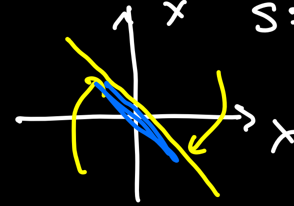
$\boxed{\text{Announcements}}$

- Extra OH 11.17.22 3-4pm

Attitude Control

$\vec{s}$: Sliding variable



$s = \dot{x} + \lambda x$

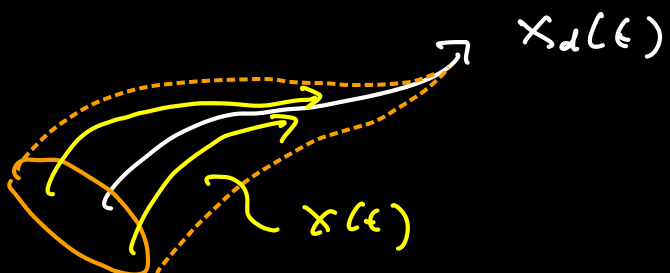
Let $\vec{s} \triangleq \vec{\omega}_e + \lambda \, \mathrm{sgn}(q_e^{\circ}) \, \vec{q}_e$ where $q_e = q_d^* \otimes q$

$\vec{\omega}_e = \vec{\omega} - q_e^* \otimes \vec{\omega}_d \otimes q_e$

Showed If $\vec{s} = 0$ Indefinitely ($S = \{ \vec{s} \mid \vec{s} = 0 \}$ Is Invariant)
Then $\vec{q}_e \to 0$ Exponentially

If $\vec{s} = 0$ & $\| \vec{q}_e \| = 0$ Then $\vec{\omega}_e = \vec{0}$

$\Rightarrow$ If $\vec{s} = 0$ Then $q \to q_d(t)$ & $\vec{\omega} \to \vec{\omega}_d(t)$



Exponentially Convergent Tracking
$\Rightarrow$ "Forget" Initial Conditions

Corollary: If $\vec{s} \to 0$ Exponentially Then $\vec{q}_e \to \vec{0}$ Exponentially

Must Now Find $\vec{M}_B$ Such That $\vec{s} \to 0$ Exponentially

$$\vec{s} = \vec{\omega}_e + \lambda \,\mathrm{sgn}(q_e^\circ)\, \vec{q}_e = \vec{\omega} - \vec{\omega}_r + \lambda \,\mathrm{sgn}(q_e^\circ)\, \vec{q}_e$$

$$\vec{\omega}_r = q_e^* \otimes \vec{\omega}_d \otimes q_e$$

Claim: If $\vec{M}_B = \vec{\omega} \times J\vec{\omega} + J\dot{\vec{\omega}}_r - J\lambda \,\mathrm{sgn}(q_e^\circ)\, \dot{\vec{q}}_e - KJ\vec{s}$
Then $\vec{s} \to \vec{0}$ Exponentially where $K > 0$

Proof: Recall $\vec{s} = \vec{\omega} - \vec{\omega}_r + \lambda \,\mathrm{sgn}(q_e^\circ)\, \vec{q}_e$

$$J\dot{\vec{s}} = J\dot{\vec{\omega}} - J\dot{\vec{\omega}}_r + J\lambda \,\mathrm{sgn}(q_e)\, \dot{\vec{q}}_e$$

$$= -\vec{\omega} \times J\vec{\omega} + \vec{M}_B - J\dot{\vec{\omega}}_r + J\lambda \,\mathrm{sgn}(q_e^\circ)\, \dot{\vec{q}}_e$$

Consider $V(\vec{s}) = \vec{s}^T J \vec{s} \quad > 0 \quad \forall \vec{s} \neq 0$

$\qquad\qquad\qquad\qquad = 0 \quad \vec{s} = 0$ ✓

$$\mathring{V}(\vec{s}) = 2\vec{s}^T J \dot{\vec{s}}$$

$$= 2s^T\left[-\vec{\omega} \times J\vec{\omega} + \vec{M}_B - J\dot{\vec{\omega}}_r + J\lambda\,\mathrm{sgn}(q_e^{\circ})\dot{\vec{q}}_e\right]$$

Find $\vec{M}_B$ Such That $\mathring{V}(\vec{s}) \leq -cV(\vec{s})$

If $\vec{M}_B = \vec{\omega} \times J\vec{\omega} + J\dot{\vec{\omega}}_r - J\lambda\,\mathrm{sgn}(q_e^{\circ})\dot{\vec{q}}_e - KJ\vec{s}$

Then $\mathring{V}(\vec{s}) = 2s^T\{-KJ\vec{s}\}$

$$= -2Ks^T J\vec{s} = -2KV(\vec{s})$$ ✓

$\Rightarrow V(\vec{s}) \to 0$ Exponentially with Rate $-2K$

& Since $v(\vec{S}) = \vec{S} J \vec{S}$ Then $\vec{S} \to 0$ Exponentially with Same Rate

$$\|\vec{s}(t)\|_J^2 \le \|s(0)\|_J^2 \, e^{-2kt}$$

Corollary: If $\vec{S} = \vec{\omega}_e + \lambda \,\mathrm{sgn}(q_e^\circ)\, \vec{q}_e$ &

$\vec{M}_B = \vec{\omega} \times J\vec{\omega} + J\dot{\vec{\omega}}_r - \lambda J \,\mathrm{sgn}(q_e^\circ)\, \dot{\vec{q}}_e - KJ\vec{S}$

Then $q_e \to (\pm 1, 0)$ & $\vec{\omega}_e \to \vec{0}$ Exponentially

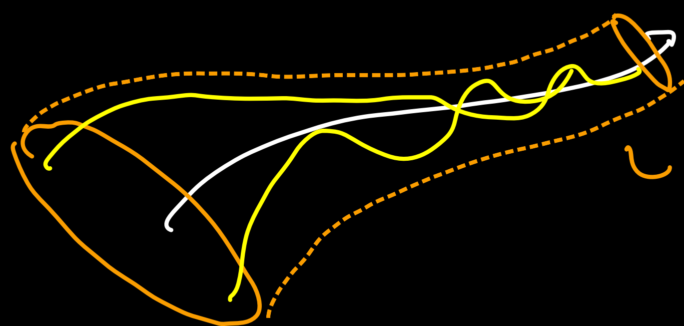
Given Any Dynamically Feasible $q_d(t)$ & $\vec{\omega}_d(t)$ Then $q \to q_d(t)$ & $\vec{\omega} \to \vec{\omega}_d(t)$ ✓

Controller Very Nonlinear, Also Requires Knowledge of Model Parameters, e.g., J

Arises From Model / Dynamics Cancellation

∃ Robust Versions Of The Above Controller $\vec{M}_B$ To Handle Model Error / Disturbances

⇒ Exponential Convergence But To Region Around $q_d(t)$ (Tube)



Instructive To Compare $\vec{M}_B$ To Stabilizing Controller

$$\vec{M}_B' = -K_p \,\mathrm{sgn}(q_e^0)\,\vec{q}_e - K_d\,\vec{\omega} \quad \leftarrow \text{Model Agnostic}$$

$\vec{M}_B'$ Stable For $q_d(t) = q_d$ (Constant) $\Rightarrow \vec{\omega}_d = \vec{0}$

↳ $V(\vec{q}_e, \vec{\omega}) = K_p\,\vec{q}_e^{\,T}\vec{q}_e + \frac{1}{2}\vec{\omega}^T J\,\vec{\omega}$

In Practice $\vec{M}_b' = -k_p \,\mathrm{sgn}(q_e^\circ)\,\vec{q}_e - K_d \vec{\omega}_e$ where $\vec{\omega}_e = \vec{\omega} - \vec{\omega}_d$ & $q_d(t), \omega_d(t)$ Works Pretty well

However, we Lose Convergence Guarantees That $\vec{M}_B$ Posseses

A Nice Compromise Is To Let (Pretty Approx. If $J = cI$)

$$\vec{M}_B = \dot{\vec{\omega}}_r - \lambda\,\mathrm{sgn}(q_e^\circ)\,\dot{\vec{q}}_e - K\vec{s}$$

$$= \underbrace{\dot{\vec{\omega}}_r}_{\text{Feedforward}} - \underbrace{\lambda\,\mathrm{sgn}(q_e^\circ)\,\dot{\vec{q}}_e}_{\text{Feedback on } \dot{\vec{q}}_e} - \underbrace{K\left(\vec{\omega}_e + \lambda\,\mathrm{sgn}(q_e^\circ)\,\vec{q}_e\right)}_{\text{PD Control}}$$

What About Unwinding ?!?

We Showed That If $\vec{s} = \vec{\omega}_e + \lambda \, \mathrm{sgn}(q_e^0) \vec{q}_e$

Then $\frac{d}{dt} \| \vec{q}_e \|^2 = -\lambda |q_e^0| \, \| \vec{q}_e \|^2 < 0 \quad \forall \; q_e^0 \neq 0$

What If $\vec{s}' = \vec{\omega}_e + \lambda \vec{q}_e$. Then If $\vec{s}' = 0$

$$\frac{d}{dt} \| \vec{q}_e \|^2 = -\lambda q_e^0 \, \| \vec{q}_e \|^2$$

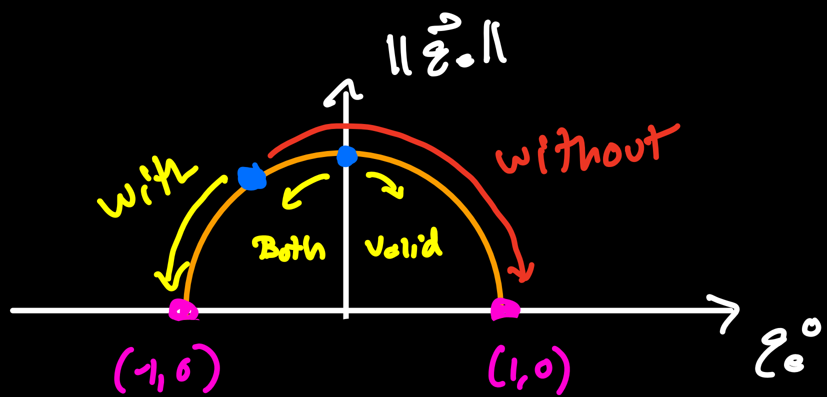
If $q_e^0 < 0$ (Error $> 180°$) Then $\frac{d}{dt} \| \vec{q}_e \|^2 > 0$
(Error Increasing)

$\Rightarrow$ $q_e = (-1, \vec{0})$ Is <u>Unstable</u> But $q_e = -q_e$

$\mathrm{sgn}(q_e^0)$ Necessary So $\frac{d}{dt} \| \vec{q}_e \|^2 < 0 \quad \forall \; q_e^0$

$\Rightarrow$ $q_e = (1, \vec{0})$ & $q_e = (-1, \vec{0})$ Are Stable

<u>Intuition</u>: Controller Picks which Equilibrium It Converges To. Selection Based On which Ever Equil. Is Closest.



Position Tracking

We Have Shown $q \to q_d(t)$ & $\vec{\omega} = \vec{\omega}_d(t)$

We Now Find Another Controller To Track $\vec{r}_d(t), \dot{\vec{r}}_d(t)$

Recall $m\ddot{\vec{r}} = \vec{F}_I + q \otimes \vec{F}_B \otimes q^*$

where $\vec{F}_B \approx \begin{bmatrix} 0 \\ 0 \\ T \end{bmatrix} = \vec{T}$ & $\vec{T}$ Is Considered Control Input

What If we Set $q \otimes \vec{T} \otimes q^* = m\vec{u}$?

$$m\ddot{\vec{r}} = \vec{F}_I + m\vec{u} \Rightarrow \underbrace{\ddot{\vec{r}} = \vec{a}_I + \vec{u}}_{\text{Just Linear System} \Rightarrow \text{PD Control?}}$$

Pick $\vec{u} = \ddot{\vec{r}}_d - \vec{a}_I - K_p \vec{r}_e - K_d \dot{\vec{r}}_e$

where $\vec{r}_e \triangleq \vec{r} - \vec{r}_d$ & $\dot{\vec{r}}_e = \dot{\vec{r}} - \dot{\vec{r}}_d$

$$\ddot{\vec{r}} = \ddot{\vec{r}}_d - K_p \vec{r}_e - K_d \dot{\vec{r}}_e \Rightarrow \ddot{\vec{r}}_e + K_d \dot{\vec{r}}_e + K_p \vec{r}_e = 0$$

Appropriate Selection of K_p & K_d Yields Exponentially Convergent Position Tracking Error ($\vec{r}_e \to 0$ Exponentially)

Result Relies On $q \otimes \vec{T} \otimes q^* = \vec{u}$ where
$\vec{u} = \ddot{\vec{r}}_d - \vec{c}_z - K_p \vec{r}_e - K_d \dot{\vec{r}}_e$

<u>Key Insights</u>

1) If $q \to q_d$ & $\vec{\omega} \to \vec{\omega}_d$ Exponentially Then
$q \otimes \vec{T} \otimes q^* \to q_d \otimes \vec{T} \otimes q_d^*$

↳ If $q \approx q_d$ Then $q \otimes \vec{T} \otimes q^* \approx q_d \otimes \vec{T} \otimes q_d^*$

2) Given $\vec{u}$, Can Determine $\vec{T}$, q_d, $\vec{\omega}_d$, $\dot{\vec{\omega}}_d$