NUMERICAL METHODS

23/7/18

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$AX = B$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$X = A^{-1}B$$

–x–

$$f(x) = a_1x^5 + a_2x^4 + a_3x^3 + \dots + a_6$$

$$\frac{df(x)}{dx} = 5a_1x^4 + 4a_2x^3 + 3a_3x^2 + 2a_4x + a_5 = 0$$

–x–

Ex :- $x^3 - x + 1 = 0$

$x^3 = x - 1$

or $x = (x-1)^{1/3}$

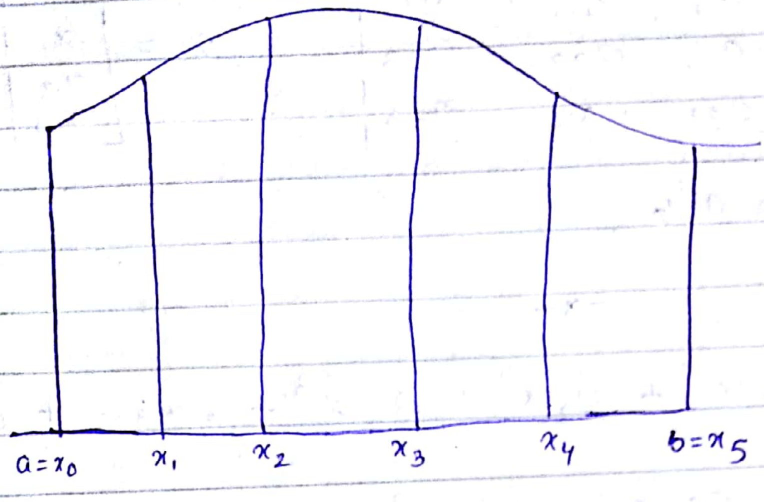
Let $x = 2$ and substitute $x = 2$ on the RHS of above eq^n

$$x_1 = (2-1)^{1/3} = 1$$
$$x_2 = (1-1)^{1/3} = 0$$
$$x_3 = (0-1)^{1/3} = -1$$
$$x_4 = (-1-1)^{1/3} = -1.26$$
$$x_5 = (-1.26-1)^{1/3} = (-2.26)^{1/3} = -1.312$$
$$x_6 = (-1.312-1.000)^{1/3} = (-2.312)^{1/3} = -1.32.$$

$$x = -\log_e x \rightarrow x = e^{-x}$$

—x—

$$\int_a^b f(x)\,dx$$



$$A_1 = \frac{1}{2}\{f(x_0) + f(x_1)\} \times d$$

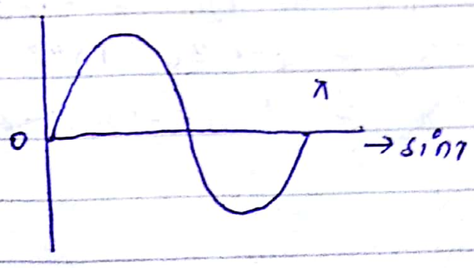
$$A_2 =$$

$$\vdots$$

$$A_5 = \frac{1}{2}\{f(x_4) + f(x_5)\} \times d$$

$$A = A_1 + A_2 + A_3 + A_4 + A_5$$
$$= \frac{1}{2} \times d \{f(x_1) + 2f(x_1) + 2f(x_2) + 2f(x_3)$$
$$+ 2f(x_4) + f(x_5)\}$$

$$\int_0^{\pi} \sqrt{1+\cos^2 x}\,dx$$



- Computer can perform four basic arithmatic operations the addition, subtraction, multiplication, ~~&~~ division and a logic operation which is comparision.
- Computer has a finite word size, it can represent with finite precision. to be contined.
- Computer has to finished within a finite time, we cannot leave computer doing the same compilation for an infinite period of time.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

1) Any number with infinite precision has to be rounded off after a finite number of digits after decimal point. Round of error; $\pi = 3.14159$

2) Transistion error - Since we truncate infinite series after a finite no. of lines.

3) Inderent error - This is due to the improper formulation of problem and non-availaibility of accurate data.

—x—

24/7/18

$x = \bar{x} + e_x$
$y = \bar{y} + e_y$

$z = x + y = (\bar{x} + \bar{y}) + (e_{\bar{x}} + e_{\bar{y}})$
$P = x + y = \bar{x}\bar{y} + (\bar{x}e_y + \bar{y}e_x + e_x e_y)$

absolute error = |computed value - exact value|

relative error = $\frac{\text{absolute error}}{\text{exact error}}$

ex : 12345.678 - Computed value
12345.675 - exact value

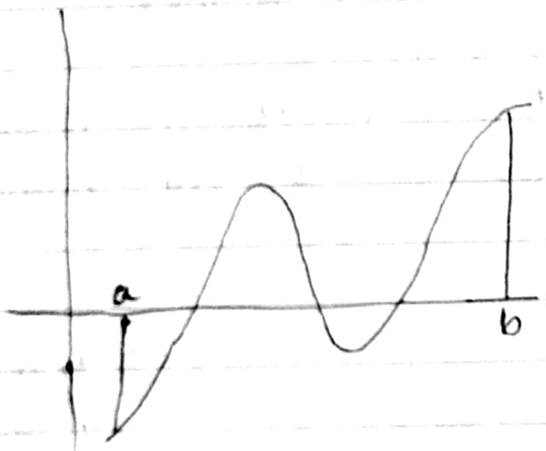
absolute error = 0.003

ex : 0.123 - Computed value
0.120 exact value.

absolute error : 0.003

$$\frac{\text{absoute}}{\text{exact}} = \frac{0.003}{0.120} = \frac{3}{120} = 0.025$$ = relative error.

$y = f(x)$ $[a,b]$



if $f(a)$ and $f(b)$ are of opposite sign, then there exist a number ξ in (a,b) such that $f(\xi) = 0$.

Solution of Non-linear equation

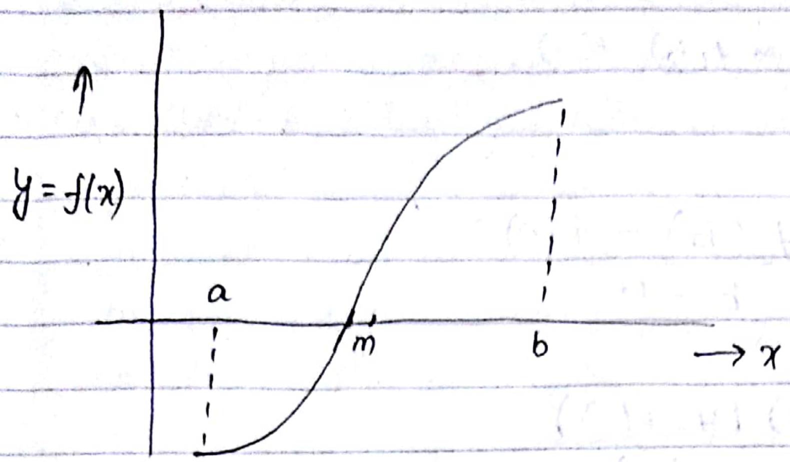
$ax^2 + bx + c = 0$ $\quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a_0 x^m + a_1 x^n + \cdots + a_{m+1} = 0$

$x + \log_e x + mx = 0$ Transidental eq^n

Scanned by CamScanner

Bisection Method



Drawback:

i) If it will be a parabola i.e when it is just touching the x-axis.

Inputs a and b are so chosen that f(a) and f(b) are of opposite sign.

$$f(a) * f(b) < 0$$

1) $m = \frac{a+b}{2}$

2) if $f(m) == 0$. Then m is a root, exit

3) if $f(a) * f(m) < 0$ then $b = m$, else $a = m$

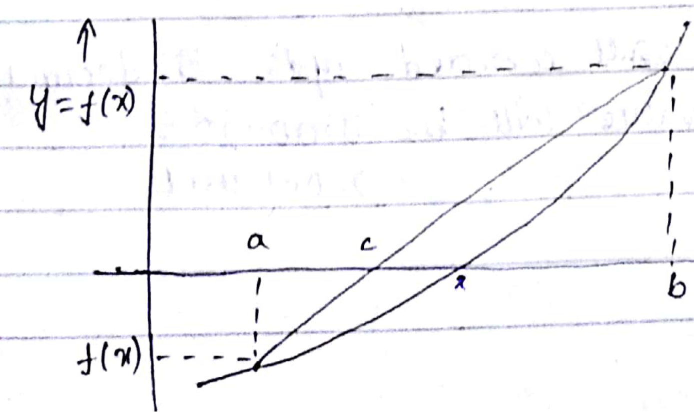
4) Continue steps 1) 2) 3) until the size of the search interval $[a,b]$ is less than the required precision

5) $m + (a+b)/2$

6) print m

7) exit.

Method of false position or regular falsi method



Advantage over bisection

1) More iteration → bisection

2) Less iteration → False rule

Inputs a and b are so chosen that $f(a)$ and $f(b)$ are of opp. sign

$$f(a) * f(b) < 0$$

$$\frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow x - a = \frac{(b-a)(y - f(a))}{f(b) - f(a)}$$

$$\Rightarrow c - a = \frac{(b-a)(-f(a))}{f(b) - f(a)}$$ [substituting $(x,y)=(c,$]

$$\Rightarrow c = \frac{af(b) - \cancel{af(a)} - bf(a) + \cancel{af(a)}}{f(b) - f(a)}$$

1) $c = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

2) If $f(c) == 0$ then c is a root.

3) if $f(a) * f(c) < 0$ then $b = c$

$a = c$

Stopping condition

if $|c_i - c_{i+1}|$ < required precision then c_{i+1} is the root.

Precision

for result corrected upto 5 decimal places. Precision value will be 0.000005

0.000001

Analysis of any numerical algorithm considers the following facts

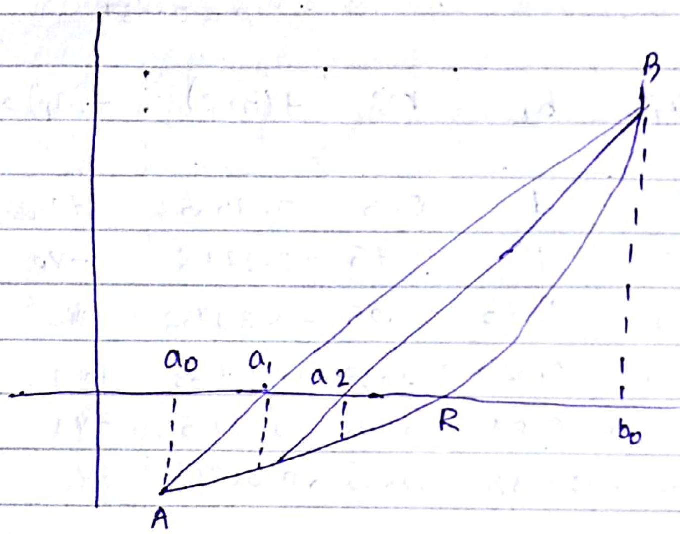
1. Whether the method is guaranteed to provide solution.

2. How fast will it give solution.

3. Amount of error associated with the solution.

$\cancel{(a_2-a_1) = (a_1-a_0) \frac{\{|f(a_1)| / |f(a_0)|\}}{\{1+\frac{f(a_1)}{f(b_0)}\}}}$

0)



From Δs $a_1 a_0 A$ and $a_0 b_0 B$

$$\frac{a_1 - a_0}{|f(a_0)|} = \frac{b_0 - a}{f(b_0)|} \quad \text{— ①}$$

From Δs $a_2 a_1 P$ and $a_2 b_0 B$.

$$\frac{a_2 - a_1}{|f(a_1)|} = \frac{b_0 - a_2}{f(b_0)} \quad \text{— ②}$$

or, $(a_2-a_1)|f(b_0)| + a_2|f(a_1)| = b_0|f(a_1)|$

or, $(a_2-a_1)|f(b_0)| + a_2|f(a_1)| - a_1|f(a_1)| = (b_0-a_1)|f(a_1)|$

or, $(a_2-a_1)(|f(b_0)| + f(a_1)) = (b_0-a_1)|f(a_1)|$

$$= (a_1 - a_0)\frac{|f(b_1)||f(a_1)|}{|f(a_0)|}$$

$$(a_2 - a_1) = (a_1 - a_0)\left\{\frac{|f(a_1)| / |f(a_0)|}{\left(1 + \frac{|f(a_1)|}{|f(b_0)|}\right)}\right\} < 1$$

1. Solve $e^{-x} - x = 0$ using bisection Method.

Let $a = 0$, $b = 1$. $f(0) = 1$, $f(1) = -0.632$

$R = 0.567$

~~Intersection~~

iteration no. (i)	a_i	b_i	m_i	$f(m_i)$	$f(a_i) * f(m_i)$
1	0	1	0.5	0.10653	+ve
2	0.5	1	0.75	-0.2776	-ve
3	0.5	0.75	0.625	-0.89738	-ve
4	0.5	0.625	0.5625	0.00728	+ve
5	0.5625	0.625	0.59375	-0.0415	-ve
6	0.5625	0.59375	0.578125	-0.01717	-ve

Scanned by CamScanner

Regular falsi Method

How fast the method converge to the root ??

Order of convergence : n

$$e_{i+1} = K e_i^{n}$$

Let $e_i = 0.01$, and $n = 2$

$$e_{i+1} \propto (0.01)^n$$
$$\propto 0.0001$$

$$|e_{i+1}| = |K| \, |e_i|^n$$

$$\log|e_{i+1}| = \log|K| + n \log|e_i|$$

$$n = \left| \frac{\log|e_{i+1}| - \log|K|}{\log|e_i|} \right|$$

$$\text{or,} \quad n = \left| \frac{\log|e_{i+1}|}{\log|e_i|} \right|$$

Fixed point iteration method/ Iterative method using substitution

Given $f(x) = 0$, whose root has be computed.
We rewrite this eq^n as $x = g(x)$
Let us assume the intial approximation to the root of $f(x) = 0$ be x_0 we generate a sequence of approximation to the root as follows:

$$x_1 = g(x_0)$$
$$x_2 = g(x_1)$$
$$x_3 = g(x_3)$$
$$\vdots$$
$$x_{n+1} = g(x_n)$$

continued until $|x_{n+1} - x_n| <$ required permission.

1. $x^2 - x - 6 = 0$

$f(x) = 0$

$\Rightarrow x^2 - 3x + 2x - 6 = 0$

$\Rightarrow (x-3)(x+2) = 0$

$x = 3, -2$

$x = \pm\sqrt{x+6}$ — (1)

$x^2 = x + 6$

$\Rightarrow x = 1 + 6/x$ — (2)

$x = x^2 - 6$ — (3).

$x = \pm\sqrt{x+6}$

Let $x_0 = 1$

$x_1 = \sqrt{x_0 + 6} = \sqrt{7} = 2.645$

$x_2 = \sqrt{2.645 + 6} = 2.94$

$x_3 = \sqrt{2.94 + 6} = 2.989$

$x_4 = \sqrt{2.989 + 6} = 2.998$.

$x = -\sqrt{x+6}$

$x_0 = 1$

$x_1 = -\sqrt{1+6} = 2.645$

$x_2 = -\sqrt{-2.645+6} = -1.831$

$x_3 = -\sqrt{-1.831+6} = -2.04$

$x_4 = -\sqrt{-2.04+6} = -1.989$

$x_5 = -\sqrt{-1.989+6} = -2.002$

Scanned by CamScanner

$x = 1 + 6/x$

$r_0 = 1$

$r_1 = 1 + 6/1 = 7$

$r_2 = 1 + 6/7 = 1.857$

$r_3 = 1 + 6/1.857 = 4.23$

$r_4 = 1 + 6/4.23 = 2.418$

$r_5 = 1 + 6/2.418 = 3.481$

$\vdots$

$x = x^2 - 6.$

$r_0 = 1$

$r_1 = 1 - 6 = -5$

$r_2 = (-5)^2 - 6 = 19$

$r_3 = (19)^2 - 6 =$

$\vdots$

Diverges.

Theorem: Let $x = R$ be a root of the eq^n $f(x) = 0$, which is rewritten as $x = g(x)$. Let both $g(x)$ and $g'(x)$ exist and continous over an interval (a, b) containing $x = R$. If $|g'(x)| < 1$ over the interval (a, b) and the initial approximation to the root r_0 is also in (a, b), then the sequence of approximations $r_1, r_2, r_3 \ldots$ will converge to the root $x = R$

e.g:- ① $x = \sqrt{x + 6} \Rightarrow g(x) = \sqrt{x + 6}$

$g'(x) = \frac{1}{2\sqrt{x+6}}$ at $x = 3$, $g'(x) < 1$

② $x = -\sqrt{x + 6} \Rightarrow g(x) = -\sqrt{x+6}$

$g'(x) = -\frac{1}{2\sqrt{x+6}}$ at $x = 2$,

$g'(x) = -\frac{1}{4}$ $|g'(x)| < 1$

Scanned by CamScanner

3. $x = 1 + 6/x \qquad g(x) = 1 + 6/x$

$g'(x) = -\frac{6}{x^2}$

at $x = 3$, $g'(x) = \frac{-6}{9} = -2/3 \quad |g'(x)| < 1$

4. $x = x^2 - 6$, $g(x) = x^2 - 6$

$g'(x) = 2x$

Proof :- Let $x = R$ be a root of $x = g(x)$

$\therefore R = g(R) \quad$ — ①

Let x_0 be the initial approximation to the root, then the sequence of approximations can be generated as

$$\left.\begin{aligned} x_1 &= g(x_0) \\ x_2 &= g(x_1) \\ &\vdots \\ x_{n+1} &= g(x_n) \end{aligned}\right\} \text{— ②}$$

Let $e_0, e_1, e_2, \ldots, e_{n+1}$ be the errors associated with $x_0, x_1, x_2, \ldots, x_{n+1}$

Then,

$e_0 = R - x_0$

from eqn ① and the set of eq^n in ②

$$\left.\begin{aligned} e_1 &= R - x_1 \\ &= g(R) - g(x_0) \\ e_2 &= R - x_2 \\ &= g(R) - g(x_1) \\ &\vdots \end{aligned}\right\} ③$$

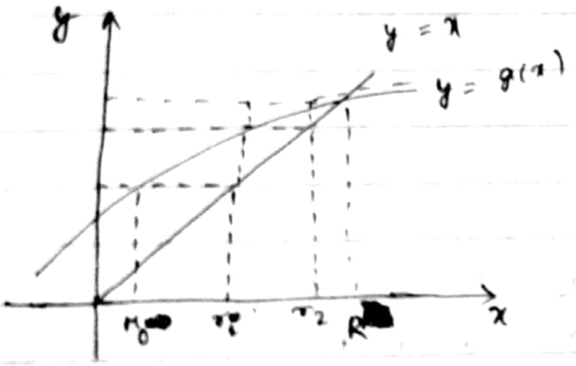
$e_{n+1} = R - x_{n+1} = g(R) - g(x_n)$

Using Mean value theorem, we can rewrite the eq^n in (3) as

$$\left.\begin{aligned} e_0 &= R - x_0 \\ e_1 &= (R - x_0)\, g(c_1) \quad x_0 < c_1 < R \\ e_2 &= (R - x_1)\, g'(c_2) \quad x_1 < c_1 < R \\ &\vdots \\ e_{n+1} &= (R - x_n)\, g'(c_{n+1}) \quad x_{n+1} < c_{n+1} < R \end{aligned}\right\} \quad (4)$$

For convergence $|e_{n+1}| < |e_n| < \ldots < |e_2| < |e_1| < |e_0|$
This will be possible if $|g'(c_i)| < 1$

Graphical Representation

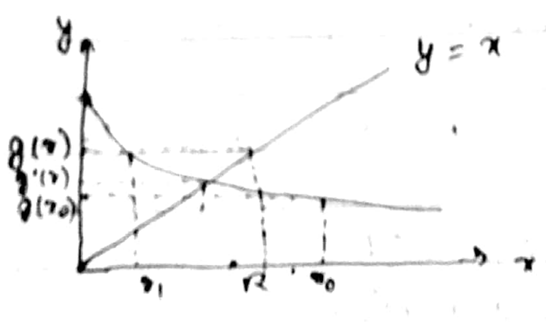


$0 < g'(x) < 1$ over the interval containing R.

We represent $x = g(x)$ as a combination of two functions $y = x$ and $y = g(x)$ intersecting at $x = R$

$x_1 = g(x_0)$

$x_2 = g(x_1)$



$|g'(x)| < 1$

$0 < g'(x) < 1$

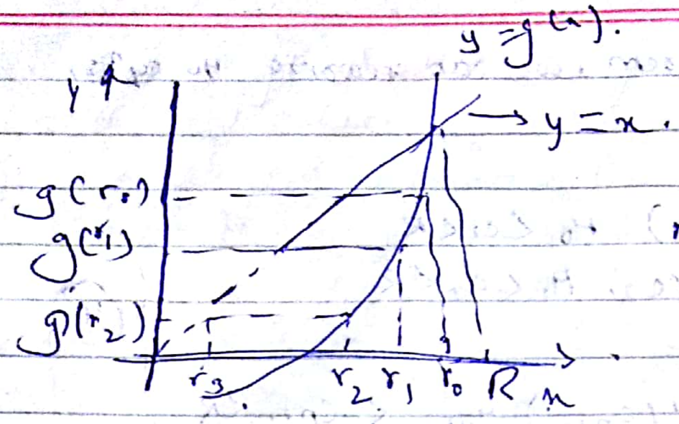
$0 > g'(x) > -1$

$|g'(x)| > 1$

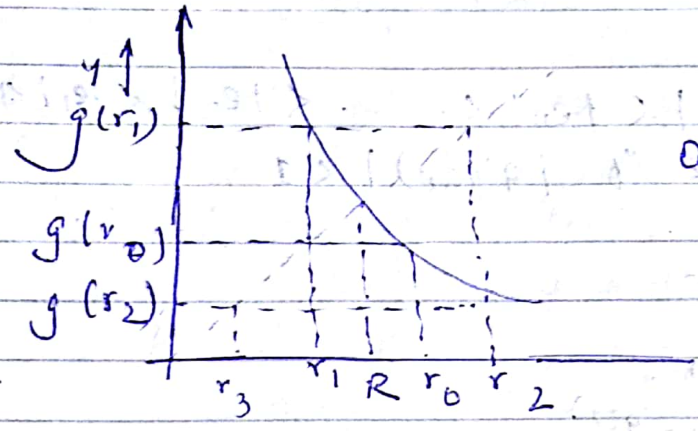
$g'(x) > 1$

$g'(x) < -1$

$0 > g'(x) > -1$



monotonic divergence



Oscillating divergence

Newton - Raphson Method

Let $f(x)=0$ has a root at $x = R$
Let the initial approximation to the root be r_0
Let e_0 be the error associated with r_0

$$R = r_0 + e_0$$

now, $f(R) = 0$
or $f(r_0 + e_0) = 0$
Using Taylor series expansion,

$$f(r_0) + e_0 f'(r_0) + \frac{e_0^2}{2!} f''(r_0) + \dots = 0$$

we assume $|e_0| << 1$
$\therefore$ we neglect e_0^2 and higher order terms in the series.

$$\therefore f(r_0) + e_0 f'(r_0) = 0$$

$$e_0 = \frac{-f(r_0)}{f'(r_0)}$$

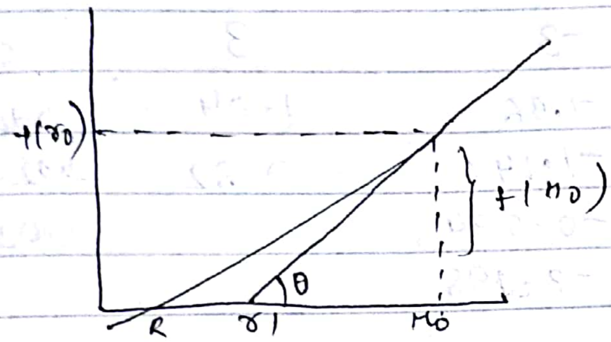
continuously in the way, we generate a sequence of approximations as

$$r_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$r_2 = x_1 - \frac{f(x_1)}{f'(r_1)}$$ continuous until

$$\vdots$$

$$r_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ $|r_{n+1} - r_n| <$ required precision

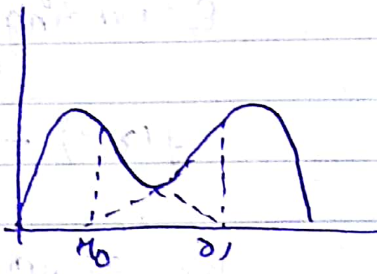


$$\tan\theta = f'(x_0) = \frac{f'(r_0)}{r_0 - r_1}$$

$$\Rightarrow r_0 - r_1 = \frac{f(x_0)}{f'(r_0)}$$

$$\Rightarrow r_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Let $f(x)=0$ has a root at $x = R$



N-R method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

iterative method $x_{n+1} = g(x_n)$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$|g'(x)| < 1$$

$$g'(x) = 1 - \frac{f'(x)}{f(x)} + \frac{f(x)\,f'(x)}{\{f'(x)\}^2}$$

$$= \left|\frac{f(x)\,f''(x)}{\{f'(x)\}^2}\right| < 1$$

$x_{i+1} = x_i - \dots$

x_i

x_{i+2}

Solve $f(x) = 0$ Using Newton - Raphson method. Take $x_0 = 0$

$f(x) = x^3 - x - 3$, $f'(x) = 3x^2 - 1$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	$\left\{ \frac{f(x_i) f''(x_i)}{f'(x_i)^2} \right\}$
0	0	-3	3	0.[illegible] < 1
1	-3	-1.96	1.04	0.[illegible]18 < 1
2	-1.96	-1.146	0.82	0.91 < 1
3	-1.146	-0.274	----	2.02 > 1
4	-0.274	-3.198		

Order of convergence for N-R method

$$x_{i+1} = R - e_{i+1} \qquad f(R) = 0$$
$$x_i = R - R_i$$

$$f(x_i + R_i) = 0$$

Expanding by taylor's formula

$$f(x_i) + e_i f'(x_i) + \frac{e_i^2}{2!} f''(x_i) + \frac{e_i^3}{3!} f'''(x_i) + \dots = 0$$

We assume $|e_i| << 1$, so we neglect e_i^3 and higher order terms.

$$f(x_i) + e_i f'(x_i) + \frac{e_i^2}{2!} f''(x_i) = 0$$

$$\text{or} \quad \frac{f(x_i)}{f'(x_i)} + e_i + e_i^2 \frac{f''(x_i)}{2 f'(x_i)} = 0$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$-\frac{f(x_i)}{f'(r_i)} = (R - x_i) + e_i^2 \frac{f''(x_i)}{2f'(r_i)}$$

or, $$x_i - \frac{f(x)}{f'(x_i)} = R + e_i^2 \frac{f''(x_i)}{2f'(x_i)}$$

or, $$x_{i+1} - R = e_i^2 \frac{f''(x_i)}{2f'(x_i)}$$

or, $$-e_{i+1} = e_i^2 \frac{f''(x_i)}{2f'(x_i)}$$

or, $$|e_{i+1}| = |e_i|^2 \left|\frac{f''(x_i)}{2f'(r_i)}\right| = |K|\,|e_i^2|$$

$$\log|e_{i+1}| = \log|K| + 2\log|e_i|$$

$$\Rightarrow \frac{\log|e_{i+1}|}{\log|e_i|} = \frac{\log|K|}{\log|e_i|} + 2$$

1. $f(x) = x^2 - x - 6 = 0$

$f'(x) = 2x - 1$

$x_0 = 2.5$

i	x_i	$\left\lvert\frac{f(x_i)f''(x_i)}{\{f'(x_i)\}^2}\right\rvert$	x_{i+1}	$e_i = \lvert x_{i+1} - r_i\rvert$	$n = \left\lvert\frac{\log e_{i+1}}{\log e_i}\right\rvert$
0	2.5	<1	3.0625	0.5625	—
1	3.0625	<1	3.000762	0.0617	4.84
2	3.00076	<1	2.99695	0.0038	2.00
3	2.997	<1	3.000001	0.00305	1.00
4	3.000001	<1	2.999996	0.000005	2.107
5	2.999996	<1	3.000000	0.000004	1.02

For fixed point iteration method

$$R - x_{i+1} = (R - x_i)\, g'(c_i) \qquad x_i < c_i < R$$

Let $|g'(c_i)| < 1$ for all c_i in the interval $[x_0, R]$ and let $|g'(c_i)| = k$

∴ we can write,

$$R - x_{i+1} = k(R - x_i) \quad \text{—} (1)$$

and $$R - x_{i+2} = k(R - x_{i+1}) \quad \text{—} (2)$$

(1) ÷ (2)

$$\frac{R - x_{i+1}}{R - x_{i+2}} = \frac{R - x_i}{R - x_{i+1}}$$

$$\Rightarrow (R - x_{i+1})^2 = (R - x_i)(R - x_{i+2})$$

$$R = \frac{x_{i+1}^2 - x_i x_{i+2}}{2x_{i+1} - x_i - x_{i+2}}$$

i	x_i	$x_{i+1} = g(x_i)$	
0	1	2.64575	x_0, x_1, x_2, 3.0046
1	2.64575	2.94037	
2	3.0046	3.00076	3.000006
3	3.00076	3.00013	
4	3.000006	3.000001	

$f(x) = x^3 - 4x^2 - 3x + 18$

$= x^2(x+2)$

$= (x+2)(x^2 - 6x + 9) = (x+2)(x-3)^2$

$f(x) = x^3 - 4x^2 - 3x + 18$

$f'(x) = 3x^2 - 8x - 3$

$x_0 = 2.5$

i	x_i	x_{i+1}	$e_i = \lvert x_{i+1} - x_i \rvert$	n
0	2.5	2.7647	0.264706	—
1	2.7647	2.86974	0.105035	1.695
2	2.86974	2.93575	0.660	1.206

Let $f(x) = (x-a)^n g(x)$

i.e there exist multiple roots of order n at $x = a$

$f^{n-1}(a) = f^{n-2}(a) = \dots = f''(a) = f'(a) = f(a) = 0$.

$f'(x) = n(x-a)^{n-1} g(x) + (x-a)^n g'(x)$

$(x-a) f'(x) = n(x-a)^n g(x) + (x-a)^{n+1} g'(x)$

$= n f(x) + \underbrace{(x-a)^{n+1} g'(x)}_{\to 0}$

$(x-a) = n \cdot \dfrac{f(x)}{f'(x)}$

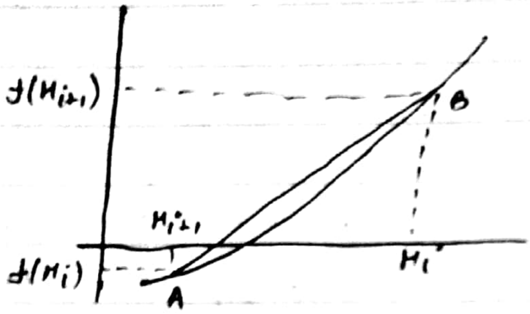
or $= x - n \dfrac{f(x)}{f'(x)}$

i	x_i	$x_{i+1} = x_i - 2\dfrac{f(x_i)}{f'(x_i)}$	$e_i = \lvert x_{i+1} - x_i \rvert$	n
0	2.5	3.0294	0.5294	~~—~~ —
1	3.0294	3.00009	0.00293	5.55
2	3.00009	3.00000	0.00009	2.639

Secant Method

Let N-R method we have to provide the functional representation of both $f(x)$ and $f'(x)$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$



$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{f(x_i)}{f(x_i) - f(x_{i-1})}$$

$$= x_i - \frac{f(x_i)(x_i x_{i-1})}{f(x_i) - f(x_{i-1})}$$

$$= \frac{x_i f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

$\cancel{f(x) = x^2 e^x \sin x}$

$\cancel{f'(x) =}$

R is the actual root of $f(x) = 0$

$\therefore f(x) = 0$

Let $x_{i-1} = R - e_{i-1}$

$x_i = R - e_i$

$x_{i+1} = R - e_{i+1}$;

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Scanned by CamScanner

$$R - H_{i+1} = \frac{(R - R_{i+1}) f(R - e_i) - (R - e_i) f(R - e_{i+1})}{f(R - e_i) - f(R - e_{i+1})}$$

$$= \frac{(R - e_{i+1}) \{ f(R) - e_i f'(R) + \frac{e_i^2}{2} f''(R) \cdots \} - (R - e_i) \{ f(R) - e_{i+1} f'(R) + \frac{e_{i+1}^2}{2} f''(R) - \cdots \}}{f(R)' - e_i f'(R) - f(R)' + e_{i+1} f'(R)}$$

$$= \frac{R(e_{i+1} - e_i) f'(R) - \frac{e_{i+1} e_i}{2} (e_{i+1} - e_i) f''(R)}{(e_{i+1} - e_i) f'(R)}$$

$$R - e_{i+1} \approx R - \frac{e_{i+1} e_i}{2} \frac{f''(R)}{f'(R)}$$

$$e_{i+1} \simeq \frac{e_{i+1} e_i}{2} \frac{f''(R)}{f'(R)}$$

Let us assume the order of convergence be n

$$e_{i+1} = K e_i^n \quad \text{—} \ (2)$$

and $e_i = K e_{i+1}^n$ — (3)

$$e_{i+1} = \left(\frac{e}{k}\right)^{1/n} \quad \text{—} \ (4)$$

with (1), (2) and (4), we get

$$K e_i^n = \left(\frac{e_i}{k}\right)^n e \cdot \frac{f''(R)}{2 f'(R)}$$

$$K e_i^n = (e_i)^{1 + 1/n} \frac{f''(R)}{2 K^{1/n} f'(R)}$$

$$e_i^n = e_i^{1 + 1/n} \frac{f''(R)}{2 f'(R) K^{1/n}}$$

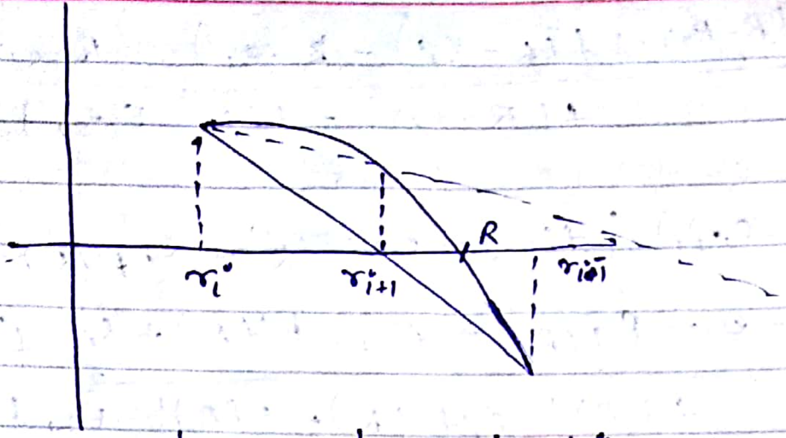
Equating powers of e_i from both sides.

$$n = 1 + 1/n$$

or, $n^2 - n + 1 = 0$

$$n = \frac{1 \pm \sqrt{1 + 4}}{2} = \frac{1 \pm \sqrt{5}}{2} = \frac{1 \pm 2.236}{2}.$$

$n = 1.618, -0.61$



$$\left| f'(x_i) \right| = \left| \frac{f(x_i) - f(x_{i+1})}{x_i - x_{i-1}} \right|$$

Math-point iteration method

Let R be the actual root of $f(x) = 0$ and r_i be an approximation to the root.

$$x_i = R - e_i$$

or $R = x_i + e_i$

Now, $f(R) = f(x_i + e_i) = 0$

Expanding by Taylor's formula,

$$f(x_i) + e_i f'(r_i) + \frac{e_i^2}{2} f''(x_i) + \frac{e^3}{3!} f'''(x_i) + \dots = 0$$

or,

$$f(x_i) + e_i \underbrace{\left\{ f'(x_i) + \frac{e_i}{2} f''(x_i) + \frac{e^3}{3!} f'''(r_i) + \dots \right\}}_{f'(x_i + e_i/2)} = 0$$

$$-f(x_i) = e_i f'(x_i + e_i/2)$$

or

$$e_i = \frac{-f(x_i)}{f'(x_i + e_i/2)}$$

$$= -\frac{f(x_i)}{f'\left(x_i - \frac{f(x_i)}{2f'(x_i)}\right)}$$

$\therefore\ x_{i+1} = x_i + e_i$

$$= x_i - \frac{f(x_i)}{f'\left[x_i - \frac{f(x_i)}{2f'(x_i)}\right]} = x_i - \frac{f(x_i)}{f'(x^*_{i+1})}$$

where $x^*_{i+1} = x_i - \frac{f(x_i)}{2f'(x_i)}$

for

for a given x_i,

$$x^*_{i+1} = x_i - \frac{f(x_i)}{2f'(x)}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x^*_{i+1})}$$

— x —

Multiple iteration Method

$$x^*_{k+1} = x_k - \frac{1}{2}\frac{f(x_k)}{f'(x_k)}$$

$R = x_k + e_k$

$R = x_{k+1} + e_{k+1}$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$x^*_{k+1} = (R - e_k) - \frac{1}{2}\frac{f(R-e_k)}{f'(R-e_k)}$$

$$\frac{f(R-e_k)}{f'(R-e_k)} = \frac{f(R) - e_k \cdot f'(R) + \frac{e_k^2}{2} f''(R) - \frac{e_k^3}{3} f'''(R) \ldots}{f'(R) - e_k f''(R) + \frac{e_k^2}{2} f'''(R) - \ldots\ldots}$$

$$= \frac{-e_k + \frac{e_k^2}{2}\frac{f''(R)}{f'(R)} - \frac{e_k^3}{3!}\frac{f'''(R)}{f'(R)}}{1 - e_k\frac{f''(R)}{f'(R)} + \frac{e_k^2}{2}\frac{f'''(R)}{f'(R)}}$$

Scanned by CamScanner

Let $\dfrac{f''(R)}{f'(R)} = a_1$ and $\dfrac{f'''(R)}{f'(R)} = a_2$

$$= \left(-e_k + \frac{a_1}{2!} e_k^2 - \frac{a_2}{3!} e_k^3\right)\left(1 - a_1 e_k + \frac{a_2}{2} e_k^2\right)^{-1}$$

$$= \left(-e_k + \frac{a_1}{2} e_k^2 - \frac{a_2}{3!} e_k^3\right)\left(1 + a_1 e_k - \frac{a_2}{2} e_k^2\right)^{-1}$$

$$= \left(-e_k + \frac{a_1}{2} e_k^2 - \frac{a_2}{3!} e_k^3\right)\left(1 + a_1 e_k - \frac{a_2}{2} e_k^2\right)$$

$$= -e_k + \frac{a_1}{2} e_k^2 - \frac{a_2}{3!} e_k^3 - a_1 e_k^2 + \frac{a_1^2}{2} e_k^3 + \frac{a_2}{2} e_k^3 + \dots$$

$$= -e_k - \frac{a_1}{2} e_k^2 + \left(\frac{a_1^2}{2} + \frac{a_2}{3}\right) e_k^3$$

$$H^*_{k+1} = (R - e_k) - \frac{1}{2} \frac{f(R - e_k)}{f'(R - e_k)}$$

$$= R - e_k - \frac{1}{2}\left\{-e_k - \frac{a}{2} e_k^2 + \left(\frac{a_1^2}{2} + \frac{a_2}{3}\right) e_k^3\right\}$$

$$= R - \frac{e_k}{2} + \frac{a_1}{4} e_k^2 - \left(\frac{a_1^2}{4} + \frac{a_2}{6}\right) e_k^3$$

$$H_{k+1} = H_k - \frac{f(H_k)}{f'(H^*_{k+1})}$$

$$R - e_{k+1} = R - e_k - \frac{f(R - e_k)}{f\left[R - \left\{\frac{e_k}{2} + \frac{a_1}{4} e_k^2 - \left(\frac{a_1^2}{4} + \frac{a_2}{6}\right) e_k^3\right\}\right]}$$

$$R = H_k + e_k$$

$$= H_{k+1} - e_{k+1}$$

$$= R - e_k - \frac{f(R) - e_k f'(R) + \frac{e_k^2}{2} f''(R) - \frac{e_k^3}{6} f'''(R)}{f'(R) - \left\{\frac{e_k}{2} - \frac{a_1}{4} e_k^2 + \left(\frac{a_1^2}{4} + \frac{a_2}{6}\right) e_k^3\right\} f''(R)}$$

$$= R - e_k - \frac{-e_k + \frac{e_k^2}{2} a_1 - \frac{e_k^3}{6} a_2}{1 - \left\{\frac{e_k}{2} - \frac{a_1}{4} e_k^2 + \left(\frac{a_1^2}{4} - \frac{a_2}{6}\right) e_k^3\right\} a_1}$$

$$= R - e_k - \left(-e_k + a_1 \frac{e_k^2}{2} - a_2 \frac{e_k^3}{6}\right)\left[1 + \left\{\frac{e_k}{2} - \frac{a_1}{4} e_k^2 + \left(\frac{a_1^2}{4} + \frac{a_2}{6}\right) e_k^3\right\} a_1\right]$$

$$= R - e_k - \left(-e_k + a_1 \frac{e_k^2}{2} - a_2 \frac{e_k^3}{6}\right)\left[1 + \left\{\frac{e_k}{2} - \frac{a_1}{4} e_k^2 + \left(\frac{a_1^2}{4} + \frac{a_2}{6}\right) e_k^3\right\} a_1\right]$$

$$= R - e_k - \left\{-e_k + \frac{a_1}{2} e_k^2 - \frac{a_2}{6} e_k^3 - \frac{a_1}{2} e_k^2 + a_1^2 \frac{e_k^3}{4} + \frac{a_1^3}{8} e_k^4 - \frac{a_1 a_2}{12} e_k^4 + \dots\right\}$$

$$= R - e_k + e_k + \left(\frac{a_2}{6} - \frac{a_1^2}{4}\right) e_k^3 - \left(\frac{a_1 a_2}{12} - \frac{a_1^3}{8}\right) e_k^4 + \dots$$

$$e_{k+1} = \left(\frac{a_2}{6} - \frac{a_1^2}{4}\right) e_k^3 - \left(\frac{a_1 a_2}{12} - \frac{a_1^3}{8}\right) e_k^4 + \dots$$

Chebyshev Method

Let x_k be the initial approximation to the root R of $f(x) = 0$

$$\therefore R = x_k + e_k$$

$$f(R) = f(x_k + e_k) = 0$$

or, $f(x_k) + e_k f'(x_k) + \frac{e_k^2}{2} f''(x_k) + \frac{e_k^3}{6} f'''(x_k) \dots = 0$

Neglecting e_k^3 and higher order terms

$$e_k f'(x_k) = f(x_k) + e_k^2 f''(x_k)$$

or $$-e_k = \frac{f(x_k)}{f'(x_k)} + \frac{e_k^2}{2} \frac{f''(x_k)}{f'(x_k)}$$

$$\therefore R = x_k + e_k$$

$$R = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{e_k^2}{2} \frac{f''(x_k)}{f'(x_k)} \quad \text{--- (2)}$$

From N-R method,

$$e_k = -\frac{f(x_k)}{f'(x_k)}$$

From ①,

$$e_k = -\frac{f(x_k)}{f'(x_k)}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{\{f(x_k)\}^2 f''(x_k)}{2\{f'(x_k)\}^3}$$

$$|f'(x_k)| < \epsilon.$$

—x—

Finding complex roots of a polynomial equation (Lin's method)

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0$$

$$(x^2 + px + q)(x^{n-2} + b_1 x^{n-3} + b_2 x^{n-4} + \dots + b_{n-3} x + b_{n-2})$$

$$+ \underbrace{Rx + S}_{} = 0 \quad — (1)$$

linear Remainder term, this should be equal to zero if the original polynomial is exactly devisible by $(x^2 + px + q)$.

$$x^n + a_1 x^{n+1} + a_2 x^{n+2} + \dots + a_{n-1} x + a_n \qquad b_3 x^{n-5} +$$

$$= (x^2 + px + q)(x^{n-2} + b_1 x^{n-3} + b_2 x^{n-4} + \dots + b_{n-3} x + b_{n-2}) + Rx + S$$

$$= x^n + (b_1 + p) x^{n-1} + (b_2 + pb_1 + q) x^{n-2} + (b_3 + pb_2 + qb_1) x^{n-3} \quad — (2)$$

equating the co-efficients from both sides of the eq^n :

$a_1 = b_1 + p \Rightarrow b_1 = a_1 - p$

$a_2 = b_2 + pb_1 + q \Rightarrow b_2 = a_2 - pb_1 - q$

$a_3 = b_3 + pb_2 + qb_1 \Rightarrow b_3 = a_3 - pb_2 - qb_1$ — (3)

$\vdots$

$a_j = b_j + pb_{j-1} + qb_{j-2} \Rightarrow b_j = a_j - pb_{j-1} - qb_{j-2}$

$j = 1, 2, 3, \dots n-2$

$R = a_{n-1} - pb_{n-2} - qb_{n-3} = 0$ — (4)

$S = a_n - qb_{n-2} = 0$ — (5).

Since the a_i and b_i values are known solving (4) and (5). we get a new set of value for p and q (say p_1, q_1).

Now, $(|p_0 - p_1|, |q_0 - q_1|) < \epsilon$

$\text{Max}(|p_{i-1} - p_i|, |q_{i-1} - q_i|) < \epsilon.$

Bairstow's Method

$R(p, q) = 0$

$S(p, q) = 0$

Let (p_0, q_0) be the approximations to (p, q) and Δp and Δq be the corrections needed to get the true values of (p, q).

$R(p_0 + \Delta p, q_0 + \Delta q) = 0$

$\Rightarrow R(p_0, q_0) + \Delta p \cdot \frac{\partial R}{\partial p}\Big|_{(p_0, q_0)} + \Delta q \frac{\partial R}{\partial q}\Big|_{(p_0, q_0)} + \dots = 0$

$S(p_0 + \Delta p, q_0 + \Delta q) = 0$

$\Rightarrow S(p_0, q_0) + \Delta p \frac{\partial S}{\partial p}\Big|_{(p_0, q_0)} + \Delta p \frac{\partial S}{\partial q}\Big|_{(p_0, q_0)} + \dots = 0$

Neglecting the higher order terms:

$$\Delta p \frac{\partial R}{\partial p}\Big|_{(p_0,q_0)} + \Delta q \frac{\partial R}{\partial q}\Big|_{(p_0,q_0)} = -R(p_0,q_0)$$

$$\Delta p \frac{\partial S}{\partial p}\Big|_{(p_0,q_0)} + \Delta q \frac{\partial S}{\partial q}\Big|_{(p_0,q_0)} = -S(p_0,q_0)$$

$$\Delta p = \frac{S(p_0,q_0)\frac{\partial S}{\partial q}\Big| - R(p_0,q_0)\frac{\partial R}{\partial q}\Big|}{\frac{\partial R}{\partial p}\Big|\frac{\partial S}{\partial q}\Big| - \frac{\partial S}{\partial p}\Big|\frac{\partial R}{\partial q}\Big|}$$

$\Delta q =$

If $\max(|\Delta p|, |\Delta q|) < \epsilon$. then stop

Equating the co-efficients from both sides of the eqn.

$$b_1 = a_1 - p$$
$$b_2 = a_2 - pb_1 - q$$
$$b_3 = a_3 - pb_2 - qb_1$$

$$\Rightarrow b_j = (a_j - pb_{j-1} - qb_{j-2} + \dots \quad j = 3, 4, \dots, n-2$$

$$R = a_{n-1} - pb_{n-2} - qb_{n-3} = 0 \quad - (4)$$
$$S = a_n - qb_{n-2} = 0 \quad - (5)$$

$$c_1 = \frac{\partial b_1}{\partial p} = -1$$

$$c_2 = \frac{\partial b_2}{\partial p} = -p\frac{\partial b_1}{\partial p} - b_1 = -p - b_1$$

$$c_3 = -p \frac{\partial b_2}{\partial p} - b_2 = q \frac{\partial b_1}{\partial p}$$

$$= -p c_2 - b_2 - q c_1$$

$$\vdots$$

$$c_j = -b_{j-1} - p c_{j-2} - q c_{j-3}$$

$$\vdots$$

$$\left.\frac{\partial R}{\partial p}\right|_{(p_0, q_0)} = -b_{n-2} - p c_{n-2} - q c_{n-3}$$

$$\left.\frac{\partial S}{\partial p}\right|_{(p_0, q_0)} = -q c_{n-2}$$

$$\left.\frac{\partial R}{\partial q}\right|_{(p_0, q_0)} = -p d_{n-2} - b_{n-3} - q d_{n-3}.$$

$$\left.\frac{\partial S}{\partial q}\right|_{(p_0, q_0)} = -b_{n-2} - q d_{n-2}$$

$$d_1 = \frac{\partial b_1}{\partial q} = 0$$

Solution of linear simultaneous eq^n :

$$x_1 + 5x_2 + 3x_3 = 10 \quad - (1)$$
$$x_1 + 3x_2 + 2x_3 = 5 \quad - (2)$$
$$2x_1 + 4x_2 - 6x_3 = -4 \quad - (3)$$

(1) - (2)

$$2x_2 + x_3 = 5$$

(2) $\times 2$ - (3)

$$2x_1 + 6x_2 + 4x_3 = 10$$
$$2x_1 + 4x_2 - 6x_3 = -4$$
$$- \quad - \quad + \quad +$$

$$2x_2 + 10x_3 = 14$$

$$2x_2 + x_3 = 5 \quad - (4)$$
$$2x_2 + 10x_3 = 14 \quad - (5)$$

$$-9x_3 = -9$$
$$\text{or} \quad x_3 = 1$$

From (4)

$$2x_2 + 1 = 5$$
$$\text{or} \quad x_2 = 2.$$

Gaussian elimination method.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots + a_{1(n-1)}x_{n-1} + a_{1n}x_n = a_{1(n+1)}$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots + a_{2(n-1)}x_{(n-1)} + a_{2n}x_n = a_{2(n+1)}$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots + a_{n(n-1)} + a_{nn}x_n = a_{n(n+1)}$$

(n-1) different linear

n → simultaneous eq^n w/ (n-1) unknown.

$$m_{i1} = \frac{a_{i1}}{a_{11}} \qquad i = 2, \dots, n.$$

$$a_{ij} = a_{ij} - m_{i1} * a_{1j} \qquad \begin{matrix} i = 2, \dots, n \\ j = 1, 2, \dots, n+1 \end{matrix}$$

$$m_{i2} = \frac{a_{i2}}{a_{22}}$$

$$a_{ij} = a_{ij} - m_{i2} * a_{2j} \qquad \begin{matrix} i = 3, \dots, n \\ j = 2, \dots, n+1 \end{matrix}$$

$$a_{kk} x_k + a_{k(k+1)} x_{k+1} + \dots + a_{k(n-1)} x_{(n-1)} + a_{kn} x_n = a_{k(n+1)}$$

$$k = 1, 2, \dots, (n-1)$$

$$m_{ik} = \frac{a_{ik}}{a_{kk}} \qquad i = (k+1), \dots, n$$

$$a_{ij} = a_{ij} - m_{ik} * a_{kj} \qquad \begin{matrix} i = (k+1), \dots, n \\ j = k, \dots, (n+1) \end{matrix}$$

$$a_{nk} x_k + a_{n(k+1)} x_{k+1} + \dots + a_{n(n-1)} x_{n-1} + a_{mn} x_n = a_{n(n+1)}.$$

$$|a_{11}| << |a_{i1}|$$

$$|a_{kk}| << |a_{ik}|$$

$$a_{1(n+1)}$$

$$a_{2(n+1)}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & a_{1(n+1)} \\ a_{21} & a_{22} & \dots & a_{2n} & a_{2(n+1)} \\ \vdots & & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} & a_{n(n+1)} \end{array}\right]$$

$$n \times (n+1)$$

$x_n =$

$n+1$

$$\left[\begin{array}{cccccc} a_{11} & a_{12} & a_{13} \cdots & a_{i(n+1)} & a_{1n} & a_{1(n+1)} \\ \cdot & a_{22} & a_{23} \cdots & a_{i(n+1)} & a_{2n} & a_{2(n+1)} \\ & & & \vdots & & \\ & & & a_{(n-1)(n-1)} & a_{(n-1)n} & a_{(n-1)(n+1)} \\ & & & & a_{nn} & a_{n(n+1)} \end{array}\right]$$

back substitution

$$x_n = \frac{a_{n(n+1)}}{a_{11}}$$

$$x_{n-1} = \frac{a_{(n-1)(n+1)} - a_{(n-1)n} x_n}{a_{(n-1)(n-1)}}$$

In general,

$$x_i = \frac{a_{i(n+1)} - \sum_{j=i+1}^{n} a_{ij} * x}{a_{ij}} \qquad i = (n-1)(n-2)$$

Input the co-efficient matric $A[n][n+1]$

// finding co-efficient with largest magnitude along the column A.

```
for (k = 1 ; k < n ; k++)
{
    big = 0 ;
    for(i = k ; i < = n ; i++)
    {
        if fals ( a [i] [k] > big )
        {
            big = a [i][k] ;
            p = i ;
        }
    }
}
for (j = k ; j < = n+1 ; j++)
{
    temp = a [k][j] ;
    a[k] [j] = a[p] [j] ;
    a[p] [j] = Temp ;
}
```

Scanned by CamScanner

```
for(i=k+1 ; i<=n ; i++)
{
      m[i][k] = a[i][k] / a[k][k];
      for (j=k ; j<=(n+1) ; j++)
            a[i][j] = a[j][j]-m[i][k]*a[k][j];

}

// elimation process is complete

x[n] = a[n][n+1] / a[n][n];
for( i=(n-1); i>=1 ; i-- )
{
      sum = 0 ;
      for(j=n ; j<i; j--)
         sum += a[i][j] * x[j];
         x[i] = (a[i][x+1]-sum) /a[i][j] ;
}
```

-2)

INOO
INDD

Gaussian Elimination

$$x_1 + 5x_2 + 3x_3 = 10$$
$$x_1 + 3x_2 + 2x_3 = 5$$
$$2x_1 + 4x_2 - 6x_3$$

assigned co-efficient matrix

$$\begin{bmatrix} 1 & 5 & 3 & 10 \\ 1 & 3 & 2 & 5 \\ 2 & 4 & -6 & -4 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 2 & 4 & -6 & -4 \\ 1 & 3 & 2 & 5 \\ 1 & 5 & 3 & 10 \end{bmatrix} \quad R_1 \longleftrightarrow R_3$$

$$\begin{bmatrix} 2 & 4 & -6 & -4 \\ 1-1 & 3-2 & 2+3 & 5+2 \\ 1-1 & 5-2 & 3+3 & 10+2 \end{bmatrix} \quad R_2 \leftarrow R_2 - R_1/2$$

$$\Rightarrow \begin{bmatrix} 2 & 4 & -6 & -4 \\ 0 & 1 & 5 & 7 \\ 0 & 3 & 6 & 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 4 & 6 & -4 \\ 0 & 3 & 6 & 12 \\ 0 & 1 & 5 & 7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 4 & 6 & -4 \\ 0 & 3 & 6 & 12 \\ 0 & 1-1 & 5-2 & 7-4 \end{bmatrix} \quad R_3 \leftarrow R_3 - R_2/3$$

$$\Rightarrow \begin{bmatrix} 2 & 4 & 6 & -4 \\ 0 & 3 & 6 & 12 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

$x_3 = 1$

$3x_2 + 6 = 12$

or, $x_2 = 2$

—x—

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1(n-1)}x_{(n-1)} + a_{1n}x_n = a_{1(n+1)}$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2(n-1)}x_{(n-1)} + a_{2n}x_n = a_{2(n+1)}$

$\vdots$

$a_{(n-1)1}x_1 + a_{(n-2)2}x_2 + a_{(n-1)3}x_3 + \dots + a_{(n-1)(n-1)}x_{n-1} +$

$a_{(n-1)n}x_n = a_{(n-1)(n-1)}$

$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{n(n-1)}x_{n-1} + a_{nn}x_n = a_{n(n+1)}$

$m_{i1} = \dfrac{a_{i1}}{a_{11}}$. $i = 2, 3, 4, \dots n$.

$m_{ik} = \dfrac{a_{i1}}{a_{11}}$ $i = k, 3, \dots, n$

Total no. of divisions required for getting the multiples

$(n-1) + (n-2) + \dots + 1$

$= \dfrac{(n-1)n}{2}$

No. of multiplications required for the elimination process for eliminating the co-eff. of x_1 from 2^{nd} row to n^{th} row.

no. of multiplication $= (n+1)(n-1) = n^2 - 1$

Suppose at any stage of elimination we are left with k ... eq^ns with k unknown for eliminating the co-efficients along first column, no. of multiplication required will be (k^2-1).

∴ Total no. of multiplications required for the entire elimination process to reduce the system of eq^n in upper triangular form

$$= \sum_{n=k}^{2} (k^2-1) = (n^2+(n-1)^2+\dots+2^2) - (n-1)$$

$$= \underbrace{n^2+(n-1)^2+\dots+2^2+1-n}$$

$$= \frac{1}{6} n(n+1)(2n+1) - n$$

For entire elimination, $\frac{1}{6} n(n+1)(2n+1)$

$$\frac{1}{6} n(n+1)(2n+1) - n$$

After elimation, For back substitution.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{n-1}x_{n-1} + a_{1n}x_n = a_{1(n+1)}$$

$$+ a_{22}x_2 + a_{23}x_3 + \dots + a_{2(n-1)}x_{n-1} + a_{2n}x_n = a_{2(n+1)}$$

⋮

$$a_{(n-1)}x_{n-1} + a_{(n-1)n}x_n = a_{(n+1)(n+1)}$$

$$+ a_{nn}x_n = a_{n(n+1)}$$

$$1+2+3+\dots+n$$

$$= \frac{n(n+1)}{2}$$

Scanned by CamScanner

Gauss Jordan Elimination

$$\therefore a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = a_{1(n+1)}$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = a_{2(n+1)}$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nm}x_n = a_{n(n+1)}.$$

no. of multiplications required for the elimination process

for eliminating the co. eff. of x_1 from 2^{nd} row to n^{th} row.

no. of multiplications $= (n+1)(n-1) = n^2 - 1$

Suppose at any stage of elimination we are left with k - eq^n with unknown for eliminating the co-efficients along first column, no. of multiplication required will be $(n-1)(n+1)$

$\therefore$ Total no. of multiplions required for the entire elimination process to reduce the system of eq^n in upper triangular form

$$= \sum_{k=n}^{2} ((n-1)(k+1))$$

$$\left[\begin{array}{cccccc|c} a_{11} & 0 & 0 & \dots & 0 & 0 & a_{1(n+1)} \\ 0 & a_{22} & 0 & & 0 & 0 & a_{2(n+1)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & & a_{(n-1)(n-1)} & 0 & a_{(n-1)(n+1)} \\ 0 & 0 & 0 & & 0 & a_{nn} & a_{n(n+1)} \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & \dots & 0 & 0 & a_{1(n+1)}/a_{11} \\ 0 & 1 & 0 & \dots & 0 & 0 & a_{2(n+1)}/a_{12} \\ \vdots & \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & a_{(n-1)(n+1)} / a_{(n-1)(n+1)} \\ 0 & 0 & 0 & \dots & 0 & 1 & a_{n(n+1)}/a_{nn} \end{array}\right]$$

Gauss - Jordan Elimination.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Let the solⁿ of this system of linear simultaneous eqⁿ be = $\begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{n1} \end{bmatrix}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{(I)}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_{12} \\ c_{22} \\ \vdots \\ c_{n2} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} c_{12} \\ c_{22} \\ \vdots \\ c_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \text{(II)}$$

Scanned by CamScanner

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} c_{1n} \\ c_{2n} \\ \vdots \\ c_{nn} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{12} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} c_{1n} \\ c_{1n} \\ \vdots \\ c_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \end{bmatrix} \quad \text{— (iii)}$$

combining (i), (ii) and (iii):

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}}_{n \times n \text{ identity matrix}}$$

Inverse of a matrix

$$\left[\overbrace{\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix}}^{n \times n} \,\middle|\, \overbrace{\begin{matrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{matrix}}\right]$$

$$\Downarrow$$

by applying Gauss-Jordan elimination

$$\left[\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{matrix} \,\middle|\, \begin{matrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{matrix}\right]$$

LU decomposition method

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad AX = B$$

$$A = LU$$

$$L = \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & 0 \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \qquad U = \begin{bmatrix} 1 & u_{12} & u_{13} \cdots & u_{1n} \\ 0 & 1 & u_{23} \cdots & u_{2n} \\ \vdots & & & \\ 0 & 0 & 0 \cdots & 1 \end{bmatrix}$$

$$AX = B$$

$$\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$UX = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$$\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \\ l_{n1} & l_{n2} & \cdots & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

using forward substitution we get $\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$

$$\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \\ l_{n1} & l_{n2} & \cdots & l_{nm} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$UX = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$$\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \\ l_{n1} & l_{n2} & \cdots & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

using forward substitution we get $\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$

$l_{i1} = a_{i1} \quad i = 1, 2, \ldots n.$

$u_{ij} = a_{ij} / l_{11} \quad j = 2, 3, \ldots n.$

$$l_{ip} = a_{ip} - \sum_{k=1}^{p-1} l_{ik} k_{ki} \quad i = p, \ldots, n$$

$$u_{pj} = \left(a_{pj} - \sum_{k=1}^{p-1} l_{pk} u_{kj}\right) / l_{pp} \quad j = p+1, \ldots, n$$

$p = 2, 3, \ldots, n$

Scanned by CamScanner

Iterative Method for soln. of linear simultaneous eq^n

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$[x_1^{(0)}, x_2^{(0)} \dots x_n^{(0)}]^T = [x^{(0)}]$$

$$x^{(1)} = [x_1^{(1)}, x_2^{(1)}, \dots x_n^{(1)}]^T$$

$$a_{11}x_1^{(1)} = b_1 - (a_{12}x_2^{(0)} + \dots + a_{1n}x_n^{(0)}).$$
$$a_{22}x_2^{(1)} = b_2 - (a_{21}x_{1}^{(0)} + \dots + a_{2n}x_n^{(0)})$$
$$\vdots$$
$$a_{nn}x_n^{(1)} = b_n - (a_{n1}x_1^{(0)} + a_{n2}x_2^{(0)} + \dots + a_{n(n-1)}x_{n-1}^{(0)})$$

check if

$$\max_{1 \le i \le n} \{ |x_i^{(1)} - x_i^{(0)}| \} < \epsilon$$

then stop.

else contains the iterations.

Jacobi's Method

$$a_{11}x_1^{(k+1)} = b_1 - (a_{12}x_2^{k} + \dots + a_{1n}x_n^{(k)})$$
$$a_{21}x_2^{(k+1)} = b_2 - (a_{21}x_1^{k} + \dots + a_{2n}x_n^{(k)})$$
$$\vdots$$
$$a_{nn}x_n^{(k+1)} = b_n - (a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \dots + a_{n(n-1)}x_{n-1}^{(k)})$$

$$x_1^{(0)} = x_2^{(0)} = x_n^{(0)} = 0$$

In matrix notation

$$X^{(k+1)} = HX^{(k)} + c$$

where $X^{(k+1)}$ and $X^{(k)}$ are $n \times 1$ (i.e a column) vector. H is a $n \times n$ matrix depending on A. and c is again a column vector depending on A and B.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots \\ a_{(n-1)1} & a_{(n-1)2} & a_{(n-1)3} & \cdots & a_{(n-1)n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0 & \cdots & & 0 & 0 \\ a_{21} & 0 & & & 0 & 0 \\ a_{31} & a_{32} & 0 & \cdots & 0 & 0 \\ \vdots \\ a_{n1} & a_{n2} & & \cdots & a_{n(n-1)} & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} a_{11} & 0 & 0 & & 0 \\ 0 & a_{22} & 0 & & 0 \\ \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & \cdots & a_{2n} \\ \vdots \\ 0 & 0 & \cdots & & a_{(n-1)n} \\ 0 & 0 & \cdots & & 0 \end{bmatrix}$$

In matrix notation, the set of eq^{n}s in (1) can be written as

$$DX^{(K+1)} = B - (L+U)X^{(K)}$$

or, $X^{(K+1)} = D^{-1}\{B - (L+U)X^{(K)}\}$

or, $X^{(K+1)} - X^{(K)} = D^{-1}\{B - (L+U)X^{(K)}\} - X^{(K)}$

$$= D^{-1}\{B - (L+U)X^{(K)} - DX^{(K)}\}$$

$$= D^{-1}\{B - (L+U+D)X^{(K)}\}$$

$$X^{(K+1)} - X^{(K)} = D^{-1}\{B - AX^{(K)}\}$$

$$X^{(K+1)} = D^{-1}\{B - (L+U)X^{(K)}\}$$

Let $X^{(K+1)} = X + E^{(K+1)}$

and $X^{(K)} = X + E^{(K)}$ where X is the true sol^n vector, $E^{(K+1)}$ and $E^{(K)}$ are the error vectors associated with $X^{(K+1)}$ and $X^{(K)}$ respectively.

$$X + E^{(K+1)} = D^{-1}\{B - (L+U)(X + E^{(K)})\}$$

or, $E^{(K+1)} = D^{-1}\{B - (L+U)X - (L+U)E^{K} - DX\}$

or, $E^{(K+1)} = D^{-1}\{B - AX - (L+U)E^{(K)}\}$

$$= D^{-1}(L+U)E^{(K)}$$

In matrix notation, the set of eq^n's in (1) can be written as

$$DX^{(K+1)} = B - (L+U)X^{(K)}$$

for convergence,

$$\cancel{\|-D^{-1}} \quad \|-D^{-1}(L+U)\| < 1$$

$$\text{or, } \|-D^{-1}(A-D)\| < 1$$

$$\text{or, } \|I - D^{-1}A\| < 1$$

Let $(I - D^{-1}A)Y = \lambda Y$ where

or, $(D-A)Y = \lambda DY$

Scalar λ is the eigen value and

vector Y is the corresponding eigen vector non-zero column vector.

$$\text{or, } -\sum_{\substack{i=1 \\ i\neq j}}^{n} a_{ij}\, y_j = \lambda a_{ii}\, y_i \qquad i = 1, \dots, n \quad j = 1, \dots, n.$$

Since Y is a non-zero vector, $|y_i| \neq 0$

$$\therefore |\lambda| = \left\{-\sum_{\substack{i=1 \\ i\neq j}}^{n} a_{ij}\right\} \Big/ |a_{ii}|$$

for convergence $|\lambda| < 1$

$$\left\{\sum_{i=1}^{n} |a_{ij}|\right\} \Big/ |a_{ii}| < 1$$

$$\text{or } \sum_{\substack{j=1 \\ i\neq j}}^{n} |a_{ij}| < |a_{ii}| \qquad i = 1, \dots, n.$$

Example

$$x + 6y = 4$$
$$x - 2y - 6z = 14$$
$$9x + 4y + z = -17$$

$R_1 \leftrightarrow R_3$

$$\cancel{A = \begin{bmatrix} 1 & 6 & 0 \\ 1 & -2 & -6 \\ 9 & 4 & 1 \end{bmatrix}} \qquad A = \left[\begin{array}{ccc:c} 1 & 6 & 0 & 4 \\ 1 & -2 & -6 & 14 \\ 9 & 4 & 1 & -17 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 9 & 4 & 1 & -17 \\ 1 & -2 & -6 & 14 \\ 1 & 6 & 0 & 4 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 9 & 4 & 1 & -17 \\ 1 & 6 & 0 & 4 \\ 1 & -2 & -6 & 14 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 9 & 4 & 1 & -17 \\ 1 & 6 & 0 & 4 \\ 1 & -2 & -6 & 14 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$A = \begin{bmatrix} 9 & 4 & 1 \\ 1 & 6 & 0 \\ 1 & -2 & -6 \end{bmatrix} \quad B = \begin{bmatrix} -17 \\ 4 \\ 14 \end{bmatrix} \quad D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} 1/9 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & -1/6 \end{bmatrix}$$

Gauss-seidal method.

$$a_{11}x_1^{(k+1)} = b_1 - (a_{12}x_2^{(k)} + \dots + a_{1n}x_n^{(k)})$$
$$a_{22}x_2^{(k+1)} = b_2 - (a_{21}x_1^{(k+1)} + \dots + a_{2n}x_n^{(k)})$$
$$a_{33}x_3^{(k+1)} = b_3 - (a_{31}x_1^{(k+1)} + a_{32}x_2^{(k+1)} + \dots a_{3n}x_n^{(k)})$$
$$\vdots$$
$$a_{nn}x_n^{(k+1)} = b_n - (a_{n1}x_1^{(k+1)} + a_{n2}x_2^{(k+1)} + \dots + a_{n(n-1)}x_{n-1}^{(k+1)}) \quad (1)$$

$$x_1^{(0)} = x_2^{(0)} = \dots = x_n^{(0)} = 0$$

A = same L = same D same U = sa-

$$a_{11}x_1^{(k+1)} = b_1 - (a_{12}x_2^{(k)} + a_{13}x_3^{(k)} + \dots + a_{1n}x_n^{(k)})$$
$$a_{21}x_1^{(k+1)} + a_{22}x_2^{(k+1)} = b_2 - (a_{23}x_3^{(k)} + a_{24}x_4^{(k)} + \dots + a_{2n}x_n^{(k)})$$
$$a_{31}x_1^{(k+1)} + a_{32}x_2^{(k+1)} + a_{33}x_3^{(k+1)} = b_3 - (a_{34}x_4^{(k)} + \dots + a_{3n}x_n^{(k)})$$
$$\vdots$$
$$a_{n1}x_1^{(k+1)} + a_{n2}x_2^{(k+1)} + \dots + a_{nn}x_n^{(k+1)} = b_n$$

$$(L+D)X^{(k+1)} = B - UX^{(k)}$$

or, $X^{(k+1)} = (L+D)^{-1}(B - UX^{(k)})$

$$\cancel{X^{(k+1)} = (L+D)^{-1}B - (L+D)^{-1}UX^{(k)}}$$

or, $X^{(k+1)} - X^{(k)} = (L+D)^{-1}(B - UX^{(k)}) - X^{(k)}$

$$= (L+D)^{-1}\{B - UX^{(k)} - (L+D)X^{(k)}\}$$

Iterative method to determine A.

Let B be an approximate inverse of A.

$\therefore AB \neq I$

Let $AB = I + E \Rightarrow E = AB - I$.

$\therefore$ from multiplying both sides with A^{-1}

$A^{-1}AB = A^{-1}(I+E)$

& $B = A^{-1}(I+E)$

Post multiplying both sides with $(I+E)^{-1}$

$$A^{-1} = B(I+E)^{-1} \simeq B(I - E + E^2 - E^3 \ldots)$$

If $\|E\| << 1$ we can write

$$A^{-1} \simeq B(I-E).$$
$$\simeq B(I - AB + I) \simeq B(2 I - AB)$$

Let $B^{(k+1)}$ and $B^{(k)}$ be the approximate inverse at $(k+1)$ and k iteration.

$$B^{(k+1)} = B^{(k)} (2 I - A B^{(k)})$$

$$k = 1, 2, 3, \ldots, n.$$

premultiplying both sides with A.

$$AB^{(k+1)} = AB^{(k)}(2I - AB^{(k)}) = I$$

$$= 2I\{AB^{(k)}\} - AB^{(k)}\}^2$$

or $AB^{(k+1)} - I = 2I\{AB^{(k)}\} - \{AB^{(k)}\}^2 - I$

$$= -\underbrace{\{AB^{(k)} - I\}}_{E^{(k)}}{}^2$$

Solution of nonlinear simultaneous eqⁿ

$f_1(x_1, x_2 \ldots x_n) = 0 \Rightarrow x_1^{(k+1)} = F_1(x_1^{(k)}, x_2^{(k)} \ldots x_n^{(k)})$

$f_2(x_1, x_2 \ldots x_n) = 0 \Rightarrow x_2^{(k+1)} = F_2(x_1^{(k)}, x_2^{(k)} \ldots x_n^{(k)})$

$$\max_{1 \le i \le n} \{ | x_i^{(k+1)} - x_i^{k} | \} < \epsilon$$

for convergence $\sum_{i=1}^{n} \left| \frac{\partial F_i}{\partial x_i} \right| < 1 \quad i = 1, 2 \ldots n$

Newton Ralphson method

Let $x_1^{(0)}, x_2^{(0)}, \ldots x_n^{(0)}$ be the intial approximations, and $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ be the corrections needed

$\therefore \; f_1(x_1^{(0)} + \Delta x_1, x_2^{(0)} + \Delta x_2, \ldots, x_n^{(0)} + \Delta x_n) = 0$

$f_2(x_1^{(0)} + \Delta x_1, x_2^{(0)} + \Delta x_2, \ldots, x_n^{(0)} + \Delta x_n) = 0$

$\vdots$

$f_n(x_1^{(0)} + \Delta x_1, x_2^{(0)} + \Delta x_2, \ldots, x_n^{(0)} + \Delta x_n) = 0$

$f_1(x^{0}) + \Delta x_1 \frac{\partial f_1}{\partial x_1}\Big|_{x^{(0)}} + \Delta x_2 \cdot \frac{\partial f_1}{\partial x_2}\Big|_{x^{(0)}} \ldots + \Delta x_n \frac{\partial f_1}{\partial x_n} = 0$

$$\max_{1 \le i \le n} (|\Delta x_i|) < \epsilon$$