Classical defn of prob:

If a random experiment has n possible outcome, which are mutually exclusive, exhaustive and equally likely and m of those are favourable to an event of A, then the prob. of the event is $P(A) = \frac{m}{n}$

i.e Prob. of an event $=$ $\dfrac{\text{No. of outcome favourable to an event}}{\text{Total no. of mutually exclusive, exhaustive and equally likely, outcomes of random exp.}}$

Defects: (i) It is based on the feasibility of subdividing the possible outcomes of the experiments into mutually exclusive, exhaustive and equally likely cases. Unless this can be done, the formula is ~~inapplic~~ inapplicable.

(ii) The defination fails when no. of possible outcomes ~~for~~ is infinitely large.

The definition is only limited to coin theory, dice throwing, etc.

Axiomatic definition of probability:

Consider an experiment whose sample space is S. For each event E of the sample space, we assume that a number P(E) is defined and satisfies the following three axioms.

Ax. 1: $0 \leq P(E) \leq 1$

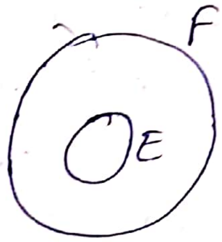
Ax. 2: $P(S) = 1$

Ax 3: For any seqn of mutually exclusive events $E_1, E_2 \ldots$ (that is events for which $E_i \cdot E_j = \phi, i \neq j$)

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

2) Prove that for $E \subset F$, $P(E) < P(F)$

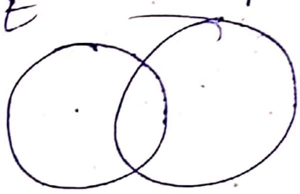
$$F = E \cup (E^c \cap F)$$



$$P(F) = P(E) + \underline{P(E^c F) > 0}$$

$$\therefore P(F) > P(E)$$

Prove that $P(E \cup F) = P(E) + P(F) - P(E \cap F)$



$$E \cup F = E \cup (E^c \cap F)$$

$$P(E \cup F) = P(E) + P(E^c \cap F) \rightarrow (I)$$

$$F = (E \cap F) \cup (E^c \cap F)$$

$$P(F) = P(E \cap F) + P(E^c F)$$

$$\Rightarrow P(E^c F) = P(F) - P(EF) \rightarrow (II)$$

$\therefore$ Putting $P(E^c F)$ in (I):

$$P(E \cup F) = P(E) + P(F) - P(E \cdot F)$$

(i) A Committe of 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the comittee Consists of 3 men & 2 women?

$$P = \frac{{}^{6}C_3 \times {}^{9}C_2}{{}^{15}C_5}$$

2) In the game of bridge, the entire deck of 52 cards is dealt out of 4 players, what is the probability that:
(i) One of them receives all 13 spades.

$$Ans = \frac{4}{^{52}C_{13}}$$

13 cards to be ~~given~~ distributed to a player out of 52 = $^{52}C_{13}$
For each player possibility for all cards to be spade = 1
∴ For each player prob. $= \frac{1}{\binom{52}{13}}$

∴ For 4 players prob. $= \frac{4}{\binom{52}{13}}$ (as event is mutually exclusive)

3) A die is thrown k times in succession.
Find the prob. of obtaining 6 at least once.
P(Getting 6 at least once) = 1 − P(Not get six at all)
Getting a six after k repeated throws ~~i.e.~~: where six should not be obtained in 1st and 2nd ... 3th throw
$= \left(\frac{5}{6}\right)^k$

$$P(E) = 1 - \left(\frac{5}{6}\right)^k$$

4) Find the prob. P_N that a natural number, chosen at random from the set $\{1, 2 .. N\}$ is divisible by a fixed natural no. k.

$N = q_n k + r_n \quad [0 \le r_n \le k-1]$

$k, 2k, 3k \dots q_N k$

$1 = \frac{q_n}{N} k + \frac{r_n}{N}, \quad N \to \infty$

r_n bounded, $\frac{r_n}{N} \to 0$

$1 = P_N k$

$\Rightarrow \boxed{P_N = 1/k}$

5) Two numbers x and y are chosen without replacement from the set $\{1, 2 \ldots N\}$. Find the prob. that is $|x-y| \geq m$, a fixed natural no.

If $x<y$

$x = 1$

$y = m+1, m+2 \ldots, N$

If $x=2$, $y = m+2, m+3 \ldots, N$

$x=3$, $y = m+3, m+4, \ldots N$

$x = N-m$, $y = N$

$(N-m) + (N-m-1) + \ldots + 1$

$= \frac{(N-m)(N-m-1)}{2}$

If $x>y$

Event points $= \frac{(N-m)(N-m+1)}{N(N-1)}$

6) A box contains 20 tickets of identical appearance, ticket numbered as $1, 2 \ldots 20$. If 3 tickets are drawn at random. Find the prob. that the nos. on drawn tickets are in A.P.

Range of differences: 1–9

If diff. = 9	1, 10, 19	2, 11, 20 $\rightarrow 2$		
" " = 8	1, 9, 17	2, 10, 18	3, 11, 19	4, 12, 20 $\rightarrow 4$

⋮

If diff = 1 1, 2, 3, 2, 3, 4 ... 18, 19, 20 $\rightarrow 18$

$$= \frac{2(1+2+3 \cdots 9)}{^{20}C_3} = \frac{9 \times 10}{^{20}C_3} = \frac{9 \times 10 \times 6}{20 \times 19 \times 18} = \frac{3}{38}$$

7) A group of $2n$ boys & $2n$ girls is divided at random into two equal batches. Find the prob. that each batch will be equally divided into boys and girls.

$$= \frac{^{2n}C_n \times {}^{2n}C_n}{^{4n}C_{2n}}$$

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⊛ $P(E_1 \cup E_2 \cup \cdots E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{i<j} P(E_iE_j) + \cdots$
$(-1)^{n+1} P(E_1E_2\cdots E_n)$

$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1E_2) - P(E_2E_3)$
$- P(E_1E_3) + P(E_1E_2E_3)$

Ex: Prove the Boole's Inequality:

$$P\left(\bigcup_{i=1}^{n} E_i\right) \le \sum_{i=1}^{n} P(E_i)$$

A seq^n of events $\{E_n, n \geq 1\}$. in monotonically increasing in.

$E_1 \subset E_2 \subset E_3 \cdots \subset E_n \subset E_{n+1} \cdots$ $\bigcup_{j=1}^{n} E_i = E_n$,

monotonically decr. $E_1 \supset E_2 \supset E_3 \cdots \supset E_n \supset E_{n+1}$

$\bigcup_{i=1}^{n} E_i = E_1$ | $\bigcap_{i=1}^{n} E_i = E_n$

If $\{E_n, n \geq 1\}$ is either an increasing or decreasing seq^n of events, then:

Prove that: $\lim_{n\to\infty} P(E_n) = P(\lim_{n\to\infty} E_n)$

$\Downarrow$

$\left(\bigcup_{i=1}^{\infty} E_i\right)$

Proof:

suppose the sequence is mono. incr.

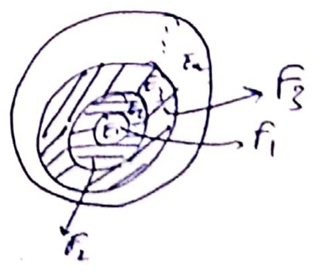
s.t $F_1 = E_1$

$F_2 = E_2 - E_1 = E_2E_1^c$

$F_3 = E_3 - E_2 = E_3E_2^c$

$\vdots$

$F_n = E_n - E_{n-1} = E_nE_{n-1}^c$



F_n is mutually exclusive.

$\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i = E_n$

$P(\lim_{n\to\infty} E_n) = P\left(\bigcup_{i=1}^{\infty} E_i\right) = P\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} P(F_i)$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} P(F_i)$$

$$= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} F_i\right)$$

$$= \lim_{n\to\infty} P(E_n)$$

Suppose the seq^n is mono. dec:

$\{E_n^c, n \geq 1\}$ seq. mono. inc. or inc.

$$P\left(\bigcup_{i=1}^{\infty} E_i^c\right) = \lim_{n\to\infty} P(E_n^c)$$

$$\text{as } \bigcup_{i=1}^{\infty} E_i^c = \left(\bigcap_{i=1}^{\infty} E_i\right)^c$$

$$P\left(\left(\bigcap_{i=1}^{\infty} E_i\right)^c\right) = 1 - P\left(\bigcap_{i=1}^{\infty} E_i\right)$$

$$= \lim_{n\to\infty} [1 - P(E_n)]$$

$$P\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} P(E_n)$$

$$\Rightarrow P\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} P(E_n)$$

$$P(A/B) = \frac{P(AB)}{P(B)}$$

$$P(E_i/A) = \frac{P(E_iA)}{P(A)} = \frac{\cancel{P(E_i/A)\cdot P(A)}}{} \quad \frac{P(A/E_i)P(E_i)}{\sum_{k=1}^{n} P(A/E_k)\cdot P(E_k)}$$

Q1) In a certain day clan, 30% of the children have grey eye 50% of them have blues and the other 20% eyes are in other clan.

One day they play a game together. In the first team 65% of the grey eyeds ones, 82% of the blue eyed ones and 50% of the children with other eye color were selected. Now if a child is selected randomly from the class and we know that he/she was not in the first game, what is the prob. that the child was blue eyed?

$P(G) = 0.3,\quad P(B) = 0.5,\quad P(O) = 0.2$

$P(S/G) = 0.65,\quad P(S/B) = 0.82,\quad P(S/O) = 0.5$

$P(\bar{S}/G) = 0.35 \qquad P(\bar{S}/B) = 0.18,\quad P(\bar{S}/O) = 0.5$

Sol

$$P(B/\bar{S}) = \frac{P(\bar{S}/B)\,P(B)}{\Sigma P(\bar{S}/B)\cdot P(B)}$$

$$= \frac{0.18 \times 0.5}{0.18 \times 0.5 + 0.35 \times 0.3 + 0.5 \times 0.2}$$

$$= \frac{0.09}{0.295}$$

$= \cancel{0.03} = \cancel{0.0}$

$= 0.305$

Q.2) The Gambler's Ruin Problem:

Two gamblers, A and B bet on the outcomes of successive flips of a coin on each flip. If the coin comes up heads, A collects from B 1 unit. If the coin " " tails, A pays to B 1.

They continue to do this untill one of them runs out of money.

If it is assumed that the successive flips of the coin are independent and each flip results in a head with prob. p. What is the prob. that A winds up with all the money if he starts with i units and B starts with $(N-i)$ units.

E = event that A winds up all the money when A starts with i units and B starts with $N-i$ units.

$P(H) = p$
$P(H^c) = q$
$p + q = 1$

$P(E) = P_i \qquad P(H)P(E/H) + P(H^c)P(E/H^c)$

$P(E) = P(H)P(E/H) + q\,P(E/H^c)$

$= p \cdot P(E/H) + q\,P(E/H^c)$

$P(E/H) = P_{i+1},\quad P(E/H^c) = P_{i-1}$

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$$(q+p)P_i = pP_{i+1} + qP_{i-1}$$

$$P_i = pP_{i+1} + qP_{i-1}$$

$$(p+q)P_i = pP_{i+1} + qP_{i-1}$$

$$pP_{i+1} - pP_i = qP_i - qP_{i-1}$$

$$(P_{i+1} - P_i) = q/p\,(P_i - P_{i-1})$$

$P_0 = 0$
$P_N = 1$

$$P_2 - P_1 = q/p\,(P_1 - P_0) = q/p\,(P_1)$$

$$P_3 - P_2 = q/p\,(P_2 - P_1)$$

$$= q/p \cdot q/p\,(P_1 - P_0) = (q/p)^2 P_1$$

$$P_i - P_{i-1} = (q/p)^{i-1} P_1$$

$$P_n - P_{n-1} = (q/p)^{n-1} P_1$$

$$P_i - P_1 = [q/p + (q/p)^2 + \dots + (q/p)^{i-1}]P_1$$

$$P_N = [1 + (q/p) + (q/p)^2 + \dots + (q/p)^{N-1}]P_1$$

$$P_i = \frac{1-(q/p)^i}{1-(q/p)} \cdot P_1, \quad \text{if } q/p \neq 1$$

$$= iP_1; \quad \text{when } q/p = 1$$

$$P_N = \frac{(1-(q/p)^N)}{(1-(q/p))} P_1 \quad \text{if } q/p \neq 1$$

$$P_1 = \frac{1-q/p}{1-(q/p)^N} \quad \text{as } P_N = 1$$

$$P_i = \frac{(1-(q/p)^i)}{(1-(q/p)^N)}, \quad \text{if } q/p \neq 1$$

when $q/p = 1$

$p_n = n p_1$

$p_1 = 1/n$

So, if $q/p = 1$

$p_i = i p_1 = i/n$

H/W/ ③ Ex: John starts with \$2 and $p = 0.6$. What is the prob. that John obtains a fortune of $N=4$, without going broke.

4) Independent trails consisting of flipping of a coin having prob p of coming up heads, are continually performed untill either a head occurs or a total no of n flips are made. Find the prob. of stop the game?

5) If 10 married couples are seated at random at a round table. Compute the prob. that no wife sits next to her husband.

each other.

$E_i \to$ ith couple sit next to each other.

$$P\left(\bigcup_{i=1}^{10} E_i\right) = \sum_{i=1}^{10} P(E_i) - \sum_{i<j}^{10} P(E_i E_j) + \dots + (-1)^{n+1} \sum_{i_1<i_2\cdots i_n} P(E_{i_1} E_{i_2} \cdots E_{i_n})$$

$$- P(E_1 E_2 \cdots E_{10})$$

$$\boxed{\text{Req}: 1 - P\left(\bigcup_{i=1}^{10} E_i\right)}$$

$$\sum P(E_i) = \frac{2! \times 18!}{19!} \times \binom{10}{1}$$

$$\sum P(E_i E_j) = \binom{10}{2} \cdot \frac{2!\, 2! \times 17!}{19!}$$

$$P\left(\bigcup_{i=1}^{10} E_i\right) = \binom{10}{1} \frac{2!\, 18!}{19!} - \binom{10}{2} 2^2 \frac{17!}{19!} + \binom{10}{3} 2^3 \frac{16!}{19!} \cdots - \binom{10}{10} 2^{10} \frac{9!}{19!}$$

6) 3 balls are to be randomly selected without replacement from an urn containing 20 balls numbered 1 through 20. If we bet that atleast one of the drawn balls has a number as large or larger than 17, what is the prob. that we win the bet.

$K \to K \to$ largest no selected, $P(K=20) = \binom{19}{2} \Big/ \binom{20}{3}$, $P(K=19) = \dfrac{\binom{18}{2}}{\binom{20}{3}}$, $P(K=18) = \dfrac{\binom{17}{2}}{\binom{20}{3}}$

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$$P(X=17) = \frac{\binom{16}{2}}{\binom{20}{3}}$$

7) Three balls are randomly chosen from an urn containing 3 white, 3 red and 5 black balls. Suppose that we win \$1 for each white ball selected and lose \$1 for each red ball selected. If we let X denote our total winnings from the exp. then X is random variable taking on the possible values 0, ±1, ±2, ±3 with resp. prob. Find the prob. that we win money.

$$P(X=0) = \frac{{}^5C_3 + {}^3C_1 \cdot {}^3C_1 \cdot {}^5C_1}{{}^{11}C_3}$$

$$P(X=1) = \frac{{}^3C_2 \cdot {}^3C_1 + {}^3C_1 \cdot {}^5C_2}{{}^{11}C_3}$$

$$P(X=-1) = \frac{\binom{3}{2}\binom{3}{1} + \binom{3}{1}\binom{5}{2}}{{}^{11}C_3} = P(X=1)$$

$$P(X=2) = \frac{\binom{3}{2}\binom{5}{1}}{\binom{11}{3}} = P(X=-2)$$

$$P(X=3) = \frac{\binom{3}{3}}{\binom{11}{3}}$$

$$\text{Req.} = \sum_{i=1}^{3} P(X=i) = ?$$

→ Cont. in Copy 2.

1) A laboratory blood test is 95% effective in detecting a certain disease when it is infact present. However the test also yields a false positive for 1% of the healthy person tested. If the 0.5% of the population actually has the disease. What is the prob. a person has the disease given that his test result is +ve?

D → Person has disease, E → Test result is +ve.

$P(D) = 0.95$ | $P(\bar{D}) = 0.105$

$P(+/D) = 0.01$ | $P(P/\bar{D}) = 0.99$

$P(D/P) = \frac{P(P/D)\cdot P(D)}{\Sigma P(P/D)P(D)} = \frac{0.95 \times 0.01}{0.95\times0.01 + 0.05\times0.99}$

$= 0.161$

$P(E/D) = 0.95$

$P(D) = 0.005$

$P(E/\bar{D}) = 0.001$

$P(\bar{D}) = 0.995$

$P(D/E) = \frac{0.95 \times 0.005}{(0.95\times0.005 + 0.995\times0.01)} = 0.323$

2) Balls numbered 1 through 20 are placed in a bag. Three balls are drawn out of the bag without replacement. What is the prob. that all the balls have odd numbers on them?

$= {}^{10}C_3 / {}^{20}C_3 = 2/19.$

3) A family has 2 children. Given that one of the children is boy. What is the prob. that both children are boys?

H/W

$$P\left(\frac{BB}{B}\right) = \frac{P(BB \cap B)}{P(B)} = \frac{1/4}{3/4} = 1/3$$

$\{BB, BG, GB, GG\}$

Random Variable

If corresponding to every point U of an event space S, a unique real value of $X = X(U)$. Then X is called a random or stochastic variable.

The range of the fn. X, i.e the set of all values which it takes will be called the spectrum of r.v.

The spectrum may be descrete or continous accordingly r.v is said to be discrete or continous.

1) $P(-\infty < X < \infty) = 1$

2) Consider the random exp. of throwing a coin,

Suppose $X = 0 \rightarrow$ Denotes event of tail and

$X = 1 \rightarrow$ " " " head.

then $P(X=0) = 1/2 = P(X=1)$

$(-\infty < X < \infty) = (X=0) + (X=1) = 1/2 + 1/2 = 1$

H/W Consider Bernoulli's seq. of 3 trails. Then the event space contains 2^3 points.

For tossing a coin the event space contains $2^3 = 8$ points.

(X is the random variable) $\rightarrow$ number of successes occuring head is the

$p(X=0) = q^3$

$P(X=1) = 3pq^2$

$p(X=2) = 3p^2q$

$p(X=3) = p^3$

$\therefore (-\infty < x < \infty) = \sum_{i=0} P(X=i) = (p+q)^3 = 1$

3) If a ticket is drawn at random from an urn containing n tickets numbered 1, 2 ... n and X denotes the number of tickets drawn,

Find $P(X = i) = 1/n$.

4) Let r tickets be drawn successively with replacement from an urn containing n tickets, numbered 1, 2 .. n. If X denotes the greatest number drawn, then the spectrum of r.v consists of points 1, 2 .. n. then find.

Prob. that max. number = i after r draws. Ans = $\left(\frac{i^r}{n^r} - \frac{(i-1)^r}{(n)^r}\right)$

5) If balls are successively drawn without replacement from an urn containing N_1 white & N_2 black balls ($N = N_1 + N_2$). Then the no. of black balls preceding the first white ball is a r.v which can take values 0, 1, 2 ... N_2

$$\text{then } P(X = i) = \frac{N_1}{N} \cdot \frac{N_2}{(N-1)} \cdot \frac{N_2 - 1}{(N-2)} \cdots \frac{(N_2 - i + 1)}{(N - i)}$$

$$P(X = 0) = \frac{N_1}{N_1 + N_2}$$

~~P(X = i) =~~

6) If n balls are drawn without replacement from an urn containing N_1 white and N_2 black balls $(n \leq N_1 + N_2)$, then the number of white balls among the balls drawn is r.v x, then

$$P(x = i) = ? \quad \binom{N_1}{i}\binom{N_2}{n-i} \Big/ \binom{N}{n}$$

for the boundary fi

$0 \leq i \leq N_1$

$0 \leq n - i \leq N_2$

$n - i \leq N_2$

$i \geq n - N_2$

$$\boxed{\max(0, n - N_2) \leq i \leq \min(N_1, n)}$$

Spectrum of x

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Distribution func.

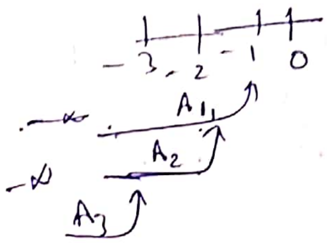
$F(x) = P(-\infty < X \le x)$

$(-\infty < X \le a) + (a < X \le b) = (-\infty < X \le b)$

$P(-\infty < X \le a) + P(a < X \le b) = P(-\infty < X \le b)$

$F(a) + P(a < X \le b) = F(b)$

$\Rightarrow P(a < X \le b) = F(b) - F(a)$



as $P(a < X \le b) \geq 0$

$\Rightarrow F(b) \geq F(a)$

F is increasing fn.

Let A_n denote the event $-\infty < X \le -n$

then the ~~seq~~ seqn. $\{A_n\} \rightarrow$ decreasing sequence

$\lim_{n \to \infty} A_n = 0$

$P(\lim_{n \to \infty} A_n) = P(0) = 0$

$P(A_n) = P(-\infty \le X \le -n) = F(-n)$

$\lim P(A_n) = P(\lim A_n)$

$\lim F(-n) = P(0) = 0$

$\Rightarrow \boxed{F(-\infty) = 0} \rightarrow$ 2

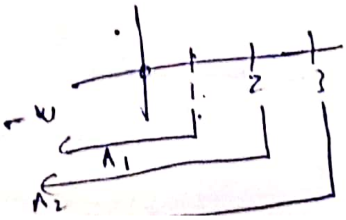
$\Rightarrow$ Let A_n denote the event $-\infty < x \le n$,

So, $\{A_n\}$ is increasing seqn.

$\lim A_n = (-\infty < n < \infty)$

Hence $P(\lim A_n) = \lim P(A_n) = \lim (F(n))$

$\Rightarrow \lim F(n) = P(-\infty < X < \infty) \Rightarrow \boxed{F(\infty) = 1} \rightarrow$ 3



For any fixed point a

take $A_n = (a - 1/n < x \le a)$; $n = 1, 2 \dots$

So, A_n is decreasing seqn.

& $\lim A_n = (x = a)$

$P(\lim A_n) = \lim P(A_n)$

$\Rightarrow P(x = a) = \lim [F(a) - F(a - 1/n)]$

$= F(a) - F(a-0) \rightarrow (4)$

Take $A_n = (a < x \le a + 1/n)$

$\{A_n\}$ contracting seqn $(n = 1, 2 \dots)$

$\lim A_n = 0$

$P(\lim A_n) = P(0) = 0$

$P(\lim A_n) = \lim P(A_n)$

$= \lim (F(a + 1/n) - F(a))$

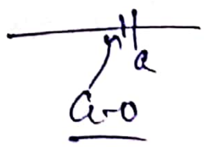
$\Rightarrow F(a+0) - F(a) = 0$

$\Rightarrow \boxed{F(a+0) = F(a)} \rightarrow (5)$

1) $F(x) = \begin{cases} 0 & -\infty < x < 0 \\ 1/5 & 0 \le x < 1 \\ 3/5 & 1 \le x < 3 \\ 1 & 3 \le x < \infty \end{cases}$

Find $P(X=0)$, $P(X=1)$, $P(X=3)$.

$\lim_{n \to 0} P(n \le x < 0) = \lim_{n \to 0} F($



$P(x = 0) = F(a) - F(a - 0)$

$= F(0) - F(0-0)$

$= 1/5 - 0 = 1/5$

$P(x = 1) = 3/5 - 1/5 = 2/5$

$P(x = 3) = 1 - 3/5 = 2/5$

Distribution

Discrete

$x = 0, 1, 2 \ldots$

Probability Mass function:

$f_i = p(X = x_i)$

→ Binomial Distribution:

$x_i = i$

$$f_i = p(X = x_i) = \binom{n}{i} p^i q^{n-i}, \quad q + p = 1$$

$\quad \& \; 0 < p < 1$

→ Poison Distribution

$$f_i = \frac{e^{-\mu} \cdot \mu^i}{i!}$$

Continuous

$x = (a, b]$

$[a, b]$

$(0, \infty)$

$[-\infty, 0)$

Probability density func:

$\cancel{f_i = p} \; f(x)$.

Cont. Distribution

$$p(x = a) = F(a) - F(a - 0)$$

$$p(a < x \leq b) = \int_a^b F'(x)\, dx = F(b) - \underbrace{F(a)}_{\text{Mass func.}}$$

$$= \int_a^b \underbrace{f(x)}_{\text{density func}} dx$$

$$\boxed{F'(x) = f(x)}$$

$$\boxed{F(x) = \int_{-\infty}^{x} f(x)\, dx}$$

Q) $f(x) = kx(1-x), \quad 0 < x < 1$

$= 0$ elsewhere

Find the value of k &

Find the distribution function. Find $P(X > 1/2)$

$$\int_{-\infty}^{x} f(x) = \int_{-\infty}^{x} kx(1-x)$$

$$= k \int_0^1 x(1-x) + \int_1^x x(1-x)$$

$$\int_{-\infty}^{\infty} f(x) = 1$$

$$= k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \left(\frac{1}{2} - \frac{1}{3}\right)k = 1$$

$$\Rightarrow k = 6$$

$$\therefore F(x) = \int_{-\infty}^{x} = \int_0^x (\cdot) = 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)$$

$$= 3x^2 - 2x^3 \quad (0 < x < 1)$$

$$P(x > 1/2) = \int_0^\infty f(x)\,dx$$

$$= \int_0^1 f(x)\,dx = \int_{1/2}^{1} (3x^2 - 2x^3)$$

$$= \frac{3x^3}{3} - (2\cdot$$

$$= 6\int_{1/2}^{1} x(1-x)$$

$$= 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_{1/2}^{1}$$

$$= 6\left(\frac{1}{6} - \left(\frac{1}{8} - \frac{1}{24}\right)\right)$$

$$= 6\left(1/6 - \frac{2}{24\,12}\right)$$

$$= 1 - 1/2 = 1/2$$

3) Q) $F(x) = 1 - e^{-x}$, $0 \le x < \infty$

Show that $F(x)$ is a possible distribution func.

Find its density func.

$f(x) = F'(x) = e^{-x}$

$F(-\infty) = 0$

$F(\infty) = 1 - e^{-\infty} = 1$

$\int_0^\infty f(x) = 1$

$\int_0^\infty e^{-x} = \left[\frac{e^{-x}}{-1}\right]_0^\infty$

$= 1$

Regular / Uniform Distribution $\longleftrightarrow$ dist

density

$f(x) = 0, \quad -\infty < x < a$

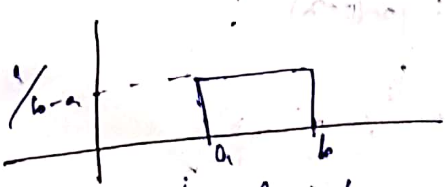
$= \frac{1}{b-a}, \quad a \le x \le b$

$= 0, \quad b < x < \infty$

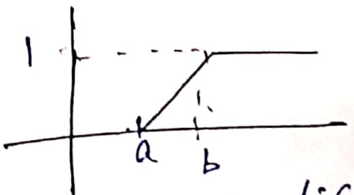
$F(x) = 0 \quad -\infty < x < a$

$= \frac{x-a}{b-a} \quad a \le x \le b$

$= 1 \quad b < x < \infty$



Regular density func.



Regular distribution function.

Q) A point X is chosen at random in the interval $a \le x \le b$, in such a way that the probability it lies in any subinterval is proportional to the length of the subinterval. Then, show that X is uniformly distributed over the interval (a, b).

Proof: Let us construct the distribution func. from the problem. $F(x) = 0$. $-\infty < x < a$.

$F = \lambda(x-a)$ $\quad a \le x \le b$ (λ is a constant)

$= 1$. $\quad b < x < \infty$

To find λ, we use R.C.S property:

$F(b+0) = F(b)$

$\therefore 1 = \lambda(b-a)$

$\Rightarrow \lambda = \frac{1}{(b-a)}$

$f(x) = F'(x)$ $\quad a < x < b$

$= \lambda$

$= \frac{1}{b-a}$ $\quad a < x < b$

$\therefore$ x is uniformly distributed over the interval $[a, b]$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\Omega} e^{-(x-m)^2/2\Omega^2}, \quad -\infty < x < \infty$$

$m, \Omega \to$ parameters.

mean $= m$

Variance $= \Omega^2$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sqrt{2\pi}\,\Omega}\int_{-\infty}^{\infty} e^{-\frac{(x-m)^2}{2\Omega^2}}dx$$

$= 1$ (To prove)

$(x-m)^2 = t$

$x = m + \sqrt{t}$

$\int e^{-x^2} dx$

$x = \sqrt{t}$

$dx = \frac{1}{2\sqrt{t}}dt$

$\int \frac{e^{-t}\,t}{2\sqrt{t}}\,dt$

$= e^{-t}\sqrt{t} + \int e^{-t}\sqrt{t}$

$\frac{1}{2\sqrt{t}} e^{-t}$ $\quad \int \frac{1}{2\sqrt{t}} e^{-t}$

$\int e^{-x^2} dx$ $\quad \int e^{-x} \cdot x^{1/2}$

$= \frac{x-m}{\sqrt{2}\,\Omega} = z$

$\Rightarrow \frac{dx}{\sqrt{2}\,\Omega} = dz$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}\,dz = \frac{2}{\sqrt{\pi}}\int_{0}^{\infty} e^{-z^2}\,dz$$

$$= \frac{2}{\sqrt{\pi}}\int_{0}^{\infty} e^{-w}\cdot w^{-1/2}\,dw \longrightarrow \cancel{\beta}(\tfrac{1}{2})\,\Gamma(\tfrac{1}{2})$$

$$= \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

When $m=0,\ \Omega=1$,

$\phi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2}, \quad -\infty < x < \infty$

Standard Normal Distribution

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-x^2/2}\,dx$$ mean $= 0$, Var $= 1$

~~Cau~~

Cauchy Dist.

$$f(x) = \frac{1}{\pi}\cdot\frac{\lambda}{\lambda^2+(x-\mu)^2}, \quad -\infty < x < \infty$$

$(\lambda > 0), \mu$ parameters

$$F(x) = \frac{\lambda}{\mu}\int_{-\infty}^{x}\frac{dx}{\lambda^2+(x-\mu)^2}$$

Normal Gamma Dist.

$$f(x) = \frac{e^{-x}\,x^{l-1}}{\Gamma(l)}, \quad 0 < x < \infty$$

$= 0$ elsewhere.

$$\Gamma(l) = \int_0^\infty e^{-x} x^{l-1} dx$$

So, $\int_{-\infty}^{\infty} f(x) dx$

$$= \int_0^\infty f(x) \, dx$$

$$= \frac{1}{\Gamma(l)} \cdot \int_0^\infty e^{-x} x^{l-1} dx = \frac{\Gamma(l)}{\Gamma(l)} = 1$$

Beta dist. of 1st kind:

Denoted by: $\beta(l, m)$

$$f(x) = \frac{x^{l-1}(1-x)^{m-1}}{\beta(l, m)} \quad 0 < x < 1$$

$$= 0 \quad \text{elsewhere}$$

$$\int_{-\infty}^{\infty} f(x) \, dx = \frac{1}{\beta(l, m)} \int_0^1 x^{l-1}(1-x)^{m-1} dx$$

$$= 1$$

Beta dist. of 2nd kind:

$\beta_2(l, m)$

$$f(x) = \frac{x^{l-1}}{\beta(l, m)(1+x)^{l+m}} \quad 0 < x < \infty \quad (l, m > 0)$$

Another form of $\beta(l, m)$:

$$\beta = \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}} dx$$

$$\int_{-\infty}^{\infty} f(x) dx = \int_0^\infty f(x) \, dx$$

$$= \int_0^\infty \frac{x^{l-1}}{\beta(l, m)(1+x)^{l+m}} dx$$

$$= \frac{1}{\beta(l, m)} \cdot \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}$$

$$= \frac{\beta(l, m)}{\beta(l, m)} = 1$$

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Transformation of Random Variable

R.V → X,

Dens. fun : $f_x(x)$

Dist. func. : $F_x(x)$

Given $y = g(x)$,

Using $f_x(x), F_x(x)$, how shall we ~~transfor~~ get :

$f_y(y), F_y(y)$?

$y = g(x) \rightarrow$ ① Cont., ② diff (neccesary) for conversion.

③ Inverse $x = g^{-1}(y)$ should exist.

$(y = g(x) \rightarrow$ strictly monotone)

$\frac{dy}{dx} > 0$ (inc.)

$\frac{dy}{dx} < 0$ (dec.

Case 1: $\frac{dy}{dx} > 0$

$y = g(x) \rightarrow$ Cont. strictly monotonically inc. func.

$x = g^{-1}(y)$ exists

Hence $(X \leq x) \Rightarrow \{g(X) \leq g(x)\}$

$\Rightarrow Y \leq y$

So, the events $X \leq x$ & $Y \leq y$ are identical.

$\cancel{P(X)} = P(X \leq x) = P(Y \leq y)$

$\therefore F_x(x) = F_y(y)$

$d F_x(x) = d F_y(y)$

$$\Rightarrow f_x(x)\,dx = f_y(y)\,dy$$

$$f_x(x)\frac{dx}{dy}\,dy = f_y(y)\,dy$$

$$\Rightarrow f_y(y) = f_x(x)\frac{dx}{dy}$$

Case II

$\frac{dy}{dx} < 0$, $y = g(x) =$ Cont. str. mono. decreasing func.

$$(X \le x) = \{g(X) \geq g(x)\}$$
$$= \{Y \geq y\}$$

$$\Rightarrow P(X \le x) = P(Y \geq y)$$
$$= 1 - P(Y < y)$$

$$F_x(x) = 1 - F_y(y)$$
$$dF_x(x) = -dF_y(y)$$
$$\Rightarrow f_x(x)\,dx = -f_y(y)\,dy$$
$$\Rightarrow f_x(x)\cdot\frac{dx}{dx}\cdot dy = -f_y(y)\,dy$$
$$\Rightarrow f_y(y) = -f_x(x)\frac{dx}{dy}$$

ie for both the cases

$$\boxed{f_y(y) = f_x(x)\left|\frac{dx}{dy}\right|}$$

Discrete Case

$$Y_i = g(x_i)$$
$$(X = x_i) = \{g(X) = g(x_i)\} = (Y = y_i)$$
$$P(X = x_i) = P(Y = y_i) \Rightarrow \boxed{f_{x_i} = f_{y_i}}$$

Q) R.V X is normal (m, Ω). Find the dist. $Y = aX + b$ where a, b = Consts..

$$y = ax + b \rightarrow x = \frac{y-b}{a} \quad \begin{vmatrix} -\infty < y < \infty \\ -\infty < x < \infty \end{vmatrix}$$

$$\frac{dy}{dx} = a$$

$$\left|\frac{dx}{dy}\right| = \left|\frac{1}{a}\right|$$

$$f_x(x) = \frac{1}{\sqrt{2\pi}\cdot\Omega} e^{-(x-m)^2/2\Omega^2}$$

$$\therefore f_y(y) = f_x(x) \cdot \left|\frac{dx}{dy}\right|$$

$$= \frac{1}{\sqrt{2\pi}\,\Omega|a|} \cdot e^{-\left(\frac{y-b}{a} - m\right)^2/2\Omega^2}$$

$$= \frac{1}{\sqrt{2\pi}\,\Omega|a|} \cdot e^{-(y-(b+am))^2/2a^2\Omega^2}$$

y is $N(am+b, |a|\Omega)$

Q) Suppose X is $N(m, \Omega)$ show that $Z = \frac{X-m}{\Omega}$ is $N(0,1)$

$$f_x(x) = \frac{1}{\sqrt{2\pi}\,\Omega}$$

Q) If x is a $\beta_2(l, m)$ variate, then show that $y = 1/x$ is a $\beta_2(m, l)$ variate.

Proof: $y = 1/x$,

$$\frac{dy}{dx} = -1/x^2$$

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$$dF_x(x) = f_x(x)\,dx = f_x(x)\cdot\left|\frac{dx}{dy}\right|dy$$

$$= \frac{x^{l-1}\cdot x^2}{\beta(l,m)\,(1+x)^{l+m}}\,dy$$

$$= \frac{y^{m-1}}{\beta(m,l)(1+y)^{m+l}}\,dy, \quad 0<y<\infty$$

Q) If X is $N(0,1)$. then prove that $Y = \frac{1}{2}X^2$, is a $\gamma(1/2)$ variate.

Proof: Let $y = \frac{x^2}{2}$, $\frac{dy}{dx} = x$,

So, here Y is not strictly inc. every where

as $-\infty < x < \infty$,

but $0 < y < \infty$

So,

$$(y < Y \le y+dy) = \{x^2 < X^2 \le (x+dx)^2\}$$

$$= (-x-dx < x \le -x) + (x < X \le x+dx)$$

$$P(y < Y \le y+dy) = P(-x-dx < X \le -x) + P(x < X \le x+dx)$$

$$= 2P(x < X \le x+dx)$$

$$\Rightarrow f_y(dy) = 2f_x(x)\,dx = 2f(x)\frac{dx}{dy}\,dy \;\cancel{= f_y(y) = 2}$$

$$\Rightarrow 2f_x(x)\cdot\frac{dx}{dy}$$

$$f_x(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \quad f_y(y) = 2f_x(x)\cdot\frac{dx}{dy}$$

$$= \frac{2}{x}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

from, $Y = \frac{x^2}{2}$

$$f_Y(y) = \frac{e^{-y}\cdot y^{-1/2}}{\Gamma(1/2)} \quad 0<y<\infty$$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$,

So, Y is a $\gamma(\frac{1}{2})$ Variate.

Q) If X is a binomial (n,p) variate

$$Y = aX+b$$

$$x_i = i$$

So, spectrum of Y is given by $y = ai+b$

$$f_{y_i} = f_{y_i} = f_{x_i} = \binom{n}{i} p^i q^{n-i}$$

Q) If X is uniformly distributed over the interval $(-2,2)$. Find the dist. of the r.v. $Z = \min\{X, 1\}$.

(Ans. after 8 pages)

If two variables used:

$$F(x,y) = P(-\infty < X \leq x, -\infty < Y \leq y)$$

$$F(x_1, x_2, x_3 \ldots x_n) = \underbrace{P(-\infty < X \leq x_1, -\infty < X \leq x_2, -\infty < X \leq x_3 \ldots -\infty < X_n \leq x_n)}$$

N-dimensional Distribution Function

(multi- ")

1) $F(x,y)$ is monotonically non-decreasing in both the variables x and y.

Proof: Let y be kept fixed, let $x_2 > x_1$

Then $(-\infty < X \leq x_2, -\infty < Y \leq y)$

$$= (-\infty < X \leq x_1, -\infty < Y \leq y) + (x_1 < X \leq x_2, -\infty < Y \leq y)$$

$P(-\infty < X \leq x_2, -\infty < Y \leq y)$

$= P(-\infty < X \leq x_1, -\infty < Y \leq y) + P(x_1 < X \leq x_2, -\infty < Y \leq y)$

$\Rightarrow F(x_2, y) = F(x_1, y) + P(x_1 < X \leq x_2, -\infty < Y \leq y)$

$\Rightarrow F(x_2, y) - F(x_1, y) = \underline{P(x_1 < X \leq x_2, -\infty < Y \leq y)} \geq 0$

$\therefore F(x_2, y) \geq F(x_1, y)$

Let x be kept fixed & $y_2 > y_1$

$(-\infty < X \leq x, -\infty < Y \leq y_2)$

$= (-\infty < X \leq x, -\infty < Y \leq y_1) + (-\infty < X \leq x, y_1 < Y \leq y_2)$

$\Rightarrow F(x, y_2) - F(x, y_1) = \underbrace{P(-\infty < X \leq x, y_1 < Y \leq y_2)}_{\geq 0}$

$\therefore F(x, y_2) \geq F(x, y_1)$

Prop-2

$0 \leq F(x,y) \leq 1, \; \forall x$ and y

as $0 \leq P(-\infty < X \leq x, -\infty < Y \leq y) \leq 1$

Prop

3) $P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = F(x_2, y_2) + F(x_1, y_1) - F(x_1, y_2) - F(x_2, y_1)$

Proof:

(*) $(-\infty < X \leq x_2, -\infty < Y \leq y_2)$

$= (-\infty < X \leq x_1, -\infty < Y \leq y_2) + (x_1 < X \leq x_2, -\infty < Y \leq y_2)$

$\Rightarrow F(x_2, y_2) = F(x_1, y_2) + P(x_1 < X \leq x_2, -\infty < Y \leq y_2)$

$\Rightarrow F(x_2, y_2) - F(x_1, y_2) = P(x_1 < X \leq x_2, -\infty < Y \leq y_2)$

(*) $(-\infty < X \leq x_2, -\infty < Y \leq y_1)$

$= (-\infty < X \leq x_1, -\infty < Y \leq y_1) + \cancel{(x_1 < X \leq x_2, y_1 < Y \leq y_2)}$

$+ (x_1 < X \leq x_2, -\infty < Y \leq y_1)$

$\Rightarrow F(x_2, y_1) = F(x_1, y_1) + P(x_1 < X \leq x_2, -\infty < Y \leq y_1)$

(*) $(x_1 < X \leq x_2, -\infty < Y \leq y_2)$

$= (x_1 < X \leq x_2, -\infty < Y \leq y_1) + (x_1 < X \leq x_2, y_1 < Y \leq y_2)$

$\Rightarrow F(x_2, y_2) - F(x_1, y_2) = F(x_2, y_1) - F(x_1, y_1) + P(x_1 < X \leq x_2, y_1 < Y \leq y_2)$

$\Rightarrow P(x_1 < X \leq x_2, y_1 < Y \leq y_2)$

$= F(x_2, y_2) + F(x_1, y_1) - F(x_1, y_2) - F(x_2, y_1)$. $\square$

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Prop-4

$F(\infty,\infty) = 1$. $F(-\infty, Y) = 0$, $F(X, -\infty) = 0$

Marginal Distribution (expr. of single dim. from multidim. func.)

$F(x, y)$ is given,

$F_x(x) = P(-\infty < X \leq x) = P(-\infty < X \leq x; -\infty < Y \leq \infty)$
$= F(x, \infty)$

$F_y(y) = P(-\infty < Y \leq y) = P(-\infty < X \leq \infty, -\infty < Y \leq y)$
$= F(\infty, y)$

→ When two r.v X & Y are independent:

$F(x, y) = F_x(x) F_y(y)$. $\boxed{\text{iff}}$

→ When x & y are independent:

$P(a < X \leq b, c < Y \leq d)$

$= P(a < X \leq b) P(c < Y \leq d)$

$\Rightarrow F(b,d) + F(a,c) - F(b,c) - F(a,d)$

$= (F(b) - F(a))(F(d) - F(c))$

Marginal distribution Discrete r.v in 2 dimension

$f_i = P(X = x_i)$ $\quad f_{ij} = P(X = x_i, Y = y_j)$

$$\sum_i \sum_j f_{ij} = 1$$

Marginal

$f_{ij} = P(X = x_i, Y = y_j)$ is given. $P(X = x_i)$

$P(X = x_i)$	$P(Y = y_j)$
$= \sum_{j=-\infty}^{\infty} f_{ij}$	$= \sum_{i=-\infty}^{\infty} f_{ij}$
$= f_i = f_{xi}$	$= f_j = f_{yj}$
denoted by	denoted by

$$F(x,y) = \sum_{(\alpha,\beta)\in B'}\sum f_{\alpha\beta} \quad \text{if } x_i \le x < x_{i+1}$$
$$y_j \le y < y_{j+1}$$

$$B' = \{(\alpha,\beta) ;\ x_\alpha \le x_i ,\ y_\beta \le y_j\}$$

$$F_x(x) = F(x,\infty) = \sum_{x_i \le x} \sum_{j=-\infty}^{\infty} f_{ij}$$
$$= \sum_{x_i \le x} f_i = \sum_{x_i \le x} f_{x_i}$$

$$F_y(y) = F(\infty, y) = \sum_{i=-\infty}^{\infty} \sum_{y_j = y} f_{ij} = \sum_{y_j \le y}\left(\sum_{i=-\infty}^{\infty} f_{ij}\right) = \sum_{y_j \le y} f_j$$
$$= \sum_{y_j \le y} f_{y_j}$$

$$\sum_{i=-\infty}^{\infty} f_i = \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} f_{ij} = 1$$

$$\sum_{j=-\infty}^{\infty} f_j = \sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} f_{ij} = 1$$

Q) The iff condition for two discrete r.v X and Y to be independent is that $f_{ij} = f_i \cdot f_j = f_{x_i} f_{y_j}$

Proof: Let X & Y are independent.

$$P(X = x_i, Y = y_j) = P(X = x_i) P(Y = y_j)$$
$$\Rightarrow f_{ij} = f_i \cdot f_j = f_{x_i} f_{y_j}$$

Conversely let $f_{ij} = f_i f_j$

$$F(x,y) = \sum_{(\alpha,\beta)\in B'}\sum f_{\alpha\beta} \quad \text{if } x_i \le x < x_{i+1}$$
$$y_j \le y < y_{j+1}$$

$$= \sum_{(\alpha,\beta)\in B'}\sum f_\alpha f_\beta \qquad B' = \{(\alpha,\beta) :\ x_\alpha \le x_i ,\ y_\beta \le y_j\}$$

$$= \left(\sum_{x_\alpha \le x_i} f_\alpha\right)\left(\sum_{y_\beta \le y_j} f_\beta\right) = F_x(x) \cdot F_y(y)$$

So, X & Y are independent r.v.

Q) The joint distribution func. of X & Y are given by the following table:

X \ Y	1	2	3	
0	0.1	0.3	0.1	0.5
2	0.2	0.1	0.2	0.5
	0.3	0.3	0.3	1

$X = 0, 2$

$Y = 1, 2, 3$

$P(X=0, Y=1) = f_{01} = 0.1$

$f_{02} = 0.3$

$f_{03} = 0.1$

$f_{21} = 0.2$

$f_{22} = 0.1$

$f_{23} = 0.2$

$(-\infty < x < \infty)$

$\hookrightarrow x = 0, 2$

Find $F(2,2)$, $F(0,2)$, $F(2,3)$

Find $F(0.5, 2.1) = 0.4$

Find $f_{x0}, f_{x2}, f_{y1}, f_{y2}, f_{y3}$

& X & Y are independent or not?

$F(2,2) = 0.7 \quad F(0,2) = 0.4$

$F(2,3) = 1$

$$\boxed{\sum_{i=-\infty}^{\infty} f_{x_i} = 1 \qquad \sum_{j=-\infty}^{\infty} f_{y_j} = 1}$$

$f_{x0} = 0.5 \quad = f_{01} + f_{02} + f_{03}$

$f_{x2} = 0.5$

$f_{y1} = 0.3 \quad = f_{01} + f_{21}$

$f_{y2} = 0.4$

$f_{y3} = 0.3$

To prove ind.

$F(x, y) = F_x(x) \cdot F_y(y)$

$f(x, y) = f_{x_i} f_{y_j}$

$f_{02} = 0.3 \qquad f_{x0} = 0.5$

$f_{y2} = 0.4$

$f_{02} \neq f_{x0} \cdot f_{y2} = 0.02$

$\therefore$ Not independent.

Continuous 2-dim r.v

Suppose $F(x, y)$ diff.

$$\int_{y=b}^{d}\int_{x=a}^{c} \frac{\partial^2 F}{\partial x \partial y}\, dx\, dy$$

$$= \int_{y=b}^{d} \left(\frac{\partial F}{\partial y}\Big|_{x=c} - \frac{\partial F}{\partial y}\Big|_{x=a} \right) dy$$

$$= F(c, d) - F(c, b) - [F(a, d) - F(a, b)]$$

$$= F(c, d) + F(a, b) - F(c, b) - F(a, d)$$

$$= P(a < X \leq c,\ b < Y \leq d)$$

$$f(x, y) = \frac{\partial^2 F}{\partial x \partial y}$$

$$\Rightarrow P(a < x \leq c,\ b < Y \leq d) = \int_b^d \int_a^c \frac{\partial^2 F}{\partial x \partial y}\, dx\, dy$$

$$= \int_b^d \int_a^c f(x, y)\, dx\, dy$$

Marginal Distribution Function

$$F_x(x) = P(-\infty < X \leq x, -\infty < Y < \infty)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{x} f(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{x} \left(\int_{-\infty}^{\infty} f(x, y)\, dy \right) dx$$

$$= \int_{-\infty}^{x} f_x(x)\, dx \quad \text{where } f_x(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

marginal density func.

$$F_y(y) = P(-\infty < X < \infty, -\infty < Y \leq y)$$

$$= \int_{-\infty}^{y} \left(\int_{-\infty}^{\infty} f(x, y)\, dx \right) dy = \int_{-\infty}^{y} f_y(y)\, dy.$$

$$f_y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

Cont. 2 dim r.v

X, Y are ind.

iff $f_x(x) \cdot f_y(y) = f(x, y)$

$F_x(x) \cdot F_y(y) = F(x, y)$

Prop
$$F_x'(x) = f_x(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$
$$F_y'(y) = f_y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

Q iff condition for X & Y ind. is that $f(x,y) = f_x(x) f_y(y)$

Proof: Let X and Y be ind.

$$F(\cancel{x}, y) = F_x(x) F_y(y)$$
$$\frac{\partial F}{\partial x} = F_x'(x) \cdot F_y(y)$$
$$\frac{\partial^2 F}{\partial y \partial x} = F_x'(x) \cdot F_y'(y) = f_x(x) f_y(y)$$
$$\Rightarrow f(x, y) = f_x(x) \cdot f_y(y)$$

Conversely:

Let $f(x, y) = f_x(x) \cdot f_y(y)$
$$\int_{-\infty}^{y}\int_{-\infty}^{x} f(x, y)\, dx \cdot dy = \int_{-\infty}^{y}\int_{-\infty}^{x} f_x(x) f_y(y)\, dx\, dy$$
$$= \left(\int_{-\infty}^{y} f_y(y)\, dy\right) \cdot \int_{-\infty}^{x} f_x(x)\, dx = F_Y(y) F_x(x)$$

So, X & Y independent.

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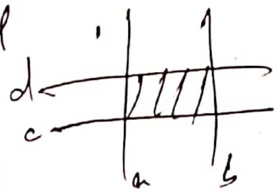
Imp. Distⁿˢ

$f(x, y) = \frac{1}{R}$, where $(x, y) \in Q$ } R is the area of the region Q
elsewhere:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dx\,dy = \iint_{Q} f(x, y)\,dx\,dy = 1$$

If region is rect: $a < x < b,\ c < y < d$



$$\therefore f(x, y) = \frac{1}{(b-a)(d-c)},\quad \begin{matrix} a<x<b \\ c<y<d \end{matrix}$$

$= 0$ elsewhere

Rectangular Distribution.

If Q' be the sub-region of Q having area R'

then $P\{(x, y) \in Q'\} = \frac{R'}{R}$

$$= \iint_{Q'} f(x, y)\,dx\,dy$$

$$f_x(x) = \int_{-\infty}^{\infty} f(x, y)\,dy = \int_{c}^{d} \frac{1}{(b-a)(d-c)}\,dy = \frac{1}{b-a}$$

$$f_y(y) = \int_{-\infty}^{\infty} f(x, y)\,dx = \int_{a}^{b} \frac{1}{(b-a)(d-c)}\,dx = \frac{1}{d-c}$$

$f_x(x) = \frac{1}{b-a}$, $a < x < b$

0, elsewhere

$f_y(y) = \frac{1}{d-c}$, $c < y < d$

0 elsewhere

$*$ $\therefore$ It is independent

$f(x, y) = \frac{1}{(b-a)(d-c)} = f_x(x) \cdot f_y(y)$

Bivariate Normal Dist.

$$f(x,y) = \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left\{ \frac{(x-m_x)^2}{\sigma_x^2} + \frac{(y-m_y)^2}{\sigma_y^2} - \frac{2\rho(x-m_x)(y-m_y)}{\sigma_x \sigma_y} \right\}}$$

$$-\infty < x < \infty, \quad -\infty < y < \infty$$

$$-\infty < m_x, m_y < \infty$$

$$0 < \sigma_x, \sigma_y < \infty$$

$$-1 < \rho < 1$$

5 parameters: $x, y, m_x, m_y, \sigma_x, \sigma_y, \rho$.

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{(x-m_x)^2}{2\sigma_x^2}}.$$

H/W If $\rho = 0$, check that

$$f(x,y) = f_x(x) f_y(y)$$

when $\rho = 0$, X & Y are independent.

Q) If Quest (before 8 pages)

Answer

Let $Z = \min\{X, 1\}$

Suppose $x < 1$, then the events:

$x < X \le x + dx$ and

$x < Z \le x + dx$ are identical

$$f(x) = \frac{1}{b-a} \quad \text{(uniform)}$$

$$= \frac{1}{4}$$

$f_Z(x)dx = f_X(x)dx = \frac{1}{4}dx$

$\Rightarrow f_Z(x) = \frac{1}{4}$ for $-2 < x < 1$

~~For~~ $1 < X < 2$ & $Z = 1$ are identical

$P(Z=1) = P(1 < X < 2)$

$= \frac{1}{4}(2-1)$

$= 1/4$

So, Z has cont. dist. in $(-2, 1)$ with const. prob. density $1/4$, and a discrete prob. mass func. $1/4$ at 1.

Q) The spectrum of r.v X consists of the points $1, 2 \ldots n$ and $P(X=i)$ is proportional to $1/i(i+1)$.

Determine, the dist. func. of X & compute $P(3 < X \le n)$ & $P(X > 5)$

Expectation:

Cont. $E(X) = \int_{-\infty}^{\infty} x f(x)dx$

Discrete $E(X) = \sum_{i=-\infty}^{\infty} i f_i = \sum_{i=-\infty}^{\infty} i P(X=i)$

Prop. ① $E(a) = a$, $a =$ const.

② $E\{a, g(x)\} = a E\{g(x)\}$

③ $E(g_1(x) + g_2(x) + \ldots + g_n(x)\} = E\{g_1(x)\} + E\{g_2(x)\} + \ldots E\{g_n(x)\}$

④ $|E(g(x))| \le E\{|g(x)|\}$

⑤ if $g(x) \geq 0$, (everywhere)

$E\{g(x)\} \geq 0$

Ⓡ $Y = g(x)$

$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)dy = \int_{-\infty}^{\infty} g(x) f_X(x) \frac{dx}{dy} \cdot dy = \int_{-\infty}^{\infty} g(x) f_X(x)dx$

$= E\{g(x)\}$

$E(X) \to$ Mean.

For Bionomial Dist.:

$$P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$$

$$m = E(X) = \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} i \binom{n}{i} p^i (1-p)^{n-i} = \sum_{i=1}^{n} i \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} \frac{(n-1)!}{(i-1)!\,[(n-1)-(i-1)]} p^{i-1} (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i}$$

$$= np \sum_{i=0}^{n-1} \binom{n-1}{i} p^{i} (1-p)^{n-1-i} = np$$

Poisson

$$m = E(X) = \sum_{i=0}^{\infty} i e^{-\mu} \frac{\mu^i}{i!}$$

$$= e^{-\mu} \sum_{i=0}^{\infty} \frac{\mu^i}{(i-1)!} = \mu e^{-\mu} \sum_{i=0}^{\infty} \frac{\mu^i}{i!} = \mu e^{-\mu} e^{\mu} = \mu$$

Normal

$$m = E(x) = \frac{1}{\sqrt{2\pi}\,\Omega} \int_{-\infty}^{\infty} x e^{-(x-m)^2/2\Omega^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\Omega} \int_{-\infty}^{\infty} (x-m+m) e^{-(x-m)^2/2\Omega^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\Omega} \int_{-\infty}^{\infty} (x-m) e^{-(x-m)^2/2\Omega^2} dx +$$

$$m \frac{1}{\sqrt{2\pi}\,\Omega} \int_{-\infty}^{\infty} e^{-(x-m)^2/2\Omega^2} dx$$

Let $x - m = z$

$dx = dz$

$$= \frac{1}{\sqrt{2\pi}\,\Omega} \int_{-\infty}^{\infty} z e^{-z/2\Omega^2} dz + m$$

$$= 0 + m = m$$

$E\{|X-a|^k\} \rightarrow k^{th}$ absolute moment of X about the pnt. $x=a$

k^{th} central moment is defined by:

$$\mu_k = E\{(X-m)^k\}$$

$\alpha_K = E(X^k) \rightarrow k^{th}$ moment about the origin.

$\underline{\mu_k(ax+b)} = E\{(ax+b-am_x-b)^k\}$

$Y = aX+b$ $\quad = E\{(aX-am_x)^k\}$

$m_y = am_x+b$ $\quad = a^k E\{(X-m_x)^k\}$

$E(Y)$ $\quad = a^k \mu_k(X)$

$= E(aX+b)$

$= a[E(X)] + E(b)$

$= am_x + b$

Variance

$$E\{(x-m)^2\} = \mu_2 = \sigma^2$$

$$\mu_2(aX+b) = a^2\mu_2(X)$$

Standard Deviation $= \sqrt{Var(X)}$

$(X-m)^2 \geq 0 \;\forall\; x$, $Var\, X = 0$

$\Rightarrow X = m$

The 2nd moment about any pnt. is min when taken about mean

$(X-a)^2 = (X-m+m-a)^2$

$= (X-m)^2 + 2(X-m)(m-a) + (m-a)^2$

$E(X-a)^2 = E(X-m)^2 + 2(m-a)E(X-m) + (m-a)^2$

$\geq \mu_2$

Variance:

$Var(aX+b) = a^2 Var(X)$ | s.d $= \sqrt{Var\, X}$

$s.d\,(ax+b) = |a|\,sd\,(x)$

$\Omega(ax+b) = |a|\,\Omega(x)$

$E(m^2) = m^2$

$\Omega^2 = E(x(x-1) - m(m-1))$ $\quad [(x-m)^2 = x(x-1) - 2mx + x + m^2$

Proof: ~~Ω^2~~ $\Omega^2 = E(x-m)^2 = E\{x(x-1)\} - 2mE(x) + E(x) + m^2$

$= E\{x(x-1)\} - m^2 + m$

$= E\{x(x-1)\} - m(m-1)$ Proved

~~$= E(x-m) + 2$~~

Banach's Match Box Problem

A mathematician always carries two match boxes each containing n matches. Whenever he needs, he chooses a box at random and draws a match from it. Find the probability that when the first box is found to be empty for the first time, the 2nd box will contain exactly i matches.

Soln: The event means that the first box is chosen $(n+1)$ times. First n times choose the first box to draw n matches.

Then chose the second box to draw $(n-i)$ matches.

Finally choose first box to find its empty.

The we have Bernoullian seqn of $n+n-i+1 = 2n-i+1$ trails with probability of success $p = 1/2$, where to choose the first box is the event of success

$\Rightarrow q = 1-p = 1/2$

So, the required prob. is: $\binom{2n-i}{n}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{n-i}\cdot \frac{1}{2}$

$= \binom{2n-i}{n}\left(\frac{1}{2}\right)^{2n-i+1}$

Bernoulli trials: If a random experiment be such that its event space ~~cannot be~~ consists of only two points, which are usually called success & failure, then a seq^n of independent trails of the experiment will be called a sequence of bernoullian trails.

Binomial

$$P(x=i) = \binom{n}{i} p^i q^{n-i} \quad | \quad q=(1-p)$$

Accidents on road → Poisson Dist. as $n \to \infty$, also $p \to 0$

$\mu \to$ Constant

$\mu = np$

$P = \frac{\mu}{n}$

as $n \to \infty$, $p \to 0$

$$P(A_i) = \binom{n}{i} p^i (1-p)^{n-i} = \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i}$$

$$= \frac{n(n-1)(n-2)\cdots(n-i)}{i!} p^i (1-p)^{n-i}$$

$$= \frac{(np)^i \underbrace{(1-1/n)(1-2/n)}_{\to 1} \cdots}{i!} \cdot \frac{(1-\frac{i-1}{n}) \left\{\left(1-\frac{\mu}{n}\right)^{-n/\mu}\right\}^{-\mu}}{\left(1-\frac{\mu}{n}\right)^i}$$

$$= \frac{\mu^i e^{-\mu}}{i!} \quad \Big| \quad \frac{1}{n} \to 0, \quad 1 - 1/n \to 1$$

Q) An urn contains 1 white & 99 black balls. If 1000 draws are made with replacement. What is the probability of 10 white balls.

$| \; p = 1/100$

$np = \lambda$

$\lambda = 1000 \times \frac{1}{100} = 10$

$$P(X=10) = \frac{\lambda \cdot 10^{10} e^{-10}}{10!} = 0.1251$$

Q) A urn contains 4 white balls numbered 0, 1, 2, 3. 3 red balls numbered 0, 1, 2 and 2 black balls are numbered 0 & 1. The random experiment consists i.

drawing a ball at random from the urn, and the r.v.s X & Y are defined as follows:

X takes values 0, 1 & 2 respectively for white, red and black balls, & Y denotes the number of the ball. Find the joint dist. of X & Y. & deduce the marginal dist. of X & Y & find conditional dist.

Y \ X	0	1	2
0	1/9	1/9	1/9
1	1/9	1/9	1/9
2	1/9	1/9	—
3	1/9	—	—

$f(0,0) = 1/9$

$f_{x_0} = 4/9$

$f_{x_1} = 3/9$

$f_{x_2} = 2/9$

$f_{y_0} = 3/9 \quad f_{y_1} = 3/9 \quad f_{y_2} = 2/9, \quad f_{y_3} = 1/9$

Q) Suppose X & Y are two poisson variates, having parameter μ_1 & μ_2. Then $f(i,j) = e^{-(\mu_1+\mu_2)} \cdot \frac{\mu_1^i}{i!} \frac{(\mu_2)^j}{j!}$ when X & Y are independent

Find f_{x_i} & f_{y_j} which are Poisson Dist.

$$\cancel{f_{x_i} = \int_{-\infty}^{\infty} e^{-(\mu_1+\mu_2)} \frac{\mu_1^i}{i!} \left(\frac{\mu_2^j}{j!}\right) d\mu_j}$$

$$f_{x_i} = e^{-\mu_1} \frac{\mu_1^i}{i!} \sum_{j=0}^{\infty} \frac{(\mu_2)^j}{j!} e^{-\mu_2}$$

$$= \frac{e^{-\mu_1}(\mu_1)^i}{i!}$$

Q) The joint prob. density func is given by $K(1-x-y)$, inside the rectangle formed by the axes and the line $x+y=1$ and zero elsewhere.
Find the value of K and $P(X<\frac{1}{2}, Y>\frac{1}{4})$ elsewhere.
Find the marginal dist. of X & Y & determine whether the r.v.s are independent or not. & find conditional dist. $f_X(x/y)$ & compute $P(X<\frac{1}{2} \mid Y=\frac{1}{4})$

$$f(x,y) = k(1-x-y), \; x>0, \; y>0, \; x+y<1$$

Using $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$x+y=1$
$y=1-x$

$$\int_{x=0}^{1}\int_{y=0}^{1-x} k(1-x-y)\,dx\,dy = 1$$

$$\int_{x=0}^{1} k\left(y-xy-\frac{y^2}{2}\right)_0^{1-x} = k\int_{x=0}^{1}\left(1-\cancel{x}-\cancel{x}+x^2-\frac{1+x^2-2x}{2}\right)dx$$

$$= k\int_{x=0}^{1}\left(1-0-2x+x^2-\frac{1}{2}-\frac{x^2}{2}+x\right)dx$$

$$= k\int_{x=0}^{1}\left(\frac{1}{2}+\frac{x^2}{2}-x\right)$$

$$= k\int_{x=0}^{1}\frac{(1-x)^2}{2}dx$$

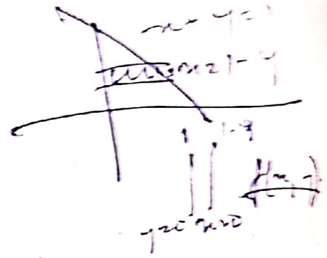
$$= k\left(\frac{(1-x)^3}{-6}\right)_0^1 = k\frac{1}{6} = 1$$

$\Rightarrow k = 6$

if $f(x,y) = f_X(x)\cdot f_Y(y)$

then X & Y ind.

else X & Y, not ind.



$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = 6\int_{y=0}^{1-x} (1-x-y)\,dy$$

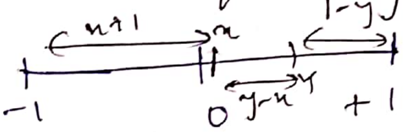
$$= 6\left[y - xy - \frac{y^2}{2}\right]_0^{1-x} = 6(1-x-\ldots)$$

Q)

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{x=0}^{1-y} 6(1-x-y)\,dx = 3(1-x)^2$$

$$P(X<\tfrac{1}{2}, Y>\tfrac{1}{4}) = \int_{x=0}^{1/2}\int_{y=1/4}^{1-x} 6(1-x-y)\,dy\,dx$$

Q) Two points are independently chosen at random, in the interval $(-1,1)$. Find the prob. that the three parts ~~must~~ into which the interval is divided can form the sides of triangle.



Let the two points be represented by the r.v.s X & Y which are independent, and each uniformly distributed in the interval $(-1,1)$.

If $Y>X$, the required event is represented by the following inequalities:

$X+1+Y-X > 1-Y \Rightarrow Y>0$

$\cancel{Y+X}\ Y-X+1-Y > X+1 \Rightarrow X<0$

$X+1+1-Y > Y-X \Rightarrow Y<X+1$

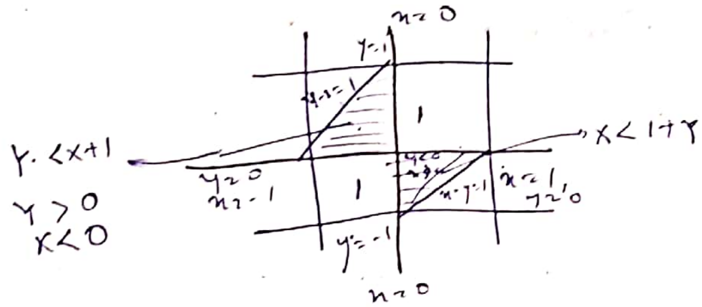
i.e (X,Y) lies in the triangular region

R_1, $\quad Y<X+1$

$\Rightarrow Y = X+1,\ Y-X=1$

where area of $R_1 = 1/2$

~~So, the total region is R = 4~~



If $Y < X$,

$Y+1+X-Y > 1-X$

$X-Y+1-X > Y+1$

$Y+1-1+1-X > X-Y$

$\Rightarrow Y < 0$

$X > 0$

$X < 1+Y$

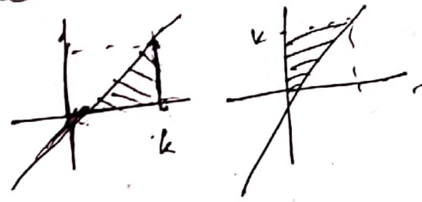
So, we get the region R_2, which has area $= 1/2$

So, the total region is $R = 4$

Q) X & Y are ind. variates, each uniformly distributed over the region (0, 1). Find the prob. that the greater of X, Y is less than a fixed number K. $(0 < K < 1)$

① : $x > y$

$x - y > 0$



$\frac{1}{2}k(k) + \frac{1}{2}kk$

$= k^2$

Sol^n: The two dimensional R.V X, Y, ie uniformly dist. over the interval (0, 1)

$R : 0 < x < 1, 0 < Y < 1$

So, this is a unit square

So, the area of square R is $A = 1$

So, the event is:

If $x > Y$ then the event is

$R_2 : X > Y, 0 < X < k ; 0 < y < k$

If $Y > X$, then the event is:

$R_1 : Y > X, 0 < Y < k, 0 < X < k$.

Now, R_1 & R_2 together form a square of area k^2

So, the required prob. : $k^2/1$

$P(Y = y_j | X = x_i) = \frac{P(X = x_i; Y = y_j)}{P(X = x_i)} = \frac{f_{ij}}{f_{xi}}$

$\rightarrow f$.

Conditional Dist.

Discrete case :

$$P(X = x_i \mid Y = y_j) = \frac{P(X = x_i, Y = y_j)}{P(Y = y_j)}$$

$$= \frac{f_{ij}}{f_{y_j}} = f_{i/j} = \frac{f_{ij}}{f_{y_j}}$$

$P(Y = y_j \mid X = x_i)$

$= P($

$$\sum_{i=-\infty}^{\infty} f_{i/j} = \sum_{i=-\infty}^{\infty} \frac{f_{ij}}{f_{y_j}} = \frac{f_{y_j}}{f_{y_j}} = 1$$

Similarly $\sum_{j=-\infty}^{\infty} f_{j/i} = 1$

Conti Case

$P(a < X \le b \mid y < Y < y + \Delta y)$

$$= \frac{P(a < X \le b, y < Y < y + \Delta y)}{P(y < Y < y + \Delta y)} \quad \text{I } \int$$

$$= \frac{\int_{y}^{y+\Delta y}\int_{a}^{b} f(x,y)\,dx\,dy}{\int_{y}^{y+\Delta y} f_Y(y)\,dy}$$

Now making $\Delta y \to 0$

$$\lim_{\Delta y \to 0} P(a < X \le b \mid y < Y < y + \Delta y)$$

$$= P(a < X \le b \mid Y = y) = \frac{\int_a^b f(x,y)\,dx}{f_Y(y)}$$

~~where~~ $= \int_a^b f_x(x|y)\,dx$, where $f_x(x|y) = \dfrac{f(x,y)}{f_y(y)} \geq 0$

Conditional Distr. function

$$F_x(x|y)$$
$$= P(-\infty < X \leq x \mid Y = y)$$
$$= \int_{-\infty}^{x} f_x(x/y)\,dx$$

$$f_y(y/x) = \frac{f(x,y)}{f_x(x)}$$

$$\int_{-\infty}^{\infty} f_y(y/x)\,dy = 1 \Rightarrow F_y(y/x) = \int_{-\infty}^{y} f_y(y/x)\,dy$$

$f_x(x|y) \rightarrow$ Cond. dens. fn.

$$P(x < X \leq x+dx \mid y < Y \leq y+dy)$$
$$= \frac{P(x < X \leq x+dx,\ y < Y \leq y+dy)}{P(y < Y \leq y+dy)}$$
$$= \frac{f(x,y)\,dx\,dy}{f_y(y)\,dy} = f_x(x/y)\,dx$$

Now find $f_{i/0}$, i.e $f_{0/0}$, $f_{1/0}$, $f_{2/0}$ from the given problem i.e. Ans: $f_{0/0} = f_{1/0} = f_{2/0} = 1/3$ & find $f_x(x/y)$ & compute $P(X < 1/2 \mid Y \leq 1/4)$

$f_x(x/y) = \dfrac{2(1-x-y)^{ans}}{(1-y)^2}$, $0 < x < 1-y$ & $P(X < 1/2 \mid Y \leq 1/4) = 8/9$

$P(X < 1/2 \mid Y = 1/4)$ of discrete dist. from the last given prob. of cont. dist.

Bivariate Normal Dist.

$$f_x(x/y) = \frac{f(x,y)}{f_y(y)}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x\sqrt{1-\rho^2}}\, e^{-\frac{1}{2\sigma_x^2(1-\rho^2)}\left\{x - m_x - \rho\frac{\sigma_x}{\sigma_y}(y-m_y)\right\}^2}$$

So, the cond. dist. of X on the hypothesis Y = y is also normal dist. having parameter:

$$\left\{m_x + \rho\frac{\sigma_x}{\sigma_y}(y-m_y),\ \sigma_x\sqrt{1-\rho^2}\right\}$$

Similarly cond. dist. of Y on the hypothesis is X = x is also normal dist. with parameters:

$$\left\{m_y + \frac{\rho\sigma_y}{\sigma_x}(x-m_x),\ \sigma_y\sqrt{1-\rho^2}\right\}$$