Optimization Techniques

Linear Programming: Model Formulation and Graphical Solution

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TOPICS

- Model Formulation
- A Maximization Model Example
- Graphical Solutions of Linear Programming Models
- A Minimization Model Example
- Irregular Types of Linear Programming Models
- Characteristics of Linear Programming Problems

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LINEAR PROGRAMMING: AN OVERVIEW

- Objectives of business decisions frequently involve *maximizing profit* or *minimizing costs*.
- Linear programming uses *linear algebraic relationships* to represent a firm's decisions, given a business *objective*, and resource *constraints*.
- Steps in application:
  1. Identify problem as solvable by linear programming.
  2. Formulate a mathematical model of the unstructured problem.
  3. Solve the model.
  4. Implementation

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MODEL COMPONENTS

- **Decision variables** - mathematical symbols representing levels of activity of a firm.
- **Objective function** - a linear mathematical relationship describing an objective of the firm, in terms of decision variables - this function is to be maximized or minimized.
- **Constraints** – requirements or restrictions placed on the firm by the operating environment, stated in linear relationships of the decision variables.
- **Parameters** - numerical coefficients and constants used in the objective function and constraints.

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SUMMARY OF MODEL FORMULATION STEPS

Step 1 : Clearly define the decision variables

Step 2 : Construct the objective function

Step 3 : Formulate the constraints

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LP MODEL FORMULATION A MAXIMIZATION EXAMPLE (1 OF 4)

- Product mix problem - Beaver Creek Pottery Company
- How many bowls and mugs should be produced to maximize profits given labor and materials constraints?
- Product resource requirements and unit profit:

Resource Requirements			
Product	Labor (Hr./Unit)	Clay (Ib./Unit)	Profit ($/Unit)
Bowl	1	4	40
Mug	2	3	50

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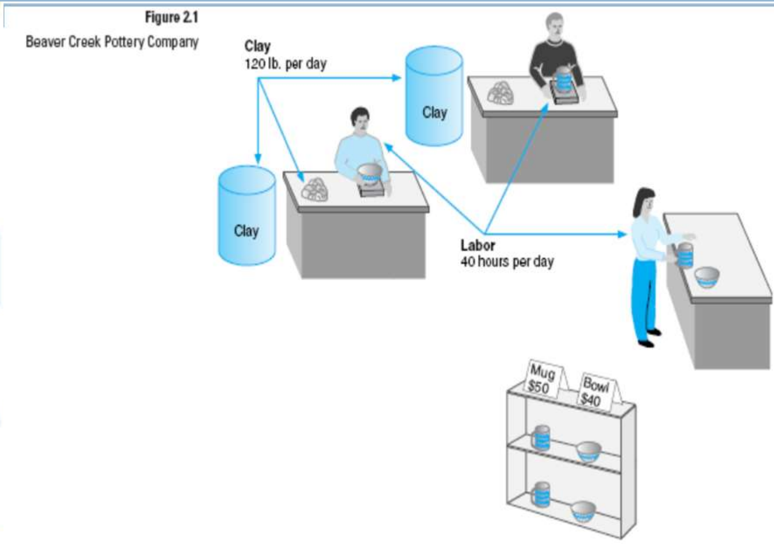
LP MODEL FORMULATION
A MAXIMIZATION EXAMPLE (2 OF 4)



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LP MODEL FORMULATION
A MAXIMIZATION EXAMPLE (3 OF 4)

Resource Availability:	40 hrs of labor per day 120 lbs of clay
Decision Variables:	x_1 = number of bowls to produce per day x_2 = number of mugs to produce per day
Objective Function:	Maximize $Z = \$40x_1 + \$50x_2$ Where Z = profit per day
Resource Constraints:	$1x_1 + 2x_2 \leq 40$ hours of labor $4x_1 + 3x_2 \leq 120$ pounds of clay
Non-Negativity Constraints:	$x_1 \geq 0$; $x_2 \geq 0$

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LP MODEL FORMULATION
A MAXIMIZATION EXAMPLE (4 OF 4)

Complete Linear Programming Model:

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$
$4x_1 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

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FEASIBLE SOLUTIONS

A *feasible solution* does not violate *any* of the constraints:

Example: x_1 = 5 bowls
x_2 = 10 mugs
$Z = \$40x_1 + \$50x_2 = \$700$

Labor constraint check: $1(5) + 2(10) = 25 < 40$ hours
Clay constraint check: $4(5) + 3(10) = 50 < 120$ pounds

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INFEASIBLE SOLUTIONS

An *infeasible solution* violates *at least one* of the constraints:

Example: x_1 = 10 bowls
x_2 = 20 mugs
$Z = \$40x_1 + \$50x_2 = \$1400$

Labor constraint check: 1(10) + 2(20) = 50 > 40 hours

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GRAPHICAL SOLUTION OF LP MODELS

- Graphical solution is limited to linear programming models containing ***only two decision variables*** (can be used with three variables but only with great difficulty).
- Graphical methods provide ***visualization of how*** a solution for a linear programming problem is obtained.

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COORDINATE AXES GRAPHICAL SOLUTION OF MAXIMIZATION MODEL (1 OF 12)

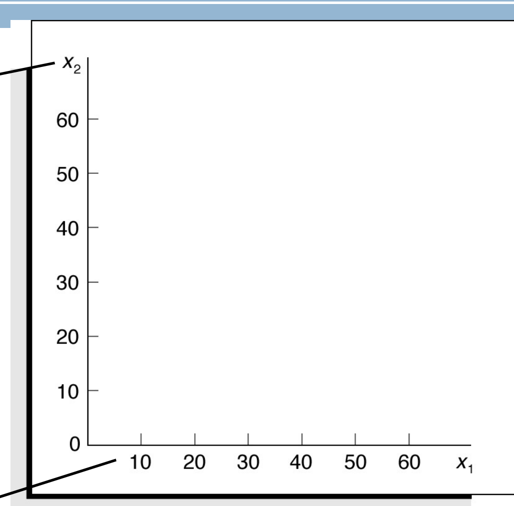
X_2 is mugs

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$



X_1 is bowls

Figure 2.2 Coordinates for Graphical Analysis

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LABOR CONSTRAINT GRAPHICAL SOLUTION OF MAXIMIZATION MODEL (2 OF 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

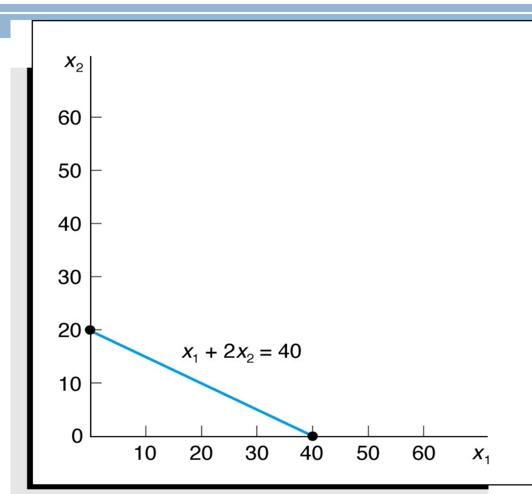

Figure 2.3 Graph of Labor Constraint

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LABOR CONSTRAINT AREA GRAPHICAL SOLUTION OF MAXIMIZATION MODEL (3 OF 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

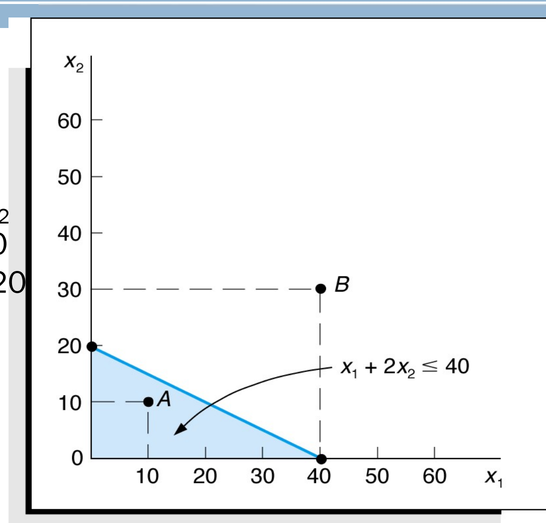

Figure 2.4 Labor Constraint Area

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CLAY CONSTRAINT AREA GRAPHICAL SOLUTION OF MAXIMIZATION MODEL (4 OF 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

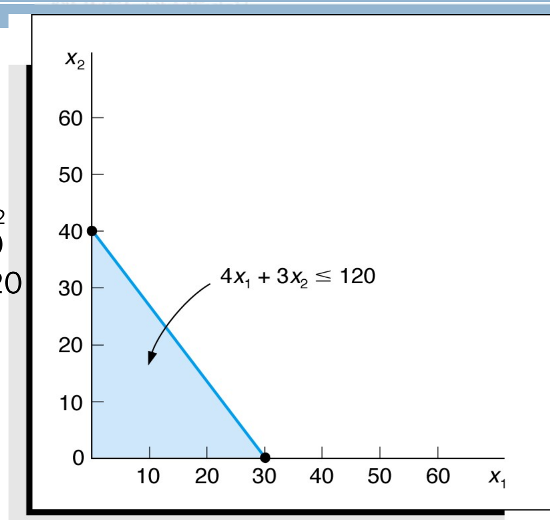

Figure 2.5 Clay Constraint Area

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BOTH CONSTRAINTS GRAPHICAL SOLUTION OF MAXIMIZATION MODEL (5 OF 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

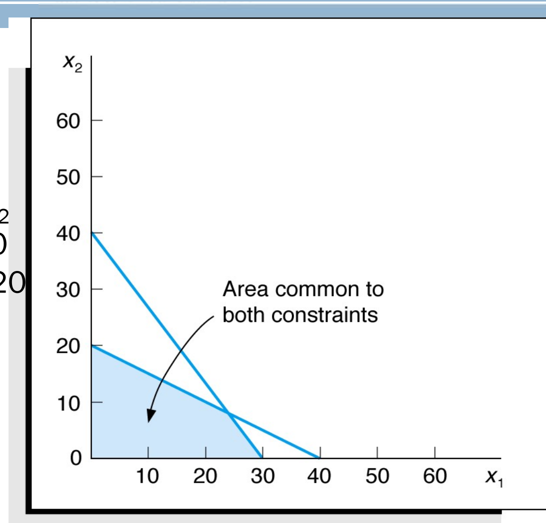

Figure 2.6 Graph of Both Model Constraints

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FEASIBLE SOLUTION AREA GRAPHICAL SOLUTION OF MAXIMIZATION MODEL (6 OF 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

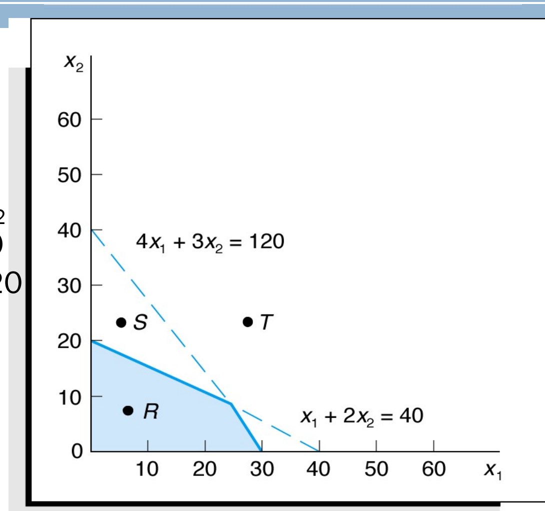

Figure 2.7 Feasible Solution Area

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OBJECTIVE FUNCTION SOLUTION = $800 GRAPHICAL SOLUTION OF MAXIMIZATION MODEL (7 OF 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

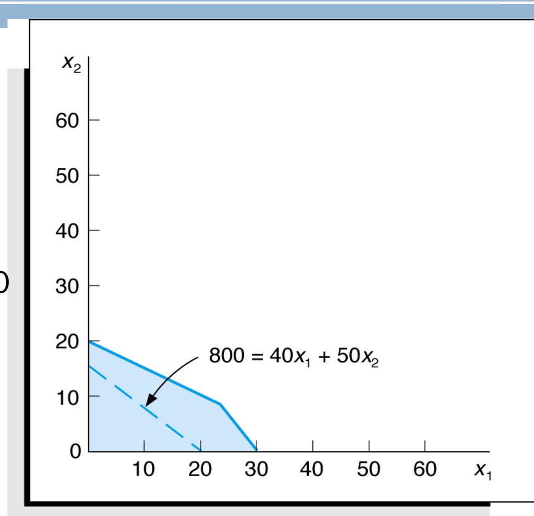

Figure 2.8 Objection Function Line for Z = $800

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ALTERNATIVE OBJECTIVE FUNCTION SOLUTION LINES GRAPHICAL SOLUTION OF MAXIMIZATION MODEL (8 OF 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

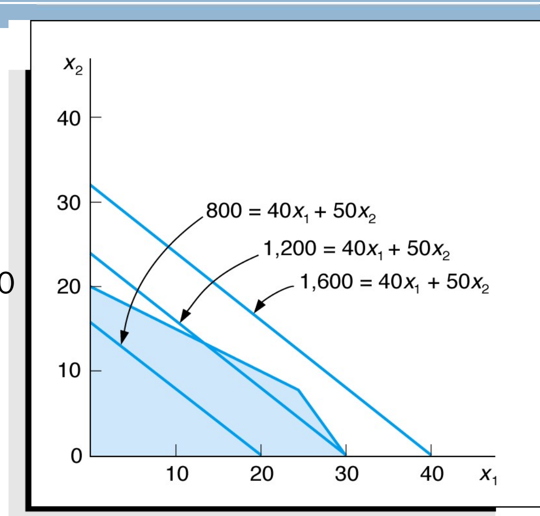

Figure 2.9 Alternative Objective Function Lines

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OPTIMAL SOLUTION GRAPHICAL SOLUTION OF MAXIMIZATION MODEL (9 OF 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_2 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

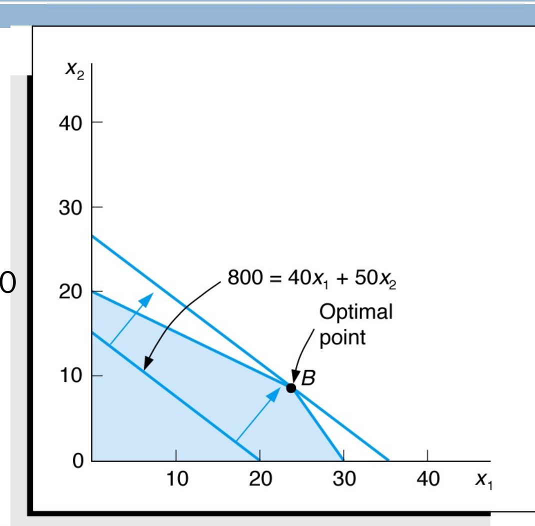

Figure 2.10 Identification of Optimal Solution Point

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OPTIMAL SOLUTION COORDINATES GRAPHICAL SOLUTION OF MAXIMIZATION MODEL (10 OF 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

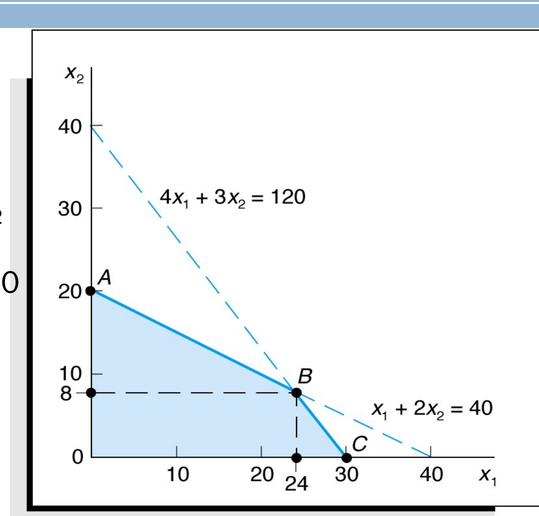

Figure 2.11 Optimal Solution Coordinates

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EXTREME (CORNER) POINT SOLUTIONS GRAPHICAL SOLUTION OF MAXIMIZATION MODEL (11 OF 12)

Maximize $Z = \$40x_1 + \$50x_2$
subject to: $1x_1 + 2x_2 \leq 40$
$4x_1 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

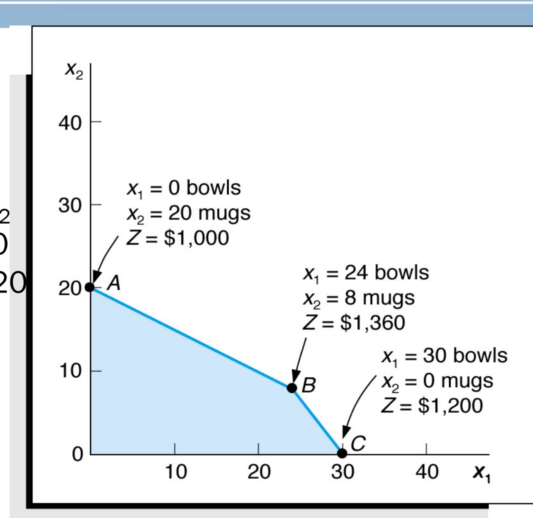

Figure 2.12 Solutions at All Corner Points

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OPTIMAL SOLUTION FOR NEW OBJECTIVE FUNCTION GRAPHICAL SOLUTION OF MAXIMIZATION MODEL (12 OF 12)

Maximize $Z = \$70x_1 + \$20x_2$
subject to: $1x_1 + 2x_2 \leq 40$
$4x_1 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

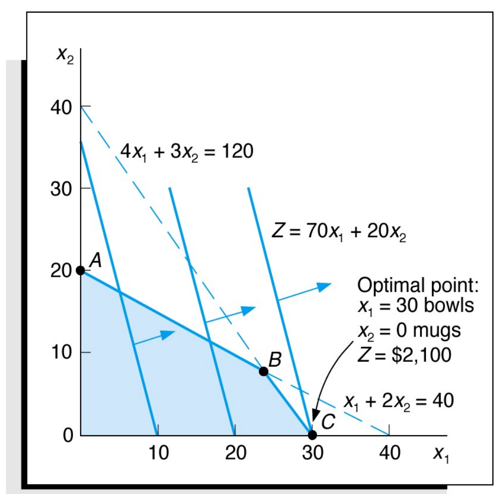

Figure 2.13 Optimal Solution with $Z = 70x_1 + 20x_2$

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SLACK VARIABLES

- Standard form requires that all constraints be in the form of equations (equalities).
- A slack variable is *added to a ≤ constraint* (weak inequality) to convert it to an equation (=).
- A slack variable typically represents an *unused resource*.
- A slack variable *contributes nothing* to the objective function value.

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LINEAR PROGRAMMING MODEL: STANDARD FORM

Max $Z = 40x_1 + 50x_2 + s_1 + s_2$

subject to: $1x_1 + 2x_2 + s_1 = 40$

$4x_1 + 3x_2 + s_2 = 120$

$x_1, x_2, s_1, s_2 \geq 0$

Where:

x_1 = number of bowls

x_2 = number of mugs

s_1, s_2 are slack variables

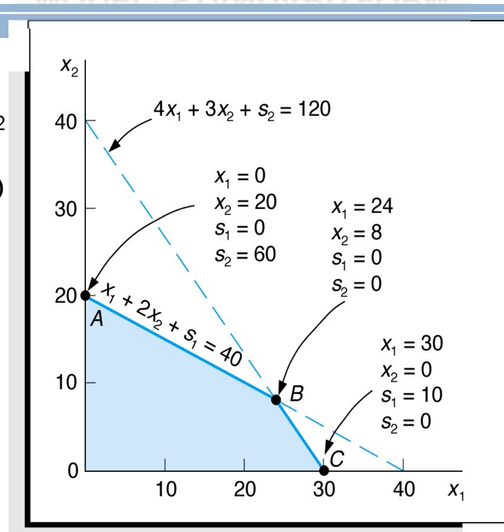

Figure 2.14 Solution Points A, B, and C with Slack

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LP MODEL FORMULATION – MINIMIZATION (1 OF 8)

- Two brands of fertilizer available - Super-gro, Crop-quick.
- Field requires at least 16 pounds of nitrogen and 24 pounds of phosphate.
- Super-gro costs \$6 per bag, Crop-quick \$3 per bag.
- Problem: How much of each brand to purchase to minimize total cost of fertilizer given following data ?

	Chemical Contribution	
Brand	Nitrogen (lb/bag)	Phosphate (lb/bag)
Super-gro	2	4
Crop-quick	4	3

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LP MODEL FORMULATION – MINIMIZATION (2 OF 8)

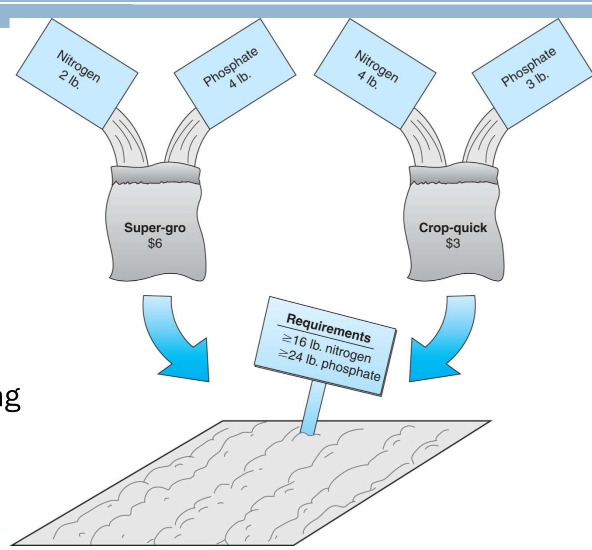

Figure 2.15 Fertilizing farmer's field

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LP MODEL FORMULATION – MINIMIZATION (3 OF 8)

Decision Variables:

x_1 = bags of Super-gro
x_2 = bags of Crop-quick

The Objective Function:

Minimize $Z = \$6x_1 + 3x_2$
Where: $\$6x_1$ = cost of bags of Super-Gro
$\$3x_2$ = cost of bags of Crop-Quick

Model Constraints:

$2x_1 + 4x_2 \geq 16$ lb (nitrogen constraint)
$4x_1 + 3x_2 \geq 24$ lb (phosphate constraint)
$x_1, x_2 \geq 0$ (non-negativity constraint)

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CONSTRAINT GRAPH – MINIMIZATION (4 OF 8)

Minimize $Z = \$6x_1 + \$3x_2$
subject to: $2x_1 + 4x_2 \geq 16$
$4x_1 + 3x_2 \geq 24$
$x_1, x_2 \geq 0$

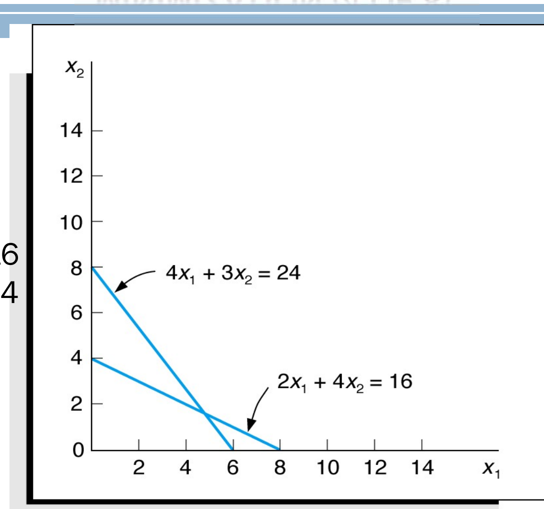

Figure 2.16 Graph of Both Model Constraints

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FEASIBLE REGION– MINIMIZATION (5 OF 8)

Minimize $Z = \$6x_1 + \$3x_2$
subject to: $2x_1 + 4x_2 \geq 16$
$4x_1 + 3x_2 \geq 24$
$x_1, x_2 \geq 0$

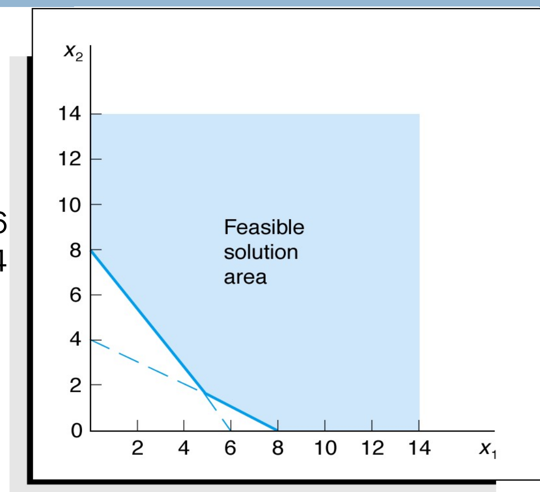

Figure 2.17 Feasible Solution Area

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OPTIMAL SOLUTION POINT – MINIMIZATION (6 OF 8)

Minimize $Z = \$6x_1 + \$3x_2$
subject to: $2x_1 + 4x_2 \geq 16$
$4x_1 + 3x_2 \geq 24$
$x_1, x_2 \geq 0$

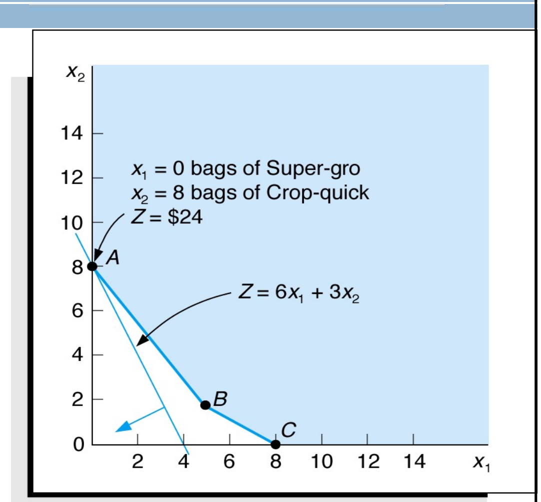

Figure 2.18 Optimum Solution Point

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SURPLUS VARIABLES – MINIMIZATION (7 OF 8)

- A surplus variable is *subtracted from a ≥ constraint* to convert it to an equation (=).
- A surplus variable *represents an excess* above a constraint requirement level.
- A surplus variable *contributes nothing* to the calculated value of the objective function.
- Subtracting surplus variables in the farmer problem constraints:

$$2x_1 + 4x_2 - s_1 = 16 \text{ (nitrogen)}$$
$$4x_1 + 3x_2 - s_2 = 24 \text{ (phosphate)}$$

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GRAPHICAL SOLUTIONS – MINIMIZATION (8 OF 8)

Minimize $Z = \$6x_1 + \$3x_2 + 0s_1 + 0s_2$

subject to: $2x_1 + 4x_2 - s_1 = 16$

$4x_1 + 3x_2 - s_2 = 24$

$x_1, x_2, s_1, s_2 \geq 0$

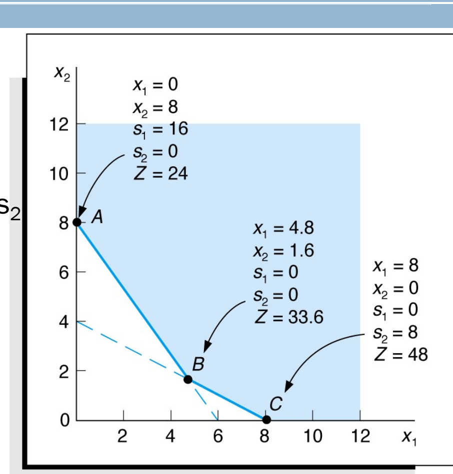

Figure 2.19 Graph of Fertilizer Example

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IRREGULAR TYPES OF LINEAR PROGRAMMING PROBLEMS

For some linear programming models, the general rules do not apply.

- Special types of problems include those with:
  - Multiple optimal solutions
  - Infeasible solutions
  - Unbounded solutions

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MULTIPLE OPTIMAL SOLUTIONS BEAVER CREEK POTTERY

The objective function is parallel to a constraint line.

Maximize $Z=\$40x_1 + 30x_2$
subject to: $1x_1 + 2x_2 \leq 40$
$4x_1 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

Where:
x_1 = number of bowls
x_2 = number of mugs

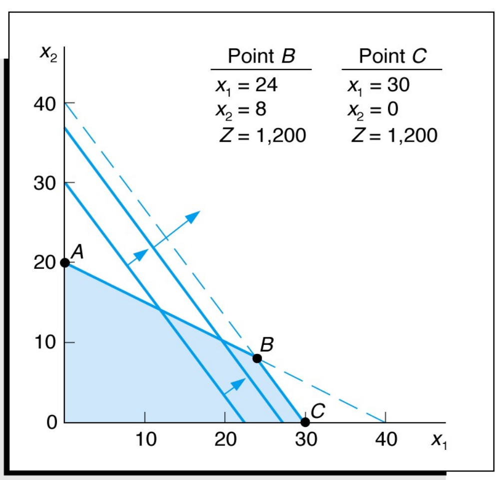

Figure 2.20 Example with Multiple Optimal Solutions

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AN INFEASIBLE PROBLEM

Every possible solution violates at least one constraint:

Maximize $Z = 5x_1 + 3x_2$
subject to: $4x_1 + 2x_2 \leq 8$
$x_1 \geq 4$
$x_2 \geq 6$
$x_1, x_2 \geq 0$

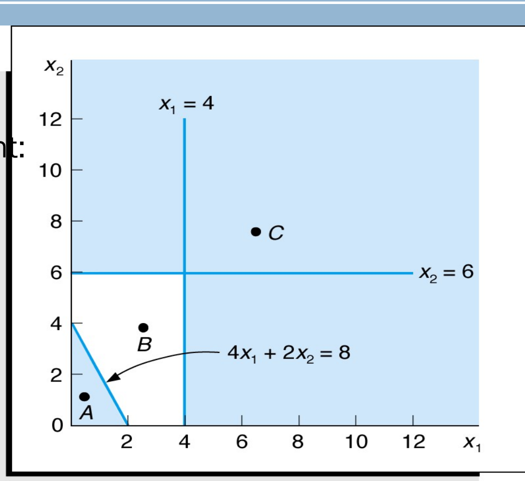

Figure 2.21 Graph of an Infeasible Problem

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AN UNBOUNDED PROBLEM

Value of the objective function increases indefinitely:

Maximize $Z = 4x_1 + 2x_2$
subject to: $x_1 \geq 4$
$x_2 \leq 2$
$x_1, x_2 \geq 0$

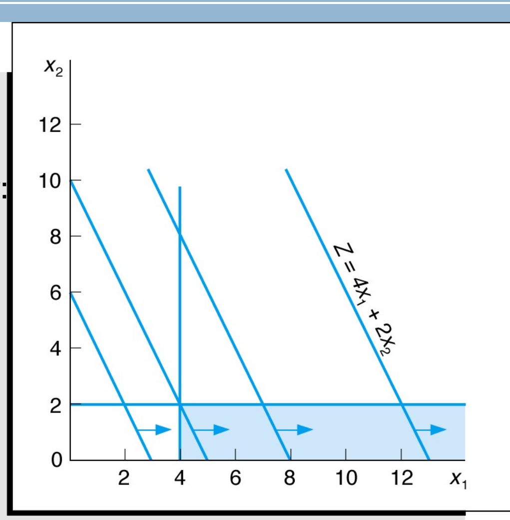

Figure 2.22 Graph of an Unbounded Problem

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CHARACTERISTICS OF LINEAR PROGRAMMING PROBLEMS

- A decision amongst alternative courses of action is required.
- The decision is represented in the model by **decision variables**.
- The problem encompasses a goal, expressed as an **objective function**, that the decision maker wants to achieve.
- Restrictions (represented by **constraints**) exist that limit the extent of achievement of the objective.
- The objective and constraints must be definable by **linear** mathematical functional relationships.

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PROPERTIES OF LINEAR PROGRAMMING MODELS

- **Proportionality** - The rate of change (slope) of the objective function and constraint equations is constant.
- **Additivity** - Terms in the objective function and constraint equations must be additive.i.e. *the combined effect of the decision variables in any one equation is the algebraic sum of their individual weighted effects. (The weighting, of course, is due to the proportionality constants.)*
- **Divisibility** -Decision variables can take on any fractional value and are therefore continuous as opposed to integer in nature.
- **Certainty** - Values of all the model parameters are assumed to be known with certainty (non-probabilistic).

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PROBLEM STATEMENT EXAMPLE PROBLEM NO. 1 (1 OF 3)

- Hot dog mixture in 1000-pound batches.
- Two ingredients, chicken ($3/lb) and beef ($5/lb).
- Recipe requirements:

  at least 500 pounds of "chicken"

  at least 200 pounds of "beef"
- Ratio of chicken to beef must be at least 2 to 1.
- Determine optimal mixture of ingredients that will minimize costs.

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SOLUTION EXAMPLE PROBLEM NO. 1 (2 OF 3)

Step 1:

Identify decision variables.

x_1 = lb of chicken in mixture

x_2 = lb of beef in mixture

Step 2:

Formulate the objective function.

Minimize $Z = \$3x_1 + \$5x_2$
where Z = cost per 1,000-lb batch
$\$3x_1$ = cost of chicken
$\$5x_2$ = cost of beef

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SOLUTION EXAMPLE PROBLEM NO. 1 (3 OF 3)

Step 3:

Establish Model Constraints

$x_1 + x_2 = 1{,}000$ lb

$x_1 \geq 500$ lb of chicken

$x_2 \geq 200$ lb of beef

$x_1/x_2 \geq 2/1$ or $x_1 - 2x_2 \geq 0$

$x_1, x_2 \geq 0$

The Model: Minimize $Z = \$3x_1 + 5x_2$

subject to: $x_1 + x_2 = 1{,}000$ lb

$x_1 \geq 500$

$x_2 \geq 200$

$x_1 - 2x_2 \geq 0$

$x_1, x_2 \geq 0$

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EXAMPLE PROBLEM NO. 2 (1 OF 3)

Solve the following model graphically:

Maximize $Z = 4x_1 + 5x_2$

subject to: $x_1 + 2x_2 \leq 10$

$6x_1 + 6x_2 \leq 36$

$x_1 \leq 4$

$x_1, x_2 \geq 0$

Step 1: Plot the constraints as equations

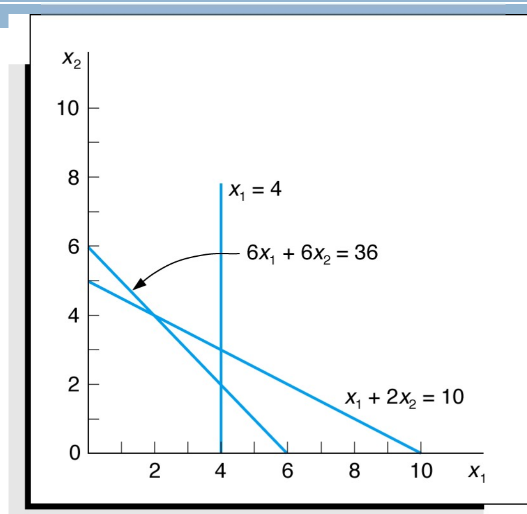

Figure 2.23 Constraint Equations

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EXAMPLE PROBLEM NO. 2 (2 OF 3)

Maximize $Z = 4x_1 + 5x_2$
subject to: $x_1 + 2x_2 \leq 10$
$6x_1 + 6x_2 \leq 36$
$x_1 \leq 4$
$x_1, x_2 \geq 0$

Step 2: Determine the feasible solution space

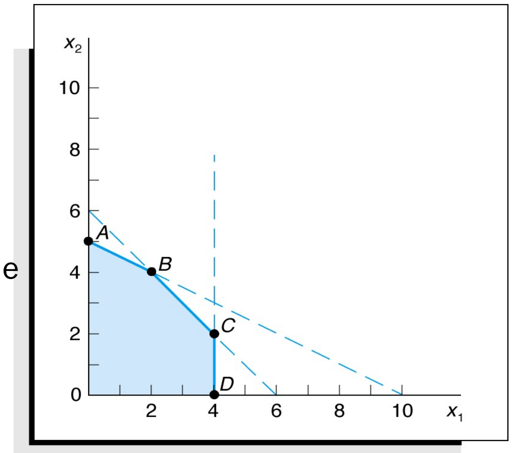

Figure 2.24 Feasible Solution Space and Extreme Points

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EXAMPLE PROBLEM NO. 2 (3 OF 3)

Maximize $Z = 4x_1 + 5x_2$
subject to: $x_1 + 2x_2 \leq 10$
$6x_1 + 6x_2 \leq 36$
$x_1 \leq 4$
$x_1, x_2 \geq 0$

Step 3 and 4: Determine the solution points and optimal solution

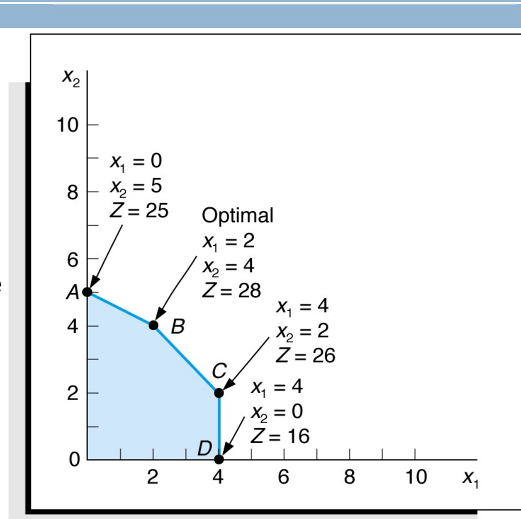

Figure 2.25 Optimal Solution Point