MA Courses Review Notes
MA2503 Linear Algebra

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Contents

1 Linear Equation

2 Rectangular System and Echelon Form

GE: Gaussian elimination; G-J: Gauss-Jordan; U: row echelon form; E_A: reduced row echelon form.

Task	Method needed
Determining the rank of A	$A \xrightarrow{GE} U$, rank(A)= number of pivot in U
Solving linear system $Ax = b$	$[A\|b] \xrightarrow{GE} [U\|c]$ or $[A\|b] \xrightarrow{GJ} [E_A\|d]$ noted: if rank(U) $\neq$ rank($[U\|c]$), then inconsistent system
Determining the column relation	$A \xrightarrow{GJ} E_A$
Computing the inverse of $A_{n\times n}$	$[A\|I_n] \xrightarrow{GJ} [I_n\|A^{-1}]$, noted: if A can not reduce to I_n, then A singular
Testing whether $A \sim B$	$A \xrightarrow{GE} U_A$ $B \xrightarrow{GE} U_B$, then compare rank(U_A) equal to rank(U_B) or not
Testing whether $A \overset{row}{\sim} B$	$A \xrightarrow{GJ} E_A$ $B \xrightarrow{GJ} E_B$, then check whether $E_A = E_B$
Testing whether $A \overset{col}{\sim} B$	$A^T \xrightarrow{GJ} E_{A^T}$ $B \xrightarrow{GJ} E_{B^T}$, then check whether $E_{A^T} = E_{B^T}$
Testing whether $b \in span\{v_1, \cdots, v_n\}$	$[v_1\|\cdots\|v_n\|b] = [A\|b] \xrightarrow{GE} [U\|c]$
Testing whether $span\{v_1, \cdots, v_n\} = \mathbb{R}^m$	$[v_1\|\cdots\|v_n] = A_{m\times n} \xrightarrow{GE} U$, then check whether $rank(A) = m$
Finding the fundamental subspaces of $A_{m\times n}$	$[A\|I_m] \xrightarrow{GE} [U\|P]$, then the bases are: $R(A)$: basic columns of A $R(A^T)$: nonzero rows of U $N(A)$: h_i's in the general solution of $Ax = 0$ $N(A^T)$: last $m - r$ rows of P, where $r = rank(A)$
Testing whether $R(A^T) = R(B^T)$ $N(A) = N(B)$	$A \xrightarrow{G-J} E_A$, $B \xrightarrow{G-J} E_B$, then check whether $E_A = E_B$ (ie. whether $A \overset{row}{\sim} B$)
Testing whether $R(A) = R(B)$ $N(A^T) = N(B^T)$	$A^T \xrightarrow{G-J} E_{A^T}$, $B^T \xrightarrow{G-J} E_{B^T}$, then check whether $E_{A^T} = E_{B^T}$ (ie. whether $A \overset{col}{\sim} B$)
Testing whether $N(A_{m\times n}) = 0$	$A \xrightarrow{GE} U$, then check whether $rank(A) = n$
Testing whether $N(A_{m\times n})^T = 0$	$A \xrightarrow{GE} U$, then check whether $rank(A) = m$
Testing whether $span\{a_1, \cdots, a_n\} = span\{b_1, \cdots, b_n\}$ in $\mathbb{R}^n$	$(a_1^T \cdots a_n^T)^T = A_{r\times n} \xrightarrow{G_J} E_A$, $(b_1^T \cdots b_n^T)^T = B_{r\times n} \xrightarrow{G_J} E_B$, then check whether the nonzero rows of E_A and E_B coincide
Testing whether $\{v_1, \cdots, v_n\}$ is linearly independent	$[v_1\|\cdots\|v_n] = A \xrightarrow{GE} U$, then check whether $rank(A) = n$
Finding linear relationship among $\{v_1, \cdots, v_n\}$	$[v_1\|\cdots\|v_n] = A \xrightarrow{G-J} E_A$, then read off the relationships from E_A
Find a basis for $span\{v_1, \cdots, v_n\}$	$[v_1\|\cdots\|v_n] = A \xrightarrow{GE} U$, then the basic columns of A form a basis
Extending $\{v_1, \cdots, v_n\}$ $(r < n)$ to a basis for $\mathbb{R}^n$	$[v_1\|\cdots\|v_r\|e_1\|\cdots\|e_n] = A \xrightarrow{GE} U$, then the basic columns of A form a basis

Table 1: Summary of applications of Gaussian and Gauss-Jordan elimination

3 Matrix Algebra

3.1 Addition and Transposition

Theorem 1 (Symmetries)**.** *Let A be an $n \times n$ square matrix:*

- *symmetric:* $A^T = A$
- *skew-symmetric:* $A^T = -A$
- *hermitian:* $A^* = A$
- *skew-hermitian:* $A^* = -A$

3.2 Linearity

Linear Function . Suppose that $\mathcal{D}$ and $\mathcal{R}$ are two sets equipped with an addition and a scalar multiplication operation (consider, for example, $\mathcal{D} = \mathbb{C}^n$ and $\mathcal{R} = \mathbb{C}^m$). A function f that maps points in $\mathcal{D}$ to points in $\mathcal{R}$ is said to be a linear function if f satisfies:

$$f(\alpha x + y) = \alpha f(x) + f(y)$$

for all $x, y \in \mathcal{D}$ and all scalars α.

3.3 Matrix Multiplication

General Definition of Matrix Multiplication . If matrices $A_{m\times p}$ and $B_{p\times n}$ are conformable, the matrix product AB is defined to be the $m \times n$ matrix as following:

$$[AB]_{ij} = A_{i*}B_{*j} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

$$\begin{pmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \vdots & a_{ip} \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{pmatrix}_{m\times p} \begin{pmatrix} * & \cdots & b_{1j} & \cdots & * \\ * & \cdots & b_{2j} & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & b_{pj} & \cdots & * \end{pmatrix}_{p\times n} = \begin{pmatrix} * & \cdots & * & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & [AB]_{ij} & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & * & \cdots & * \end{pmatrix}_{m\times n}$$

Rows and Columns of a Matrix Product . To express the individual columns and rows of a matrix product:

$$[AB]_{*j} = \begin{pmatrix} [AB]_{1j} \\ [AB]_{2j} \\ \vdots \\ [AB]_{mj} \end{pmatrix} = \begin{pmatrix} a_{11}b_{1j} + a_{12}b_{2j} + \cdots + a_{1p}b_{pj} \\ a_{21}b_{1j} + a_{22}b_{2j} + \cdots + a_{2p}b_{pj} \\ \vdots \\ a_{m1}b_{1j} + a_{m2}b_{2j} + \cdots + a_{mp}b_{pj} \end{pmatrix} = \begin{pmatrix} a_{11}b_{1j} \\ a_{21}b_{1j} \\ \vdots \\ a_{m1}b_{1j} \end{pmatrix} + \cdots + \begin{pmatrix} a_{1p}b_{pj} \\ a_{2p}b_{pj} \\ \vdots \\ a_{mp}b_{pj} \end{pmatrix}$$
$$= A_{*1}b_{1j} + A_{*2}b_{2j} + \cdots + A_{*p}b_{pj}$$

3.4 Matrix Inversion

Matrix Inversion . For a given square matrix $A_{n\times n}$, the matrix $B_{n\times n}$ that satisfied the conditions

$$AB = I_n, \qquad BA = I_n$$

is called the inverse of A and is denoted by $B = A^{-1}$. an invertible matrix is said to be nonsingular, and a square matrix with no inverse is called a singular matrix.

Theorem 2 (4.**Characterization of nonsingular matrices**). *For an $n \times n$ matrix A, the following statements are equivalent:*

- A^{-1} *exists (A is nonsingular);*
- $rank(A) = n$;
- $A \xrightarrow{Gauss-Jordan} I$;
- $Ax = 0$ *has only the trivial solution* $x = 0$.

3.5 Elementary Matrices and Equivalence

Elementary Matrix . Matrices of the form $I - uv^T$ where u and v are $n \times 1$ column vectors with $v^T u \neq 1$ are called elementary matrices.

Equivalence . whenever B can be derived from A by a combination of elementary row and column operations, we say that A and B are equivalent matrices and write $A \sim B$; in matrix terms,

$$\begin{aligned} A \sim B &\iff PAQ = B && \textit{for nonsingular } P \textit{ and } Q \\ A \overset{row}{\sim} B &\iff PAQ = B && \textit{for nonsingular } P \\ A \overset{col}{\sim} B &\iff PAQ = B && \textit{for nonsingular } Q \end{aligned}$$

and note that if $A \overset{row}{\sim} B$, then:

$$B_{*k} = \sum_{j=1}^{n} \alpha_j B_{*j} \iff A_{*k} = \sum_{j=1}^{n} \alpha_j A_{*j}$$

Same as the column equivalence, in summary, row equivalence preserves column relationships, and column equivalence preserves row relationships.

Theorem 3 (6.**Rank Normal Form**). *If A is an $m \times n$ matrix such that $rank(A) = r$, then*

$$A \sim N_r = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

where N_r is called the rank normal form of A. It is the end product of a complete reduction of A by using both row and column operations.

Theorem 4 (7.**Testing for equivalence**). *For $m \times n$ matrices A and B the following statement are true:*

- $A \sim B \iff rank(A) = rank(B)$
- $A \sim^{row} B \iff E_A = E_B$
- $A \sim^{col} B \iff E_{A^T} = E_{B^T}$

Note*: in particular, that:*

- *either* $A \sim^{row} B$ *or* $A \sim^{col} B$ *implies* $A \sim B$, *but not vice versa*
- *multiplication by nonsingular matrices doesn't change rank.*

3.6 LU Factorization

$$PA = LU$$

Follow the process of Gaussian Elimination.

$$Ax = b : \quad Ly = b \to y = Ux$$

4 Vector Space

Theorem 5 (3 **Characterization of subspaces**). *The range of every linear function $f\colon \mathcal{R}^n \to \mathcal{R}^m$ is a subspace of $\mathcal{R}^m$, and every subspace of $\mathcal{R}^m$ is the range of some linear function $g\colon \mathcal{R}^r \to \mathcal{R}^m$ $(r \leq m)$.*

Theorem 6 (4 **Testing for equal ranges**). *For $m \times n$ matrices A and B the following statements are true.*

- $R(A^T) = R(B^T)$ *if and only if* $A \overset{row}{\sim} B$;
- $R(A) = R(B)$ *if and only if* $A \overset{col}{\sim} B$.

Theorem 7 (5 **Testing for equal null spaces**). *For $m \times n$ matrices A and B the following statements are true.*

- $N(A) = N(B)$ *if and only if* $A \overset{row}{\sim} B$;
- $N(A^T) = N(B^T)$ *if and only if* $A \overset{col}{\sim} B$.

Theorem 8 (7 **Linear independence and rank**). *If A is $m \times n$, then:*

- *the columns of A form a linearly independent set if and only if either of the following holds: (i) $N(A) = \{0\}$, or (ii) $rank(A) = n$;*
- *the rows of A form a linearly independent set if and only if either of the following holds: (i) $N(A^T) = \{0\}$, or (ii) $rank(A) = m$;*
- *if A is a square matrix, then A is nonsingular if and only if:*
  - *the columns of A form a linearly independent set, or*
  - *the rows of A form a linearly independent set.*

Theorem 9 (8 **Maximal independent subsets**). *if A is an $m \times n$ matrix and $rank(A) = r$, then:*

- *any maximal independent subset of columns (rows) from A contains exactly r columns (rows);*
- *in particular, the r basic columns in A constitute one maximal independent subset of columns from A.*

Theorem 10 (9 **Basic facts of independence**). *For a nonempty set of vectors $\mathcal{S} = \{u_1, u_2, \cdots, u_n\}$ in a space $\mathcal{V}$, the following are true:*

- *if $\mathcal{S}$ contains a linearly dependent subset, then $\mathcal{S}$ itself must be linearly dependent; conversely, if $\mathcal{S}$ is linearly independent, then every subset of $\mathcal{S}$ must also be linearly independent;*
- *if $\mathcal{S}$ is linearly independent and if $v \in \mathcal{V}$, then the extension set $\mathcal{S}_{ext} = \mathcal{S} \cup \{v\}$ is linearly independent if and only if $v \notin span(\mathcal{S})$;*
- *if $\mathcal{S} \subseteq \mathcal{R}^m$ and if $n > m$, then $\mathcal{S}$ must be linearly dependent.*

Theorem 11 (11 **Characterizations of a basis**). *Let $\mathcal{V}$ be a subspace of $\mathcal{R}^m$, and let $\mathcal{B} = \{b_1, b_2, \cdots, b_n\} \subseteq \mathcal{V}$. The following statement are equivalent:*

- *$\mathcal{B}$ is a basis for $\mathcal{V}$;*
- *$\mathcal{B}$ is a minimal spanning set for $\mathcal{V}$;*
- *$\mathcal{B}$ is a maximal linearly independent subset of $\mathcal{V}$.*

Theorem 12 (12 **Dimension theorem**). *Let $\mathcal{V}$ be a subspace of $\mathcal{R}^m$. Then any two linearly independent spanning sets (i.e. any two bases) for $\mathcal{V}$ must have the same number of elements.*

Theorem 13 (13 **Rank plus nullity theorem**). *if A is an $m \times n$ matrix, then:*

$$dimR(A) + dimN(A) = n$$

$$dimR(A^T) + dimN(A^T) = m$$

Theorem 14 (14 **Dimension of a sum**). *If $\mathcal{X}$ and $\mathcal{Y}$ are subspaces of a vector space $\mathcal{V}$, then:*

$$dim(\mathcal{X} + \mathcal{Y}) = dim\mathcal{X} + dim\mathcal{Y} - dim(\mathcal{X} \cap \mathcal{Y})$$

Summary of the rank . if A is an $m \times n$ matrix and $rank(A) = r$, then:

$$\begin{aligned} r &= \textit{the number of nonzero rows in any row echelon form of } A \\ &= \textit{the number of pivots in any row echelon form of } A \\ &= \textit{the number of basic columns in } A \\ &= \textit{the size of a maximal independent set of columns from } A \\ &= \textit{the size of a maximal independent set of rows from } A \\ &= dim\mathcal{R}(A) \\ &= dim\mathcal{R}(A^T) \\ &= n - dim\mathcal{N}(A) \\ &= m - dim\mathcal{N}(A^T) \\ &= \textit{the size of the largest nonsingular submatrix in } A \end{aligned}$$

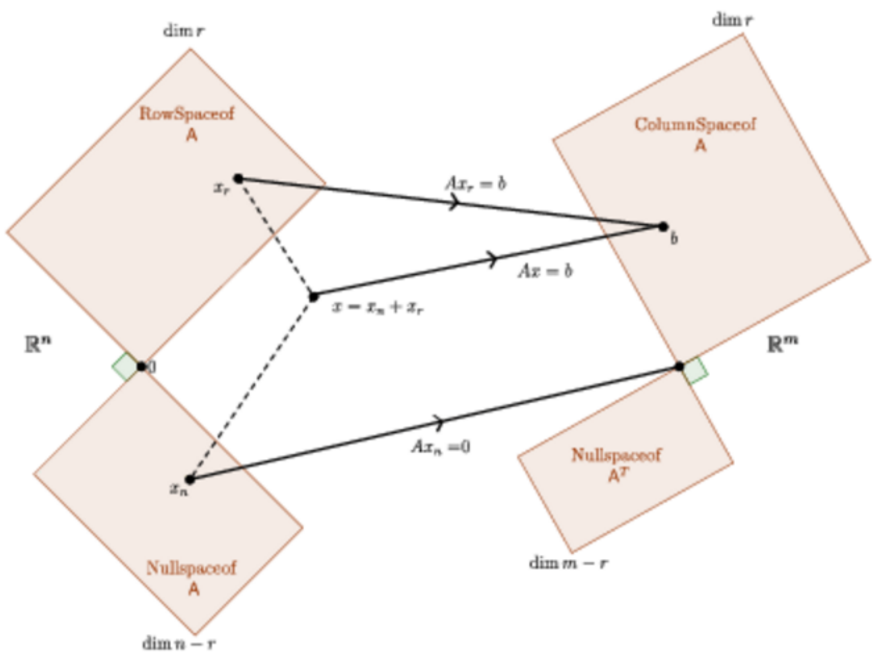

Figure 1: Four Fundamental Subspace of Matrix

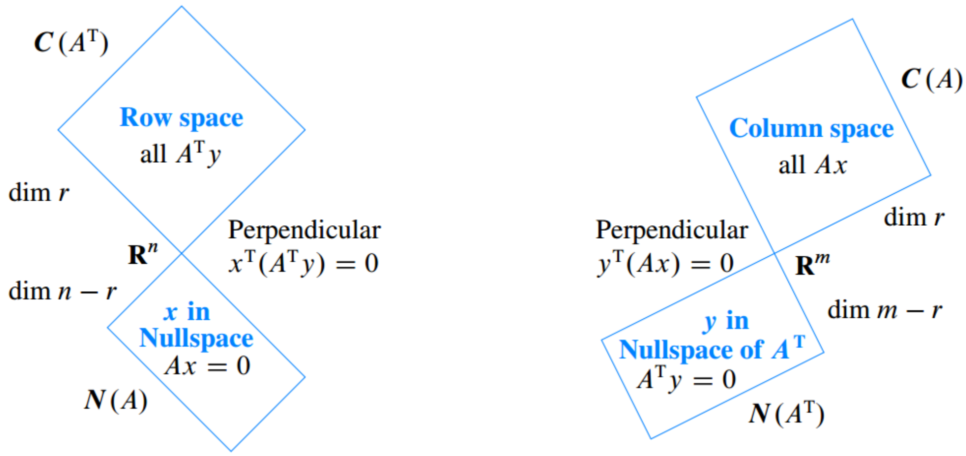

Figure 2: Dimensions and orthogonality for any m by n matrix A of rank r

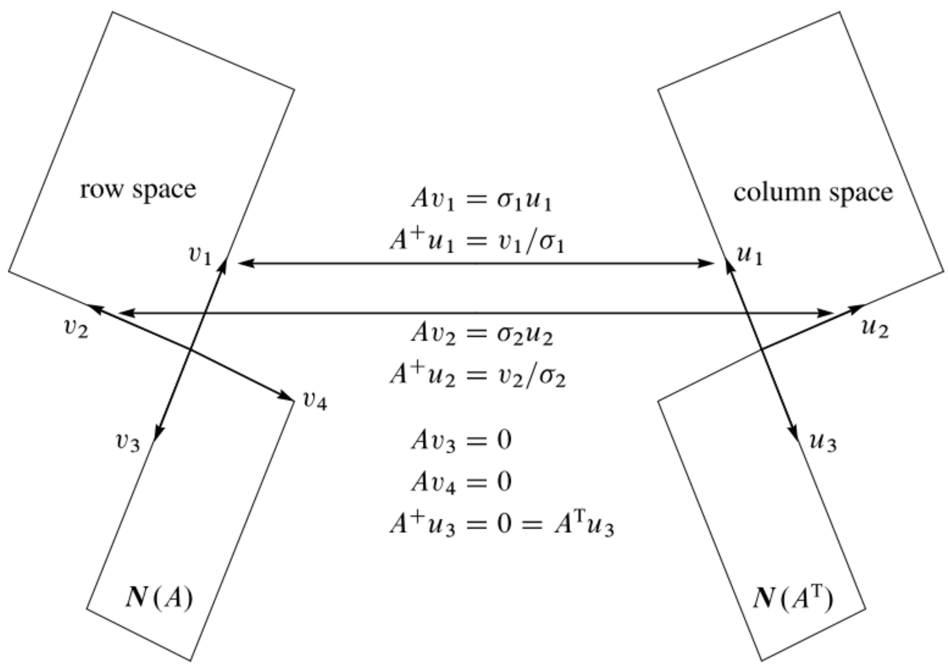

Figure 3: Orthonormal bases that diagonalize A (3 by 4) and A$^+$ (4 by 3)

5 Norms. Inner Product, and Orthogonality

Norm . For an $n \times 1$ vector x, the Euclidean norm of x is defined to be:

$$||x|| = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} = \sqrt{x^T x}$$

whenever $x \in \mathcal{R}^{n \times 1}$, or

$$||x|| = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} = \sqrt{x^* x}$$

whenever $x \in \mathcal{C}^{n \times 1}$

The p-norm can also defined as:

$$||x||_p = \left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} = (x^* x)^{\frac{1}{p}}$$

whenever $x \in \mathcal{C}^{n \times 1}$, $p \geq 1$

Standard Inner Product . The Euclidean vector norm can be viewed as a norm induced by the standard inner product:

$$\langle x, y \rangle = x^T y = \sum_{i=1}^{n} x_i y_i \qquad \textit{whenever } x, y \in \mathbb{R}^{n \times 1}$$

$$\langle x, y \rangle = x^* y = \sum_{i=1}^{n} \bar{x}_i y_i \qquad \textit{whenever } x, y \in \mathbb{C}^{n \times 1}$$

with the Euclidean norm defined by $||x|| = \langle x, x \rangle^{1/2}$

Orthogonality . Let $\mathcal{V}$ be either $\mathbb{R}^n$ or $\mathbb{C}^n$, Two vectors $x, y \in \mathcal{V}$ are said to be orthogonal (to each other) if $< x, y >= 0$, and this is denoted by writing $x \perp y$.

And note that:

- For $\mathbb{R}^n$ with the standard product, $x \perp y \implies x^T y = 0$
- For $\mathbb{C}^n$ with the standard product, $x \perp y \implies x^* y = 0$

General Angle . According to the law of cosine:

$$||u - v||^2 = ||u||^2 + ||v||^2 - 2||u||||v|| \cos \theta$$

in general, it implies that:

$$\cos \theta = \frac{||u||^2 + ||v||^2 - ||u - v||^2}{2||u||||v||} = \frac{u^T v}{||u||||v||}$$

Therefore, the radian measure of the angle between two nonzero vectors $x, y \in \mathbb{R}^n$ is defined to be the number $\theta \in [0, \pi]$ such that:

$$\cos \theta = \frac{< x, y >}{||x||||y||}$$

Orthogonal Set . Let $\mathcal{B} = \{u_1, u_2, \cdots, u_n\}$ be a set of vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$, $\mathcal{B}$ is called an orthogonal set if $||u_i|| = 1$ for each i and $u_i \perp u_j$ for all $i \neq j$. In other words, $\mathcal{B}$ is orthogonal if:

$$< u_i, u_j >= \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

where δ_{ij} is the classical Kronecker delta symbol.

Properties of Orthogonal Sets . Let $\mathcal{B} = \{u_1, u_2, \cdots, u_r\}$ be an orthogonal set in $\mathbb{R}^n(\mathbb{C}^n)$, then:

- $\mathcal{B}$ is linearly independent;
- if $r = n$, then $\mathcal{B}$ forms an orthogonal basis for the $\mathbb{R}^n(\mathbb{C}^n)$.

Fourier Expansion . For an orthogonal basis $\mathcal{B} = \{u_1, u_2, \cdots, u_r\}$:

- the expression:
$$x = \sum_{i=1}^{n} < u_i, x > u_i$$
is called the Fourier expansion of x (with respect to the basis $\mathcal{B}$), and the scalars $\xi_i =< u_i, x >$ are called the Fourier coefficients of x.
- geographically, the Fourier expansion resolves x into n mutually orthogonal vectors $< u_i, x > u_i$, each of which represents the orthogonal projection of x onto the space (line) spanned by u_i.

Theorem 15 (**Gram-Schmidt Procedure**). *The complete Gram-Schmidt procedure proceeds as follows, where $\mathcal{V} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$, and after the procedure into the orthogonal basis $\mathcal{B} = \{\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \cdots, \boldsymbol{\eta}_n\}$*

$$\begin{aligned}
\boldsymbol{\beta}_1 &= \boldsymbol{v}_1 && \Longrightarrow \boldsymbol{\eta}_1 = \frac{\boldsymbol{\beta_1}}{||\boldsymbol{\beta_1}||} \\
\boldsymbol{\beta}_2 &= \boldsymbol{v}_2 - \langle \boldsymbol{v}_2, \boldsymbol{\eta}_1 \rangle \boldsymbol{\eta}_1 && \Longrightarrow \boldsymbol{\eta}_2 = \frac{\boldsymbol{\beta_2}}{||\boldsymbol{\beta_2}||} \\
\boldsymbol{\beta}_3 &= \boldsymbol{v}_3 - \langle \boldsymbol{v}_3, \boldsymbol{\eta}_1 \rangle \boldsymbol{\eta}_1 - \langle \boldsymbol{v}_3, \boldsymbol{\eta}_2 \rangle \boldsymbol{\eta}_2 && \Longrightarrow \boldsymbol{\eta}_3 = \frac{\boldsymbol{\beta_3}}{||\boldsymbol{\beta_3}||} \\
&\vdots && \vdots \\
\boldsymbol{\beta}_k &= \boldsymbol{v}_k - \sum_{i=1}^{k-1} \langle \boldsymbol{v}_k, \boldsymbol{\eta}_i \rangle \boldsymbol{\eta}_i && \Longrightarrow \boldsymbol{\eta}_k = \frac{\boldsymbol{\beta_k}}{||\boldsymbol{\beta_k}||} \\
&\vdots && \vdots \\
\boldsymbol{\beta}_n &= \boldsymbol{v}_n - \sum_{i=1}^{n-1} \langle \boldsymbol{v}_n, \boldsymbol{\eta}_i \rangle \boldsymbol{\eta}_i && \Longrightarrow \boldsymbol{\eta}_n = \frac{\boldsymbol{\beta_n}}{||\boldsymbol{\beta_n}||}
\end{aligned}$$

Theorem 16 (**Characteristic of Direct Sums**). *Let $\mathcal{V}$ be a vector space and $\mathcal{X}$, $\mathcal{Y}$ be subspaces of $\mathcal{V}$ with respective basis $\mathcal{B}_{\mathcal{X}}$ and $\mathcal{B}_{\mathcal{Y}}$. Then the following statements are equivalent:*

- $\mathcal{V} = \mathcal{X} \oplus \mathcal{Y}$
- *for each $v \in \mathcal{V}$, there are unique $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, such that $v = x + y$*
- *$\mathcal{B}_{\mathcal{X}} \cap \mathcal{B}_{\mathcal{Y}} = \emptyset$(empty set) and $\mathcal{B}_{\mathcal{X}} \cup \mathcal{B}_{\mathcal{Y}}$is a basis for $\mathcal{V}$*

Orthogonal Complement . Let $\mathcal{V}$ be either $\mathbb{R}^n$ or $\mathbb{C}^n$ and let $\mathcal{M}$ be a subset of $\mathcal{V}$. The orthogonal complement $\mathcal{M}^\perp$ of $\mathcal{M}$ is the set of all vectors in $\mathcal{V}$ that are orthogonal to every vector in $\mathcal{M}$. In other words:
$$\mathcal{M}^\perp = \{y \in \mathcal{V} :< x, y >= 0, \forall x \in \mathcal{M}\}$$

Theorem 17 (**Orthogonal Complementary Subspaces**). *Let $\mathcal{V}$ be either $\mathbb{R}^n$ or $\mathbb{C}^n$. If $\mathcal{M}$ is a subspace of $\sqsubseteq$, then:*
$$\mathcal{V} = \mathcal{M} \oplus \mathcal{M}^\perp$$
Furthermore, if $\mathcal{N}$ is a subspace such that $\mathcal{V} = \mathcal{M} \oplus \mathcal{M}^\perp$ and $\mathcal{N} \perp \mathcal{M}$, then:
$$\mathcal{N} = \mathcal{M}^\perp$$

Theorem 18 (**Perp Operation**). *Let $\mathcal{V}$ be either $\mathbb{R}^n$ or $\mathbb{C}^n$ and let $\mathcal{M}$ be a subset of $\mathcal{V}$. Then the following statements are true:*

- $$dim\mathcal{M}^{\perp} = n - dim\mathcal{M}$$

- $$(\mathcal{M}^{\perp})^{\perp} = \mathcal{M}$$

Theorem 19 (**Orthogonal Decomposition Theorem**). *For every $A \in \mathbb{R}^{m \times n}$, the following statements are true:*

$$\mathcal{R}(A)^{\perp} = \mathcal{N}(A^T)$$
$$\mathcal{N}(A)^{\perp} = \mathcal{R}(A^T)$$

Consequently, every matrix $A \in \mathbb{R}^{m \times n}$ produces an orthogonal decomposition of $\mathbb{R}^m$ and $\mathbb{R}^n$ in th sense that:

$$\mathbb{R}^m = \mathcal{R}(A) \oplus \mathcal{R}(A)^{\perp} = \mathcal{R}(A) \oplus \mathcal{N}(A^T)$$

$$\mathbb{R}^n = \mathcal{N}(A) \oplus \mathcal{N}(A)^{\perp} = \mathcal{N}(A) \oplus \mathcal{R}(A^T)$$

6 Determinants

Determinant . Let $A = [a_{ij}]$ be an $n \times n$ matrix. the determinant of A is defined to be the scalar:

$$det(A) = \sum_p \sigma(p) a_{1_{p_1}} a_{2_{p_2}} \cdots a_{n_{p_n}}$$

where the sum is taken over the $n!$ permutations $p = (p_1, p_2, \cdots, p_n)$ of $(1, 2, \cdots, n)$. The determinant of A can be denoted by $det(A)$ or $|A|$.

Triangular Determinants . For a triangular matrix, it determinant is equal to the product of its diagonal entries:

$$det(T) = \begin{vmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{nn} \end{vmatrix} = t_{11} t_{22} \cdots t_{nn}$$

Theorem 20 (2.**Effects of row operations**). *Let B be the matrix obtained from $A_{n \times n}$ by one of the three elementary row operations:*

- *type I: interchange rows i and j*
- *type II: multiply row i by $\alpha \neq 0$*
- *type III: add α times row i to row j*

Then, the determinant $det(B)$ is given by:

- $det(B) = -det(A)$ *for the type I operations*
- $det(B) = \alpha det(A)$ *for the type II operations*
- $det(B) = det(A)$ *for the type III operations*

Theorem 21 (3.**Invertibility and Determinants**). *For every $n \times n$ matrix A, the following statements are true:*

- *A is non-singular if and only if $det(A) \neq 0$, or equivalently*
- *A is singular if and only if $det(A) = 0$*

Theorem 22 (4.**Product Rule**). *For all $n \times n$ matrices A and B,*

$$det(AB) = det(A) det(B)$$

Cofactor Expansion . Let A be an $n \times n$ matrix, with $n \geq 2$. The (i, j)-minor of A, M_{ij}, is defined to be the determinant of the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the i-th row and the j-th column of A. The (i, j)-cofactor of A, A_{ij}, is defined to be $(-1)^{i+j} M_{ij}$. And follow the definition, we can express the determinant as the following:

$$det(A) = a_{i1} A_{i1} + a_{i2} A_{i2} + \cdots + a_{in} A_{in}, \qquad (\textit{about the row } i)$$

$$det(A) = a_{1j} A_{1j} + a_{2j} A_{2j} + \cdots + a_{nj} A_{nj}, \qquad (\textit{about the column } j)$$

Characteration of Nonsingular Matrices . If A is a $n \times n$ matrix, the the following statement are equivalent:

- A^{-1}n exists (A is nonsingular)
- $rank(A) = n$

- $A \xrightarrow{G-J} I$
- $Ax = 0$ has only the trivial solution $x = 0$
- A is the product of elementary matrices of type I, II or III
- the columns of A forms a linearly independent set
- the rows of A forms a linearly independent set
- $det(A) \neq 0$

7 Eigenvalue and Eigenvectors

Eigenvalue and Eigenvector . For an $n \times n$ matrix A, scalar λ and vectors $x_{n\times 1} \neq 0$ satisfying $Ax = \lambda x$ are called the eigenvalue and eigenvectors of A, and any pair, (λ, x), is called an eigenpair for A. The set of distinct eigenvalues, denoted by $\sigma(A)$, is called the spectrum of A.

Similarity Transformation . Two $n \times n$ matrices A and B are said to be similar if there exists a non-singular matrix P such that $P^{-1}AP = B$. The product $P^{-1}AP$ is called a similarity transformation of A.

Diagonalizability . Let A be an $n \times n$ matrix

- A is said to be diagonalizable if A is similar to a diagonal matrix.
- A i9s said to have a complete set of eigenvectors if A has a set of n linearly independent eigenvectors, if A fails to process a complete set of eigenvectors, then A is called deficient or defective.

Theorem 23 (4.**Schur's triangularization theorem**). *Every square matrix is unitarily similar to an upper-triangular matrix. That is, for each $n \times n$ matrix A, there exists a unitary matrix $\mathcal{U}$ (not unique) and an upper-triangular matrix T (not unique) such that $\mathcal{U}^* A \mathcal{U} = T$, and the diagonal entries of T are the eigenvalues of A.*

Theorem 24 (5.**The Cayley-Hamilton Theorem**). *Every square matrix satisfies its own characteristic equation $p(\lambda) = 0$*

Multiplicities . Let $\lambda \in \sigma(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_s\}$

- The algebraic multiplicity of λ, denoted by $alg\ mult_A(\lambda)$, is the number of times it is repeated as a root of the characteristic polynomial.
- when $alg\ mult_A(\lambda) = 1$, λ is called a simple eigenvalue.
- the geometric multiplicity of λ, denoted by $geo\ mult_A(\lambda)$, is the dimension of the eigenspace $\mathcal{N}(A - \lambda I)$.
- Eigenvalues such that $alg\ mult_A(\lambda) = geo\ mult_A(\lambda)$ are called semisimple eigenvalues of A.

Theorem 25 (7.**Diagonalizality and multiplicities**). *An $n \times n$ matrix A is diagonalizable if and only if:*

$$geo\ mult_A(\lambda) = alg\ mult_A(\lambda)$$

for each $\lambda \in \sigma(A)$, i.e. if and only if every eigenvalue is semisimple.