Graph

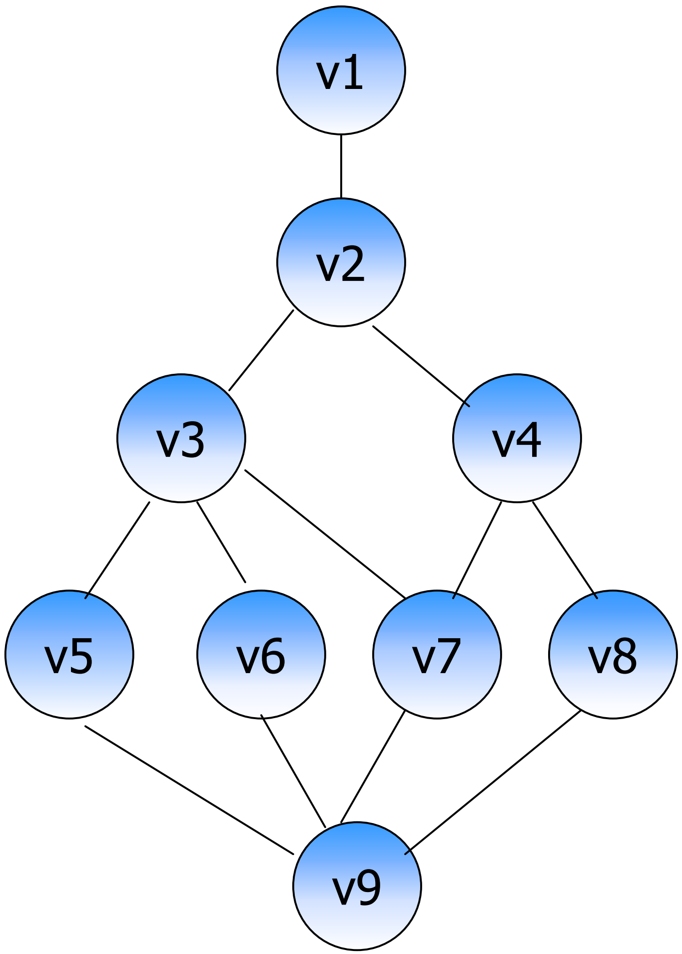
Graph - Definitions

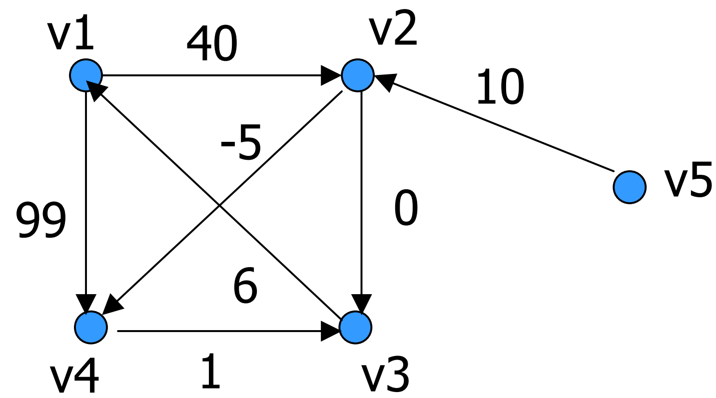
- A graph G is represented by a tuple (V, E) where V is a set of vertices $V=(v_1, v_2, v_3, v_4,\ldots)$ and E is a set of edges $E=(e_1, e_2, e_3, e_4,\ldots)$
- An element of E, say e_i is a pair of vertices (v_m, v_n). So $E \subseteq V \times V$
- If $e_i = (v_m, v_n)$ is an ordered pair, then the graph is a directed one; v_m is the start vertex and v_n is the end vertex. We call such graphs as Digraphs.
- If $e_i = (v_m, v_n)$ is an edge of G, then v_m and v_n are said to be adjacent.

Graph – Definitions …

- The number of edges incident on a vertex is said to be degree. In a digraph, in-degree and out-degree are differentiated.
- A graph $G' = (V', E')$ is called a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.
- A path from vertex v_1 to v_n is a sequence of vertices $v_1 v_2 v_3 \ldots v_k$, such that (v_i, v_{i+1}) is an edge for $i = 1$ to $k-1$.
- Paths may have cycles and edges may have associated weights.

Examples





Graph Representation – Adjacency Matrix

- Let G = (V, E) be a graph of n vertices. Its adjacency matrix is an n*n matrix M consisting of 0s and 1s.
- M[i][j] = 1 iff (v_i, v_j) is an edge of G
  = 0 otherwise
- For undirected graphs, adjacency matrix is necessarily symmetric.

- For a weighted graph,
  M[i][j] = weight((v_i, v_j)), if $(v_i, v_j) \in E$
  = infinity, otherwise

Adjacency Matrix

```
#define maxnovertices  100

typedef      struct {
             int      novertices;

             float    M[maxnovertices][maxnovertices];
             }        graph;
```

Graph Representation – Adjacency List

Here a list is maintained for each vertex.

The list for any vertex contains the vertices adjacent to it.

For weighted graphs, additionally, the weight information is stored.

Adjacency List

```
#define     maxnovertices        100

typedef struct node        {
            int      to_vertex;
            float    weight;
      struct node    * neighbour;
                     }        nodetype;
typedef     struct  {
            int               novertices;
            nodetype     * List[maxnovertices];
                     }        graph;
```

Adjacency List …

Total number of nodes required for a graph (undirected) of n vertices and e edges is n+2e.

For digraph, the number is n+e.

For sparse graph, this representation is efficient.

Graph Traversal – Depth-First Search

```
rec-dfs (v)
 {
        visit(v);
        for (each vertex u adjacent to v)
        if (u is not yet visited)
              rec-dfs(u);
}
```

Iterative Depth-First Search

```
iter_dfs ( graph v){
 int                      i,u1,u2;
stack-of-vertices         stak;
 unsigned char            visited[maxnovertices];

for (i=0; i< maxnovertices; i++)  visited[i] = false;
  s_create(stak);
  push(v, stak);
  do{
    u1 = pop(stak);
    if (!visited[u1])  {visit(u1); visited[u1] = true;}
    for (each u2 adjacent to u1){
    if (!(visited[u2]))  push(u2, stak);}
   }while (! empty(stak));
}
```

Graph Traversal – Breadth-First Search

```
BFS(graph v){
int                         i,u1,u2 ;
queue-of-vertices           q;
unsigned char               visited[maxnovertices];

for (i=0; i< maxnovertices; i++)  visited[i] = false;
init-q(q);  enqueue(v, q);
 do{
         u1 = dequeue(q);
         if (!(visited[u1])) {visit(u1); visited[u1] = true;}
         for (each u2 adjacent to u1){
         if  (!(visited[u2]))  enqueue(u2, q);}
  }while (!empty_q(q));
}
```

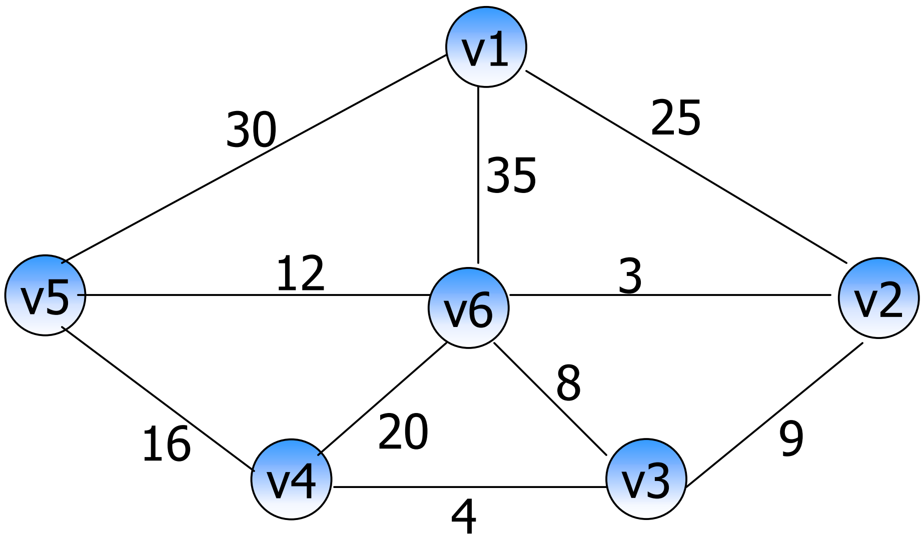
Spanning Tree

- A subgraph G′ of G which contains all the vertices of G and is a tree is called a spanning tree of G. There may be more than one spanning trees of a graph.

- In case of a weighted graph one can assign a cost to a spanning tree, defined as the sum of the weights of its edges.

- An important problem is to find out the minimum cost spanning tree (MCST) of a graph which may not be unique.

Find the Minimum Cost Spanning Tree



Kruskal's Algorithm

```
MCST-K(graph G){                //G=(V,E)
tree    T;
        init_t(T);
        sort E in non-decreasing order according to the weight of
    the edges;
        while(T contains less than (n-1) edges and E is not empty)  {
            e = edge(u,v) ∈ E having least weight;
            delete e from E;
            if (inclusion of e in T does not form a cycle)  add e to T;
        }
            if  (T contains less than (n-1) edges)
            error ("no spanning tree exists")
           else       output T as the MCST;
}
```

Complexity O(|E|log |E|)

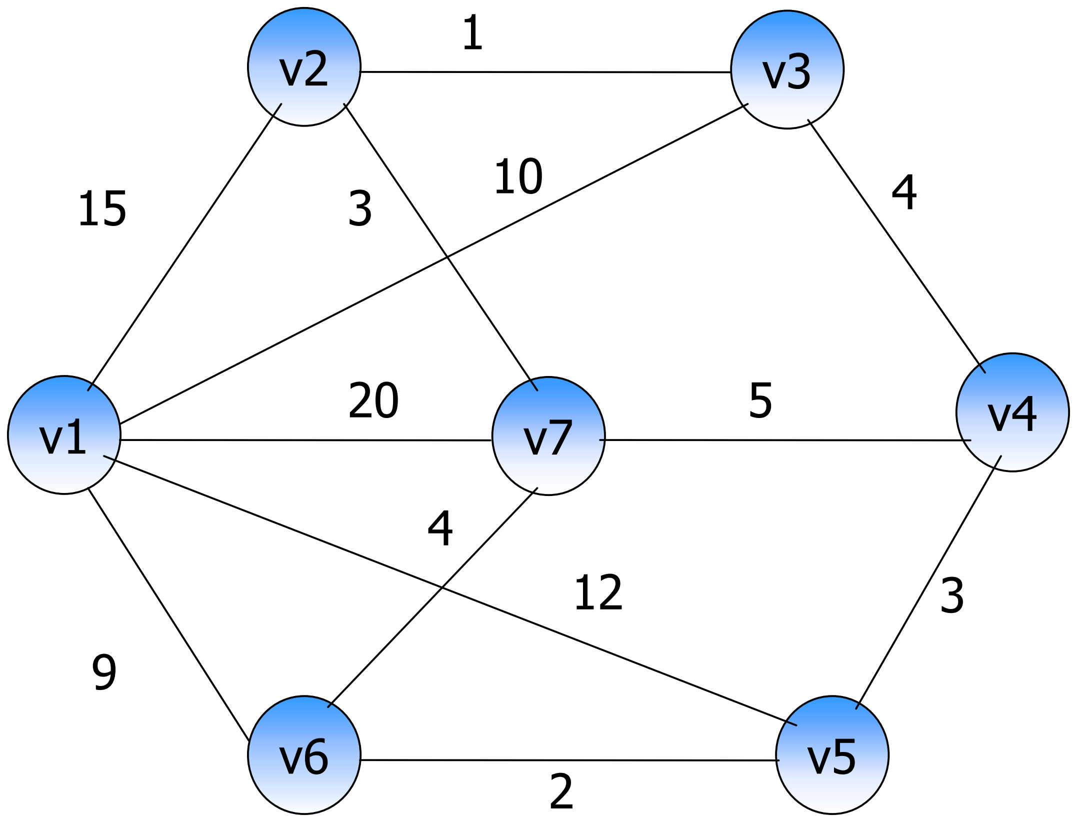
Prim's Algorithm

```
TV = { v1};
for (T=Φ; T contains fewer than n-1 edge; add (u,v) to T)
{
    Let (u,v) be a least-cost edge  such that u ∈ TV and !(v ∈ TV);
    if (there is no such edge) break;
    add v to TV;
}
if (T contains fewer than n-1 edges)
        print (“no spanning tree exists”);
```

Dijkstra's shortest path algorithm (Greedy)

SP-D(G)
//Find the shortest paths from v1 to all the vertices
{
 $l_1^*=0$;
 for (i =2; i <= n; i++} $l_i = C_{1i}$;
do{
 $l_k = \min\{l_i\}$; // among all temporary labels
 $l_k^* = l_k$;
 Let v_k be the node which just received a permanent label;
 for all adjacent vertices of v_k having temporary labels do
 $l_i = \min(l_i, l_k^* + C_{ki})$;}
while there are vertices without permanent labels;
}

Time Complexity=$O(n^2)$



v1
v2
v3
v4
v5
v6
v7
1
15
3
10
4
20
5
4
12
3
9
2

All pair Shortest Paths

```
APSP  (graph  G){
float  A[n][n];
for (i = 0; i< n; i++)
        for ( j =0; j < n; j++)
A[i][j]:=M[i][j];
for ( k = 0; k < n; k++)
    for (i = 0; i< n, i++)
            for ( j =0; j < n; j++)
 A[i][j] = min(A[i][j], A[i][k] + A[k][j]);
 }
```

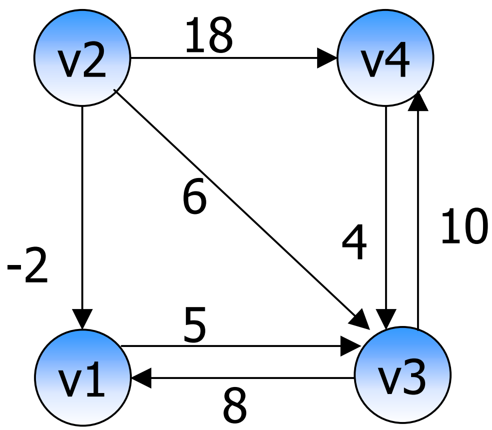
Time Complexity=O(n^3)

Transitive Closure Matrix

```
APSP (graph G){
boolean T[n][n];
for (i = 0; i< n; i++)
        for ( j =0; j < n; j++)
if (M[i,j] <> ∝)  T[i][j] =true  else T[i][j] = false;
for ( k = 0; k < n; k++)
    for (i = 0; i< n, i++)
            for ( j =0; j < n; j++)
T[i][j] = T[i][j] OR ( T[i][k] AND T[k][j])
}
```

Transitive Closure Matrix – A matrix of boolean values where a true at (i,j) represents the existence of a path between nodes i and j

Digraph with cycles and negative weight



	v1	v2	v3	v4
V1	0	∝	5	∝
V2	-2	0	6	18
V3	∝	8	0	10
V4	∝	∝	4	0