LEARNING GRAPHS FROM DATA: A GENERAL OVERVIEW

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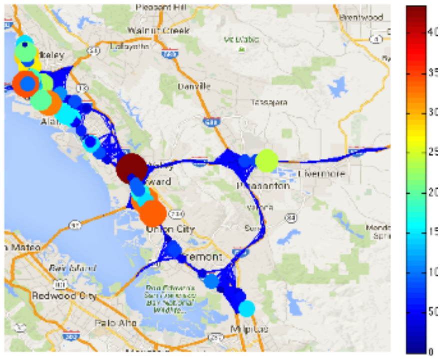
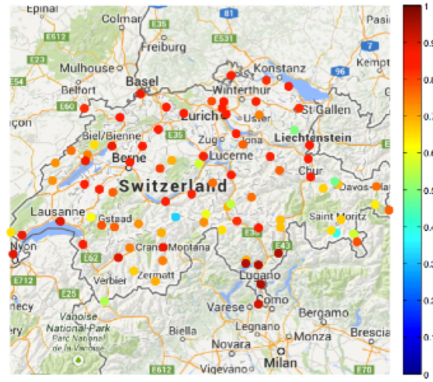
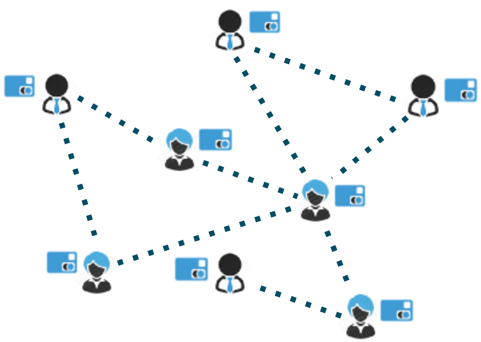
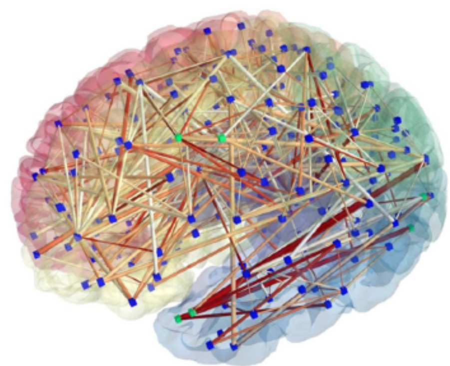
Data is often structured









Graphs: flexible tools to represent the geometric structure of signals defined on irregular domains

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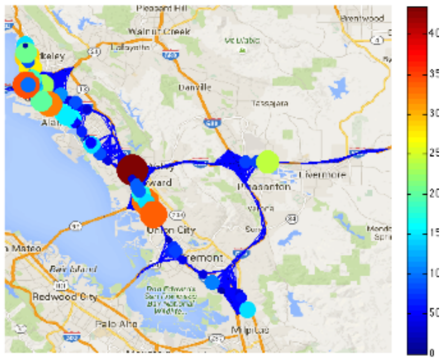
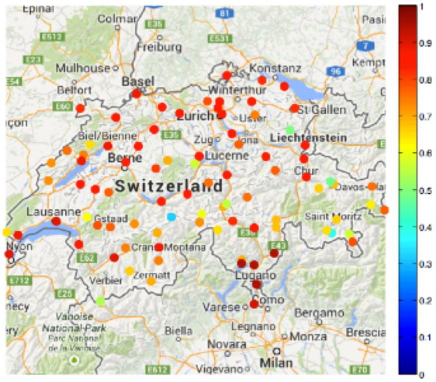
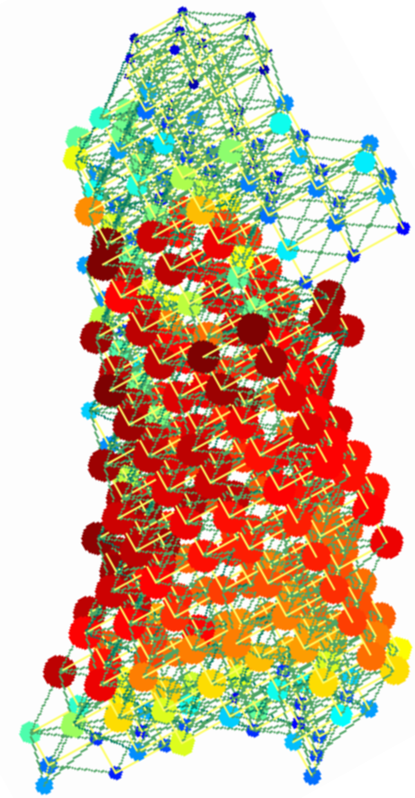
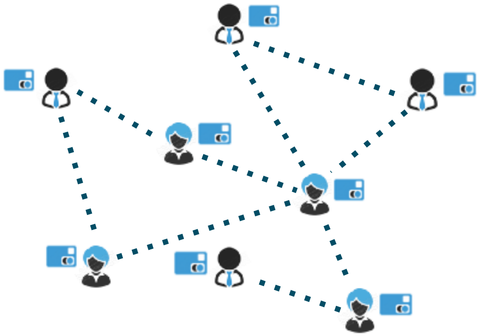
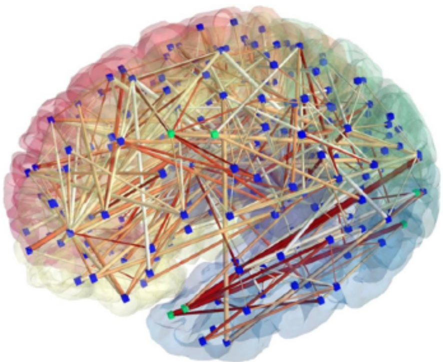






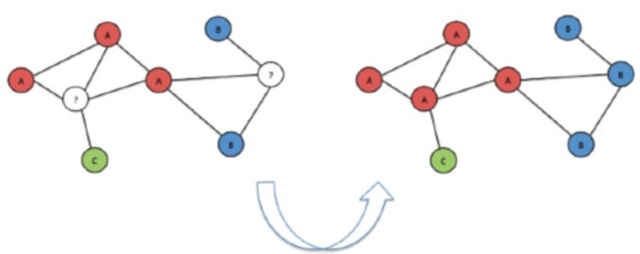




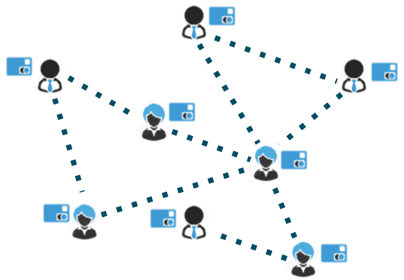
Graphs: flexible tools to represent the geometric structure of signals defined on irregular domains

The importance of the graph

- Graph signal models define the interplay between signals and structure
  - Such models are used for **effective data processing or understanding**
  - Useful for **future prediction/decision making**



Objective: Inference of missing values



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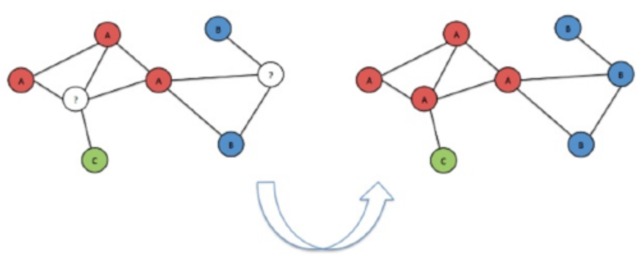

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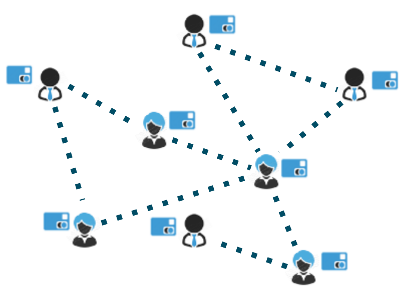
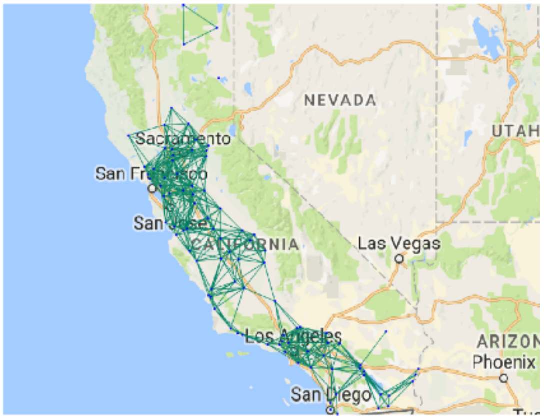


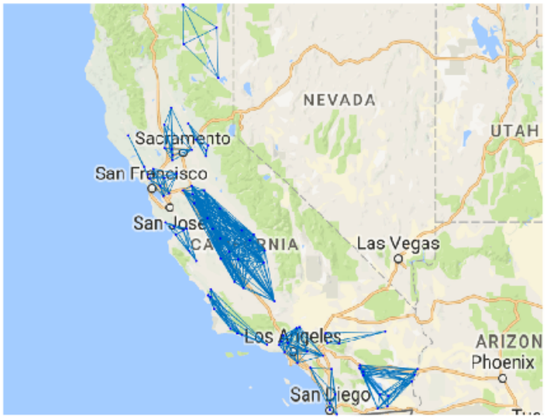
Objective: behavioural similarity/ influence between people

- The graph might (often) not be known a priori
- It becomes important to infer the proper structure

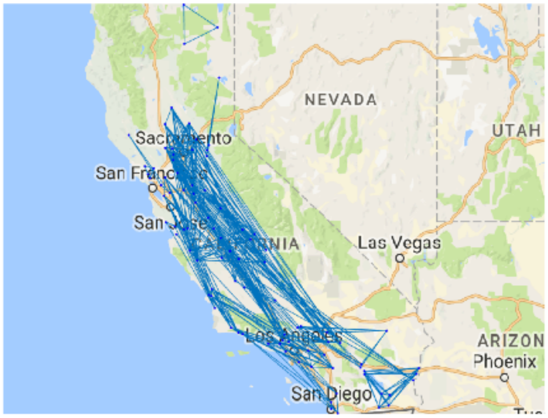
A simple example



Distance based

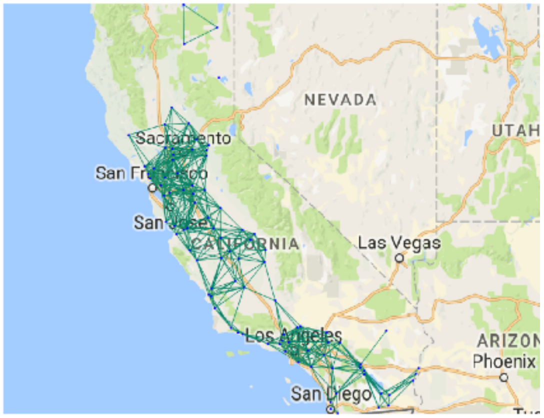


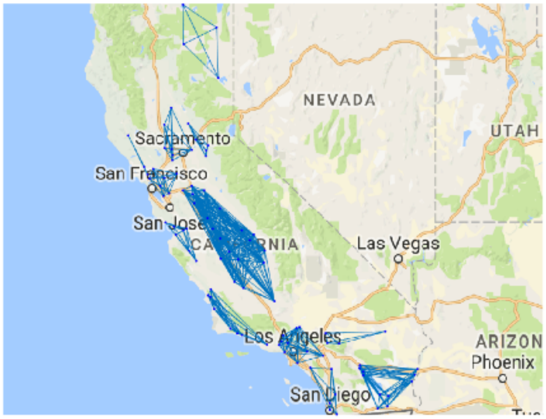
Region based

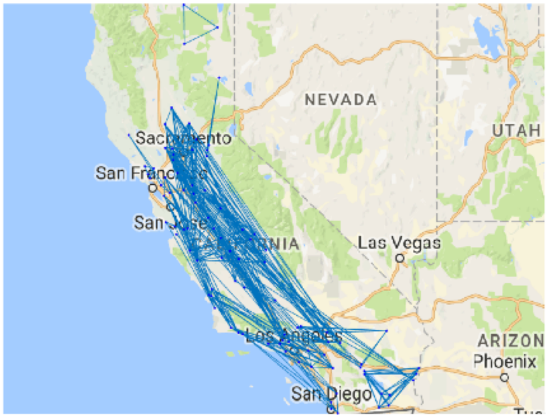


Evapotranspiration zone based

A simple example



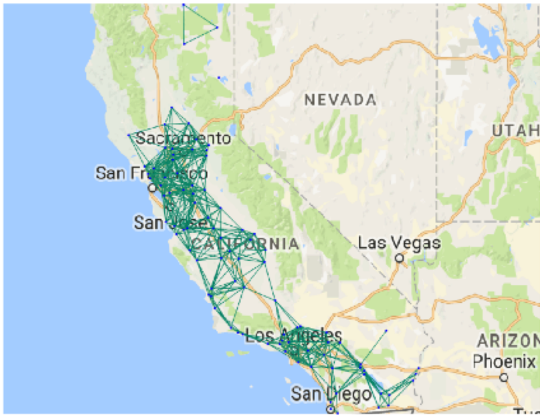




Distance based **Region based** **Evapotranspiration zone based**

- What is the structure of my data?

A simple example



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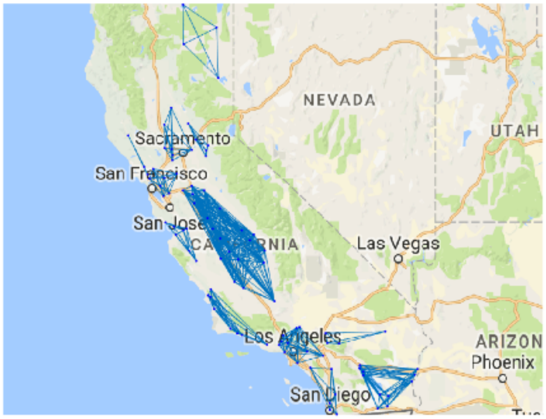

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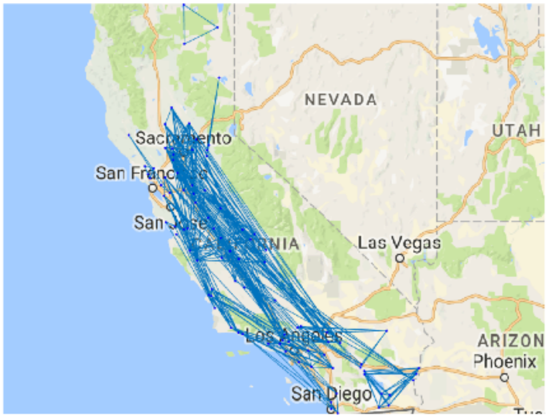

Evapotranspiration zone based

- What is the structure of my data?
- Can we use observations (e.g., temperature, humidity) to infer the right graph?

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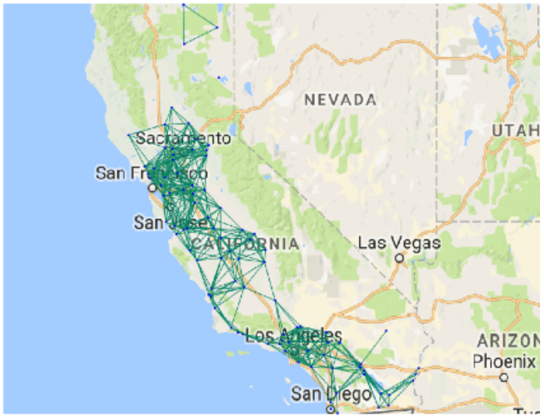

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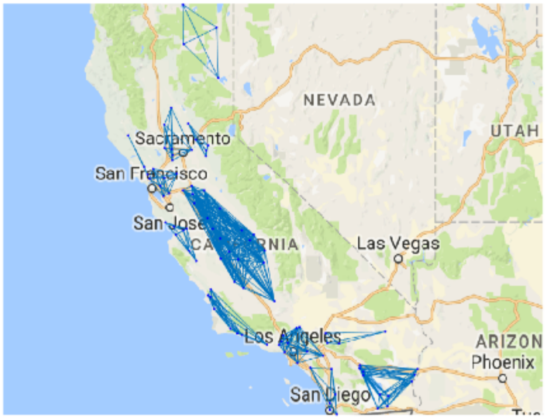

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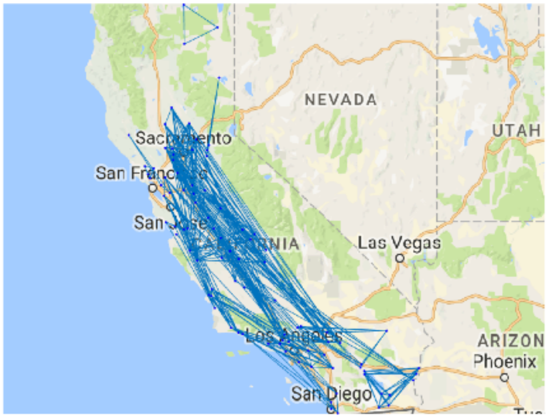
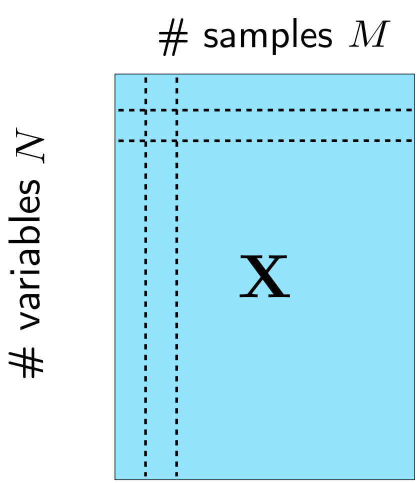


Evapotranspiration zone based

- What is the structure of my data?
- Can we use observations (e.g., temperature, humidity) to infer the right graph?
- What info about the data should the 'optimal' graph reveal?

The topology inference problem

- **Different terminologies** in the literature:
  - covariance estimation
  - learning graphical models
  - network inference/graph learning
  - …



The topology inference problem

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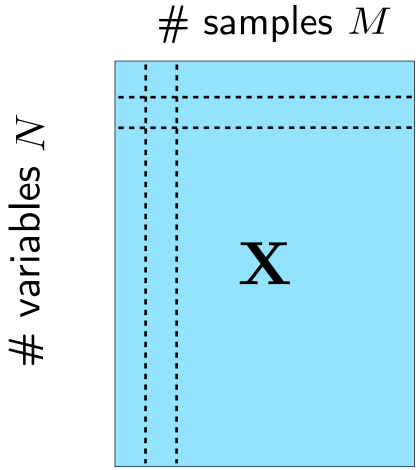
- **Common goal:** Discover structure in the data by building a measure of relations between variables



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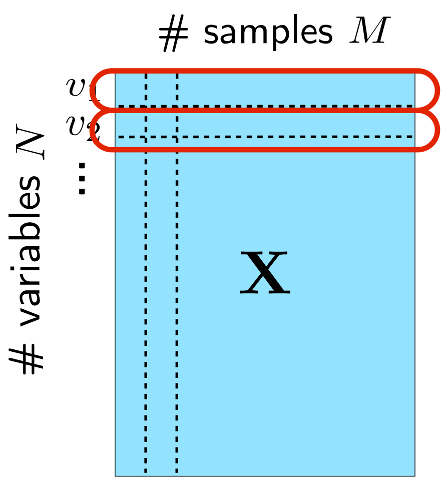

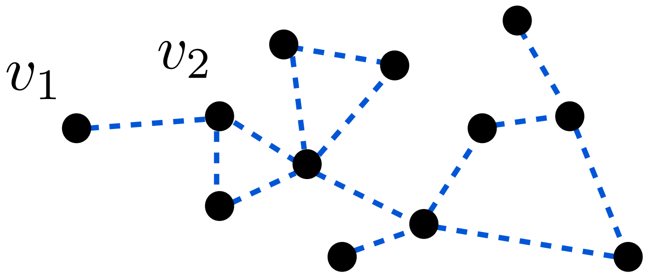

How do we build or learn a graph?

Topology inference methods

- **Model-free:**
  - do not assume any mechanism to generate the data
  - usually use statistical measures
- **Model-based:**
  - assume a mathematical model to generate the data/ signals
  - determine the model parameters to fit observed data

Model-free methods

- **Sample correlation/Pearson correlation:**

$$\rho_{ij} = \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}}$$

Rate of concurring values in two nodes

Model-free methods

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Rate of concurring values in two nodes

- **K-NN graph:** Connect each node to its K nearest neighbors

Model-free methods

- **Sample correlation/Pearson correlation:**

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Rate of concurring values in two nodes

- **K-NN graph:** Connect each node to its K nearest neighbors

- **Kernel-based graph:** Define a similarity measure based on a graph kernel (e.g., Gaussian)

$$W_{i,j} = \begin{cases} exp\left(-\frac{[dist(i,j)]^2}{2\theta^2}\right), & \text{if } dist(i,j) \leq \kappa \\ 0, & \text{otherwise.} \end{cases}$$

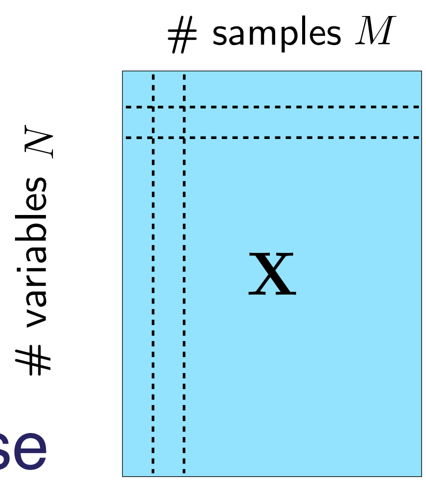
Model-free methods

✓ Computationally simple
✓ Intuitive

- Usually requires the definition of a good distance measure
- Need for parameter tuning
- Address pre/post-processing separately from connectivity inference
- Sensitive to noise

Model-based: Formal definition

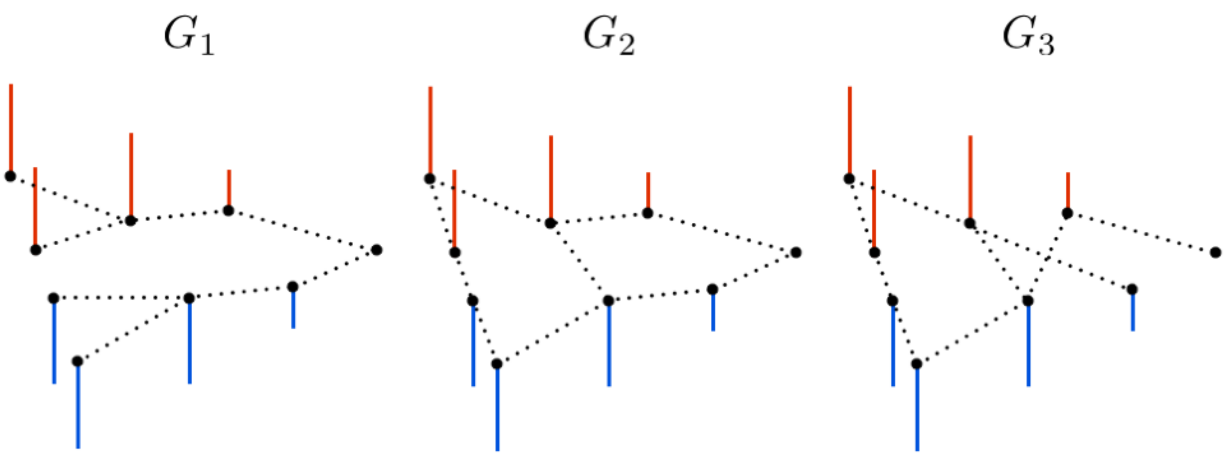
- **Given:**
  - M observations on N variables
  - some prior knowledge (e.g., distribution, data model, etc) about the data
- **Goal:**
  - Build or infer relationships between these variables that take the form of a graph



$$X = \mathcal{F}(G)$$

Model-based methods

- ✓ Easier to incorporate priors
- ✓ More robust to noise



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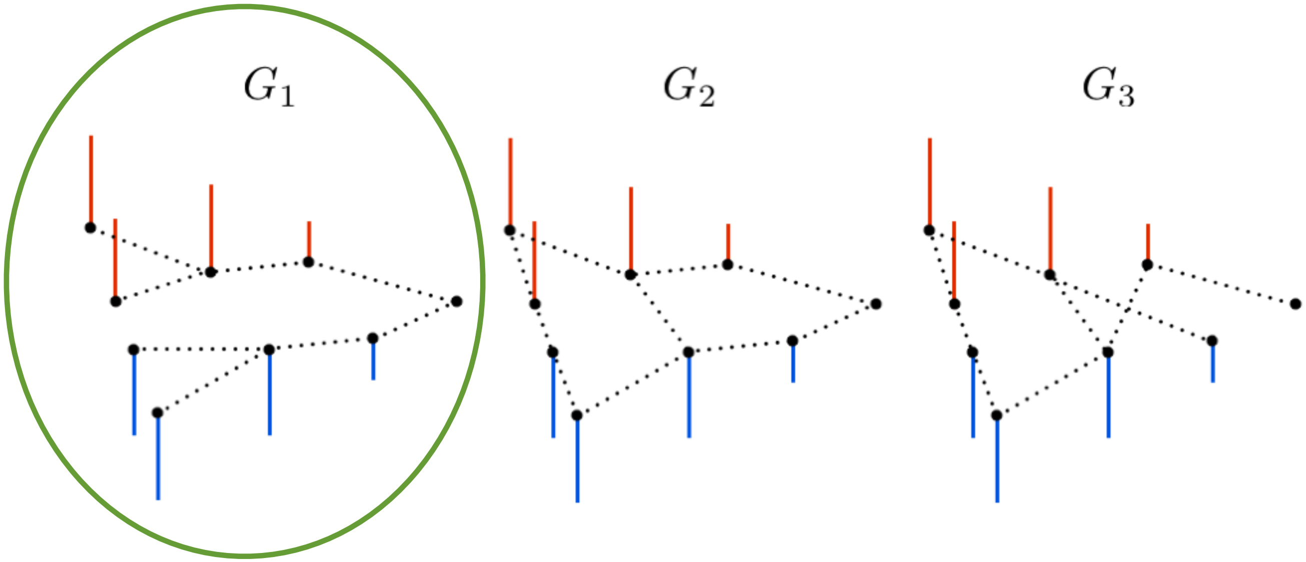

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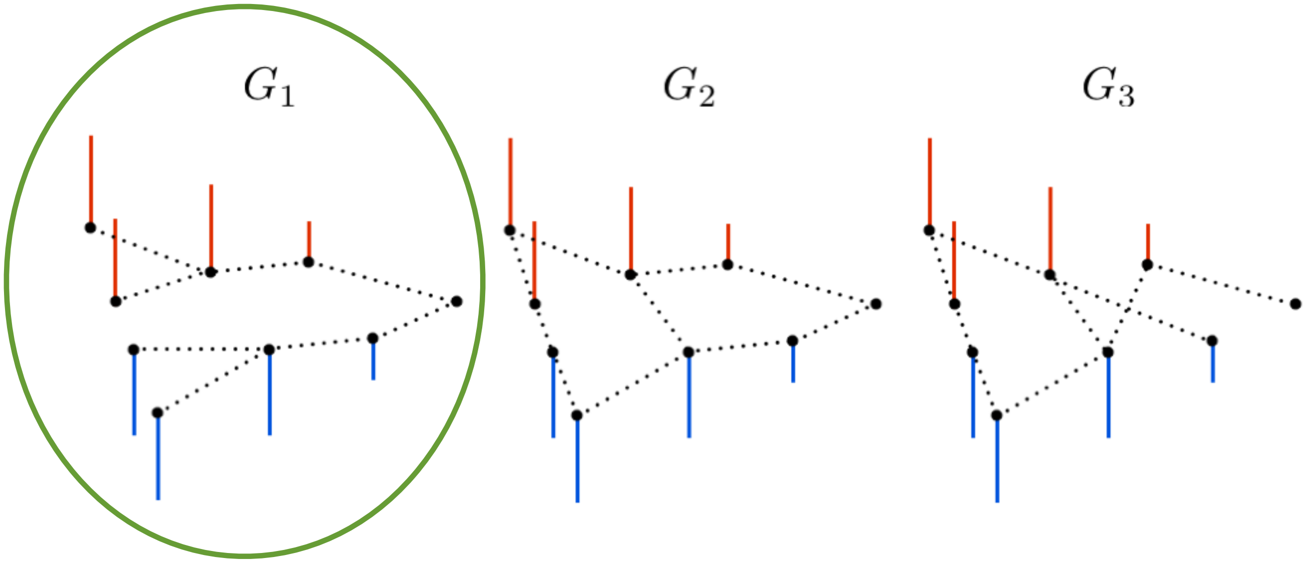
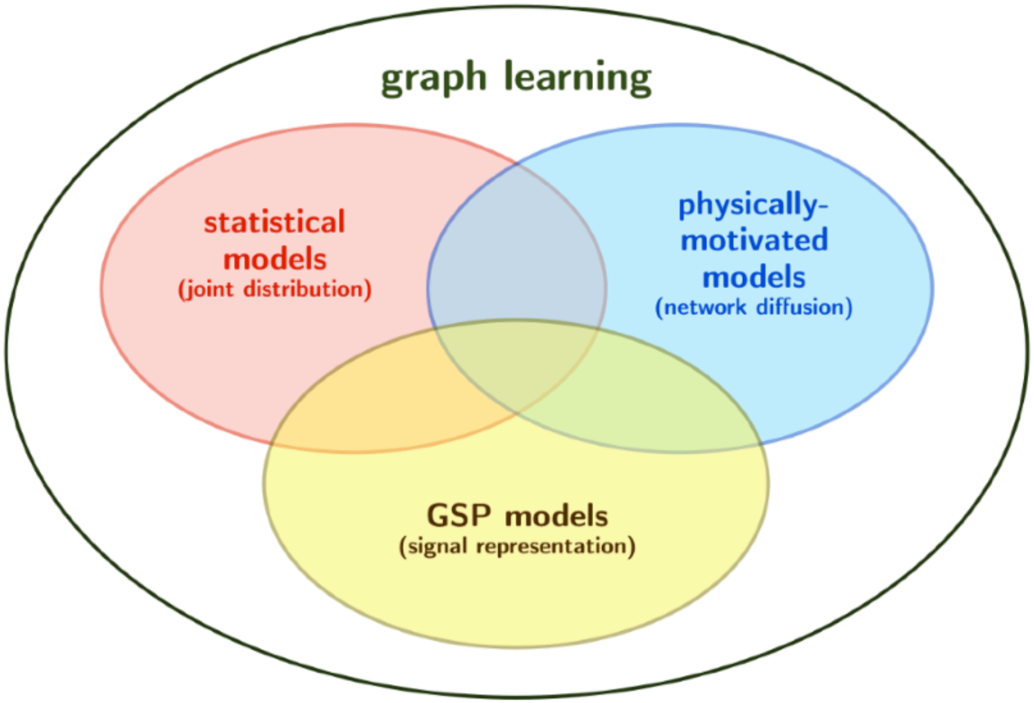


- Scalability issues: The solution of a complex optimization problem is usually required

Model-based methods in a nutshell

- Different approaches…



- … with however some similar assumptions

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Outline

- A (partial) historical overview of model-based methods
- A graph signal processing perspective
  - A quick reminder of GSP
  - GSP for topology inference
- Applications/Concluding remarks

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Historical overview

- Learning **graphical models**: probabilistic models for which the graph represent the conditional dependence between variables

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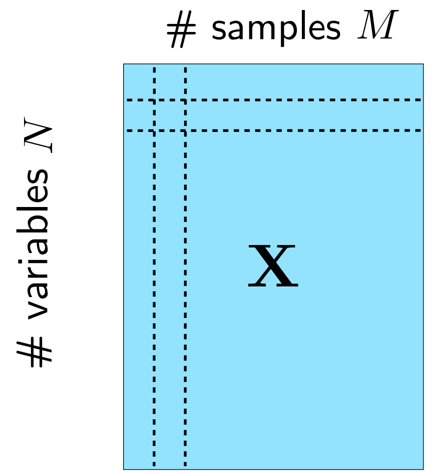
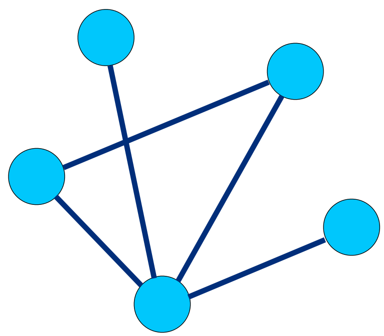
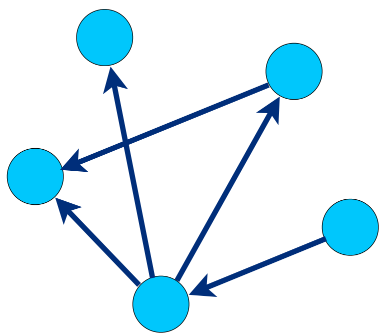




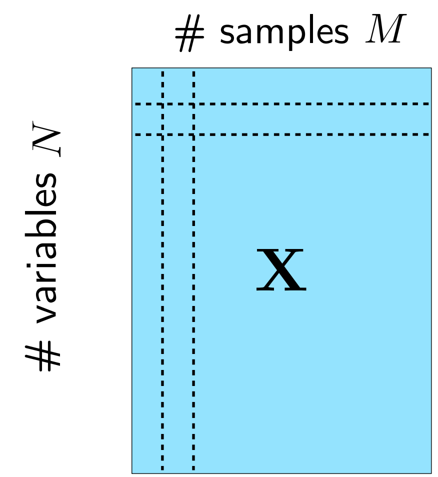
Undirected graphical models:
Markov random fields (MRF)

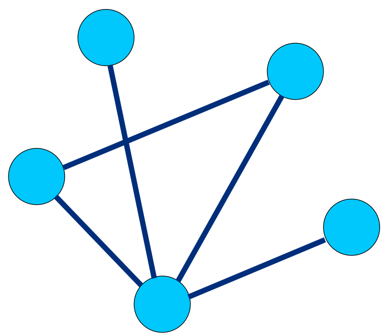


Directed graphical models:
Bayesian networks (BN)

Historical overview

- Learning **graphical models**: probabilistic models for which the graph represent the conditional dependence between variables





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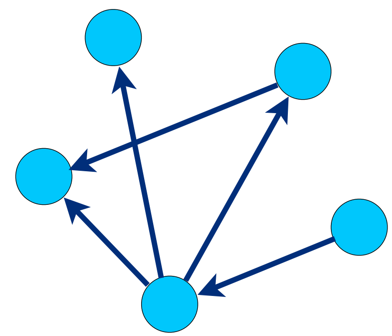

Directed graphical models:
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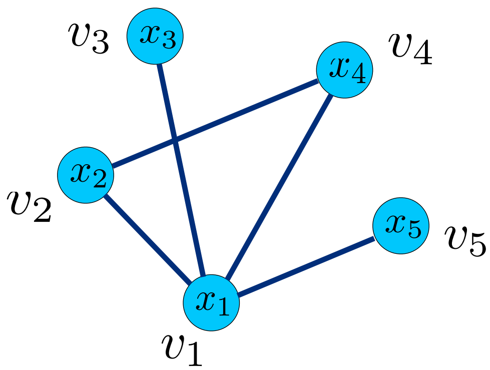
Historical overview

- Learning pairwise MRF



Historical overview

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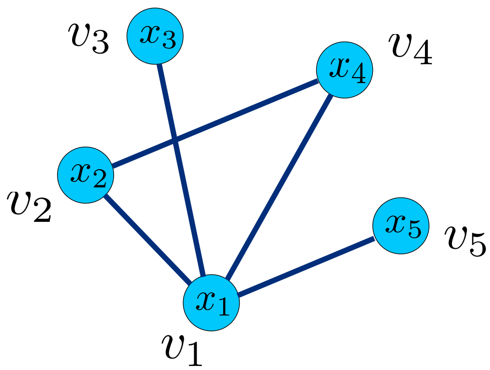

Conditional independence:

$$(i,j) \notin E \Leftrightarrow x_i \perp x_j \mid \mathbf{x} \setminus \{x_i, x_j\}$$

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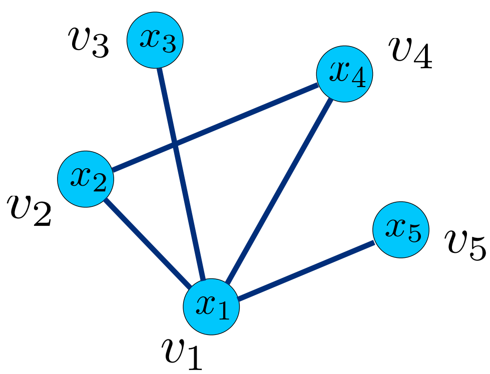

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Probability parametrized by the precision matrix (inverse covariance) Θ:

$$P(\mathbf{x}|\boldsymbol{\Theta}) = \frac{1}{Z(\boldsymbol{\Theta})} \exp\Big(\sum_{i \in V} \theta_{ii} x_i^2 + \sum_{(i,j) \in E} \theta_{ij} x_i x_j\Big)$$

Historical overview

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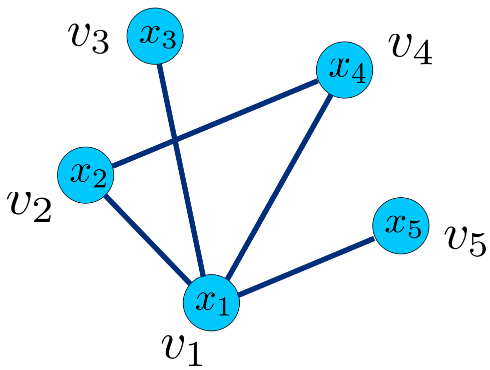

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Gaussian MRF with precision Θ:

$$P(\mathbf{x}|\boldsymbol{\Theta}) = \frac{|\boldsymbol{\Theta}|^{1/2}}{(2\pi)^{N/2}} \exp\Big(-\frac{1}{2} \mathbf{x}^T \boldsymbol{\Theta} \mathbf{x} \Big)$$

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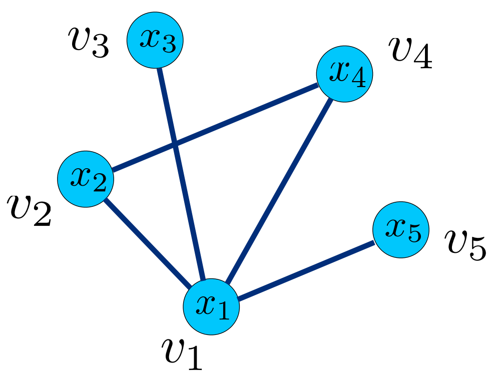

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Learning a sparse Θ :

- **interactions are mostly local**
- **computationally more tractable**

Historical overview

Prune the smallest elements in precision (inverse covariance) matrix

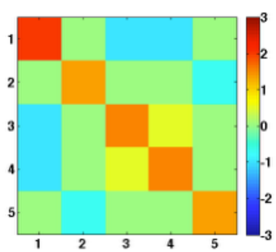
Historical overview

covariance selection

Dempster

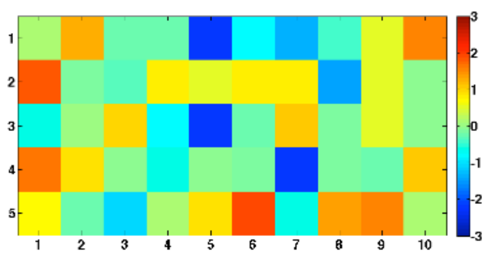
1972

Prune the smallest elements in precision (inverse covariance) matrix



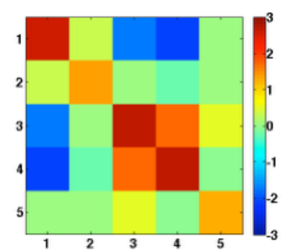
$\boldsymbol{\Theta}$

ground-truth precision



$\mathbf{X} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Theta})$

data matrix

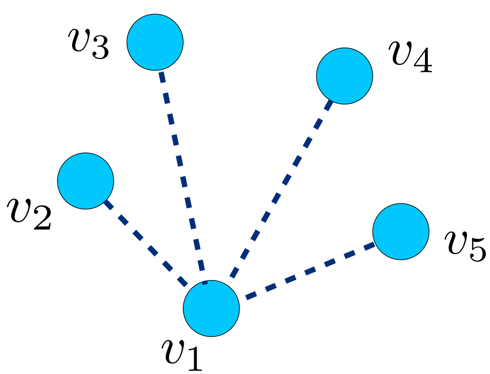


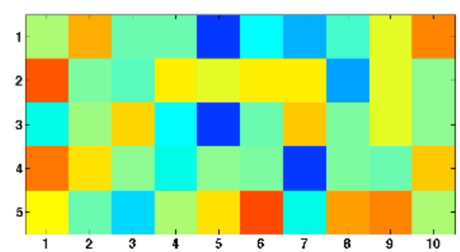
$\mathbf{S}^{-1}$

inverse of sample covariance

Historical overview

Learning a graph = learning neighbourhood of each node





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Historical overview

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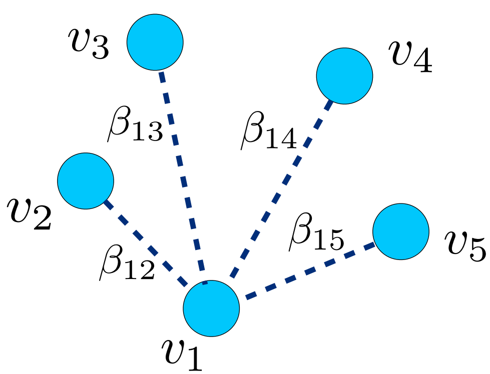

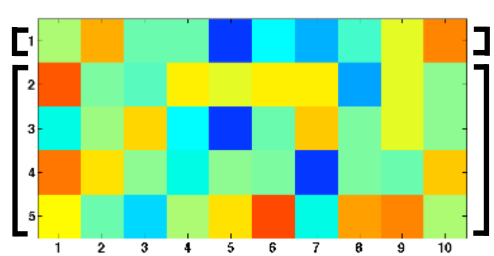

$\mathbf{X}_1{}^T$

$\mathbf{X}_{\backslash 1}{}^T$

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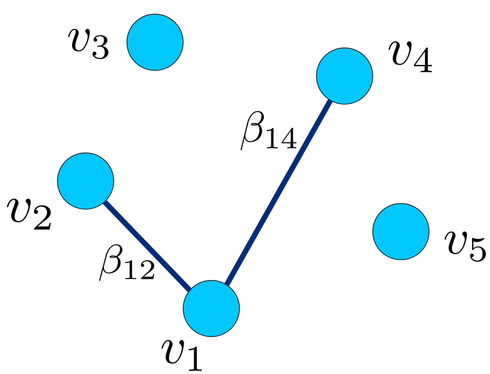

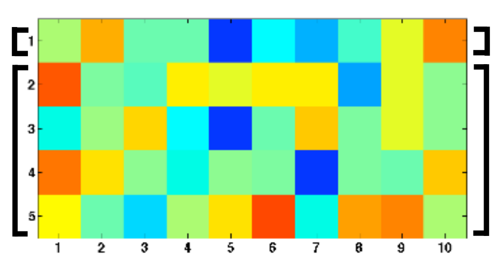

$\mathbf{X}_1{}^T$

$\mathbf{X}_{\backslash 1}{}^T$

LASSO regression: $\min_{\boldsymbol{\beta}_1} ||\mathbf{X}_1 - \mathbf{X}_{\backslash 1}\boldsymbol{\beta}_1||^2 + \lambda||\boldsymbol{\beta}_1||_1$

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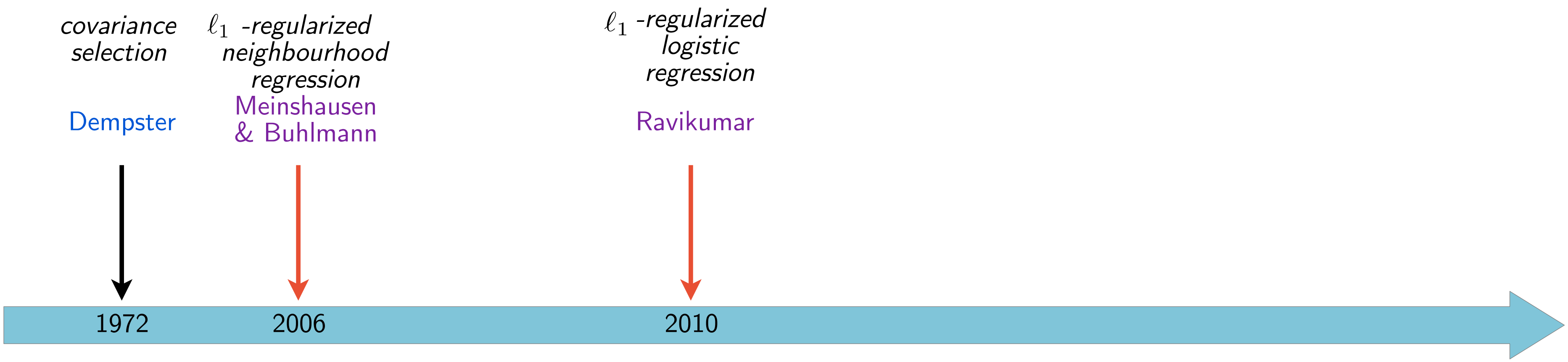
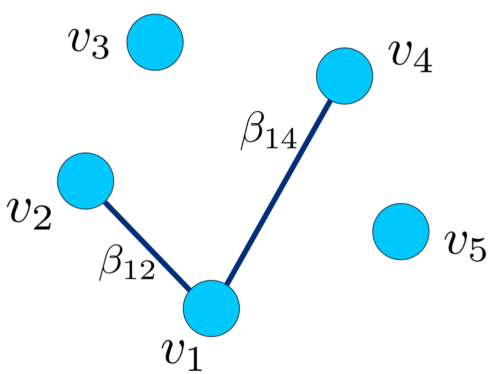
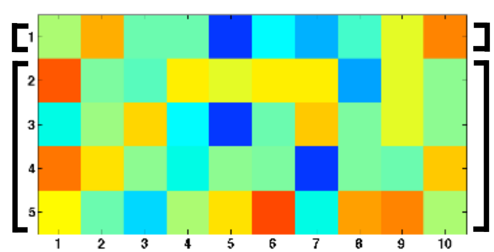
Historical overview



Learning a graph = learning neighbourhood of each node





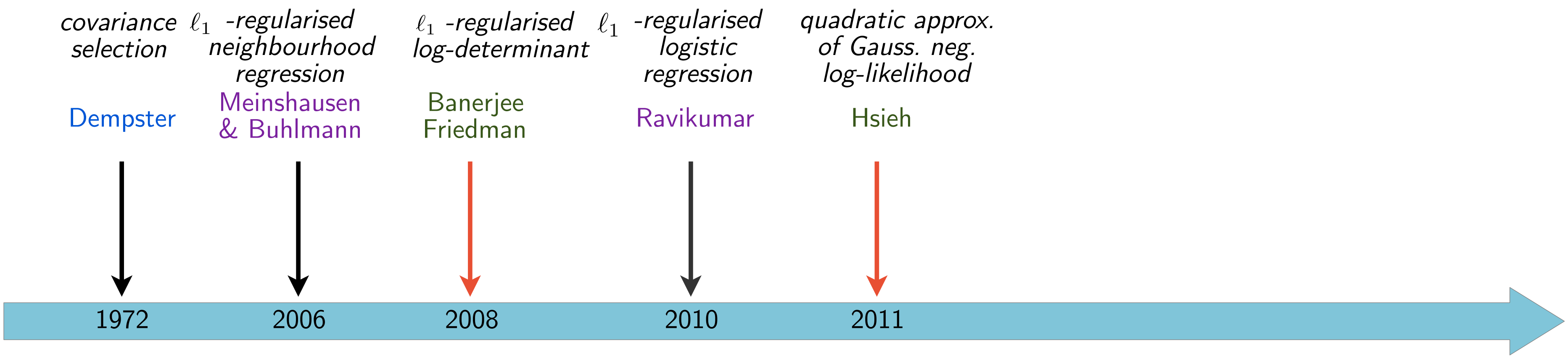
$\mathbf{X}_1{}^T$

$\mathbf{X}_{\backslash 1}{}^T$

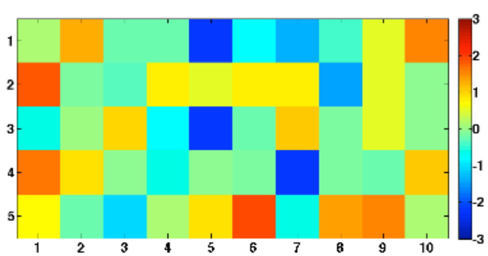
LASSO regression: $\min_{\boldsymbol{\beta}_1} ||\mathbf{X}_1 - \mathbf{X}_{\backslash 1}\boldsymbol{\beta}_1||^2 + \lambda||\boldsymbol{\beta}_1||_1$

Logistic regression for discrete variables

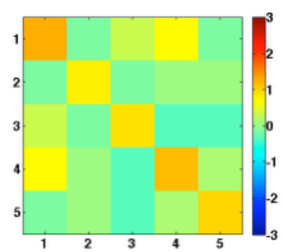
Historical overview



Estimation of sparse precision matrix



$\mathbf{X}$

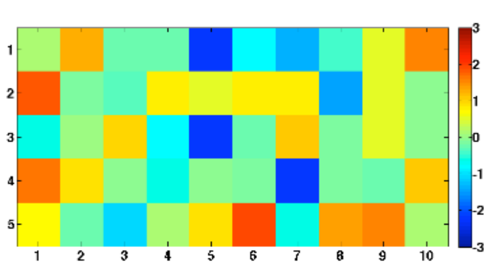


$\mathbf{S}$

Historical overview

- 1972 — *covariance selection* — Dempster
- 2006 — *ℓ_1 -regularised neighbourhood regression* — Meinshausen & Buhlmann
- 2008 — *ℓ_1 -regularised log-determinant* — Banerjee Friedman
- 2010 — *ℓ_1 -regularised logistic regression* — Ravikumar
- 2011 — *quadratic approx. of Gauss. neg. log-likelihood* — Hsieh

Estimation of sparse precision matrix



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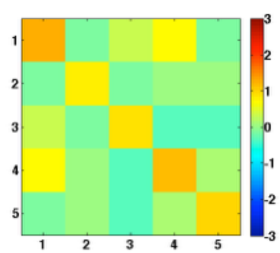

$\mathbf{S}$

graphical LASSO maximizes likelihood of precision matrix $\boldsymbol{\Theta}$:

$$|\boldsymbol{\Theta}|^{M/2}\exp\Big(-\sum_{m=1}^{M}\frac{1}{2}\mathbf{X}(m)^T\boldsymbol{\Theta}\mathbf{X}(m)\Big)$$

Historical overview

covariance selection	ℓ_1 -regularised neighbourhood regression	ℓ_1 -regularised log-determinant	ℓ_1 -regularised logistic regression	quadratic approx. of Gauss. neg. log-likelihood
Dempster	Meinshausen & Buhlmann	Banerjee Friedman	Ravikumar	Hsieh
1972	2006	2008	2010	2011

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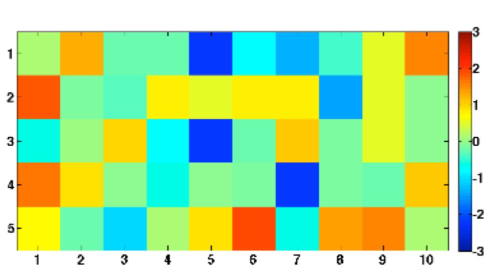

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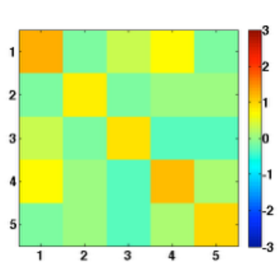

$\mathbf{S}$

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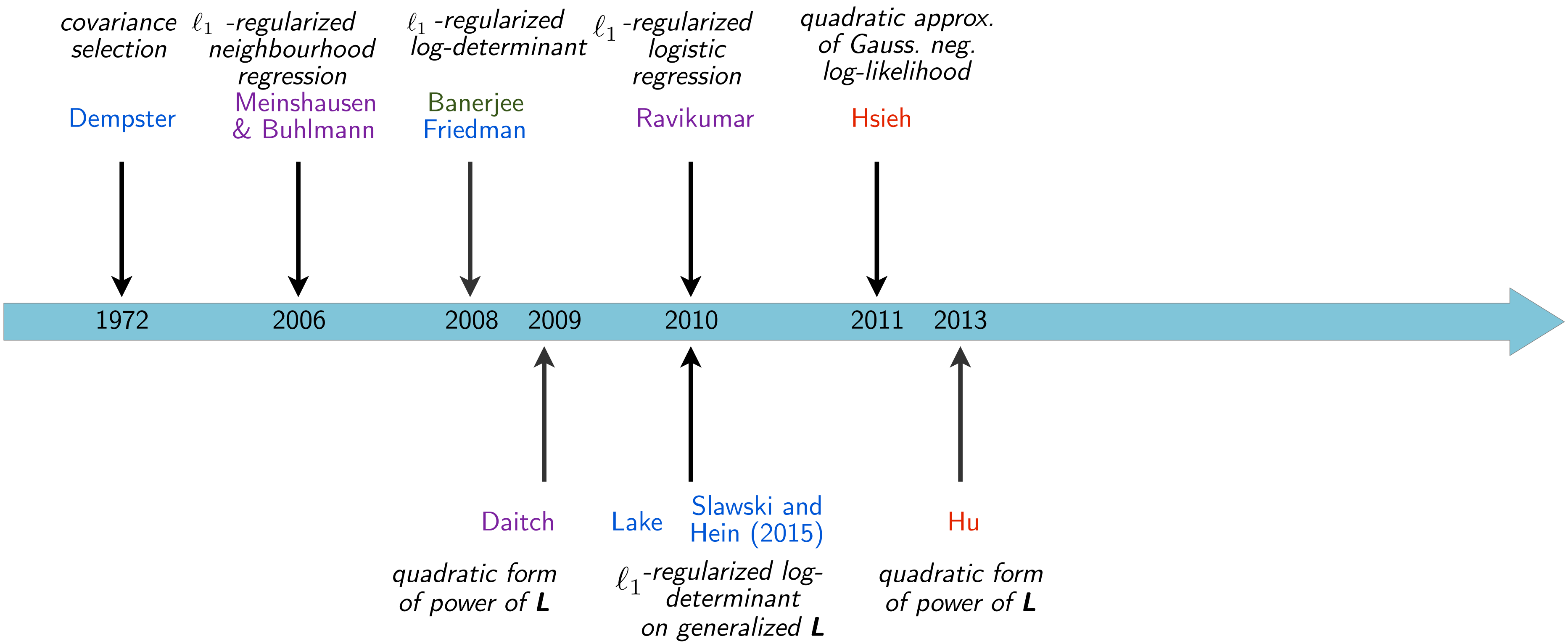
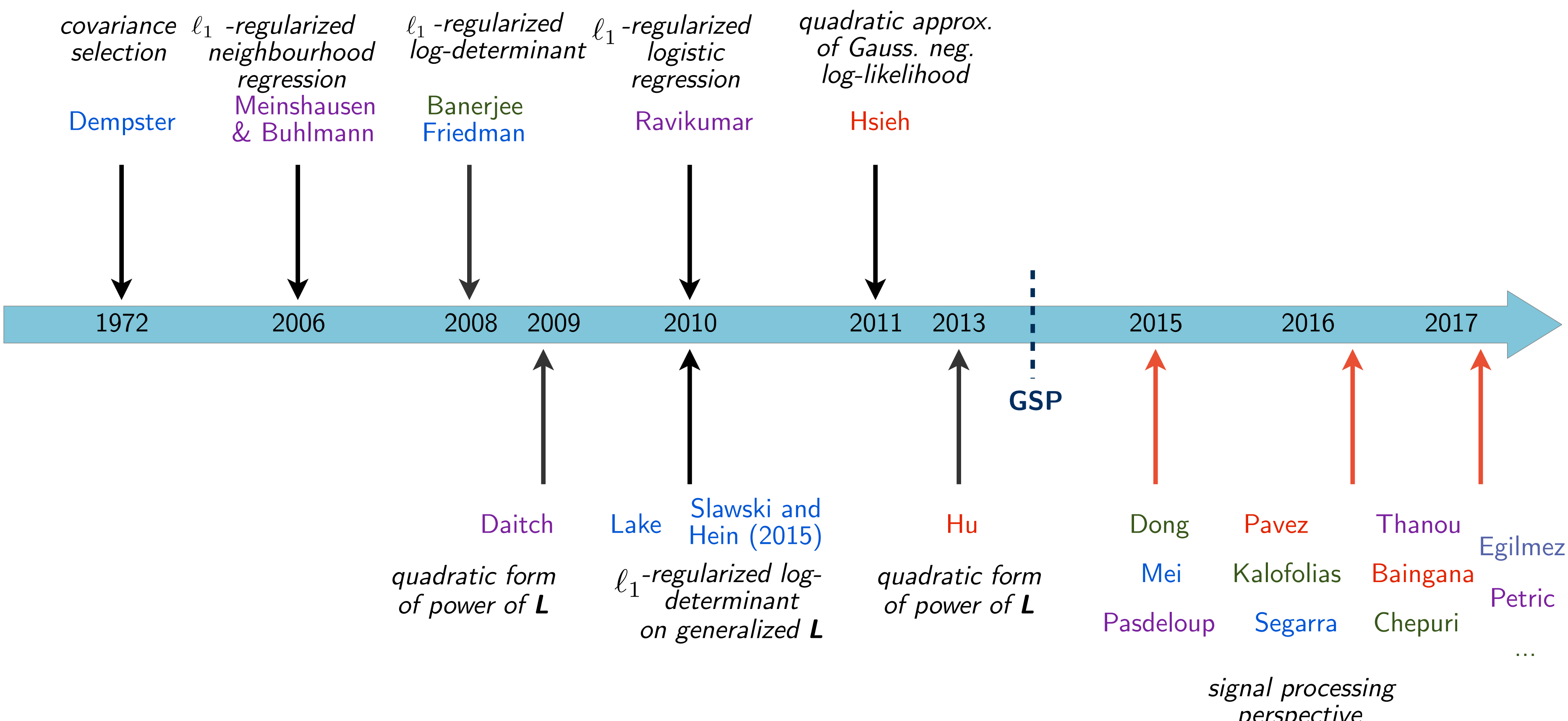
$$\max_{\boldsymbol{\Theta}} \underbrace{\log \det \boldsymbol{\Theta} - \mathrm{tr}(\mathbf{S}\boldsymbol{\Theta})}_{\text{log-likelihood function}} - \rho ||\boldsymbol{\Theta}||_1$$

Historical overview



From arbitrary precision matrix to graph Laplacian

Historical overview



From arbitrary precision matrix to graph Laplacian

Common setting in graph signal processing

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A signal processing approach?

- Existing approaches have limitations
  - Simple correlation/similarity functions are simplistic
  - There is no strong emphasis on signal/graph interaction with spectral/frequency domain interpretation

Outline

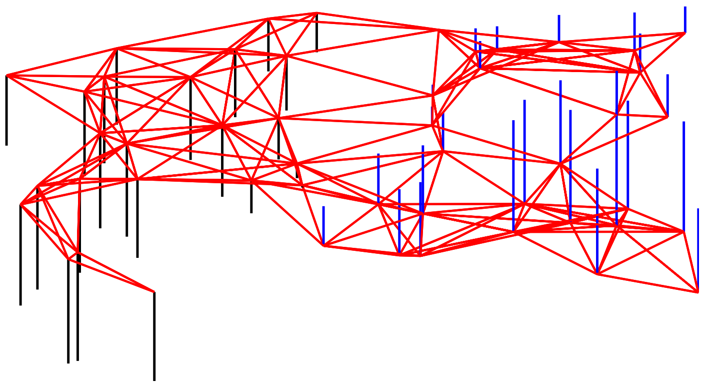
- A partial historical overview

- **A graph signal processing perspective**
  - **A quick reminder of GSP**
  - **GSP for topology inference**

- Applications/Concluding remarks

Graphs and signals on graphs

- **Graph:** A mathematical tool $G = (\mathcal{V}, \mathcal{E}, W)$ for modelling the irregular domain
  - directed/**undirected**
  - **weighted**/unweighted



- Connectivity pattern captured by:
  - Adjacency A or weight matrix W (positive weights)
  - Laplacian matrix L

- **Graph signal:** A function $x : \mathcal{V} \to \mathbb{R}$ that assigns real values to each vertex of the graph

The graph Laplacian operator

- It is defined as $L = D - W$
  - Complete set of eigenvectors $\chi = [\chi_0, \chi_1, ..., \chi_{N-1}]$
  - Non-negative eigenvalues $0 = \lambda_0 <= ... <= \lambda_{N-1}$

- Defines global smoothness on graphs

$$x^T L x = \sum_{m \in \mathcal{V}} \sum_{m \in \mathcal{N}_m} W_{m,n} [x(m) - x(n)]^2$$

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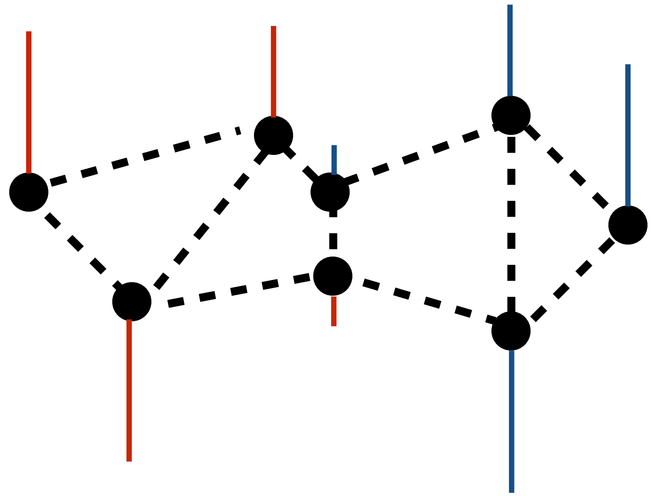

globally smooth signal

SDSC

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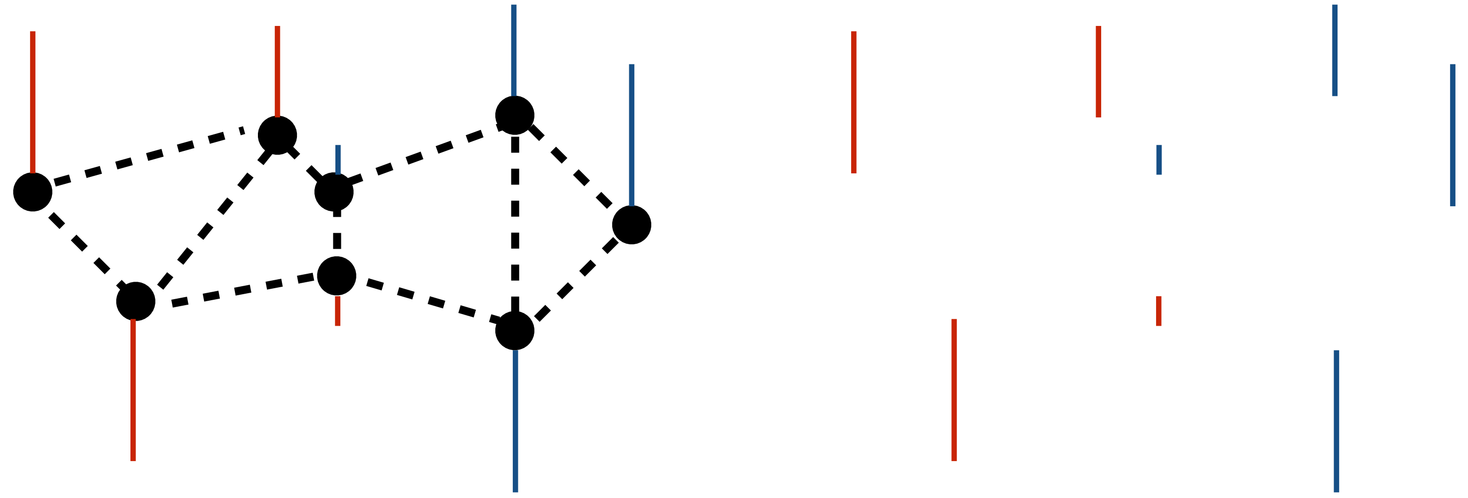

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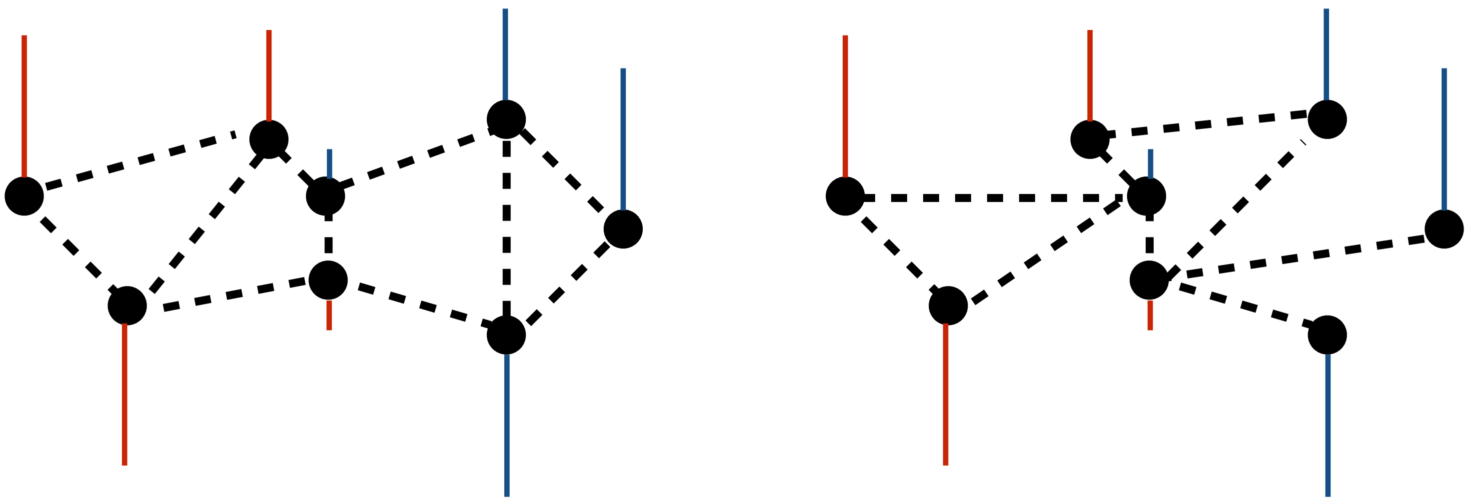

globally smooth signal

SDSC

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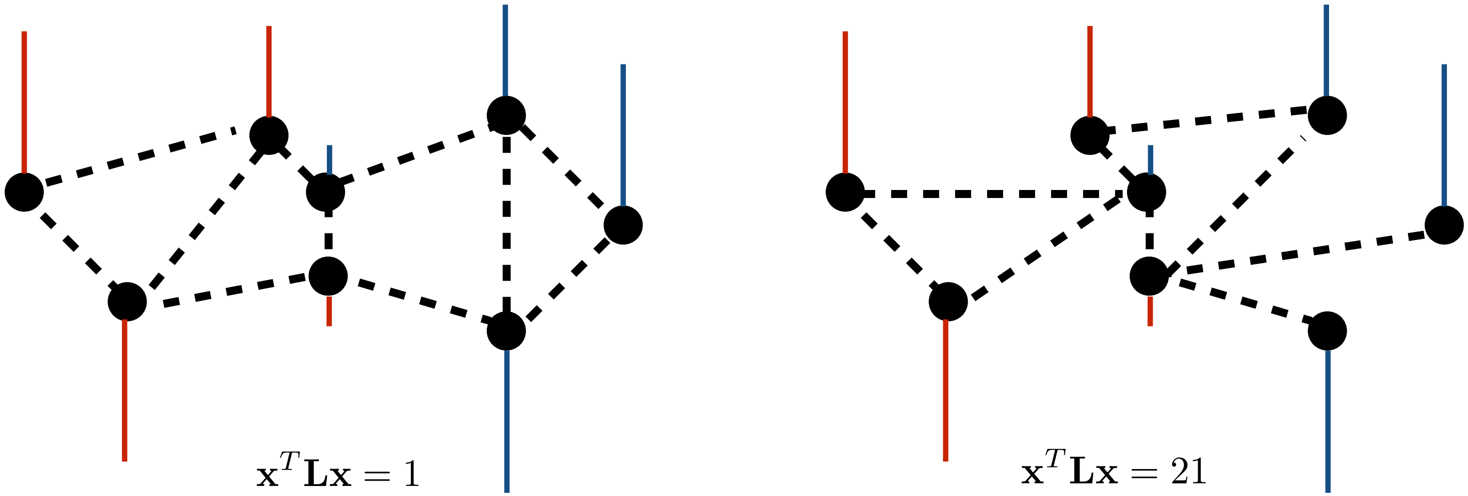


$\mathbf{x}^T \mathbf{L} \mathbf{x} = 1$

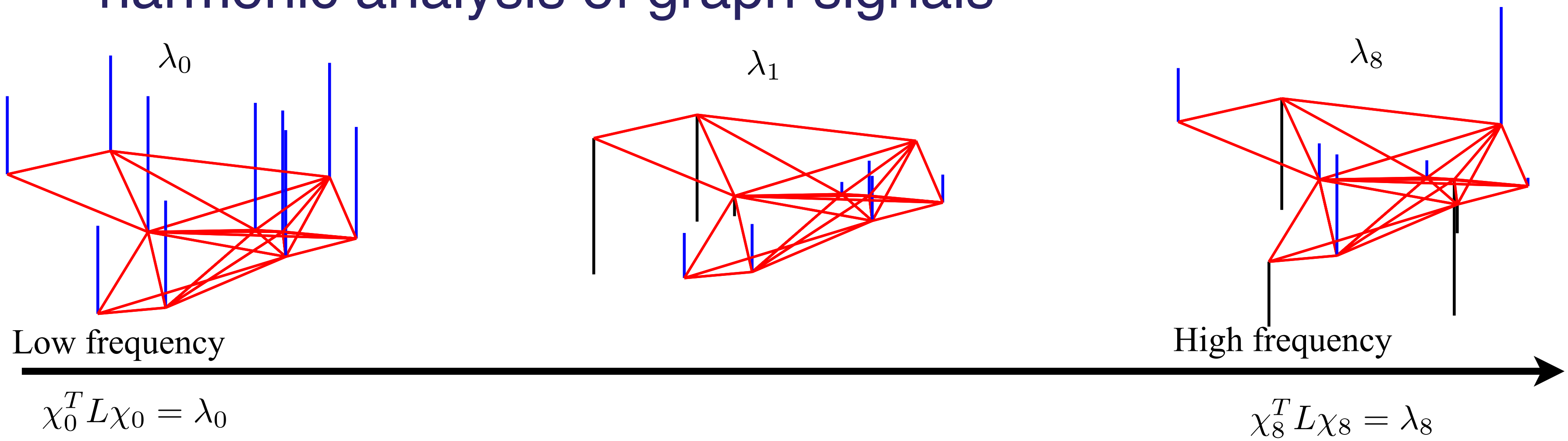
globally smooth signal

$\mathbf{x}^T \mathbf{L} \mathbf{x} = 21$

high frequency signal

The graph Fourier transform

- The eigenvectors of the Laplacian provide an harmonic analysis of graph signals



GFT $$\hat{x}(\lambda_\ell) = < x, \chi_\ell > = \sum_{n=1}^{N} x(n)\chi_\ell(n)$$

IGFT $$x(n) = \sum_{\ell=0}^{N-1} \hat{x}(\lambda_\ell)\chi_\ell(n)$$

Filtering on graphs

- Filtering in the spectral domain with a transfer function $\hat{g}(\cdot)$

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x_{in}

Filtering on graphs

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$$x_{in} \xrightarrow{\text{GFT}} \chi^T x_{in}$$

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Filtering on graphs

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$$x_{in} \xrightarrow{\text{GFT}} \chi^T x_{in} \xrightarrow{\hat{g}(\Lambda)} \widehat{g}(\Lambda)\chi^T x_{in} \xrightarrow{\text{IGFT}} \chi\widehat{g}(\Lambda)\chi^T x_{in}$$

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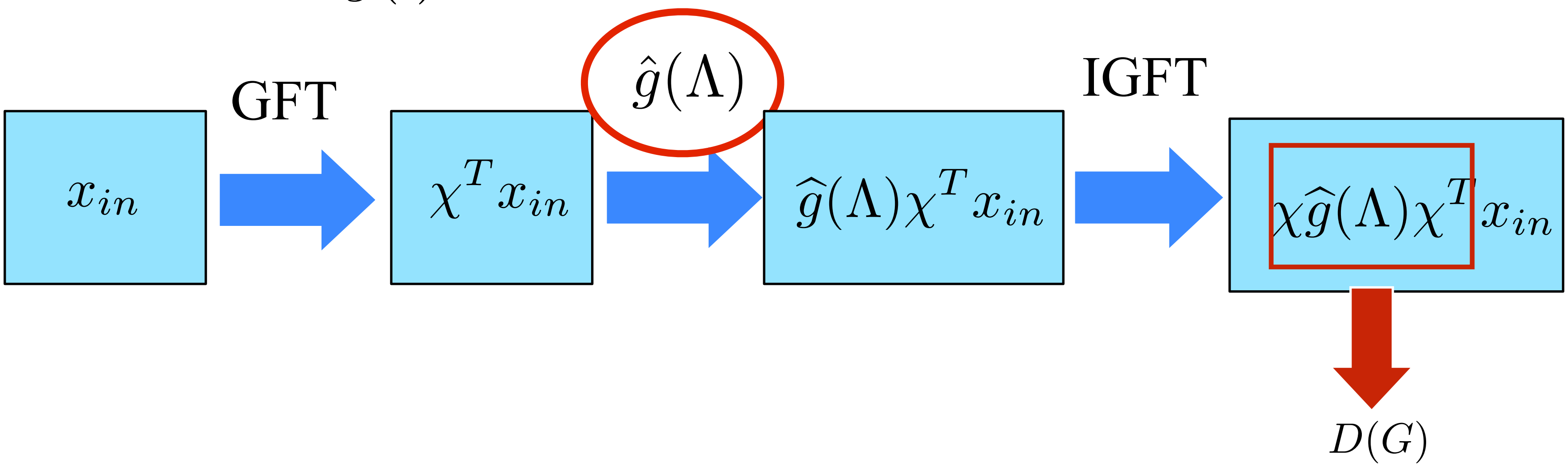
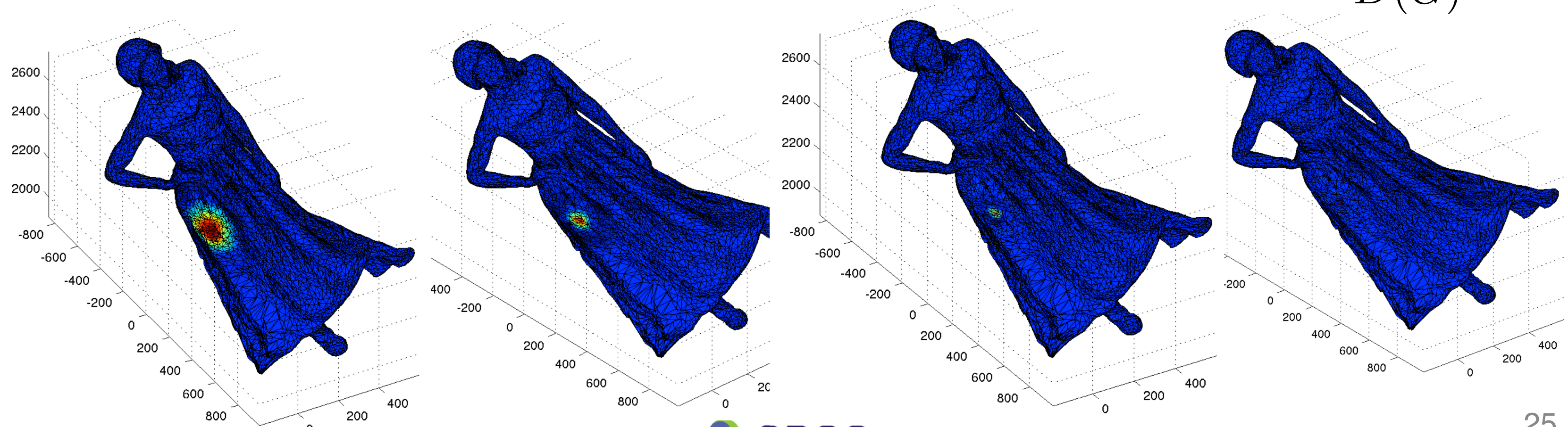


Filtering on graphs

- Filtering in the spectral domain with a transfer function $\hat{g}(\cdot)$

$$x_{in} \xrightarrow{\text{GFT}} \chi^T x_{in} \xrightarrow{\hat{g}(\Lambda)} \hat{g}(\Lambda)\chi^T x_{in} \xrightarrow{\text{IGFT}} \boxed{\chi\hat{g}(\Lambda)\chi^T} x_{in}$$

$$\boxed{\chi\hat{g}(\Lambda)\chi^T} \longrightarrow D(G)$$



Topology inference revisited: A GSP perspective

- Define graph signal representation models
- Strong emphasis on signal/graph interaction with spectral/frequency-domain interpretation

SDSC

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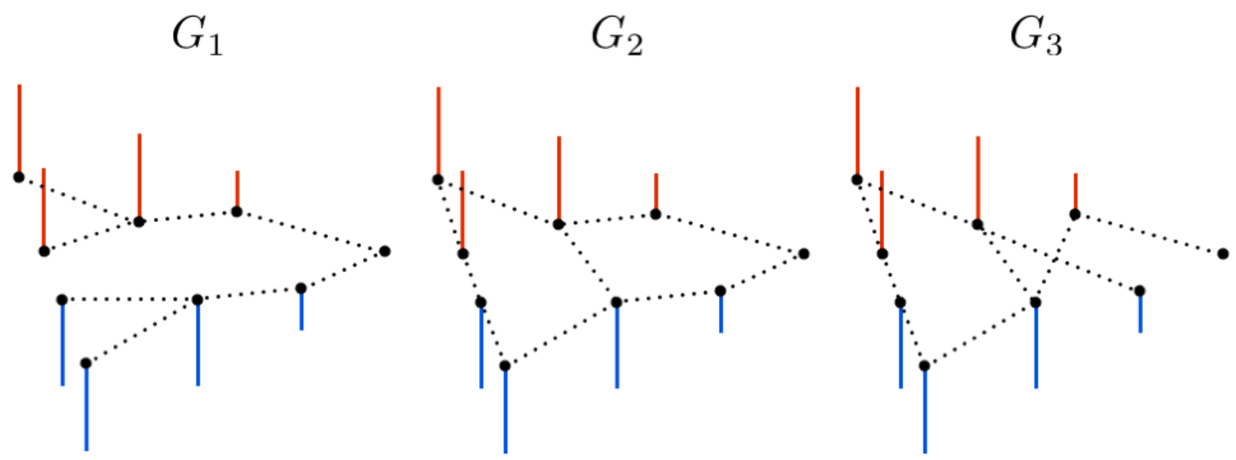

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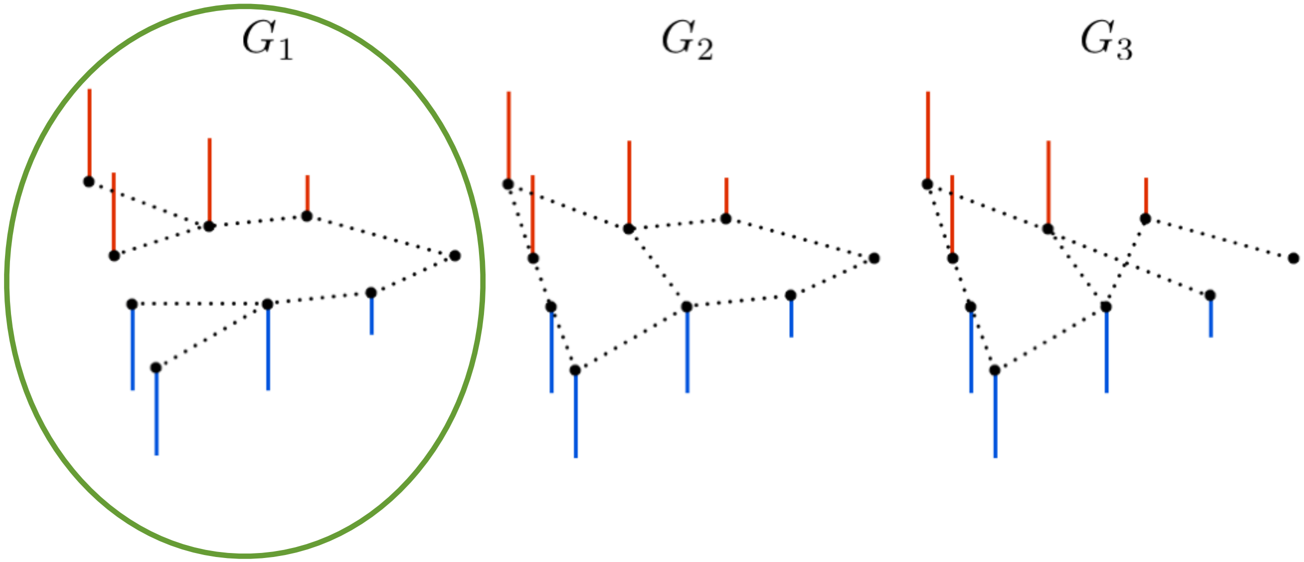

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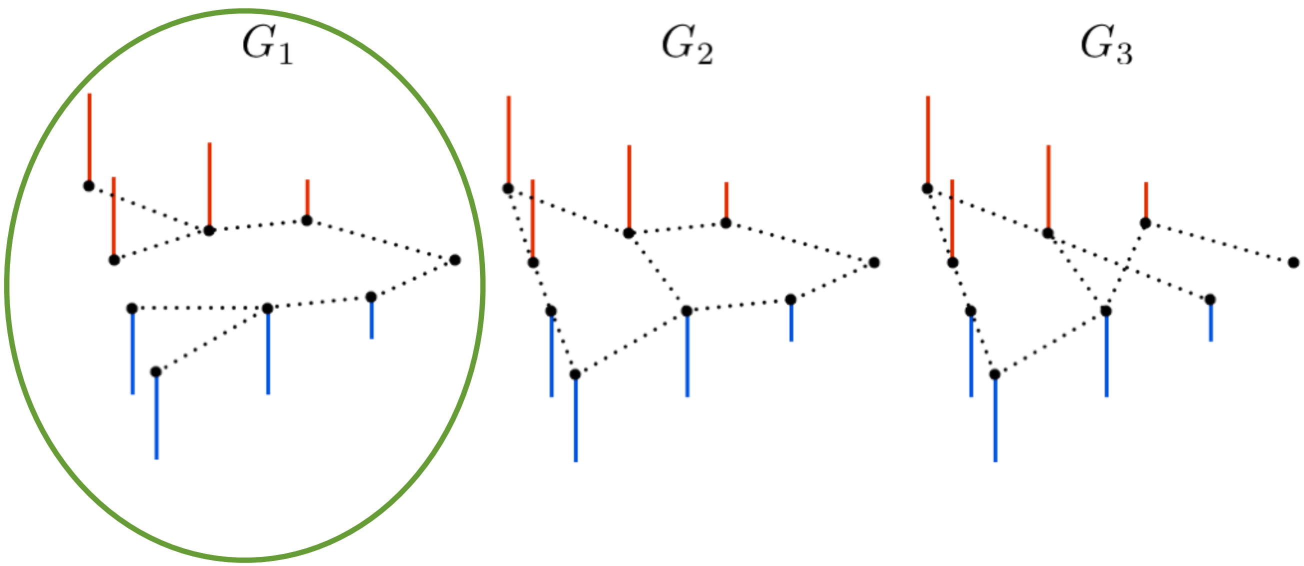
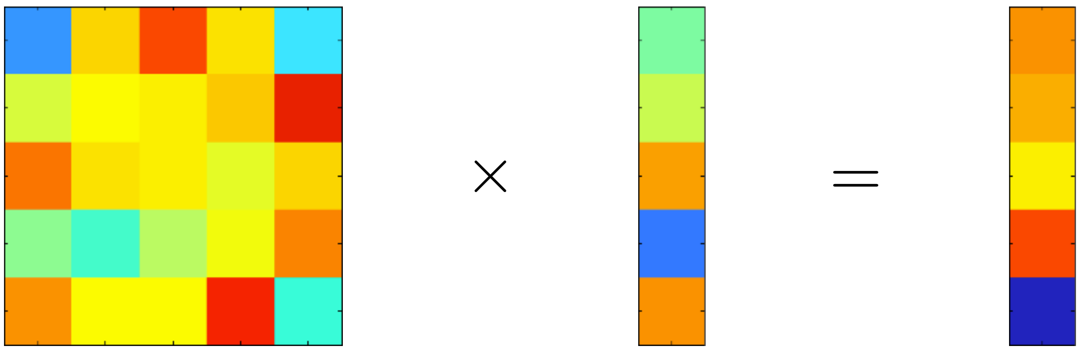


Can we exploit the interplay between graphs and signals on graphs to discover the topology?

Graph signal representations

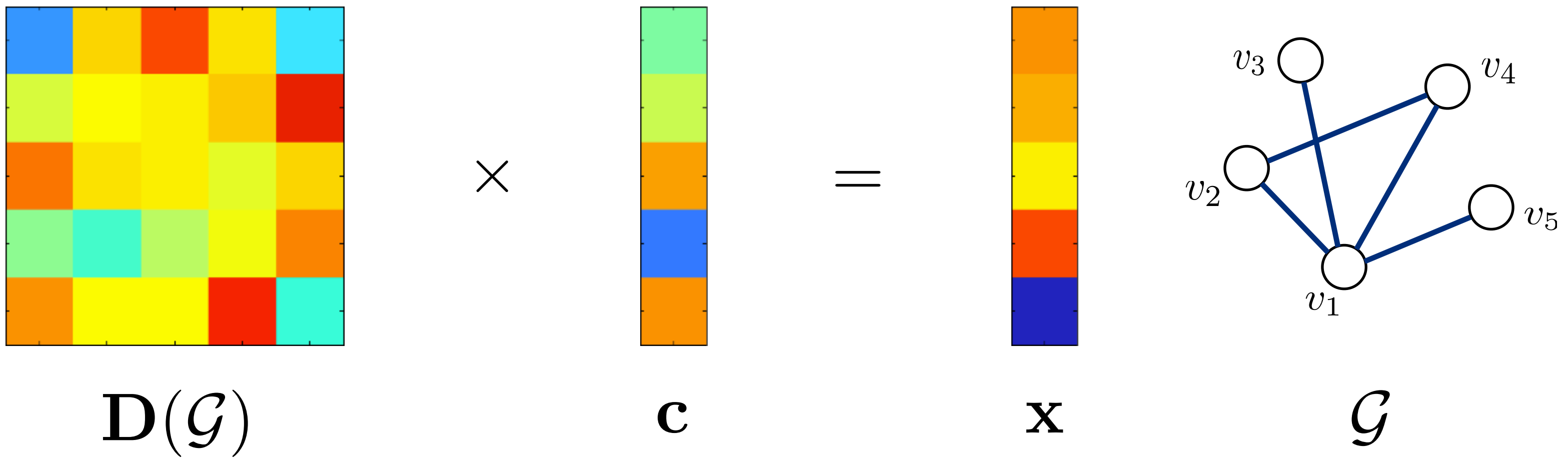
- Signal processing is about $Dc \approx x$



$$\mathbf{D} \times \mathbf{c} = \mathbf{x}$$

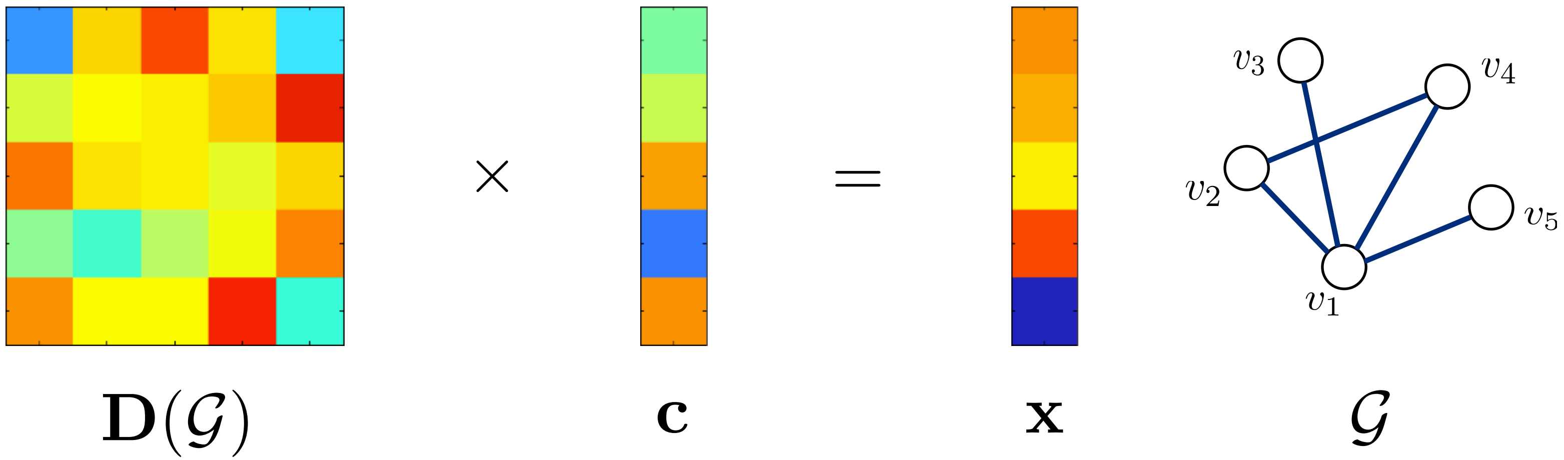
Graph signal representations

- Signal processing is about $Dc \approx x$
- Graph signal processing is about $D(G)c \approx x$



Graph signal representations

- Forward step: Given x, G, design D to study c



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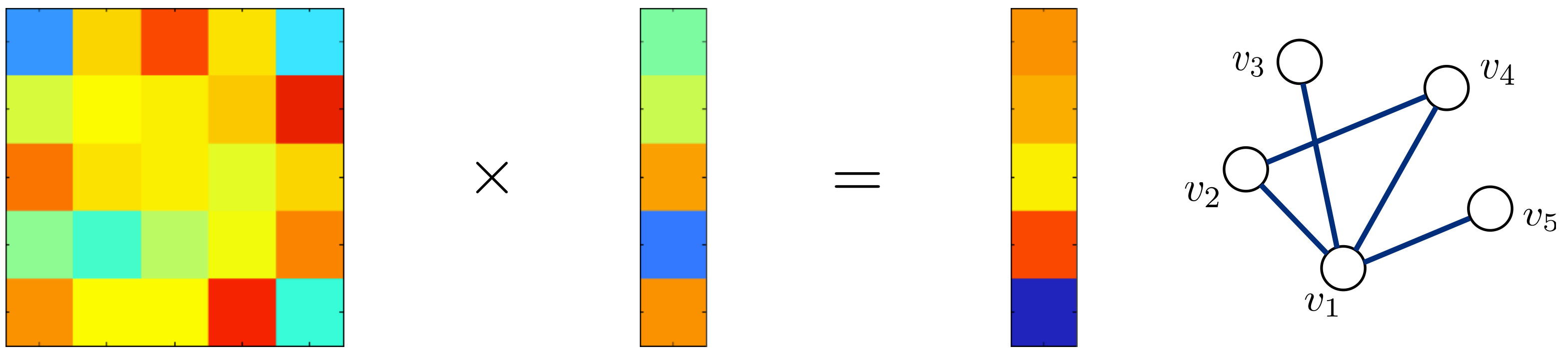

$$\mathbf{D}(\mathcal{G}) \quad \mathbf{c} \quad \mathbf{x} \quad \mathcal{G}$$

Fourier/wavelet atoms	graph Fourier/ wavelet coefficient	[Coifman06,Narang09,Hammond11,Shuman13,Sandryhaila13]
trained dictionary atoms	graph dictionary coefficient	[Zhang12,Thanou14]

Graph signal representations

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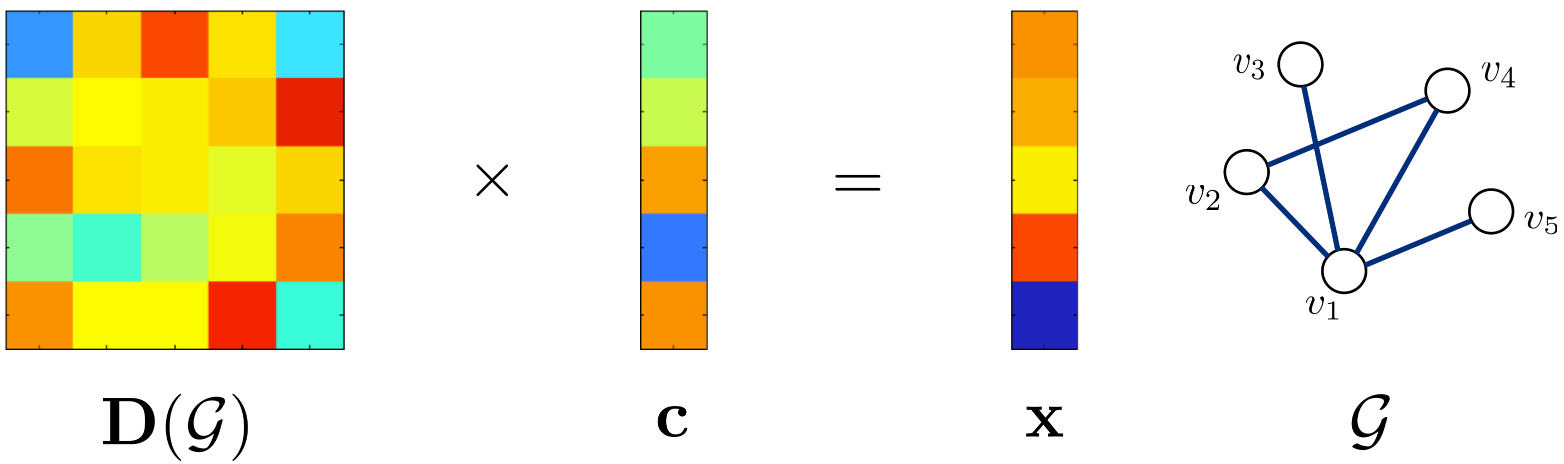
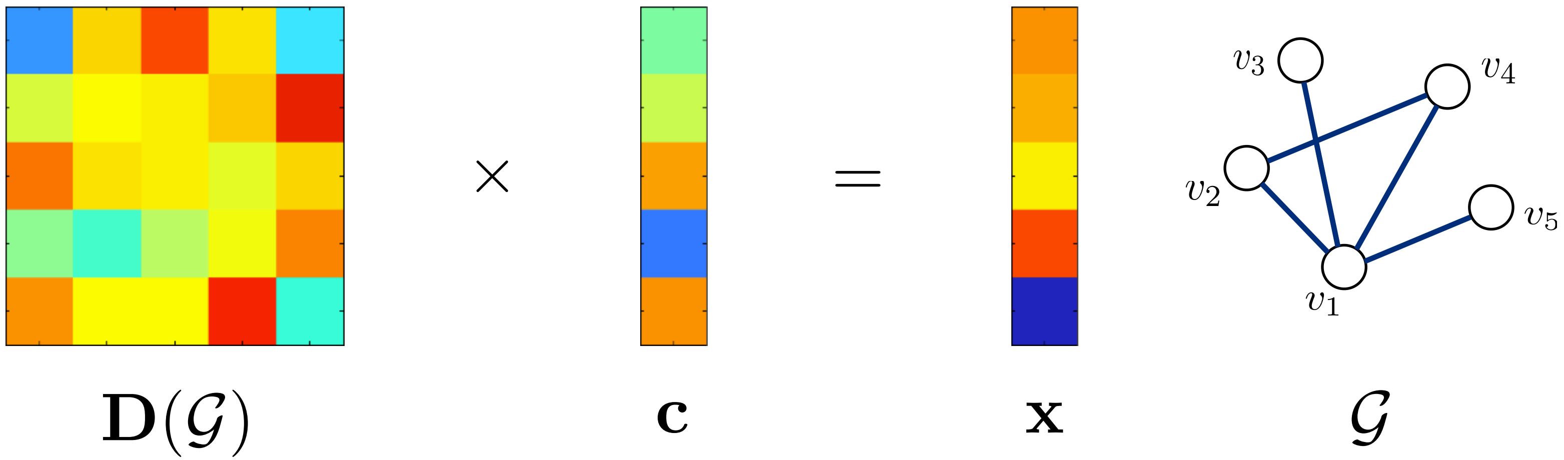


Graph signal representations

- Forward step: Given x, G, design D to study c
- Backward (topology inference): Given x, design D, c to infer G



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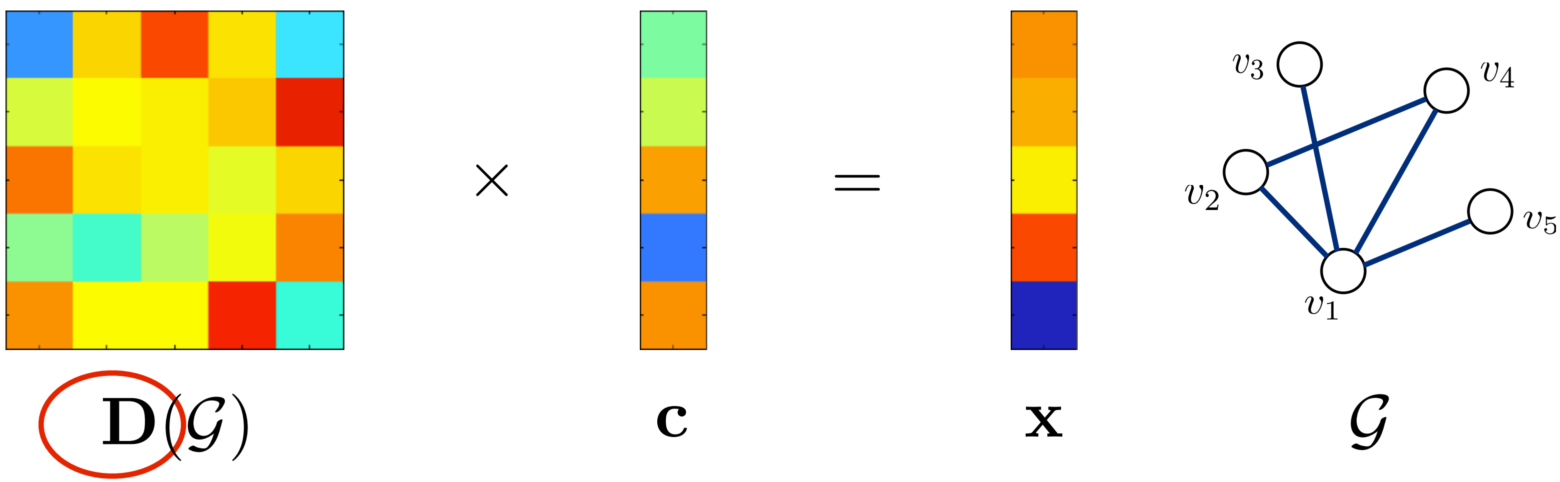

- The key is a signal/graph model behind **D**
- Designed around graph operators (adjacency/Laplacian matrices, shift operators)

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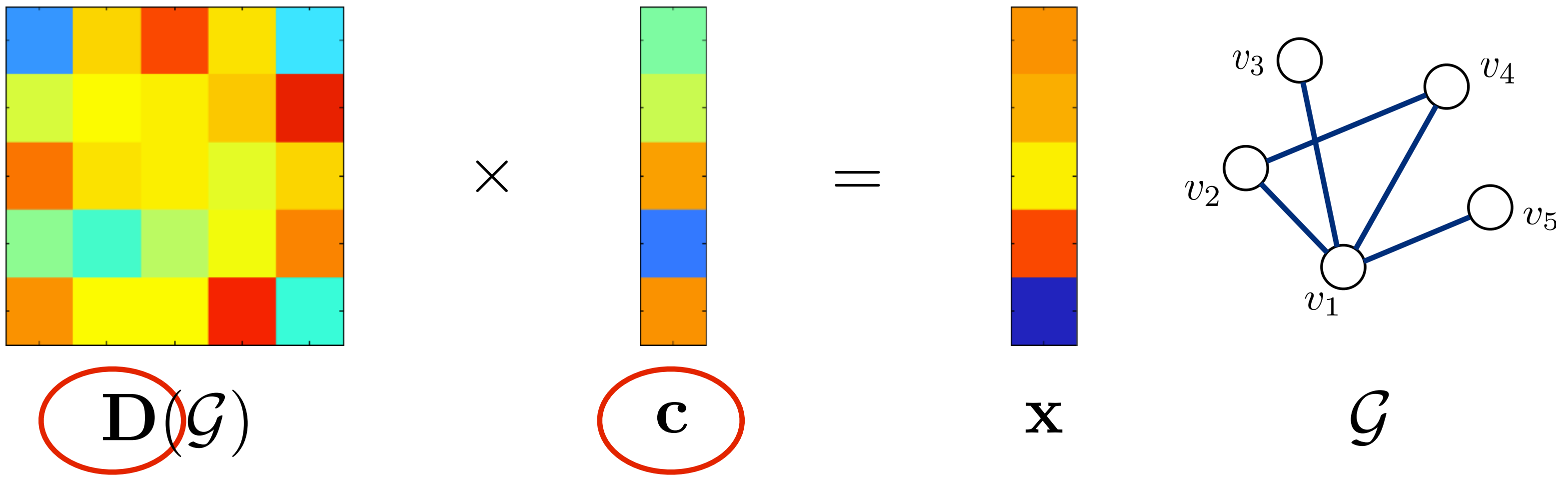

- The key is a signal/graph model behind **D**
- Designed around graph operators (adjacency/Laplacian matrices, shift operators)
- Choice of/assumption on **c** often determines signal characteristics

Model 1: Global smoothness

- Connected nodes on the graph have similar signal values
- Usually quantified by the Laplacian quadratic from:

$$\mathbf{x}^T\mathbf{L}\mathbf{x} = \frac{1}{2}\sum_{i,j}\mathbf{W}_{ij}\left(\mathbf{x}(i) - \mathbf{x}(j)\right)^2$$

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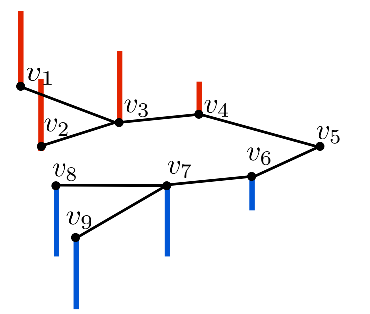
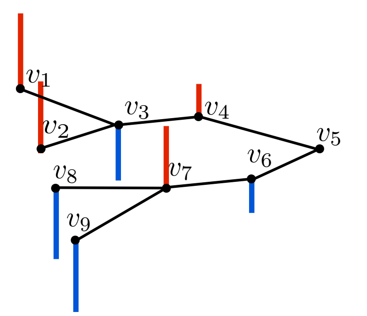


$$\mathbf{x}^T \mathbf{L} \mathbf{x} = 1$$



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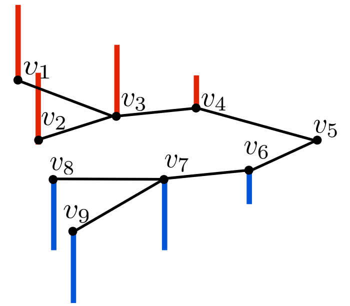


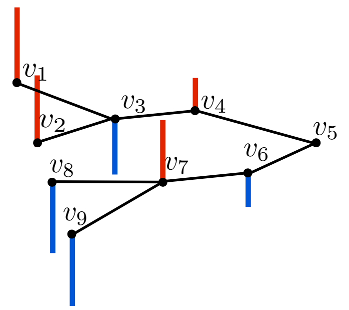
$\mathbf{x}^T\mathbf{L}\mathbf{x} = 1$

Similar to previous approaches:

Lake (2010): $\max_{\mathbf{\Theta}=\mathbf{L}+\frac{1}{\sigma^2}\mathbf{I}} \log\det\mathbf{\Theta} - \frac{1}{M}\mathrm{tr}(\mathbf{X}\mathbf{X}^T\mathbf{\Theta}) - \rho||\mathbf{\Theta}||_1$

Daitch (2009): $\min_{\mathbf{L}} \mathbf{X}^T\mathbf{L}^2\mathbf{X}$

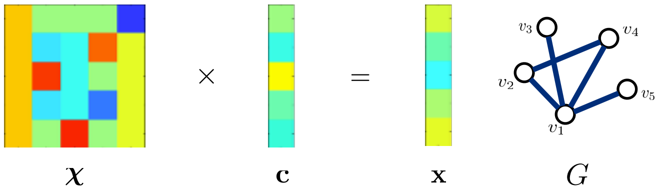
Hu (2013): $\min_{\mathbf{L}} \mathrm{tr}(\mathbf{X}^T\mathbf{L}^s\mathbf{X}) - \beta||\mathbf{W}||_F$



$\mathbf{x}^T\mathbf{L}\mathbf{x} = 21$

Model 1: Global smoothness

- Dong et al (2015), Kalofolias (2016)
  - $D(G) = \chi$ (eigenvector matrix of **L**)
  - Gaussian assumption on **c**: $\mathbf{c} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Lambda})$
  - Gaussian Markov Random Field
    $$x \sim \mathcal{N}(0, L^{\dagger} + \sigma_{\epsilon}^{2} I)$$



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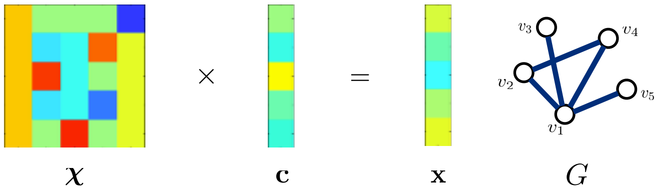
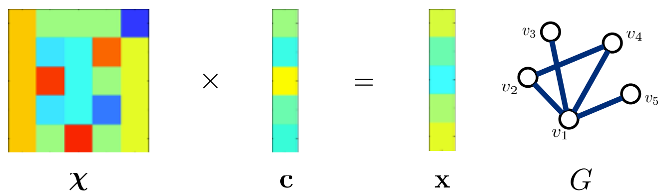


$$
\begin{aligned}
c_{\mathrm{MAP}}(x) &:= \arg\max_{c} p(c|x) = \arg\max_{c} p(x|c)p(c) \\
&= \arg\min_{c} \left(-\log\, p_{\epsilon}(x - \chi c) - \log\, p_{C}(c)\right) \\
&= \arg\min_{c} \|x - \chi c\|_2^2 + \alpha c^T \Lambda c
\end{aligned}
$$

Model 1: Global smoothness

- Dong et al (2015), Kalofolias (2016)



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$$\min_{\mathbf{L},\mathbf{Y}} \underbrace{||\mathbf{X} - \mathbf{Y}||_F^2}_{\text{data fidelity}} + \alpha \underbrace{\mathrm{tr}(\mathbf{Y}^T \mathbf{L} \mathbf{Y})}_{\text{smoothness on } \mathbf{Y}} + \beta \underbrace{||\mathbf{L}||_F^2}_{\text{regularization}}$$

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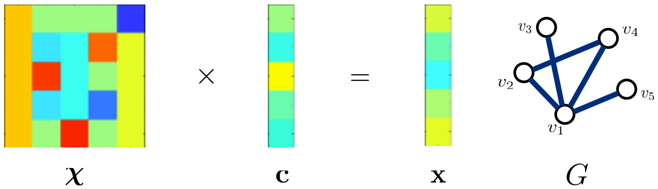

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$$\min_{\mathbf{L},\mathbf{Y}} ||\mathbf{X} - \mathbf{Y}||_F^2 + \alpha\, \mathrm{tr}(\mathbf{Y}^T\mathbf{L}\mathbf{Y}) + \beta ||\mathbf{L}||_F^2$$

data fidelity smoothness on **Y** regularization

Learning enforces signal property (global smoothness)!

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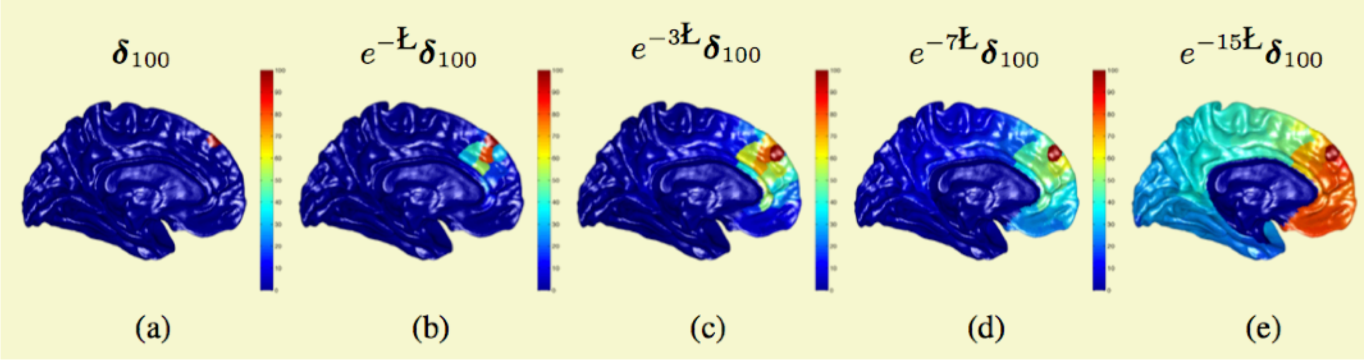
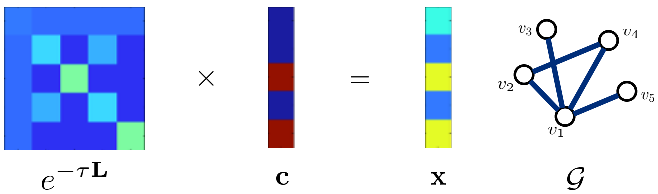
Model 2: Diffusion process

- Signals are the outcome of a diffusion process on the graph (more of local smoothness than global one!)
- Example: Movement of people/ vehicles
- Fully characterized by diffusion operators (graph filters)



Heat diffusion operators on the brain [7]

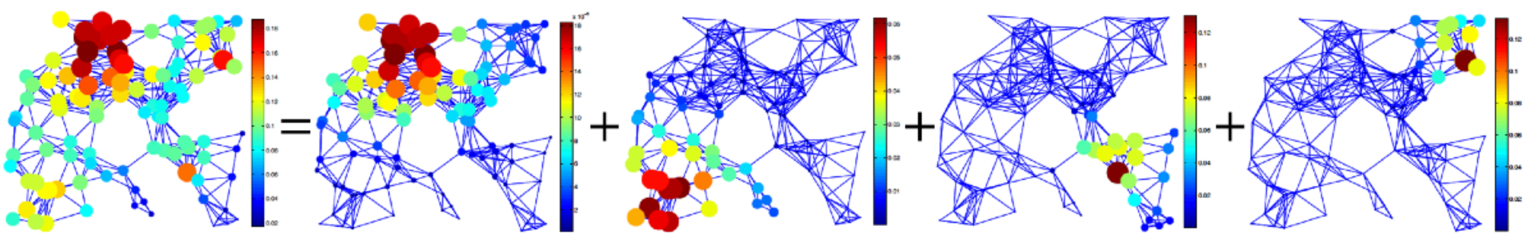
Model 2: Diffusion process



- Thanou et al. (2016)
  - Dictionary of heat diffusion operators

$$D(G) = e^{-\tau L}$$

  - Sparsity assumption on c
  - Each signal is a linear combination of heat diffusion processes



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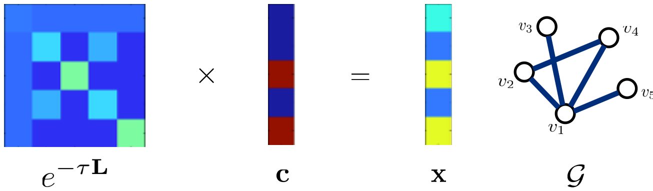

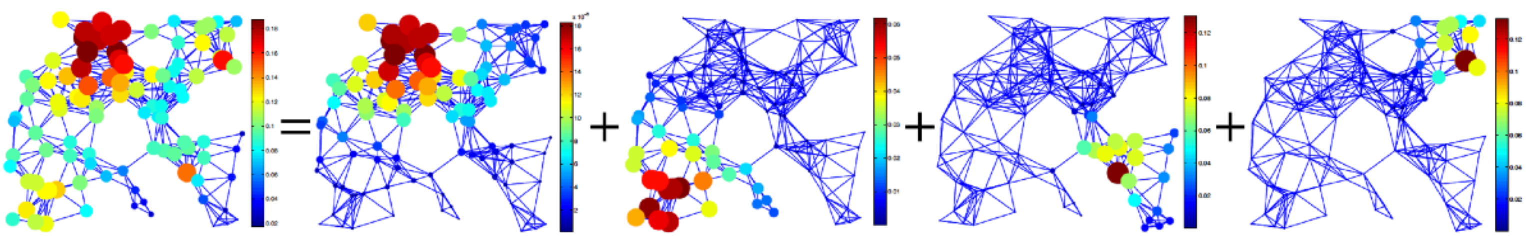

$$
\begin{aligned}
c_{\mathrm{MAP}}(x) &:= \arg\max_c p(c|x) = \arg\max_c p(x|c)p(c) \\
&= \arg\min_c \left(-\log\, p_\epsilon(x - e^{-\tau L}c) - \log\, p_C(c)\right) \\
&= \arg\min_c \|x - e^{-\tau L}c\|_2^2 + \alpha\|c\|_1
\end{aligned}
$$

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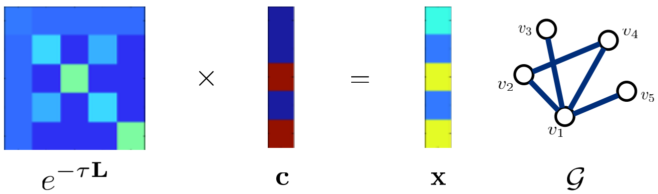

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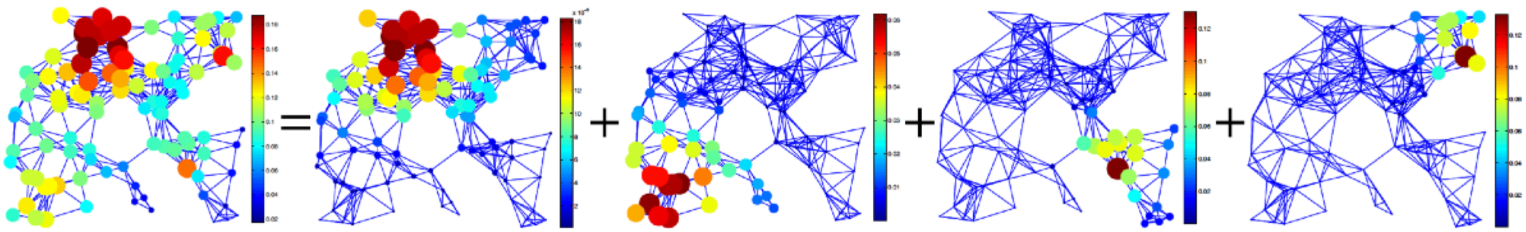

$$\min_{\mathbf{L},\mathbf{C},\tau} ||\mathbf{X} - \mathbf{D}(\mathbf{L})\mathbf{C}||_F^2 + \alpha \sum_{m=1}^{M} ||\mathbf{c}_m||_1 + \beta ||\mathbf{L}||_F^2 \quad \text{s.t.} \quad \mathbf{D} = [e^{-\tau_1 \mathbf{L}}, ..., e^{-\tau_S \mathbf{L}}]$$

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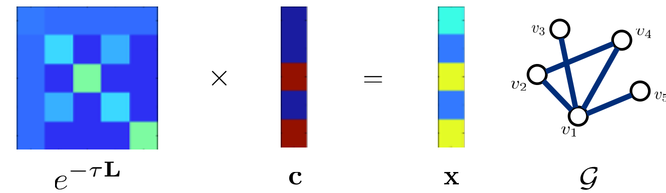

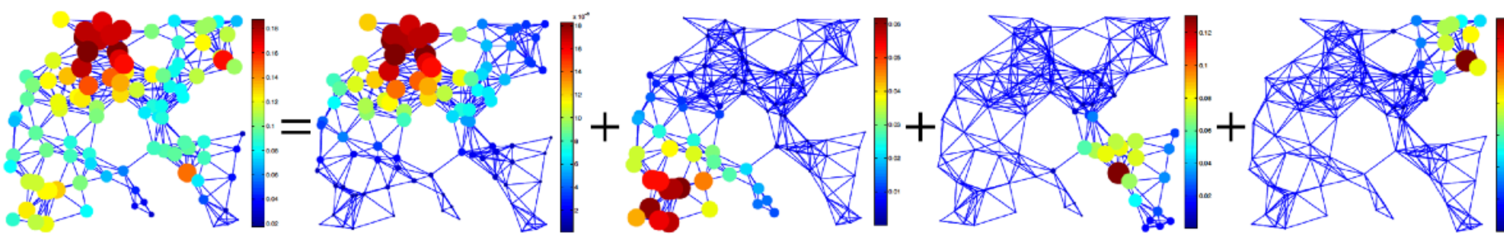

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data fidelity　　sparsity on **c**　　regularization

SDSC

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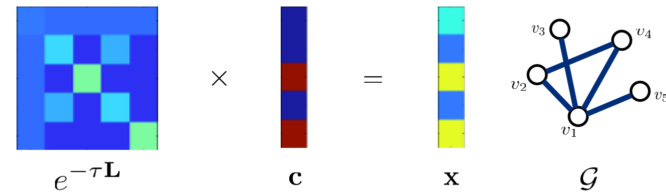

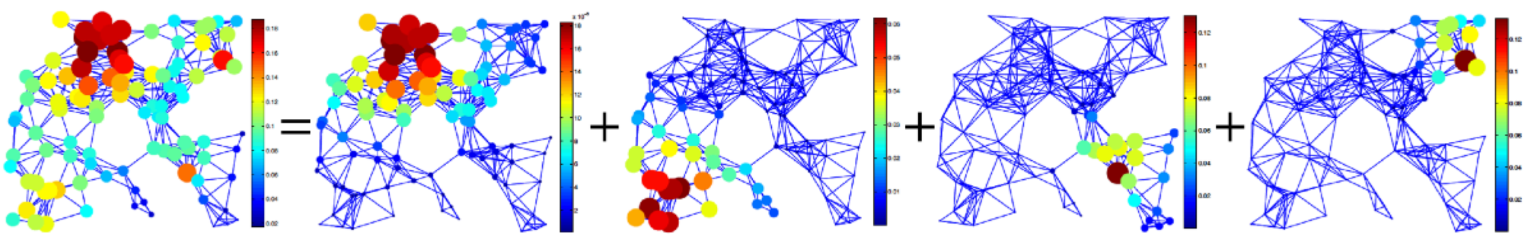

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data fidelity   sparsity on $\mathbf{c}$ regularization

Can be extended to general polynomial case (Petric Maretic et al. 2017)

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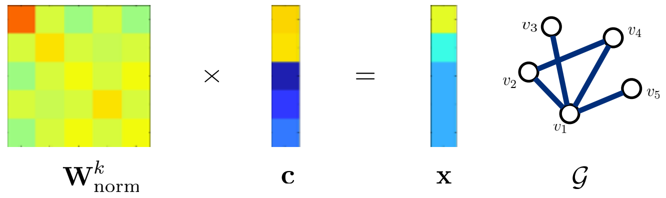
Model 2: Diffusion process

- Pasdeloup et al. (2016), Segarra et al. (2016)
  - Dictionary: powers of the normalized weight matrix

$$\mathbf{D}(\mathcal{G}) = \mathbf{T}^{\mathbf{k}(m)} = \mathbf{W}_{\text{norm}}^{\mathbf{k}(m)}$$



  - i.i.d. assumption on the coefficients c

  - Two stage approach for estimating the topology:

    - Estimate the eigenvectors from the sample covariance

$$\mathbf{\Sigma} = \mathbb{E}\Big[\sum_{m=1}^{M} \mathbf{X}(m)\mathbf{X}(m)^T\Big] = \sum_{m=1}^{M} \mathbf{W}_{\text{norm}}^{2\mathbf{k}(m)}$$

    - Optimize the eigenvalues given constraints on the topology (sparsity, non-negativity)

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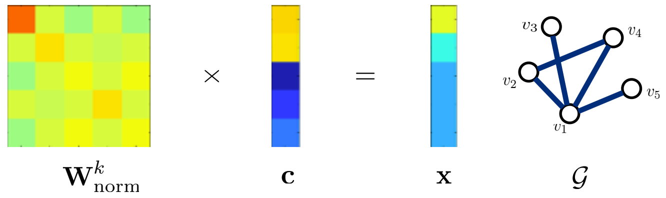

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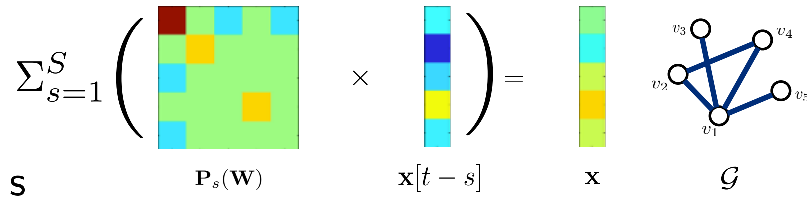
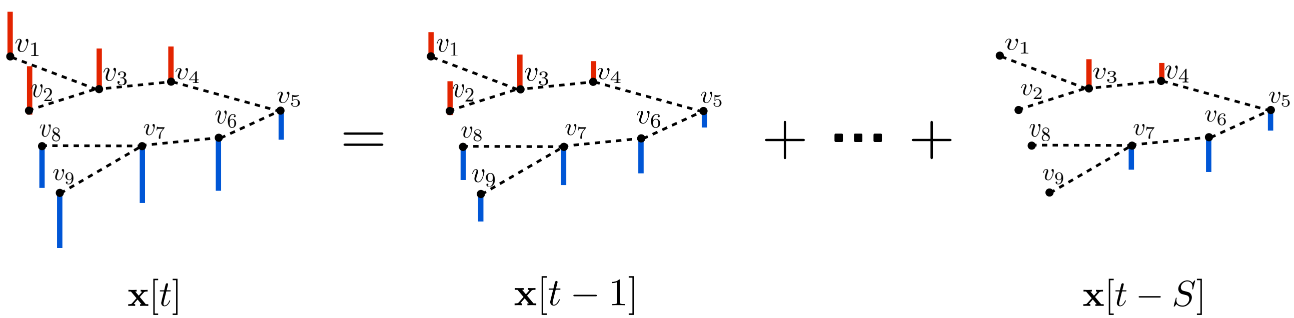
More a "graph-centric" framework: Cost on graph components instead of signals

Model 3: Time varying signals



- Mei et al. (2016)
  - $D_s(G) = P_s(W)$: polynomial of **W** of degree s
  - Define $c_s = x[t-s]$



$$\min_{\mathbf{W},\mathbf{a}} \frac{1}{2} \sum_{k=S+1}^{K} ||\mathbf{x}[k] - \sum_{s=1}^{S} \mathbf{P}_s(\mathbf{W})\mathbf{x}[k-s]||_2^2 + \lambda_1 ||\text{vec}(\mathbf{W})||_1 + \lambda_2 ||\mathbf{a}||_1$$

data fidelity — sparsity on **W** — sparsity on **a**

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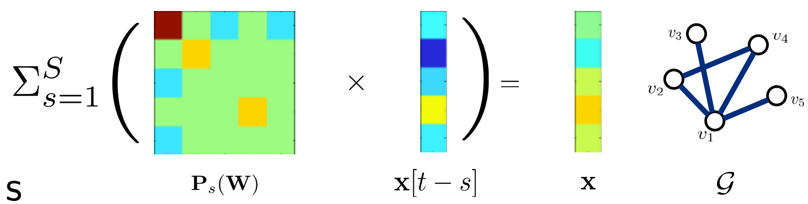

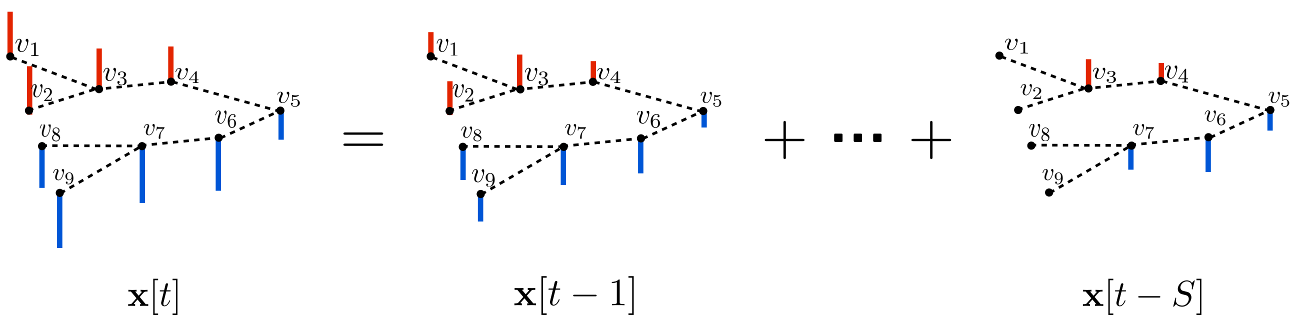

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data fidelity   sparsity on **W**   sparsity on **a**

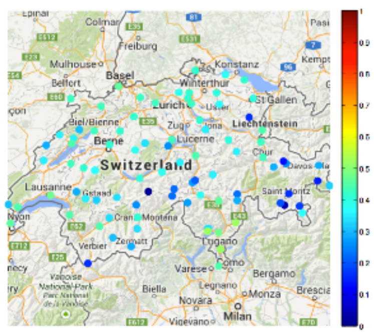
Good for inferring causal relations between signals

Kernel version (nonlinear): Shen et al. (2016)

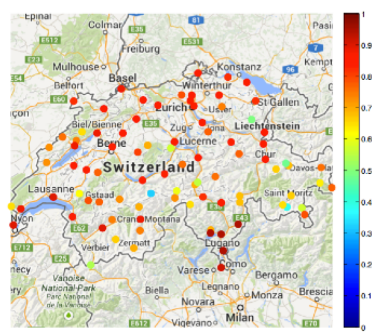
Application 1: Understanding meteorological data

- Signals: 12 monthly temperature records at 89 measuring stations [9]

February

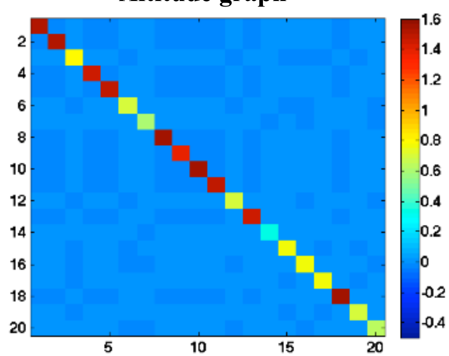


June

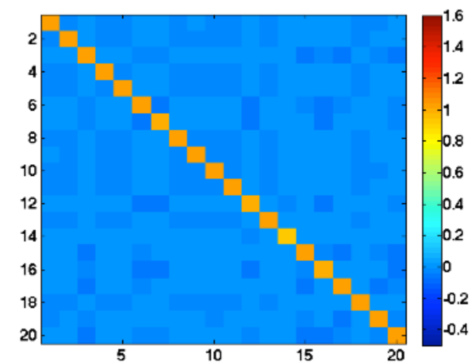


October

Altitude graph



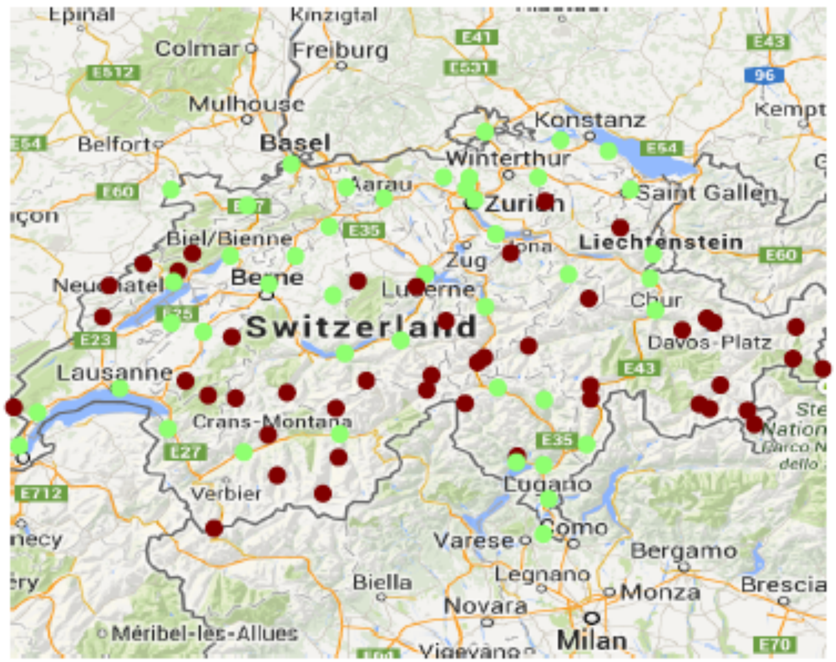
Learned graph



The learned graph captures the altitude relation between stations

Application 1: Understanding meteorological data

- The learned graph can be used for partitioning nodes into several clusters:

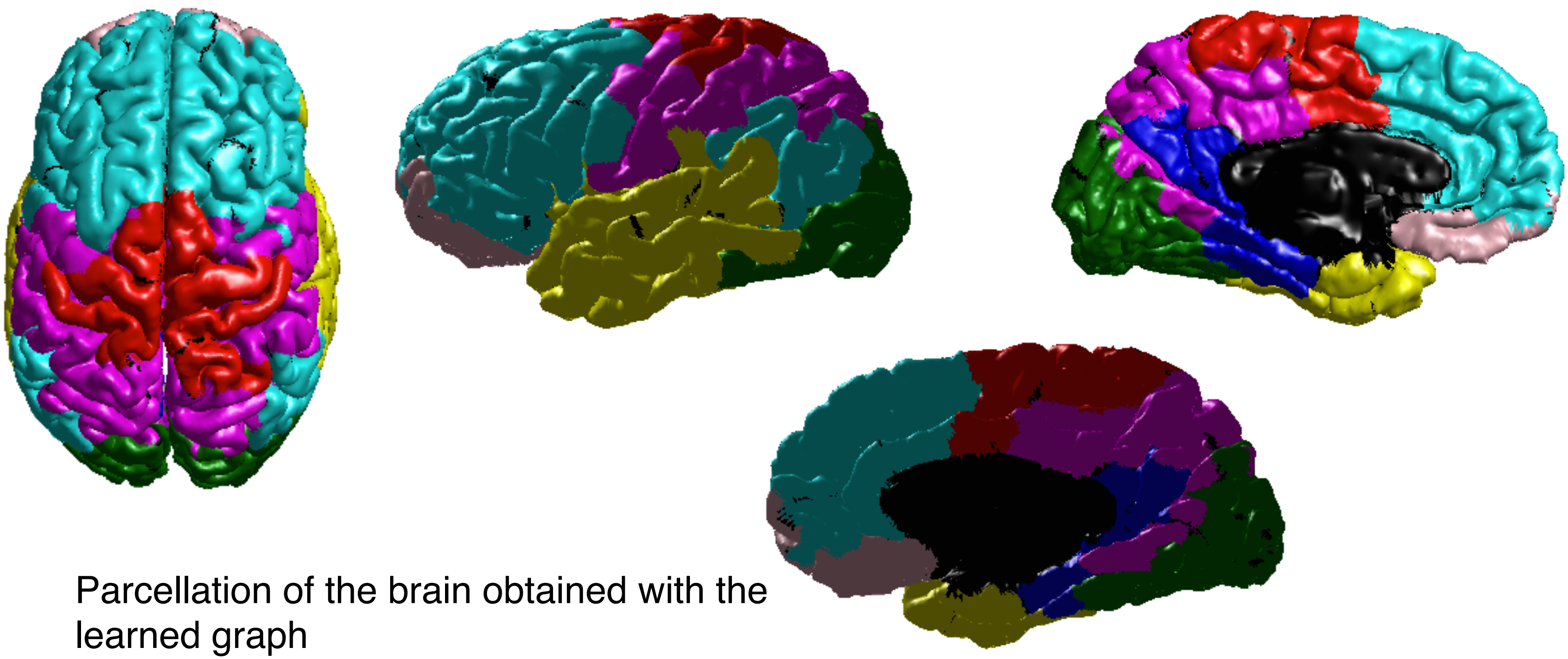


Mountain regions (Jura, Alps)

Flat regions and valleys

Application 2: Infer brain connections

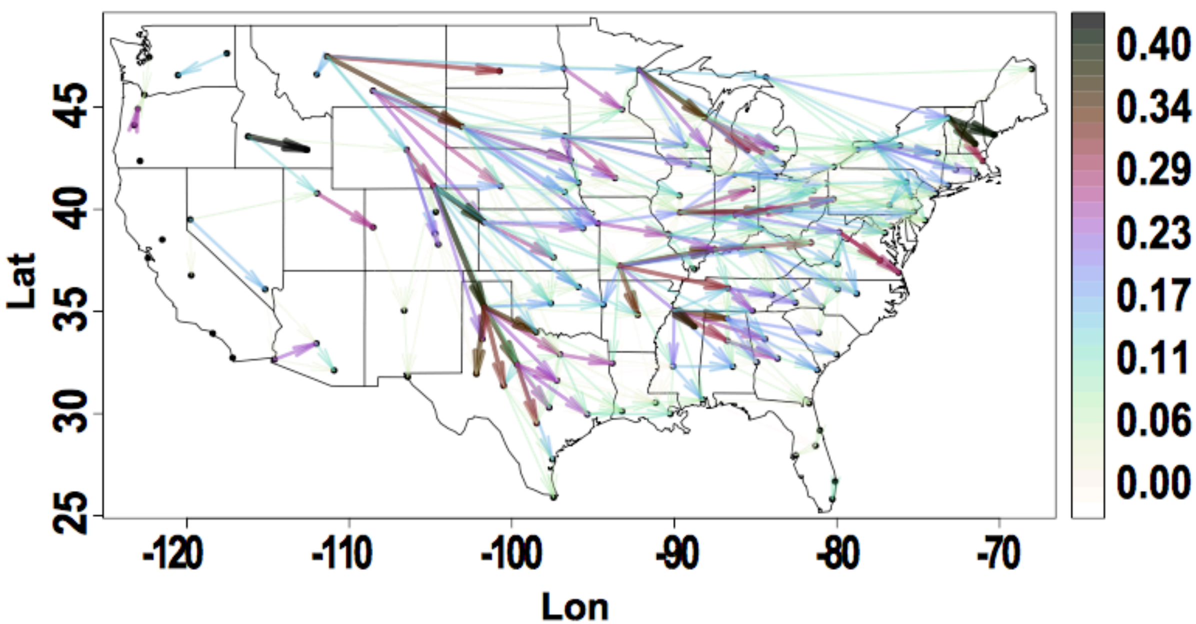
- Signals: time series recorded in MRI scan while the subjects are at rest



Parcellation of the brain obtained with the learned graph

Application 3: Learning wind directions

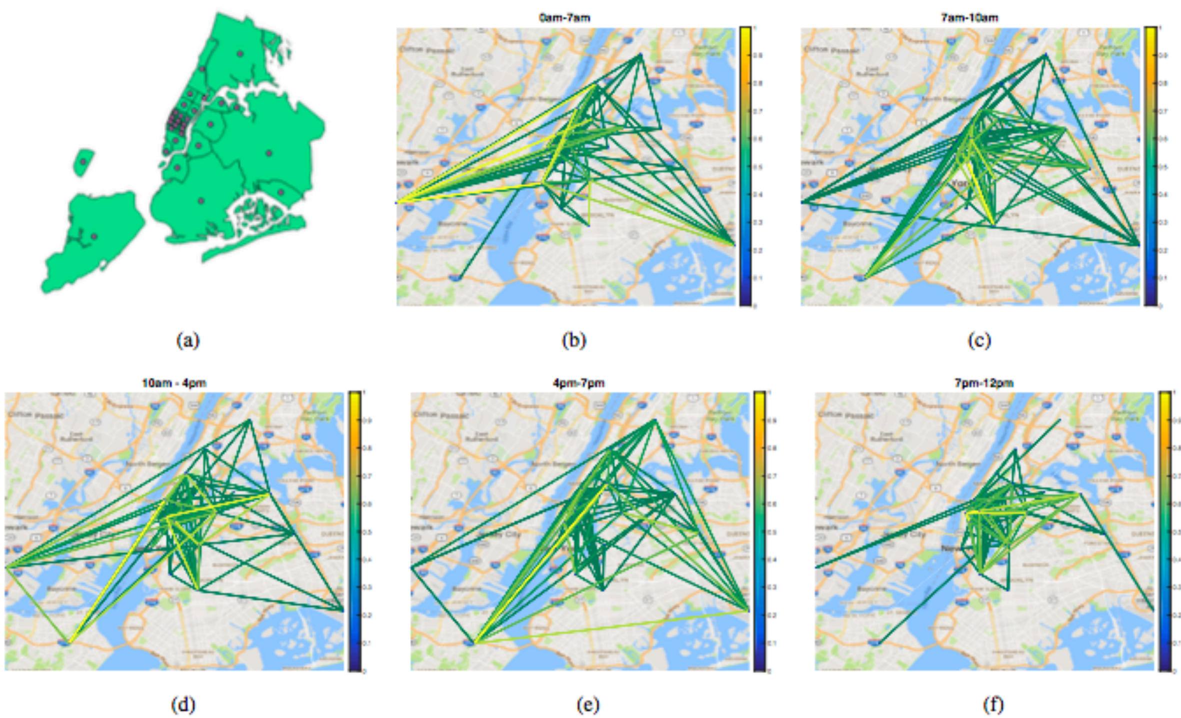
- Signals: daily temperature taken over 365 days at 150 cities in USA [20]



- The graph learned the predominant west-to-east and north-northwest-to-south-southeast directions

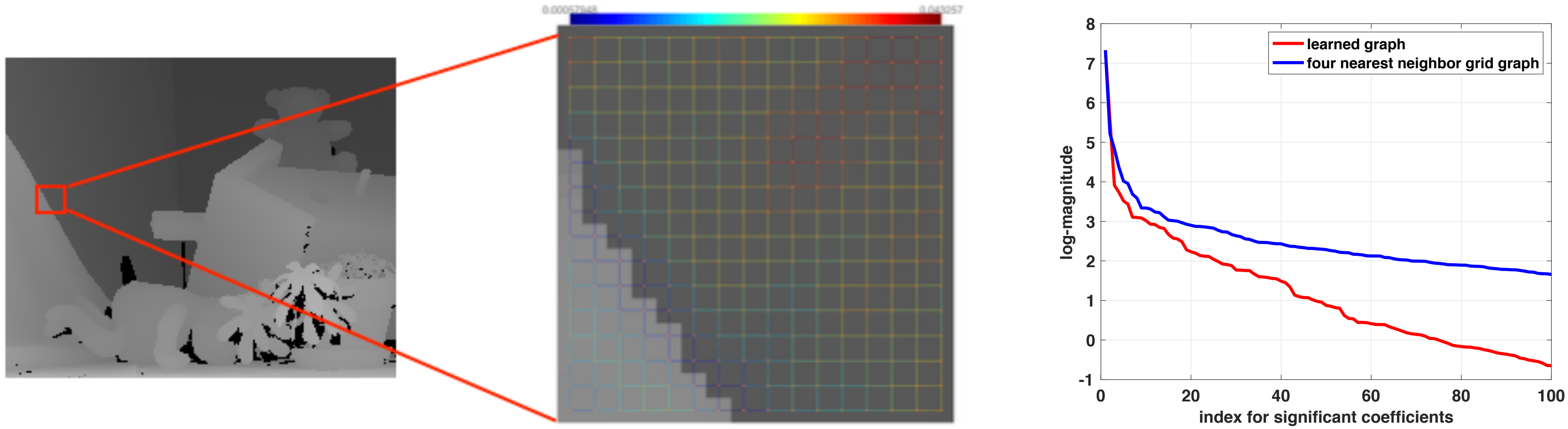
Application 4: Mining traffic behavior

- Learn the traffic patterns in NCY using a heat diffusion model [35]
  - nodes: centroids of taxi zones
  - signals: the hourly number of Uber pickups in each zone



Application 5: Image coding

- Graph learning can be used for efficient coding of image signals
- Additional constraints can be used to reduce the transmission cost of the graph



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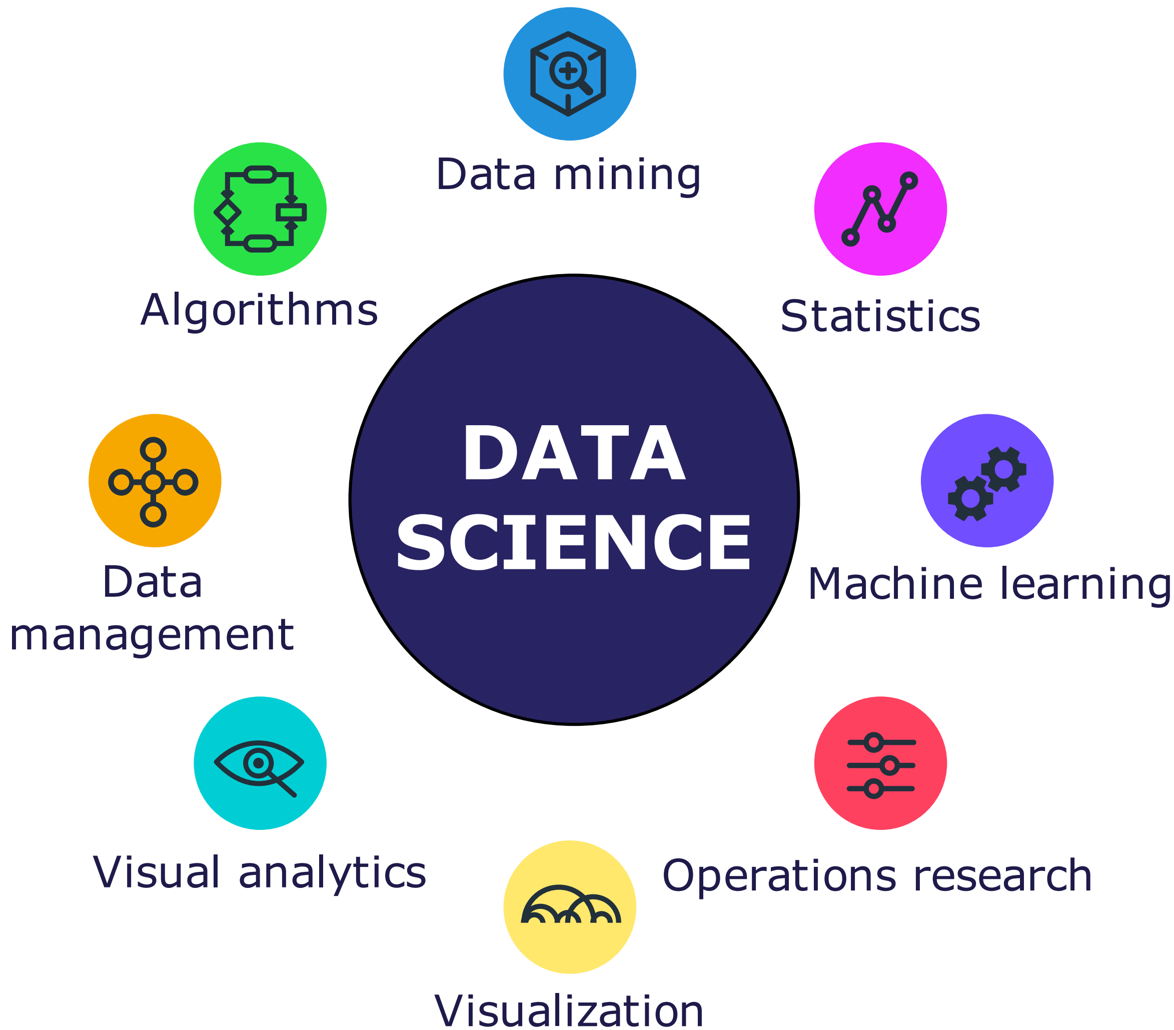
Comparison of GSP methods

Methods	Signal model	Assumption	Learning output	Edge direction	Inference
Dong (2015)	Global smoothness	Gaussian	Laplacian	Undirected	Signal-centric
Thanou (2016)	Heat diffusion	Sparsity	Laplacian	Undirected	Signal-centric
Pasdeloup (2015)	Diffusion by Adj. matrix	Stationary	Normalised Adj./ Laplacian	Undirected	Graph-centric
Mei (2015)	Time-varying	Dependent on previous states	Adj. matrix	Directed	Signal-centric

Take-home messages

- Data is often structured
- **Graphs**: a flexible tool for modeling many real-world data
- If the structure if not known, **learn** it from the data
- The choice of the learning algorithm depends on many parameters (signal model, application)

Network inference: Powerful tool for Data Science



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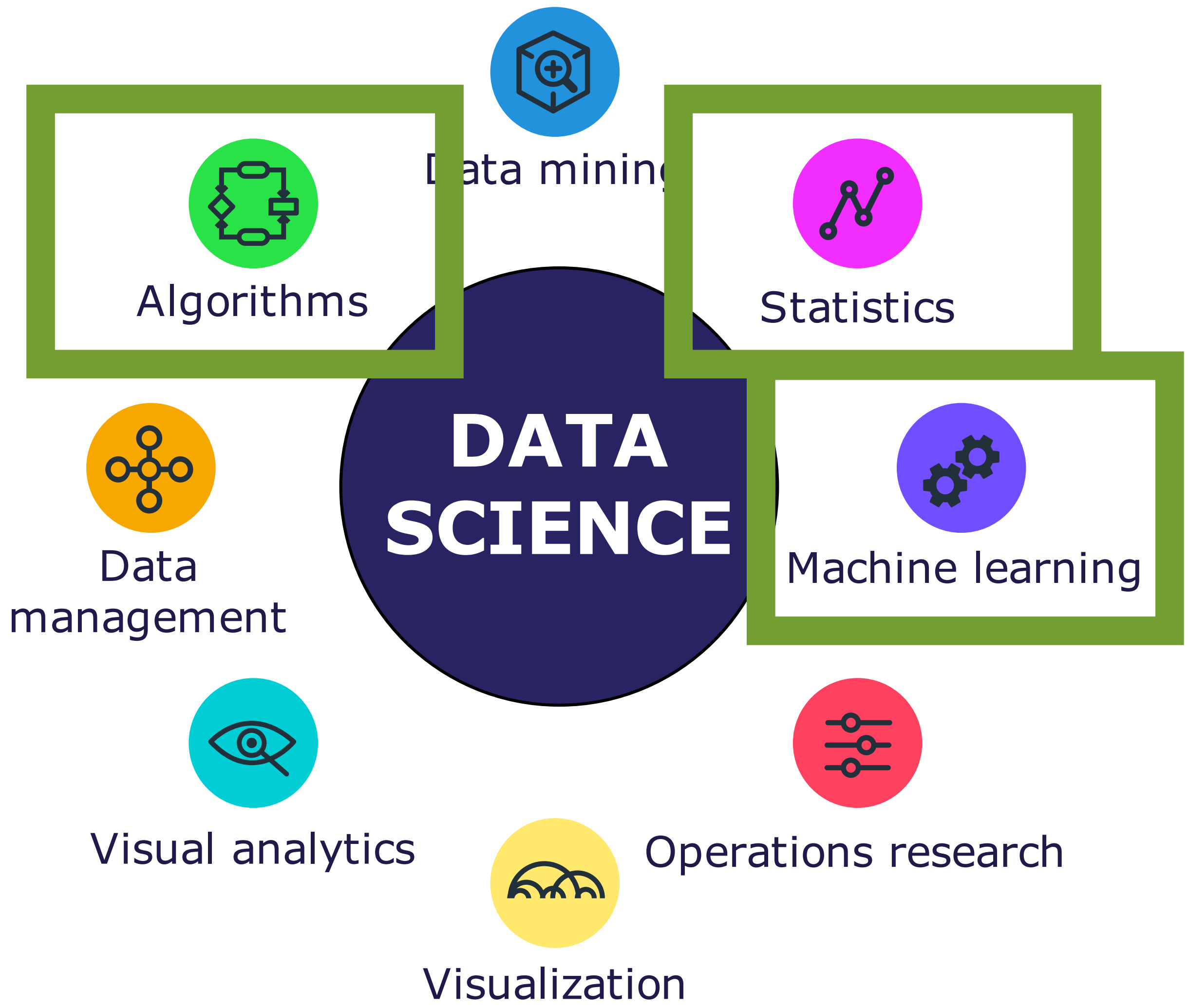

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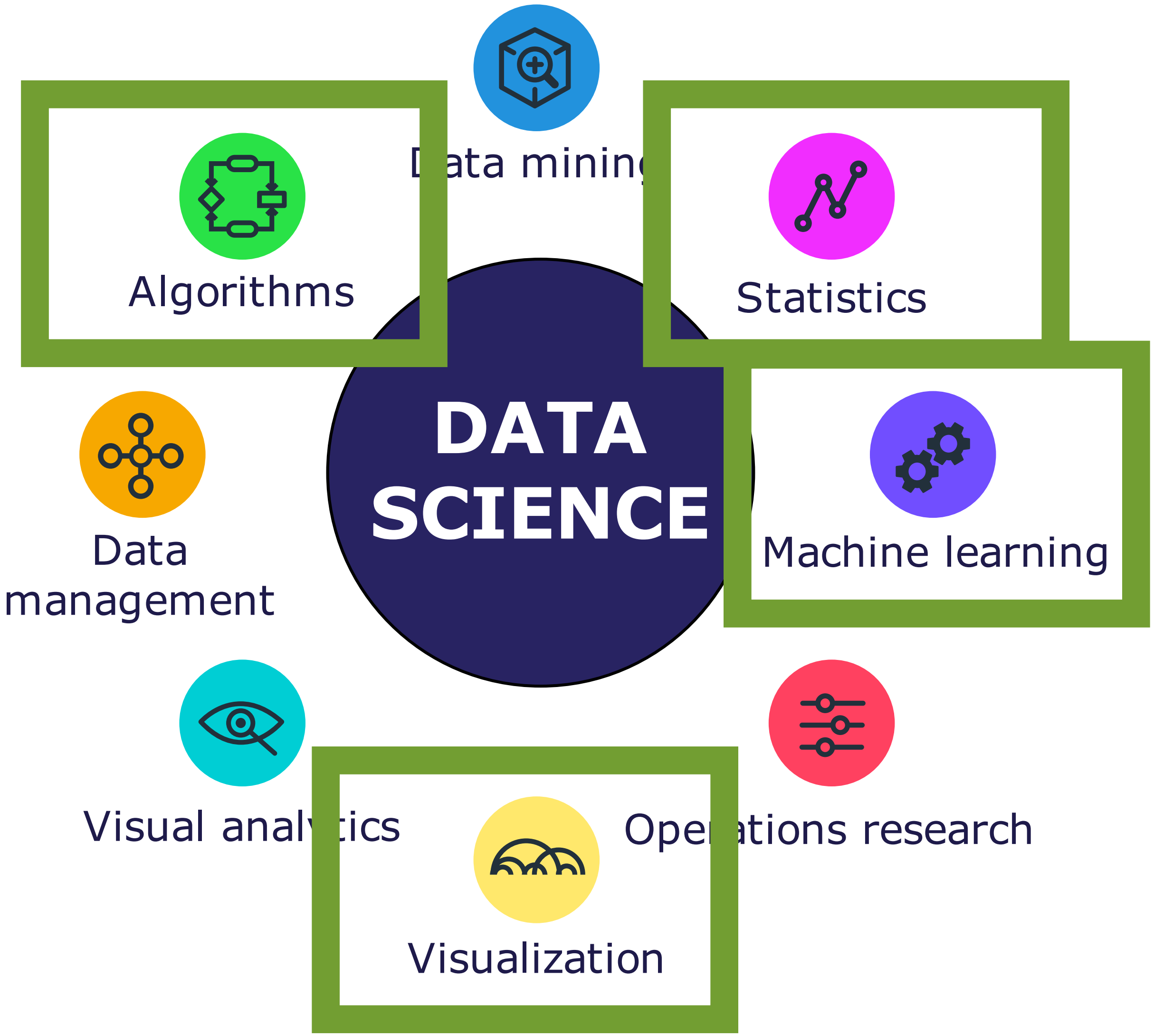

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Overview paper

Learning Graphs from Data: A Signal Representation Perspective

Xiaowen Dong*, Dorina Thanou*, Michael Rabbat, and Pascal Frossard

The construction of a meaningful graph topology plays a crucial role in the effective representation, processing, analysis and visualization of structured data. When a natural choice of the graph is not readily available from the datasets, it is thus desirable to infer or learn a graph topology from the data. In this tutorial overview, we survey solutions to the problem of graph learning, including classical viewpoints from statistics and physics, and more recent approaches that adopt a graph signal processing (GSP) perspective. We further emphasize the conceptual similarities and differences between classical and GSP graph inference methods and highlight the potential advantage of the latter in a number of theoretical and practical scenarios. We conclude with several open issues and challenges that are keys to the design of future signal processing and machine learning algorithms for learning graphs from data.

I. Introduction

Modern data analysis and data processing tasks typically involve large sets of structured data, where the structure carries critical information about the nature of the data. One can find numerous examples of such datasets in a wide diversity of application domains, including transportation networks, social networks, computer networks, and brain networks. Typically, graphs are used as mathematical tools to describe the structure of such data. They provide a flexible way for representing relationships between data entities. Numerous signal processing and machine learning algorithms have been introduced in the past decade for analyzing structured data on *a priori* known graphs [1], [2]. However, there are often settings where the graph is not readily available, and the structure of the data has to be estimated in order to permit effective representation, processing, analysis or visualization of graph data. In this case, a crucial task is to infer a graph topology that describes the characteristics of the data observations, hence capturing the underlying relationship between these entities.

The problem of graph learning is the following: given M observations of N variables or data entities, represented in a data matrix $\mathbf{X} \in \mathbb{R}^{N \times M}$, and given some prior knowledge (e.g., distribution, data model,

*Authors contributed equally.

QUESTIONS?

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