

3.107 compound propositions

negation

conjunction

~~disj~~ disjunction

exclusive - or

Precedence:

7	1		
^	2		
v	3		

3.202 logical implication

$P \rightarrow q$, conditional statement / implication

P: hypothesis / antecedent / premise

q: conclusion / consequence

P	q	$P \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

A : the conditional statement $p \rightarrow q$

$q \rightarrow p$: converse of A

$\neg q \rightarrow \neg p$: contrapositive of A

$\neg p \rightarrow \neg q$: inverse of A

3.204 logical equivalence

bio-biconditional $P \leftrightarrow q$

$P \equiv q$: logically equivalent

$P \leftrightarrow q$ is always true
operator Precedence

7	1
\wedge	2
\vee	3
\rightarrow	4
\leftrightarrow	5

4.103

definition of a predicate

The statement "x squared is equal to 4" has two parts:

— The variable x, which is subject of the statement

— The predicate "square is equal to 4", which is the property that the subject of the statement can have

This statement can be formalised as $P(x)$ where P is the predicate "squared = 4" and x is the variable.

P is said to be the propositional function

Once a value is assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value.

e.g.: $P(2)$ is True

$P(3)$ is False

4.105 Quantifiers

The universal quantification of a predicate $P(x)$ is the proposition:

" $P(x)$ is true for all values of x in the universal of discourse"

notation: $\forall x P(x) \Rightarrow$ "for all x "

$$\forall x P(x) \Leftrightarrow P(n_1) \wedge P(n_2) \wedge \dots$$

eg: s: "For every x and every y ,

$$x + y > 10$$

$$\Rightarrow \forall x, y P(x, y)$$

Existential quantifier

"There exists a value x in the universe

$$\exists x P(x) \Rightarrow$$
 "there exists x "

$$\exists x P(x) \Leftrightarrow P(n_1) \vee P(n_2) \vee \dots$$

Uniqueness quantifier

"There exists a unique value x in the universe of discourse such that $P(x)$ is true."

$$\text{notation: } \exists! x P(x)$$

There exists a unique x

$$\text{eg. } P(x) "x^2 = 4"$$

$$E: \exists! x P(x)$$

If the universe of discourse is positive integers, E is TRUE ($x=2$)
 If _____ integers, E is FALSE
 $(x=2, -2)$

4.107 Nested quantifiers

$\forall x \exists y P(x, y)$, for every x , there is a y for which $P(x, y)$ is True

$\exists x \forall y P(x, y)$, there is an x for which $P(x, y)$ is True for every y .

$$\exists x P(x, y)$$

x is bound

y is free

Logical operations:

eg. $P(x)$ denotes " $x > 3$ "

$Q(x)$ denotes " x squared is even"

$$\exists x (P(x) \wedge Q(x)) \equiv T \quad (\text{ex. } x=4)$$

$$\forall x (P(x) \rightarrow Q(x)) \equiv F \quad (\text{ex. } x=8)$$

order of operations:

$$\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$$

$$\exists x \exists y \equiv \exists y \exists x$$

$$\forall x \exists y \neq \exists y \forall x$$

precedence:

$\wedge \exists$ have a higher precedence priority than all logical operators

4.201 De Morgan's law for quantifiers for negating quantifiers:

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

$$\exists x \neg \exists y \forall z P(x, y, z)$$

$$\equiv \exists x \forall y \neg \forall z P(x, y, z)$$

$$\equiv \exists x \forall y \exists z \neg P(x, y, z)$$

4.203 Rules of inference

Tautology: $(P \wedge (P \rightarrow q)) \rightarrow q$

The rule of inference:

$$P \rightarrow q$$

$$P$$

$$\therefore q$$

Tautology: $(\neg q \wedge (P \rightarrow q)) \rightarrow \neg P$

$$\neg q$$

$$P \rightarrow q$$

$$\therefore \neg P$$

$((P) \wedge (q)) \rightarrow (P \wedge q)$

$$P$$

$$q$$

$$\therefore P \wedge q$$

$(P \wedge q) \rightarrow P$

$$P \wedge q$$

$$\therefore P$$

$\frac{P(A)}{P(A) \times P(x)}$
Universal instantiation, UI
Quantifiers
4.2.5 Rules of inference with
Universal generalization, UG
 $\frac{P(C)}{P(C) \times P(x)}$
Universal instantiation, UI
for some domain
for the domain
Existential quantification, EG

$\frac{P(A) \vee P(B)}{P(A \vee B)}$
 $(A \wedge B) \leftarrow ((A \wedge C) \vee (B \wedge C))$
 $\frac{P(A) \wedge P(B)}{P(A \wedge B)}$
 $B \leftarrow (\neg A \vee B) \wedge (\neg B \vee B)$
 $\frac{P(A \wedge B)}{P(A)}$
 $(A \wedge B) \leftarrow ((A \wedge C) \vee (B \wedge C))$
 $\frac{P(A \wedge B)}{P(B)}$
 $(A \wedge B) \leftarrow ((A \wedge C) \vee (B \wedge C))$

Universal modus ponens

$$\forall x P(x) \rightarrow Q(x)$$

$\exists a$ for some element of
the domain

$$\therefore Q(a)$$

Universal modus tollens

$$\forall x P(x) \rightarrow Q(x)$$

$\neg Q(a)$ for some element of
the domain

$$\therefore \neg P(a)$$

Expressing complex statement

5.101

AND: $x \cdot y$, $\bar{x} \bar{y}$, $\bar{x} y$, $x \bar{y}$

OR: $x+y$, $\bar{x} y$, $x \bar{y}$

NOT: \bar{x} , $\bar{x} \bar{y}$

\neg NOT > AND > OR

5.103 Postulates of Boolean algebra

Axioms:

$$x(y+z) = (x \cdot y) + (x \cdot z)$$

$$x+(y \cdot z) = (x+y) \cdot (x+z)$$

Basic theorems:

$$x+(\bar{x} y) = x$$

$$x \cdot (x+y) = x$$

$$\text{if } y+x=1, y \cdot x=0 \Rightarrow x=y$$

(SEE NEXT PAGE)

De Morgan's theorems

$$\overline{x \cdot y} = \overline{x} + \overline{y}$$

$$\overline{x+y} = \overline{x} \cdot \overline{y}$$

Principle of duality

OR \rightarrow AND

AND \rightarrow OR

0 \rightarrow 1, 1 \rightarrow 0

$$+ \rightarrow \cdot$$

$$\cdot \rightarrow +$$

$$0 \rightarrow 1, 1 \rightarrow 0$$

Huntington's postulates

closure: any result $\in \{0, 1\}$

identity:

$$x+0=x, x \cdot 1=x$$

commutativity

$$x+y=y+x, x \cdot y=y \cdot x$$

distributivity

$$x(y+z) = (x \cdot y) + (x \cdot z)$$

$$x+(y \cdot z) = (x+y) \cdot (x+z)$$

complements

$$x+x'=1, x \cdot x'=0$$

distinct elements: 0 \neq 1

Basic theorems:

idempotent laws

$$x+x=x, x \cdot x=x$$

tautology and contradiction

$$x+x'=1, x \cdot 0=0$$

involution

$$(x')' = x$$

associative laws

$$(x+y)+z = x+(y+z), (xy)z = x \cdot (yz)$$

absorption laws

$$x + (x \cdot y) = x$$

$$x \cdot (x + y) = x$$

uniqueness of complement

if $y + y' = 1$, and $y \cdot y' = 0$

then $y = y'$

TM

$$1' = 0, \quad 0' = 1$$

5.105 Boolean functions

$$x \oplus y = x'y + x'y'$$

$$x \overline{\cdot} y = x'y + y$$

sum-of-products:

$$f(x, y, z) = xy + xz + yz$$

product-of-sums:

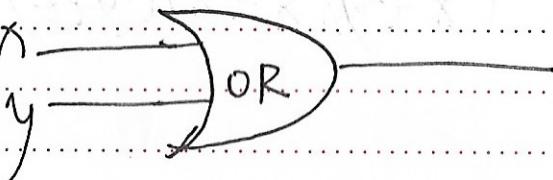
$$(x+y)(x+z)(y+z)$$

5.201 Logic gates

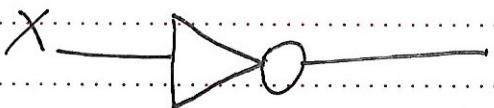
AND



OR

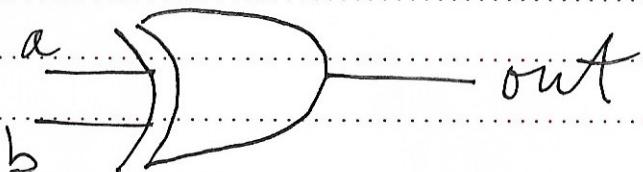


NOT/inverter gate

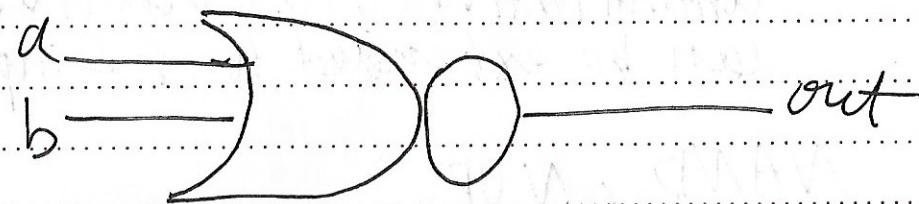


$$f = \overline{x}$$

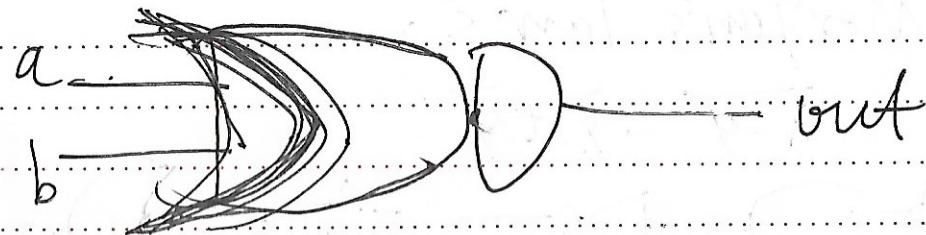
XOR/~~other~~



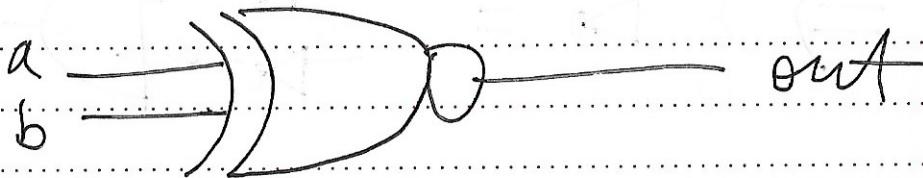
NAND (not AND)



NOR (not OR)



XNOR (not XOR)



AND, OR, XOR, XNOR:

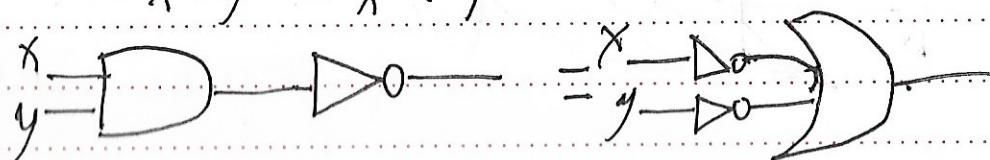
commutative, associative,
can be extended to ≥ 2 inputs.

NAND, NOR:

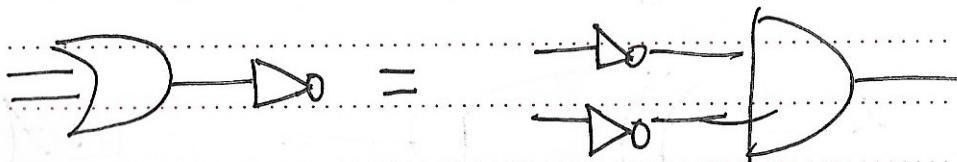
only commutative

De Morgan's laws:

$$\overline{x \cdot y} = \overline{x} + \overline{y}$$



$$\overline{x+y} = \overline{x} \cdot \overline{y}$$



6.10 | Intro to proofs

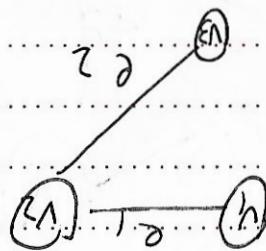
theorem 定理

axiom 公理

lemma 引理

corollary 推论

7.103 Definition of a graph



7.104 Adjacency matrix
Edges
Vertices
 e_1, e_2
 v_1, v_2, v_3, v_4
and v_i are incident

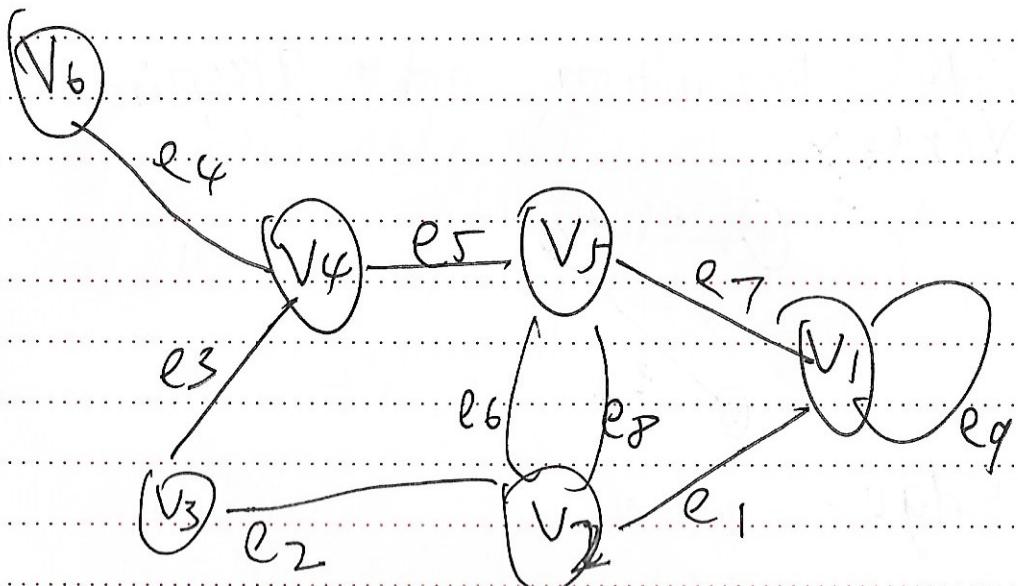
eg. $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$

$$a_0 = 1, a_1 = -2, a_2 = -1$$
$$a_3 = 0, a_4 = 1$$
$$a_5 = -2, a_6 = 1$$
$$a_7 = -1, a_8 = 0$$
$$a_9 = 1, a_{10} = -2$$
$$a_{11} = -1, a_{12} = 0$$
$$a_{13} = 1, a_{14} = -2$$
$$a_{15} = -1, a_{16} = 0$$
$$a_{17} = 1, a_{18} = -2$$
$$a_{19} = -1, a_{20} = 0$$
$$a_{21} = 1, a_{22} = -2$$
$$a_{23} = -1, a_{24} = 1$$
$$a_{25} = 1, a_{26} = -2$$
$$a_{27} = -1, a_{28} = 0$$
$$a_{29} = 0, a_{30} = 1$$
$$a_{31} = 1, a_{32} = -2$$
$$a_{33} = -1, a_{34} = 1$$
$$a_{35} = 1, a_{36} = -2$$
$$a_{37} = -1, a_{38} = 0$$
$$a_{39} = 0, a_{40} = 1$$
$$a_{41} = 1, a_{42} = -2$$
$$a_{43} = -1, a_{44} = 1$$
$$a_{45} = 1, a_{46} = -2$$
$$a_{47} = -1, a_{48} = 0$$
$$a_{49} = 0, a_{50} = 1$$
$$a_{51} = 1, a_{52} = -2$$
$$a_{53} = -1, a_{54} = 1$$
$$a_{55} = 1, a_{56} = -2$$
$$a_{57} = -1, a_{58} = 0$$
$$a_{59} = 0, a_{60} = 1$$
$$a_{61} = 1, a_{62} = -2$$
$$a_{63} = -1, a_{64} = 1$$
$$a_{65} = 1, a_{66} = -2$$
$$a_{67} = -1, a_{68} = 0$$
$$a_{69} = 0, a_{70} = 1$$
$$a_{71} = 1, a_{72} = -2$$
$$a_{73} = -1, a_{74} = 1$$
$$a_{75} = 1, a_{76} = -2$$
$$a_{77} = -1, a_{78} = 0$$
$$a_{79} = 0, a_{80} = 1$$
$$a_{81} = 1, a_{82} = -2$$
$$a_{83} = -1, a_{84} = 1$$
$$a_{85} = 1, a_{86} = -2$$
$$a_{87} = -1, a_{88} = 0$$
$$a_{89} = 0, a_{90} = 1$$
$$a_{91} = 1, a_{92} = -2$$
$$a_{93} = -1, a_{94} = 1$$
$$a_{95} = 1, a_{96} = -2$$
$$a_{97} = -1, a_{98} = 0$$
$$a_{99} = 0, a_{100} = 1$$

6.206 Solving recurrence relations

eg. Fibonacci sequence

$$f_0 = 0, f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2}$$
$$\text{solution}$$
$$r^2 = r^2 - r + 1 = 0$$
$$r = \frac{1 \pm \sqrt{5}}{2}$$
$$r_1 = \frac{1 + \sqrt{5}}{2}, r_2 = \frac{1 - \sqrt{5}}{2}$$
$$f_n = C_1 r_1^n + C_2 r_2^n$$
$$f_0 = 0, f_1 = 1$$
$$C_1 = 1, C_2 = 0$$
$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n + \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$



e₆, e₈: parallel edges

e₉: a loop

Directed graph: Digraph

7.105

Walk

$V_1 \rightarrow V_6$:

$V_1, V_2, V_2, V_3, V_3, V_4, V_4, V_6$

= e_1, e_2, e_3, e_4

= V_1, V_2, V_3, V_4, V_6

length = 4

or: V_1, V_2, V_3, V_2, V_5

(Pass twice through e_2)

Trail: a walk, no edge is repeated.

Path: a trail, neither vertices nor edges are repeated.

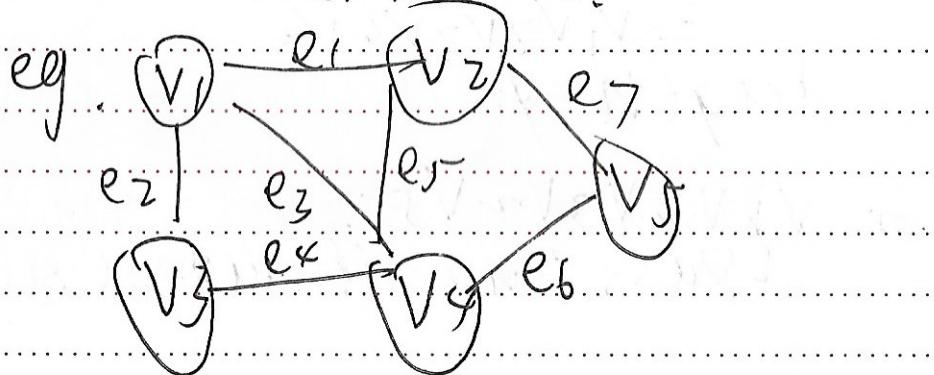
Cycle: V_1, V_2, V_5, V_1

Euler path:

A Eulerian path : a path that uses each edge precisely once.

If such path exists,

\Rightarrow traversable.



V₁V₃V₄V₅V₂ V₄V₁ V₂

Hamiltonian path:

(a traceable path) : a path that ~~visits~~ visits each vertex exactly once.

A graph that contains Hamiltonian path: a traceable graph.

Hamiltonian cycle:

a cycle that visits each vertex exactly once, except for the starting vertex.

Hamiltonian graph:

a graph contains Hamiltonian cycle.

Connectivity: (undirected graph)

any two nodes are connected by a path

Strong connectivity:

a directed graph is strongly connected if there is a directed path from any node to any other node.

Transitive Closure:

G^* has the same vertices as G .
If G has a directed path from u to v ,
 G^* has a directed edge from u to v .
($u \neq v$)

7.107 The degree sequence of a graph.

$\deg(v)$: number of edges incident on V
 $\text{In-deg}(v)$: v is the terminal vertex
 $\text{Out}(v) - \deg(v)$: v is initial vertex

Degree sequence of a graph.

an undirected graph G : a monotonic nonincreasing sequence of the vertex degrees of all the vertices of G .

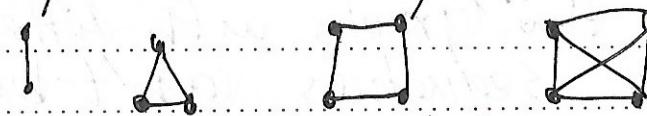
Property:

1. The sum of the degree sequence of a graph is always even.
2. ~~number~~ $|E| = \frac{1}{2} \sum \deg(v)$

7.109 Special graphs

Simple graphs: without loops and parallel edges
property: $\deg(v) \leq n-1$, $n = |V|$

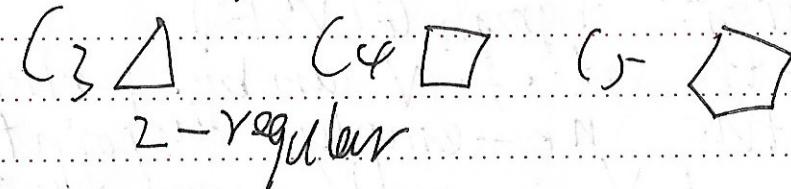
Regular graphs: all local degrees are the same number
 r -regular, $\deg(v) = r$



degree sequence of G : r, r, \dots, r
 $\sum \deg(v) = r \times n$

$$|E| = \frac{1}{2} rn$$

Special regular graphs: cycles



complete graphs: every pair of vertices are adjacent

K_n

properties: $\deg(v) = n-1$
 $\sum \deg(v) = n(n-1)$
 $|E| = \frac{1}{2} n(n-1)$

7.201 Isomorphic graphs

Definition of isomorphism:

there is a bijection (invertible function)

$$f: G_1 \rightarrow G_2$$

that preserves adjacency and non-adjacency.

If uv is in $E(G_1)$, then

$$f(u)f(v)$$
 is in $E(G_2)$.

Properties: two graphs with different degree sequences can't be isomorphic.

7.203 Bipartite graphs

Definition: A graph $G(V, E)$:

if the set of V can be partitioned into two non-empty disjoint sets V_1 and V_2 in such a way that each edge e in G has one endpoint in V_1 and another endpoint in V_2 .

The graph is 2 colorable
No odd-length cycle

Matching: a matching is a set of pairwise non-adjacent edges, none of which are loops; that is, no two edges share a common endpoint.

A vertex is matched / saturated if it is an endpoint of one of the edges in the matching. Otherwise the vertex is unmatched.

The Hopcroft - Karp algorithm
Solving the maximum matching problem in a bipartite graph

augmenting path:

breadth-first search

depth-first search

1. Initialize $M = \emptyset$

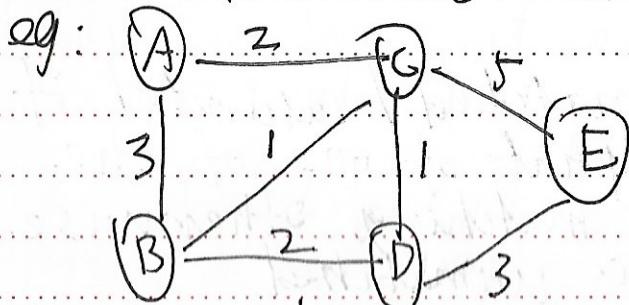
2. while there exists an augmenting path P

1) use BFS to build layers that terminate at free vertices

2) start at the free vertices in C , use DFS

3. Return M

7.207 Dijkstra's algorithm



Vertex	Shortest distance from A	Previous vertex
A	0	
B	∞	
C	∞	
D	∞	
E	∞	

Initialisation

unvisited = [] ABCDE

for each v in G:

shortest-dis[v] <= 0

shortest-dis[v] <= ∞

previous-vertex[v] <= undefined

add v to unvisited

Iteration 1

while unvisited is not empty:

u < vertex in unvisited with
min shortest-dis[u]

remove u from unvisited

for each neighbor v of u

alt <

shortest-dis[u] +

length(u, v)

if alt < shortest-dis[v]

shortest-dis[v] = alt

previous-vertex[v] = u

A	0	
B	3	A
C	2	A
D	inf	un
E	inf	un

unvisited = [B, C, D, E]

A	0	
B	3	A
C	2	A
D	3	C
E	7	C

unvisited =

[B, D, E]

8.103 Definition of a free acyclic graph: if and only if G has no cycles.

A tree is a connected acyclic undirected graph.

A disconnected graph containing no cycles is called a forest.

Theorem 1: An undirected graph is a tree if and only if there is unique simple path between any two of its vertices.

Theorem 2: A tree with n vertices has $n-1$ edges.

Rooted trees: a rooted tree is when one vertex has been designated as the root and every edge is directed away from root.

8.105 Spanning trees of a graph

Definition: a spanning tree of a graph G is a connected subgraph of G which contains all vertices of G , but with no cycles.

Constructing a spanning tree

1. keep all vertices
2. break all the cycles but keep the tree connected

Non-isomorphic spanning trees.

Two spanning trees are said isomorphic if there is a bijection preserving adjacency between two trees.

8.107 Minimum spanning tree

A minimum-cost spanning tree is a spanning tree that has the lowest weight / cost.

Kruskal's algorithm

Start with the cheapest edges in the spanning tree.

Repeatedly add the cheapest edge that does not create a cycle.

Prim's algorithm

Start with any one node in the spanning tree.

Repeatedly add the cheapest edge and the node it leads to, for which the node is not already in the spanning tree.

8.20 | Rooted trees

A rooted tree is a directed tree having one distinguished vertex r , called a root, such that for every vertex v there is a directed path from r to v .

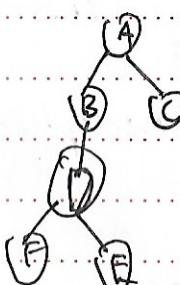
Theorem: a directed tree is represented as

a rooted tree if and only if one vertex has in-degree 0 whereas all other vertices have in-degree 1.

8.20 | Rooted trees

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Theorem: A directed tree is represented as a rooted tree if and only if one vertex has in-degree 0 whereas all other vertices have in-degree 1.



A: root

B: parent of D

E, F: children of D

B, A: ancestors of E, F
E, F are siblings

B,D: internal nodes

C,E,F: external nodes

The depth or path length of a node in a tree is the number of edges from the root to that node.

The height of a node in a tree is the longest path from that node to a leaf.

Special trees:

Binary Tree

Ternary Tree

m-any Tree

regular rooted trees.

An m-any tree is regular if every one of its internal nodes has exactly m children.

Properties:

An m-any tree has at most m^h vertices at level h.

Isomorphic trees:

Two trees T_1, T_2 are isomorphic if there is a bijection $f: v(T_1) \rightarrow v(T_2)$

which preserves adjacency

and non-adjacency.

That is, if uv is in $E(T_1)$

and $f(u), f(v)$ is in $E(T_2)$

Notation: $T_1 \cong T_2$

Properties:

Two trees with different degree sequences are not isomorphic.

Isomorphic rooted trees:

Two isomorphic trees are ~~isomorphic~~ isomorphic as rooted trees,

if and only if there is a bijection that maps the root of one tree to the root of the other.

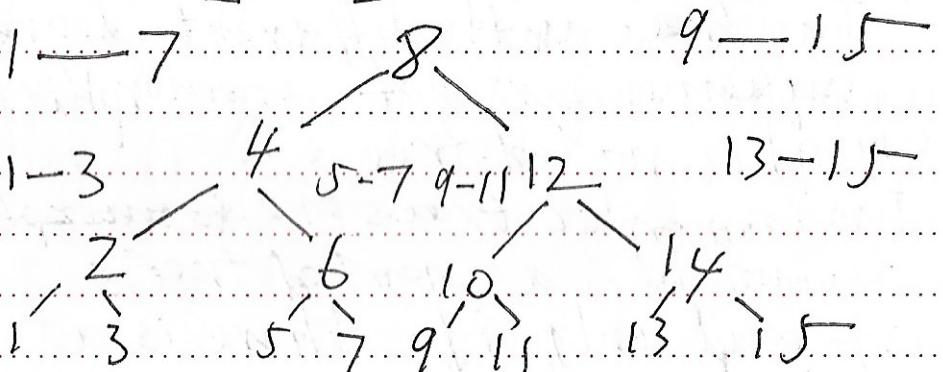
8.203 Binary search trees

A binary search tree is a binary tree in which the vertices are labelled with items so that a label of a vertex is greater than the labels of all vertices in the left subtree of this vertex and is less than the labels of

all vertices in the right subtree of
this vertex.

Application

$$\text{root} = \left\lfloor \frac{1+15}{2} \right\rfloor = 8$$



Height of the tree

$$① \quad 2^{h-1} < 1 + N \leq 2^h$$

$$h-1 \leq \log_2(1+N) \leq h$$

$$h = \lceil \log_2(N+1) \rceil$$

$$\text{eg. } N=15, h=4 \\ 2^{4-1} < 1+15 \leq 2^4$$

$$h = \lceil \log_2(15+1) \rceil = 4$$

29. find the first 3 level of a binary search tree to store

, 4,000 records.

Find the height of the tree

$$\begin{array}{r}
 1 \dots 4000 \\
 \text{Root} = L \left\lceil \frac{4000}{2} \right\rceil = 2000 \\
 1 - 1999 \quad 2000 \quad 2001 - 4000 \\
 \text{Root} = L \left\lceil \frac{2001 + 4000}{2} \right\rceil \\
 h = \lceil \log_2 (N+1) \rceil = \lceil \log_2 4017 \rceil = 12
 \end{array}$$

Binary search algorithm

root - sub - sub sub -

9103

What is a relation?

Let A, B be sets.

Let R be a relation linking elements of A to elements of B .

Let $x \in A$ and $y \in B$

We say that x is related to

y with respect to the relation R , and we write $x R y$.

A relation is a link between two elements of a set.

e.g.: X is a 'SON OF' Y

$R = \{ \text{SON OF} \}$

$x R y, \quad y R x$

Cartesian product.

Let A and B be two sets.

The Cartesian product $A \times B$ is defined by a set of pairs (x, y) such that $x \in A$ and $y \in B$.

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$$

A binary relation from A to B is a subset of a Cartesian product $A \times B$.

$R \subseteq A \times B$ means R is a set of ordered pairs of the form (x, y) where $x \in A$ and $y \in B$.

$(x, y) \in R$ means $x R y$

eg. $A = \{a, b, c\}, \quad B = \{1, 2, 3\}$
 $R = \{(a, 1), (b, 2), (c, 3)\}$

$a R 1, b R 2, c R 3$

When $A = B, \quad R \subseteq A \times A$

eg. $A = \{1, 2, 3, 4\}$

Let R be a relation on set A .
 $x, y \in A, \quad x R y$ if and only if $x < y$

$$\begin{aligned} & 1R2, 1R3, 1R4, 2R3, 2R4, 3R4 \\ & R = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\} \end{aligned}$$

9.105: representations of a relation

$$n_a = |A|, \quad n_b = |B|$$

The matrix of R is $n_a \times n_b$ matrix,

$$M_R = [m_{ij}] \quad n_a \times n_b$$

$$m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R \end{cases}$$

combining relations

Union: $R \cup S, \quad R$ or S

Intersection: $R \cap S, \quad R$ and S

$$R \cap S = \{(a, b) : (a, b) \in R \text{ and } (a, b) \in S\}$$

Via Boolean operators

Join: $M_R \cup M_S = M_R \vee M_S$

Meet: $M_R \cap M_S = M_R \wedge M_S$

using directed graphs

$(a, b) \in R$ arrows are drawn $a \rightarrow b$

eg. $A = \{1, 2, 3, 4\}$

$$R = \{(x, y) \mid x \text{ divides } y\}$$



9.107

reflexivity: A relation R in a set S is said to be reflexive, if and only if $\forall x \in S$
 $(x, x) \in R$, $\forall x \in S$

$$\text{eg: } R = \{(a, b) \in \mathbb{Z}^2 \mid a \leq b\}$$

Digraph of reflexive relations on elements in a set S contains a loop on every element of S

Matrix of reflexive relation: M_R . All the values of the diagonal of M_R are equal to 1.

Symmetry: $\forall a, b \in S$, if aRb , then bRa

$$\text{eg: } R = \{(a, b) \in \mathbb{Z}^2 \mid a \bmod 2 = b \bmod 2\}$$

Digraph: no single connection between any two different vertices.
 Matrix M_R is symmetric with respect to the diagonal

Anti-symmetric: $\forall a, b \in S$, if aRb and bRa , then $a=b$

$$\text{eg: } R = \{(a, b) \in \mathbb{Z}^2 \mid a \leq b\}$$

- Let $a, b \in S$, aRb and bRa
- $a \leq b$ and $b \leq a$
- $a = b$

• R is anti-symmetric
 Digraph: contains no parallel edges between any two different vertices
 Matrix: if $i \neq j$ and $m_{ij} \neq 0$, then $m_{ji} = 0$

9.109 transitivity

A relation R on set S is called transitive if and only if $\forall a, b, c \in S$,
 $\text{if } (aRb \text{ and } bRc) \text{ then } aRc$

Transitive closure of a relation

$$R = \{(a, b), (b, c), (c, d)\}$$

$$R_{\text{enhanced}} = \{(a, b), (b, c), (a, c), (c, d), (b, d), (a, d)\}$$

(transitive closure of R)

9.201 Equivalence relations / classes
 R is an equivalence relation if and only if R is reflexive, symmetric, and transitive

Eg: $R = \{(a,b) \in \mathbb{Z}^2 \mid a \bmod 2 = b \bmod 2\}$

Let R be an equivalence relation on a set S . Then the equivalence class of $a \in S$ is: the subset of S containing all the elements related to a through R .

$[a] = \{x \in S \mid xRa\}$

Eg. Let $S = \{1, 2, 3, 4\}$ and R be a relation on elements in S :

$R = \{(a,b) \in S^2 \mid a \bmod 2 = b \bmod 2\}$

R is an equivalence relation with 2 equivalence classes:

$[1] = [3] = \{1, 3\}$

$[2] = [4] = \{2, 4\}$

Q.203

R is a partial order, iff R is reflexive, anti-symmetric, and transitive.

Eg. $R = \{(a,b) \in \mathbb{Z}^2 \mid a \leq b\}$

R is total order, iff R is partial order, and for all a, b elements in S , we have either $a R b$, or $b Ra$.

Eg. $R = \{(a,b) \in \mathbb{Z}^2 \mid a \leq b\}$

$\forall a, b \in \mathbb{Z}, a \leq b \text{ or } b \leq a$ is true

10.103 The basics of counting
Product rule:

A and B are disjoint

$$|A \times B| = |A| \cdot |B|$$

Addition rule

A and B are disjoint

$$|A \cup B| = |A| + |B|$$

Subtraction rule

Principle of inclusion-exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Division rule

Suppose a task can be done using a procedure that can be carried out in n ways, and for every way w , exactly d of the n ways correspond to w . Then this task can be done in n/d ways.

In term of sets: if the finite set A is the union of n pairwise disjoint subsets each with d elements, then $n = |A|/d$.

10.105 Pigeonhole principle

10.107 Permutations and combinations
 $P(n,r)$
 $C(n,r)$

$$10.20 | (x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

$$2^n = \sum_{k=0}^n \binom{n}{k}$$

Pascal's identity: $n \geq k \geq 1: \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

10.204
 Permutations with repetition
 The number of r-permutations of a set of n objects with repetition allowed is n^r .

Combination with repetition $\binom{k+n-1}{k}$
 = the total number of ways to put k identical balls into n distinct boxes / the total number of functions from a set of k identical elements to a set of n distinct elements

	order matters	order not matters
repetition not permitted	$\frac{n!}{(n-k)!}$ $P(n,k)$	$\frac{n!}{k!(n-k)!}$ $\binom{n}{k}$
repetition permitted	$\binom{n}{k}$	$\binom{k+n-1}{k}$

10.206 Distinguishable objects and boxes with exclusion:

distribute k balls, numbered from 1 to k, into n boxes, numbered from 1 to n, in such a way that no box receives more than one ball
 $P(n,k)$

without exclusion: $(1-\frac{1}{n})(1-\frac{1}{n-1})\dots(1-\frac{1}{2})(1-\frac{1}{1})$
 distribute k balls, into n boxes without restrictions on the number of ~~key~~ balls in each box
 n^k

indistinguishable objects and distinguishable boxes with exclusion:
 distribute k balls into n boxes

numbered $(1-n)$, no box receives more than one ball

$$\binom{n}{k}$$

without exclusion: without restrictions on the number of balls in each box

$$\binom{n+k-1}{k}$$

e.g. place 8 indistinguishable balls into 6 distinguishable boxes

$$\binom{8+6-1}{8}$$