BASIC MULTIVARIABLE CALCULUS - MATH 226 NOTES

基本多元微积分——MATH 226 笔记

Author
Wenyou (Tobias) Tian
田文友
University of British Columbia
英属哥伦比亚大学
2023

Contents

1 Vectors and Coordinate Geometry, 向量与坐标系几何

1.1 Coordinates and Sets, 坐标与集合

1.1.1 Cartesian Coordinates, 笛卡尔坐标系

In $\mathbb{R}^2$, we denote points as (x, y); in $\mathbb{R}^3$, we denote points as (x, y, z).
If $P = (x_1, y_1, z_1), Q = (x_2, y_2, z_2)$, then the *Euclidean* distance between P and Q is

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Typically, 1 equation in $\mathbb{R}^3$ represents a plane or surface, while 2 equations in $\mathbb{R}^3$ represent a curve or line. However, this is not always the case as 2 equations define the intersection of two surfaces, which may not have intersections at all.

1.1.2 Topology, 拓扑

Definition 1.1. If $P \in \mathbb{R}^n$, then a ***neighborhood***（邻域）of P is a "ball" denoted as

$$B_r(P) := \{Q \in \mathbb{R}^n : |PQ| < r\}$$

centered at P, with radius r, for some $r > 0$.
This set DOES NOT contain the surface of the "ball".

Definition 1.2. A set $S \subset \mathbb{R}^n$ is ***open***（开集）if

$$\forall P \in S, \exists r > 0, B_r(P) \subset S$$

S is ***closed***（闭集）if its complement S^C is open.

By this definition, there are sets that are *neither open nor closed*.

Definition 1.3. A point P is on the boundary of a set S if

$$\forall P, \forall r > 0, \exists Q_1 \in S, Q_2 \in S^C, Q_1, Q_2 \in B_r(P)$$

1.2 Vectors and 3D Geometry, 向量与三维几何

Consider a general vector

$$\vec{v} = \langle v_1, v_2, v_3 \rangle = \overrightarrow{PQ}$$

Then with $P = (x_1, y_1, z_1), Q = (x_2, y_2, z_2)$, we have

$$\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle = \langle v_1, v_2, v_3 \rangle$$

We can have vector addition: if we denote $\vec{w} = \langle w_1, w_2, w_3 \rangle$, then

$$\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$$

In $\mathbb{R}^3$, the standard basis consists of 3 vectors

$$\begin{aligned} \vec{i} &= \langle 1, 0, 0 \rangle \\ \vec{j} &= \langle 0, 1, 0 \rangle \\ \vec{k} &= \langle 0, 0, 1 \rangle \end{aligned}$$

Then, we can also write

$$\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$$

The length of a vector is defined to be

$$|\vec{v}| := \sqrt{v_1^2 + v_2^2 + v_3^2}$$

and if $\vec{v} \neq \mathbf{0}$, then the unit vector of $\vec{v}$ is

$$\vec{u} := \frac{\vec{v}}{|\vec{v}|}$$

Given two vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle, \vec{w} = \langle w_1, w_2, w_3 \rangle$, the dot product is defined to be

$$\vec{v} \cdot \vec{w} := v_1 w_1 + v_2 w_2 + v_3 w_3$$

If we further have the angle between these two vectors to be θ, then

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}$$

The two vectors are **perpendicular** if their dot product is 0. $\mathbf{0}$ is **orthogonal**（正交）to every vector.

Let $\vec{a}, \vec{b}$ be two vectors with $\vec{b} \neq \mathbf{0}$, then we can decompose $\vec{a}$ as follows:

1. $\vec{a} = \vec{v} + (\vec{a} - \vec{v})$
2. $\vec{v} \parallel \vec{b}$, $\vec{a} - \vec{v} \perp \vec{b}$

Then $\vec{v}$ is the **vector projection**（向量投影）of $\vec{a}$ on $\vec{b}$, denoted as $\vec{v} = \vec{a}_{\vec{b}}$. The **scalar projection**（标量投影）is thus $\pm|\vec{v}|$.
To calculate this vector projection, we have

$$\vec{v} = \vec{a}_{\vec{b}} = |\vec{a}| \cos\theta \frac{\vec{b}}{|\vec{b}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}$$

where the scalar projection is

$$s = |\vec{v}| = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

so the vector projection can also be

$$\vec{v} = s\frac{\vec{b}}{|\vec{b}|}$$

Given two vectors $\vec{u}, \vec{v}$, the cross product $\vec{u} \times \vec{v} = \vec{w}$ has the properties where

1. $\vec{w} \perp \vec{u}, \vec{v}$
2. $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$
3. $\vec{u}, \vec{v}, \vec{u} \times \vec{v}$ forms a **right-hand triad**

In terms of coordinates,

$$\vec{u} \times \vec{v} := \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

Consider a parallelogram $ABCD$, denote vectors $\overrightarrow{AB}, \overrightarrow{AD}$, then

$$Area_{ABCD} = |\overrightarrow{AB} \times \overrightarrow{AD}|$$

Then the triangle $\triangle ABC$ will have area

$$A_{\triangle ABC} = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$$

Furthermore, if we have a parallelepiped $ABCD-EFGH$, with three vectors $\overrightarrow{FG}, \overrightarrow{FE}, \overrightarrow{FB}$, we have the volume to be

$$V_{ABCD-EFGH} = |(\overrightarrow{FG} \times \overrightarrow{FE}) \cdot \overrightarrow{FB}|$$

1.3 Lines and Planes, 线与面

Planes can be defined in various ways, however, the most convenient way in $\mathbb{R}^3$ is to have a point (x_0, y_0, z_0) on the plane and a vector $\langle A, B, C\rangle$ perpendicular to this plane that gives us the equation of the plane

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

A line, on the other hand, requires a point on the plane (x_0, y_0, z_0), and a direction vector (a, b, c), which for every point (x, y, z) on this line, we should have a system of linear equations satisfied

$$\begin{aligned} x - x_0 &= ta \\ y - y_0 &= tb \\ z - z_0 &= tc \end{aligned}$$

This equivalent to the *symmetric form*,

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t$$

To find the intersection line of two planes, if the two planes are not parallel, then we can **cross product** their normal vectors to find the direction vector of the line. Then with one point on the line, we can get the parametric equation of the line:

$$\langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

1.4 Quadric Surfaces, 二次曲面

A general sphere（球）centered at (x_0, y_0, z_0), with radius r has the equation

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

An ellipsoid（椭球）centered at (x_0, y_0, z_0), with semi-axes a, b, c, has the equation

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} + \frac{(z - z_0)^2}{c^2} = 1$$

A circular cylinder（圆柱）centred about the z-axis with radius r can be defined with

$$x^2 + y^2 = r^2$$

An elliptic cylinder（椭圆柱）centred about the z-axis with semi-axes a, b can be defined with

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

While a parabolic cylinder can be defined as like

$$y = ax^2$$

A circular paraboloid（抛物面）is defined as like

$$z = x^2 + y^2$$

An elliptic paraboloid is defined as like

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

A hyperbolic paraboloid is defined as like

$$z = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

By setting $x = 0, y = 0, z = 0$, many properties of the hyperbolic paraboloid can be found. However, in general, since $z = (\frac{y}{b} - \frac{x}{a})(\frac{y}{b} + \frac{x}{a})$, then given an arbitrary $z \neq 0$, we can have the linear system

$$\begin{aligned} \frac{y}{b} - \frac{x}{a} &= \frac{z}{c} \\ \frac{y}{b} + \frac{x}{a} &= c \\ c &\neq 0 \end{aligned}$$

This is an example of *doubly ruled surface* where every point on a hyperbolic paraboloid is contained by two distinct lines that are contained in the surface.
There are 3 types of hyperboloids（双曲面）, the one-sheet, the two-sheet, and the cone.
One-sheet hyperboloid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Two-sheet hyperboloid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

Cone:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

2 Functions of Several Variables, Limits, Continuity, Partial Derivatives, 多元函数，极限，连续性，偏微分

2.1 Functions and Surfaces, 函数与表面

Functions of several variables in $\mathbb{R}^n$ maps

$$(x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n)$$

where $(x_1, \ldots, x_n) \in \mathcal{D}(f)$. The natural domain of f is the set of $(x_1, \ldots, x_n)$ where f is naturally defined. The range of f is then set of all values of f, where $f(x_1, \ldots, x_n) \in \mathbb{R}$.
The corresponding graph has the set

$$\{(x_1, \ldots, x_n, f(x_1, \ldots, x_n))\}$$

which is a subset of $\mathbb{R}^{n+1}$.
We can visualize high-dimension graphs using level curves. For a function of 2 variables, level curves on the $x - y$ plane is formed by having $f(x, y) = c$, similarly, for a function of 3 variables, level curves in the xyz space is formed by having $f(x, y, z) = c$.

2.2 Limits and Continuity, 极限与连续性

For functions of 2 variables,

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if:

1. Every neighbourhood of (a, b) contains points of $\mathcal{D}(f)$ other than (a, b)
2. $\forall \epsilon > 0, \exists \delta > 0, (x, y) \in \mathcal{D}(f) \wedge 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x, y) - L| < \epsilon$

The rules for limits from single-variable calculus still apply.

Definition 2.1. $f(x, y)$ is continuous at (a, b) if:
1. $(a, b) \in \mathcal{D}(f)$,
2.

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

Theorem 2.1. The Squeeze Theorem for multivariable calculus states that:
If f, g, h are defined on some $B_r(a, b)$, except at (a, b) and

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{(x,y)\to(a,b)} h(x, y) = L$$

if g lies between f, h on that neighbourhood, then

$$\lim_{(x,y)\to(a,b)} g(x, y) = L$$

For a multivariable function to have a limit at a certain point, then whichever path we choose to approach this point, the limit should all be the same; thus, if for two paths we choose to approach this point, we evaluated to have different limits, the limit at this point does not exist.
For functions that do have a limit at a point, we usually use rules of limits and the Squeeze Theorem to evaluate such a limit, $\epsilon - \delta$ proof is usually not required.
Note the useful inequality

$$|a+b| \leq |a| + |b|$$

2.3 Partial Derivatives, 偏微分

Definition 2.2. $f_i(x_1, \ldots, x_n)$ is the partial derivative with respect to x_i with $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ fixed, where

$$f_i(x_1, \ldots, x_n) = \lim_{h \to 0} \frac{1}{h}(f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_n))$$

Notation-wise, we have

$$f_1(x,y) = f_x(x,y) = \frac{\partial f(x,y)}{\partial x} = D_1(x,y) = D_x(x,y)$$

Different from single-variable calculus, if f_1, f_2 exists, this does not imply that f is continuous at (x,y).
A function being differentiable at a point implies that we can approximate $f(x,y)$ by a linear function.
Given Cantor lines, we can also estimate partial derivatives.

2.4 Tangent Planes, 切面

Given a function f, if $f_x(P), f_y(P)$ exist and f is continuous on a neighbourhood of P, then a plane tangent to f at P exists.
For two variables, where $z = f(x,y)$, a tangent plane at (a,b) can be calculated to be

$$z = f(a,b) + f_1(x,y)(x-a) + f_2(x,y)(y-b)$$

The line through P that is perpendicular to the surface has the direction vector

$$\vec{n} = \langle -f_1(x,y), -f_2(x,y), 1 \rangle$$

2.5 Higher Order Derivatives, 高阶导数

Notation-wise, consider the following equality

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{21}(x,y) = f_{yx}(x,y)$$

If we assume $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are continuous at $P = (a, b)$, and $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are continuous on neighbourhoods of P, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

A *partial differential equation* is an equation involving the partial derivatives of some function. At this current stage, we can verify some function to be the solution to a PDE.

2.6 Chain Rule, 链式法则

If $f(x_1, \ldots, x_n)$ is C^k, then f and its partial derivatives up to order k are continuous on $\mathcal{D}(f)$.

Recall the single-variable chain rule, consider a function $f(x, y)$, where $x = x(t)$, and $y = y(t)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x(t), y(t)) = \frac{\partial f}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t}$$

For variables, that is, if $x = x(u, v), y = y(u, v)$, then

$$\frac{\partial}{\partial u} f(x, y) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial}{\partial v} f(x, y) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

Moreover, if we are differentiating with respect to a variable where $x = x(t), y = y(t), f(x, y, t)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x, y, t) = \frac{\partial f}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial t}$$

For higher-order partials, we iterate the chain rule.

3 Topics in Differentiation, 微分

3.1 Linear Approximation and Differentiability, 线性逼近与可微

For $f(x, y)$, if we approximate the surface near (a, b) with a linear function $L(x, y)$, then this linear function has the form

$$z = L(x, y) = f(a, b) + f_1(x, y)(x - a) + f_2(x, y)(y - b)$$

Then, f is differentiable at (a, b) if

$$f(a, b) - L(a, b) = o(\sqrt{(x - a)^2 + (y - b)^2})$$

which is equivalent to

$$\lim_{(x,y)\to(a,b)} \frac{|f(x, y) - L(x, y)|}{\sqrt{(x - a)^2 + (y - b)^2}} = 0$$

Furthermore, if $f(x, y)$ is differentiable at (a, b), then $f(x, y)$ is continuous at (a, b), and f_1, f_2 exists at (a, b).
If $f(x, y)$ is C^1 on a neighbourhood of (a, b), then it is differentiable at (a, b).
The linear approximation can be interpreted as **differentials**（微分）where if $z = f(x, y)$ then

$$\mathrm{d}z = \mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$$

We then can prove whether or not a function is differentiable at a point from the definition.

3.2 Gradients and Directional Derivatives, 梯度与方向导数

Given a function $f(x_1, \dots, x_n)$, then the ***gradient*** of the function is a vector where

$$\nabla f = \langle f_1, \dots, f_n \rangle$$

In general, ∇f is always perpendicular/normal to the level curve/surface of f, that is $\nabla f \cdot \vec{T} = 0$ To find the direction derivative of f, we first normalize the direction vector to $\vec{u} = \langle u_1, u_2 \rangle$, where $|\vec{u}| = 1$, then if f is differentiable at (a, b), we have

$$D_{\vec{u}} f(x, y)|_{(x,y)=(a,b)} = \nabla f(a, b) \cdot \vec{u}$$

In general, we define such a derivative with a limit definition where,

$$D_{\vec{u}} f(x, y)|_{(x,y)=(a,b)} := \lim_{h\to 0} \frac{1}{h}(f(a + hu_1, b + hu_2) - f(a, b))$$

To find those directions with the steepest ascent/descent, we want to maximize/minimize $|D_{\vec{u}} f(x, y)|_{(x,y)=(a,b)}|$, since it is equal to $\nabla f \cdot \vec{u} = |\nabla f||\vec{u}| \cos\theta$,

1. To maximize, we choose $\theta = 0$, where $\vec{u} = \frac{\nabla f}{|\nabla f|}$
2. To minimize, we choose $\theta = \pi$, where $\vec{u} = -\frac{\nabla f}{|\nabla f|}$

3.3 Implicit Differentiation, 隐函数求导

Given equation $F(x, y) = c$, if we define y as a function of x, find $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}}{\mathrm{d}x}c = \frac{\mathrm{d}F}{\mathrm{d}x} = F_1(x, y(x)) + F_2(x, y(x))\frac{\mathrm{d}y}{\mathrm{d}x}$$

Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_1}{F_2}$$

If $F_2 = 0$, then either $\nabla F_1(P) = F_1(P)i$, or the tangent line is vertical.
Given equation $F(x, y, z) = c$, if we define z as a function of x, y, find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$:

$$\frac{\partial}{\partial x}c = \frac{\partial F}{\partial x} = F_1(x, y, z) + F_3(x, y, z)\frac{\partial z}{\partial x}$$

$$\frac{\partial}{\partial y}c = \frac{\partial F}{\partial y} = F_1(x, y, z) + F_3(x, y, z)\frac{\partial z}{\partial y}$$

Then

$$\frac{\partial z}{\partial x} = -\frac{F_1}{F_3}$$

$$\frac{\partial z}{\partial y} = -\frac{F_2}{F_3}$$

If $F_3 = 0$, then there is a vertical plane, and the result is inconclusive.

Given a system of equations

$$u = f(x, y)$$
$$v = g(x, y)$$

Find $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$.
Assume $x = x(u, v), y = y(u, v)$, then

$$\frac{\partial}{\partial u}u = \frac{\partial}{\partial u}f(x, y)$$
$$\frac{\partial}{\partial v}u = \frac{\partial}{\partial v}f(x, y)$$

This gives us

$$1 = f_1\frac{\partial x}{\partial u} + f_2\frac{\partial y}{\partial u}$$
$$0 = g_1\frac{\partial x}{\partial u} + g_2\frac{\partial y}{\partial u}$$

Then by either solving directly or using Cramer's Rule, we can find $\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}$, with Cramer's Rule, we have

$$\frac{\partial x}{\partial u} = \frac{\begin{vmatrix} 1 & f_2 \\ 0 & g_2 \end{vmatrix}}{\begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}}, \frac{\partial y}{\partial u} = \frac{\begin{vmatrix} f_1 & 1 \\ g_1 & 0 \end{vmatrix}}{\begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}}$$

Then, a 2×2 Jacobian is written as

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}$$

3.4 Taylor Polynomial, 泰勒多项式

The first-order Taylor approximation is the linear approximation mentioned previously, thus, we focus on the second-order Taylor approximation, where we approximate surfaces using quadric surfaces.

$$\begin{aligned} p_2(x,y) = f(a,b) \\ &+ (f_1(a,b)(x-a) + f_2(a,b)(y-b)) \\ &+ \frac{1}{2}(f_{11}(a,b)(x-a)^2 + f_{12}(a,b)(x-a)(y-b) + f_{21}(a,b)(x-a)(y-b) + f_{22}(a,b)(y-b)^2) \end{aligned}$$

Then with the *Hessian matrix*:

$$\mathcal{H}f(a,b) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

The second-order Taylor polynomial becomes

$$p_2(x,y) = f(a,b) + \nabla f(a,b) \cdot \langle x-a, y-b \rangle + \frac{1}{2} \begin{pmatrix} x-a & y-b \end{pmatrix} \mathcal{H}f(a,b) \begin{pmatrix} x-a \\ y-b \end{pmatrix}$$

For error estimation of a second-order Taylor polynomial, we would have

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - p_2(x,y)}{(\sqrt{(x-a)^2 + (y-b)^2})^2} = 0$$

For higher dimensions, suppose $f = f(x_1, \ldots, x_n)$ in a neighbourhood of $\mathbf{a} = (a_1, \ldots, a_n)$, then we have the Taylor polynomial to be

$$f(\mathbf{x}) \approx p_2(\mathbf{x}) := f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a})\mathcal{H}f(\mathbf{a})(\mathbf{x} - \mathbf{a})^T$$

where

$$\mathcal{H}f(\mathbf{a}) = \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix}$$

4 Critical Points and Extreme Values, 关键点与极值

4.1 Critical Points and Local Extrema, 关键点与极值

Let $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}^n$,

1. A local minimum at $\mathbf{a}$ indicates:

$$\forall \mathbf{x} \in B_r(\mathbf{a}), f(\mathbf{x}) \geq f(\mathbf{a})$$

2. A local maximum at $\mathbf{a}$ indicates:

$$\forall \mathbf{x} \in B_r(\mathbf{a}), f(\mathbf{x}) \leq f(\mathbf{a})$$

If it is global extrema, we replace the neighbourhood of $\mathbf{a}$ to be the given domain $\mathcal{D}$

The necessary conditions for local minimum and maximum:

At least one of the following happens:
1. $\nabla f(\mathbf{a}) = \mathbf{0}$, "critical point"
2. $\nabla f(\mathbf{a})$ DNE, "critical/singular point"
3. $\mathbf{a}$ is a boundary point of the domain

There are 3 types of critical points:

1. Local minimums
2. Local maximums
3. Neither, a saddle point

Now consider the second-order Taylor polynomial for f, denote

$$Q(\mathbf{x}) = \frac{1}{2}(\mathbf{x} - \mathbf{a})\mathcal{H}f(\mathbf{a})(\mathbf{x} - \mathbf{a})^T$$

Thus,

1. If $Q(x) > 0$ for all $\mathbf{x} \neq \mathbf{a}$, then $f(\mathbf{a})$ is a local minimum: positive definite
2. If $Q(x) < 0$ for all $\mathbf{x} \neq \mathbf{a}$, then $f(\mathbf{a})$ is a local maximum: negative definite
3. If $Q(x) > 0$ and $Q(x) < 0$ both occur, then given $\det(\mathcal{H}f) \neq 0$, then it is indefinite, there is a saddle point.
4. If $\det(\mathcal{H}f) = 0$, then this test is inconclusive.

Consider the Hessian matrix,

$$\mathcal{H}f(\mathbf{a}) = \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix}$$

Let

$$
\begin{aligned}
D_1 &= \begin{vmatrix} f_{11} \end{vmatrix} \\
D_2 &= \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} \\
&\vdots \\
D_n &= \det(\mathcal{H}f(\mathbf{a}))
\end{aligned}
$$

Then,

1. If $D_1 > 0, D_2 > 0, \ldots, D_n > 0$, then it is positive definite, giving us a local minimum;
2. If $D_1 < 0, D_2 > 0, D_3 < 0, \ldots,$ then it is negative definite, giving us a local maximum;
3. If $D_n \neq 0$ and any other pattern then it is indefinite, giving us a saddle point;
4. If $D_n = 0$, the test is inconclusive.

4.2 Global Extrema on Restricted Domains, 限定定义域上的最值

Given a domain $X \subset \mathbb{R}^n$, it is ***compact*** if X is **bounded** and **closed**.
If f is continuous on X, and X is compact, then f attains a minimum or maximum on X.
The procedure to find the minimum and maximum on these compact domains:

1. Find all critical and singular points, and evaluate the function at those points.
2. Find the minimum and maximum on the boundary
3. Choose the largest/smallest value

There is no need to classify.

If the region is not compact, then, an additional step is needed to justify global extremes, that is finding

$$\lim_{(x,y)\to(\pm\infty,\pm\infty)} f(x,y)$$

Depending on the limit, there may or may not be global extrema.

4.3 Lagrange Multiplier, 拉格朗日乘数

We use Lagrange multipliers when we need to maximize or minimize $f(\mathbf{x})$ subject to a constraint $g(\mathbf{x}) = c$.
Assume f, g are both C^1 on some open set containing P, and $\nabla g(P) \neq 0$. Let $\mathcal{C} = \{(x, y) : g(x, y) = c\}$, if f restricted to $\mathcal{C}$ has a local minimum/maximum at P, then

$$\nabla f(P) = \lambda \nabla g(P)$$

Thus, the procedure to minimize or maximize f on $\mathcal{C}$:

1. Find $\nabla f = \lambda \nabla g$
2. Find $\nabla g = 0$
3. Find ∇f or ∇g DNE.

4. Evaluate at these points and end points of $\mathcal{C}$
5. Choose the largest/smallest value accordingly

5 Integration in Several Variables, 多元积分

5.1 Double Integrals, 双重积分

Let $f(x, y)$ defined on some $\mathcal{D}$, consider a partition $\mathcal{P}$ over $\mathcal{D}$, that is a collection of rectangles $\{R_{ij}\}$ with the choice of points P_{ij}^*, then the ***Riemann sum*** associated with f and the partition $\mathcal{P}$ is

$$\mathcal{R}(f, \mathcal{P}) = \sum_{i=0}^{m} \sum_{j=0}^{n} f(P_{ij}^*) \Delta A_{ij}$$

where $\Delta A_{ij} = \Delta x_i \Delta y_j$.
f is then **integrable** on $\mathcal{D}$ if

$$\lim_{\text{diam}(\mathcal{P}) \to 0} \mathcal{R}(f, \mathcal{P}) = L$$

where

$$\text{diam}(\mathcal{P}) = \max \sqrt{\Delta x_i^2 + \Delta y_j^2}$$

We then write the double integral as

$$\iint_{\mathcal{D}} f(x, y) \mathrm{d}A$$

The integral exists when:

1. f is continuous on $\mathcal{D}$.
2. f is continuous except for a finite number of curves of finite length.
3. f is bounded on $\mathcal{D}$ and its set of discontinuities has **Jordan area** of 0. This is saying the upper bound and the lower bound converge to the same area.

For more general domains, assume f is defined on $X \subset D$, where X is bounded and closed, then if D is a rectangle, we can extend f to $\tilde{f}$ on D, where

$$\tilde{f}(x, y) = \begin{cases} f(x, y) & (x, y) \in X \\ 0 & (x, y) \in D - X \end{cases}$$

Then

$$\iint f(x, y) \mathrm{d}A = \iint \tilde{f}(x, y) \mathrm{d}A$$

For iterated integrals, let $f(x, y)$ be bounded on a closed and bounded rectangle, if f is integrable on this rectangle, then

$$\iint_D f(x, y) \mathrm{d}A = \int_a^b \int_c^d f(x, y) \mathrm{d}y \mathrm{d}x = \int_c^d \int_a^b f(x, y) \mathrm{d}x \mathrm{d}y$$

If one variable can be written as a function in another variable, then we could have

$$\int_a^b \int_{c(x)}^{d(x)} f(x, y) \mathrm{d}y \mathrm{d}x$$

$$\int_c^d \int_{a(y)}^{b(y)} f(x,y)\mathrm{d}x\mathrm{d}y$$

To interpret double integrals, it is the volume under the graph of $z = f(x,y)$ above D. Therefore, the average of f on D can be computed to be

$$A = \frac{\iint_D f(x,y)\mathrm{d}A}{\iint_D 1\mathrm{d}A}$$

We can also find the centroid of the region D:

$$\bar{x} = \frac{\iint_D x\mathrm{d}A}{\iint_D 1\mathrm{d}A}$$

$$\bar{y} = \frac{\iint_D y\mathrm{d}A}{\iint_D 1\mathrm{d}A}$$

5.2 Improper Integrals, 反常积分

Improper integrals are when either $\mathcal{D}$ or f is not bounded. If $f \geq 0$, and it is continuous except possibly at the boundary, then

$$\iint_D f(x,y)\mathrm{d}A = \begin{cases} L & \text{convergent} \\ \pm\infty & \text{divergent} \end{cases}$$

We can determine the convergence or divergence of improper integrals by comparisons without evaluation.
Consider a general integral $\iint_D f(x,y)\mathrm{d}A$, and another integral $\iint_D g(x,y)\mathrm{d}A$, if $f, g \geq 0$, then

1. If $f \leq g$, and $\iint g\mathrm{d}A$ is convergent, then $\iint f\mathrm{d}A$ is convergent.
2. If $f \geq g$, and $\iint g\mathrm{d}A$ is divergent, then $\iint f\mathrm{d}A$ is divergent.

5.3 Polar Coordinates, 极坐标

In $\mathbb{R}^2$, we can use another coordinate system to the Cartesian system. Given a point (x,y) in the Cartesian plane, we can denote this point as (r,θ) given that

$$x = r\cos\theta$$
$$y = r\sin\theta$$

r is the distance between the point and the origin and θ is the degree in radians rotated counterclockwise from the positive x-axis.
Thus, $\theta \in [0, 2\pi]$, or $\theta \in [-\pi, \pi]$, depending on whichever is more convenient.
In the partition with respect to r, θ, we have $\Delta A_{ij} = r_j \Delta\theta_j \Delta r_j$, thus, the Riemann sum of polar coordinates is in the form:

$$\sum_{ij} f(P_{ij}^*) r_j \Delta\theta_j \Delta r_j$$

Thus, the integral in polar coordinates will be of the form

$$\iint_D f(r,\theta)r\mathrm{d}\theta\mathrm{d}r$$

For a more general change of variables, consider we want to change the coordinate system from x, y to u, v, where we know $x = x(u, v)$, $y = y(u, v)$, then

$$\mathrm{d}A = \mathrm{d}x\mathrm{d}y = |\frac{\partial(x,y)}{\partial(u,v)}|\mathrm{d}u\mathrm{d}v$$

where $\frac{\partial(x,y)}{\partial(u,v)}$ represents the *Jacobian*:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

However, it is possible that we know $u = u(x, y)$ and $v = v(x, y)$ but still want to convert the system from x, y to u, v, then we consider the formula

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}}$$

5.4 Triple Integrals, 三重积分

A triple integral is of the form

$$\iiint_D f(x,y,z)\mathrm{d}V, D \subset \mathbb{R}^3$$

The Riemann sum would be the corresponding form

$$\sum_{i,j,k} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta x_i \Delta y_j \Delta z_k$$

An example of an iterated integral in 3D is

$$\int_a^b \int_{c(x)}^{d(x)} \int_{e(x,y)}^{f(x,y)} g(x,y,z)\mathrm{d}z\mathrm{d}y\mathrm{d}x$$

Similar to double integral definitions, the volume of D is evaluated to be

$$\iiint_D 1\mathrm{d}V$$

The centroids are also defined similarly.
In certain situations, the use of other coordinate systems is more convenient, considering

the cylindrical coordinates, where

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z
\end{aligned}
$$

The Jacobian would then be $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z = r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$.

5.5 Spherical Coordinates, 球坐标

In $\mathbb{R}^3$, another set of coordinates can be used if the volume we are integrating has certain symmetry, then

$$
\begin{aligned}
x &= R\sin\phi\cos\theta \\
y &= R\sin\phi\sin\theta \\
z &= R\cos\phi
\end{aligned}
$$

where R is the distance between the point and the origin, ϕ is the angle rotated from positive z-axis to negative z-axis ($\phi \in [0,\pi]$), θ is the angle rotated counterclockwise from positive x-axis ($\theta \in [0, 2\pi]$).
The Jacobian for spherical coordinates is thus

$$\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z = R^2 \sin\phi \mathrm{d}R\mathrm{d}\phi\mathrm{d}\theta$$

Given a volume D, if we denote the center of mass to be (x_{CM}, y_{CM}, z_{CM}), with the density of D satisfy a function $\rho = \rho(x,y,z)$, then

$$
\begin{aligned}
x_{CM} &:= \frac{\iiint_D x\rho(x,y,z)\mathrm{d}V}{\iiint_D \rho(x,y,z)\mathrm{d}V} \\
y_{CM} &:= \frac{\iiint_D y\rho(x,y,z)\mathrm{d}V}{\iiint_D \rho(x,y,z)\mathrm{d}V} \\
z_{CM} &:= \frac{\iiint_D z\rho(x,y,z)\mathrm{d}V}{\iiint_D \rho(x,y,z)\mathrm{d}V}
\end{aligned}
$$