傅立葉級數 (Fourier Series)

① 實數

$f(x) = f(x+p)$ ⇒ 則 P 為 f(x) 的週期函數 ($P=2\pi$)

$$\Rightarrow f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

且
$$\begin{cases} a_0 = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x)\,dx \\ a_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad n = 1, 2, \cdots \\ b_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \end{cases}$$

If 推廣到任意週期 $P = 2L$

$$\Rightarrow f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x \right), \quad f(x) = f(x+2L)$$

$$\begin{cases} a_0 = \dfrac{1}{2L} \displaystyle\int_{-L}^{L} f(x)\,dx \\ a_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \end{cases}$$

② 複數

by $e^{it} = \cos t + i\sin t \Rightarrow \cos t = \frac{1}{2}(e^{it} + e^{-it})$, $\sin t = \frac{1}{2}(e^{it} - e^{-it})$

代回實數 Fourier $\overset{(t=nx)}{\Rightarrow}$ $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$, $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$

③ 離散時間傅立葉轉換 (Discrete Time Fourier Transform, DTFT)

ⓐ 產生資料 ⇒ If 有一週期函數 $f(x)$ at $0 \le x \le 2\pi$

以 N為間隔取出x的點 ⇒ $x_k = \frac{2\pi}{N}k$, $k=0,\cdots,N-1$

ⓑ Let $g(x) = \sum_{n=0}^{N-1} c_n e^{inx}$

In order to make $g(x_k) \overset{\text{已知}}{=} f(x_k) \Rightarrow \overbrace{f_k = f(x_k)}^{\text{已知}} = g(x_k) = \sum_{n=0}^{\infty} c_n e^{inx_k}$

同乘 e^{-inx_k} 且k從0加到N-1 $\Rightarrow c_n = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-inx_k}$

將 x_k 以 $\frac{2\pi}{N}k$ 取代 $\Rightarrow e^{-inx_k} = e^{\frac{-2\pi i}{N}kn} = W_N^{kn}$, $W_N = e^{-\frac{2\pi i}{N}}$

故 訊號 $f=[f_0, \cdots f_{N-1}]^T$ 的 DFT 轉換結果 $\hat{f} = [\hat{f}_0 \cdots \hat{f}_{N-1}]^T$

$$\Rightarrow \hat{f}_n = N c_n = \sum_{k=0}^{N-1} W_N^{kn} f_k$$

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④ FFT (Fast Fourier Transform)

equal. ($\hat{f}_n = F_k$, 只是代號改)

Let $N = 2^M$ 又 by 上面知 $F_k = \sum_{n=0}^{N-1} f_n e^{-\frac{2\pi i}{N}nk} = \sum_{n=0} f_n W^{nk}$, $W = e^{-\frac{2\pi i}{N}}$, $k=0,1,\cdots,N-1$

Suppose half and last $N/2$ point $\Rightarrow F_k = \sum_{n=0}^{\frac{N}{2}-1} f_n e^{-\frac{2\pi i}{N}kn} + \sum_{n=\frac{N}{2}}^{N-1} f_n e^{-\frac{2\pi i}{N}kn}$

Let $m = n - \frac{N}{2} \Rightarrow F_k = \sum_{n=0}^{\frac{N}{2}-1} f_n e^{-\frac{2\pi i}{N}kn} + \sum_{m=0}^{\frac{N}{2}-1} f_{m+\frac{N}{2}} e^{-i(\frac{2\pi}{N})k(m+\frac{N}{2})}$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[f_n + e^{-i\pi k} f_{n+\frac{N}{2}} \right] e^{-\frac{2\pi i}{N}kn}.$$

由於 $e^{-i\pi k} = (-1)^k$

For even $\Rightarrow F_{2k} = \sum_{n=0}^{\frac{N}{2}-1} \left(f_n + f_{n+\frac{N}{2}} \right) e^{-\frac{2\pi i}{N} 2kn}$

For odd $\Rightarrow F_{2k+1} = \sum_{n=0}^{\frac{N}{2}-1} \left(f_n - f_{n+\frac{N}{2}} \right) e^{-\frac{2\pi i}{N} (2k+1)n}$, $k=0, \dots, \frac{N}{2}-1$

☆ Example we have 8 points. $(f_0, f_1, f_2, \cdots f_7)$

$N=8 \Rightarrow \frac{N}{2}=4$, $\frac{N}{2}-1=3$.

故 for $F_0 = \sum_{n=0}^{3} \left(f_n + f_{n+4} \right) e^{0} = \left[(f_0+f_4)+(f_1+f_5)+(f_2+f_6)+(f_3+f_7) \right]$

$F_1 = \sim$, $F_2 = \sim$ $\cdots$ $F_7 = \sim$

F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7

↓

sum of data

☆ why is conjugate. (共軛)

假設目前在 k 處 $\Rightarrow F_k = \sum_{n=0}^{N-1} f_n e^{-\frac{2\pi i}{N} kn}$

If 共軛在 N-k 處 $\Rightarrow F_{N-k} = \sum_{n=0}^{N-1} f_n \, e^{-\frac{2\pi i}{N}[N-k]n}$.

$= \sum_{n=0}^{N-1} f_n \, e^{-in\left[2\pi - \frac{2\pi k}{N}\right]}$

$e^{-2\pi i} = \cos(-2\pi) + i\sin(-2\pi) = 1$ $\Rightarrow$ $= \sum_{n=0}^{N-1} f_n \, e^{\frac{2\pi i}{N} kn}$.

$\Rightarrow$ 故 $e^{-\frac{2\pi i}{N} kn}$ 與 $e^{\frac{2\pi i}{N} kn}$ 為共軛

$\Rightarrow F_k = F_{N-k}$

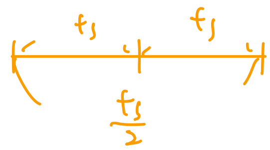
⑤ Discussion of frequency.

by DFT ⇒ $F(k\Delta f) = \sum_{n=0}^{N-1} f(n\Delta t)\, e^{i(2\pi k\Delta f)(n\Delta t)}$, $k = 0, \cdots, N-1$

N ⇒ total sampling number , T ⇒ sample time ,

$\Delta t \Rightarrow \frac{T}{N}$ time between data points

$f_s \Rightarrow \frac{1}{\Delta t} = \frac{N}{T}$ sampling frequency.



$\Delta f \Rightarrow \frac{1}{T} = \Delta f$ frequency increment (resolution)

∵ 3个点才能解析一个波 ⇒ ∴ $\underbrace{k\Delta f}_{\text{决定要哪一个频率的波}} = \frac{f_s}{2} \Rightarrow k = \frac{f_s}{2\Delta f} = \frac{N/T}{2(1/T)} = \frac{N}{2}$

⇒ $k = \frac{N}{2}$ ⇒ 最大的波数