3.3 NORMAL RANDOM VARIABLES

A continuous random variable X is said to be **normal** or **Gaussian** if it has a PDF of the form (see Fig. 3.9)

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2},$$

where μ and σ are two scalar parameters characterizing the PDF, with σ assumed positive. It can be verified that the normalization property

$$\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\,dx = 1$$

holds (see the end-of-chapter problems).

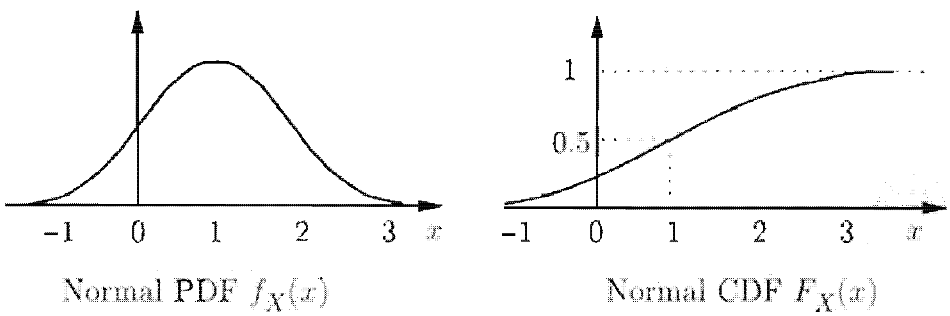

Figure 3.9: A normal PDF and CDF, with $\mu = 1$ and $\sigma^2 = 1$. We observe that the PDF is symmetric around its mean μ, and has a characteristic bell shape. As x gets further from μ, the term $e^{-(x-\mu)^2/2\sigma^2}$ decreases very rapidly. In this figure, the PDF is very close to zero outside the interval $[-1, 3]$.

The mean and the variance can be calculated to be

$$\mathbf{E}[X] = \mu, \qquad \text{var}(X) = \sigma^2.$$

To see this, note that the PDF is symmetric around μ, so the mean can only be μ. Furthermore, the variance is given by

$$\text{var}(X) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2}\,dx.$$

Using the change of variables $y = (x-\mu)/\sigma$ and integration by parts, we have

$$\begin{aligned}
\text{var}(X) &= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-y^2/2}\,dy \\
&= \frac{\sigma^2}{\sqrt{2\pi}} \left(-y e^{-y^2/2}\right)\Big|_{-\infty}^{\infty} + \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\,dy \\
&= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\,dy \\
&= \sigma^2.
\end{aligned}$$

The last equality above is obtained by using the fact

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy = 1,$$

which is just the normalization property of the normal PDF for the case where $\mu = 0$ and $\sigma = 1$.

A normal random variable has several special properties. The following one is particularly important and will be justified in Section 4.1.

Normality is Preserved by Linear Transformations

If X is a normal random variable with mean μ and variance σ^2, and if $a \neq 0$, b are scalars, then the random variable

$$Y = aX + b$$

is also normal, with mean and variance

$$\mathbf{E}[Y] = a\mu + b, \qquad \mathrm{var}(Y) = a^2\sigma^2.$$

The Standard Normal Random Variable

A normal random variable Y with zero mean and unit variance is said to be a **standard normal**. Its CDF is denoted by Φ:

$$\Phi(y) = \mathbf{P}(Y \leq y) = \mathbf{P}(Y < y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2}\, dt.$$

It is recorded in a table (given in the next page), and is a very useful tool for calculating various probabilities involving normal random variables; see also Fig. 3.10.

Note that the table only provides the values of $\Phi(y)$ for $y \geq 0$, because the omitted values can be found using the symmetry of the PDF. For example, if Y is a standard normal random variable, we have

$$\begin{aligned}\Phi(-0.5) = \mathbf{P}(Y \leq -0.5) &= \mathbf{P}(Y \geq 0.5) = 1 - \mathbf{P}(Y < 0.5)\\ &= 1 - \Phi(0.5) = 1 - .6915 = 0.3085.\end{aligned}$$

More generally, we have

$$\Phi(-y) = 1 - \Phi(y), \qquad \text{for all } y.$$