Lagrange Multiplier Method

Solve the multi-objective minimization problem:

$$min_x E1(x), E2(x), \ldots, Ek(x) \tag{1}$$

where

$$E_i = 0.5x^T H_i x + x^T f_i \tag{2}$$

and E_i is deemed "more important" than E_{i+1} (lexicographical ordering).

Formulation

$$C_0 = H_0 \tag{3}$$

$$d_0 = f_0 \tag{4}$$

$$C_{i+1} = \begin{bmatrix} H_{i+1} & C_i^T \\ C_i & 0 \end{bmatrix} \tag{5}$$

$$d_i = \begin{bmatrix} f_{i+1} \\ d_i \end{bmatrix} \tag{6}$$

Where H_{i+1} is padded by zeros to match the size of C_i, and f_{i+1} is padded by zeros to match the size of d_i.

If C_{i+1} is full rank we solve the equation $C_{i+1}\vec{z} = \vec{d}_{i+1}$ The final solution is the first n elements of $\vec{z}$. Otherwise we then find the row space of C_{i+1} and use that as C_{i+1}. To find the row space of C_{i+1} we preform QR decomposition on C_{i+1}^T.

$$PC_{i+1}^T = QR \tag{7}$$

$$row(C_{i+1}) = col(C_{i+1}^T) = col(Q) \tag{8}$$

From this QR decomposition we can find the rank of C_{i+1}, $r = rank(C_{i+1})$, which is the index of the first row of all zeros in Q. The column space of Q is spanned by the first r columns of Q.

$$C_{i+1} \leftarrow (Q_{:,1:r} R_{1:r,1:r})^T \tag{9}$$

Then we shrink d_{i+1} to include only the rows corresponding to the row space of C_{i+1}

$$d_{i+1} = (Pd)_{:rx} \tag{10}$$

If there are no more Energies we solve the equation $C_{i+1}\vec{z} = \vec{d}_{i+1}$. The final solution is the first n elements of $\vec{z}$.

$$\min_{x,\lambda_1,\lambda_2,\ldots,\lambda_{i-1}} \frac{1}{2} x^T H_i x + x^T f_i + 0 \cdot \lambda_1 + 0 \cdot \lambda_2 + ... + 0 \cdot \lambda_{i-1} \tag{11}$$

such that $C_{i-1} \cdot [x^T \lambda_1 \lambda_2 ... \lambda_{i-1}]^T = D_{i-1}$ or

$$y = [x^T \lambda_1 \lambda_2 ... \lambda_{i-1}]^T \tag{12}$$

$$\min_y \frac{1}{2} y^T A y - y^T B \tag{13}$$

such that $C_{i-1} \cdot [x^T \lambda_1 \lambda_2 ... \lambda_{i-1}]^T = D_{i-1}$.

Example

For example, on the third iteration we have the following:

$$C_2 = \begin{bmatrix} H_2 & 0 & H_1 & H_0^T \\ 0 & 0 & H_0 & 0 \\ H_1^T & H_0^T & 0 & 0 \\ H_0 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

where each H_i and 0 are $n \times n$ matrices

$$d_2 = \begin{bmatrix} f_2 \\ 0 \\ f_1 \\ f_0 \end{bmatrix} \tag{15}$$

where each f_i and 0 are $n \times 1$ vectors.

On the fourth iteration we have the following

$$C_3 = \begin{bmatrix} H_3 & 0 & 0 & 0 & H_2^T & 0 & H_1 & H_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & H_0 & 0 \\ 0 & 0 & 0 & 0 & H_1^T & H_0^T & 0 & 0 \\ 0 & 0 & 0 & 0 & H_0 & 0 & 0 & 0 \\ H_2 & 0 & H_1 & H_0^T & 0 & 0 & 0 & 0 \\ 0 & 0 & H_0 & 0 & 0 & 0 & 0 & 0 \\ H_1^T & H_0^T & 0 & 0 & 0 & 0 & 0 & 0 \\ H_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

$$d_3 = \begin{bmatrix} f_3 \\ 0 \\ 0 \\ 0 \\ f_2 \\ 0 \\ f_1 \\ f_0 \end{bmatrix} \tag{17}$$

As can been seen the C matrix doubles in size each iteration. The size of C_i is $(n2^i) \times (n2^i)$.

Limitations

QR factorization on the ith set of constraints will be very expensive because the number of non-zeros will be $O(n2^i)$.