1. **(ECE 818 only).** Consider two-link planar manipulator with $m_1 = m_2 = 1$, $\ell_1 = \ell_2 = 1$, and assume that each link is a thin rod.

(a) Assume that $q_1(0) = q_2(0) = 0$ and the manipulator is initially at rest. Write a MATLAB program that implements a controller that settles the manipulator at $q_1 = \pi/4, q_2 = \pi/2$.

(b) Assume that $q_1(0) = 0, q_2(0) = \pi/4$ and the manipulator is initially at rest. Write a MATLAB program that implements a controller that tracks trajectories $q_1(t) = 2.5t^2 - 1.5t^3$ and $q_2(t) = 3.5t^2 - 2.5t^3$.

(a)

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} l_1 \cos\theta_1 \\ l_1 \sin\theta_1 \end{bmatrix} \qquad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} l_1\cos\theta_1 + l_2 \cos(\theta_1+\theta_2) \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} -l_1 \sin\theta_1 \dot{\theta}_1 \\ l_1 \cos\theta_1 \dot{\theta}_1 \end{bmatrix} \qquad \Rightarrow \begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -l_1 \sin\theta_1 \dot{\theta}_1 - l_2 \sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) \\ l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) \end{bmatrix}$$

$$\therefore K_1 = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) = \frac{1}{2} m_1 (l_1^2 \dot{\theta}_1^2)$$

$$\begin{aligned} K_2 &= \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2} m_2 [l_1^2 \dot{\theta}_1^2 + l_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ &\quad + 2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \sin\theta_1 \sin(\theta_1+\theta_2) \\ &\quad + 2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \cos\theta_1 \cos(\theta_1+\theta_2)] \\ &= \frac{1}{2} m_2 [l_1^2 \dot{\theta}_1^2 + l_2^2 (\dot{\theta}_1+\dot{\theta}_2)^2 + 2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1+\dot{\theta}_2) \cos\theta_2] \\ &= \frac{1}{2} m_2 [l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_1^2 + 2 l_2^2 \dot{\theta}_1 \dot{\theta}_2 + l_2^2 \dot{\theta}_2^2 \\ &\quad + 2 l_1 l_2 \cos\theta_2 \dot{\theta}_1^2 + 2 l_1 l_2 \cos\theta_2 \dot{\theta}_1 \dot{\theta}_2] \\ &= \frac{1}{2} m_2 [(l_1^2 + 2 l_1 l_2 \cos\theta_2 + l_2^2) \dot{\theta}_1^2 + 2(l_2^2 + l_1 l_2 \cos\theta_2) \dot{\theta}_1 \dot{\theta}_2 \\ &\quad + l_2^2 \dot{\theta}_2^2] \end{aligned}$$

$$K_1 + K_2 = \overbrace{\left(\frac{1}{2} m_1 l_1^2 + \frac{1}{2} m_2 l_1^2 + m_2 l_1 l_2 \cos\theta_2 + \frac{1}{2} m_2 l_2^2\right)}^{\frac{1}{2} M_{11}} \dot{\theta}_1^2 + \overbrace{m_2 (l_2^2 + l_1 l_2 \cos\theta_2)}^{M_{12}} \dot{\theta}_1 \dot{\theta}_2 + \overbrace{\frac{1}{2} m_2 l_2^2}^{\frac{1}{2} M_{22}} \dot{\theta}_2^2$$

$$P_1 = m_1 g y_1 = m_1 g l_1 \sin\theta_1 \qquad P_2 = m_2 g y_2 = m_2 g (l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2))$$

$$P = (m_1 g l_1 + m_2 g l_1) \sin\theta_1 + m_2 g l_2 \sin(\theta_1 + \theta_2)$$

$$\therefore \frac{\partial K}{\partial \dot{q}} = \begin{bmatrix} (m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2 + 2 m_2 l_1 l_2 \cos\theta_2) \dot{\theta}_1 + m_2 (l_2^2 + l_1 l_2 \cos\theta_2) \dot{\theta}_2 \\ m_2 l_2^2 \dot{\theta}_2 + m_2 (l_2^2 + l_1 l_2 \cos\theta_2) \dot{\theta}_1 \end{bmatrix}$$

$$\frac{\partial K}{\partial q} = \begin{bmatrix} 0 \\ -m_2 l_1 l_2 \dot{\theta}_1^2 \sin\theta_2 - m_2 \dot{\theta}_1 \dot{\theta}_2 l_1 l_2 \sin\theta_2 \end{bmatrix}$$

$$\therefore \frac{d}{dt} \frac{\partial K}{\partial \dot{q}} = \begin{bmatrix} (m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2 + 2 m_2 l_1 l_2 \cos\theta_2) \ddot{\theta}_1 - 2 m_2 l_1 l_2 \dot{\theta}_1 \sin\theta_2 \dot{\theta}_2 + m_2 (l_2^2 + l_1 l_2 \cos\theta_2) \ddot{\theta}_2 \\ m_2 l_2^2 \ddot{\theta}_2 + m_2 (l_2^2 + l_1 l_2 \cos\theta_2) \ddot{\theta}_1 - m_2 l_1 l_2 \dot{\theta}_1 \sin\theta_2 \dot{\theta}_2 - m_2 l_1 l_2 \dot{\theta}_2^2 \sin\theta_2 \end{bmatrix}$$

$$\frac{\partial P}{\partial q} = \begin{bmatrix} (m_1 g l_1 + m_2 g l_1) \cos\theta_1 + m_2 g l_2 \cos(\theta_1 + \theta_2) \\ m_2 g l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

$$\frac{\partial P}{\partial q} = \begin{bmatrix} (m_1 g l_1 + m_2 g l_1)\cos\theta_1 + m_2 g l_2 \cos(\theta_1+\theta_2) \\ m_2 g l_2 \cos(\theta_1+\theta_2) \end{bmatrix} \qquad \frac{\partial K}{\partial q} = \begin{bmatrix} 0 \\ -m_2 l_1 l_2 \dot\theta_1^2 \sin\theta_2 - m_2 \dot\theta_1 \dot\theta_2 l_1 l_2 \sin\theta_2 \end{bmatrix}$$

$$\frac{d}{dt}\frac{\partial K}{\partial \dot q} = \begin{bmatrix} (m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 \cos\theta_2)\ddot\theta_1 - 2m_2 l_1 l_2 \dot\theta_1 \sin\theta_2 \dot\theta_2 + m_2(l_2^2 + l_1 l_2 \cos\theta_2)\ddot\theta_2 \\ m_2 l_2^2 \ddot\theta_2 + m_2(l_2^2 + l_1 l_2 \cos\theta_2)\ddot\theta_1 - m_2 l_1 l_2 \dot\theta_1 \sin\theta_2 \dot\theta_2 \qquad - m_2 l_1 l_2 \dot\theta_2^2 \sin\theta_2 \end{bmatrix}$$

$$M_{11} = \overset{P_1}{m_1 l_1^2} + \overset{P_2}{m_2 l_1^2} + \overset{P_3}{m_2 l_2^2} + 2m_2 l_1 l_2 \cos\theta_2 \qquad M_{12} = m_2(l_2^2 + l_1 l_2 \cos\theta_2)$$

$$M_{22} = m_2 l_2^2 \quad P_2$$

$$\therefore \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix} \begin{bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{bmatrix} + \begin{bmatrix} -2m_2 l_1 l_2 \sin\theta_2 \dot\theta_1 \dot\theta_2 - m_2 l_1 l_2 \sin\theta_2 \dot\theta_2^2 \\ m_2 l_1 l_2 \sin\theta_2 \dot\theta_1^2 \end{bmatrix}$$

$$+ \begin{bmatrix} (m_1+m_2) l_1 g \cos\theta_1 + m_2 g l_2 \cos(\theta_1+\theta_2) \\ m_2 g l_2 \cos(\theta_1+\theta_2) \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

$$\begin{bmatrix} -2m_2 l_1 l_2 \sin\theta_2 \dot\theta_2 & -m_2 l_1 l_2 \sin\theta_2 \dot\theta_2 \\ m_2 l_1 l_2 \sin\theta_2 \dot\theta_1 & 0 \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix}$$

when $m_1 = m_2 = 1 \qquad l_1 = l_2 = 1$

$$\therefore \begin{bmatrix} 3 + 2\cos\theta_2 & 1 + \cos\theta_2 \\ 1 + \cos\theta_2 & 1 \end{bmatrix} \begin{bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{bmatrix} + \begin{bmatrix} -2\sin\theta_2 \dot\theta_1 \dot\theta_2 - \sin\theta_2 \dot\theta_2^2 \\ \sin\theta_2 \dot\theta_1^2 \end{bmatrix}$$

$$+ \begin{bmatrix} g\cos\theta_1 + g\cos(\theta_1+\theta_2) \\ g\cos(\theta_1+\theta_2) \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

S-Function of simulink:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix}$$

$$D(q)\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + C(q,\dot{q})\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + G(q) = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = u$$

$$S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \ddot{q}_1 \\ \dot{q}_2 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ S_1 \\ x_4 \\ S_2 \end{bmatrix}$$

I use PD controller in lecture 19

$$u = -K_P \tilde{q} - K_D \dot{q} + g(q)$$

I use S-Function and simulink in matlab

(b) $u = M(q)\, a_q + C(q,\dot{q})\,\dot{q} + g(q)$

$$a_q = \ddot{q}^d(t) - K_p \tilde{q} - K_D \dot{\tilde{q}}$$

$q_1^d = 2.5t^2 - 1.5t^3$ $\quad \dot{q}_1^d = 5t - 4.5t^2$ $\quad \ddot{q}_1^d = 5 - 9t$

$q_2^d = 3.5t^2 - 2.5t^3$ $\quad \dot{q}_2^d = 7t - 7.5t^2$ $\quad \ddot{q}_2^d = 7 - 15t$

see code in details!