Asymptotic order

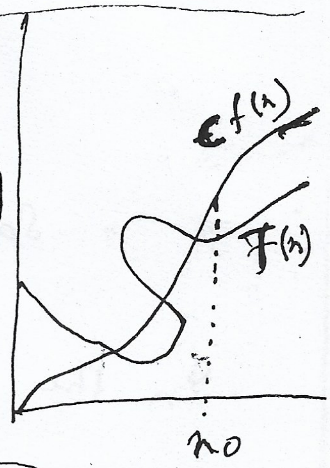
BCSE-2019

$O \rightarrow$ Big Oh

$\Omega \rightarrow$ omega

$\Theta \rightarrow$ Theta

Big-Oh : O



Def: $T(n) = O(f(n))$ if there exists constants $c > 0$ & $n_0 \geq 0$ so that for all $n \geq n_0$ we have $T(n) \leq c f(n)$

- $T(n)$ is asymptotically upper bounded by f.

- Note that c cannot depend on n

Ex: $T(n) = pn^2 + qn + r$ (p, q, r are positive constants)

We can claim that

$$T(n) = O(n^2).$$

proof: for all $n \geq 1$

we have $qn < qn^2$

$r \leq rn^2$

So, $T(n) \leq pn^2 + qn^2 + rn^2$

$\leq (p+q+r)n^2$

$T(n) = O(n^2)$, where $c = p+q+r$

$n_0 = 1$

It is also correct that

$$T(n) = O(n^3)$$

because

$$T(n) \leq (p+q+r)\, n^2$$

& we have $n^2 \leq n^3$

So, $T(n) \leq (p+q+r)\, n^3$

In the first case, $T(n) = O(n^2)$

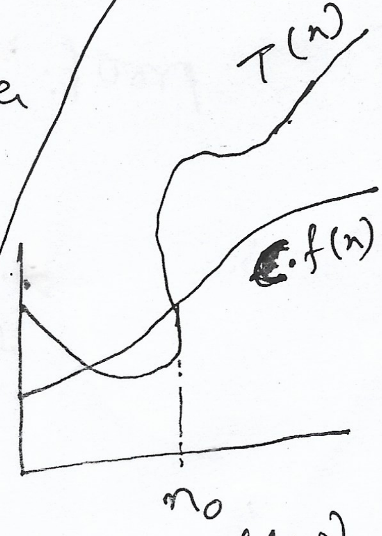
This running time is the tight upper bound (Tight possible running time)

Omega : Ω

Def: $T(n) = \Omega(f(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ so that for all $n \geq n_0$ we have $T(n) \geq c \cdot f(n)$

- $T(n)$ is asymptotically lower bounded by f.
- c is independent of n

(Bounding from below)



$T(n) = \Omega(f(n))$

Ex. $T(n) = pn^2 + qn + r$

$\geq pn^2$

[we are reducing the size of $T(n)$ until it looks like a constant times n^2]

$T(n) = \Omega(n^2)$

with

$c = p > 0$

$n_0 = 0$

In contrast, to establish the upper bound, we need to involve "inflating" the terms in $T(n)$]

Theta : Θ

(Asymptotically tight bound)

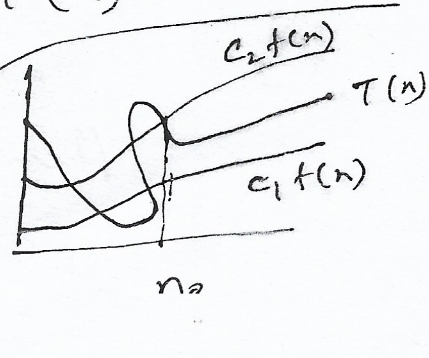
Def: $T(n) = \Theta(f(n))$ if $T(n)$ is both $O(f(n))$ and also $\Omega(f(n))$

Note: $T(n)$ grows exactly like $f(n)$ to within a constant factor

Ex: $T(n) = pn^2 + qn + r$ & $T(n) = \Omega(n^2)$

$T(n) = O(n^2)$

$T(n) = \Theta(n^2)$



Limit Method : Big-oh

$$T(n) = O(f(n)) \text{ if}$$

$$\lim_{n \to \infty} \left(\frac{T(n)}{f(n)} \right) = c < \infty$$

including the case in which limit is 0

Intuition: T(n) becomes insignificant relative to f(n) ~~as n approaches infinity~~, or ~~that is~~, the ratio converges to ~~[illegible]~~ positive constant ~~as~~ ~~$\lim_{n\to\infty}$~~ as n goes to infinity.

EX

Say $T(n) = 4 \cdot n^3 + 10 \cdot n^2 + 5 \cdot n + 1$

Consider $f(n) = n^4$

$$\lim_{n \to \infty} \left(\frac{4 \cdot n^3 + 10 \cdot n^2 + 5 \cdot n + 1}{n^4} \right)$$

$$= \lim_{n \to \infty} \left(\frac{4}{n} + \frac{10}{n^2} + \frac{5}{n^3} + \frac{1}{n^4} \right)$$

$$= 0$$

Hence $T(n) = O(n^4)$.

~~Limit method~~

~~Ω (omega)~~

~~(Lower~~

Limit method:

Ω (omega)

(Lower bound that is not asymptotically tight)

$$T(n) = \Omega(f(n)) \text{ if}$$

$$\lim_{n \to \infty} \left(\frac{T(n)}{f(n)} \right) = c > 0$$

including the case in which the limit is ∞

Intuition:

T(n) becomes arbitarily large relative to f(n) or the ratio converges to a positive constant as n goes to infinity.

Ex.

Say

$T(n) = 4n^3 + 10n^2 + 5n + 1$

$g(n) = n^2$

$$\lim_{n \to \infty} \left(\frac{4n^3 + 10n^2 + 5n + 1}{n^2} \right)$$

$$= \infty$$

Hence $T(n) = \Omega(n^2)$.

Limit method:

Theta (Θ).

(asymptotically tight bound)

$T(n) = \Theta(f(n))$ if

$$\lim_{n \to \infty} \left(\frac{T(n)}{f(n)} \right) = c$$

where c is some constant s.t.

$0 < c < \infty$.

Intuition: The ratio of $T(n)$ and $f(n)$ ~~goes to infi~~ converges to a positive constant as n goes to infinity.

When $T(n) = \Theta(f(n))$

$T(n) = O(f(n))$ and

$T(n) = \Omega(f(n))$

That is, for some $c_1 > 0$ & $c_2 > 0$ and $n > n_0$

$T(n) \le c_1 f(n)$ &

$T(n) \geq c_2 f(n)$ $\Rightarrow$ $\cancel{c_2 f(n) \le T(n) \le c_1 f}$

$\Rightarrow c_2 f(n) \le T(n) \le c_1 f(n).$

$\Rightarrow \dfrac{T(n)}{f(n)} \le c_1$ & $\dfrac{T(n)}{f(n)} \geq c_2$

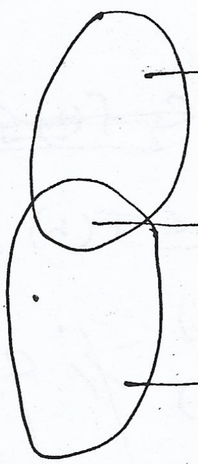
$$c_2 f(n) \le T(n) \le c_1 f(n)$$

$$c_2 \le \frac{T(n)}{f(n)} \le c_1.$$

When $n \to \infty$

$$\frac{T(n)}{f(n)} = c \quad \text{where} \quad c_1 \le c \le c_2$$

$T(n)$ grows exactly like $f(n)$ to within a constant factor.



$\Omega(f(n))$: class of functions that grows at least as fast as $f(n)$.

$\Theta(f(n))$: class of functions that grow at the same rate as $f(n)$

$O(f(n))$

↓ class of functions that grow no faster than $f(n)$.

$T(n) \in \Theta(f(n)) \Rightarrow T(n)$ belong to a class of functions that grow at the same rate as $f(n)$.

$$T(n) = pn^2 + qn + r$$

$\cancel{f(n) = n^2}$ show that

$$T(n) = \Theta(n^2)$$

$$\lim_{n \to \infty} \left(\frac{T(n)}{n^2} \right)$$

$$= \lim_{n \to \infty} \left(\frac{pn^2 + qn + r}{n^2} \right)$$

$$= \lim_{n \to \infty} \left(p + \frac{q}{n} + \frac{r}{n} \right)$$

$$= p$$

Hence $T(n) = \Theta(n^2)$

proved.

Little oh

$$T(n) = o(f(n))$$

/ $$\lim_{n \to \alpha} \frac{T(n)}{f(n)} = 0$$

Little omega (ω)

$$T(n) = \omega(f(n))$$

if $$\lim_{n \to \alpha} \frac{T(n)}{f(n)} = \alpha$$

Show That

for $f_1(n) \in O(g_1(n))$

and $f_2(n) \in O(g_2(n))$

$f_1(n) + f_2(n) \in O(\max(g_1(n), g_2(n)))$

proof

$$f_1(n) \le c_1 g_1(n) \;\&\; f_2(n) \le c_2 g_2(n)$$

$$\Rightarrow f_1(n) + f_2(n) \le (c_1 + c_2)(g_1(n) + g_2(n))$$

$$\Rightarrow f_1(n) + f_2(n) \le 2(c_1 + c_2) \max(g_1(n), g_2(n))$$

$$\Rightarrow f_1(n) + f_2(n) = O(\max(g_1(n), g_2(n)))$$

This implies that we can ignore lower order terms

that is, $p(n) = a_n n^k + \dots a_0$

$p(n) = O(n^k)$

Asymptotic Proof Techniques
Definitional Proof - Example II

Proof.

- If $n \geq 1$ it is clear that $n \leq n^3$ and $n^2 \leq n^3$.
- Therefore, we have that

$$n^2 + n \leq n^3 + n^3 = 2n^3$$

- Thus, for $n_0 = 1$ and $c = 2$, by the definition of Big-O, we have that $f(n) \in O(g(n))$. □

Asymptotic Proof Techniques
Definitional Proof - Example III

Example

Let $f(n) = n^3 + 4n^2$ and $g(n) = n^2$. Find a tight bound of the form $f(n) \in \Delta(g(n))$.

Here, our intuition should tell us that

$$f(n) \in \Omega(g(n))$$

Asymptotic Proof Techniques
Definitional Proof - Example III

Proof.

- If $n \geq 0$ then

$$n^3 \leq n^3 + 4n^2$$

- As before, if $n \geq 1$,

$$n^2 \leq n^3$$

- Thus, when $n \geq 1$,

$$n^2 \leq n^3 \leq n^3 + 3n^2$$

- Thus by the definition of Big-Ω, for $n_0 = 1, c = 1$, we have that $f(n) \in \Omega(g(n))$. □

Asymptotic Proof Techniques
Trick for polynomial of degree 2

If you have a polynomial of degree 2 such as $an^2 + bn + c$, you can prove it is $\Theta(n^2)$ using the following values:

- $c_1 = \frac{a}{4}$
- $c_2 = \frac{7a}{4}$
- $n_0 = 2 \cdot max(\frac{|b|}{a}, \sqrt{\frac{|c|}{a}})$

Limit Method

Now try this one:

$$\begin{aligned} f(n) &= n^{50} + 12n^3 \log^4 n - 1243n^{12} + \\ &\quad 245n^6 \log n + 12 \log^3 n - \log n \\ g(n) &= 12n^{50} + 24 \log^{14} n^4 3 - \frac{\log n}{n^5} + 12 \end{aligned}$$

Using the formal definitions can be *very* tedious especially when one has very complex functions. It is much better to use the *Limit Method* which uses concepts from calculus.

Limit Method Process

Say we have functions $f(n)$ and $g(n)$. We set up a limit quotient between f and g as follows:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & \text{then } f(n) \in O(g(n)) \\ c > 0 & \text{then } f(n) \in \Theta(g(n)) \\ \infty & \text{then } f(n) \in \Omega(g(n)) \end{cases}$$

- Justifications for the above can be proven using calculus, but for our purposes the limit method will be sufficient for showing asymptotic inclusions.
- Always try to look for algebraic simplifications *first*.
- If f and g *both* diverge or converge on zero or infinity, then you need to apply l'Hôpital's Rule.

For all $k > 0$

Show that

$$(\ln n)^k = O(n^\alpha)$$

<u>limit method</u>

$$\lim_{n \to \infty} \frac{(\ln n)^k}{n^\alpha}$$

$$= \left(\lim_{n \to \infty} \frac{\ln n}{n^{\alpha/k}} \right)^k$$

$$= \left(\lim_{n \to \infty} \frac{1/n}{(\alpha/k)\, n^{\alpha/k - 1}} \right)^k \quad \text{(using L'Hopital rule)}$$

$$= \lim_{n \to \infty} \left(\frac{1}{(\alpha/k)\, n^{\alpha/k}} \right)^k = 0$$

Hence $(\ln n)^k = O(n^\alpha)$

Let $T(n) = 2^n$

$f(n) = 3^n$

Show that

$T(n) = O(3^n)$.

<u>Limit method.</u>

$$\lim_{n \to \infty} \frac{T(n)}{f(n)} = \lim_{n \to \infty} \frac{2^n}{3^n}$$

Using L' Hopital Rule

$$\lim_{n \to \infty} \frac{T(n)}{g(n)} = \lim_{n \to \infty} \frac{2^n}{3^n} = \frac{(\ln 2)\, 2^n}{(\ln 3)\, 3^n}.$$

(still diverse)

Using algebra

$$\lim_{n \to \infty} \frac{2^n}{3^n} = \lim_{n \to \infty} \left(\frac{2}{3}\right)^n$$

$= 0$

[Since $\frac{2}{3} < 1$.

Theorem:

$$\lim_{n \to \infty} (\alpha)^n = \begin{cases} 0 & \text{if } \alpha < 1 \\ 1 & \text{if } \alpha = 1 \\ \infty & \text{if } \alpha > 1 \end{cases}$$

Hence $T(n) = O(3^n)$.

Efficiency Classes

Constant	$\mathcal{O}(1)$
Logarithmic	$\mathcal{O}(\log(n))$
Linear	$\mathcal{O}(n)$
Polylogarithmic	$\mathcal{O}(\log^k(n))$
Quadratic	$\mathcal{O}(n^2)$
Cubic	$\mathcal{O}(n^3)$
Polynomial	$\mathcal{O}(n^k)$ for any $k > 0$
Exponential	$\mathcal{O}(2^n)$
Super-Exponential	$\mathcal{O}(2^{f(n)})$ for $f(n) = n^{(1+\epsilon)}, \epsilon > 0$ For example, $n!$

Table: Some Efficiency Classes

Summary

Asymptotics is easy, but remember:

- Always look for algebraic simplifications
- You *must always* give a rigorous proof
- Using the limit method is always the best
- Always show l'Hôpital's Rule if need be
- Give as simple (and tight) expressions as possible

Say you have computed $T(n)$ for an algorithm and found that $T(n) = \Theta(f(n))$

What does it mean actually?

→ $T(n)$ belongs to a class of functions that grow at the same rate as $f(n)$.

Similarly $T(n) = O(f(n))$

$T(n)$ belongs to a class of functions that grow no faster than $f(n)$

$T(n) = \Omega(f(n))$

$T(n)$ belongs to a class of functions that grow at least as fast as $f(n)$.

If $T(n) = O(n^2)$

means

$T(n)$ belongs to a class of functions that grow no faster than n^2.

If $T(n) = \Theta(n^2)$

means

$T(n)$ belongs to a class of the function that grow at the same rate as n^2

if $T(n) = \Omega(n^2)$

means

$T(n)$ belongs to a class of functions that grow at least as fast as $f(n)$

How to Compare two algorithms?

for the same problem. $\begin{cases} \text{Algorithm 1} \longrightarrow & T(n) = O(n^2) \\ \text{Algorithm 2} \longrightarrow & T(n) = \Theta(n^2) \end{cases}$

Which one is better?

→ Algorithm 2 → due to its asymptotically tight upper bound.

Actually Big-theta is more informative than Big-O.

But it is harder to prove $T(n)$ is $\Theta(f(n))$ than $O(f(n))$.