11. a) Let $\vec{A}$ and $\vec{B}$ be two vector point functions. then prove that $\operatorname{div}(\vec{A}\times\vec{B}) = \vec{B}\cdot\operatorname{Curl}\vec{A} - \vec{A}\cdot\operatorname{Curl}\vec{B}$

b) Evaluate $\iint_S \vec{F}\cdot\hat{n}ds$, where $\vec{F} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is that part of the plane $2x + 3y + 6z = 12$ which is located in the first octant. 4+6

12. a) State and prove Stoke's theorem.

b) If $\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$, then evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the curve C in the xy-plane, $y = x^3$ from the point (1, 1) to (2, 8). 6+4

13. Verify the divergence theorem for $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$, taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$. 10

14. Verify Green's theorem in the plane for

$$\oint_C \{(2x - y^3)dx - xydy\},$$

where C is the boundary of the region enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$. 10

Ex/ME/Math/T/121/2018

Bachelor of Engineering in Mechanical Engineering Examination, 2018

(1st Year, 2nd Semester)

Mathematics - II

Time : Three hours Full Marks : 100

(50 marks for each part)

Use a separate Answer-Script for each part

PART - I

Answer ***any five*** questions. 5×10=50

All questions carry equal marks.

1. a) Show that every square matrix can be uniquely expressed as a sum of a symmetric and a skew-symmetric matrices.

Hence or otherwise express the matrix A as the sum of a symmetric and a skew-symmetric matrices where

$$A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$$

b) Prove that

$$\begin{vmatrix} (b+c)^2 & c^2 & b^2 \\ c^2 & (c+a)^2 & a^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix} = 2(bc + ca + ab)^3$$

2. a) Solve the following system of equations by matrix inversion method :

$$x_1 + x_2 + x_3 = 1$$
$$x_1 + 2x_2 + 3x_3 = 6$$
$$x_1 + 3x_2 + 4x_3 = 6$$

b) Define rank of a matrix. Determine the rank of the following matrix :

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

3. a) Prove that a skew-symmetric determinant of fourth order is a perfect square.

b) Find the eigen values and the corresponding eigen vectors of the following matrix :

$$\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

4. a) State and prove Cayley-Hamilton theorem.

PART - II

Answer ***any five*** questions.

8. a) Determine the values of λ and μ, for which the vectors $(-3\hat{i}+4\hat{j}+\lambda\hat{k})$ and $(\mu\hat{i}+8\hat{j}+6\hat{k})$ are collinear.

b) Prove by vector method that in a triangle the perpendiculars drawn from the vertices to the opposite sides are concurrent. 5+5

9. a) Find the principal normal $\hat{n}$ and curvature of the space curve $x = 3\cos t$, $y = 3\sin t$, $z = 4t$.

b) Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of any point and $r = |\vec{r}|$. Then prove that

i) $\nabla^2\left(\frac{1}{r}\right) = 0$

ii) $\vec{\nabla}\cdot\left\{\frac{f(r)}{r}\vec{r}\right\} = \frac{1}{r}\frac{d}{dr}\left\{r^2 f(r)\right\}$. 5+5

10. a) Show that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force field and hence find the work done in moving an object in this field from (1, –2, 1) to (3, 1, 4).

b) Find the directional derivative of $\phi(x, y, z) = xy^2 + x^3 y$ at the point (2, –1, 1) in the direction of the normal to the surface $x \log z - y^2 + 4 = 0$ at (–1, 2, 1). 5+5

b) Prove that the locus of a variable line which interests the lines $y-z=1, \quad x=0$; $z-x=1, \quad y=0$; $x-y=1$, $z=0$ is the surface whose equation is

$$x^2+y^2+z^2-2(yz+zx+xy)=1$$

7. a) A sphere of constant radius r passes through the origin and cuts the axes in A, B, C. Prove that the locus of the foot of the perpendicular from 0 to the plane ABC is given by $(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2})=4r^2$.

b) Find the equation of a sphere circumscribing the tetrahedron whose faces are $\frac{y}{b}+\frac{z}{c}=0$, $\frac{z}{c}+\frac{x}{a}=0$, $\frac{x}{a}+\frac{y}{b}=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

b) Find the characteristic equation of the matrix

$$A=\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

and use Cayley-Hamilton theorem to find its inverse.

5. a) Prove that the lines whose direction cosines are given by the relations $al+bm+cn=0$ and $mn+nl+lm=0$ are

i) Perpendicular if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$

ii) Parallel if $\sqrt{a}+\sqrt{b}+\sqrt{c}=0$.

b) Find the condition of the coplanarity of two straight lines

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$$

and $$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

and if the condition is satisfied, find the equation of the plane on which the lines lie.

6. a) Prove that the planes $x=ry+qz$, $y=pz+rx$, $z=qx+py$ pass through one line if $p^2+q^2+r^2+2qpr=1$ and show that the equation of the line is $\frac{x}{\sqrt{1-p^2}}=\frac{y}{\sqrt{1-q^2}}=\frac{z}{\sqrt{1-r^2}}$.

[Turn over

b) Prove that the locus of a variable line which interests the lines $y - z = 1, \quad x = 0$; $z - x = 1, \quad y = 0$; $x - y = 1$, $z = 0$ is the surface whose equation is

$$x^2 + y^2 + z^2 - 2(yz + zx + xy) = 1$$

7. a) A sphere of constant radius r passes through the origin and cuts the axes in A, B, C. Prove that the locus of the foot of the perpendicular from 0 to the plane ABC is given by $(x^2 + y^2 + z^2)^2 (x^{-2} + y^{-2} + z^{-2}) = 4r^2$.

b) Find the equation of a sphere circumscribing the tetrahedron whose faces are $\frac{y}{b} + \frac{z}{c} = 0$, $\frac{z}{c} + \frac{x}{a} = 0$, $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

b) Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

and use Cayley-Hamilton theorem to find its inverse.

5. a) Prove that the lines whose direction cosines are given by the relations $al + bm + cn = 0$ and $mn + nl + lm = 0$ are

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ii) Parallel if $\sqrt{a} + \sqrt{b} + \sqrt{c} = 0$.

b) Find the condition of the coplanarity of two straight lines

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$$

and $$\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$$

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