Last Time we Found $T \in \mathbb{R}^{2\times2}$ Such That

$$\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = T \begin{bmatrix} \alpha \\ q \end{bmatrix} \quad \text{where} \quad \ddot{x} = v$$

$x^*(t)$ Is Compute By Solving Some OCP
Or More Generally Given Any $x_d(t)$

$$\begin{bmatrix} \alpha_d(t) \\ q_d(t) \end{bmatrix} = T^{-1} \begin{bmatrix} x_d(t) \\ \dot{x}_d(t) \end{bmatrix}$$

$\alpha_d(t)$ & $q_d(t)$ Uniquely Described By $x_d(t)$
& Its Derivatives

<u>Differential Flatness</u>

Property That Allows One To Parameterize The System's State By Set Of <u>Indep Variables</u>

& Their Derivatives

Flat Varibles

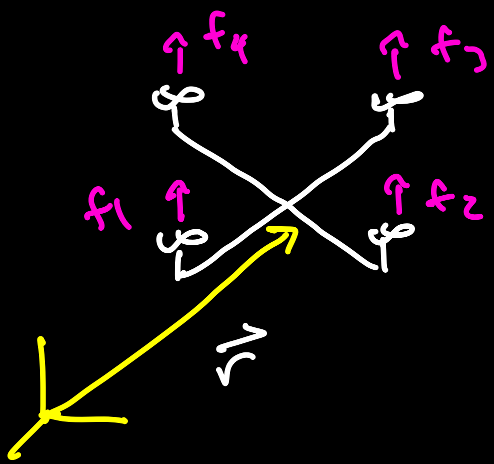
$$\dot{\vec{x}} = A\vec{x} + Bu \;\; \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \;\; z^{(n)} = v \qquad z: \text{Flat Variable}$$

Where $\begin{bmatrix} z \\ \dot{z} \\ \vdots \end{bmatrix} = T\vec{x} \qquad \vec{x} \in \mathbb{R}^n$

Differential Flatness Bypasses The Dynamics Of Original System

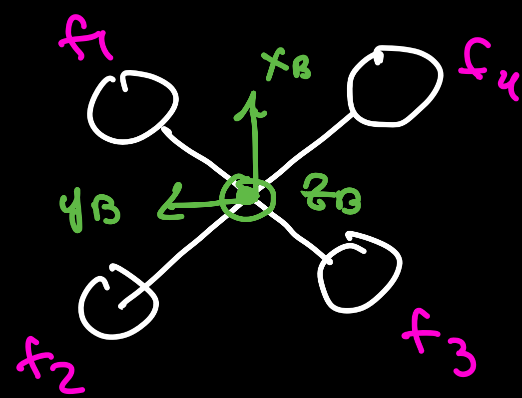
Fact: # Of Flat Variables = # Of Control Inputs

Ex. 3D Quadrotor



$q \doteq {}^{z}_{0}q$

Top-Down



States: $\vec{r}, \dot{\vec{r}}, q, \vec{\omega}$

Depends on veh. Geom.

Inputs: $f_1, f_2, f_3, f_4 \Rightarrow \vec{T} = \begin{bmatrix} 0 \\ 0 \\ \sum f_i \end{bmatrix}, \quad \vec{M}_B = \Gamma \begin{bmatrix} f_1 \\ \vdots \\ f_4 \end{bmatrix}$

Objective: Smooth Point-To-Point Trajectory + Some BC

Dynamics: $m\ddot{\vec{r}} = q \otimes \vec{T} \otimes q^* - \vec{g}$

$$\dot{q} = \frac{1}{2} q \otimes \begin{bmatrix} 0 \\ \vec{\omega} \end{bmatrix}$$

$$J \dot{\vec{\omega}} = -\vec{\omega} \times J \vec{\omega} + \vec{M}_B$$

$$\min_{\vec{T}, \vec{m}_B} \quad J$$

Sub To Dynamics + BC

Non-Convex Optimization B/C NonLinear Dynamics $\Rightarrow$ Hard...

Non-Convex $\Rightarrow$ Trajectory Design with Original States Is very Challenging

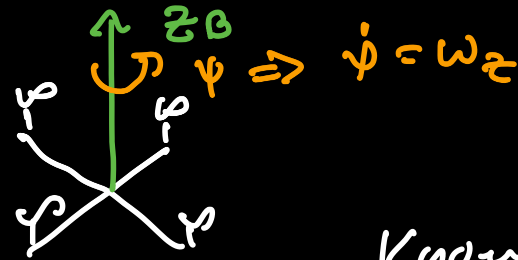
Key Insight: System Is Differentially Flat

$\Rightarrow \quad \vec{r}, \dot{\vec{r}}, q, \vec{\omega} \quad \rightleftarrows \quad z_1, z_2, \ldots \quad$ s.t. $\quad z_i^{(m)} = v_i$

Nonlinear System

Linear System

Fact: $\vec{r}$ & ψ (Yaw Angle) Are Flat Variables



Let $\vec{r}_d(t)$ & $\psi_d(t)$ Be Known Sufficiently Smooth Trajectories

Differentiating $r_d(t)$ Twice

$$m\ddot{\vec{r}}_d = q_d \otimes \vec{T}_d \otimes q_d^* - M\vec{g}$$

Let's Solve For $\vec{T}_d$

Recall That Rotations Perserve Length

$$m\|\ddot{\vec{r}}_d + \vec{g}\| = \|q_d \otimes \vec{T}_d \otimes q_d^*\| = \|\vec{T}_d\| = \sum f_{i,d} = T_d$$

$$\Rightarrow \boxed{T_d = m\|\ddot{\vec{r}}_d + g\|} \Rightarrow \vec{T}_d = \begin{bmatrix} 0 \\ 0 \\ m\|\ddot{\vec{r}}_d + g\| \end{bmatrix}$$

$$q_d \otimes \vec{T}_d \otimes q_d^* = m(\ddot{\vec{r}}_d + \vec{g})$$

<u>Formula:</u> Given $\hat{v}_B$ & $\hat{v}_I$ $\|\hat{v}_B\| = \|\hat{v}_I\| = 1$ Then The Quaternion That Aligns $\hat{v}_B$ & $\hat{v}_I$ Is

$$q = \frac{1}{\sqrt{2(1+\hat{v}_B^T\hat{v}_I)}} \begin{bmatrix} 1+\hat{v}_B^T\hat{v}_I \\ \hat{v}_B \times \hat{v}_I \end{bmatrix}$$

Recall That $\vec{T}_d = \begin{bmatrix} 0 \\ 0 \\ T \end{bmatrix} \Rightarrow \hat{T}_d = \frac{\vec{T}_d}{T_d}$

$$\boxed{q_d = \frac{1}{\sqrt{2(1+\hat{T}_d^T\hat{F}_I)}} \begin{bmatrix} 1+\hat{T}_d^T\hat{F}_I \\ \hat{T}_d \times \hat{F}_I \end{bmatrix} \quad \text{Where } \vec{F}_I = m(\ddot{\vec{r}}_d + g)}$$

$$m\ddot{\vec{r}}_d = q_d \otimes \vec{T}_d \otimes q_d^* - \vec{g}$$

$$\Big\Downarrow \ \frac{d}{dt}(\cdot)$$

$$m\vec{r}_d^{\,(3)} = \dot{q}_d \otimes \vec{T}_d \otimes q_d^* + q_d \otimes \vec{T}_d \otimes \dot{q}_d^* + q_d \otimes \dot{\vec{T}}_d \otimes q_d^*$$

$$\dot{q}_d = \frac{1}{2} q_d \otimes \begin{bmatrix} 0 \\ \vec{\omega}_d \end{bmatrix} \qquad \dot{q}_d^* = -\frac{1}{2} \begin{bmatrix} 0 \\ \vec{\omega}_d \end{bmatrix} \otimes q_d^*$$

$$m\vec{r}_d^{\,(3)} = q_d \otimes \left\{ \frac{1}{2} \begin{bmatrix} 0 \\ \vec{\omega}_d \end{bmatrix} \otimes \begin{bmatrix} 0 \\ \vec{T}_d \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ \vec{T}_d \end{bmatrix} \otimes \begin{bmatrix} 0 \\ \vec{\omega}_d \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\vec{T}}_d \end{bmatrix} \right\} \otimes q_d^*$$

Can show $\begin{bmatrix} 0 \\ \vec{\omega}_d \end{bmatrix} \otimes \begin{bmatrix} 0 \\ \vec{T}_d \end{bmatrix} - \begin{bmatrix} 0 \\ \vec{T}_d \end{bmatrix} \otimes \begin{bmatrix} 0 \\ \vec{\omega}_d \end{bmatrix} = \begin{bmatrix} 0 \\ 2\,\vec{\omega}_d \times \vec{T}_d \end{bmatrix}$

$$m\vec{r}_d^{\,(3)} = q_d \otimes \left\{ \begin{bmatrix} 0 \\ \vec{\omega}_d \times \vec{T}_d \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\vec{T}}_d \end{bmatrix} \right\} \otimes q_d^*$$

Want $\underline{\underline{\vec{\omega}_d}}$

$$\vec{T}_d = \begin{bmatrix} 0 \\ 0 \\ T_d \end{bmatrix} \quad \& \quad \vec{\omega}_d = \begin{bmatrix} \omega_{xd} \\ \omega_{yd} \\ \omega_{zd} \end{bmatrix}$$

$$\vec{\omega}_d \times \vec{T}_d = \begin{bmatrix} \omega_{yd} T_d \\ -\omega_{xd} T_d \\ 0 \end{bmatrix} = T_d \begin{bmatrix} \omega_{yd} \\ -\omega_{xd} \\ 0 \end{bmatrix}$$

Rearranging, we Get

$$\boxed{\begin{bmatrix} \omega_{yd} \\ -\omega_{xd} \\ 0 \end{bmatrix} = \frac{m}{T_d} q_d^* \otimes \vec{r}_d^{(3)} \otimes q_d - \frac{\dot{\vec{T}}_d}{T_d}}$$

ω_{zd} Is Absent $\Rightarrow$ ω_{zd} Can Be Freely Picked!
$\Rightarrow$ ψ_d where $\dot{\psi}_d = \omega_{zd}$ Is

Indeed Flat Variable

Yaw Angle $\psi_d(t)$ Can Be Arbitrarily Picked

$\Rightarrow$ q_d Doesn't Restrict what $\psi_d(t)$ Should Be

Given Some $\psi_d(t)$ Then

$$\bar{q}_d = q_d \otimes \underbrace{\left(c(\psi_d/2), 0, 0, s(\psi_d/2) \right)}_{\substack{\text{Body Frame Rotation} \\ \text{About } (0,0,1) \text{ Through } \psi_d}}$$

Observe $\quad \ddot{\vec{r}}_d \rightarrow q_d \ \& \ \vec{T}_d$ (Up To Yaw Angle)

$$\vec{r}_d^{(3)}, \ddot{\vec{r}}_d, \dot{\psi}_d \Rightarrow \vec{\omega}_d$$

$$\vec{r}_d^{(4)}, \ldots, \ddot{\psi}_d \Rightarrow \dot{\vec{\omega}}_d = \vec{\alpha}_d \rightarrow \vec{M}_{B_d}$$

$$\Downarrow$$

$$J\dot{\vec{\omega}}_d = -\vec{\omega}_d \times J\vec{\omega}_d + \vec{M}_{Bd}$$

So Given $\vec{r}_d(t), \ldots, \vec{r}_d^{(4)}(t)$ + $\psi_d(t), \ldots, \ddot{\psi}_d(t)$

Then we Know $\vec{r}_d, \dot{\vec{r}}_d, q_d, \vec{\omega}_d$ + $f_{1d}, f_{2d}, f_{3d}, f_{4d}$

$(\vec{T}_d, \vec{M}_{Bd})$

For Instance If $\vec{r}_d(t) = \begin{bmatrix} \sum a_i t^i \\ \sum b_i t^i \\ \sum c_i t^i \end{bmatrix}$ $\psi_d(t) = \sum d_i t^i$

Then we Know $\vec{r}_d, \dot{\vec{r}}_d, q_d, \vec{\omega}_d$ + $\vec{T}_d, \vec{M}_{Bd}$