NUMERICAL METHODS

23/7/18

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$AX = B$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$X = A^{-1}B$$

–x–

$$f(x) = a_1x^5 + a_2x^4 + a_3x^3 + \dots + a_6$$

$$\frac{df(x)}{dx} = 5a_1x^4 + 4a_2x^3 + 3a_3x^2 + 2a_4x + a_5 = 0$$

–x–

Ex :- $x^3 - x + 1 = 0$

$x^3 = x - 1$

or $x = (x-1)^{1/3}$

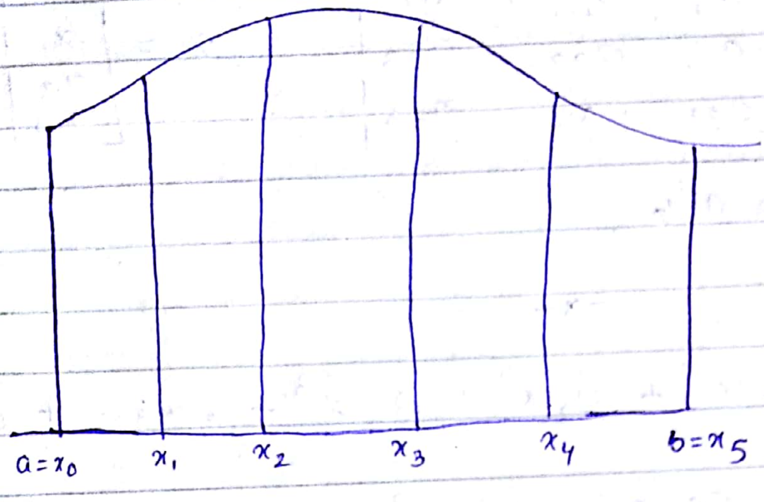
Let $x = 2$ and substitute $x = 2$ on the RHS of above eq^n.

$$x_1 = (2-1)^{1/3} = 1$$
$$x_2 = (1-1)^{1/3} = 0$$
$$x_3 = (0-1)^{1/3} = -1$$
$$x_4 = (-1-1)^{1/3} = -1.26$$
$$x_5 = (-1.26-1)^{1/3} = (-2.26)^{1/3} = -1.312$$
$$x_6 = (-1.312-1.000)^{1/3} = (-2.312)^{1/3} = -1.32.$$

$x = -\log_e x \rightarrow x = e^{-x}$

— x —

$$\int_a^b f(x)\,dx$$



$$A_1 = \frac{1}{2}\{f(x_0) + f(x_1)\} \times d$$

$A_2 =$

$\vdots$

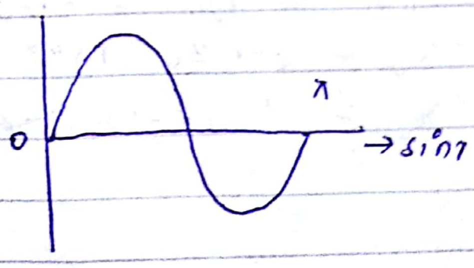
$$A_5 = \frac{1}{2}\{f(x_4) + f(x_5)\} \times d$$

$$A = A_1 + A_2 + A_3 + A_4 + A_5$$

$$= \frac{1}{2} \times d\{f(x_1) + 2f(x_1) + 2f(x_2) + 2f(x_3)$$

$$+ 2f(x_4) + f(x_5)\}$$

$$\int_0^{\pi} \sqrt{1+\cos^2 x}\; dx$$



- Computer can perform four basic arithmatic operations the addition, subtraction, multiplication, ~~&~~ division and a logic operation which is comparision.
- Computer has a finite word size, it can represent with finite precision. ~~*~~ to be contined.
- Computer has to finished within a finite time, we cannot leave computer doing the same compilation for an infinite period of time.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

1) ~~*~~ Any number with infinite precision has to be Rounded off after a finite number of digits after decimal point. Round of error; $\pi = 3.14159$

2) Transistion error - Since we truncate infinite series after a finite no. of lines.

3) Inderent error - This is due to the improper formulation of problem and non-availaibility of accurate data.

—x—

24/7/18

$x = \bar{x} + e_x$
$y = \bar{y} + e_y$

$Z = x+y = (\bar{x}+\bar{y}) + (e_{\bar{x}}+e_{\bar{y}})$
$P = x+y = \bar{x}\bar{y} + (\bar{x}e_y + \bar{y}e_x + e_x e_y)$

absolute error = |computed value - exact value|

$$\text{relative error} = \frac{\text{absolute error}}{\text{exact error}}$$

ex : 12345.678 - Computed value
12345.675 - exact value

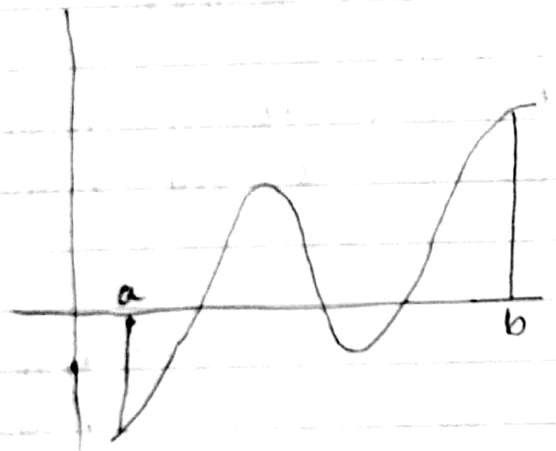
absolute error = 0.003

ex : 0.123 - Computed value
0.120 exact value.

absolute error : 0.003

$\frac{\text{absolute}}{\text{exact}} = \frac{0.003}{0.120} = \frac{3}{120} = 0.025$ = relative error.

$y = f(x)$ $[a,b]$



if $f(a)$ and $f(b)$ are of opposite sign, then there exist a number ρ in (a,b) such that $f(\rho) = 0$.

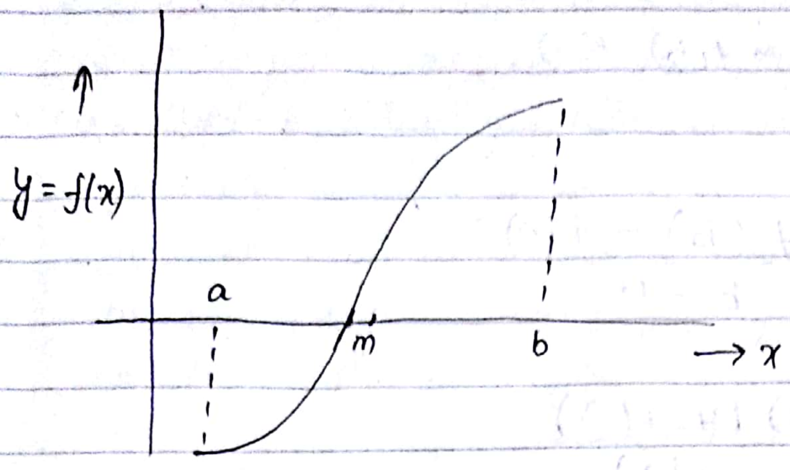
Solution of Non-linear equation

$ax^2 + bx + c = 0$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a_n x^m + a_1 x^n + \dots + a_{m+1} = 0$

$x + \log_e x + mx = 0$ Transidental eq^n

Bisection Method



Drawback:

i) If it will be a parabola i.e when it is just touching the x-axis.

Inputs a and b are so chosen that f(a) and f(b) are of opposite sign.

$$f(a) * f(b) < 0$$

1) $m = \frac{a+b}{2}$

2) if $f(m) == 0$. Then m is a root, exit

3) if $f(a) * f(m) < 0$ then $b = m$, else $a = m$

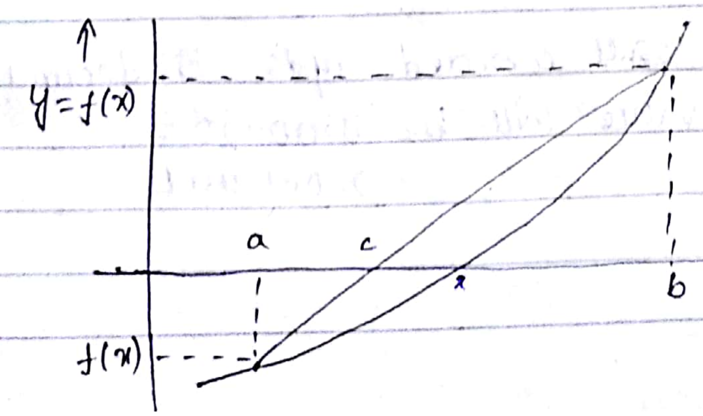
4) Continue steps 1) 2) 3) until the size of the search interval [a, b] is less than the required precision

5) $m + (a+b)/2$

6) print m

7) exit.

Method of false position or regular falsi method



Advantage over bisection

1) More iteration → bisection

2) Less iteration → False rule

Inputs a and b are so chosen that f(a) and f(b) are of opp. sign

$$f(a) * f(b) < 0$$

$$\frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow x - a = \frac{(b-a)(y - f(a))}{f(b) - f(a)}$$

$$\Rightarrow c - a = \frac{(b-a)(-f(a))}{f(b) - f(a)}$$ [substituting $(x, y) = (c, 0)$]

$$\Rightarrow c = \frac{a f(b) - \cancel{a f(a)} - b f(a) + \cancel{a f(a)}}{f(b) - f(a)}$$

1) $c = \dfrac{a f(b) - b f(a)}{f(b) - f(a)}$

2) If $f(c) == 0$ then c is a root.

3) if $f(a) * f(c) < 0$ then $b = c$

$a = c$

Stopping condition

if $|c_i - c_{i+1}|$ < required precision then c_{i+1} is the root.

Precision

for result corrected upto 5 decimal places. Precision value will be 0.000005

0.000001

Analysis of any numerical algorithm considers the following facts

1. Whether the method is guaranteed to provide solution.

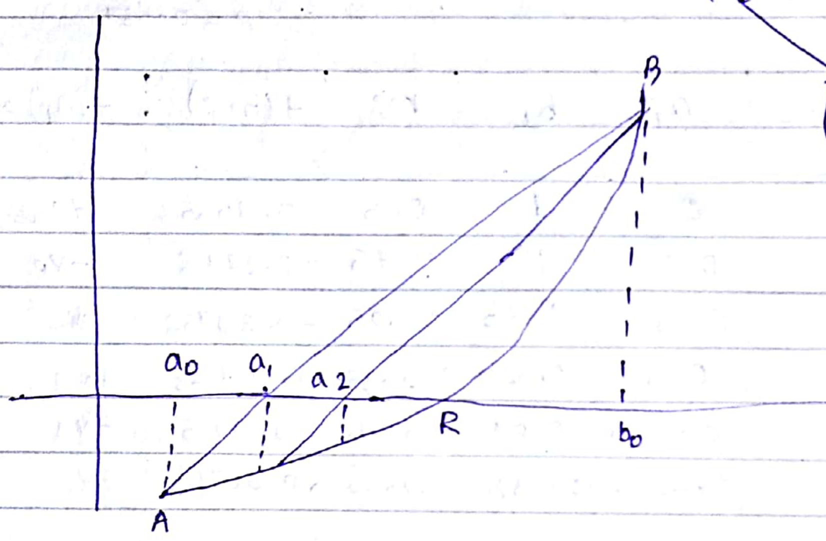
2. How fast will it give solution.

3. Amount of error associated with the solution.

$\cancel{(a_2-a_1) = (a_1-a_0)}$

$\cancel{\dfrac{|f(a_1)|\,/|f(a_0)|}{\left(1+\dfrac{f(a_1)}{f(b_0)}\right)}}$

o)



From Δs $a_1 a_0 A$ and $a_0 b_0 B$

$$\frac{a_1 - a_0}{|f(a_0)|} = \frac{b_0 - a}{f(b_0)|} \quad \text{— (1)}$$

From Δs $a_2 a_1 P$ and $a_2 b_0 B$.

$$\frac{a_2 - a_1}{|f(a_1)|} = \frac{b_0 - a_2}{f(b_0)} \quad \text{— (2)}$$

or, $(a_2-a_1)|f(b_0)| + a_2|f(a_1)| = b_0|f(a_1)|$

or, $(a_2-a_1)\ |f(b_0)| + a_2\ |f(a_1)| - a_1|f(a_1)| = (b_0-a_1)|f(a_1)|$

or, $(a_2-a_1)(|f(b_0)| + f(a_1)) = (b_0-a_1)|f(a_1)|$

$$= (a_1 - a_0)\frac{|f(b_1)|\,|f(a_1)|}{|f(a_0)|}$$

$$(a_2 - a_1) = (a_1 - a_0)\left\{\frac{|f(a_1)| / |f(a_0)|}{\left(1 + \frac{|f(a_1)|}{|f(b_0)|}\right)}\right\} < 1$$

1. Solve $e^{-x} - x = 0$ using bisection Method.

Let $a = 0$, $b = 1$. $f(0) = 1$, $f(1) = -0.632$

$R = 0.567$

~~intersection~~

iteration no. (i)	a_i	b_i	m_i	$f(m_i)$	$f(a_i) * f(m_i)$
1	0	1	0.5	0.10653	+ve
2	0.5	1	0.75	-0.2776	-ve
3	0.5	0.75	0.625	-0.89738	-ve
4	0.5	0.625	0.5625	0.00728	+ve
5	0.5625	0.625	0.59375	-0.0415	-ve
6	0.5625	0.59375	0.578125	-0.01717	-ve

Regular Falsi Method

How fast the method converge to the root ??

Order of convergence : n

$$e_{i+1} = K e_i^{n}$$

Let $e_i = 0.01$, and $n = 2$

$$e_{i+1} \propto (0.01)^{n}$$

$$\propto 0.0001$$

$$|e_{i+1}| = |K|\,|e_i|^{n}$$

$$\log|e_{i+1}| = \log|K| + n\log|e_i|$$

$$n = \left|\frac{\log|e_{i+1}| - \log|K|}{\log|e_i|}\right|$$

or, $$n = \left|\frac{\log|e_{i+1}|}{\log|e_i|}\right|$$

Fixed point iteration method / Iterative method using substitution

Given $f(x) = 0$, whose root has be computed. We rewrite this eq^n as $x = g(x)$

Let us assume the intial approximation to the root of $f(x) = 0$ be x_0 we generate a sequence of approximation to the root as follows:

$$x_1 = g(x_0)$$

$$x_2 = g(x_1)$$

$$x_3 = g(x_3)$$

$$\vdots$$

$$x_{n+1} = g(x_n)$$

continued until $|x_{n+1} - x_n| <$ required permission.

1. $x^2 - x - 6 = 0$

$f(x) = 0$

$\Rightarrow x^2 - 3x + 2x - 6 = 0$

$\Rightarrow (x-3)(x+2) = 0$

$x = 3, -2$

$x = \pm\sqrt{x+6}$ — (1)

$x^2 = x + 6$

$\Rightarrow x = 1 + 6/x$ — (2)

$x = x^2 - 6$ — (3).

$x = \pm\sqrt{x+6}$

Let $x_0 = 1$

$x_1 = \sqrt{x_0 + 6} = \sqrt{7} = 2.645$

$x_2 = \sqrt{2.645 + 6} = 2.94$

$x_3 = \sqrt{2.94 + 6} = 2.989$

$x_4 = \sqrt{2.989 + 6} = 2.998$.

$x = -\sqrt{x+6}$

$x_0 = 1$

$x_1 = -\sqrt{1+6} = 2.645$

$x_2 = -\sqrt{-2.645+6} = -1.831$

$x_3 = -\sqrt{-1.831+6} = -2.04$

$x_4 = -\sqrt{-2.04+6} = -1.989$

$x_5 = -\sqrt{-1.989+6} = -2.002$

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$x = 1 + 6/x$

$r_0 = 1$

$r_1 = 1 + 6/1 = 7$

$r_2 = 1 + 6/7 = 1.857$

$r_3 = 1 + 6/1.857 = 4.23$

$r_4 = 1 + 6/4.23 = 2.418$

$r_5 = 1 + 6/2.418 = 3.481$

$\vdots$

$x = x^2 - 6.$

$r_0 = 1$

$r_1 = 1 - 6 = -5$

$r_2 = (-5)^2 - 6 = 19$

$r_3 = (19)^2 - 6 =$

$\vdots$

Diverges.

Theorem: Let $x = R$ be a root of the eq^n $f(x) = 0$, which is rewritten as $x = g(x)$. Let both $g(x)$ and $g'(x)$ exist and continous over an interval (a, b) containing $x = R$. If $|g'(x)| < 1$ over the interval (a, b) and the initial approximation to the root r_0 is also in (a, b), then the sequence of approximations $r_1, r_2, r_3 \dots$ will converge to the root $x = R$

e.g:- ① $x = \sqrt{x+6} \Rightarrow g(x) = \sqrt{x+6}$

$g'(x) = \frac{1}{2\sqrt{x+6}}$ at $x = 3$, $g'(x) < 1$

② $x = -\sqrt{x+6} \Rightarrow g(x) = -\sqrt{x+6}$

$g'(x) = -\frac{1}{2\sqrt{x+6}}$ at $x = 2$,

$g'(x) = -\frac{1}{4}$ $|g'(x)| < 1$

3. $x = 1 + 6/x$ $\quad g(x) = 1 + 6/x$

$g'(x) = -\frac{6}{x^2}$

at $x = 3$, $g'(x) = \frac{-6}{9} = -2/3$ $\quad |g'(x)| < 1$

4. $x = x^2 - 6$, $g(x) = x^2 - 6$

$g'(x) = 2x$

Proof :- Let $x = R$ be a root of $x = g(x)$

$\therefore R = g(R)$ — (1)

Let x_0 be the intial approximation to the root, then the sequence of approximations can be generated as

$$\left.\begin{aligned} x_1 &= g(x_0) \\ x_2 &= g(x_1) \\ &\vdots \\ x_{n+1} &= g(x_n) \end{aligned}\right\} \text{— (2)}$$

Let $e_0, e_1, e_2, \ldots, e_{n+1}$ be the errors associated with $x_0, x_1, x_2, \ldots, x_{n+1}$

Then,

$e_0 = R - x_0$

from eqn (1) and the set of eq^n in (2)

$$\left.\begin{aligned} e_1 &= R - x_1 \\ &= g(R) - g(x_0) \\ e_2 &= R - x_2 \\ &= g(R) - g(x_1) \\ &\vdots \end{aligned}\right\} (3)$$

$e_{n+1} = R - x_{n+1} = g(R) - g(x_n)$

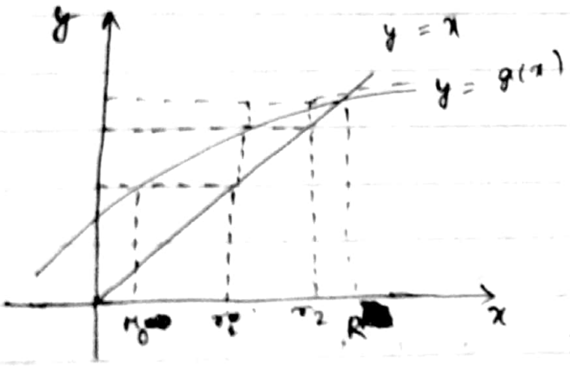
Using Mean value theorem, we can rewrite the eq^n in (3) as

$$\left.\begin{aligned} e_0 &= R - x_0 \\ e_1 &= (R - x_0)\, g'(c_1) \quad x_0 < c_1 < R \\ e_2 &= (R - x_1)\, g'(c_2) \quad x_1 < c_1 < R \\ &\vdots \\ e_{n+1} &= (R - x_n)\, g'(c_{n+1}) \quad x_{n+1} < c_{n+1} < R \end{aligned}\right\} \quad (4)$$

For convergence $|e_{n+1}| < |e_n| < \dots < |e_2| < |e_1| < |e_0|$

This will be possible if $|g'(c_i)| < 1$

Graphical Representation

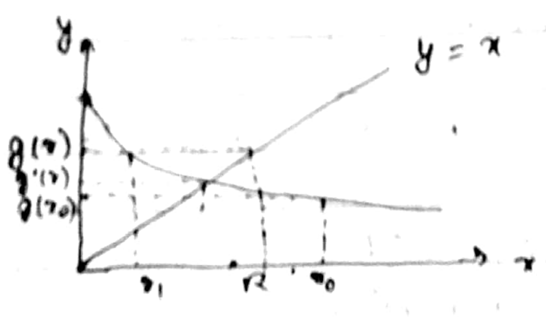


$0 < g'(x) < 1$ over the interval containing R.

We represent $x = g(x)$ as a combination of two functions $y = x$ and $y = g(x)$ intersecting at $x = R$

$x_1 = g(x_0)$

$x_2 = g(x_1)$



$0 > g'(x) > -1$

$|g'(x)| < 1$

$0 < g'(x) < 1$

$0 > g'(x) > -1$

$|g'(x)| > 1$

$g'(x) > 1$

$g'(x) < -1$

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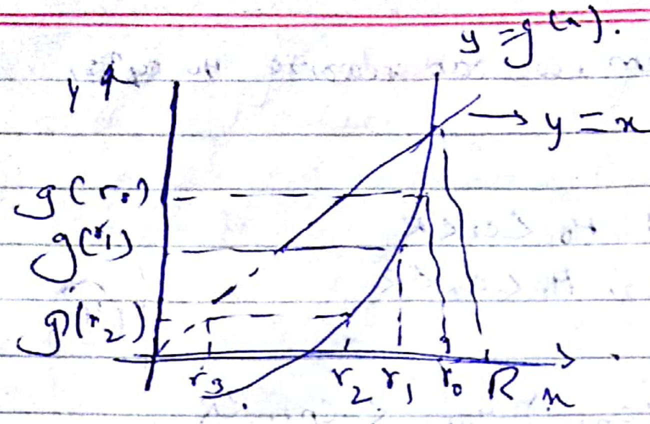
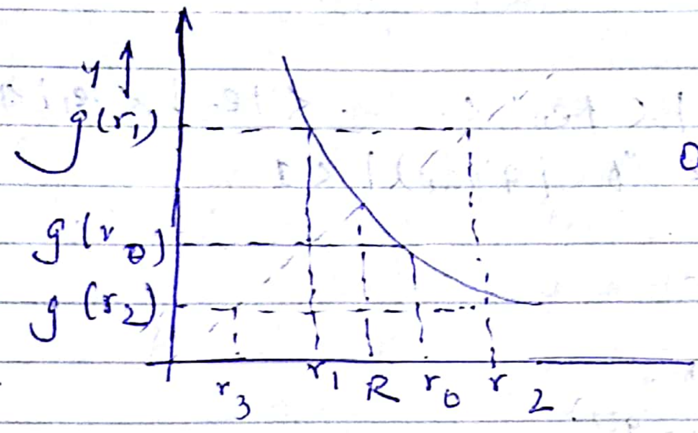


monotonic divergence



Oscillating divergence

Newton - Raphson Method

Let $f(x)=0$ has a root at $x = R$
Let the initial approximation to the root be x_0
Let e_0 be the error associated with x_0

$$R = x_0 + e_0$$

now, $f(R) = 0$

or, $f(x_0 + e_0) = 0$

Using Taylor series expansion,

$$f(x_0) + e_0 f'(x_0) + \frac{e_0^2}{2!} f''(x_0) + \dots = 0$$

we assume $|e_0| << 1$

$\therefore$ we neglect e_0^2 and higher order terms in the series.

$$\therefore f(x_0) + e_0 f'(x_0) = 0$$

$$e_0 = \frac{-f(x_0)}{f'(x_0)}$$

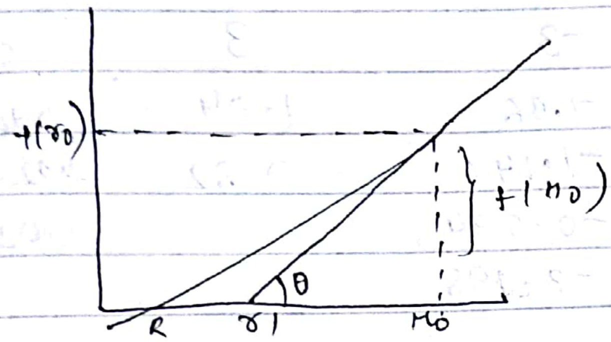
continuously in the way, we generate a sequence of approximations as

$$r_1 = H_0 - \frac{f(H_0)}{f'(H_0)}$$

$$r_2 = H_1 - \frac{f(H_1)}{f'(r_1)}$$ continous until

$$\vdots$$

$$r_{n+1} = H_n - \frac{f(H_n)}{f'(H_n)}$$ $|r_{n+1} - r_n| <$ required precision

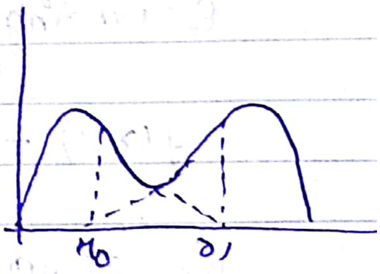


$$\tan\theta = f'(H_0) = \frac{f(r_0)}{r_0 - r_1}$$

$$\Rightarrow r_0 - r_1 = \frac{f(H_0)}{f'(r_0)}$$

$$\Rightarrow r_1 = H_0 - \frac{f(H_0)}{f'(H_0)}$$

Let $f(x) = 0$ has a root at $x = R$



N-R method $H_{n+1} = H_n - \frac{f(H_n)}{f'(H_n)}$

iterative method $H_{n+1} = g(H_n)$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$|g'(x)| < 1$$

$$g'(x) = 1 - \frac{f'(x)}{f(x)} + \frac{f(x)\,f'(x)}{\{f'(x)\}^2}$$

$$= \left|\frac{f(x)\,f''(x)}{\{f'(x)\}^2}\right| < 1$$

$x_i \quad x_{i+1} = x_i - \dots$

x_{i+2}

Solve $f(x) = 0$ Using Newton - Raphson method. Take $x_0 = 0$

$f(x) = x^3 - x - 3$, $f'(x) = 3x^2 - 1$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	$\left\{\frac{f(x_i) f''(x_i)}{f'(x_i)^2}\right\}$
0	0	-3	3	0.0[illegible] < 1
1	-3	-1.96	1.04	0.[illegible]18 < 1
2	-1.96	-1.146	0.82	0.92 < 1
3	-1.146	-0.274	----	2.027 > 1
4	-0.274	-3.198		

Order of convergence for N-R method

$x_{i+1} = R - e_{i+1}$ $\qquad f(R) = 0$

$x_i = R - R_i$

$f(x_i + R_i) = 0$

Expanding by taylor's formula

$$f(x_i) + e_i f'(x_i) + \frac{e_i^2}{2!} f''(x_i) + \frac{e_i^3}{3!} f'''(x_i) + \dots = 0$$

We assume $|e_i| << 1$, so we neglect e_i^3 and higher order terms.

$$f(x_i) + e_i f'(x_i) + \frac{e_i^2}{2!} f''(x_i) = 0$$

or $$\frac{f(x_i)}{f'(x_i)} + e_i + \frac{e_i^2 f''(x_i)}{2 f'(x_i)} = 0$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$-\frac{f(x_i)}{f'(r_i)} = (R - x_i) + e_i^2 \frac{f''(x_i)}{2f'(r_i)}$$

or, $$x_i - \frac{f(x)}{f'(x_i)} = R + e_i^2 \frac{f''(x_i)}{2f'(x_i)}$$

or, $$x_{i+1} - R = e_i^2 \frac{f''(x_i)}{2f'(x_i)}$$

or, $$-e_{i+1} = e_i^2 \frac{f''(x_i)}{2f'(x_i)}$$

or, $$|e_{i+1}| = |e_i|^2 \left|\frac{f''(x_i)}{2f'(r_i)}\right| = |K| \, |e_i^2|$$

$$\log|e_{i+1}| = \log|K| + 2\log|e_i|$$

$$\Rightarrow \frac{\log|e_{i+1}|}{\log|e_i|} = \frac{\log|K|}{\log|e_i|} + 2$$

1. $f(x) = x^2 - x - 6 = 0$

$f'(x) = 2x - 1$

$x_0 = 2.5$

i	x_i	$\left\lvert\frac{f(x_i)f''(x_i)}{\{f'(x_i)\}^2}\right\rvert$	x_{i+1}	$e_i = \lvert x_{i+1} - x_i\rvert$	$n = \left\lvert\frac{\log\lvert e_{i+1}\rvert}{\log\lvert e_i\rvert}\right\rvert$
0	2.5	<1	3.0625	0.5625	—
1	3.0625	<1	3.000762	0.0617	4.84
2	3.00076	<1	2.99695	0.0038	2.00
3	2.997	<1	3.000001	0.00305	1.00
4	3.000001	<1	2.999996	0.000005	2.107
5	2.999996	<1	3.000000	0.000004	1.02

For fixed point iteration method

$$R - x_{i+1} = (R - x_i)\, g'(c_i) \qquad x_i < c_i < R$$

Let $|g'(c_i)| < 1$ for all c_i in the interval $[x_0, R]$ and let $|g'(c_i)| = k$

∴ we can write,

$$R - x_{i+1} = k(R - x_i) \quad \text{—} (1)$$

and $$R - x_{i+2} = k(R - x_{i+1}) \quad \text{—} (2)$$

$(1) \div (2)$

$$\frac{R - x_{i+1}}{R - x_{i+2}} = \frac{R - x_i}{R - x_{i+1}}$$

$$\Rightarrow (R - x_{i+1})^2 = (R - x_i)(R - x_{i+2})$$

$$R = \frac{x_{i+1}^2 - x_i x_{i+2}}{2x_{i+1} - x_i - x_{i+2}}$$

i	x_i	$x_{i+1} = g(x_i)$	
0	1	2.64575	x_0, x_1, x_2, 3.0046
1	2.64575	2.94037	
2	3.0046	3.00076	3.000006
3	3.00076	3.00013	
4	3.000006	3.000001	

$f(x) = x^3 - 4x^2 - 3x + 18$

$= x^2(x+2)$

$= (x+2)(x^2 - 6x + 9) = (x+2)(x-3)^2$

$f(x) = x^3 - 4x^2 - 3x + 18$

$f'(x) = 3x^2 - 8x - 3$

$x_0 = 2.5$

i	x_i	x_{i+1}	$e_i = \lvert x_{i+1} - x_i \rvert$	n
0	2.5	2.7647	0.264706	—
1	2.7647	2.86974	0.105035	1.695
2	2.86974	2.93575	0.660	1.206

Let $f(x) = (x-a)^n g(x)$

i.e there exist multiple roots of order n at $x = a$

$f^{n-1}(a) = f^{n-2}(a) = \dots = f''(a) = f'(a) = f(a) = 0$

$f'(x) = n(x-a)^{n-1} g(x) + (x-a)^n g'(x)$

$(x-a) f'(x) = n(x-a)^n g(x) + (x-a)^{n+1} g'(x)$

$= n f(x) + \underbrace{(x-a)^{n+1} g'(x)}_{0}$

$(x-a) = n \cdot \dfrac{f(x)}{f'(x)}$

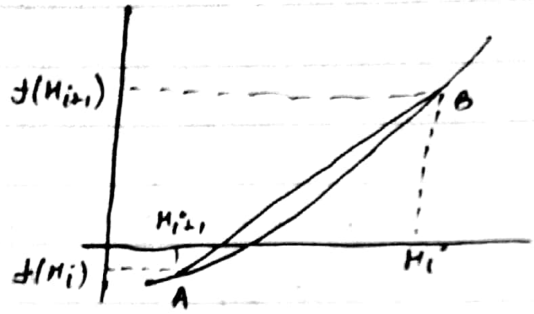
$\text{or} = x - n \dfrac{f(x)}{f'(x)}$

i	x_i	$x_{i+1} = x_i - 2\dfrac{f(x_i)}{f'(x_i)}$	$e_i = \lvert x_{i+1} - x_i \rvert$	n
0	2.5	3.0294	0.5294	~~5~~ –
1	3.0294	3.00009	0.0293	8.55
2	3.00009	3.00000	0.00009	2.639

Secant Method

Let N-R method we have to provide the functional representation of both $f(x)$ and $f'(x)$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$



$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{f(x_i)}{f(x_i) - f(x_{i+1})}$$

$$= x_i - \frac{f(x_i)(x_i x_{i+1})}{f(x_i) - f(x_{i+1})}$$

$$= \frac{x_i f(x_i) - x_i f(x_{i+1})}{f(x_i) - f(x_{i+1})}$$

~~$f(x) = x^2 e^x \sin x$~~

~~$f'(x) =$~~

R is the actual root of $f(x) = 0$

$$\therefore f(x) = 0$$

Let $x_{i-1} = R - e_{i-1}$

$x_i = R - e_i$

$x_{i+1} = R - e_{i+1}$;

$$x_{i+1} = \frac{x_{i+1} f(x_i) - x_i f(x_{i+1})}{f(x_i) - f(x_{i+1})}$$

$$R - H_{i+1} = \frac{(R - R_{i+1})\, f(R - e_i) - (R - e_i)\, f(R - e_{i+1})}{f(R - e_i) - f(R - e_{i+1})}$$

$$= \frac{(R - e_{i+1})\left\{f(R) - e_i f'(R) + \frac{e_i^2}{2} f''(R) \cdots\right\} - (R - e_i)\left\{f(R) - e_{i+1} f'(R) + \frac{e_{i+1}^2}{2} f''(R) - \cdots\right\}}{f(R)' - e_i f'(R) - f(R)' + e_{i+1} f'(R)}$$

$$= \frac{R(e_{i+1} - e_i) f'(R) - \frac{e_{i+1} e_i}{2} (e_{i+1} - e_i) f'(R)}{(e_{i+1} - e_i) f'(R)}$$

$$R - e_{i+1} \approx R - \frac{e_{i+1} e_i}{2} \frac{f''(R)}{f'(R)}$$

$$e_{i+1} \simeq \frac{e_{i+1} e_i}{2} \frac{f''(R)}{f'(R)}$$

Let us assume the order of convergence be n

$$e_{i+1} = K e_i^n \quad \text{--- (2)}$$

and $$e_i = K e_{i+1}^n \quad \text{--- (3)}$$

$$e_{i+1} = \left(\frac{e}{k}\right)^{1/n} \quad \text{--- (4)}$$

with (1), (2) and (4), we get

$$K e_i^n = \left(\frac{e_i}{k}\right)^n e \cdot \frac{f''(R)}{2f'(R)}$$

$$K e_i^n = (e_i)^{1+1/n} \frac{f''(R)}{2K^{1/n} f'(R)}$$

$$e_i^n = e_i^{1+1/n} \frac{f''(R)}{2f'(R) K^{1/n}}$$

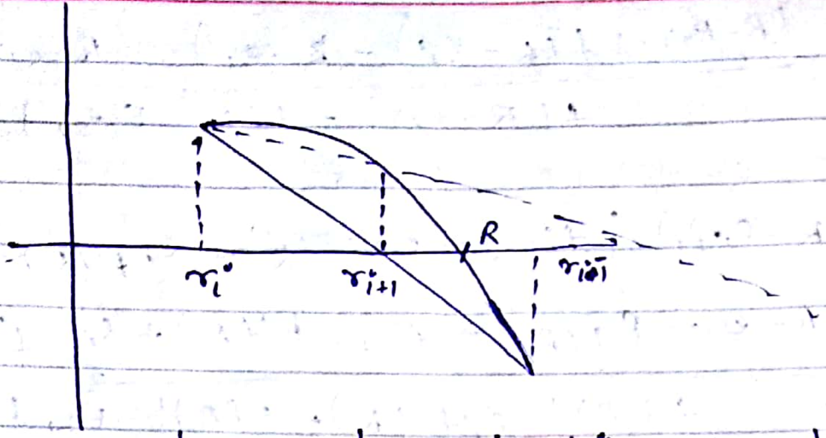
Equating powers of e_i from both sides.

$$n = 1 + 1/n$$

or, $$n^2 - n + 1 = 0$$

$$n = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} = \frac{1 \pm 2.236}{2}$$

$$n = 1.618, -0.61$$



$$\left|f'(x_i)\right| = \left|\frac{f(x_i) - f(x_{i+1})}{x_i - x_{i-1}}\right|$$

Math-point iteration method

Let R be the actual root of $f(x) = 0$
and r_i be an approximation to the root.

$$x_i = R - e_i$$

or $R = x_i + e_i$

Now, $f(R) = f(x_i + e_i) = 0$

Expanding by Taylor's formula,

$$f(x_i) + e_i f'(x_i) + \frac{e_i^2}{2} f''(x_i) + \frac{e^3}{3!} f'''(x_i) + \dots = 0$$

or,
$$f(x_i) + e_i \underbrace{\left\{ f'(x_i) + \frac{e_i}{2} f''(x_i) + \frac{e^3}{3!} f'''(x_i) + \dots \right\}}_{f'(x_i + e_i/2)} = 0$$

$$-f(x_i) = e_i f'(x_i + e_i/2)$$

or
$$e_i = \frac{-f(x_i)}{f'(x_i + e_i/2)}$$

$$= -\frac{f(x_i)}{f'\left(x_i - \frac{f(x_i)}{2f'(x_i)}\right)}$$

$$\therefore x_{i+1} = x_i + e_i$$

$$= x_i - \frac{f(x_i)}{f'\left[x_i - \frac{f(x_i)}{2f'(x_i)}\right]} = x_i - \frac{f(x_i)}{f'(x^*_{i+1})}$$

$$= \frac{f(x_i)}{f'\left(x_i - \frac{f(x_i)}{2f'(x_i)}\right)}$$

where $x^*_{i+1} = x_i - \frac{f(x_i)}{2f'(x_i)}$

for a given x_i,

$$x^*_{i+1} = x_i - \frac{f(x_i)}{2f'(x)}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x^*_{i+1})}$$

—x—

Multiple iteration Method

$$x^*_{k+1} = x_k - \frac{1}{2}\frac{f(x_k)}{f'(x_k)}$$

$R = x_k + e_k$

$R = x_{k+1} + e_{k+1}$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$x^*_{k+1} = (R - e_k) - \frac{1}{2}\frac{f(R-e_k)}{f'(R-e_k)}$$

$$\frac{f(R-e_k)}{f'(R-e_k)} = \frac{f(R) - e_k \cdot f'(R) + \frac{e_k^2}{2} f''(R) - \frac{e_k^3}{3} f'''(R) - \cdots}{f'(R) - e_k f''(R) + \frac{e_k^2}{2} f'''(R) - \cdots\cdots}$$

$$= \frac{-e_k + \frac{e_k^2}{2}\frac{f''(R)}{f'(R)} - \frac{e_k^3}{3!}\frac{f'''(R)}{f'(R)}}{1 - e_k\frac{f''(R)}{f'(R)} + \frac{e_k^2}{2}\frac{f'''(R)}{f'(R)}}$$

Let $\dfrac{f''(R)}{f'(R)} = a_1$ and $\dfrac{f'''(R)}{f'(R)} = a_2$

$$= \left(-e_K + \frac{a_1}{2!} e_K^2 - \frac{a_2}{3!} e_K^3\right)\left(1 - a_1 e_K + \frac{a_2}{2} e_K^2\right)^{-1}$$

$$= \left(-e_K + \frac{a_1}{2} e_K^2 - \frac{a_2}{3!} e_K^3\right)\left(1 + a_1 e_K - \frac{a_2}{2} e_K^2\right)^{-1}$$

$$= \left(-e_K + \frac{a_1}{2} e_K^2 - \frac{a_2}{3!} e_K^3\right)\left(1 + a_1 e_K - \frac{a_2}{2} e_K^2\right)$$

$$= -e_K + \frac{a_1}{2} e_K^2 - \frac{a_2}{3!} e_K^3 - a_1 e_K^2 + \frac{a_1^2}{2} e_K^3 + \frac{a_2}{2} e_K^3 + \cdots$$

$$= -e_K - \frac{a_1}{2} e_K^2 + \left(\frac{a_1^2}{2} + \frac{a_2}{3}\right) e_K^3$$

$$H^*_{K+1} = (R - e_K) - \frac{1}{2}\frac{f(R-e_K)}{f'(R-e_K)}$$

$$= R - e_K - \frac{1}{2}\left\{-e_K - \frac{a}{2} e_K^2 + \left(\frac{a_1^2}{2} + \frac{a_2}{3}\right) e_K^3\right\}$$

$$= R - \frac{e_K}{2} + \frac{a_1}{4} e_K^2 - \left(\frac{a_1^2}{4} + \frac{a_2}{6}\right) e_K^3$$

$$H_{K+1} = H_K - \frac{f(H_K)}{f'(H^*_{K+1})}$$

$$R - e_{K+1} = R - e_K - \frac{f(R - e_K)}{f\left[R - \left\{\frac{e_K}{2} + \frac{a_1}{4} e_K^2 - \left(\frac{a_1^2}{4} + \frac{a_2}{6}\right) e_K^3\right\}\right]}$$

$$R = H_K + e_K$$

$$= H_{K+1} - e_{K+1}$$

$$= R - e_K - \frac{f(R) - e_K f'(R) + \frac{e_K^2}{2} f''(R) - \frac{e_K^3}{6} f'''(R)}{f'(R) - \left\{\frac{e_K}{2} - \frac{a_1}{4} e_K^2 + \left(\frac{a_1^2}{4} + \frac{a_2}{6}\right) e_K^3\right\} f''(R)}$$

$$= R - e_k - \frac{-e_k + \frac{e_k^2}{2} a_1 - \frac{e_k^3}{6} a_2}{1 - \left\{ \frac{e_k}{2} - \frac{a_1}{4} e_k^2 + \left(\frac{a_1^2}{4} - \frac{a_2}{6}\right) e_k^3 \right\} a_1}$$

$$= R - e_k - \left(-e_k + a_1 \frac{e_k^2}{2} - a_2 \frac{e_k^3}{6}\right)\left[1 + \left\{\frac{e_k}{2} - \frac{a_1}{4} e_k^2 + \left(\frac{a_1^2}{4} + \frac{a_2}{6}\right) e_k^3\right\} a_1\right]$$

$$= R - e_k - \left(-e_k + a_1 \frac{e_k^2}{2} - a_2 \frac{e_k^3}{6}\right)\left[1 + \left\{\frac{e_k}{2} - \frac{a_1}{4} e_k^2 + \left(\frac{a_1^2}{4} + \frac{a_2}{6}\right) e_k^3\right\} a_1\right]$$

$$= R - e_k - \left\{ -e_k + \frac{a_1}{2} e_k^2 - \frac{a_2}{6} e_k^3 - \frac{a_1}{2} e_k^2 + a_1^2 \frac{e_k^3}{4} + \frac{a_1^3}{8} e_k^4 - \frac{a_1 a_2}{12} e_k^4 + \dots \right\}$$

$$= R - e_k + e_k + \left(\frac{a_2}{6} - \frac{a_1^2}{4}\right) e_k^3 - \left(\frac{a_1 a_2}{12} - \frac{a_1^3}{8}\right) e_k^4 + \dots$$

$$e_{k+1} = \left(\frac{a_2}{6} - \frac{a_1^2}{4}\right) e_k^3 - \left(\frac{a_1 a_2}{12} - \frac{a_1^3}{8}\right) e_k^4 + \dots$$

Chebyshev Method

Let x_k be the initial approximation to the root R of $f(x) = 0$

$$\therefore \quad R = x_k + e_k$$

$$f(R) = f(x_k + e_k) = 0$$

or,

$$f(x_k) + e_k f'(x_k) + \frac{e_k^2}{2} f''(x_k) + \frac{e_k^3}{6} f'''(x_k) \dots = 0$$

Neglecting e_k^3 and higher order terms

$$e_k f'(x_k) = f(x_k) + e_k^2 f''(x_k)$$

$$\text{or} \quad -e_k = \frac{f(x_k)}{f'(x_k)} + \frac{e_k^2}{2} \frac{f''(x_k)}{f'(x_k)}$$

$$\therefore \quad R = x_k + e_k$$

$$R = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{e_k^2}{2} \frac{f''(x_k)}{f'(x_k)} \quad \text{--- (1)}$$

From N-R method,

$$e_k = -\frac{f(x_k)}{f'(x_k)}$$

From ①,

$$e_k = -\frac{f(x_k)}{f'(x_k)}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{\{f(x_k)\}^2 f''(x_k)}{2\{f'(x_k)\}^3}$$

$$|f'(x_k)| < \epsilon .$$

— x —

Finding complex roots of a polynomial equation (Lin's method)

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n = 0$$

$$(x^2 + px + q)(x^{n-2} + b_1 x^{n-3} + b_2 x^{n-4} + \cdots + b_{n-3} x + b_{n-2})$$

$$+ \underbrace{Rx + S}_{} = 0 \quad — (1)$$

linear Remainder term, this should be equal to zero if the original polynomial is exactly devisible by $(x^2 + px + q)$.

$$x^n + a_1 x^{n+1} + a_2 x^{n+2} + \cdots + a_{n-1} x + a_n$$

$$= (x^2 + px + q)(x^{n-2} + b_1 x^{n-3} + b_2 x^{n-4} + b_3 x^{n-5} + \cdots + b_{n-3} x + b_{n-2}) + Rx + S$$

$$= x^n + (b_1 + p) x^{n-1} + (b_2 + pb_1 + q) x^{n-2} + (b_3 + pb_2 + qb_1) x^{n-3} \quad — (2)$$

equating the co-efficients from both sides of the eq^n :

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$$a_1 = b_1 + p \Rightarrow b_1 = a_1 - p$$

$$a_2 = b_2 + pb_1 + q \Rightarrow b_2 = a_2 - pb_1 - q$$

$$a_3 = b_3 + pb_2 + qb_1 \Rightarrow b_3 = a_3 - pb_2 - qb_1$$

$$\vdots$$

$$\left.a_j = b_j + pb_{j-1} + qb_{j-2} \Rightarrow b_j = a_j - pb_{j-1} - qb_{j-2}\right\} \quad (3).$$

$j = 1, 2, 3, \ldots n-2$

$$R = a_{n-1} - pb_{n-2} - qb_{n-3} = 0 \quad - (4)$$
$$S = a_n - qb_{n-2} = 0 \quad - (5).$$

Since the a_i and b_i values are known solving (4) and (5). we get a new set of value for p and q (say p_1, q_1).

$$\text{Now, } (|p_0 - p_1|, |q_0 - q_1|) < \epsilon$$

$$\text{Max}(|p_{i-1} - p_i|, |q_{i-1} - q_i|) < \epsilon.$$

<u>Bairstow's Method</u>

$$R(p,q) = 0$$
$$S(p,q) = 0$$

Let (p_0, q_0) be the approximations to (p,q) and Δp and Δq be the corrections needed to get the true values of (p,q).

$$R(p_0 + \Delta p, q_0 + \Delta q) = 0$$
$$\Rightarrow R(p_0, q_0) + \Delta p \cdot \frac{\partial R}{\partial p}\Big|_{(p_0, q_0)} + \Delta q \frac{\partial R}{\partial q}\Big|_{(p_0, q_0)} + \ldots = 0$$

$$S(p_0 + \Delta p, q_0 + \Delta q) = 0$$
$$\Rightarrow S(p_0, q_0) + \Delta p \frac{\partial S}{\partial p}\Big|_{(p_0, q_0)} + \Delta p \frac{\partial S}{\partial q}\Big|_{(p_0, q_0)} + \ldots = 0$$

Neglecting the higher order terms:

$$\Delta p \left.\frac{\partial R}{\partial p}\right|_{(p_0,q_0)} + \Delta q \left.\frac{\partial R}{\partial q}\right|_{(p_0,q_0)} = -R(p_0,q_0)$$

$$\Delta p \left.\frac{\partial S}{\partial p}\right|_{(p_0,q_0)} + \Delta q \left.\frac{\partial S}{\partial q}\right|_{(p_0,q_0)} = -S(p_0,q_0)$$

$$\Delta p = \frac{S(p_0,q_0)\left.\frac{\partial S}{\partial q}\right| - R(p_0,q_0)\left.\frac{\partial R}{\partial q}\right|}{\left.\frac{\partial R}{\partial p}\right|\left.\frac{\partial S}{\partial q}\right| - \left.\frac{\partial S}{\partial p}\right|\left.\frac{\partial R}{\partial q}\right|}$$

$$\Delta q =$$

If $\max(|\Delta p|, |\Delta q|) < \epsilon$. then stop

Equating the co-efficients from both sides of the eqn.

$$b_1 = a_1 - p$$
$$b_2 = a_2 - pb_1 - q$$
$$b_3 = a_3 - pb_2 - qb_1$$

$$\Rightarrow b_j = (a_j - pb_{j-1} - qb_{j-2} + \ldots \quad j = 3, 4, \ldots, n-2$$

$$R = a_{n-1} - pb_{n-2} - qb_{n-3} = 0 \quad - (4)$$
$$S = a_n - qb_{n-2} = 0 \quad - (5)$$

$$c_1 = \frac{\partial b_1}{\partial p} = -1$$

$$c_2 = \frac{\partial b_2}{\partial p} = -p\frac{\partial b_1}{\partial p} - b_1 = -p - b_1$$

$$c_3 = -p\frac{\partial b_2}{\partial p} - b_2 = q\frac{\partial b_1}{\partial p}$$

$$= -pc_2 - b_2 - qc_1$$

$\vdots$

$$c_j = -b_{j-1} - pc_{j-2} - qc_{j-3}$$

$\vdots$

$$\left.\frac{\partial R}{\partial p}\right|_{(p_0,q_0)} = -b_{n-2} - pc_{n-2} - qc_{n-3}$$

$$\left.\frac{\partial S}{\partial p}\right|_{(p_0,q_0)} = -qc_{n-2}$$

$$\left.\frac{\partial R}{\partial q}\right|_{(p_0,q_0)} = -pd_{n-2} - b_{n-3} - qd_{n-3}.$$

$$\left.\frac{\partial S}{\partial q}\right|_{(p_0,q_0)} = -b_{n-2} - qd_{n-2}$$

$$d_1 = \frac{\partial b_1}{\partial q} = 0$$

$q_0)$

$o)$

g^n.

$n-2$

Solution of linear simultaneous eq^n :

$$x_1 + 3x_2 + 3x_3 = 10 \quad -(1)$$
$$x_1 + x_2 + 2x_3 = 5 \quad -(2)$$
$$2x_1 + 4x_2 - 6x_3 = -4 \quad -(3)$$

(1) - (2)

$$2x_2 + x_3 = 5$$

(2) $\times 2$ - (3)

$$2x_1 + 6x_2 + 4x_3 = 10$$
$$\underline{2x_1 + 4x_2 - 6x_3 = -4}$$
$$2x_2 + 10x_3 = 14$$

$$2x_2 + x_3 = 5 \quad -(4)$$
$$\underline{2x_2 + 10x_3 = 14 \quad -(5)}$$
$$-9x_3 = -9$$
$$\text{or} \quad x_3 = 1$$

From (4)

$$2x_2 + 1 = 5$$
$$\text{or} \quad x_2 = 2.$$

Gaussian elimination method.

(n-1) different linear

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1(n-1)}x_{n-1} + a_{1n}x_n = a_{1(n+1)}$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2(n-1)}x_{(n-1)} + a_{2n}x_n = a_{2(n+1)}$$
$$\vdots$$
$$n \to a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{n(n-1)} + a_{nn}x_n = a_{n(n+1)}$$

Simultaneous eq^n w/ (n-1) unknown.

$$m_{i1} = \frac{a_{i1}}{a_{11}} \qquad i = 2, \ldots, n.$$

$$a_{ij} = a_{ij} - m_{ij} * a_{ij} \qquad \begin{matrix} i = 2, \ldots, n \\ j = 1, 2, \ldots, n+1 \end{matrix}$$

$$m_{i2} = \frac{a_{i2}}{a_{22}}$$

$$a_{ij} = a_{ij} - m_{i2} * a_{2j} \qquad \begin{matrix} i = 3, \ldots, n \\ j = 2, \ldots, n+1 \end{matrix}$$

$$a_{kk} x_k + a_{k(k+1)} x_{k+1} + \cdots + a_{k(n-1)} x_{(n-1)} + a_{kn} x_n = a_{k(n+1)}$$

$$k = 1, 2, \ldots, (n-1)$$

$$m_{ik} = \frac{a_{ik}}{a_{kk}} \qquad i = (k+1), \ldots, n$$

$$a_{ij} = a_{ij} - m_{ik} * a_{kj} \qquad \begin{matrix} i = (k+1), \ldots, n \\ j = k, \ldots, (n+1) \end{matrix}$$

$$a_{nk} x_k + a_{n(k+1)} x_{k+1} + \cdots + a_{n(n-1)} x_{n-1} + a_{mn} x_n = a_{n(n+1)}.$$

$$|a_{11}| << |a_{i1}|$$

$$|a_{kk}| << |a_{ik}|$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & a_{1(n+1)} \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & a_{2(n+1)} \\ \vdots & & & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} & \vdots & a_{n(n+1)} \end{bmatrix}$$

$n \times (n+1)$

$a_{1(n+1)}$

$a_{2(n+1)}$

$x_n =$

$n+1$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{i(n+1)} & a_{1n} & a_{1(n+1)} \\ & a_{22} & a_{23} & \cdots & a_{i(n+1)} & a_{2n} & a_{2(n+1)} \\ & & & & \vdots & & \\ & & & & a_{(n-1)(n+1)} & a_{(n-1)n} & a_{(n-1)(n+1)} \\ & & & & & a_{nn} & a_{n(n+1)} \end{bmatrix}$$

back substitution

$$x_n = \frac{a_{x(x+1)}}{a_{11}}$$

$$x_{n-1} = \frac{a_{(n-1)(n+1)} - a_{(n-1)n}\, x_n}{a_{(n-1)(n-1)}}$$

In general,

$$x_i = \frac{a_{i(n+1)} - \sum_{j=i+1}^{n} a_{ij} * x}{a_{ij}} \qquad i = (n-1)(n-2)$$

Input the co-efficient matrix $A[n][n+1]$

```
for (k = 1 ; k < n ; k++)
{
    big = 0 ;
    for(i = k ; i < = n ; i++)
    {
        if fals (a[i][k] > big)
        {
            big = a[i][k];
            p = i ;
        }
    }
}
for (j = k ; j < = n+1 ; j++)
{
    temp = a[k][j];
    a[k][j] = a[p][j] ;
    a[p][j] = Temp ;
}
```

// finding co-efficient with largest magnitude along the column A.

```
for(i=k+1 ; i<=n ; i++)
{
    m[i][k] = a[i][k] / a[k][k];
    for (j=k ; j<=(n+1) ; j++)
        a[i][j] = a[i][j] - m[i][k] * a[k][j];
}

// elimation process is complete

x[n] = a[n][n+1] / a[n][n];
for( i=(n-1); i>=1 ; i--)
{
    sum = 0 ;
    for(j=n ; j<i; j--)
        sum += a[i][j] * x[j];
        x[i] = (a[i][x+1] - sum) /a[i][j] ;
}
```

-2)

INOO
INDD

Gaussian Elimination

$$x_1 + 5x_2 + 3x_3 = 10$$
$$x_1 + 3x_2 + 2x_3 = 5$$
$$2x_1 + 4x_2 - 6x_3$$

assigned co-efficient matrix

$$\begin{bmatrix} 1 & 5 & 3 & 10 \\ 1 & 3 & 2 & 5 \\ 2 & 4 & -6 & -4 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 2 & 4 & -6 & -4 \\ 1 & 3 & 2 & 5 \\ 1 & 5 & 3 & 10 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} 2 & 4 & -6 & -4 \\ 1-1 & 3-2 & 2+3 & 5+2 \\ 1-1 & 5-2 & 3+3 & 10+2 \end{bmatrix} \quad R_2 \leftarrow R_2 - R_1/2$$

$$\Rightarrow \begin{bmatrix} 2 & 4 & -6 & -4 \\ 0 & 1 & 5 & 7 \\ 0 & 3 & 6 & 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 4 & 6 & -4 \\ 0 & 3 & 6 & 12 \\ 0 & 1 & 5 & 7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 4 & 6 & -4 \\ 0 & 3 & 6 & 12 \\ 0 & 1-1 & 5-2 & 7-4 \end{bmatrix} \quad R_3 \leftarrow R_3 - R_2/3$$

$$\Rightarrow \left[\begin{array}{cccc} 2 & 4 & 6 & -4 \\ 0 & 3 & 6 & 12 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

$x_3 = 1$

$3x_2 + 6 = 12$

or, $x_2 = 2$

—x—

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1(n-1)}x_{(n-1)} + a_{1n}x_n = a_{1(n+1)}$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2(n-1)}x_{(n-1)} + a_{2n}x_n = a_{2(n+1)}$$

$\vdots$

$$a_{(n-1)1}x_1 + a_{(n-1)2}x_2 + a_{(n-1)3}x_3 + \dots + a_{(n-1)(n-1)}x_{n-1} + a_{(n-1)n}x_n = a_{(n-1)(n+1)}$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{n(n-1)}x_{n-1} + a_{nn}x_n = a_{n(n+1)}$$

$$m_{i1} = \frac{a_{i1}}{a_{11}} \quad i = 2, 3, 4, \dots n.$$

$$m_{ik} = \frac{a_{i1}}{a_{11}} \quad i = k, 3, \dots, n$$

Total no. of divisions required for getting the multiples

$$(n-1) + (n-2) + \dots + 1$$

$$= \frac{(n-1)n}{2}$$

No. of multiplications required for the elimination process for eliminating the co-eff. of x_1 from 2^{nd} row to n^{th} row.

no. of multiplication $= (n+1)(n-1) = n^2 - 1$

Suppose at any stage of elimination we are left with k ... eq^{n}s with k unknown for eliminating the co-efficients along first column, no. of multiplication required will be (k^2-1).

$\therefore$ Total no. of multiplications required for the entire elimination process to reduce the system of eq^n in upper triangular form

$$= \sum_{n=k}^{2} (k^2-1) = (n^2+(n-1)^2+\dots+2^2) - (n-1)$$

$$= \underbrace{n^2+(n-1)^2+\dots+2^2+1-n}$$

$$= \frac{1}{6} n(n+1)(2n+1) - n$$

For entire elimination, $\frac{1}{6} n(n+1)(2n+1)$

$$\frac{1}{6} n(n+1)(2n+1) - n$$

After elimation, For back substitution,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{n-1}x_{n-1} + a_{1n}x_n = a_{1(n+1)}$$

$$+ \quad a_{22}x_2 + a_{23}x_3 + \dots + a_{2(n-1)}x_{n-1} + a_{2n}x_n = a_{2(n+1)}$$

$\vdots$

$$a_{(n-1)}x_{n-1} + a_{(n-1)n}x_n = a_{(n+1)(n-1)}$$

$$+ a_{nn}x_n = a_{n(n+1)}$$

$1 + 2 + 3 + \dots + n$

$$= \frac{n(n+1)}{2}$$

Gauss Jordan Elimination

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = a_{1(n+1)}$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = a_{2(n+1)}$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nm}x_n = a_{n(n+1)}.$$

no. of multiplications required for the elimination process

for eliminating the co-eff. of x_1 from 2^{nd} row to n^{th} row.

no. of multiplications $= (n+1)(n-1) = n^2 - 1$

Suppose at any stage of elimination we are left with k-eq^n with unknown for eliminating the co-efficients along first column, no. of multiplication required will be $(n-1)(n+1)$

$\therefore$ Total no. of multipliers required for the entire elimination process to reduce the system of eq^n in upper triangular form

$$= \sum_{k=n}^{2} ((n-1)(k+1))$$

$$\left[\begin{array}{cccccc|c}
a_{11} & 0 & 0 & \cdots & 0 & 0 & a_{1(n+1)} \\
0 & a_{22} & 0 & & 0 & 0 & a_{2(n+1)} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & 0 & & a_{(n-1)(n-1)} & 0 & a_{(n-1)(n+1)} \\
0 & 0 & 0 & & 0 & a_{nn} & a_{n(n+1)}
\end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{cccccc|c}
1 & 0 & 0 & \cdots & 0 & 0 & a_{1(n+1)}/a_{11} \\
0 & 1 & 0 & \cdots & 0 & 0 & a_{2(n+1)}/a_{12} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & a_{(n-1)(n+1)}/a_{(n-1)(n+1)} \\
0 & 0 & 0 & \cdots & 0 & 1 & a_{n(n+1)}/a_{nn}
\end{array}\right]$$

Gauss- Jordan Elimination.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Let the solⁿ of this system of linear simultaneous eqⁿ be = $\begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{n1} \end{bmatrix}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{(I)}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_{12} \\ c_{22} \\ \vdots \\ c_{n2} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} c_{12} \\ c_{22} \\ \vdots \\ c_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \text{(II)}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} c_{1n} \\ c_{2n} \\ \vdots \\ c_{nn} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{12} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} c_{12} \\ c_{1n} \\ \vdots \\ c_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \end{bmatrix} \quad \text{---} \ (iii)$$

combining (i), (ii) and (ii):

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}}_{n \times n \text{ identity matrix}}$$

Inverse of a matrix

$$\left[\overbrace{\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix}}^{n \times n} \,\middle|\, \overbrace{\begin{matrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{matrix}} \right]$$

$$\Downarrow$$

by applying Gauss-Jordan elimination

$$\left[\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{matrix} \,\middle|\, \begin{matrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{matrix}\right]$$

LU decomposition method

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad AX = B$$

$$A = LU$$

$$L = \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & 0 \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} \cdots & u_{1n} \\ 0 & 1 & u_{23} \cdots & u_{2n} \\ \vdots & & & \\ 0 & 0 & 0 \cdots & 1 \end{bmatrix}$$

$$AX = B$$

$$\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$UX = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$$\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \\ l_{n1} & l_{n2} & \cdots & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

using forward substitution we get $\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$

$$\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \\ l_{n1} & l_{n2} & \cdots & l_{nm} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$UX = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$$\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \\ l_{n1} & l_{n2} & \cdots & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

using forward substitution we get $\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$

$l_{i1} = a_{i1} \quad i = 1, 2, \ldots n.$

$u_{1j} = a_{1j} / l_{11} \quad j = 2, 3, \ldots n.$

$l_{ip} = a_{ip} - \sum_{k=1}^{p-1} l_{ik} k_{ki} \quad i = p, \ldots, n$

$u_{pj} = (a_{pj} - \sum_{k=1}^{p-1} l_{pk} u_{kj}) / l_{pp} \quad j = p+1, \ldots, n$

$p = 2, 3, \ldots, n$

Iterative Method for soln. of Linear simultaneous eq^n

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots \dots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\left[x_1^{(0)}, x_2^{(0)} \dots x_n^{(0)}\right]^T = \left[x^{(0)}\right]^T$$

$$x^{(1)} = \left[x_1^{(1)}, x_2^{(1)}, \dots x_n^{(1)}\right]^T$$

$$a_{11}x_1^{(1)} = b_1 - \left(a_{12}x_2^{(0)} + \dots + a_{1n}x_n^{(0)}\right).$$
$$a_{22}x_2^{(1)} = b_2 - \left(a_{21}x_1^{(0)} + \dots + a_{2n}x_n^{(0)}\right)$$
$$\vdots$$
$$a_{nn}x_n^{(1)} = b_n - \left(a_{n1}x_1^{(0)} + a_{n2}x_2^{(0)} + \dots + a_{n(n-1)}x_{n-1}^{(0)}\right)$$

check if

$$\max_{1 \le i \le n} \left\{ |x_i^{(1)} - x_i^{(0)}| \right\} < \epsilon$$

then stop.

else contains the iterations.

Jacobi's Method

$$a_{11}x_1^{(k+1)} = b_1 - \left(a_{12}x_2^{k} + \dots + a_{1n}x_n^{(k)}\right)$$
$$a_{21}x_2^{(k+1)} = b_2 - \left(a_{21}x_1^{k} + \dots + a_{2n}x_n^{(k)}\right)$$
$$\vdots$$
$$a_{nn}x_n^{(k+1)} = b_n - \left(a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \dots + a_{n(n-1)}x_{n-1}^{(k)}\right)$$

$$x_1^{(0)} = x_2^{(0)} = x_n^{(0)} = 0$$

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In matrix notation

$$X^{(K+1)} = HX^{(K)} + c$$

where $X^{(K+1)}$ and $X^{(K)}$ are $n \times 1$ (i.e a column) vector. H is a $n \times n$ matrix depending on A. and c is again a column vector depending on A and B.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{(n-1)1} & a_{(n-1)2} & a_{(n-1)3} & \cdots & a_{(n-1)n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0 & \cdots & & 0 & 0 \\ a_{21} & 0 & & & 0 & 0 \\ a_{31} & a_{32} & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \\ a_{n1} & a_{n2} & \cdots & & a_{n(n-1)} & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} a_{11} & 0 & 0 & & 0 \\ 0 & a_{22} & 0 & & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ 0 & 0 & \cdots & & a_{(n-1)n} \\ 0 & 0 & \cdots & & 0 \end{bmatrix}$$

In matrix notation, the set of eq^{n}s in (1) can be written as

$$DX^{(K+1)} = B - (L+U)X^{(K)}$$

or, $X^{(K+1)} = D^{-1}\{B-(L+U)X^{(K)}\}$

or, $X^{(K+1)} - X^{(K)} = D^{-1}\{B-(L+U)X^{(K)}\} - X^{(K)}$

$$= D^{-1}\{B-(L+U)X^{(K)} - DX^{(K)}\}$$

$$= D^{-1}\{B - (L+U+D)X^{(K)}\}$$

$$X^{(K+1)} - X^{(K)} = D^{-1}\{B - AX^{(K)}\}$$

$$X^{(K+1)} = D^{-1}\{B-(L+U)X^{(K)}\}$$

Let $X^{(K+1)} \doteq X + E^{(K+1)}$

and $X^{(K)} = X + E^{(K)}$ where X is the true sol^n vector, $E^{(K+1)}$ and $E^{(K)}$ are the error vectors associated with $X^{(K+1)}$ and $X^{(K)}$ respectively.

$$X + E^{(K+1)} = D^{-1}\{B-(L+U)(X+E^{(K)})\}$$

or, $E^{(K+1)} = D^{-1}\{B - (L+U)X - (L+U)E^{K} - DX\}$

or, $E^{(K+1)} = D^{-1}\{B-AX - (L+U)E^{(K)}\}$

$$= D^{-1}(L+U)E^{(K)}$$

In matrix notation, the set of eq^{n}s in (1) can be written as

$$DX^{(K+1)} = B - (L+U)X^{(K)}$$

for convergence,

$$\|-D^{-1}(L+U)\| < 1$$

or, $\|-D^{-1}(A-D)\| < 1$

or, $\|I - D^{-1}A\| < 1$

Let $(I-D^{-1}A)Y = \lambda Y$ where

or, $(D-A)Y = \lambda DY$

λ is the eigen value and (scalar)

Y is the corresponding eigen vetor non-zero column vector. (vector)

or, $-\sum_{\substack{i=1 \\ i\neq j}}^{n} a_{ij} y_j$

$= \lambda a_{ii} y_i$ $\quad i = 1, \ldots, n$; $j = 1, \ldots, n$.

Since Y is a non-zero vector, $|y_i| \neq 0$

$$\therefore |\lambda| = \left\{-\sum_{\substack{i=1 \\ i\neq j}}^{n} a_{ij}\right\} \Big/ |a_{ii}|$$

for convergence $|\lambda| < 1$

$$\left\{\sum_{i=1}^{n} |a_{ij}|\right\} \Big/ |a_{ii}| < 1$$

or $\sum_{\substack{j=1 \\ i\neq j}}^{n} |a_{ij}| < |a_{ii}| \quad i = 1, \ldots, n.$

Example

$$x + 6y = 4$$
$$x - 2y - 6z = 14$$
$$9x + 4y + z = -17$$

$R_1 \leftrightarrow R_3$

$A = \begin{bmatrix} 1 & 6 & 0 \\ 1 & -2 & -6 \\ 9 & 4 & 1 \end{bmatrix}$ $\quad A = \left[\begin{array}{ccc|c} 1 & 6 & 0 & 4 \\ 1 & -2 & -6 & 14 \\ 9 & 4 & 1 & -17 \end{array}\right]$

$$\Rightarrow \left[\begin{array}{ccc|c} 9 & 4 & 1 & -17 \\ 1 & -2 & -6 & 14 \\ 1 & 6 & 0 & 4 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 9 & 4 & 1 & -17 \\ 1 & 6 & 0 & 4 \\ 1 & -2 & -6 & 14 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 9 & 4 & 1 & -17 \\ 1 & 6 & 0 & 4 \\ 1 & -2 & -6 & 14 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$A = \begin{bmatrix} 9 & 4 & 1 \\ 1 & 6 & 0 \\ 1 & -2 & -6 \end{bmatrix} \quad B = \begin{bmatrix} -17 \\ 4 \\ 14 \end{bmatrix} \quad D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} 1/9 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & -1/6 \end{bmatrix}$$

Gauss-seidal method.

$$a_{11} x_1^{(k+1)} = b_1 - (a_{12} x_2^{(k)} + \dots + a_{1n} x_n^{(k)})$$
$$a_{22} x_2^{(k+1)} = b_2 - (a_{21} x_1^{(k+1)} + \dots + a_{2n} x_n^{(k)})$$
$$a_{33} x_3^{(k+1)} = b_3 - (a_{31} x_1^{(k+1)} + a_{32} x_2^{(k+1)} + \dots a_{3n} x_n^{(k)})$$
$$\vdots$$
$$a_{nn} x_n^{(k+1)} = b_n - (a_{n1} x_1^{(k+1)} + a_{n2} x_2^{(k+1)} + \dots + a_{n(n-1)} x_{n-1}^{(k+1)})$$

(1)

$$x_1^{(0)} = x_2^{(0)} = \dots = x_n^{(0)} = 0$$

A = same L = same D same U = sa-

$$a_{11} x_1^{(k+1)} = b_1 - (a_{12} x_2^{(k)} + a_{13} x_3^{(k)} + \dots + a_{1n} x_n^{(k)})$$
$$a_{21} x_1^{(k+1)} + a_{22} x_2^{(k+1)} = b_2 - (a_{23} x_3^{(k)} + a_{24} x_4^{(k)} + \dots + a_{2n} x_n^{(k)})$$
$$a_{31} x_1^{(k+1)} + a_{32} x_2^{(k+1)} + a_{33} x_3^{(k+1)} = b_3 - (a_{34} x_4^{(k)} + \dots + a_{3n} x_n^{(k)})$$
$$\vdots$$
$$a_{n1} x_1^{(k+1)} + a_{n2} x_2^{(k+1)} + \dots + a_{nn} x_n^{(k+1)} = b_n$$

$$(L+D) X^{(k+1)} = B - UX^{(k)}$$

or, $$X^{(k+1)} = (L+D)^{-1}(B - UX^{(k)})$$

$$\cancel{X^{(k+1)} = (L+D)^{-1}B - (L+D)^{-1}UX^{(k)}}$$

or, $$X^{(k+1)} - X^{(k)} = (L+D)^{-1}(B - UX^{(k)}) - X^{(k)}$$

$$= (L+D)^{-1}\{B - UX^{(k)} - (L+D)X^{(k)}\}$$

Iterative method to determine A.

Let B be an approximate inverse of A.

$\therefore AB \neq I$

Let $AB = I + E \Rightarrow E = AB - I$.

$\therefore$ from multiplying both sides with A^{-1}

$$A^{-1}AB = A^{-1}(I+E)$$

& $B = A^{-1}(I+E)$

Post multiplying both sides with $(I+E)^{-1}$

$$A^{-1} = B(I+E)^{-1} \simeq B(I - E + E^2 - E^3 \cdots)$$

If $\|E\| << 1$ we can write

$$A^{-1} \simeq B(I-E).$$

$$\simeq B(I - AB + I) \simeq B(2I - AB)$$

Let $B^{(k+1)}$ and $B^{(k)}$ be the approximate inverse at $(k+1)$ and k iteration.

$$B^{(k+1)} = B^{(k)} (2 I - A B^{(k)})$$

$$k = 1, 2, 3, \ldots, n.$$

premultiplying both sides with A.

$$AB^{(k+1)} = AB^{(k)}(2I - AB^{(k)}) = I$$

$$= 2I\{AB^{(k)}\} - \{AB^{(k)}\}^2$$

or $$AB^{(k+1)} - I = 2I\{AB^{(k)}\} - \{AB^{(k)}\}^2 - I$$

$$= -\{\underbrace{AB^{(k)} - I}_{E^{(k)}}\}^2$$

Solution of nonlinear simultaneous eq^n

$$f_1(x_1, x_2 \ldots x_n) = 0 \Rightarrow x_1^{(k+1)} = F_1(x_1^{(k)}, x_2^{(k)} \ldots x_n^{(k)})$$

$$f_2(x_1, x_2 \ldots x_n) = 0 \Rightarrow x_2^{(k+1)} = F_2(x_1^{(k)}, x_2^{(k)} \ldots x_n^{(k)})$$

$$\max_{1 \le i \le n} \left\{ \left| x_i^{(k+1)} - x_i^{k} \right| \right\} < \epsilon$$

for convergence $\sum_{i=1}^{n} \left| \frac{\partial F_i}{\partial x_i} \right| < 1 \quad i = 1, 2 \ldots n$

Newton Ralphson method

Let $x_1^{(0)}, x_2^{(0)}, \ldots x_n^{(0)}$ be the intial approximations, and $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ be the corrections needed

$\therefore$

$$f_1(x_1^{(0)} + \Delta x_1, x_2^{(0)} + \Delta x_2, \ldots, x_n^{(0)} + \Delta x_n) = 0$$

$$f_2(x_1^{(0)} + \Delta x_1, x_2^{(0)} + \Delta x_2, \ldots, x_n^{(0)} + \Delta x_n) = 0$$

$\vdots$

$$f_n(x_1^{(0)} + \Delta x_1, x_2^{(0)} + \Delta x_2, \ldots, x_n^{(0)} + \Delta x_n) = 0$$

$$f_1(x^{0}) + \Delta x_1 \frac{\partial f_1}{\partial x_1}\bigg|_{x^{(0)}} + \Delta x_2 \cdot \frac{\partial f_1}{\partial x_2}\bigg|_{x^{(0)}} \cdots + \Delta x_n \frac{\partial f_1}{\partial x_n} = 0$$

$$\max_{1 \le i \le n} (|\Delta x_i|) < \epsilon$$

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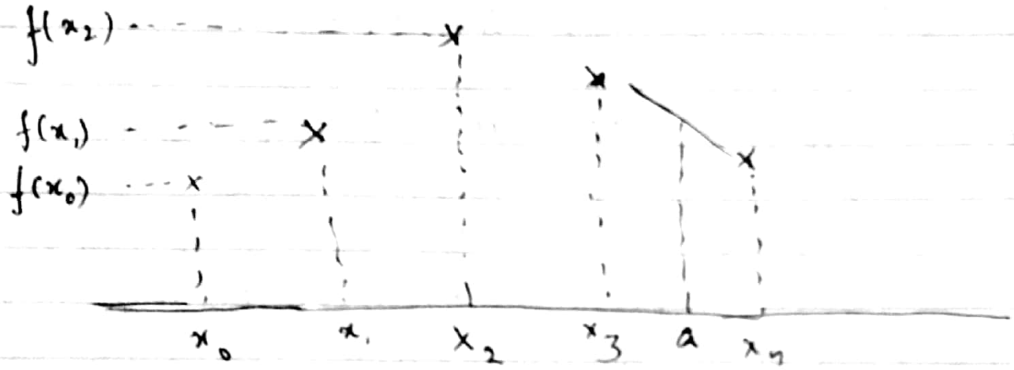
Interpolation

x_i	$y_i = f(x_i)$
x_0	y_0
x_1	y_1
$\vdots$	$\vdots$
x_n	y_n



$$\frac{y - y_k}{x - x_k} = \frac{y_{k+1} - y_k}{x_{k+1} - x_k}.$$

$$y = y_k + \left(\frac{y_{k+1} - y_k}{x_{k+1} - x_k} \right) (x - x_k)$$

$$y \Big|_{x=a} = y_k + \left(\frac{y_{k+1} - y_k}{x_{k+1} - x_k} \right) (a - x_k).$$

2 types of situation may arrive.

The independent variable (ie x_i values) are
1) equispaced   2) not equispaced.

For equispaced x_i values we follow different finite difference formula like Newton forward / backward difference formulae, gauss central difference formulae.

For non equispaced caseses, we folow, divided difference method. Lagrange interpolation method. Iterative interpolation method.

Finite difference operators :

1) forward difference (Δ)

First order forward difference is defined as

$$\Delta y_i = y_{i+1} - y_i$$

Second order forward difference

$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i$$

k^{th} order forward difference

$$\Delta^k y_i = \Delta^{k-1} y_{i+1} - \Delta^{k-1} y_i$$

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$
x_0	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_1$
x_1	y_1	Δy_1	$\Delta^2 y_1$	$\Delta^3 y_1$	
x_2	y_2	Δy_2	$\Delta^2 y_2$		
x_3	y_3	Δy_3			
x_4	y_4				

First order forward difference is defined as

$$\Delta y_i = y_{i+1} - y_i$$

Second order forward difference

$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i = (y_{i+2} - y_{i+1}) - (y_{i+1} - y_i)$$

$$= y_{i+2} - 2y_{i+1} + y_i$$

$$\Delta^3 y_i = \Delta^2 y_{i+1} - \Delta^2 y_i = (y_{i+3} - 2y_{i+2} + y_{i+1}) - (y_{i+2} - 2y_{i+1} + y_i)$$

$$= y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$$

$$\Delta^k y_i = \Delta^{k-1} y_{i+1} - \Delta^{k-1} y_i$$

Backward difference (∇)

First order backward difference $\nabla y_{i+1} = y_{i+1} - y_i$

Second order backward difference

$$\nabla^2 y_{i+1} = \nabla y_{i+1} - \nabla y_i$$

k^{th} order backward differences

$$\nabla^k y_{i+1} = \nabla^{k-1} y_{i+1} - \nabla^{k-1} y_i$$

- Backward difference Table

x_i	y_i	∇y_i	∇y_i^2	∇y_i^3	∇y_i^4
x_0	y_0				
x_1	y_1	∇y_1			
x_2	y_2	∇y_2	∇y_2^2		
x_3	y_3	∇y_3	∇y_3^2	∇y_3^3	
x_4	y_4	∇y_4	∇y_4^2	∇y_4^3	∇y_4^4

Shift Operator (E)

This operator shifts a functional value of a higher index

$$E y_i = y_{i+1}$$

Inverse shift operator (E^{-1})

$$y_i = E^{-1} y_{i+1}$$

Equivalence amount finite difference operators.

$$\Delta y_i = y_{i+1} - y_i$$
$$= E y_i - y_i$$
$$= (E-1) y_i$$
$$\Delta = (E-1).$$
$$E = 1 + \Delta$$

$$\Delta^r y_k = \nabla^r y_{k+r}$$

for $r=1$

$$\Delta y_k = y_{k+1} - y_k.$$
$$= \nabla y_{k+1}$$

for $r=2$.

$$\Delta^2 y_k = \Delta y_{k+1} - \Delta y_k$$
$$= \nabla y_{k+2} - \Delta y_{k+1}$$
$$= \nabla^2 y_{k+2}$$

Similarly

$$\nabla y_{i+1} = y_{i+1} - y_i$$
$$= y_{i+1} - E^{-1} y_{i+1}$$
$$= (1 - E^{-1}) y_{i+1}.$$

$$\nabla = (1 - E^{-1}).$$
$$E^{-1} = 1 - \nabla$$
$$E = \frac{1}{1-\nabla}.$$

Divided difference Method

First order divided difference is defined as

$$y[x_i, x_{i+1}] = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

Similarly 2^{nd} order ~~derivative~~ divided difference

$$y[x_i, x_{i+1}, x_{i+2}] = \frac{y[x_{i+1}, x_{i+2}] - y[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

K to the order divided difference

$$y[x_i, x_{i+1}, \dots, x_{i+K}] = \frac{y[x_{i+1}, x_{i+2}, \dots, x_{i+K}] - y[x_i, x_{i+1}, \dots, x_{i+K-1}]}{(x_{i+K} - x_i)}$$

x_i	y_i	$y[x_i, x_{i+1}]$	$y[x_i, x_{i+1}, x_{i+2}]$
x_0	y_0	$y[x_0, x_1]$	$y[x_0, x_1, x_2]$
x_1	y_1	$y[x_1, x_2]$	$y[x_1, x_2, x_3]$
x_2	y_2	$y[x_2, x_3]$	$y[x_2, x_3, x_4]$
$\vdots$	$\vdots$	$\vdots$	
x_{n-1}	y_{n-1}	$y[x_{n-1}, x_n]$	
x_n	y_n		

First order divided difference is defined as

$$y[x_i, x_{i+1}] = \frac{y_{x_{i+1}} - y_i}{x_{i+1} - x_i} = \frac{y_{i+1}}{(x_{i+1} - x_i)} + \frac{y_i}{(x_i - x_{i+1})}$$

Similarly, 2nd Order divided difference

$$y[x_i, x_{i+1}, x_{i+2}] = \frac{y[x_{i+1}, x_{i+2}] - y[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

$$= \frac{\left\{\frac{y_{i+2}}{(x_{i+2} - x_{i+1})} - \frac{y_{i+1}}{(x_{i+1} - x_{i+2})}\right\} - \left\{\frac{y_{i+1}}{(x_{i+1} - x_i)} + \frac{y_i}{(x_i - x_{i+1})}\right\}}{(x_{i+2} - x_i)}$$

$$= \frac{\left[\frac{y_{i+2}}{(x_{i+2} - x_{i+1})} + \frac{y_{i+1}\{\cancel{x_{i+1}} - x_i - \cancel{x_{i+1}} - x_{i+2}\}}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})} - \frac{y_i}{(x_i - x_{i+1})}\right]}{(x_{i+2} - x_i)}$$

$$= \frac{y_{i+2}}{(x_{i+2} - x_{i+1})(x_{i+2} - x_i)} + \frac{y_{i+1}}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})} + \frac{y_i}{(x_i - x_{i+1})(x_i - x_{i+2})}$$

K^{th} order divided difference.

$$y[x_i, x_{i+1}, \dots, x_{i+k}] = \frac{y[x_{i+1}, x_{i+2}, \dots x_{i+k}] - y[x_i, x_{i+1}, \dots, x_{i+k-1}]}{(x_{i+k} - x_i)}$$

$$= \frac{y_{i+k}}{(x_{i+k}-x_{i+k-1})\cdots(x_{i+k}-x_i)} + \frac{y_{i+k-1}}{(x_{i+k-1}-x_{i+k})(x_{i+k-1}-x_{i+k-2})\cdots(x_{i+k-1}-x_i)}$$

$$+ \frac{y_i}{(x_i-x_{i+k})\cdots(x_i-x_{i+1})}$$

$$y[x, x_0] = \frac{y-y_0}{x-x_0}$$

or, $y - y_0 = (x-x_0)\, y[x, x_0]$

or, $y = y_0 + (x-x_0)\, y[x, x_0]$ — (1)

$$y[x, x_0, x_1] = \frac{y[x_0, x_1] - y[x, x_2]}{x_1 - x}$$

$\Rightarrow y[x_0, x_1] - y[x, x_0] = (x_1 - x)\, y[x, x_0, x_1]$

$\Rightarrow y[x, x_0] = y[x_0, x_1] + (x-x_1)\, y[x, x_0, x_1]$ — (2).

$$y[x, x_0, x_1, x_2] = \frac{y[x_0, x_1, x_2] - y[x, x_0, x_1]}{(x_2 - x)}$$

$\Rightarrow y[x, x_0, x_1] = y[x_0, x_1, x_2] + (x - x_2)\, y[x, x_0, x_1, x_2]$

From (1) and (2),

$$y = y_0 + (x-x_0)\{y[x_0, x_1] + (x-x_1)\, y[x, x_0, x_1]\}$$

$$= y_0 + (x-x_0)\, y[x_0, x_1] + (x-x_0)(x-x_1)\, y[x, x_0, x_1]$$

$$= y_0 + (x-x_0)\, y[x_0, x_1] + (x-x_0)(x-x_1)\, y[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2)\, y[x, x_0, x_1, x_2]$$

$$= y_0 + (x-x_0)\, y[x_0, x_1] + (x-x_0)(x-x_1)\, y[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2)\, y[x_0, x_1, x_2, x_3] + \cdots + (x-x_0)(x-x_1)\cdots(x-x_n)\, y[x_0, x_1, x_2, \ldots, x_n]$$

No. of multiplication and division required

$$\{ n + (n-1) \cdots + 1 \} = n(n+1)/2$$

Lagrange's method

Here we assume the enterpolarity as an weighted sum of $(n+1)$ n^{th} degree polynomial.

$$y = y_0 b_0(x) + y_1 b_1(x) + \cdots + y_n b_n(x)$$

Each $b_i(x)$ is a polynomial of degree n

Since this polynomial satisfied the condition ~~y(x)~~

$$y(x_i) = y_i \quad i = 0, 1, \ldots, n$$

$$y_0 = y_0 b_0(x_0) + y_1 b_1(x_0) + \cdots + y_n b_n(x_0)$$

$$y_1 = y_0 b_0(x_1) + y_1 b_1(x_1) + \cdots + y_n b_n(x_1)$$

$$\vdots$$

$$y_n = y_0 b_0(x_n) + y_1 b_1(x_n) + \cdots + y_n b_n(x_n)$$

we choose $b_i(x)$ in such a way that

$$b_i(x_j) = 1 \quad \text{for } i = j$$
$$= 0 \quad \text{otherwise}$$

$$b_i(x) = C_i (x - x_0)(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)$$

$$b_i(x_i) = 1$$

$$b_i(x_i) = C_i (x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n) = 1$$

$$C_i = \frac{1}{\prod_{\substack{j=0 \\ j \neq i}}^{n} (x_i - x_j)}$$

$$b_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \left\{ \frac{x - x_j}{x_i - x_j} \right\}$$

$$y(x) = \sum_{i=0}^{n} y_i \prod_{\substack{j=0 \\ j \neq i}}^{n} \left\{ \frac{(x - x_j)}{(x_i - x_j)} \right\}$$

No. of multiplications and divisions required

$$(2n+2)(n+1) = 2(n+1)^2$$

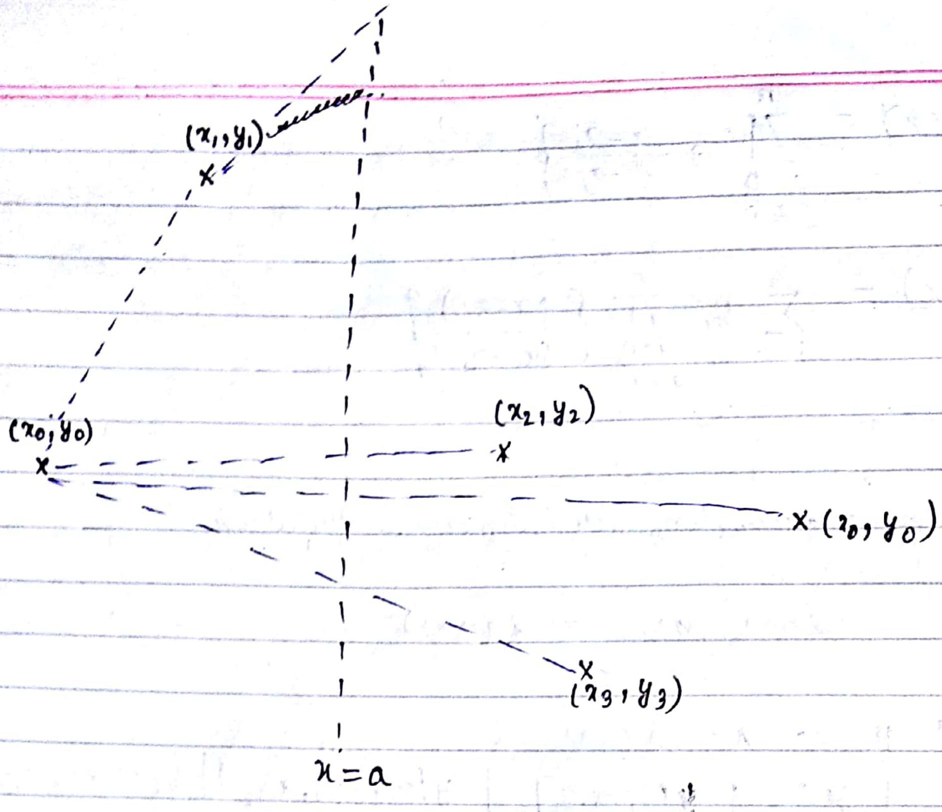
Find y at $x = 3$

x_i	y_i	$y[x_i, x_{i+1}]$	$y[x_i, x_{i+1}, x_{i+2}]$	$y[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$	...
0	1	13	$(1-13)/2 = -6$	⋮	0
1	14	1	$(-5-1)/3 = -2$		0
2	15	$(5-15)/2 = -5$	2		
4	5	1	6		
5	6	13			
6	19				

$$y(3) = y_0 + (3 - x_0)\, y[x_0, x_1] + (3 - x_0)(3 - x_1)\, y[x_0, x_1, x_2] + (3 - x_0)(3 - x_1)(3 - x_2) \cdot y[x_0, x_1, x_2, x_3]$$

$$= 1 + (3-0) \times (3) + (3-0)(3-1)(-6) + (3-0)(3-1)(3-2) \times 1$$

$$= 10$$



Eq^n of each st. line :

$$\frac{y - y_i}{x - x_i} = \frac{y_i - y_0}{x_i - x_0}$$

$$\frac{y_{(0,i)}(a) - y_i}{a - x_i} = \frac{y_i - y_0}{x_i - x_0}$$

$$\Rightarrow y_{0,i}(a) = y_i + (a - x_i)(y_i - y_0)/(x_i - x_0)$$

$$= \frac{1}{(x_i - x_0)}\left\{ \cancel{y_i x_i} - y_i x_0 + a y_i - a y_0 - \cancel{x_i y_i} + x_i y_0 \right\}$$

$$= \frac{1}{(x_i - x_0)}\left\{ a(y_i - y_0) + x_i y_0 - x_0 y_i \right\}$$

next iteration we replace the values of y_i with $y_{0,i}(a)$

Then we fit st. lines between $(x_1, y_{0,i}(a))$ and $(x_i, y_{0,i}(a))$ $i = 2, 3, \ldots, n$

Let the interplation result for each of the lines be represented as $y_{0,1,i}$

$$\therefore\ y_{0,1,i}(a) = \frac{1}{(x_i - x_1)} \{ a(y_{0,i}(a) - y_{0,i}(a)) + x_i y_{0,i}(a) - x_0 y_{0,i}(a) \}$$

x_i	y_i	$y_{0,i}(3)$	$y_{0,1,i}(3)$	$y_{0,1,2,i}(3)$
0	1			
1	14	40		
2	15	22	4	
3	5	4	16	10
4	6	4	22	10
6	19	10	28	10

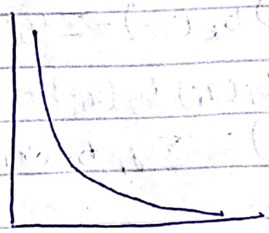
Curve fitting using least square error method

Let the equation of the curve to be fitted by

$g(x) = a_1 b_1(x) + a_2 b_2(x) + \ldots + a_k b_k(x)$

where each $b_i(x)$ is a chosen polynomial $(x, x^2, \ldots)$

or transcental term $(\sin x, \cos x, \log x, x^2)$



$pv^{\gamma} = c$

$xy^{\gamma} = c$

$\log x + \gamma \log y = \log c$

or $\log x + \gamma \log y - \log c = 0$

$\gamma = \frac{1}{\gamma}(c - x)$

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let $\log x = X$

$\log y = Y$

and $\log c = C$

For each of the given $(n+1)$ tabular value, $y(x_i) = y_i$, $i = 0, 1, \ldots, n$ the vertical distance between the fitted curve and tabular y_i value is expressed as

$$e_i = \{g(x_i) - y_i\} \quad (i = 0, 1, \ldots n)$$

Sum of the squares of all terms

$$E = \sum_{i=0}^{n} e_i^2 = \sum_{i=0}^{n} \{g(x_i) - y_i\}^2$$

$$E = \sum_{i=0}^{n} \{a_1 b_i(x_i) + a_2 b_2(x_i) + \ldots + a_k b_k(x_i) - y_i\}^2$$

For minimum E,

$$\frac{\partial E}{\partial a_1} = \frac{\partial E}{\partial a_2} = \ldots = \frac{\partial E}{a_k} = 0$$

$$\frac{\partial E}{\partial a_1} = 2\{a \sum b_i(x_i) + a_2 \sum b_2(x_i) + \ldots + a_k \sum b_k x_i - \sum y_i\} \sum b_1(x_i)$$

$$= 2\{a_1 \{\sum b_1(x_i)\}^2 + a_2 \sum b_1(x_i) b_2(x_i) + \ldots + a_k \sum b_1(x_i) b_k(x_i) - \sum y_i b_1(x_i)\} = 0$$

or $a_1 \{\sum b_1(x_1)\}^2 + a_2 \sum b_1(x_i) b_2(x_i) + \ldots + a_k \sum b_1(a_i) b_k(x_i) = \sum y_i b_i(x_i)$

$$\frac{\partial E}{\partial a_2} \rightarrow a_1 \sum b_1(x_i) b_2(x_i) + a_2 \{\sum b_2(x_i)\}^2 + \cdots + a_K \sum b_2(x_i) b_K(x_i) = \sum y_i b_2(x_i)$$

$\vdots$

$$\frac{\partial E}{\partial a_K} \rightarrow a_1 \sum b_1(x_i) + a_2 \{\sum b_2(x_i) b_k(x_i)\} + \cdots + a_K \{\sum b_K(x_i)\}^2 = \sum y_i b_K(x_i)$$

—x—

1. Fit a curve of the form $y = ab^x$ to the following tabular values :

i	x_i	y_i	$Y_i = \log_{10} y_i$	x_i^2	$x_i Y_i$
0	2	8.3	0.9191	4	1.8382
1	3	15.4	1.875	9	3.5625
2	4	33.1	1.5198	16	6.0792
3	5	65.2	1.8142	25	9.0710
4	6	127.4	2.1052	36	12.6310
	$\Sigma x_i = 20$		$\Sigma x_i = 7.5458$	$\Sigma x_i^2 = 90$	$\Sigma x_i Y_i = 33.1821$

$$y = a_1 g_1(x) + a_2 g_2(x) + \cdots + a_K g_K(x)$$

$$y = ab^x$$

$$Y = \log_{10} y = \underbrace{\log_{10} a}_{A} + x \underbrace{\log_{10} b}_{B}$$

$$Y = A + Bx$$

$g_1(x) = 1$

$g_2(x) = x$

$$y = a_1 g_1(x) + a_2 g_2(x) + \dots + a_k g_k(x)$$

$$a_1 \Sigma [g_1(x_i)]^2 + a_2 \Sigma g_1(x_i) g_2(x_i) + \dots + a_k \Sigma g(x_i) g_k(x_i)$$
$$= \Sigma y_i g_i(x_i)$$

$$a_1 \Sigma g_1(x_i) g_2(x_i) + a_2 \{\Sigma g_2(x_i)\}^2 + \dots +$$
$$a_k \Sigma g_2(x_i) g_k(x_i) = \Sigma y_i g_2(x_i)$$

$$\vdots$$

$$a_1 \Sigma g_1(x_i) g_k(x_i) + a_2 \Sigma g_2(x_i) g_k(x_i) + \dots +$$
$$a_k \Sigma g_1(x_i) g_k(x_i) \cancel{+ a_2 \Sigma g_2} = \Sigma y_i g_k(x)$$

$$A \Sigma \{g_1 x_i\}^2 + B \Sigma g_1(x_i) g_2(x_i) = \Sigma y_i g_i(x)$$
$$\downarrow$$
$$A \Sigma (1)^2 + B \Sigma x_i = \Sigma y_i$$

$$A \Sigma g_1(x_i) g_2(x_i) + B \Sigma \{g_2(x_i)\}^2 = \Sigma y_i g_2(x_i)$$
$$\downarrow$$
$$A \Sigma x_i + B \Sigma x_i^2 = \Sigma x_i y_i$$

$$\left.\begin{array}{l} 5A + 20B = 7.5458 \\ 20A + 90B = 33.1821 \end{array}\right\} \text{solving these}$$

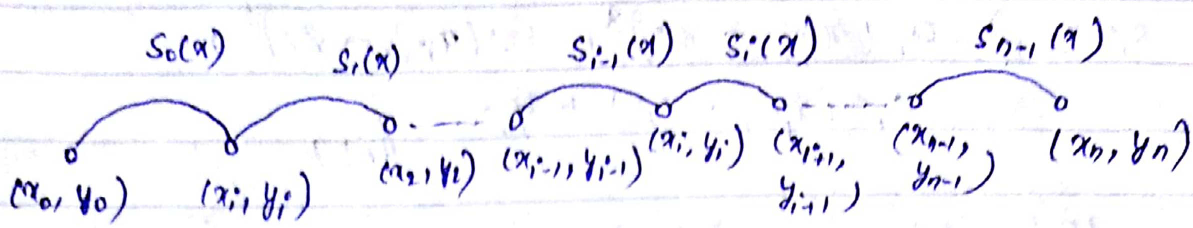
$A = 0.3096$

$B = 0.2999$

$$y = 2.0396(1.9948)^{x_i}$$

$$a = 10^A = (10)^{0.3096} = 2.0396$$
$$b = 10^B = 1.9948$$

Spline approximation



minimum degree of polynomial to approximate $S_i(x)$ will be three.

Let $S_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i$

For $(n+1)$ tabular data, there will be n different $S_i(x)$, $2, n$ unknowns which require $4n$ condⁿs.

$$\left.\begin{aligned} S_i(x_i) &= y_i \quad -① \\ S_i(x_{i+1}) &= y_{i+1} \quad -② \quad i = 0, 1, \ldots, n-1 \end{aligned}\right\} 2n \text{ cond}^n\text{s.}$$

Slope of the neighbouring segments at the common knot point are equal

i.e.

$$\left.\begin{aligned} S_{i+1}'(x_i) &= S_i'(x_i) \quad -③ \\ S_{i-1}''(x_i) &= S_i''(x_i) \quad -④ \end{aligned}\right\} \begin{aligned} &i = 1, 2, \ldots, (n-1) \\ &(2n-2) \text{ cond}^n\text{s.} \end{aligned}$$

Other 2 condⁿ are

$$S_0''(x_0) = 0 \quad -⑤$$
$$S_{n-1}''(x_n) = 0 \quad -⑥$$

$$S_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i$$

at $x = x_i$, $y = y_i$

$y_i = d_i$ or $d_i = y_i$ $\quad -⑦$

$\therefore$ Putting ⑦ in ②

Putting (7) in (2)

$$s_i(x_{i+1}) = a_i(x_{i+1} - x_i)^3 + b_i(x_{i+1} - x_i)^2 + c_i(x_{i+1} - x_i) + y_i = y_{i+1}$$

Let $(x_{i+1} - x_i) = \Delta x_i$ and $y_{i+1} - y_i = \Delta y_i$

$$\therefore \Delta y_i = a_i(\Delta x_i)^3 + b_i(\Delta x_i)^2 + c_i \Delta x_i$$

or $$\frac{\Delta y_i}{\Delta x_i} = a_i(\Delta x_i)^2 + b_i(\Delta x_i) + c_i$$

$i = 0, 1, \dots, n-1$

From (3),

$$s_i'(x) = 3a_i(x - x_i)^2 + 2b_i(x - x_i) + c_i$$

Let $s_i'(x_i) = m_i$, $i = 0, 1, \dots, n-1$.

$s_i'(x_i) = m_i = c_i$

$s_i'(x_{i+1}) = m_{i+1} = 3a_i(x_{i+1} - x_i)^2 + 2b_i(x_{i+1} - x_i) + c_i$

or, $$3a_i \Delta x_i^2 + 2b_i \Delta x_i + m_i = m_{i+1} \quad (9)$$

solving (8) and (9).

$$a_i = \frac{1}{\Delta x_i^2}\left\{m_i + m_{i+1} - \frac{2\Delta y_i}{\Delta x_i}\right\}$$

$$b_i = \frac{1}{\Delta x_i}\left\{-2m_i - m_{i+1} - \frac{3\Delta y_i}{\Delta x_i}\right\}$$

$$s_i''(x) = 6a_i(x - x_i) + 2b_i$$

$$s_i''(x_{i+1}) = s_{i+1}''(x_i)$$

$$6a_i(x_{i+1} - x_i) + 2b_i = 2b_{i+1}$$

$$6a_i \Delta x_i + 2b_i = 2b_{i+1}$$

$i = 0, 1, \dots, n-2$

$$\frac{6}{\Delta x_i^2}\left\{m_i + m_{i+1} - \frac{2\Delta y_i}{\Delta x_i}\right\}\Delta x_i + 2\frac{1}{\Delta x_i}\left\{-2m_i - 2m_{i+1} + 3\frac{\Delta y_i}{\Delta x_i}\right\}$$

$$= \frac{2}{\Delta x_{i+1}}\left\{-2m_{i+1} - 2m_{i+2} + 3\frac{\Delta y_{i+1}}{\Delta x_{i+1}}\right\}$$

$$\therefore \frac{m_i}{\Delta x_i} + 2\left\{\frac{1}{\Delta x_i} + \frac{1}{\Delta x_{i+1}}\right\} + \frac{m_{i+2}}{\Delta x_{i+1}} = 3\left(\frac{\Delta y_{i+1}}{\Delta x_{i+1}} + \frac{\Delta y_i}{\Delta x_i}\right)$$

$i = 0, 1, \ldots, n-2.$

other 2 condn are

$S_0''(x_0) = 0$ — (5) $\rightarrow 2m_0 + m_1 = \frac{3\Delta_0}{\Delta x_m}$

$S_{n-1}''(x_n) = 0$ — (6)

$$x_{n-1} + 2m_n = \frac{3\Delta y_{n-1}}{\Delta x_{n-1}}$$

— x —

Numerical Differentiation

x_i	y_i
x_0	y_0
x_1	y_1
$\vdots$	$\vdots$
x_n	y_n

Fit a n^{th} degree polynomial to these tabular data.

$$y = y_0 + p\Delta x_0 + \frac{p(p-1)}{2!}\Delta^2 x_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 x_0 + \ldots\ldots$$

where $x = x_0 + ph$ or $p = \frac{(x - x_0)}{h} \Rightarrow \frac{dp}{dx} = \frac{1}{h}$

$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\frac{d}{dp}\left\{ y_0 + p\Delta x_0 + \frac{p(p-1)}{2!}\Delta^2 x_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 x_0 + \cdots \right\}$$

$$= \frac{1}{h}\left\{ \Delta x_0 + \frac{2p-1}{2!}\Delta^2 x_0 + \frac{3p^2-6p+2}{3!}\Delta^3 x_0 + \cdots \right\}$$

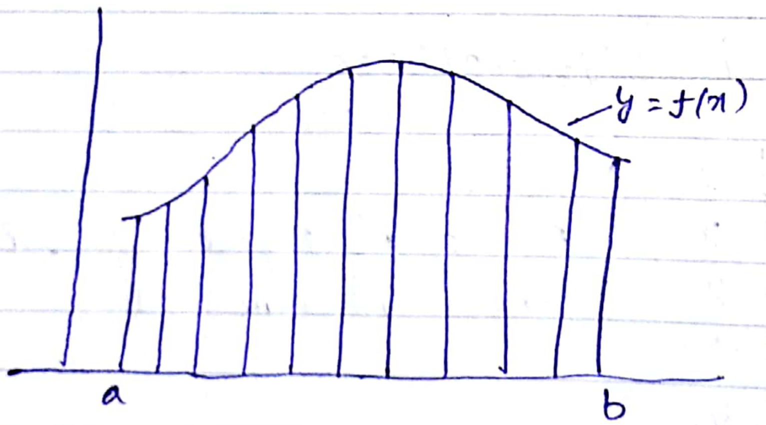
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx}$$

$$= \frac{1}{h}\frac{d}{dp}\left(\frac{dy}{dx}\right)$$

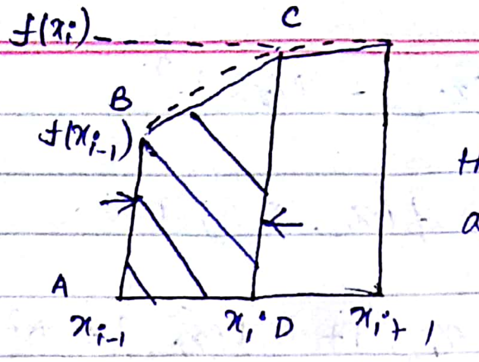
$$= \frac{1}{h^2}\frac{d}{dp}\left\{ \Delta x_0 + \frac{2p-1}{2!}\Delta^2 x_0 + \frac{3p^2-6p+2}{3!}\Delta^3 x_0 + \cdots \right\}$$

Numerical Integration

$$\int_a^b f(x)\,dx$$



when the strips are of equal width, the corresponding integration formulas are called Newton-cotes formulae other wise the formulae are called gaussian formulae.



for the interval $[x_{i-1}, x_i]$ the function $f(x)$ is approximated as a st. line

∴ The area under the curve over the interval $[x_{i-1}, x_i]$ will be equal to the area of the trapezium ABCD

$$A_{Li} = A\{f(x_{i-1}) + f(x_i)\}/2$$

$$= \frac{A}{2}\{f(x_{i-1}) + f(x_i)\}$$

for the interval $[x_{i-1}, x_i]$

$$I_{trapezoid} = \frac{h}{2}\{f(x_{i-1}) + f(x_i)\}$$

$$I_{actual} = \int_{x_{i-1}}^{x_i} f(x)\,dx$$

$$= [F(x)]_{x_{i-1}}^{x_i} \quad \text{where } \int f(x)\,dx = F(x)$$

since $x_i = x_{i-1} + h$

$$I_{actual} = F(x_{i-1} + h) - F(x_{i-1})$$

$$= F(x_{i-1}) + hF'(x_{i-1}) + \frac{h^2}{2!}F''(x_{i-1}) + \frac{h^3}{3!}F'''(x) + \ldots$$

$$= hf(x_{i-1}) + \frac{h^2}{2}f'(x_{i-1}) + \frac{h^3}{3!}f''(x_{i-1}) + \ldots \quad (1)$$

~~Neglecting the higher order terms in (1) and (3).~~

$$I_{trapezoid} = \frac{h}{2}\{f(x_{i-1}) + f(x_i)\}$$

$$= \frac{h}{2}\{f(x_{i-1}) + f(x_{i-1}+h)\}$$

$$= \frac{h}{2}\{f(x_{i-1}) + f(x_{i-1}) + h f'(x_{i-1}) + \frac{h^2}{2} f''(x_{i-1}) + \dots\}$$

$$= h f(x_{i-1}) + \frac{h^2}{2} f'(x_{i-1}) + \frac{h^3}{4} f''(x_{i-1}) + \dots \quad (2)$$

Neglecting the higher order terms in (1) and (2)

$$I_{actual} - I_{trapezoid} = \left(\frac{h^3}{6} - \frac{h^3}{4}\right) f''(x_{i-1})$$

$$= -\frac{h^3}{12} f''(x_{i-1})$$

Summarizing up the errors for individual intervals the total error over $[x_0, x_n]$ is

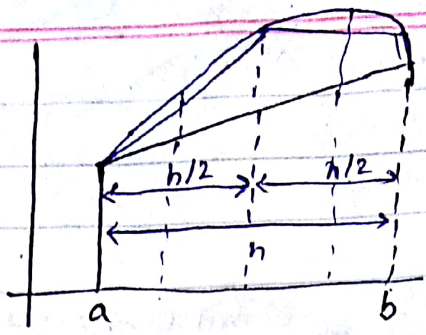
$$E = \sum_{i=1}^{n} -\frac{h^3}{12} f''(x_{i-1}) = -\frac{h^3}{12} \sum_{i=1}^{n} f''(x_{i-1})$$

Let $\bar{F}''$ be the the average second order derivative of $f(x)$

or $\therefore \bar{f}'' = \sum_{i=1}^{n} f''(x_{i-1}) / n$

or $E = -\frac{h^3}{12} n\bar{f}'' = -\frac{h^2}{12}(nh)\bar{f}''$

$$= -\frac{(b-a)}{12} h^2 \bar{f}''$$



$$I_0 = \frac{h}{2}\{f(a)+f(b)\}$$

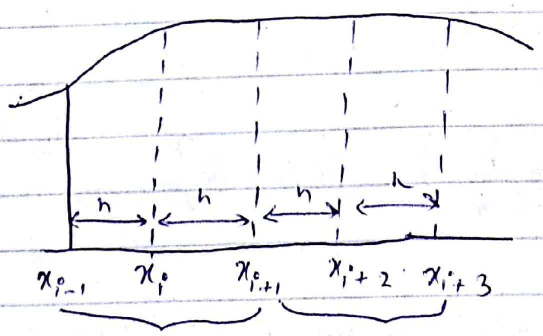
$$= \frac{h}{4}\{f(a)+2f(c)+f(b)\}$$

$$I_0 = I_1$$

h = (b-a)
n = 1 , S1 = 0
S0 = f(a) + f(b)
I1 = (h * S0) ,

Repeat
{
S1 = 0 ; I0 = I1 ;
h = h/2 ;
for { i = 1; i <= n ; i = i + 2 }
S1 = S1 + f(a + i * h)
n = 2 * n ;
S0 = S0 + 2 * S1 ; I1 = (h + S0)/2 ;
until (Fabs (I1 - I0) < ε)

Simpson's 1/3 Rule



x_{i-1} $f(x_{i-1})$
x_i $f(x_i)$
x_{i+1} $f(x_{i+1})$

x_{i-1} $f(x_{i-1})$ $\Delta f(x_{i-1})$ $\Delta^2 f(x_{i-1})$
x_i $f(x_i)$ $\Delta f(x_i)$
x_{i+1} $f(x_{i+1})$

Let us fit a newton's forward difference polynomial through the above 3 points

$$y = f(x_{i-1}) + p\,\Delta f(x_{i-1}) + \frac{p(p-1)}{2}\Delta^2 f(x_{i-1})$$

where $p = (x - x_{i-1})/h$

$$\int_{x_{i-1}}^{x_{i+1}} = h\int_{x_{i-1}}^{x_{i+1}} y\,dp = h\int_0^2 y\,dp$$

at $x = x_{i-1}$, $p = 0$
at $x = x_{i+1}$, $p = 2$

$$= h\int_0^2 \left\{ f(x_{i-1}) + p\,\Delta f(x_{i-1}) + \frac{p(p-1)}{2}\Delta^2 f(x_{i-1}) \right\} dp$$

$$= \frac{h}{3}\left\{ f(x_{i-1}) + 4f(x_i) + f(x_{i+1}) \right\}$$

$\Delta f(x_{i-1}) = f(x_i) - f(x_{i-1})$
$\Delta^2 f(x_{i-1}) = f(x_{i+1}) - 2f(x_i) + f(x_{i-1})$

$$= \left[p\, f(x_{i-1}) \right]_0^2 + \left[\frac{p^2}{2}\Delta f(x_{i-1}) \right]_0^2 + \left[\frac{p^3}{6} - \frac{p^2}{4} \right]_1^2 \Delta^2 f(x_{i+1})$$

$$= 2f(x_{i-1}) + 2\{f(x_i) - f(x_{i-1})\} + \frac{1}{3}\{f(x_{i-1}) - 2f(x_i) + f(x_{i+1})\}$$

$$= \frac{1}{3}\{ f(x_{i-1}) + 4f(x_i) + f(x_{i+1}) \}$$

$$A_{i-1} = \frac{h}{3}\{ f(x_{i-1}) + 4f(x_i) + f(x_{i+1}) \}$$

$$A = \sum_{i=1,3} h/3 \{ f(x_{i-1}) + 4f(x_i) + f(x_{i+1}) \}$$

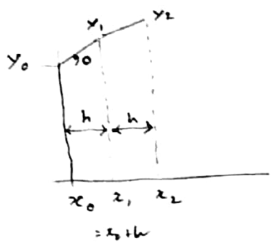
30-1...

Solution of differential eq^ns

$$\frac{dy}{dx} = f(x,y) \quad \text{with some specified cond}^{ns}$$

$$y(x_0) = y_0 \qquad ; \qquad y = g(x,y)$$

Euler's method :-



$$s_1 = f(x_0, y_0)$$

$$\tan\theta = \frac{y_1 - y_0}{x_1 - x_0} = s_1$$

$$\Rightarrow y_1 = y_0 + s_1(x_1 - x_0)$$

$$= y_0 + h s_1$$

$$(x_{k-1}, y_{k-1})$$

$$(x_{k-1}+h, \; y_{k-1} + hf(x_{k-1}, y_{k-1}))$$

Error Estimate for Euler's method

$$(x_{k-1}, y_{k-1})$$

$$x_k = x_{k-1} + h$$

$$y_k = y(x_{k-1} + h)$$

$$= y(x_{k-1}) + h y'(x_{k-1}) + \frac{h^2}{2} y''(x_{k-1}) + \cdots$$

$$= \underbrace{y(x_{k-1}) + h f(x_{k-1}, y_{k-1})} + \underbrace{\frac{h^2}{2} y''(x_{k-1}) + \cdots}_{\text{Error}}$$

Truncation Error associated with E.M. $= \frac{h^2}{2} y''(x_{k-1})$ + higher order terms

Let the error associated with y_{k-1} and y_k be e_{k-1} and e_k respectively whereas the true values are y'_{k-1} and y'_k resp.

$$\left.\begin{aligned} y_{k-1} &= y'_{k-1} + e_{k-1} \\ y_k &= y'_k + e_k \end{aligned}\right\} \quad y'_k + e_k = y'_{k-1} + e_{k-1} + hf(x_{k-1}, y'_{k-1} + e_{k-1}) + \cdots$$

$$= y'_{k-1} + hf(x_{k+1}, y'_{k-1}) + e_{k-1}\{1 + hf'(x_{k-1}, y'_{k-1})\} + \cdots$$

$$\therefore \; e_k = e_{k-1}\{1 + hf'(x_{k-1}, y'_{k-1})\} + \text{higher order terms}$$

Neglecting higher order terms

$$|e_k| = |e_{k-1}| \, |1 + hf'(x_{k-1}, y'_{k-1})|$$

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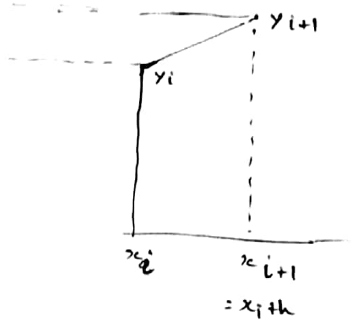
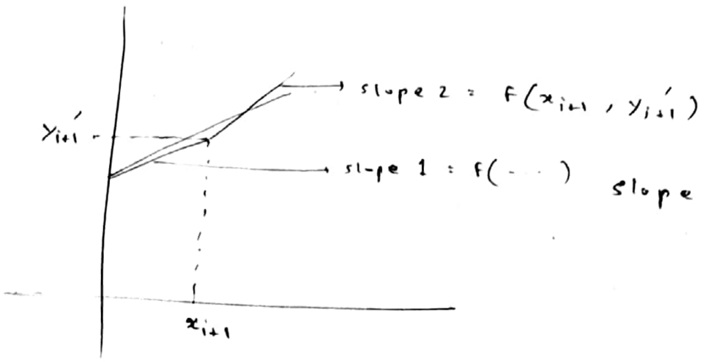
for stability of the method,

$$\left| 1 + h f'(x_{k-1}, y'_{k-1}) \right| < 1$$



single step method :-

Modified Euler :-



slope = $\frac{\text{slope1} + \text{slope2}}{2}$

$$y_{i+1} = y_i + h\left(\frac{\text{slope1} + \text{slope 2}}{2}\right)$$

$$y_{i+1}^{(1)} = y_i + \frac{h}{2}\left\{ f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(0)}) \right\}$$

continue iteration

$$y_{i+1}^{(k+1)} = y_i + \frac{h}{2}\left\{ f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(k)}) \right\}$$

until

$$\left| y_{i+1}^{k+1} - y_i^{(k)} \right| < \varepsilon$$

Error Estimate :-

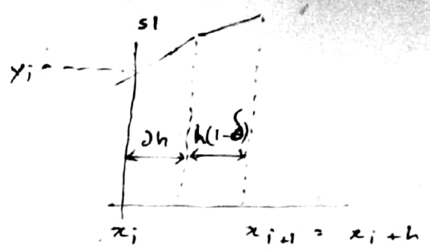
$$y_k =$$

$$= y_{k-1} + h f(x_{k-1}, y_{k-1}) + \frac{h^2}{2}\left\{ \frac{f(x_k, y_k) - f(x_k, y_{k-1})}{h} \right\}$$

$$= y_{k-1} + \underbrace{\frac{h}{2}\left\{ f(x_{k-1}, y_{k-1}) + f(x_k, y_k) \right\}}_{\text{Simple Error}} + \text{higher order term}$$

h^3

Runge - Kutta (2nd order)

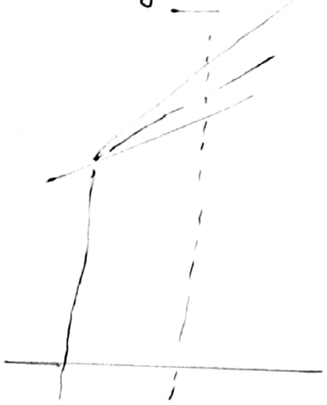


$$s = w_1 s1 + w_2 s2$$

if $\delta = \frac{1}{2}$, $w_2 = 2/3$ (3/4), $w_1 = 1/3$ (1/4)

$$\frac{w_2}{w_1} = \frac{1+\delta}{1-\delta} \qquad \frac{?}{1}$$

Runge Kutta (4th order)



$$s1 = f(x_i, y_i)$$
$$s2 = f\left(x_i + \frac{h}{2},\ y_i + \frac{h}{2} \cdot s1\right)$$
$$s3 = f\left(x_i + \frac{h}{2},\ y_i + \frac{h}{2} \cdot s2\right)$$
$$s4 = f(x_i + h,\ y_i + h\, s3)$$

$$s = \frac{1}{6}\left[s1 + 2s2 + 2s3 + s4 \right]$$

$$y_{i+1} = y_i + h s$$

$$y_{i+1}^{k} = y_i + \frac{h}{6}\left\{ s1 + 4s2 + 4s3 + f\left(x_i + h,\ y_i + h F(x_{i+1}, y_{i+1}^{(k-1)})\right) \right\}$$

continue until

$$\left| y_{i+1}^{(k)} - y_{i+1}^{(k-1)} \right| < \varepsilon$$

Sol^n of 2nd order

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$ with initial cond $y(x_0) = y_0$

and $\left.\frac{dy}{dx}\right|_{x_0} = z_0$

$$\frac{dy}{dx} = z$$

$$\frac{dz}{dx} = f(x, y, z)$$

$$\frac{dy}{dx} = z$$

with initial cond. $y(x_0) = y_0$, $z(x_0) = z_0$

For any known pt (x_i, y_i, z_i) on the sol^n curve next pt can be given as

$$x_{i+1} = x_i + h$$

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$z_{i+1} = z_i + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

where $k_1 = h\,g(x_i, y_i, z_i)$

$$l_1 = h\,f(x_i, y_i, z_i)$$

$$k_2 = h\,g\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}, z_i + \frac{l_1}{2}\right)$$

$$l_2 = h\,f\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right)$$

$$k_3 = h\,g\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}, z_i + \frac{l_2}{2}\right)$$

$$l_3 = h\,f\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}, z_i + \frac{l_2}{2}\right)$$

$$k_4 = h\,g(x_i + h, y_i + k_3, z_i + l_3)$$

$$l_4 = h\,f(x_i + h, y_i + k_3$$

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1) Adams Bashforth method

2) Milne's method

3) Hamming's method

Shift Operator

This operator shifts a functional value y_i to a higher index

$E y_i = y_{i+1}$

Inverse shift operator E^{-1}

$y_i = E^{-1} y_{i+1}$

Equivalence amt of finite diff operator

$$\Delta y_i = y_{i+1} - y_i$$
$$= E y_i - y_i$$
$$= (E-1) y_i$$

$$\Delta \equiv E - 1$$
$$\boxed{E = 1 + \Delta}$$

Similarly,

$$\Delta y_{i+1} = y_{i+1} - y_i$$
$$= (1 - E^{-1}) y_i$$

$$\nabla \equiv 1 - E^{-1}$$

2) $$\boxed{E = (1 - \nabla)^{-1} = \frac{1}{1-\nabla}}$$

$$E = 1 + \Delta = \frac{1}{1-\nabla}$$

$$1 - \nabla = \frac{1}{E}$$

$$1 - \frac{1}{E} = \frac{E-1}{E}$$

1. $\Delta = E \nabla$

2. $\nabla - \Delta = -\Delta \nabla$ ✓

$$-(E-1)\left(\frac{E-1}{E}\right)$$
$$-(E-1)\left(1 - \frac{1}{E}\right)$$
$$-\left(E - 1 - 1 + \frac{1}{E}\right)$$
$$-\Delta + \nabla$$

3. $\Delta + \nabla = \Delta / \nabla - \nabla / \Delta$

$$RHS = \frac{E-1}{(E-1)/E} - \frac{(E-1)/E}{E-1}$$

$$= E - \frac{1}{E} = (1+E) - \frac{1}{E} + 1$$

$$= \Delta + \nabla$$

$$\boxed{\Delta^r y_k = \nabla^r y_{k+1}}$$

$r = 1$, $\Delta y_k = y_{k+1} - y_k = \nabla y_{k+1}$

$r = p$ $\quad \Delta^p y_k = \nabla^p y_k$, $\quad r = p+1$ $\quad \Delta^{p+1} y_k = \Delta^p y_{k+1} - \Delta^p y_k = \nabla^p y_{k+p+1} - \nabla^p y_{k+p}$

$= RHS$

$x_i = x_0 + ih$

$$y_p = E^p y_0$$
$$= (1+\Delta)^p y_0$$
$$= \left(1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \dots\right) y_0$$
$$= \left(1 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots\right)$$

Newton's Forward Diff formula

useful for interpolation near first few values
- top end of tabular values

Newton's backward diff. interpolation formula for interpolation near the backward end or bottom of tabular values

$$y_p = E^{-p} y_n = (1-\nabla)^p y_n$$
$$= \left(1 - p\nabla + \frac{p(p-1)}{2!}\nabla^2 - \frac{p(p-1)(p-2)}{3!}\nabla^3 + \dots\right) y_n$$
$$= \left(y_n - p\nabla y_n + \frac{p(p-1)}{2!}\nabla^2 y_n - \frac{p(p-1)(p-2)}{3!}\nabla^3 y_n + \dots\right)$$

Int. near bottom end

Gauss central Diff formula :-

central ~~vors~~ Gauss backward version is used for near central p...

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$	$\Delta^5 y_i$	$\Delta^6 y_i$
x_{-3}	y_{-3}	Δy_{-3}					
x_{-2}	y_{-2}	Δy_{-2}				$\Delta^5 y_{-3}$	$\Delta^6 y_{-3}$
x_{-1}	y_{-1}	Δy_{-1}				$\Delta^5 y_{-2}$	
x_0	y_0	Δy_0					
x_1	y_1	Δy_1					
x_2	y_2	Δy_2					
x_3	y_3						

Let the interpolating poly be

$$y_p = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_{-1})$$
$$+ a_3(x-x_0)(x-x_{-1})(x-x_1)$$
$$+ a_4(x-x_0)^2(x-x_1)(x-x_{-2})(x-x_{-1})$$

at $x = x_i$ $y_p = y_i$ $i = -n, -(n-1), \ldots, 0, 1, 2, \ldots, n$

$x = x_0$, $a_0 = y_0$

$x = x_{-1}$, $y_p = y_{-1}$

$\therefore y_{-1} = y_0 + a_1(x_{-1} - x_0)$

or $y_{-1} - y_0 = a_1(-h)$

or $-\Delta y_{-1} = -a_1 h$ or $a_1 = \dfrac{\Delta y_{-1}}{h}$

at $x = x_1$, $y = y_1$

$y_1 = a_0 + a_1(x_1 - x_0) + a_2(x_1 - x_0)(x_1 - x_{-1})$

$= y_0 + \dfrac{\Delta y_{-1}}{h} h + a_2 2h^2$

or $a_2 2h^2 = y_1 - y_0 - \Delta y_{-1} = \Delta y_0 - \Delta y_{-1} = \Delta^2 y_{-1}$

or $a_2 = \dfrac{\Delta^2 y_{-1}}{2h^2}$

$a_3 = \dfrac{\Delta^3 y_{-2}}{3h^3}$

$a_4 = \dfrac{\Delta^4 y_{-2}}{4!\,h^4}$

$$y_p = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$
$$+ a_3(x - x_0)(x - x_1)(x - x_{-1})$$
$$+ a_4(x - x_0)(x - x_1)(x - x_{-1})(x - x_2)$$
$$+ \cdots$$

$x = x_0 \qquad y = y_0$

$a_0 = y_0$

$x = x_1 \qquad y = y_1$

$$a_1 = \frac{\Delta y_0}{h}$$

$$a_2 = \frac{\Delta^2 y_{-1}}{h^2}$$

$$a_3 = \frac{\Delta^3 y_{-1}}{h^3}$$

Let the temp v/s sp. heat of Ethyl Alcohol
find heat at 15°C, 25°C

x°C	sp heat	Δ	Δ^2	Δ^3	Δ^4
0	0.51	0.04	-0.02	0.02	-0.03
10	0.55	0.02	0	0.01	
→ 20	0.57	0.02	0.01		
30	0.59	0.03			
40	0.62				

$a_0 = 0.57$

$a_1 =$

x	y	Δ	Δ^2	Δ^3
1	8	9	-2	0
2	17	7	-2	0
3	24	5	-2	0
4	29	3	-2	
5	32	1		
6	33			

$\left[\underline{\underline{\text{degree}}} = 2\right]$

x	$y = e^x$
1.12	3.064854
1.14	3.126768
1.16	3.189933
1.18	3.254374
1.20	3.320116

$\longrightarrow e^{1.15} = 3.1581929\ldots$

x_i	y_i
x_0	y_0
x_1	y_1
x_2	y_2
$\vdots$	$\vdots$
x_n	y_n

when x_i values are not equispaced

1) divided diff method
2) Lagrange interpolation method
3) Aitken's iterative interpolation method

Divided difference method

First order divided difference is defined as

$$y[x_i, x_{i+1}] = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} = \frac{y_{i+1}}{x_{i+1} - x_i} - \frac{y_i}{x_{i+1} - x_i}$$

Similarly 2^{nd} order divided difference

$$y[x_i, x_{i+1}, x_{i+2}] = \frac{y[x_{i+1}, x_{i+2}] - y[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

k^{th} order divided difference

$$y[x_i, x_{i+1}, \dots, x_{i+k}] = \frac{y[x_{i+1}, x_{i+2}, \dots, x_{i+k}] - y[x_i, x_{i+1}, \dots, x_{i+k-1}]}{x_{i+k} - x_i}$$

x_i	y_i	$y[x_i, x_{i+1}]$	$y[x_i, x_{i+1}, x_{i+2}]$
x_0	y_0	$y[x_0, x_1]$	$y[x_0, x_1, x_2]$
x_1	y_1	$y[x_1, x_2]$	$y[x_1, x_2, x_3]$
x_2	y_2	$y[x_2, x_3]$	$y[x_2, x_3, x_4]$
$\vdots$	$\vdots$		
x_{n-1}	y_{n-1}	$y[x_{n-1}, x_n]$	
x_n	y_n		

Divisions $= n(n-1) + (n-2) + \dots + 1$
$= n(n+1)$ } Table