Optimization Techniques

SIMPLEX METHOD

- *Simplex method* is the most popular method used for the solution of *Linear Programming Problems* (LPP).
- Objectives
  - To discuss the motivation of simplex method
  - To discuss Simplex algorithm
  - To demonstrate the construction of simplex tableau

Optimization Techniques

MOTIVATION OF SIMPLEX METHOD

- Solution of a LPP, if exists, lies at one of the vertices of the feasible region.
- All the basic solutions can be investigated one-by-one to pick up the optimal solution.
- For 10 equations with 15 variables there exists 15*C*10= 3003 basic feasible solutions!
- Too large number to investigate one-by-one.
- This can be overcome by simplex method

Optimization Techniques

SIMPLEX METHOD: CONCEPT IN 3D CASE

- In 3D, a feasible region (i.e., volume) is bounded by several surfaces
- Each vertex (a basic feasible solution) of this volume is connected to the three other adjacent vertices by a straight line to each, being intersection of two surfaces.
- *Simplex algorithm* helps to move from one vertex to another adjacent vertex which is closest to the optimal solution among all other adjacent vertices.
- Thus, it follows the shortest route to reach the optimal solution from the starting point.

Optimization Techniques

SIMPLEX METHOD

- *Simplex*: a linear-programming algorithm that can solve problems having more than two decision variables.
- The simplex technique involves generating a series of solutions in tabular form, called tableaus. By inspecting the bottom row of each tableau, one can immediately tell if it represents the optimal solution. Each tableau corresponds to a corner point of the feasible solution space. The first tableau corresponds to the origin. Subsequent tableaus are developed by shifting to an adjacent corner point in the direction that yields the highest (smallest) rate of profit (cost). This process continues as long as a positive (negative) rate of profit (cost) exists.

Optimization Techniques

SIMPLEX ALGORITHM

The key solution concepts

- Solution Concept 1: the simplex method focuses on CPF solutions.
- Solution concept 2: the simplex method is an iterative algorithm (a systematic solution procedure that keeps repeating a fixed series of steps, called, an iteration, until a desired result has been obtained) with the following structure:

Optimization Techniques

SIMPLEX ALGORITHM

Initialization: setup to start iterations, including finding an initial CPF solution

Optimality test: is the current CPF solution optimal?

if no — if yes → stop

Iteration: Perform an iteration to find a better CFP solution

Optimization Techniques

SIMPLEX ALGORITHM

- Solution concept 3: whenever possible, the initialization of the simplex method chooses the origin point (all decision variables equal zero) to be the initial CPF solution.
- Solution concept 4: given a CPF solution, it is much quicker computationally to gather information about its adjacent CPF solutions than about other CPF solutions. Therefore, each time the simplex method performs an iteration to move from the current CPF solution to a better one, it always chooses a CPF solution that is adjacent to the current one.

Optimization Techniques

SIMPLEX ALGORITHM

- Solution concept 5: After the current CPF solution is identified, the simplex method examines each of the edges of the feasible region that originate from this CPF solution. Each of these edges leads to an adjacent CPF solution at the other end, but the simplex method doesn't even take the time to solve for the adjacent CPF solution. Instead it simply identifies the rate of improvement in Z that would be obtained by moving along the edge. And then chooses to move along the one with largest positive rate of improvement.

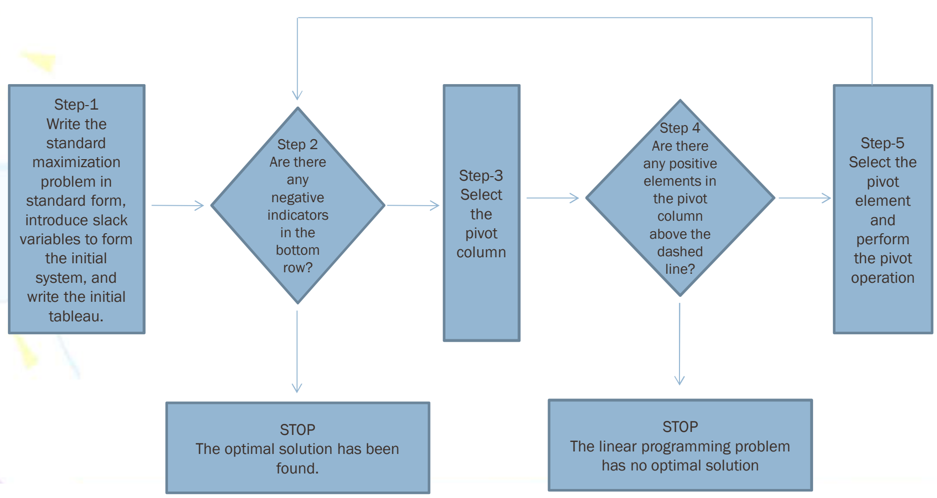
Optimization Techniques

SIMPLEX ALGORITHM

- Solution concept 6: A positive rate of improvement in Z implies that the adjacent CPF solution is better than the current one, whereas a negative rate of improvement in Z implies that the adjacent CPF solution is worse. Therefore, the optimality test consists simply of checking whether any of the edges give a positive rate of improvement in Z. if none do, then the current CPF solution is optimal.

Optimization Techniques

SIMPLEX METHOD



Simplex algorithm for standard maximization problems

Optimization Techniques

TO SOLVE A LINEAR PROGRAMMING PROBLEM IN STANDARD FORM, USE THE FOLLOWING STEPS.

1- Convert each inequality in the set of constraints to an equation by adding slack variables.

2- Create the initial simplex tableau.

3- Select the pivot column. (The column with the "most negative value" element in the last row.)

4- Select the pivot row. (The row with the smallest non-negative result when the last element in the row is divided by the corresponding in the pivot column.)

5-Use elementary row operations calculate new values for the pivot row so that the pivot is 1 (Divide every number in the row by the pivot number.)

6- Use elementary row operations to make all numbers in the pivot column equal to 0 except for the pivot number. If all entries in the bottom row are zero or positive, this the final tableau. If not, go back to step 3.

7- If you obtain a final tableau, then the linear programming problem has a maximum solution, which is given by the entry in the lower-right corner of the tableau.

Optimization Techniques

THE SIMPLEX METHOD IN TABULAR FORM

- Steps:

1. Initialization:

a. transform all the constraints to equality by introducing slack, surplus, and artificial variables as follows:

Constraint type	Variable to be added
$\geq$	+ slack (s)
$\leq$	- Surplus (s) + artificial (A)
$=$	+ Artificial (A)

Optimization Techniques

DIFFERENT VARIABLES

Slack Variables :

- Slack variable represents an unused quaintly of resources ; it is added to less than or equal ($\leq$) to type constraints in order to get an equality constraint.

Surplus Variables :

- A surplus variable represents the amount by which solution values exceed a resource. These variables are also called 'Negative Slack Variables' . Surplus variables like slack variables carry a zero coefficient in the objective function. it is added to greater than or equal to ($\geq$) type constraints in order to get an equality constraint.

Artificial Variables :

- Artificial variables are added to those constraints with equality (=) and greater than or equal to ($\geq$) sign. An Artificial variable is added to the constraints to get an initial solution to an LP problem. Artificial variables have no meaning in a physical sense and are not only used as a tool for generating an initial solution to an LP problem.

Optimization Techniques

WHICH VARIABLES AND WHEN?

Particulars	Slack Variable	Surplus Variable	Artificial Variable
Mean	Unused resources of the idle resources.	Excess amount of resources utilized.	No physical or economic meaning. It is Fictitious.
When used ?	With $\leq$ Constraints	With $\geq$ Constraints	With $\geq$ And = constraints
Coefficient	+1	-1	+1
Co-efficient in the Z – objective function	0	0	-M for Maximization and +M for minimization
As Initial Program variable	Used as starting point.	Can't be used since unit matrix condition is not satisfied	It is initially used but later on eliminated.
In Optimal Table	Used to help for interpreting idle & key resources.	–	It indicates the Infeasible Solution

Optimization Techniques

SIMPLEX METHOD IN TABULAR FORM

b. Construct the initial simplex tableau

Basic variable	X_1	…	X_n	S_1	……	S_n	A_1	….	A_n	RHS
S	Coefficient of the constraints									b_1
⋮										⋮
A										b_m
Z	Objective function coefficient In different signs									Z value

Optimization Techniques

SIMPLEX METHOD IN TABULAR FORM

2. Test for optimality:

Case 1: Maximization problem

the current BF solution is optimal if every coefficient in the objective function row is nonnegative

Case 2: Minimization problem

the current BF solution is optimal if every coefficient in the objective function row is nonpositive

SIMPLEX METHOD IN TABULAR FORM

3. Iteration

Step 1: determine the entering basic variable by selecting the variable (automatically a nonbasic variable) with the most negative value (in case of maximization) or with the most positive (in case of minimization) in the last row (Z-row). Put a box around the column below this variable, and call it the "pivot column"

Optimization Techniques

SIMPLEX METHOD IN TABULAR FORM

- Step 2: Determine the leaving basic variable by applying the minimum ratio test as following:

1. Pick out each coefficient in the pivot column that is strictly positive (>0)
2. Divide each of these coefficients into the right hand side entry for the same row
3. Identify the row that has the smallest of these ratios
4. The basic variable for that row is the leaving variable, so replace that variable by the entering variable in the basic variable column of the next simplex tableau. Put a box around this row and call it the "pivot row"

Optimization Techniques

SIMPLEX METHOD IN TABULAR FORM

- Step 3: Solve for the new BF solution by using elementary row operations (multiply or divide a row by a nonzero constant; add or subtract a multiple of one row to another row) to construct a new simplex tableau, and then return to the optimality test. The specific elementary row operations are:

1. Divide the pivot row by the "pivot number" (the number in the intersection of the pivot row and pivot column)
2. For each other row that has a negative coefficient in the pivot column, add to this row the product of the absolute value of this coefficient and the new pivot row.
3. For each other row that has a positive coefficient in the pivot column, subtract from this row the product of the absolute value of this coefficient and the new pivot row.

Optimization Techniques

SIMPLEX METHOD

- Example (All constraints are $\leq$)

Solve the following problem using the simplex method

- Maximize

$$Z = 3X_1 + 5X_2$$

Subject to

$$X_1 \leq 4$$

$$2X_2 \leq 12$$

$$3X_1 + 2X_2 \leq 18$$

$$X_1, X_2 \geq 0$$

Optimization Techniques

SIMPLEX METHOD

- Solution
- Initialization

1. Standard form

Maximize Z,

Subject to

$$Z - 3X_1 - 5X_2 = 0$$

$$X_1 + S_1 = 4$$

$$2X_2 + S_2 = 12$$

$$3X_1 + 2X_2 + S_3 = 18$$

$$X_1, X_2, S_1, S_2, S_3 \geq 0$$

Sometimes it is called the augmented form of the problem because the original form has been augmented by some supplementary variables needed to apply the simplex method

Optimization Techniques

DEFINITIONS

- A basic solution is an augmented corner point solution.
- A basic solution has the following properties:

1. Each variable is designated as either a nonbasic variable or a basic variable.
2. The number of basic variables equals the number of functional constraints. Therefore, the number of nonbasic variables equals the total number of variables minus the number of functional constraints.
3. The nonbasic variables are set equal to zero.
4. The values of the basic variables are obtained as simultaneous solution of the system of equations (functional constraints in augmented form). The set of basic variables are called “basis”
5. If the basic variables satisfy the nonnegativity constraints, the basic solution is a Basic Feasible (BF) solution.

Optimization Techniques

INITIAL TABLEAU

2. Initial tableau

Entering variable

Basic variable	X_1	X_2	S_1	S_2	S_3	RHS
S_1	1	0	1	0	0	4
S_2	0	2	0	1	0	12
S_3	3	2	0	0	1	18
Z	-3	-5	0	0	0	0

Leaving variable

Pivot column

Pivot number

Pivot row

Optimization Techniques

SIMPLEX TABLEAU

Notes:

- The basic feasible solution at the initial tableau is (0, 0, 4, 12, 18) where:

$X_1 = 0$, $X_2 = 0$, $S_1 = 4$, $S_2 = 12$, $S_3 = 18$, and $Z = 0$

Where S_1, S_2, and S_3 are basic variables

X_1 and X_2 are nonbasic variables

- The solution at the initial tableau is associated to the origin point at which all the decision variables are zero.

Optimization Techniques

OPTIMALITY TEST

- By investigating the last row of the initial tableau, we find that there are some negative numbers. Therefore, the current solution is not optimal

Optimization Techniques

ITERATION

- Step 1: Determine the entering variable by selecting the variable with the most negative in the last row.
- From the initial tableau, in the last row (Z row), the coefficient of X_1 is -3 and the coefficient of X_2 is -5; therefore, the most negative is -5. consequently, X_2 is the entering variable.
- X_2 is surrounded by a box and it is called the pivot column

Optimization Techniques

ITERATION

- Step 2: Determining the leaving variable by using the minimum ratio test as following:

Basic variable	Entering variable X_2 (1)	RHS (2)	Ratio (2)÷(1)
S_1	0	4	None
S_2 Leaving	**2**	**12**	**6 Smallest ratio**
S3	2	18	9

Optimization Techniques

ITERATION

- Step 3: solving for the new BF solution by using the eliminatory row operations as following:

1. New pivot row = old pivot row ÷ pivot number

Basic variable	X_1	X_2	S_1	S_2	S_3	RHS
S_1						
X_2	0	1	0	1/2	0	6
S_3						
Z						

Note that X_2 becomes in the basic variables list instead of S_2

Optimization Techniques

ITERATION

2. For the other row apply this rule:
New row = old row – the coefficient of this row in the pivot column (new pivot row).

For S_1

	1	0	1	0	0	4
-						
0	(0	1	0	1/2	0	6)
	1	0	1	0	0	4

For S_3

	3	2	0	0	1	18
-						
2	(0	1	0	1/2	0	6)
	3	0	0	-1	1	6

for Z

	-3	-5	0	0	0	0
-						
-5	(0	1	0	1/2	0	6)
	-3	0	0	5/2	0	30

Substitute this values in the table

Optimization Techniques

ITERATION

This solution is not optimal, since there is a negative numbers in the last row

Basic variable	X_1	X_2	S_1	S_2	S_3	RHS
S_1	1	0	1	0	0	4
X_2	0	1	0	1/2	0	6
S_3	3	0	0	-1	1	6
Z	-3	0	0	5/2	0	30

The most negative value; therefore, X_1 is the entering variable

The smallest ratio is 6/3 =2; therefore, S_3 is the leaving variable

Optimization Techniques

ITERATION

- Apply the same rules we will obtain this solution:

Basic variable	X_1	X_2	S_1	S_2	S_3	RHS
S_1	0	0	1	1/3	-1/3	2
X_2	0	1	0	1/2	0	6
X1	1	0	0	-1/3	1/3	2
Z	0	0	0	3/2	1	36

This solution is optimal; since there is no negative solution in the last row: basic variables are X_1 = 2, X_2 = 6 and S_1 = 2; the nonbasic variables are S_2 = S_3 = 0

Z = 36

Optimization Techniques

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Optimization Techniques

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Optimization Techniques

SPECIAL CASES OF LINEAR PROGRAMMING

- Infeasible solution
- Multiple solution (infinitely many solution)
- Unbounded solution
- Degenerated solution

Optimization Techniques

NOTES ON THE SIMPLEX TABLEAU

1. In any Simplex tableau, the intersection of any basic variable with itself is always one and the rest of the column is zeroes.
2. In any simplex tableau, the objective function row (Z row) is always in terms of the nonbasic variables. This means that under any basic variable (in any tableau) there is a zero in the Z row. For the non basic there is no condition (it can take any value in this row).
3. If there is a zero under one or more nonbasic variables in the last tableau (optimal solution tableau), then there is a multiple optimal solution.
4. When determining the leaving variable of any tableau, if there is no positive ratio (all the entries in the pivot column are negative and zeroes), then the solution is unbounded.
5. If there is a tie (more than one variables have the same most negative or positive) in determining the entering variable, choose any variable to be the entering one.
6. If there is a tie in determining the leaving variable, choose any one to be the leaving variable. In this case a zero will appear in RHS column; therefore, a "cycle" will occur, this means that the value of the objective function will be the same for several iterations.
7. A Solution that has a basic variable with zero value is called a "degenerate solution".
8. If there is no Artificial variables in the problem, there is no room for "infeasible solution"

Optimization Techniques

"BIG M METHOD"

- Simplex method incase of Artificial variables
- Solve the following linear programming problem by using the simplex method:
- Min Z =2 X_1 + 3 X_2

S.t.

$\frac{1}{2} X_1 + \frac{1}{4} X_2 \leq 4$

$X_1 + 3X_2 \geq 20$

$X_1 + X_2 = 10$

$X_1, X_2 \geq 0$

Optimization Techniques

BIG M METHOD

- Solution

Step 1: standard form

Min Z,

s.t.

$$Z - 2X_1 - 3X_2 - MA_1 - MA_2 = 0$$
$$\tfrac{1}{2}X_1 + \tfrac{1}{4}X_2 + S_1 = 4$$
$$X_1 + 3X_2 - S_2 + A_1 = 20$$
$$X_1 + X_2 + A_2 = 10$$
$$X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$$

Where: M is a very large number

Optimization Techniques

BIG M METHOD

- Notes

M, a very large number, is used to ensure that the values of A_1 and A_2, …, and A_n will be zero in the final (optimal) tableau as follows:

1. If the objective function is Minimization, then A_1, A_2, …, and A_n must be added to the RHS of the objective function multiplied by a very large number (M).

Example: if the objective function is Min $Z = X_1+2X_2$, then the obj. function should be Min $Z = X_1 + X_2 + MA_1 + MA_2 + \ldots + MA_n$

OR

$$Z - X_1 - X_2 - MA_1 - MA_2 - \ldots - MA_n = 0$$

2. If the objective function is Maximization, then A_1, A_2, …, and A_n must be subtracted from the RHS of the objective function multiplied by a very large number (M).

Example: if the objective function is Max $Z = X_1+2X_2$, then the obj. function should be Max $Z = X_1 + X_2 - MA_1 - MA_2 - \ldots - MA_n$

OR

$$Z - X_1 - X_2 + MA_1 + MA_2 + \ldots + MA_n = 0$$

N.B.: When the Z is transformed to a zero equation, the signs are changed

Optimization Techniques

BIG M METHOD

- Step 2: Initial tableau

Basic variables	X_1 2	X_2 3	S_1 0	S_2 0	A_1 M	A_2 M	RHS
S_1	½	¼	1	0	0	0	4
A_1	1	3	0	-1	1	0	20
A_2	1	1	0	0	0	1	10
Z	-2	-3	0	0	-M	-M	0

Note that one of the simplex rules is violated, which is the basic variables A_1, and A_2 have a non zero value in the z row; therefore, this violation must be corrected before proceeding in the simplex algorithm as follows.

Optimization Techniques

BIG M METHOD

- To correct this violation before starting the simplex algorithm, the elementary row operations are used as follows:

New (Z row) = old (z row) ± M (A_1 row) ± M (A_2 row)

In our case, it will be positive since M is negative in the Z row, as following:

Old (Z row):	-2	-3	0	0	-M	-M	0
M (A1 row):	M	3M	0	-M	M	o	20M
M (A2 row):	M	M	0	0	0	M	10M
New (Z row):	2M-2	4M-3	0	-M	0	0	30M

It becomes zero

Optimization Techniques

BIG M METHOD

- The initial tableau will be:

Basic variables	X_1 2	X_2 3	S_1 0	S_2 0	A_1 M	A_2 M	RHS
S_1	1/2	1/4	1	0	0	0	4
A_1	1	3	0	-1	1	0	20
A_2	1	1	0	0	0	1	10
Z	2M-2	4M-3	0	-M	0	0	30M

- Since there is a positive value in the last row, this solution is not optimal
- The entering variable is X_2 (it has the most positive value in the last row)
- The leaving variable is A_1 (it has the smallest ratio)

Optimization Techniques

BIG M METHOD

- First iteration

Basic variables	X_1 2	X_2 3	S_1 0	S_2 0	A_1 M	A_2 M	RHS
S_1	5/12	0	1	1/12	-1/12	0	7/3
X_2	1/3	1	0	-1/3	1/3	0	20/3
A_2	2/3	0	0	1/3	-1/3	1	10/3
Z	2/3M-1	0	0	1/3M-1	1-4/3M	0	20+10/3M

- Since there is a positive value in the last row, this solution is not optimal
- The entering variable is X_1 (it has the most positive value in the last row)
- The leaving variable is A_2 (it has the smallest ratio)

Optimization Techniques

BIG M METHOD

- Second iteration

Basic variables	X_1	X_2	S_1	S_2	A_1	A_2	RHS
S_1	0	0	1	-1/8	1/8	-5/8	1/4
X_2	0	1	0	-1/2	1/2	-1/2	5
X_1	1	0	0	1/2	-1/2	3/2	5
Z	0	0	0	-1/2	½-M	3/2-M	25

This solution is optimal, since there is no positive value in the last row. The optimal solution is:

$X_1 = 5$, $X_2 = 5$, $S_1 = ¼$

$A_1 = A_2 = 0$ and $Z = 25$

Optimization Techniques

NOTE FOR THE BIG M METHOD

- In the final tableau, if one or more artificial variables (A_1, A_2, …) still basic and has a nonzero value, then the problem has an infeasible solution.
- All other notes are still valid in the Big M method.

Optimization Techniques

SPECIAL CASES

- In the final tableau, if one or more artificial variables (A_1, A_2, …) still basic and has a nonzero value, then the problem has an infeasible solution
- If there is a zero under one or more nonbasic variables in the last tableau (optimal solution tableau), then there is a multiple optimal solution.
- When determining the leaving variable of any tableau, if there is no positive ratio (all the entries in the pivot column are negative and zeroes), then the solution is unbounded.

EXAMPLE: MINIMIZE $Z= 600X_1+500X_2$
SUBJECT TO CONSTRAINTS,
$2X_1+ X_2$ >OR= 80
X_1+2X_2 >OR= 60 AND X_1,X_2 >OR= 0

STEP1: CONVERT THE LP PROBLEM INTO A SYSTEM OF LINEAR EQUATIONS.

WE DO THIS BY REWRITING THE CONSTRAINT INEQUALITIES AS EQUATIONS BY SUBTRACTING NEW "SURPLUS & ARTIFICIAL VARIABLES" AND ASSIGNING THEM *ZERO & +M* COEFFICIENTSRESPECTIVELY IN THE OBJECTIVE FUNCTION AS SHOWN BELOW.
SO THE OBJECTIVE FUNCTION WOULD BE:
$Z=600X_1+500X_2+0.S_1+0.S_2+MA_1+MA_2$
SUBJECT TO CONSTRAINTS,
$2X_1+ X_2-S_1+A_1 = 80$
$X_1+2X_2-S_2+A_2 = 60$
X_1,X_2,S_1,S_2,A_1,A_2 >OR= 0

STEP 2: OBTAIN A BASIC SOLUTION TO THE PROBLEM.
WE DO THIS BY PUTTING THE DECISION VARIABLES
$X_1=X_2=S_1=S_2=0$,
SO THAT $A_1= 80$ AND $A_2=60$.
THESE ARE THE INITIAL VALUES OF *ARTIFICIAL VARIABLES*.

STEP 3: FORM THE INITIAL TABLEAU AS SHOWN.

		C_j	600	500	0	0	M	M	
C_B	Basic Variable (B)	Basic Soln(X_B)	X_1	X_2	S_1	S_2	A_1	A_2	**Min.Ratio (X_B/Pivotal Col.)**
M	A1	80	2	1	-1	0	1	0	80
M	A2	60	1	2 ☆	0	-1	0	1	60 →
		Z_j	3M	3M	−M	−M	M	M	
		Cj - Zj	600-3M	500-3M ↑	M	M	0	0	

IT IS CLEAR FROM THE TABLEAU THAT X_2 WILL ENTER AND A_2 WILL LEAVE THE BASIS. HENCE 2 IS THE KEY ELEMENT IN PIVOTAL COLUMN.
NOW,THE NEW ROW OPERATIONS ARE AS FOLLOWS:
R_2(NEW) = R_2(OLD)/2
R_1(NEW) = R_1(OLD) - 1*R_2(NEW)

		C_j	600	500	0	0	M	
C_B	Basic Variable (B)	Basic Soln(X_B)	X_1	X_2	S_1	S_2	A_1	**Min.Ratio (X_B/Pivotal Col.)**
M	A_1	50	3/2 ☆	0	-1	1/2	1	100/3 →
500	X_2	30	1/2	1	0	- 1/2	0	60
		Z_j	3M/2+250	500	−M	M/2-250	M	
		Cj - Zj	**350-3M/2** ↑	0	M	250-M/2	0	

IT IS CLEAR FROM THE TABLEAU THAT X_1 WILL ENTER AND A_1 WILL LEAVE THE BASIS. HENCE 2 IS THE KEY ELEMENT IN PIVOTAL COLUMN. NOW,THE NEW ROW OPERATIONS ARE AS FOLLOWS:

$R_1(NEW) = R_1(OLD)*2/3$

$R_2(NEW) = R_2(OLD) – (1/2)*R_1(NEW)$

		C_j	600	500	0	0	Min. Ratio (X_B/Pivotal Col.)
C_B	Basic Variable (B)	Basic Soln(X_B)	X_1	X_2	S_1	S_2	
600	X_1	100/3	1	0	−2/3	1/3	
500	X_2	40/3	0	1	1/3	−2/3	
		Z_j	600	500	-700/3	-400/3	
		$C_j - Z_j$	0	0	700/3	400/3	

Optimization Techniques

SINCE ALL THE VALUES OF (C_J-Z_J) ARE EITHER ZERO OR POSITIVE AND ALSO BOTH THE ARTIFICIAL VARIABLES HAVE BEEN REMOVED, AN OPTIMUM SOLUTION HAS BEEN ARRIVED AT WITH X_1=100/3 , X_2=40/3 AND Z=80,000/3.

Optimization Techniques

Four Special cases in Simplex

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Optimization Techniques

SIMPLEX ALGORITHM – SPECIAL CASES

- **There are four special cases arise in the use of the simplex method.**

1. Degeneracy
2. Alternative optimal
3. Unbounded solution
4. infeasible solution

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Optimization Techniques

DEGENERACY (NO IMPROVEMENT IN OBJECTIVE)

- **Degeneracy: It is situation when the solution of the problem degenerates.**
- **Degenerate Solution: A Solution of the problem is said to be degenerate solution if value or values of basic variable(s) become zero**
- **It occurs due to redundant constraints.**

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Optimization Techniques

Degeneracy – Special cases (cont.)

- This is in itself not a problem, but making simplex iterations form a degenerate solution, give rise to cycling, meaning that after a certain number of iterations without improvement in objective value the method may turn back to the point where it started.

Optimization Techniques

DEGENERACY – SPECIAL CASES (CONT.)

Example:

Max $f = 3x_1 + 9x_2$

Subject to:

$x_1 + 4x_2 \leq 8$

$X_1 + 2x_2 \leq 4$

$X_1, x_2 \geq 0$

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Optimization Techniques

Degeneracy – Special cases (cont.)

The solution:

Step 1. write inequalities in equation form

Let S_1 and S_2 be the slack variables

$X_1 + 4X_2 + s_1 = 8$

$X_1 + 2X_2 + s_2 = 4$

$X_1, X_2, s_1, s_2 \geq 0$

Let X1=0, X2=0

f=0, S1=8, S2=4

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Optimization Techniques

Degeneracy – Special cases (cont.)

- Initial Tableau

X_2 will enter the basis because of having minimum objective function coefficient.

Entering Variable

Leaving Variable

Basis	X_1	X_2	S_1	S_2	RHS	Ratio
S_1	1	4	1	0	8	8/4=2
S_2	1	2	0	1	4	4/2=2
f	-3	-9	0	0	0	

S_1 and S_2 tie for leaving variable(with same minimum ratio 2) so we can take any one of them arbitrary.

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Optimization Techniques

Degeneracy – Special cases (cont.)

- **Tableau 1**

Basis	X_1	X_2	S_1	S_2	RHS	Ratio
X_2	1/4	1	1/4	0	2	**8**
S_2	½	0	-1/2	1	0	0 min
f	-3/4	0	9/4	0	18	

Here basic variable S_2 is 0 resulting in a degenerate basic solution.

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Optimization Techniques

Degeneracy – Special cases (cont.)

Tableau 2

Basis	X_1	X_2	S_1	S_2	RHS
X_2	0	1	1/2	-1/2	2
X_1	1	0	-1	2	0
f	0	0	3/2	3/2	18

Same objective

- Same objective function no change and improvement (cycle)

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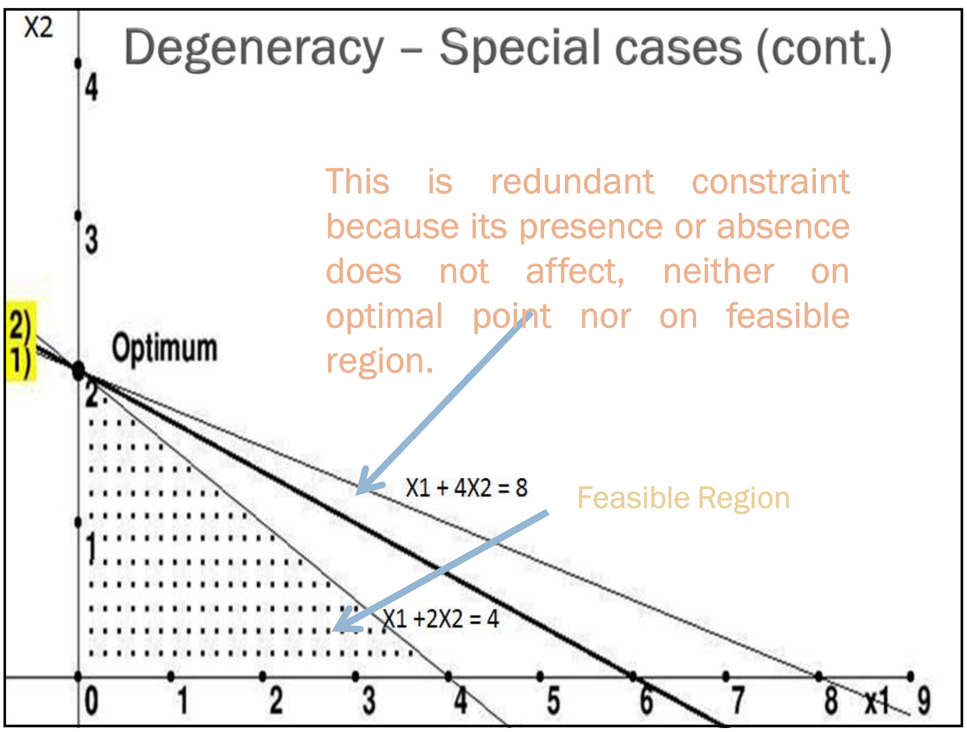
Degeneracy – Special cases (cont.)



This is redundant constraint because its presence or absence does not affect, neither on optimal point nor on feasible region.

Optimization Techniques

ALTERNATIVE OPTIMAL

- If the f-row value for one or more non basic variables is 0 in the optimal tableau, alternate optimal solution exists.
- When the objective function is parallel to a binding constraint, objective function will assume same optimal value. So this is a situation when the value of optimal objective function remains the same.
- We have infinite number of such points

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Optimization Techniques

Alternative optimal – Special cases (cont.)

Example:

Max $2x_1 + 4x_2$

ST

$x_1 + 2x_2 \leq 5$

$x_1 + x_2 \leq 4$

$x_1, x_2 \geq 0$

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Optimization Techniques

Alternative optima – Special cases (cont.)

The solution

Max $2x_1 + 4x_2$

Let S_1 and S_2 be the slack variables

$x_1 + 2x_2 + s_1 = 5$

$x_1 + x_2 + s_2 = 4$

$x_1, x_2, s_1, s_2 \geq 0$

Initial solution

Let $x_1 = 0, x_2 = 0,$

$f=0, s_1 = 5, s_2 = 4$

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Optimization Techniques

Alternative optima – Special cases (cont.)

- **Initial Tableau**

Entering Variable

Leaving Variable

Basis	X1	X2	S1	S2	RHS	Ratio
S_1	1	2	1	0	5	5/2=2.5
S_2	1	1	0	1	4	4/1=4
f	-2	-4	0	0	0	

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Optimization Techniques

Alternative optima – Special cases (cont.)

- Optimal solution is 10 when x2=5/2, x1=0.

Basis	X1	X2	S1	S2	RHS	Ratio
X_2	1/2	1	1/2	0	5/2	5
S_2	1/2	0	-1/2	1	3/2	3 min
f	0	0	2	0	10	

- How do we know that alternative optimal exist ?

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Optimization Techniques

Alternative optima – Special cases (cont.)

Entering Variable

- By looking at f-row coefficient of the nonbasic variable.

Leaving Variable

Basis	X1	X2	S1	S2	RHS	Ratio
X_2	1/2	1	1/2	0	5/2	5
S_2	1/2	0	-1/2	1	3/2	3
f	0	0	2	0	10	

hat x1 can enter the basic solution without changing the value of f. Optimal sol. f=10 x1=0, x2=5/2

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Optimization Techniques

Alternative optima – Special cases (cont.)

- The second alternative optima is:

Basis	X1	X2	S1	S2	RHS
X_2	0	1	1	-1	1
X_1	1	0	-1	2	3
f	0	0	2	0	10

- The new optimal solution is 10 when x1=3, x2=1

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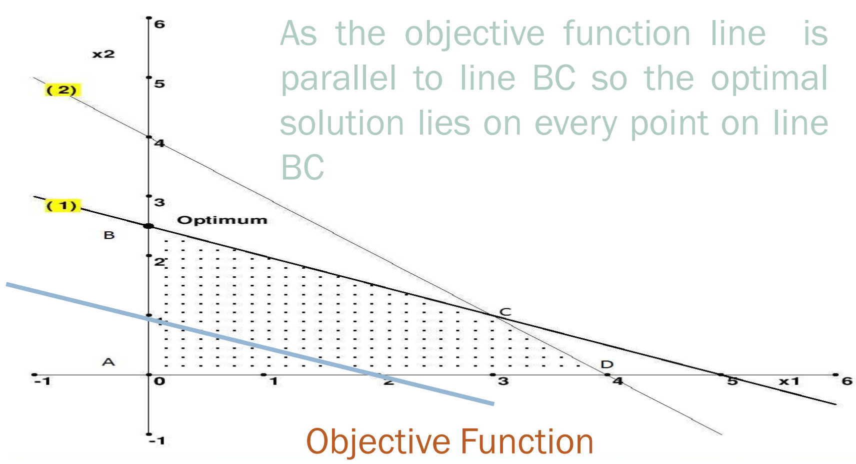
Optimization Techniques

ANY POINT ON BC REPRESENTS AN ALTERNATE OPTIMUM WITH F=10

As the objective function line is parallel to line BC so the optimal solution lies on every point on line BC



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Optimization Techniques

Alternative optimal – Special cases (cont.)

- In practice alternate optimals are useful as they allow us to choose from many solutions experiencing deterioration in the objective value.

Optimization Techniques

Unbounded Solution

When determining the leaving variable of any tableau, if there is no positive ratio (all the entries in the pivot column are negative and zeros), then the solution is unbounded.

Optimization Techniques

Unbounded Solution – Special cases (cont.)

Example

Max $2x_1 + x_2$

Subject to

$x_1 - x_2 \leq 10$

$2x_1 \leq 40$

$x_1, x_2 \geq 0$

Optimization Techniques

Unbounded Solution – Special cases (cont.)

Solution

Max $2x_1 + x_2$

Let S_1 and S_2 be the slack variables

$x_1 - x_2 + s_1 = 10$

$2x_1 + 0x_2 + s_2 = 40$

$x_1, x_2, s_1, s_2 \geq 0$

Initial Solution: $x_1 = 0$, $x_2 = 0$,

$f = 0$, $s_1 = 10$, $s_2 = 40$

Optimization Techniques

Unbounded Solution – Special cases (cont.)

Basis	X1	X2	S1	S2	RHS	Ratio
S_1	1	-1	1	0	10	10 min
S_2	2	0	0	1	40	20
f	-2	-1	0	0	0	

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Optimization Techniques

Unbounded Solution – Special cases (cont.)

Basis	X1	X2	S1	S2	RHS	Ratio
X_1	1	-1	1	0	10	-
S_2	0	2	-2	1	20	10
f	0	-3	2	0	20	

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Optimization Techniques

Unbounded Solution – Special cases (cont.)

Basis	X1	X2	S1	S2	RHS	Ratio
X_1	1	0	0	½	20	-
X_2	0	1	-1	1/2	10	-
f	0	0	-3	3/2	50	

- The values of non basic variable are either zero or negative.
- So, solution space is unbounded

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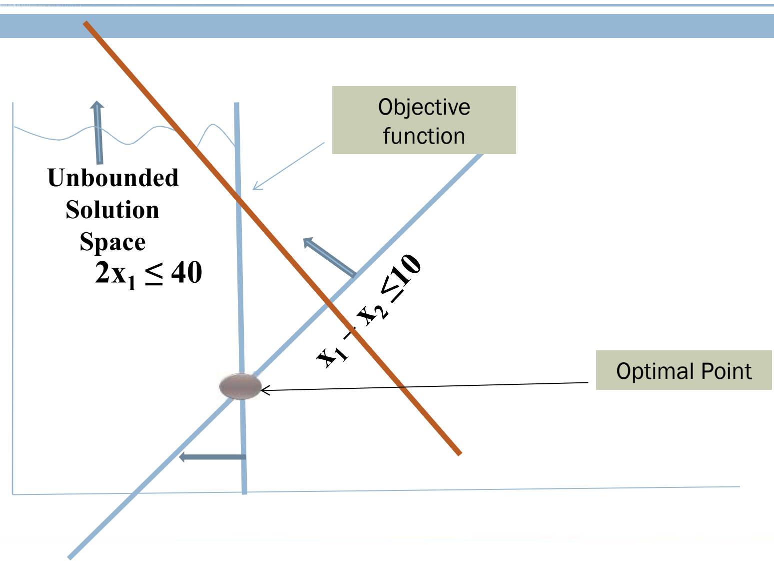
Optimization Techniques

Unbounded Solution – Special cases (cont.) Graphical Solution



Optimization Techniques

INFEASIBLE SOLUTION

- **A problem is said to have infeasible solution if there is no feasible optimal solution is available**

Optimization Techniques

EXAMPLE:

Max $f=3x_1+2x_2$

S.T.

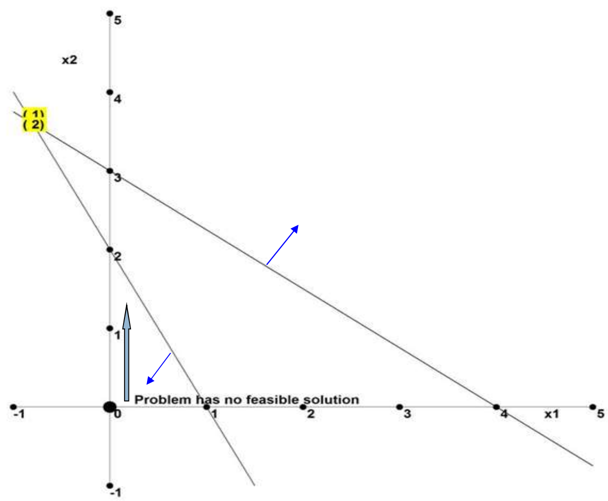
$2x_1+x_2 \leq 2$

$3x_1+4x_2 \geq 12$

$x_1,\ x_2=0$

Optimization Techniques

SOLUTION (GRAPHICAL):



Optimization Techniques

TWO PHASE SIMPLEX METHOD

Optimization Techniques

ILLUSTRATION

Minimize **$f=4x_1+x_2$** (Objective Function)

Subject to: (Constraints)

$3x_1+x_2 = 3$

$4x_1+3x_2 \geq 6$

$x_1+2x_2 \leq 4$

$x_1, x_2 \geq 0$ (Non-Negativity Constraints)

Optimization Techniques

INEQUALITIES CONSTRAINTS IN EQUATION FORM

Let S_1 and S_2 be the surplus and slack variables for second and third constraints, respectively.

Minimize $f=4x_1+x_2$ (Objective Function)

Subject to: (Constraints)

$3x_1+x_2 = 3$ …………. (1)

$4x_1+3x_2-S_1 = 6$ …………. (2)

$x_1+2x_2+S_2 = 4$ …………. (3)

$x_1, x_2, S_1, S_2 \geq 0$

Optimization Techniques

INITIAL SOLUTION

Let x1= 0, x2 = 0

Putting above values in objective function ($f = 3x_1 + x_2$) and equation 1-3,

$f = 0$

$0 = 3$ (Contradiction)

$S_1 = -6$ (Violation)

$S_2 = 4$

Optimization Techniques

INITIAL SOLUTION

Let x1= 0, x2 = 0

Putting above values in objective function

($f = 3x_1 + x_2$) a

This is against the basic rules of Mathematics as the 0 cannot be equal to 3.

$f = 0$

$0 = 3$ (Contradiction)

$S_1 = -6$ (Violation)

$S_2 = 4$

This is against the non negativity constraint that X must be non zero.

Optimization Techniques

INITIAL SOLUTION

Let x1= 0 x2 = 0

This is against the basic rules of

This situation cannot be called as initial feasible solution because it is not satisfying the condition. In order to make it feasible we need to add Artificial variables In equations having contradictions and violation

that X must be non zero.

Optimization Techniques

ARTIFICIAL VARIABLES

Let A_1 and A_2 be the artificial variables for first and second equation respectively.

Minimize: $f = 4x_1 + x_2$ (Objective Function)

Subject to: (Constraints)

$3x_1 + x_2 + A_1 = 3$ .... (4)

$4x_1 + 3x_2 - S_1 + A_2 = 6$ .... (5)

$x_1 + 2x_2 + S_2 = 4$ .... (6)

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

Optimization Techniques

SOLUTION OF ARTIFICIAL VARIABLE

Questions which involve artificial variables can not solve straight away. We have to use following two methods to solve it

1. **Penalty Method or M-Method (Big M Method)**
2. **Two Phase Method**

In this lecture we will solve this problem by Two Phase Method

Optimization Techniques

SOLUTION BY TWO PHASE METHOD

Phase I:

In Phase I we introduce artificial objective function Capital "F" as the sum of artificial variables introduced in the constraints. We substitute the values of artificial variables from the constraints and get artificial objective function.

We have Two artificial variables so the artificial objective function would be as under;

F = A1 + A2

Optimization Techniques

ARTIFICIAL OBJECTIVE FUNCTION

- **A1**

Use Equation no 4 to find the value of A1

3x1+ x2 + A1= 3

A1 = 3 - 3x1- x2

- **A2**

Use Equation no 5 to find the value of A2

$4x_1 + 3x_2 - S_1 + A_2 = 6$

$A_2 = 6 - 4x_1 - 3x_2 + S_1$

Optimization Techniques

$F = A_1 + A_2$

Putting the values of A1 and A2

F = (3 – 3X1- X2) + (6 – 4X1- 3X2 + S1)

$= 9 - 7x_1 - 4x_2 + S_1$

Minimize f = 4x1+ x2

$F = 9 - 7x_1 - 4x_2 + S_1$

Subject to:

3x1+ x2 + A1 = 3 ………….. (4)

4x1+ 3x2 – S1 + A2= 6………….. (5)

x1+ 2x2 + S2 = 4 ………….. (6)

$x_1, x_2, S_1, S_2, A_1 + A_2 \geq 0$

Optimization Techniques

INITIAL FEASIBLE SOLUTION

Arbitrary values =# of Variables -- # of Equation

6 - 3 = 3

This solution is called initial feasible solution because it satisfies the all Non negativity constraint and also do not have any contradiction or violation

f = 0

A1 = 3

A2 = 6

S2 = 4

Optimization

For leaving variable the rule is same as maximization, the variable with minimum ratio;A1

Basics	X_1	X_2	S_1	S_2	A1	A2	RHS	Ratio
A1	3	1	0	0	1	0	3	3/3=1(Min)
A2	4	3	-1	0	0	1	6	6/4=1.5
S2	1	2	0	1	0	0	4	4/1=4
f	-4	-1	0	0	0	0	0	
F	7	4	-1	0	0	0	9	

In artificial function of minimization we select the maximum positive as the entering variable which is X1

Optimization Techniques

OPTIMAL SOLUTION FOR ARTIFICIAL OBJECTIVE FUNCTION

The solution of artificial objective function is said to be optimal when artificial objective functions coefficients become non-positive or zero

Optimization Techniques

CALCULATION

$$\text{New Pivot Row} = \frac{1}{\text{Pivot No.}} \text{ X } \text{Old Pivot Row}$$

$$\text{New Pivot Row} = \frac{1}{3} \text{ X } [3 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 3]$$

$$[1 \quad 1/3 \quad 0 \quad 0 \quad 1/3 \quad 0 \quad 1]$$

Optimization Techniques

CALCULATION

New Row = Old Row – Pivot Column Coefficient x New Pivot Row

New A2 Row =	[4	3	-1	0	0	1	6]
- (4)	[1	1/3	0	0	1/3	0	1]
=	**[0**	**5/3**	**-1**	**0**	**-4/3**	**1**	**2]**
New S2 Row =	[1	2	0	1	0	0	4]
- (1)	[1	1/3	0	0	1/3	0	1]
=	**[0**	**5/3**	**0**	**1**	**-1/3**	**0**	**3]**
New f Row =	[-4	-1	0	0	0	0	0]
- (-4)	[1	1/3	0	0	1/3	0	1]
=	**[0**	**1/3**	**0**	**0**	**4/3**	**0**	**4]**
New F Row =	[7	4	-1	0	0	0	9]
- (7)	[1	1/3	0	0	1/3	0	1]
=	**[0**	**5/3**	**-1**	**0**	**-7/3**	**0**	**2]**

Optimization Techniques

TABLEAU I

Basics	X1	X_2	S_1	S_2	A1	A2	RHS	Ratio
X1	1	1/3	0	0	1/3	0	1	1÷1/3=3
A2	0	5/3	-1	0	-4/3	1	2	2 ÷5/3=1.2 (Min)
S2	0	5/3	0	1	-1/3	0	3	3/5 ÷ 3=1.8
f	0	1/3	0	0	4/3	0	4	
F	0	5/3	-1	0	-7/3	0	2	

Still there is one positive coefficient so we need to make further tableau

Optimization Techniques

CALCULATION

$$\text{New Pivot Row} = \frac{1}{\text{Pivot No.}} \text{ X Old Pivot Row}$$

$$\text{New Pivot Row} = \frac{1}{5/3} \text{ X } [0 \quad 5/3 \quad -1 \quad 0 \quad -4/3 \quad 1 \quad 2]$$

$$[0 \quad 1 \quad -3/5 \quad 0 \quad -4/5 \quad 3/5 \quad 6/5]$$

Optimization Techniques

CALCULATION

New Row = Old Row – Pivot Column Coefficient x New Pivot Row

New X1 Row =

	[1	1/3	0	0	1/3	0	1]
-(1/3)	[0	1	-3/5	0	-4/5	3/5	6/5]
=	**[1**	**0**	**1/5**	**0**	**3/5**	**-1/5**	**3/5]**

New S2 Row =

	[0	5/3	0	1	-1/3	0	3]
-(5/3)	[0	1	-3/5	0	-4/5	3/5	6/5]
=	**[0**	**0**	**1**	**1**	**1**	**-1**	**1]**

Optimization Techniques

CALCULATION

New f Row =

	[0	1/3	0	0	4/3	0	4]
-(1/3)	[0	1	-3/5	0	-4/5	3/5	6/5]
=	**[0**	**0**	**1/5**	**0**	**8/5**	**-1/5**	**18/5]**

New F Row

	[0	5/3	-1	0	-7/3	0	2]
-(5/3)	[0	1	-3/5	0	-4/5	3/5	6/5]
=	**[0**	**0**	**0**	**0**	**-1**	**-1**	**0]**

Optimization Techniques

TABLEAU II

Basics	X1	X_2	S_1	S_2	A1	A2	RHS
X1	1	0	1/5	0	3/5	-1/5	3/5
X2	0	1	-3/5	0	-4/5	3/5	6/5
S2	0	0	1	1	1	-1	1
f	0	0	1/5	0	8/5	-1/5	18/5
F	0	0	0	0	-1	-1	0

Now there is no positive value in the artificial function so this is the end of Phase I

Optimization Techniques

PHASE II

In phase two rewrite the previous tableau by dropping the artificial objective function and artificial variables

Basics	X1	X_2	S_1	S_2	RHS	Ratio
X1	1	0	1/5	0	3/5	3/5÷1/5=3
X2	0	1	-3/5	0	6/5	
S2	0	0	1	1	1	1/1=1(Min)
f	0	0	1/5	0	18/5	

↑

Optimization Techniques

CALCULATION

$$\text{New Pivot Row} = \frac{1}{\text{Pivot No.}} \text{ X } \text{Old Pivot Row}$$

$$\text{New Pivot Row} = \frac{1}{1} \text{ X } [0 \quad 0 \quad 1 \quad 1 \quad 1]$$

[0 0 1 1 1]

Optimization Techniques

CALCULATION

New Row = Old Row – Pivot Column Coefficient x New Pivot Row

New X1 Row =

	[1	0	1/5	0	3/5]
-(1/5)	[0	0	1	1	1]
=	**[1**	**0**	**0**	**-1/5**	**2/5]**

New X2 Row =

	[0	1	-3/5	0	6/5]
-(-3/5)	[0	0	1	1	1]
=	**[0**	**1**	**0**	**3/5**	**9/5]**

Optimization Techniques

CALCULATION

New f Row =

	[0	0	1/5	0	18/5]
-(1/5)	[0	0	1	1	1]
=	**[0**	**0**	**0**	**-1/5**	**17/5]**

Optimization Techniques

TABLEAU

Basics	X1	X_2	S1	S_2	RHS
X1	1	0	0	-1/5	2/5
X2	0	0	0	3/5	9/5
S1	0	0	1	1	1
f	0	0	0	-1/5	17/5

Now there is no positive value in the objective function so this is the optimal point

Optimization Techniques

OPTIMAL SOLUTION

X1 = 2/5

X2 = 9/5

f = 17/5

Cross checking of maximization point

put values of X1 and X2 from above solution into original objective function

$f=4x_1+ x_2$

$=4\ (2/5) + (9/5)$

$=17/5$

Optimization Techniques

Thank You