

# Two multiply 2 big integer numbers:

## Basic Algo

$$\begin{array}{r}
 4167 \quad n \\
 \times 9832 \quad m \\
 \hline
 8334 \\
 125010 \\
 \dots 600 \\
 \dots 00 \\
 \hline
 \end{array}$$

Time complexity:

~~(n \* m)~~

Per digit = 1 mul / 1 add<sup>n</sup>

multiplic<sup>n</sup> =  $n^2$

addit<sup>n</sup> = (n rows) (n columns)

=  $n^2$

Net =  $O(n^2)$

## Divide and Conquer

$$\begin{array}{lcl}
 x = \boxed{x_L} \boxed{x_R} & = & 2^{n/2} x_L + x_R \\
 y = \boxed{y_L} \boxed{y_R} & = & 2^{n/2} y_L + y_R
 \end{array}$$

$\frac{n}{2}$  bits       $\frac{n}{2}$  bits

$$\begin{aligned}
 xy &= (2^{n/2} x_L + x_R) \cdot (2^{n/2} y_L + y_R) \\
 &= 2^n x_L y_L + 2^{n/2} (x_L y_R + x_R y_L) + x_R y_R
 \end{aligned}$$

$$T(n) = T(n/2) \quad T(n/2) \quad T(n/2) \quad T(n/2)$$

Product of n bit No.s      Product of 2 numbers of size  $n/2$  bits

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$$T(n) \leq 4T(n/2) + O(n)$$

↓

$2^n \times$  Product  
shifting  $n$  bits

Recursive Equat<sup>n</sup>: time complexity of recursive algo shown with recursive equat<sup>n</sup>

$$T(n) \leq 4T(n/4) + Cn/2$$

$$T(n) \leq 4 \{ 4T(n/4) + Cn/2 \} + Cn$$

$$\leq 4^2 T(n/2^2) + 3C \cdot n$$

⋮

$$\text{At step } x \leq 4^x T(n/2^x) + (2^x - 1)C \cdot n$$

$$\text{At some } x: n/2^x = 1$$

$$x = \log_2(n)$$

$$T(n) = 4^{\log_2(n)} T(1) + (2^{\log_2(n)} - 1)Cn$$

$$= n^2 + O(n \log(n)) (n-1)Cn$$

$$= O(n^2)$$

$$= O(n^2) \quad \langle \text{No gain} \rangle$$

All through the long winter, I dream of my garden. On the first day of spring, I dig my fingers deep into the soft earth. I can feel its energy, and my spirits soar. - Helen Hayes



December 2018

Tuesday

338 - 027

4

WEEK 49

Appointments

9  $xy = 2^n x_L y_L + 2^{n/2} (x_L y_R + x_R y_L) + x_R y_R$

10  $x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R$

11

function multiply( $x, y$ )

12 input:  $n$ -bit positive integer numbers  
 $x$  and  $y$

13 output: their product

14 if ( $n=1$ ): return  $xy$

$x_L, x_R$  = leftmost  $\lceil n/2 \rceil$ , rightmost  $\lfloor n/2 \rfloor$   
of  $x$ .

15  $y_L, y_R$  = leftmost  $\lceil n/2 \rceil$ , rightmost  $\lfloor n/2 \rfloor$  of  $y$ .



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WEEK

December 2018

Wednesday

300-000

Solving Recurrence Equat<sup>n</sup> with Recursion TreeMaster Theorem

if  $T(n) = a T(n/b) + O(n^d)$   
 for some constant  $a > 1$ ,  $b > 1$   
 and  $d \geq 0$  then

$$\begin{aligned}
 T(n) &= O(n^d) & \text{if } d > \log_b a \\
 &O(n^d \log n) & \text{if } d = \log_b a \\
 &O(n^{\log_b a}) & \text{if } d < \log_b a
 \end{aligned}$$

Proof:

Assumptions:  $n$  is power of  $b$   
 $n = b^m$

Total work done at level depth  $K$

$$= a^K \times O(n/b^K)^d$$

$$= O(n^d) \times (a/b^d)^K$$

$$T(n) = \sum_{K=0}^{\log_b n} O(n^d) \times (a/b^d)^K$$

Case 1:

$$a/b^d > 1$$

$$b^d < a$$

$$d < \log_b a$$



December 2018

Thursday

340 - 025

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WEEK 49

Appointments

9 Case 1: if ratio  $< 1$

10 Then the series is decreasing and its  
 11 sum is just given by the first term  
 $O(n^d)$

$$P(n) = a_k n^k + a_{k-1} n^{k-1} + \dots + a_0$$

$$P(n) = O(n^k)$$

$$a/b^d < 1$$

$$d > \log_b a$$

15 Case 2: if ratio  $= 1$   
 $a/b^d = 1$   
 $d = \log_b a$

$$T(n) = O(n^d) \times \log_b n$$

$$= O(n^d \log_b n)$$

18 Case 3: ratio  $> 1$   
 $a/b^d > 1$   
 $d < \log_b a$

20 The series is increasing and its sum  
 is given by the last term  
 $O(n^d) \times (a/b^d)^{\log_b n}$

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$$\text{last term} = c n^d \times \left(\frac{a}{b}\right)^{\log_b n}$$

$$= c n^d \times \frac{a^{\log_b n}}{b^{d \log_b n}}$$

$$= c n^d \times \frac{a^{\log_b n}}{n^d}$$

$$= c a^{\log_b n}$$

$$\approx c n$$

$$\left[ \log_{ab} n = \frac{\log_a n}{\log_a b} \right]$$

$$= c \cdot a^{(\log_a n) \cdot (\log_b a)}$$

$$= c n \cdot a^{(\log_b a)}$$

$$= O(n^{\log_b a})$$

### Merge Sort

$$T(n) = T(n/2) + T(n/2) + \text{Running time for merge}$$

Combination step should be  $O(n^d)$  for divide and conquer algo's

$$= 2T(n/2) + O(n)$$

$$= O(n^d \log n)$$

I'm sure I've been a toad, one time or another. With bats, weasels, worms...I rejoice in the kinship. Even the caterpillar I can love, and the various vermin. Theodore Roethke



## Appointments

9 Prove that for all  $n \geq 0$   

$$\sum_{i=1}^n \frac{i(i+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

10 The proof is by induction on  $n$ . (the upper limit)

11 Base case:  
 $n = 0$

12 In this case both side of the equation is 0

13 Induction step:

For  $n \geq 0$ , assume that

14 
$$\sum_{i=0}^k \frac{i(i+1)}{2} = \frac{k(k+1)(k+2)}{6} \text{ holds for } k < n$$

15 
$$\sum_{i=0}^{n-1} \frac{i(i+1)}{2} = \frac{(n-1)(n)(n+1)}{6}$$

16 
$$\sum_{i=0}^n \frac{i(i+1)}{2} = \sum_{i=0}^{n-1} \frac{i(i+1)}{2} + \frac{n(n+1)}{2}$$

$$= \frac{(n-1)(n)(n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2} \left( \frac{n-1}{3} + 1 \right)$$

$$= \frac{n(n+1)(n+2)}{6}$$

The Day Maldives Embraced Islam (Maldives)

Sunday 9

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