# Sampling-based Inference for Large Linear Models, with Application to Linearised Laplace

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**Problem:** **Posterior inference** and **hyperparameter selection** is intractable with millions of observations and millions of parameters due to **cubic scaling**.

**Solution:** We cast inference and hyperparameter selection as a sequence of **quadratic optimisation problems**. We can solve these relatively easily for high dimensional problems with *roughly* **linear scaling**.

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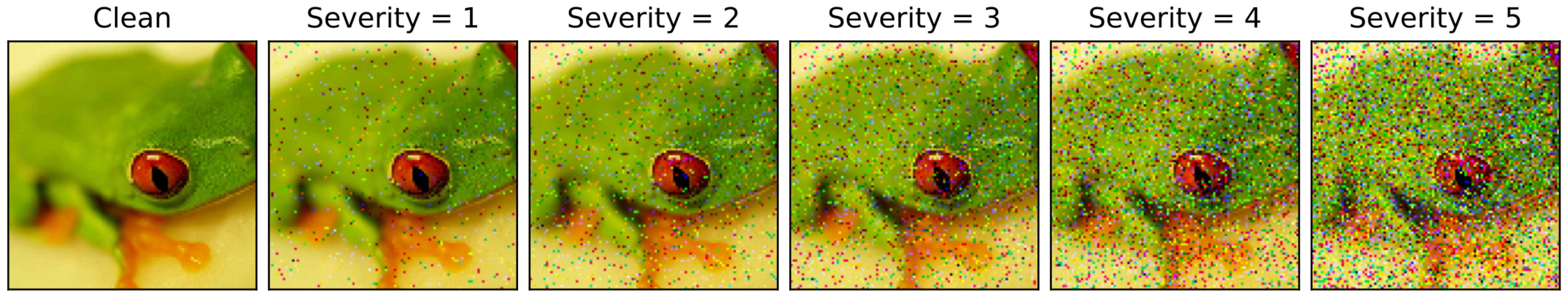
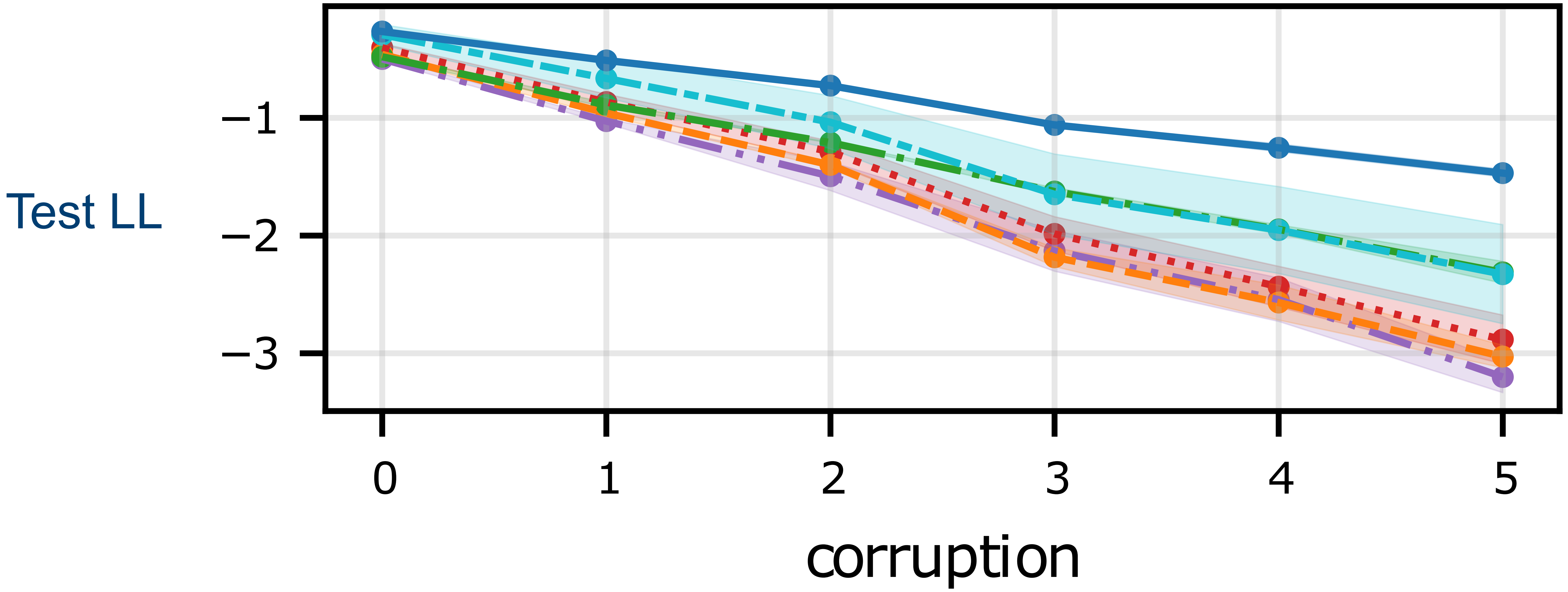
- We can generalise this to non-Gaussian likelihoods (i.e. classification) by 'Gaussianising' with the **Laplace** approximation

# Linearised NNs work well

**Corrupted CIFAR10** (Ovadia 2019)




**Model:**
ResNet-18 with **11M** weights

**Inference:**
**Lin Laplace Subnetwork**
(Daxberger et. al. 2021)

**"Bayesian Deep Learning via Subnetwork Inference"**

**Baselines:**

- MAP
- Diagonal Laplace
- MC Dropout (Gal 2016)
- Deep Ensembles (Lakshminarayanan 2017)
- SWAG (Maddox 2019)

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$\theta \sim \mathcal{N}(0, A^{-1})$

$\eta_i \sim \mathcal{N}(0, B_i^{-1})$ → $B = B_i \otimes I_n \in \mathbb{R}^{nm \times nm}$

# Consider: Bayesian Linear Models

- Number of parameters is large $d > 1e6$
- Observation space is large $n \cdot m > 1e6$

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4.5TB (resnet18)

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1. Noise-fit term.
   - Depends on each observation's feature expansion so it needs to be **minibatched**.
     - Very large variance when estimated stochastically.
2. Regularisation term.
   - We can compute its gradient in closed form.

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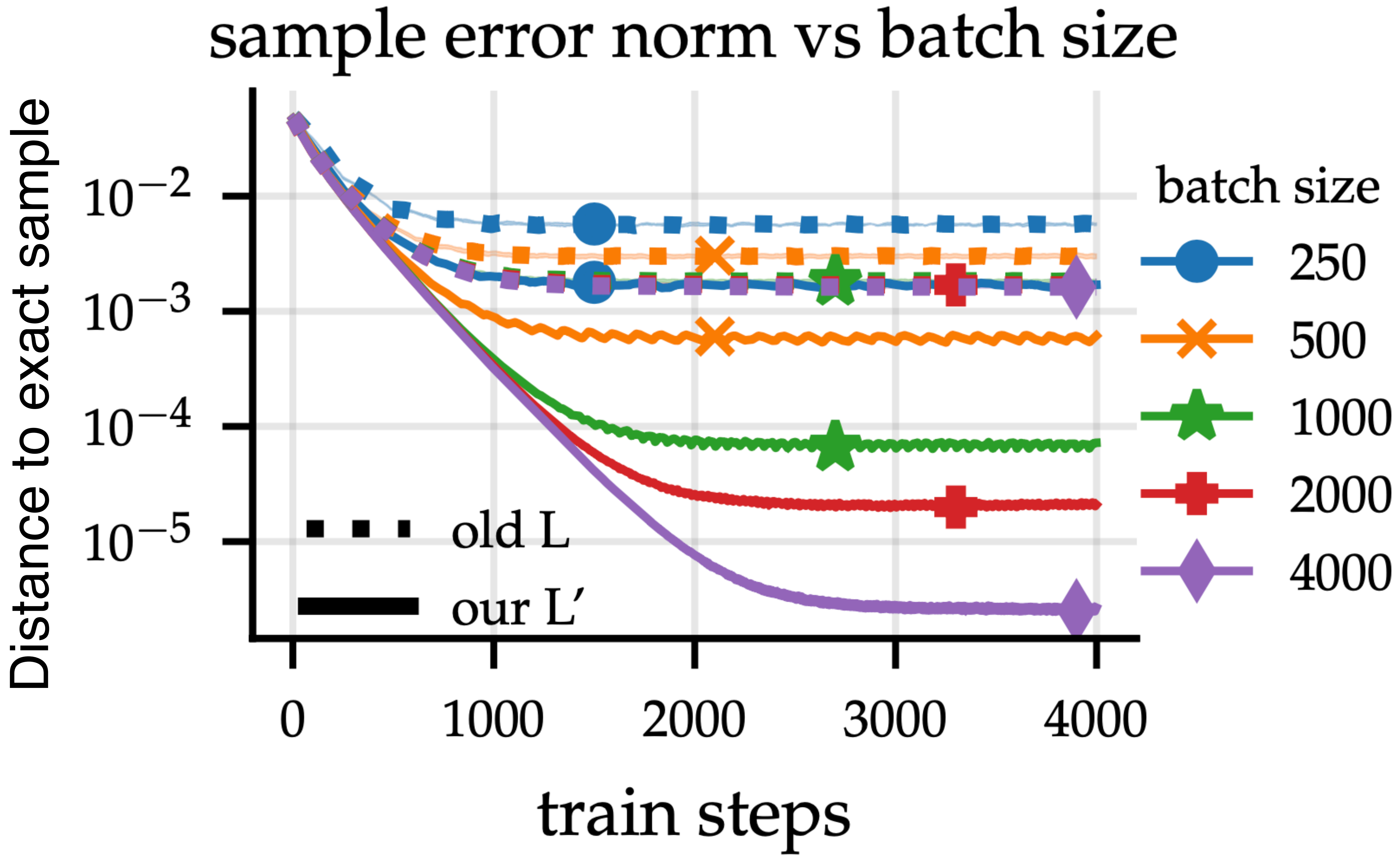

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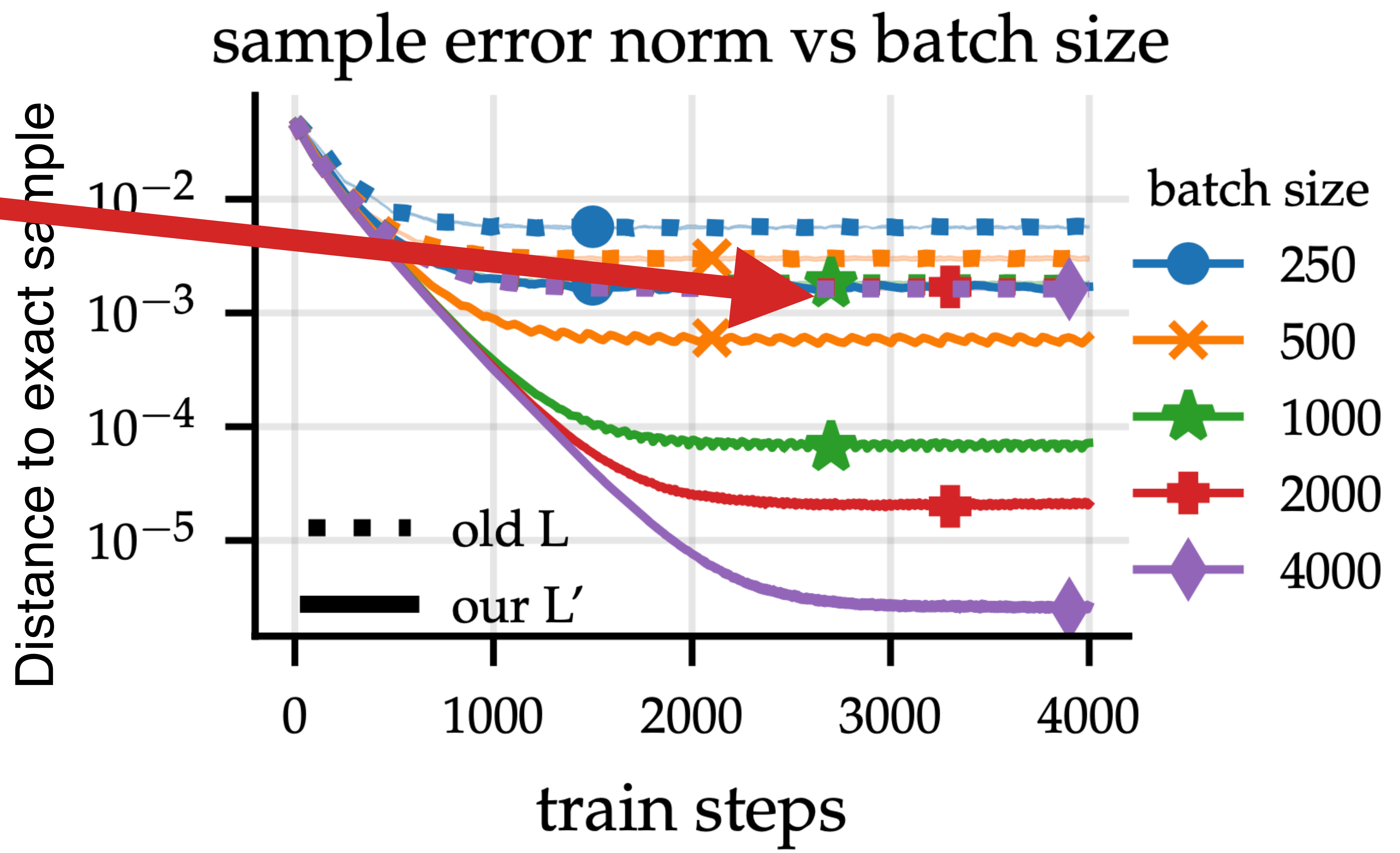
16x reduction in batch size!



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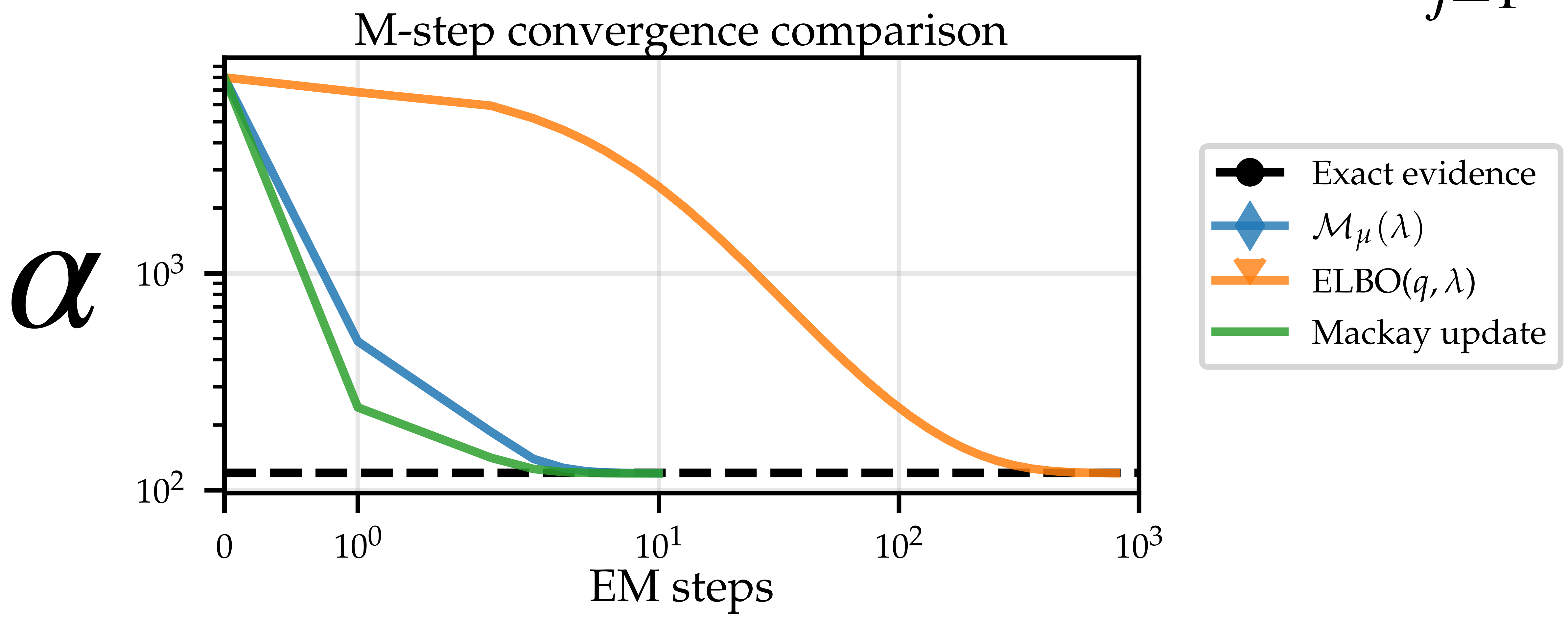

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$\mathcal{O}(kdnm)$

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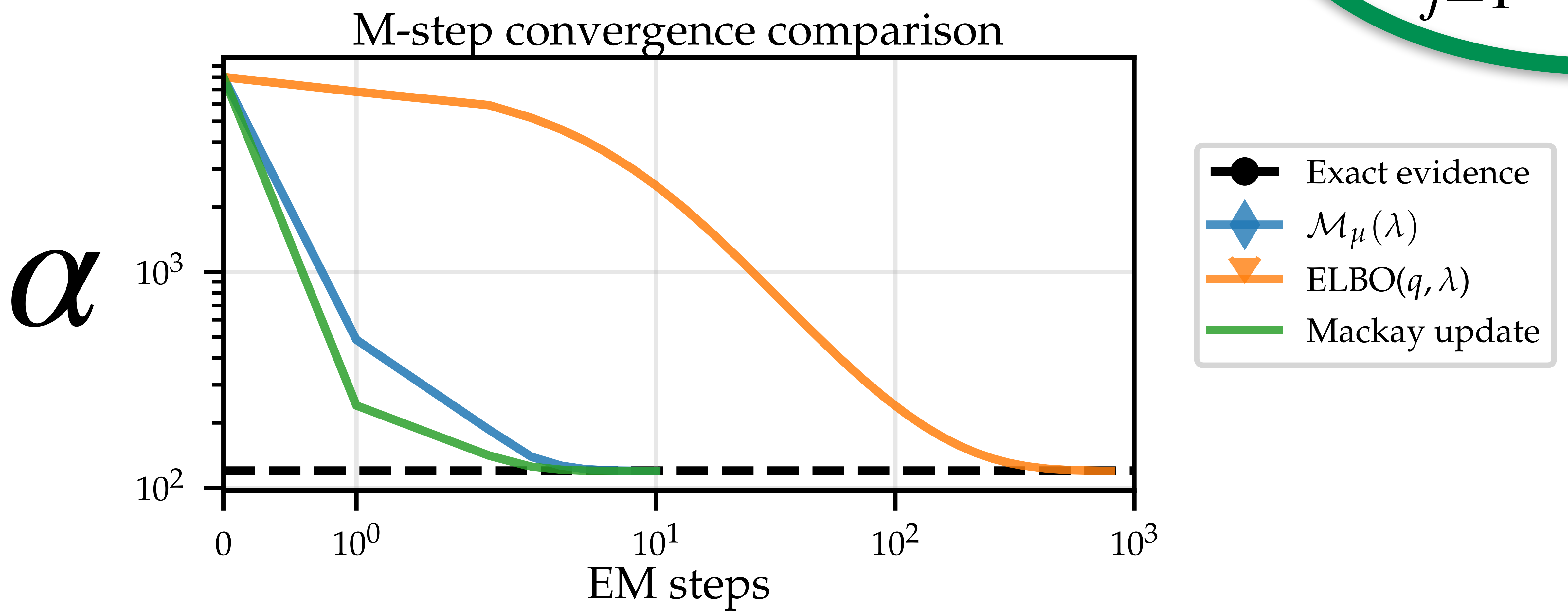
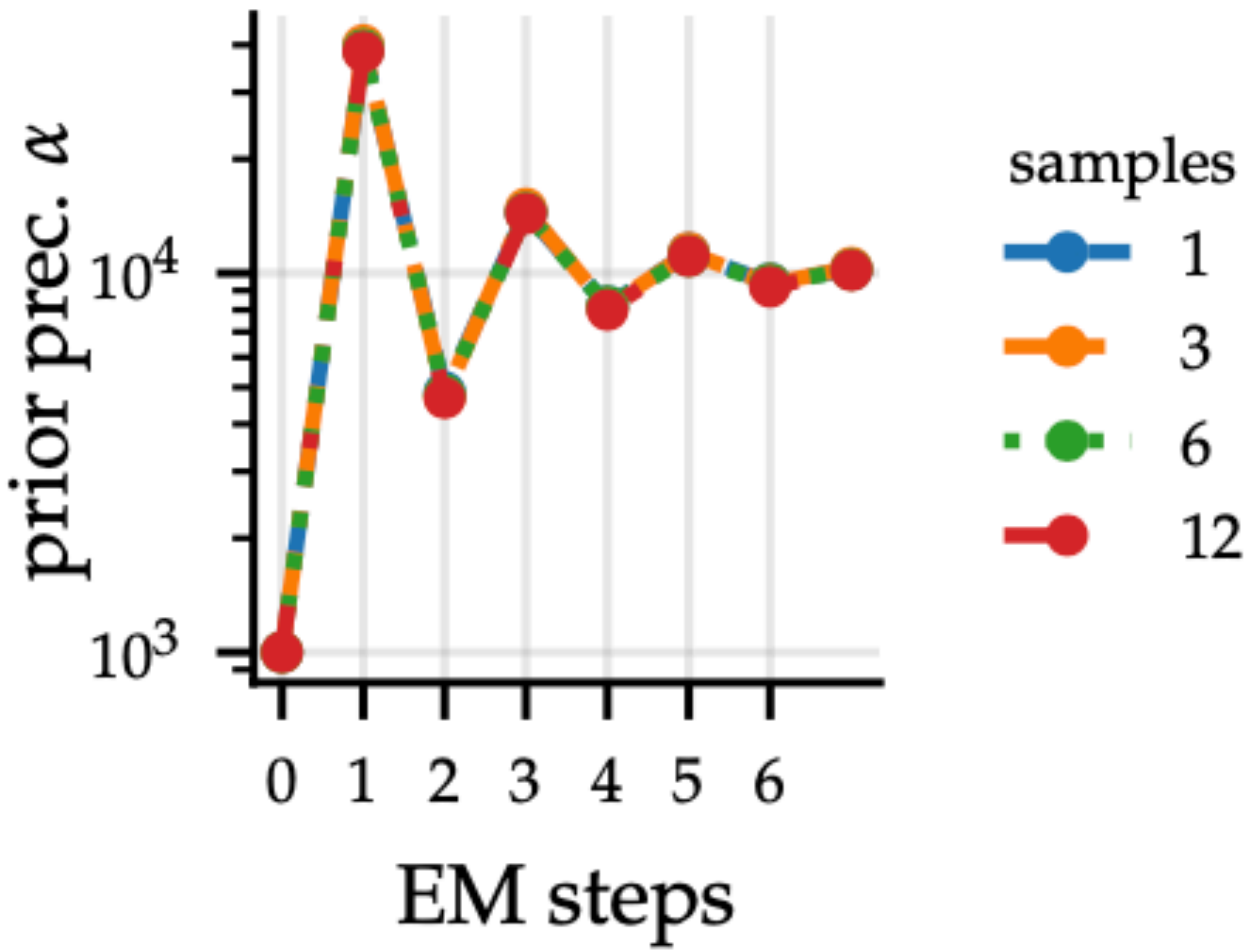


# Stability of estimator: 1 sample is enough

ResNet-18 ($d = 11M$) on CIFAR-100 ($nm = 5M$)



# Demonstration: Scalable Uncertainty Estimation in NNs

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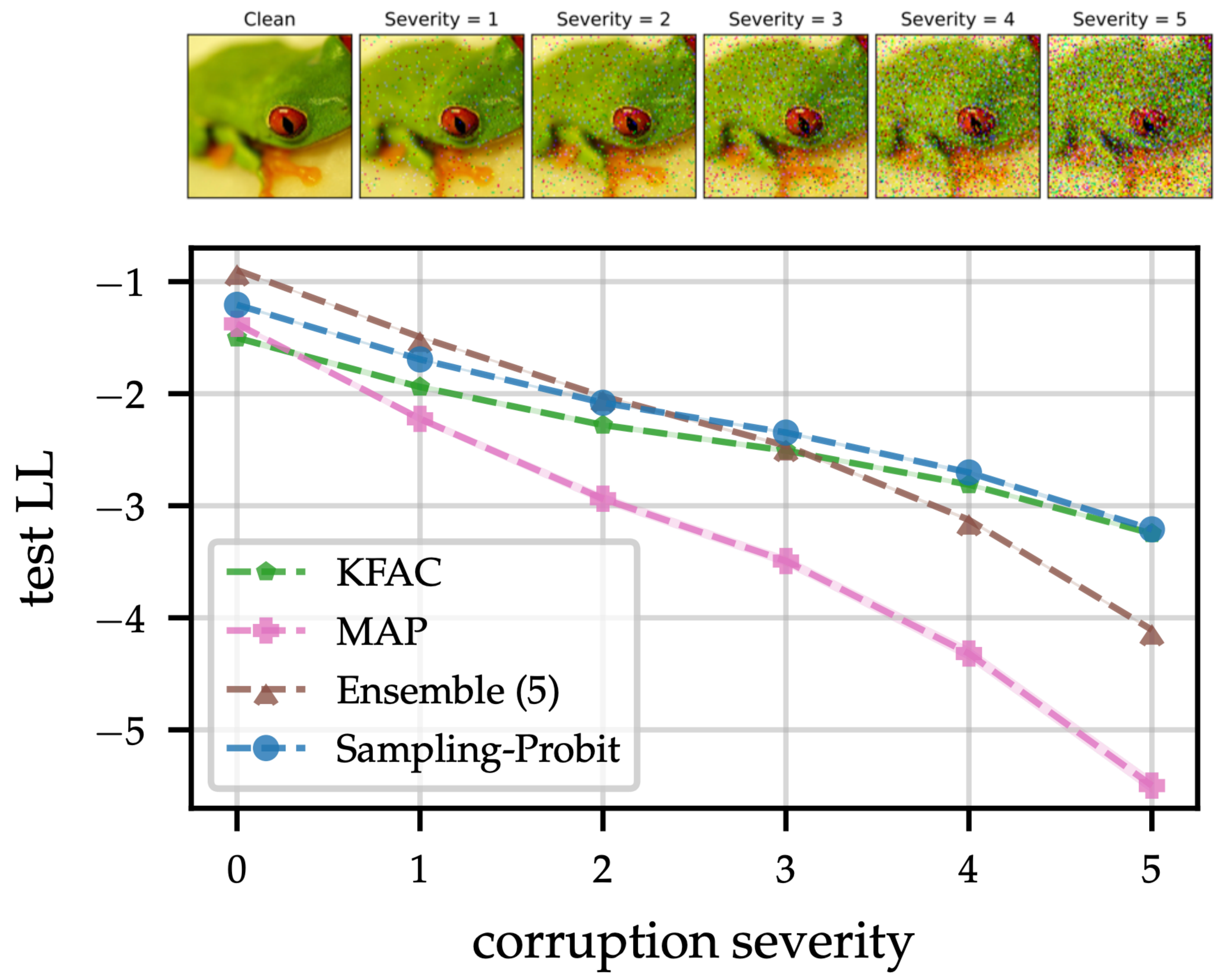

# Thank you to my collaborators!

Javier Antorán

Riccardo Barbano

Eric Nalisnick

David Janz

José Miguel
Hernández-Lobato