Null Space Method

Solve the multi-objective minimization problem:

$$min_x E1(x), E2(x), \dots, Ek(x) \tag{1}$$

where

$$E_i = 0.5x^T H_i x + x^T f_i \tag{2}$$

and E_i is deemed "more important" than E_{i+1} (lexicographical ordering).

Computing the Affine Null Space

First we need to find a basis for the null space and a particular solution x_i to the equation $Ax = b$.

QR

First, we compute the QR decomposition of A^T

$$PA^T = QR = \begin{bmatrix} Q_1 Q_2 \end{bmatrix} \begin{bmatrix} R_1 R_2 \\ 0 \end{bmatrix} \tag{3}$$

The columns of Q_1 span the column space of A, $col(A^T) = row(A)$, and the columns of Q_2 span the null space of A, $null(A)$. R_1 is a $r \times r$ matrix where r is the rank of A. We use QR decomposition with a column pivoting to get a permutation matrix, P, so we can easily compute the rank of A.

So finding N a matrix whose columns span the null space of A is simple.

$$N = Q_2 = Q_{:,r:} \tag{4}$$

To find a particular solution to $Ax = b$ we solve a linear system

$$x_0 = Q_1 y = Q_1 (R_1^T)^{-1} (P^T b) \tag{5}$$

That is we find the solution to $R_1^T y = P^T b$ and transform it to the column space of A.

Proof: $A^T Q_2^T y \equiv 0 \; \forall y$

$A^T = QR \Leftrightarrow A = R^T Q^T = \begin{bmatrix} \hat{R} \; 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$

$\begin{bmatrix} \hat{R} \; 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} Q_2^T y = \begin{bmatrix} \hat{R} \; 0 \end{bmatrix} \begin{bmatrix} Q_1 Q_2^T \\ Q_2 Q_2^T \end{bmatrix} y = \begin{bmatrix} \hat{R} \; 0 \end{bmatrix} \begin{bmatrix} 0 \\ Q_2 Q_2^T \end{bmatrix} y = 0y = 0$

$Q_1 Q_2^T = 0$ because Q is an orthogonal matrix.

SVD

First, we compute the singular value decomposition of A

$$A = U\Sigma V^T \tag{6}$$

where Σ is a diagonal matrix containing the singular values of A. The null space of A is spanned by the vectors in V corresponding to zero values in Σ.

$$N = V_{:,s} \tag{7}$$

where s is the set of indices for zeros along the diagonal of Σ.

Next to find a particular solution to $H_i x = b$ we invert the SVD.

$$x_0 = A^{-1}b = (U\Sigma V^T)^{-1}b = (V\Sigma^+ U^T)b \tag{8}$$

where Σ^+ is the Moore-Penrose pseudoinverse of Σ. Note, U and V are orthogonal matrices, so their transpose is their inverse.

LUQ

(See `luq-decomposition.pdf`)

Multi-Objective Optimization

Using one of the above method for computing the affine null space we can perform multi-objective optimization on all E_i.

$$N_0 = I \tag{9}$$

$$z_0 = 0 \tag{10}$$

$$\bar{N_i}, x_i = AffineNullSpace(N_{i-1}^T H_i N_{i-1},\ -N_{i-1}^T(H_i z_{i-1} + f_i)) \tag{11}$$

$$z_i = N_{i-1}x_i + z_{i-1} \tag{12}$$

$$N_i = N_{i-1}\bar{N_i} \tag{13}$$

Where `AffineNullSpace` is one of the functions defined in section one. We repeat this processes until either we have run out of energies or $\bar{N_i}$ is of size (0×0). The resulting solution is the final z.

For example,

$$
\begin{gathered}
\bar{N_1}, x_1 = AffineNullSpace(H_1,\ -f_1) \\
z_1 = Ix_1 + 0 = x_1 \\
N_1 = I\bar{N_1} = \bar{N_1} \\
\\
\bar{N_2}, x_2 = AffineNullSpace(N_1 H_2 N_1,\ -N_1^T(H_2 z_1 + f_2)) \\
z_2 = N_1 x_2 + z_1 = \bar{N_1} x_2 + x_1 \\
N_2 = N_1 \bar{N_2} = \bar{N_1}\bar{N_2} \\
\\
\bar{N_3}, x_3 = AffineNullSpace(N_2 H_3 N_2,\ -N_2^T(H_3 z_2 + f_3)) \\
z_3 = N_2 x_3 + z_2 = \bar{N_1}\bar{N_2} x_3 + \bar{N_1} x_2 + x_1 \\
N_3 = N_2 \bar{N_3} = \bar{N_1}\bar{N_2}\bar{N_3}
\end{gathered} \tag{14}
$$

With each iteration we find a minimum solution for the current energy. Importantly, this new solution preserves the energy value of the previous solution for all preceding energies.

Proof

$\left(\frac{d}{dx}E_1(x)\right) = H_1 x + f_1 = 0$

x_1 is a particular solution to $H_1 x = -f_1$ and a minimal energy solution to $E_1(x)$.

$N_1 y + x_1$ is a parameterization of all minimal energy solutions for $E_1(x)$.

$H_1(N_1 y + x_1) = H_1 N_1 y + H_1 x_1 = 0 + H_1 x_1 = -f_1$

Prove that z_2 is a minimal energy solution to $E_1(x)$:

$E_2(x) = \frac{1}{2}x^T H_2 x + x^T f_2 + c_2$

$E_2(N_1 y + x_1) = \frac{1}{2}(N_1 y + x_1)^T H_2 (N_1 y + x_1) + (N_1 y + x_1)^T f_2 + c_2$

$E_2(N_1 y + x_1) = \frac{1}{2}y^T N_1^T H_2 N_1 y + y^T N_1^T H_2 x_1 + y^T N_1^T f_2 + \frac{1}{2}x_1^T H_2 x_1 + x_1^T f_2 + c_2$

$\left(\frac{d}{dx}E_2(N_1 y + x_1)\right) = N_1^T H_2 N_1 y + N_1^T(H_2 x_1 + f_2)$

x_2 is a particular solution to $N_1^T H_2 N_1 y = -N_1^T(H_2 x_1 + f_2)$

$z_2 = N_1 x_2 + x_1$

$H_1 z_2 = H_1 N_1 x_2 + H_1 x_1 = -f_1$

Example

To illustrate this better let us take two energies in 3D

$E_1(x, y) = z = (y + 7)^2$

$E_2(x, y) = z = x^2 + y^2$

The minimal solutions to E_1 can be paramaterized as $N_1\vec{w} + \vec{x_1} = \begin{bmatrix}1\\0\end{bmatrix}\vec{w} + \begin{bmatrix}-7\\-7\end{bmatrix}$.

Substituting this parameterization for (x, y) in E_2 we get the following:

$E_2(x, y) = (w_0 - 7)^2 + (-7)^2 = w_0^2 - 14w_0 + 49 + 49$

$\frac{d}{dw}E_2 = 2w_0 - 14 = 0 \Rightarrow w_0 = 7$

$\therefore z_2 = N_1\begin{bmatrix}7\end{bmatrix} + z_1 = \begin{bmatrix}0\\-7\end{bmatrix}$

z_2 is the minimal energy value for E_2 that is in the null space of E_1.