Complex Analysis

Prob: Show that the function

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0. \\ 0, & z = 0 \end{cases}$$

satisfies the Cauchy - Riemann equations at $z=0$ but $f'(0)$ doesnot exist.

Ans) Writing $f(z) = u(x,y) + iv(x,y)$, we get

$$u(x,y) = \frac{x^3 - y^3}{x^2+y^2} \quad \text{and} \quad v(x,y) = \frac{x^3+y^3}{x^2+y^2}, \ (x,y) \neq (0,0)$$

Since $f(0) = 0$, we have $u(0,0) = v(0,0) = 0$. Now as $z \to 0$ we obtain at the point $z=0$.

$$u_x = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{x}{x} = 1.$$

$$u_y = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \to 0} \frac{-y}{y} = -1.$$

$$v_x = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x \to 0} \frac{x}{x} = 1.$$

$$v_y = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y \to 0} \frac{y}{y} = 1.$$

Therefore, at $z=0$, $u_x = v_y$ and $u_y = -v_x$. Thus the Cauchy-Riemann equations are satisfied at $z=0$.

Now, $$\lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{x^3 - y^3 + i(x^3+y^3)}{(x^2+y^2)(x+iy)} = \lim_{z \to 0} \frac{(1+i)(x^3+iy}{(x^2+y^2)(x+i}$$

$$= \lim_{z \to 0} \frac{(1+i)(x^3+iy^3)(x-iy)}{(x^2+y^2)(x+iy)(x-iy)} = \lim_{z \to 0} \frac{(1+i)(x^3+iy^3)(x-iy)}{(x^2+y^2)^2}$$

Choosing the path $y = mx$, we get

$$\lim_{x \to 0} \frac{(1+i)(1+im^3)(1-im)x^4}{(1+m^2)^2 x^4} = \frac{(1+i)(1+im^3)(1-im)}{(1+m^2)^2}$$

— which depends on m.

Therefore, the limit doesnot exist. Hence $f'(0)$ doesnot exist.

Prob: Show that an analytic function with constant modulus is constant.

Ans) Let $f(z) = u + iv$ be the given analytic function.

Then u and v satisfy C-R equations.

i.e. $u_x = v_y$ and $u_y = -v_x$. — (1)

We have $|f(z)| = c$ (const.) $\Rightarrow u^2 + v^2 = c^2$.

Diff. partially w.r.to x and y, we get

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad , \quad u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0.$$

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0 \quad \text{and} \quad u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0 \text{ [using (}$$

Eliminating $\frac{\partial u}{\partial y}$, we get,

$$(u^2+v^2)\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0 \quad \text{provided } u+iv \neq$$

Similarly, $\frac{\partial u}{\partial y} = 0$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$.

Since, u_x, u_y, v_x, v_y are all zero, therefore u and v are constant. Hence $f(z) = u + iv$ is a constant function.

Prob: If $f(z) = u + iv$ is analytic function and $u - v = e^x(\cos y - \sin y)$. Find $f(z)$.

Ans) We have $u - v = e^x(\cos y - \sin y)$.

$$u_x - v_x = e^x(\cos y - \sin y) \quad \text{--- (1)}$$

and $u_y - v_y = e^x(-\sin y - \cos y)$

$$\Rightarrow -v_x - u_x = e^x(-\sin y - \cos y)$$

$$\Rightarrow u_x + v_x = e^x(\sin y + \cos y) \quad \text{--- (2)}$$

Solving (1) & (2),

$$u_x = e^x \cos y \quad \text{--- (3)}$$

$$v_x = e^x \sin y \quad \text{--- (4)}.$$

Integrating (4), $v = e^x \sin y + f(y)$.

Differentiating it partially w.r.t. y; $v_y = e^x \cos y + f'(y)$ --- (5)

From C-R eqn, $u_x = v_y$.

$$\therefore e^x \cos y = e^x \cos y + f'(y) \quad [\text{from (3) \& (5)}]$$

$$\Rightarrow f'(y) = 0.$$

$$\Rightarrow f(y) = c.$$

$$\therefore v = e^x \sin y + c. \quad \text{--- (6)}$$

$$u = e^x(\cos y - \sin y) + v$$

$$\therefore u = e^x \cos y + c \quad \text{--- (7)}$$

$$\therefore f(z) = u + iv$$

$$= e^x \cos y + c + i(e^x \sin y + c)$$

$$= e^x(\cos y + i \sin y) + c + ic$$

$$= e^{x+iy} + \alpha$$

$$= e^z + \alpha. \quad (\underline{\text{Ans}})$$

\# **Prob:** If $u = e^x(x\cos y - y\sin y)$, find the analytic function

Ans) $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ [By C-R eq^{n}s]

$= e^x(x\sin y + \sin y + y\cos y)\,dx$ (M)

$+ e^x(x\cos y - y\sin y + \cos y)\,dy$ (N)

$\int M\,dx = \int e^x(x\sin y + \sin y + y\cos y)\,dx$

$= e^x(\cancel{x\sin y +}\sin y + y\cos y) + \sin y(xe^x - e^x)$ (takin y as co

$= e^x \cancel{\sin y} + e^x y\cos y + xe^x\sin y - e^x \cancel{\sin y}$

$= e^x(x\sin y + y\cos y)$

$\int N\,dy = \int e^x(x\cos y - y\sin y + \cos y)\,dy$

$= e^x.x\sin y + e^x\sin y + (y\cos y - \sin y)e^x$ (Taking x as const.)

$= e^x(x\sin y + y\cos y)$

$\therefore v = e^x(x\sin y + y\cos y) + c.$

$f(z) = u + iv$

$= e^x(x\cos y - y\sin y) + ie^x(x\sin y + y\cos y) + ic.$

$= e^x[x(\cos y + i\sin y) + iy(\cos y + i\sin y)] + ic$

$= e^x(x+iy)(\cos y + i\sin y) + ic.$

$= e^{x+iy}.(x+iy) + ic$

$= ze^z + ic.$ (Ans)

Prob: If $f(z)$ is an analytic function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\mathrm{Re}\, f(z)|^2 = 2|f'(z)|^2$$

Ans) Let $f(z) = u + iv$

Then $\mathrm{Re}\, f(z) = u$.

Now, $\dfrac{\partial u^2}{\partial x} = 2u\dfrac{\partial u}{\partial x}$ | $\dfrac{\partial u^2}{\partial y} = 2u\dfrac{\partial u}{\partial y}$

$\dfrac{\partial^2 u^2}{\partial x^2} = 2\left(\dfrac{\partial u}{\partial x}\right)^2 + 2u\dfrac{\partial^2 u}{\partial x^2}$ | $\dfrac{\partial u^2}{\partial y^2} = 2\left(\dfrac{\partial u}{\partial y}\right)^2 + 2u\dfrac{\partial^2 u}{\partial y^2}$

$$\therefore \frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2 + u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\right]$$

or, $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)u^2 = 2\left[\left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial u}{\partial y}\right)^2\right] + 0.$ [since $f(z)$ is analytic, u is a harmonic f

or, $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)u^2 = 2\left[\left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial v}{\partial x}\right)^2\right]$ [By C-R eq^{n}s]

or, $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)|u|^2 = 2|f'(z)|^2$ [$\because f'(z) = u_x + iv_x$]

Hence,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\mathrm{Re}\, f(z)|^2 = 2|f'(z)|^2$$

Prob: Evaluate $\int_{(0,1)}^{(3,10)} [(3x+y)\,dx + (2y-x)\,dy]$ along the curve $y = x^2$.

Ans) Along the curve $y = x^2 + 1$, we have

$$dy = 2x\,dx.$$

and x varies from 0 to 3.

$$\therefore \int_{(0,1)}^{(3,10)} [(3x+y)\,dx + (2y-x)\,dy]$$

$$= \int_0^3 [(3x+x^2+1)\,dx + (2x^2-2-x)\,2x\,dx] \quad [y = x^2$$

$$= \int_0^3 (4x^3 - x^2 + 7x + 1)\,dx = \left[x^4 - \frac{x^3}{3} + 7.\frac{x^2}{2} + x\right]_0^3$$

$$= \frac{213}{2}.$$

Prob: Evaluate $\int_{(0,1)}^{(2,5)} [(3x+y)\,dx + (2y-x)\,dy]$ along the line joining $(0,1)$ and $(2,5)$.

Ans) Equation of the line joining $(0,1)$ and $(2,5)$ is

$$\frac{y-1}{x-0} = \frac{1-5}{0-2} \Rightarrow y = 2x+1$$

$$\Rightarrow dy = 2\,dx.$$

$$\therefore \int_{(0,1)}^{(2,5)} [(3x+y)\,dx + (2y-x)\,dy] = \int_0^2 [(3x+2x+1)\,dx + (4x+2-x)\,2\,dx]$$

$$= \int_0^2 (11x+5)\,dx.$$

$$= \left[\frac{11x^2}{2} + 5x\right]_0^2$$

$$= 32.$$

Prob : If $f(z) = \frac{z^2+5z+6}{z-2}$, does Cauchy's theorem applicable when C is a circle of radius 1 with origin as centre.

Ans) $f(z)$ is not analytic at $z=2$.

When C is the circle $|z|=1$, the point $z=2$ lies outside C as a result $f(z)$ is analytic within and on C. Hence Cauchy's theorem is applicable i.e.

$$\int_C \frac{z^2+5z+6}{z-2}\,dz = 0.$$

Prob : Evaluate $\int_C \frac{e^z}{z-2}\,dz$ where C is the circle

(a) $|z|=3$ and (b) $|z|=1$

Ans) (a) Let $f(z)=e^z$.

e^z is analytic in the z-plane.

$z=2$ lies inside the circle $|z|=3$.

$\therefore$ By Cauchy's ~~theorem~~ integral formula,

$$\frac{1}{2\pi i}\int_C \frac{e^z}{z-2}\,dz = f(2) = e^2. \quad \text{where } C: |z|=3$$

$$\therefore \int_C \frac{e^z}{z-2}\,dz = 2\pi i e^2.$$

(b) $\frac{e^z}{z-2}$ is analytic within and on the circle $|z|=1$ as $z=2$ lies outside $|z|=1$.

$\therefore$ From Cauchy's integral theorem $\int_C \frac{e^z}{z-2}\,dz = 0$

where $C: |z|=1$.

Prob: Expand $\dfrac{1}{z(z^2-3z+2)}$ in a Laurent's series valid for the regions

(i) $0<|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$

Ans) Let $f(z) = \dfrac{1}{z(z^2-3z+2)} = \dfrac{1}{z(z-1)(z-2)}$

Resolving into partial fractions, we get

$$f(z) = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$$

(i) $0<|z|<1$.

We have $f(z) = \dfrac{1}{2z} + (1-z)^{-1} - \dfrac{1}{4}\left(1-\dfrac{z}{2}\right)^{-1}$

$$= \frac{1}{2z} + (1+z+z^2+z^3+\cdots) - \frac{1}{4}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$$

$$= \frac{1}{2z} + \frac{3}{4} + \frac{7}{8}z + \frac{15}{16}z^2 + \frac{31}{32}z^3 + \cdots \quad \text{(Ans.)}$$

(ii) $1<|z|<2$. Then $\dfrac{1}{|z|}<1$ and $\dfrac{|z|}{2}<1$.

$$f(z) = \frac{1}{2z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{4}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{2z} - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) - \frac{1}{4}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3\right]$$

$$= \left(-\frac{1}{2z}-\frac{1}{z^2}-\frac{1}{z^3}-\cdots\right) - \frac{1}{4}\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots\right) \quad \text{(Ans.)}$$

(iii) $|z|>2$. Then $\dfrac{2}{|z|}<1$

$$\therefore f(z) = \frac{1}{2z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{1}{2z} - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) + \frac{1}{2z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}\cdots\right)$$

$$= (2-1)\frac{1}{z^3} + (2^2-1)\frac{1}{z^4} + (2^3-1)\frac{1}{z^5} + \cdots \quad \text{(Ans.)}$$

2

Prob: Classify the singular point $z=0$ of the functions

(i) $\dfrac{e^z}{z+\sin z}$ (ii) $\dfrac{e^z}{z-\sin z}$

Hence obtain the principal part of the Laurent series expansion of $f(z)$ in each case.

Ans) (i) Write the denominator of $f(z)$ as

$$z+\sin z = z + \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$

$$= 2z - \frac{z^3}{3!} + \cdots = z\,g(z)$$

where, $g(z) = 2 - \dfrac{z^2}{3!} + \cdots$ and $g(0) \neq 0$

Therefore, we can write, $f(z) = \dfrac{e^z}{z\,g(z)}$, $g(0)\neq 0$

Hence $z=0$ is a simple pole.

To obtain the principal part of the Laurent series expansion of $f(z)$ about $z=0$, we write

$$f(z) = \frac{e^z}{z+\sin z} = \frac{1+z+(z^2/2!)+\cdots}{2z - \frac{z^3}{3!}+\cdots}$$

$$= \frac{1+z+(z^2/2!)+\cdots}{2z\left[1-(z^2/12)+\cdots\right]}$$

$$= \frac{1}{2z}\left[1+z+\frac{z^2}{2}+\cdots\right]\left[1-\left(\frac{z^2}{12}+\cdots\right)\right]^{-1}$$

$$= \frac{1}{2z}\left[1+z+\frac{z^2}{2}+\cdots\right]\left[1+\frac{z^2}{12}+\cdots\right]$$

$$= \frac{1}{2z}\left[1+z+\frac{7z^2}{12}+\cdots\right]$$

$$= \frac{1}{2z}+\frac{1}{2}+\frac{7z}{24}+\cdots$$

Therefore, the principal part of the Laurent series is $1/(2z)$.

/ Ans.

(ii) Write the denominator of $f(z)$ as

$$z - \sin z = z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = \frac{z^3}{3!} - \frac{z^5}{5!} + \cdots$$

$$= z^3 g(z)$$

where $g(z) = \frac{1}{6}\left(1 - \frac{z^2}{20} + \frac{z^4}{840} - \cdots\right)$ and $g(0) \neq 0$

Therefore, we can write

$$f(z) = \frac{e^z}{z^3 g(z)} \quad , g(0) \neq 0 .$$

Hence, $z=0$ is pole of order 3.

To determine the principal part of the Laurent series expansion of $f(z)$ about $z=0$, we write

$$f(z) = \frac{e^z}{z - \sin z} = \frac{6\left[1 + z + (z^2/2!) + (z^3/3!) + \cdots\right]}{z^3\left(1 - \frac{z^2}{20} + \frac{z^4}{840} - \cdots\right)}$$

$$= \frac{6}{z^3}\left[1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots\right]\left[1 - \left(\frac{z^2}{20} - \frac{z^4}{840} + \cdots\right)\right]^{-1}$$

$$= \frac{6}{z^3}\left[1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots\right]\left[1 + \frac{z^2}{20} + \frac{11 z^4}{8400} + \cdots\right]$$

$$= \frac{6}{z^3}\left[1 + z + \frac{11}{20}z^2 + \frac{13}{60}z^3 + \cdots\right]$$

Therefore, the principal part of the Laurent series is

$$\frac{6}{z^3} + \frac{6}{z^2} + \frac{33}{10z} \cdot \text{(Ans)}.$$

Prob: Show that the function $\frac{(z^2+4)}{e^z}$ has an isolated essential singularity at $z = \infty$.

Solⁿ: $f(z) = \frac{z^2+4}{e^z}$

Putting $z = 1/y$,

$$\therefore f(1/y) = \left(4 + \frac{1}{y^2}\right) e^{-1/y}$$

$$= \left(4 + \frac{1}{y^2}\right)\left(1 - \frac{1}{y} + \frac{1}{2!}\cdot\frac{1}{y^2} - \frac{1}{3!}\cdot\frac{1}{y^3} + \right.$$

$$= 4 + \left[-\frac{4}{y} + (1+2)\frac{1}{y^2} + \left(-1-\frac{2}{3}\right)\frac{1}{y^3} + \left(\frac{1}{2}+\frac{1}{6}\right)\frac{1}{y^4}\right.$$

$$= 4 + \left[-\frac{4}{y} + \frac{3}{y^2} - \frac{5}{3y^3} + \frac{2}{3y^4} - \cdots\right]$$

We have infinite number of terms in the negative powers of y in the principal part of the expansion of $f(1/y)$, therefore $f(1/y)$ has an isolated essential singularity at $y = 0$.

Hence $f(z)$ has an isolated essential singularity at $z = \infty$ (Ans.)

Prob: What kind of singularity have the function $f(z) = \tan\frac{1}{z}$ at $z = 0$.

Ans> $f(z) = \tan(1/z) = \frac{\sin(1/z)}{\cos(1/z)}$.

Poles of $f(z)$ are given by $\cos(1/z) = 0$

$$\frac{1}{z} = 2n\pi \pm \pi/2$$

$$z = \frac{1}{(2n \pm 1/2)\pi}$$, where n is any integer

Since, $z = 0$ is a limit point of these poles, therefore $z = 0$ is a non-isolated essential singularity. (Ans.)

Prob: Show that the function e^z has an isolated essential singularity at $z=\infty$.

Ans) Let $f(z)=e^z$. Then $f(1/w)=e^{1/w}$

The behaviour of $f(z)$ at $z=\infty$ will be the same as that of $f(1/w)$ at $w=0$.

Now, $f(1/w)=e^{1/w}=1+\frac{1}{w}+\frac{1}{2!}\cdot\frac{1}{w^2}+\frac{1}{3!}\cdot\frac{1}{w^3}+\cdots$

We have infinite number of terms in the negative powers of w in the principal part of the expansion of $f(1$

therefore $e^{1/w}$ has an isolated essential singularity at $w=0$. Hence e^z has an isolated essential singularity at $z=\infty$.

Prob: Find the residue at all the singular points of $f(z)=\frac{1}{z^3+z^5}$

Ans) $f(z)=\frac{1}{z^3+z^5}=\frac{1}{z^3(1+z^2)}=\frac{1}{z^3(z+i)(z-i)}$

has simple poles at $z=\pm i$ and a pole of order 3 at $z=0$

Hence, $\operatorname{Res}_{z=i} f(z)=\lim_{z\to i}[(z-i)f(z)]=\lim_{z\to i}\left[\frac{1}{z^3(z+i)}\right]=\frac{1}{2}$.

$\operatorname{Res}_{z=-i} f(z)=\lim_{z\to -i}[(z+i)f(z)]=\lim_{z\to -i}\left[\frac{1}{z^3(z-i)}\right]=\frac{1}{2}$

$\operatorname{Res}_{z=0} f(z)=\frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left[z^3f(z)\right]$

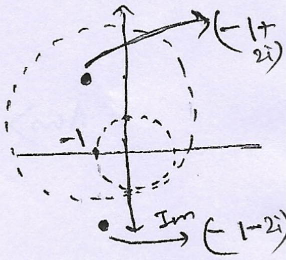
$=\frac{1}{2}\cdot\lim_{z\to 0}\frac{d^2}{dz^2}\left[\frac{1}{1+z^2}\right]$

$=\frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\left[1-z^2+z^4-\cdots\right]$

$=\frac{1}{2}\lim_{z\to 0}\left[-2+12z^2-\cdots\right]=-1$ (Ans)

Prob: Evaluate $\int_C \frac{z-3}{z^2+2z+5}\,dz$ where C is the circle

(i) $|z|=1$ (ii) $|z+1-i|=2$ (iii) $|z+1+i|=2$



Ans) Let $f(z) = \frac{z-3}{z^2+2z+5}$

Poles of $f(z)$ are given by $z^2+2z+5=0$ or, $z=-1\pm 2i$

(i) The poles $z=-1+2i$, $-1-2i$ lie outside the circle $|z|=1$ so that $f(z)$ is analytic everywhere within C.

$\therefore$ By Cauchy's theorem, we have

$$\int_C \frac{z-3}{z^2+2z+5}\,dz = 0.$$

(ii) Only one pole $z=-1+2i$ lies inside the circle $|z+1-i|=2$

Thus $f(z)$ is analytic at every point within C except at $z=-1+2i$.

Now, residue at $z=-1+2i$ is

$$\lim_{z\to -1+2i}\left[\{z-(-1+2i)\}f(z)\right]$$

$$= \lim_{z\to -1+2i} \frac{(z+1-2i)(z-3)}{z^2+2z+5} = \lim_{z\to -1+2i} \frac{z-3}{z+1+2i}$$

$$= \frac{1}{2}+i$$

$\therefore$ By residue theorem, we have

$$\int_C f(z)\,dz = 2\pi i \times \text{Residue at } (-1+2i) = \pi(-2+i)$$

(iii) Only one pole $z=-1-2i$ lies inside the circle C given by $|z+1+i|=2$ so that $f(z)$ is analytic within C except at $z=-1$.

Residue at $z=-1-2i$ is

$$\lim_{z\to -1-2i} \frac{(z+1+2i)(z-3)}{z^2+2z+5} = \frac{1}{2}-i$$

$\therefore$ By Residue theorem, $\int f(z)\,dz = 2\pi i \times \left(\frac{1}{2}-i\right) = \pi(2+i)$.

Fourier Series

\# Find the Fourier cosine and sine series of the function

$$f(x) = 1 \quad , \quad 0 \leq x \leq 2.$$

Ans) Fourier cosine series is $f(x) = 1$

Fourier sine series is

$$f(x) = \frac{4}{\pi}\left[\sin\left(\frac{\pi x}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{2}\right) + \cdots\right]$$

\# Find the Fourier cosine and sine series of the function

$$f(x) = x, \quad 0 \leq x \leq \pi.$$

Ans) Fourier cosine : $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left\{\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2}\right.$

Fourier sine : $f(x) = 2\left\{\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right.$

\# Expand in Fourier series $x + x^2$ on $-\pi \leq x \leq \pi$ and deduce that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

Ans) $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\cancel{\frac{\pi^2}{2}} + \frac{\pi^3}{3} - \cancel{\frac{\pi^2}{2}} + \frac{\pi^3}{3}\right.$$

$$= \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\cos nx\,dx = \frac{4}{n^2}\cos n\pi = \begin{cases} 4/n^2, & n \text{ even} \\ -4/n^2, & n \text{ odd} \end{cases}$$

Similarly, $b_n = \begin{cases} -2/n & , n = \text{even} \\ 2/n & , n = \text{odd.} \end{cases}$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi^2}{3} - 4\left\{\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right\}$$

$$+ 2\left\{\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right\} \text{ (Ans).}$$

At $x = \pm\pi$, the series converges to $\frac{1}{2}\{f(-\pi+) + f(\pi-)\}$

$$= \frac{1}{2}\{-\pi + \pi^2 + \pi + \pi^2\} = \pi^2.$$

$\therefore$ At $x = \pi$,

$$\pi^2 = \frac{1}{3}\pi^2 - 4\left\{-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \cdots\right\}$$

or, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{1}{6}\pi^2$ (proved)