Power Series Method

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Power series about $x_0 = 0$.

Series :-

1) $e^x : 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{m=0}^{\infty} \frac{x^m}{m!}$

Convergence of Power series :

1) convergence on a point
2) convergence on whole plane
3) convergence on a circular plane with Radius R ($|x| < R$)
   OR Radius of convergence.

Radius of Convergence :

Cauchy's theorem on limits :

i) $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$ $\quad$ * ii) $\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

2) $\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \dots$

3) $\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \dots$

1. $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots, \; |x| < R$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots\dots, \; |x| < R$

$a_1 + 2a_2 x + 3a_3 x^2 + \dots\dots = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots,$
$|x| < R$

$\Rightarrow a_1 = a_0$: coeff. of x^0

$2a_2 = a_1$: coef of x^1

$\rightarrow a_2 = \frac{a_1}{2} = \frac{a_0}{2} = \frac{a_0}{2!}$

coeff. of x^2 : $3a_3 = a_2$

$\rightarrow a_3 = \frac{a_2}{3} = \frac{a_0}{3 \cdot 2} = \frac{a_0}{3!}$

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

$= a_0 + a_0 x + \frac{a_0}{2!} x^2 + \frac{a_3}{3!} x^3 + \dots$

$= a_0 \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right]$

$= a_0 e^x$

2. $\frac{d^2y}{dx^2} + y = 0$

$\boxed{\cancel{y = A\cos x + B\sin x}}$

$\cancel{y' = \sum_{n=0}^{\infty} a_n x^n}$

$\cancel{y' = -A\sin x + B\cos x = B\cos x - A\sin x}$.

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n'=0}^{\infty} (n'+2)(n'+1) a_{n'+2} x^{n'} + \sum_{n=0}^{\infty} a_n x^n = 0 \qquad [n-2 = n']$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + a_n] x^n = 0 \qquad |x| < R$$

$$\Rightarrow (n+2)(n+1) a_{n+2} + a_n = 0 \qquad \forall n = 0, 1, 2, \ldots$$

↑ Recursion formula

For $n = 0$.

$$2 \cdot 1 a_2 + a_0 = 0 \Rightarrow a_2 = -\frac{a_0}{2!}$$

For $n = 1$

$$3 \cdot 2 \cdot a_3 + a_1 = 0 \Rightarrow a_3 = -\frac{a_1}{3!}$$

For $n = 2$

$$4 \cdot 3 \cdot a_4 + a_2 = 0 \Rightarrow a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4!}$$

For $n = 3$

$$5 \cdot 4 \cdot a_5 + a_3 = 0 \Rightarrow a_5 = \frac{a_1}{5!}$$

$$y = a_0 + a_1 x_1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$= a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a}{5!} x^5 + \cdots$$

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$$= a_0\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots\right]+a_1\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\right]$$

$$= a_0\cos x + a_1\sin x.$$

3. Find radius of convergence of : (Converges for all) Answer

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$$

4. Find radius of convergence of :

$$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots$$

$$R=\lim_{n\to\infty}\left|\frac{a_{2n}}{a_{2n+2}}\right|$$

" $y=\sum_{n=0}^{\infty} a_n(x-x_0)^n$ p as about $x=x_0$ "

5. $4y''-4y'+y=0$

$y(1)=0,\quad y'(1)=1$

$$y=\sum_{n=0}^{\infty} a_n(x-1)^n$$

$$y'=\sum_{n=1}^{\infty} n\,a_n(x-1)^{n-1}$$

$$y''=\sum_{n=2}^{\infty} n(n-1)\,a_n(x-1)^{n-2}$$

$$4\sum_{n=2}^{\infty} n(n-1)\,a_n(x-1)^{n-2}-4\sum_{n=1}^{\infty} n\,a_n(x-1)^{n-1}+\sum_{n=0}^{\infty} a_n(x-1)^n$$

$$=0.$$

$$\cancel{4\sum_{n'=0}^{\infty} 2(n+1)\, a_n (x-1)^{n'} - 4\sum_{n}}$$

$n-2 = m$

$n-1 = p$

$$4\sum_{m=0}^{\infty} (m+2)(m+1)\, a_{m+2} (x-1)^m - 4\sum_{p=0}^{\infty} (p+1)\, a_{p+1}$$

$$(x-1)^p + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$4 \cdot 2 \cdot 1\, a_2 (x-1)^0 - 4 \cdot 1\, a_1 (x-1)^0 +$$

$$a_0 (x-1)^0 = 0$$

For $n=0$, $8a_2 - 4a_1 + a_0 = 0$.

For $n=1$,

$$4 \cdot 3 \cdot 2\, a_3 (x-1)^1 - 4 \cdot 2 \cdot a_2 (x-1)^1 + a_1 (x-1) = 0$$

$$4(n+2)(n+1)\, a_{n+2} - 4(n+1)\, a_{n+1} + a_n = 0, \quad n = 0, 1, 2$$

$n=0$

$$4 \cdot 2 \cdot 1 \cdot a_2 - 4 \cdot a_1 + a_0 = 0$$

$$\Rightarrow 8a_2 = a_0 + 4a_1$$

$$a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{8}, \quad a_4 = \frac{1}{48}$$

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n, \quad |x-1| < R$$

$$y = a_0 + a_1 (x-1) + a_2 (x-1)^2 + a_3 (x-1)^3 + \ldots$$

$y = 0$ at $x = 1$

$$\therefore \boxed{a_0 = 0, \quad a_1 = 1}$$

$$y = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{8}(x-1)^3 + \frac{1}{48}(x-1)^4 + \ldots$$

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6. $xy' = y$

~~$y' = e^{xy}$~~

$$x\frac{dy}{dx} = y.$$

$$\Rightarrow \frac{dy}{y} = \int\frac{dx}{x}$$

$$\Rightarrow \ln y = \ln x + c.$$

$$\Rightarrow x =$$

3. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{\angle n}$$

$$a_n = \frac{1}{\angle n} \qquad a_{n+1} = \frac{1}{\angle n+1}$$

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\angle n}{\angle n+1}\right|$$

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{\cancel{\angle n}}{(n+1)\cancel{\angle n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{\infty}\right|$$

$$\frac{1}{R} = 0$$

$$R = \frac{1}{0} = \infty. \quad \boxed{R = \infty}$$

4. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\cdots$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{|\underline{2n}}$$

$$a_{2n} = \frac{(-1)^n}{|\underline{2n}} \qquad a_{2n+1} = \frac{(-1)^{n+1}}{|\underline{2n+2}}$$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{|\underline{2n+2}} \times \frac{|\underline{2n}}{(-1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(-1)^n}\cdot(-1)}{(2n+2)(2n+1)\cancel{|\underline{2n}}} \times \frac{\cancel{|\underline{2n}}}{\cancel{(-1)^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)}{(2n+2)(2n+1)} \right| = \lim_{n\to\infty} \left| \frac{}{2\cancel{(\ldots)}} \right.$$

$$= \lim_{n\to\infty} \left| \frac{(-1)}{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)} \right| = \frac{1}{4}$$

$R = 4$

1. $y'' + y' + x^2 y = 0$

$y(0) = 1 \quad y'(0) = 2$

$$y = 1 + 2x - \frac{x^2}{} + \frac{x^3}{3} + \frac{x^4}{6} + \cdots\cdots$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

2. $x^2 y' = y$

$$y = \sum_{n=0}^{\infty} a_n x^n \quad , \ |x| < R$$

$$\Rightarrow y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

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3. $y' = 1 + y^2$

$y = \sum_{n=0}^{\infty} a_n x^n$ $\qquad y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$ $\qquad n-1 = n'$, $n = n'+1$

$= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$

$= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$

$= 1 + \left(\sum_{n=0}^{\infty} a_n x^n\right)^2 = 1 + (a_0 + a_1 x + a_2 x^2 + \dots)^2$

$= 1 + a_0^2 + a_1^2 x^2 + 2a a_0 a_2 x^2 + \dots$

$a_1 + 2a_2 x + 3a_3 x^2 + \dots = 1 + a_0^2 + a_1^2 x^2 + 2 a_0 a_2 x^2$

$a_1 = 1 + a_1^2$

$2a_2 = a_1^2 + 2a_1 a_2$

Power Series apply only on

1. linear homogenous equation
2. we apply power series method about an ordinary point of the given equation.

* General second order linear homogenous diff. eqn

$$y'' + P(x) y' + Q(x) y = 0 \quad \text{—} ①$$

A point $x = x_0$ is called an ordinary point of ①, if both $P(x)$ and $Q(x)$ were analytic at $x = x_0$.

Example of analytic function are $f(x) = x^2, \sin x, \cos x, e^x$

New analytics function $f(x) = \underbrace{\frac{1}{x}, e^{1/x}}_{x=0}, \underbrace{\frac{x}{x-1}}_{x=1}$

<u>Definition.</u>

If any one of $P(x)$ or $Q(x)$ or both $P(x)$ and $Q(x)$ finds to be analytic at some point, $x = x_0$, then $x = x_0$ is called a singular point of the differentitation equation ①.

1. $x^2y'' + xy' + y = 0$

$\Rightarrow y'' + \frac{y'}{x} + \frac{y}{x^2} = 0$; $P(x) = \frac{1}{x}$, $Q(x) = \frac{1}{x^2}$

$\Rightarrow x = 0$ is a singular pt.

$\Rightarrow x(x-1)y'' + xy' + y = 0$

$\Rightarrow y'' + \frac{1}{x-1}y' + \frac{1}{x(x-1)}y = 0$

$\Rightarrow P(x) = \frac{1}{x-1}$, $Q(x) = \frac{1}{x(x-1)}$

$\Rightarrow x = 0, 1$ are singular points.

<u>Note.</u>

We can always assume a power series sol^n of a given diff. eq^n about an ordinary point.

ex: $x^2y' = y$

$\Rightarrow y' - \frac{1}{x^2}y = 0$.

$$y'' + P(x)y' + q(x)y = 0 \quad -(1).$$

Defination.

Suppose $x = x_0$ is a singular point of (1). Now $x = x_0$ is called a regular singular point of (1) if both

i) $\lim_{x \to x_0} (x - x_0) P(x)$

ii) $\lim_{x \to x_0} (x - x_0)^2 Q(x)$ exist.

~~1. $x(x-1)y'' + xy' + y = 0$.~~

Note

If $x = x_0$ is a regular singular point of (1), we can assume Frobenius series sol^n of (1) about $x = x_0$

$$y = (x - x_1)^m \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad — (A)$$

(A) becomes

If $m = 0$, Power series

~~$m > 0$~~, $n = 0$, +ve, -ve integer fraction exponent

If $x = 2$,

~~$y = (x - x_2)^2$~~.

$$y = (x - x_1)^2 [a_0 + a_1 (x - x_0) + a_2 (x - x_1)^2 + \dots]$$

$$= b_0 + b_1 (x - x_0) + b_2 (x - x_0)^2$$

$$\therefore b = 0, \quad b_1 = 0, \quad b_2 = a_0$$

Power series is a special case of Frobenius series

1. $x(x-1)y'' + xy' + y$

$P(x) = \frac{1}{x-1} \qquad Q(x) = \frac{1}{x(x-1)}$

$x_0 = 0.$

$\lim_{x \to 0} x \cdot \frac{1}{x-1} = 0$

$\lim_{x \to 0} x^2 \cdot \frac{1}{x(x-1)} = 0$

$x = 0$ is a regular singular point of B.

<u>Frobenius series sol^n</u>

$2x^2y'' + x(2x+1)y' - y = 0 \quad -(1).$

$x = 0$ is singular pt of.

$\lim_{x \to 0} x \cdot \left(\frac{2x+1}{2x}\right) = \frac{1}{2}$

$\lim_{x \to 0} x^2\left(-\frac{1}{2x^2}\right) = -\frac{1}{2}$

$x = 0$ is a regular singular pt.

$y = \sum_{n=0}^{\infty} a_n x^{n+m}, \quad a_0 \neq 0$

$y' = \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$

$y'' = \sum_{n=0}^{\infty} (n+m)(n+m-1) x^{n+m-2}$

$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+1}$

$y' = ma_0 x^{m-1} + (m+1) a_1 x^m + \dots$

$$y = a_0 + a_1 x + a_2 x^2$$
$$y' = a_1 + 2a_2 x^2$$

$$2x^2 y'' + x(2(x+1)) y' - y = 0 \quad — (1).$$

$$\Rightarrow 2x^2 \sum_{n=0}^{\infty} (m+n-1) a_n x^{m+n-2} + 2x^2 \sum (m+n) a_n x^{n-m-1}$$
$$+ x \sum_{n=0}^{\infty} (m+n) a_n x^{n+m-1} - \sum_{n=0}^{\infty} x^{n+m} = 0$$

$$\Rightarrow 2 \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m} + 2 \sum_{n=0}^{\infty} a_n (n+m) x^{n+m+1}$$
$$+ \sum_{n=0}^{\infty} (n+m) a_n (x^{n+m} - \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$= \sum_{n=0}^{\infty} [2(n+m)(n+m-1) + n+m) - 1] a_n x^{n+m}$$
$$+ 2 \sum_{n=1}^{\infty} a_{n-1} (n-1+m) x^{m+n} = 0.$$

$$\Rightarrow 2 \sum_{n=1}^{\infty} \left[\{ 2(n+m)^2 - (n+m) - 1 \{ a_n + 2(n+m-1) a_{n-1} \right]$$
$$+ (2m^2 - m - 1) a_0 x^m = 0.$$

Co-eff. of lowered order term in x for the series

$$= (2m^2 - m - 1) a_n = 0$$
$$= 2m^2 - m - 1 = 0 \quad \because a_0 \neq 0$$

Indices. eq^n.

$$n = 1, -\frac{1}{2}$$

Coeff. of $x^{n+m} = 0$. $n = 1, 2, \dots$

$$2(n+m) + 1 \{ (n+m-1) a_n + 2(n+m-1) a_{n-1} = 0$$

Recursion formula

$$a_n = \frac{-2}{2n+2m+1} a_{n-1} \ \forall \ a_n \geq$$

$m=1$,

$$a_n = -\frac{2}{2n+3} a_{n-1}, \ n \geq 1$$

$\cancel{n+m-1 = n+1}$

$m=1$,

$$a_n = \frac{-2}{2n+3} a_{n-1}$$

$n=1$

$$a_1 = -\frac{2}{5} a_0$$

$$a_2 = \frac{-2a_1}{7} = -\frac{2}{7}\left(-\frac{2}{5}a_0\right)$$

$$= \frac{4}{35} a_0.$$

$$y = a_0 x\left(1 - \frac{2}{5}x + \frac{4}{35}x^2 - \dots\right)$$

$$y = x^m\left(a_0 + a_1 x + a_2 x^2 + \dots\right)$$

$$= x^1\left(a_0 - \frac{2}{5}a_0 x + \frac{4}{35}a_0 x^2 + \dots\right)$$

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General soln:

If $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$, linearly independent.

$m = -1/2$

$a = -\frac{1}{n} a_{n-1}$

$a_1 = -a_0 \quad a_2 = a_0/2 \quad a_3 = -a_0/6$

$y = \frac{a_0}{\sqrt{x}}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \ldots\right)$

$= a_0 x^{-1/2} e^{-x}$ $\quad \boxed{y_2(x) = x^{-1/2} e^{-x}}$

2. $4xy'' + 2y' + y = 0$

$P(x) = \frac{1}{2x} \quad Q(x) = \frac{1}{4x}$

$y = \sum_{n=0}^{\infty} a_n x^{n+m}, \quad a_0 \neq 0.$

$y' = \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$

$y'' = \sum_{n=0}^{\infty} (n+m)(n+m-1) x^{n+m-2}$

$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+1}$

$y' = m a_0 x^{m-1} + (m+1) a_1 x^m + \ldots$

$y = a_0 + a_1 x + a_2 x^2$

$y = a_1 + 2a_2 x^2$

$4xy'' + 2y' + y = 0$

$\Rightarrow 4x \sum_{n=0}^{\infty} (n+m)(n+m-1)x^{m+n-2} + 2\left(\sum_{n=0}^{\infty} (n+m)a_n x^{n+m}\right.$

$+ \sum_{n=0}^{\infty} a_n x^{n+m}$

$\Rightarrow \sum_{n=0}^{\infty} 4(n+m)(n+m-1)$

$\vdots$

$m = 0, 1/2.$

$$a_n = -\frac{1}{2(m+n)(2m+2n-1)} a_{n-1}$$

$m = 0,$

$$a_n = \frac{1}{2n(2n-1)} a_{n-1}$$

$n = 1$

$$a_1 = \frac{1}{2 \times 1} = \frac{1}{2} a_0$$

$n = 2$

$$a_2 = \frac{1}{4(2\times 2-1)} a_{2-1}$$

$$= \frac{1}{12} a_1$$

$$y_1(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} - \cdots$$

$$= 1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} - \cdots$$

$$= \cos\sqrt{x}$$

$$y_2(x) = x^{1/2}\left(1 - \frac{x}{3!} + \frac{x^2}{5!} \cdots\right)$$

$$= \sqrt{x} - \frac{(\sqrt{x})^3}{3!} + \frac{(\sqrt{x})^5}{5!}$$

$y'' - -1$

$$= \sin\sqrt{x}$$

H|W.

1. $2x^2y'' - xy' + (1-x^2)y = 0$
2. $2x^2y'' + xy' - (x+1)y = 0$
3. $9x(1-x)y'' - 12y' + 4y = 0$
4. $(2x+x^3)y'' - y' - 6xy = 0$.

1. $2x^2y'' - xy' + (1-x^2)y = 0$

$P(x) = -\frac{1}{x}$ $\qquad Q(x) = \frac{1-x^2}{2x^2}$

$$y = \sum_{n=0}^{\infty} a_n x^{n+m} \quad [|x| < R]$$

$\lim_{x\to 0} xP(x) = -1$

$\lim_{x\to 0} x^2 Q(x) = \frac{1}{2}$

singular

Regularity

$$\therefore\ 2x^2 \sum_{n=0}^{\infty} (n+m)(n+m-1)a_n x^{n+m-2} - x\sum_{n=0}^{\infty}(n+m)a_m x^{n+m-1}$$

$$+ (1-x^2)\sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[2(n+m)(n+m-1) - (n+m) + 2\right]a_n x^{n+m} - \sum_{n=0}^{\infty} a_n x^{n+m+2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}(2n+2m-1)(n+m-1)a_n x^{n+m} - \sum_{n=2}^{\infty} a_{n'-2} x^{n'+m} = 0$$

$$\Rightarrow \sum_{n=2}^{\infty}\left[(2n+2m-1)(n+m-1)a_n - a_{n-2}\right]x^{n+m}$$

$$+ (2m-1)(m-1)a_0 x^m + (2m+1)m a_1 x^{m+1} = 0$$

$(2m-1)(m-1) = 0 \quad \because a_0 \neq 0$

$(2m+1)m a_1 = 0 \Rightarrow a_1 = 0$

$m = 1 \,/\, m = 1/2$

$$a_n = \frac{a_{n-2}}{(2n+2m-1)(n+m-1)}$$

when $m = 1$,

$$a_n = \frac{a_{n-2}}{(2n+1)n} \Rightarrow a_1 = 0$$

$$a_2 = a_0/10 \ ; \ a_3 = 0 \ ; \ a_4 = \frac{a_0}{360} \ ; \ a_5 = 0$$

$$y_1 = x\left(a_0 + \frac{a_0}{10}x^2 + \frac{a_0}{360}x^4 + \dots\right)$$

$$= a_0 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + \dots\right)$$

ii) when $m = \frac{1}{2}$

$$a_n = \frac{a_{n-2}}{2n(n-\frac{1}{2})} = \frac{a_{n-2}}{n(2n-1)}$$

$$a_1 = 0$$

$$a_2 = \frac{a_0}{6} \qquad a_4 = \frac{a_0}{168}$$

$$a_3 = 0 \qquad a_5 = 0$$

$$\therefore y = x^{1/2}\left(a_0 + \frac{a_0}{6}x^2 + \frac{a_0}{168}x^4 + \dots\right)$$

$$= a_0\left(x^{1/2} + \frac{x^{5/2}}{6} + \frac{x^{3/2}}{168} + \dots\right)$$

$$y_2(x) = x^{1/2} + \frac{1}{6}x^{5/2} + \frac{1}{168}x^{3/2} + \dots$$

$$y = Ay_1(x) + By_2(x)$$

$$= A\left(x + \frac{x^3}{10} + \frac{x^5}{360} + \dots\right) + B\left(\frac{1}{x^2} + \frac{1}{6}x^{5/2} + \frac{1}{168}x^{9/2} + \dots\right)$$

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Cauchy euler eqⁿ.

$$a_1 (ax+b)^n \frac{d^n y}{dx^n} + a_2 (ax+b)^{n-1} \frac{d^{n-1}}{dx^{n-1}} y + \dots$$

$$+ a_n (ax+b) \frac{dy}{dx} + a_{n+1} = 0$$

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+m}$$

1. $4x^2 y'' - 8x^2 y' + (4x^2+1) y = 0$

$$y'' - 2y' + \left(\frac{4x^2+1}{4x^2}\right) y = 0$$

$$P(x) = -2 \qquad Q(x) = 1 + \frac{1}{4x^2}$$

$$\lim_{x \to 0} x P(x) = 0$$

$$\lim_{x \to 0} x^2 Q(x) = x^2 \left(1 + \frac{1}{4x^2}\right)$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+m}$$

$y' =$

$y'' =$

$$\Rightarrow 4x^2 \sum_{n=0}^{\infty} (n+m)(m+m-1) a_n x^{n+m-2} - 8x^2 \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$$

$$+ (4x^2+1) \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$\Rightarrow \cancel{4(n+m)(n+m-1) a_n - 8(n+m) a_{n-1} + 4a_{n-2} = 0}$$

$$\cancel{a_0 = 0 ; a_1 = 0}$$

$$\cancel{a_{n-2} = \frac{8(n+m) a_{n-1} - 4(n+m)(n+m-1) a_n}{4}}$$

$$\Rightarrow 4x^2 \sum_{n=0}^{\infty} (n+m)(m+n-1) a_n x^{n+m-2} - 8x^2 \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$$

$$+ (4x^2) \sum_{n=0}^{\infty} a_n x^{n+m} + \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$\Rightarrow 4 \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m} - 8 \sum_{n=0}^{\infty} (n+m) a_n x^{n+m+1}$$

$$+ 4 \sum_{n=0}^{\infty} a_n x^{n+m+2} + \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$\Rightarrow 4 \sum_{n=0}^{\infty} (n+m)(n+m+1) a_n x^{n+m} - 8 \sum_{n=1}^{\infty} (n+m-1) a_{n-1} x^{n}$$

$$+ 4 \sum_{n=2}^{\infty} a_{n-2} x^{n+m} + \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} [4(n+m)(n+m-1) a_n - 8(n+m-1) a_{n-1} + 4a_{n-2} + a_n] x^{1+m} + 4m(m-1) a_0 x^m + 4(m+1) m a_1 x^{m+1}$$

$$- 8 m a_0 x^{m+1} + a_0 x^m + a_1 x^{m+1} = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} [\{4(n+m)^2 - 4(n+m) + 1\} a_n - 8\{(n+m-1) a_{n-1} + 4a_{n-2}] x^{n+m}$$

$$+ a_0\{4m(m-1) + 1\} x^m + \{4m(m+1) a_1 + a_1 - 8m a_0\} x^{m+1} = 0$$

Indicial eq^n :

$$4m(m-1) + 1 = 0$$

$$\because a_0 \neq 0$$

$$m = 1/2, 1/2$$

Co-eff $x^{m+1} = 0$

$\Rightarrow \{4m(m+1)+1\} a_1 - 8m\, a_0 = 0$

$m = 1/2$

$2\left(\frac{1}{2}+1\right)+1 \{ a_1 - 8 \times \frac{1}{2} a_0 = 0$

$\Rightarrow 4a_1 - 4a_0 = 0$

$\Rightarrow a_1 = a_0$

Recursion formula :

coeff. of $x^{n+m} = 0$

$\Rightarrow \{2(n+m)-1\}^2 a_n - 8(n+m-1) a_{n-1} + 4a_{n-2} = 0$

$\forall n \geq 2$

$\Rightarrow n^2 a_n - (2n-1) a_{n-1} + a_{n-2} = 0$

$a_2 = \frac{a_0[3-1]}{4} = \frac{a_0}{2}$

$a_3 = \frac{a_0}{6}, \quad a_4 = \frac{a_0}{24}$

$4x^2y'' - 8x^2y' + (4x^2+1)y = 0$ — (1)

$y = x^m (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots)$

$= x^{1/2}\left(a_0 + a_0 x + \frac{a_0 x^2}{2!} + \frac{a_0}{3!} x^3 + \frac{a_0}{4!} x^4 + \dots\right)$

$= a_0 x^{1/2} e^x$

$\boxed{y_2(x) = x^{1/2} e^x}$ — (2)

$$y = y_1(x)\,v(x)$$

$$y' = y_1'v + y_1v'$$

$$y'' = y_1''v + 2y_1'v' + y_1v''$$

$$4x^2(y_1''v + 2y_1'v' + y_1v'') - 8x^2(y_1'v + y_1v')$$
$$+ (4x^2+1)y_1v = 0$$

$$\Rightarrow \underbrace{[4x^2y_1'' - 8x^2y_1' + (4x^2+1)y_1]}\,v + [8x^2y_1' - 8x^2y_1]v'$$
$$+ 4x^2y_1v'' = 0$$

$$\Rightarrow 8x^2(y_1' - y_1)v' + 4x^2y_1v'' = 0$$

$$\Rightarrow 2(y_1' - y_1)v' + y_1v'' = 0$$

$$\Rightarrow \frac{2(y_1'-y)}{y_1} + \frac{v''}{v'} = 0, \quad x \neq 0$$

$$\Rightarrow \frac{y_1'}{y_1} - 2 + \frac{v''}{v'} = 0$$

Integrating

$$2\log y_1 - 2x + \ln v' = \text{constant } c,$$

$$\Rightarrow \ln(v'y_1^2) = 2x + c$$

$$\Rightarrow v'y_1^2 = ke^{2x} \qquad c = -\ln k$$

$$\Rightarrow v' = \frac{ke^{2x}}{y_1^2} = \frac{ke^{2x}}{xe^{2x}} = \frac{k}{x}$$

$$\Rightarrow \frac{dv}{dx} = \frac{k}{x}$$

$$\Rightarrow dv = k\frac{dx}{x}$$

$$\Rightarrow \boxed{v = k\ln x}$$

$y = k \ln x \, y_1(x)$

$y_2(x) = y_1(x) \ln x$

General soln:

H/W

$y = A y_1(x) + B y_2(x)$

$xy'' + y' - xy = 0$

2. $x^2 y'' - xy' + \frac{3}{4} y = 0$

$P(x) = -\frac{1}{x}$ $\quad Q(x) = \frac{3}{4x^2}$

$y = \sum_{n=0}^{\infty} a_n x^{n+m} = 0$

$x^2 \sum_{n=0}^{\infty} (n+m)(m+n-1) a_n x^{n+m-2} - x \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$

$+ \frac{3}{4} \sum_{n=0}^{\infty} a_n x^{n+m} = 0$

$\Rightarrow \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m} - \sum_{n=0}^{\infty} (n+m) a_n x^{n+m}$

$+ \frac{3}{4} \sum_{n=0}^{\infty} a_n x^{n+m} = 0$

$a_n (n+m)(n+m-1) - a_n (n+m) + \frac{3}{4} a_n = 0$

$a_n = 0$

let $n+m = t$

$a_{t-m} \, t(t-1) - a_{t-m} \, t + \frac{3}{4} a_{t-m} = 0$

$\Rightarrow a_{t-m} \left[t(t-1) - t + \frac{3}{4} \right] = 0$

$\Rightarrow a_{t-m} \left[t^2 - t - t + \frac{3}{4} \right] = 0.$

$$t^2 - 2t + \frac{3}{4} = 0$$

$$t = \frac{1}{2}, \frac{3}{2}$$

~~Recursion~~

$n = 0 \Rightarrow m = \frac{1}{2} / 3/2$

Recursion formula :

coeff. of $x^{n+m} = 0$

$$\Rightarrow \left[(n+m)^2 - 2(n+m) + \frac{3}{4}\right] a_n = 0 \ \forall n \geq 1$$

$$\Rightarrow \left[(n+m-\frac{3}{2})(n+m-\frac{1}{2}) a_n\right] = 0 \ \forall n \geq 1$$

$m = 3/2$

$n(n+1) a_n = 0 \quad , n \geq 1$

$\Rightarrow a_n = 0 \ \forall n \geq 1$

$m = 1/2$

$(n-1) n a_n = 0 \quad , n \geq 1$

$n = 1, \quad 0 \cdot 1 \cdot a_1 = 0$

a_1 is arbitary.

$n \geq 2, \quad n(n-1) \neq 0$

$\Rightarrow a_n = 0.$

$$y = x^m (a_0 + a_1 x + a_2 x^2 + \dots)$$

$$= x^{3/2}(a_0) = a_0 x^{3/2}$$

$$\boxed{y_1(x) = x^{3/2}}$$

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$m = 1/2$

$y = x^m(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)$

$= x^{1/2}(a_0 + a_1 x)$

$= a_0 x^{1/2} + a_1 x^{3/2}$

$$\boxed{y_2(x) = x^{1/2}}$$

General sol^n :

$$y = A y_1 + B y_2$$

3. $xy'' - y' + 4x^3 y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^{n+m}$$

$$y' = \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$$

$$y'' = \sum_{n=0}^{\infty} (m+n-1)(n+m) a_n x^{n+m-2}$$

$$= x \sum_{m=0}^{\infty} (m+n-1)(n+m) a_n x^{n+m-2} - \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$$
$$+ 4x^3 \sum_{n=0}^{\infty} a_n x^{n+m}$$

$$\sum_{n=0}^{\infty} (m+n-1)(n+m) a_n x^{n+m-1} - \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$$
$$+ 4 \sum_{n=0}^{\infty} a_n x^{n+m+3}$$

$$\Rightarrow \left[(m+n-1)(n+m) - (n+m)\right] a_n x^{n+m-1}$$
$$+ 4 a_n x^{n+m+3} = 0$$

$n+4+m-$

$\Rightarrow \left[(m+n-1)(n+m) - (n+m)\right] a_n x^{n+m-1} + 4a_{n-4} x^{n+m-1} =$

$\Rightarrow (m+n-1)(n+m)a_n - (m+n)\, a_n + 4a_{n-4} = 0.$

~~Put $n=4$~~

~~$(m+3)(m+4) - (m+4)\, a_4 + 4a_0 = 0$~~

~~$(m+4)\left[(m+3) - a_4\right] + 4a_0 = 0$~~

Put $n = 0$,

$(m-1)\, m - m = 0.$

$m^2 - m - m = 0$

$m^2 - 2m = 0$

$m = 0, 2$

If $m = 0$

$(n-1)\, n a_n - n a_n + 4a_{n-4} = 0.$

$n=0$,

a_0 is arbitrary.

$n=1$,

$0.1 - 1.a_n + 4a_{0-4} = 0$

$a_1 = 0$

$n=2$,

$a_0 = 0$

$n=3$,

$(3-1).3a_3 - 3a_3$

$- a_3 = 0.$

$n=4$

$3.4a_4 - 4.a_4 + 4a_0 = 0$

$8a_4 + 4a_0 = 0 \qquad a_4 = -\frac{4}{8} a_0$

$m = 2$,

$\cancel{(m-1)}$ $\sin^2 x$

~~H/w~~

Case - I :-

If $m_1 - m_2$ is not an integer or zero: The eq^n has two linearly independent solutions $y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n$ $a_0 \neq 0$

$y_2 = x^{m_2} \sum_{n=0}^{\infty} b_n x^n$, $b_0 \neq 0$

Ex :- $2x^2 y'' + x(2x+1) y' - y = 0$

Case - II :-

If $m_1 - m_2 = 0$, there ~~exist~~ exist second Frobenius Series sol^n

$y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n$

$y_2 = (y_1 \log x + x^{m_1} \sum_{n=0}^{\infty} c_n x^n$, $c \neq 0$

Ex :- $4x^2 y'' - 8x^2 y' + (4x^2 + 1) y = 0$

Case III :-

If $m_1 - m_2$ = integer two linearly independent sol^n are $y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n$

$y_2 = c y_1 \log x + x^{m_2} \sum_{n=0}^{\infty} c_n x^n$

A) In some cases $c \neq 0$, so the sol^n. contains a logarithm term. This case corresponds to when second Frobenius series sol^n can't exist

$$Ex :- x(1+x)y'' + 3xy' + y = 0.$$

B) In some cases $c = 0$, so the sol^n does not contain algorithm term. In this case there does exist a second Frobenius series sol^n.

$$Ex :- x^2y'' - xy' + \frac{3}{4}y = 0.$$

Legendre differential eq^n

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

n is a constant

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m a_m x^{m-1} +$$
$$n(n+1)\sum_{m=2}^{\infty} a_m x^m = 0$$

$$\Rightarrow \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - 2\sum_{m=1}^{\infty} m a_m x^m +$$
$$n(n+1)\sum_{m=0}^{\infty} a_n x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m=0}^{\infty} m(m-1)a_m x^m - 2\sum_{m=0}^{\infty} m a_m x^n$$

$$\Rightarrow \sum_{m=0}^{\infty}\left[(m+2)(m+1)a_{m+2} - \left\{m(m+1) + 2m - n(n+1)\right\}a_n\right]x^m = 0$$

Coeff of $x^m = 0$, $\forall\ m \geq 0$

$$\Rightarrow a_{m+2} = \frac{m(m-1) + 2m - n(n+1)}{(m+2)(m+1)}a_m$$

$$= \frac{(m-n)(m+n+1)}{(m+2)(m+1)}a_m$$

$$\boxed{a_{m+2} = -\frac{(n-m)(n+m+1)}{(m+2)(m+1)} a_m \, , \quad m \geq 0}$$

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$

$m = 0,$

$$a_2 = -\frac{n(n+1)}{2 \cdot 1}$$

$m = 1,$

$$a_3 = -\frac{(n-1)(n+2)}{3 \cdot 2} a_1$$

$m = 2,$

$$a_4 = -\frac{(n-3)(n+3)}{4 \cdot 3} a_2$$

$$= +\frac{n(n-2)(n+1)(n+3)}{4!} a_0$$

$m = 4,$

$$a_5 = +\frac{(n-1)(n-3)(n+2)(n+4)}{5!} a_1$$

$y =$ $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 +$

$$= a_0 + a_1 x - \frac{n(n+1)}{2!} a_0 x^2 - \frac{(n-1)(n+2)}{3!} x^3 a_1 + \frac{n(n-2)(n+1)(n+3)}{4!} x^4 a_0$$

$$= a_0 \left[1 - \frac{n(n+1)}{2} x^2 + \frac{n(n-2)(n+1)(n+3)}{4!} x^4 - \frac{n(n-2)(n-4)(n+1)(n+3)(n+5)}{4!} + \cdots \right]$$

$$+ a_1 \left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!} - \frac{(n-1)(n-3)(n-5)(n+2)(n+4)(n+6)}{7!} + \cdots \right]$$

$= a_0 y_0(x) + a_1 y_1(x)$

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$$y_0(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 \dots$$

$$y_1(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 \dots$$

i) n is zero or an integer.

Both y_0 and y_1 are infinite series.

$$R = \lim_{m \to \infty} \left| \frac{a_m}{a_{m+2}} \right|$$

$$= 1$$

ii) n is zero or an integer

If n is zero or an integer, then one of the infinite series terminates and the other remains infinite. The terminated series give a polynomial f^n, these polynomials $(P_n(x))$ satisfying the cond^n $P_n(1) = 1$ are called Legendre Polynomial.

$n = 0$, $\quad (1-x^2)y'' - 2xy' = 0$

$y_0(x) = 1$

$P_0(x) = 1 \times m_0$

$P_0(1) = 1$

$\Rightarrow m_0 = 1$

$$\boxed{P_0(x) = 1}$$

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$n=1,$

$y_1(x) = x \rightarrow (1-x^2)y'' - 2xy' + 1\cdot 2y = 0$

$P_1(x) = m_1 \times x$

$P_1(1) = 1$

$\Rightarrow m_1 = 1$

$$\boxed{P_1(x) = x}$$

$n=2,$

$y_0(x) = 1 - \frac{2\cdot 3}{2} x^2 = 1 - 3x^2$

$P_2(x) = m_2 \times (1-3x^2)$

$P_2(x) = 1$

$m_2 = -1/2$

$$\boxed{P_2(x) = \frac{3x^2-1}{2}}$$

$n=3,$

$y_1(x) = x - \frac{2\cdot 5}{3!} x^3$

$= x - \frac{\cancel{10}\,5}{\cancel{6}\,3} x^3 = \frac{3x - 5x^3}{3}$

$P_3(x) = m_3 \left(\frac{3x-5x^3}{3}\right)$

$P_3(x) = 1$

$m_3 = -\frac{3}{\ldots}$

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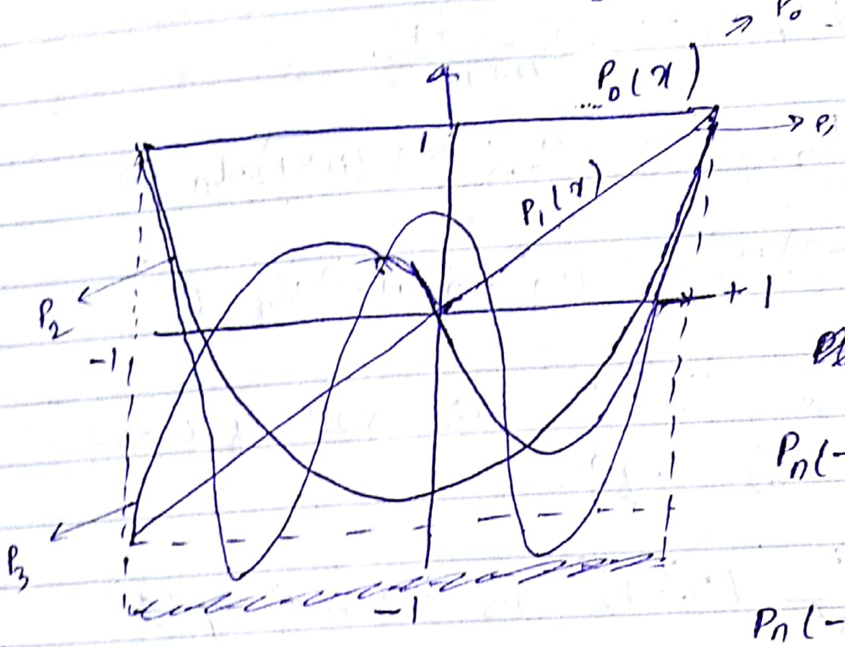
$P_3(x) = -\frac{3}{2}\left(x - \frac{5}{3}x^3\right)$

$n=4,$

$y_0(x) = 1 - \frac{4.5}{2!}x^2 + 4.2.e.$

$\vdots$

$P_4(x) = \frac{35x^4 - 30x^2 + 3}{8}$



$P_n(-1) = 1,\ n \to \text{even}$
$= -1,\ n \to \text{odd}$

$P_n(-1) = (-1)^n$
$P_n(1) = 1$

Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Proof : $u = (x^2-1)^n$

$u_1 = \frac{d}{dx}(x^2-1)^n$

$= n(x^2-1)^{n-1} \times 2x = \frac{2nx(x^2-1)^n}{(x^2-1)}$

$\Rightarrow (x^2-1)u_1 = 2nxu$

Leibnitz formula

$$(uv)_n = \sum_{i=0}^{n} {}^nC_i \, u_{n-i} \, v_i$$

Differentiating $(n+1)$ times,

$$(u_1)_{n+1}(x^2-1) + {}^{n+1}C_1 (u_1)_n \times 2x + {}^{n+1}C_2 (u_1)_{n-1} \times 2$$

$$= 2n\left[(u)_{n+1}\, x + {}^{n+1}C_1 (u)_n\right]$$

$\Rightarrow$ ~~$(1-x^2) U_{n+2} - 2x U_{n+1} +$~~

$$\Rightarrow (1-x^2) U_{n+2} - 2x U_{n+1} + n(n+1) u_n = 0$$

$$(1-x^2) U_n'' - 2x U_n' + n(n+1) U_n = 0$$

$y = U_n$ is the sol^n of legendre diff eq^n.

$y = \dfrac{d^n}{dx^n}(x^2-1)^n$ is the sol^n of (1).

$y = a \times \dfrac{d^n}{dx^n}(x^2-1)^n$ is also a sol^n of (1)

Now $y = 1$ at $x = 1$

$$1 = a\left[\frac{d^n}{dx^n}(x^2-1)^n\right]_{x=1}$$

$$= a\left[\frac{d^n}{dx^n}\left\{(x+1)^n (x-1)^n\right\}\right]$$

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$$(uv)_n = \sum_{i=0}^{n} {}^nC_i \cdot u_{n-i} \cdot v_i$$

$$= {}^nC_0 u_n v_0 + {}^nC_1 u_{n-1} v_1$$

$$1 = a\left[n!\,(x+1)^n + n(n-1)!\,(x-1)\,n(x+1)^{n-1} + \cdots(\ )(x-1)^2 + \cdots + (\ )(x-1)^n\right]_{n=1}$$

$$1 = a \times n!\, 2^n$$

$$\Rightarrow a = \frac{1}{2^n n!}$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \cdot (x^2-1)^n$$

$$P_0(x) = \frac{1}{2^0} \frac{d^0}{dx^0} (x^2-1)^0$$

$$= 1$$

$$P_1(x) = \frac{1}{2^1 1!} \frac{d^1}{dx^1} (x^2-1)^1$$

$$= \frac{1}{2} \times 2x = x$$

$$P_2(x) = \frac{1}{8}(12x^2 - 4)$$

$$(x^2-1)^n = \sum_{r=0}^{n} {}^nC_r (x^2)^{n-r} (-1)^r$$

$$\frac{d^n}{dx^n}(x^2-1)^n = \frac{d}{dx^n} \sum {}^nC_r\, x^{2n-2r} (-1)^r$$

$$P_n(x) = \frac{1}{2^n n!} \sum_{r=0}^{n} {}^nC_r (-1)^r (2n-2r)(2n-2r-1)\ldots(2n-2r-n+1)\, x^{2n-2r-1}$$

$$= \frac{1}{2^n n!} \sum_{r=0}^{N} {}^nC_r (-1)^r (2n-2r) \cdots (n-r$$

$$P_n(x) = \frac{1}{2^n n!} \sum_{r=0}^{N} {}^nC_r \, \frac{(-1)^r (2n-2r) \cdots (n-2r+1) \; (n-2r) \cdots 2 \cdot 1}{(n-2r) \cdots 2 \cdot 1} \, x^{n-2r}$$

$$= \frac{1}{2^n n!} \sum_{r=0}^{N} {}^nC_r \, (-1)^r \frac{(2n-2r)!}{(n-2r)!} \, x^{n-2r}$$

$\cancel{n - 2r > 0}$

$n - 2r \geq 0$

$r \leq \frac{n}{2}$, $N = \left[\frac{n}{2}\right]$

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Orthogonal func

A set of function $\{\phi_i(x)\}$ is said to be orthogonal in $[a,b]$ w.r.t weight function $W(x) > 0$, if

$$\int_a^b W(x)\,\phi_i(x)\,\phi_j(x)\,dx = 0 \text{ for } i \neq j \quad — (1)$$

The above set is called orthonormal if (1) is satisfied together with the following condition

$$\int_a^b W(a)\,\phi_i(x)\,\phi_i(x)\,dx = 1$$

$$\bar{a} = (a_1, a_2, a_3)$$
$$\bar{b} = (b_1, b_2, b_3)$$
$$\bar{a}\cdot\bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$$

* Legendre polynomials $\{P_n(x)\}_{n=0}^{\infty}$ are orthogonal in $[-1,1]$ w.r.to $W(x) = 1$

We have to prove $\int_{-1}^{1} P_n(x) P_m(x)\,dx = 0$, $m \neq n$

Let $P_n(x)$ and $P_m(x)$ are the solⁿ of Legendre diff eqⁿ given below

$$(1-x^2)y'' + 2xy' + n(n+1)y = 0$$

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0 \quad — (i) \times P_m$$

$$(1-x^2)P_n'' - 2xP_m' + m(m+1)P_m = 0 \quad — (ii) \times P_n$$

$$① \times P_m - ② \times P_n$$

$$(1-x^2)(P_m P_n'' - P_n P_m'') - 2x(P_n' P_m - P_m' P_n') + \{n(n+1) - m(m+1)\}\, P_m P_n = 0$$

$$\Rightarrow \frac{d}{dx}\left[(1-x^2)(P_n' P_m - P_m' P_n)\right] + (n-m)(n+m+1) P_m P_n = 0$$

Integrating w.r.t x for $[-1, 1]$

$$\Rightarrow (1-x^2)(P_n' P_m - P_m' P_n)\Big|_{-1}^{1} + (-n-m)(n+m+1)\int_{-1}^{1} P_m P_n \, dx = 0$$

$$\Rightarrow (n-m)(n+m+1)\int_{-1}^{1} P_m P_n \, dx = 0$$

$$\Rightarrow \int_{-1}^{1} P_m P_n \, dx = 0 \quad \text{if } m \neq n.$$

ii) $$\int_{-1}^{1} \{P_n(x)\}^2 \, dx = ??$$

$$\int_{-1}^{1} P_n(x) P_n(x) \, dx = \frac{1}{2^n n!} \int_{-1}^{1} \underset{\text{II}}{\frac{d^n (x^2-1)^n}{dx^n}} \; \underset{\text{I}}{P_n(x)}\, dx$$

$$= \frac{1}{2^n n!}\left[P_n(x) \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \Big|_{-1}^{1} - \int_{-1}^{1} P_n'(x) \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \, dx \right]$$

$$= \frac{1}{2^n n!}(-1)\left[P_n'(x)\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n\Big|_{-1}^{1} - \int_{-1}^{1} P_n''(x)\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n\,dx\right]$$

$$= \frac{1\times(-1)^2}{2^n n!}\int_{-1}^{1} P_n''(x)\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n\,dx$$

$$\vdots$$

$$= \frac{1}{2^n n!}(-1)^n\int_{-1}^{1} P_n^n(x)(x^2-1)^n\,dx$$

$$P_n = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$\Rightarrow \quad P_n^n(x) = \frac{1}{2^n n!}\frac{d^{2n}(x^2-1)^n}{dx^{2n}}$$

$$= \frac{1}{2^n n!}(2n)!$$

$$\frac{(-1)^n}{2^n n!}\int_{-1}^{1}\frac{1}{2^n n!}(2n)!\times(-1)^n(1-x^2)^n\,dx.$$

$$= \frac{(-1)^{2n}(2n)!}{(2^n n!)^2}\int_{-1}^{1}(1-x^2)^n\,dx.$$

$$I_1 = \frac{(2n)!}{(2^n n!)^2} \times 2 \times I_2.$$

$$I_1 = \int_{-1}^{1} (1-x^2)^n dx = 2\underbrace{\int_0^1 (1-x^2)^n dx}_{I_2} = 2I_2$$

$x = \sin\theta \Rightarrow dx = \cos\theta d\theta.$

$$I_2 = \int_0^1 (1-x^2)^n dx.$$

$$= \int_0^{\pi/2} \cos^{2n+1}\theta \, d\theta$$

$$= \frac{1}{2}\beta\left(\frac{0+1}{2}, \frac{2n+1+1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{1}{2}, n+1\right).$$

$$= \frac{1}{2} \frac{\Gamma(1/2)\Gamma(n+1)}{\Gamma\left(n+1+\frac{1}{2}\right)}.$$

* $$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right).$$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

$$\Gamma\left(n+1+\frac{1}{2}\right) = \left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right).$$

$$= \left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right) \cdots \frac{3}{2} \, \frac{1}{2}\Gamma\left(\frac{1}{2}\right).$$

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$$= \frac{(2n+1)\cancel{(2n)}(2n-1)\cdots 3\cdot 1\sqrt{\pi}}{2^{n+1}}$$

$$= \frac{(2n+1)(2n)(2n-1)(2n-2)\cdots 3\cdot 2\cdot 1\sqrt{\pi}}{2n(2n-2)\cdots 4\cdot 2\times 2^{n+1}}$$

$$= \frac{(2n+1)!}{2^n\, n! \times 2^{n+1}}\sqrt{\pi}$$

$$= \frac{(2n+1)!}{n!\, 2^{2n+1}}\sqrt{\pi}.$$

$$I = \frac{(2n)!}{(2^n n!)^2} \times 2 \times I_2 = \frac{2}{2n+1}$$

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n. \end{cases}$$

Q Find Legendre series expansion of

$f(x) = x^3 + x$ in $[-1, 1]$.

$$f(x) = \sum_{0}^{\infty} a_n P_n(x) = a_0P_0 + a_1P_1 + a_2P_2 \ldots$$

$$a_n \times \frac{2}{2n+1} = \int_{-1}^{1} f(x)P_n(x)\,dx = a_0\cancelto{0}{\int_{-1}^{1} P_0(x)P_n(x)\,dx}$$

$$+ a_1\cancelto{0}{\int_{-1}^{1} P_1(x)P_n(x)\,dx} \ldots \cancelto{0}{\ldots}$$

$$+ a_n\int_{-1}^{1} P_n(x)P_n(x)\,dx + \cancelto{}{\ldots}$$

$f(x) = x^3 + x$.

$$a_0 = \frac{2\times 0+1}{2}\int_{-1}^{1}(x^3+x)\,\cancel{P_0}(x) = 0$$

$$= \frac{1}{2}\int_{-1}^{1}(x^3+x)\,dx = 0.$$

$a_2 = 0$.

$$a_1 = \frac{2\times 1+1}{2}\int_{-1}^{1} x^3+x\ \cancel{P_1}(x)\,dx.$$

$$= \frac{3}{2}\int_{-1}^{1}(x^3+x)\,x\,dx = \frac{8}{5}.$$

$a_4 = 0$

$$a_3 = \frac{2\times 3+1}{2}\int_{-1}^{1}(x^3+x)\,P_3(x)\,dx = \frac{2}{5}.$$

$$P_n(x) = \frac{1}{2^n n!}\cdot\frac{d^n}{dx^n}(x^2-1)^n.$$

→ If $f(x)$ is continuous over $[-1, 1]$ then $f(x)$ can be written as linear combination of legendre polynomial

Q Find the Legendre Series expantion

i) $f(x) = e^x$.

ii) $f(x) = \begin{cases} 0 & [-1, 0) \\ x & [0, 1] \end{cases}$.

M II $P_0(x) = 1$ $\quad 1 = P_0$

$P_1(x) = x$ $\quad x = P_1$

$P_2(x) = \frac{3x^2-1}{2}$ $\quad x^2 = \frac{2P_2+1}{3} = \frac{2P_2+P_0}{3}$

$P_3(x) = \frac{5x^3-3x}{2}$ $\quad x^3 = \frac{2P_3+3P_1}{5}$.

eg $f(x) = x^3 + x$

$= \frac{2P_3+3P_1}{5} + P_1 = \frac{8}{5}P_1 + \frac{2}{5}P_3$

M III

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)\, P_n(x)\, dx$$

$$= \frac{2n+1}{2}\int_{-1}^{1} f(x)\, \frac{1}{2^n n!}\, \frac{d^n (x^2-1)^n}{dx^n}\, dx$$

$$= \frac{2n+1}{2}\, \frac{1}{2^n n!}\int_{-1}^{1} \underset{\mathrm{I}}{f(x)}\, \underset{\mathrm{II}}{\frac{d^n}{dx^n}(x^2-1)^n}\, dx$$

$$a_n = \frac{2n+1}{2}\, \frac{1}{2^n n!}\, (-1)^n \int_{-1}^{1} f^{(n)}(x)\, (x^2-1)^n\, dx.$$

$f(x) \longrightarrow$ odd , $n \longrightarrow$ even

$f^{(n)}(x) \longrightarrow$ odd

Generating f^n

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_0^{\infty} P_n(x)\, t^n.$$

Proof

$$\frac{1}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2)^{-1/2}$$

$$= [1+\{-t(2x+t)\}]^{-1/2}.$$

$$= \sum_0^{\infty} (-1)^r \frac{1\cdot 3 \cdots (2r+1)}{r!\, 2^r} (-1)^r (2x-t)^r$$

$$= \sum_0^{\infty} \frac{1\cdot 2\cdot 3 \quad (2r)(2r+1)}{r!\, 2^r \cdot 2^r \cdot r!} t^r (2x-t)^r$$

$$= \sum_0^{\infty} \frac{1\cdot 2\cdot 3 \cdots (2r)}{r!\, 2^r \cdot 2^r\, r!} t^r (2x-t)^r.$$

$$= \sum_0^{\infty} \frac{(2r)!}{2^{2r}(r!)^2} t^r (2x-t)^r$$

$$(2x-t)^r = \sum_{p=0}^{r} {}^rC_p\, (2x)^{r-p}\, t^p (-1)^p.$$

$$(1-2xt+t^2)^{-1/2} = \sum_{r=0}^{\infty} \frac{(2r)!}{2^{2r}(r!)^2} \sum_{p=0}^{r} {}^rC_p (-1)^p (2x)^{r-p} t^{r+p} \quad -①.$$

Let $r+p = n$.

For a fixed r $\quad p = n-r$

$0 \le p \le r \Rightarrow 0 \le n-r \le r \Rightarrow r \le n,\ r \ge \frac{n}{2}$

If n is even $\quad \frac{n}{2} \le r \le n.$

If n is odd $\quad \frac{n+1}{2} \le r \le n.$

Coefficient of t^n in ①.

$$= \sum_{r=m}^{n} \frac{(2r)!}{2^{2r}(r!)^2}\, {}^{r}C_{n-r}\, (-1)^{n-r} (2x)^{2r-n}.$$

Now change the variable from r to $k = n-r$.

Coeff of t^n

$$= \sum_{k=m'}^{0} \frac{(2(n-k))!}{2^{2(n-k)}((n-k)!)^2}\, {}^{n-k}C_{k}\, (-1)^k (2x)^{n-2k}$$

$$m' = \begin{cases} n - \frac{n}{2} & \text{even} \\ n - \frac{n+1}{2} & \text{odd} \end{cases}$$

$m' = n - m.$

$$= \sum_{k=0}^{m'} \frac{(2n-2k)!}{2^{2n-2k}\{(n-k)!\}^2} \cdot \frac{(n-k)!\,(-1)^k\, 2^{n-2k}}{k!\,(n-2k)!} \times x^{n-2k}$$

$$= \sum_{k=0}^{m=[n/2]} \frac{(-1)^k \, (2n-2k)!}{2^n \, k! \, (n-k)! \, (n-2k)!} x^{n-2k}$$

$$= P_n(x).$$

— x —

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n, \quad |t|<1$$

i) $P_n(1) = ?$
ii) $P_n(-1) = ?$
iii) $P_n(0) = ??$

i) $x = 1,$

$$LHS = \frac{1}{\sqrt{1-2t+t^2}} = \frac{1}{\sqrt{(1-t)^2}} = (1-t)^{-1}$$

$$= 1+t+t^2+t^3+\ldots\ldots$$

$$= \sum_{n=0}^{\infty} t^n$$

$$RHS = \sum_{n=0}^{\infty} P_n(1) t^n = P_0(1) + P_1(1) + P_2(1) t^2 + \ldots\ldots$$

ii) $x = -1$

$$LHS = \frac{1}{\sqrt{1+2t+t^2}} = (1+t)^{-1}$$

$$= \sum_{n=0}^{\infty} (-1)^n t^n$$

iii) $x = 0$,

$$\text{LHS} = \frac{1}{\sqrt{1+t^2}} = (1+t^2)^{-1/2}$$

$$\cancel{\text{RHS} = \sum_{n=0}^{\infty} P_n(0)\, t^n}$$

$$P_{2n}(0) = \frac{(-1)^n\, 1\cdot 3 \cdots (2n-1)}{2^n \cdot n!}$$

$$P_{2n+1}(0) = 0$$

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n, \quad |t| < 1$$

Differentiating w.r.t. t

$$\left(-\frac{1}{2}\right)(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_n n P_n(n)\, t^{n-1}$$

$$\Rightarrow \frac{x-t}{1-2xt+t^2} \cdot \frac{1}{\sqrt{1-2xt+t^2}} = \sum_n n P_n(n)\, t^{n-1}$$

$$\Rightarrow (n-t)\sum_n P_n(n)\, t^n = (1-2xt+t^2)\sum_n n P_n(x)\, t^{n+1}$$

Equating co-eff. of t^n from both sides

$$x P_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2x_n P_n(x) + (n-1) P_{n-1}(x)$$

$$\Rightarrow \boxed{(n+1) P_{n+1}(x) + (2n+1)\, x P_n(x) + n P_{n-1}(x) = 0}$$

$$P_0(x) = 1$$
$$P_1(x) = x$$

$$2P_2(x) - 3\cdot x P_1(x) + P_0(x) = 0$$

$$\Rightarrow P_2 = \frac{3x^2-1}{2}$$

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n \quad , |t|<1 \text{ --- } (1)$$

Diff. w.r.t t :

$$\left(-\frac{1}{2}\right)(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_n nP_n(x)t^{n-1} \text{ --- } (2)$$

Diff. (1) w.r.t x

$$\left(-\frac{1}{2}\right)(1-2xt+t^2)^{-3/2}(-2t) = \sum_n P_n'(x) t^n \text{ --- } (3)$$

$$(-2t) \times (2) - (-2x+2t) \times (3) = 0$$

$$\Rightarrow (-2t)\sum_n nP_n(x)t^{n-1} = (-2x+2t)\sum_n P_n'(x)t^n$$

$$\Rightarrow \sum_n nP_n(x)t^n = (x-t)\sum_n P_n'(x)t^n$$

Equating coeff of t^n from both sides

$$\boxed{nP_n(x) = xP_n'(x) - P_{n-1}'(x)}$$

Tchebyshev differential eq^n

$$(1-x^2)y'' - xy' + n^2y = 0$$

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

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