Lagrange Multiplier Method

Solve the multi-objective minimization problem:

$$min_x E1(x), E2(x), \ldots, Ek(x) \tag{1}$$

where

$$E_i = 0.5 * x.T * H_i * x + x.T * f_i \tag{2}$$

and E_i is deemed "more important" than E_{i+1} (lexicographical ordering).

Algorithm

$$C_0 = H_0 \tag{3}$$

$$d_0 = f_0 \tag{4}$$

$$C_{i+1} = \begin{bmatrix} H_{i+1} & C_i^T \\ C_i & 0 \end{bmatrix} \tag{5}$$

$$d_i = \begin{bmatrix} f_{i+1} \\ d_i \end{bmatrix} \tag{6}$$

Where H_{i+1} is padded by zeros to match the size of C_i, and f_{i+1} is padded by zeros to match the size of d_i.

When there are no more Energies or C_i is full-rank the equation $C\vec{z} = \vec{d}$ is solved. The final solution is the first n elements of $\vec{z}$.

Example

For example, on the third iteration we have the following:

$$C_2 = \begin{bmatrix} H_2 & 0 & H_1^T & H_0^T \\ 0 & 0 & H_0 & 0 \\ H_1^T & H_0^T & 0 & 0 \\ H_0 & 0 & 0 & 0 \end{bmatrix} \tag{7}$$

where each H_i and 0 are $n \times n$ matrices

$$C_2 = \begin{bmatrix} f_2 \\ 0 \\ f_1 \\ f_0 \end{bmatrix} \tag{8}$$

where each f_i and 0 are $n \times 1$ vectors.

On the fourth iteration we have the following

$$C_2 = \begin{bmatrix} H_3 & 0 & 0 & 0 & H_2^T & 0 & H_1 & H_0^T \\ 0 & 0 & 0 & 0 & 0 & 0 & H_0 & 0 \\ 0 & 0 & 0 & 0 & H_1 & H_0^T & 0 & 0 \\ 0 & 0 & 0 & 0 & H_0 & 0 & 0 & 0 \\ H_2 & 0 & H_1^T & H_0^T & 0 & 0 & 0 & 0 \\ 0 & 0 & H_0 & 0 & 0 & 0 & 0 & 0 \\ H_1^T & H_0^T & 0 & 0 & 0 & 0 & 0 & 0 \\ H_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

$$C_3 = \begin{bmatrix} f_3 \\ 0 \\ 0 \\ 0 \\ f_2 \\ 0 \\ f_1 \\ f_0 \end{bmatrix} \tag{10}$$

As can been seen the C matrix doubles in size each iteration. The size of C_i is $(n2^i) \times (n2^i)$.