}\,', \vec{c}\,' \in \mathcal{V}$ *şi*

$$(\vec{a}_1, \vec{a}_2, \vec{a}_3) = sgn(\sigma)(\vec{a}_{\sigma(1)}, \vec{a}_{\sigma(2)}, \vec{a}_{\sigma(3)}),\ \ \forall\ \vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathcal{V} \text{ şi } \forall\, \sigma \in S_3 \tag{1.3}$$

Remark 1.3. One can rewrite the relations (1.3) as follows:

$$\begin{array}{rcl} (\vec{a}_1, \vec{a}_2, \vec{a}_3) & = & (\vec{a}_2, \vec{a}_3, \vec{a}_1) = (\vec{a}_3, \vec{a}_1, \vec{a}_2) \\ & = & -(\vec{a}_2, \vec{a}_1, \vec{a}_3) = (\vec{a}_1, \vec{a}_3, \vec{a}_2) = -(\vec{a}_3, \vec{a}_2, \vec{a}_1), \end{array}$$

$\forall\ \vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathcal{V}$

Corollary 1.4. 1. $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})\ \forall\ \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$.

2. *For every $\vec{a}, \vec{b}, \vec{c}, \vec{d} \in \mathcal{V}$ the* Laplace formula

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

holds.

Proof. 1. $(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{a}, \vec{b}, \vec{c}) = (\vec{b}, \vec{c}, \vec{a}) = (\vec{b} \times \vec{c})\ \vec{a} = \vec{a} \cdot (\vec{b} \times \vec{c})$.

2. Indeed, we have successively:

$$\begin{aligned}(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) &= (\vec{a},\vec{b},\vec{c}\times\vec{d}) = (\vec{c}\times\vec{d},\vec{a},\vec{b}) = [(\vec{c}\times\vec{d})\times\vec{a}]\cdot\vec{b} \\ &= (\vec{a}\cdot\vec{c})\,\vec{d} - (\vec{a}\cdot\vec{d})\,\vec{c}]\cdot\vec{b} = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c}] \\ &= \begin{vmatrix}\vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d}\\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d}\end{vmatrix}.\end{aligned}$$

□

Definition 1.5. The basis $[\vec{a},\vec{b},\vec{c}]$ of the space $\mathcal{V}$ is said to be *directe* if $(\vec{a},\vec{b},\vec{c}) > 0$. If, on the contrary, $(\vec{a},\vec{b},\vec{c}) < 0$, we say that the basis $[\vec{a},\vec{b},\vec{c}]$ is *inverse*

Definition 1.6. The *oriented volume* of the parallelipiped constructed on the noncoplanar vectors $\vec{a},\vec{b},\vec{c}$ is $\varepsilon\cdot V$, where V is the volume of this parallelepiped and $\varepsilon = +1$ or -1 insomuch as the basis $[\vec{a},\vec{b},\vec{c}]$ is directe or inverse respectively.

Propoziţia 1.7. *The triple scalar product $(\vec{a},\vec{b},\vec{c})$ of the noncoplanar vectors $\vec{a},\vec{b},\vec{c}$ equals the oriented volume of the parallelepiped constructed on these vectors.*

1.2 Applications of the triple scalar product

1.2.1 The distance between two straight lines

If $d_1,\ d_2$ are two straight lines, then the distance between them, denoted by $\delta(d_1,d_2)$, is being defined as

$$\min\{||\overrightarrow{M_1M_2}|| \mid M_1\in d_1,\ M_2\in d_2\}.$$

1. If $d_1\cap d_2\neq\emptyset$, then $\delta(d_1,d_2)=0$.
2. If $d_1||d_2$, then $\delta(d_1,d_2) = ||\overrightarrow{MN}||$ where $\{M\} = d\cap d_1,\ \ \{N\} = d\cap d_2$ and d is a straight line perpendicular to the lines d_1 and d_2. Obviously $||\overrightarrow{MN}||$ is independent on the choice of the line d.
3. We now assume that the straight lines d_1, d_2 are noncoplanar (skew lines). In this case there exits a unique straight line d such that $d\perp d_1,d_2$ and $d\cap d_1=\{M_1\},\ d\cap d_2 = \{M_2\}$. The straight line d is called the *common perpendicular* of the lines d_1, d_2 and obviously $\delta(d_1,d_2) = ||\overrightarrow{M_1M_2}||$.

Assume that the straight lines d_1, d_2 are given by their points $A_1(x_1,y_1,z_1)$, $A_2(x_2,y_2,z_2)$ and their vectors şi au vectorii directori $\vec{d}_1\ (p_1,q_1,r_1)\ \vec{d}_2\ (p_2,q_2,r_2)$, that is, thei equations are

$$d_1 : \frac{x-x_1}{p_1} = \frac{y-y_1}{q_1} = \frac{z-z_1}{r_1}$$

$$d_2 : \frac{x-x_2}{p_2} = \frac{y-y_2}{q_2} = \frac{z-z_2}{r_2}.$$

The common perpendicular of the lines d_1,d_2 is the intersection line between the plane containing the line d_1 which is parallel to the vector $\vec{d}_1\times\vec{d}_2$, and the plane containing the line d_2 which is parallel to $\overrightarrow{d}_1 \times \overrightarrow{d}_2$. Since

$$\overrightarrow{d}_1 \times \overrightarrow{d}_2 = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} \overrightarrow{i} + \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} \overrightarrow{j} + \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \overrightarrow{k}$$

it follows that the equations of the common perpendicular are

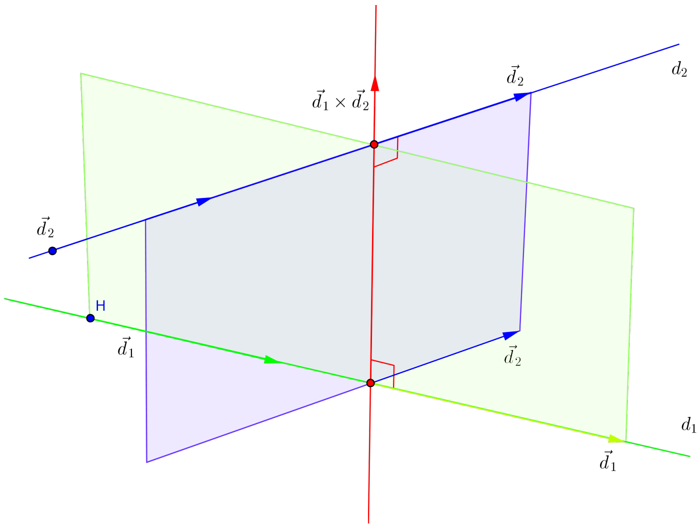


Figure 1: Prependiculara comună a dreptelor d_1 şi d_2

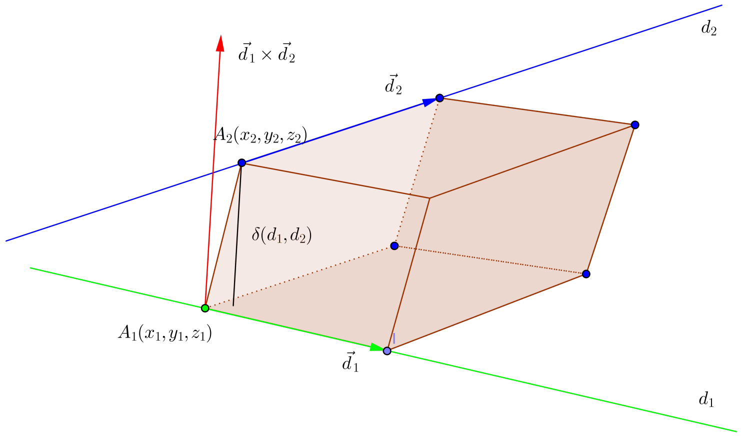
$$\begin{cases} \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ p_1 & q_1 & r_1 \\ \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} & \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} & \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \end{vmatrix} = 0 \\ \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ p_2 & q_2 & r_2 \\ \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} & \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} & \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \end{vmatrix} = 0. \end{cases} \tag{1.4}$$

The distance between the straight lines d_1, d_2 can be also regarded as the height of the parallelogram constructed on the vectors $\overrightarrow{d}_1, \overrightarrow{d}_2, \overrightarrow{d}_1 \times \overrightarrow{d}_2$. Thus

$$\delta(d_1, d_2) = \frac{|(\overrightarrow{A_1A_2}, \overrightarrow{d}_1, \overrightarrow{d}_2)|}{||\overrightarrow{d}_1 \times \overrightarrow{d}_2||}. \tag{1.5}$$

Therefore we obtain

$$\delta(d_1, d_2) = \frac{\left| \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} \right|}{\sqrt{\begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix}^2 + \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix}^2 + \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}^2}} \tag{1.6}$$



1.2.2 The coplanarity condition of two straight lines

Using the notations of the previous section, observe that the straight lines d_1, d_2 are coplanar if and only if the vectors $\overrightarrow{A_1A_2}, d_1, d_2$ are linearly dependent (coplanar), or equivalently $(\overrightarrow{A_1A_2}, \vec{d}_1, \vec{d}_2) = 0$. Consequently the stright lines d_1, d_2 are coplanar if and only if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0 \tag{1.7}$$

1.3 Problems

1. Compute the distance between the lines

$$(d_1) \begin{cases} 3x - 2y = 5 \\ x - 2y - 1 = 0 \end{cases} \quad (d_2) \begin{cases} 4x - 3y + 4 = 0 \\ x - z + 2 = 0. \end{cases}$$

2. Show the following identities:

   (a) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a}, \vec{c}, \vec{d})\, \vec{b} - (\vec{b}, \vec{c}, \vec{d})\, \vec{a} = (\vec{a}, \vec{b}, \vec{d})\, \vec{c} - (\vec{a}, \vec{b}, \vec{c})\, \vec{d}$.

   (b) $(\vec{u} \times \vec{v}, \vec{v} \times \vec{w}, \vec{w} \times \vec{u}) = (\vec{u}, \vec{v}, \vec{w})^2$.

3. The *reciprocal vectors* of the noncoplanar vectors $\vec{u}, \vec{v}, \vec{w}$ are defined by

$$\vec{u}' = \frac{\vec{v} \times \vec{w}}{(\vec{u}, \vec{v}, \vec{w})}, \quad \vec{v}' = \frac{\vec{w} \times \vec{u}}{(\vec{u}, \vec{v}, \vec{w})}, \quad \vec{w}' = \frac{\vec{u} \times \vec{v}}{(\vec{u}, \vec{v}, \vec{w})}.$$

   Show that:

   (a)

$$\begin{aligned} \vec{a} &= (\vec{a} \cdot \vec{u}')\, \vec{u} + (\vec{a} \cdot \vec{v}')\, \vec{v} + (\vec{a} \cdot \vec{w}')\, \vec{w} \\ &= \frac{(\vec{a}, \vec{v}, \vec{w})}{(\vec{u}, \vec{v}, \vec{w})}\, \vec{u} + \frac{(\vec{u}, \vec{a}, \vec{w})}{(\vec{u}, \vec{v}, \vec{w})}\, \vec{v} + \frac{(\vec{u}, \vec{v}, \vec{a})}{(\vec{u}, \vec{v}, \vec{w})}\, \vec{w}. \end{aligned}$$

(b) the reciprocal vectors of $\vec{u}'$, $\vec{v}'$, $\vec{w}'$ are the vectors $\vec{u}, \vec{v}, \vec{w}$.

4. Find the value of the parameter α for which the pencil of planes through the straight line AB has a common plane with the pencil of planes through the straight line CD, where $A(1, 2\alpha, \alpha)$, $B(3, 2, 1)$, $C(-\alpha, 0, \alpha)$ and $D(-1, 3, -3)$.

5. Find the value of the parameter λ for which the straight lines

$$(d_1)\ \frac{x-1}{3} = \frac{y+2}{-2} = \frac{z}{1},\ (d_2)\ \frac{x+1}{4} = \frac{y-3}{1} = \frac{z}{\lambda}$$

are coplanar. Find the coordinates of their intersection point in that case.

2 Week 8: Conics

2.1 The Ellipse

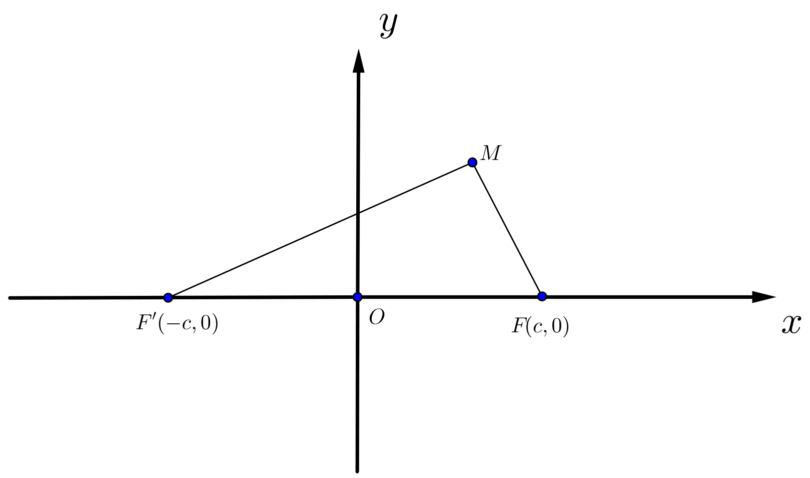
Definition 2.1 2.1. *An* ellipse *is the locus of points in a plane, the sum of whose distances from two fixed points, say F and F', called* foci *is constant.*

The distance between the two fixed points is called the *focal distance*

Let F and F' be the two foci of an ellipse and let $|FF'| = 2c$ be the focal distance. Suppose that the constant in the definition of the ellipse is $2a$. If M is an arbitrary point of the ellipse, it must verify the condition

$$|MF| + |MF'| = 2a.$$

One may chose a Cartesian system of coordinates centered at the midpoint of the segment $[F'F]$, so that $F(c, 0)$ and $F'(-c, 0)$.



Remark 2.2. *In $\Delta MFF'$ the following inequality $|MF| + |MF'| > |FF'|$ holds. Hence $2a > 2c$. Thus, the constants a and c must verify $a > c$.*

Thus, for the generic point $M(x,y)$ of the ellipse we have succesively:

$$\begin{aligned}
|MF|+|MF'|=2a \quad \Leftrightarrow \quad & \sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a \\
& \sqrt{(x-c)^2+y^2}=2a-\sqrt{(x+c)^2+y^2} \\
& x^2-2cx+c^2+y^2=4a^2-4a\sqrt{(x+c)^2+y^2}+(x+c)^2+y^2 \\
& a\sqrt{(x+c)^2+y^2}=cx+a^2 \\
& a^2(x^2+2xc+c^2)+a^2y^2=c^2x^2+2a^2cx+a^2 \\
& (a^2-c^2)x^2+a^2y^2-a^2(a^2-c^2)=0.
\end{aligned}$$

Denote a^2-c^2 by b^2, as $(a>c)$. Thus $b^2x^2+a^2y^2-a^2b^2=0$, i.e.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0 \tag{2.1}$$

Remark 2.3. *The equation (2.1) is equivalent to*

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}; \qquad x=\pm\frac{a}{b}\sqrt{b^2-y^2},$$

which means that the ellipse is symmetric with respect to both the x and the y axes. In fact, the line FF', determined by the foci of the ellipse, and the perpendicular line on the midpoint of the segment $[FF']$ are axes of symmetry for the ellipse. Their intersection point, which is the midpoint of $[FF']$, is the center of symmetry of the ellipse, or, simply, its center.

Remark 2.4. *In order to sketch the graph of the ellipse, observe that it is enough to represent the function*

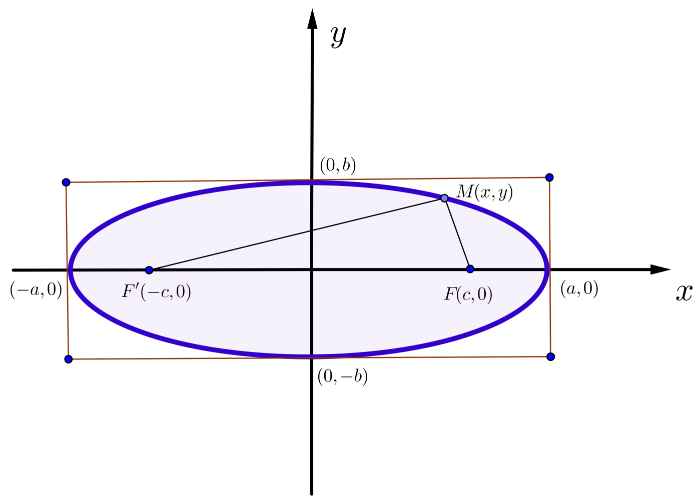
$$f:[-a,a]\to\mathbb{R}, \qquad f(x)=\frac{b}{a}\sqrt{a^2-x^2},$$

and to complete the ellipse by symmetry with respect to the x-axis.

One has

$$f'(x)=-\frac{b}{a}\frac{x}{\sqrt{a^2-x^2}}, \qquad f''(x)=-\frac{ab}{(a^2-x^2)\sqrt{a^2-x^2}}.$$

x	$-a$		0		a
$f'(x)$	$\mid$	$+\ +\ +$	0	$-\ -\ -$	$\mid$
$f(x)$	0	$\nearrow$	b	$\searrow$	0
$f''(x)$	$\mid$	$-\ -\ -$	$-$	$-\ -\ -$	$\mid$



2.2 The Hyperbola

Definiţia 2.5. *The* hyperbola *is defined as the geometric locus of the points in the plane, whose absolute value of the difference of their distances to two fixed points, say F and F' is constant.*

The two fixed points are called the *foci* of the hyperbola, and the distance $|FF'| = 2c$ between the foci is the *focal distance*.

Suppose that the constant in the definition is $2a$. If $M(x,y)$ is an arbitrary point of the hyperbola, then

$$||MF| - |MF'|| = 2a.$$

Choose a Cartesian system of coordinates, having the origin at the midpoint of the segment $[FF']$ and such that $F(c,0)$, $F'(-c,0)$.

Remark 2.6. *In the triangle $\Delta MFF'$, $||MF| - |MF'|| < |FF'|$, so that $a < c$.*

Let us determine the equation of a hyperbola. By using the definition we get $|MF| - |MF'| = \pm 2a$, namely

$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = \pm 2a,$$

or, equivalently

$$\sqrt{(x-c)^2 + y^2} = \pm 2a + \sqrt{(x+c)^2 + y^2}.$$

We therefore have successively

$$\begin{aligned}
&x^2 - 2cx + c^2 + y^2 = 4a^2 \pm 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \\
&cx + a^2 = \pm a\sqrt{(x+c)^2 + y^2} \\
&c^2x^2 + 2a^2cx + a^4 = a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2 \\
&(c^2 - a^2)x^2 - a^2y^2 - a^2(c^2 - a^2) = 0.
\end{aligned}$$

By using the notation $c^2 - a^2 = b^2$ ($c > a$) we obtain the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0. \tag{2.2}$$

The equation (2.2) is equivalent to

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}; \qquad x = \pm\frac{a}{b}\sqrt{y^2 + b^2}.$$

Therefore, the coordinate axes are axes of symmetry of the hyperbola and the origin is a center of symmetry equally called the *center of the hyperbola*.

Remark 2.7. *To sketch the graph of the hyperbola, is it enough to represent the function*

$$f : (-\infty, -a] \cup [a, \infty) \to \mathbb{R}, \qquad f(x) = \frac{b}{a}\sqrt{x^2 - a^2},$$

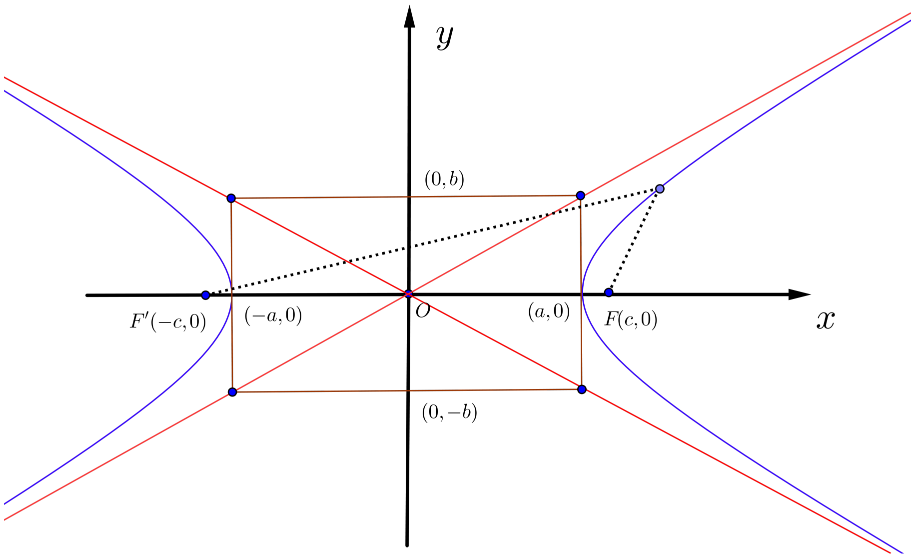
by taking into account that the hyperbola is symmetric with respect to the x-axis.

Since $\lim\limits_{x\to\infty} \frac{f(x)}{x} = \frac{b}{a}$ and $\lim\limits_{x\to-\infty} \frac{f(x)}{x} = -\frac{b}{a}$, it follows that $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ are asymptotes of f.

One has, also

$$f'(x) = \frac{b}{a}\frac{x}{\sqrt{x^2 - a^2}}, \qquad f''(x) = -\frac{ab}{(x^2 - a^2)\sqrt{x^2 - a^2}}.$$

x	$-\infty$		$-a$		a		∞
$f'(x)$	$-$	$- - -$	$\vert$	$/ / /$	$\vert$	$+ + +$	$+$
$f(x)$	∞	$\searrow$	$0\vert$	$/ / /$	$\vert 0$	$\nearrow$	∞
$f''(x)$	$-$	$- - -$	$\vert$	$/ / /$	$\vert$	$- - -$	$-$



2.3 The Parabola

Definiţia 2.8. *The* parabola *is a plane curve defined to be the geometric locus of the points in the plane, whose distance to a fixed line d is equal to its distance to a fixed point F.*

The line d is the *director line* and the point F is the *focus*. The distance between the focus and the director line is denoted by p and represents the *parameter* of the parabola.

Consider a Cartesian system of coordinates xOy, in which $F\left(\frac{p}{2},0\right)$ and $d : x = -\frac{p}{2}$. If $M(x,y)$ is an arbitrary point of the parabola, then it verifies

$$|MN| = |MF|,$$

where N is the orthogonal projection of M on Oy.

Thus, the coordinates of a point of the parabola verify

$$\sqrt{\left(x+\frac{p}{2}\right)^2+0} = \sqrt{\left(x-\frac{p}{2}\right)^2+y^2}$$

$$\left(x+\frac{p}{2}\right)^2 = \left(x-\frac{p}{2}\right)^2 = y^2$$

$$x^2+px+\frac{p^2}{4} = x^2-px+\frac{p^2}{4}+y^2,$$

and the equation of the parabola is

$$y^2 = 2px. \tag{2.3}$$

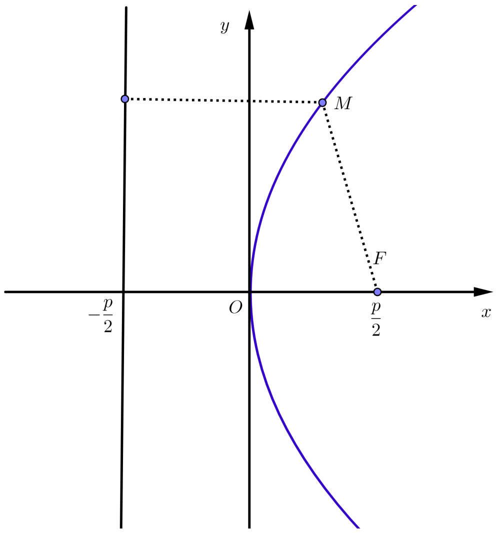
Remark 2.9. *The equation (2.3) is equivalent to $y = \pm\sqrt{2px}$, so that the parabola is symmetric with respect to the x-axis.*

Representing the graph of the function $f : [0, \infty) \to [0, \infty)$ and using the symmetry of the curve with respect to he x-axis, one obtains the graph of the parabola.

One has

$$f'(x) = \frac{p}{\sqrt{2px_0}}; \qquad f''(x) = -\frac{p}{2x\sqrt{2x}}.$$

x	0		∞
$f'(x)$	$\vert$	$+ + +$	$+$
$f(x)$	0	$\nearrow$	∞
$f''(x)$	$-$	$- - -$	$-$



Theorem 2.10. *(The preimage theorem) If $I \subseteq \mathbb{R}$ is an open set, $f : U \longrightarrow R$ is a C^1-smooth function and $a \in Imf$ is a regular value*[1] *of f, then te inverse image of a through f,*

$$f^{-1}(a) = \{(x, y) \in U | f(x, y) = a\}$$

is a planar regular curve called the regular curve of implicit cartezian equation $f(x, y) = a$.

Proposition 2.11. *The equation of the tangent line $T_{(x_0,y_0)}(C)$ of the planar egular curve C of implicit cartezian equation $f(x, y) = a$ at the point $p = (x_0, y_0) \in C$, is*

$$T_{(x_0,y_0)}(C) : f'_x(p)(x - x_0) + f'_y(p)(y - y_0) = 0,$$

and the equation of the normal line $N_{(x_0,y_0)}(C)$ of C at p is

$$N_{(x_0,y_0)}(C) : \frac{x - x_0}{f'_x(p)} = \frac{y - y_0}{f'_y(p)}.$$

[1]The value $a \in \mathrm{Im}(f)$ of the function f is said to be *regular* if $(\nabla f)(x, y) \neq 0$, $\forall (x, y) \in f^{-1}(a)$

2.4 Problems

1. Find the equations of the tangent lines to the ellipse $\mathcal{E}: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$ having a given angular coefficient $m \in \mathbb{R}$.
2. Find the equations of the tangent lines to the ellipse $\mathcal{E}: x^2+4y^2-20=0$ which are orthogonal to the line $d: 2x-2y-13=0$.
3. Find the equations of the tangent lines to the ellipse $\mathcal{E}: \dfrac{x^2}{25}+\dfrac{y^2}{16}-1=0$, passing through $P_0(10,-8)$.
4. If $M(x,y)$ is a point of the tangent line $T_{M_0}(E)$ of the ellipse $\mathcal{E}: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at one of its points $M_0(x_0,y_0) \in \mathcal{E}$, show that $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} \geq 1$.
5. Find the equations of the tangent lines to the hyperbola $\mathcal{H}: \dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-1=0$ having a given angular coefficient $m \in \mathbb{R}$.
6. Find the equations of the tangent lines to the hyperbola $\mathcal{H}: \dfrac{x^2}{20}-\dfrac{y^2}{5}-1=0$ which are orthogonal to the line $d: 4x+3y-7=0$.
7. Find the equations of the tangent lines to the parabola $\mathcal{P}: y^2=2px$ having a given angular coefficient $m \in \mathbb{R}$.
8. Find the equation of the tangent line to the parabola $\mathcal{P}: y^2-8x=0$, parallel to $d: 2x+2y-3=0$.
9. Find the equation of the tangent line to the parabola $\mathcal{P}: y^2-36x=0$, passing through $P(2,9)$.
10. Find the locus of the orthogonal projections of the center $O(0,0)$ of the ellipse
$$E: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$
on its tangents.
11. Find the locus of the orthogonal projections of the center $O(0,0)$ of the hyperbola
$$H: \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$
on its tangents.
12. Show that a ray of light through a focus of an ellipse reflects to a ray that passes through the other focus (optical property of the ellipse).

Solution. Let $F_1(c,0)$, $F_2(-c,0)$ be the foci of the ellipse $\mathcal{E}: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Recall that the gradient $\nabla f(x_0,y_0)=(f_x(x_0,y_0), f_y(x_0,y_0))$ is a normal vector of the ellipse $\mathcal{E}$ to its point $M_0(x_0,y_0)$, where
$$f(x,y)=\delta(F_1,M)+\delta(F_2,M)=\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}$$

and $M(x,y)$, as the ellipse is a level set of f. Note that

$$f_x(x_0,y_0) = \frac{x_0-c}{\delta(F_1,M_0)} + \frac{x_0+c}{\delta(F_2,M_0)} \text{ and } f_y(x_0,y_0) = \frac{y}{\delta(F_1,M_0)} + \frac{y}{\delta(F_2,M_0)},$$

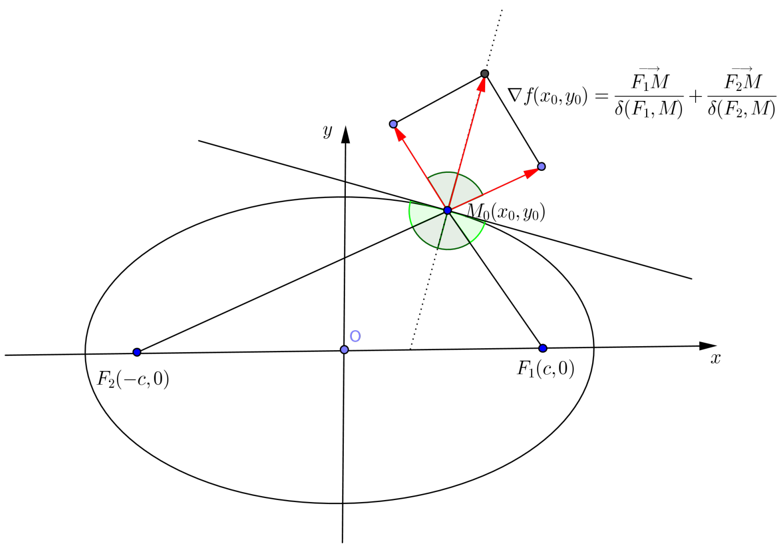
and shows that

$$\begin{aligned}\nabla f &= (f_x(x_0,y_0), f_y(x_0,y_0)) = \left(\frac{x_0-c}{\delta(F_1,M_0)} + \frac{x_0+c}{\delta(F_2,M_0)}, \frac{y_0}{\delta(F_1,M_0)} + \frac{y_0}{\delta(F_2,M_0)}\right)\\ &= \frac{(x_0-c,y)}{\delta(F_1,M_0)} + \frac{(x_0+c,y)}{\delta(F_2,M_0)} = \frac{\overrightarrow{F_1M_0}}{\delta(F_1,M_0)} + \frac{\overrightarrow{F_2M_0}}{\delta(F_2,M_0)}.\end{aligned}$$

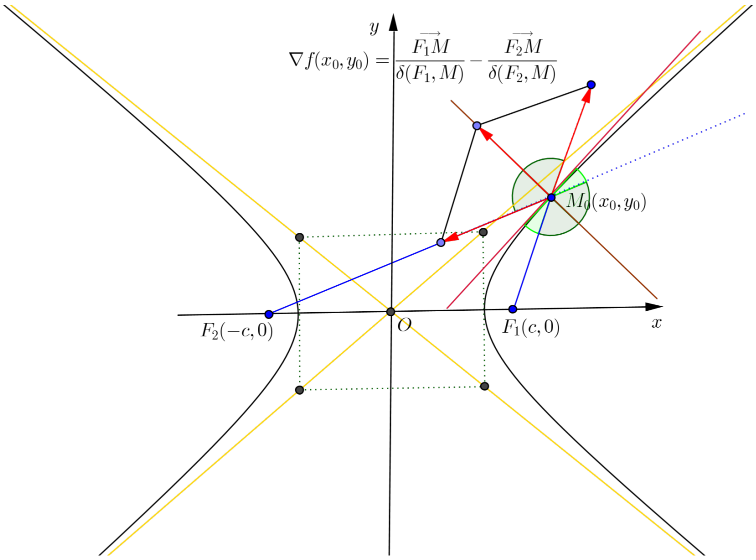
The sum ∇f of the versors

$$\frac{\overrightarrow{F_1M_0}}{\delta(F_1,M_0)} \text{ and } \frac{\overrightarrow{F_2M_0}}{\delta(F_2,M_0)}$$

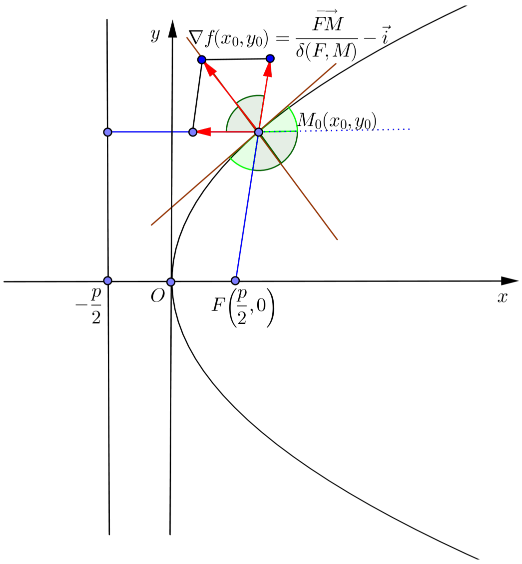
make obviously equal angles with the directions of the vectors $\overrightarrow{F_1M_0}$ and $\overrightarrow{F_2M_0}$ and it is also orthogonal to the tangent $T_{M_0}(\mathcal{E})$ of the ellipse at $M_0(x_0,y_0)$. This shows that the angle between the ray F_1M and the tangent $T_{M_0}(\mathcal{E})$ equals the angle between the ray F_2M and the tangent $T_{M_0}(\mathcal{E})$.



13. Show that a ray of light through a focus of a hyperbola reflects to a ray that passes through the other focus (optical property of the hyperbola). (Hint: A similar agument applied to the function $f(x,y) = \delta(F_1,M) - \delta(F_2,M) = \sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2}$, where $F_1(c,0)$, $F_2(-c,0)$ are the foci of the hyperbola $\mathcal{H} : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.)



14. Show that a ray of light through a focus of a parabola reflects to a ray parallel to the axis of the parabola (optical property of the parabola).



3 Week 9: Quadrics

3.1 The ellipsoid

The *ellipsoid* is the quadric surface given by the equation

$$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \qquad a, b, c \in \mathbb{R}_+^*. \tag{3.1}$$

- The coordinate planes are all planes of symmetry of $\mathcal{E}$ since, for an arbitrary point $M(x, y, z) \in \mathcal{E}$, its symmetric points with respect to these planes, $M_1(-x, y, z)$, $M_2(x, -y, z)$ and $M_3(x, y, -z)$ belong to $\mathcal{E}$; therefore, the coordinate axes are axes of symmetry for $\mathcal{E}$ and the origin O is the center of symmetry of the ellipsoid (3.1);

- The traces in the coordinates planes are ellipses of equations

$$\begin{cases} \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \\ x = 0 \end{cases}, \begin{cases} \frac{x^2}{a^2} + \frac{z^2}{c^2} - 1 = 0 \\ y = 0 \end{cases}, \begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \\ z = 0. \end{cases}$$

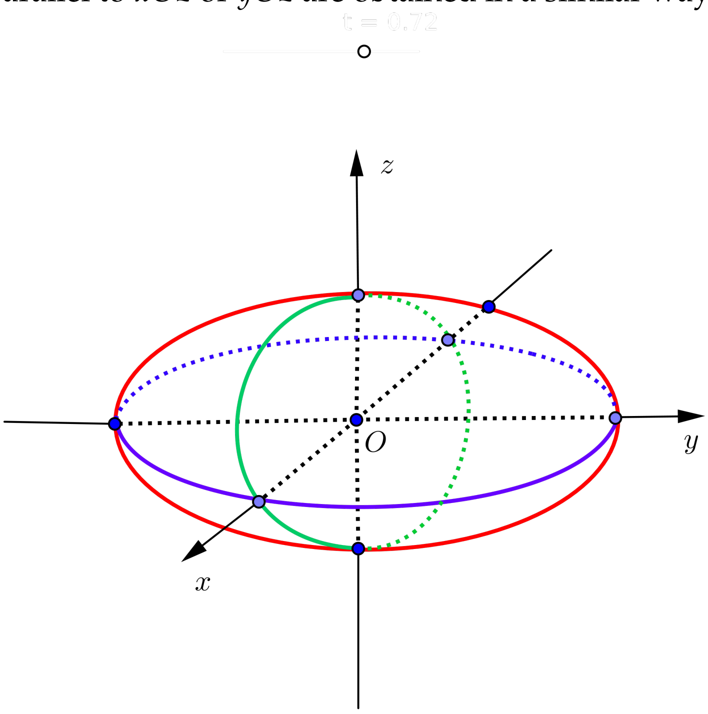
- The sections with planes parallel to xOy are given by setting $z = \lambda$ in (3.1). Then, a section is of equations $\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 - \dfrac{\lambda^2}{c^2} \\ z = \lambda \end{cases}$.

- If $|\lambda| < c$, the section is an ellipse

$$\begin{cases} \dfrac{x^2}{\left(a\sqrt{1 - \dfrac{\lambda^2}{c^2}}\right)^2} + \dfrac{y^2}{\left(b\sqrt{1 - \dfrac{\lambda^2}{c^2}}\right)^2} = 1 \\ z = \lambda \end{cases};$$

- If $|\lambda| = c$, the intersection is reduced to one (tangency) point $(0, 0, \lambda)$;
- If $|\lambda| > c$, the plane $z = \lambda$ does not intersect the ellipsoid $\mathcal{E}$.

The sections with planes parallel to xOz or yOz are obtained in a similar way.



3.2 The hyperboloid of one sheet

The surface of equation

$$\mathcal{H}_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0, \qquad a, b, c \in \mathbb{R}_+^*, \tag{3.2}$$

is called *hyperboloid of one sheet*.

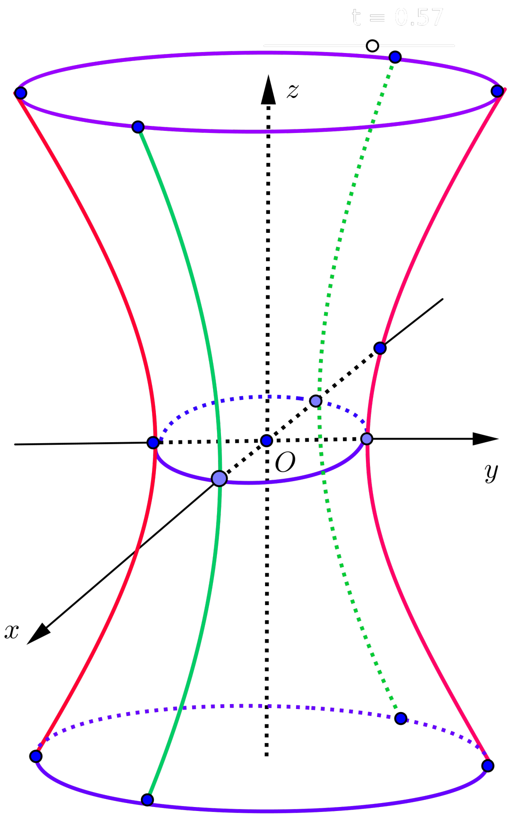
- The coordinate planes are planes of symmetry for $\mathcal{H}_1$; hence, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{H}_1$;

- The intersections with the coordinates planes are, respectively, of equations

$$\left\{\begin{array}{c}\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}-1=0\\ x=0\\ \text{a hyperbola}\end{array}\right. ;\left\{\begin{array}{c}\dfrac{x^2}{a^2}-\dfrac{z^2}{c^2}-1=0\\ y=0\\ \text{a hyperbola}\end{array}\right. ;\left\{\begin{array}{c}\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0\\ z=0\\ \text{an ellipse}\end{array}\right. ;$$

- The intersections with planes parallel to the coordinate planes are

$$\left\{\begin{array}{c}\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1-\dfrac{\lambda^2}{a^2}\\ x=\lambda\\ \text{hyperbolas}\end{array}\right. ;\left\{\begin{array}{c}\dfrac{x^2}{a^2}-\dfrac{z^2}{c^2}=1-\dfrac{\lambda^2}{b^2}\\ y=\lambda\\ \text{hyperbolas}\end{array}\right. ;\left\{\begin{array}{c}\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1+\dfrac{\lambda^2}{c^2}\\ z=\lambda\\ \text{ellipses}\end{array}\right. ;$$



Remark: The surface $\mathcal{H}_1$ contains two families of lines, as

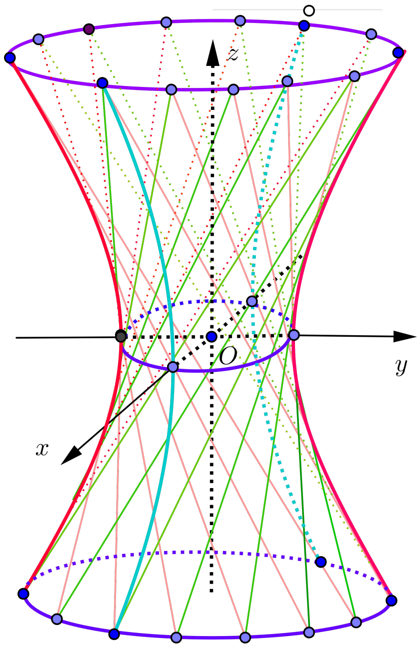
$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{y^2}{b^2}\Leftrightarrow\left(\frac{x}{a}+\frac{z}{c}\right)\left(\frac{x}{a}-\frac{z}{c}\right)=\left(1+\frac{y}{b}\right)\left(1-\frac{y}{b}\right).$$

The equations of the two families of lines are:

$$d_\lambda:\left\{\begin{array}{l}\lambda\left(\dfrac{x}{a}+\dfrac{z}{c}\right)=1+\dfrac{y}{b}\\[2ex] \dfrac{x}{a}-\dfrac{z}{c}=\lambda\left(1-\dfrac{y}{b}\right)\end{array}\right.,\ \lambda\in\mathbb{R},$$

$$d'_\mu:\left\{\begin{array}{l}\mu\left(\dfrac{x}{a}+\dfrac{z}{c}\right)=1-\dfrac{y}{b}\\[2ex] \dfrac{x}{a}-\dfrac{z}{c}=\mu\left(1+\dfrac{y}{b}\right)\end{array}\right.,\ \mu\in\mathbb{R}.$$

Through any point on $\mathcal{H}_1$ pass two lines, one line from each family.



3.3 Th hyperboloid of two sheets

The *hyperboloid of two sheets* is the surface of equation

$$\mathcal{H}_2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0, \qquad a, b, c \in \mathbb{R}_+^*. \tag{3.3}$$

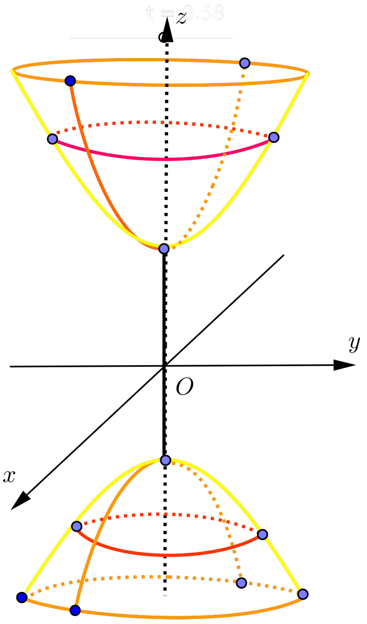
- The coordinate planes are planes of symmetry for $\mathcal{H}_1$, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{H}_1$;
- The intersections with the coordinates planes are, respectively,

$$\left\{\begin{array}{c} \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0 \\ x = 0 \\ \text{a hyperbola;} \end{array}\right., \left\{\begin{array}{c} \frac{x^2}{a^2} - \frac{z^2}{c^2} + 1 = 0 \\ y = 0 \\ \text{a hyperbola} \end{array}\right., \left\{\begin{array}{c} \frac{x^2}{a^2} + \frac{y^2}{b^2} + 1 = 0 \\ z = 0 \\ \text{the empty set} \end{array}\right.;$$

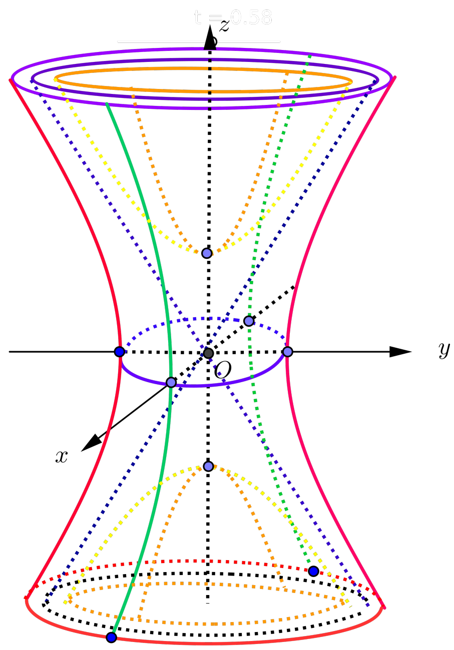
- The intersections with planes parallel to the coordinate planes are

$$\left\{\begin{array}{c} \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 - \frac{\lambda^2}{a^2} \\ x = \lambda \\ \text{hyperbolas} \end{array}\right., \left\{\begin{array}{c} \frac{x^2}{a^2} - \frac{z^2}{c^2} = -1 - \frac{\lambda^2}{b^2} \\ y = \lambda \\ \text{hyperbolas} \end{array}\right., \left\{\begin{array}{c} \frac{x^2}{a^2} + \frac{y^2}{b^2} = -1 + \frac{\lambda^2}{c^2} \\ z = \lambda. \end{array}\right.$$

  - If $|\lambda| > c$, the section is an ellipse;
  - If $|\lambda| = c$, the intersection reduces to the point of coordinates $(0, 0, \lambda)$;
  - If $|\lambda| < c$, one obtains the empty set.



The hyperboloid of two sheets



The hyperboloids of one and two sheets and their common asymptotic cone

4 Week 10

4.1 The hyperboloid of two sheets

The *hyperboloid of two sheets* is the surface of equation

$$\mathcal{H}_2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0, \qquad a, b, c \in \mathbb{R}_+^*. \tag{4.1}$$

- The coordinate planes are planes of symmetry for $\mathcal{H}_1$, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{H}_1$;

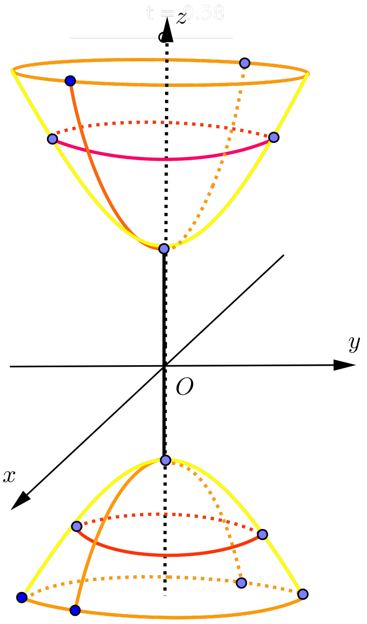
- The intersections with the coordinates planes are, respectively,

$$\left\{\begin{array}{c} \dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}+1=0 \\ x=0 \\ \text{a hyperbola;} \end{array}\right., \left\{\begin{array}{c} \dfrac{x^2}{a^2}-\dfrac{z^2}{c^2}+1=0 \\ y=0 \\ \text{a hyperbola} \end{array}\right., \left\{\begin{array}{c} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+1=0 \\ z=0 \\ \text{the empty set} \end{array}\right.;$$

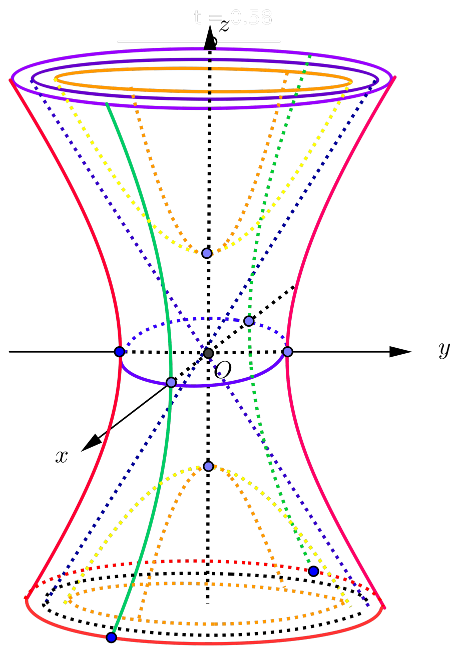
- The intersections with planes parallel to the coordinate planes are

$$\left\{\begin{array}{c} \dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=-1-\dfrac{\lambda^2}{a^2} \\ x=\lambda \\ \text{hyperbolas} \end{array}\right., \left\{\begin{array}{c} \dfrac{x^2}{a^2}-\dfrac{z^2}{c^2}=-1-\dfrac{\lambda^2}{b^2} \\ y=\lambda \\ \text{hyperbolas} \end{array}\right., \left\{\begin{array}{c} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=-1+\dfrac{\lambda^2}{c^2} \\ z=\lambda. \end{array}\right.$$

  - If $|\lambda| > c$, the section is an ellipse;
  - If $|\lambda| = c$, the intersection reduces to a point $(0,0,\lambda)$;
  - If $|\lambda| < c$, one obtains the empty set.



The hyperboloid of two sheets



The hyperboloids of one and two sheets and their common asymptotic cone

4.2 Elliptic Cones

The surface of equation

$$\mathcal{C} : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \qquad a, b, c \in \mathbb{R}_+^*, \tag{4.2}$$

is called *elliptic cone*.

- The coordinate planes are planes of symmetry for $\mathcal{C}$, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{C}$;

- The intersections with the coordinates planes are

$$\left\{\begin{array}{c}\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=0\\ x=0\\ \text{two lines}\end{array}\right.,\left\{\begin{array}{c}\dfrac{x^2}{a^2}-\dfrac{z^2}{c^2}-1=0\\ y=0\\ \text{two lines}\end{array}\right.,\left\{\begin{array}{c}\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=0\\ z=0\\ \text{the point } O(0,0,0).\end{array}\right.$$

- The intersections with planes parallel to the coordinate planes are

$$\left\{\begin{array}{c}\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=-\dfrac{\lambda^2}{a^2}\\ x=\lambda\\ \text{hyperbolas}\end{array}\right.;\quad\left\{\begin{array}{c}\dfrac{x^2}{a^2}-\dfrac{z^2}{c^2}=-\dfrac{\lambda^2}{b^2}\\ y=\lambda\\ \text{hyperbolas.}\end{array}\right.,\left\{\begin{array}{c}\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\dfrac{\lambda^2}{c^2}\\ z=\lambda.\\ \text{ellipses}\end{array}\right.$$

4.3 Elliptic Paraboloids

The surface of equation

$$\mathcal{P}_e:\frac{x^2}{p}+\frac{y^2}{q}=2z,\qquad p,q\in\mathbb{R}_+^*,\tag{4.3}$$

is called *elliptic paraboloid*.

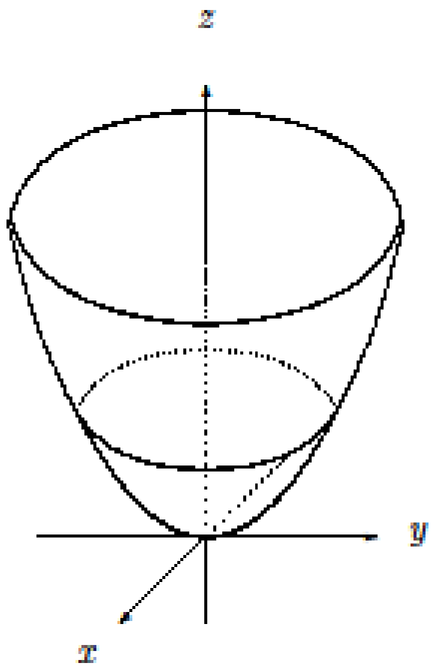
- The planes xOz and yOz are planes of symmetry;
- The traces in the coordinate planes are

$$\left\{\begin{array}{c}\dfrac{y^2}{q}=2z\\ x=0\\ \text{a parabola}\end{array}\right.,\left\{\begin{array}{c}\dfrac{x^2}{p}=2z\\ y=0\\ \text{a parabola}\end{array}\right.,\left\{\begin{array}{c}\dfrac{x^2}{p}+\dfrac{y^2}{q}=0\\ z=0\\ \text{the point } O(0,0,0).\end{array}\right.$$

- The intersection with the planes parallel to the coordinate planes are $\left\{\begin{array}{c}\dfrac{x^2}{p}+\dfrac{y^2}{q}=2\lambda\\ z=\lambda\end{array}\right.,$
  - If $\lambda>0$, the section is an ellipse;
  - If $\lambda=0$, the intersection reduces to the origin;
  - If $\lambda<0$, one has the empty set;

  and

$$\left\{\begin{array}{c}\dfrac{y^2}{q}=2z-\dfrac{\lambda^2}{p}\\ x=\lambda\\ \text{parabolas}\end{array}\right.;\qquad\left\{\begin{array}{c}\dfrac{x^2}{p}=2z-\dfrac{\lambda^2}{q}\\ y=\lambda\\ \text{parabolas}\end{array}\right.;$$



4.4 Hyperbolic Paraboloids

The *hyperbolic paraboloid* is the surface given by the equation

$$\mathcal{P}_h : \frac{x^2}{p} - \frac{y^2}{q} = 2z, \qquad p, q > 0. \tag{4.4}$$

- The planes xOz and yOz are planes of symmetry;
- The traces in the coordinate planes are, respectively,

$$\left\{\begin{array}{c} -\dfrac{y^2}{q} = 2z \\ x = 0 \\ \text{a parabola} \end{array}\right. ; \quad \left\{\begin{array}{c} \dfrac{x^2}{p} = 2z \\ y = 0 \\ \text{a parabola} \end{array}\right. ; \quad \left\{\begin{array}{c} \dfrac{x^2}{p} - \dfrac{y^2}{q} = 0 \\ z = 0 \\ \text{two lines.} \end{array}\right. ;$$

- The intersection with the planes parallel to the coordinate planes are

$$\left\{\begin{array}{c} \dfrac{y^2}{q} = -2z + \dfrac{\lambda^2}{p} \\ x = \lambda \\ \text{parabolas} \end{array}\right. ; \quad \left\{\begin{array}{c} \dfrac{x^2}{p} = 2z + \dfrac{\lambda^2}{q} \\ y = \lambda \\ \text{parabolas.} \end{array}\right.$$

$$\left\{\begin{array}{c} \dfrac{x^2}{p} - \dfrac{y^2}{q} = 2\lambda \\ z = \lambda \\ \text{hyperbolas} \end{array}\right.$$

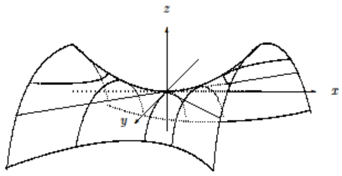
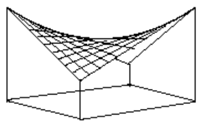
Remark: The hyperbolic paraboloid contains two families of lines. Since

$$\left(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}}\right)\left(\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}}\right) = 2z,$$

then the two families are, respectively, of equations

$$d_\lambda : \begin{cases} \dfrac{x}{\sqrt{p}} - \dfrac{y}{\sqrt{q}} = \lambda \\ \lambda \left(\dfrac{x}{\sqrt{p}} + \dfrac{y}{\sqrt{q}} \right) = 2z \end{cases}, \lambda \in \mathbb{R} \text{ and}$$

$$d'_\mu : \begin{cases} \dfrac{x}{\sqrt{p}} + \dfrac{y}{\sqrt{q}} = \mu \\ \mu \left(\dfrac{x}{\sqrt{p}} - \dfrac{y}{\sqrt{q}} \right) = 2z \end{cases}, \mu \in \mathbb{R}.$$

4.5 Singular Quadrics

Elliptic Cylinder, Hyperbolic Cylinder, Parabolic Cylinder

- The *elliptic cylinder* is the surface of equation

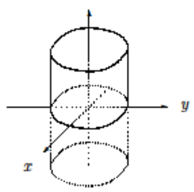
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \quad a, b > 0 \text{ or } \frac{x^2}{a^2} + \frac{z^2}{c^2} - 1 = 0, \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0. \tag{4.5}$$

- The *hyperbolic cylinder* is the surface of equation

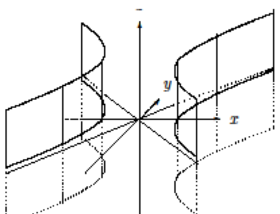
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0, \quad a, b > 0 \text{ or } \frac{x^2}{a^2} - \frac{z^2}{c^2} - 1 = 0, \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0. \tag{4.6}$$

- The *parabolic cylinder* is the surface of equation

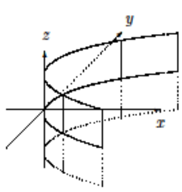
$$y^2 = 2px, \quad p > 0, \quad (\text{or an alternative equation}). \tag{4.7}$$



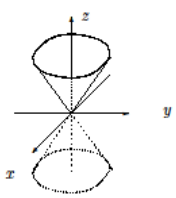
Cilindrul eliptic
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Cilindrul hiperbolic
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



Cilindrul parabolic
$y^2 = 2px$



Conul $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

4.6 Problems

1. Find the intersection points of the ellipsoid
$$\frac{x^2}{16}+\frac{y^2}{12}+\frac{z^2}{4}=1$$
with the line
$$\frac{x-4}{2}=\frac{y+6}{-3}=\frac{z+2}{-2}.$$

2. Find the rextilinear generatrices of the quadric $4x^2-9y^2=36z$ which passes through the point $P(3\sqrt{2},2,1)$.

3. Find the rectilinear generatrices of the hyperboloid of one sheet
$$(\mathcal{H}_1)\ \frac{x^2}{36}+\frac{y^2}{9}-\frac{z^2}{4}=1$$
which are parallel to the plane $(\pi)\ x+y+z=0$.

4. Find the locus of points on the hyperbolic paraboloid $(\mathcal{P}_h)\ y^2-z^2=2x$ through which the rectilinear generatrices are perpendicular.

5 Generated Surfaces

Consider the 3-dimensional Euclidean space $\mathcal{E}_3$, together with a Cartesian system of coordinates $Oxyz$. Generally, the set

$$S=\{M(x,y,z):F(x,y,z)=0\},$$

where $F:D\subseteq\mathbb{R}^3\to\mathbb{R}$ is a real function and D is a domain, is called *surface* of implicit equation $F(x,y,z)=0$. For example the quadric surfaces, defined in the previous chapter for F a polynomial of degree two, are such of surfaces. On the other hand, the set

$$S_1=\{M(x,y,z):x=x(u,v),y=y(u,v),z=z(u,v)\},$$

where $x,y,z:D_1\subseteq\mathbb{R}^2\to\mathbb{R}$, is a *parameterized surface*, of parametric equations

$$\begin{cases} x=x(u,v)\\ y=y(u,v)\\ z=z(u,v)\end{cases},\quad (u,v)\in D_1.$$

The intersection between two surfaces is a *curve* in 3-space (remember, for instance, that the intersection between a quadric surface and a plane is a conic section, hence the conics are plane curves). Then, the set

$$C=\{M(x,y,z):F(x,y,z)=0,G(x,y,z)=0\},$$

where $F,G:D\subseteq\mathbb{R}^3\to\mathbb{R}$, is the curve of *implicit* equations

$$\begin{cases} F(x,y,z)=0\\ G(x,y,z)=0\end{cases}.$$

As before, one can parameterize the curve. The set

$$C_1 = \{M(x,y,z) : x = x(t), y = y(t), z = z(t)\},$$

where $x, y, z : I \subseteq \mathbb{R} \to \mathbb{R}$ and I is open, is called *parameterized curve* of parametric equations

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}, \quad t \in I.$$

Let be given a family of curves, depending on one single parameter λ,

$$\mathcal{C}_\lambda : \begin{cases} F_1(x,y,z;\lambda) = 0 \\ F_2(x,y,z;\lambda) = 0 \end{cases}.$$

In general, the family $\mathcal{C}_\lambda$ does not cover the entire space. By eliminating the parameter λ between the two equations of the family, one obtains the equation of the surface *generated* by the family of curves.

Suppose now that the family of curves depends on two parameters λ, μ,

$$\mathcal{C}_{\lambda,\mu} : \begin{cases} F_1(x,y,z;\lambda,\mu) = 0 \\ F_2(x,y,z;\lambda,\mu) = 0 \end{cases},$$

and that the parameters are related through $\varphi(\lambda,\mu) = 0$ If it can be obtained an equation which does not depend on the parameters (by eliminating the parameters between the three equations), then the set of all the points which verify it is called surface *generated* by the family (or the sub-family) of curves.

5.1 Cylindrical Surfaces

Definition 5.1. *The surface generated by a variable line, called* generatrix, *which remains parallel to a fixed line d and intersects a given curve $\mathcal{C}$, is called* cylindrical surface. *The curve $\mathcal{C}$ is called the* director curve *of the cylindrical surface.*

Theorem 5.2. *The cylindrical surface, with the generatrix parallel to the line*

$$d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases},$$

which has the director curve

$$\mathcal{C} : \begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases},$$

(d and $\mathcal{C}$ are not coplanar), is characterized by an equation of the form

$$\varphi(\pi_1, \pi_2) = 0. \tag{5.1}$$

Proof. The equations of an arbitrary line, which is parallel to

$$d : \begin{cases} \pi_1(x,y,z) = 0 \\ \pi_2(x,y,z) = 0 \end{cases}, \text{ are } d_{\lambda,\mu} : \begin{cases} \pi_1(x,y,z) = \lambda \\ \pi_2(x,y,z) = \mu \end{cases}.$$

Not every line from the family $d_{\lambda,\mu}$ intersects the curve $\mathcal{C}$. This happens only when the system of equations

$$\begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \\ \pi_1(x,y,z) = \lambda \\ \pi_2(x,y,z) = \mu \end{cases}$$

is compatible. By eliminating λ and μ between four equations of the system, one obtains a *necessary condition* $\varphi(\lambda,\mu) = 0$ for the parameters λ and μ in order to nonempty intersection between the line $d_{\lambda,\mu}$. The equation of the surface can be determined now from the system

$$\begin{cases} \pi_1(x,y,z) = \lambda \\ \pi_2(x,y,z) = \mu \\ \varphi(\lambda,\mu) = 0 \end{cases},$$

and it is immediate that $\varphi(\pi_1,\pi_2) = 0$. □

Remark 5.3. *Any equation of the form (5.1), where π_1 and π_2 are linear function of x, y and z, represents a cylindrical surface, having the generatrices parallel to* $d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases}$.

Example 5.4. *Let us find the equation of the cylindrical surface having the generatrices parallel to*

$$d : \begin{cases} x + y = 0 \\ z = 0 \end{cases}$$

and the director curve given by

$$\mathcal{C} : \begin{cases} x^2 - 2y^2 - z = 0 \\ x - 1 = 0 \end{cases}.$$

The equations of the generatrices d are

$$d_{\lambda,\mu} : \begin{cases} x + y = \lambda \\ z = \mu \end{cases}.$$

They must intersect the curve $\mathcal{C}$, i.e. the system

$$\begin{cases} x^2 - 2y^2 - z = 0 \\ x - 1 = 0 \\ x + y = \lambda \\ z = \mu \end{cases}$$

has to be compatible. A solution of the system can be obtained using the three last equations

$$\begin{cases} x = 1 \\ y = \lambda - 1 \\ z = \mu \end{cases}$$

and, replacing in the first one, one obtains the compatibility condition

$$2(\lambda - 1)^2 + \mu - 1 = 0.$$

Thus, the equation of the required cylindrical surface is

$$2(x + y - 1)^2 + x - 1 = 0.$$

6 Week 11

6.1 Conical Surfaces

Definition 6.1. *The surface generated by a variable line, called* generatrix, *which passes through a fixed point V and intersects a given curve $\mathcal{C}$, is called* conical surface. *The point V is called the* vertex *of the surface and the curve $\mathcal{C}$* director curve.

Theorem 6.2. *The conical surface, of vertex $V(x_0, y_0, z_0)$ and director curve*

$$\mathcal{C} : \begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases},$$

(V and $\mathcal{C}$ are not coplanar), is characterized by an equation of the form

$$\varphi\left(\frac{x - x_0}{z - z_0}, \frac{y - y_0}{z - z_0}\right) = 0. \tag{6.1}$$

Proof. The equations of an arbitrary line through $V(x_0, y_0, z_0)$ are

$$d_{\lambda\mu} : \begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \end{cases}.$$

A generatrix has to intersect the curve $\mathcal{C}$, hence the system of equations

$$\begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \\ F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases}$$

must be compatible. This happens for some values of the parameters λ and μ, which verify a *compatibility condition*

$$\varphi(\lambda, \mu),$$

obtained by eliminating x, y and z in the the previous system of equations. In these conditions, the equation of the conical surface rises from the system

$$\begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \\ \varphi(\lambda, \mu) = 0 \end{cases},$$

i.e.

$$\varphi\left(\frac{x - x_0}{z - z_0}, \frac{y - y_0}{z - z_0}\right) = 0.$$

□

Remark 6.3. *If φ is a polynomial function, then the equation (6.1) can be written in the form*

$$\phi(x - x_0, y - y_0, z - z_0) = 0,$$

where ϕ is homogeneous with respect to $x - x_0$, $y - y_0$ and $z - z_0$. If φ is polynomial and V is the origin of the system of coordinates, then the equation of the conical surface is $\phi(x, y, z) = 0$, with ϕ a homogeneous polynomial. Conversely, an algebraic homogeneous equation in x, y and z represents a conical surface with the vertex at the origin.

Example 6.4. *Let us determine the equation of the conical surface, having the vertex $V(1,1,1)$ and the director curve*

$$\mathcal{C} : \begin{cases} (x^2+y^2)^2 - xy = 0 \\ z = 0 \end{cases}.$$

The family of lines passing through V has the equations

$$d_{\lambda\mu} : \begin{cases} x-1 = \lambda(z-1) \\ y-1 = \mu(z-1) \end{cases}.$$

The system of equations

$$\begin{cases} (x^2+y^2)^2 - xy = 0 \\ z = 0 \\ x-1 = \lambda(z-1) \\ y-1 = \mu(z-1) \end{cases}$$

must be compatible. A solution is

$$\begin{cases} x = 1-\lambda \\ y = 1-\mu \\ z = 0 \end{cases},$$

and, replaced in the first equation of the system, gives the compatibility condition

$$[(1-\lambda)^2 + (1-\mu)^2]^2 - (1-\lambda)(1-\mu) = 0.$$

The equation of the conical surface is obtained by eliminating the parameters λ and μ in

$$\begin{cases} x-1 = \lambda(z-1) \\ y-1 = \mu(z-1) \\ ((1-\lambda)^2 + (1-\mu)^2)^2 - (1-\lambda)(1-\mu) = 0 \end{cases}.$$

Expressing $\lambda = \dfrac{x-1}{z-1}$ and $\mu = \dfrac{y-1}{z-1}$ and replacing in the compatibility condition, one obtains

$$\left[\left(\frac{z-x}{z-1}\right)^2 + \left(\frac{z-y}{z-1}\right)^2\right]^2 - \left(\frac{z-x}{z-1}\right)\left(\frac{z-y}{z-1}\right) = 0,$$

or

$$[(z-x)^2 + (z-y)^2]^2 - (z-x)(z-y)(z-1)^2 = 0.$$

6.2 Conoidal Surfaces

Definition 6.5. *The surface generated by a variable line, which intersects a given line d and a given curve $\mathcal{C}$, and remains parallel to a given plane π, is called* conoidal surface. *The curve $\mathcal{C}$ is the* director curve *and the plane π is the* director plane *of the conoidal surface.*

Theorem 6.6. *The conoidal surface whose generatrix intersects the line*

$$d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases}$$

and the curve

$$\mathcal{C} : \begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases}$$

and has the director plane $\pi = 0$, (π is not parallel to d and that C is not contained into π), is characterized by an equation of the form

$$\varphi\left(\pi, \frac{\pi_1}{\pi_2}\right) = 0. \tag{6.2}$$

Proof. An arbitrary generatrix of the conoidal surface is contained into a plane parallel to π and, on the other hand, comes from the bundle of planes containing d. Then, the equations of a generatrix are

$$d_{\lambda\mu} : \begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \end{cases}.$$

Again, the generatrix must intersect the director curve, hence the system of equations

$$\begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \\ F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases}$$

has to be compatible. This leads to a compatibility condition

$$\varphi(\lambda, \mu) = 0,$$

and the equation of the conoidal surface is obtained from

$$\begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \\ \varphi(\lambda, \mu) = 0 \end{cases}.$$

By expressing λ and μ, one obtains (6.2). □

Example 6.7. *Let us find the equation of the conoidal surface, whose generatrices are parallel to xOy and intersect Oz and the curve*

$$\begin{cases} y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases}.$$

The equations of xOy and Oz are, respectively,

$$xOy : z = 0, \quad \text{and} \quad Oz : \begin{cases} x = 0 \\ z = 0 \end{cases},$$

so that the equations of the generatrix are

$$d_{\lambda,\mu} : \begin{cases} x = \lambda y \\ z = \mu \end{cases}.$$

From the compatibility of the system of equations

$$\begin{cases} x = \lambda y \\ z = \mu \\ y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases},$$

one obtains the compatibility condition

$$2\lambda^2\mu - 2\lambda^2 - 2\mu + 1 = 0,$$

and, replacing $\lambda = \frac{y}{x}$ and $\mu = z$, the equation of the conoidal surface is

$$2x^2z - 2y^2z - 2x^2 + y^2 = 0.$$

6.3 Revolution Surfaces

Definition 6.8. *The surface generated after the rotation of a given curve $\mathcal{C}$ around a given line d is said to be a* revolution surface.

Theorem 6.9. *The equation of the revolution surface generated by the curve*

$$\mathcal{C} : \begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases},$$

in its rotation around the line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r},$$

is of the form

$$\varphi((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2, px + qy + rz) = 0. \tag{6.3}$$

Proof. An arbitrary point on the curve $\mathcal{C}$ will describe, in its rotation around d, a circle situated into a plane orthogonal on d and having the center on the line d. This circle can be seen as the intersection between a sphere, having the center on d and of variable radius, and a plane, orthogonal on d, so that its equations are

$$\mathcal{C}_{\lambda,\mu} : \begin{cases} (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \lambda. \\ px + qy + rz = \mu \end{cases}$$

The circle has to intersect the curve $\mathcal{C}$, therefore the system

$$\begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \\ (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \lambda \\ px + qy + rz = \mu \end{cases}$$

must be compatible. One obtains the compatibility condition

$$\varphi(\lambda, \mu) = 0,$$

which, after replacing the parameters, gives the equation of the surface (6.3). □

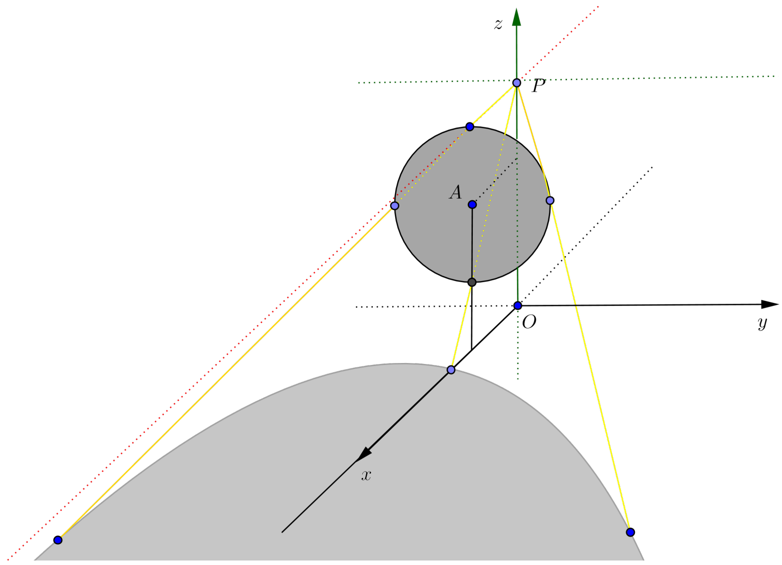
6.4 Problems

1. Find the equation of the cylindrical surface whose director curve is the planar curve

$$(C) \begin{cases} y^2 + z^2 = x \\ x = 2z \end{cases}$$

   and the generatrix is perpendicular to the plane of the director curve.

2. A disk of radius 1 is centered at the point $A(1, 0, 2)$ and is parallel to the plane yOz. A source of light is placed at the point $P(0, 0, 3)$. Characterize analitically the shadow of the disk rushed over the plane xOy.



3. Consider a circle and a line parallel with the plane of the circle. Find the equation of the conoidal surface generated by a variable line which intersects the line (d) and the circle (C) and remains orthogonal to (d). (The Willis conoid)

4. Find the equation of the revolution surface generated by the rotation of a variable line through a fixed line.

5. The *torus* is the revolution surface obtained by the rotation of a circle C about a fixed line (d) within the plane of the circle such that $d \cap C = \emptyset$. Find the equation of the torus.

7 Week 12. Transformations

7.1 Transformations of the plane

Definition 7.1. *An affine transformation of the plane is a mapping*

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ L(x,y) = (ax + by + c, dx + ey + f), \tag{7.1}$$

for some constant real numbers a, b, c, d, e, f.

By using the matrix language, the action of the map L can be written in the form

$$L(x,y) = [x\ y] \left[\begin{array}{cc} a & d \\ b & e \end{array} \right] + [c\ f].$$

The affine transformation L can be also identified with the map $L^c : \mathbb{R}^{2\times 1} \longrightarrow \mathbb{R}^{2\times 1}$ given by

$$\begin{aligned} L^c\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right) &= \left[\begin{array}{c} ax + by + c \\ dx + ey + f \end{array}\right] = \left[\begin{array}{cc} a & b \\ d & e \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right] + \left[\begin{array}{c} c \\ f \end{array}\right] \\ &= [L]\left[\begin{array}{c} x \\ y \end{array}\right] + \left[\begin{array}{c} c \\ f \end{array}\right], \text{ where } [L] = \left[\begin{array}{cc} a & b \\ d & e \end{array}\right]. \end{aligned}$$

Lemma 7.2. *If $(aB - bA)^2 + (dB - eA)^2 > 0$, then the affine transformation (9.2) maps the line (d) $Ax + By + C = 0$ to the line*

$$(eA - dB)x + (aB - bA)y + (bf - ce)A - (af - cd)B + (ae - bd)C = 0.$$

If $aB - bA = dB - eA = 0$, then $ae - bd = 0$ and L is the constant map $\left(\frac{cB - bC}{B}, \frac{fB - eC}{B}\right)$.

Definition 7.3. *An affine transformation (9.2) is said to be* singular *if*

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix} = 0 \text{ i.e. } ae - bd = 0.$$

and non-singular otherwise.

7.1.1 Translations

Note that the affine transformation L is nonsingular if and only if it is invertible. In such a case the inverse L^{-1} is a non-singular affine transformation and $[L^{-1}] = [L]^{-1}$.

Definition 7.4. *The* translation *of vector* $(h,k) \in \mathbb{R}^2$ *is the affine transformation*

$$T(h,k) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ [T(h,k)]\,(x,y) = (x+h, y+k).$$

Thus

$$[T(h,k)^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+h \\ y+k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix},$$

i.e.

$$[T(h,k)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that the translation $T(h,k)$ is non-singular (invertible) and $(T(h,k))^{-1} = T(-h,-k)$.

7.1.2 Scaling about the origin

Definition 7.5. *The* scaling about the origin *by non-zero scaling factors* $(s_x, s_y) \in \mathbb{R}^2$ *is the affine transformation*

$$S(s_x,s_y) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ \left[S(s_x,s_y)\right](x,y) = (s_x \cdot x, s_y \cdot y).$$

Thus

$$\left[S(s_x,s_y)^c\right]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} s_x \cdot x \\ s_y \cdot y \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

i.e.

$$[S(s_x,s_y)] = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}.$$

Note that the scaling about the origin by non-zero scaling factors $(s_x, s_y) \in \mathbb{R}^2$ is non-singular (invertible) and $(S(s_x,s_y))^{-1} = S(s_x^{-1}, s_y^{-1})$.

7.1.3 Reflections

Definition 7.6. *The* reflections about the x-axis *and the y-axis respectively are the affine transformation*

$$r_x, r_y : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ r_x(x,y) = (x,-y),\ r_y = (-x,y).$$

Thus

$$[r_x^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

i.e.

$$[r_x] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \text{ Similarly } [r_y] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that $r_x = S(-1,1)$ and $r_y = S(1,-1)$. Thus the two reflections are non-singular (invertible) and $r_x^{-1} = r_x$, $r_y^{-1} = r_y$.

Definition 7.7. *The* reflection $r_l : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ about the line *l maps a given point M to the point M' defined by the property that l is the perpendicular bisector of the segment MM'. One can show that the action of the reflection about the line $l : ax + by + c = 0$ is*

$$r_l(x,y)=\left(\frac{b^2-a^2}{a^2+b^2}x-\frac{2ab}{a^2+b^2}y-\frac{2ac}{a^2+b^2}, -\frac{2ab}{a^2+b^2}x+\frac{a^2-b^2}{a^2+b^2}y-\frac{2bc}{a^2+b^2}\right).$$

Thus

$$\begin{aligned}[r_l^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} \frac{b^2-a^2}{a^2+b^2}x-\frac{2ab}{a^2+b^2}y-\frac{2ac}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2}x+\frac{a^2-b^2}{a^2+b^2}y-\frac{2bc}{a^2+b^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{2ac}{a^2+b^2} \\ \frac{2bc}{a^2+b^2} \end{bmatrix},\end{aligned}$$

i.e.

$$[r_l] = \frac{1}{a^2+b^2}\begin{bmatrix} b^2-a^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix}.$$

Note that the reflection r_l is non-singular (invertible) and $r_l^{-1} = r_l$.

7.1.4 Rotations

Definition 7.8. *The* rotation $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *about the origin through an angle θ maps a point $M(x,y)$ into a point $M'(x',y')$ with the properties that the segments $[OM]$ and $[OM']$ are congruent and the $m(\widehat{MOM'}) = \theta$. If $\theta > 0$ the rotation is supposed to be* anticlockwise *and for $\theta < 0$ the rotation is* clockwise. *If $(x,y) = (r\cos\varphi, r\sin\varphi)$, then the coordinates of the rotated point are $\big(r\cos(\theta+\varphi), r\sin(\theta+\varphi)\big) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$, i.e.*

$$R_\theta(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

Thus

$$\begin{aligned}[R_\theta^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},\end{aligned}$$

i.e.

$$[R_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Note that the rotation R_θ is non-singular (invertible) and $R_\theta^{-1} = R_{-\theta}$.

8 Week 13

8.0.5 Reflections

Definition 8.1. *The* reflections about the x-axis *and the y-axis respectively are the affine transformation*

$$r_x, r_y : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ r_x(x,y) = (x,-y),\ r_y = (-x,y).$$

Thus

$$[r_x^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

i.e.

$$[r_x] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Similarly

$$[r_y] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that $r_x = S(-1,1)$ and $r_y = S(1,-1)$. Thus the two reflections are non-singular (invertible) and $r_x^{-1} = r_x$, $r_y^{-1} = r_y$.

Definition 8.2. *The* reflection $r_l : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ about the line l *maps a given point M to the point M' defined by the property that l is the perpendicular bisector of the segment MM'. One can show that the action of the reflection about the line $l : ax + by + c = 0$ is*

$$r_l(x,y)=\left(\frac{b^2-a^2}{a^2+b^2}x-\frac{2ab}{a^2+b^2}y-\frac{2ac}{a^2+b^2}, -\frac{2ab}{a^2+b^2}x+\frac{a^2-b^2}{a^2+b^2}y-\frac{2bc}{a^2+b^2}\right).$$

$$\begin{aligned} \textit{Thus}\ [r_l^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} \frac{b^2-a^2}{a^2+b^2}x-\frac{2ab}{a^2+b^2}y-\frac{2ac}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2}x+\frac{a^2-b^2}{a^2+b^2}y-\frac{2bc}{a^2+b^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{2ac}{a^2+b^2} \\ \frac{2bc}{a^2+b^2} \end{bmatrix}, \end{aligned}$$

i.e. $[r_l] = \frac{1}{a^2+b^2}\begin{bmatrix} b^2-a^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix}$. *Note that the reflection r_l is non-singular (invertible) and $r_l^{-1} = r_l$.*

8.0.6 Shears

Definition 8.3. *Given a fixed direction in the plane specified by a unit vector $v = (v_1, v_2)$, consider the lines d with direction v and the oriented distance d from the origin. The* shear *about the origin of factor r in the direction v is defined to be the transformation which maps a point $M(x,y)$ on d to the point $M' = M + rdv$. The equation of the line through M of direction v is $v_2X - v_1Y + (v_1y - v_2x) = 0$. The oriented distacnce from the origin to this line is $v_1y - v_2x$. Thus the action of the shear $Sh(v,r) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ about the origin of factor r in the direction v is*

$$\begin{aligned} Sh(v,r)(x,y) &= (x,y) + rd(v_1,v_2) \\ &= (x,y) + (r(v_1y - v_2x)v_1, r(v_1y - v_2x)v_2) \\ &= (x,y) + \left(-rv_1v_2x + rv_1^2y, -rv_2^2x + rv_1v_2y\right) \\ &= \left((1 - rv_1v_2)x + rv_1^2y, -rv_2^2x + (1 + rv_1v_2)y\right) \end{aligned}$$

Thus

$$\begin{aligned}[Sh(v,r)^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} (1-rv_1v_2)x + rv_1^2y \\ -rv_2^2x + (1+rv_1v_2)y \end{bmatrix} \\ &= \begin{bmatrix} 1-rv_1v_2 & rv_1^2 \\ -rv_2^2 & 1+rv_1v_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},\end{aligned}$$

i.e. $[Sh(v,r)] = \begin{bmatrix} 1-rv_1v_2 & rv_1^2 \\ -rv_2^2 & 1+rv_1v_2 \end{bmatrix}$.

8.1 Homogeneous coordinates

The affine transformation

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ L(x,y) = (ax+by+c, dx+ey+f)$$

can be written by using the matrix language and by equations:

1. (a) indentifying the vectors $(x,y) \in \mathbb{R}^2$ with the line matrices $[x\ y] \in \mathbb{R}^{1\times 2}$ and implicitely $\mathbb{R}^2$ with $\mathbb{R}^{1\times 2}$:

$$L[x\ y] = [x\ y]\begin{bmatrix} a & d \\ b & e \end{bmatrix} + [c\ f].$$

   (b) indentifying the vectors $(x,y) \in \mathbb{R}^2$ with the column matrices $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2\times 1}$ and implicitely $\mathbb{R}^2$ cu $\mathbb{R}^{2\times 1}$:

$$L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}.$$

2. $\begin{cases} x' = ax+by+c \\ y' = dx+ey+f. \end{cases} \Leftrightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$

Observe that the representation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$

is equivalent to

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

In this lesson we identify the points $(x,y) \in \mathbb{R}^2$ with the points $(x,y,1) \in \mathbb{R}^3$ and even with the punctured lines of $\mathbb{R}^3$, (rx, ry, r), $r \in \mathbb{R}^*$. Due to technical reasons we shall actually identify the points $(x,y) \in \mathbb{R}^2$ with the punctured lines of $\mathbb{R}^3$ represented in the form

$$\begin{bmatrix} rx \\ ry \\ r \end{bmatrix},\ r \in \mathbb{R}^*,$$

and the latter ones we shall call *homogeneous coordinates* of the point $(x,y) \in \mathbb{R}^2$. The set of homogeneous coordinates (x,y,w) will be denoted by $\mathbb{RP}^2$ and call it the real *projective plane*. The homogeneous coordinates $(x,y,w) \in \mathbb{RP}^2$, $w \neq 0$ şi $(\frac{x}{w}, \frac{y}{w}, 1)$represent the same element of $\mathbb{RP}^2$.

Observation 8.4. *The projective plane $\mathbb{RP}^2$ is actually the quotient set $(\mathbb{R}^3 \setminus \{0\})/\sim$, where $'\sim'$ is the following equivalence relation on $\mathbb{R}^3 \setminus \{0\}$:*

$$(x, y, w) \sim (\alpha, \beta, \gamma) \Leftrightarrow \exists r \in \mathbb{R}^* \text{ a.î. } (x, y, w) = r(\alpha, \beta, \gamma).$$

Observe that the equivalence classes of the equivalence relation $\sim'$ are the punctured lines of $\mathbb{R}^3$ through the origin without the origin itself, i.e. the elements of the real projective plane $\mathbb{RP}^2$.

Definition 8.5. *A* projective transformation *of the projective plane $\mathbb{RP}^2$ is a transformation*

$$L : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2,\ L\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} ax+by+cw \\ dx+ey+fw \\ gx+hy+kw \end{bmatrix}, \tag{8.1}$$

where $a, b, c, d, e, f, g, h, k \in \mathbb{R}$. Note that

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$$

is called the homogeneous transformation matrix *of L.*

Observe that a projective transformation (9.3) is well defined since

$$L\begin{bmatrix} rx \\ ry \\ rw \end{bmatrix} = \begin{bmatrix} arx+bry+crw \\ drx+ery+frw \\ grx+hry+krw \end{bmatrix} = \begin{bmatrix} r(ax+by+cw) \\ r(dx+ey+fw) \\ r(gx+hy+kw) \end{bmatrix}.$$

If $g = h = 0$ and $k \neq 0$, then the projective transformation (9.3) is said to be *affine*. The restriction of the affine transformation (9.3), which corresponds to the situation $g = h = 0$ and $k = 1$, to the subspace $w = 1$, has the form

$$L\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax+by+cw \\ dx+ey+fw \\ 1 \end{bmatrix}, \tag{8.2}$$

i.e.

$$\begin{cases} x' = ax+by+c \\ y' = dx+ey+f. \end{cases} \tag{8.3}$$

Observation 8.6. *If $L_1, L_2 : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$ are two projective applications, then their product (concatenation) transformation $L_1 \circ L_2$ is also a projective transformation and its homogeneous transformation matrix is the product of the homogeneous transformation matrices of L_1 and L_2.*

Indeed, if

$$L_1\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & k_1 \end{bmatrix}\begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

and

$$L_2\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & k_2 \end{bmatrix}\begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

then

$$(L_1 \circ L_2)\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \left(\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & k_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & k_2 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Observation 8.7. *If $L_1, L_2 : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$ are two affine applications, then their product $L_1 \circ L_2$ is also an affine transformation.*

8.2 Problems

1. Consider a quadrilateral with vertices $A(1,1)$, $B(3,1)$, $C(2,2)$, and $D(1.5,3)$. Find the image quadrilaterals through the translation $T(1,2)$, the scaling $S(2,2.5)$, the reflections about the x and y-axes, the clockwise and anticlockwise rotations through the angle $\pi/2$ and the shear $Sh\left(\left(2/\sqrt{5}, 1/\sqrt{5}\right), 1.5\right)$.

2. Find the concatenation (product) of an anticlockwise rotation about the origin through an angle of $\frac{3\pi}{2}$ followed by a scaling by a factor of 3 units in the x-direction and 2 units in the y-direction. (Hint: $S(3,2)R_{3\pi/2}$)

3. Find the homogeneous matrix of the product (concatenation) $S(3,2) \circ R_{\frac{3\pi}{2}}$.

4. Find the equations of the rotation $R_\theta(x_0, y_0)$ about the point $M_0(x_0, y_0)$ through an angle θ.

   Solution The homogeneous transformation matrix of the rotation $R_\theta(x_0, y_0)$ about the point $M_0(x_0, y_0)$ through an angle θ is

$$\begin{aligned} R_\theta(x_0, y_0) &= T(x_0, y_0) R_\theta T(-x_0, -y_0) \\ &= \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta & -x_0\cos\theta + y_0\sin\theta + x_0 \\ \sin\theta & \cos\theta & -x_0\sin\theta - y_0\cos\theta + y_0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

   Thus, the equations of the required rotation are:

$$\begin{cases} x' &= x\cos\theta - y\sin\theta - x_0\cos\theta + y_0\sin\theta + x_0 \\ y' &= x\sin\theta + y\cos\theta - x_0\sin\theta - y_0\cos\theta + y_0. \end{cases}.$$

5. Show that the concatenation (product) of two rotations, the first through an angle θ about a point $P(x_0, y_0)$ and the second about a point $Q(x_1, y_1)$ (distinct from P) through an angle $-\theta$ is a translation.

9 Week 14

9.1 Transformations of the plane in homogeneous coordinates

In this section we shall identify an affine transformation of $\mathbb{RP}^2$ with its homogeneous transformation matrix

9.1.1 Translations and scalings

• The homogeneous transformation matrix of the translation $T(h,k)$ is

$$T(h,k) = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}.$$

• The homogeneous transformation matrix of the scaling $S(s_x, s_y)$ is

$$S(s_x, s_y) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

9.1.2 Reflections

• The homogeneous transformation matrix of reflection r_x about the x-axis is

$$r_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

• The homogeneous transformation matrix of reflection r_y about the y-axis is

$$r_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

• The homogeneous transformation matrix of reflection r_l about the line $l : ax + by + c = 0$ is

$$r_l = \begin{bmatrix} \dfrac{b^2 - a^2}{a^2 + b^2} & -\dfrac{2ab}{a^2 + b^2} & -\dfrac{2ac}{a^2 + b^2} \\ -\dfrac{2ab}{a^2 + b^2} & \dfrac{a^2 - b^2}{a^2 + b^2} & -\dfrac{2bc}{a^2 + b^2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Since in homogeneous coordinates multiplication by a factor does not affect the result, the above matrix can be multiplied by a factor $a^2 + b^2 + c^2$ to give the homogeneous matrix of a general reflection

$$\begin{bmatrix} b^2 - a^2 & -2ab & -2ac \\ -2ab & a^2 - b^2 & -2bc \\ 0 & 0 & a^2 + b^2 \end{bmatrix}.$$

Example 9.1. *Consider a line (d) $ax + by + c$ whose slope is* $\mathrm{tg}\theta = -\dfrac{a}{b}$. *By using the observation that the reflection r_d in the line d is the following concatenation (product)*

$$T(0, -c/b) \circ R_\theta \circ r_x \circ R_{-\theta} \circ T(0, c/b),$$

one can show that the homogeneous transformation matrix of r_d is

$$\begin{bmatrix} \dfrac{b^2 - a^2}{a^2 + b^2} & -\dfrac{2ab}{a^2 + b^2} & -\dfrac{2ac}{a^2 + b^2} \\ -\dfrac{2ab}{a^2 + b^2} & \dfrac{a^2 - b^2}{a^2 + b^2} & -\dfrac{2bc}{a^2 + b^2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution. The homogeneous matrix of the concatenation

$$T(0,-c/b)\circ R_\theta \circ r_x \circ R_{-\theta}\circ T(0,c/b)$$

is

$$\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & -c/b\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & c/b\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta-\sin^2\theta & 2\sin\theta\cos\theta & 2\frac{c}{b}\sin\theta\cos\theta\\ 2\sin\theta\cos\theta & \sin^2\theta-\cos^2\theta & \frac{c}{b}\left(\sin^2\theta-\cos^2\theta-1\right)\\ 0 & 0 & 1\end{bmatrix}. \tag{9.1}$$

Since $\mathrm{tg}\theta=-\frac{a}{b}$, it follows that $\frac{a^2}{b^2}=\frac{\sin^2\theta}{\cos^2\theta}=\frac{\sin^2\theta}{1-\sin^2\theta}=\frac{1-\cos^2\theta}{\cos^2\theta}$, namely

$$\sin^2\theta=\frac{a^2}{a^2+b^2} \text{ and } \cos^2\theta=\frac{b^2}{a^2+b^2}.$$

Thus

$$\sin\theta=\pm\frac{a}{\sqrt{a^2+b^2}} \text{ and } \cos\theta=\mp\frac{b}{\sqrt{a^2+b^2}}, \text{ as } \frac{\sin\theta}{\cos\theta}=\mathrm{tg}\theta=-\frac{a}{b}.$$

Therefore $\sin\theta\cos\theta=-\frac{ab}{a^2+b^2}$ and the matrix (9.1) becomes

$$\begin{bmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} & -\frac{c}{b}\frac{2ab}{a^2+b^2}\\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} & \frac{c}{b}\left(\frac{a^2-b^2}{a^2+b^2}-1\right)\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} & -\frac{2ac}{a^2+b^2}\\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} & -\frac{2bc}{a^2+b^2}\\ 0 & 0 & 1\end{bmatrix}.$$

9.1.3 Rotations

The homogeneous transformation matrix of the rotation R_θ about the origin through an angle θ is

$$R_\theta=\begin{bmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Example 9.2. *The homogeneous transformation matrix of the product (concatenation) homogeneous transformation $T(h,k)\circ R_\theta$ is the product*

$$\begin{bmatrix} 1 & 0 & h\\ 0 & 1 & k\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix} \cos\theta & -\sin\theta & h\\ \sin\theta & \cos\theta & k\\ 0 & 0 & 1\end{bmatrix}.$$

In order to find the homogeneous transformation matrix of the inverse transformation

$$(T(h,k)\circ R_\theta)^{-1}=R_\theta^{-1}\circ T(h,k)^{-1}=R_{-\theta}\circ T(-h,-k)$$

of the product (concatenation) homogeneous transformation $T(h,k)\circ R_\theta$ we can either multiply the homogeneous transformation matrices of the inverse transformations $R_\theta^{-1} = R_\theta^{-1}$ and $T(h,k)^{-1} = T(-h,-k)$ or use the next proposition. The product of the homogeneous transformation matrices of the inverse transformations $R_\theta^{-1} = R_\theta^{-1}$ and $T(h,k)^{-1} = T(-h,-k)$ is

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix} =$$
$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix} =$$
$$= \begin{bmatrix} \cos\theta & \sin\theta & -h\cos\theta - k\sin\theta \\ -\sin\theta & \cos\theta & h\sin\theta - k\cos\theta \\ 0 & 0 & 1 \end{bmatrix}.$$

Proposition 9.3. *A homogeneous transformation L is invertible if and only if its homogeneous transformation matrix, say T, is invertible and T^{-1} is the transformation matrix of L^{-1}.*

Proof. Suppose that L has an inverse L^{-1} with transformation matrix T_1. The product transformation $L\circ L^{-1} = id$ has the transformation matrix $TT_1 = I_3$. Similarly, $L^{-1}\circ L = I_3$ has the transformation matrix $T_1T = I_3$. Thus $T_1 = T^{-1}$. Conversely, assume that T has an inverse T^{-1}, and let L_1 be the homogeneous transformation defined by T^{-1}. Since $TT^{-1} = I_3$ and $T^{-1}T = I_3$, it follows that $L\circ L_1 = I$ and $L_1\circ L = I$. Hence L_1 is the inverse transformation of L.

Example 9.4. *The homogeneous transformation matrix of inverse*

$$(T(h,k)\circ R_\theta)^{-1} = R_\theta^{-1}\circ T(h,k)^{-1} = R_{-\theta}\circ T(-h,-k)$$

of the product (concatenation) homogeneous transformation $T(h,k)\circ R_\theta$ is the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta & h \\ \sin\theta & \cos\theta & k \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & -h\cos\theta - k\sin\theta \\ -\sin\theta & \cos\theta & h\sin\theta - k\cos\theta \\ 0 & 0 & 1 \end{bmatrix}.$$

9.2 Transformations of the space

Definition 9.5. *An affine transformation of the plane is a mapping*

$$L:\mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ T(x,y,z) = (ax+by+cz+d, ex+fy+gz+h, kx+ly+mz+n), \quad (9.2)$$

for some constant real numbers a,b,c,d,e,f,g,h,k,l,m,n.

By using the matrix language, the action of the map L can be written in the form

$$L(x,y,z) = [x\ y\ z]\begin{bmatrix} a & e & k \\ b & f & l \\ c & g & m \end{bmatrix} + [d\ h\ n].$$

The affine transformation L can be also identified with the map $L^c : \mathbb{R}^{3\times 1} \longrightarrow \mathbb{R}^{3\times 1}$ given by

$$L^c\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} ax+by+cz+d \\ ex+fy+gz+h \\ kx+ly+mz+n \end{bmatrix} = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix}$$
$$= [L]\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix}, \text{ where } [L] = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix}.$$

Definition 9.6. *An affine transformation (9.2) is said to be* singular *if*

$$\begin{vmatrix} a & b & c \\ e & f & g \\ k & l & m \end{vmatrix} = 0.$$

and non-singular otherwise.

9.2.1 Translations

The *translation of* $\mathbb{R}^3$ *of vector* $(h,k,l) \in \mathbb{R}^3$ is the affine transformation

$$T(h,k,l) : \mathbb{R}^3 \to \mathbb{R}^3,\ T(h,k,l)(x_1,x_2,x_3) = (x_1+h, x_2+k, x_3+l).$$

Its associated transformation is

$$T(h,k,l)^c : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1},\ T(h,k,l)^c\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ -x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} h \\ k \\ l \end{bmatrix},$$

which shows that its standard matrix and equations are:

$$[T(h,k,l)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{cases} w_1 = x_1 + h \\ w_2 = x_2 + k \\ w_3 = x_3 + l \end{cases}.$$

9.2.2 Scaling about the origin

The *scaling about the origin* by non-zero scaling factors $(s_x, s_y, s_z) \in \mathbb{R}^3$ is the affine transformation

$$S(s_x,s_y,s_z) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ \left[S(s_x,s_y,s_z)\right](x,y,z) = (s_x\cdot x, s_y\cdot y, s_z\cdot z).$$

Thus

$$\left[S(s_x,s_y,s_z)^c\right]\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} s_x\cdot x \\ s_y\cdot y \\ s_z\cdot z \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

i.e.

$$[S(s_x,s_y,s_z)] = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}.$$

Note that the scaling about the origin by non-zero scaling factors $(s_x,s_y,s_z) \in \mathbb{R}^3$ is non-singular (invertible) and $(S(s_x,s_y,s_z))^{-1} = S(s_x^{-1}, s_y^{-1}, s_z^{-1})$.

9.2.3 Reflections about planes

1. The *reflection of* $\mathbb{R}^3$ *through the xy-plane is* $r_{xy} : \mathbb{R}^3 \to \mathbb{R}^3$, $r_{xy}(x_1, x_2, x_3) = (x_1, x_2, -x_3)$. Its associated transformation is

$$r^c_{xy} : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1},\ r_{xy}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ x_2 \\ -x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

which shows that its standard matrix and equations are:

$$[r_{xy}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{cases} w_1 = x_1 \\ w_2 = x_2 \\ w_3 = -x_3 \end{cases}.$$

2. The *reflection of* $\mathbb{R}^3$ *through the xz-plane is* $r_{xz} : \mathbb{R}^3 \to \mathbb{R}^3$, $r_{xz}(x_1, x_2, x_3) = (x_1, -x_2, x_3)$. Its associated transformation is

$$r_{xz} : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1},\ r^c_{xz}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ -x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

which shows that its standard matrix and equations are:

$$[r_{xz}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{cases} w_1 = x_1 \\ w_2 = -x_2 \\ w_3 = x_3 \end{cases}.$$

3. The *reflection of* $\mathbb{R}^3$ *through the yz-plane is* $r_{yz} : \mathbb{R}^3 \to \mathbb{R}^3$, $r_{yz}(x_1, x_2, x_3) = (-x_1, x_2, x_3)$. Its associated transformation is

$$r^c_{yz} : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1},\ r^c_{yz}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} -x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

which shows that its standard matrix and equations are:

$$[r_{yz}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{cases} w_1 = -x_1 \\ w_2 = x_2 \\ w_3 = x_3 \end{cases}.$$

4. The *reflection of* $\mathbb{R}^3$ *through an arbitrary plane* $\pi : ax_1 + bx_2 + cx_3 + d = 0$ *is* $r_\pi : \mathbb{R}^3 \to \mathbb{R}^3$, given by

$$r_\pi(x, y, z) = \left(\frac{(-a^2 + b^2 + c^2)x - 2aby - 2acz - 2ad}{a^2 + b^2 + c^2}, \right.$$
$$\frac{-2abx + (a^2 - b^2 + c^2)y - 2bcz - 2bd}{a^2 + b^2 + c^2},$$
$$\left. \frac{-2acx - 2bcy + (a^2 + b^2 - c^2)z - 2cd}{a^2 + b^2 + c^2} \right).$$

Its associated transformation $r_\pi : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1}$ is given by

$$r_\pi^c\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} \dfrac{(-a^2+b^2+c^2)x - 2aby - 2acz - 2ad}{a^2+b^2+c^2} \\ \dfrac{-2abx + (a^2-b^2+c^2)y - 2bcz - 2bd}{a^2+b^2+c^2} \\ \dfrac{-2acx - 2bcy + (a^2+b^2-c^2)z - 2cd}{a^2+b^2+c^2} \end{bmatrix}$$
$$= \frac{1}{a^2+b^2+c^2}\left(\begin{bmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} - 2d\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right).$$

which shows that its standard matrix and equations are:

$$[r_\pi] = \frac{1}{a^2+b^2+c^2}\begin{bmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}$$

and

$$\begin{cases} w_1 = \dfrac{(-a^2+b^2+c^2)x - 2aby - 2acz - 2ad}{a^2+b^2+c^2} \\ w_2 = \dfrac{-2abx + (a^2-b^2+c^2)y - 2bcz - 2bd}{a^2+b^2+c^2} \\ w_3 = \dfrac{-2acx - 2bcy + (a^2+b^2-c^2)z - 2cd}{a^2+b^2+c^2} \end{cases}$$

9.2.4 Rotations

The *rotation operator of* $\mathbb{R}^3$ *through a fixed angle* θ *about an oriented axis*, rotates about the axis of rotation each point of $\mathbb{R}^3$ in such a way that its associated vector sweeps out some portion of the cone determine by the vector itself and by a vector which gives the direction and the orientation of the considered oriented axis. The angle of the rotation is measured at the base of the cone and it is measured clockwise or counterclockwise in relation with a viewpoint along the axis looking toward the origin. As in $\mathbb{R}^2$, the positives angles generates counterclockwise rotations and negative angles generates clockwise roattions. The counterclockwise sense of rotaion can be determined by the right-hand rule: If the thumb of the right hand points the direction of the direction of the oriented axis, then the cupped fingers points in a counterclockwise direction. The rotation operators in $\mathbb{R}^3$ are linear.

For example

1. The counterclockwise rotation about the positive x-axis through an angle θ has the equations

$$\begin{aligned} w_1 &= x \\ w_2 &= y\cos\theta - z\sin\theta \\ w_3 &= y\sin\theta + z\cos\theta \end{aligned},$$

its standard matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

2. The counterclockwise rotation about the positive y-axis through an angle θ has the equations
$$\begin{array}{lcl} w_1 & = & x\cos\theta + z\sin\theta \\ w_2 & = & y \\ w_3 & = & -x\sin\theta + z\cos\theta \end{array},$$
its standard matrix is
$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

3. The counterclockwise rotation about the positive z-axis through an angle θ has the equations
$$\begin{array}{lcl} w_1 & = & x\cos\theta - y\sin\theta \\ w_2 & = & x\sin\theta + y\cos\theta \\ w_3 & = & z \end{array},$$
its standard matrix is
$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

9.3 Homogeneous coordinates

The affine transformation
$$L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ T(x,y,z) = (ax+by+cz+d, ex+fy+gz+h, kx+ly+mz+n),$$
can be written by using the matrix language and by equations:

1. (a) indentifying the vectors $(x,y,z) \in \mathbb{R}^3$ with the line matrices $[x\ y\ z] \in \mathbb{R}^{1\times 3}$ and implicitely $\mathbb{R}^3$ with $\mathbb{R}^{1\times 3}$. With this identification, the action of L is given by
$$L[x\ y\ z] = [x\ y\ z]\begin{bmatrix} a & e & k \\ b & f & l \\ c & g & m \end{bmatrix} + [d\ h\ n].$$

   (b) indentifying the vectors $(x,y,z) \in \mathbb{R}^3$ with the column matrices $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^{3\times 1}$ and implicitely $\mathbb{R}^3$ with $\mathbb{R}^{3\times 1}$. We de note by $L^c : \mathbb{R}^{3\times 1} \longrightarrow \mathbb{R}^{3\times 1}$ the associated map via this identification, and its action is given by
$$\begin{aligned} L^c\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) &= \begin{bmatrix} ax+by+cz+d \\ ex+fy+gz+h \\ kx+ly+mz+n \end{bmatrix} = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix} \\ &= [L]\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix}, \text{ where } [L] = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix}. \end{aligned}$$

2. $\begin{cases} x' & = ax+by+cz+d \\ y' & = ex+fy+gz+h \\ z' & = kx+ly+mz+n \end{cases} \Leftrightarrow \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix}$

Observe that the representation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix}$$

is equivalent to

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

In this section we identify the points $(x, y, z) \in \mathbb{R}^3$ with the points $(x, y, z, 1) \in \mathbb{R}^4$ and even with the punctured lines of $\mathbb{R}^4$, (rx, ry, rz, r), $r \in \mathbb{R}^*$. Due to technical reasons we shall actually identify the points $(x, y, z) \in \mathbb{R}^3$ with the punctured lines of $\mathbb{R}^4$ represented in the form

$$\begin{bmatrix} rx \\ ry \\ rz \\ r \end{bmatrix}, \ r \in \mathbb{R}^*,$$

and the latter ones we shall call *homogeneous coordinates* of the point $(x, y, z) \in \mathbb{R}^3$. The set of homogeneous coordinates (x, y, z, w) will be denoted by $\mathbb{RP}^3$ and call it the real *projective space*. The homogeneous coordinates $(x, y, z, w) \in \mathbb{RP}^3$, $w \neq 0$ and $\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}, 1\right)$ represent the same element of $\mathbb{RP}^3$.

Observation 9.7. *The projective space $\mathbb{RP}^3$ is actually the quotient set $(\mathbb{R}^4 \setminus \{0\})/\sim$, where $'\sim'$ is the following equivalence relation on $\mathbb{R}^4 \setminus \{0\}$:*

$$(x, y, z, w) \sim (\alpha, \beta, \gamma, \delta) \Leftrightarrow \exists r \in \mathbb{R}^* \text{ a.î. } (x, y, z, w) = r(\alpha, \beta, \gamma, \delta).$$

Observe that the equivalence classes of the equivalence relation $\sim'$ are the punctured lines of $\mathbb{R}^3$ through the origin without the origin itself, i.e. the elements of the real projective plane $\mathbb{RP}^3$.

Definition 9.8. *A* projective transformation *of the projective space $\mathbb{RP}^3$ is a transformation*

$$L : \mathbb{RP}^3 \longrightarrow \mathbb{RP}^3,\ L\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ p & q & r & s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} ax + by + cz + dw \\ ex + fy + gz + hw \\ kx + ly + mz + nw \\ px + qy + rz + sw \end{bmatrix}, \tag{9.3}$$

where $a, b, c, d, e, f, g, h, k, l, m, n, p, q, r, s \in \mathbb{R}$. Note that

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ p & q & r & s \end{bmatrix}$$

is called the homogeneous transformation matrix *of L.*

Observe that a projective transformation (9.3) is well defined since

$$L\begin{bmatrix} tx \\ ty \\ tz \\ tw \end{bmatrix} = \begin{bmatrix} atx+bty+ctz+dtw \\ etx+fty+gtz+htw \\ ktx+lty+mtz+ntw \\ ptx+qty+rtz+tsw \end{bmatrix} = \begin{bmatrix} t(ax+by+cz+dw) \\ t(ex+fy+gz+hw) \\ t(kx+ly+mz+nw) \\ t(px+qy+rz+sw) \end{bmatrix}.$$

If $p=q=r=0$ and $s\neq 0$, then the projective transformation (9.3) is said to be *affine*. The restriction of the affine transformation (9.3), which corresponds to the situation $p=q=r=0$ and $s=1$, to the subspace $w=1$, has the form

$$L\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} ax+by+cz+d \\ ex+fy+gz+h \\ kx+ly+mz+n \\ 1 \end{bmatrix}, \tag{9.4}$$

i.e.

$$\begin{cases} x' &= ax+by+cz+d \\ y' &= ex+fy+gz+h \\ z' &= kx+ly+mz+n. \end{cases} \tag{9.5}$$

Observation 9.9. *If $L_1, L_2 : \mathbb{RP}^3 \longrightarrow \mathbb{RP}^3$ are two projective applications, then their product (concatenation) transformation $L_1 \circ L_2$ is also a projective transformation and its homogeneous transformation matrix is the product of the homogeneous transformation matrices of L_1 and L_2 .*

Indeed, if

$$L_1\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ e_1 & f_1 & g_1 & h_1 \\ k_1 & l_1 & m_1 & n_1 \\ p_1 & q_1 & r_1 & s_1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

and

$$L_2\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a_2 & b_2 & c_2 & d_2 \\ e_2 & f_2 & g_2 & h_2 \\ k_2 & l_2 & m_2 & n_2 \\ p_2 & q_2 & r_2 & s_2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

then

$$(L_1\circ L_2)\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \left(\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ e_1 & f_1 & g_1 & h_1 \\ k_1 & l_1 & m_1 & n_1 \\ p_1 & q_1 & r_1 & s_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 & c_2 & d_2 \\ e_2 & f_2 & g_2 & h_2 \\ k_2 & l_2 & m_2 & n_2 \\ p_2 & q_2 & r_2 & s_2 \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Observation 9.10. *If $L_1, L_2 : \mathbb{RP}^3 \longrightarrow \mathbb{RP}^3$ are two affine applications, then their product $L_1 \circ L_2$ is also an affine transformation.*

9.4 Transformations of the space in homogeneous coordinates

9.4.1 Translations

The homogeneous transformation matrix of the translation

$$T(h,k,l):\mathbb{R}^3\to\mathbb{R}^3,\ T(h,k,l)(x_1,x_2,x_3)=(x_1+h,x_2+k,x_3+l)$$

is

$$\begin{bmatrix} 1 & 0 & 0 & h \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

9.4.2 Scaling about the origin

The homogeneous transformation matrix of the scaling

$$S(s_x, s_y, s_z) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \ \left[S(s_x, s_y, s_z)\right](x, y, z) = (s_x \cdot x, s_y \cdot y, s_z \cdot z)$$

is

$$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

9.4.3 Reflections about planes

1. The homogeneous transformation matrix of the reflection

$$r_{xy} : \mathbb{R}^3 \to \mathbb{R}^3, \ r_{xy}(x_1, x_2, x_3) = (x_1, x_2, -x_3)$$

of $\mathbb{R}^3$ through the xy-plane is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. The homogeneous transformation matrix of the reflection

$$r_{yz} : \mathbb{R}^3 \to \mathbb{R}^3, \ r_{yz}(x_1, x_2, x_3) = (-x_1, x_2, x_3)$$

of $\mathbb{R}^3$ through the yz-plane is

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. The homogeneous transformation matrix of the reflection

$$r_{xz} : \mathbb{R}^3 \to \mathbb{R}^3, \ r_{yz}(x_1, x_2, x_3) = (x_1, -x_2, x_3)$$

of $\mathbb{R}^3$ through the xz-plane is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. The homogeneous transformation matrix of the reflection $r_\pi : \mathbb{R}^3 \to \mathbb{R}^3$

$$r_\pi(x,y,z) = \left(\frac{(-a^2+b^2+c^2)x - 2aby - 2acz - 2ad}{a^2+b^2+c^2}, \frac{-2abx + (a^2-b^2+c^2)y - 2bcz - 2bd}{a^2+b^2+c^2}, \frac{-2acx - 2bcy + (a^2+b^2-c^2)z - 2cd}{a^2+b^2+c^2} \right).$$

through an arbitrary plane $\pi : ax_1 + bx_2 + cx_3 + d = 0$ is

$$\begin{bmatrix} \frac{-a^2+b^2+c^2}{a^2+b^2+c^2} & \frac{-2ab}{a^2+b^2+c^2} & \frac{-2ac}{a^2+b^2+c^2} & \frac{-2ad}{a^2+b^2+c^2} \\ \frac{-2ab}{a^2+b^2+c^2} & \frac{a^2-b^2+c^2}{a^2+b^2+c^2} & \frac{-2bc}{a^2+b^2+c^2} & \frac{-2bd}{a^2+b^2+c^2} \\ \frac{-2ac}{a^2+b^2+c^2} & \frac{-2bc}{a^2+b^2+c^2} & \frac{a^2+b^2-c^2}{a^2+b^2+c^2} & \frac{-2cd}{a^2+b^2+c^2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since in homogeneous coordinates multiplication by a factor does not affect the result, the above matrix can be multiplied by a factor $a^2+b^2+c^2$ to give the homogeneous matrix of a general reflection

$$\begin{bmatrix} -a^2+b^2+c^2 & -2ab & -2ac & -2ad \\ -2ab & a^2-b^2+c^2 & -2bc & -2bd \\ -2ac & -2bc & a^2+b^2-c^2 & -2cd \\ 0 & 0 & 0 & a^2+b^2+c^2 \end{bmatrix}.$$

9.4.4 Rotations

1. The homogeneous transformation matrix of the counterclockwise rotation about the positive x-axis through an angle θ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2. The homogeneous transformation matrix of the counterclockwise rotation about the positive y-axis through an angle θ is

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. The homogeneous transformation matrix of the counterclockwise rotation about the positive z-axis through an angle θ is

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

9.5 Problems

1. Find the homogeneous transformation matrix of the product (concatenation)

$$T(1,1,-2) \circ Rot_y(\pi/6),$$

   where $Rot_y(\pi/6)$ stands for the rotation about the positive y-axis through an angle θ.

2. Find the homogeneous transformation matrix of the rotation through an angle θ, of the space, about an arbitrary line.

3. Find the homogeneous transformation matrix of the rotation through an angle θ about the line PQ, where $P(2,1,5)$ and $Q(4,7,2)$.

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