Math Final Summary Note

Definite Integral

$$\int_a^b f(x)\mathrm{d}x$$

$$\int_a^b -f(x)\mathrm{d}x = -\int_a^b f(x)\mathrm{d}x$$

Given $f(x)$ defined on $[a,b]$ and $\exists s \in [a,b]$,

$$\int_a^b f(x)\mathrm{d}x = \int_a^s f(x)\mathrm{d}x + \int_s^b f(x)\mathrm{d}x$$

Riemann Sum/Integral

- Given f defined on $[a,b]$
    - n subdivisions of equal length
    - approximate each vertical strip using a rectangle
    - $\sum$ Area of strip
    - $n \to \infty$
- f is continuous on $[a,b]$, consider $\int_a^b f(x)\mathrm{d}x$
    - n strips: $\Delta x = \frac{b-a}{n}$
        - $[x_0,x_1],[x_1,x_2],\ldots,[x_{k-1},x_k],..,[x_{n-1},x_n]$
    - consider sample point $x_k^* \in [x_{k-1},x_k]$ and build rectangle with height $f(x_k^*)$, and area $\Delta x \cdot f(x_k^*)$
    - $$S_n = \Delta x \sum_{k=1}^{n} f(x_k^*)$$
    - Possible x_k^*
        - Right Riemann sum: $x_k^* = x_k = a + k\Delta x$
            - $$R_n = \Delta x \sum_{k=1}^{n} f(x_k)$$
        - Left Riemann sum: $x_k^* = x_{k-1} = a + (k-1)\Delta x$

- $$L_n = \Delta x \sum_{k=1}^{n} f(x_{k-1})$$
- Midpoint Riemann sum: $x_k^* = \frac{x_{k-1}+x_k}{2}$
  - $$M_n = \Delta x \sum_{k=1}^{n} f(x_k^M)$$
- Trapezoidal Riemann sum
  - $$T_n = \Delta x \sum_{k=1}^{n} \frac{f(x_{k-1}) + f(x_k)}{2}$$

Properties of Integral

Definition of Definite Integral

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \int_a^b f(x)\mathrm{d}x$$

- If the limit exists, the limit takes the same value $\forall x_k^* \in [x_{k-1}, x_k]$

When does $\lim\limits_{n\to\infty} S_n$ exist?

- $f(x)$ is defined on $[a, b]$
  - continuous on $[a, b]$, or,
  - finite number of jump discontinuities
- $f(x)$ is integrable on $[a, b]$

Properties of the Definite Integral

$$\int_a^b [f(x) \pm g(x)]\mathrm{d}x = \int_a^b f(x)\mathrm{d}x \pm \int_a^b g(x)\mathrm{d}x$$

$$\int_a^b k\mathrm{d}x = k(b-a)$$

$$\int_a^b [Af(x) \pm Bg(x)]\mathrm{d}x = A\int_a^b f(x)\mathrm{d}x \pm B\int_a^b g(x)\mathrm{d}x$$

$$\int_a^b -f(x)\mathrm{d}x = -\int_a^b f(x)\mathrm{d}x$$

$$\int_a^a f(x)\mathrm{d}x = 0$$

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$$

Properties of Summation

$$\sum_{i=1}^{n} k \cdot x_i = k \sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n}(x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$$

$$\sum_{i=a}^{n} k = k(n - a + 1)$$

$$\sum_{i=1}^{n} = \frac{n(n+1)}{2}$$

Reverse Engineering

- Find Δx
- $\deg(i) = \deg(n)$
- Guess a and b
- Guess f
- $$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5} = \int_a^{a+1} (x-a)^4 dx$$

Fundamental Theorem of Calculus

Definition of Integral Function

Given **continuous** function f on $[a, b]$, $\forall x \in [a, b]$, let

$$F(x) = \int_a^x f(t)\mathrm{d}t$$

Function composition

$$F(g(x)) = \int_a^{g(x)} f(t)\mathrm{d}t$$

Derivatives

$$F'(x) = f(t)$$

$$[F(g(x))]' = f(g(x)) \cdot g'(x)$$

Fundamental Theorem of Calculus

Let f be continuous on I, $\exists a \in I$

- Part 1: Define $F(x) = \int_a^x f(t)\mathrm{d}t$ on $I \to F'(x) = f(x)$ on I
- Part 2: G be any antiderivative of f on I, then $\forall b \in I$, $\int_a^b f(t)\mathrm{d}t = G(b) - G(a)$

Area Between Curves

If $f(x)$ and $g(x)$ are continuous, and $f(x) \geq g(x)$ on $[a, b]$, then the area between curve is

$$Area = \int_a^b [f(x) - g(x)]\mathrm{d}x$$

Depending on the situation, it could be easier to do $\int[f(y) - g(y)]\mathrm{d}y$

Also make sure the different intervals of different inequality relationships

Modelling with ODEs

- $\frac{\mathrm{d}y}{\mathrm{d}t} = ay + b$
- separable if $y' = g(y)f(t)$
- if separable, $\int \frac{1}{g(y)}\mathrm{d}y = \int f(t)\mathrm{d}t$
- IVP: ODE with initial conditions
- Solving an ODE is to find the functions that satisfy the ODE

Techniques of Integration

Substitution - Reverse Chain Rule

Assume: $f(x)$ and $g(x)$ are continuous, $f(g(x))$ is defined. If $u = g(x)$, then

$$\int f'g(x) \cdot g'(x)\mathrm{d}x = \int f'(u)\mathrm{d}x$$

$$\int_a^b f'g(x)\cdot g'(x)\mathrm{d}x = \int_{g(a)}^{g(b)} f'(u)\mathrm{d}x$$

Integration by Parts - Reversing Product Rule

If u and v are differentiable,

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u, \int_a^b u\mathrm{d}v = uv\Big|_a^b - \int_a^b v\mathrm{d}u$$

where $\mathrm{d}v = \frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x$, and $\mathrm{d}u = \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$.

Make sure $\int v\mathrm{d}u$ can be computed with existing techniques

Choosing u and v

- easy to either differentiate or integrate
    - $\ln(x)$ is easy to differentiate, not to integrate
    - $\arctan(x)$ is easy to differentiate, not to integrate
    - $\frac{1}{1+x^2}$ is easy to integrate, not to differentiate
    - $e^x, \sin(x), \cos(x), \ldots$

Partial Fraction

$$\int \frac{a}{bx+c}\mathrm{d}x = \frac{a}{b}\ln|bx+c| + C$$

$$\text{Case 1}: \Delta > 0, \int \frac{dx+e}{ax^2+bx+c}\mathrm{d}x = \int (\frac{A}{x-m} + \frac{B}{x-n})\mathrm{d}x$$

$$\text{Case 2}: \Delta = 0, \int \frac{dx+e}{ax^2+bx+c}\mathrm{d}x = \int [\frac{A}{x-m} + \frac{B}{(x-m)^2}]\mathrm{d}x$$

$$\text{Case 3}: \Delta < 0, \text{ complete the square for denominator}$$

$$\frac{P(x)}{(x-r)(x-s)(x-t)} = \frac{A}{x-r} + \frac{B}{x-s} + \frac{C}{x-t}, A = \frac{P(r)}{(r-s)(r-t)}, etc$$

$$\frac{P(x)}{(x-r)(x^2+bx+c)} = \frac{A}{x-r} + \frac{B}{x^2+bx+c}$$

$$\text{By long division, } \frac{P(x)}{Q(x)} = s(x) + \frac{r(x)}{Q(x)}, \text{ if } \deg(P(x)) > \deg(Q(x))$$

Trig Sub

- Trig Integrals
    - $\int \sin(x)\mathrm{d}x = -\cos(x) + C$
    - $\int \cos(x)\mathrm{d}x = \sin(x) + C$
    - $\int \sec^2(x)\mathrm{d}x = \tan(x) + C$
    - $\int \sec(x)\tan(x)\mathrm{d}x = \sec(x) + C$
    - $\int \tan(x)\mathrm{d}x = -\ln|\cos(x)| + C$
    - $\int \sec(x)\mathrm{d}x = \ln|\sec(x) + \tan(x)| + C$
- Trig Identities
    - $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$
    - $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
    - $\sin(2x) = 2\sin(x)\cos(x)$
    - $\cos(2x) = \cos^2(x) - \sin^2(x)$
    - $\sin^2(x) + \cos^2(x) = 1$
    - Prosthaphaeresis
        - $\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
        - $\sin(\alpha)\cos(\beta) = \frac{1}{2}[\sin(\alpha+\beta)\sin(\alpha-\beta)]$
    - $\sec^2(x) = 1 + \tan^2(x)$
- $\int \sin^n(x)\cos^m(x)\mathrm{d}x$
    - n odd, $u = \cos(x) \to -\int(1+u^2)^{\frac{n-1}{2}}u^m\mathrm{d}u$
    - m odd, $u = \sin(x) \to \int u^n(1-u^2)^{\frac{m-1}{2}}\mathrm{d}u$
    - m and n even, integration of sum of even power of $\sin(x)$
- $\int \sec^m(x)\tan^n(x)\mathrm{d}x$
    - $\int \sec^2(x)\mathrm{d}x = \tan(x) + C$
    - $\int \sec(x)\tan(x)dx = \sec(x) + C$
- Trig sub
    - $\sqrt{a^2 - x^2} \to x = a\sin(\theta)$
    - $\sqrt{a^2 + x^2} \to x = a\tan(\theta)$

Solids

$$\text{Volume} = \lim_{n\to\infty}\sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\mathrm{d}x$$

Washer: $L \perp$ axis of rotation

$$\text{Area} = \pi(y_2^2 - y_1^2)$$

$$\text{Volume} = \int_a^b \pi[(f(x))^2 - (g(x))^2]\mathrm{d}x$$

Cylinder: $L \parallel$ axis of rotation

$$\text{Area} = 2\pi x(y_2 - y_1)$$

$$\text{Volume} = \int_a^b 2\pi x[f(x) - g(x)]\mathrm{d}x$$

Improper Integrals

Type I

Let f be continuous on $[a, \infty)$, or $(-\infty, a]$

$$\int_a^\infty f(x)\mathrm{d}x = \lim_{R\to\infty}\int_a^R f(x)\mathrm{d}x, \int_{-\infty}^a f(x)\mathrm{d}x = \lim_{R\to-\infty}\int_R^a f(x)\mathrm{d}x$$

- If the limit exists, the integral is convergent
- If the limit DNE, the integral is divergent, if $lim = \pm\infty$, it diverges to $\pm\infty$
- If f is continuous on $[d, \infty)$, s.t. $a \in [d, \infty)$ and $b \in [d, \infty)$, $\int_a^\infty f(x)\mathrm{d}x$ converges $\iff \int_b^\infty f(x)\mathrm{d}x$ converges, since $\int_a^\infty f(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^\infty f(x)\mathrm{d}x$

Type II

If f is continuous on $(a, b]$ or $[a, b)$, and $c \in (a, b]$ or $c \in [a, b)$,

$$\int_a^b f(x)\mathrm{d}x = \lim_{c\to a^+}\int_c^b f(x)\mathrm{d}x, \int_a^b f(x)\mathrm{d}x = \lim_{c\to b^-}\int_a^c f(x)\mathrm{d}x$$

Comparison Theorem

$-\infty \le a \le b \le \infty$, assume f and g are continuous on (a, b), and $\forall x \in (a, b)$, $0 \le f(x) \le g(x)$,

- If $\int_a^b g(x)\mathrm{d}x$ is convergent, $\int_a^b f(x)\mathrm{d}x$ is convergent
- If $\int_a^b f(x)\mathrm{d}x$ is divergent (to ∞), $\int_a^b g(x)\mathrm{d}x$ is divergent (to ∞)

Continuous Probability

Definition: A continuous random variable X is an object that records **outcome** of an experiment as one of a continuous set of values

$p(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f(x)\mathrm{d}x$

$f(x)$: Probability Density Function (PDF)

- $f(x) \ge 0$
- $\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$

Statistical Tools:

- Mean: $\mu = \int_{-\infty}^{\infty} xf(x)\mathrm{d}x$
- Variance: $\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)f(x)\mathrm{d}x$
- Standard deviation: $\sigma = \sqrt{\sigma^2}$
- Expectation: $E(X = x) = \int_{-\infty}^{\infty} xf(x)\mathrm{d}x$
- $\sigma^2 = E(x^2) - \mu^2$

Distributions

- Uniform
    - $f(x) = \frac{1}{b-a}, \forall x \in [a, b]$, $f(x) = 0$, otherwise
    - $\mu = \frac{a+b}{2}$, $\sigma^2 = \frac{a^2+ab+b^2}{3} - (\frac{a+b}{2})^2$
- Exponential
    - $f(x) = ke^{-kx}, \forall x \in [0, \infty)$, $f(x) = 0$, otherwise
    - $\mu = \sigma = \frac{1}{k}$
- Standard
    - $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

- $\mu = 0,\ \sigma = 1$
- $f_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
- $\int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi}$

Cumulative Density Function

- Given a PDF, define a CDF

$$F(t) = \int_{-\infty}^{t} f(x) \mathrm{d}x$$

$$\lim_{t \to \infty} F(t) = 1$$

- Median
  - Uniform: $\frac{a+b}{2}$
  - Exponential: $\frac{\ln 2}{k}$
  - Standard: 0

$$t \text{ when } F(t) = \frac{1}{2}$$

Work

$$W = \int_{a}^{b} F(r) \mathrm{d}r$$

Can also integrate over time, be flexible regarding which integral to use.

Key to find the force $F(r)$ over a small displacement Δr.

Sequence and Series

Sequence

- A sequence is an ordered list of real numbers with a first element in the list but no last element.
- A sequence is a function where the domain is set of $\mathbb{Z}^+$
- $a_n = f(n), \forall n \in \mathbb{Z}^+$
- A sequence a_n is said to converge to L ($a_n \to L$) if as n gets larger and larger, a_n gets closer and closer to L

- If a_n does not converge, it diverges; if $a_n \to \pm\infty$, then the sequence diverges to $\pm\infty$
- Theorem
    - Suppose $a_n \le c_n \le b_n$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$, then $\lim_{n\to\infty} c_n = L$
- $n^p \le r^n \le n! \le n^n$
- Rigorous Definition
    - $\forall \epsilon > 0, \exists N \in \mathbb{Z}^+$, if $n \ge N$, $|a_n - L| < \epsilon$
- Every convergent sequence is bounded
- Every bounded increasing sequence is convergent

Series

- Infinite series: A formal sum $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$, where $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an infinite sequence, written as
- $$\sum_{n=1}^{\infty} a_n$$
- nth partial sum of a series is
- $$S_n = \sum_{i=1}^{n} a_i$$
- Therefore,
- $$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = S$$
    - In this case, we say the series converges to that limit S
- Geometric Series
    - $$\sum_{n=1}^{\infty} ar^{n-1}$$
    - If $a = 0$, $S = 0$
    - If $|r| < 1$, $S = \frac{a}{1-r}$
    - If $r \ge 1$ and $a > 0$, diverges to ∞
    - If $a < 0$, diverges to $-\infty$
    - If $r = -1$, series diverges
- Telescoping Series

    - $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$
    - Partial sum: $S_n = 1 - \frac{1}{n+1}$
    - Converges to 1
- Harmonic Series
    - $$\sum_{n=1}^{\infty} \frac{1}{n}$$
    - Diverges to ∞
- p-Series
    - $$\sum_{n=1}^{\infty} \frac{1}{n^p}$$
    - If $p > 1$, converges
    - If $p = 1$, harmonic series
    - If $p < 1$, diverges

Tests for Convergence and Divergence

- Divergence Test
    - If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$
- If $a_n \geq 0$ and S_n are bounded, $\sum_n a_n$ converges (S_n is bounded and increasing)
- Integral Test
    - Let f be **continuous**, **non-negative**, and **decreasing**, on some interval $[N, \infty)$
    - Let $a_n = f(n)$
    - Then $\sum_n a_n$ converges $\iff$ $\int_N^\infty f(x)\mathrm{d}x$ converges
- Elementary Comparison Tests
    - Let $\forall n : 0 \leq a_n \leq b_n$
    - If $\sum_n b_n$ converges, $\sum_n a_n$ converges
    - If $\sum_n a_n$ diverges to ∞, $\sum_n b_n$ diverges to ∞
    - If $\sum_n |a_n|$ converges, then $\sum_n a_n$ converges
- Limit Comparison Test
    - Let a_n, b_n be infinite sequences, s.t. each $b_n > 0$, and $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L$, L is finite

- If $\sum_n b_n$ converges, so does $\sum_n a_n$
- If $\sum_n a_n$ diverges and $L \neq 0$, so does $\sum_n b_n$

- Ratio Test
    - $$\lim_{n \to \infty} |\frac{a_{n+1}}{a_n}| = L$$
    - If $L < 1$, the series converges
    - If $L > 1$, the series diverges
    - Otherwise, the series can go either way
- An infinite series $\sum_n a_n$ is absolutely convergent if $\sum_n |a_n|$ converges
    - If $\sum_n a_n$ is absolutely convergent, then $\sum_n a_n$ is convergent, since $\sum_n a_n \leq \sum_n |a_n|$
    - An infinite series is **conditionally convergent**, if $\sum_n a_n$ is convergent but $\sum_n |a_n|$ diverges
- Alternating Series Test
    - Let a_n be a sequence, $\forall a_n \geq 0$, a_n is decreasing, $\lim_{n \to \infty} a_n = 0$
        - Then $\sum_n (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 \ldots$ is convergent

Power Series

Definition: A power series is an object of the form

$$\sum_{n=0}^{\infty} A_n (x-c)^n = A_0 + A_1(x-c) + A_2(x-c)^2 + \ldots$$

A_n is the coefficient, and c is the center

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}, |x| < 1$$

Can differentiate and integrate power series

$$f(x) = \sum_{n=0}^{\infty} A_n (x-c)^n$$

$$f'(x) = \sum_{n=0}^{\infty} n A_n (x-c)^{n-1}$$

$$\int f(x) \mathrm{d}x = C + \sum_{n=0}^{\infty} A_n \frac{(x-c)^{n+1}}{n+1}$$

Let R be the radius of convergence, $A = \lim\limits_{n \to \infty} |\frac{A_{n+1}}{A_n}|$, we have $R = \frac{1}{A}$

- $R = \infty$, converges everywhere
- $R = 0$, converges only at $x = c$, $\sum_{n=0}^{\infty} A_n(x-c)^n = A_0$

Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

If $f(x) = \sum_{n=0}^{\infty} A_n(x-c)^n$ holds true $\forall x$ in some open interval containing c, then that power series is a Taylor Series,

$$A_n = \frac{f^{(n)}(c)}{n!}$$

This does not indicate that $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ holds true throughout the radius of convergence.

Remainder:

$$\exists a \in [c, x], R_n(x) = \frac{f^{(n+1)}(a)}{(n+1)!}(x-c)^{n+1}$$

To prove a power series hold for all x, just prove $\lim_{n \to \infty} R_n(x) = 0$

In a Taylor series, the nth derivative is always obtained from the coefficient of the term x^n.

Function with no analytical solutions to its integral can be expressed in a power series. $\int \frac{\sin(x)}{x} \mathrm{d}x$

Common Taylor Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, |x| < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$