# Computer Graphics 2: Maths Preliminaries

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# Introduction

Computer graphics is all about maths!

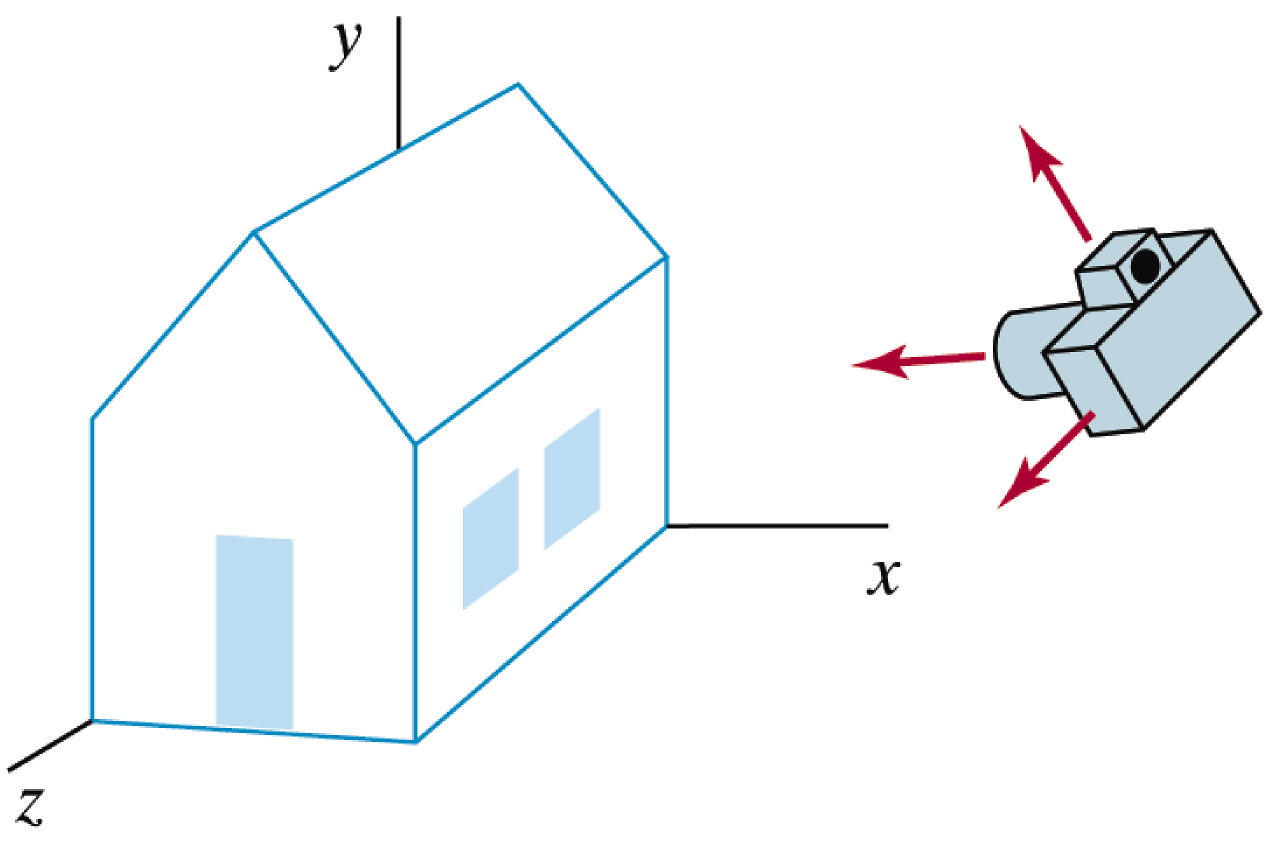
None of the maths is hard, but we need to understand it well in order to be able to understand certain techniques

Today we'll look at the following:

- Coordinate reference frames
- Points & lines
- Vectors
- Matrices

# Big Idea



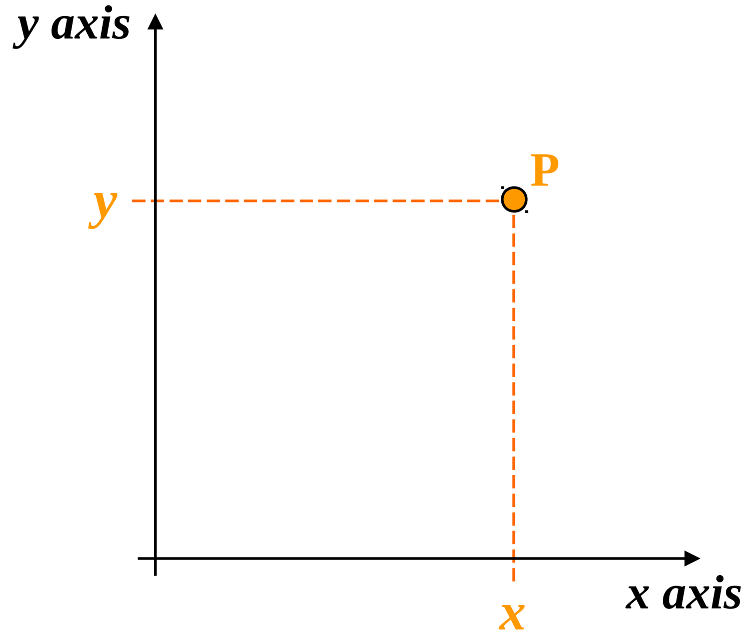


# Coordinate Reference Frames – 2D

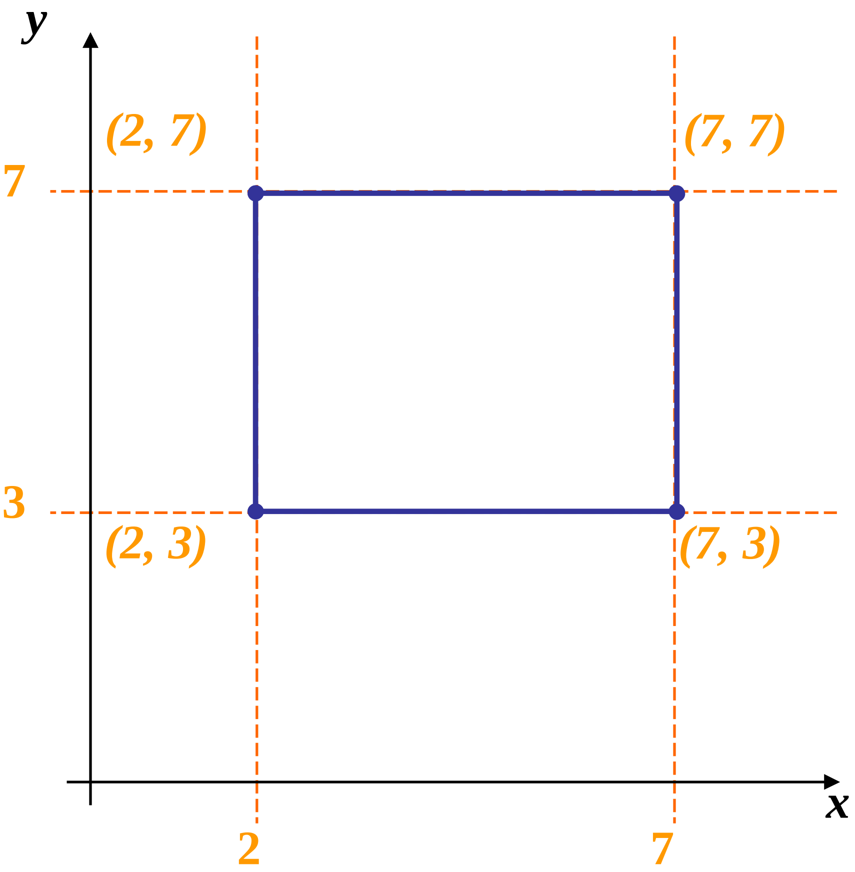
When setting up a scene in computer graphics we define the scene using simple geometry

For 2D scenes we use simple two dimensional Cartesian coordinates

All objects are defined using simple coordinate pairs

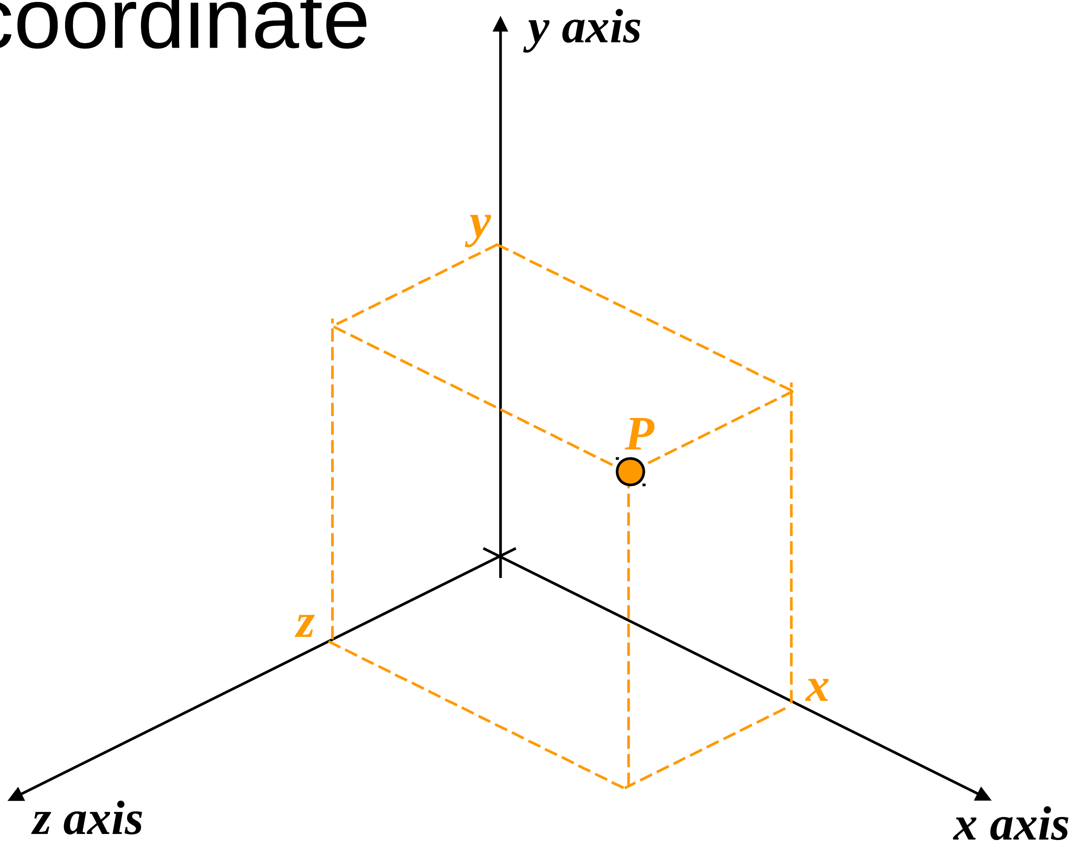


# Coordinate Reference Frames – 2D (cont…)



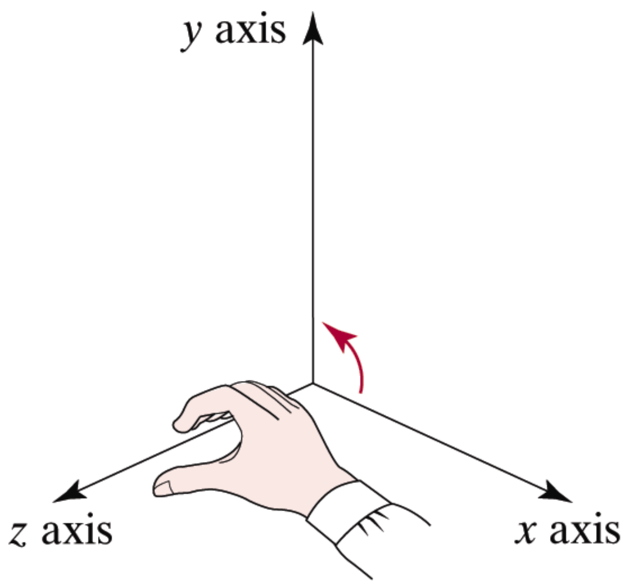
# Coordinate Reference Frames – 3D

For three dimensional scenes we simply add an extra coordinate



# Left Handed Or Right Handed?

There are two different ways in which we can do 3D coordinates – *left handed* or *right handed*




Right-Hand
Reference System


**We will mostly use the right-handed system**

Left-Hand
Reference System

Images taken from Hearn & Baker, "Computer Graphics with OpenGL" (2004)

# Points & Lines

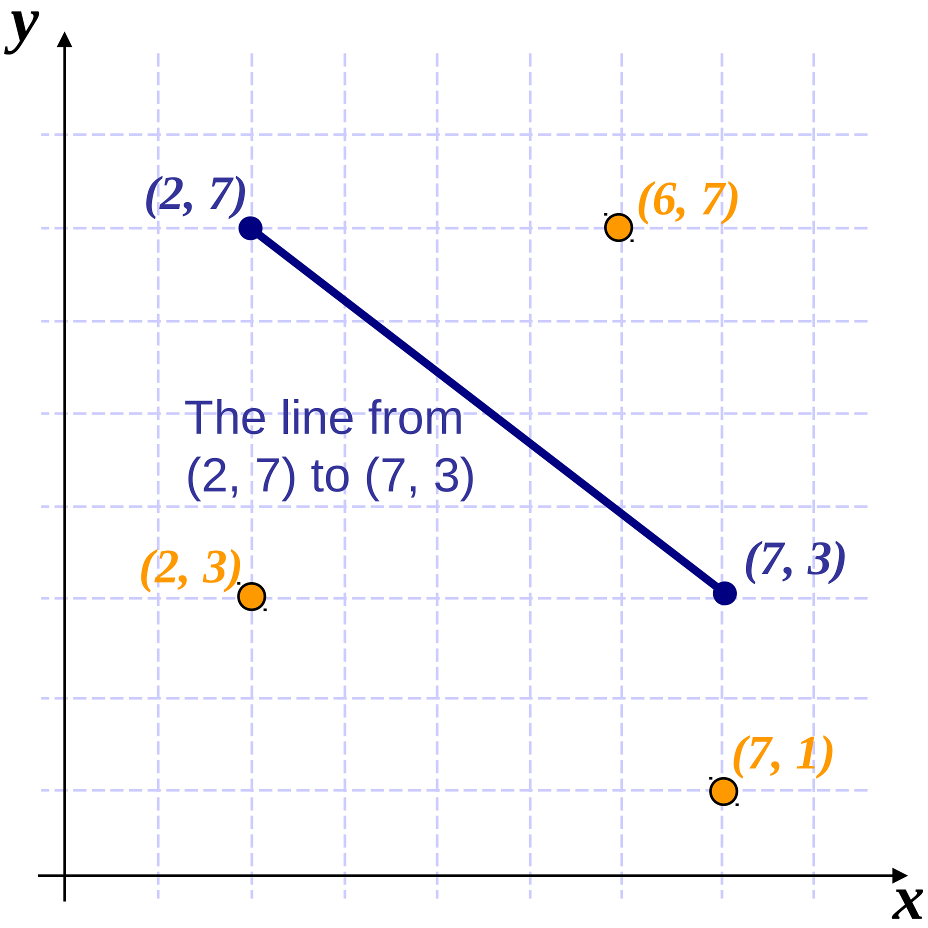
## Points:

- A point in two dimensional space is given as an ordered pair $(x, y)$
- In three dimensions a point is given as an ordered triple $(x, y, z)$

## Lines:

- A line is defined using a start point and an end-point
  - In 2d: $(x_{start}, y_{start})$ to $(x_{end}, y_{end})$
  - In 3d: $(x_{start}, y_{start}, z_{start})$ to $(x_{end}, y_{end}, z_{end})$

# Points & Lines (cont…)



# The Equation of A Line

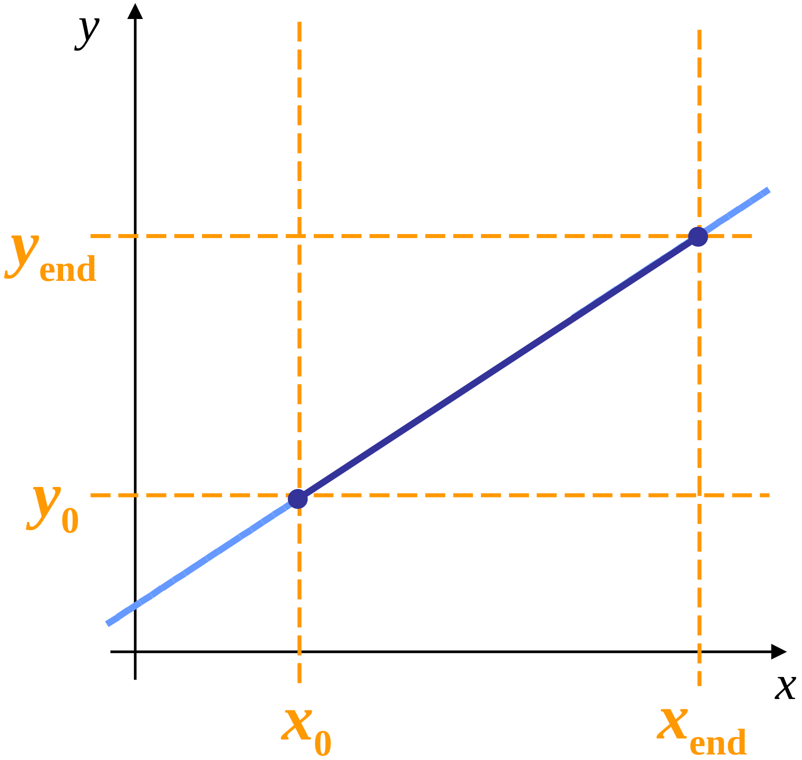
The *slope-intercept equation of a line* is:

$$y = m \cdot x + b$$

where:

$$m = \frac{y_{end} - y_0}{x_{end} - x_0}$$

$$b = y_0 - m \cdot x_0$$




The equation of the line gives us the corresponding *y* point for every *x* point

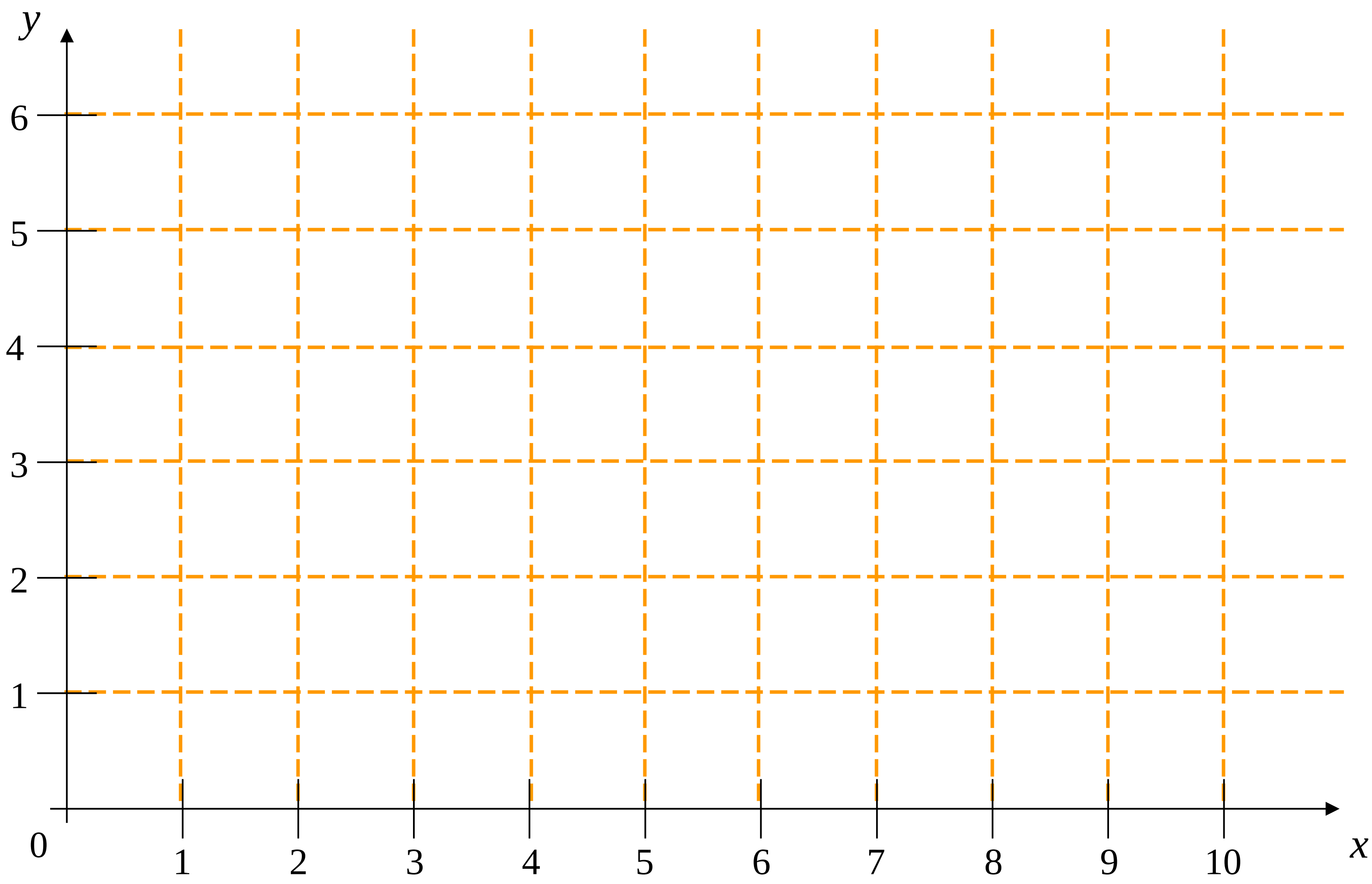
# A Simple Example

Let's draw a portion of the line given by the equation:

$$y = \frac{3}{5}x + \frac{4}{5}$$

Just work out the $y$ coordinate for each $x$ coordinate

# A Simple Example (cont…)



# A Simple Example (cont…)

For each $x$ value just work out the $y$ value:

$$y(2) = \frac{3}{5} \cdot 2 + \frac{4}{5} = 2$$

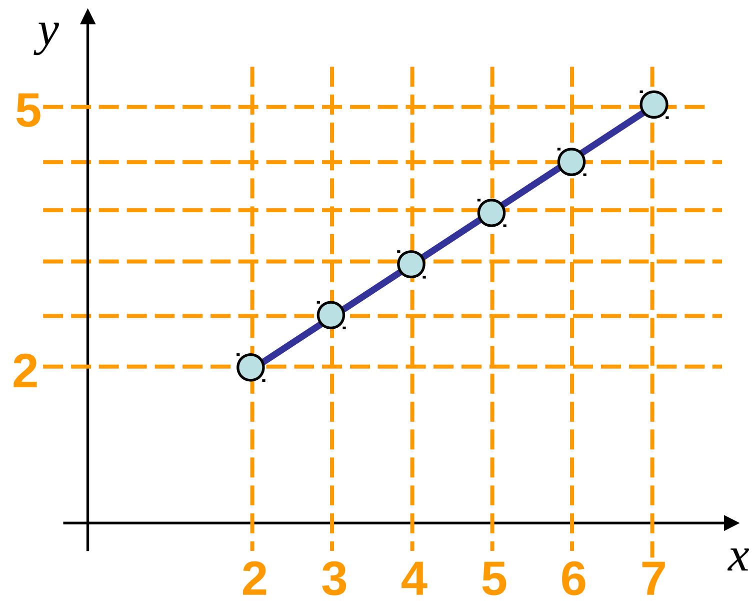
$$y(3) = \frac{3}{5} \cdot 3 + \frac{4}{5} = 2\frac{3}{5}$$

$$y(4) = \frac{3}{5} \cdot 4 + \frac{4}{5} = 3\frac{1}{5}$$

$$y(5) = \frac{3}{5} \cdot 5 + \frac{4}{5} = 3\frac{4}{5}$$

$$y(6) = \frac{3}{5} \cdot 6 + \frac{4}{5} = 4\frac{2}{5}$$

$$y(7) = \frac{3}{5} \cdot 7 + \frac{4}{5} = 5$$



# Vectors

Vectors:

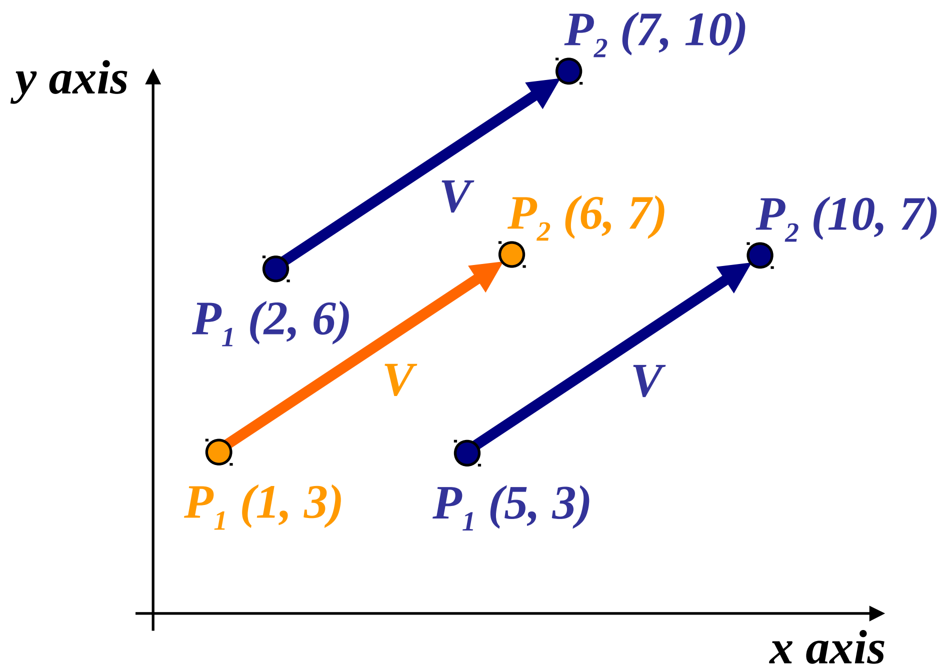
- A vector is defined as the difference between two points
- The important thing is that a vector has a direction and a length

What are vectors for?

- A vector shows how to move from one point to another
- Vectors are very important in graphics - especially for transformations

# Vectors (2D)

To determine the vector between two points simply subtract them

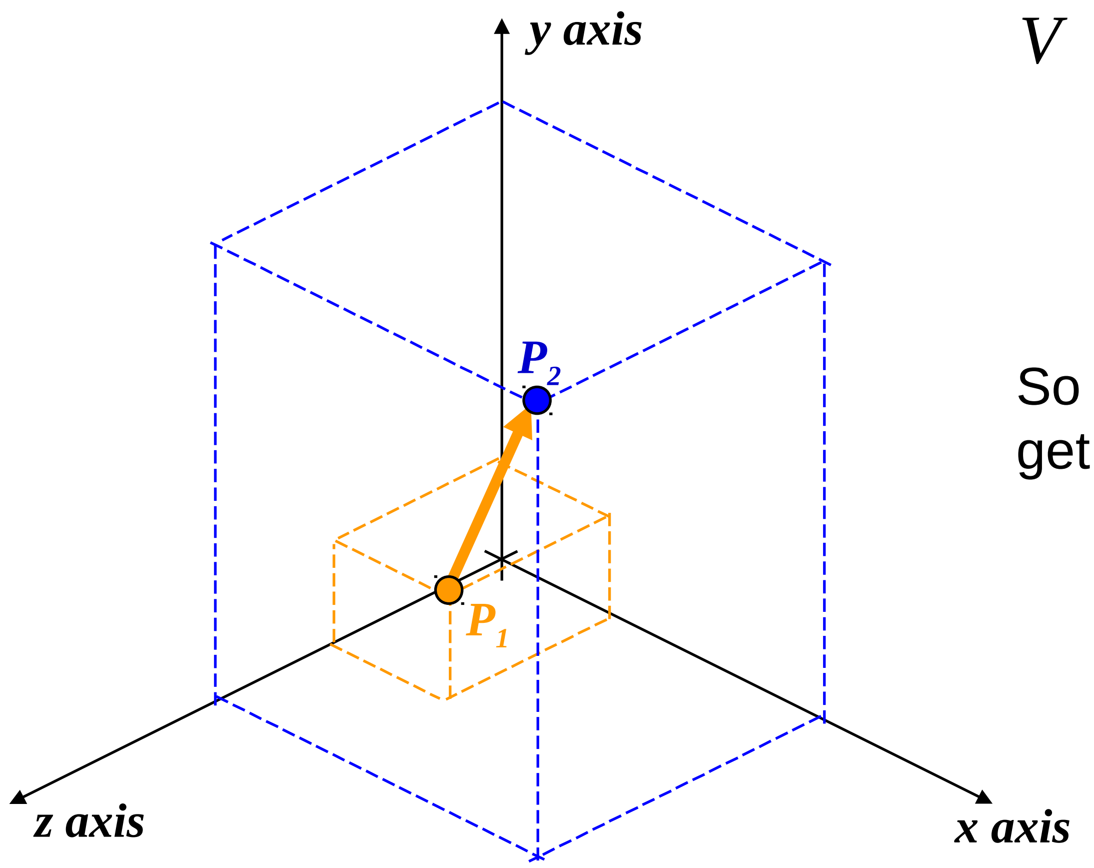



$$\begin{aligned} V &= P_2 - P_1 \\ &= (x_2 - x_1, y_2 - y_1) \\ &= (6 - 1, 7 - 3) \\ &= (5, 4) \end{aligned}$$

**WATCH OUT:** Lots of pairs of points share the same vector between them

# Vectors (3D)

In three dimensions a vector is calculated in much the same way




$$V = P_2 - P_1$$
$$= (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$
$$= (V_x, V_y, V_z)$$

So for (2, 1, 3) to (7, 10, 5) we get

$$= (7 - 2, 10 - 1, 5 - 3)$$
$$= (5, 9, 2)$$

# Vector Operations

There are a number of important operations we need to know how to perform with vectors:

- Calculation of vector length
- Vector addition
- Scalar multiplication of vectors
- Scalar product
- Vector product

# Vector Operations: Vector Length

Vector lengths are easily calculated in two dimensions:

$$|V| = \sqrt{V_x^2 + V_y^2}$$

and in three dimensions:

$$|V| = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

# Direction of a Vector

The direction of a vector is the measure of the angle it makes with a horizontal line .

One of the following formulas can be used to find the direction of a vector:

$$\tan\theta = \frac{y}{x}$$

where $x$ is the horizontal change and $y$ is the vertical change or

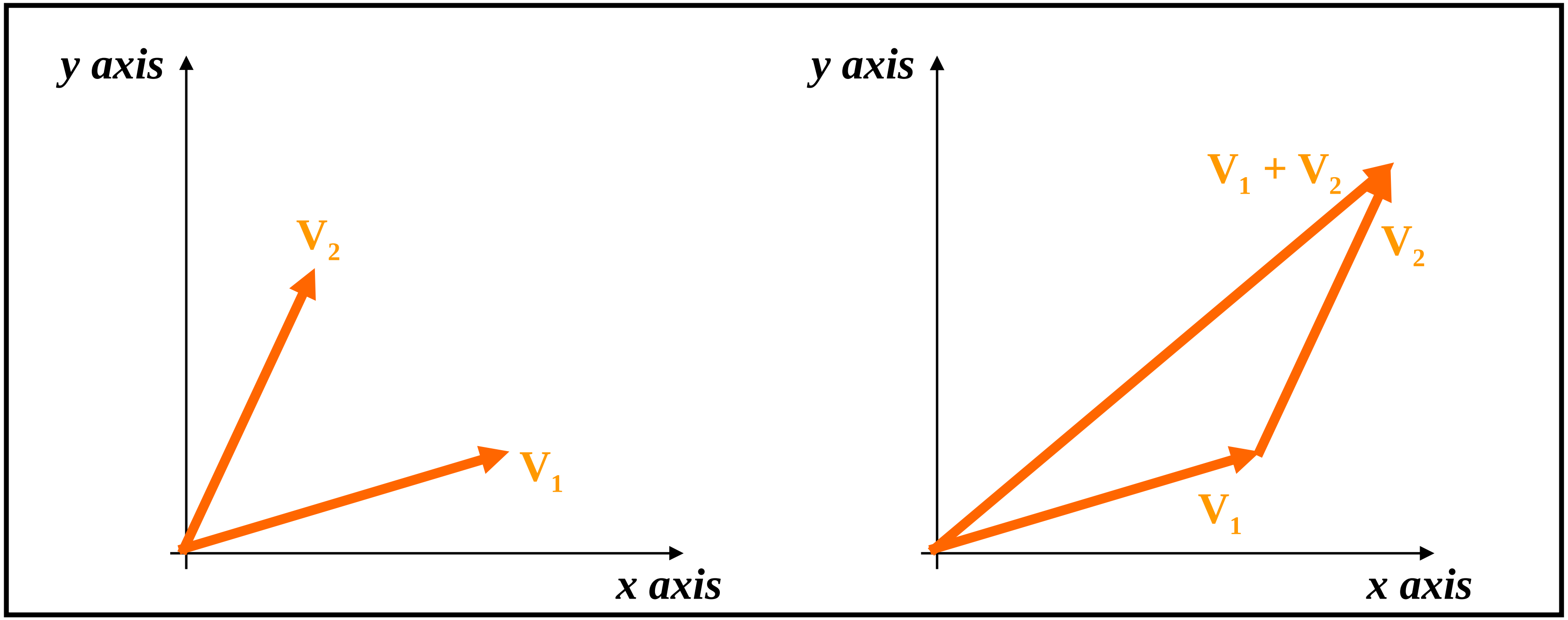
$$\tan\theta = (y2 - y1)/(x2 - x1)$$

where $(x1, y1)$ is the initial point and $(x2, y2)$ is the terminal point.

# Vector Operations: Vector Addition

The sum of two vectors is calculated by simply adding corresponding components

$$V_1 + V_2 = (V_{1x} + V_{2x}, V_{1y} + V_{2y})$$

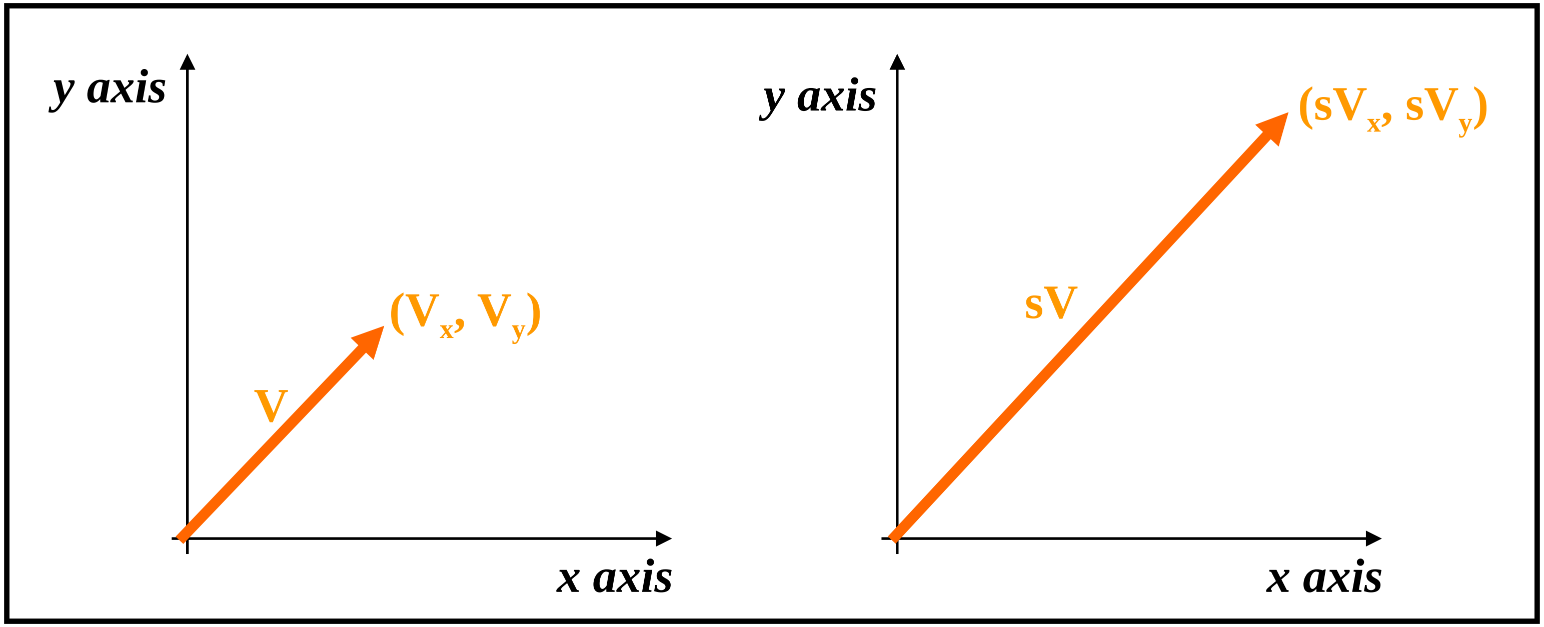



Performed similarly in three dimensions

# Vector Operations: Scalar Multiplication

Multiplication of a vector by a scalar proceeds by multiplying each of the components of the vector by the scalar

$$sV = (sV_x, sV_y)$$



# Other Vector Operations

There are other important vector operations that we will cover as we come to them

These include:

- Scalar product (dot product)
- Vector product (cross product)

# Matrices

A matrix is simply a grid of numbers

$$\begin{bmatrix} 1 & 11 & 13 \\ 10 & 4 & -3 \\ 2 & 0 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 4.3 \\ 6.7 \\ 1.2 \end{bmatrix} \quad \begin{bmatrix} 4 & 8 & 15 \\ 16 & 23 & 42 \end{bmatrix}$$

However, by using matrix operations we can perform a lot of the maths operations required in graphics extremely quickly

# Matrix Operations

The important matrix operations for this course are:

- Scalar multiplication
- Matrix addition
- Matrix multiplication
- Matrix transpose
- Determinant of a matrix
- Matrix inverse

# Matrix Operations: Scalar Multiplication

To multiply the elements of a matrix by a scalar simply multiply each one by the scalar

$$s*\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} s*a & s*b & s*c \\ s*d & s*e & s*f \\ s*g & s*h & s*i \end{bmatrix}$$

Example:

$$3*\begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix} = \begin{bmatrix} 6 & 12 & 18 \\ 24 & 30 & 36 \\ 42 & 48 & 54 \end{bmatrix}$$

# Matrix Operations: Addition

To add two matrices simply add together all corresponding elements

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} a+r & b+s & c+t \\ d+u & e+v & f+w \\ g+x & h+y & i+z \end{bmatrix}$$

Example:

$$\begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix} + \begin{bmatrix} 3 & 5 & 7 \\ 9 & 11 & 13 \\ 15 & 17 & 19 \end{bmatrix} = \begin{bmatrix} 5 & 9 & 13 \\ 17 & 21 & 25 \\ 29 & 33 & 37 \end{bmatrix}$$

**Both matrices have to be the same size**

# Matrix Operations: Matrix Multiplication

We can multiply two matrices **A** and **B** together as long as the number of columns in **A** is equal to the number of rows in **B**

So, if we have an $m$ by $n$ matrix **A** and a $p$ by $q$ matrix **B** we get the multiplication:

$$\mathbf{C}=\mathbf{AB}$$

where **C** is a $m$ by $q$ matrix whose elements are calculated as follows:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{ki}$$

# Matrix Operations: Matrix Multiplication (cont…)

Examples:

$$\begin{bmatrix} 0 & -1 \\ 5 & 7 \\ -2 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0*1+(-1)*3 & 0*2+(-1)*4 \\ 5*1+7*3 & 5*2+7*4 \\ -2*1+8*3 & -2*2+8*4 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 26 & 38 \\ 22 & 28 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1*4+2*5+3*6 \end{bmatrix} = \begin{bmatrix} 32 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4*1 & 4*2 & 4*3 \\ 5*1 & 5*2 & 5*3 \\ 6*1 & 6*2 & 6*3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix}$$

# Matrix Operations: Matrix Multiplication (cont…)

**Watch Out!** Matrix multiplication is not commutative, so:

$$AB \neq BA$$

# Matrix Operations: Transpose

The transpose of a matrix $\mathrm{M}$, written as $\mathrm{M}^{\mathrm{T}}$ is obtained by simply interchanging the rows and columns of the matrix

For example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

# Other Matrix Operations

There are some other important matrix operations that we will explain as we need them

These include:

- – Determinant of a matrix
- – Matrix inverse

# Summary

In this lecture we have taken a brief tour through the following:

– Basic idea

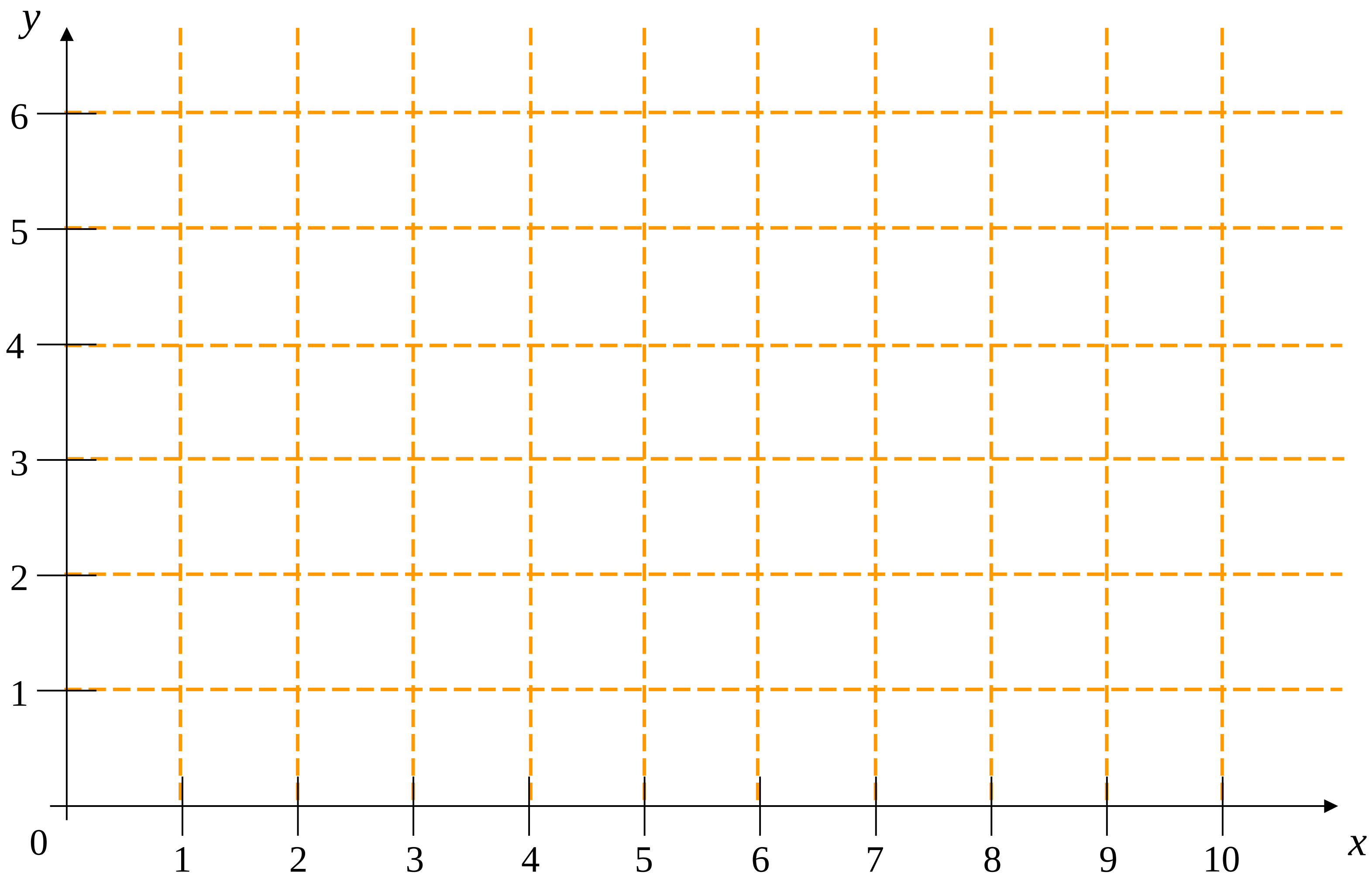
– The mathematics of points, lines and vectors

– The mathematics of matrices

These tools will equip us to deal with the computer graphics techniques that we will begin to look at, starting next time

# Exercises 1

Plot the line $y = \frac{1}{2}x + 2$ from $x = 1$ to $x = 9$



# Exercises 2

Perform the following matrix additions:

$$\begin{bmatrix} -11 & -19 & -15 & 5 \end{bmatrix} + \begin{bmatrix} -1 & -14 & -5 & 1 \end{bmatrix} = \begin{bmatrix} \_\_ & \_\_ & \_\_ & \_\_ \end{bmatrix}$$

$$\begin{bmatrix} 3 & 20 & -11 & 0 \\ -14 & 5 & -3 & 6 \\ 15 & 2 & 9 & -18 \\ -19 & 4 & -15 & 10 \end{bmatrix} + \begin{bmatrix} 16 & 10 & -12 & -11 \\ 10 & -15 & 15 & 5 \\ -1 & 14 & -9 & 0 \\ 3 & -3 & -16 & -5 \end{bmatrix} = \begin{bmatrix} \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \end{bmatrix}$$

$$\begin{bmatrix} 13 & -3 & -4 \\ 19 & 9 & 8 \end{bmatrix} + \begin{bmatrix} -1 & -15 & -7 \\ -14 & 5 & 17 \end{bmatrix} = \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix}$$

# Exercises 3

Perform the following matrix multiplications:

$$\begin{bmatrix} 8 & 15 & 19 & 4 \\ 7 & -4 & 12 & 3 \end{bmatrix} * \begin{bmatrix} -15 & 19 \\ -12 & -19 \\ 0 & -13 \\ 10 & 7 \end{bmatrix} = \begin{bmatrix} \underline{\hspace{4cm}} & \underline{\hspace{4cm}} \\ \underline{\hspace{4cm}} & \underline{\hspace{4cm}} \end{bmatrix} = \begin{bmatrix} \underline{\hspace{1cm}} & \underline{\hspace{1cm}} \\ \underline{\hspace{1cm}} & \underline{\hspace{1cm}} \end{bmatrix}$$

$$\begin{bmatrix} 16 & 10 & -12 & -11 \\ 10 & -15 & 15 & 5 \\ -1 & 14 & -9 & 0 \\ 3 & -3 & -16 & -5 \end{bmatrix} * \begin{bmatrix} 4 \\ 11 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} \underline{\hspace{6cm}} \\ \underline{\hspace{6cm}} \\ \underline{\hspace{6cm}} \\ \underline{\hspace{6cm}} \end{bmatrix} = \begin{bmatrix} \underline{\hspace{1cm}} \\ \underline{\hspace{1cm}} \\ \underline{\hspace{1cm}} \\ \underline{\hspace{1cm}} \end{bmatrix}$$

# Exercises 4

Perform the following multiplication of a matrix by a scalar

$$6*\begin{bmatrix} 15 & 19 \\ 2 & 5 \\ 0 & -1 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \\ \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$$

Calculate the transpose of the following matrix

$$\begin{bmatrix} 3 & 11 \\ 6 & 4 \\ 23 & 7 \end{bmatrix}^T = \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix}$$