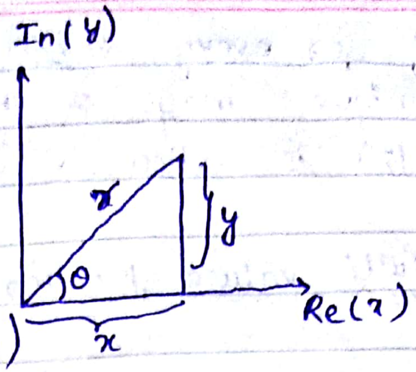
Complex Analysis

$Z = a + ib$

$z = x + iy$

$\bar{z} = x - iy$



$z = x + iy = r(\cos\theta + i\sin\theta)$

$r = \sqrt{x^2 + y^2}$

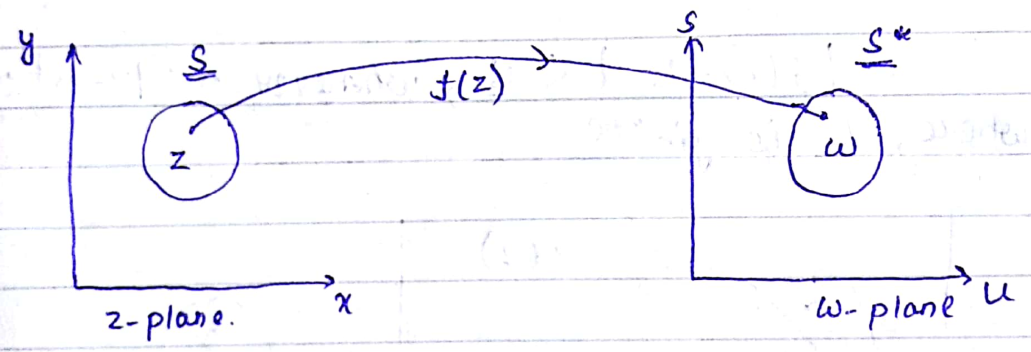
$\theta = \tan^{-1}(y/x)$ $\qquad \cos\theta = \frac{x}{r}$; $\sin\theta = \frac{y}{r}$

Ch-I Function of Complex analysis

Def : Function of complex variable :

Let S and S^* be two non-empty sets of complex numbers.

$z = x + y$ $\qquad\qquad \omega = s + u$



If there is a rule f ; which assigns a complex no. w in S^* for each z in S, then f is said to be a complex valued solution of a complex variable z and is written as

$$\omega = f(z) = u(x,y) + iv(x,y).$$

The set S is called domain of defination of f

The set of all images in S^* is called the range.

If for every $z \in D$ (Domain set), then exists an unique image in the w-plane, then the function $f(z)$ is called a single-valued function.

Multi-valued function

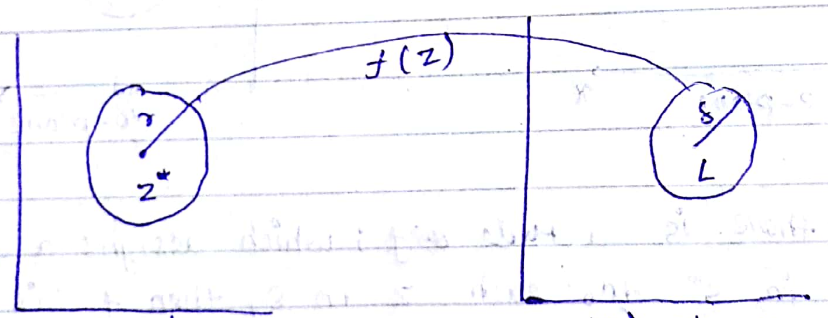
$$f(z) = sqrt(t)$$
$$z = -1$$
$$f(z) = \pm i$$

Def : Limit of a function

Let $f(z)$ be a single-valued function of z defined on δ which includes the δ-neighbourhood of a ~~pts~~ point $z = z_0$. The function $f(z)$ is such to have a limit $L \in \mathbb{C}$ as $z \to z_0$, if given arbitary small real number $\epsilon > 0$, there exist a real number $\delta > 0$, such that

$$|f(z) - L| < \epsilon \text{, wherever } 0 < |z - z_0| < \delta$$

where, L is finite



$$0 < |z - z^*| < \delta \qquad |f(z) - L| < \epsilon$$

$$f(z) \to L \text{ as } z \to z_0$$

If limit exist, then we can write

$$\boxed{\lim_{z \to z_0} f(z) = L}$$

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Th. :- If $\lim_{z \to z_0} f(z)$ exists, then ~~we can write~~

it is unique.

Def : continuity of a function

Let $f(z)$ be a single valued function of z defined in some neighbourhood of a point z_0 including the z_0. Then $f(z)$ is said to be continous at a point z_0, if for a given real number $\epsilon > 0$, we can find a real number $\delta > 0$, such that

$$|f(z) - f(z_0)| < \epsilon, \text{ wherever } |z - z_0| < \delta.$$

If the function $f(z)$ is continous at a point z_0, then we can write,

$$\lim_{z \to z_0} f(z) = f(z_0)$$

1. Using the defination of limits, show that

$$\lim_{z \to \infty} \left[\frac{1}{z^2}\right] = 0$$

Substitute $z = 1/\xi$

$$\lim_{z \to 0} \left[\frac{1}{z^2}\right] = 0 \qquad f(z) = \frac{1}{z^2}$$

$$f(1/\xi) = \xi^2$$

$$\lim_{\xi \to 0} [\xi^2] = 0$$

We need to determine a real number $\delta > 0$ such that for a given real number $\xi > 0$, we have

$$|f(z) - L| < \epsilon \quad \text{wherever } 0 < |z - z_0| < \delta$$

$$|f(1/\xi) - 0| < \epsilon \quad \text{wherever } 0 < |\xi - 0| < \delta$$

(1)

$$|f(1/\xi) - 0| = |\xi^2| < \epsilon$$

$$\boxed{|\xi| < \sqrt{\epsilon}}$$

Then, we take $\delta < \sqrt{\epsilon}$
with the choice of δ, we find that

$|\xi^2 - 0| < \epsilon$, wherever $0 < |\xi| < \delta$

[cond (i)]

$$\therefore \lim_{\xi \to 0} \xi^2 = 0$$

$$\boxed{\lim_{z \to 0} \left(\frac{1}{z^2}\right) = 0} \quad \text{(Proved)}$$

2. Find $\lim_{z \to 2i} [3x + iy^2]$

$$z = x + iy = 2i$$
$$x = 0 \; ; \; y = 2$$

Now, $\lim_{z \to 2} [3x + iy^2]$

$$= \lim_{\substack{x \to 0 \\ y \to 2}} [3x + iy^2]$$

$$= 4i \quad \text{(Ans)}$$

3. Find $\lim_{z \to \infty} \left[\frac{z}{2 - iz}\right]$

$$= \lim_{\xi \to 0} \left[\frac{1}{2\xi - i}\right] = -\frac{1}{i} = i \quad \text{(Ans)}$$

4. Show that the following limits doesnot exist.

i) $\lim\limits_{z\to 0} \frac{z}{|z|}$ ii) $\lim\limits_{z\to 0} \frac{[\operatorname{Re} z - \operatorname{Im} z]^2}{|z|^2}$ iii) $\lim\limits_{z\to 0} \left(\frac{\bar{z}}{z}\right)$

Ans = i). $f(z) = \frac{z}{|z|}$

Suppose, we choose the path $y \to 0$ followed by $x \to 0$, we get

$$\lim_{z\to 0} \frac{z}{|z|} = \lim_{x\to 0}\left[\lim_{y\to 0} \frac{x+y}{\sqrt{x^2+y^2}}\right]$$

$$= \lim_{x\to 0} \frac{x}{\sqrt{x^2}} = \lim_{x\to 0} \frac{x}{|x|} = \pm 1$$

Now, we choose the path $x \to 0$ followed by $y \to 0$, we get

$$\lim_{z\to 0} \frac{z}{|z|} = \lim_{y\to 0}\left[\lim_{x\to 0} \frac{x+iy}{\sqrt{x^2+y^2}}\right] = \pm i$$

Hence, the limit doesnot exist.

iii) $\lim\limits_{z\to 0} \left(\frac{\bar{z}}{z}\right)$

$$= \frac{x-iy}{x+iy} = \frac{(x-iy)^2}{(x+iy)(x-iy)}$$

$$f(x+iy) = \frac{x^2 - y^2 - 2ixy}{x^2+y^2}$$

We can choose diff. paths in many ways to examine the limit as $z \to 0$. Let m be a number and $z \to 0$ along the line $y = mx$

$$f(x+mx) = \left(\frac{1-m^2}{1+m^2}\right) - i\left(\frac{2m}{1+m^2}\right)$$

which taken diff. values for diff. m.

$\lim_{z \to z_0} \left(\frac{\bar{z}}{z}\right)$ doesnot exist.

ii) $\lim_{z \to 0} \frac{[\operatorname{Re} z - \operatorname{Im} z]^2}{|z|^2}$ Let $z = x + iy$

$$= \frac{[x - y]^2}{x^2 + y^2}$$

$$= \frac{x^2 + y^2 - 2xy}{x^2 + y^2}$$

Put $y = mx$

$$= \frac{x^2 + m^2 x^2 - 2x^2 m}{x^2 + m^2 x^2}$$

$$= \frac{1 + m^2 - 2m}{1 + m^2}$$

$\therefore$ Dependent on m. So, It does not exist

5. Show that, the function $f(z)$ is not continous at zero, where

$$f(z) = \begin{cases} \frac{\ln(z)}{|z|}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

At $z = z_0 = 0$, $f(z_0) = 0$

At $z \neq 0$, $f(z) = \frac{m}{\sqrt{1 + m^2}}$ (choose the path $y = mx$)

which is independent on m.

Hence, the limit doesnot exist at $z = 0$.

$\therefore$ the function is not continous at $z = 0$.

Def : Derivative of function

Let $f(z)$ be a single-valued function defined in a ~~[illegible]~~ Domain D. The function $f(z)$ is said to be differentiable at a point z_0, if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exist : This limit is called the derivative of $f(z)$ at the point $z = z_0$ and is denoted by $f'(z_0)$

$$f'(z_0) = \lim_{x \to x_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Def : Analytic : A function $f(x)$ of a complex variable z is used to be analytic at a point z_0, if it is differentiable at the point z_0 and also at each point in some neighbourhood of the point z_0.

A function $f(z)$ is such to be analysis in a domain D, if it is analytic at every point in D.

Note that, analytically implies different ability but not vice-versa.

Limit

$$|f(z) - L| < \epsilon, \quad 0 < |z - z_0| < \delta$$

$$\text{or, } \lim_{z \to z_0} f(z) = L$$

Continuity

$$|f(z) - f(z_0)| < \epsilon \qquad |z - z_0| < \delta$$

$$\lim_{z \to z_0} f(z) = f(z_0)$$

H/W

2. Show that the function $f(z) = \bar{z}$ is continous at the point $z = 0$ but not differentiable at $z = 0$

1. Function $f(z) = |z|^2$ is differentiable at $z = 0$ and no where else

$$f(z) = |z|^2 = z\bar{z}$$

At $z_0 = 0$

$$\lim \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{f(z)}{z}$$

$$= \frac{z\bar{z}}{z} = \bar{z} = 0$$

$\therefore$ Differentiable at $z = 0$

when $z_0 \neq 0$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Let $z = z_0 + \Delta z$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)(\bar{z_0} + \Delta \bar{z}) - z_0\bar{z_0}}{\Delta z}$$

$$= \lim_{\Delta z \to z} \left[z_0 \frac{\Delta \bar{z}}{\Delta z} + \bar{z_0} + \Delta \bar{z} \right]$$

$$= \lim \left[z_0 \frac{\Delta \bar{z}}{\Delta z} \right] + \lim_{\Delta z \to 0} \left[\bar{z_0} + \Delta \bar{z} \right]$$

$$= \lim_{\Delta z \to 0} \left[z_0 \frac{\Delta \bar{z}}{\Delta z} \right] + \bar{z_0}$$

$\therefore$ Does not exist.

Hence the function $|z|^2$ is not differentiable at $z \neq 0$.

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Note : $f(z) = |z|^2$ is differentiable at $z=0$ and no where else

$\therefore f^n$ is differentiable at $z=0$ but not analytic everywhere.

—— x ——

Cauchy - Rieman Equation

Necessary condition for function to be analytic :

Suppose that the function $f(z) = U(x,y) + iv(x,y)$ is continous in some neighbourhood of point $z = x + iy$ and is diff at z. Then the first order partial derivative of $U(x,y)$ and $v(x,y)$, exists.

$$U_x = V_y \quad \text{and} \quad U_y = -V_x$$

1. Show that the $f(z) = \sqrt{xy}$ is not analytic at origin. Although, cauchy - Reiman eq^n is satisfied at that point.

$$f(z) = u + iv = \sqrt{|xy|}$$

$$u = \sqrt{|xy|} \qquad v = 0$$

At the origin, we have

$$U_x = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x}$$

$$= \lim_{x \to 0} \frac{0-0}{x} = 0$$

$$U_y = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \to 0} \frac{0-0}{y}$$

$$= 0$$

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$v_x = 0$

$v_y = 0$

$\therefore\ u_x = v_y\ ;\ v_y = -u_y$

Hence, the C-R eqⁿ are satisfied at the origin.

$$f'(0) = \frac{f(z) - f(0)}{z}$$

$$= \lim_{z\to 0} \frac{\sqrt{|xy|} - 0}{x + iy}$$

Suppose, $z \to 0$ along $y = mx$, then we have

$$f'(0) = \lim_{x\to 0} \frac{\sqrt{|mx^2|}}{x + imx}$$

$$= \frac{\sqrt{|m|}}{1 + im}$$ — which depends on m.

So, the function is not analytic at origin.

for a function ~~to~~ be a

Sufficient condition for the function to be analytic

Suppose that the real and imaginary part $u(x,y)$ and $v(x,y)$ of the $f(z) = u(x,y) + iv(x,y)$ are continous ~~and have continous $f(z) = u(x,y) + iv(x,y)$ are~~ first order partial derivatives in a domain D. if u and v satisfy for CR eq^n for all points in D. then the $f(z)$ ~~the eqn~~ is analytic in D and $f'(z) = u_x + iv_x = v_y - iu_y$

i) $f(z)$ satisfies C-R eq^n

ii) u_x, u_y, v_x, v_y are continous

1. Using the C-R eq^n show that $f(z) = \bar{z}$ is not analytical at any point.

$$f(z) = \bar{z} = x - iy$$

$$u(x,y) = x$$

$$u_x = 1 \qquad v(x,y) = -y.$$

$$u_y = 0 \qquad v_x = 0$$

$$v_y = -1$$

CR eq^n is not satisfied at any point

Harmonic eq^n :

A real value function $\phi(x,y)$ of two variables x and y that has continous second order partial derivative in a domain D and satisfies the Laplace eq^n

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$ is said to be

harmonic.

Theorem

1. Real and imaginary parts of a laplace eqⁿ $U_{xx} + U_{yy} = 0$ and $V_{xx} + V_{yy} = 0$ respectively in D. i.e $u(x,y)$ and $v(x,y)$ are harmonic in D.

Harmonic Conjugate

Let $u(x,y)$ be a harmonic function then a function $v(x,y)$ is said to harmonic conjugate of $u(x,y)$, if i) $v(x,y)$ is harmonic ii) $u_x = v_y$ and $u_y = -v_x$ i.e u and v are satisfy the C.R eqⁿ.

~~If f(z) analytic~~ in D. ~~f(z) is harmonic~~

1) If $f(z) = u + iv$ is analytic in D then v is the h.c of u. Conversely if v is the h.c of u in a D then $f(z) = u + iv$ is analytic in D.

Determination of the conjugate function

Suppose $f(z) = u + iv$ is analytic function. u, v are conjugate function.

$u(x,y)$ = known function. $v(x,y) = ??$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \quad \text{[By C-R eq}^n\text{]} \quad (1)$$

~~for(i=0; i<1000; i++)~~

~~1432~~

~~1434~~

~~Syntax~~

Ordinary diff. eqⁿ

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{--- (1)}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{--- (2)}$$

If (2) is satisfied then we say that eqⁿ(1) is exact diff. eqⁿ.

$$M = -\frac{\partial v}{\partial y} = -\frac{\partial^2 v}{\partial x^2}$$

$$N = \frac{\partial v}{\partial x} = \frac{\partial^2 v}{\partial x^2}$$

v is h.c

$\therefore$ by Laplace $\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = 0\right)$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Therefore v can be determined by integrating.

(1) DO $\int M\,dx = a$

(2) DO $\int N\,dy = b$

(3) sum of $a+b$ (deleting those terms which are already taken from (1))

ex.

$\int M\,dx = x^3 + xy$ $\quad \int N\,dy = y^3 + xy$

$$V = x^3 + xy + y^3 + c.$$

$$dV = M\,dx + N\,dy$$

Das and Mukherjee.

1. Prove that $u = y^3 - 3x^2y$ is a harmonic function. Determine its harmonic conjugate and find the corresponding function $f(z)$ in terms of z.

$$u = y^3 - 3x^2y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial x} = -6xy \qquad \frac{\partial u}{\partial y} = 3y^2 - 3x^2$$

$$\frac{\partial^2 u}{\partial x^2} = -6y \qquad \frac{\partial u^2}{\partial y^2} = 6y$$

$$-6y + 6y = 0$$

$\therefore$ It is a harmonic function.

Say, $v(x,y)$ is the harmonic conjugate of $u(x,y)$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \text{ [By C-R eq}^n\text{]}$$

$$dv = \underbrace{-(3y^2 - 3x^2)}_{M} dx + \underbrace{(-6xy)}_{N} dy$$

$$\int M\, dx = \int (3x^2 - 3y^2)\, dx$$

$$= x^3 - 3xy^2 \text{ (Taking } y \text{ as const)}$$

$$\int N\, dy = -\int 6xy\, dy = -3xy^2 \text{ (Taking } x \text{ as const)}$$

$v = x^3 - 3xy^2 + c$, where c is a real const

$f(z) = u + iv$

$= y^3 - 3x^2y + i(x^3 - 3xy^2 + c)$

$= i(x+iy)^3 + ic$

$f(z) = i(z^3 + c)$

— x —

Complex integration

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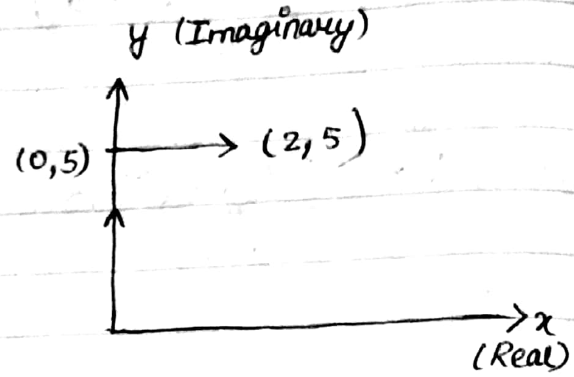
1. Evalute : $I = \int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$

along the line from (0,1) to (0,5) and then from (0,5) to (2,0)



For $x = 0$, y axis from 1 to 5.

The eq^n of the line is

$x = 0$

$\therefore\ dx = 0$

$$I_1 = \int_1^5 2y\,dy = \cdots = 24$$

Along (0,5) to (2,5), $y = 5 \Rightarrow dy = 0$

x varies from 0 to 2

$$I_2 = \int_0^2 (3x+5)\,dx = 16$$

So, our given integral

$$I = I_1 + I_2 = 24 + 16 = 40.$$

2. Evalute the following integrals:

$\int_C z^2 dz$, where z is the arc of circle

$|z| = 2$ from $\theta = 0$ to $\theta = \pi/2$

Let $z = 2e^{i\theta}$

$dz = 2ie^{i\theta}d\theta$

$$I = \int_c z^2 dz = \int_0^{\pi/3} [2e^{i\theta}]^2 \cdot 2ie^{i\theta} d\theta$$

$$= 8\int_0^{\pi/3} [e^{i\theta}]^2 ie^{i\theta} d\theta$$

$$= -\frac{16}{3}$$

3. Find the value of the integral

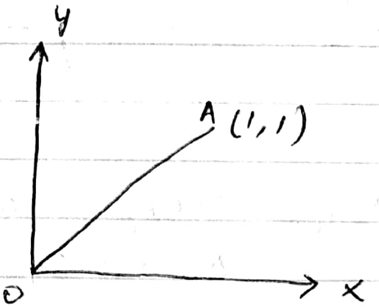
$$\int_0^{1+i} (x - y + ix^2) dz$$

i) Along the St. Line from $z=0$ to $z = 1+i$

ii) Along the real axis from $z=0$ to $z=1$ and then along a line parallel to the imaginary axis from $z=1$ to $z = 1 = 1+i$

Let $z = x+iy$

$dz = dx + idy$



The eq^n of the line OA is $y = x \Rightarrow dy = dx$

$$\int_0^{1+i} x - y + ix^2 \, dz$$

$$= \int_0^1 (x - x + ix^2)(1+i)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^2}{2} + \frac{ix^3}{3}\right]_0^1 (1+i) = \left[\frac{i}{3}(\right.$$

Cauchy Integral Theorem

Let $f(z)$ be analytic and $f'(z)$ be continous in a simple connected domain D. Then

$\oint f(z)\,dz = 0$, along every simple

and smooth (piecewise smooth) ~~along every~~ closed curve C contained in D

1. If $f(z) = \frac{z^3+5z+6}{z-2}$, does Cauchy's theorem apply when the path of integration C is a circle of radius 3 with origin on centre.

NO

$f(z)$ is not analytic at $z=2$
when the path of integration is the circle $|z|=3$ the pt. $z=2$ lies inside C so $f(z)$ is not analytic within C. Cauchy's theorem is not applicable i.e $\int \frac{z^2+5z+c}{z-2}\,dz \neq 0$.

2. Verify Cauchy's theorem for the function $z^3 - iz^2 - 5z + 2$ if C is the circle $|z-1|=2$

Ans) $f(z) = z^3 - iz^2 - 5z + 2$

Since $f(z)$ is $p(x)$ in z, ∴ it is analytic within C.

On C, we can ~~dem~~ close $z-1 = 2e^{i\theta}$ $0 \le \theta \le 2\pi$

$z = 1 - 2e^{i\theta}$

$dz = 2e^{i\theta} d\theta$

Now,

$$\int_{i} f(z)\,dz = \int_0^{2\pi} (z^3 - iz^2 - 5z + 2)\, 2e^{i\theta} d\theta$$

$$= 0$$

$\therefore$ It follows cauchy Integral theorem.

Cauchy Integral Formula

Simple connected domain.

A connected domain is single if every simple closed curve inside D enclose only points of D.

In a simple connected domain (D) let z_0 be a point D and C be any single closed curve in D enclosed in the point $z = z_0$ where

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0}\, d(z)$$

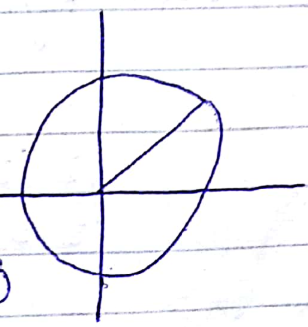
where C is transversed in the anti clockwise direction.

1. Evalute $\int_C \frac{1}{z(z-1)} dz$ when C is the circle $|z| = 3$

$$z(z-1) = 0$$

$$\Rightarrow z = 0, \; z = 1$$

Since these points $z = 0$ and $z = 1$ as within C.



we write $f(z) = 1$ —— (1)

$$\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{(z-1)}$$

$$1 = A(z-1) + Bz$$

$$z = 0 \Rightarrow A = -1$$
$$z = 1 \Rightarrow B = 1$$

$$\frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1} \quad — (2)$$

$$\int_c \frac{1}{z(z-1)} dz = \int \frac{f(z)}{z(z-1)} \quad \text{[By using (1)]} \quad — (3)$$

Now, by cachy integral formula, we have

$$\int_c \frac{f(z)}{z-1} dz = 2\pi i f(1) = 2\pi i$$

$$\int \frac{f(z)}{z} dz = \int_c \frac{f(z)}{z-0} = 2\pi f(0) = 2\pi i$$

From 3,

$$\int \frac{1}{z(z-1)} dz = 2\pi i - 2\pi i$$
$$= 0$$

2. Evalute $\int_c \frac{z^2-4}{z(z^2+9)}$ when c is the circle $|z| = 3$

$$z(z^2+9) = 0$$
$$z = \pm 3i, \; z = 0$$

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Since the point $z=0$ as within O only.

$$\cancel{\frac{z^2-4}{z(z^2+9)}} \qquad f(z) = \frac{z^2-4}{z^2+9}$$

$$\int_C \frac{z^2-4}{z(z^2+9)}\,dz = \int_C \frac{f(z)}{z-0}\,dz$$

$$= 2\pi i f(0)$$

$$= 2\pi i(-4/9)$$

$$= -\frac{8\pi i}{9}$$

Series Equation

Taylor's theorem

Let $f(z)$ be analytic at all points within a circle C_0 with centre z_0 and radius r_0. Then for every point z within C_0 we have

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f'(z_0)}{2!}(z-z_0)^2 + \dots + \frac{f''(z_0)}{n!}(z-z_0)^n + \dots$$

$$= f(z_0) - \sum_{n=1}^{\infty} \frac{(z-z_0)^n}{n!} f^n(z_0)$$

$$= \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} f''(z_0)$$

Laurent's theorem

Let $f(z)$ be analytic in a ring shaped region B bounded by two concentric circles C_1 and C_2 with center z_0 and radii r_1 and r_2 where $(r_1 > r_2)$ and z be any point of B.

Then,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^n$$

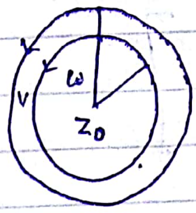
where,

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w-z_0)^{n+1}} \, dw$$

and

$$b_n = \frac{1}{2\pi i} \int_{C_2} (w-z_0)^{n-1} f(w) \, dw,$$

$n = 1, 2, 3, \dots$



Uniqueness theorem

Suppose that we have obtained in any manner or as a defination of $f(z)$ the formula

$$f(z) = \sum_{n=-\infty}^{\infty} P_n (z-z_0)^n$$

which is analytic in the region $(r_2 < |z-z_0| < r_1)$ then the series in necessarily identical with Laurent's series of $f(z)$.

1. Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent's series valid for the regions

i) $|z| < 1$

ii) $1 < |z| < 3$

iii) $|z| > 3$

iv) $0 < |z+1| < 2$

Ans = We have $f(z) = \frac{1}{(z+1)(z+3)}$

Let, $\frac{1}{(z+1)(z+3)} = \frac{A}{(z+1)} + \frac{B}{(z+3)}$.

$A = 1/2$, $B = -1/2$.

i) $|z| < 1$

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$$= \frac{1}{2}(z+1)^{-1} - \frac{1}{6}\left(1 + \frac{z}{3}\right)^{-1}$$

$\left[\frac{1}{1+x} = 1 + x + x^2 + x^3 + \cdots, \ |x| < 1. \right.$

$\left. \frac{1}{1-x} = 1 - x + x^2 - x^3 \cdots \quad |x| < 1 \right]$.

$= \frac{1}{2}\left(1 - z + z^2 - z^3 + \dots\right) - \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \dots\right)$

$= \frac{1}{3} - \frac{4}{9} z + \frac{13}{27} z^2 \dots$

ii) $1 < |z| < 3$

Then we have, $\frac{1}{|z|} < 1$ and $\frac{|z|}{3} < 1$

$\frac{1}{2(z+1)} = \frac{1}{2z}\left(1 + \frac{1}{z}\right)^{-1}$

$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \dots\right)$

$\frac{1}{2(z+3)} = \frac{1}{6\left(1 + \frac{z}{3}\right)} = \frac{1}{6}\left(1 + \frac{z}{3}\right)^{-1}$

$= \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \dots\right)$

$f(z) = \dots + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{1}{18} z - \frac{1}{54} z^2 + \frac{1}{162} z^3 - \dots$

iii) $|z| > 3 \Rightarrow \frac{3}{|z|} < 1$

$\frac{1}{|z|} < \frac{3}{|z|} < 2$

$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)} = \frac{1}{2z}\left(1 + \frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1 + \frac{3}{z}\right)^{-1}$

$=$

iv) $0 < |z+1| < 2$

Let $u = z+1$

$$0 < |u| < 2$$

~~Then, we have~~

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)} = \frac{1}{2u}\left(1+\frac{u}{2}\right)^{-1}$$

$$= \cdots$$

$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{z+1}{8} - \frac{(z+1)^2}{16} + \cdots$$

H/w

Expand $\frac{(z-2)(z+2)}{(z+1)(z+4)}$ for

i) $|z|<1$ ii) $1<|z|<4$ iii) $|z|>4$

Hint: $f(z) = 1 - \frac{5z+8}{(z+1)(z+4)}$

$$= 1 - \frac{1}{z+1} - \frac{4}{z+4}$$

Singularities of analytic function and calculus of Residue:

Zeroes of a analytic function

The value of z for which the analytic function $f(z)$ becomes zero is ~~call~~ said to be the zeros of $f(z)$

$$f(z) = z^2+2.$$

Singularities

If a function is analytic at points of a bounded domain except at a ~~bound~~ finite numbers of points. Then these exception points are called singularity.

1) Isolated singular point: A singular point z_0 is said to isolated if there is a deleted neighbourhood $0 < |z-z_0| < \epsilon$, for $\epsilon > 0$ of z_0 throughout which f is analytic. Other wise the function is called

2) ~~Non-isol~~ Non-isolated singular Point: (Limit point of sequence of poles).

$$\text{eg}: f(z) = \frac{z^2+5}{z(z-3)(z^2-1)}$$

The func is analytic except the points $z = 0, 3, \pm 1$

These are the isolated singular points.

E.g $f(z) = \log(z)$

$z=0 \rightarrow$ Singular point

Every neighbourhood of zero contains points on the negative real ones where $\log z$ is not analytic

So, $z=0 \rightarrow$ non-isolated singular pt.

2)

Types of isolated singular points :-

Let $z=z_0$ be an isolated singular point of a function singular point then $f(z)$ has Laurent

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \underbrace{\sum_{n=1}^{\infty} b_n(z-z_0)^{-n}}_{\text{principal part of } f \text{ at } z_0.}$$

a) Pole

If the principal part of $f(z)$ at ~~z_0~~ z_0 consists of a finite number of term say m, then $z=z_0$ is called a pole of order m, ~~of~~ $f(z)$ has an expansion of the ~~the~~ form

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots$$

$$\dots + \frac{b_m}{(z-z_0)^m}$$

A pole of order $m=1$ is called a simple pole.

e.g. $f(z) = \dfrac{e^z}{z^2+4}$

singular point $z = \pm 2$

$$= e^{2}\left[\frac{A}{z+2i} + \frac{B}{z-2i}\right]$$

b) Removable singular point.

If all the term b_n in the laurent expansion $f(z)$ about the isolated singular point $z=z_0$ is zero. Then the isolated single point is called removable singular point.

$$f(z) = \frac{\sin(z-1)}{z-1)}$$

$$\frac{\sin(z-1)}{z-1} = \frac{1}{(z-1)}\left[(z-1) - \frac{(z-1)^3}{3!} + \frac{(z-1)^5}{5!} - \cdots\right]$$

$$= \left[1 - \frac{(z-1)^2}{2!} + \frac{(z-1)^4}{5!} - \cdots\right]$$

Here $z=1$ is removable ~~singular~~ singular point.

This singularity can be made disappear by defining $\frac{\sin(z-1)}{z-1)} = 1$ at $z=1$ so f^n become analytic at $z=1$.

c) Isolated Essential Singularity:

If an infinite number of the coefficients b_n in the principal part of Laurent's expansion of $f(z)$ about the isolated singular point z_0 then z_0 is said to be an essential singular point of z_0.

Note that the Laurent part of zero is called

Isolated essential singularities of $f(z)$

e.g.: $f(z) = \sin\frac{1}{z}$ → singular point $z=0$

$$\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} + \dots$$

This expansion contain infinite number of term in negative powers of z.

10 Find the zero's and nature of singularities of the function $f(z) = \frac{z-2}{z^2}\sin\frac{1}{z-1}$

Zeros

$f(z) = 0$

$\Rightarrow (z-2)\sin\frac{1}{z-1} = 0$

$z = 2, \quad \frac{1}{z-1} = n\pi, \quad n = 0, 1, 2.$

$\Rightarrow z = 1 + \frac{1}{n\pi}$

$z = 2, \; 1 + \frac{1}{n\pi}$ (Ans)

singularities

0, 1

$z = 0$: pole of order 2

$z = 1$: Isolated essential singularity

2. $f(z) = \frac{1}{\tan(\pi/z)}$

It has infinite of isolated singularities which lie on the real axis from $z = -1$ to 1.
The isolated singular points are given by

$$z = \pm \frac{1}{n} \quad (n = 1, 2, 3, \ldots\ldots)$$

The origin is also a singular point (non-isolated type) such in every nbd of 0 then are infinite numbers of other singularities.

3. $f(z) = \frac{\cot \pi z}{(z-a)^2} = \frac{\cos \pi z}{\sin(\pi z)(z-a)^2}$

$(z-a)^2 \cdot \sin \pi z = 0$

$\Rightarrow z = a,\ z = n\ (0, \pm 1, \pm 2)$
↳ singular points

$z = a \rightarrow$ pole of order 2

$z = n \rightarrow$ Simple pole

$z = \pm \infty \rightarrow$ non-isolated singular point

$f(z) = \operatorname{cosec}\left(\frac{1}{z}\right) = \sin\left(\frac{1}{z}\right)$

$\frac{1}{z} = n\pi \Rightarrow z = \left(\frac{1}{n\pi}\right)$ n is any integer.

Since, $z = 0$ is a limit point of these poles.
therefore $z = 0$ is non-isolated essential singularities.

Residue

Let $f(z)$ be a function analytic in a Domain D except at some isolated points. Let $a \in D$ be a point of isolated singularity of the function $f(z)$. Then there is an annulus $0 < |z-a| < R$ is in which $f(z)$ can be expanded in Laurent's series as follows

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}.$$

The co-efficient b_1 is known as the residue of $f(z)$ at the point 'a'

$$r_1 < |z-a| < r_2$$

$$f(z) - \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{f(w-a)^{n+1}} dw$$

$$b_n = \frac{1}{2\pi i} \int_C (w-a)^{n-1} f(w)\, dw$$

where $c = |z - z_0| = \rho$, $r_2 < \rho < r_1$ are traverse in the anti-clockwise direction.

Computation of Residue at a finite order pole:

i) Residue at a simple Pole:

If $z = a$, is a simple pole of $f(z)$ then the residue at the simple pole $z=a$ is given by $\lim_{z \to a} [(z-a) f(z)]$

ii) Residue at a pole of order m:

If $z = a$ is a pole of order n then the residue at $z=a$ is given by

$$\frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)]$$

Cauchy's Residue theorem:

let a function $f(z)$ be analytic and single value inside and on a closed contour C except at a finite number of poles.

$$\int_C f(z)\, dz = 2\pi i \sum R$$

where $\sum R$ is the sum of the residues of $f(z)$ at its poles within C.

1. Find the residue of $\frac{1}{(z^3+1)^3}$ at $z=i$

we have

$$f(z) = \frac{1}{(z^2+1)^3} = \frac{1}{(z+i)^3(z-i)^3}$$

$z = i, -i$ are poles of order 3

$\therefore$ Residue of $z=i$ is $\frac{1}{2!} \lim_{z \to i} \frac{d^2}{dz^2} [(z-i)^3 f(z)]$

$$= \frac{1}{2} \lim_{z \to i} \frac{d^2}{dz^2} \left[\frac{1}{(z+1)^3}\right]$$

$$= \frac{1}{2} \lim_{z \to i} \frac{d}{dz} \left[\frac{(-3)}{(z+i)^4} \right]$$

$m = 3$

$$= \frac{1}{2} \lim_{z \to i} \frac{12}{(z+i)^5}$$

$$= \frac{1}{2} \times \frac{12}{2^5 i^5} = \frac{6}{2^5 i} = \frac{3i}{16} \text{ (Ans)}$$

2. Determine the poles of the function $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ and the residue at each point. Hence evalute $\int_c f(z)\,dz$, where c is the circle $|z| = 2.5$

Here, $z = 1$ is a pole of order 2 and $z = -2$ is a simple pole of $f(z)$.

Residue of $z = 1$ is

$m = 2$.

$$\frac{1}{1!} \lim_{z \to 1} \frac{d'}{(dz)'} \left[(z-1)^2 f(z) \right]$$

$$\lim_{z \to 1} \frac{d}{dz} \left((z-1 \right.$$

$$= \frac{5}{9}$$

The function $f(z)$ is analytic on $|z| = 2.5$ at all points inside it ~~or~~ except at $z = 1, -2$,

$$\int_{C: |z| = 2.5} f(z)\,dz = 2\pi i \sum R = 2\pi i \,(\text{Residue at } z = 1 + \text{Residue at } z = -2)$$

$$= 2\pi i \times \left(\frac{5}{9} + \frac{4}{9}\right)$$

$$= 2\pi i$$

H/W

Find ~~Heuris on~~ residue $f(z) = \dfrac{z^2 - 2z}{(z+1)^2 (z^2+4)}$

Singular Point.

A point $z = z_0$ at which the function $f(z)$ is not analytic is called a singular point of $f(z)$.

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Fourier Series

Fourier series corresponding to a function $f(x)$ in an interval $-\pi < x < \pi$ under certain condition (Dirichlet's conditions) is the trignometric series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where a_0, $\{a_n\}$, $\{b_n\}$ are constants determined from

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{n} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{n} \int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

The conditions under which a function $f(x)$ on an interval $-\pi \leq x \leq \pi$ can be expanded in an convergent fourier series are known as Dirichlet's conditions. These are as follows:

* $f(x)$ is bounded and integrable on $-\pi \leq x \leq \pi$
* $f(x)$ has a finite number of discontinous in any given bounded interval and the discontinuties cannot be infinite

- The interval can be broken up into a finite number of open partial intervals in each of which $f(x)$ is montonic (piecewise monotonic).

- the integral $\int_{-\pi}^{\pi} f(x)\,dx$ is to satisfied convergence

convergence theorem : If the function $f(x)$ satisfies Dirichlet's condition the fourier series of $f(x)$ on the interval $[-\pi, \pi]$ converge to $f(x)$ at a point of continuity.
At point of continuity the fourier series converges to $\frac{1}{2}[f(x+) + f(x-)]$, where $f(x+)$ and $f(-x)$ are right and left limit respe.

At the both end points of the interval $[-\pi, \pi]$ the fourier series converges to $\frac{1}{2}[f(-\pi+) + f(\pi-)]$

1. Fourier series expansion

$$f(x) = \begin{cases} \pi + x , & -\pi \leq x < 0 \\ 0 , & 0 \leq x \leq \pi \end{cases}$$

The fourier coefficients are obtained as follows

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (\pi + x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} 0\,dx = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} x\cos nx\,dx + \int_{-\pi}^{0} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi \sin nx}{n} + \frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{\cos n\pi}{n^2}\right]$$

$$= \frac{1}{\pi n^2}\left[1 - \cos n\pi\right]$$

$$= \frac{1}{\pi n^2}\left[1 - (-1)^n\right]$$

$$= \begin{cases} 0, & n = \text{even} \\ \frac{2}{\pi n^2}, & n = \text{odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} \pi \sin nx \, dx + \int_{-\pi}^{0} x \sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi \cos nx}{n} + \left(-\frac{x \cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\right] = -\frac{1}{n}$$

Therefore, the fourier series expansion is given by

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{\pi n^2}\left\{1 - (-1)^n\right\} \cos nx - \frac{1}{n} \sin nx\right]$$

$$= \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right]$$

$$-\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots\right]$$

1) $f(x)$ is even function implies $f(-x) = f(x)$

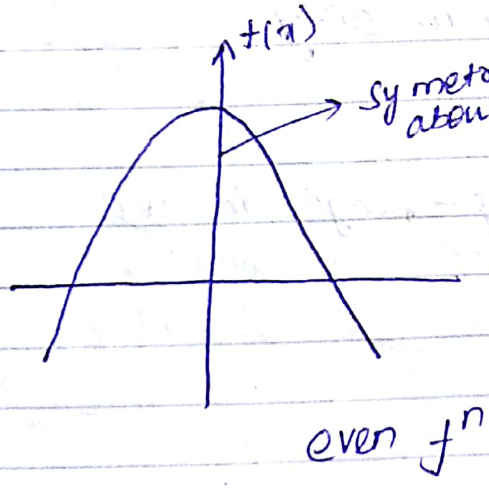
and $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

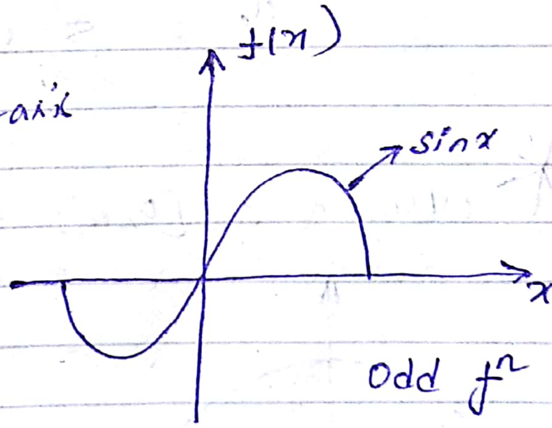
e.g: $f(x) = \cos(-x)$
$= \cos x$
x^2,

2) $f(x)$ is odd function implies $f(-x) = -f(x)$

$$\int_{-a}^{a} f(x)\,dx = 0$$

e.g: $\sin x$, x





3) when $f(x)$ is odd function, $f(x)\cos nx$ is odd
, $f(x)\sin nx$ is even

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} \underbrace{f(x)\cos nx}_{\text{odd}}\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

4) when $f(x)$ is even fn, then $f(x)\cos nx$ is even, $f(x)\sin nx$ is odd.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

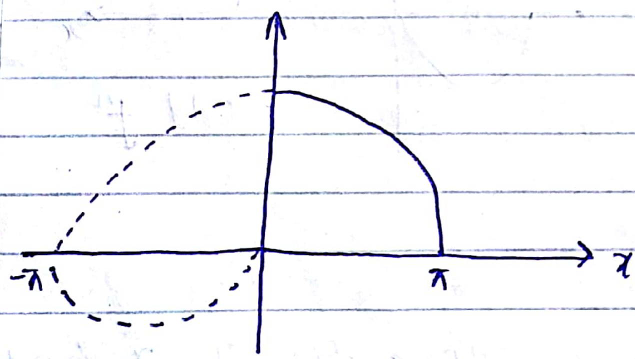
$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

which indicating a cosine series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \text{ only.}$$

* when $x \in [0, \pi]$ half-range interval



$f(-x) = f(x) \rightarrow$ even
$f(-x) = -f(x) \rightarrow$ odd

Fourier Cosine Series:

Let $f(x)$ satisfies Dirichlet's condition on $0 \le x \le \pi$. Then the fourier cosine series expansion on half-range interval $[0, \pi]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where, $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$ and

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx$$

Fourier sine series:

Let $f(x)$ satisfies Dirichlet's condition on $0 \le x \le \pi$. Then the fourier series comparison on half-rang interval $[0, \pi]$ is given by.

$$\sum_{n=1}^{\infty} b_n \sin nx$$

where, $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx$.

1. Expand $f(x) = x$ in fourier series in the interval $-\pi \le x \le \pi$.

$$a_0 = \frac{2}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{2}{\pi} \int_{-\pi}^{\pi} x\, dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{2} \right]_{-\pi}^{\pi}$$

$$= \frac{2}{\pi} \times 0 = 0$$

$a_0 = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\cos nx}_{odd} = 0,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\sin nx}_{even}$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\, dx$$

$$= \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(-\frac{\pi\cos n\pi}{n}\right)$$

$$= -\frac{2}{n}\cos n\pi = \begin{cases} -2/n, & n \text{ even} \\ 2/n, & n \text{ odd} \end{cases}$$

Fouriers Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= 2\left\{\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right\}$$

Ans

(A) $f(x)$ is defined for $x \in [-l, l]$

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right)$$

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where, $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$; $a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

The expressions for the co-efficient are called Euler formulas.

B) $f(x)$ is defined on $[0, l]$

Fourier Cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$ and $a_n = \int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

c) $f(x)$ is defined on $[0, l]$

fourier sine series :

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where, $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

1) Find the fourier cosine series of the function $f(x) = \begin{cases} x^2, & 0 \le x \le 2 \\ 4, & 2 \le x \le 4 \end{cases}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{4}\int_0^2 x^2\,dx + \frac{2}{4}\int_2^4 4\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 + \frac{1}{2}\left[4x\right]_2^4$$

$$= \frac{1}{2}\times\frac{2^3}{3} + \frac{1}{2}\times\left[4\times4 - 4\times2\right]$$

$$\cancel{= \frac{16}{3} + \frac{1}{2}\times 8\,4}$$

$$\cancel{= \frac{16}{3} + 4 = \frac{16+}{3}}$$

$$\frac{4}{3} + 4$$

$$= \frac{4+12}{3} = \frac{16}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{4}\int_0^2 x^2\cos\left(\frac{n\pi x}{4}\right)dx + \int_2^4 4\cos\left(\frac{n\pi x}{4}\right)dx$$

$$= \frac{1}{2}\left[\underbrace{\int_0^2 x^2\cos\left(\frac{n\pi x}{4}\right)dx}_{I} + \int_2^4 4\cos\frac{n\pi x}{4}\,dx\right]$$

$$I = \int x^2 \cos\left(\frac{n\pi x}{4}\right) dx = \cancel{\frac{x^2 \sin\left(\frac{n\pi x}{4}\right)}{(n\pi/4)}}$$

$$= \frac{x^2 \sin\left(\frac{n\pi x}{4}\right)}{(n\pi/4)} - \int \frac{2x \sin\left(\frac{n\pi x}{4}\right)}{(n\pi/4)}$$

$$= \frac{x^2 \sin\left(\frac{n\pi x}{4}\right)}{n\pi/4} + \frac{2x \cos\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)^2} - \frac{2\sin\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)^3}$$

$$a_n = \frac{1}{2}\left[\left[\frac{x^2 \sin\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)} + \frac{2x \cos\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)^2} - \frac{2\sin\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)^3}\right]_0^2 + 4\left[\frac{\sin\frac{n\pi x}{4}}{\frac{n\pi}{4}}\right]_2^4\right]$$

$$= \frac{1}{2}\left[\cancel{\frac{4\sin(n\pi/2)}{(n\pi/4)}} + \frac{4\cos(n\pi/2)}{(n\pi/4)^2} - \frac{2\sin(n\pi/2)}{(n\pi/4)^3} - \cancel{\frac{4\sin(n\pi/2)}{(n\pi/4)}}\right]$$

$$= \frac{1}{2} \times \frac{4}{(n\pi)^2}\left[\cos(n\pi/2) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{32}{\pi^2 n^2}\left[\cos\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]$$

Fourier Cosine series is

$$f(x) = \frac{8}{3} + \frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\cos\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]\cos\left(\frac{n\pi x}{4}\right)$$

1. Find the fourier cosine series of the function $f(x) = x\sin x$ on $[0,\pi]$

Hence, deduce that

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7}$$

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx = \ldots\ldots = 2$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^\pi x\sin x\cos nx\,dx$$

$$= \frac{2}{2\pi}\int_0^\pi x\{\sin(n+1)x - \sin(n-1)x\}\,dx$$

$$= \frac{1}{\pi}\left[-\frac{x\cos(n+1)x}{(n+1)} + x\frac{\cos(n-1)x}{(n-1)}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos(n+1)\pi}{(n+1)} + \frac{\pi\cos(n-1)\pi}{(n-1)}\right]$$

$$= \frac{(-1)^{n-1}}{(n-1)} - \frac{(-1)^{n+1}}{(n+1)}$$

$$= (-1)^{n-1}\left\{\frac{1}{n-1} - \frac{1}{n+1}\right\}$$

$$= (-1)^{n-1}\frac{2}{n^2-1} \quad (n \neq 1)$$

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$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx = \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$= \cdots = -1/2$$

Fourier cosines series

$$f(x) \simeq 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty} \frac{(-1)\cos nx}{n^2 1}$$

$$x \sin x = 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty} \frac{(-1)^n \cos nx}{n^2 - 1}$$

$$x = \pi/2 = 1 - 0 - 2\sum_{n=2}^{\infty} \frac{(-1)^n \cos n\pi/2}{n^2 - 1}$$

$$\pi/2 = 1 + 2\left[\frac{1}{2^2-1} + 0 - \frac{1}{4^2-1} + 0 + \frac{1}{6^2-1} + \cdots\right]$$

$$= 1 + 2\left[\frac{1}{2^2-1} - \frac{1}{4^2 1} + \frac{1}{6^2 - 1} - \cdots\right]$$

$$\pi/4 = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 2} \quad \text{(Proved)}$$