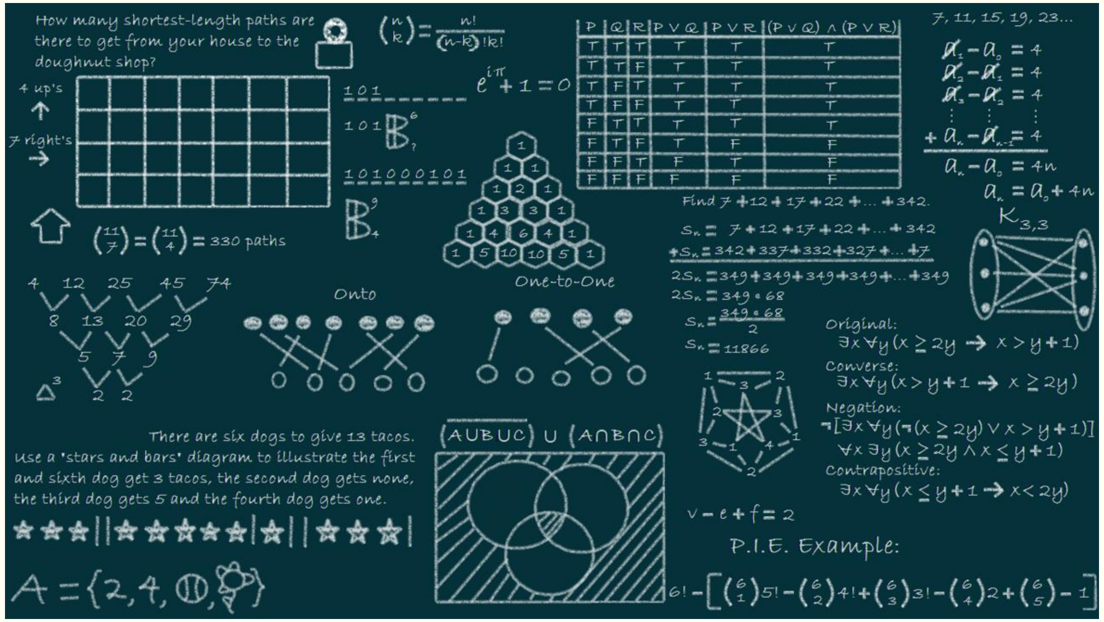
CALCULATING BIG-O EXPERIMENTALLY & MATHEMATICALLY



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Portland State University

Lecture Outline

- Find Duplicate Algorithm
  - A first analysis on time complexity
- Experimental Analysis
- Mathematical Analysis
  - Closed form runtime using sums
  - Asymptotic Analysis
  - Calculating Big-O

FIND DUPLICATE ALGORITHM

A first analysis on time complexity

Why Algorithms? An Example

- Let's discuss a problem that frequently arises when developing a web search engine.
- Every day, we have a list A of web pages that we have to visit.

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  - What does “visiting” a web page mean?

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- Question: are there any duplicate links in list A?

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- Question: are there any duplicate links in list A?
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  - Most times list A contains a great amount of duplicate links.
  - It is a **huge waste of resources** to revisit those links.
  - If we can find duplicate links eventually we can create a new list with unique entries.

Why Algorithms? An Example

- Recap:
  - A list A that contains strings representing webpages.
  - List A can have duplicate entries.
  - Define FindDuplicate(A), return true if there is a duplicate, false otherwise.
- Question: how do we compute FindDuplicate(A)?
- Any ideas?

FindDuplicate(A) – First Version

Input: a length *n* array A of strings

Output: *true* if the array contains a duplicate, *false* otherwise

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for i=1 to n:
      for j=1 to n:
            if  (i != j) AND A[i] == A[j]
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- What can we say about how fast this would run? How many comparisons?

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- runtime function $f(n) \approx n^2$, with input variable n
- This is our first analysis of **time complexity!**

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- What if the first two elements are duplicates? This is the <u>best case</u>. Constant running time.
  - Always the same regardless of input size *n*

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  - Running time increases with *n*
- What about average case running time? Often difficult to determine but can be useful.

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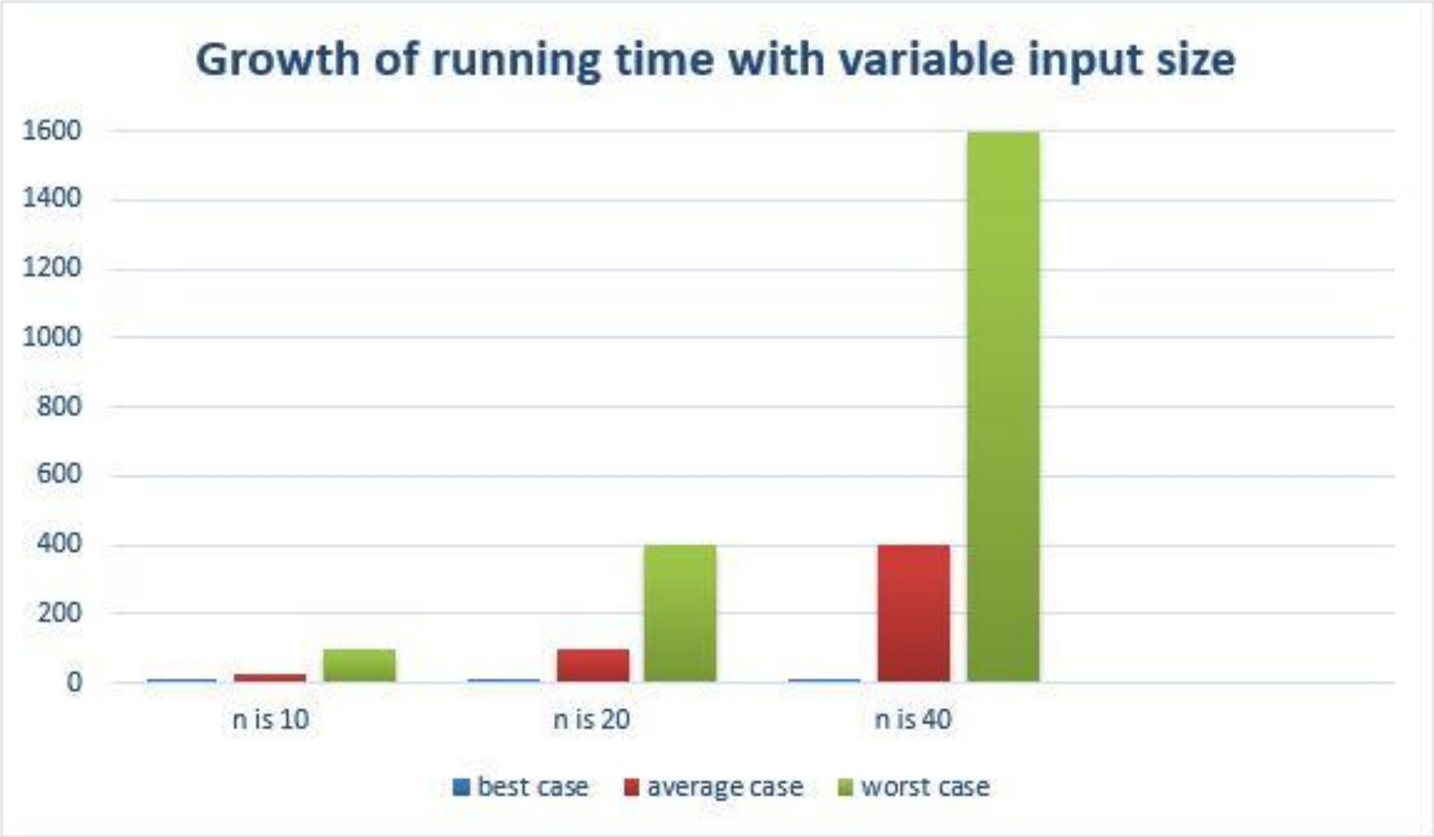

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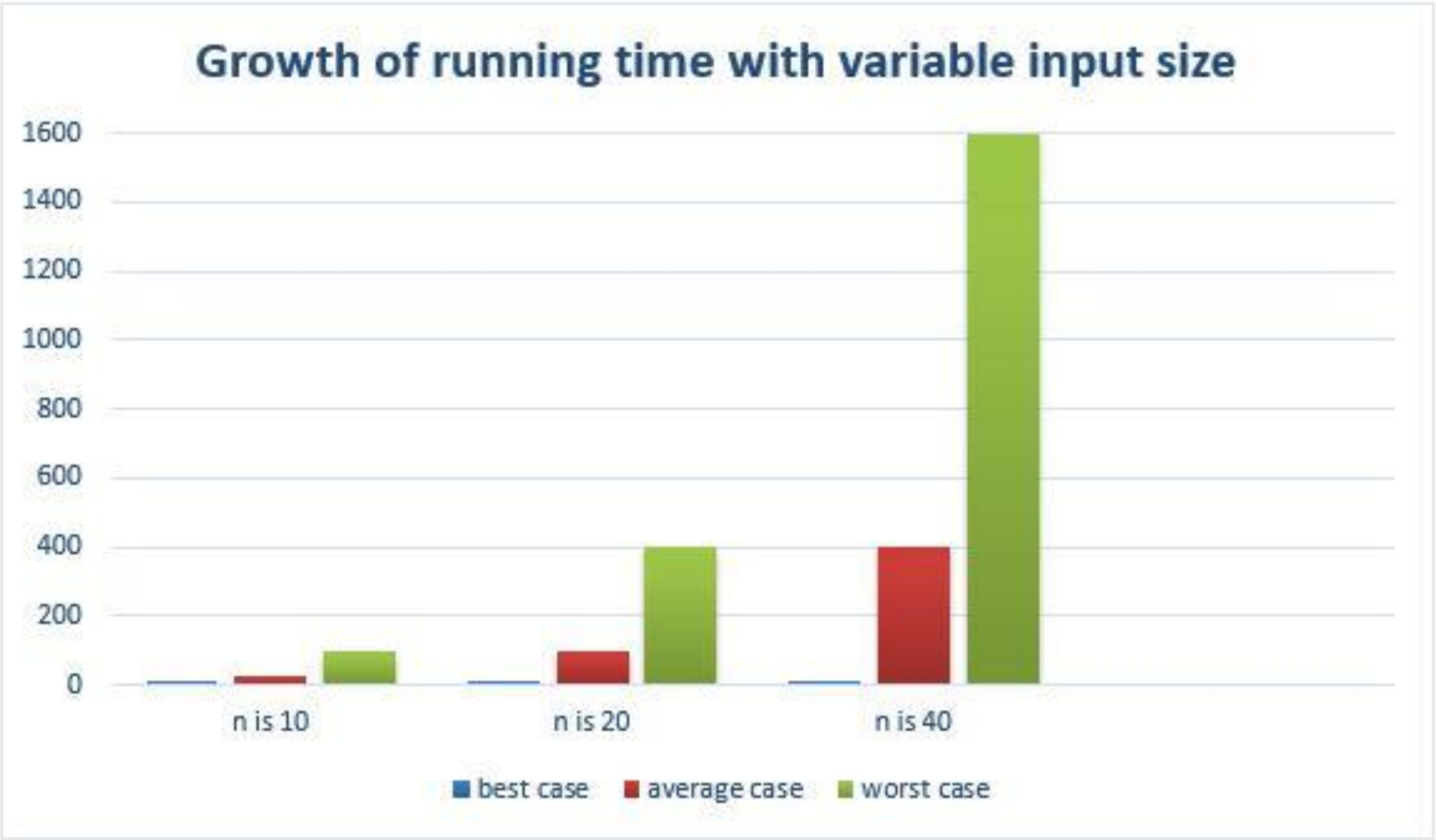

- Worst case complexity is what we will use mostly from now on.
- Easier to analyze
- **Very useful as input *n* grows very large**
- Asymptotic analysis

FindDuplicate(A) – First Version Speed

- What does all this mean in practice?
- We need n^2 comparisons in the worst case.
- Suppose A has 5 million items. We need to do 25 trillion comparisons.

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- Suppose A has 5 million items. We need to do 25 trillion comparisons.
- On a computer that can do 1 billion comparisons per second
  - This would take about 7 hours
  - This is very optimistic, in practice, it would be at least a few dozen hours

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  - **CAN WE DO BETTER?**
  - Hint: sort list A

FindDuplicate(A) – Second Version

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sort A
for i=1 to n-1:
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- We need to: sort A and execute the for loop.

FindDuplicate(A) – Second Version Speed

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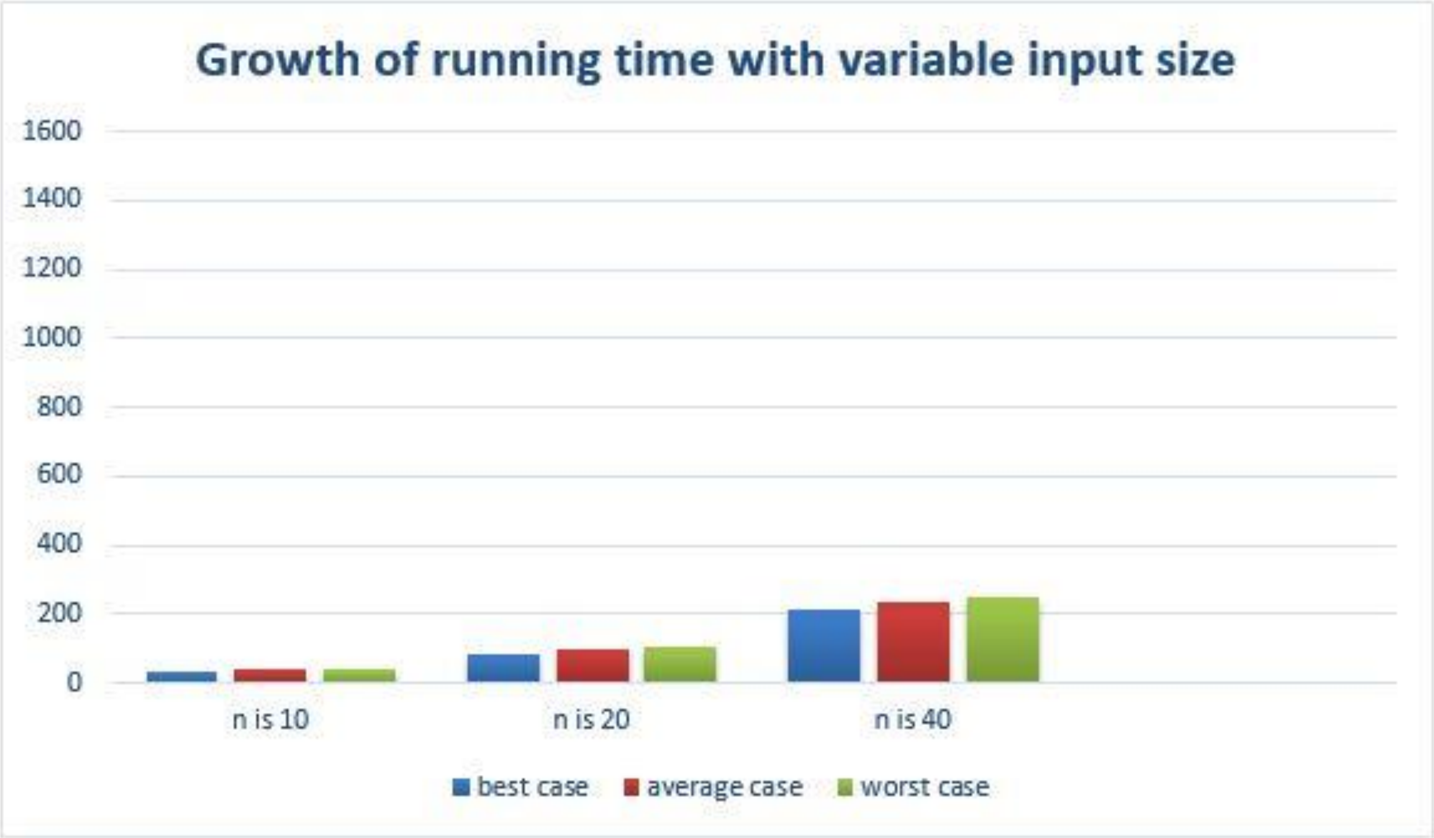
- How many iterations does the for loop take?
  - no more than n.
- Total iterations n *(log(n)+1) ≈ n*log(n), constants do not matter!

FindDuplicate(A) – Second Version Speed

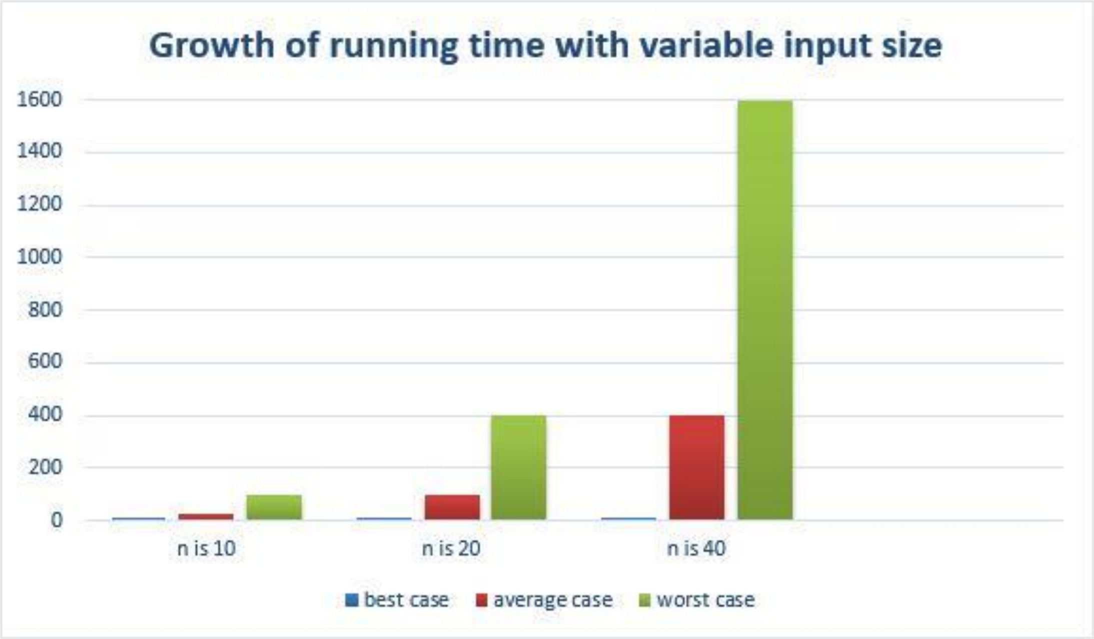
- Will it always run all the way through?
- What if the first two elements of sorted array are duplicates? This is the best case.
  - ≈ n*log(n) + 1, we always have to sort the array
- What if the last two elements of sorted array are duplicates? This is the worst case
  - ≈ n*log(n) + n
- What about average case running time? Proportional to n*log(n)

FindDuplicate(A) – Second Version Speed

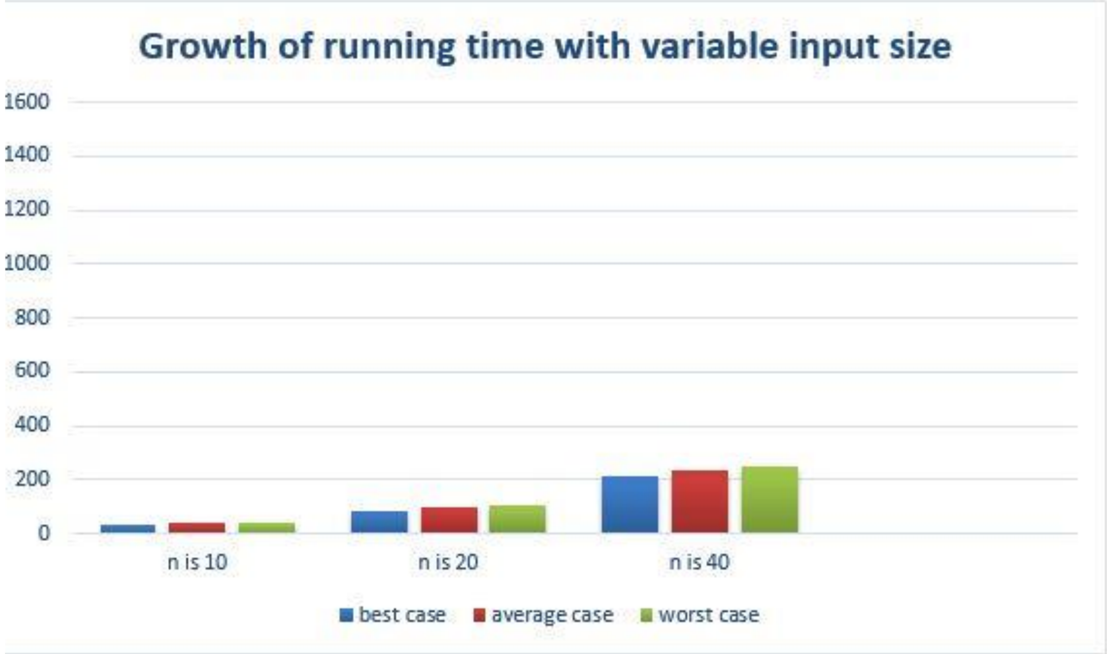


FindDuplicate(A) – Second Version Speed

$f_1(n) \approx n^2$



$f_2(n) \approx n*log(n)$



FindDuplicate(A) – Second Version Speed

- The worst case number of calculations is approximately *n*log(n)*
- Suppose A has 5 million items.
  - log(*n)* is 22.253
  - We need to do 111.27 million comparisons.

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- Suppose A has 1 billion items. On a computer that can do 1 billion comparisons per second, this would take about 30 seconds

FindDuplicate(A) – Second Version Speed

	Worst case running time for first version ≈ n^2	Worst case running time for second version ≈ $n*log(n)$
Input size $n = 5*10^6$	7 hours	110 milliseconds
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  - When comparing running times of two implementations, we must make sure the comparison is fair. Same hardware and software environments must be used

EXPERIMENTAL ANALYSIS

Experimental Analysis

- According to our time complexity analysis from the previous section, the brute-force implementation for FindDuplicate(A) requires n^2 comparisons.

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- How accurately does this approximation represent the actual running time function?
- Can we experimentally come up with a more accurate representation for the running time function?

Experimental Analysis

Code a program implementing the given algorithm

↓

Generate datasets with varying size/composition

↓

Run the program with those datasets as input

↓

Measure of the actual running time

↓

Plot results

↓

Curve fitting

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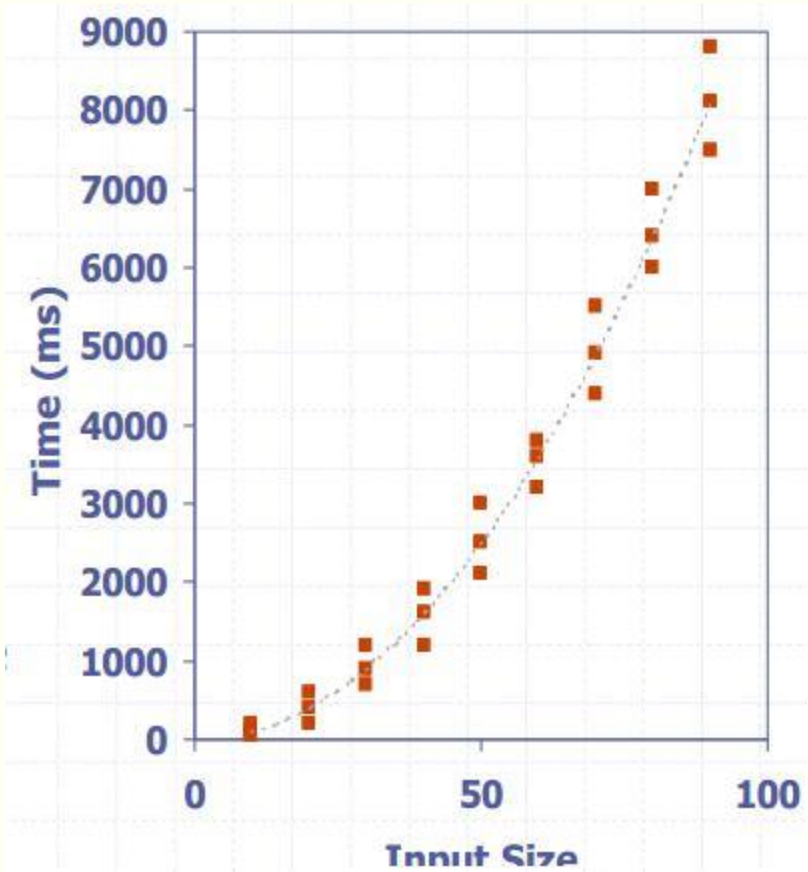

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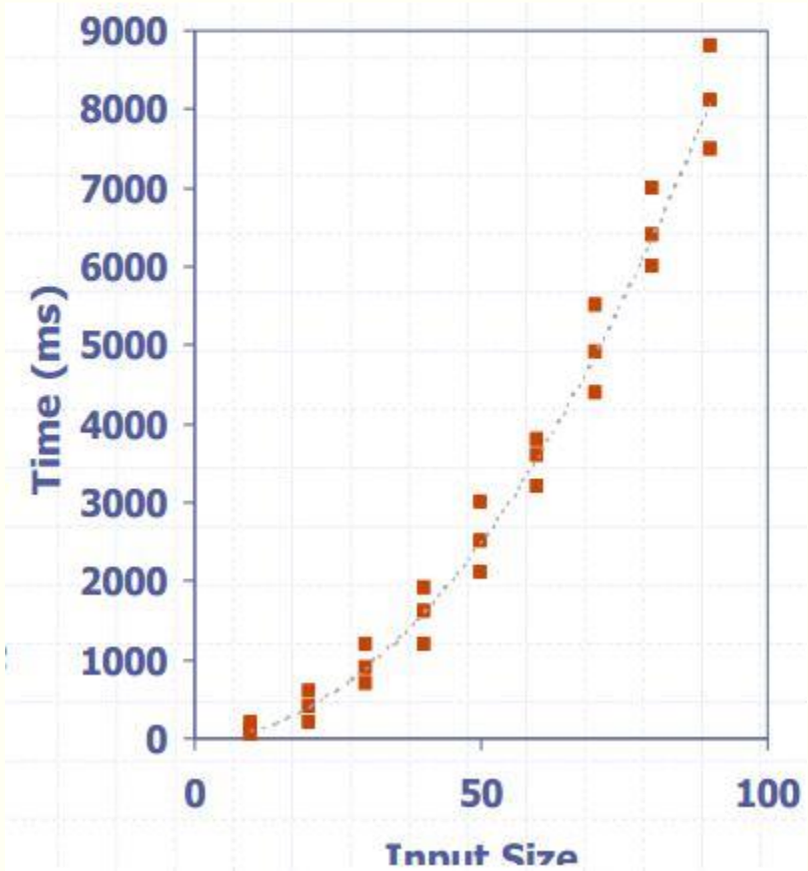
- Use curve fitting techniques to approximate the function (e.g., least squares fitting)



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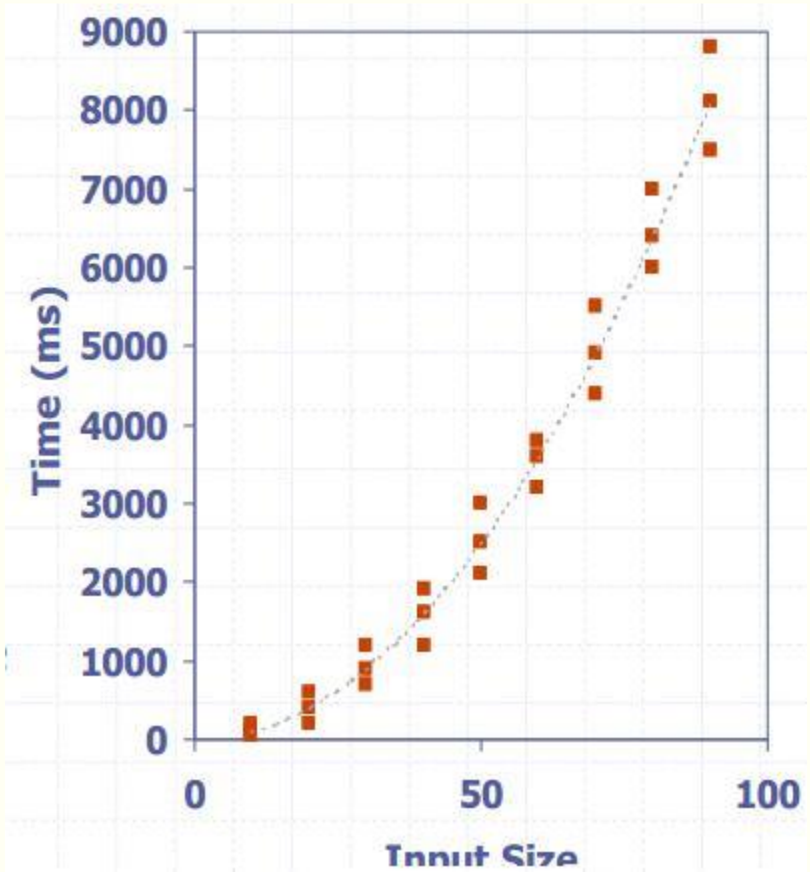
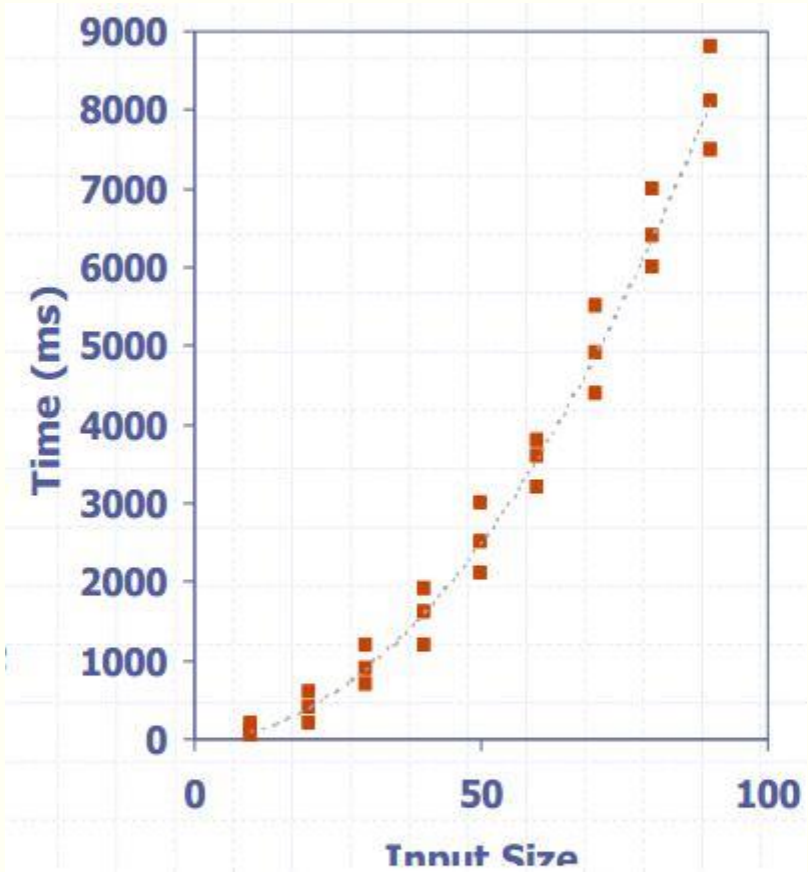
- Use curve fitting techniques to approximate the function (e.g., least squares fitting)
- $f_1(n) = c_1 * n^2 + c_2 * (n+1)$



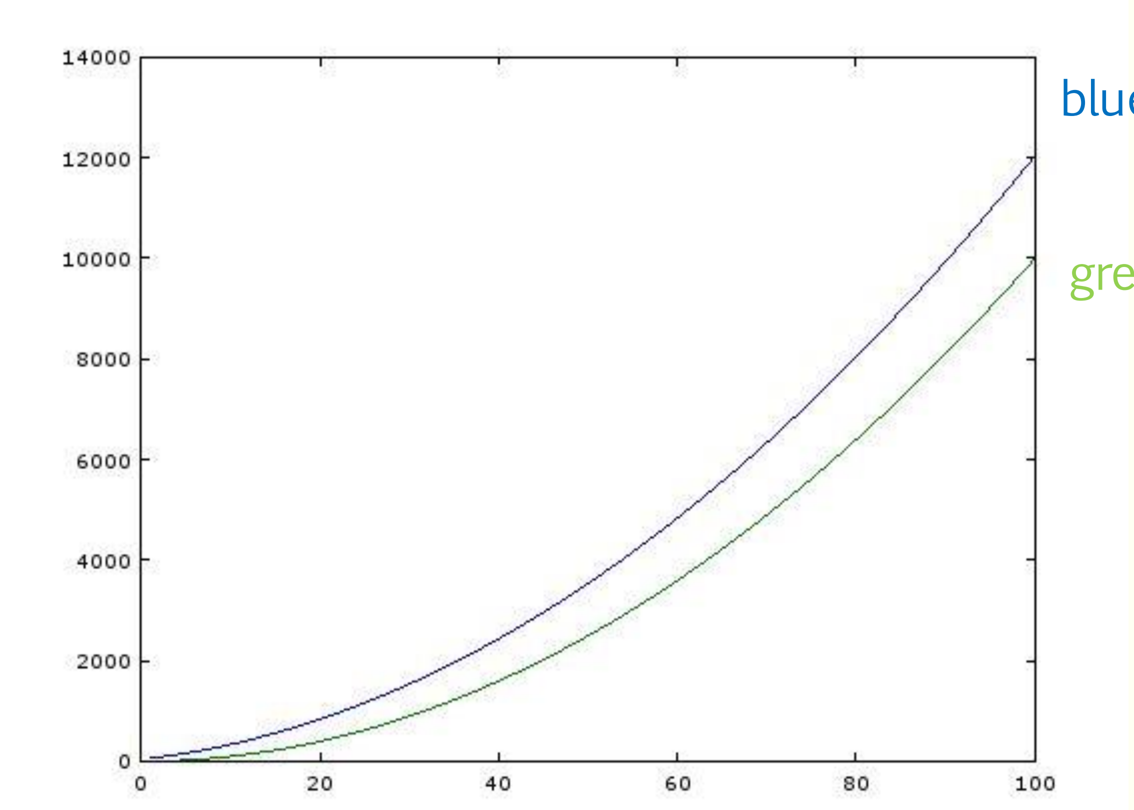
Experimental Analysis

- Use curve fitting techniques to approximate the function (e.g., least squares fitting)
- $f_1(n) = c_1 * n^2 + c_2 * (n+1)$
- How does this compare with time complexity $\approx n^2$ from the previous section?



Experimental Analysis

Running time



blue curve: $c_1 * n^2 + c_2 * (n+1)$

green curve: n^2

Size of input n

Experimental Analysis

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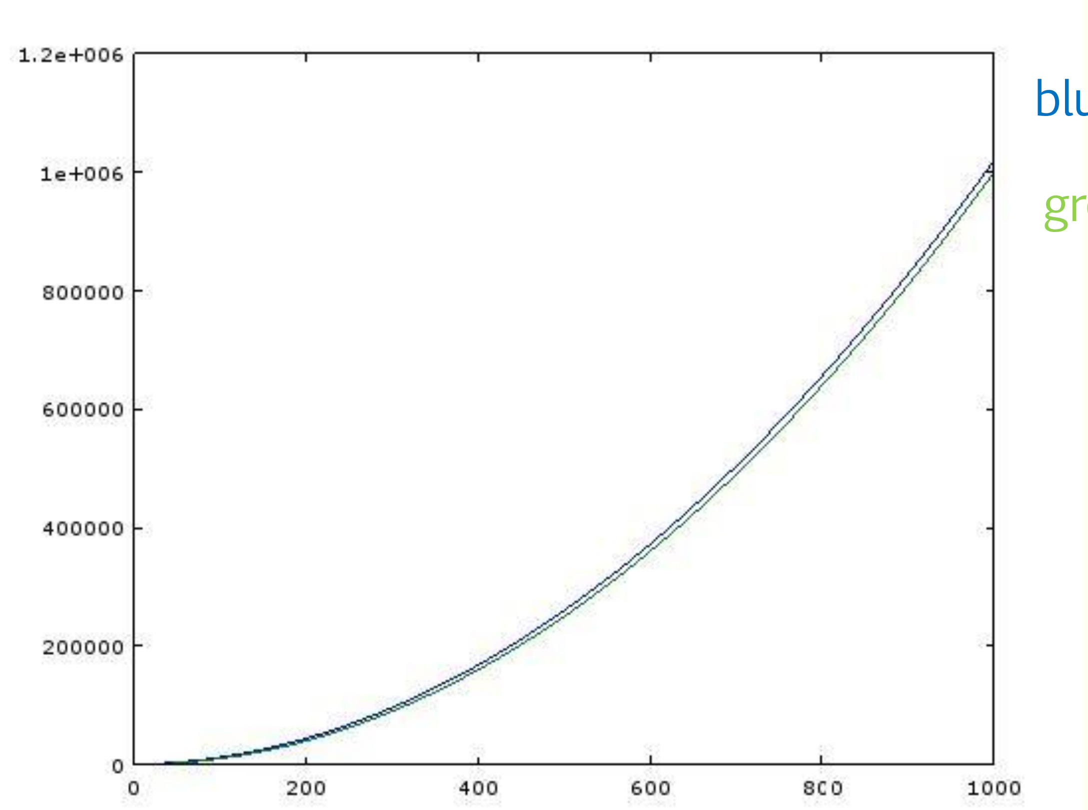

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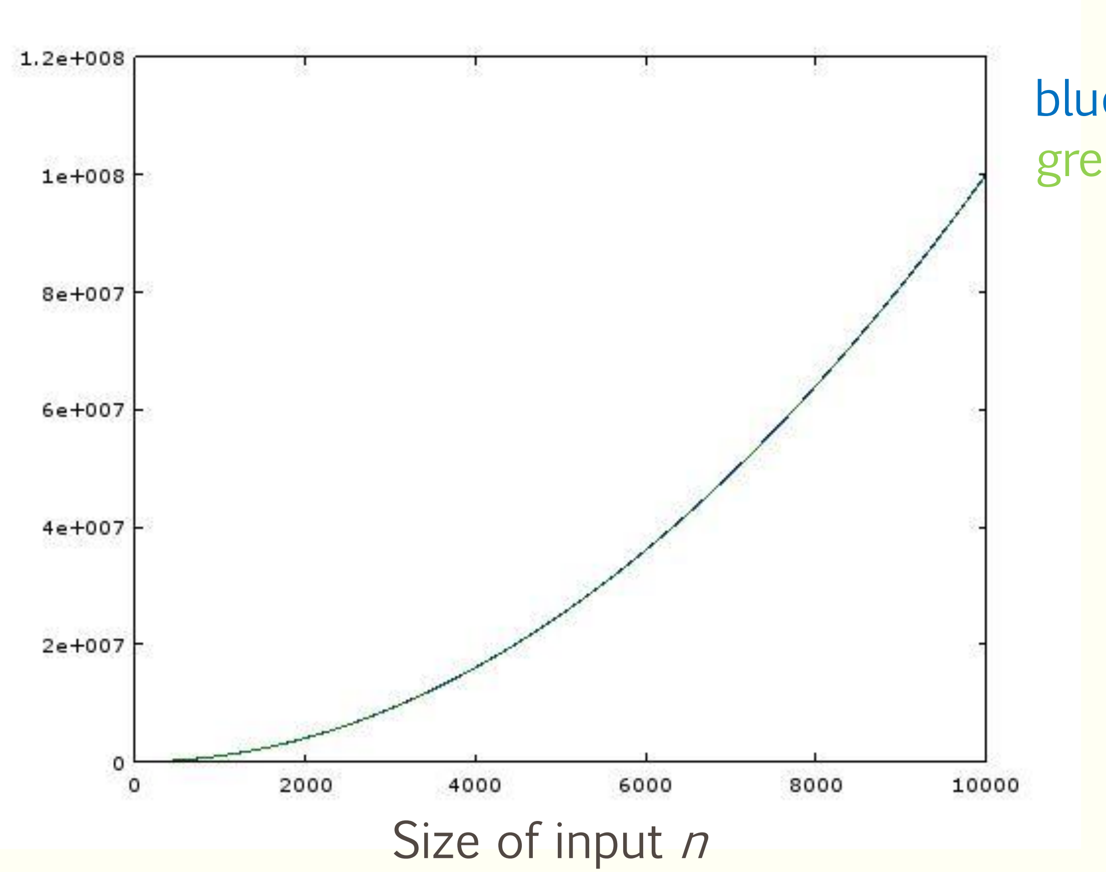

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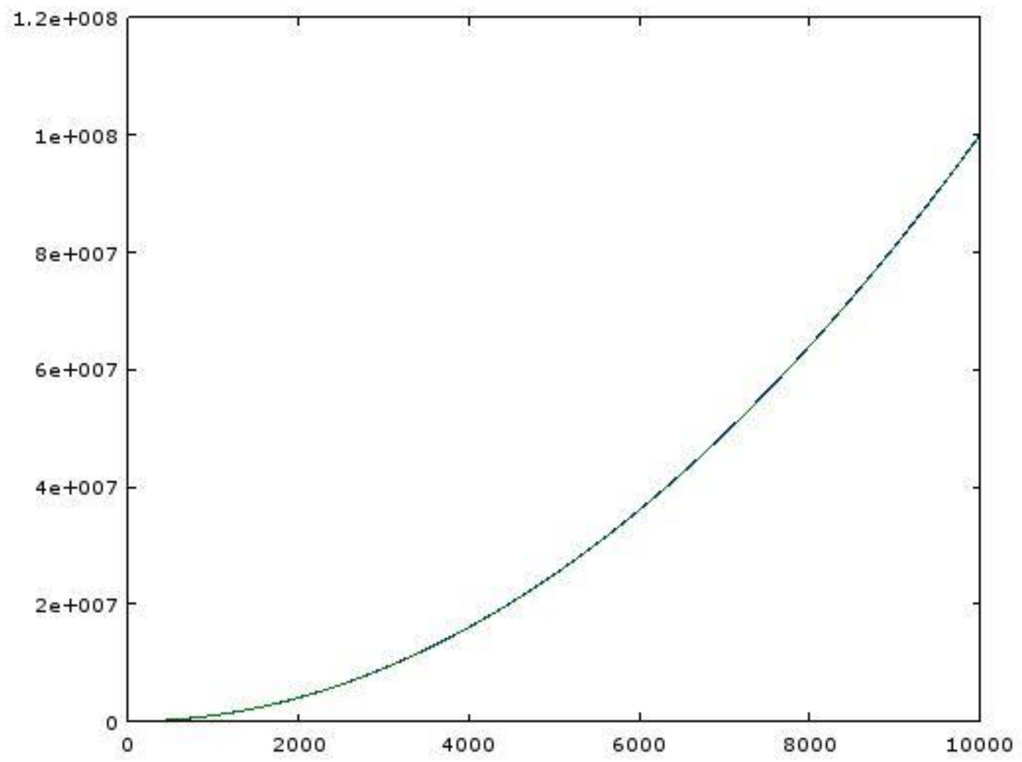

Size of input *n*

- Plotting the functions can provide us quickly with some useful information
- As n grows very large the two functions have similar behavior
- Plotting functions is not an accurate scientific method
- Need for mathematical analysis

Experimental Analysis Conclusion

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  - Collecting or generating the datasets can be challenging. Results may not be indicative of the running time on other inputs not included in the experiment.

Experimental Analysis Conclusion

- However, experimental analysis is not always a good option.
  - For example, if it would take days or weeks to run the programs
  - Collecting or generating the datasets can be challenging. Results may not be indicative of the running time on other inputs not included in the experiment.
  - Same hardware and software environments must be used
  - Implementations using different programming languages may tell us more about the difference between the languages than the difference between implementations.

MATHEMATICAL ANALYSIS

Mathematical Analysis

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- <u>Measure input size</u>
  - find key quantity to measure the **size of the data** that determine the running time.
- Sometimes obvious: length of array *n*
- For graph problems can be nodes and edges

Mathematical Analysis

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- <u>Identify Algorithm's basic operation</u>
  - Identify the important operations/instructions in the program
  - <u>Basic operation</u>: the operation that uses the most runtime overall (typically, most expensive operation of innermost loop)

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- Identify Algorithm's basic operation
  - integer comparison - c_0
  - Array access - c_1
  - Conditional - c_2
  - Loop - c_l

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- Identify Algorithm’s basic operation
  - Sometimes more useful to clump operations:

$$c_{op} = 2 * c_0 + c_1 + \ \ldots$$

Mathematical Analysis

Decide on a parameter (or parameters) for measuring the size of the input

Identify the Algorithm's basic operation

Determine if best, worst, and average case will be different

Set up a sum representing the number of times the operation happens

Find a closed form formula for the expression

Mathematical Analysis

- Closed form runtime:

$$C_{\text{OUTERLOOP}} = C_l + \sum_{i=1}^{n} C_{INNERLOOP} = C_l + \sum_{i=1}^{n} (C_l + \sum_{j=1}^{n} C_{op})$$

$$= C_{op} * n^2 + C_l * (n + 1)$$

Mathematical Analysis

- We just proved mathematically that:

  running time function $f_1(n) = C_{op} * n^2 + C_l * (n + 1)$

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- We just proved mathematically that:

  running time function $f_1(n) = C_{op} * n^2 + C_l * (n + 1)$

- Now lets prove that $f_1(n) \approx n^2$ or more formally $f_1(n) \in O(n^2)$

Motivation for Big-O Notation

- When we analyze an algorithm, our goal is to find a function **g(*n*)**, such that the running time of the algorithm is **proportional** to **g(*n*)**.

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Motivation for Big-O Notation

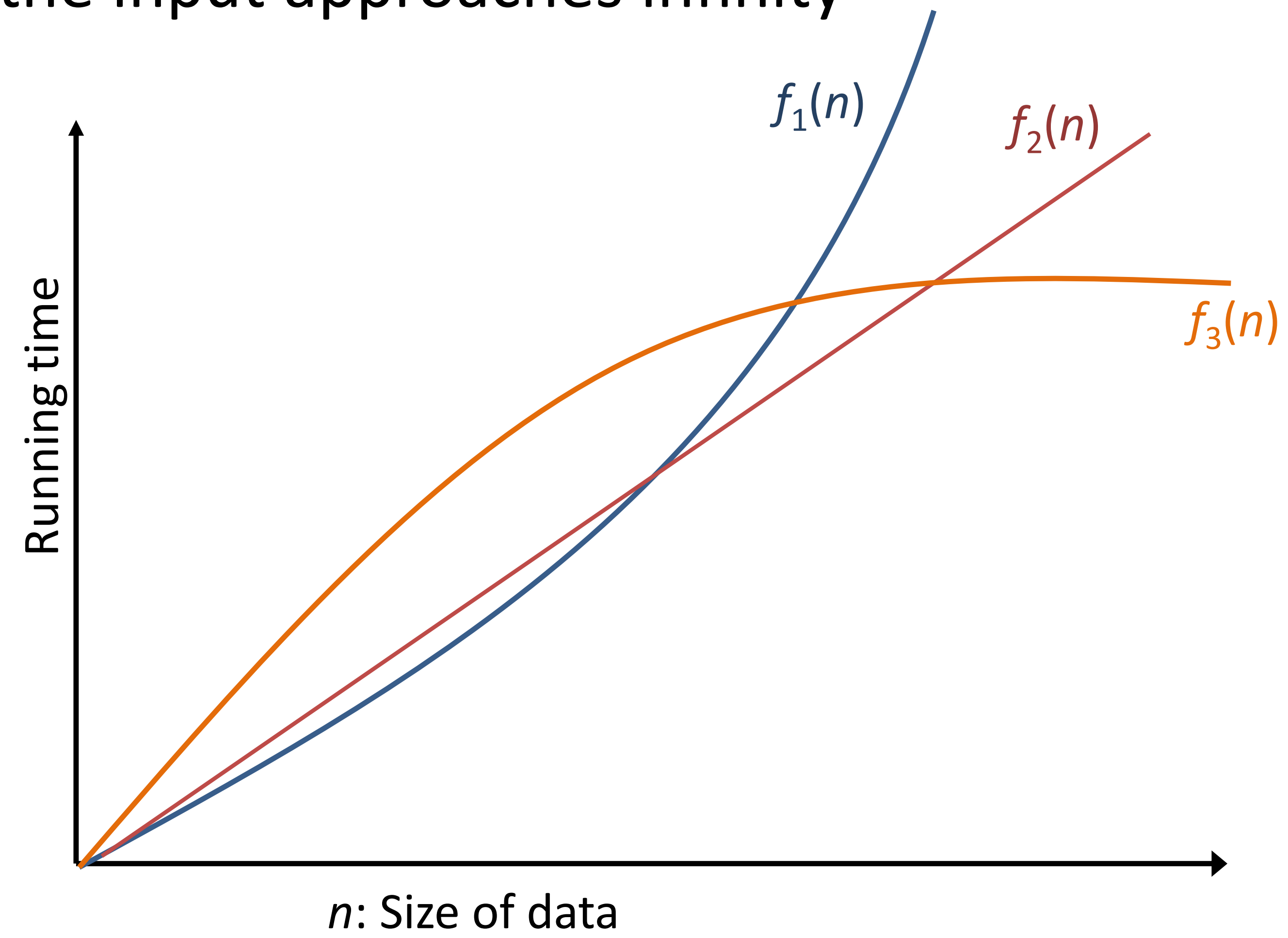
- When we analyze an algorithm, our goal is to find a function **g(*n*)**, such that the running time of the algorithm is **proportional** to **g(*n*)**.
- Why **proportional** and not **equal**?
- Because **the actual running time is not a defining characteristic of an algorithm**
    - Running time depends on programming language, actual implementation, compiler used, machine executing the code, …

Motivation for Big-O Notation

- There are some details that we would actually NOT want **g(*n*)**to include, because they can make a function unnecessarily complicated.
  - Constants, are not important!
  - Behavior fluctuations on small data.
- The Big-Oh notation, which we will see in a few slides, achieves that, and greatly simplifies algorithmic analysis.

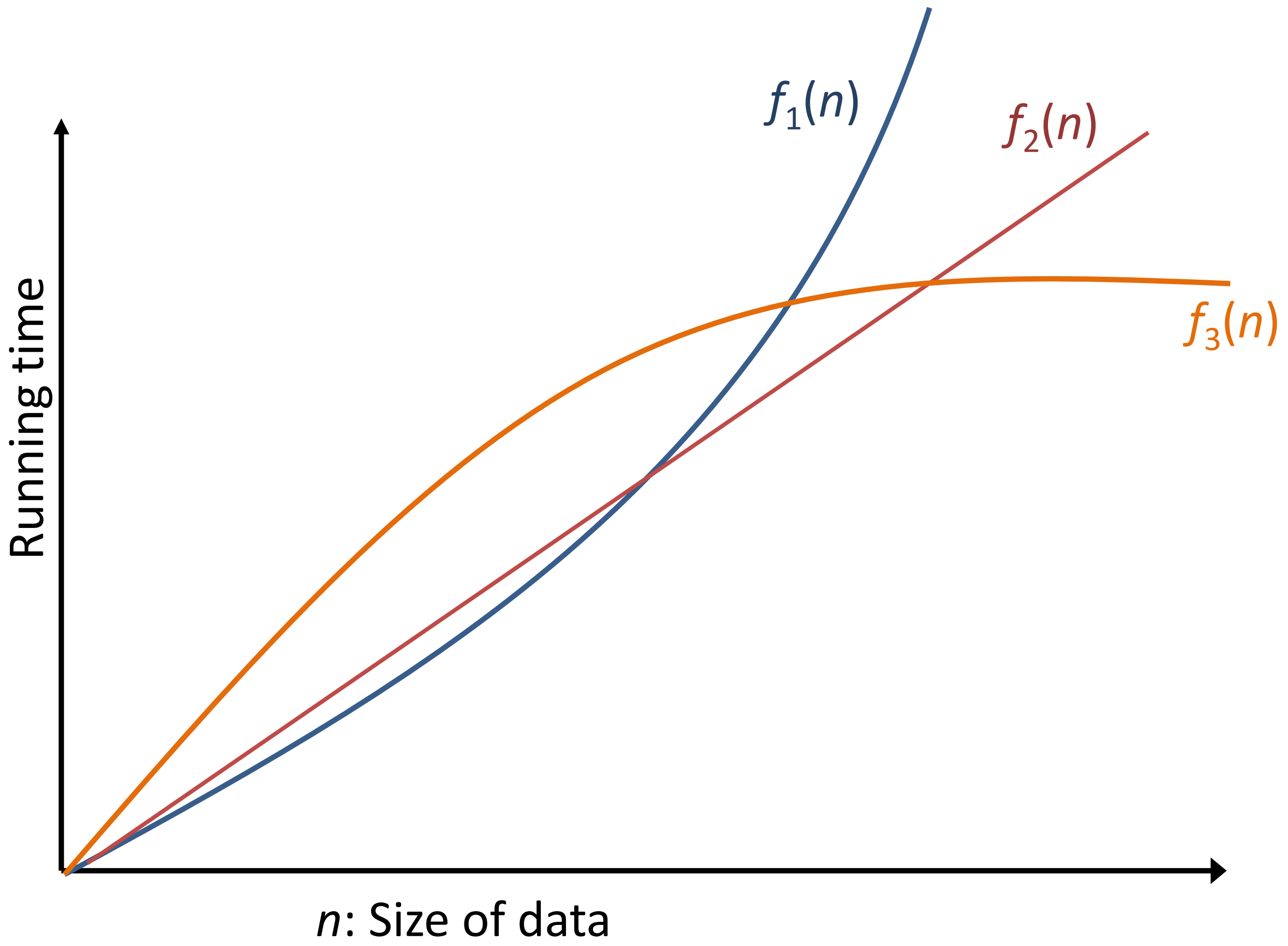
Why Asymptotic Behavior Matters

- Asymptotic behavior: The behavior of a function as the input approaches infinity



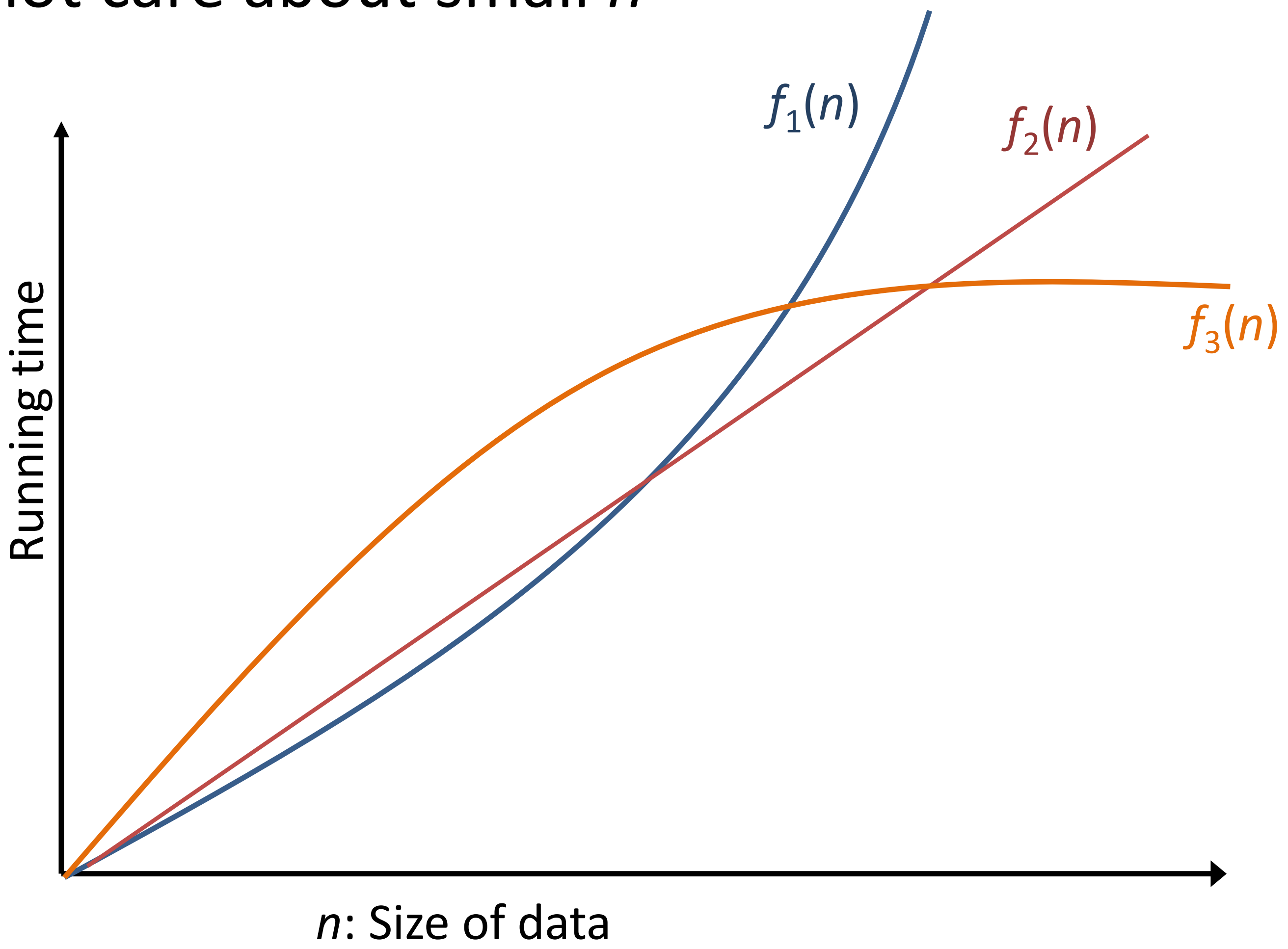
Why Asymptotic Behavior Matters

- Which of these functions works best asymptotically?



Why Asymptotic Behavior Matters

- f_3 seems to grow VERY slowly after a while. We do not care about small n



Definition of Big-O

- A function *f*(*n*) is said to be in O(*g(n)*) if there exist constants *c* and n_0 such that:

$$f(n) \leq c * g(n),\ \ for\ all\ n \geq n_0$$

Definition of Big-O

- A function $f(n)$ is said to be in $O(g(n))$ if there exist constants c and n_0 such that:

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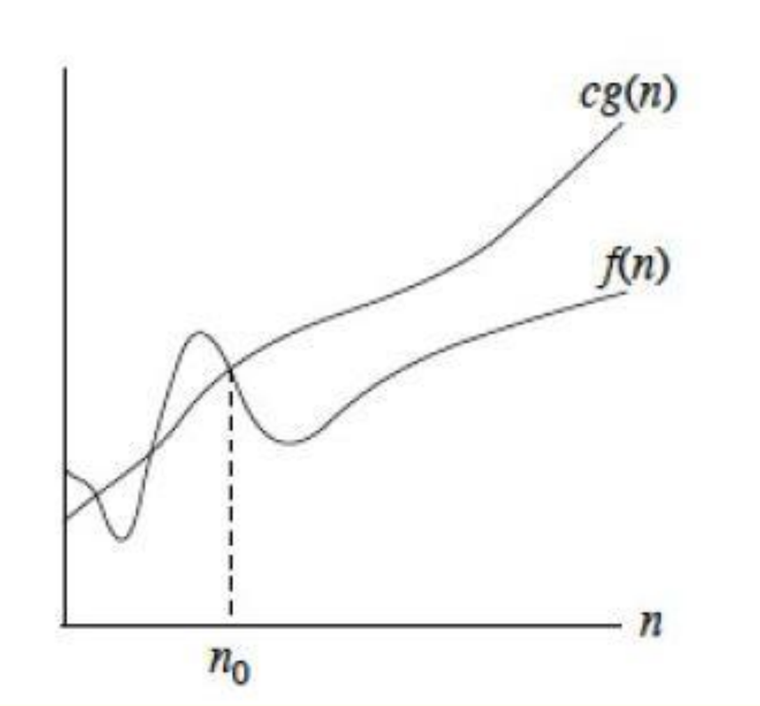

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- Typically, *f(n)* is the running time of an algorithm, in your favorite units, implementation, and machine. This can be a rather complicated function.
- In algorithmic analysis, we try to find a *g(n)* that is simple, and such that $f(n) \in O(g(n))$

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$$f(n) \leq c * g(n), \; for \; all \; n \geq n_0$$

- The Big-Oh notation greatly simplifies the analysis task, by:
  1. Ignoring constant factors. How is this achieved?
     - By the *C* in the definition. We are free to choose ANY constant *C* we want, to make the formula work.
     - Thus, Big-Oh notation is independent of programming language, compiler, machine performance, and so on…

Definition of Big-O

$$f(n) \leq c * g(n),\ for\ all\ n \geq n_0$$

- The Big-Oh notation greatly simplifies the analysis task, by:

2. Ignoring behavior for small inputs. How is this achieved?
   - By the N_0 in the implementation. If a finite number of values are not compatible with the formula, just ignore them.
   - Thus, big-Oh notation focuses on asymptotic behavior.

Definition of Big-O

$$f(n) \leq c * g(n), \; for\ all\ n \geq n_0$$

- The Big-Oh notation greatly simplifies the analysis task, by:
  3. Allowing us to describe complex running time behaviors of complex algorithms with simple functions, such as N, log N, N^2, 2^N, and so on.
     - Such simple functions are sufficient for answering many important questions, **once you get used to Big-Oh notation**.

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  - Example 1: if we know that the algorithm will only be applied to relatively small N, we may prefer a running time of N^2 over $N\log(N)$.
  - Example 2: even constant factors can be important. For many applications, we strongly prefer a running time of $3N$ over $1500N$.

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  - Example 2: even constant factors can be important. For many applications, we strongly prefer a running time of $3N$ over $1500N$.
- Big-Oh notation is not meant to tells us everything about running time.
- But, Big-Oh notation tells us a lot, and is often much easier to compute than actual running times.

Using Big-O

- We have mathematically proven that:

$$f_1(n) = C_{op} * n^2 + C_l * (n + 1)$$

- How we prove that $f_1(n) \in O(n^2)$?

Using Big-O

- We have mathematically proven that:

$$f_1(n) = C_{op} * n^2 + C_l * (n + 1)$$

- How we prove that $f_1(n) \in O(n^2)$?
- This is where the Big-Oh definition comes into play. We can find an n_0 such that, for all $n > n_0$:

$$f_{1(n)} \leq (C_{op} + 1) * n^2$$

- If you don't believe this, do the calculations for practice

Using Big-O

- Another way to show correctness: as *n* goes to infinity, what is the limit of $f(n) / n^2$?
  - $C_{op} > 0$
  - This shows that the non-quadratic terms become negligible as *n* gets larger

Using Big-O

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  - $C_{op} > 0$
  - This shows that the non-quadratic terms become negligible as *n* gets larger

Let f(n) and g(n) be two real function then, $f(n) = O(g(n))$ is equivalent to

$$\exists c \in \mathbb{R} : \lim_{n \to \infty} \frac{f(n)}{g(n)} = c$$

where, $c \geqslant 0$

(*Here c is finite or* ***can be zero***)

Properties of Big Oh:

- For all constants c>0 and a>1, and monotonically increasing functions f(n):

$$f(n)^c \text{ is } O(a^{f(n)})$$

- For example:
  - n^c is $O(a^n)$
  - n^{256} is $O(1.0001^n)$
  - $(\log_a n)^c$ is $O(a^{\log_a n})$, *which* is $O(n)$.

More Properties of Big Oh:

- O notation is additive and multiplicative:

  If $f(n)$ is $O(s(n))$ and $g(n)$ is $O(t(n))$, then:

  - $f(n) + g(n)$ is $O(s(n) + t(n))$
  - $f(n)g(n)$ is $O(s(n)t(n))$.

- O notation is transitive:

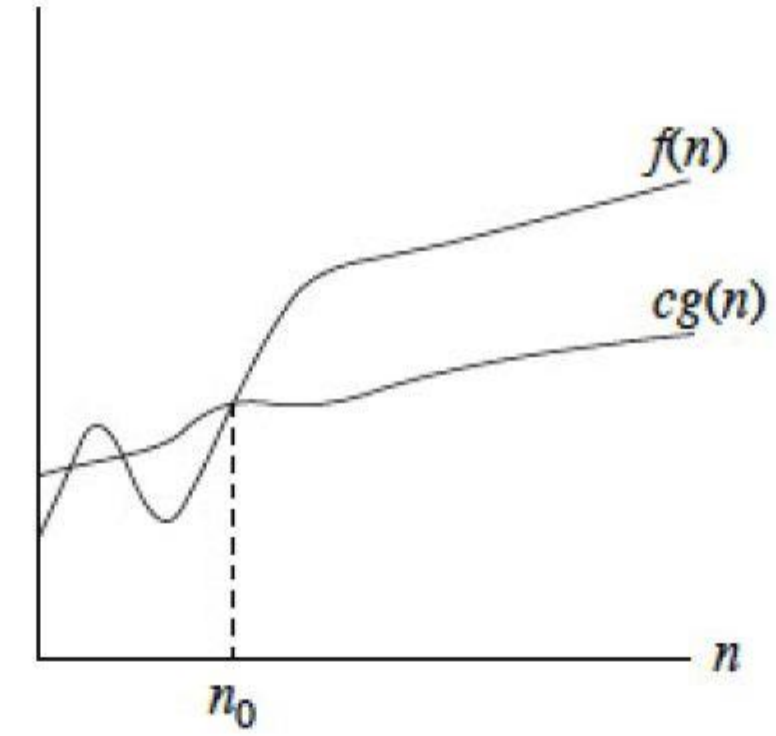
  If $f(n)$ is $O(g(n))$, and $g(n)$ is $O(h(n))$, then

  $f(n)$ is $O(h(n))$.

Omega (Ω) Notation

$$f(n) \in \Omega(n)$$

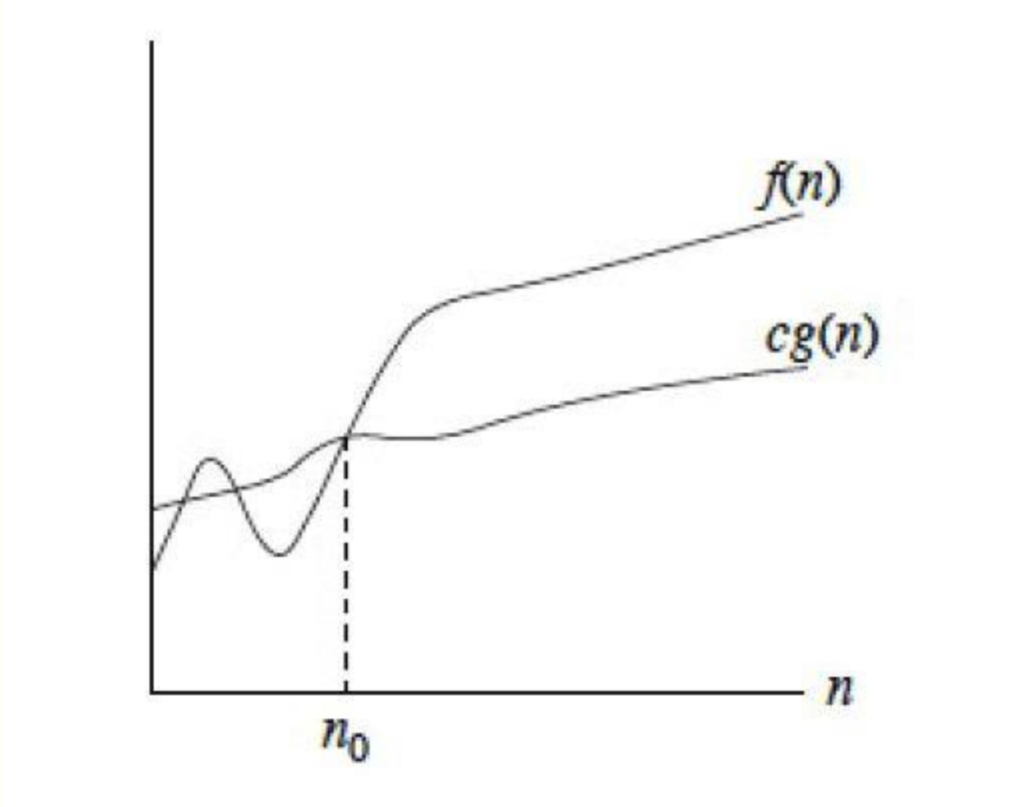
If *f(n)* is bounded below by some constant multiple of *g(n)* for all large *n*



Omega example

- $n*(n\text{-}1)/2 \in \Omega(n^2)$

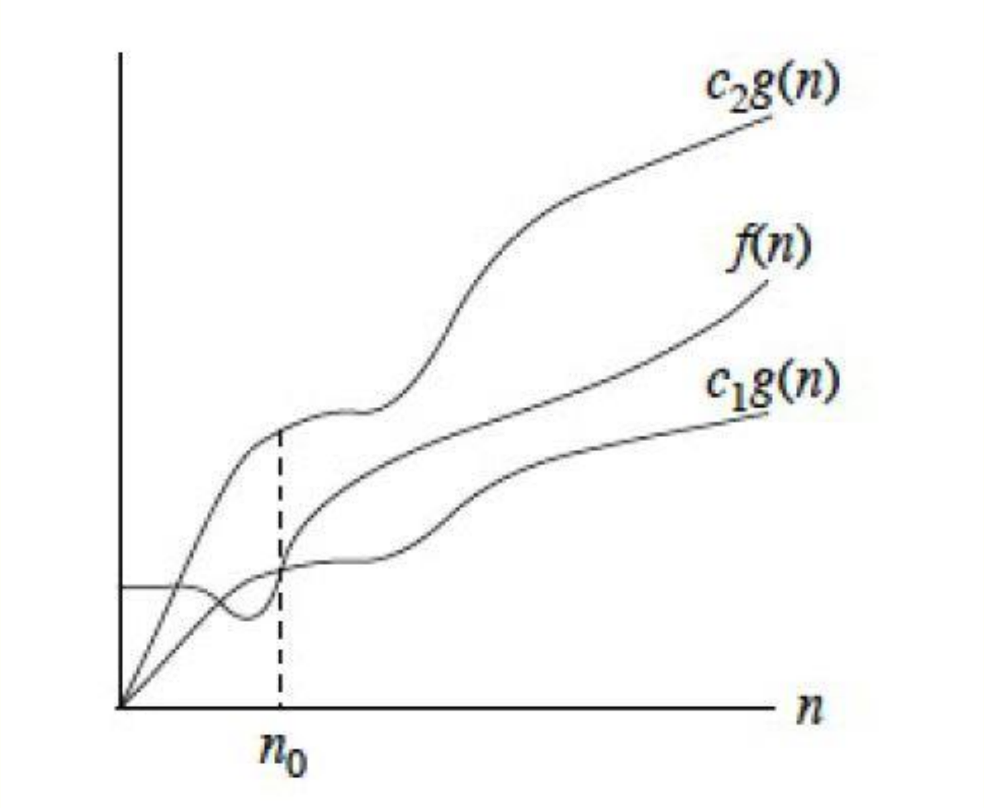
- $f(n) \geq c * g(n),\ for\ all\ n \geq n_0$



Theta (Θ) Notation

$$f(n) \in \Theta(n)$$

If *f(n)* is bounded both above and below by some constant multiple of *g(n)* for all large *n*



Omega (Ω) and Theta (Θ) Notations

- If $f(n) \in O(g(n))$, then we also say that $g(n) \in \Omega(f(n))$.

- If $f(n) \in O(g(n))$ **and** $f(n) \in \Omega(g(n))$, then we say that $f(n) \in \Theta(g(n))$.
- The Theta notation is clearly stricter than the Big-Oh notation:
  - We can say that $n^2 \in O(n^{100})$.
  - We **cannot** say that $n^2 \in \Theta(n^{100})$.
  - We can say that $2^{n+1} \in \Theta(2^n)$

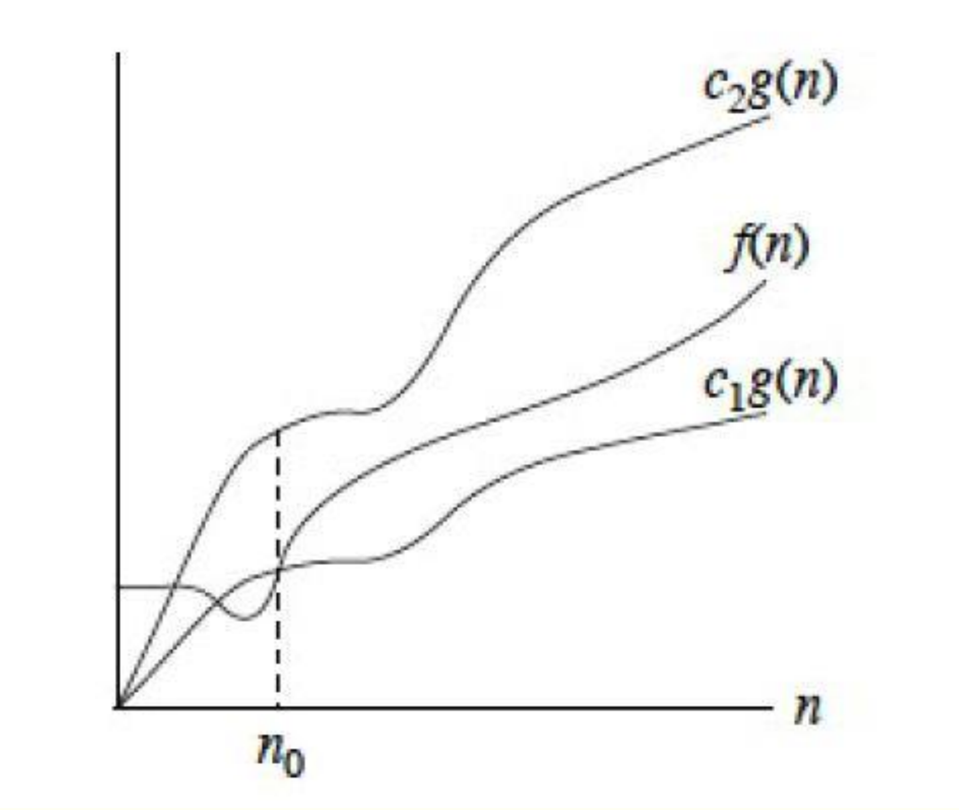
Theta example

- $2^{n+1} \in \Theta(2^n)$

- $f(n) \geq c_1 * g(n),\ for\ all\ n \geq n_0$

AND

- $f(n) \leq c_2 * g(n),\ for\ all\ n \geq n_0$



Using Limits for Comparing Order of Growth

- $\lim_{n\to\infty} \frac{t(n)}{g(n)} = 0$, implies that *t(n)* has a smaller order of growth than *g(n)*
- $\lim_{n\to\infty} \frac{t(n)}{g(n)} = c, c > 0$, implies that *t(n)* has the same order of growth as *g(n)*
- $\lim_{n\to\infty} \frac{t(n)}{g(n)} = \infty$, implies that *t(n)* has a larger order of growth than *g(n)*
- Note that the first two cases mean that: $t(n) \in O(g(n))$
- Note that the last two cases mean that: $t(n) \in \Omega(g(n))$
- Note that the second case means that: $t(n) \in \Theta(g(n))$

Using Limits - Comments

- The previous formulas relating limits to big-Oh notation show once again that big-Oh notation ignores:
  - constants
  - behavior for small values of N.
- How do we see that?
  - In the previous formulas, it is sufficient that the limit is equal to a constant. **The value of the constant does not matter.**
  - In the previous formulas, only **the limit at infinity** matters. This means that we can ignore behavior up to any finite value, if we need to.