Null Space Method

Solve the multi-objective minimization problem:

$$min_x E1(x), E2(x), \ldots, Ek(x) \tag{1}$$

where

$$E_i = 0.5x^T H_i x + x^T f_i \tag{2}$$

and E_i is deemed "more important" than E_{i+1} (lexicographical ordering).

Computing the Affine Null Space

First we need to find a basis for the null space and a particular solution x_i to the equation $Ax = b$.

QR

First, we compute the QR decomposition of A^T

$$PA^T = QR = \begin{bmatrix} Q_1 Q_2 \end{bmatrix} \begin{bmatrix} R_1 R_2 \\ 0 \end{bmatrix} \tag{3}$$

The columns of Q_1 span the column space of A, $col(A^T) = row(A)$, and the columns of Q_2 span the null space of A, $null(A)$. R_1 is a $r \times r$ matrix where r is the rank of A. We use QR decomposition with a column pivoting to get a permutation matrix, P, so we can easily compute the rank of A.

So finding N a matrix whose columns span the null space of A is simple.

$$N = Q_2 = Q_{:,r:} \tag{4}$$

To find a particular solution to $Ax = b$ we solve a linear system

$$x_0 = Q_1 y = Q_1 (R_1^T)^{-1} (P^T b) \tag{5}$$

That is we find the solution to $R_1^T y = P^T b$ and transform it to the column space of A.

Proof: $A^T Q_2^T y \equiv 0 \ \forall y$

$A^T = QR \Leftrightarrow A = R^T Q^T = \begin{bmatrix} \hat{R} \ 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$

$\begin{bmatrix} \hat{R} \ 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} Q_2^T y = \begin{bmatrix} \hat{R} \ 0 \end{bmatrix} \begin{bmatrix} Q_1 Q_2^T \\ Q_2 Q_2^T \end{bmatrix} y = \begin{bmatrix} \hat{R} \ 0 \end{bmatrix} \begin{bmatrix} 0 \\ Q_2 Q_2^T \end{bmatrix} y = 0y = 0$

$Q_1 Q_2^T = 0$ because Q is an orthogonal matrix.

SVD

First, we compute the singular value decomposition of A

$$A = U \Sigma V^T \tag{6}$$

where Σ is a diagonal matrix containing the singular values of A. The null space of A is spanned by the vectors in V corresponding to zero values in Σ.

$$N = V_{:,s} \tag{7}$$

where s is the set of indices for zeros along the diagonal of Σ.

Next to find a particular solution to $H_i x = b$ we invert the SVD.

$$x_0 = A^{-1} b = (U \Sigma V^T)^{-1} b = (V \Sigma^+ U^T) b \tag{8}$$

where Σ^+ is the Moore-Penrose pseudoinverse of Σ. Note, U and V are orthogonal matrices, so their transpose is their inverse.

LUQ

(See `luq-decomposition.pdf`)

Multi-Objective Optimization

Using one of the above method for computing the affine null space we can perform multi-objective optimization on all E_i.

$$N_0 = I \tag{9}$$

$$z_0 = 0 \tag{10}$$

$$\bar{N}_i, x_i = AffineNullSpace(N_i^T H_i N_i,\ N_i^T (H_i z_i + f_i)) \tag{11}$$

$$z_{i+1} = N_i x_i + z_i \tag{12}$$

$$N_{i+1} = N_i \bar{N}_i \tag{13}$$

Where `AffineNullSpace` is one of the functions defined in section one. We repeat this processes until either we have run out of energies or $\bar{N}_i$ is of size (0×0). The resulting solution is the final z.

For example,

$$
\begin{gathered}
\bar{N}_0, x_0 = AffineNullSpace(H_0,\ f_0) \\
z_1 = x_0 \\
N_1 = \bar{N}_0 \\
\\
\bar{N}_1, x_1 = AffineNullSpace(\bar{N}_0^T H_1 \bar{N}_0, \bar{N}_0^T (H_1 x_0 + f_1)) \\
z_2 = N_1 x_1 + z_1 = \bar{N}_0 x_1 + x_0 \\
N_2 = N_1 \bar{N}_1 = \bar{N}_0 \bar{N}_1 \\
\\
\bar{N}_2, x_2 = AffineNullSpace(...) \\
z_3 = N_2 x_2 + z_2 = \bar{N}_0 \bar{N}_1 x_2 + \bar{N}_0 x_1 + x_0 \\
N_3 = N_2 \bar{N}_2 = \bar{N}_0 \bar{N}_1 \bar{N}_2
\end{gathered} \tag{14}
$$

With each iteration we find a minimum solution for the current energy. Importantly, this new solution preserves the energy value of the previous solution for all preceding energies.