

# **Solving Initial Value Problems for ODEs**

using Fixed-Step Solvers

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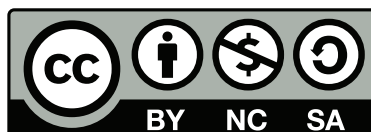
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# PART I

## **Basics**

# Initial Value Problems

## 1.1 Definition

There are many classes of problems involving ordinary differential equations (ODEs) and many different ways to solve them numerically. However, colloquially, **IVP solvers**<sup>1</sup> are tools that can (numerically) solve initial value problems (IVPs) for ODEs. In these types of problems, we are given a differential equation along with an initial condition. In the single-variable case,

$$\frac{dy}{dt} = f(t, y), \quad x(t_0) = x_0$$

Our goal is to solve for  $y(t)$ . The above equation indicates that the rate of change is  $y$  is dependent on (a) the instantaneous value of  $y$  and (b) the independent variable,  $t$ <sup>2</sup>.

The previous equation represented a 1st-order IVP. However, IVP solvers can also be designed to be able to solve higher-order IVPs. For example, we can express a 2nd-order IVP as

$$\frac{d^2y}{dt^2} = f\left(t, x, \frac{dy}{dt}\right), \quad y(t_0) = y_0, \quad \left.\frac{dy}{dt}\right|_{t=t_0} = y'_0$$

For a 1st-order IVP, we know  $f$  and  $y_0$  and wish to find  $y(t)$ . For a 2nd-order IVP, we know  $f$ ,  $y_0$ , and  $y'_0$ , and wish to find  $y(t)$ . In some (extremely rare) cases, it is possible to solve these problems explicitly. However, even then, it is often easier to just use a numerical method. Once again, it is *vital* to note that these numerical methods are applicable to vector-valued IVPs as well. If we consider a vector-valued variable  $\mathbf{y} = (y_1, \dots, y_n)^T$ , then the 1st and 2nd-order IVPs become

$$\begin{aligned} \frac{d\mathbf{y}}{dt} &= f(t, \mathbf{y}), \quad \mathbf{y}(t_0) = \mathbf{y}_0 \\ \frac{d^2\mathbf{y}}{dt^2} &= f\left(t, \mathbf{y}, \frac{d\mathbf{y}}{dt}\right), \quad \mathbf{y}(t_0) = \mathbf{y}_0, \quad \left.\frac{d\mathbf{y}}{dt}\right|_{t=t_0} = \mathbf{y}'_0 \end{aligned}$$

<sup>1</sup> Sometimes also referred to as “ODE solvers”

<sup>2</sup> The independent variable is denoted as  $t$  because in most cases IVP solvers are used in situations where time is the independent variable.



## 1.2 Converting Higher-Order IVPs to First-Order IVPs: The Scalar Case

Consider the general form of a 2nd-order IVP. Noting that the second derivative of a variable can be a function of the independent variable, the variable itself, and the first derivative of the variable, we write (in the general case)

$$\frac{d^2 z}{dt^2} = g\left(t, z, \frac{dz}{dt}\right), \quad \left.\frac{dz}{dt}\right|_{t=t_0} = z'_0, \quad z(t_0) = z_0 \quad (1.1)$$

Our goal is to solve such IVPs using an IVP solver. *However, IVP solvers are only suitable for solving 1st-order IVPs.* The solution to this issue is to define a **vector-valued differential equation** using a vector,  $\mathbf{y}$ , to represent the dependent variable. The general form of such a vector-valued differential equation is

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y}), \quad \mathbf{y}(t_0) = \mathbf{y}_0 \quad (1.2)$$

At first glance, going from Eq. (1.1) to Eq. (1.2) seems like an impossible undertaking. However, the conversion can be accomplished in three steps, in a rather algorithmic fashion. Thus, we first introduce Algorithm 1 to define the procedure for an  $m$ th-order IVP, and then use this algorithm to show how to convert Eq. (1.1) to Eq. (1.2).

### Algorithm 1:

Converting a higher-order IVP to a first-order IVP (scalar dependent variable).

#### Given:

- $\frac{d^m z}{dt^m} = g\left(t, z, \frac{dz}{dt}, \frac{d^2 z}{dt^2}, \dots, \frac{d^{m-1} z}{dt^{m-1}}\right)$  -  $m$ th-order ODE
- $z_0, z'_0, z''_0, \dots, z_0^{(m-1)}$  - initial conditions

#### Procedure:

1. Define the vector  $\mathbf{y}$  as the original dependent variable  $z$  and its derivatives up to the  $(m-1)$ th derivative.

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} z \\ dz/dt \\ d^2 z/dt^2 \\ \vdots \\ d^{m-1} z/dt^{m-1} \end{bmatrix}$$

2. Find the derivative of  $\mathbf{y}$ .

$$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} dz/dt \\ d^2 z/dt^2 \\ d^3 z/dt^3 \\ \vdots \\ d^m z/dt^m \end{bmatrix}$$

3. Note that the original differential equation was defined as  $d^m z/dt^m = g(\dots)$ .

Thus, we can substitute the function  $g$  into the last component of  $dy/dt$ .

$$\frac{dy}{dt} = \begin{bmatrix} dz/dt \\ d^2z/dt^2 \\ d^3z/dt^3 \\ \vdots \\ g\left(t, z, \frac{dz}{dt}, \frac{d^2z}{dt^2}, \dots, \frac{d^{m-1}z}{dt^{m-1}}\right) \end{bmatrix}$$

4. From step 1, we can see that  $z = y_1$ ,  $dz/dt = y_2$ ,  $d^2z/dt^2 = y_3$ , etc. Make these substitutions into the RHS of the equation from step 3. This essentially gives us  $\mathbf{f}(t, \mathbf{y})$ .

$$\frac{dy}{dt} = \mathbf{f}(t, \mathbf{y}) = \begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ \vdots \\ g(t, y_1, y_2, y_3, \dots, y_n) \end{bmatrix}$$

5. At this point, we have the function  $\mathbf{f}(t, \mathbf{y})$  defining the vector-valued differential equation. Now, we just need to define its initial conditions. Once again exploiting the fact that  $z = y_1$ ,  $dz/dt = y_2$ ,  $d^2z/dt^2 = y_3$ , etc., we can write

$$\mathbf{y}_0 = \begin{bmatrix} y_{1,0} \\ y_{2,0} \\ y_{3,0} \\ \vdots \\ y_{n,0} \end{bmatrix} = \begin{bmatrix} z_0 \\ z'_0 \\ z''_0 \\ \vdots \\ z_0^{(m-1)} \end{bmatrix}$$

**Return:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ ) defining the ODE  $dy/dt = \mathbf{f}(t, \mathbf{y})$
- $\mathbf{y}_0 \in \mathbb{R}^m$  - initial condition

Let's return to the 2nd-order case and apply Algorithm 1 to write it as a 1st-order IVP. Since  $z$  is a scalar variable, we need  $\mathbf{y} \in \mathbb{R}^2$ ; this is because one component of  $\mathbf{y}$  will store  $z$ , while the other will store its derivative with respect to  $t$ .

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} z \\ dz/dt \end{bmatrix}$$

Differentiating  $\mathbf{y}$ ,

$$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} dz/dt \\ d^2z/dt^2 \end{bmatrix}$$

Noting that  $d^2z/dt^2 = g(t, z, dz/dt)$ ,

$$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} dz/dt \\ g(t, z, dz/dt) \end{bmatrix}$$

From our definition of  $\mathbf{y}$ , we can see that  $z = y_1$  and  $dz/dt = y_2$ . Making these substitutions,

$$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} y_2 \\ g(t, y_1, y_2) \end{bmatrix}$$

Thus, we have

$$\mathbf{f}(t, \mathbf{y}) = \begin{bmatrix} y_2 \\ g(t, y_1, y_2) \end{bmatrix}$$

Finally, defining  $\mathbf{y}_0$  using the initial conditions for  $z$  and  $dz/dt$ ,

$$\mathbf{y}_0 = \begin{bmatrix} z_0 \\ z'_0 \end{bmatrix} \quad (1.3)$$

For additional understanding, consider Example 1.2.1, which also demonstrates implementation of a vector-valued differential equation in MATLAB.

### Example 1.2.1: Converting a 3<sup>rd</sup>-order IVP to a 1<sup>st</sup>-order IVP.

Consider the following 3<sup>rd</sup>-order IVP:

$$\frac{d^3b}{dx^3} = 25x^2b^3 \frac{d^2b}{dx^2} + xb, \quad b(x_0) = 5, \quad \left. \frac{db}{dx} \right|_{x=x_0} = 0, \quad \left. \frac{d^2b^2}{dx^2} \right|_{x=x_0} = -10$$

Write this IVP as a 1<sup>st</sup>-order IVP and implement the resulting vector-valued differential equation in MATLAB.

#### ■ SOLUTION

First, let's define the vector variable:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b \\ db/dx \\ d^2b/dx^2 \end{bmatrix}$$

Differentiating,

$$\frac{d\mathbf{y}}{dx} = \begin{bmatrix} db/dx \\ d^2b/dx^2 \\ d^3b/dx^3 \end{bmatrix}$$

Substituting the original ODE into the last element of  $d\mathbf{y}/dx$ ,

$$\frac{d\mathbf{y}}{dx} = \begin{bmatrix} db/dx \\ d^2b/dx^2 \\ 25x^2b^3(d^2b/dx^2) + xb \end{bmatrix}$$

Noting that  $y_1 = b$ ,  $y_2 = db/dx$ , and  $y_3 = d^2b/dx^2$ , we get

$$\frac{d\mathbf{y}}{dx} = \begin{bmatrix} y_2 \\ y_3 \\ 25x^2y_1^3y_3 + xy_1 \end{bmatrix}$$

Since, by convention, we write  $d\mathbf{y}/dx = \mathbf{f}(x, \mathbf{y})$ , we get

$$\mathbf{f}(x, \mathbf{y}) = \begin{bmatrix} y_2 \\ y_3 \\ 25x^2y_1^3y_3 + xy_1 \end{bmatrix}$$

Assembling the initial conditions into the vector  $\mathbf{y}_0$  (and recalling that  $y_1 = b$ ,  $y_2 = db/dx$ , and  $y_3 = d^2b/dx^2$ ),

$$\mathbf{y}_0 = \begin{bmatrix} 5 \\ 0 \\ -10 \end{bmatrix}$$

To define the vector-valued differential equation and the corresponding initial condition in MATLAB,

```

% vector-valued differential equation
f = @(x,y) [y(2);y(3);
            25*x^2*y(1)^3*y(3)+x*y(1)];

% initial condition
y0 = [ 5;
       0;
      -10];

```

### 1.3 Converting Higher-Order IVPs to First-Order IVPs: The Vector Case

Higher order IVPs are not restricted to scalar variables; in fact, most of the time, the IVPs we deal with our higher-order IVPs where the dependent variable is a vector. The general form of such an IVP is shown in Eq. (1.4) below, where we assume that  $\mathbf{z} \in \mathbb{R}^p$ .

$$\frac{d^m \mathbf{z}}{dt^m} = \mathbf{g} \left( t, \mathbf{z}, \frac{d\mathbf{z}}{dt}, \dots, \frac{d^{m-1} \mathbf{z}}{dt^{m-1}} \right), \quad \mathbf{z}(t_0) = \mathbf{z}_0, \quad \left. \frac{d\mathbf{z}}{dt} \right|_{t=t_0} = \mathbf{z}'_0, \quad \dots, \quad \left. \frac{d^{m-1} \mathbf{z}}{dt^{m-1}} \right|_{t=t_0} = \mathbf{z}_0^{(m-1)} \quad (1.4)$$

To convert Eq. (1.4) to a 1st-order IVP, we introduce Algorithm 2, which is essentially identical to Algorithm 1, just extended to the case of a vector dependent variable.

#### Algorithm 2:

Converting a higher-order IVP to a first-order IVP (vector dependent variable).

##### Given:

- $\frac{d^m \mathbf{z}}{dt^m} = \mathbf{g} \left( t, \mathbf{z}, \frac{d\mathbf{z}}{dt}, \frac{d^2 \mathbf{z}}{dt^2}, \dots, \frac{d^{m-1} \mathbf{z}}{dt^{m-1}} \right)$  -  $m$ th-order ODE
- $\mathbf{z}_0, \mathbf{z}'_0, \mathbf{z}''_0, \dots, \mathbf{z}_0^{(m-1)} \in \mathbb{R}^p$  - initial conditions

##### Procedure:

1. Define the vector  $\mathbf{y}$  as the original dependent variable  $\mathbf{z}$  and its derivatives up to the  $(m-1)$ th derivative.

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_{1:p} \\ \mathbf{y}_{(n+1):2p} \\ \mathbf{y}_{(2n+1):3p} \\ \vdots \\ \mathbf{y}_{((m-1)n+1):mp} \end{bmatrix} = \begin{bmatrix} \mathbf{z} \\ d\mathbf{z}/dt \\ d^2 \mathbf{z}/dt^2 \\ \vdots \\ d^{m-1} \mathbf{z}/dt^{m-1} \end{bmatrix}$$

In an expanded form, we write

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_p \\ y_{n+1} \\ \vdots \\ y_{2p} \\ y_{2n+1} \\ \vdots \\ y_{3p} \\ \vdots \\ y_{(m-1)n+1} \\ \vdots \\ y_{mp} \end{bmatrix} = \begin{bmatrix} z_1 \\ \vdots \\ z_p \\ dz_1/dt \\ \vdots \\ dz_p/dt \\ d^2 z_1/dt^2 \\ \vdots \\ d^2 z_p/dt^2 \\ \vdots \\ d^m z_1/dt^m \\ \vdots \\ d^m z_p/dt^m \end{bmatrix}$$

2. Find the derivative of  $\mathbf{y}$ .

$$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} dz/dt \\ d^2 \mathbf{z}/dt^2 \\ d^3 \mathbf{z}/dt^3 \\ \vdots \\ d^m \mathbf{z}/dt^m \end{bmatrix} = \begin{bmatrix} dz_1/dt \\ \vdots \\ dz_p/dt \\ d^2 z_1/dt^2 \\ \vdots \\ d^2 z_p/dt^2 \\ d^3 z_1/dt^3 \\ \vdots \\ d^3 z_p/dt^3 \\ \vdots \\ d^{m-1} z_1/dt^{m-1} \\ \vdots \\ d^{m-1} z_p/dt^{m-1} \end{bmatrix}$$

3. Note that the original differential equation was defined as  $d^m \mathbf{z}/dt^m = \mathbf{g}(\dots)$ .

Thus, we can substitute the function  $g$  into the last component of  $dy/dt$ .

$$\frac{dy}{dt} = \begin{bmatrix} dz_1/dt \\ \vdots \\ dz_p/dt \\ \hline d^2 z_1/dt^2 \\ \vdots \\ d^2 z_p/dt^2 \\ \hline d^3 z_1/dt^3 \\ \vdots \\ d^3 z_p/dt^3 \\ \hline \vdots \\ \hline g\left(t, \mathbf{z}, \frac{d\mathbf{z}}{dt}, \frac{d^2 \mathbf{z}}{dt^2}, \dots, \frac{d^{m-1} \mathbf{z}}{dt^{m-1}}\right) \end{bmatrix}$$

4. From step 1, we can see that  $\mathbf{z} = (y_1, \dots, y_p)^T$ ,  $d\mathbf{z}/dt = (y_{n+1}, \dots, y_{2p})^T$ ,  $d^2 \mathbf{z}/dt^2 = (y_{2n+1}, \dots, y_{3p})^T$ , etc. Make these substitutions into the RHS of the equation from step 3. This essentially gives us  $\mathbf{f}(t, \mathbf{y})$ .

$$\frac{dy}{dt} = \begin{bmatrix} y_{n+1} \\ \vdots \\ y_{2p} \\ \hline y_{2n+1} \\ \vdots \\ y_{3p} \\ \hline y_{3n+1} \\ \vdots \\ y_{4p} \\ \hline \vdots \\ \hline g\left(t, \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}, \begin{bmatrix} y_{n+1} \\ \vdots \\ y_{2p} \end{bmatrix}, \begin{bmatrix} y_{2n+1} \\ \vdots \\ y_{3p} \end{bmatrix}, \dots, \begin{bmatrix} y_{(m-1)n+1} \\ \vdots \\ y_{mp} \end{bmatrix}\right) \end{bmatrix}$$

5. At this point, we have the function  $\mathbf{f}(t, \mathbf{y})$  defining the vector-valued differential equation. Now, we just need to define its initial conditions.

$$\mathbf{y}_0 = \begin{bmatrix} \mathbf{z}_0 \\ \mathbf{z}'_0 \\ \mathbf{z}''_0 \\ \vdots \\ \mathbf{z}_0^{(m-1)} \end{bmatrix}$$

**Return:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R}^{mp} \rightarrow \mathbb{R}^{mp}$ ) defining the ODE  $dy/dt = \mathbf{f}(t, \mathbf{y})$
- $\mathbf{y}_0 \in \mathbb{R}^{mp}$  - initial condition

**Example 1.3.1:** Converting a 2nd-order vector IVP to a 1st-order IVP

Consider the following 2nd-order IVP:

$$\frac{d^2 \mathbf{x}}{dt^2} = (x_1 + x_2) \frac{d\mathbf{x}}{dt} + \mathbf{x} \frac{dx_1}{dt}, \quad \mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}'_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where  $\mathbf{x} = (x_1, x_2)^T$ . Write this IVP as a 1st-order IVP and implement the resulting differential equation in MATLAB.

**■ SOLUTION**

First, let's define our new vector variable:

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_{1:2} \\ \mathbf{y}_{3:4} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ d\mathbf{x}/dt \end{bmatrix}$$

In an expanded form,

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ dx_1/dt \\ dx_2/dt \end{bmatrix}$$

Differentiating,

$$\frac{d\mathbf{y}}{dx} = \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ d^2 x_1/dt^2 \\ d^2 x_2/dt^2 \end{bmatrix}$$

Expanding the original ODE,

$$\begin{cases} \frac{dx_1^2}{dt^2} = (x_1 + x_2) \frac{dx_1}{dt} + x_1 \frac{dx_1}{dt} \\ \frac{dx_2^2}{dt^2} = (x_1 + x_2) \frac{dx_2}{dt} + x_2 \frac{dx_1}{dt} \end{cases}$$

Replacing the last two element of  $d\mathbf{y}/dx$  with the above equation,

$$\frac{d\mathbf{y}}{dx} = \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ (x_1 + x_2) \frac{dx_1}{dt} + x_1 \frac{dx_1}{dt} \\ (x_1 + x_2) \frac{dx_2}{dt} + x_2 \frac{dx_1}{dt} \end{bmatrix}$$

Noting that  $y_1 = x_1$ ,  $y_2 = x_2$ , and  $y_3 = dx_1/dt$ , and  $y_4 = dx_2/dt$ , we get

$$\frac{d\mathbf{y}}{dx} = \begin{bmatrix} y_3 \\ y_4 \\ (y_1 + y_2)y_3 + y_1 y_3 \\ (y_1 + y_2)y_4 + y_2 y_3 \end{bmatrix}$$

Since, by convention, we write  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$ , we get

$$\mathbf{f}(t, \mathbf{y}) = \begin{bmatrix} y_3 \\ y_4 \\ (y_1 + y_2)y_3 + y_1 y_3 \\ (y_1 + y_2)y_4 + y_2 y_3 \end{bmatrix}$$

Assembling the initial conditions into the vector  $\mathbf{y}_0$  (and recalling that  $\mathbf{y}_{1:2} = \mathbf{x}$  and  $\mathbf{y}_{3:4} = d\mathbf{x}/dt$ ,

$$\mathbf{y}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

To define the vector-valued differential equation and the corresponding initial condition in MATLAB,

```
% vector-valued differential equation
f = @(t,y) [y(3);
            y(4);
            (y(1)+y(2))*y(3)+y(1)*y(3);
            (y(1)+y(2))*y(4)+y(2)*y(3)];

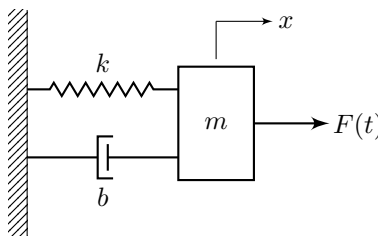
% initial condition
y0 = [0;
      0;
      1;
      1];
```



# Defining Systems Mathematically and Computationally

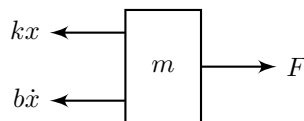
## 2.1 State Vectors and State Space

In physics and engineering, most differential equations we deal with are second-order or higher. For example, consider the generic mass-spring-damper system shown in Fig. 2.1. If we assume that  $x$  is defined such that  $x = 0$  when the



**Figure 2.1:** Mass-spring-damper system.

spring is at equilibrium, then (ignoring gravity) we can draw the free body diagram shown in Fig. 2.2. From Newton's



**Figure 2.2:** Free body diagram of mass.

second law,

$$F - kx - b\dot{x} = m\ddot{x}$$

Solving for  $\ddot{x}$ ,

$$\ddot{x} = -\left(\frac{b}{m}\right)\dot{x} - \left(\frac{k}{m}\right)x + \left(\frac{1}{m}\right)F$$

If we know the initial conditions  $x(t_0) = x_0$  and  $\dot{x}(t_0) = \dot{x}_0$ , then we have the following 2nd-order IVP:

$$\ddot{x} = -\left(\frac{b}{m}\right)\dot{x} - \left(\frac{k}{m}\right)x + \left(\frac{1}{m}\right)F, \quad x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0 \quad (2.1)$$

Since IVP solvers are only suitable for solving 1st-order IVPs (as discussed previously), we need to convert the 2nd-order IVP given by Eq. (2.1) to a 1st-order IVP. In the context of engineering, we do this by defining a **state vector**, which fully describes the state of the physical system. Essentially, the vector  $\mathbf{y}$  in Section 1.3 served as the state vector. There are two commonly used conventions for denoting a state vector as outlined in Convention 1 below.

**Convention 1: State vector nomenclature.**

- $\mathbf{y}$  – used when defining/discussing IVP solvers, as in mathematics, “ $y$ ” is typically used to denote the dependent variable
- $\mathbf{x}$  – used in engineering contexts to denote the state of a physical system

The components of the state vector are referred to as **state variables**. A state variable is just a variable that helps describe some component of the mathematical state of a physical system. Thus, the state vector is essentially the set of state variables that fully describes the system.

But what counts as “fully describing” the system? Intuitively, we want the state variables (and thereby the state vector) to describe enough about the system for us to determine its future behavior *in the absence of external forces acting on the system*<sup>1</sup>. By “future behavior”, we mean its position (i.e. trajectory) and its velocity. In the case of unforced motion in the mass-spring-damper system above, we only need the mass’s position and velocity at one instant to completely determine its position and velocity at any time in the future [11, 12]. Therefore, our state variables in this case are

$$\text{state variables} = x, \dot{x}$$

Our state vector (i.e. collection of state variables) is then

$$\mathbf{y} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

Note that in this case, we violate Convention 1 and denote our state vector as  $\mathbf{y}$ . We do this because when we go to input the math into a computer, it will be easier to do if we don’t have so many  $x$ ’s floating around.

Now that we’ve decided on our state-vector, we want to rewrite the 2nd-order IVP given by Eq. (2.1) as a 1st-order IVP in terms of the state vector. Mathematically<sup>2</sup>, the goal is to write

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}), \quad \mathbf{y}(t_0) = \mathbf{y}_0 \quad (2.2)$$

To do this, we can use an identical procedure to that taken in Section 1.2. If  $\mathbf{y} = (x, \dot{x})^T$  is our state vector, then the derivative of the state vector is  $\dot{\mathbf{y}} = (\dot{x}, \ddot{x})^T$ . Therefore, we can write the following system of differential equations,

<sup>1</sup> While the force,  $F$ , certainly does affect how the mass will move in the *future*, it does not affect how the mass is moving *in this instant* (i.e. it affects the acceleration instantaneously, but the position and velocity cannot change instantaneously). As an example, we can consider two cars who, at some instant, are exactly side by side traveling at the exact same speed. In one of the cars, the driver is pushing the gas pedal, while in the other car, the driver has slammed on the brakes. The fact that the drivers are applying different forces to their car has no bearing on the fact that at the instant considered, the state of the cars are the same; they are both at the same position, going at the same speed. The forces that the drivers are applying certainly do, however, affect how the car will be moving in the *following* instant.

<sup>2</sup> Since this system is linear, we could also write it as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where we would find that

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix}}^{\mathbf{A}} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \overbrace{\begin{bmatrix} 1 \\ 1/m \end{bmatrix}}^{\mathbf{B}} \overbrace{[f]}^{\mathbf{u}}$$

In the general case,  $\mathbf{u}$  is typically a function of  $\mathbf{x}$  and  $t$ , so we could define the function  $\mathbf{f}(t, \mathbf{x})$  as

$$\mathbf{f}(t, \mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}(t, \mathbf{x})$$

in order to simulate the system with an IVP solver.

together with their initial conditions:

$$\begin{cases} \dot{x} = \dot{x} \\ \ddot{x} = -\left(\frac{b}{m}\right)\dot{x} - \left(\frac{k}{m}\right)x + \left(\frac{1}{m}\right)F \end{cases}, \quad \begin{cases} x(t_0) = x_0 \\ \dot{x}(t_0) = \dot{x}_0 \end{cases}$$

In matrix form, these can be written as

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ -\left(\frac{b}{m}\right)\dot{x} - \left(\frac{k}{m}\right)x + \left(\frac{1}{m}\right)F \end{bmatrix}, \quad \begin{bmatrix} x(t_0) \\ \dot{x}(t_0) \end{bmatrix} = \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} \quad (2.3)$$

Since we know  $\mathbf{y} = (x, \dot{x})^T$  and  $\dot{\mathbf{y}} = (\dot{x}, \ddot{x})^T$ , we can simply compare Eq. (2.3) to Eq. (2.2) to find that

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) = \begin{bmatrix} \dot{x} \\ -\left(\frac{b}{m}\right)\dot{x} - \left(\frac{k}{m}\right)x + \left(\frac{1}{m}\right)F \end{bmatrix}, \quad \mathbf{y}_0 = \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix}$$

Another source of confusion at this point can be the fact that in the equation above, we wrote the left hand side as  $\mathbf{f}(t, \mathbf{y})$ , but the right hand side is not explicitly a function of  $\mathbf{y}$ ; rather, it is a function of the *components* (i.e.  $x$  and  $\dot{x}$ ) of  $\mathbf{y}$ . If we wanted to be overly-rigorous, we *could* distinguish the two as

$$\mathbf{f}(t, x, \dot{x}) = \begin{bmatrix} \dot{x} \\ -\left(\frac{b}{m}\right)\dot{x} - \left(\frac{k}{m}\right)x + \left(\frac{1}{m}\right)F \end{bmatrix}, \quad \mathbf{f}(t, \mathbf{y}) = \begin{bmatrix} \mathbf{y}^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ -\left(\frac{b}{m}\right)\mathbf{y}^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \left(\frac{k}{m}\right)\mathbf{y}^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \left(\frac{1}{m}\right)F \end{bmatrix}$$

However, making this distinction is unnecessary in both a mathematical sense and a practical sense; when programming these equations, it is trivial to access an element of a vector – we do *not* need to use dot products to do this.

This brings us to the topic of actually programming these functions. As with many concepts, there can be a substantial disconnect between writing these equations down on paper and actually implementing them in a computer. Consequently, the best way to explain how to do it is to show by example. There are two main approaches we use, which we outline in Convention 2. Both conventions are used in Example 2.1.1.

### Convention 2: Programming a differential equation.

There are **two** main approaches to programming a differential equation:

#### 1. Using a change of variables.

This is manageable for simple systems (this is the approach we took previously in Examples 1.2.1 and 1.3.1). We use this method under “SOLUTION” in the example below.

#### 2. “Unpacking the state vector”, performing calculations, and “assembling the state vector derivative”.

Results in slightly longer code, but *much* easier/more intuitive to read; this is the method we use most often, especially for more complex systems. We use this method under “ALTERNATE SOLUTION” in the example below.

**Example 2.1.1:** Implementing the mass spring damper IVP in MATLAB.

Recall the 2nd-order IVP (Eq. (2.1)) governing the mass-spring-damper system shown in Fig. 2.1.

$$\ddot{x} = -\left(\frac{b}{m}\right)\dot{x} - \left(\frac{k}{m}\right)x + \left(\frac{1}{m}\right)F, \quad x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0 \quad ((2.1))$$

For this example, assume the following values:

$$b = 5 \text{ (N} \cdot \text{s)/m}, \quad k = 1 \text{ N/m}, \quad m = 2 \text{ kg}, \quad x_0 = 1 \text{ m}, \quad \dot{x}_0 = 0 \text{ m/s}$$

Additionally, let  $F(t)$  be given by

$$F(t) = \cos \pi t$$

Implement a vector-valued differential equation  $\mathbf{f}(t, \mathbf{y})$  and initial condition  $\mathbf{y}_0$  (as shown below) in MATLAB that describes the evolution of this system.

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}), \quad \mathbf{y}(t_0) = \mathbf{y}_0$$

**■ SOLUTION**

We can begin by defining known quantities and the forcing function.

```
% parameters
b = 5;           % damping constant [N.s/m]
k = 1;           % spring constant [N/m]
m = 2;           % mass [kg]
x0 = 1;          % initial position [m]
xdot0 = 0;       % initial velocity [m/s]

% forcing function
F = @(t) cos(pi*t);
```

From our work before this example, we already found that

$$\mathbf{y} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \quad \mathbf{f}(t, \mathbf{y}) = \begin{bmatrix} \dot{x} \\ -\left(\frac{b}{m}\right)\dot{x} - \left(\frac{k}{m}\right)x + \left(\frac{1}{m}\right)F \end{bmatrix}, \quad \mathbf{y}_0 = \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} \quad (2.4)$$

Writing the initial condition in MATLAB,

```
% initial condition
y0 = [x0;
      xdot0];
```

Now, we must bridge the gap between the mathematical definition of  $\mathbf{y}$  and how the computer “sees”  $\mathbf{y}$ . In the computer,  $\mathbf{y}$  is just a  $2 \times 1$  array that has the indices 1 and 2. Therefore, what the computer sees is

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad (2.5)$$

If we compare Eqs. (2.4) (which is the mathematical definition of the state vector) and (2.5) (which is the state vector used by the computer), we can see that

$$x = y_1, \quad \dot{x} = y_2$$

This essentially represents a change of variables; we now use  $y_1$  to denote  $x$  and  $y_2$  to denote  $\dot{x}$ . Performing this change of variables in Eq. (2.4),

$$\mathbf{f}(t, \mathbf{y}) = \begin{bmatrix} y_2 \\ -\left(\frac{b}{m}\right)y_2 - \left(\frac{k}{m}\right)y_1 + \left(\frac{1}{m}\right)F \end{bmatrix}$$

In MATLAB, we can write

```
% differential equation
f = @(t,y) [ y(2);
            -(b/m)*y(2)-(k/m)*y(1)+(1/m)*F(t) ];
```

### ■ ALTERNATE SOLUTION

While the solution above is adequate for simple functions, it is more cumbersome to use for complicated functions. Additionally, it makes the code less “readable” – all the state variables are written as elements of the state vector, so it is difficult to see what they physically represent. A better solution, especially when dealing with more complex systems, is to “unpack” the state vector, perform all calculations, and then assemble the state vector derivative. Again, this is best explained by just showing the solution in MATLAB.

There are multiple things to note, however, when defining the differential equation:

1. We first define a local function `f_extra`, which must be located at the very end of the script.
2. To be able to later use `f_extra` in an IVP solver, we assign it the function handle `f`.
3. We named the local function `f_extra` because it has some *extra* input parameters beyond what `f` will have (`f` can only have the input parameters `t` and `y` due to how IVP solvers are defined). See Section 2.3.1 for more information on passing extra parameters.

```
% parameters
b = 5;           % damping constant [N.s/m]
k = 1;           % spring constant [N/m]
m = 2;           % mass [kg]
x0 = 1;          % initial position [m]
xdot0 = 0;       % initial velocity [m/s]

% forcing function
F = @(t) cos(pi*t);

% initial condition
y0 = [x0;
      xdot0];

% assigns function handle to differential equation
f = @(t,y) f_extra(t,y,b,k,m,F);

% defines differential equation
function ydot = f_extra(t,y,b,k,m,F)

    % unpacks state vector
    x = y(1);
    xdot = y(2);

    % preallocates state vector derivative
```

```

ydot = zeros(size(y));

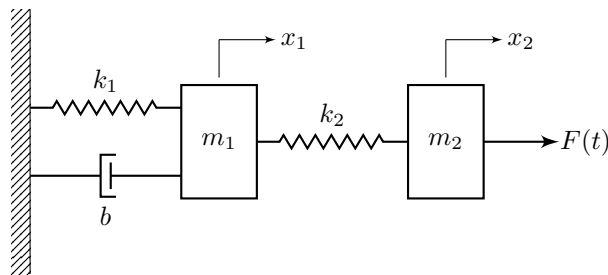
% assembles state vector derivative
ydot(1) = xdot;
ydot(2) = -(b/m)*xdot-(k/m)*x+(1/m)*F(t);

end

```

## 2.2 Coupled ODEs

Consider the coupled mass-spring-damper system shown in Fig. 2.3. Using Newton's second law, one can derive the



**Figure 2.3:** Coupled mass-spring-damper system.

following equations of motion for the two blocks:

$$\begin{aligned}\ddot{x}_1 &= -\left(\frac{k_1 + k_2}{m_1}\right)x_1 + \left(\frac{k_2}{m_1}\right)x_2 - \left(\frac{b}{m_1}\right)\dot{x}_1 \\ \ddot{x}_2 &= \left(\frac{k_2}{m_2}\right)x_1 - \left(\frac{k_2}{m_2}\right)x_2 + F(t)\end{aligned}$$

$\ddot{x}_1$  and  $\ddot{x}_2$  are both 2nd-order ODEs, and they are **coupled** to one another ( $x_1$  and  $x_2$  appear in both equations). Since they are coupled, we cannot solve one without solving the other simultaneously. In the context of IVP solvers, we will still need to represent the system using a single 1st-order vector-valued ODE. To do this, we can simply define the state vector for the overall system as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}$$

Then the time derivative of the state vector is

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_2 \end{bmatrix}$$

Since  $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ , we have

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) = \begin{bmatrix} \dot{x}_1 \\ -\left(\frac{k_1 + k_2}{m_1}\right)x_1 + \left(\frac{k_2}{m_1}\right)x_2 - \left(\frac{b}{m_1}\right)\dot{x}_1 \\ \dot{x}_2 \\ \left(\frac{k_2}{m_2}\right)x_1 - \left(\frac{k_2}{m_2}\right)x_2 + F(t) \end{bmatrix}$$

## 2.3 Extra Parameters

### 2.3.1 Passing Extra Parameters

Consider the function  $f(x, y) = -a(x - b)^2 - c(y - d)^2$  where  $x$  and  $y$  are independent variables, and  $a$ ,  $b$ ,  $c$ , and  $d$  are constants. In MATLAB, we can program this in two ways:

1. as an **anonymous** function
2. as a **MATLAB** function

As an anonymous function, we can program  $f(x, y)$  as

```
f = @(x,y) -a*(x-b)^2-c*(y-d)^2;
```

In the above function definition,  $a$ ,  $b$ ,  $c$ , and  $d$  must already have been initialized. Whenever the function  $\mathbf{f}(\mathbf{x}, \mathbf{y})$  is called, it will then use the values of  $a$ ,  $b$ ,  $c$ , and  $d$  at the time  $\mathbf{f}$  was defined. Consider the snippet of code below:

```
% function definition
a = 5; b = 5; c = 5; d = 5;
f = @(x,y) -a*(x-b)^2-c*(y-d)^2;

% call function, update "a", then call function again
f(2,2)
a = 20;
f(2,2)
```

Both calls of  $\mathbf{f}(2, 2)$  will evaluate to the same result because  $\mathbf{f}(2, 2)$  will still be using  $a = 5$  both times. Now, say that we want to update the values of the constants in  $\mathbf{f}(\mathbf{x}, \mathbf{y})$ . We *could* do the following:

```
% original function definition
a = 5; b = 5; c = 5; d = 5;
f = @(x,y) -a*(x-b)^2-c*(y-d)^2;

% update values of constants
a = 10; b = 10; c = 10; d = 10;
f = @(x,y) -a*(x-b)^2-c*(y-d)^2;
```

An alternate way to do this is

```
% define function where constants can vary as well
f_extra = @(x,y,a,b,c,d) -a*(x-b)^2-c*(y-d)^2;

% define original function by assigning function handle to f_extra
f = @(x,y) f_extra(x,y,5,5,5,5);

% update values of constants
f = @(x,y) f_extra(x,y,10,10,10,10);
```

The code above demonstrates the concept of **passing extra parameters** [8]. Typically, our functions will have constants in them whose values are stored in variables. However, for IVP solvers in particular, we can only pass in

functions of the form  $f(t, y)$ . Therefore, if we want to be able to easily vary the parameter values later, we first define a function `f_extra` (where the “extra” denotes that “extra” parameters are being passed), and then assign it a new **function handle**.

In many cases, we have really complicated functions that could be calculated in steps across 10+ lines of code. In these cases, the functions must be defined as *MATLAB* functions. For the same equation as considered above, we could write

```
function f = f_extra(x,y,a,b,c,d)
    f = -a*(x-b)^2-c*(y-d)^2;
end
```

We can assign function handles to MATLAB functions as well, and essentially treat them like anonymous functions thereafter. Assigning `f_extra` a function handle just like before (and with the same parameter values),

```
a = 10; b = 10; c = 10; d = 10;
f = @(x,y) f_extra(x,y,a,b,c,d);
```

### 2.3.2 Recovering Extra Parameters

From the block diagrams in Section 3.4, we can see that IVP solvers only output (a) the time vector and (b) a matrix storing the solution at each time in the time vector. However, oftentimes, the functions we pass to the IVP solvers will be very complicated functions within which many other quantities are calculated. As an example, for rocket simulations, the function defining the differential equation will be calculating the force due to drag at every time step. The drag force is a function of both the rocket’s position (drag depends on air density, air density decreases with altitude) and velocity. However, there is no way for the IVP solver to directly output the drag force.

The workaround is to define the differential equation so that it returns more than just the state vector derivative. An example of how we typically define functions representing differential equations is

```
f = @(t,y) f_extra(t,y,a);
```

where

```
function f = f_extra(t,y,a);
    x = 2*a-5;
    f = x^2;
end
```

However, consider the case where we want to recover a variable  $x$  that is calculated *inside of* `f_extra`. Then we can instead define `f_extra` as

```
function [f,x] = f_extra(t,y,a);
    x = 2*a-5;
    f = x^2;
end
```

We can still assign a function handle to `f_extra`, just like before. When we do so, only the *first* output parameter (`f`) of `f_extra` will be returned.

At this point, we have all the ingredients we need in order to be able to recover extra parameters. The overall process is best explained using an example – see Example 2.3.1 below.

#### Example 2.3.1: Recovering extra parameters from an IVP solution.

Solve the following IVP on the interval  $[0, 10]$ :

$$\frac{dy}{dt} = xy \quad \text{where} \quad x = 2a - 5t, \quad a = 5, \quad y_0 = (1, 1)^T$$

Then, plot  $x(t)$  over this interval.



## ■ SOLUTION

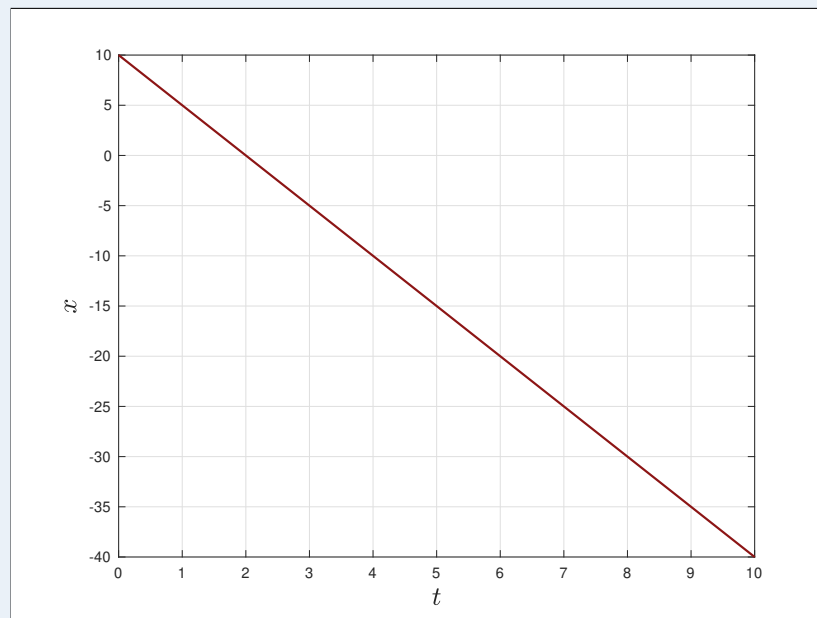
```
% assign function handle and pass extra parameter
f = @(t,y) f_extra(t,y,5);

% solve IVP
y0 = [1;1];
t0 = 0;
tf = 10;
[t,y] = ode45(f,[t0,tf],y0);

% recover "x" at every time step
x = zeros(size(t));
for i = 1:length(t)
    [~,x(i)] = f(t(i),y(i,:));
end

% plot x vs. t
figure;
plot(t,x,'linewidth',1.5,'color',[0.5490,0.0824,0.0824]);
grid on;
xlabel('$t$', 'Interpreter','latex','FontSize',18);
ylabel('$x$', 'Interpreter','latex','FontSize',18);

% MATLAB function must be declared at end
function [f,x] = f_extra(t,y,a)
    x = 2*a-5*t;
    f = x*y;
end
```





# IVP Solvers

## 3.1 The General Solution

Consider the initial value problem

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y}), \quad \mathbf{y}(t_0) = \mathbf{y}_0$$

Solving for  $\mathbf{y}(t)$ ,

$$\mathbf{y}(t) - \underbrace{\mathbf{y}(t_0)}_{\mathbf{y}_0} = \int_{t_0}^t \mathbf{f}(t, \mathbf{y}) dt$$

$$\boxed{\mathbf{y}(t) = \mathbf{y}_0 + \int_{t_0}^t \mathbf{f}(t, \mathbf{y}) dt} \quad (3.1)$$

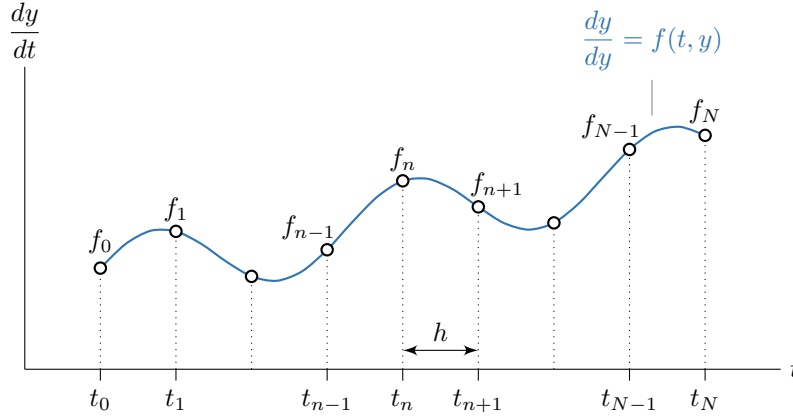
## 3.2 Discretization and Solution Arrays

In many cases, we cannot analytically evaluate the integral in Eq. (3.1) (hence our need for IVP solvers). The solution is to approximate the result by discretizing time, sample  $\mathbf{f}(t, \mathbf{y})$  at those discrete points in time, and then assume that  $\mathbf{f}(t, \mathbf{y})$  behaves a certain way between the two sample times<sup>1</sup>. A visual representation of discretizing a scalar ODE in time is shown in Fig. 3.1.

Using IVP solvers, we find the solution,  $\mathbf{y}(t)$ , at these discrete points in time, which we call **sample times**. The corresponding **time vector** storing the sample times,  $\mathbf{t} \in \mathbb{R}^{N+1}$ , is defined as

$$\boxed{\mathbf{t} = \begin{bmatrix} t_0 \\ t_1 \\ t_2 \\ \vdots \\ t_N \end{bmatrix} = \begin{bmatrix} t_0 \\ t_0 + h \\ t_0 + 2h \\ \vdots \\ t_0 + (N-1)h \\ t_N \end{bmatrix}} \quad (3.2)$$

<sup>1</sup> An example of a behavior we could assume between consecutive sample times is that  $f(t, y) = f(t_n, y_n) \forall t \in [t_n, t_{n+1})$  (i.e. the ODE is constant between the two sample times). This results in the Euler method; see Section 5.2.



**Figure 3.1:** Discretization of an ODE in time.

The **step size** or **time step** between two points in time,  $h$ , is defined as

$$h = t_{n+1} - t_n \quad (3.3)$$

For **fixed-step solvers** (what we cover in this text),  $h$  is constant.

The **solution matrix**,  $[y]$ , stores the solution  $\mathbf{y}(t)$  at the discrete points in time specified by the time vector. There are two different forms the solution matrix can be written in, as shown in Eq. (3.4). These different forms are discussed in more detail in Sections 3.2.1 and 3.2.2.

$$[y] = \underbrace{\begin{bmatrix} \mathbf{y}_0^T \\ \mathbf{y}_1^T \\ \mathbf{y}_2^T \\ \vdots \\ \mathbf{y}_N^T \end{bmatrix}}_{\text{row form}} = \underbrace{[\mathbf{y}_0 \quad \mathbf{y}_1 \quad \mathbf{y}_2 \quad \cdots \quad \mathbf{y}_N]}_{\text{column form}} \quad (3.4)$$

### 3.2.1 Solution Matrix: Row Form

In **row form**, the  $n$ th *row* of  $[y]$  stores  $\mathbf{y}_n^T$ , where  $\mathbf{y}_n = \mathbf{y}(t_n) \in \mathbb{R}^p$  is the solution of the IVP at time  $t_n$ .

$$[y] = \begin{bmatrix} \mathbf{y}_0^T \\ \mathbf{y}_1^T \\ \mathbf{y}_2^T \\ \vdots \\ \mathbf{y}_N^T \end{bmatrix} \in \mathbb{R}^{(N+1) \times p} \quad (3.5)$$

This is the form in which the IVP solver (Algorithm 3) returns the solution to an IVP. This is also the form that MATLAB's ODE suite returns the solution to IVPs.

### 3.2.2 Solution Matrix: Column Form

In **column form**, the  $n$ th *column* of  $[y]$  stores  $\mathbf{y}_n$ , where  $\mathbf{y}_n = \mathbf{y}(t_n) \in \mathbb{R}^p$  is the solution of the IVP at time  $t_n$ .

$$[y] = [\mathbf{y}_0 \quad \mathbf{y}_1 \quad \mathbf{y}_2 \quad \cdots \quad \mathbf{y}_N] \in \mathbb{R}^{p \times (N+1)} \quad (3.6)$$

During the solution process, it is more convenient to work with  $[\mathbf{y}]$  in column form, as shown in Eq. (3.6). During most of Algorithm 3, the solution matrix is in column form. Additionally, Algorithm 4 also assumes that the solution matrix is in this form. However, for the IVP solver to return the solution matrix in the row form defined by Eq. 3.5, we need to remember to take its transpose at the end of the solution process.

$$[\mathbf{y}] \leftarrow [\mathbf{y}]^T$$

### 3.3 A Brute Force Solution Method

One of the primary applications of IVP solvers is trajectory simulation. As a first introduction to the numerical solution to a trajectory, undergraduate courses often start with the fact that (a) velocity,  $\mathbf{v}$ , is the derivative of position,  $\mathbf{r}$ , and (b) acceleration,  $\mathbf{a}$ , is the derivative of velocity.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}, \quad \mathbf{a} = \frac{d\mathbf{v}}{dt}$$

If we know the initial position and velocity ( $\mathbf{r}_0 = \mathbf{r}(t_0)$  and  $\mathbf{v}_0 = \mathbf{v}(t_0)$ , respectively) at the initial time  $t_0$ , and if we know the acceleration as a function of time, position, and acceleration (i.e.  $\mathbf{a} = \mathbf{a}(t, \mathbf{r}, \mathbf{v})$ ), then to find the position and velocity at any time  $t$ , we must evaluate the following expressions:

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{v}(t_0) + \int_{t_0}^t \mathbf{a}(t, \mathbf{r}, \mathbf{v}) dt \\ \mathbf{r}(t) &= \mathbf{r}(t_0) + \int_{t_0}^t \mathbf{v}(t, \mathbf{r}, \mathbf{v}) dt \end{aligned}$$

In simple cases, such as when  $\mathbf{a}$  is a constant or is an integrable function of  $t$ , then we can evaluate these integrals analytically. However, in most cases,  $\mathbf{a}$  is a very complicated function, and it is impossible or very difficult to solve these integrals analytically.

A first approach to solving such a problem numerically is discretizing time into very small increments (i.e.  $t_{n+1} = t_n + \Delta t$ , as illustrated in Figure 3.1), and assuming that the velocity and acceleration remain constant over those small time intervals. Mathematically, if we know the position,  $\mathbf{r}_n = \mathbf{r}(t_n)$ , velocity,  $\mathbf{v}_n = \mathbf{v}(t_n)$ , and acceleration,  $\mathbf{a}_n = \mathbf{a}(t_n)$ , at the  $n$ th sample time,  $t_n$ , then at the next (i.e.  $(n+1)$ th) sample time,  $t_{n+1}$ , we have

$$\begin{aligned} \mathbf{v}_{n+1} &= \mathbf{v}_n + \int_{t_n}^{t_{n+1}} \mathbf{a}_n dt \\ \mathbf{r}_{n+1} &= \mathbf{r}_n + \int_{t_n}^{t_{n+1}} \mathbf{v}_n dt \end{aligned}$$

Since we have assumed that  $\mathbf{a}_n$  and  $\mathbf{v}_n$  are constant over a short interval of time  $\Delta t$ , and since  $\Delta t = t_{n+1} - t_n$ ,

$$\begin{aligned} \mathbf{v}_{n+1} &= \mathbf{v}_n + \mathbf{a}_n \Delta t \\ \mathbf{r}_{n+1} &= \mathbf{r}_n + \mathbf{v}_n \Delta t \end{aligned}$$

For physical systems, we typically have an expression of the acceleration as  $\mathbf{a} = \mathbf{a}(t, \mathbf{r}, \mathbf{v})$  from Newton's second law (i.e. it is our equation of motion/dynamics model). Thus, we calculate  $\mathbf{a}_n$  using

$$\mathbf{a}_n = \mathbf{a}(t_n, \mathbf{r}_n, \mathbf{v}_n)$$

See Example 3.3.1 below for an example of this approach to solving a simple problem involving Newton's laws. We refer to this approach as a **brute force** approach for two main reasons:

1. It uses the simplest and least accurate approximation available (this is actually the Euler method; see Section 5.2).
2. It is problem-specific; that is, as you will see in the code in the example, for every single problem, you'd have to manually edit some aspects of the "solver".

**Example 3.3.1:** Brute force solution to the motion of a car.

Consider a car of mass  $m = 1500$  kg, cross-sectional area  $A = 2.5$  m<sup>2</sup>, and drag coefficient  $C_D = 0.25$  that starts from rest at  $x = 0$  m. A constant force of  $F_{\text{eng}} = 4500$  N is supplied by the engine to move the car. Find the velocity and acceleration of the car, as well as the drag force acting on the car, when the car is at a position of  $x = 300$  m. Assume a constant air density of  $\rho = 1.2$  kg/m<sup>3</sup>.

**■ SOLUTION**

Applying Newton's Second Law to find the acceleration yields

$$a = \frac{F_{\text{eng}} - F_D}{m}$$

where the drag force,  $F_D$ , is given by

$$F_D = \frac{1}{2} C_D A \rho v^2$$

Implementing a “brute force” approach in MATLAB to numerically solve for the resulting motion,

```
% car parameters
m = 1500;           % mass [kg]
A = 2.5;            % cross-sectional area [m^2]
CD = 0.25;          % drag coefficient [-]
F_eng = 4500;        % applied force [N]

% IVP parameters
x0 = 0;             % initial position [m]
v0 = 0;             % initial velocity [m/s]
xf = 300;           % final position [m]

% physical parameters
rho = 1.2;          % air density [kg/m^3]

% computational parameters
dt = 0.1;           % time step [s]

% preallocate arrays
t = zeros(10000,1);
x = zeros(size(t));
v = zeros(size(t));
a = zeros(size(t));

% solves initial value problem
n = 1;
while x(n) < xf

    % calculates drag force [N]
    FD = 0.5*CD*A*rho*v(n)^2;

    % calculates acceleration [m/s^2]
    a(n) = (F_eng-FD)/m;

    % integrates acceleration to find velocity [m/s]
    v(n+1) = v(n)+a(n)*dt;

    % integrates velocity to find position [m]
    x(n+1) = x(n)+v(n)*dt;
```

```

        % increments loop index
        n = n+1;

    end

    % trims arrays
    t = t(1:(n-1));
    x = x(1:(n-1));
    v = v(1:(n-1));
    a = a(1:(n-1));

    % velocity and acceleration of car when position is x = 300 m
    v300 = v(end);
    a300 = a(end);

    % prints results
    fprintf("    velocity = %.3f m/s\n",v300)
    fprintf("acceleration = %.3f m/s^2\n",a300)
    fprintf("  drag force = %.3f N\n\n",FD)

```

The code above produces the following results:

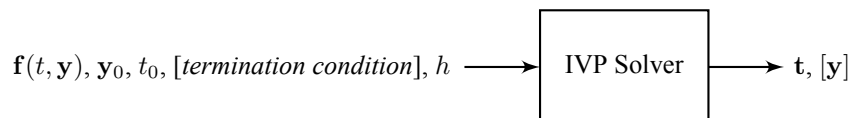
```

    velocity = 40.853 m/s
acceleration = 2.583 m/s^2
  drag force = 625.858 N

```

## 3.4 General Form of an IVP solver

An **IVP solver** (where “IVP” stands for “initial value problem”) is a computational function that solves *initial value problems* defined using ordinary differential equations. IVP solvers act as a black box; we pass them a very complicated, vector-valued differential equation, an initial condition, an initial time, some termination condition (more on this later), and a step size, and it magically returns the solution over some interval of time. This basic functionality is depicted as a block diagram in Fig. 3.2. Figure 3.2 depicts the block diagram of a generic IVP solver. The input and



**Figure 3.2:** IVP solver block diagram.

output parameters of this generic IVP solver are defined in more detail in Tables 3.1 and 3.2. Note that the termination condition is somewhat ambiguous here, as it depends on what “mode” the IVP solver is operating in. Specifically, there are two basic “modes” of IVP solvers, discussed in Sections 3.6 and 3.7, that differ in how the solver is terminated:

1. time detection mode – the IVP is solved until some specified final time (see Section 3.6)
2. event detection mode – the IVP is solved until some event occurs (see Section 3.7)

**Table 3.1:** Inputs to an IVP solver.

Variable	Name	Description
$\mathbf{f}(t, \mathbf{y}) \in \mathbb{R}^{N+1}$	ODE	multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ordinary differential equation $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
$\mathbf{y} \in \mathbb{R}^p$	initial condition	initial condition $\mathbf{y}_0 = \mathbf{y}(t_0)$
<i>[termination condition]</i>	termination condition	condition to terminate the IVP solver; see Sections 3.6 and 3.7
$t_0 \in \mathbb{R}$	initial time	initial time
$h \in \mathbb{R}$	step size	step size between sample times ( $t_{n+1} = t_n + h$ )

**Table 3.2:** Outputs of an IVP solver.

Variable	Name	Description
$\mathbf{t} \in \mathbb{R}^{N+1}$	time vector	stores sequence of sample times (see Section 3.2)
$[\mathbf{y}] \in \mathbb{R}^{(N+1) \times p}$	solution matrix	Stores the solution $\mathbf{y}(t)$ at discrete points in time. Specifically, the $n$ th row of $[\mathbf{y}]$ stores $\mathbf{y}_n^T$ , where $\mathbf{y}_n = \mathbf{y}(t_n) \in \mathbb{R}^p$ is the solution of the IVP at time $t_n$ (where time $t_n$ is stored in the $n$ th element of $\mathbf{t}$ ).

## 3.5 Propagation Functions

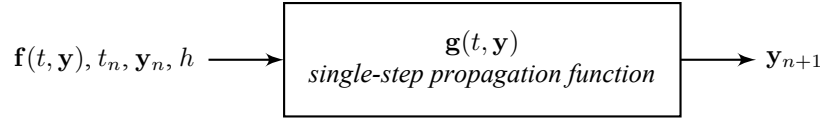
In Section 3.4, we discussed solving IVPs over some time span  $[t_0, t_f]$ , obtaining the results at intermediate sample times. To obtain the solutions at these intermediate sample times, we use **propagation functions**. Specifically, we use a propagation function to find the solution at the next sample time, given the solution at the current sample time, and maybe at previous sample times as well. There are two general types of propagation functions; **single-step propagation functions** (for single-step methods; see Section 3.5.1), and **multistep propagation functions** (for multistep methods; see Section 3.5.2).

### 3.5.1 Single-Step Methods and One-Step Propagation

Single-step methods (see Chapter 5) require only the solution at the current sample time to find the solution at the next sample time. The general operation of a single-step propagation function is defined by Eq. (3.7) and illustrated in Fig. 3.3.

$$\boxed{\mathbf{y}_{n+1} = \mathbf{g}(t_n, \mathbf{y}_n)} \quad (3.7)$$

The *IVP Solver Toolbox* uses propagation functions to build up the solution to an IVP over a time span. However, in some cases (such as discretizing a function), it is useful to just return the solution,  $\mathbf{y}$ , at the very next sample time (i.e.  $\mathbf{y}_{n+1}$  at time  $t_{n+1}$ ), given the solution at the current sample time (i.e.  $\mathbf{y}_n$  at time  $t_n$ ). We refer to this as **one-step propagation**, since the state vector (i.e. the solution) is being propagated forward “one step” in time. Single-step



**Figure 3.3:** Single-step propagation function.

solvers are ideally suited to serve as “one-step propagators”, since the solution at the subsequent sample time only depends on the solution at the current sample time.

### 3.5.2 Multistep Methods

In contrast to single-step methods (which only require the solution at the current sample time to find the solution at the next sample time), **multistep methods** (see Chapters 6 and 7) utilize the solutions at previous sample times to approximate the solution at the next sample time. Specifically, a multistep method of order  $m$  will evaluate  $\mathbf{f}(t, \mathbf{y})$  at  $m$  sample times:

1.  $\mathbf{f}(t_n, \mathbf{y}_n)$
2.  $\mathbf{f}(t_{n-1}, \mathbf{y}_{n-1})$
3.  $\mathbf{f}(t_{n-2}, \mathbf{y}_{n-2})$
- $\vdots$
- $m$ .  $\mathbf{f}(t_{n-m+1}, \mathbf{y}_{n-m+1})$

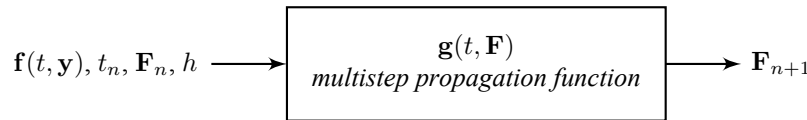
Typically, function evaluations are the most expensive computation in an IVP solver. To avoid repeating function evaluations unnecessarily, we define the matrix  $\mathbf{F}_n \in \mathbb{R}^{p \times (m+1)}$  at the  $n$ th sample time as

$$\mathbf{F}_n = \left[ \begin{array}{c|c|c|c|c|c} \mathbf{f}(t_n, \mathbf{y}_n) & \mathbf{f}(t_{n-1}, \mathbf{y}_{n-1}) & \mathbf{f}(t_{n-2}, \mathbf{y}_{n-2}) & \cdots & \mathbf{f}(t_{n-m+1}, \mathbf{y}_{n-m+1}) & \mathbf{y}_n \end{array} \right] \quad (3.8)$$

The first column of  $\mathbf{F}$  stores  $\mathbf{f}$  evaluated the current sample time, the next  $m - 1$  columns store the evaluations of  $\mathbf{f}$  for the previous  $m - 1$  sample times, and the  $(m + 1)$ th column of  $\mathbf{F}$  stores the solution at the current sample time,  $\mathbf{y}_n$ . Storing  $\mathbf{y}_n$  as part of  $\mathbf{F}_n$  is essential for defining propagation functions for multistep methods.

Instead of propagating  $\mathbf{y}_n$  from one sample time to the next, a **multistep propagation function** propagates  $\mathbf{F}_n$  from one sample time to the next. This behavior is defined by Eq. (3.9) and illustrated in Fig. 3.4.

$$\mathbf{F}_{n+1} = \mathbf{g}(t_n, \mathbf{F}_n) \quad (3.9)$$



**Figure 3.4:** Multistep propagation function.

When defining the multistep propagation functions in Sections 6.3 and 7.4, it will be necessary to refer to specific columns of  $\mathbf{F}$ . Since we are using a subscript to denote the sample number, we will use a superscript in parentheses to denote a specific column.



**Convention 3:** Columns of the  $\mathbf{F}$  matrix.

The  $k$ th column of the  $\mathbf{F}$  matrix at sample time  $n$  is denoted as

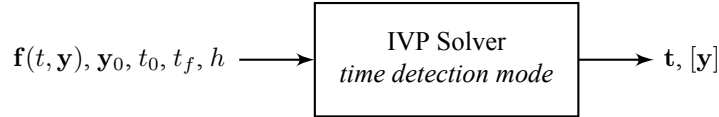
$$\mathbf{F}_n^{(k)}$$

The submatrix consisting of columns  $k$  through  $l$  of the  $\mathbf{F}$  matrix at sample time  $n$  is denoted as

$$\mathbf{F}_n^{(k:\ell)}$$

## 3.6 Time Detection Mode

In general, we use IVP solvers in what we refer to as the **time detection mode**. In these cases, we are given an initial time,  $t_0$ , and wish to solve the IVP until some final time,  $t_f$ . The reason we refer to this as the time detection mode is that the algorithm will continue solving the IVP until it *detects* that it has reached the final *time*,  $t_f$ . A block diagram of an IVP solver in the time detection mode is shown in Fig. 3.5.



**Figure 3.5:** IVP solver block diagram (time detection mode).

Recall from Eq. (3.4) that the IVP solver should produce a time vector  $\mathbf{t} \in \mathbb{R}^{N+1}$ , where the first element of  $\mathbf{t}$  is equal to  $t_0$ . In this case, for the time detection mode, we also want the last element of  $\mathbf{t}$  to be equal to the specified final time,  $t_f$  (i.e.  $t_N = t_f$ ). However, we also specify the step size,  $h$  (see Section 3.2). This presents an issue because  $t_f - t_0$  may not be an exact multiple of  $h$ . If we want to solve the IVP until  $t_f$ , we need to choose the number of sample times  $N + 1$  such that the time at the last sample time is *greater than or equal to*  $t_f$ . To do this, we simply use the ceiling function (i.e. rounding up).

$$N = \left\lceil \frac{t_f - t_0}{h} \right\rceil \quad (3.10)$$

Thus, the last element of  $\mathbf{t}$  is

$$t_N = t_0 + Nh \quad (3.11)$$

The issue now is that  $t_N \neq t_f$  (and consequently  $\mathbf{y}_N \neq \mathbf{y}(t_f)$ ). To solve this issue, in the last step of an IVP solver algorithm, we can use linear interpolation to estimate  $\mathbf{y}_f = \mathbf{y}(t_f)$ .

$$\mathbf{y}_f = \mathbf{y}_{N-1} + \left[ \frac{\mathbf{y}_N - \mathbf{y}_{N-1}}{t_N - t_{N-1}} \right] (t_f - t_{N-1}) \quad (3.12)$$

Finally, we replace the last row of  $[\mathbf{y}]$  with  $\mathbf{y}_f^T$ .

$$[\mathbf{y}] = \begin{bmatrix} \mathbf{y}_0^T \\ \mathbf{y}_1^T \\ \mathbf{y}_2^T \\ \vdots \\ \mathbf{y}_N^T \end{bmatrix} \xrightarrow[\text{obtained via linear interpolation}]{\text{replacing last row with the } \mathbf{y}_f^T} [\mathbf{y}] = \begin{bmatrix} \mathbf{y}_0^T \\ \mathbf{y}_1^T \\ \mathbf{y}_2^T \\ \vdots \\ \mathbf{y}_f^T \end{bmatrix}$$

For the overall IVP solver algorithm (Algorithm 3 in Section 3.9), we do *not* directly implement a time detection mode. Rather, we formulate a condition function to terminate integration once the final time is reached, in effect casting it into event detection mode (see Section 3.7). This “conversion” is detailed in Section 3.8.

### 3.6.1 Integrating Backwards in Time

In some cases, the final time,  $t_f$ , may be less than the initial time,  $t_0$ . For example, this is the case when solving the continuous-time Riccati differential equation. In these cases, we need to be given the **final condition**  $\mathbf{y}_f = \mathbf{y}(t_f)$ . However, this is still the same problem as solving an IVP; we are essentially solving the IVP *backward in time*, and in our IVP solvers, the final condition will be substituted for the initial condition. In practice, our IVP solver (Algorithm 3) checks to see if  $t_f$  is less than  $t_0$ , and if so, it switches the sign on the step size,  $h$  (i.e. it makes it negative).

$$h \leftarrow -h$$

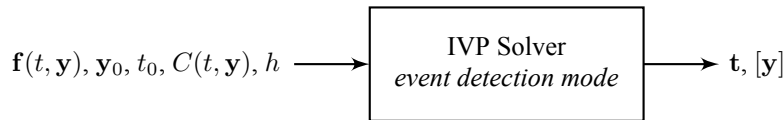
We can then solve an IVP using Algorithm 3 as

$$[t, [y]] = \text{solve\_ivp}(f, t_0, t_f, \mathbf{y}_f, h)$$

## 3.7 Event Detection Mode

To motivate the need for an event detection mode, consider a satellite slowly deorbiting due to atmospheric drag. We want to keep solving for its motion until it impacts the Earth. In the time detection mode, we specified a final time until which we wished to keep solving the IVP. However, in this case, we don't know ahead of time when the satellite will impact the Earth. We could make some guesses for  $t_f$ , but either we underestimate it and the satellite never impacts the Earth before the solver terminates, or we overestimate it and waste time and computational effort calculating motion that (a) we don't need to and (b) doesn't make sense (we would essentially be simulating the trajectory of the satellite as if the Earth isn't there).

In **event detection mode**, an IVP solver terminates the solver at the occurrence of an *event* rather than at a certain time. Instead of passing a final time  $t_f$  to the IVP solver, we pass it a **condition function**  $C(t, \mathbf{y})$ .



**Figure 3.6:** IVP solver block diagram (event detection mode).

Based on the time,  $t$ , and/or the current value of the state vector,  $\mathbf{y}$ , the condition function  $C(t, \mathbf{y})$  determines whether the solver should continue its solution process or if the solver should be terminated. Therefore, the condition function is evaluated at each and every iteration of the IVP solver. Below, we first introduce a more mathematical definition of a condition function, to keep in line with our practice thus far of introducing everything from a mathematical standpoint.

The condition function is defined in a piecewise manner, where it will evaluate to some value while the IVP solver should continue integrating. However, at some point, the condition function will switch to a different value, signaling to the IVP solver that it should be terminated. Specifically, the condition function returns 1 while the condition is satisfied, and 0 after it is no longer satisfied.

#### Convention 4: Mathematical definition of the condition function.

A condition function  $C(t, \mathbf{y})$  (in a mathematical setting) is defined such that it returns 1 *until* the event (which defines the point at which the solver should be terminated) occurs, after which it will return 0.

$$C(t, \mathbf{y}) = \begin{cases} 1, & \text{event has not occurred; continue integration} \\ 0, & \text{event has occurred; terminate integration} \end{cases}$$

Thus, while  $C(t, \mathbf{y}) = 1$ , the solver should continue integrating to find the solution.

In a computational setting, we have the condition function  $C(t, y)$ , which returns `true` if integration should continue and `false` if it should be terminated.

**Convention 5:** Computational definition of the condition function.

A condition function  $C(t, y)$  (in a computational setting) is defined such that it returns `true` *until* the event (which defines the point at which the solver should be terminated) occurs, after which it will return `false`.

Thus, while  $C(t, y) = \text{true}$ , the solver should continue integrating to find the solution.

In MATLAB, the integer 1 can be used interchangeably with the logical `true`. Similarly, the integer 0 can be used interchangeably with the logical `false`.

**Example 3.7.1:** Defining a condition function.

Implement a condition function  $C(t, y)$  in MATLAB that can be used to terminate an IVP solver when  $y_2$  is greater than 300 and  $y_5$  is negative (these two conditions should be satisfied simultaneously for the solver to terminate).

■ **SOLUTION**

We can write the condition function in a single line as an anonymous function.

```
C = @(t,y) ~((y(2) > 300) && (y(5) < 0));
```

■ **ALTERNATE SOLUTION**

Alternatively, we can use a MATLAB function with an if/else structure, and then assign a function handle to the MATLAB function. While in this case it is (a) not as efficient and (b) unnecessary to do this, for more complicated condition functions this may be necessary.

```
% assign function handle
C = @(t,y) condition(t,y);

% MATLAB functions must be at end of script
function C = condition(t,y)
    if (y(2) > 300) && (y(5) < 0)
        C = false;
    else
        C = true;
    end
end
```

■ **ALTERNATE SOLUTION**

There exists yet another alternate solution, which is identical to the previous solution, but uses 1 instead of `true` and 0 instead of `false` (since these values are interchangeable in MATLAB).

```

% assign function handle
C = @(t,y) condition(t,y);

% MATLAB functions must be at end of script
function C = condition(t,y)
    if (y(2) > 300) && (y(5) < 0)
        C = 0;
    else
        C = 1;
    end
end
end

```

Note that in this example, we never used the time variable. However, in some cases we will need to, and by convention, we must still include it in our function definition.

### 3.8 Converting Time Detection Mode to Event Detection Mode

In the *IVP Solver Toolbox*, we use a single algorithm (Algorithm 3) to solve IVPs. While we want it to be able to operate in both time detection and event detection modes, to simplify the code, it is useful to “convert” a time detection problem into an event detection problem. Recall that a condition function,  $C(t, \mathbf{y})$ , is a function of time; thus, we can use the condition function to check whether the time has reached a certain time. Therefore, we can use a condition function to terminate the solution routine once the final time has been reached. This particular condition function can be written as

$$C(t, \mathbf{y}) = \begin{cases} 1, & t \leq t_f \\ 0, & t > t_f \end{cases}$$

In a computational setting, this can be written in a single line as

$$C(t, \mathbf{y}) = (t \leq t_f)$$

However, we must also account for the case where we are integrating backwards in time. In that case, we have that  $t_f < t_0$ . The condition function then becomes

$$C(t, \mathbf{y}) = \begin{cases} 1, & t \leq t_f \text{ and } t_0 < t_f \\ 0, & t > t_f \text{ and } t_0 < t_f \\ 1, & t \geq t_f \text{ and } t_0 > t_f \\ 0, & t < t_f \text{ and } t_0 > t_f \end{cases} \quad (3.13)$$

In a computational setting, this can be written as

$$\begin{array}{l} \text{if } t_f > t_0 \\ \quad C(t, \mathbf{y}) = (t \leq t_f) \\ \text{else} \\ \quad C(t, \mathbf{y}) = (t \geq t_f) \\ \text{end} \end{array} \quad (3.14)$$

### 3.8.1 Premature Termination Due to Numerical Issues

Due to numerical issues, in some cases this method will only solve for the solution up until  $t_f - h$ . For example, let's consider the case where our step size is  $h = 0.001$  and the final time is  $t_f = 10$ . The time vector that we want our solution to have is

$$\mathbf{t} = \begin{bmatrix} \vdots \\ 9.998 \\ 9.999 \\ 10.000 \end{bmatrix}$$

However, due to numerical errors, the time vector we get might be

$$\mathbf{t} = \begin{bmatrix} \vdots \\ 9.998000000000000000000001 \\ 9.999000000000000000000001 \end{bmatrix}$$

Essentially, we never reach the final time; we refer to this as **premature termination**. This occurs because when the iteration with  $t = 9.999000000000000000000001$  has finished, the solver will attempt an iteration at  $t + h$ , which in this case is  $t = 10.000000000000000000000001$ . However, for  $t = 10.000000000000000000000001$ , the condition function will return `false`, and the *desired* final iteration will never be executed.

To fix this issue, we can subtract  $h$  from the left hand side of the inequalities defining the condition function. The side effect of this is in some cases, we will now have two solutions corresponding to essentially the same time. Numerically, they will be different, but ever so slightly (on the order of  $10^{-15}$ - $10^{-16}$ ). In practice, we can set a threshold, and if the difference between the final two times is within that threshold, we know one of the solutions is redundant. We *could* delete either of these redundant solutions. However, when a final time is specified, we will be interpolating/extrapolating for the solution at that final time, in a manner identical to that discussed in Section 3.6. Therefore, since the last solution in that case will be at the *exact* final time, we delete the penultimate solution.

## Summary

In summary, we update the condition function to

```

if tf-h > t0
    C(t,y) = (t-h <= tf)
else
    C(t,y) = (t-h >= tf)
end

```

If the times at the final two iterations are within  $10^{-10}$  of each other, i.e. if

$$|t_N - t_{N-1}| < 10^{-10}$$

then delete the penultimate element of  $\mathbf{t}$  and the penultimate column of  $[\mathbf{y}]$  (while  $[\mathbf{y}]$  is still in the column form presented in Section 3.2.2).

### 3.8.2 Integrating Backwards in Time

In event detection mode, we do not specify a final time, so the IVP solver cannot determine on its own if it needs to solve the IVP backwards in time. Nonetheless, we can still solve IVPs backwards in time using Algorithm 3 by manually specifying a negative step size.

$$[\mathbf{t}, [\mathbf{y}]] = \text{solve\_ivp}(\mathbf{f}, t_0, C, \mathbf{y}_f, -h)$$

## 3.9 IVP Solver Algorithm

**Convention 6:** Indexing the time vector and solution matrix.

The  $n$ th element of the time vector is denoted

$$\mathbf{t}_n$$

The subvector corresponding to elements  $k$  through  $\ell$  of the time vector is denoted

$$\mathbf{t}_{k:\ell}$$

The  $n$ th column of the solution matrix is denoted

$$[\mathbf{y}]_{:,n}$$

The submatrix corresponding to columns  $k$  through  $\ell$  of the solution matrix is denoted

$$[\mathbf{y}]_{:,k:\ell}$$

**Algorithm 3:** `solve_ivp`

Solve initial value problems using fixed-step IVP solvers.

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ )  
defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t_0 \in \mathbb{R}$  - initial time
- $t_f \in \mathbb{R}$  or  $C(t, \mathbf{y})$  - final time or condition function ( $C : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{B}$ )
- $\mathbf{y}_0 \in \mathbb{R}^p$  - initial condition
- $h \in \mathbb{R}$  - step size
- `method` - (*OPTIONAL*) integration method (one of the methods from Chapters 5–7)

**Procedure:**

1. Determine if inputs correspond to time detection mode or event detection mode.  
If the problem is specified as a time detection problem, convert it to an event detection problem.

if  $t_f$  is input (time detection mode)

- (a) Make the step size negative if  $t_0 > t_f$  (integrating backward in time).

```

if  $t_0 > t_f$ 
  |    $h = -h$ 
end

```

- (b) Define the condition function to convert time detection problem to event detection problem.

```

if  $t_0 > t_f$ 
  |    $C(t, \mathbf{y}) = \begin{cases} 1, & t - h \leq t_f \\ 0, & t - h > t_f \end{cases}$ 
else
  |    $C(t, \mathbf{y}) = \begin{cases} 1, & t - h \geq t_f \\ 0, & t - h < t_f \end{cases}$ 
end

```

- (c) Indicate that the final time is known.

```

  final_time_known = true

```

```

else

```

Indicate that the final time is unknown (the problem has been specified in event detection mode).

```

  final_time_known = false

```

```

end

```

2. Default the integration method to RK4 if it is not input.

```

  method = 'RK4'

```

3. If `method` specifies a single-step method, define the propagation function as

```

   $g(t, \mathbf{y}) = \text{RKX}(\mathbf{f}(t, \mathbf{y}), t, \mathbf{y}, h)$ 

```

Note that `RKX` refers to any of the algorithms presented in Section 5.7.

4. If `method` specifies a multistep predictor method, define the propagation function as

```

   $g(t, \mathbf{F}) = \text{ABX}(\mathbf{f}(t, \mathbf{F}), t, \mathbf{y}, h)$ 

```

Note that `ABX` refers to any of the algorithms presented in Section 6.3.

5. If `method` specifies a multistep predictor-corrector method, define the propagation function as

```

   $g(t, \mathbf{F}) = \text{ABMX}(\mathbf{f}(t, \mathbf{F}), t, \mathbf{y}, h)$ 

```

Note that `ABMX` refers to any of the algorithms presented in Section 7.4.

6. Keep track of whether we are using a single-step or multistep method.

```

  if method specifies a single-step method

```

```

|   single_step = true
else
|   single_step = false
end

```

7. If using a multistep method, determine its order,  $m$ .

```

if !(single_step)
|   Store the order of the multistep method specified by method as  $m$ .
end

```

8. Determine the state dimension ( $p$ ) given that  $\mathbf{y}_0 \in \mathbb{R}^p$ .

9. Preallocate the time vector,  $\mathbf{t} \in \mathbb{R}^{10000}$ , and the solution matrix,  $\mathbf{Y} \in \mathbb{R}^{10000 \times p}$ .

```

t = 0_{10000 \times 1}
[y] = 0_{p \times 10000}

```

10. Store the initial conditions in  $\mathbf{t}$  and  $[\mathbf{y}]$ .

```

t_1 = t_0
[y]_{1:p,1} = y_0

```

11. Solve the IVP.

```

if single_step
|
|   n = 1
|   while C(t_n, [y]_{:,n})
|       (a) Expand t and [y] if needed (Algorithm 4).
|
|       if (n + 1) > length(t)
|           [t, [y]] =
|               expand_ivp_arrays(t, [y])
|       end
|
|       (b) Propagate state vector to next sample time.
|
|       [y]_{:,n+1} = g(t_n, [y]_{:,n})
|
|       (c) Increment time and loop index.
|
|       t_{n+1} = t_n + h
|       n = n + 1
|   end
else

```



(a) Populate the first  $m + 1$  elements of the time vector.

$$\mathbf{t}_{1:(m+1)} = \begin{bmatrix} t_0 \\ t_0 + h \\ t_0 + 2h \\ \vdots \\ t_0 + (m-1)h \\ t_0 + mh \end{bmatrix}$$

(b) Propagate state vector using RK4 (Algorithm 20) for first  $m$  sample times.

```

for  $n = 1$  to  $m$ 
|    $[\mathbf{y}]_{:,n+1} = \text{RK4}(\mathbf{f}, \mathbf{t}_n, [\mathbf{y}]_{:,n}, h)$ 
end

```

(c) Initialize the  $\mathbf{F}$  matrix (stores function evaluations for first  $m$  sample times in first  $m$  columns, state vector at  $(m + 1)$ th sample time in  $(m + 1)$ th column).

$$\mathbf{F}_n = \left[ \begin{array}{c|c|c} \mathbf{f}(t_1, [\mathbf{y}]_{:,1}) & \cdots & \mathbf{f}(t_m, [\mathbf{y}]_{:,m}) \end{array} \right] \left| \begin{array}{c} [\mathbf{y}]_{:,m+1} \end{array} \right]$$

(d) State vector propagation while condition is satisfied.

```

 $n = m + 1$ 
while  $C(\mathbf{t}_n, [\mathbf{y}]_{:,n})$ 
|   i. Expand  $\mathbf{t}$  and  $[\mathbf{y}]$  if needed (Algorithm 4).
|
|       if  $(n + 1) > \text{length}(\mathbf{t})$ 
|       |    $[\mathbf{t}, [\mathbf{y}]] =$ 
|       |    $\text{expand\_ivp\_arrays}(\mathbf{t}, [\mathbf{y}])$ 
|       |   end
|
|   ii. Updates  $\mathbf{F}$  matrix (propagates to next sample
|       time).
|
|        $\mathbf{F} = \mathbf{g}(\mathbf{t}_n, \mathbf{F})$ 
|
|   iii. Extract the state vector at the next sample
|        time and store it in the solution matrix.
|
|        $[\mathbf{y}]_{:,n+1} = \mathbf{F}^{(m+1)}$ 
|
|   iv. Increment time and loop index.
|
|        $\mathbf{t}_{n+1} = \mathbf{t}_n + h$ 
|        $n = n + 1$ 
|
end

```

**end**

12. Trim arrays.

$$\begin{aligned} [\mathbf{y}] &= [\mathbf{y}]_{:,1:n} \\ \mathbf{t} &= \mathbf{t}_{1:n} \end{aligned}$$

13. Determine the number of subintervals ( $N$ ) given that  $\mathbf{t} \in \mathbb{R}^{N+1}$ .

14. Linearly interpolate to find solution at desired final time (if known).

**if** final\_time\_known

(a) Linearly interpolate to find the solution at  $t_f$ .

$$[\mathbf{y}]_{:,N+1} = [\mathbf{y}]_N + \left[ \frac{[\mathbf{y}]_{N+1} - [\mathbf{y}]_N}{\mathbf{t}_{N+1} - \mathbf{t}_N} \right] (t_f - \mathbf{t}_N)$$

(b) Replace last element of  $\mathbf{t}$  with  $t_f$ .

$$\mathbf{t}_{N+1} = t_f$$

**end**

15. Delete the penultimate solution if it corresponds to (almost exactly) the same time as the last solution.

**if**  $|\mathbf{t}_{N+1} - \mathbf{t}_N| < 10^{-10}$

Delete the penultimate element of  $\mathbf{t}$  and the penultimate column of  $[\mathbf{y}]$ .

**end**

16. Transpose solution matrix so it is returned in “standard form”.

$$[\mathbf{y}] = [\mathbf{y}]^T$$

**Return:**

- $\mathbf{t} \in \mathbb{R}^{N+1}$  - time vector
- $[\mathbf{y}] \in \mathbb{R}^{(N+1) \times p}$  - solution matrix

**Note:**

- The  $n$ th row of  $[\mathbf{y}]$  stores the *transpose* of the state vector (i.e. the solution) corresponding to the  $n$ th time in  $\mathbf{t}$ . This convention is chosen to match the convention used by MATLAB’s ODE suite.

### 3.9.1 Starting Multistep Solvers

Recall the definition of the  $\mathbf{F}$  matrix (repeated below) from Eq. (3.8) in Section 3.5.2.

$$\mathbf{F}_n = \left[ \begin{array}{c|c|c|c|c} \mathbf{f}(t_n, \mathbf{y}_n) & \mathbf{f}(t_{n-1}, \mathbf{y}_{n-1}) & \mathbf{f}(t_{n-2}, \mathbf{y}_{n-2}) & \cdots & \mathbf{f}(t_{n-m+1}, \mathbf{y}_{n-m+1}) \\ \hline & & & & \mathbf{y}_n \end{array} \right]$$

From the definition of  $\mathbf{F}$ , we can immediately notice that we need the solutions at  $m$  sample times to populate  $\mathbf{F}$ . Therefore, we need to use a single-step solver for the first  $m$  iterations of a multistep solver (i.e. multistep methods

are *not* self-starting) [6, pp. 133-134]. In Algorithm 3 (which is implemented in the *IVP Solver Toolbox*), the RK4 integrator (see Algorithm 20) is used for the first  $m$  iterations.

### 3.9.2 Expanding Solution Arrays

The general IVP solver outlined in Algorithm 3 operates in event detection mode, which means it does not know ahead of time how big the time vector and solution matrix need to be. By default, the arrays to store these quantities are initialized so that they can store results for 10,000 sample times. However, sometimes we will need to increase the size of these arrays. This is done by initializing new arrays that have twice the size of the previous arrays, and copying the results from the old arrays into the new arrays. This procedure is formalized in Algorithm 4.

#### Algorithm 4: `expand_ivp_arrays`

Expands the arrays storing the IVP solution.

##### Given:

- $\mathbf{t} \in \mathbb{R}^{N+1}$  - time vector
- $[\mathbf{y}] \in \mathbb{R}^{p \times (N+1)}$  - solution matrix

##### Procedure:

1. Determine the number of subintervals, given that  $\mathbf{t} \in \mathbb{R}^{N+1}$ .
2. Determine the state dimension, given that  $[\mathbf{y}] \in \mathbb{R}^{p \times (N+1)}$ .
3. Expand the time vector.

$$\mathbf{t}_{\text{new}} = \begin{bmatrix} \mathbf{t} \\ \mathbf{0}_{N+1} \end{bmatrix}$$

4. Expand the solution matrix.

$$[\mathbf{y}]_{\text{new}} = \begin{bmatrix} [\mathbf{y}] & \mathbf{0}_{p \times (N+1)} \end{bmatrix}$$

##### Return:

- $\mathbf{t}_{\text{new}} \in \mathbb{R}^{2(N+1)}$  - expanded time vector
- $[\mathbf{y}]_{\text{new}} \in \mathbb{R}^{p \times 2(N+1)}$  - expanded solution matrix

### 3.9.3 0- vs. 1-Based Indexing

The time vector,  $\mathbf{t}$ , is defined using 0-based indexing (i.e. its first element has index 0 and stores  $t_0$ ). Similarly, the solution matrix,  $[\mathbf{y}]$ , is also defined using 0-based indexing (i.e. its first row has index 0 and stores  $\mathbf{y}_0^T$ ).

However, MATLAB uses 1-based indexing. Thus, Algorithm 3 is written using 1-based indexing, which can get a little confusing; for example, element  $\mathbf{t}_5$  of the time vector,  $\mathbf{t}$ , stores time  $t_4$ .

# 4

## Matrix-Valued Differential Equations

### 4.1 Definition

Until now, we have considered vector-valued ODEs of the form

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y})$$

where  $\mathbf{y} \in \mathbb{R}^p$  is a *vector* and where  $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ . However, differential equations of the form

$$\frac{d\mathbf{M}}{dt} = \mathbf{F}(t, \mathbf{M})$$

are also common, where  $\mathbf{M} \in \mathbb{R}^{p \times r}$  is a matrix and where  $\mathbf{F} : \mathbb{R} \times \mathbb{R}^{p \times r} \rightarrow \mathbb{R}^{p \times r}$  is a **matrix-valued ODE**.

For vector-valued ODEs, we referred to  $\mathbf{y}$  as the state vector. In the context of matrix-valued ODEs, we refer to  $\mathbf{M}$  as the **state matrix**<sup>1</sup>.

### 4.2 Transforming a Matrix-Valued IVP into a Vector-Valued IVP

Consider writing the matrix  $\mathbf{M} \in \mathbb{R}^{p \times r}$  in terms of its column vectors:

$$\mathbf{M} = \begin{bmatrix} | & & | \\ \mathbf{m}_1 & \cdots & \mathbf{m}_r \\ | & & | \end{bmatrix} \quad (4.1)$$

We know that

$$\begin{aligned} \frac{d\mathbf{M}}{dt} &= \mathbf{F}(t, \mathbf{M}) \\ \begin{bmatrix} | & & | \\ \frac{d\mathbf{m}_1}{dt} & \cdots & \frac{d\mathbf{m}_r}{dt} \\ | & & | \end{bmatrix} &= \mathbf{F} \left( t, \begin{bmatrix} | & & | \\ \mathbf{m}_1 & \cdots & \mathbf{m}_r \\ | & & | \end{bmatrix} \right) \end{aligned}$$

---

<sup>1</sup> Not to be confused with the matrix  $\mathbf{A}$  in the dynamics equation  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ , which is also often called the state matrix (in the context of linear systems and control theory).

where  $\mathbf{F} : \mathbb{R} \times \mathbb{R}^{p \times r} \rightarrow \mathbb{R}^{p \times r}$ .

Our goal is to write the matrix-valued ODE,  $\mathbf{F}$ , as a vector-valued ODE,  $\mathbf{f}$ , where

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y})$$

Let's imagine stacking all the column vectors of  $\mathbf{M}$  on top of each other. Then our goal becomes to write a function of the form

$$\begin{bmatrix} d\mathbf{m}_1/dt \\ \vdots \\ d\mathbf{m}_r/dt \end{bmatrix} = \mathbf{f} \left( t, \begin{bmatrix} \mathbf{m}_1 \\ \vdots \\ \mathbf{m}_r \end{bmatrix} \right) \quad (4.2)$$

Since the matrix  $\mathbf{M}$  has  $pr$  entries, stacking all the column vectors of  $\mathbf{M}$  on top of one another will produce a vector of length  $pr$ . Thus, we have  $\mathbf{f} : \mathbb{R} \times \mathbb{R}^{pr} \rightarrow \mathbb{R}^{pr}$ .

In an IVP solver setting, our vector-valued function  $\mathbf{f}$  should have the inputs  $t$  and  $\mathbf{y} \in \mathbb{R}^{pr}$ , where

$$\mathbf{y} = \begin{bmatrix} \mathbf{m}_1 \\ \vdots \\ \mathbf{m}_r \end{bmatrix} \quad (4.3)$$

It follows that essentially, given  $\mathbf{y} \in \mathbb{R}^{pr}$ , we construct  $\mathbf{M} \in \mathbb{R}^{p \times r}$ , evaluate the matrix-valued ODE  $\mathbf{F}$  to find  $d\mathbf{M}/dt \in \mathbb{R}^{p \times r}$ , and then construct  $d\mathbf{y}/dt \in \mathbb{R}^{pr}$  from  $d\mathbf{M}/dt$ . We formalize this procedure for defining a vector-valued ODE  $\mathbf{f}(t, \mathbf{y})$  in terms of a matrix-valued ODE  $\mathbf{F}(t, \mathbf{M})$  as Algorithm 6. Note that Algorithm 6 relies on a “helper” algorithm, Algorithm 5.

#### Algorithm 5: state\_vector\_derivative

Finds the state vector derivative given a matrix-valued ODE.

##### Given:

- $\mathbf{F}(t, \mathbf{M})$  - multivariate, matrix-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^{p \times r} \rightarrow \mathbb{R}^{p \times r}$ ) defining the ODE  $d\mathbf{M}/dt = \mathbf{F}(t, \mathbf{M})$
- $t \in \mathbb{R}$  - current time
- $\mathbf{y} \in \mathbb{R}^{pr}$  - current state vector, where  $\mathbf{y}$  is a vector formed by stacking the column vectors of  $\mathbf{M}$  on top of one another
- $p \in \mathbb{R}$  - number of rows of  $\mathbf{M}$

##### Procedure:

1. Determine the product  $pr$ , given that  $\mathbf{y} \in \mathbb{R}^{pr}$ .
2. If  $p$  is not specified, assume that  $\mathbf{M}$  is square (i.e.  $r = p$ , so  $\mathbf{M} \in \mathbb{R}^{p^2}$ ), and calculate  $p$  accordingly.

```

if ( $p$  is input as an empty vector)
    |    $p = \sqrt{pr}$ 
end

```

3. Determine  $r$ .

$$r = \frac{pr}{p}$$

4. Obtain  $\mathbf{M} \in \mathbb{R}^{p \times r}$  from  $\mathbf{y} \in \mathbb{R}^{pr}$ .

$$\mathbf{y} = \begin{bmatrix} \mathbf{m}_1 \\ \vdots \\ \mathbf{m}_r \end{bmatrix} \rightarrow \mathbf{M} = \begin{bmatrix} | & & | \\ \mathbf{m}_1 & \cdots & \mathbf{m}_r \\ | & & | \end{bmatrix}$$

This can be done in MATLAB using the `reshape` function:

$$\mathbf{M} = \text{reshape}(\mathbf{y}, [p, r])$$

5. Evaluate the matrix-valued ODE.

$$\frac{d\mathbf{M}}{dt} = \mathbf{F}(t, \mathbf{M})$$

6. Obtain  $d\mathbf{y}/dt \in \mathbb{R}^{pr}$  from  $d\mathbf{M}/dt \in \mathbb{R}^{p \times r}$ .

$$\frac{d\mathbf{M}}{dt} = \begin{bmatrix} | & & | \\ \frac{d\mathbf{m}_1}{dt} & \cdots & \frac{d\mathbf{m}_r}{dt} \\ | & & | \end{bmatrix} \rightarrow \frac{d\mathbf{y}}{dt} = \begin{bmatrix} d\mathbf{m}_1/dt \\ \vdots \\ d\mathbf{m}_r/dt \end{bmatrix}$$

This can be done in MATLAB as

$$\frac{d\mathbf{y}}{dt} = \frac{d\mathbf{M}}{dt}(:)$$

**Return:**

- $\frac{d\mathbf{y}}{dt} \in \mathbb{R}^{pr}$  - state vector derivative

**Algorithm 6:** `mat2vec_ode`

Transforms a matrix-valued ODE into a vector-valued ODE.

**Given:**

- $\mathbf{F}(t, \mathbf{M})$  - multivariate, matrix-valued function ( $\mathbf{F} : \mathbb{R} \times \mathbb{R}^{p \times r} \rightarrow \mathbb{R}^{p \times r}$ ) defining the ODE  $d\mathbf{M}/dt = \mathbf{F}(t, \mathbf{M})$
- $p \in \mathbb{R}$  - (*OPTIONAL*) number of rows of  $\mathbf{M}$

**Procedure:**

1. Default  $p$  to an empty vector if it is not input.
2. Create a function defining the corresponding vector-valued ODE utilizing Algorithm 5.

$$\mathbf{f}(t, \mathbf{y}) = \text{state\_vector\_derivative}(\mathbf{F}, t, \mathbf{y}, p)$$

**Return:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^{pr} \rightarrow \mathbb{R}^{pr}$ ) defining corresponding vector-valued ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$

Similarly, given an IVP involving a matrix-valued ODE, the associated initial condition will also be a matrix:

$$\mathbf{M}(t_0) = \mathbf{M}_0 \in \mathbb{R}^{p \times r}$$

Algorithm 6 essentially defines  $\mathbf{f}(t, \mathbf{y})$  as a calculation procedure. To obtain the associated vector initial condition  $\mathbf{y}_0$  for use in an IVP solver, we can simply note that

$$\mathbf{M}_0 = \begin{bmatrix} | & & | \\ \mathbf{m}_{1,0} & \cdots & \mathbf{m}_{r,0} \\ | & & | \end{bmatrix} \rightarrow \mathbf{y}_0 = \begin{bmatrix} \mathbf{m}_{1,0} \\ \vdots \\ \mathbf{m}_{r,0} \end{bmatrix}$$

Defining this step as Algorithm 7,

#### Algorithm 7: mat2vec\_IC

Transforms a matrix initial condition into a vector initial condition.

##### Given:

- $\mathbf{M}_0 \in \mathbb{R}^{p \times r}$  - initial condition for matrix-valued IVP

**Procedure:** Obtain  $\mathbf{y}_0 \in \mathbb{R}^{pr}$  from  $\mathbf{M} \in \mathbb{R}^{p \times r}$ .

$$\mathbf{M}_0 = \begin{bmatrix} | & & | \\ \mathbf{m}_{1,0} & \cdots & \mathbf{m}_{r,0} \\ | & & | \end{bmatrix} \rightarrow \mathbf{y}_0 = \begin{bmatrix} \mathbf{m}_{1,0} \\ \vdots \\ \mathbf{m}_{r,0} \end{bmatrix}$$

This can be done in MATLAB as

$$\mathbf{y}_0 = \mathbf{M}_0(:)$$

##### Return:

- $\mathbf{y}_0 \in \mathbb{R}^{pr}$  - initial condition for corresponding vector-valued IVP

Finally, if we are solving the IVP in event detection mode, then we will have a condition function<sup>2</sup>  $C(t, \mathbf{M})$  (where  $C : \mathbb{R} \times \mathbb{R}^{p \times r} \rightarrow \mathbb{B}$ ) that we must also transform to be compatible with the corresponding vector-valued IVP. This can be done in an almost identical way to how we transformed  $\mathbf{F}(t, \mathbf{M})$  to  $\mathbf{f}(t, \mathbf{y})$ ; see Algorithm 9 below. Note that Algorithm 9 also depends on a helper algorithm, Algorithm 8.

#### Algorithm 8: vector\_condition\_function

Evaluates the condition function for a matrix-valued IVP given the current time and corresponding state vector.

##### Given:

- $\mathbf{C}_m(t, \mathbf{M})$  - condition function for matrix-valued IVP ( $\mathbf{C}_m : \mathbb{R} \times \mathbb{R}^{p \times r} \rightarrow \mathbb{B}$ )
- $t \in \mathbb{R}$  - current time
- $\mathbf{y} \in \mathbb{R}^{pr}$  - current state vector, where  $\mathbf{y}$  is a vector formed by stacking the column vectors of  $\mathbf{M}$  on top of one another
- $p \in \mathbb{R}$  - number of rows of  $\mathbf{M}$

##### Procedure:

1. Determine the product  $pr$ , given that  $\mathbf{y} \in \mathbb{R}^{pr}$ .

<sup>2</sup> See Section 3.7 for a discussion of condition functions.

2. If  $p$  is not input, assume that  $\mathbf{M}$  is square (i.e.  $r = p$ , so  $\mathbf{M} \in \mathbb{R}^{p^2}$ ), and calculate  $p$  accordingly.

```

if ( $p$  is input as an empty vector)
    |    $p = \sqrt{pr}$ 
end

```

3. Determine  $r$ .

$$r = \frac{pr}{p}$$

4. Obtain  $\mathbf{M} \in \mathbb{R}^{p \times r}$  from  $\mathbf{y} \in \mathbb{R}^{pr}$ .

$$\mathbf{y} = \begin{bmatrix} \mathbf{m}_1 \\ \vdots \\ \mathbf{m}_r \end{bmatrix} \rightarrow \mathbf{M} = \begin{bmatrix} | & & | \\ \mathbf{m}_1 & \cdots & \mathbf{m}_r \\ | & & | \end{bmatrix}$$

This can be done in MATLAB using the `reshape` function:

$$\mathbf{M} = \text{reshape}(\mathbf{y}, [p, r])$$

5. Evaluate the condition function.

$$C = C_m(t, \mathbf{M})$$

**Return:**

- $C \in \mathbb{B}$  - evaluation of condition function

**Algorithm 9:** `mat2vec_C`

Transforms the condition function for a matrix-valued IVP into the condition function for its corresponding vector-valued IVP.

**Given:**

- $C_m(t, \mathbf{M})$  - condition function for matrix-valued IVP ( $C_m : \mathbb{R} \times \mathbb{R}^{p \times r} \rightarrow \mathbb{B}$ )
- $p \in \mathbb{R}$  - (*OPTIONAL*) number of rows of  $\mathbf{M}$

**Procedure:**

1. Default  $p$  to an empty vector if it is not input.
2. Create a function defining the condition function for the corresponding vector-valued IVP using Algorithm 8.

$$C_v(t, \mathbf{y}) = \text{vector\_condition\_function}(C_m, t, \mathbf{y}, p)$$

**Return:**

- $C_v(t, \mathbf{y})$  - condition function ( $C_v : \mathbb{R} \times \mathbb{R}^{pr} \rightarrow \mathbb{R}^{pr}$ ) for corresponding vector-valued IVP



## 4.3 Obtaining the Matrix Results

`mat2vec_ode` and `mat2vec_IC` (Algorithms 6 and 7, respectively) from the previous section essentially help us define the IVP

$$\frac{dy}{dt} = \mathbf{f}(t, \mathbf{y}), \quad \mathbf{y}(t_0) = \mathbf{y}_0$$

Running an IVP solver on this IVP will give us the solution matrix  $[\mathbf{y}] \in \mathbb{R}^{(N+1) \times pr}$ . The  $n$ th row of  $[\mathbf{y}]$  stores  $\mathbf{y}_n^T$ , where  $\mathbf{y}_n = \mathbf{y}(t_n) \in \mathbb{R}^{pr}$  is the solution of the IVP at time  $t_n$ . We want the solution array  $[\mathbf{M}] \in \mathbb{R}^{p \times r \times (N+1)}$ , where the  $n$ th page<sup>3</sup> of  $[\mathbf{M}]$  stores the solution,  $\mathbf{M}(t_n)$ , at time  $t_n$  to the IVP

$$\frac{d\mathbf{M}}{dt} = \mathbf{F}(t, \mathbf{M}), \quad \mathbf{M}(t_0) = \mathbf{M}_0$$

To obtain  $[\mathbf{M}]$ , we can use Algorithm 10.

### Algorithm 10: `vec2mat_sol`

Transforms the solution matrix for a vector-valued IVP into the solution array for the corresponding matrix-valued IVP.

#### Given:

- $[\mathbf{y}] \in \mathbb{R}^{(N+1) \times pr}$  - solution matrix (the  $n$ th row of  $[\mathbf{y}]$  stores  $\mathbf{y}_n^T$ , where  $\mathbf{y}_n \in \mathbb{R}^{pr}$  is the solution at  $t = t_n$ )
- $p \in \mathbb{R}$  - (OPTIONAL) number of rows of  $\mathbf{M}$

#### Procedure:

1. Determine the product  $pr$  and the scalar  $N$ , given that  $[\mathbf{y}] \in \mathbb{R}^{(N+1) \times pr}$ .
2. If  $p$  is not input, assume that  $\mathbf{M}$  is square (i.e.  $r = p$  so  $\mathbf{M} \in \mathbb{R}^{p^2}$ ), and calculate  $p$  accordingly.

```

if (p is not input or is input as an empty vector)
    |   p = sqrt(pr)
end

```

3. Determine  $r$ .

$$r = \frac{pr}{p}$$

4. Preallocate the solution array  $[\mathbf{M}] \in \mathbb{R}^{p \times r \times (N+1)}$ .

$$[\mathbf{M}] = \mathbf{0}_{p \times r \times (N+1)}$$

5. Populate the solution array.

```

for n = 1 to N + 1
    |   ([M])1:p,1:r,n = reshape([y]n,1:pr, [p, q])
end

```

#### Return:

- $[\mathbf{M}] \in \mathbb{R}^{p \times r \times (N+1)}$  - solution array ( $n$ th page of  $[\mathbf{M}]$  stores the solution,  $\mathbf{M}(t_n) = \mathbf{M}_n$ , at time  $t_n$ )

<sup>3</sup> See [7] for the definition of a page of an array.

**Note:**

- If  $p$  is not input, it is assumed that the state matrix ( $\mathbf{M}$ ) is a square matrix.
- $\mathbf{M} \in \mathbb{R}^{p \times r}$
- $\mathbf{y} \in \mathbb{R}^{pr}$
- $\mathbf{t} \in \mathbb{R}^{N+1}$
- The  $n$ th page of  $[\mathbf{M}]$  stores the state matrix (i.e. the solution) corresponding to the  $n$ th time in  $\mathbf{t}$ .

## 4.4 One-Step Propagation for Matrix-Valued ODEs

Recall (from Section 3.5.1) that the single-step methods (algorithms are summarized in Section 5.7) are ideally suited for use in “one-step propagation”. The documentation of the *IVP Solver Toolbox* functions corresponding to these algorithms is written specifically for the case of vector-valued ODEs. However, the exact same functions can be used for matrix-valued ODEs, since the functions make no assumptions regarding the dimensions of the inputs.

## 4.5 IVP Solver Algorithm for Matrix-Valued IVPs

To solve a matrix-valued IVP, we can first transform to a vector-valued IVP using Algorithms 6 and 7 from Section 4.2, solving the corresponding vector-valued IVP using Algorithm 3 from Section 3.9, and finally transforming the solution back to a matrix-valued form using Algorithm 10 from Section 4.3. This general procedure is formalized as Algorithm 11 below.

**Algorithm 11: solve\_ivp\_matrix**

Solve matrix-valued initial value problems using fixed-step IVP solvers.

**Given:**

- |  |   |
|--|---|
| • $\mathbf{F}(t, \mathbf{M})$                | - multivariate, matrix-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^{p \times r} \rightarrow \mathbb{R}^{p \times r}$ ) defining the ODE $d\mathbf{M}/dt = \mathbf{F}(t, \mathbf{M})$ |
| • $t_0 \in \mathbb{R}$                       | - initial time  |
| • $t_f \in \mathbb{R}$ or $C(t, \mathbf{y})$ | - final time or condition function ( $C : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{B}$ )  |
| • $\mathbf{M}_0 \in \mathbb{R}^{p \times r}$ | - initial condition   |
| • $h \in \mathbb{R}$                         | - step size   |
| • $p \in \mathbb{Z}$                         | - (OPTIONAL) number of rows of state matrix   |
| • <b>method</b>                              | - (OPTIONAL) integration method (one of the methods from Chapters 5–7)  |

**Procedure:**

1. If  $p$  is not input, initialize it as an empty vector. Algorithms 6, 9, and 10 should be implemented such that inputting  $p$  as an empty vector will be interpreted as  $p$  not being input at all.
2. Convert the matrix-valued ODE to a vector-valued ODE (Algorithm 6).

$$\mathbf{f} = \text{mat2vec\_ode}(\mathbf{F}, p)$$

3. Convert matrix initial condition to vector initial condition (Algorithm 7).

$$\mathbf{y}_0 = \text{mat2vec\_IC}(\mathbf{M}_0)$$

4. Convert condition function for matrix-valued IVP into the condition function for the corresponding vector-valued IVP (Algorithm 9).

```

if (condition function given)
    |    $C = \text{mat2vec\_C}(C, p)$ 
end

```

5. Solve the corresponding vector-valued IVP (Algorithm 3).

$$[t, [y]] = \text{solve\_ivp}(\mathbf{f}, t_0, t_f \text{ or } C, \mathbf{y}_0, h, \text{method})$$

6. Transform the solution matrix for the vector-valued IVP into the solution array for the matrix-valued IVP (Algorithm 10).

$$[\mathbf{M}] = \text{vec2mat\_sol}([y], p)$$

**Return:**

- $\mathbf{t} \in \mathbb{R}^{N+1}$  - time vector
- $[\mathbf{M}] \in \mathbb{R}^{(N+1) \times p}$  - solution array

**Note:**

- If  $p$  is not input, it is assumed that the state matrix ( $\mathbf{M}$ ) is a square matrix.
- The  $n$ th page of  $[\mathbf{M}]$  stores the state matrix (i.e. the solution) corresponding to the  $n$ th time in  $\mathbf{t}$ .

# PART II

## **Fixed-Step IVP Solvers**

# 5

## *Explicit Runge-Kutta (Single-Step) Methods*

### 5.1 General Form of the Explicit Runge-Kutta Method

In general, an  $s$ -stage **Runge-Kutta method** is defined as

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h \sum_{i=1}^s b_i \mathbf{k}_i \quad (5.1)$$

For *explicit* Runge-Kutta methods,

$$\begin{aligned} \mathbf{k}_1 &= \mathbf{f}(t_n, \mathbf{y}_i) \\ \mathbf{k}_2 &= \mathbf{f}(t_n + c_2 h, \mathbf{y}_i + h(a_{21} \mathbf{k}_1)) \\ \mathbf{k}_3 &= \mathbf{f}(t_n + c_3 h, \mathbf{y}_i + h(a_{31} \mathbf{k}_1 + a_{32} \mathbf{k}_2)) \\ &\vdots \\ \mathbf{k}_i &= \mathbf{f} \left( t_n + c_i h, \mathbf{y}_i + h \sum_{j=1}^{i-1} a_{ij} \mathbf{k}_j \right) \\ &\vdots \\ \mathbf{k}_s &= \mathbf{f} \left( t_n + c_s h, \mathbf{y}_i + h \sum_{j=1}^{s-1} a_{sj} \mathbf{k}_j \right) \end{aligned} \quad (5.2)$$

The order,  $m$ , of a Runge-Kutta method is not necessarily equal to the number of stages.

$$p \neq s \quad (\text{in general})$$

However, for the explicit Runge-Kutta methods summarized in this document, we will in fact have that

$$p = s \quad (\text{for the RK methods summarized in this document})$$

Note that

$$\mathbf{y}_n, \mathbf{y}_{n+1}, \mathbf{k}_i \in \mathbb{R}^p$$

$$h, a_{ij}, b_i, c_i \in \mathbb{R}$$

For a given method, the coefficients  $a_{ij}$ ,  $b_i$ , and  $c_i$  are often defined using a **Butcher tableau**:

$$\begin{array}{c|ccc} 0 & & & \\ c_2 & a_{21} & & \\ c_3 & a_{31} & a_{32} & \\ \vdots & \vdots & \vdots & \ddots \\ c_s & a_{s1} & a_{s2} & \dots & a_{s,s-1} \\ \hline & b_1 & b_2 & \dots & b_{s-1} & b_s \end{array}$$

We refer to Eq. (5.1) as the **propagation equation**, since it propagates the state vector  $\mathbf{y}$  from one sample time to the next (i.e. from time  $t_n$  to time  $t_{n+1}$ ). The `tableau2eqns`<sup>1</sup> function in the *IVP Solver Toolbox* can be used to generate the propagation equation, as well as all the  $\mathbf{k}$  vector coefficients. The input to this function is a matrix  $\mathbf{T} \in \mathbb{R}^{(s+1) \times (s+1)}$  storing the Butcher tableau.

$$\mathbf{T} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ c_2 & a_{21} & 0 & 0 & \dots & 0 \\ c_3 & a_{31} & a_{32} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ c_s & a_{s1} & a_{s2} & \dots & a_{s,s-1} & 0 \\ \hline 0 & b_1 & b_2 & \dots & b_{s-1} & b_s \end{bmatrix}$$

`tableau2eqns` first extracts the **Runge-Kutta matrix**  $\mathbf{A} \in \mathbb{R}^{s \times s}$ , the **weight vector**  $\mathbf{b} \in \mathbb{R}^{1 \times s}$ , and the **node vector**  $\mathbf{c} \in \mathbb{R}^s$  [5, 9].

$$\mathbf{A} = \begin{bmatrix} a_{11} & \dots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{s1} & \dots & a_{ss} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ a_{21} & 0 & 0 & \dots & 0 \\ a_{31} & a_{32} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{s1} & a_{s2} & \dots & a_{s,s-1} & 0 \end{bmatrix}$$

$$\mathbf{b} = [b_1 \quad \dots \quad b_s]$$

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_s \end{bmatrix} = \begin{bmatrix} 0 \\ c_2 \\ \vdots \\ c_s \end{bmatrix}$$

A list of the Butcher tableaus used to generate the equations in Sections 5.2-5.5 can be found in Section 5.6.

## 5.2 Euler Method

Consider the initial value problem

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y}), \quad \mathbf{y}(t_0) = \mathbf{y}_0$$

<sup>1</sup> [https://tamaskis.github.io/IVP\\_Solver\\_Toolbox-MATLAB/tableau2eqns\\_doc.html](https://tamaskis.github.io/IVP_Solver_Toolbox-MATLAB/tableau2eqns_doc.html)

Recall from Eq. (3.1) that the solution of this IVP is

$$\mathbf{y}(t) = \mathbf{y}_0 + \int_{t_0}^t \mathbf{f}(t, \mathbf{y}) dt$$

As discussed in Section 3.2, in general, it is either very difficult or mathematical impossible to evaluate the integral above in closed form. However, note that we can pick any point in time to be point  $t_0$ . Therefore, given  $\mathbf{y}_n = \mathbf{y}(t_n)$  at sample time  $t_n$ , we can find  $\mathbf{y}_{n+1} = \mathbf{y}(t_{n+1})$  at the next sample time as

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \int_{t_n}^{t_{n+1}} \mathbf{f}(t, \mathbf{y}) dt$$

The function  $\mathbf{f}$  has some value  $\mathbf{f}(t_n, \mathbf{y}_n)$  at time  $t_n$ . To define the Euler method (`RK1_euler`), let's assume that  $\mathbf{f}$  maintains this value until the next sample time,  $t_{n+1}$  [2].

$$\mathbf{f}(t, \mathbf{y}) \approx \mathbf{f}(t_n, \mathbf{y}_n) \quad \forall t \in [t_n, t_{n+1})$$

Since  $\mathbf{f}(t_n, \mathbf{y}_n)$  is now treated as a constant over the time interval  $[t_n, t_{n+1})$ , we can pull it out from under the integrand.

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \mathbf{f}(t_n, \mathbf{y}_n) \int_{t_n}^{t_{n+1}} dt = \mathbf{y}_n + \mathbf{f}(t_n, \mathbf{y}_n) \underbrace{(t_{n+1} - t_n)}_h$$

$$\boxed{\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(t_n, \mathbf{y}_n)} \quad (5.3)$$

## 5.3 Runge-Kutta Second-Order Methods

Method	Abbreviation	Equations	Reference(s)
Midpoint Method	RK2	$\mathbf{k}_1 = \mathbf{f}(t_n, \mathbf{y}_n)$ $\mathbf{k}_2 = \mathbf{f}\left(t_n + \frac{h}{2}, \mathbf{y}_n + \frac{h\mathbf{k}_1}{2}\right)$ $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{k}_2$	[5, 9]
Heun's Second-Order Method	RK2_heun	$\mathbf{k}_1 = \mathbf{f}(t_n, \mathbf{y}_n)$ $\mathbf{k}_2 = \mathbf{f}(t_n + h, \mathbf{y}_n + h\mathbf{k}_1)$ $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{2}(\mathbf{k}_1 + \mathbf{k}_2)$	[5, 9]
Ralston's Second-Order Method	RK2_ralston	$\mathbf{k}_1 = \mathbf{f}(t_n, \mathbf{y}_n)$ $\mathbf{k}_2 = \mathbf{f}\left(t_n + \frac{2h}{3}, \mathbf{y}_n + \frac{2h\mathbf{k}_1}{3}\right)$ $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{4}(\mathbf{k}_1 + 3\mathbf{k}_2)$	[5, 9]

## 5.4 Runge-Kutta Third-Order Methods

Method	Abbreviation	Equations	Reference(s)
Classic (Kutta's) Third-Order Method	RK3	$\mathbf{k}_1 = \mathbf{f}(t_n, \mathbf{y}_n)$ $\mathbf{k}_2 = \mathbf{f}\left(t_n + \frac{h}{2}, \mathbf{y}_n + \frac{h\mathbf{k}_1}{2}\right)$ $\mathbf{k}_3 = \mathbf{f}(t_n + h, \mathbf{y}_n - h\mathbf{k}_1 + 2h\mathbf{k}_2)$ $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{6}(\mathbf{k}_1 + 4\mathbf{k}_2 + \mathbf{k}_3)$	[3, pp. 129–131][5]
Heun's Third-Order Method	RK3_heun	$\mathbf{k}_1 = \mathbf{f}(t_n, \mathbf{y}_n)$ $\mathbf{k}_2 = \mathbf{f}\left(t_n + \frac{h}{3}, \mathbf{y}_n + \frac{h\mathbf{k}_1}{3}\right)$ $\mathbf{k}_3 = \mathbf{f}\left(t_n + \frac{2h}{3}, \mathbf{y}_n + \frac{2h\mathbf{k}_2}{3}\right)$ $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{4}(\mathbf{k}_1 + 3\mathbf{k}_3)$	[1, p. 287][3, pp. 129–131][5]
Ralston's Third-Order Method	RK3_ralston	$\mathbf{k}_1 = \mathbf{f}(t_n, \mathbf{y}_n)$ $\mathbf{k}_2 = \mathbf{f}\left(t_n + \frac{h}{2}, \mathbf{y}_n + \frac{h\mathbf{k}_1}{2}\right)$ $\mathbf{k}_3 = \mathbf{f}\left(t_n + \frac{3h}{4}, \mathbf{y}_n + \frac{3h\mathbf{k}_2}{4}\right)$ $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{9}(2\mathbf{k}_1 + 3\mathbf{k}_2 + 4\mathbf{k}_3)$	[5]
Strong Stability Preserving Runge-Kutta Third-Order Method	SSPRK3	$\mathbf{k}_1 = \mathbf{f}(t_n, \mathbf{y}_n)$ $\mathbf{k}_2 = \mathbf{f}(t_n + h, \mathbf{y}_n + h\mathbf{k}_1)$ $\mathbf{k}_3 = \mathbf{f}\left(t_n + \frac{h}{2}, \mathbf{y}_n + \frac{h\mathbf{k}_1}{4} + \frac{h\mathbf{k}_2}{4}\right)$ $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{6}(\mathbf{k}_1 + \mathbf{k}_2 + 4\mathbf{k}_3)$	[5]

## 5.5 Runge-Kutta Fourth-Order Methods

Method	Abbreviation	Equations	Reference(s)
Classic Fourth-Order Method	RK4	$\mathbf{k}_1 = \mathbf{f}(t_n, \mathbf{y}_n)$ $\mathbf{k}_2 = \mathbf{f}\left(t_n + \frac{h}{2}, \mathbf{y}_n + \frac{h\mathbf{k}_1}{2}\right)$ $\mathbf{k}_3 = \mathbf{f}\left(t_n + \frac{h}{2}, \mathbf{y}_n + \frac{h\mathbf{k}_2}{2}\right)$ $\mathbf{k}_4 = \mathbf{f}(t_n + h, \mathbf{y}_n + h\mathbf{k}_3)$ $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4)$	[1, p. 288][3, p. 131][5]



Ralston's Fourth-Order Method	RK4_ralston	$\begin{aligned} \mathbf{k}_1 &= \mathbf{f}(t_n, \mathbf{y}_n) \\ \mathbf{k}_2 &= \mathbf{f}(t_n + 0.4h, \mathbf{y}_n + 0.4h\mathbf{k}_1) \\ \mathbf{k}_3 &= \mathbf{f}(t_n + 0.45573725h, \mathbf{y}_n + 0.29697761h\mathbf{k}_1 \\ &\quad + 0.15875964h\mathbf{k}_2) \\ \mathbf{k}_4 &= \mathbf{f}(t_n + h, \mathbf{y}_n + 0.21810040h\mathbf{k}_1 \\ &\quad - 3.05096516h\mathbf{k}_2 + 3.83286476h\mathbf{k}_3) \\ \mathbf{y}_{n+1} &= \mathbf{y}_n + h(0.17476028\mathbf{k}_1 - 0.55148066\mathbf{k}_2 \\ &\quad + 1.20553560\mathbf{k}_3 + 0.17118478\mathbf{k}_4) \end{aligned}$	[5]
3/8-Rule Fourth-Order Method	RK4_38	$\begin{aligned} \mathbf{k}_1 &= \mathbf{f}(t_n, \mathbf{y}_n) \\ \mathbf{k}_2 &= \mathbf{f}\left(t_n + \frac{h}{3}, \mathbf{y}_n + \frac{h\mathbf{k}_1}{3}\right) \\ \mathbf{k}_3 &= \mathbf{f}\left(t_n + \frac{2h}{3}, \mathbf{y}_n - \frac{h\mathbf{k}_1}{3} + h\mathbf{k}_2\right) \\ \mathbf{k}_4 &= \mathbf{f}(t_n + h, \mathbf{y}_n + h\mathbf{k}_1 - h\mathbf{k}_2 + h\mathbf{k}_3) \\ \mathbf{y}_{n+1} &= \mathbf{y}_n + \frac{h}{8}(\mathbf{k}_1 + 3\mathbf{k}_2 + 3\mathbf{k}_3 + \mathbf{k}_4) \end{aligned}$	[5]

## 5.6 List of Butcher Tableaus

Euler Method [5] (RK1_euler)	$\begin{array}{c c} 0 & 0 \\ \hline & 1 \end{array}$
Midpoint Method [5] (RK2)	$\begin{array}{c cc} 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ \hline & 0 & 1 \end{array}$
Heun's Second-Order Method [5] (RK2_heun)	$\begin{array}{c cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$
Ralston's Second-Order Method [5] (RK2_ralston)	$\begin{array}{c cc} 0 & 0 & 0 \\ 2/3 & 2/3 & 0 \\ \hline & 1/4 & 3/4 \end{array}$
Classic (Kutta's) Third-Order Method [3, p. 131][5] (RK3)	$\begin{array}{c ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1 & -1 & 2 & 0 \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$

Heun's Third-Order  
Method [3, p. 131][5]  
(RK3\_heun)

0	0	0	0
1/3	1/3	0	0
2/3	0	2/3	0
	1/4	0	3/4

Ralston's Third-Order  
Method [5]  
(RK3\_ralston)

0	0	0	0
1/2	1/2	0	0
3/4	0	3/4	0
	2/9	1/3	4/9

Strong Stability  
Preserving Runge-Kutta  
Third-Order Method [5]  
(SSPRK3)

0	0	0	0
1	1	0	0
1/2	1/4	1/4	0
	1/6	1/6	2/3

Classic Fourth-Order  
Method [3, p. 131][5]  
(RK4)

0	0	0	0	0
1/2	1/2	0	0	0
1/2	0	1/2	0	0
1	0	0	1	0
	1/6	1/3	1/3	1/6

Ralston's Fourth-Order  
Method [5]  
(RK4\_ralston)

0	0	0	0	0
0.4	0.4	0	0	0
0.45573725	0.29697761	0.15875964	0	0
1	0.21810040	-3.05096516	3.83286476	0
	0.17476028	-0.55148066	1.20553560	0.17118478

3/8-Rule Fourth-Order  
Method [5]  
(RK4\_38)

0	0	0	0	0
1/3	1/3	0	0	0
2/3	-1/3	1	0	0
1	1	-1	1	0
	1/8	3/8	3/8	1/8

## 5.7 Algorithms

### Algorithm 12: RK1\_euler

Propagates the state vector forward one time step using the Euler (first-order) method.

#### Given:

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{y} \in \mathbb{R}^p$  - state vector (i.e. solution) at current sample time
- $h \in \mathbb{R}$  - step size

#### Procedure:

$$\mathbf{y}_{\text{next}} = \mathbf{y} + h\mathbf{f}(t, \mathbf{y})$$

#### Return:

- $\mathbf{y}_{\text{next}} \in \mathbb{R}^p$  - state vector (i.e. solution) at the next sample time (i.e.  $t + h$ )

### Algorithm 13: RK2

Propagates the state vector forward one time step using the mid-point method (Runge-Kutta second-order method).

#### Given:

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{y} \in \mathbb{R}^p$  - state vector (i.e. solution) at current sample time
- $h \in \mathbb{R}$  - step size

#### Procedure:

1. k terms.

$$\mathbf{k}_1 = \mathbf{f}(t, \mathbf{y})$$

$$\mathbf{k}_2 = \mathbf{f}\left(t + \frac{h}{2}, \mathbf{y} + \frac{h\mathbf{k}_1}{2}\right)$$

2. State vector propagated to next sample time.

$$\mathbf{y}_{\text{next}} = \mathbf{y} + h\mathbf{k}_2$$

#### Return:

- $\mathbf{y}_{\text{next}} \in \mathbb{R}^p$  - state vector (i.e. solution) at the next sample time (i.e.  $t + h$ )

**Algorithm 14:** RK2\_heun

Propagates the state vector forward one time step using Heun's second-order method [Runge-Kutta second-order method].

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{y} \in \mathbb{R}^p$  - state vector (i.e. solution) at current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1. k terms.

$$\mathbf{k}_1 = \mathbf{f}(t, \mathbf{y})$$

$$\mathbf{k}_2 = \mathbf{f}(t + h, \mathbf{y}_n + h\mathbf{k}_1)$$

2. State vector propagated to next sample time.

$$\mathbf{y}_{\text{next}} = \mathbf{y} + \frac{h}{2} (\mathbf{k}_1 + \mathbf{k}_2)$$

**Return:**

- $\mathbf{y}_{\text{next}} \in \mathbb{R}^p$  - state vector (i.e. solution) at the next sample time (i.e.  $t + h$ )

**Algorithm 15:** RK2\_ralston

Propagates the state vector forward one time step using Ralston's second-order method [Runge-Kutta second-order method].

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{y} \in \mathbb{R}^p$  - state vector (i.e. solution) at current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1. k terms.

$$\mathbf{k}_1 = \mathbf{f}(t, \mathbf{y})$$

$$\mathbf{k}_2 = \mathbf{f}\left(t + \frac{2h}{3}, \mathbf{y}_n + \frac{2h\mathbf{k}_1}{3}\right)$$

2. State vector propagated to next sample time.

$$\mathbf{y}_{\text{next}} = \mathbf{y} + \frac{h}{4} (\mathbf{k}_1 + 3\mathbf{k}_2)$$

**Return:**

- $\mathbf{y}_{\text{next}} \in \mathbb{R}^p$  - state vector (i.e. solution) at the next sample time (i.e.  $t + h$ )

**Algorithm 16: RK3**

Propagates the state vector forward one time step using [Kutta's] Runge-Kutta third-order method.

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{y} \in \mathbb{R}^p$  - state vector (i.e. solution) at current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1. k terms.

$$\mathbf{k}_1 = \mathbf{f}(t, \mathbf{y})$$

$$\mathbf{k}_2 = \mathbf{f}\left(t + \frac{h}{2}, \mathbf{y} + \frac{h\mathbf{k}_1}{2}\right)$$

$$\mathbf{k}_3 = \mathbf{f}(t + h, \mathbf{y} - h\mathbf{k}_1 + 2h\mathbf{k}_2)$$

2. State vector propagated to next sample time.

$$\mathbf{y}_{\text{next}} = \mathbf{y} + \frac{h}{6} (\mathbf{k}_1 + 4\mathbf{k}_2 + \mathbf{k}_3)$$

**Return:**

- $\mathbf{y}_{\text{next}} \in \mathbb{R}^p$  - state vector (i.e. solution) at the next sample time (i.e.  $t + h$ )

**Algorithm 17: RK3\_heun**

Propagates the state vector forward one time step using Heun's third-order method [Runge-Kutta third-order method].

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{y} \in \mathbb{R}^p$  - state vector (i.e. solution) at current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1. k terms.

$$\mathbf{k}_1 = \mathbf{f}(t, \mathbf{y})$$

$$\mathbf{k}_2 = \mathbf{f}\left(t + \frac{h}{3}, \mathbf{y} + \frac{h\mathbf{k}_1}{3}\right)$$

$$\mathbf{k}_3 = \mathbf{f}\left(t + \frac{2h}{3}, \mathbf{y} + \frac{2h\mathbf{k}_2}{3}\right)$$

2. State vector propagated to next sample time.

$$\mathbf{y}_{\text{next}} = \mathbf{y} + \frac{h}{4} (\mathbf{k}_1 + 3\mathbf{k}_3)$$

**Return:**

- $\mathbf{y}_{\text{next}} \in \mathbb{R}^p$  - state vector (i.e. solution) at the next sample time (i.e.  $t + h$ )

**Algorithm 18: RK3\_ralston**

Propagates the state vector forward one time step using Ralston's third-order method (Runge-Kutta third-order method).

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{y} \in \mathbb{R}^p$  - state vector (i.e. solution) at current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1.  $\mathbf{k}$  terms.

$$\mathbf{k}_1 = \mathbf{f}(t, \mathbf{y})$$

$$\mathbf{k}_2 = \mathbf{f}\left(t + \frac{h}{2}, \mathbf{y} + \frac{h\mathbf{k}_1}{2}\right)$$

$$\mathbf{k}_3 = \mathbf{f}\left(t + \frac{3h}{4}, \mathbf{y} + \frac{3h\mathbf{k}_2}{4}\right)$$

2. State vector propagated to next sample time.

$$\mathbf{y}_{\text{next}} = \mathbf{y} + \frac{h}{9} (2\mathbf{k}_1 + 3\mathbf{k}_2 + 4\mathbf{k}_3)$$

**Return:**

- $\mathbf{y}_{\text{next}} \in \mathbb{R}^p$  - state vector (i.e. solution) at the next sample time (i.e.  $t + h$ )

**Algorithm 19: SSPRK3**

Propagates the state vector forward one time step using the strong stability preserving Runge-Kutta third-order method.

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{y} \in \mathbb{R}^p$  - state vector (i.e. solution) at current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1.  $k$  terms.

$$\mathbf{k}_1 = \mathbf{f}(t, \mathbf{y})$$

$$\mathbf{k}_2 = \mathbf{f}\left(t + h, \mathbf{y} + h\mathbf{k}_1\right)$$

$$\mathbf{k}_3 = \mathbf{f}\left(t + \frac{h}{2}, \mathbf{y} + \frac{h\mathbf{k}_1}{4} + \frac{h\mathbf{k}_2}{4}\right)$$

2. State vector propagated to next sample time.

$$\mathbf{y}_{\text{next}} = \mathbf{y} + \frac{h}{6} (\mathbf{k}_1 + \mathbf{k}_2 + 4\mathbf{k}_3)$$

**Return:**

- $\mathbf{y}_{\text{next}} \in \mathbb{R}^p$  - state vector (i.e. solution) at the next sample time (i.e.  $t + h$ )

**Algorithm 20: RK4**

Propagates the state vector forward one time step using the (classic) Runge-Kutta fourth-order method.

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{y} \in \mathbb{R}^p$  - state vector (i.e. solution) at current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1.  $k$  terms.

$$\mathbf{k}_1 = \mathbf{f}(t, \mathbf{y})$$

$$\mathbf{k}_2 = \mathbf{f}\left(t + \frac{h}{2}, \mathbf{y} + \frac{h\mathbf{k}_1}{2}\right)$$

$$\mathbf{k}_3 = \mathbf{f}\left(t + \frac{h}{2}, \mathbf{y} + \frac{h\mathbf{k}_2}{2}\right)$$

$$\mathbf{k}_4 = \mathbf{f}(t + h, \mathbf{y} + h\mathbf{k}_3)$$

2. State vector propagated to next sample time.

$$\mathbf{y}_{\text{next}} = \mathbf{y} + \frac{h}{6} (\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4)$$

**Return:**

- $\mathbf{y}_{\text{next}} \in \mathbb{R}^p$  - state vector (i.e. solution) at the next sample time (i.e.  $t + h$ )

**Algorithm 21: RK4\_ralston**

Propagates the state vector forward one time step using Ralston's fourth-order method [Runge-Kutta fourth-order method].

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{y} \in \mathbb{R}^p$  - state vector (i.e. solution) at current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1. k terms.

$$\mathbf{k}_1 = \mathbf{f}(t, \mathbf{y})$$

$$\mathbf{k}_2 = \mathbf{f}(t + 0.4h, \mathbf{y} + 0.4h\mathbf{k}_1)$$

$$\mathbf{k}_3 = \mathbf{f}(t + 0.45573725h, \mathbf{y} + 0.29697761h\mathbf{k}_1 + 0.15875964h\mathbf{k}_2)$$

$$\mathbf{k}_4 = \mathbf{f}(t + h, \mathbf{y} + 0.21810040h\mathbf{k}_1 - 3.05096516h\mathbf{k}_2 + 3.83286476h\mathbf{k}_3)$$

2. State vector propagated to next sample time.

$$\mathbf{y}_{\text{next}} = \mathbf{y} + h(0.17476028\mathbf{k}_1 - 0.55148066\mathbf{k}_2 + 1.20553560\mathbf{k}_3 + 0.17118478\mathbf{k}_4)$$

**Return:**

- $\mathbf{y}_{\text{next}} \in \mathbb{R}^p$  - state vector (i.e. solution) at the next sample time (i.e.  $t + h$ )

**Algorithm 22: RK4\_38**

Propagates the state vector forward one time step using the 3/8-rule fourth-order method [Runge-Kutta fourth-order method].

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{y} \in \mathbb{R}^p$  - state vector (i.e. solution) at current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1. k terms.

$$\mathbf{k}_1 = \mathbf{f}(t, \mathbf{y})$$

$$\mathbf{k}_2 = \mathbf{f}\left(t + \frac{h}{3}, \mathbf{y} + \frac{h\mathbf{k}_1}{3}\right)$$

$$\mathbf{k}_3 = \mathbf{f}\left(t + \frac{2h}{3}, \mathbf{y} - \frac{h\mathbf{k}_1}{3} + h\mathbf{k}_2\right)$$

$$\mathbf{k}_4 = \mathbf{f}(t + h, \mathbf{y} + h\mathbf{k}_1 - h\mathbf{k}_2 + h\mathbf{k}_3)$$



2. State vector propagated to next sample time.

$$\mathbf{y}_{\text{next}} = \mathbf{y} + \frac{h}{8} (\mathbf{k}_1 + 3\mathbf{k}_2 + 3\mathbf{k}_3 + \mathbf{k}_4)$$

**Return:**

- $\mathbf{y}_{\text{next}} \in \mathbb{R}^p$  - state vector (i.e. solution) at the next sample time (i.e.  $t + h$ )

# 6

## *Adams-Bashforth (Multistep Predictor) Methods*

### 6.1 Adams-Bashforth Predictor

The general Adams-Bashforth predictor is defined as

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h \sum_{i=1}^m b_i \mathbf{f}(t_{n-i+1}, \mathbf{y}_{n-i+1}) \quad (6.1)$$

where  $m$  is the order of the method.

To find the vector  $\mathbf{b} = (b_1, \dots, b_m)^T$  storing the vectors for this predictor, we first define the matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$  and the vector  $\mathbf{c} \in \mathbb{R}^m$  as

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 2 & \dots & m & (m-1) \\ 0 & 1 & 2^2 & \dots & m^2 & (m-1)^2 \\ 0 & 1 & 2^3 & \dots & m^3 & (m-1)^3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & 2^{m-1} & \dots & m^{m-1} & (m-1)^{m-1} \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{3} \\ -\frac{1}{4} \\ \vdots \\ \frac{(-1)^{m-1}}{m} \end{bmatrix}$$

This can be more compactly expressed as

$$\mathbf{A} = [a_{ij}] \quad \text{where} \quad a_{ij} = (j-1)^{i-1} \quad (\text{for } i, j = 1, 2, \dots, n) \quad (6.2)$$

$$\mathbf{c} = [c_i] \quad \text{where} \quad c_i = \frac{(-1)^{i-1}}{i} \quad (\text{for } i = 1, 2, \dots, n) \quad (6.3)$$

Next, we just need to solve the linear system  $\mathbf{A}\mathbf{b} = \mathbf{c}$  for  $\mathbf{b}$  [10].

$$\mathbf{b} = \mathbf{A}^{-1}\mathbf{c} \quad (6.4)$$

## 6.2 Adams-Bashforth Methods

Order	Abbrev.	Equations	Reference(s)
2	AB2	$y_{n+1} = y_n + \frac{h}{2}(3f_n - f_{n-1})$	[4][6, pp. 135–136]
3	AB3	$y_{n+1} = y_n + \frac{h}{12}(23f_n - 16f_{n-1} + 5f_{n-2})$	[4][6, pp. 135–136]
4	AB4	$y_{n+1} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3})$	[4][6, pp. 135–136]
5	AB5	$y_{n+1} = y_n + \frac{h}{720}(1901f_n - 2774f_{n-1} + 2616f_{n-2} - 1274f_{n-3} + 251f_{n-4})$	[4][6, pp. 135–136]
6	AB6	$y_{n+1} = y_n + \frac{h}{1440}(4277f_n - 7923f_{n-1} + 9982f_{n-2} - 7298f_{n-3} + 2877f_{n-4} - 475f_{n-5})$	[6, pp. 135–136]
7	AB7	$y_{n+1} = y_n + \frac{h}{60480}(198721f_n - 447288f_{n-1} + 705549f_{n-2} - 688256f_{n-3} + 407139f_{n-4} - 134472f_{n-5} + 19087f_{n-6})$	[6, pp. 135–136]
8	AB8	$y_{n+1} = y_n + \frac{h}{120960}(434241f_n - 1152169f_{n-1} + 2183877f_{n-2} - 2664477f_{n-3} + 2102243f_{n-4} - 1041723f_{n-5} + 295767f_{n-6} - 36799f_{n-7})$	[6, pp. 135–136]

## 6.3 Algorithms

These algorithms are written in terms of the  $\mathbf{F}$  matrix discussed in Section 3.5.2.

### Algorithm 23: AB2

Propagates the state vector forward one time step using the Adams-Bashforth 2nd-order method.

#### Given:

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{F} \in \mathbb{R}^{p \times 3}$  -  $\mathbf{F}$  matrix for current sample time
- $h \in \mathbb{R}$  - step size

#### Procedure:

1. Updates  $\mathbf{F}$  matrix with new function evaluation.

$$\mathbf{F} = [ \mathbf{F}^{(2)} \mid \mathbf{f}(t, \mathbf{F}^{(3)}) \mid \mathbf{F}^{(3)} ]$$

2. State vector propagated to next sample time.

$$\mathbf{F}^{(3)} = \mathbf{F}^{(3)} + \frac{h}{2} (3\mathbf{F}^{(2)} - \mathbf{F}^{(1)})$$

**Return:**

- $\mathbf{F} \in \mathbb{R}^{p \times 3}$  -  $\mathbf{F}$  matrix updated for next sample time

**Algorithm 24: AB3**

Propagates the state vector forward one time step using the Adams-Bashforth 3rd-order method.

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{F} \in \mathbb{R}^{p \times 4}$  -  $\mathbf{F}$  matrix for current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1. Updates  $\mathbf{F}$  matrix with new function evaluation.

$$\mathbf{F} = [ \mathbf{F}^{(2:3)} \mid \mathbf{f}(t, \mathbf{F}^{(4)}) \mid \mathbf{F}^{(4)} ]$$

2. State vector propagated to next sample time.

$$\mathbf{F}^{(4)} = \mathbf{F}^{(4)} + \frac{h}{12} (23\mathbf{F}^{(3)} - 16\mathbf{F}^{(2)} + 5\mathbf{F}^{(1)})$$

**Return:**

- $\mathbf{F} \in \mathbb{R}^{p \times 4}$  -  $\mathbf{F}$  matrix updated for next sample time

**Algorithm 25: AB4**

Propagates the state vector forward one time step using the Adams-Bashforth 4th-order method.

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{F} \in \mathbb{R}^{p \times 5}$  -  $\mathbf{F}$  matrix for current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1. Updates  $\mathbf{F}$  matrix with new function evaluation.

$$\mathbf{F} = [ \mathbf{F}^{(2:4)} \mid \mathbf{f}(t, \mathbf{F}^{(5)}) \mid \mathbf{F}^{(5)} ]$$

2. State vector propagated to next sample time.

$$\mathbf{F}^{(5)} = \mathbf{F}^{(5)} + \frac{h}{24} (55\mathbf{F}^{(4)} - 59\mathbf{F}^{(3)} + 37\mathbf{F}^{(2)} - 9\mathbf{F}^{(1)})$$

**Return:**

- $\mathbf{F} \in \mathbb{R}^{p \times 5}$  -  $\mathbf{F}$  matrix updated for next sample time

**Algorithm 26: AB5**

Propagates the state vector forward one time step using the Adams-Bashforth 5th-order method.

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{F} \in \mathbb{R}^{p \times 6}$  -  $\mathbf{F}$  matrix for current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1. Updates  $\mathbf{F}$  matrix with new function evaluation.

$$\mathbf{F} = [ \mathbf{F}^{(2:5)} \mid \mathbf{f}(t, \mathbf{F}^{(6)}) \mid \mathbf{F}^{(6)} ]$$

2. State vector propagated to next sample time.

$$\mathbf{F}^{(6)} = \mathbf{F}^{(6)} + \frac{h}{720} \left( 1901\mathbf{F}^{(5)} - 2774\mathbf{F}^{(4)} + 2616\mathbf{F}^{(3)} - 1274\mathbf{F}^{(2)} + 251\mathbf{F}^{(1)} \right)$$

**Return:**

- $\mathbf{F} \in \mathbb{R}^{p \times 6}$  -  $\mathbf{F}$  matrix updated for next sample time

**Algorithm 27: AB6**

Propagates the state vector forward one time step using the Adams-Bashforth 6th-order method.

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{F} \in \mathbb{R}^{p \times 7}$  -  $\mathbf{F}$  matrix for current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1. Updates  $\mathbf{F}$  matrix with new function evaluation.

$$\mathbf{F} = [ \mathbf{F}^{(2:6)} \mid \mathbf{f}(t, \mathbf{F}^{(7)}) \mid \mathbf{F}^{(7)} ]$$

2. State vector propagated to next sample time.

$$\mathbf{F}^{(7)} = \mathbf{F}^{(7)} + \frac{h}{1440} \left( 4277\mathbf{F}^{(6)} - 7923\mathbf{F}^{(5)} + 9982\mathbf{F}^{(4)} - 7298\mathbf{F}^{(3)} + 2877\mathbf{F}^{(2)} - 475\mathbf{F}^{(1)} \right)$$

**Return:**

- $\mathbf{F} \in \mathbb{R}^{p \times 7}$  -  $\mathbf{F}$  matrix updated for next sample time

**Algorithm 28: AB7**

Propagates the state vector forward one time step using the Adams-Bashforth 7th-order method.

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{F} \in \mathbb{R}^{p \times 8}$  -  $\mathbf{F}$  matrix for current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1. Updates  $\mathbf{F}$  matrix with new function evaluation.

$$\mathbf{F} = [ \mathbf{F}^{(2:7)} \mid \mathbf{f}(t, \mathbf{F}^{(8)}) \mid \mathbf{F}^{(8)} ]$$

2. State vector propagated to next sample time.

$$\begin{aligned} \mathbf{F}^{(8)} = \mathbf{F}^{(8)} + \frac{h}{60480} & \left( 198721\mathbf{F}^{(7)} - 447288\mathbf{F}^{(6)} + 705549\mathbf{F}^{(5)} \right. \\ & - 688256\mathbf{F}^{(4)} + 407139\mathbf{F}^{(3)} - 134472\mathbf{F}^{(2)} \\ & \left. + 19087\mathbf{F}^{(1)} \right) \end{aligned}$$

**Return:**

- $\mathbf{F} \in \mathbb{R}^{p \times 8}$  -  $\mathbf{F}$  matrix updated for next sample time

**Algorithm 29: AB8**

Propagates the state vector forward one time step using the Adams-Bashforth 8th-order method.

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{F} \in \mathbb{R}^{p \times 9}$  -  $\mathbf{F}$  matrix for current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1. Updates  $\mathbf{F}$  matrix with new function evaluation.

$$\mathbf{F} = [ \mathbf{F}^{(2:8)} \mid \mathbf{f}(t, \mathbf{F}^{(9)}) \mid \mathbf{F}^{(9)} ]$$

2. State vector propagated to next sample time.

$$\begin{aligned} \mathbf{F}^{(9)} = \mathbf{F}^{(9)} + \frac{h}{120960} & \left( 434241\mathbf{F}^{(8)} - 1152169\mathbf{F}^{(7)} + 2183877\mathbf{F}^{(6)} \right. \\ & - 2664477\mathbf{F}^{(5)} + 2102243\mathbf{F}^{(4)} - 1041723\mathbf{F}^{(3)} \\ & \left. + 295767\mathbf{F}^{(2)} - 36799\mathbf{F}^{(1)} \right) \end{aligned}$$

**Return:**

- $\mathbf{F} \in \mathbb{R}^{p \times 9}$  -  $\mathbf{F}$  matrix updated for next sample time

# *Adams-Bashforth-Moulton (Multistep Predictor-Corrector) Methods*

## 7.1 Adams-Moulton Corrector

The general Adams-Moulton corrector is defined as

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h \sum_{i=1}^m b_i \mathbf{f}(t_{n-i+2}, \mathbf{y}_{n-i+2}) \quad (7.1)$$

where  $m$  is the order of the method.

To find the vector  $\mathbf{b} = (b_1, \dots, b_m)^T$  storing the vectors for this corrector, we first define the matrix  $\mathbf{A} \in \mathbb{R}^{m \times m}$  and the vector  $\mathbf{c} \in \mathbb{R}^m$  as

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ -1 & 0 & 1 & 2 & \dots & (m-3) & (m-2) \\ 1 & 0 & 1 & 2^2 & \dots & (m-3)^2 & (m-2)^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{m-2} & 0 & 1 & 2^{m-2} & \dots & (m-3)^{m-2} & (m-2)^{m-2} \\ (-1)^{m-1} & 0 & 1 & 2^{m-1} & \dots & (m-3)^{m-1} & (m-2)^{m-1} \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{3} \\ \vdots \\ \frac{(-1)^{m-2}}{m-1} \\ \frac{(-1)^{m-1}}{m} \end{bmatrix}$$

This can be more compactly expressed as

$$\mathbf{A} = [a_{ij}] \quad \text{where} \quad a_{ij} = (j-2)^{i-1} \quad (\text{for } i, j = 1, 2, \dots, m) \quad (7.2)$$

$$\mathbf{c} = [c_i] \quad \text{where} \quad c_i = \frac{(-1)^{i-1}}{i} \quad (\text{for } i = 1, 2, \dots, m) \quad (7.3)$$

Next, we just need to solve the linear system  $\mathbf{A}\mathbf{b} = \mathbf{c}$  for  $\mathbf{b}$  [10].

$$\mathbf{b} = \mathbf{A}^{-1}\mathbf{c} \quad (7.4)$$

## 7.2 Predictor-Corrector (PECE) Algorithms

Let  $\mathbf{f}_k$  represent the evaluation of the ODE  $dy/dt = \mathbf{f}(t, \mathbf{y})$  at time  $t_k$  and state  $\mathbf{y}_k$ .

$$\mathbf{f}_k = \mathbf{f}(t_k, \mathbf{y}_k)$$

The Adams-Bashforth methods discussed in Chapter 6 were *explicit* methods, since the state vector at the next sample time could be calculated *explicitly* from evaluating the ODE using the state vector at previous sample times. In general, this can be expressed as

$$\mathbf{y}_{k+1} = \mathbf{g}(\mathbf{f}_k, \mathbf{f}_{k-1}, \dots, \mathbf{f}_0)$$

However, the Adams-Moulton corrector introduced in Section 7.1 defines *implicit* methods, since the state vector at the next sample time is expressed as a function of  $\mathbf{f}_{k+1}$ , where  $\mathbf{f}_{k+1} = \mathbf{f}(t_{k+1}, \mathbf{y}_{k+1})$  (i.e.  $\mathbf{y}_{k+1}$  is essentially a function of itself):

$$\mathbf{y}_{k+1} = \mathbf{g}(\mathbf{f}_{k+1}, \mathbf{f}_k, \mathbf{f}_{k-1}, \dots, \mathbf{f}_0)$$

**Predictor-corrector algorithms** combine predictor and corrector methods and are defined using the following four steps:

1. **Predict**  $\mathbf{y}_{k+1}$  at time  $t_{k+1}$  using an *explicit* method of order  $m$ . Since this is a *predicted* value, we denote it using a hat as  $\hat{\mathbf{y}}_{k+1}$ .

$$\hat{\mathbf{y}}_{k+1} = \mathbf{g}(\mathbf{f}_k, \mathbf{f}_{k-1}, \dots, \mathbf{f}_0)$$

2. **Evaluate** the ODE  $dy/dt = \mathbf{f}(t, \mathbf{y})$  at  $t_{k+1}$  and  $\hat{\mathbf{y}}_{k+1}$ .

$$\hat{\mathbf{f}}_{k+1} = \mathbf{f}(t_{k+1}, \hat{\mathbf{y}}_{k+1})$$

3. **Correct** the prediction for  $\mathbf{y}_{k+1}$  using an *implicit* method of order  $m$ . Since this is the corrected value (that we are assuming to be the true value, i.e. the solution), we do *not* denote it using a hat.

$$\mathbf{y}_{k+1} = \mathbf{g}(\hat{\mathbf{f}}_{k+1}, \mathbf{f}_k, \mathbf{f}_{k-1}, \dots, \mathbf{f}_0)$$

4. **Evaluate** the ODE at the corrected value for the state vector at the  $(k+1)$ th sample time.

$$\mathbf{f}_{k+1} = \mathbf{f}(t_{k+1}, \mathbf{y}_{k+1})$$

Due to the four steps above (predict, evaluate, correct, evaluate), predictor-corrector algorithms are also known as **PECE** algorithms [3, pp. 112–113][6, p. 138]. In their implementation in the *IVP Solver Toolbox* (general implementation outlined in Section 3.9), the steps are slightly condensed/reordered as follows:

1. **Evaluate** the ODE  $dy/dt = \mathbf{f}(t, \mathbf{y})$  at  $t_k$  and  $\mathbf{y}_k$ .

$$\mathbf{f}_k = \mathbf{f}(t_k, \mathbf{y}_k)$$

2. **Predict**  $\mathbf{y}_{k+1}$  at time  $t_{k+1}$  using an *explicit* method of order  $m$ . Since this is a *predicted* value, we denote it using a hat as  $\hat{\mathbf{y}}_{k+1}$ .

$$\hat{\mathbf{y}}_{k+1} = \mathbf{g}(\mathbf{f}_k, \mathbf{f}_{k-1}, \dots, \mathbf{f}_0)$$

3. **Correct** the prediction for  $\mathbf{y}_{k+1}$  using an *implicit* method of order  $m$ . Since this is the corrected value (that we are assuming to be the true value, i.e. the solution), we do *not* denote it using a hat.

$$\mathbf{y}_{k+1} = \mathbf{g}(\mathbf{f}(t_{k+1}, \hat{\mathbf{y}}_{k+1}), \mathbf{f}_k, \mathbf{f}_{k-1}, \dots, \mathbf{f}_0)$$



## 7.3 Adams-Bashforth-Moulton Methods

Order	Abbrev.	Equations	Reference(s)
2	ABM2	$\hat{\mathbf{y}}_{n+1} = \mathbf{y}_n + \frac{h}{2}(3\mathbf{f}_n - \mathbf{f}_{n-1})$ $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{2}[\mathbf{f}(t_{n+1}, \hat{\mathbf{y}}_{n+1}) + \mathbf{f}_n]$	[4][6, pp. 135–138]
3	ABM3	$\hat{\mathbf{y}}_{n+1} = \mathbf{y}_n + \frac{h}{12}(23\mathbf{f}_n - 16\mathbf{f}_{n-1} + 5\mathbf{f}_{n-2})$ $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{12}[5\mathbf{f}(t_{n+1}, \hat{\mathbf{y}}_{n+1}) + 8\mathbf{f}_n - \mathbf{f}_{n-1}]$	[4][6, pp. 135–138]
4	ABM4	$\hat{\mathbf{y}}_{n+1} = \mathbf{y}_n + \frac{h}{24}(55\mathbf{f}_n - 59\mathbf{f}_{n-1} + 37\mathbf{f}_{n-2} - 9\mathbf{f}_{n-3})$ $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{24}[9\mathbf{f}(t_{n+1}, \hat{\mathbf{y}}_{n+1}) + 19\mathbf{f}_n - 5\mathbf{f}_{n-1} + \mathbf{f}_{n-2}]$	[4][6, pp. 135–138]
5	ABM5	$\hat{\mathbf{y}}_{n+1} = \mathbf{y}_n + \frac{h}{720}(1901\mathbf{f}_n - 2774\mathbf{f}_{n-1} + 2616\mathbf{f}_{n-2} - 1274\mathbf{f}_{n-3} + 251\mathbf{f}_{n-4})$ $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{720}[251\mathbf{f}(t_{n+1}, \hat{\mathbf{y}}_{n+1}) + 646\mathbf{f}_n - 264\mathbf{f}_{n-1} + 106\mathbf{f}_{n-2} - 19\mathbf{f}_{n-3}]$	[4][6, pp. 135–138]
6	ABM6	$\hat{\mathbf{y}}_{n+1} = \mathbf{y}_n + \frac{h}{1440}(4277\mathbf{f}_n - 7923\mathbf{f}_{n-1} + 9982\mathbf{f}_{n-2} - 7298\mathbf{f}_{n-3} + 2877\mathbf{f}_{n-4} - 475\mathbf{f}_{n-5})$ $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{1440}[475\mathbf{f}(t_{n+1}, \hat{\mathbf{y}}_{n+1}) + 1427\mathbf{f}_n - 798\mathbf{f}_{n-1} + 482\mathbf{f}_{n-2} - 173\mathbf{f}_{n-3} + 27\mathbf{f}_{n-4}]$	[6, pp. 135–138]
7	ABM7	$\hat{\mathbf{y}}_{n+1} = \mathbf{y}_n + \frac{h}{60480}(198721\mathbf{f}_n - 447288\mathbf{f}_{n-1} + 705549\mathbf{f}_{n-2} - 688256\mathbf{f}_{n-3} + 407139\mathbf{f}_{n-4} - 134472\mathbf{f}_{n-5} + 19087\mathbf{f}_{n-6})$ $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{60480}[19087\mathbf{f}(t_{n+1}, \hat{\mathbf{y}}_{n+1}) + 65112\mathbf{f}_n - 46461\mathbf{f}_{n-1} + 37504\mathbf{f}_{n-2} - 20211\mathbf{f}_{n-3} + 6312\mathbf{f}_{n-4} - 863\mathbf{f}_{n-5}]$	[6, pp. 135–138]
8	ABM8	$\hat{\mathbf{y}}_{n+1} = \mathbf{y}_n + \frac{h}{120960}(434241\mathbf{f}_n - 1152169\mathbf{f}_{n-1} + 2183877\mathbf{f}_{n-2} - 2664477\mathbf{f}_{n-3} + 2102243\mathbf{f}_{n-4} - 1041723\mathbf{f}_{n-5} + 295767\mathbf{f}_{n-6} - 36799\mathbf{f}_{n-7})$ $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{120960}[36799\mathbf{f}(t_{n+1}, \hat{\mathbf{y}}_{n+1}) + 139849\mathbf{f}_n - 121797\mathbf{f}_{n-1} + 123133\mathbf{f}_{n-2} - 88547\mathbf{f}_{n-3} + 41499\mathbf{f}_{n-4} - 11351\mathbf{f}_{n-5} + 1375\mathbf{f}_{n-6}]$	[6, pp. 135–138]

## 7.4 Algorithms

These algorithms are written in terms of the  $\mathbf{F}$  matrix discussed in Section 3.5.2.

**Algorithm 30: ABM2**

Propagates the state vector forward one time step using the Adams-Bashforth-Moulton 2nd-order method.

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{F} \in \mathbb{R}^{p \times 3}$  -  $\mathbf{F}$  matrix for current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1. Updates  $\mathbf{F}$  matrix with new function evaluation.

$$\mathbf{F} = [ \mathbf{F}^{(2)} \mid \mathbf{f}(t, \mathbf{F}^{(3)}) \mid \mathbf{F}^{(3)} ]$$

2. Predictor step.

$$\hat{\mathbf{y}} = \mathbf{F}^{(3)} + \frac{h}{2} (3\mathbf{F}^{(2)} - \mathbf{F}^{(1)})$$

3. Corrector step.

$$\mathbf{F}^{(3)} = \mathbf{F}^{(3)} + \frac{h}{2} [ \mathbf{f}(t + h, \hat{\mathbf{y}}) + \mathbf{F}^{(2)} ]$$

**Return:**

- $\mathbf{F} \in \mathbb{R}^{p \times 3}$  -  $\mathbf{F}$  matrix updated for next sample time

**Algorithm 31: ABM3**

Propagates the state vector forward one time step using the Adams-Bashforth-Moulton 3rd-order method.

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{F} \in \mathbb{R}^{p \times 4}$  -  $\mathbf{F}$  matrix for current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1. Updates  $\mathbf{F}$  matrix with new function evaluation.

$$\mathbf{F} = [ \mathbf{F}^{(2:3)} \mid \mathbf{f}(t, \mathbf{F}^{(4)}) \mid \mathbf{F}^{(4)} ]$$

2. Predictor step.

$$\hat{\mathbf{y}} = \mathbf{F}^{(4)} + \frac{h}{12} (23\mathbf{F}^{(3)} - 16\mathbf{F}^{(2)} + 5\mathbf{F}^{(1)})$$

3. Corrector step.

$$\mathbf{F}^{(4)} = \mathbf{F}^{(4)} + \frac{h}{12} [ 5\mathbf{f}(t + h, \hat{\mathbf{y}}) + 8\mathbf{F}^{(3)} - \mathbf{F}^{(2)} ]$$

**Return:**

- $\mathbf{F} \in \mathbb{R}^{p \times 4}$  -  $\mathbf{F}$  matrix updated for next sample time

**Algorithm 32: ABM4**

Propagates the state vector forward one time step using the Adams-Bashforth-Moulton 4th-order method.

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{F} \in \mathbb{R}^{p \times 5}$  -  $\mathbf{F}$  matrix for current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1. Updates  $\mathbf{F}$  matrix with new function evaluation.

$$\mathbf{F} = [ \mathbf{F}^{(2:4)} \mid \mathbf{f}(t, \mathbf{F}^{(5)}) \mid \mathbf{F}^{(5)} ]$$

2. Predictor step.

$$\hat{\mathbf{y}} = \mathbf{F}^{(5)} + \frac{h}{24} (55\mathbf{F}^{(4)} - 59\mathbf{F}^{(3)} + 37\mathbf{F}^{(2)} - 9\mathbf{F}^{(1)})$$

3. Corrector step.

$$\mathbf{F}^{(5)} = \mathbf{F}^{(5)} + \frac{h}{24} [9\mathbf{f}(t+h, \hat{\mathbf{y}}) + 19\mathbf{F}^{(4)} - 5\mathbf{F}^{(3)} + \mathbf{F}^{(2)}]$$

**Return:**

- $\mathbf{F} \in \mathbb{R}^{p \times 5}$  -  $\mathbf{F}$  matrix updated for next sample time

**Algorithm 33: ABM5**

Propagates the state vector forward one time step using the Adams-Bashforth-Moulton 5th-order method.

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{F} \in \mathbb{R}^{p \times 6}$  -  $\mathbf{F}$  matrix for current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1. Updates  $\mathbf{F}$  matrix with new function evaluation.

$$\mathbf{F} = [ \mathbf{F}^{(2:5)} \mid \mathbf{f}(t, \mathbf{F}^{(6)}) \mid \mathbf{F}^{(6)} ]$$

2. Predictor step.

$$\hat{\mathbf{y}} = \mathbf{F}^{(6)} + \frac{h}{720} (1901\mathbf{F}^{(5)} - 2774\mathbf{F}^{(4)} + 2616\mathbf{F}^{(3)} - 1274\mathbf{F}^{(2)} + 251\mathbf{F}^{(1)})$$

3. Corrector step.

$$\mathbf{F}^{(6)} = \mathbf{F}^{(6)} + \frac{h}{720} \left[ 251\mathbf{f}(t+h, \hat{\mathbf{y}}) + 646\mathbf{F}^{(5)} - 264\mathbf{F}^{(4)} + 106\mathbf{F}^{(3)} - 19\mathbf{F}^{(2)} \right]$$

**Return:**

- $\mathbf{F} \in \mathbb{R}^{p \times 6}$  -  $\mathbf{F}$  matrix updated for next sample time

#### Algorithm 34: ABM6

Propagates the state vector forward one time step using the Adams-Bashforth-Moulton 6th-order method.

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{F} \in \mathbb{R}^{p \times 7}$  -  $\mathbf{F}$  matrix for current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1. Updates  $\mathbf{F}$  matrix with new function evaluation.

$$\mathbf{F} = [ \mathbf{F}^{(2:6)} \mid \mathbf{f}(t, \mathbf{F}^{(7)}) \mid \mathbf{F}^{(7)} ]$$

2. Predictor step.

$$\hat{\mathbf{y}} = \mathbf{F}^{(7)} + \frac{h}{1440} \left( 4277\mathbf{F}^{(6)} - 7923\mathbf{F}^{(5)} + 9982\mathbf{F}^{(4)} - 7298\mathbf{F}^{(3)} + 2877\mathbf{F}^{(2)} - 475\mathbf{F}^{(1)} \right)$$

3. Corrector step.

$$\mathbf{F}^{(7)} = \mathbf{F}^{(7)} + \frac{h}{1440} \left[ 475\mathbf{f}(t+h, \hat{\mathbf{y}}) + 1427\mathbf{F}^{(6)} - 798\mathbf{F}^{(5)} + 482\mathbf{F}^{(4)} - 173\mathbf{F}^{(3)} + 27\mathbf{F}^{(2)} \right]$$

**Return:**

- $\mathbf{F} \in \mathbb{R}^{p \times 7}$  -  $\mathbf{F}$  matrix updated for next sample time

#### Algorithm 35: ABM7

Propagates the state vector forward one time step using the Adams-Bashforth-Moulton 7th-order method.

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{F} \in \mathbb{R}^{p \times 8}$  -  $\mathbf{F}$  matrix for current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1. Updates  $\mathbf{F}$  matrix with new function evaluation.

$$\mathbf{F} = [ \mathbf{F}^{(2:7)} \mid \mathbf{f}(t, \mathbf{F}^{(8)}) \mid \mathbf{F}^{(8)} ]$$

2. Predictor step.

$$\begin{aligned} \hat{\mathbf{y}} = \mathbf{F}^{(8)} + \frac{h}{60480} & \left( 198721\mathbf{F}^{(7)} - 447288\mathbf{F}^{(6)} + 705549\mathbf{F}^{(5)} \right. \\ & - 688256\mathbf{F}^{(4)} + 407139\mathbf{F}^{(3)} - 134472\mathbf{F}^{(2)} \\ & \left. + 19087\mathbf{F}^{(1)} \right) \end{aligned}$$

3. Corrector step.

$$\begin{aligned} \mathbf{F}^{(8)} = \mathbf{F}^{(8)} + \frac{h}{60480} & \left[ 19087\mathbf{f}(t+h, \hat{\mathbf{y}}) + 65112\mathbf{F}^{(7)} - 46461\mathbf{F}^{(6)} \right. \\ & \left. + 37504\mathbf{F}^{(5)} - 20211\mathbf{F}^{(4)} + 6312\mathbf{F}^{(3)} - 863\mathbf{F}^{(2)} \right] \end{aligned}$$

**Return:**

- $\mathbf{F} \in \mathbb{R}^{p \times 8}$  -  $\mathbf{F}$  matrix updated for next sample time

**Algorithm 36: ABM8**

Propagates the state vector forward one time step using the Adams-Bashforth-Moulton 8th-order method.

**Given:**

- $\mathbf{f}(t, \mathbf{y})$  - multivariate, vector-valued function ( $\mathbf{f} : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ ) defining the ODE  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$
- $t \in \mathbb{R}$  - current sample time
- $\mathbf{F} \in \mathbb{R}^{p \times 9}$  -  $\mathbf{F}$  matrix for current sample time
- $h \in \mathbb{R}$  - step size

**Procedure:**

1. Updates  $\mathbf{F}$  matrix with new function evaluation.

$$\mathbf{F} = [ \mathbf{F}^{(2:8)} \mid \mathbf{f}(t, \mathbf{F}^{(9)}) \mid \mathbf{F}^{(9)} ]$$

2. Predictor step.

$$\begin{aligned} \hat{\mathbf{y}} = \mathbf{F}^{(9)} + \frac{h}{120960} & \left( 434241\mathbf{F}^{(8)} - 1152169\mathbf{F}^{(7)} + 2183877\mathbf{F}^{(6)} \right. \\ & - 2664477\mathbf{F}^{(5)} + 2102243\mathbf{F}^{(4)} - 1041723\mathbf{F}^{(3)} \\ & \left. + 295767\mathbf{F}^{(2)} - 36799\mathbf{F}^{(1)} \right) \end{aligned}$$

3. Corrector step.

$$\begin{aligned}\mathbf{F}^{(9)} = \mathbf{F}^{(9)} + \frac{h}{120960} & \left[ 36799\mathbf{f}(t+h, \hat{\mathbf{y}}) + 139849\mathbf{F}^{(8)} - 121797\mathbf{F}^{(7)} \right. \\ & + 123133\mathbf{F}^{(6)} - 88547\mathbf{F}^{(5)} + 41499\mathbf{F}^{(4)} \\ & \left. - 11351\mathbf{F}^{(3)} + 1375\mathbf{F}^{(2)} \right]\end{aligned}$$

**Return:**

- $\mathbf{F} \in \mathbb{R}^{p \times 9}$  -  $\mathbf{F}$  matrix updated for next sample time

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