BACHELOR OF PRINTING ENGINEERING EXAMINATION, 2018

(2nd Year, 2nd Semester)

MATHEMATICS - IV R

Time : Three Hours

Full Marks : 100

The figures in the margin indicate full marks.

Symbols / Notations have their usual meanings.

Answer **any five** questions.

1. (a) Prove that the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is a monotone increasing sequence and bounded above.

(b) Prove that the sequence $\{u_n\}$ converges to 7, where $u_1 = \sqrt{7}$ and $u_{n+1} = \sqrt{7u_n}$ $\forall$ $n \geq 1$.

(c) Test the convergence of the sequence $\{u_n\}$, where $u_n = \sqrt[n]{n}$.

(d) Prove that three distinct points A, B, C are collinear if and only if there exist three scalars x, y, z, not all zero, such that

$$x\overrightarrow{OA} + y\overrightarrow{OB} + z\overrightarrow{OC} = \overrightarrow{0} \text{ and } x + y + z = 0,$$

where O is any base point.

5+5+5+5

2. (a) Discuss the convergence of the series

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2+2}$$

(b) Test the convergence of the series :

$$2 + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \ldots$$

(c) Test the convergence of the series :

$$1 + \frac{1}{2}.\frac{1}{3} + \frac{1.3}{2.4}.\frac{1}{5} + \frac{1.3.5}{2.4.6}.\frac{1}{7} + \ldots$$

(d) If D, E and F are the midpoints of the sides BC, CA and AB respectively of a triangle ABC, prove that

$$\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \vec{0}.$$

5+5+5+5

3. (a) Find the Fourier series for the function $f(x) = e^{-ax}$, $-\pi < x < \pi$. Hence prove that

$$\frac{\pi}{\sinh \pi} = 2\left[\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} + \ldots\right].$$

(b) Determine the half-range Fourier cosine series for

$$f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \pi \end{cases}$$

10+10

4. (a) If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three non-coplanar vectors, then show that the three points having position vectors $\vec{a} - 2\vec{b} + 3\vec{c}$, $-2\vec{a} + 3\vec{b} + 2\vec{c}$, $-8\vec{a} + 13\vec{b}$ are collinear.

(b) Prove that three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar if and only if $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$.

(c) If A, B, C, D are any four points in space, show that

$$\overrightarrow{AB} \times \overrightarrow{CD} + \overrightarrow{BC} \times \overrightarrow{AD} + \overrightarrow{CA} \times \overrightarrow{BD}$$

is independent of D.

(d) A force $-2\vec{i} + 3\vec{j} + 2\vec{k}$ acts through the point $6\vec{i} + 11\vec{j} + 2\vec{k}$. Find the moment of the force about the point $\vec{i} + \vec{j} + \vec{k}$.

5+5+5+5

5. (a) Prove that a vector function $\vec{f}$ on a scalar variable t remains parallel to fixed direction if and only if

$$\vec{f} \times \frac{d\vec{f}}{dt} = \vec{0}.$$

(b) Find the directional derivatives of $\phi = x^2yz + 4xz^2$ at $(1,\ -2,\ -1)$ in the direction $2\vec{i} - \vec{j} - 2\vec{k}$.

(c) Prove that $r^n\vec{r}$ is irrotational for all values of n but it is solenoidal if $n = -3$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$.

(d) State Stokes theorem. 6+6+6+2

6. (a) Show that $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ is irrotational. Find a scalar function ϕ such that $\vec{F} = \vec{\nabla}\phi$.

(b) Find the work done in moving a particle in the force field $\vec{f} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along (i) the straight line from $(0,0,0)$ to $(2,1,3)$ and (ii) the curve defined by $x^2 = 4y$ and $3x^2 = 8z$ from $x = 0$ to $x = 2$.

(c) Evaluate

$$\iiint_V \vec{\nabla}.\vec{f}\, dV,$$

where $\vec{f} = 4xy\hat{i} + yz\hat{j} - xy\hat{k}$ and V is bounded by $x = 0$, $x = 2$, $y = 0$, $y = 2$, $z = 0$ and $z = 2$.

7+7+6

7. (a) Evaluate

$$\iint_S \vec{f}.\hat{n}dS,$$

where $\vec{f} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is the surface $2x + 3y + 6z = 12$ in the first octant.

(b) Evaluate

$$\iint_S \vec{f}.\hat{n}dS,$$

where $\vec{f} = ax\hat{i} + by\hat{j} + cz\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = 1$.

(c) Use Green's theorem to evaluate

$$\oint_C \Big[\Big(y - \sin x\Big)dx + \cos x dy\Big],$$

where C is the triangle with vertices $(0,0)$, $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 1\right)$.

7+6+7

8. (a) Use Gauss divergence theorem to prove the following identity:

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)dV = \int_S (\phi\vec{\nabla}\psi - \psi\vec{\nabla}\phi).\vec{n}\,dS,$$

where V is the volume enclosed by the closed surface S and $\vec{n}$ is the outward drawn normal to S.

(b) At an instant t the vector from the origin to a moving point is

$$\vec{r} = \vec{a}\cos\omega t + \vec{b}\sin\omega t,$$

where $\vec{a}$ and $\vec{b}$ are two constant vectors and ω is a scalar constant.

(i) Find the velocity $\vec{v}$ and show that

$$\vec{r} \times \vec{v} = \text{a constant vector.}$$

(ii) Show that the acceleration is directed towards the origin and is proportional to the distance of the point from the origin.

(c) Evaluate

$$\int_C \left[\left(x^2 + xy\right)dx + \left(x^2 + y^2\right)dy\right],$$

where C is the square formed by the lines $y = \pm 1$ and $x = \pm 1$.

6+8+6